Bipartite Matching

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Definition (Bipartite graph). A graph $G = (V, E)$ is *bipartite* if the set of vertices V can be partitioned into two disjoint and independent sets A and B such that no edge in E has both endpoints in the same set of the partition.

Note that square-grid graphs are bipartite, but triangle-grid graphs are not.

Definition (Matching). A matching $M \subset E$ is a collection of edges such that every vertex of V is incident to at most one edge of M, i.e. a set of edges in G that do not share vertices.

1 Maximum cardinality matching problem

In this problem, G is unweighted and our goal is to find a matching M of maximum size. As a start, we want to prove optimality of a matching. To this end, we could find upper bounds on the size of any matching and hope that the smallest of these upper bounds is equal to the size of the largest matching. This is a duality concept that will prove useful.
A *vertex cover* is a set C of vertices such that all edges e of E are incident to at least on vertex of C. In other words, there is no edge completely contained in $V \setminus C$. Any matching M is at most the size of any vertex cover, since any vertex cover C must contain at least one of the endpoints of each edge in M. This shows *weak duality.* This leads to a theorem:

Theorem . For any bipartite graph G, the maximum size of a matching is equal to the minimum size of a vertex cover on G.

Definition (Alternating path). An alternating path with respect to a matching M is a path that alternates between edges in M and edges in $E \setminus M$.

Definition (Augmenting path). An augmenting path with respect to a matching M is an alternation path in which the first and last vertices are exposed.

A useful property of augmenting paths is the following: let P be an augmenting path with respefct to a matching M (so the edges in P alternate between M and $G \setminus M$), and set $M^{'} = (M \setminus P) \cup (P \setminus M)$, we get a matching $M^{'}$ with $|M^{'}| = |M| + 1$. We say that we have *augmented M along P*. This leads to a theorem:

Theorem . A matching M is maximum if and only if there are no augmenting paths with respect to M.

Proof. Let P be an augmenting path w.r.t. a matching M. Let $M^{'} = (M \setminus P) \cup (P \setminus M)$. Then $M^{'}$ is a matching with size greater than M. This contradicts the maximality of M.

Now suppose M is not maximum. Let $M^{'}$ be a maximum matching ($|M^{'}| > |M|$). Let $Q = (M \setminus M^{'}) \cup (M^{'} \setminus M)$. Then:

- Q has more edges from $M^{'}$ than from M, since $|M^{'}| > |M|$ means that $|M^{'} \setminus M| > |M \setminus M^{'}|$.
- Each vertex is incident to at most one edge in $M \cap Q$ and one edge in $M^{'} \cap Q$.
- So Q is composed of cycles and paths that alternate between edges from M and M. This path is an augmenting path w.r.t. M.

Hence there must exist an augmenting path P w.r.t M, which is a contradiction. □

This motivates the following algorithm:

Algorithm 1 Maximum unweighted matching

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Input: G = (V, E)
M = ∅
while there exists an augmenting path P w.r.t. M do
   augment M along P to get a matching M'
   set M = M'
end while
return M
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