Homework 05 - Network Science

Gabriel Luciano Gomes (265673)

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1 Average Degree in Lattice

For this exercise, we're going to consider a network $G(n)$, which is a two-dimensional lattice, an $n \times n$ grid. Nodes linked to their neighbours to their right, left, up and down.

1.1 Determine the number of nodes $N(n)$ of $G(n)$.

As the network is a two-dimensional lattice, we can calculte the number of nodes by its size. In other words, each row has n columns, so the following formula describes the amount of nodes of this network.

$$N(n) = n \times n$$

1.2 Determine the number of links $L(n)$ of $G(n)$.

Following the idea of a lattice, each node on a row is connected to the node ahead. By saying so, each row of the network has $n - 1$ links. The same is logic is applied to the columns as it can also be interpreted likewise. Hence, we got the following formula to describe the amount of links.

$$\begin{aligned}
n - 1 & \qquad \text{links for each row} \\
n - 1 & \qquad \text{links for each column} \\
n \times (n-1) + n \times (n-1) & \qquad \text{Multiplying by the network dimension} \\
n^2 - n + n^2 - n & \\
2n^2 - 2n & \\
L(n) = 2n(n-1) &
\end{aligned}$$

1.3 Determine the average degree $\langle k \rangle$ of $G(n)$ as a function of n.

The average degree $\langle k \rangle$ of a network is defined by the following formula.

$$\langle k \rangle = \frac{2L}{N}$$

Once the number of nodes and links are known, it's only necessary to replace the formula with the values. Thus, we have:

$$\langle k \rangle = \frac{2 \times 2n(n-1)}{n \times n} \tag{1}$$

$$= \frac{4n(n-1)}{n \times n} \tag{2}$$

$$= \frac{4(n-1)}{n} \tag{3}$$

1.4 Let $v(i, j)$ denote the node located at row i and column j in $G(n)$, for $i \leq i, j \leq n$. Determine a fomrula for the distance $d(v(i, j), v(r, s))$.

Once the network is a lattice, it has the same behaviours as a Cartesian plane, we can use the Manhattan distance to compute the distance between these two nodes. The following image explains the concept of it.

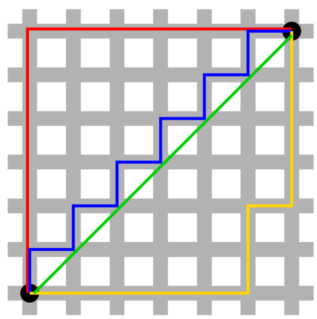

Figure 1: Distance between two points.

Hence, we have

$$d(v(i, j), v(r, s)) = |i - v| + |j - s|$$

1.5 Determine the average distance $\langle d \rangle$ of $G(n)$ as a function of n.

To calculate the average distance, we can base on the average degree formula of a connected graph:

$$\langle d \rangle = \frac{1}{2L_{max}} \sum_{i \neq j} d_{ij}$$

As L_{max} the maximum number of links in the network, it is the same as the number of $L(n)$ defined at 1.3. Furthermore, we can compute the distances among nodes on each row with the following summatory.

$$\sum_{i=1}^{n-1} i \text{ , be the distance of a node to other nodes in the same line}$$

$$n \times \sum_{i=1}^{n-1} i\text{, as we have n nodes in the row, we multiply the sum by n}$$

$$n \times \sum_{i=1}^{n-1} i + n \times \sum_{i=1}^{n-1} i\text{, the same applies to the columns}$$

Replacing values, we have:

$$\begin{aligned}
\langle d \rangle &= \left(n \times \frac{n(n-1)}{2} + n \times \frac{n(n-1)}{2} \right) \times \frac{1}{4n(n-1)} \\
&= \left(\frac{n^3 - n^2}{2} + \frac{n^3 - n^2}{2} \right) \times \frac{1}{4n(n-1)} \\
&= \frac{2n^3 - 2n^2}{2} \times \frac{1}{4n(n-1)} \\
&= \frac{2n^2(n-1)}{8n(n-1)} \\
&= \frac{2n}{8}
\end{aligned}$$

2 Rate equation

2.1 Explain Equation 5.31 of the book

For this exercise, it's required to explain the following formula :

$$\Pi(k) = \frac{k}{\sum_j k_j} = \frac{k}{2mt}$$

The formula computes the preferential attachment of the network, in other words, calculate the probability that a link of the new node connects to k-degree node. To do so, we divide k over the sum of degrees of inserted nodes. As the number of nodes within the network is related to the time t and since the network is undirected, the term $2mt$ captures that each link contributes to the degree of two nodes and the summatory is differentent for each t. Thus, the prerential attachment is related to the network growth.

2.2 Explain Equation 5.32 of the book

For this exercise, it's required to explain the following formula :

$$\frac{k}{2mt} \times Np_k(t) \times m = \frac{k}{2} p_k(t)$$

The goal of this formula is to calculate the number of links that are expected to connect to degree k nodes after the arrival of a new node.

The first term of the equation $\left(\dfrac{k}{2mt} \right)$ is the probability that new incoming node will link to a node with degree k, explained at 2.1. The second term $(N \times p_k(t))$, provides the total number of nodes with degree k at a moment t, once it has the amount of nodes (N) and the degree distribution at a specific time $(p_k(t)$. And the third term is the minimum degree of incoming node, as it receives m links at its creation. Since each node is inserted at a different t_i, $N = t$. And simplifying the formula, we have:

$$\frac{k}{2mt} \times Np_k(t) \times m$$
$$\frac{k}{2t} \times Np_k(t)$$
$$\frac{k}{2t} p_k(t)$$

2.3 Explain Equation 5.33 of the book

For this exercise, it's required to explain the following formula :

$$\frac{k}{2} p_k(t)$$

This formula is based on the fact that a new node can link to a degree-k node, turning it into a k+1 degree node, hence decreasing $N(k,t)$.

2.4 Explain Equation 5.34 of the book

For this exercise, it's required to explain the following formula :

$$\frac{k-1}{2} p_{k-1}(t)$$

This formula is based on the fact that a new node can link to a degree $(k-1)$ node, turning it into a degree k node, hence increasing $N(k,t)$.