4.16 (5)

Supongamos que $\mathbb{L} = \mathbb{L}_- \mathbb{L}_+$, con $[\mathbb{L}_-, \mathbb{L}_+] = \mathbb{I}$, demostrar que:

Si $\mathbb{L}|x\rangle = \lambda|x\rangle$ y $|y\rangle = \mathbb{L}_+|x\rangle$ entonces:

$$\mathbb{L}|y\rangle = (\lambda+1)|y\rangle$$

En efecto, tenemos:

$$\mathbb{L}|x\rangle = \lambda|x\rangle \Rightarrow \mathbb{L}|x\rangle = (\mathbb{L}_-\mathbb{L}_+)\lambda|x\rangle$$

$$\Rightarrow \mathbb{L}|x\rangle = \lambda(\mathbb{L}_-\mathbb{L}_+)|x\rangle$$

$$\Rightarrow \mathbb{L}|x\rangle = \lambda\mathbb{L}_-(\mathbb{L}_+|x\rangle) = \lambda\mathbb{L}_-|y\rangle$$

Entonces: $[\mathbb{L}_-, \mathbb{L}_+] = \mathbb{I} = \mathbb{L}_-\mathbb{L}_+ - \mathbb{L}_+\mathbb{L}_- = \mathbb{I}$

$$\mathbb{L}_-\mathbb{L}_+ = \mathbb{I} + \mathbb{L}_+\mathbb{L}_-$$

$$\mathbb{L}|y\rangle = (\mathbb{L}_-\mathbb{L}_+)|y\rangle = (\mathbb{L}_-\mathbb{L}_+)(\mathbb{L}_+|x\rangle) = (\mathbb{I} + \mathbb{L}_+\mathbb{L}_-)\mathbb{L}_+|x\rangle$$

$$= (\mathbb{I}\mathbb{L}_+ + \mathbb{L}_+\mathbb{L}_-\mathbb{L}_+)|x\rangle = \mathbb{I}\mathbb{L}_+|x\rangle + \mathbb{L}_+\mathbb{L}_-\mathbb{L}_+|x\rangle$$

$$= \mathbb{I}|y\rangle + \mathbb{L}_+(\mathbb{L}_-\mathbb{L}_+)|x\rangle = \mathbb{I}|y\rangle + \mathbb{L}_+\mathbb{L}|x\rangle$$

$$= \mathbb{I}|y\rangle + \mathbb{L}_+(\lambda|x\rangle) = \mathbb{I}|y\rangle + \lambda\mathbb{L}_+|x\rangle$$

$$= |y\rangle + \lambda|y\rangle = (\lambda+1)|y\rangle$$

Y de mismo modo demostrar que:

si $\mathbb{L}|x\rangle = \lambda|x\rangle$ y $|z\rangle = \mathbb{L}_-|x\rangle$ entonces:

$$\mathbb{L}|z\rangle = (\lambda - 1)|z\rangle$$

En efecto: $\mathbb{L}_-\mathbb{L}_+ - \mathbb{L}_+\mathbb{L}_- = \mathbb{I} \Rightarrow \mathbb{L}_+\mathbb{L}_- = \mathbb{L}_-\mathbb{L}_+ - \mathbb{I}$

$$\mathbb{L}|z\rangle = \mathbb{L}_-\mathbb{L}_+|z\rangle = \mathbb{L}_-\mathbb{L}_+(\mathbb{L}_-|x\rangle) = \mathbb{L}_-(\mathbb{L}_+\mathbb{L}_-)|x\rangle$$

$$= \mathbb{L}_-(\mathbb{L}_-\mathbb{L}_+ - \mathbb{I})|x\rangle$$

$$= (\mathbb{L}_-\mathbb{L}_-\mathbb{L}_+ - \mathbb{L}_-\mathbb{I})|x\rangle$$

$$= \mathbb{L}_-\mathbb{L}_-\mathbb{L}_+|x\rangle - \mathbb{L}_-\mathbb{I}|x\rangle \quad , \quad \mathbb{I}|x\rangle = |x\rangle$$

$$= \mathbb{L}_-\mathbb{L}|x\rangle - \mathbb{L}_-|x\rangle$$

$$= \mathbb{L}_-\lambda|x\rangle - \mathbb{L}_-|x\rangle = \lambda\mathbb{L}_-|x\rangle - \mathbb{L}_-|x\rangle$$

$$= \lambda|z\rangle - |z\rangle = (\lambda - 1)|z\rangle$$