Algorithms Homework 3

2. This formulation will go into an infinite loop with the input Search([1,2],2):

$$\text{Search}([1,2],2) = \text{SearchHelp}([1,2],2;0,1)$$

$\hat{m} = 0 + \lfloor (1-0)/2 \rfloor = 0$,

$a[0] = 1$,

$2 > 1 \Rightarrow \text{SearchHelp}([1,2],2;0,1)$

<u>Infinite Loop</u>

3. (a)

$S_1 = 6$	$P_1 = 6$	$C_{1,1} = 18$ $C_{1,2} = 14$
$S_2 = 4$	$P_2 = 8$	$C_{2,1} = 62$ $C_{2,2} = 66$
$S_3 = 12$	$P_3 = 72$	
$S_4 = -2$	$P_4 = -10$	
$S_5 = 6$	$P_5 = 48$	
$S_6 = 8$	$P_6 = -12$	
$S_7 = -2$	$P_7 = -84$	
$S_8 = 6$		
$S_9 = -6$		
$S_{10} = 14$		

$$\begin{bmatrix} 18 & 14 \\ 62 & 66 \end{bmatrix}$$

(b) $\text{Strassen}(A_{1\times1}, B_{1\times1}) = a_{1,1} \cdot B_{1,1}$

$$\text{Strassen}(A_{n\times n}, B_{n\times n}) = \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}$$

where:

$$\begin{aligned} C_{1,1} = {} & \text{Strassen}(A_{n/2, n/2}, (B_{n/2,n} - B_{n,n})) \\ & + \text{Strassen}(A_{n,n}, (B_{n,n/2} - B_{n/2,n/2})) \\ & - \text{Strassen}((A_{n/2,n/2} + A_{n/2,n}), B_{n,n}) \\ & + \text{Strassen}((A_{n/2,n} - A_{n,n}), (B_{n,n/2}, B_{n,n})) \end{aligned}$$

$$C_{1,2} = \text{Strassen}\left(A_{n/2,n/2}, \left(B_{n/2,n} - B_{n,n}\right)\right) + \text{Strassen}\left(\left(A_{n/2,n/2} + A_{n/2,n}\right), B_{n,n}\right)$$

$$C_{2,1} = \text{Strassen}\left(\left(A_{n,n/2} + A_{n,n}\right), B_{n/2,n/2}\right) + \text{Strassen}\left(A_{n,n}, \left(B_{n,n/2} - B_{n/2,n/2}\right)\right)$$

$$C_{2,2} = \text{Strassen}\left(\left(A_{n/2,n/2} + A_{n,n}\right), \left(B_{n/2,n/2} + B_{n,n}\right)\right)$$

$$- \text{Strassen}\left(\left(A_{n,n/2} + A_{n,n}\right), B_{n/2,n/2}\right) + \text{Strassen}\left(A_{n/2,n/2}, \left(B_{n/2,n} - B_{n,n}\right)\right)$$

$$+ \text{Strassen}\left(\left(A_{n,n/2} - A_{n/2,n/2}\right), \left(B_{n/2,n/2} + B_{n/2,n}\right)\right)$$

(c) $P_3 = (A_{2,1} + A_{2,2})(B_{1,1})$

$P_4 = (A_{2,2})(B_{2,1} - B_{11})$

$$C_{2,1} = P_3 + P_4 = A_{2,1}B_{1,1} + \cancel{A_{2,2}B_{11}} + A_{2,2}B_{2,1} - \cancel{A_{2,2}B_{1,1}}$$

$$= A_{2,1}B_{1,1} + A_{2,2}B_{2,1}$$

(d) $P_5 = (A_{1,1} + A_{2,2})(B_{1,1} + B_{2,2}) = \cancel{A_{1,1}B_{1,1}} + \cancel{A_{1,1}B_{2,2}} + \cancel{A_{2,2}B_{1,1}} + A_{2,2}B_{2,2}$

$P_1 = (A_{1,1})(B_{1,2} - B_{2,2}) = \cancel{A_{1,1}B_{1,2}} - \cancel{A_{1,1}B_{2,2}}$

$P_3 = (A_{2,1} + A_{2,2})(B_{1,1}) = (\cancel{A_{2,1}B_{1,1}} + \cancel{A_{2,2}B_{1,1}}) - 1$

$P_7 = (A_{1,1} - A_{2,1})(B_{1,1} + B_{1,2}) = (\cancel{A_{1,1}B_{1,1}} + \cancel{A_{1,1}B_{1,2}} - \cancel{A_{2,1}B_{1,1}} - A_{2,1}B_{1,2}) - 1$

$+$

$$C_{2,2} = A_{2,1}B_{1,2} + A_{2,2}B_{2,2}$$

(e) $T(n) = 7T(\frac{n}{2}) + \frac{9}{2}n^2$

$= 7(7T(2^{m-2}) + \frac{9}{2}(2^{m-1})^2) + \frac{9}{2}(2^m)^2$

$= 7^k T(2^{m-k}) + 7^{k-1}(2^{m-(k-1)})^2 \cdot \frac{9}{2} \dots 7^0 \frac{9}{2}(2^m)^2$ let $k = m$

$= 7^m T(1) + 7^{m-1}(2^1)^2 \cdot \frac{9}{2} \dots 7^0 \frac{9}{2}(2^m)^2$

(f) Strassen's algorithm can be expanded to matrices that are not powers of two by filling the matrix with 0s up to the closest power of 2.

Filling with zeros doesn't require any multiplication so it won't effect the time complexity of arithmetic ~~Strassen~~

(e) $P_1 = ac$

$P_2 = bd$

$P_3 = (a+b)(c+d)$

$(a+b)(c+d)$

$ac + ad + bc + bd$

$-ac \qquad -bd$

$ad + bc$

Real: $P_1 - P_2$

Imaginary: $P_3 - P_1 - P_2$

4. (a) $D(n) = \lfloor \lg(n) \rfloor + 2$ by the strong form of induction

Observe: $D(1) = 2$

$\lfloor \lg(1) \rfloor + 2 = 0 + 2 = 2$ ✓

Assume: $D(k) = \lfloor \lg(k) \rfloor + 2$ if $1 < k < n$

Prove: $D(n) = D(\lfloor n/2 \rfloor) + 1 = \lfloor \lg(n) \rfloor + 2$

$D(\lfloor n/2 \rfloor) = \lfloor \lg \lfloor n/2 \rfloor \rfloor + 2$ by our induction hypothesis

$$D(n) = D(\lfloor n/2 \rfloor) + 1$$
$$= (\lfloor \lg \lfloor n/2 \rfloor \rfloor + 2) + 1$$
$$= \lfloor \lg n \rfloor - 1 + 2 + 1$$
$$= \lfloor \lg(n) \rfloor + 2 \qquad \square$$

(b)

$$D(n) = T(n+1) - T(n)$$

$$\sum_{k=1}^{n-1} D(k) = (T(\cancel{2}) - T(1)) + (\cancel{T(3)} - \cancel{T(2)}) + (T(4) - \cancel{T(3)}) \ldots$$

Every middle term will cancel

$$\ldots (\cancel{T(n-2+1)} - \cancel{T(n-2)}) + (T(n-1+1) - \cancel{T(n-1)})$$

We're left only with $-T(1) + T(n)$

$$T(n) - T(1) = \sum_{k=1}^{n-1} D(k) \qquad \square$$

$T(1) = 0$ By definition

&

$D(k) = (\lfloor \lg k \rfloor + 2)$ by Prob (a)

$$T(n) - 0 = \sum_{k=1}^{n-1} (\lfloor \lg k \rfloor + 2)$$

(c) $T(n) = \sum_{k=1}^{n-1} (\lfloor \lg k \rfloor + 2)$

$= \sum_{k=1}^{n-1} \lfloor \lg(k) \rfloor + 2n - 2.$

$\sum_{k=1}^{n-1} \lg(k) = \lg(1) + \lg(2) + \lg(3) \ldots \lg(n-2) + \lg(n-1)$

This is $\le \underbrace{\lg(n) + \lg(n) \ldots + \lg(n)}_{n-1 \text{ times}}$ Because $\lg(x)$ is monotonically increasing

$(n-1)\lg(n) < n \lg n$ by definition

Therefore $\sum_{k=1}^{n-1} D(k) = O(n \log n)$