Differentially Private ANOVA

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1 The ANOVA Framework

The setting is that we have a dataset of k groups, each populated with an arbitrary number of individuals. Let n be the total number of individuals. The goal of ANOVA is to test if there is a statistical difference between the means of these groups. This test is ubiquitous in the social sciences, as well as in biology. Our goal is to develop a framework for executing ANOVA tests in a differentially private manner, as the data being evaluated is often sensitive. To that end, we will define the specific terms of an ANOVA.

Let $\mathcal{D}$ be our database. Let $\{\mathcal{D}_i \ : \ 1 \leq i \leq k\}$, be a partition of $\mathcal{D}$ such that $\bigcup_{i=1}^{k} \mathcal{D}_i = \mathcal{D}$. We will denote entries y_{ij}, which says that this is the jth entry in the ith group. Let $x_i = |D_i|$, $y_{ij} \in [0,1]$, $\overline{y_i}$ the mean of D_i, and $\overline{y}$ the mean of all $\overline{y_i}$ (the mean of means). We will not assume that every group has the same size. There are three important quantities in the ANOVA framework:

$$\text{SST} = \sum_{i=1}^{k} \sum_{j=1}^{x_i} (y_{ij} - \overline{y})^2;$$

$$\text{SSA} = \sum_{i=1}^{k} x_i (\overline{y_i} - \overline{y})^2;$$

$$\text{SSE} = \sum_{i=1}^{k} \sum_{j=1}^{x_i} (y_{ij} - \overline{y_i})^2.$$

These are the sum of squares total, treatment, and error, respectively. We further define two more terms:

$$\text{MSA} := \frac{1}{k-1} \sum_{i=1}^{k} \sum_{j=1}^{x_i} (\overline{y_i} - \overline{y})^2 = \frac{SSA}{k-1},$$

and

$$\text{MSE} := \frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{x_i}(y_{ij} - \overline{y}_i)^2 = \frac{SSE}{n-k}.$$

These are the mean squared treatment and mean squared error, respectively. We use these to calculate the F-ratio, $F = \frac{\text{MSA}}{\text{MSE}}$. This gives us a p-value, based on the F-distribution.
As a first attempt at making this framework differentially private, we will support releasing noisy versions of $\overline{y}$ and $\overline{y_i}$. We will also support noisy versions of the following queries:

$$d : \mathcal{D} \times [0, 1] \to \mathbb{R}$$

given by

$$d(\mathcal{D}, c) := \sum_{i=1}^{k}\sum_{j=1}^{x_i}(y_{ij} - c)^2,$$

and

$$g : \mathcal{D} \times [0, 1] \to \mathbb{R}$$

given by

$$g(\mathcal{D}, c) := \sum_{i=1}^{k} x_i(\overline{y_i} - c)^2.$$

In this way, if the data analyst chooses c as the noisy $\overline{y}$ or $\overline{y_i}$, we support noisy versions of SST, SSA, and SSE.

2 Naive Approach

Our first attempt will be to do a straightforward worst-case analysis of the sensitivity of these various parts of ANOVA, and add the corresponding Laplacian noise.

2.1 Releasing noisy $\overline{y}$

Define the mean query,

$$m : \mathcal{D} \to \mathbb{R}$$

as

$$m(\mathcal{D}) = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij}.$$

We will analyze the global sensitivity of m. In the worst case there is one group, say $\mathcal{D}_k$, with one member whose value is zero. This means there exists a neighboring database $\mathcal{D}'$ where this member is in a different group, say $\mathcal{D}_{k-1}$, with value 1. So we get the following:

$$\begin{aligned}
\Delta m &= \max_{\mathcal{D},\mathcal{D}' \text{ neighbors}} \left|\left| m(\mathcal{D}) - m(\mathcal{D}^{'}) \right|\right|_1 \\
&= \left| \frac{1}{k}\sum_{i=1}^{k}\left(\frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij}\right) - \frac{1}{k-1}\sum_{i=1}^{k-1}\left(\frac{1}{x_i^{'}}\sum_{j=1}^{x_i^{'}} y_{ij}^{'}\right) \right| \\
&= \left| \frac{1}{k}\left(\sum_{i=1}^{k-2}\frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij} + \overline{y}_{k-1}\right) - \frac{1}{k-1}\left(\sum_{i=1}^{k-2}\frac{1}{x_i^{'}}\sum_{j=1}^{x_i^{'}} y_{ij}^{'} + \overline{y}_{k-1}^{'}\right) \right| \\
&= \left| \frac{1}{k}\overline{y}_{k-1} - \frac{1}{k-1}\overline{y}_{k-1}^{'} + \frac{1}{k}\sum_{i=1}^{k-2}\frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij} - \frac{1}{k-1}\sum_{i=1}^{k-2}\frac{1}{x_i^{'}}\sum_{j=1}^{x_i^{'}} y_{ij}^{'} \right| \\
&= \left| \frac{1}{k}\overline{y}_{k-1} - \frac{1}{k-1}\overline{y}_{k-1}^{'} + \sum_{i=1}^{k-2}\frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij}\left(\frac{1}{k} - \frac{1}{k-1}\right) \right| \\
&= \left| \frac{1}{k}\overline{y}_{k-1} - \frac{1}{k-1}\overline{y}_{k-1}^{'} - \frac{1}{k(k-1)}\sum_{i=1}^{k-2}\frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij} \right| \\
&\leq \left| -\frac{1}{k-1} - \frac{1}{k(k-1)}\sum_{i=1}^{k-2}\frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij} \right| \\
&\leq \left| -\frac{1}{k-1} - \frac{1}{k-1} \right| \\
&= \left| \frac{-2}{k-1} \right| \\
&= \frac{2}{k-1} \text{ for } k > 1.
\end{aligned}$$

We can now use the Laplace Mechanism of Dwork[citation needed] to create a differentially private algorithm for releasing $\overline{y}$.

Algorithm 1

Input: Database $\mathcal{D} = \{\mathcal{D}_i \mid 1 \leq i \leq k\}$
Output: Noisy $\overline{y}$
for $i = 1$ to k **do**
  Compute $\overline{y}_i$
end for
Compute $\overline{y} = \frac{1}{k}\sum_{i=1}^{k} \overline{y}_i$
Compute $x = \overline{y} + Y$ where $Y \sim \text{Lap}(\frac{\frac{2}{k-1}}{\epsilon})$
return x

Theorem 1. ***Algorithm 1*** *preserves* $(\epsilon, 0)$*-differential privacy.*

2.2 Releasing noisy $\overline{y}_i$

We will now calculate the sensitivity of $\overline{y}_i$. Model this as a database query

$$m_i : \mathcal{D} \to \mathbb{R}$$

$$m_i(\mathcal{D}) = \frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij}.$$

In the worst case $\mathcal{D}_i$ has a member with value 1. This means there exists a neighboring database with $\mathcal{D}_i^{'}$ that is the same as $\mathcal{D}_i$ except that that one member now has value 0. This gives us the following sensitivity bound.

$$\begin{aligned}
\Delta m_i &= \max_{\mathcal{D},\mathcal{D}^{'} \text{ neighbors}} \left|\left| m_i(\mathcal{D}) - m_i(\mathcal{D}^{'}) \right|\right|_1 \\
&= \left| \frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij} - \frac{1}{x_i}\sum_{j=1}^{x_i} y_{ij}^{'} \right| \\
&= \left| \frac{1}{x_i}\left(\sum_{j=1}^{x_i-1} y_{ij} + 1\right) - \frac{1}{x_i}\left(\sum_{j=1}^{x_i-1} y_{ij} + 0\right) \right| \\
&= \frac{1}{x_i}.
\end{aligned}$$

So again, we can add the corresponding noise given by $\text{Lap}(\frac{\frac{1}{x_i}}{\epsilon})$.

2.3 Releasing noisy d

We now analyze the sensitivity of the query $d : \mathcal{D} \times [0,1] \to \mathbb{R}$ given by $d(\mathcal{D}, c) = \sum_{i=1}^{k} \sum_{j=1}^{x_i} (y_{ij} - c)^2$. In the worst case one person's value changes from 0 in $\mathcal{D}$ to 1 in a neighboring database $\mathcal{D}^{'}$

$$
\begin{aligned}
\Delta d &= \max_{\mathcal{D},\mathcal{D}^{'} \text{ neighbors}} \left|\left| d(\mathcal{D}) - d(\mathcal{D}^{'}) \right|\right|_1 \\
&= \left| \left(\sum_{i=1}^{k} \sum_{j=1}^{x_i} (y_{ij} - c)^2 \right) - \left(\sum_{i=1}^{k-1} \sum_{j=1}^{x_i^{'}} \left(y_{ij}^{'} - c \right)^2 \right) \right| \\
&= \left| \sum_{i=1}^{k-1} \sum_{j=1}^{x_i} (y_{ij} - c)^2 - \sum_{i=1}^{k-1} \sum_{j=1}^{x_i^{'}} \left(y_{ij}^{'} - c \right)^2 - c^2 \right| \\
&= \left| \left(\sum_{i=1}^{k-2} \sum_{j=1}^{x_i} (y_{ij} - c)^2 + \sum_{j=1}^{x_{k-1}} \left(y_{(k-1)j} - c \right)^2 \right) - \left(\sum_{i=1}^{k-2} \sum_{j=1}^{x_i^{'}} \left(y_{ij}^{'} - c \right)^2 + \sum_{j=1}^{x_{k-1}} \left(y_{(k-1)j}^{'} - c \right)^2 \right) - c^2 \right| \\
&= \left| \sum_{j=1}^{x_{k-1}} \left(y_{(k-1)j} - c \right)^2 - \sum_{j=1}^{x_{k-1}^{'}} \left(y_{(k-1)j}^{'} - c \right)^2 - c^2 \right| \\
&= \left| \sum_{j=1}^{x_{k-1}} \left(y_{(k-1)j} - c \right)^2 - \left(\sum_{j=1}^{x_{k-1}} \left(y_{(k-1)j} - c \right)^2 + (1 - c)^2 \right) - c^2 \right| \\
&= \left| -1 + 2c - 2c^2 \right| \\
&\leq 1 \text{ for } c \in [0,1].
\end{aligned}
$$

We can use this to get an algorithm for releasing d with added noise from $\text{Lap}(\frac{1}{\epsilon})$.

Algorithm 2

Input: Database $\mathcal{D} = \{\mathcal{D}_i \mid 1 \leq i \leq k\}$, constant $c \in [0, 1]$
Output: Noisy $d(\mathcal{D}, c)$
y = 0
for $i = 1$ to k **do**
  for $j = 1$ to x_i **do**
    Compute $y = y + (y_{ij} - c)^2$
  end for
end for
Compute $x = y + Y$ where $Y \sim \text{Lap}(\frac{1}{\epsilon})$
return x

Theorem 2. ***Algorithm 2*** *preserves* $(\epsilon, 0)$*-differential privacy.*

2.4 Releasing noisy g

We now analyze the sensitivity of g. In the worst case we have a group, say $\mathcal{D}_k$, with one member, whose value is 1. Then there exists a neighboring database $\mathcal{D}^{'}$ that is the same as $\mathcal{D}$, except that the member in $\mathcal{D}_k$ has moved to $\mathcal{D}_k^{'}$ and has value 0. This gives us the following bound.

$$
\begin{aligned}
\Delta g &= \max_{\mathcal{D},\mathcal{D}^{'} \text{ neighbors}} \left|\left| g(\mathcal{D}) - g(\mathcal{D}^{'}) \right|\right|_1 \\
&= \left| \sum_{i=1}^{k} \sum_{j=1}^{x_i} (\overline{y}_i - c)^2 - \sum_{i=1}^{k-1} \sum_{j=1}^{x_i^{'}} \left(\overline{y}_i^{'} - c\right)^2 \right| \\
&= \left| \sum_{i=1}^{x_{k-1}} \sum_{j=1}^{x_i} (\overline{y}_i - c)^2 + (1-c)^2 - \sum_{j=1}^{x_{k-1}} \left(\overline{y}_{k-1}^{'} - c\right)^2 \right| \\
&= \left| \sum_{j=1}^{x_{k-1}} \left(\overline{y}_{k-1} - c\right)^2 + (1-c)^2 - \sum_{j=1}^{x_{k-1}+1} \left(\overline{y}_{k-1}^{'} - c\right)^2 \right| \\
&= \left| x_{k-1} \left(\frac{1}{x_{k-1}} \sum_{j=1}^{x_{k-1}} y_{(k-1)j} \right)^2 - (x_{k-1}+1) \left(\frac{1}{x_{k-1}+1} \sum_{j=1}^{x_{k-1}} y_{(k-1)j} \right)^2 + (1-c)^2 \right| \\
&\leq \frac{1}{x_{k-1}+1} + 1.
\end{aligned}
$$

This gives us an algorithm for releasing g with noise added from $\text{Lap}\left(\frac{\frac{1}{x_{k-1}+1}+1}{\epsilon}\right)$.

Algorithm 3

Input: Database $\mathcal{D} = \{\mathcal{D}_i \mid 1 \leq i \leq k\}$, constant $c \in [0, 1]$
Output: Noisy $g(\mathcal{D}, c)$
y $= 0$
for $i = 1$ to k **do**
  for $j = 1$ to x_i **do**
    Compute $y = y + (\overline{y}_i - c)^2$
  end for
end for
Compute $x = y + Y$ where $Y \sim \text{Lap}\left(\frac{\frac{1}{x_{k-1}+1}+1}{\epsilon}\right)$
return x

2.5 Releasing the F-ratio

We can now use everything above to release the F-ratio. A user can first query for noisy $\overline{y}$, noisy $\overline{y}_i$s, then use these as input for the noisy d and g queries to get noisy SSE and SSA. We assume that k and n are public, which allows the analyst to compute MSA and MSE, the F-ratio, and thus the p-value.