Homework 06 - Network Science

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1 Rate Equations

1.1 Equation 5.35

$$(N+1)p_k(t+1) = Np_k(t) + \frac{k-1}{2}p_{k-1}(t) - \frac{k}{2}p_k(t)$$

This equation is related to the expected number of k-nodes after inserting a new node, thats why we have on the right side the $N+1$ and $t+1$. Following, the term $N_pk(t)$ is related to the amount of degree-k nodes before the insertion and, to complete the set, we sum the expected number of nodes that became k-degree nodes (expressed by $\frac{k-1}{2}p_{k-1}(t)$). Finally, we remove the expected number of nodes that are no longer k-degree nodes, that is, became (k+1)-degree nodes, expressed by $\frac{k}{2}p_k(t)$.

1.2 Equation 5.42

$$p_k = \frac{k-1}{2}p_{k-1} - \frac{k}{2}p_k$$

This equation is the shorten version to extimate the degree distribution. Hence, the first term is expected number of (k-1)-degree nodes and the second is the expected number of k-deree nodes.

1.3 Equation 5.43

$$2p_k = (k-1)p_{k-1} - kp_k = -p_{k-1} - k[p_k - p_{k-1}]$$

This equation is the simplification of 1.2 by multiplying both sides by 2. Following the author evidences k and then get the $k[p_k - p_{k-1}]$ term.

1.4 Equation 5.44

$$2p_k = -p_{k-1} - k\frac{p_k - p_{k-1}}{k-(k-1)} \approx -p_{k-1} - k\frac{\partial p_k}{\partial k}$$

The main idea of this formula is to create a differential equation, since it needs to compute the difference between the expected k-degree and (k-1)-degre nodes. To do so, the author includes the term $k-(k-1)$ to make it possible, nonetheless the original value doesn't change. Thus, we get the given approximation.

1.5 Equation 5.45

$$p_k = -\frac{1}{2}\frac{\partial[kp_k]}{\partial k}$$

This equation reduces equation presented at 1.4 to evaluate only the expected number of k-degree nodes.

2 Solve the following questions using the rate equation for Model A

2.1 Obtain a differential equation for the degree distribution in Model A.

For this exercise, we're going to use the following preferential attachment formula, described at 5.16.

$$\Pi(k) = \frac{1}{m_o + t - 1}$$

Applying it into given formulas, we can get:

$$\begin{aligned}
&\frac{1}{m_o + t - 1} \times Np_k \times m = \frac{mNp_k}{m_o + t - 1} \\
&\frac{mNp_k}{m_o + t - 1} \text{ (5.33)} \\
&\frac{mNp_{k-1}}{m_o + t - 1} \text{ (5.34)} \\
&(N+1)p_k = Np_k + \frac{mNp_{k-1}}{m_o + t - 1} - \frac{mNp_k}{m_o + t - 1} \text{ (5.35)} \\
&(N+1)p_k - Np_k = \frac{mNp_{k-1}}{m_o + t - 1} - \frac{mNp_k}{m_o + t - 1} \\
&Np_k + p_k - Np_k = \frac{mNp_{k-1}}{m_o + t - 1} - \frac{mNp_k}{m_o + t - 1} \\
&p_k = \frac{mN(p_{k-1} - p_k)}{m_o + t - 1} \text{ (5.42)} \\
&p_k = \frac{-mN}{m_o + t - 1}\frac{p_k - p_{k-1}}{k - (k-1)} \text{ (5.43 and 5.44)} \\
&p_k = \frac{-mN}{m_o + t - 1}\frac{\partial p_k}{\partial k} \text{ (5.45)}
\end{aligned}$$

2.2 Obtain a formula for this degree distribution by solving the equation you got in the previous item.

To obtain the formula, we have to solve the following expression, gotten from 2.1:

$$p_k = \frac{-mN}{m_o + t - 1}\frac{\partial p_k}{\partial k}$$

$$1 = \frac{-mN}{m_o + t - 1} \frac{\partial p_k}{\partial k} * \frac{1}{p_k} \qquad \text{(divide both sides by } \frac{1}{p_k}\text{)}$$

$$\partial k = \frac{-mN}{m_o + t - 1} \frac{\partial p_k}{p_k} \qquad \text{(multiply both sides by } \partial k\text{)}$$

For a better reading, we're going to use: $w = \dfrac{-mN}{(m_o + t - 1)}$

So,

$$\partial k = w \frac{\partial p_k}{p_k}$$

$$\frac{\partial k}{w} = \frac{\partial p_k}{p_k} \qquad \text{(divide both sides by w)}$$

$$\int \frac{1}{w} dk = \int \frac{1}{p_k} dp_k \qquad \text{(integrate both sides)}$$

$$\frac{k}{w} + c = \log(pk) + c \qquad \text{(the constants are irrelevant)}$$

$$\exp\left(\frac{k}{w}\right) = \exp(\log(pk)) \qquad \text{(exponentiate both sides)}$$

$$p_k = \exp\left(\frac{k}{w}\right)$$

$$p_k = \exp\left(\frac{k}{\frac{-mN}{m_o+t-1}}\right) \qquad \text{(replacing w values)}$$

$$p_k = \exp\left(-\frac{k(m_o + t - 1)}{mN}\right)$$

2.3 Does your formula in the previous item agree with the solution given in the book?

Although the formulas are not exactly the same, both equations express that the expected number of nodes follows a exponential function. So, the found solution agrees with the one given in the book. Hence, it decays much faster than a power law.