A Tractable Model of Buffer Stock Saving

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Abstract

We present an analytically tractable model of the effects of nonfinancial risk on intertemporal choice. Our framework can be adopted in contexts where modelers have until now chosen not to incorporate serious nonfinancial risk because the available methods did not yield transparent insights. Our model produces an intuitive formula for target assets, and we show how to analyze transition dynamics using a familiar Ramsey-style phase diagram. Despite its starkness, the model captures many of the key implications of nonfinancial risk for intertemporal choice.

Keywords risk, uncertainty, precautionary saving, buffer stock saving

JEL codes C61, D11, E24

PDF: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete.pdf
Web: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete/
Archive: http://econ.jhu.edu/people/ccarroll/papers/ctDiscrete.zip
(Contains Mathematica and Matlab code solving the model)

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1 Introduction

The Merton-Samuelson model of portfolio choice is the foundation for the vast literature analyzing financial risk,[1] not because it offers conclusions that cannot be obtained from other frameworks,[2] but because it is easy to use and its key insights emerge in a way that is natural, transparent, and intuitive — in a word, the Merton-Samuelson model is tractable.

Unfortunately, nonfinancial risk[3] (which is much more important than financial risk for most households)[4] has proven more difficult to analyze. Of course, a large and impressive numerical literature has carefully computed the effects of specific nonfinancial risks in a variety of particular contexts.[5] But because the computational methods necessary to solve such models are daunting and the insights that emerge are not easy to explain, much of the economic literature (and much graduate-level instruction) has dodged the question of how nonfinancial risk influences choices, by assuming perfect insurance markets or perfect foresight or risk neutrality or quadratic utility or Constant Absolute Risk Aversion, or by attempting only to match aggregate risks (which are orders of magnitude smaller than idiosyncratic risks). These approaches rob the question of its essence, either by assuming (counterfactually) that markets transform nonfinancial risk into financial risk or by making implausible assumptions crafted to generate the implausible conclusion that decisions are largely or entirely unaffected by nonfinancial risk.[6]

Our contribution is to offer a tractable model that captures the main features of realistic models of the optimal response to nonfinancial risk, but without the customary technical difficulties. The model is a natural extension of the no-risk perfect foresight framework. Its solution is characterized by simple, intuitive equations and we show how the model's results can be analyzed using a phase diagram like that of the canonical Ramsey growth model.

The trick that yields tractability is to distill all nonreturn risk into a stark and simple possibility: A one-time uninsurable permanent loss in nonfinancial income. Our view is that

[1] Merton (1969); Samuelson (1969); see Sethi and Thompson (2000) for an overview and extensions.

[2] Merton and Samuelson cite the pioneering work of Markowitz (1959), Tobin (1958), and Phelps (1960) among others whose work had already contained many of the important qualitative insights.

[3] By which we mean risk that is both imperfectly insurable and imperfectly correlated with financial risk.

[4] Nonfinancial income typically accounts for the great majority of most households' total income, while risky assets like stocks represent a relatively small percentage of total wealth. Furthermore, stock returns are poorly correlated with the return on the index portfolio of all the assets in the economy. Idiosyncratic risk is even more poorly spanned by market risk and is several orders of magnitude greater than aggregate risk (Jagannathan, Kubota, and Takehara (1998)).

[5] The literature on in heterogeneous-agents macroeconomic models includes, among many others, Carroll (1992), Aiyagari (1994), and Krusell and Smith (1998), with roots that go back to Schechtman and Escudero (1977) and Bewley (1977), with other important contributions by Clarida (1987), Zeldes (1989), and Chamberlain and Wilson (2000).

[6] The case of CARA utility with only labor income risk is included here because Carroll and Kimball (1996) show that it is a knife-edge case that is unrepresentative of the broader effects of uncertainty (notably, it fails to exhibit the consumption concavity that holds for virtually every other combination of assumptions); indeed, the addition of rate-of-return risk renders the optimal consumption function concave even under CARA utility. (The other traditional objection is that the optimal consumption plan under CARA utility generally involves setting consumption to a negative value in some states of the world; it is hard to think of a plausible economic interpretation of negative consumption.)

the consumer's response to this single, tractable risk captures most of the substantive essence of the results obtained by the numerical literature under more realistic but more complex assumptions about income dynamics. Indeed, our view is that our model matches essentially all of the qualitative and even some of the quantitative implications of such models.

The kind of real-world shock that our μ shock most closely resembles is permanent disability from which no recovery is possible. We take the liberty of interpreting the shock more loosely, as capturing the role of unemployment or forced retirement risk, for several reasons. Most importantly, compared to perfect foresight models, the key qualitative characteristic that distinguishes models with a meaningful precautionary motive is the concavity of the consumption function (see Carroll and Kimball (1996) for general proof that uncertainty induces consumption concavity), which our model generates in the simplest way we know of. While exact quantitative details of the consumption function generated by models with more realistic formulations of income dynamics will differ from those we obtain, our purpose here is more qualitative and analytical than quantitative and empirical.[7]

The same framework could be interpreted to apply in other contexts as well; for instance, the risk faced by a country whose exports are dominated by a commodity whose price might collapse permanently (e.g., oil exporters, if cold fusion had worked).[8]

The optimal response to any such risk is to aim to accumulate a buffer stock of precautionary assets. The existing literature has computed the numerical value of the target under specific parametric assumptions, but has struggled to present an intuitive picture of the determinants of that target. Here, we are able to derive an analytical formula for the target level of wealth, and show how the precautionary motive (nonlinearly) interacts with the other saving motives that have been well understood since Irving Fisher (1930)'s work: The income, substitution, and human wealth effects.

Thanks to the model's tractability, we are able to derive a simple expression that shows how the familiar perfect-foresight consumption Euler equation is modified by the presence of our risk. The effect of our risk on consumption growth is related to the probability of the risky event, to its magnitude, to the consumer's degree of risk aversion, to the consumer's wealth, and to all of the parameters of the model (including risk aversion and time preference). We

[7]The reader might also be reassured to know that the model could be modified without too much difficulty to accommodate a multi-phase life cycle in which the risk applies only in the last phase of life, in which it can be interpreted as the probability of forced early retirement. But such an extension would add complexity and opacity to our exposition and notation, undermining the paper's chief virtue, which is the transparency of its results.

[8] The model could even be interpreted as applying to the behavior of a firm controlled by a risk-neutral manager, so long as the collapse of a line of business could have the effect of reducing the firm's collateral value and therefore increasing its cost of external finance *à la* Bernanke, Gertler, and Gilchrist (1996). In this case, the convex increase in borrowing rates when cash drops plays the same role as the convexity of the marginal utility function for a consumer; see also Berk, Stanton, and Zechner (2009) for an argument that senior firm managers are not risk neutral even if shareholders are, because poor performance under their tenure will reduce their own future employment opportunities. A firm controlled by such managers may behave very much like a risk-averse household.

obtain an exact analytical expression (not a log-linearized one) for the combined value of the higher-order Euler equation terms at the target.

With this expression in hand, the intuition for why empirical Euler equation estimation is problematic comes into clear focus, and the problems that have bedeviled the Euler equation literature can be plainly articulated and understood (see section 2.2.11).

A tractable model like ours can be used as a reference point for more specialized models, complementing the perfect foresight, certainty equivalent, or perfect markets models that are currently so widely used because of their tractability. Even computation-intensive literatures like heterogeneous-agents macroeconomics may find ours a useful 'toy model' with which to exposit some of the subtle points that numerical simulations deliver without interpretation.[9]

If the consequences of nonfinancial risk were numerically trivial, a tractable model would be of little interest. But the success of the growing heterogeneous-agents literature in macroeconomics suggests that the profession is increasingly coming to the view that something essential is missed when idiosyncratic risk is ruled out.

Part, at least, of what is missing in such models is the ability to analyze the consequences of wealth heterogeneity. In the absence of nonfinancial risk, Uzawa (1968) pointed out that if infinite-horizon agents have heterogeneous preferences, the most patient agent ends up owning all aggregate wealth in equilibrium (a point that was anticipated, but not fully developed, in Ramsey (1928) and is fleshed out in Becker and Foias (1987)). Mathematically, this is because any agent who is impatient (at the prevailing interest rate) will choose to run down his wealth to its minimum possible value, while any agent who is patient relative to that interest rate will choose to accumulate indefinitely.

If we grant that there is even a small amount of heterogeneity in time preference rates, relative risk aversion, expected income growth, mortality risk, age, or any other factor that influences intertemporal choice, a model without nonfinancial risk leads to radical wealth inequality that is even greater than the substantial but not unlimited differences in wealth in the cross-sectional data.

In such a world, the representative agent model's approximation of behavior around the aggregate steady-state wealth-to-income ratio is an approximation around a wealth-to-income ratio that is arbitrarily far from the wealth-to-income ratio that anyone in the economy will actually experience. To take a simple example, suppose everybody has the same nonfinancial income, borrowing is ruled out, and suppose that 5 percent of agents are "patient" and 95 percent are "impatient." Then the patient agents will have a wealth-to-income ratio equal to about 20 times the aggregate ratio, while impatient agents will have a wealth-to-income ratio

[9] In order to assist authors in modifying our model for other purposes, we have constructed a public archive that contains Matlab and Mathematica programs that produce all the results and figures reported in this paper, along with some other examples of uses to which the model could be put. The archive is available on the first author's website.

of zero. Since no approximation is good at vast distances from the approximating point, there is no reason to suppose that approximating behavior around the aggregate wealth-to-income ratio will provide even a remotely accurate description of the behavior of the actual aggregate economy with heterogeneous agents but perfect foresight. Indeed, the logic of the model is that essentially *nobody* will hold a level of wealth close to the representative agent's level of wealth, around which *everybody's* behavior is approximated.

The appeal of introducing nonfinancial risk is that the precautionary motive prevents wealth from asymptoting to its lower bound for households who are impatient relative to the interest rate. Each kind of household's behavior can be approximated around a target level of wealth that reflects their actual wealth, rather than around the representative agent's wealth. In the presence of heterogeneity and nonfinancial risk, the only agents whose behavior is *not* captured by the model are those who are patient relative to the equilibrium interest rate. Since the wealth of those agents is heading to infinity, their behavior may be reasonably approximated by the behavior of a perfect foresight agent.

2 The Decision Problem

For concreteness, we analyze the problem of an individual consumer facing a labor income risk. Other interpretations (like those mentioned in the introduction) are left for future work or other authors.[10]

The aggregate wage rate, W_t, grows by a constant factor G from the current time period to the next, reflecting exogenous productivity growth:

$$\mathsf{W}_{t+1} = \mathsf{G}\mathsf{W}_t. \tag{1}$$

The interest rate is exogenous and constant (the economy is small and open); the interest factor is denoted R.[11] Define m as market resources (financial wealth plus current income), a as end-of-period assets after all actions have been accomplished (specifically, after the consumption decision), and b as bank balances before receipt of labor income. Individuals are subject to a dynamic budget constraint (DBC) that can be decomposed into the following elements:

$$\begin{aligned} a_t &= m_t - c_t & (2)\\ b_{t+1} &= \mathsf{R}a_t & (3)\\ m_{t+1} &= b_{t+1} + \mathsf{W}_{t+1}\ell_{t+1}\xi_{t+1} & (4) \end{aligned}$$

[10] See Toche (2005) for an explicit but brief treatment of a closely related model in continuous time.

[11] General equilibrium is not much more difficult; it requires specifying a production function and finding the level of capital for which the optimal level of saving is zero (net of depreciation). Little further insight is obtained, while several potential sources of confusion would be added.

where ℓ measures the consumer's labor productivity (hours of work for an employed consumer are assumed to be exogenous and fixed) and ξ is a dummy variable indicating the consumer's employment state: Everyone in this economy is either employed ($\xi = 1$, a state indicated by the letter 'e') or unemployed ($\xi = 0$, a state indicated by 'u'). Thus, labor income is zero for unemployed consumers.[12]

2.1 The Unemployed Consumer's Problem

There is no way out of unemployment; if an individual becomes unemployed, that individual remains unemployed forever, $\xi_t = 0 \Longrightarrow \xi_{t+1} = 0 \ \forall \ t$.

2.1.1 The Consumption Function

Consumers have a CRRA utility function $\mathrm{u}(c) = c^{1-\rho}/(1-\rho)$, with $\rho > 1$, and they discount future utility geometrically by β per period. We show below that the simplicity of the unemployed consumer's behavior is what makes employed consumer's problem tractable. The solution to the unemployed consumer's optimization problem is simply:[13]

$$c_t^u \quad = \quad \kappa^u b_t, \tag{5}$$

where

$$\kappa^u = 1 - \mathsf{R}^{-1}(\mathsf{R}\beta)^{1/\rho}. \tag{6}$$

κ^u is the 'marginal propensity to consume' out of total wealth (MPC), which for the unemployed consists in bank balances b only, because we have assumed that noncapital income of the unemployed is zero. This is for simplicity only; incorporation of an unemployment insurance system that pays a fixed proportion of the final employed wage is straightforward, because in the absence of further uncertainty the value of those future payments is equivalent to receipt of a lump sum equal to the present discounted value of the payments. The accompanying solution code for the model, in fact, incorporates a parameter measuring the generosity of the unemployment insurance system, and the solutions presented in the paper are those produced when that parameter is set to zero.

[12] This is without loss of generality. On this, see also footnote 4 in Toche (2005).

[13] This is a standard result, which follows from the first-order condition and the budget constraint:

$$\mathrm{u}'(c_t^u) = \mathsf{R}\beta\mathrm{u}'(c_{t+1}^u) \Rightarrow (c_t^u)^{-\rho} = \mathsf{R}\beta\left(c_{t+1}^u\right)^{-\rho} \Rightarrow c_{t+1}^u = (\mathsf{R}\beta)^{1/\rho} c_t^u.$$

Consumption grows at the geometric rate $(\mathsf{R}\beta)^{1/\rho}$. The present discounted value of consumption at time t must equal total wealth, so that

$$\sum_{i=0}^{\infty} R^{-i} c_{t+i}^u = \sum_{i=0}^{\infty} R^{-i} (\mathsf{R}\beta)^{i/\rho} c_t^u = \sum_{i=0}^{\infty} (1-\kappa^u)^i c_t^u = c_t^u/\kappa^u = b_t.$$

2.1.2 *Parameter Restrictions*

Table 6 summarizes our notation. We follow the terminology in Carroll (2016), where a detailed discussion of the concepts is provided. We impose a condition on parameters to ensure that the marginal propensity to consume out of total wealth is positive, $\kappa^u > 0$:

$$(\mathsf{R}\beta)^{1/\rho} \quad < \quad \mathsf{R}. \tag{7}$$

The unemployed consumer will be spending enough to make the ratio of financial wealth to human wealth decline over time. The interpretation is that the consumer must not be so patient that a boost to wealth would fail to boost current consumption. An alternative (equally correct) interpretation is that the condition guarantees that the present discounted value (PDV) of consumption for the unemployed consumer remains finite.[14]

Note that our results do not depend on the stronger[15] restriction that is often imposed:

$$\mathsf{R}\beta \quad < \quad 1. \tag{8}$$

Here the consumer will choose to spend more than the amount that would permit constant consumption; such a consumer's absolute level of wealth declines over time, together with consumption, since consumption and wealth are proportional. Carroll (2016) calls the first type of consumer "return impatient" and the second, more impatient type "absolutely impatient." The restriction we impose, therefore, is the Return Impatience Condition (RIC) defined in Carroll (2019).

2.2 The Employed Consumer's Problem

The consumer's preferences are the same in the employment and unemployment states; only exposure to risk differs. There are two effects at work. On the one hand, the precautionary saving motive is inversely related to wealth, so that 'poor' consumers choose to save more than 'rich' consumers. On the other hand, the impatience motive is independent of wealth. The two effects thus influence wealth in opposite directions. Under our parameter restrictions, there is a point at which prudence and impatience are in exact balance. This balance condition defines the target wealth-to-income ratio.

[14] 'Pathologically patient' consumers who do not satisfy this condition would hoard any increase in wealth in order to enable even more extra consumption in the distant future.

[15] Assuming $\mathsf{R} \geq 1$.

2.2.1 A Human-Wealth-Preserving Spread in Unemployment Risk

A consumer who is *employed* in the current period has $\xi_t = 1$; if this person is still employed next period ($\xi_{t+1} = 1$), market resources will be:

$$m^e_{t+1} = \mathsf{R}(m^e_t - c^e_t) + \mathsf{W}_{t+1}\ell_{t+1}. \tag{9}$$

However, there is no guarantee that the consumer will remain employed: Employed consumers face a constant risk μ of becoming unemployed.

We assume that ℓ grows by a factor $(1-\mu)^{-1}$ every period,

$$\ell_{t+1} = \ell_t/(1-\mu), \tag{10}$$

because under this assumption, for a consumer who remains employed, labor income will grow by factor $\Gamma = \mathsf{G}/(1-\mu)$, so that the *expected* labor income growth factor for employed consumers is the same G as in the no-risk perfect foresight case:

$$\begin{aligned} \mathbb{E}_t[\mathsf{W}_{t+1}\ell_{t+1}\xi_{t+1}] &= \Gamma \mathsf{W}_t\ell_t\,(\mu \times 0 + (1-\mu)\times 1) \\ \Rightarrow \frac{\mathbb{E}_t[\mathsf{W}_{t+1}\ell_{t+1}\xi_{t+1}]}{\mathsf{W}_t\ell_t} &= \Gamma(1-\mu) = \mathsf{G} \qquad = \text{no-risk growth} \end{aligned}$$

implying that an increase in μ is a pure increase in risk with no effect on the PDV of expected labor income – a mean-preserving spread. Thus, any change in behavior that results from a change in μ is cleanly interpretable as reflecting an effect of uncertainty rather than the effect of a change in human wealth. For analytical convenience, we henceforth treat the risk-adjusted growth factor Γ as a parameter, rather than the 'structural' parameter G.

2.2.2 First Order Optimality Condition

The usual steps lead to the standard consumption Euler equation. Using $i \in \{e, u\}$ to stand for the two possible states,

$$\begin{aligned} \mathrm{u}'(c^e_t) &= \mathsf{R}\beta\,\mathbb{E}_t\left[\mathrm{u}'(c^i_{t+1})\right] \\ \Rightarrow 1 &= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{c^i_{t+1}}{c^e_t}\right)^{-\rho}\right]. \end{aligned} \tag{11}$$

Henceforth nonbold variables will be used to represent the bold equivalent divided by the level of permanent labor income for an employed consumer, e.g. $c^e_t = \mathbf{c}^e_t/(\mathsf{W}_t\ell_t)$; thus we can rewrite the consumption Euler equation as:

$$\begin{aligned} 1 &= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\frac{\mathsf{W}_{t+1}\ell_{t+1}c^i_{t+1}}{\mathsf{W}_t\ell_t c^e_t}\right)^{-\rho}\right] \\ \Rightarrow 1 &= \mathsf{R}\beta\,\mathbb{E}_t\left[\left(\Gamma\,\frac{c^i_{t+1}}{c^e_t}\right)^{-\rho}\right] \end{aligned}$$

$$\Rightarrow \; 1 \;=\; \mathsf{R}\beta\Gamma^{-\rho}\left\{(1-\mu)\left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho} + \mu\left(\frac{c^u_{t+1}}{c^e_t}\right)^{-\rho}\right\}, \tag{12}$$

where the term in braces is a probability-weighted average of the growth rates of marginal utility in the case where the consumer remains employed (the first term) and the case where the consumer becomes unemployed (the second term).

2.2.3 *Analysis and Intuition of Consumption Growth Path*

An intuitive interpretation of the consumption Euler equation is readily available. Re-write equation (12) in terms of the growth rate of consumption in the employment state:

$$\left(\frac{c^e_{t+1}\Gamma}{c^e_t}\right) = \left(\frac{c^e_{t+1}}{c^e_t}\right) \;=\; \text{Þ}\left\{1+\mu\left[\left(\frac{c^e_{t+1}}{c^u_{t+1}}\right)^{\rho}-1\right]\right\}^{1/\rho}, \tag{13}$$

where

$$\text{Þ} = (\mathsf{R}\beta)^{1/\rho} \tag{14}$$

is the factor by which the level of consumption c would grow in a perfect foresight no-risk model.

To understand (13), it is useful to consider an approximation. Define $\nabla_{t+1} \equiv \left(\frac{c^e_{t+1}-c^u_{t+1}}{c^u_{t+1}}\right)$, the proportion by which the consumption ratio next period would be lower in the event of a transition into unemployment; we refer to this loosely as the size of the 'consumption risk.' Define ω, the 'excess prudence' factor, as $\omega = (\rho-1)/2$.[16] Applying a Taylor approximation to (13) (see appendix A.1) yields:

$$\left(\frac{c^e_{t+1}}{c^e_t}\right) \;\approx\; (1+\mu(1+\omega\nabla_{t+1})\nabla_{t+1})\,\text{Þ}, \tag{15}$$

which simplifies further in the logarithmic utility case (since $\rho = 1$ and thus $\omega = 0$) to

$$\left(\frac{c^e_{t+1}}{c^e_t}\right) \;\approx\; (1+\mu\nabla_{t+1})\,\text{Þ}. \tag{16}$$

The approximations in (15) or (16) capture the essence of equation (13). As a consequence of missing insurance markets, consumption growth depends on the employment outcome; the consumption ratio if employed next period c^e_{t+1} is greater than if unemployed c^u_{t+1}, so that ∇_{t+1} is positive. In the limit case, if we let the transition probability μ go to zero, consumption risk ∇ vanishes and thus c^e_{t+1}/c^e_t approaches its perfect-foresight value Þ. Equation (15) thus shows that the presence of unemployment risk boosts consumption growth by an amount proportional to the probability of becoming unemployed μ multiplied by a factor that is increasing in the

[16] It is 'excess' in the sense of exceeding the benchmark case of logarithmic utility which corresponds to $\rho = 1$. Logarithmic utility is often viewed as a lower bound on the possible degree of risk aversion.

amount of consumption risk ∇. Equation (16) shows that, in the logarithmic case, the precautionary boost to consumption growth is directly proportional to the size of the consumption risk.

The effect of risk on saving is transparent. For a given value of m_t^e, risk has no effect (by construction) on the PDV of future labor income and human wealth, but the larger is μ, the faster consumption growth must be, as equation (15) shows. For consumption growth to be faster while keeping its PDV constant, the *level* of *current* c^e must be lower. Thus, the introduction of a risk of unemployment μ induces a (precautionary) increase in saving.

In the (persuasive) case where $\rho > 1$, equation (15) implies that a consumer with a higher degree of prudence (larger ρ and therefore larger ω) will anticipate greater consumption growth. This reflects the greater precautionary saving motive induced by a higher degree of prudence.

2.2.4 The Steady State

To compute the steady state of the model, we explicitly solve for the $\Delta c_{t+1}^e = 0$ and $\Delta m_{t+1}^e = 0$ loci.

Consider a consumer who is employed in period t and at his steady-state target value of b_t^e. Substituting $c_{t+1}^e = c_t^e = c^e$ and $c_{t+1}^u = c^u$ into (13) and rearranging yields the ratio of consumption in the two states at the buffer-stock target value:

$$\begin{aligned}
(\text{Þ}/\Gamma)^{-\rho} &= 1 + \mu\left[\left(\frac{c_{t+1}^e}{c_{t+1}^u}\right)^{\rho} - 1\right] \\
\Rightarrow \frac{c^e}{c^u} &= \left[\frac{(\text{Þ}/\Gamma)^{-\rho} - (1-\mu)}{\mu}\right]^{1/\rho}.
\end{aligned} \tag{17}$$

Next, the dynamic budget constraint (4) may be used to express c^u in terms of c^e. If a transition into unemployment occurs at date t, then $m_{t+1}^u = b_{t+1}^u = b_{t+1}^e$, implying:

$$\begin{aligned}
\mathsf{m}_{t+1}^u &= \mathsf{R}(\mathsf{m}_t^e - \mathsf{c}_t^e) \\
\Rightarrow \frac{\mathsf{m}_{t+1}^u}{\mathsf{W}_{t+1}\ell_{t+1}} &= \mathsf{R}\left[\frac{\mathsf{m}_t^e}{\mathsf{W}_t\ell_t} - \frac{\mathsf{c}_t^e}{\mathsf{W}_t\ell_t}\right]\frac{\mathsf{W}_t\ell_t}{\mathsf{W}_{t+1}\ell_{t+1}} \\
\Rightarrow m_{t+1}^u &= \mathsf{R}(m_t^e - c_t^e)/\Gamma \\
\Rightarrow c_{t+1}^u &= \kappa^u\mathsf{R}(m_t^e - c_t^e)/\Gamma \\
\Rightarrow c^u &= (m^e - c^e)\kappa^u\mathsf{R}/\Gamma
\end{aligned} \tag{18}$$

where the consumption function (5) $c_{t+1}^u = \kappa^u m_{t+1}^u$ was used. Then, equations (17) and (18) may be solved for c^e in terms of m^e; straightforward algebra shows that the $\Delta c_{t+1}^e = 0$ locus is characterized by proportionality between c^e and m^e:

$$c^e = \pi m^e \tag{19}$$

where

$$\frac{\pi}{1-\pi} \equiv \kappa^u (\mathsf{R}/\Gamma)\left[\frac{(\text{Þ}/\Gamma)^{-\rho} - (1-\mu)}{\mu}\right]^{1/\rho}. \tag{20}$$

To characterize the $\Delta m^e_{t+1} = 0$ locus, we normalize the budget constraint (4) for the employment state, paralleling the normalization for the unemployment state detailed in (18),

$$\begin{aligned} m^e_{t+1} &= (m^e_t - c^e_t)\mathsf{R}/\Gamma + 1 \\ \Rightarrow c^e &= \frac{1 + (\mathsf{R}/\Gamma - 1)m^e}{\mathsf{R}/\Gamma}. \end{aligned} \tag{21}$$

The target values m^e and c^e solve the linear system formed by equations (19) and (21). Equation (19) is a steady-state version of the Euler equation of the employed worker, *conditional upon remaining employed*, where the marginal utility in the unemployment state is expressed in terms of wealth. Note that while the steady-state condition is associated with a (hypothetical) steady state in which unemployment is never actually realized, the aversion to the risk of unemployment exerts an effect on that steady state. Equation (21) is a steady-state version of the budget constaint of the employed worker. To each steady-state condition is associated a locus used to depict the phase diagram of the system in Figure 1. The buffer-stock target ratio is given by the intersection of the two loci. Since the conditions are linear, the intersection is unique (our parameter restrictions ensure existence).

The appendix details the complete solution. Before we turn to an analysis of the target level of m^e, we briefly discuss parameter restrictions needed to ensure its existence.

2.2.5 Parameter Restrictions (Continued)

In section 2.1.2, we derived a set of parameter restrictions from the problem of the unemployed consumer that ensure $\kappa^u > 0$ (the RIC). In this section, we derive a further set of restrictions from the problem of the employed consumer. The solution in employment is characterized by $m^e_t > c^e_t > 0$ because, with CRRA preferences, zero consumption carries an infinite penalty, implying that a consumer facing the risk of perpetual unemployment will never borrow. Therefore, as may be seen from equation (17), steady-state consumption is positive only if:

$$\text{Þ}/\Gamma < (1-\mu)^{-1/\rho}. \tag{22}$$

In the limit, as μ approaches zero, the condition reduces to $Þ/\Gamma < 1$, or equivalently:

$$(\mathsf{R}\beta)^{1/\rho} \quad < \quad \Gamma \tag{23}$$

It is useful to compare condition (23) with condition (7). The condition ensures that such a consumer facing no risk ($\mu = 0$) would be sufficiently impatient to choose a wealth-to-permanent-income ratio that would be falling over time. The restriction we impose, therefore, is a weak form of the Growth Impatience Condition (GIC) defined in Carroll (2016).[17]

2.2.6 *The Target Level of m^e*

The target level of the ratio of wealth to income m^e may be written in closed form:

$$m^e \quad = \quad 1 + \left(\frac{1}{\Gamma/\mathsf{R} - 1 + (1 - Þ/\mathsf{R})\left(1 + ((Þ/\Gamma)^{-\rho} - 1)/\mu\right)^{1/\rho}} \right), \tag{24}$$

where we have used the shorthand $Þ \equiv (\mathsf{R}\beta)^{1/\rho}$.

We now illustrate the analytics of the target formula with two special cases.

The first special case we consider is $\rho = 1$ (logarithmic utility). The appendix shows that an approximation of the target formula reduces to:

$$m^e \quad \approx \quad 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma + \vartheta - \mathsf{r})/\mu)} \right), \tag{25}$$

where[18] $\gamma \equiv \Gamma - 1$, $\mathsf{r} \equiv \mathsf{R} - 1$, $\vartheta \equiv (1/\beta) - 1$. This expression encapsulates several of the key intuitions of the model. The human wealth effect of labor income growth (conditional upon remaining employed) is captured by the first γ term in the denominator; for any calibration for which the denominator is positive, increasing γ reduces the target level of wealth. This reflects the fact that a consumer who anticipates being richer in the future will choose to save less in the present, and the result of lower saving is smaller wealth. The human wealth effect of interest rates is correspondingly captured by the $-\mathsf{r}$ term, which goes in the opposite direction to the effect of income growth, because an increase in the rate at which future labor income is discounted constitutes a reduction in human wealth. An increase in the rate at which future utility is discounted, ϑ, reduces the target wealth level. Finally, a reduction in unemployment risk raises $(\gamma + \vartheta - \mathsf{r})/\mu$ and therefore reduces the target wealth level.[19,20]

[17] The appendix shows that no additional restrictions are needed to guarantee positive wealth in the unemployment state — the RIC and GIC are sufficient for that.

[18] To a first-order approximation, these definitions are equivalent to $\gamma \simeq \ln(\Gamma)$, $\mathsf{r} \simeq \ln(\mathsf{R})$, $\vartheta \simeq -\ln(\beta)$

[19] $(\gamma + \vartheta - \mathsf{r}) > 0$ is guaranteed by (23) under log utility ($\rho = 1$).

[20] This discussion omits the fact that an increase in μ requires an adjustment to γ via (10) which induces a human wealth effect that goes in the opposite direction from the direct effect of uncertainty. For sufficiently large values of μ, this effect can dominate the direct effect of uncertainty and the target wealth-to-income ratio declines. See the illustration below of the effects of an increase in uncertainty for further discussion. The same qualitative results may be found by a direct analysis of the partial derivatives of equation (35).

Note that the different effects *interact* with each other, in the sense that the strength of, say, the human wealth effect of interest rates will vary depending on the values of the other parameters.

The assumption of log utility is implausible; empirical estimates from structural estimation exercises (e.g. Gourinchas and Parker (2002), Cagetti (2003), or the subsequent literature) typically find estimates considerably in excess of $\rho = 1$, and evidence from Barsky, Juster, Kimball, and Shapiro (1997) suggests that values of 5 or higher are not implausible. The next case illuminates how risk aversion affects the target formula for $\rho > 1$.

The second special case we consider is $\vartheta = \mathsf{r}$. The appendix shows that an approximation of the target formula reduces to:

$$m \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma/\mu)(1 - (\gamma/\mu)\omega))} \right). \tag{26}$$

Compare the target level in (26) with (25). The key difference is that (26) contains an extra term involving ω, which measures the amount by which relative risk aversion exceeds the logarithmic benchmark. An increase in ω reduces the denominator of (26) and thereby raises the target level of wealth, just as would be expected from an increase in the intensity of the precautionary motive.

In the $\omega > 0$ case, the interaction effects between parameter values are particularly intense for the $(\gamma/\mu)^2$ term that multiplies ω; this implies, e.g., that a given increase in unemployment risk can have a greater effect on the target level of wealth for a consumer who is more prudent.

2.2.7 *The Phase Diagram*

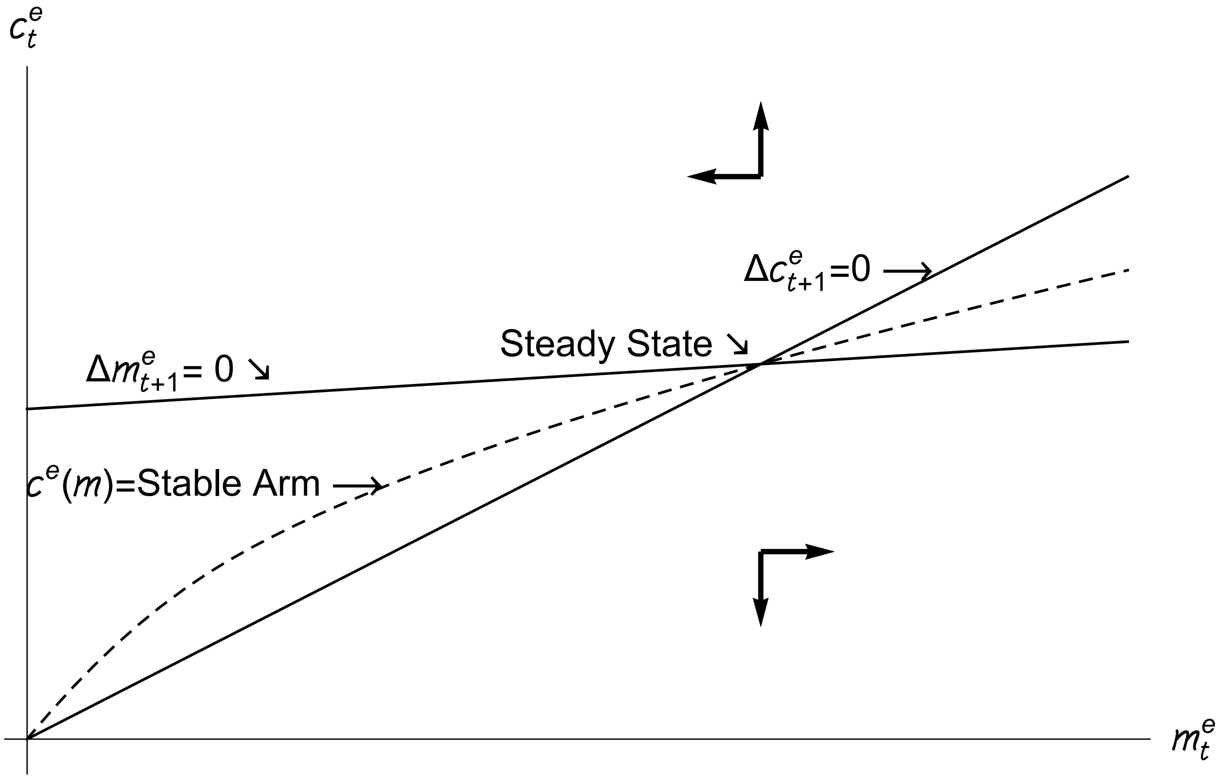
Figure 1 presents the phase diagram of system (19)-(21) under our baseline parameter values. Since the employed consumer never borrows, market resources never fall below the value of current labor income, which is the value selected for the origin of the diagram.[21] An intuitive interpretation is that the $\Delta m^e = 0$ locus characterized by (21) shows how much consumption c^e_t would be required to leave resources m^e_t unchanged so that $m^e = m^e_t$.[22] Thus, any point below the $\Delta m^e = 0$ line would have consumption below the break-even amount, implying that wealth would rise. Conversely for points above $\Delta m^e = 0$. This is the logic behind the horizontal arrows of motion in the diagram: Above $\Delta m^e = 0$ the arrows point leftward, below $\Delta m^e = 0$ the arrows point rightward.

The intuitive interpretation of the $\Delta c^e = 0$ locus characterized by (19) is more subtle.

[21] Our parameterization is not indended to maximize realism, but instead to generate well-proportioned figures that illustrate the mechanisms of the model as clearly as possible. The parameter values are encapsulated in the file `ParametersBase.m` in the online archive.

[22] Some authors refer to $\Delta m^e = 0$ as the level of 'permanent income.' However, this definition differs from Friedman (1957)'s and, moreover, is a potential source of confusion with permanent labor income' $\mathsf{W}_t \ell_t$; we prefer to describe the locus as depicting the level of 'sustainable consumption.'

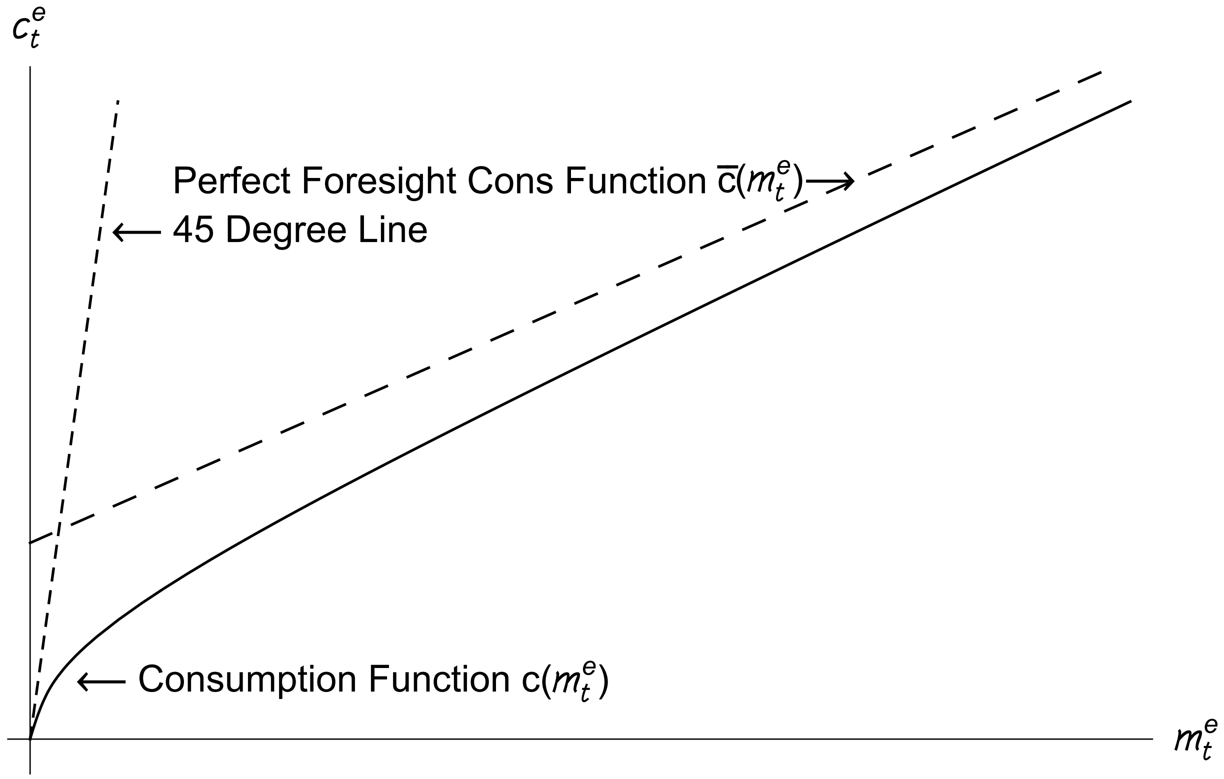
Figure 1 Phase Diagram



Note first that expected consumption growth depends on the amount by which consumption would fall if the unemployment state were realized, an amount which depends on the c^e/c^u ratio. For any level of resources, the further below the break-even level actual consumption is, the smaller the c^e/c^u ratio is, and therefore the smaller consumption growth is. Thus, points below the $\Delta c^e = 0$ locus are associated with negative values of Δc^e (and points above it are associated with positive values). This is the logic behind the vertical arrows of motion in the diagram: Above the $\Delta c^e = 0$ locus the arrows point upward; below the $\Delta c^e = 0$ locus the arrows point downward.

The slope of the locus depends essentially on the degree of impatience, as can be seen most directly by considering (20). As μ approaches zero, the proportionality between c^e and m^e approaches $\pi = 1$; this is because the vanishing risk of unemployment makes borrowing against future income more and more tempting, and (given our assumption of impatience), the target level of wealth will approach closer and closer to its minimum feasible value on the 45 degree line. On the other hand, as μ approaches 1 (so that receiving future income becomes more and more unlikely) the temptation exerted by that future income diminishes, so the pressure of impatience is smaller and the target level of wealth is greater. While the interactions between the terms in (20) are nonlinear, a similar story can be told for the other

Figure 2 The Consumption Function for the Employed Consumer



indicators of impatience: Greater impatience tilts the slope of the curve upward, and vice versa.

2.2.8 The Consumption Function

Figure 2 shows the optimal consumption function $\mathrm{c}(m)$ for an employed consumer (dropping the e superscript to reduce clutter). This is of course the stable arm of the phase diagram. Also plotted are the 45 degree line along which $c_t = m_t$ and

$$\bar{\mathrm{c}}(m) = \kappa^u(m - 1 + h), \tag{27}$$

where

$$h = \left(\frac{1}{1 - \mathsf{G}/\mathsf{R}}\right)$$

is the level of (normalized) human wealth. $\bar{\mathrm{c}}(m)$ is the solution to the no-risk (perfect foresight) version of the model; it is depicted in order to introduce another property of the model: As wealth approaches infinity, the solution to the problem with risky labor income approaches the solution to the no-risk problem arbitrarily closely.[23,24]

The consumption function $\mathrm{c}(m)$ is *concave*: The marginal propensity to consume $\boldsymbol{\kappa}(m) \equiv d\mathrm{c}(m)/dm$ is higher at low levels of m because the intensity of the precautionary motive increases as resources m decline.[25] The MPC is higher at lower levels of m because the *relaxation* in the intensity of the precautionary motive induced by a small increase in m (Kimball, 1990) is relatively larger for a consumer who starts with less than for a consumer who starts with more resources (Carroll and Kimball, 1996).

To see this important point, consider a counterfactual. Suppose the consumer were to spend all his resources in period t, i.e. $c_t = m_t$. In this situation, if the consumer were to become unemployed in the next period, he would then be left with resources $m^u_{t+1} = (m_t - c_t)\mathsf{R}/\Gamma = 0$, which would induce consumption $c^u_{t+1} = \kappa^u m^u_{t+1} = 0$, yielding negative infinite utility. A rational, optimizing consumer will always avoid such an eventuality, no matter how small its likelihood may be. Thus the consumer never spends all available resources.[26] This implication is illustrated in figure 2 by the fact that consumption function always remains below the 45 degree line.

2.2.9 *Expected Consumption Growth Is Downward Sloping in m^e*

Figure 3 illustrates some of the key points in a different way. It depicts the growth rate of consumption c^e_{t+1}/c^e_t as a function of m^e_t. Consumption growth is equal to what it would be in the absence of risk, plus a precautionary term; for algebraic verification, multiply both sides of (13) by Γ to obtain:

$$\left(\frac{c^e_{t+1}}{c^e_t}\right) = \text{Þ}\left\{1 + \mu\left[\left(\frac{c^e_{t+1}}{c^u_{t+1}}\right)^{\rho} - 1\right]\right\}^{1/\rho}. \tag{28}$$

Observe that the contribution of the precautionary motive becomes arbitrarily large as $m_t \to 0$, because $c^u_{t+1} = \kappa^u m^u_{t+1} = (m_t - \mathrm{c}(m_t))\kappa^u\mathsf{R}/\Gamma$ approaches zero as $m_t \to 0$; that is, as resources m^e_t decline, expected consumption growth approaches infinity. The point where consumption growth is equal to income growth is at the target value of m^e.

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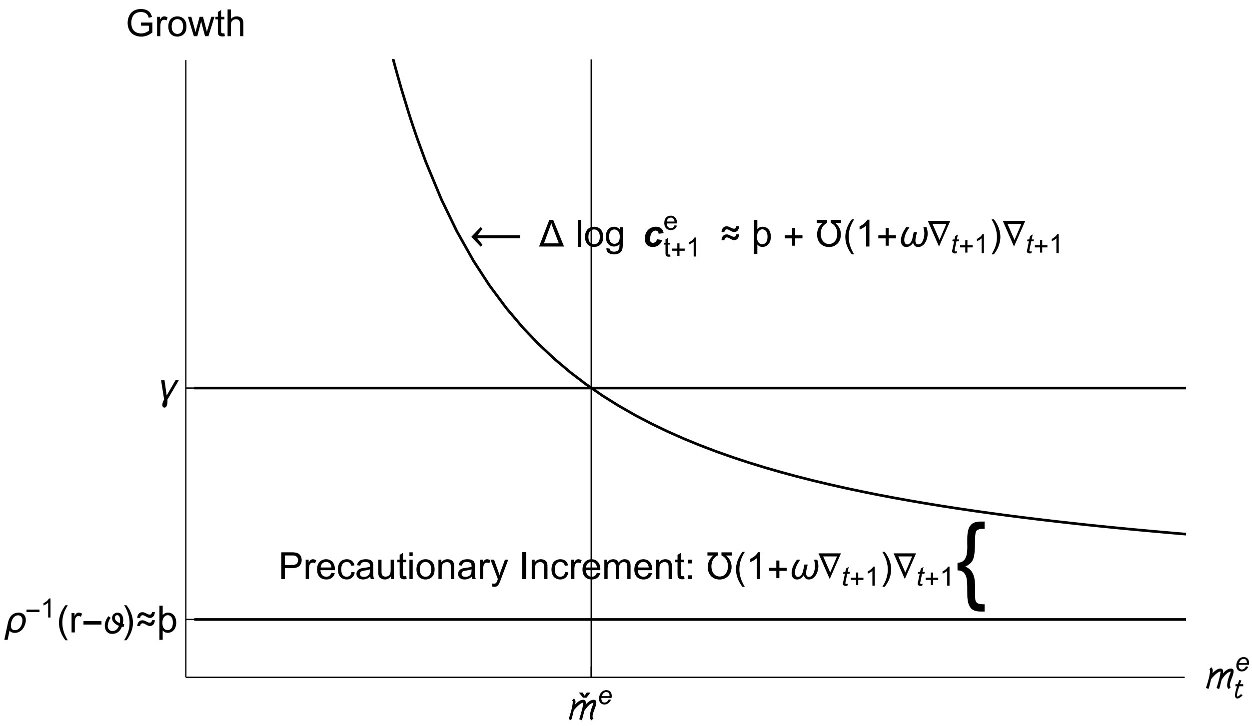
[23] This limiting result requires that we impose the additional assumption $\Gamma < \mathsf{R}$, because the no-risk consumption function is not defined if $\Gamma \geq \mathsf{R}$.

[24] If the horizontal axis is stretched far enough, the two consumption functions appear to merge (visually), with the 45 degree line merging (visually) with the vertical axis. The current scaling is chosen both for clarity and to show realistic values of wealth.

[25] Carroll and Kimball (1996) prove that the consumption function must be concave for a general class of stochastic processes and utility functions – including almost all commonly-used model assumptions except for the knife-edge cases explicitly chosen to avoid concavity.

[26] This is an implication not just of the CRRA utility function used here but of the general class of continuously differentiable utility functions that satisfy the *Inada condition* $\mathrm{u}'(0) = \infty$.

Figure 3 Income and Consumption Growth



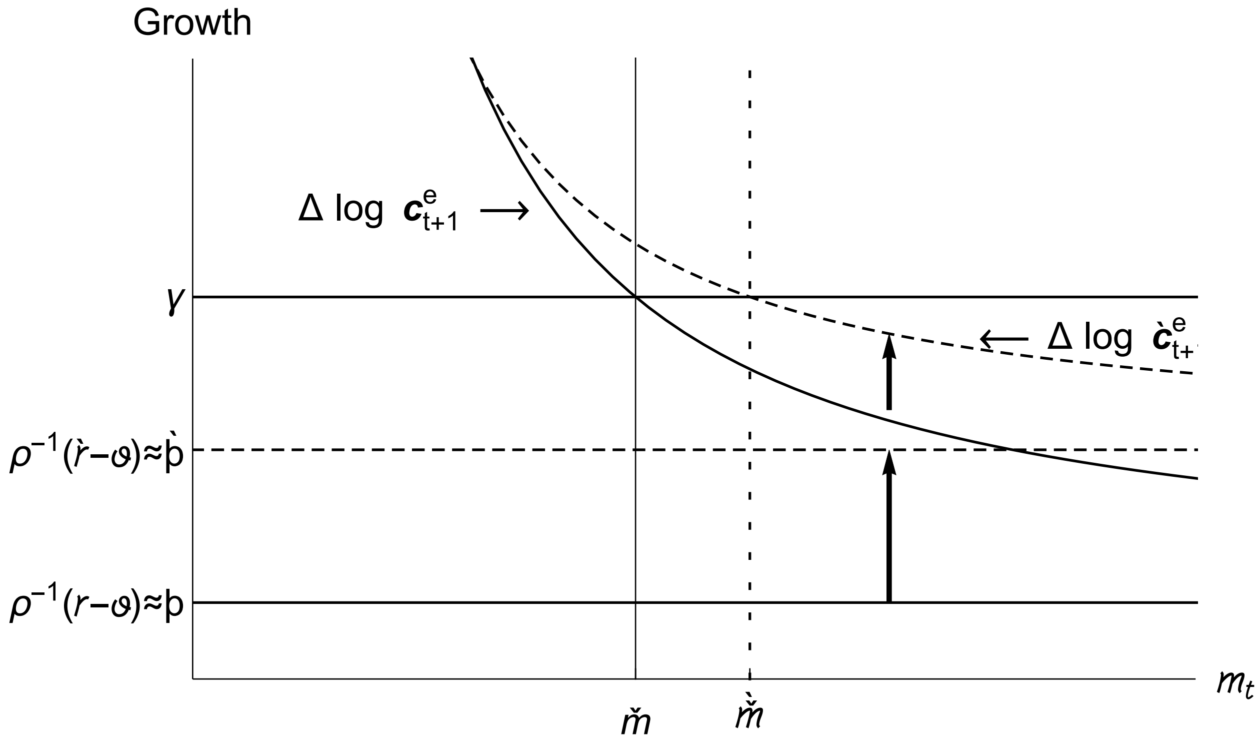
2.2.10 Summing Up the Intuition

We are finally in position to get an intuitive understanding of how the model works and why a target wealth ratio exists. On the one hand, consumers are growth-impatient: It cannot be optimal for them to let wealth become arbitrarily large in relation to income. On the other hand, the precautionary motive intensifies as the level of wealth falls. The two effects work in opposite directions. As resources fall, the precautionary motive becomes stronger, eventually offsetting the impatience motive. The point at which prudence becomes exactly large enough to match impatience defines the target wealth-to-income ratio.

It is instructive to work through a couple of comparative dynamics exercises. In doing so, we assume that all changes to the parameters are exogenous, unexpected, and permanent. Figure 4 depicts the effects of increasing the interest rate to $\grave{r} > r$. The no-risk consumption growth locus shifts up to the higher value $\grave{þ}_{r} \approx \rho^{-1}(\grave{r} - \vartheta)$, inducing a corresponding increase in the expected consumption growth locus. Since the expected growth rate of labor income remains unchanged, the new target level of resources $\grave{\check{m}}^{e}$ is higher. Thus, an increase in the interest rate raises the target level of wealth, an intuitive result that carries over to more elaborate models of buffer-stock saving with more realistic assumptions about the income process (Carroll, 2016).

Figure 4 depicts the effects of increasing the risk of unemployment μ. The principal effect

Figure 4 Effect of an Increase in r



we are interested in is the upward shift in the expected consumption growth locus to $\Delta \grave{c}_{t+1}$. If the household starts at the original target level of resources m, the size of the upward shift at that point is captured by the arrow originating at $\{m, \gamma\}$.

In the absence of other consequences of the rise in μ, the effect on the target level of m would be unambiguously positive. However, recall our adjustment to the growth rate conditional upon employment (10); this induces the shift in the income growth locus to $\grave{\gamma}$ which has an offsetting effect on the target m ratio. Under our benchmark parameter values, the target value of m is higher than before the increase in risk even after accounting for the effect of higher γ, but in principle it is possible for the γ effect to dominate the direct effect. Note, however, that even if the target value of m is lower, it is possible that the *saving rate* will be higher; this is possible because the higher γ makes a given saving rate translate into a lower ratio of wealth to income. In any case, our view is that most useful calibrations of the model are those for which an increase in uncertainty results in either an increase in the saving rate or an increase in the target ratio of resources to permanent income. This is partly because our intent is to use the model to illustate the general features of precautionary behavior, including the qualitative effects of an increase in the magnitude of transitory shocks, which unambiguously increase both target m and saving rates.

2.2.11 Death to the Log-Linearized Consumption Euler Equation!

Our simple model may help explain why the attempt to estimate preference parameters like the degree of relative risk aversion or the time preference rate using consumption Euler equations has been so signally unsuccessful (Carroll, 2001). On the one hand, as illustrated in figures 3 and 4, the steady state growth rate of consumption, for impatient consumers, is equal to the steady-state growth rate of income,

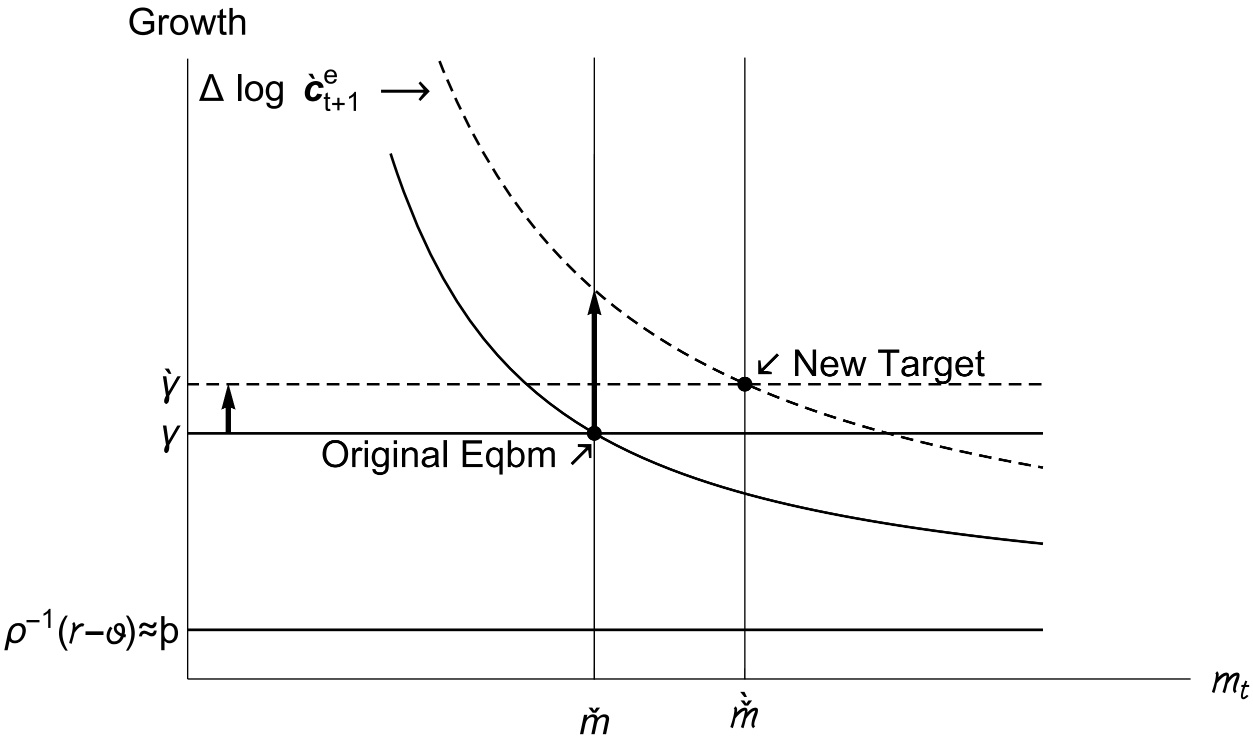
$$\Delta \log \ c^e_{t+1} \quad = \quad \gamma. \tag{29}$$

On the other hand, under logarithmic utility our approximation of the Euler equation for consumption growth, obtained from equation (28), seems to tell a different story,

$$\Delta \log \ c^e_{t+1} \quad \approx \quad \text{þ} + \mu \nabla_{t+1}, \tag{30}$$

where the last line uses the Taylor approximations used to obtain (15). The approximate Euler equation (30) does not contain any term *explicitly* involving income growth. How can we reconcile (29) and (30) and resolve the apparent contradiction? The answer is that the size of the precautionary term $\mu \nabla_{t+1}$ is *endogenous* (and depends on γ). To see this, solve (29)- (30): In steady-state,

Figure 5 Effect of an Increase in Unemployment Risk μ to $\grave{\mu}$



$$\mu\nabla \approx \gamma - þ. \quad (31)$$

The expression in (31) helps to understand the relationship between the model parameters and the steady-state level of wealth. From figure 3 it is apparent that $\nabla_{t+1}(m_t^e)$ is a downward-sloping function of m_t^e. At low levels of current wealth, much of the spending of an employed consumer is financed by current income. In the event of job loss, such a consumer must suffer a large drop in consumption, implying a large value of ∇_{t+1}.

To illustrate further the workings of the model, consider an increase in the growth rate of income. On the one hand, the right-hand side of (31) rises. But, lower wealth raises consumption risk, so that the new target level of m must be lower, and this raises the left-hand side of (31). In equilibrium, both sides of the expression rise by the same amount.

The fact that consumption growth equals income growth in the steady-state poses major problems for empirical attempts to estimate the Euler equation. To see why, suppose we had a collection of countries indexed by i, identical in all respects except that they have different interest rates r^i. In the spirit of Hall (1988), one might be tempted to estimate an equation of the form

$$\Delta \log\ c^i = \eta_0 + \eta_1 \mathsf{r}^i + \epsilon^i, \quad (32)$$

and to interpret the coefficient on r^i as an empirical estimate of the value of ρ^{-1}. This empirical strategy will fail. To see why, consider the following stylized scenario. Suppose that all the countries are inhabited by impatient workers with optimal buffer-stock target rules, but each country has a different after-tax interest rate (measured by r^i). Suppose that the workers are not far from their wealth-to-income target, so that $\Delta \log c^i = \gamma^i$. Suppose further that all countries have *the same* steady-state income growth rate and *the same* unemployment rate.[27]

A regression of the form of (32) would return the estimates

$$\begin{aligned} \eta_0 &= \gamma \\ \eta_1 &= 0. \end{aligned}$$

The regression specification suffers from an *omitted variable* bias caused by the influence of the (endogenous) $\mu\nabla^i$ term. In our scenario, the omitted term is correlated with the included variable r^i (and if our scenario is exact, the correlation is perfect). Thus, estimates obtained from the log-linearized Euler equation specification in (32) will be biased estimates of ρ^{-1}. For a thorough discussion of this econometric problem, see Carroll (2001). For a demonstration that the problem is of pratical importance in (macroeconomic) empirical studies, see Parker and Preston (2005).

[27] The key point holds if countries have different growth rates; this stylized example is merely an illustration.

3 Applications

Here we provide some illustrations of how the formulas derived above can be used to cleanly answer some questions that would be much more difficult to address with traditional techniques.

3.1 The Interest Elasticity of Saving

A substantial empirical literature attempted to measure something called ‘the interest elasticity of saving’ from Wright (1967) to the early 1980s. The inability of that literature to reach any consensus led Summers (1981) and others to turn to theoretical analysis of the problem.

As those papers showed, in a partial equilibrium infinite horizon perfect foresight context, the question ‘what is the interest elasticity of saving’ does not have a sensible answer.

In that context, an impatient consumer will run his net worth down to the sticking point defined by the maximum possible amount that he can borrow. If the impatient consumer is constrained to have a minimum wealth of zero the interest elasticity of saving well defined: It is zero, because the impatient consumer will remain at zero wealth for any variation in the interest rate such that he continues to be impatient.

The only other answer to the question (in this perfect foresight context) comes from an increase in the interest rate that switches the agent from being impatient to being patient. A consumer who is patient (relative to the prevailing interest rate) will accumulate assets forever, so in principle an arbitrarily small increment to the interest rate could (eventually) produce an arbitrarily large increase in wealth. Effectively, the interest elasticity of saving in such a case would be infinity.

It is tempting to blame these extreme results on the infinite horizon assumption. But Summers (1981) showed that while the answers obtainable from finite-horizon (life cycle) models were finite, they were still extremely – implausibly – large.

Subsequent numerical solutions showed that for specific combinations of numerical assumptions the response of saving to interest rates can be radically muted, compared to the perfect foresight benchmark. (See, e.g., Carroll (1997) or Cagetti (2001)). But such results are a leading example of the impenetrability of the numerical literature: Adepts in the literature are persuaded that the results are correct, but little intuition emerges.

In our framework, some progress can be made by thinking carefully about the results above. Define the saving rate as the amount of saving divided by market resources m^e i.e. the personal saving rate out of current disposable income. Saving is simply market resources minus consumption. Thus, the saving rate is given by:

$$s_t^e = \frac{m_t^e - c_t^e}{m_t^e} = 1 - c_t^e/m_t^e = 1 - \pi \tag{33}$$

where the quantity π is defined in (20).

Figures 14 and 15 show the value of the target saving rate s^e, as the coefficient of relative risk aversion ρ and the transition probability μ are varied. Figures 16 and 17 show the value of the elasticity of the target saving rate s^e with respect to the interest rate R, as the coefficient of relative risk aversion ρ and the transition probability μ are varied.

The intuition for these results can be understood be thinking carefully about π: [I bet you can figure out something useful to say here!]

3.2 An Empirical Application

The tractable buffer-stock model emphasizes three factors that affect saving and that might vary substantially over time. First, because the precautionary motive decreases with wealth, the saving rate decreases as market resources m increase. Secondly, because an expansion in the availability of credit reduces the target level of wealth m, the saving rate decreases as credit conditions tighten. Thirdly, because of the precautionary saving motive, the saving rate increases as unemployment risk μ rises. Carroll, Slacalek, and Sommer (2013) estimate a structural version of the tractable model on U.S. data for the 1966-2011 period. Their main findings are: increased credit availability accounts for most of the secular decline in the saving rate; the gap between target and actual wealth accounts for the bulk of the business-cycle variation (including an important part attributable to cyclical movements in the precautionary motive).

The gist of the structural estimation can be captured by a simple, linear reduced-form model:

$$s_t = \gamma_1 + \gamma_m m_t + \gamma_{\text{CEA}} \text{CEA}_t + \gamma_{\text{EU}}\, \mathbb{E}_t\, u_{t+4} + \boldsymbol{\gamma}'_X \boldsymbol{X}_t + \epsilon_t, \tag{34}$$

where market resources m are measured as the ratio of household net worth to disposable income, lagged by one period; where CEA stands for "Credit Easing Accumulated", an index measure of credit supply; where u_{t+4} is a proxy for μ based on survey responses to four-quarter-ahead expected changes in the unemployment rate; and where the vector $\boldsymbol{X}_t$ collects other factors outside the scope of the model, e.g. demographics, corporate saving, government saving. The theory suggests that the regression coefficients should have the following signs:

$$\gamma_m < 0, \gamma_{\text{CEA}} < 0, \gamma_{\text{EU}} > 0,$$

Table 5 reports the estimated coefficients from the univariate regression (34). The three coefficients have the predicted signs and are statistically significant. This parsimonious regression captures about 90 percent of the variation in saving.

The estimated coefficients on net wealth suggest a long-run marginal propensity to consume of about 1.2 cents out of a dollar of total wealth. This estimate is low compared to studies that do not explicitly account for credit conditions (the usual range of estimates is about 3 – 7 cents). Structural estimates of the model reinforce these findings, leading Carroll, Slacalek, and Sommer (2013) to suggest that much of what the existing literature has interpreted as pure "wealth effects" may instead reflect some combination of precautionary saving and credit availability.

3.3 An International Capital Flows Application

The high saving rates and rapid accumulation of foreign reserves in emerging economies and the associated financial imbalances between regions present economists with a number of puzzles. One interpretation of these global trends is that they reflect, in some important part, a precautionary accumulation against the risks associated with economic and financial globalization, e.g. increased exposure to external demand and supply shocks; increased vulnerability to exchange rate misalignments and sudden stops; international business cycle spillovers and contagion. Carroll and Jeanne (2009) build a small-open macroeconomy version of the tractable model of buffer-stock saving to explore some of these ideas. Individuals are matched to a job at birth, face a constant risk of permanent job-loss throughout their working life (equivalent to uncertain retirement), and die with a fixed probability independent of age. In aggregate equilibrium, labor market flows are in balance and there exists a well-defined distribution of wealth.

Among other things, Carroll and Jeanne (2009) analyze the long-run consequences of a fall in the desired stock of wealth outside of the United States. They show that domestic welfare is unambiguously reduced, in both the short and long run. The main effect operates via a decrease in the capital–output ratio: the world interest rate rises, the domestic real wage falls, stimulating exports during the transition, with the resulting stimulus to domestic output insufficient to offset the greater reduction in investment. Foreign welfare is also reduced in the long run, but may increase in the short run for those generations alive when net foreign assets accumulated by previous generations are used to finance the increase in consumption induced by the fall in the desired stock of wealth.

4 Conclusions

The core logic of our tractable model of buffer-stock saving, despite its simplicity, emerges in almost every aspect under more realistic assumptions that allow for transitory shocks, permanent shocks, and life-cycle labor-market transitions calibrated to match the details of the household income process, but after much more work (Carroll, 2016).

We hope that the simplicity of our framework will encourage its use in analyzing questions that have so far side-stepped the role of nonfinancial risk. For example, Carroll and Jeanne (2009) construct a fully articulated model of international capital mobility for a small open economy using our buffer-stock model as the core element. We can envision a variety of other purposes the model could serve, including applications to topical questions such as the effects of risk in a search model of unemployment.

Figure 6 Effect of Lower ϑ On Consumption Function

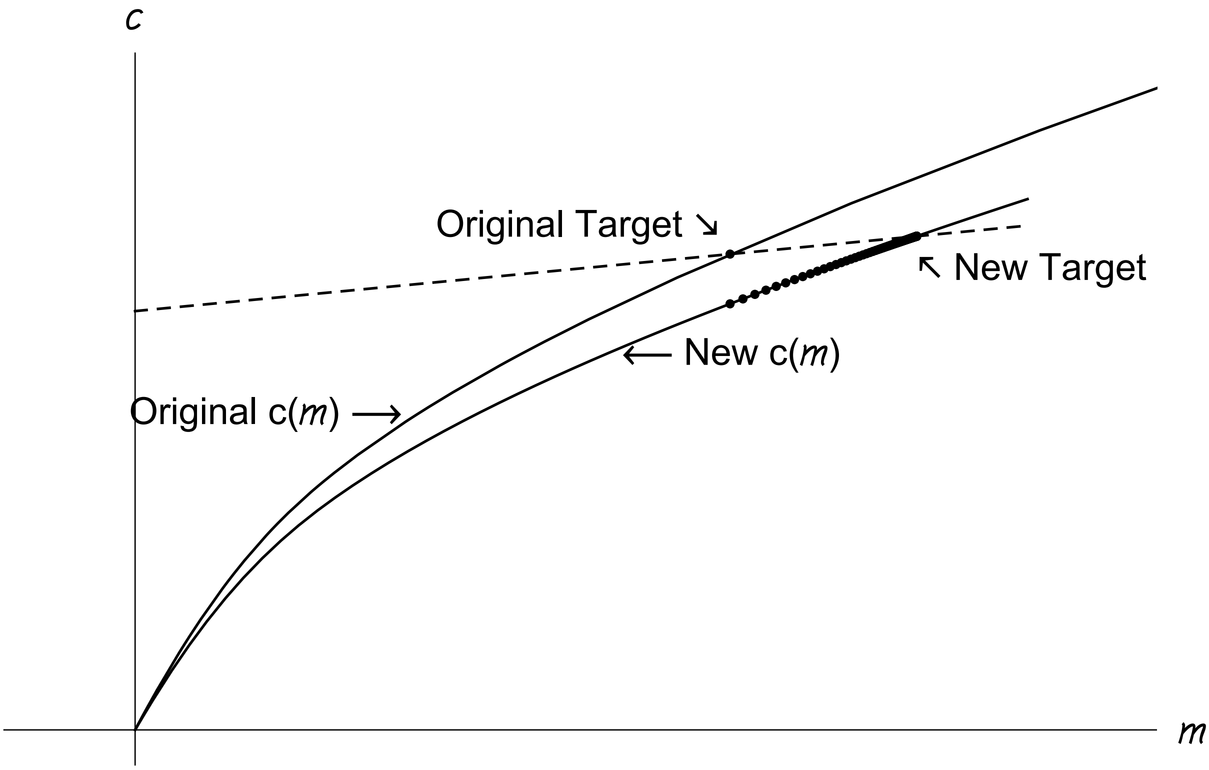


Figure 7 Path of c^e Before and After ϑ Decline

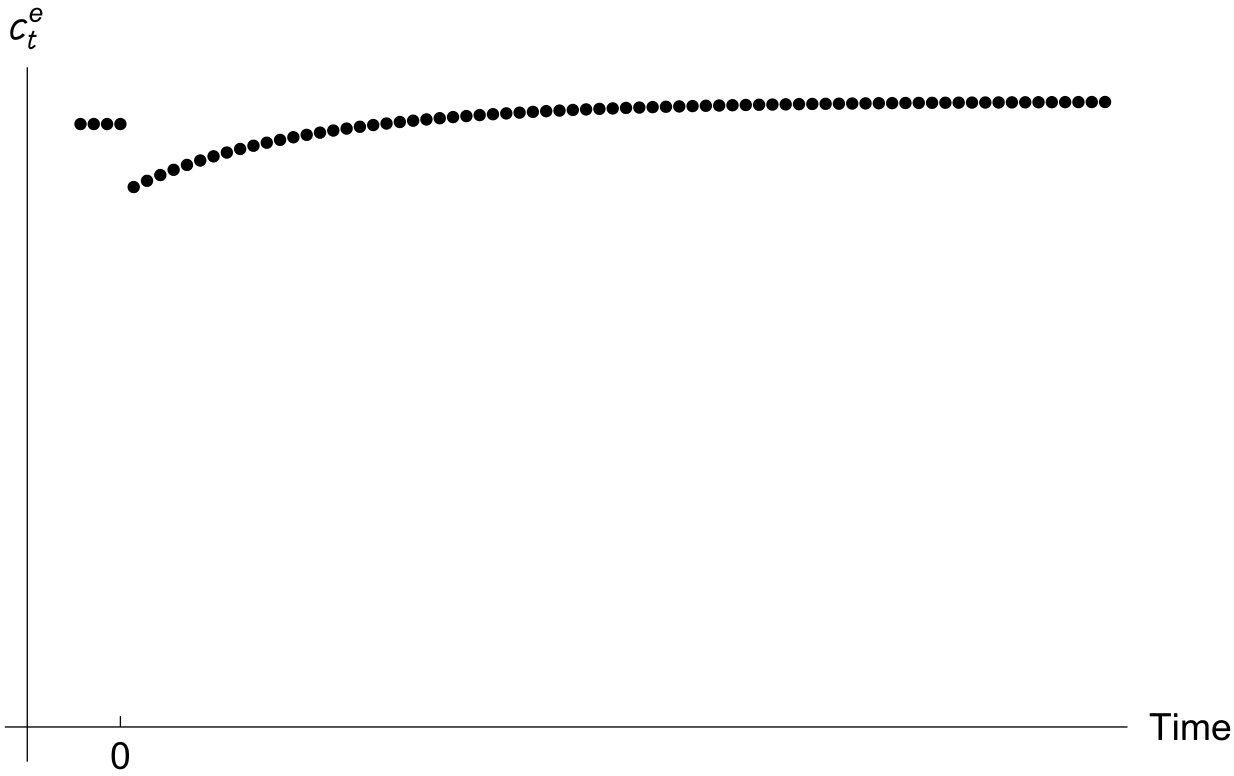


Figure 8 Path of m^e Before and After ϑ Decline

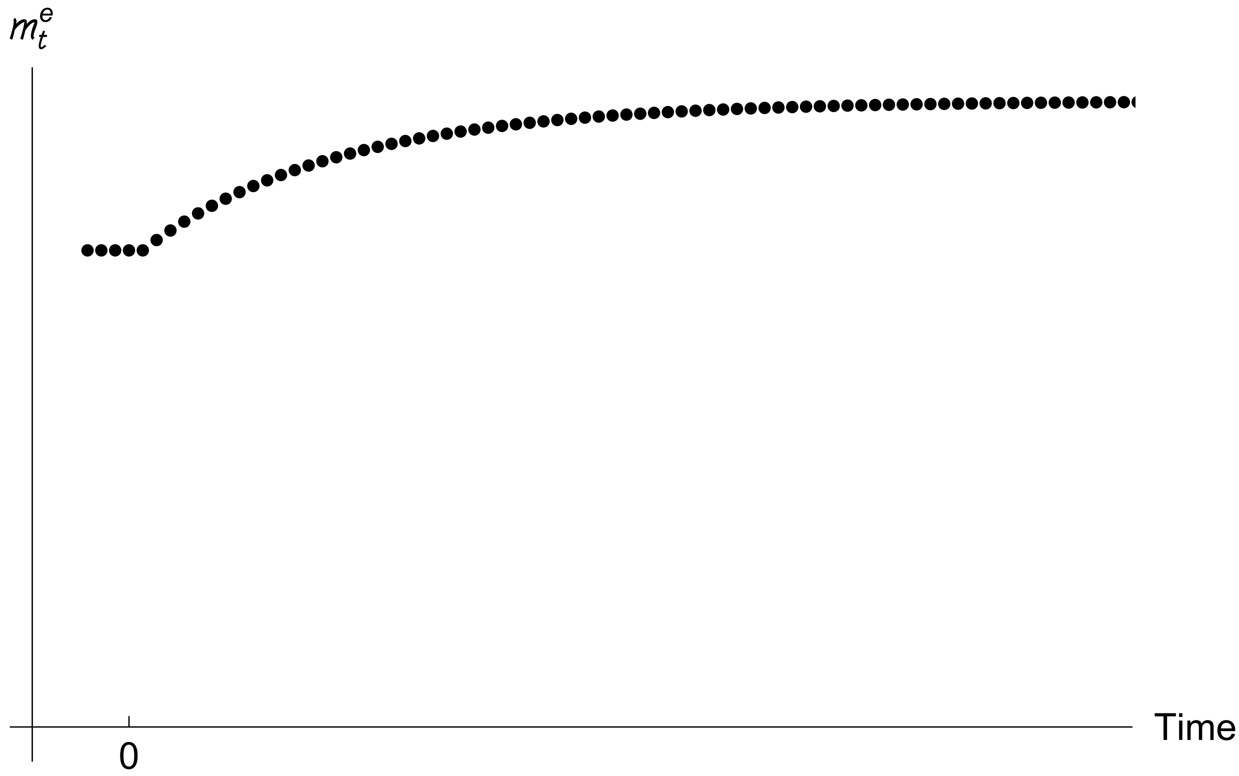


Figure 9 Marginal Propensity to Consume κ_t Before and After ϑ Decline

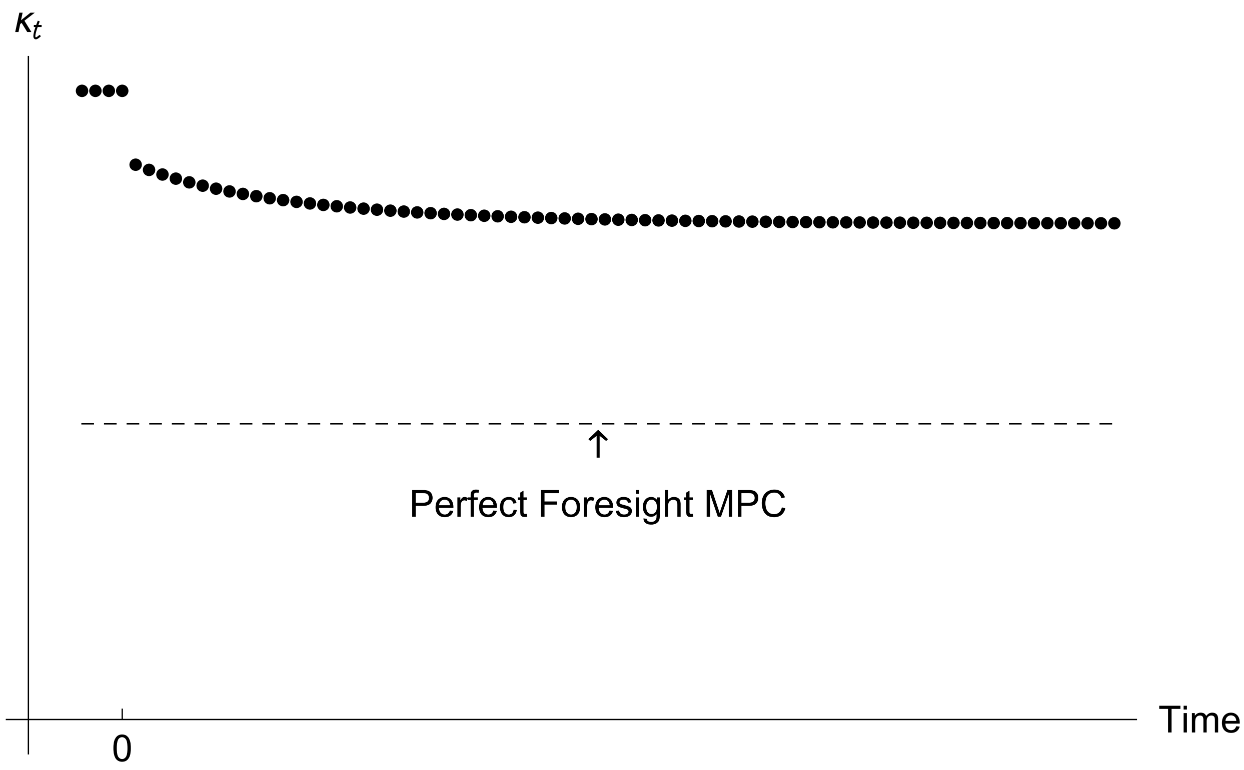


Figure 10 Target Wealth/Income Ratio, as ρ Varies with μ Fixed

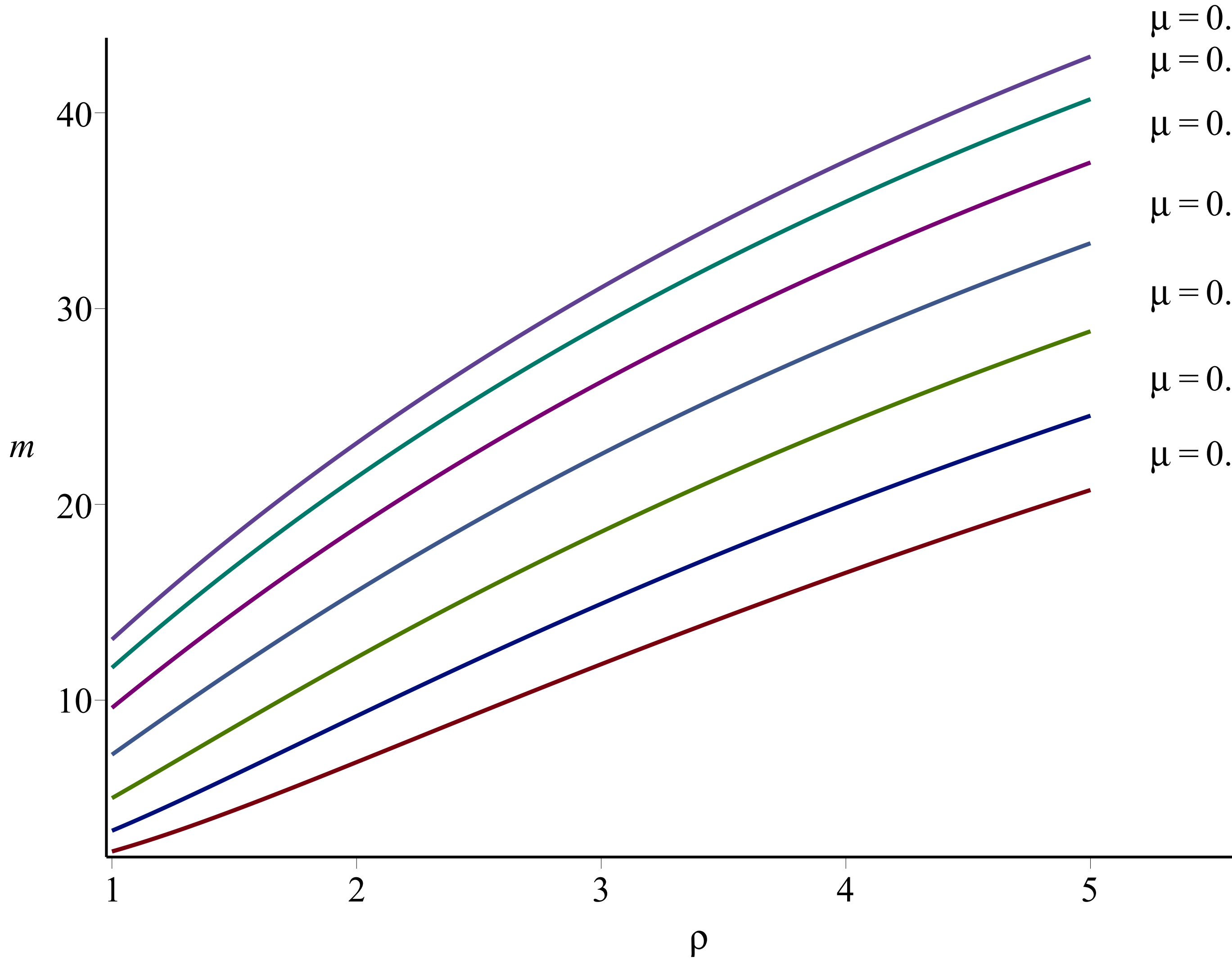


Figure 11 Target Wealth/Income Ratio, as μ Varies with ρ Fixed

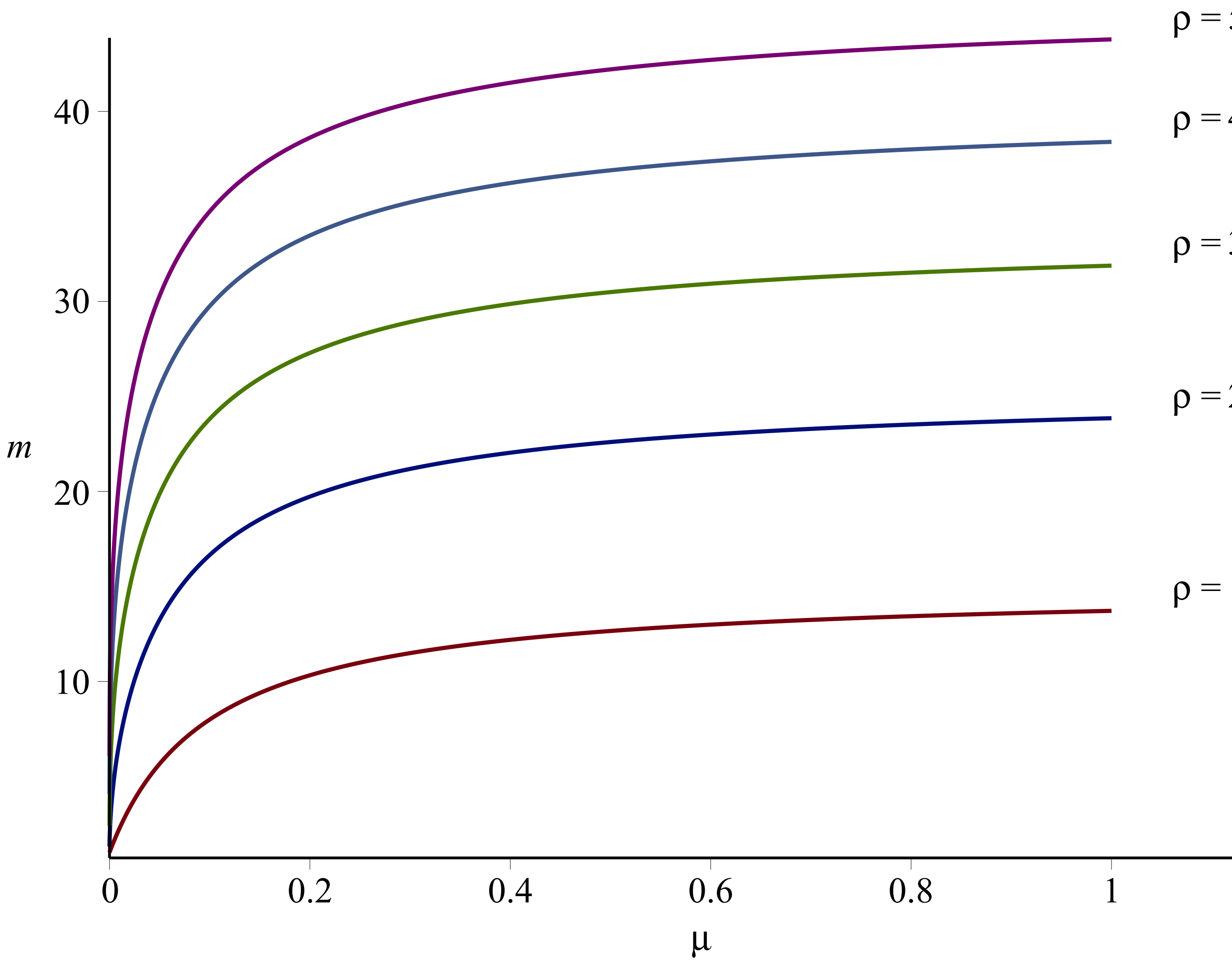


Figure 12 Marginal Change in Target Ratio, as ρ Varies with μ Fixed

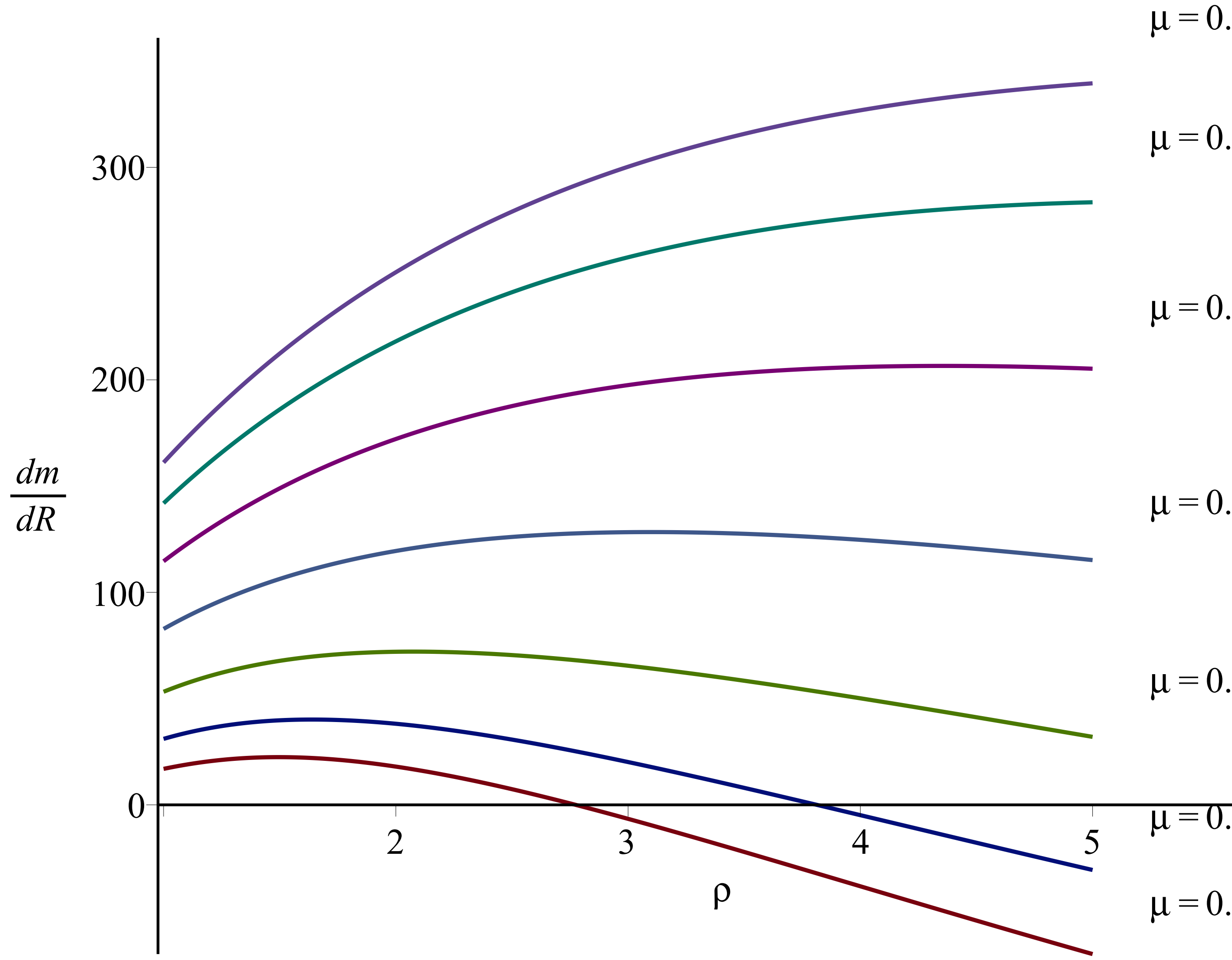


Figure 13 Marginal Change in Target Ratio, as μ Varies with ρ Fixed

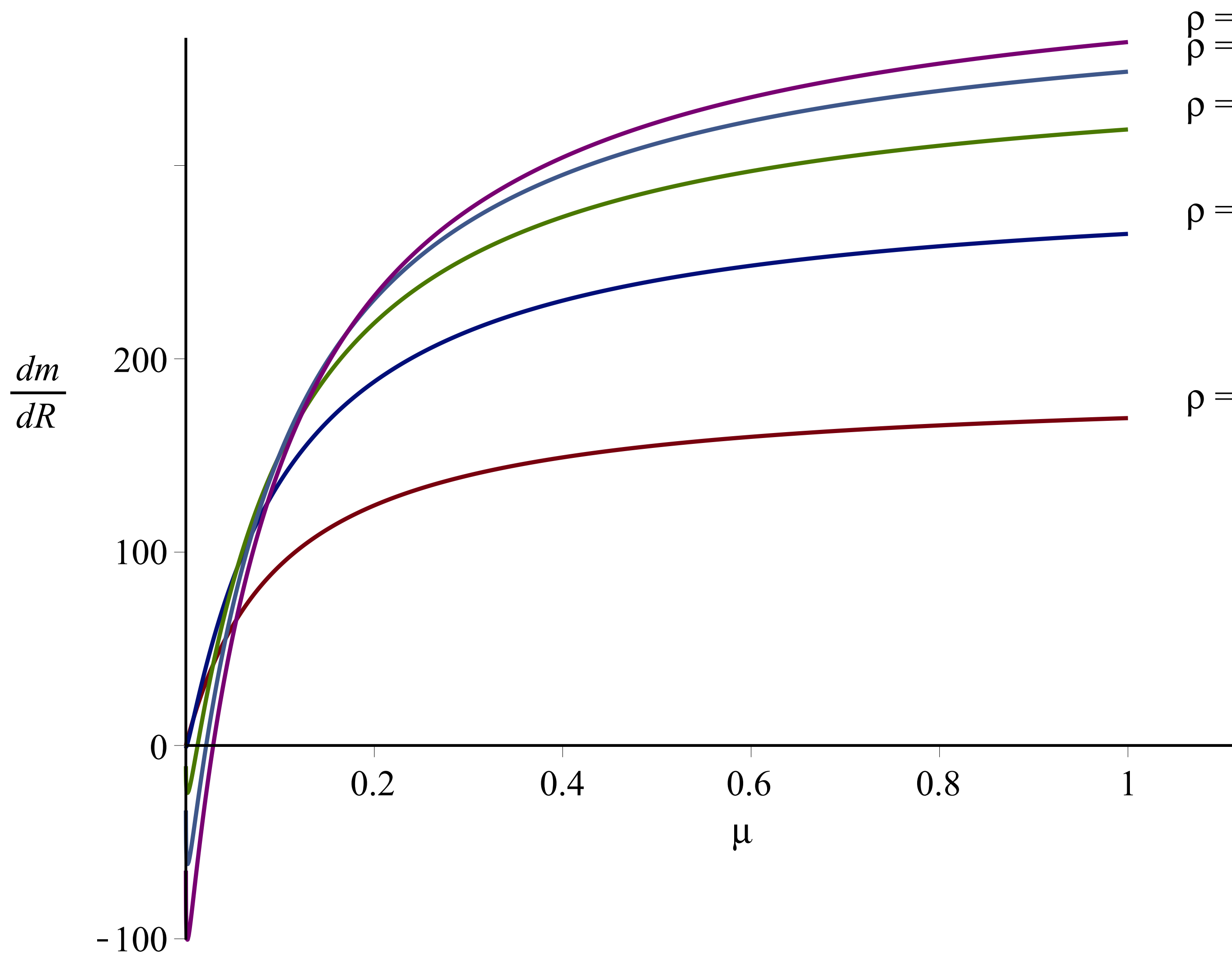


Figure 14 Target Saving Rate, as ρ Varies with μ Fixed

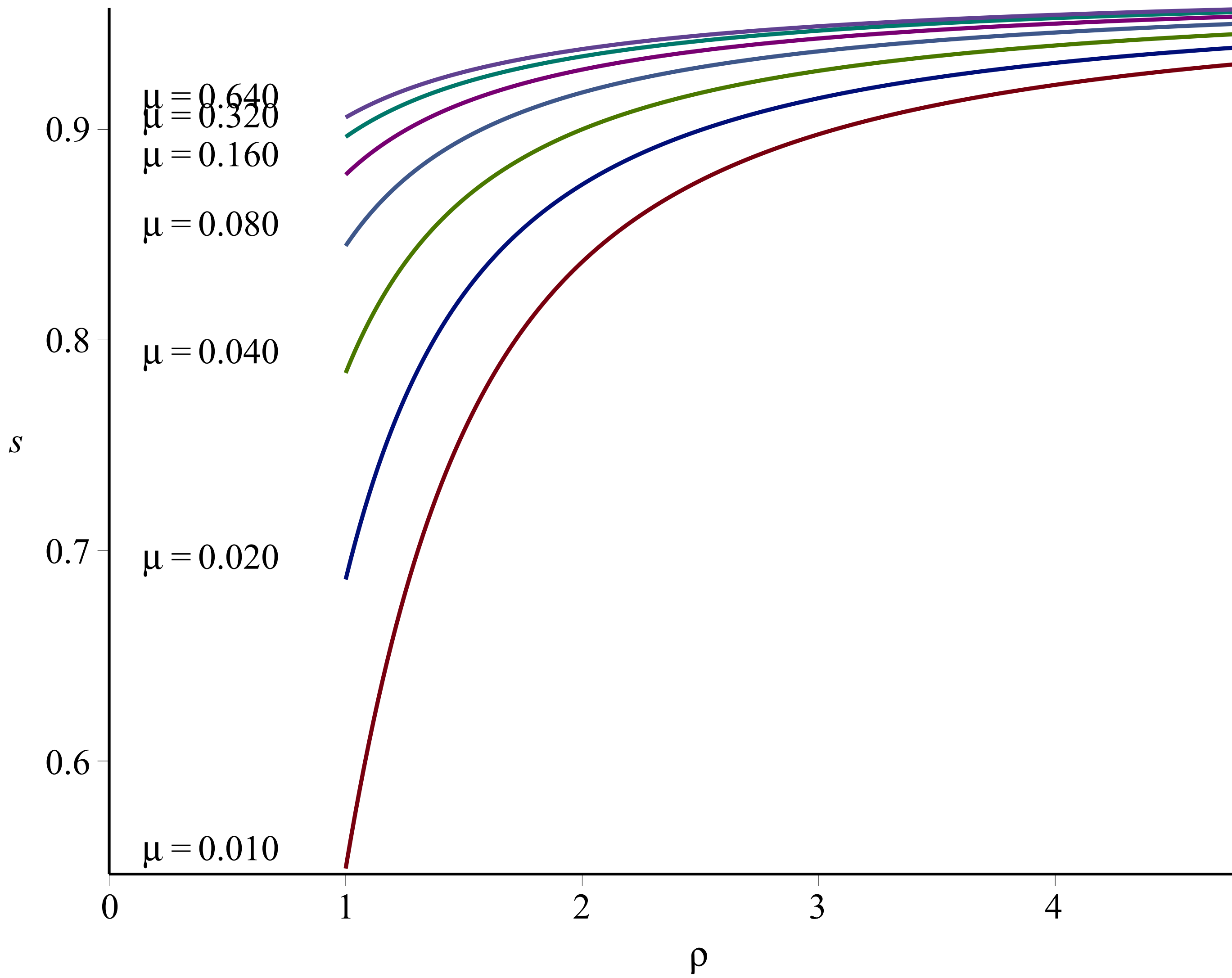


Figure 15 Target Saving Rate, as μ Varies with ρ Fixed

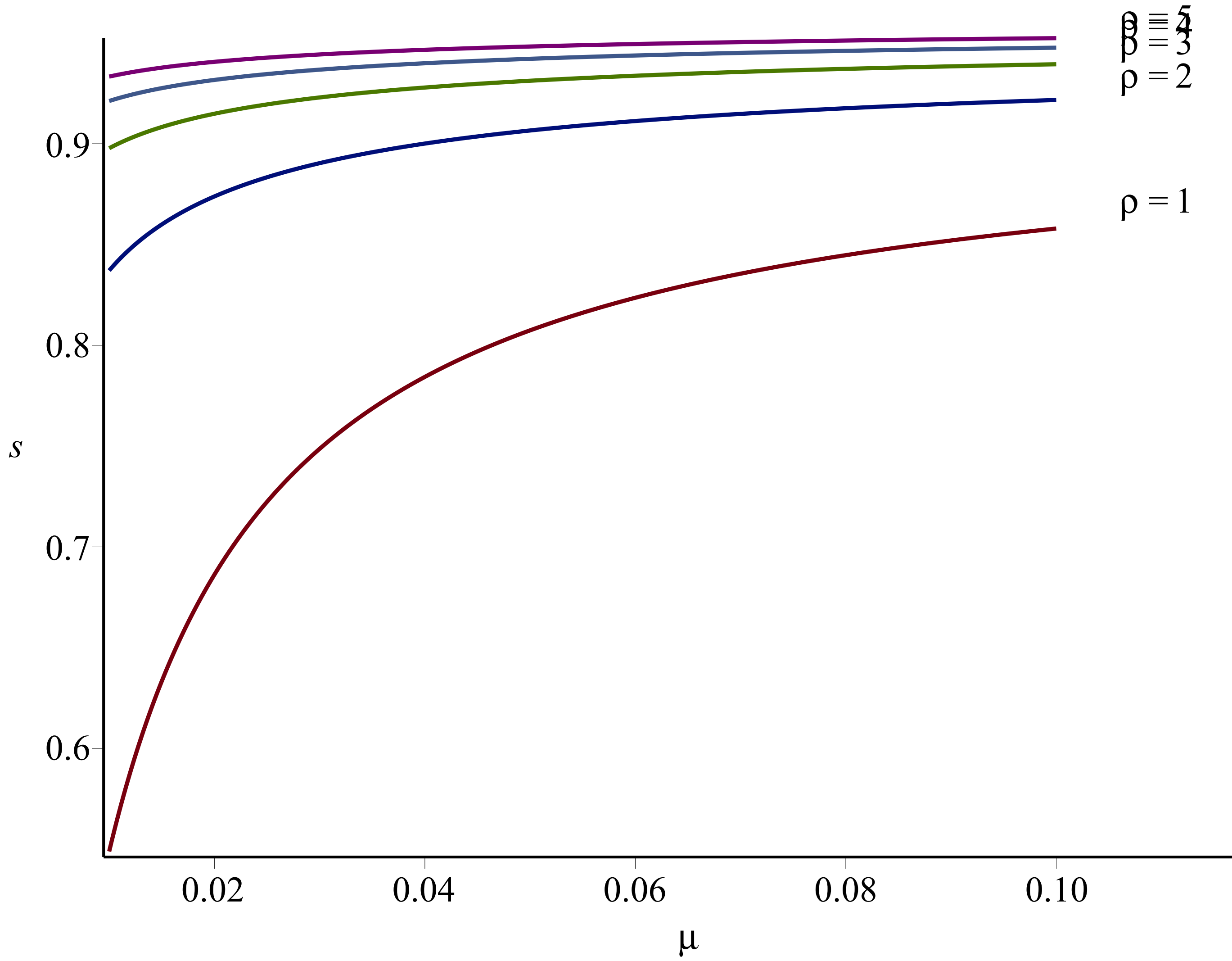


Figure 16 Elasticity of Target Saving Rate to changes in the Interest Rate, as ρ Varies with μ Fixed

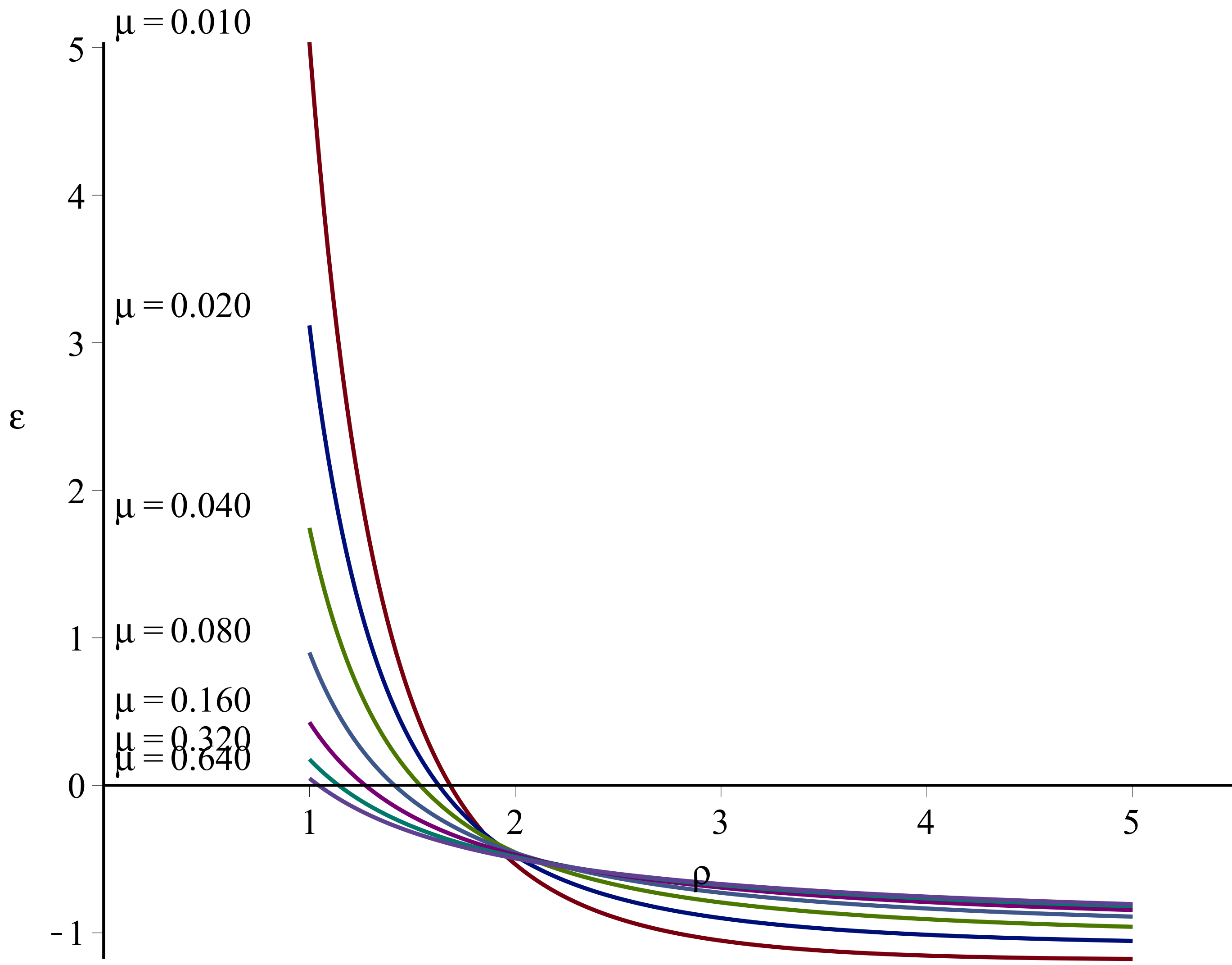


Figure 17 Elasticity of Target Saving Rate to changes in the Interest Rate, as μ Varies with ρ Fixed

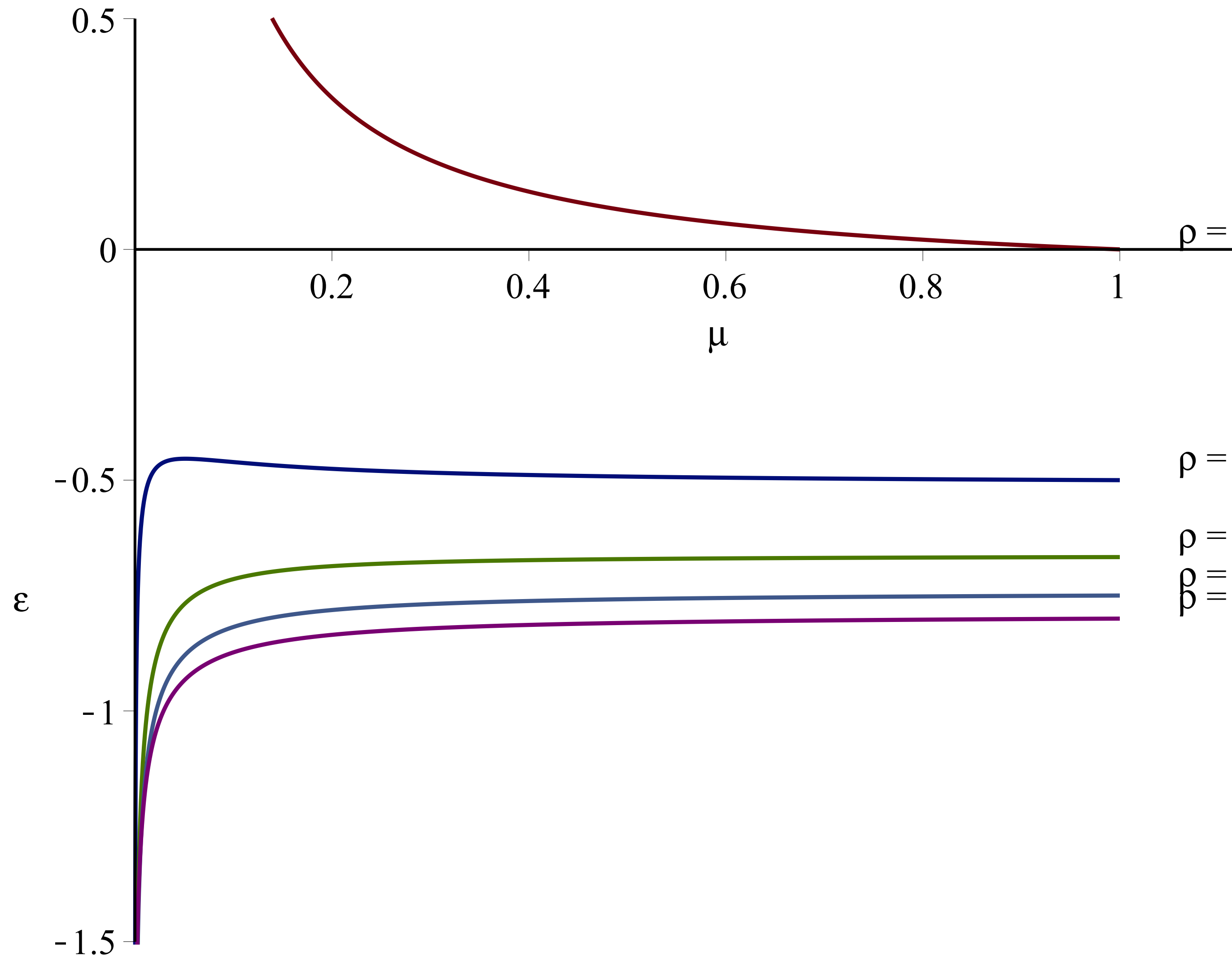


Table 1

Values of the Target Wealth/Income Ratio as ρ and μ Vary

$\mu \setminus \rho$	1	2	3	4	5	6	7	8
0.01	2.2	6.8	11.8	16.5	20.7	24.5	28.0	31.1
0.02	3.3	9.2	15.0	20.0	24.5	28.5	32.0	35.2
0.04	5.0	12.2	18.6	24.1	28.8	33.0	36.6	39.7
0.08	7.2	15.6	22.6	28.4	33.3	37.6	41.2	44.4
0.16	9.6	18.8	26.3	32.4	37.5	41.8	45.5	48.7
0.32	11.7	21.4	29.1	35.5	40.7	45.1	48.9	52.2
0.64	13.1	23.1	31.1	37.5	42.9	47.4	51.2	54.5

Source : Authors' calculations.

Table 2

Effect of Cut in After-Tax Interest Rate on Target Ratio (%)

$\mu \setminus \rho$	1	2	3	4	5	6	7	8
0.01	6.86	2.14	-1.09	-2.87	-3.93	-4.60	-5.034	-5.33
0.02	8.65	3.64	.86	-.71	-1.69	-2.35	-2.81	-3.13
0.04	9.95	5.38	3.06	1.67	.74	.07	-.42	-.80
0.08	10.76	7.13	5.24	4.01	3.13	2.45	1.92	1.49
0.16	11.21	8.60	7.08	6.00	5.17	4.50	3.96	3.50
0.32	11.46	9.64	8.40	7.44	6.66	6.02	5.48	5.015
0.64	11.58	10.28	9.22	8.35	7.62	7.00	6.47	6.02

Source : Authors' calculations.

Table 3

Values of the Target Saving Rate as ρ and μ Vary

$\mu \setminus \rho$	1	2	3	4	5	6	7	8
0.01	.549	.837	.898	.921	.933	.941	.946	.949
0.02	.686	.874	.915	.932	.941	.946	.950	.953
0.04	.784	.900	.923	.940	.947	.951	.954	.956
0.08	.845	.918	.937	.946	.951	.954	.957	.959
0.16	.879	.928	.943	.950	.954	.957	.960	.960
0.32	.896	.935	.947	.953	.956	.959	.961	.962
0.64	.906	.938	.949	.954	.958	.960	.961	.963

Source : Authors' calculations.

Table 4

Elasticity of the saving Rate to changes in the Interest Rate, as ρ and μ vary

$\mu \setminus \rho$	1	2	3	4	5	6	7	8
0.01	4.78	-.60	-1.08	-1.16	-1.17	-1.17	-1.16	-1.15
0.02	2.99	-.52	-.91	-1.01	-1.05	-1.06	-1.07	-1.07
0.04	1.68	-.48	-.80	-.90	-.95	-.97	-.99	-1.00
0.08	.87	-.47	-.73	-.83	-.88	-.91	-.93	-.94
0.16	.41	-.47	-.69	-.78	-.83	-.86	-.89	-.90
0.32	.17	-.48	-.66	-.75	-.80	-.84	-.86	-.88
0.64	.046	-.49	-.65	-.74	-.79	-.82	-.84	-.86

Source : Authors' calculations.

Table 5

Estimates of the Tractable Model on Post-War U.S. Data

	γ_1	γ_m	γ_{CEA}	γ_{EU}	Overall
	15.226	-1.183	-6.121	0.287	
	(2.157)	(0.347)	(0.573)	(0.075)	
$\bar{R}^2$					0.895
p–value					0.000
Durbin–Watson					0.933

Notes : Estimation sample 1966Q2–2011Q1. Statistically significant at the 1% level. Newey-West standard errors at 4 lags. Source : Carroll, Slacalek, and Sommer (2013).

Appendix

A.1 Taylor Approximation for Consumption Growth

PT: *Minor edits in the appendix, mostly presentation.* Applying a second-order Taylor approximation to (13), simplifying, and rearranging yields:

$$
\begin{aligned}
\left\{1+\mu\left[\left(\frac{c^e_{t+1}}{c^u_{t+1}}\right)^{\rho}-1\right]\right\}^{1/\rho} &= \left\{1+\mu\left[\left(\frac{c^u_{t+1}+c^e_{t+1}-c^u_{t+1}}{c^u_{t+1}}\right)^{\rho}-1\right]\right\}^{1/\rho} \\
&= \{1+\mu\left[(1+\nabla_{t+1})^{\rho}-1\right]\}^{1/\rho} \\
&\approx \left\{1+\mu\left[1+\rho\nabla_{t+1}+\rho(\nabla_{t+1})^2\omega-1\right]\right\}^{1/\rho} \\
&= \left\{1+\rho\mu(\nabla_{t+1}+(\nabla_{t+1})^2\omega)\right\}^{1/\rho} \\
&\approx 1+\mu\,(1+\nabla_{t+1}\omega)\,\nabla_{t+1}.
\end{aligned}
$$

A.2 The Exact Formula for m

The steady-state value of m^e, denoted m, is the solution of (19)-(21), which may be computed in closed form. To simplify some of the intermediate steps in the algebra, define the short-hand notation: $\zeta \equiv \mathcal{R}\kappa^u\Pi$ and $\mathcal{R} \equiv \mathsf{R}\Gamma^{-1}$ and $\Pi \equiv \left(\frac{\text{Þ}_{\Gamma}^{-\rho}-(1-\mu)}{\mu}\right)^{1/\rho}$. From this: $\mathsf{R}\kappa^u\Pi = \zeta\Gamma$. A series of straightforward manipulations yields:

$$
\begin{aligned}
\left(\frac{\zeta}{1+\zeta}\right)m &= (1-\mathcal{R}^{-1})m+\mathcal{R}^{-1} \\
\left(\mathcal{R}\frac{\zeta}{1+\zeta}\right)m &= (\mathcal{R}-1)m+1 \\
\left(\mathcal{R}\left\{\frac{\zeta}{1+\zeta}-1\right\}+1\right)m &= 1 \\
\left(\mathcal{R}\left\{\frac{\zeta-(1+\zeta)}{1+\zeta}\right\}+\frac{1+\zeta}{1+\zeta}\right)m &= 1 \\
\left(\frac{1+\zeta-\mathcal{R}}{1+\zeta}\right)m &= 1 \\
m &= \left(\frac{1+\zeta}{1+\zeta-\mathcal{R}}\right) \\
m &= \left(\frac{1+\zeta+\mathcal{R}-\mathcal{R}}{1+\zeta-\mathcal{R}}\right) \\
m &= 1+\left(\frac{\mathcal{R}}{1+\zeta-\mathcal{R}}\right)
\end{aligned}
$$

$$m = 1 + \left(\frac{\mathsf{R}}{\Gamma + \zeta\Gamma - \mathsf{R}}\right). \tag{35}$$

A first point about this formula is that:

$$\zeta\Gamma = \mathsf{R}\kappa^{u}\left(1 + \frac{(\text{Þ}/\Gamma)^{-\rho} - 1}{\mu}\right)^{1/\rho} \tag{36}$$

is likely to increase as μ vanishes to zero.[28] Note that (36) tends to infinity as $\mu \to 0$, which implies that $\lim_{\mu\to 0} m = 1$. This is precisely what would be expected since an impatient consumer is self-constrained to keep $m^e > 1$. Thus, as the risk gets infinitesimally small, the amount by which the target m^e exceeds its minimum possible value shrinks to zero.

We now show that the RIC and GIC ensure that the denominator of the fraction in (35) is positive:

$$\begin{aligned}
\Gamma + \zeta\Gamma - \mathsf{R} &= \Gamma + \mathsf{R}\kappa^{u}\Pi - \mathsf{R} \\
&= \Gamma + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\left(\frac{(\frac{(\mathsf{R}\beta)^{1/\rho}}{\Gamma})^{-\rho} - 1}{\mu} + 1\right)^{1/\rho} - \mathsf{R} \\
&> \Gamma + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\left(\frac{(\frac{(\mathsf{R}\beta)^{1/\rho}}{\Gamma})^{-\rho} - 1}{1} + 1\right)^{1/\rho} - \mathsf{R} \\
&= \Gamma + \mathsf{R}\left(1 - \frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)\frac{\Gamma}{(\mathsf{R}\beta)^{1/\rho}} - \mathsf{R} \\
&= \Gamma + \mathsf{R}\frac{\Gamma}{(\mathsf{R}\beta)^{1/\rho}} - \Gamma - \mathsf{R} \\
&= \mathsf{R}\left(\frac{\Gamma}{(\mathsf{R}\beta)^{1/\rho}} - 1\right) \\
&> 0.
\end{aligned}$$

A.3 An Approximation for m

We can obtain further insight into (35) by using a judicious mix of first- and second-order Taylor expansions. Define the short-hand ψ:

$$\psi = \frac{(\text{Þ}/\Gamma)^{-\rho} - 1}{\mu}.$$

First, substituting $\kappa^{u} = -\text{þ}_{\mathsf{r}}$ into (35), and computing a second-order Taylor expansion:

$$\begin{aligned}
\zeta\Gamma &= \mathsf{R}\kappa^{u}\,(1 + \psi)^{1/\rho} \\
&\approx -\mathsf{R}\text{þ}_{\mathsf{r}}\left(1 + \rho^{-1}\psi + (\rho^{-1})(\rho^{-1} - 1)(\psi^{2}/2)\right)
\end{aligned}$$

[28]The effect is not necessarily monotonic because μ affects $\text{Þ}/\Gamma$ as well as the denominator of (35); however, for plausible calibrations the effect of the denominator predominates.

$$= \quad -\mathsf{R}\text{þ}_{\mathsf{r}}\left(1 + \rho^{-1}\psi\left\{1 + \left(\frac{1-\rho}{\rho}\right)(\psi/2)\right\}\right). \tag{37}$$

Secondly, applying a first-order Taylor expansion to ψ:

$$\psi = \frac{(1 + \text{þ}_{\gamma})^{-\rho} - 1}{\mu} \approx \frac{1 - \rho\text{þ}_{\gamma} - 1}{\mu} = -\frac{\rho\text{þ}_{\gamma}}{\mu}. \tag{38}$$

Thirdly, substitute (38) into (37): PT: *removed a line in the equation below (a reminder of signs of different components is not necessary here, is it?):*

$$\zeta\Gamma \quad \approx \quad -\mathsf{R}\text{þ}_{\mathsf{r}}\left(1 - (\text{þ}_{\gamma}/\mu)(1 + (1-\rho)(-\text{þ}_{\gamma}/\mu)/2)\right) \tag{39}$$

$$. \tag{40}$$

By our definition of ω (the excess of prudence over the logarithmic benchmark):

$$\omega \quad \equiv \quad \frac{\rho - 1}{2}.$$

Equation (35) can then be approximated by:

$$\begin{aligned} m \quad &\approx \quad 1 + \left(\frac{1}{\Gamma/\mathsf{R} - \text{þ}_{\mathsf{r}}\left(1 - (\text{þ}_{\gamma}/\mu)(1 - (-\text{þ}_{\gamma}/\mu)\omega)\right) - 1}\right) \\ &\approx \quad 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + (-\text{þ}_{\mathsf{r}})\left(1 + (-\text{þ}_{\gamma}/\mu)(1 - (-\text{þ}_{\gamma}/\mu)\omega)\right)}\right) \end{aligned} \tag{41}$$

where negative signs have been preserved in front of the þ_{r} and þ_{γ} terms as a reminder that the GIC and the RIC imply these terms are themselves negative (so that $-\text{þ}_{\mathsf{r}}$ and $-\text{þ}_{\gamma}$ are positive). An increase in relative risk aversion ρ, *ceteris paribus*, raises ω and thereby lowers the denominator of (41). This reasoning suggests that greater risk aversion results in a larger target level of wealth.[29]

The formula also provides insight about how the human wealth effect works in equilibrium. All else equal, the human wealth effect is captured by the $(\gamma - \mathsf{r})$ term in the denominator of (41): it is obvious that a larger value of γ results in a smaller target value for m. However, the size of the human wealth effect also depends on the magnitude of the patience and prudence contributions to the denominator, and those terms could dominate the human wealth effect.

For (41) to make sense, we need the denominator of the fraction to be positive. Let:

$$\hat{\text{þ}}_{\gamma} \quad \equiv \quad \text{þ}_{\gamma}(1 - (-\text{þ}_{\gamma}/\mu)\omega). \tag{42}$$

The denominator of (41) is positive if:

$$(\gamma - \mathsf{r}) \quad > \quad \text{þ}_{\mathsf{r}} - \text{þ}_{\mathsf{r}}\hat{\text{þ}}_{\gamma}/\mu\left(\rho^{-1}(\mathsf{r} - \vartheta) - \mathsf{r}\right) - \text{þ}_{\mathsf{r}}\hat{\text{þ}}_{\gamma}/\mu$$

[29] "Suggests" rather than proves, because this derivation uses approximations; plausible numerical calibrations are in agreement with the suggestion.

$$\begin{aligned}
\Rightarrow \gamma &> \rho^{-1}(\mathsf{r} - \vartheta) - \text{þ}_{\mathsf{r}}\hat{\text{þ}}_{\gamma}/\mu \\
\Rightarrow 0 &> \rho^{-1}(\mathsf{r} - \vartheta) - \gamma - \text{þ}_{\mathsf{r}}(\hat{\text{þ}}_{\gamma}/\mu) \\
\Rightarrow 0 &> \text{þ}_{\gamma} - \text{þ}_{\mathsf{r}}(\hat{\text{þ}}_{\gamma}/\mu). \qquad (43)
\end{aligned}$$

From the RIC, we have $\text{þ}_{\mathsf{r}} < 0$; from the GIC, we have $\text{þ}_{\gamma} < 0$; the latter in turn gives $\hat{\text{þ}}_{\gamma} < 0$; and thus condition (43) holds. PT: *I shortened the footnote, might it be removed altogether? I couldn't follow it very well, I wasn't sure what "the above" was referring to.* [30]

The same set of derivations imply that we can replace the denominator in (41) with the negative of the RHS of (43), yielding a more compact expression for the target level of resources:

$$\begin{aligned}
m &\approx 1 + \left(\frac{1}{\text{þ}_{\mathsf{r}}(\hat{\text{þ}}_{\gamma}/\mu) - \text{þ}_{\gamma}}\right) \\
&= 1 + \left(\frac{1/(-\text{þ}_{\gamma})}{1 + (-\text{þ}_{\mathsf{r}}/\mu)(1 + (-\text{þ}_{\gamma}/\mu)\omega)}\right). \qquad (44)
\end{aligned}$$

This formula makes plain that an increase in either form of impatience raises the denominator of the fraction in (44) and thus reduces the target level of assets.

Two specializations of the formula are particularly useful. The first useful special case is $\rho = 1$ (logarithmic utility). In this case,

$$\begin{aligned}
\omega &= 0 \\
\text{þ}_{\mathsf{r}} &= -\vartheta \\
\text{þ}_{\gamma} &= \mathsf{r} - \vartheta - \gamma
\end{aligned}$$

and the approximation reduces to:

$$m \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma + \vartheta - \mathsf{r})/\mu)}\right) \qquad (45)$$

Equation (45) neatly captures the effect of an increase in human wealth (an increase in γ or a decrease in r), the effect of an increase in impatience ϑ, the effect of a decrease in unemployment risk μ: these reduce target wealth.

The second useful special case is $\mathsf{r} = \vartheta$ (but $\rho > 1$). In this case,

$$\begin{aligned}
\text{þ}_{\mathsf{r}} &= -\vartheta \\
\text{þ}_{\gamma} &= -\gamma \\
\hat{\text{þ}}_{\gamma} &= -\gamma(1 - (\gamma/\mu)\omega)
\end{aligned}$$

[30] We implicitly assume that, in the second-order Taylor approximation in (37), the absolute value of the second-order term is negligeable relative to the first-order term, i.e. $|\rho^{-1}\psi| \geq |(\rho^{-1})(\rho^{-1} - 1)(\psi^2/2)|$.

and the approximation becomes:

$$m \approx 1 + \left(\frac{1}{(\gamma - \mathsf{r}) + \vartheta(1 + (\gamma/\mu)(1 - (\gamma/\mu)\omega))} \right) \tag{46}$$

Equation (46) shows that an increase in the prudence term ω shrinks the denominator and thereby boosts the target level of wealth.[31]

[31] It would be inappropriate to use the equation to consider the effect of an increase in r because the equation was derived under the assumption $\vartheta = \mathsf{r}$, so r is not free to vary.

Table 6 Summary of Notation

a	-	end-of-period t assets (after consumption decision)
b	-	middle-of-period t balances (before consumption decision)
c	-	consumption
ℓ	-	personal labor productivity
m	-	market resources (capital, capital income, and labor income)
R, r	-	interest factor, rate
W	-	aggregate wage
G	-	growth factor for aggregate wage rate W
$\Gamma \equiv \mathsf{G}/(1-\mu)$	-	conditional (on employment) growth factor for individual labor income
γ	-	$\log \Gamma$, conditional growth *rate* for labor income
β	-	time preference factor ($= 1/(1+\vartheta)$)
ξ	-	dummy variable indicating the employment state, $\xi \in \{0, 1\}$
κ	-	marginal propensity to consume
ρ	-	coefficient of relative risk aversion
ϑ	-	time preference rate ($\approx -\log \beta$)
μ	-	probability of falling into permanent unemployment
Þ, þ	-	absolute patience factor, rate
Þ$_{\Gamma}$, þ$_{\gamma}$	-	growth patience factor, rate
Þ$_{\mathsf{R}}$, þ$_{\mathsf{r}}$	-	return patience factor, rate
ω	-	excess prudence factor ($= (\rho - 1)/2$)
∇	-	proportional consumption drop upon entering unemployment
$\mathcal{R}$	-	short for R/Γ

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