Homework 03 - Network Science

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1 Exercise

For this exercise, the following properties were considered.

- N nodes placed on a circle
- Each node connected to m neighbours on either side
- Each node has degree 2m

1.1 Calculate the average clustering coefficient $\langle C \rangle$ of this network in terms of m. Assume that $N > 2m + 1$, so the underlying graph is not complete.

Let's assume the following parameters:

- G - The current Graph
- v - A node of G
- k_v - Degree of node v
- L_i - Links between neighbours of v

To calculate the average clustering coefficient for this graph G, we got to calculate the clustering coefficient for each node $v \in G$. Therefore, as G has the same properties for each node, we only need to calculate the clustering coefficint for only one node.

The following expression explains it.

$$\langle C \rangle = \frac{\sum_{v \in G}^{N} \frac{2Li}{k_v(k_v - 1)}}{N}$$

As each node has the same properties, we can exchange the summatory for a constant N. By doing so, we stick with:

$$\langle C \rangle = \frac{N * \frac{2Li}{k_v(k_v - 1)}}{N} = \frac{2Li}{k_v(k_v - 1)}$$

The k_v value was given, so $k_v = 2m$, then we only need to calcualte the $2L_i$ value, which are the amount of links amoung neighbours of v. Furthermore, v has m neighbours on either side so, we're going to check only one direction due to its simetry, and for each visited neighbour u, on that side, u will have one less link available (not visited yet) to connect to v's neighbours, and so do u's neighbours.

By saying so, we got the following equation:

Links amoung neighbours $= (2m-1, 2m-2, 2m-3, ..., 2m-m)$. We can translate it as a summatory, like:

$$\sum_{n=1}^{m}(2m-n)$$

The following draw, exemplifies the previous ideia. Where the black links are the links to u neighbours of V, the green link are the links among u neighbours of v, the red links the rest of the graph. Also, the 2m, 2m-1 and 2m-2 are the number of possibile different links at that level.

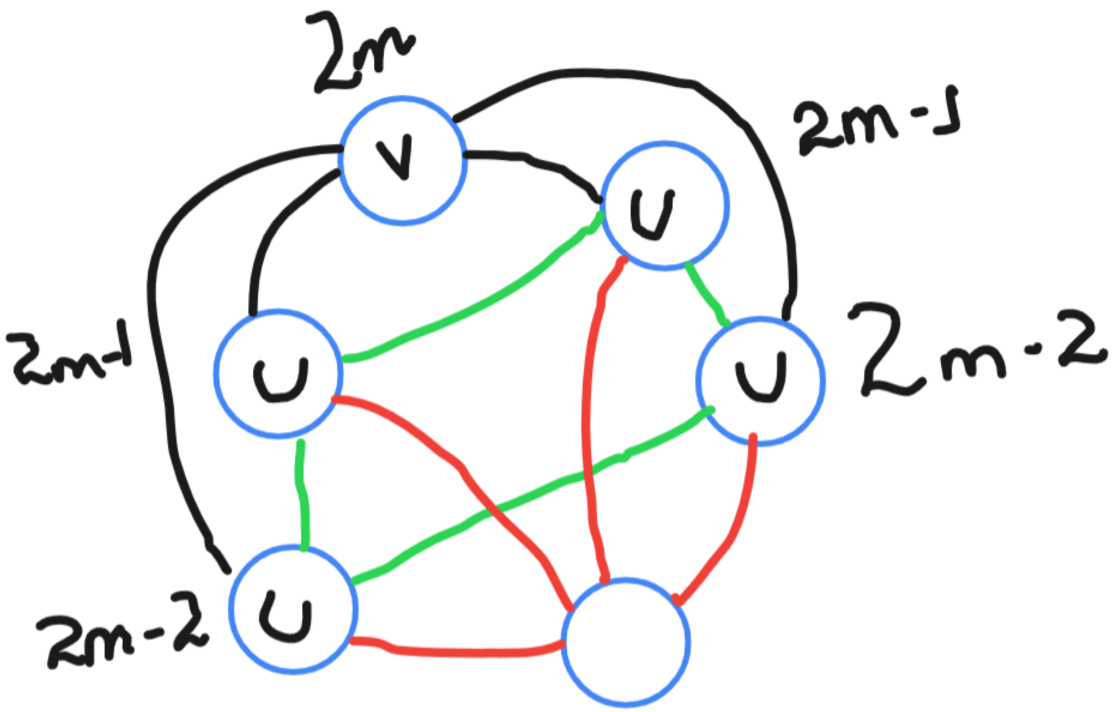

Figure 1: Example of a Graph with m=2

Nonetheless, as G is symetrical, we need to subtract the other parcel of links, whom were not computed, so we have:

$$(\sum_{n=1}^{m}(2m-n)) - m$$

We can simplify the equation and get the following:

$$\sum_{n=1}^{m}(2m-n)) - m = (\sum_{n=1}^{m}2m - \sum_{n=1}^{m}n) - m$$

$$= (m*2m) - \frac{m(m+1)}{2} - m = \frac{4m^2 - m^2 - m - 2m}{2} = \frac{3m^2 - 3m}{2}$$

Replacing it on the final formula with all values, we will have the following formula:

$$\langle C \rangle = \frac{2 * \dfrac{3m^2 - 3m}{2}}{2m(2m-1)} = \frac{3m^2 - 3m}{4m^2 - 2m}$$

1.2 What happens to $\langle C \rangle$, when $m \to \infty$?

When $m \to \infty$, we're going to get even a denser Graph. Nevertheless, it won't be 1, because it is not a complete graph. So, to compute that, we're going to use the previous formula acquired at 1.1.

$$\lim_{m \to \infty} \frac{3m^2 - 3m}{4m^2 - 2m}$$

We can simplify this limit and calculate it to get the result:

$$\lim_{m \to \infty} \frac{3m^2}{4m^2} = \lim_{m \to \infty} \frac{3}{4} \implies \langle C \rangle = \frac{3}{4}$$

1.3 How many nodes u satisfy d(v, u) = 2 for a fixed node v? Assume that $N > 4m + 1$.

As the graph has m neighbours on either side, all of its nodes have degree 2m, which was given. Also, to have $d(v, u) = 2$, means that node v has to travel exactly 2 nodes to reach its destiny. To do so, node v needs to move to any of its neighbours and then visit any other node within the neighbourhood of that reached node. Furthermore, $N > 4m + 1$ makes the graph able to have distances greater than 2. Thus, the following expression indicates the amount of nodes u satisfies the condition d(v,u) = 2.

$$d(v, u) = {\color{red} m} + {\color{blue} m} = 2m$$

Where ${\color{red} m}$ is the amount of neighbours of v, and ${\color{blue} m}$ are the neighbours of neighbours of v.

1.4 How many nodes u satisfy d(v, u) = 3 for a fixed node v? Assume that $N > 6m + 1$.

$N = 2m$. Same explanation as question 1.3

1.5 Which of the following formulas better approximate d_{max} in this network?

$$\frac{N}{M}, \frac{2N}{m}, \frac{N}{2m}$$

The d_{max} is obtained through the longest path within the graph. We saw previously that each node has the same properties, so once again, the max distance for a node is equal for the entire network. So if we take a random node $v \in G$, his m neighbours will be at distance 1, and the neighbours of his neighbours will be at distance 2 and this will be going until reach the maximum point which is when the last neighbour has an edge who was already visited. Bellow is an example with $m = 1$. The edge with $d = 3$, has two neighbours whom was already visited. So the idea will be the same for any value of m.

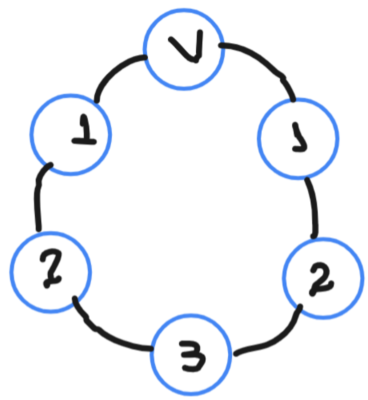

Figure 2: Example of distances

Accordingly, the value of d_{max} will be the rounded value of the fraction $\dfrac{N}{2m}$, which better aproximates the given graph.