Chapter 1

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1 Bipartite graphs and matchings

Throughout this thesis we will be interested in a specific subclass of graphs known as bipartite graphs. Unless otherwise noted, our algorithms will assume a bipartite structure.

Definition (Bipartite graph). A *bipartite graph* is a graph whose vertices can be partitioned into two set L and R such that all edges connect a vertex $l \in L$ to a vertex $r \in R$. We will denote this graph $G = (L, R, E)$, where E is the edge set.

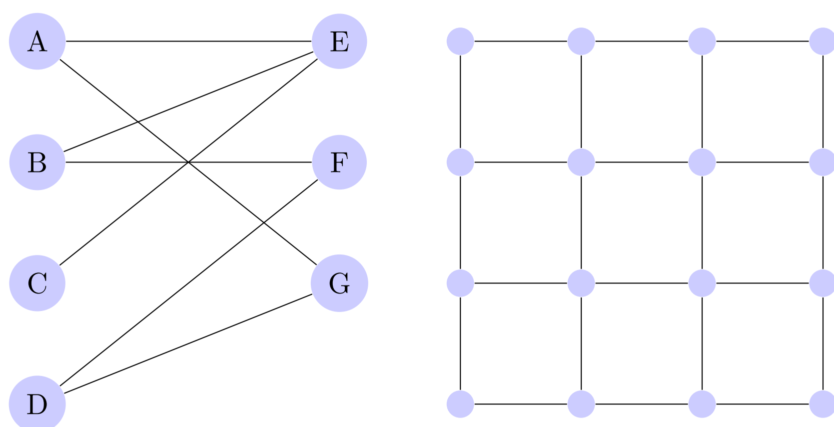


Figure 1: Examples of bipartite graphs

In Figure 1 we have a bipartite graph with vertex partition given by $L = \{A, B, C, D\}$ and $R = \{E, F, G\}$. All edges in this graph are between a node $l \in L$ and a node $r \in R$. The graph on the right is also bipartite. It may take a little more time to convince yourself that you can partition the vertices into disjoint L and R in a way that maintains the bipartite property. Try it!

Now that we know what we are working with, let's introduce a problem that we'd like to solve on these graphs. We now describe the notion of a matching on a graph. We will define it for arbitrary graphs,

Definition (Matching). Let $G = (V, E)$ be a graph. A subset $M \subset E$ is a *matching* if no two edges in M are incident to the same vertex.

We say that a vertex $v \in V$ is matched with respect to M if it is an endpoint of some edge in M. The following figure shows some examples of different matchings on one of the graphs from Figure 1.

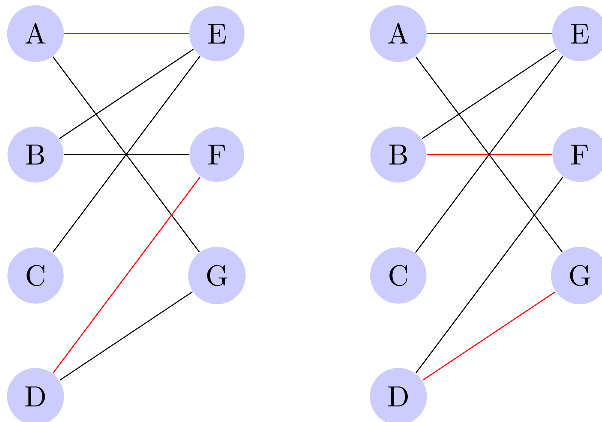


Figure 2: Examples of matchings on a bipartite graph

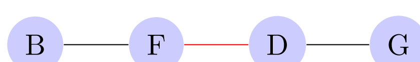
There many different valid matchings on the graph in Figure 2. Oftentimes, we want to find the largest matching on a graph. (ADD SENTENCE OR TWO ON MOTIVATION FOR MAXIMAL MATCHING.) This leads to the following definition.

Definition (Maximal matching). A *maximal matching* on G is a matching M such that if any other edge not in M is added to M, it is no longer a valid matching. Alternatively put, M is maximal if there is no matching $M^{'}$ such that $M \subset M^{'}$.

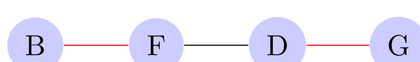
Both matchings in Figure 2 are maximal matchings; in each case there are no edges that we can add to M and have that M is still a matching. However, notice that the size of the matchings is different, even though both are maximal on G. This leads to the following definition.

Definition (Maximum matching). A matching M on a graph G is said to be a maximum matching if for all other matchings $M^{'}$ on G, $|M^{'}| \leq |M|$.

In our example, the matching on the right given by $M = \{(A, E), (B, F), (D, G)\}$ is a maximum matching (convince yourself). In general there may be many unique maximum matchings on a graph.
In this section we are interested in general methods for finding maximum matchings on bipartite graphs. One of the fundamental approaches is to look at certain subgraphs called alternating paths. Before we define what these are, let's look at a motivating example. Suppose we have the matching on the left in Figure 2. So $M = \{(A, E), (D, F)\}$. Consider the following sequence of vertices in the graph:



Call this sequence p. Let's perform an operation that we will denote $M \oplus p$, which operates like XOR: add to M each edge in p that isn't in M, and remove from M each edge in p that is in M. This gives us the following segment:



Notice that the size of our matching has grown by 1! In fact, this new matching is exactly the matching given by the graph on the right in Figure 2. This is a general technique in finding maximum matchings. We want to look for these paths that start and end at unmatched vertices, and whose edges are alternately matched and unmatched. If we can find one of these paths, we will be able to increase the size of matching. We define this formally now.

Definition (Alternating path). Let G be a graph and M some matching on G. An *alternating path* is a sequence of vertices and edges that begins with an unmatched vertex, and whose edges alternate between being in M and not in M.

Definition (Augmenting path). An *augmenting path* is an alternating path that starts and ends on unmatched vertices. When we augment M by an augmenting path p, we use the notation $M \oplus p$.