Things that we need to do

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1 ϵ calculation in primal dual method

Given the two linear programs

$$\begin{aligned} &\text{minimize } \mathbf{c}^T\mathbf{x} && (1)\\ &\text{subject to } A\mathbf{x} \geq \mathbf{b} && (2)\\ &\mathbf{x} \geq 0 && (3) \end{aligned}$$

$$\begin{aligned} &\text{maximize } \mathbf{b}^T\mathbf{y} && (4)\\ &\text{subject to } A^T\mathbf{y} \leq \mathbf{c} && (5)\\ &\mathbf{y} \geq 0. && (6) \end{aligned}$$

and the corresponding restricted primal and dual

$$\begin{aligned} &\text{minimize } \sum_{i \notin I} s_i + \sum_{j \notin J} x_j && (7)\\ &\text{subject to } \quad A_i\mathbf{x} \geq b_i \quad i \in I, && (8)\\ &A_i\mathbf{x} - b_i = s_i \quad i \notin I, && (9)\\ &\mathbf{x} \geq 0, && (10)\\ &\mathbf{s} \geq 0. && (11) \end{aligned}$$

$$\begin{aligned} &\text{maximize } \mathbf{b}^T\mathbf{y}^{'} && (12)\\ &\text{subject to } A^j\mathbf{y}^{'} \leq 0 \quad j \in J, && (13)\\ &A^j\mathbf{y}^{'} \leq 1 \quad j \notin J, && (14)\\ &y_i^{'} \geq -1 \quad i \notin I, && (15)\\ &y_i^{'} \geq 0 \quad i \in I. && (16) \end{aligned}$$

Recall that we define the sets $I = \{i | y_i = 0\}$ and $J = \{j | A_j^T y = c_j\}$. Goemans and Williamson claim that the dual solution $\mathbf{y}^{''} = \mathbf{y} + \epsilon\mathbf{y}^{'}$ is an improved, dual feasible solution when ϵ is given as follows.

- Their first claim: by definition of I, $\mathbf{y}^{''} \geq 0$ if $\epsilon \leq \min_{i \notin I : y_i^{'} < 0}(-y_i / y_i^{'})$. I'm not quite sure how they make this conclusion. Note that $y^{''} \geq 0$ is $y_i + \epsilon y_i^{'} \geq 0$. Now, there are two cases, either

$y_i^{'} \geq 0$, in which case we have $\epsilon \geq (-y_j/y_j^{'})$, which is negative. In the other case, we have that $y_i^{'} < 0$, so we have $y_i - \epsilon y_i^{'} \geq 0$, which gives us $\epsilon \leq (-y_j/y_j^{'})$. Why do we minimize ϵ over the latter, and not the former?

- Their second claim: by definition of J, $A^T y^{''} \leq c$ if $\epsilon \leq \min_{j \notin J : A_j^T y' > 0} (c_j - A_j^T y)/(A_j^T y^{'})$. It's again a straightforward calculation to get the quantity $(c_j - A_j^T y)/(A_j^T y^{'})$, but I do not understand why they are only taking the minimum over $j \notin J : A_j^T y^{'} > 0$.

I need help understanding these details in this calculation in order to perform a similar calculation for our modified restricted programs.

2 Our altered primal-dual problem

In our case for maximum weight matchings, our primal problem is a maximization problem, and the dual is a minimization problem. My candidate restricted programs for this are given as follows

$$\begin{aligned}
&\text{minimize} \sum_{i \in L} s_i + \sum_{j \in R} s_j && (17)\\
&\text{subject to} \quad \sum_j x_{ij} - s_i = 1 \qquad \forall i, && (18)\\
&\qquad \sum_i x_{ij} - s_j = 1 \qquad \forall j, && (19)\\
&\qquad x_{ij} = 0 \quad (i,j) \notin J, && (20)\\
&\qquad x_{ij} \geq 0 \quad (i,j) \in J, && (21)\\
&\qquad s \geq 0. && (22)
\end{aligned}$$

$$\begin{aligned}
&\text{maximize} \sum_{i \in L} u_i^{'} + \sum_{j \in R} v_j^{'} && (23)\\
&\text{subject to} \quad u_i^{'} + v_j^{'} \leq 0 \qquad (i,j) \in J, && (24)\\
&u_i^{'} + v_j^{'} \leq 1 \quad (i,j) \notin J \quad , && (25)\\
&\qquad u_i^{'}, v_j^{'} \geq -1 \qquad i,j \notin I, && (26)\\
&\qquad u_i^{'}, v_j^{'} \geq 0 \qquad i,j \in I. && (27)
\end{aligned}$$

These are not a mirror of what Goemans and Williamson give, but I think they reduce to the same combinatorial problem: find a maximum cardinality matching in the subgraph $G = (L, R, J)$ (ie the equality subgraph). What we need to show (I think) is that, first of all, this dual solution is greater than zero, and, furthermore, we can find an ϵ such that $\sum_i u_i + \sum_j v_j \geq \sum_i u_i + \sum_j v_j + \epsilon(\sum_i u_i^{'} + \sum_j v_j^{'})$, where the RHS of the inequality is a dual feasible solution. Showing this shows that our system improves the dual solution when we can't find an optimal primal. I think the ϵ will be something like $\min_{(i,j) \in E \setminus J}(u_i + v_j - w_{ij})$ (this would correspond to what happens in the Hungarian).

3 How is the auction algorithm primal-dual??

Original authors of the algorithm claim that it is a modified Hungarian algorithm, and authors of the blog post claim it is primal-dual. I see this in a naive way, as we are dealing with adjusting labelings and overdemanded sets as given in Hall's Marriage Theorem, but I CAN NOT figure out how this algorithm relates to the primal-dual linear programs.

4 Other less immediate things

- Brief section on flows, Ford-Fulkerson.
- More info on the economics of matchings – more in depth analysis of auctions, how the algorithm given is a good model of actual multi-item auctions.
- Possible expansion on how we can use the primal-dual method for approximating NP-hard combinatorial problems. Don't need to go into detail here, but the thesis would not feel complete without at least dedicating a couple of pages to this.