A Tractable Model of Buffer Stock Saving: Methods

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Abstract

This document briefly describes the methods used to solve the model described in the main paper.

Keywords risk, uncertainty, precautionary saving, buffer stock saving

JEL codes C61, D11, E24

PDF: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete.pdf
Web: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete/
Archive: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete.zip
(Contains Mathematica and Matlab code solving the model)

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1 Numerical Solution

1.1 The Consumption Function

To solve the model by the method of *reverse shooting*, we need c_t^e as a function of c_{t+1}^e.

$$
\begin{aligned}
\left(\frac{c_{t+1}^e}{c_t^e}\right) &= \Gamma^{-1}(\mathsf{R}\beta)^{1/\rho}\left\{1+\mu\left[\left(\frac{c_{t+1}^e}{c_{t+1}^u}\right)^{\rho}-1\right]\right\}^{1/\rho} \\
c_t^e &= \left(\frac{c_{t+1}^e}{\Gamma^{-1}(\mathsf{R}\beta)^{1/\rho}\left\{1+\mu\left[\left(\frac{c_{t+1}^e}{\kappa^u(m_{t+1}^e-1)}\right)^{\rho}-1\right]\right\}^{1/\rho}}\right) \\
&= \Gamma(\mathsf{R}\beta)^{-1/\rho}c_{t+1}^e\left\{1+\mu\left[\left(\frac{c_{t+1}^e}{\kappa^u(m_{t+1}^e-1)}\right)^{\rho}-1\right]\right\}^{-1/\rho}.
\end{aligned}
$$

CDC: *2010-11-25: There's something wrong with the argument below but I can't find the error.*

PT: *2012-09-28: Just some thoughts. Don't know if it'll help.*

You are looking for the limit of the expression as $m_{t+1}^e - 1$ tends to zero. Isn't it the case that the limit is ill-behaved at that point? We know that as m^e tends to 1, the consumer only has human capital with which to finance consumption and is vulnerable to an infinite utility loss if that human capital is destroyed.

Also, to rule out a division by zero, c_{t+1}^e must be zero as well. If $c_{t+1}^e > 0$ while $m_{t+1}^e = 1$, then the entire expression on the right-hand side, your limit, is zero (infinity is turned to zero by the negative power), isn't it?

Also, you are computing a Taylor expansion approximation of the expression raised to the power $-1/\rho$. If I'm not mistaken that would be valid if c_{t+1}^e is small relative to $m_{t+1}^e - 1$. But don't we rather expect the consumption function to be steep at that point instead of flat?

End of comment.

whose limit as $m_{t+1}^e \downarrow 1$ is

$$
\begin{aligned}
c_t^e &= \Gamma(\mathsf{R}\beta)^{-1/\rho}c_{t+1}^e\left\{\mu\left[\left(\frac{\kappa^u(m_{t+1}^e-1)}{c_{t+1}^e}\right)\right]\right\} \\
&= \Gamma(\mathsf{R}\beta)^{-1/\rho}\left\{\mu\left[\left(\kappa^u(m_{t+1}^e-1)\right)\right]\right\}
\end{aligned}
$$

which can be differentiated with respect to m_t^e yielding

$$
\begin{aligned}
\bar{\kappa}_t^e &= \Gamma(\mathsf{R}\beta)^{-1/\rho}\left\{\mu\left[\left(\kappa^u(1-\bar{\kappa}_t^e)\mathcal{R}\right)\right]\right\} && (1) \\
&= (\mathsf{R}\beta)^{-1/\rho}\left\{\mu\left[\left(\kappa^u(1-\bar{\kappa}_t^e)\mathsf{R}\right)\right]\right\} && (2) \\
&= \text{Þ}_{\mathsf{R}}^{-1}\left\{\mu\left[\left(\kappa^u(1-\bar{\kappa}_t^e)\right)\right]\right\} && (3) \\
\bar{\kappa}_t^e(1+\text{Þ}_{\mathsf{R}}^{-1}\mu\kappa^u) &= \text{Þ}_{\mathsf{R}}^{-1}\mu\kappa^u && (4) \\
\bar{\kappa}_t^e &= \left(\frac{\text{Þ}_{\mathsf{R}}^{-1}\mu\kappa^u}{1+\text{Þ}_{\mathsf{R}}^{-1}\mu\kappa^u}\right) && (5)
\end{aligned}
$$

We also need the reverse shooting equation for m_t^e:

$$m_t^e \;=\; \mathcal{R}^{-1}(m_{t+1}^e - 1) + c_t^e.$$

The reverse shooting approximation will be more accurate if we use it to obtain estimates of the marginal propensity to consume as well. These are obtained by differentiating the consumption Euler equation with respect to m_t:

$$\begin{aligned} \mathrm{u}'(\mathrm{c}^e(m_t)) &= \overbrace{\mathcal{R}\beta\Gamma^{1-\rho}}^{\beth}\,\mathbb{E}_t[\mathrm{u}'(\mathrm{c}^\bullet(m_{t+1}))] \\ \mathrm{u}''(\mathrm{c}^e(m_t))\boldsymbol{\kappa}^e(m_t) &= \beth\mathcal{R}(1-\boldsymbol{\kappa}^e(m_t))\,\mathbb{E}_t[\mathrm{u}''(\mathrm{c}^\bullet(m_{t+1}))\boldsymbol{\kappa}^\bullet(m_{t+1})] \end{aligned} \tag{6}$$

so that defining $\kappa_t^e = \boldsymbol{\kappa}^e(m_t)$ we have

$$\kappa_t^e \;=\; (1-\kappa_t^e)\underbrace{\beth\mathcal{R}(1/\mathrm{u}''(c_t^e))\,\mathbb{E}_t\left[\mathrm{u}''(c_{t+1}^\bullet)\kappa_{t+1}^\bullet\right]}_{\equiv\natural_{t+1}} \tag{7}$$

$$(1+\natural_{t+1})\kappa_t^e \;=\; \natural_{t+1} \tag{8}$$

$$\kappa_t^e \;=\; \left(\frac{\natural_{t+1}}{1+\natural_{t+1}}\right). \tag{9}$$

At the target level of m^e,

$$\overbrace{(1/\mathrm{u}''(\check{c}^e))\,\mathbb{E}_t\left[\mathrm{u}''(c^\bullet)\kappa^\bullet\right]}^{\check{\natural}/\mathcal{R}\beth} \;=\; (1-\mu)\overbrace{(\mathrm{u}''(\check{c}^e)/\mathrm{u}''(\check{c}^e))}^{=1}\kappa^e + \mu(\mathrm{u}''(\check{c}^u)/\mathrm{u}''(\check{c}^e))\kappa^u$$

so that

$$\check{\natural} \;=\; \beth\mathcal{R}((1-\mu)\kappa^e + \mu(\check{c}^u/\check{c}^e)^{-\rho-1}\kappa^u) \tag{10}$$

yielding from (8) a quadratic equation in κ^e:

$$\Big(1 + \beth\mathcal{R}((1-\mu)\kappa^e + \mu(\check{c}^u/\check{c}^e)^{-\rho-1}\kappa^u)\Big)\kappa^e \;=\; \beth\mathcal{R}((1-\mu)\kappa^e + \mu(\check{c}^u/\check{c}^e)^{-\rho-1}\kappa^u) \tag{11}$$

which has one solution for κ^e in the interval $[0, 1]$, which is the MPC at target wealth.[1]

The limiting MPC as consumption approaches zero, $\bar{\kappa}^e$, will also be useful; this is obtained by noting that utility in the employed state next year becomes asymptotically irrelevant as c_t^e approaches zero, so that

$$\begin{aligned} \lim_{c_t^e\to 0}\overbrace{\beth\mathcal{R}\kappa_{t+1}^e\left((1-\mu)(c_{t+1}^e/c_t^e)^{-\rho-1} + \mu(c_{t+1}^u/c_t^e)^{-\rho-1}\kappa^u\right)}^{\natural_{t+1}} &= \beth\mathcal{R}\mu(c_{t+1}^u/c_t^e)^{-\rho-1}\kappa^u \\ &= \beth\mathcal{R}\mu(\kappa^u\mathcal{R}a_t^e/(a_t^e(\bar{\kappa}^e/(1-\bar{\kappa}^e)))^{-\rho-1})\kappa^u \\ &= \beth\mathcal{R}\mu(\kappa^u\mathcal{R}((1-\bar{\kappa}^e)/\bar{\kappa}^e))^{-\rho-1}\kappa^u \end{aligned}$$

so that from (9) we have

$$\bar{\kappa}^e \equiv \lim_{m_t\to 0}\boldsymbol{\kappa}^e(m_t) \;=\; \left(\frac{\beth\mathcal{R}\mu(\kappa^u\mathcal{R}((1-\bar{\kappa}^e)/\bar{\kappa}^e))^{-\rho-1}\kappa^u}{1+\beth\mathcal{R}\mu(\kappa^u\mathcal{R}((1-\bar{\kappa}^e)/\bar{\kappa}^e))^{-\rho-1}\kappa^u}\right) \tag{12}$$

[1] The *Mathematica* code constructs this derivative and solves the quadratic equation analytically; the Matlab code simply copies the analytical formula generated by *Mathematica* .

which implicitly defines $\bar{\kappa}^e$. After parameter values have been defined a numerical rootfinder can calculate a solution almost instantly.

Finally, it will be useful to have an estimate of the curvature (second derivative) of the consumption function at the target. This can be obtained by a procedure analogous to the one used to obtain the MPC: differentiate the differentiated Euler equation (6) again and substitute the target values. Noting that $\kappa^{u\prime} = 0$, we can obtain:

$$(\boldsymbol{\kappa}_t^e)^2 \mathrm{u}'''(\mathrm{c}_t^e) + \boldsymbol{\kappa}_t^{e\prime}\mathrm{u}''(\mathrm{c}_t^e) = \quad \beth\mathcal{R}\{(-\boldsymbol{\kappa}_t^{e\prime})\,\mathbb{E}_t[\mathrm{u}''(\mathrm{c}_{t+1}^{\bullet})\boldsymbol{\kappa}_{t+1}^{\bullet}] \\ + \quad \mathcal{R}(1-\boldsymbol{\kappa}_t^e)^2\left(\mathbb{E}_t[(\boldsymbol{\kappa}_{t+1}^{\bullet})^2\mathrm{u}'''(\mathrm{c}_{t+1}^{\bullet})] + (1-\mu)\mathrm{u}''(\mathrm{c}_{t+1}^e)\boldsymbol{\kappa}_{t+1}^{e\prime}\right) \tag{13}$$

so that

$$\boldsymbol{\kappa}_t^{e\prime} = \left(\frac{\beth\mathcal{R}^2(1-\boldsymbol{\kappa}_t^e)^2\left(\mathbb{E}_t[(\boldsymbol{\kappa}_{t+1}^{\bullet})^2\mathrm{u}'''(\mathrm{c}_{t+1}^{\bullet})] + (1-\mu)\mathrm{u}''(\mathrm{c}_{t+1}^e)\boldsymbol{\kappa}_{t+1}^{e\prime}\right) - (\boldsymbol{\kappa}_t^e)^2\mathrm{u}'''(\mathrm{c}_t^e)}{\mathrm{u}''(\mathrm{c}_t^e) + \beth\mathcal{R}\,\mathbb{E}_t[\mathrm{u}''(\mathrm{c}_{t+1}^{\bullet})\boldsymbol{\kappa}_{t+1}^{\bullet}]}\right)$$

which can be further simplified at the target because $\boldsymbol{\kappa}_t^{e\prime}(m) = \boldsymbol{\kappa}_{t+1}^{e\prime}(m) = \kappa^{e\prime}$ so that

$$\kappa^{e\prime} = \left(\frac{\beth\mathcal{R}^2(1-\kappa^e)^2\,\mathbb{E}_t[(\kappa^{\bullet})^2\mathrm{u}'''(c^{\bullet})] - (\kappa^e)^2\mathrm{u}'''(\check{c}^e)}{\mathrm{u}''(\check{c}^e) + \beth\mathcal{R}\,\mathbb{E}_t[\mathrm{u}''(c^{\bullet})\kappa^{\bullet}] - \beth\mathcal{R}^2(1-\kappa^e)^2(1-\mu)\mathrm{u}''(\check{c}^e)}\right). \tag{14}$$

Another differentiation of (14) similarly allows the construction of a formula for the value of $\kappa^{e\prime\prime}$ at the target m; in principle, any number of derivatives can be constructed in this manner.[2]

Reverse shooting requires us to solve separately for an approximation to the consumption function above the steady state and another approximation below the steady state. Using the approximate steady-state κ^e and $\kappa^{e\prime}$ obtained above, we begin by picking a very small number for $\blacktriangle$ and then creating a Taylor approximation to the consumption function near the steady state:

$$m_{\grave{t}}^e = m + \blacktriangle \tag{15}$$

$$\tilde{\mathbf{c}}(\blacktriangle) = \check{c}^e + \blacktriangle\kappa^e + (\blacktriangle^2/2)\kappa^{e\prime} + (\blacktriangle^3/6)\kappa^{e\prime\prime} \tag{16}$$

and then iterate the reverse-shooting equations until we reach some period n in which $m_{\grave{t}-n}^e$ escapes some pre-specified interval $[\underline{m}^e, \bar{m}^e]$ (where the natural value for $\underline{m}^e$ is 1 because this is the m that would be owned by a consumer who had saved nothing in the prior period and therefore is below any feasible value of m that could be realized by an optimizing consumer). This generates a sequence of points all of which are on the consumption function. A parallel procedure (substituting $-$ for $+$ in (15) and where appropriate in (16)) generates the sequence of points for the approximation below the steady state. Taken together with the already-derived characterization of the function at the target level of wealth, these points constitute the basis for an interpolating approximation to the consumption function on the interval $[\underline{m}^e, \bar{m}^e]$.

1.2 The Value Function

As a preliminary, note that since $\mathrm{u}(xy) = \mathrm{u}(x)y^{1-\rho}$, value for an unemployed consumer is

$$\mathrm{V}_t^u = \mathrm{u}(C_t^u) + \beta\mathrm{u}(C_{t+1}^u) + \beta^2\mathrm{u}(C_{t+2}^u) + \ldots \tag{17}$$

[2] *Mathematica* permits the convenient computation of the analytical derivatives, and then the substitution of constant target values to obtain analytical expressions like (14). These solutions are simply imported by hand into the Matlab code.

$$= \mathrm{u}(C_t^u)\left(1+\beta\{(\mathsf{R}\beta)^{1/\rho}\}^{1-\rho}+\beta^2\left\{(\mathsf{R}\beta)^{2/\rho}\right\}^{1-\rho}+...\right) \tag{18}$$

$$= \mathrm{u}(C_t^u)\underbrace{\left(\frac{1}{1-\beta(\mathsf{R}\beta)^{(1/\rho)-1}}\right)}_{\equiv\mathfrak{v}} \tag{19}$$

where the RIC guarantees that the denominator in the fraction is a positive number.

From this we can see that value for the normalized problem is similarly:

$$\mathrm{v}^u(m_t) = \mathrm{u}(\kappa^u m_t)\mathfrak{v}. \tag{20}$$

Turning to the problem of the employed consumer, we have

$$\mathrm{v}^e(m_t) = \mathrm{u}(c_t^e)+\beta\Gamma^{1-\rho}\,\mathbb{E}_t[\mathrm{v}^\bullet(m_{t+1})] \tag{21}$$

and at the target level of market resources this will be unchanging for a consumer who remains employed so that

$$\check{v}^e = \mathrm{u}(\check{c}^e)+\beta\Gamma^{1-\rho}\left((1-\mu)\check{v}^e+\mu\mathrm{v}^u(a^e\mathcal{R})\right) \tag{22}$$

$$(1-\beta\Gamma^{1-\rho}(1-\mu))\check{v}^e = \mathrm{u}(\check{c}^e)+\beta\Gamma^{1-\rho}\mu\mathrm{v}^u(a^e\mathcal{R}) \tag{23}$$

$$\check{v}^e = \left(\frac{\mathrm{u}(\check{c}^e)+\beta\Gamma^{1-\rho}\mu\mathrm{v}^u(a^e\mathcal{R})}{(1-\beta\Gamma^{1-\rho}(1-\mu))}\right). \tag{24}$$

Given these facts, our recursion for generating a sequence of points on the consumption function can be used at the same time to generate corresponding points on the value function from

$$v_t^e = \mathrm{u}(c_t^e)+\beta\Gamma^{1-\rho}\left((1-\mu)v_{t+1}^e+\mu\mathrm{v}^u(a_t^e\mathcal{R})\right) \tag{25}$$

with the first iteration point generated by numerical integration from

$$v_{\grave{t}}^e = \check{v}^e+\int_0^{\blacktriangle}\mathrm{u}'(\tilde{\mathbf{c}}(\bullet))d\bullet \tag{26}$$

2 The Algorithm

With the above results in hand, the model is solved and the various functions constructed as follows. Define $\star_t = \{m_t^e, c_t^e, \kappa_t^e, v_t^e, \kappa_t^{e\prime}\}$ as a vector of points that characterizes a particular situation that an optimizing employed household might be in at any given point in time. Using the backwards-shooting functions derived above, for any point $\star_{\grave{t}}$ we can construct the sequence of points that must have led up to it: $\star_{\grave{t}-1}$ and $\star_{\grave{t}-2}$ and so on. And using the approximations near the steady state like (16), we can construct a vector-valued function $\circ(\blacktriangle)$ that generates, e.g., $\{m+\blacktriangle, \tilde{\mathbf{c}}(\blacktriangle), ...\}$.

Now define an operator $\cdots$ as follows: $\cdots$ applied to some starting point $\star_t$ uses the backwards dynamic equations defined above to produce a vector of points $\star_{t-1}, \star_{t-2}, ...$ consistent with the model until the m_{t-n}^e that is produced goes outside of the pre-defined bounds $[\underline{m}^e, \bar{m}^e]$ for solving the problem.

We can merge the points below the steady state with the steady state with the points above the steady state to produce $\ddot{\star} = \cdots(\circ(-\varepsilon))\cup\circ(0)\cup\cdots(\circ(\varepsilon))$. These points can then be used to generate appropriate interpolating approximations to the consumption function and other desired functions.

Designate, e.g., the vector of points on the consumption function generated in this manner by $\ddot{\star}[c]$, so that

$$\{\ddot{\star}[m], \{\ddot{\star}[c], \ddot{\star}[\kappa^e], \ddot{\star}[\kappa^{e\prime}]\}^\top\}^\top = \begin{pmatrix} m[1] & \{c[1], \kappa^e[1], \kappa^{e\prime}[1]\} \\ m[2] & \{c[2], \kappa^e[2], \kappa^{e\prime}[2]\} \\ \ldots & \ldots \\ m[N] & \{c[N], \kappa^e[N], \kappa^{e\prime}[N]\} \end{pmatrix} \tag{27}$$

where N is the number of points that have been generated by the merger of the backward shooting points described above.

The object (27) is not an arbitrary example; it reflects a set of values that uniquely define a fourth order piecewise polynomial spline such that at every point in the set the polynomial matches the level and first derivative included in the list. Standard numerical mathematics software can produce the interpolating function with this input; for example, the syntax in *Mathematica* is simply

$$\texttt{cE} \quad = \quad \texttt{Interpolation}[\{\ddot{\star}[m], \{\ddot{\star}[c], \ddot{\star}[\kappa^e], \ddot{\star}[\kappa^{e\prime}]\}^\top\}^\top]. \tag{28}$$

which creates a function `cE` that is a $\mathbf{C}^4$ interpolating polynomial connecting these points.

The reverse shooting algorithm terminates at some finite maximum point $\bar{m}$, but for completeness it is useful to have an approximation to the consumption function that is reasonably well behaved for any m no matter how large.[3]

Since we know that the consumption function in the presence of uncertainty asymptotes to the perfect foresight function, we adopt the following approach. Defining the level of precautionary saving as[4]

$$¢(m) = \bar{\mathrm{c}}(m) - \mathrm{c}(m), \tag{29}$$

we know (see the discussion below in appendix section 3) that

$$\lim_{m \to \infty} ¢(m) = 0. \tag{30}$$

Defining $\vec{m} = m - \bar{m}$, a convenient functional form to postulate for the propensity to precautionary-save is

$$¢(m) = e^{\phi_0 - \phi_1 \vec{m}} + e^{\gamma_0 - \gamma_1 \vec{m}} \tag{31}$$

with derivatives

$$¢'(m) = -\phi_1 e^{\phi_0 - \phi_1 \vec{m}} - \gamma_1 e^{\gamma_0 - \gamma_1 \vec{m}} \tag{32}$$

$$¢''(m) = \phi_1^2 e^{\phi_0 - \phi_1 \vec{m}} + \gamma_1^2 e^{\gamma_0 - \gamma_1 \vec{m}} \tag{33}$$

$$¢'''(m) = -\phi_1^3 e^{\phi_0 - \phi_1 \vec{m}} - \gamma_1^3 e^{\gamma_0 - \gamma_1 \vec{m}}. \tag{34}$$

Evaluated at $\bar{m}$ (for which ¢ and its derivatives will have numerical values assigned by the reverse-shooting solution method described above), this is a system of four equations in four

[3] An extrapolation of the approximating interpolation will not perform well; a polynomial approximation will inevitably "blow up" if evaluated at large enough m.

[4] Mnemonic: This is the amount of consumption that is canceled as a result of uncertainty.

unknowns and, though nonlinear, can be easily solved for values of the ϕ and γ coefficients that match the level and first three derivatives of the "true" $\cancel{c}$ function.[5]

3 Modified Formulas For Case Where $\Gamma \geq \mathsf{R}$

The text asserts that if $\Gamma < \mathsf{R}$ the consumption function for a finite-horizon employed consumer approaches the $\bar{c}_t(m)$ function that is optimal for a perfect-foresight consumer with the same horizon,

$$\lim_{m \uparrow \infty} \bar{c}_t(m) - c_t(m) \quad = \quad 0. \tag{35}$$

This proposition can be proven by careful analysis of the consumption Euler equation, noting that as m approaches infinity the proportion of consumption will be financed out of (uncertain) labor income approaches zero, and that the magnitude of the precautionary effect is proportional to the square of the proportion of such consumption financed out of uncertain labor income.

A footnote also claims that for employed consumers, $c(m)$ approaches a different, but still well-defined, limit even if $\Gamma \geq \mathsf{R}$, so long as the impatience condition holds. It turns out that the limit in question is the one defined by the solution to a perfect foresight problem with liquidity constraints. A semi-analytical solution does exist in this case, but it is omitted.

[5]The exact symmetry in the treatment of γ and ϕ means that there will actually be two symmetrical solutions; either can be used.