# Homework 11 - Network Science

Gabriel Luciano Gomes (265673)

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## 1 Communities on a Circle

For this exercise, we're going to consider a one dimensional lattice with N nodes that form a circle, where each node connects to its two neighbours. Additionally, we're going to partition the line into $n_c$ consecutive clusters of size $N_c = \frac{N}{n_c}$

### 1.1 Calculate the modularity of the obtained partition

To solve this question, we're going to use the formula presented by Barabási at chapter 10 of his book. So, given that $M_c$ is the modularity of a cluster, $L_c$ is the number of within $c$ cluster, and $k_c$ is total degree of the nodes in community $c$, we have:

$$M_c = \frac{L_c}{L} - \left(\frac{k_c}{2L}\right)^2$$

As the original network forms a circle, we know that $L = N$ and $L_c = N_c - 1$. Finally, we can write $k_c$ as $2L_c + 2$. Hence, replacing the obtained values, we have:

$$\begin{aligned}
M_c &= \frac{L_c}{N} - \left(\frac{2L_c + 2}{2N}\right)^2 && (1)\\
&= \frac{L_c}{N} - \left(\frac{L_c + 1}{N}\right)^2 && (2)\\
&= \frac{N_c - 1}{N} - \left(\frac{N_c - 1 + 1}{N}\right)^2 && (3)\\
&= \frac{N_c - 1}{N} - \frac{N_c^2}{N^2} && (4)\\
&= \frac{N * (N_c - 1) - N_c^2}{N^2} && (5)
\end{aligned}$$

Although there are more than one cluster, they're all identical. So, we have to sum all of them, resulting in:

$$
\begin{aligned}
M &= nc * \frac{N*(N_c-1)-N_c^2}{N^2} && \text{(replacing } n_c \text{ by } \frac{N}{N_c}\text{)} && (6)\\
&= \frac{N}{N_c} * \frac{N*(N_c-1)-N_c^2}{N^2} && && (7)\\
&= \frac{1}{N_c} * \frac{NN_c-N-N_c^2}{N} && && (8)\\
&= \frac{N_c(N-N_c)-N}{N_cN} && && (9)
\end{aligned}
$$

### 1.2 Obtain the community size $n_c$ corresponding to the best partition

For this questions, we're going to analyse the results found at 1.1. In order to the the maximum value of $nc$, it's necessary to compute the second derivative to get the maximum of the function. Thus, we have:

$$
\begin{aligned}
M_c &= \frac{L_c}{L} - \left(\frac{2L_c+2}{2L}\right)^2\\
&= \frac{L_c}{L} - \left(\frac{L_c+1}{L}\right)^2 \approx \frac{L_c}{L} - \left(\frac{L_c}{L}\right)^2\\
&= \frac{L_c}{L} - \frac{L_c^2}{L^2} = \frac{LL_c-L_c^2}{L^2}
\end{aligned}
$$

Now that we have the formula, we make it equals to zero and get its maximum:

$$
\begin{aligned}
&y : \frac{LL_c-L_c^2}{L^2} = 0\\
&y : LL_c - L_c^2 = 0\\
&y' : L - 2L_c \rightarrow \frac{L}{2}\\
&y'' : 2
\end{aligned}
$$

Since the maximum of $M_c$ corresponds to the best partition, then we have that $nc = 2$ for the maximum partition with $L_c = \frac{L}{2}$.

## 2 Modularity Maximum

### 2.1 Show that M cannot exceed one

For this exercise, we're going analyse the same formula used in 1.1,

$$
M_c = \frac{L_c}{L} - \left(\frac{k_c}{2L}\right)^2
$$

The first part ($\frac{L_c}{L}$) is about the number of links present in the cluster and the original network. Since $L_c$ is the number of links within the community $c$ it can have at most L links. So, in this case the maximum value obtained in this part is $\frac{L}{L} = 1$.

Following, the second part $\left(\frac{k_c}{2L}\right)^2$ is about the community degree. As $k_c$ means the maximum degree in this group, and it also can not exceed the maximum degree of the whole network ($k_c = 2L$) resulting in 1 as well.

Thus, for this example where whether $L_c$ and $k_c$ are the maximum values possible, it will result in 0. This confirms Barabási's affirmative of a single community when modularity is equal to 0. Nonetheless, even if values are different from each other, the sum of all clusters can not exceed one, since all modules are based on fragmentation of the original network. This conclusion can be observed in question 2.2, as the maximum value approaches to one, but never exceed it.

### 2.2 Is there a network with partition into cluster that achieves M=1?

No, there isn't. However, there is a network presented by Danon L. et al. in [1] where $M \to 1$. This network is composed by $m$ identical complete graphs (clicks) disjoint from each other. For this network, each click has $l$ = L/m links, and the total degree is $k_c = 2l$ since the graph is fully connected. Furthermore, there are no links between the clicks, obtaining the following formula:

$$\begin{aligned} M &= m\left[\frac{l}{L} - \left(\frac{2l}{2L}\right)^2\right] = m\left[\frac{l}{L} - \frac{l^2}{L^2}\right] \\ &= m\left[\frac{\frac{L}{m}}{L} - \frac{\frac{L^2}{m^2}}{L^2}\right] = m\left[\frac{L}{m} * \frac{1}{L} - \frac{L^2}{m^2} * \frac{1}{L^2}\right] \\ &= m\left[\frac{1}{m} - \frac{1}{m^2}\right] = 1 - \frac{1}{m} \end{aligned}$$

Since $m$ is the number of clicks present in this network, the modularity **converges** to one when it goes to infinity.

## References

[1] Leon Danon, Albert Díaz-Guilera, Jordi Duch, and Alex Arenas. Comparing community structure identification. *Journal of Statistical Mechanics: Theory and Experiment*, 2005(09):P09008–P09008, sep 2005.