

Curso de integrais duplas e triplas

César Antônio de Magalhães
ceanma@gmail.com

29 de agosto de 2016

Sumário

I	Integrais duplas	3
1	Invertendo os limites de integração - Aula 1	3
2	Determinação da região de integração - Aula 2	5
3	Cálculo de volume - Aula 3	8
4	Invertendo a ordem de integração - Aula 4	9
5	Cálculo de integrais duplas ou iteradas	12
5.1	Aula 5	12
5.2	Aula 6	13
5.3	Aula 7	14
6	Cálculo de área - Aula 8	15
7	Cálculo de volume	17
7.1	Aula 9	17
7.2	Aula 10	17
8	Coordenadas polares	18
8.1	Aula 1	18
A	Derivadas	22
B	Derivadas simples	22

C	Derivadas trigonométricas	22
D	Integrais	22
E	Integrais simples	22
F	Integrais trigonométricas	22
G	Relação entre coordenada cartesina e polar	22
H	Funções trigonométricas	24
I	Determinação do seno, cosseno e tangente	24
J	Círculo trigonométrico	24
K	Identidades trigonométricas	24
L	Relação entre trigonométricas e inversas	24
M	Substituição trigonométrica	24
N	Ângulos notáveis	24

Lista de Tabelas

1	Derivadas simples	22
2	Derivadas trigonométricas	23
3	Integrais simples	24
4	Integrais trigonométricas	26
5	Relação entre coordenada cartesina e polar	26
6	Identidades trigonométricas	27
7	Relação entre trigonométricas e inversas	27
8	Substituição trigonométrica	27
9	Ângulos notáveis	28

Lista de Figuras

1	Integrais duplas - Aula 1 - Exercício I e II	4
2	Integrais duplas - Aula 2 - Exercício I	5
3	Integrais duplas - Aula 2 - Exercício II	5

4	Integrais duplas - Aula 2 - Exercício III	6
5	Integrais duplas - Aula 2 - Exercício IV	7
6	Integrais duplas - Aula 2 - Exercício V	8
7	Integrais duplas - Aula 3 - Exercício I	8
8	Integrais duplas - Aula 3 - Exercício II	9
9	Integrais duplas - Aula 4 - Exercício II	10
10	Integrais duplas - Aula 4 - Exercício III	11
11	Integrais duplas - Aula 5 - Exercício I	12
12	Integrais duplas - Aula 5 - Exercício II	13
13	Integrais duplas - Aula 8 - Exercício I	15
14	Integrais duplas - Aula 9 - Exercício I	17
15	Integrais duplas - Aula 10 - Exercício I	18
16	Coordenadas polares - Aula 01 - Exercício I	19
17	Coordenadas polares - Aula 01 - Exercício II	21
18	Coordenada cartesina e polar	23
19	Determinação do seno, cosseno e tangente	24
20	Círculo trigonométrico	25

Resumo

Exercícios retirados do canal do Youtube, O Matematico [1].

Parte I

Integrais duplas

1 Invertendo os limites de integração - Aula 1

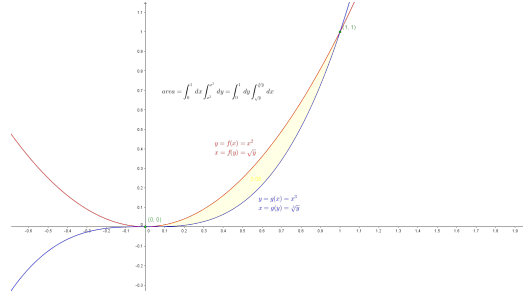
1. Exercício

$$f(x) = x^2; g(x) = x^3$$

$$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$$

$$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$$

Figura 1: Integrais duplas - Aula 1 - Exercício I e II



$$\begin{aligned}
 a &= \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx [y]_{x^3}^{x^2} = \int_0^1 dx [x^2 - x^3] = \\
 &\int_0^1 x^2 dx - \int_0^1 x^3 dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left[\frac{4x^3 - 3x^2}{12} \right]_0^1 = \frac{1}{12} [4x^3 - 3x^2]_0^1 = \\
 &\frac{1}{12} [x^2 (4x - 3)]_0^1 = \frac{1}{12} [1^2 (4 \cdot 1 - 3) - 0^2 (4 \cdot 0 - 3)] = \frac{1}{12} = 0,08\bar{3}
 \end{aligned}$$

2. Exercício

$$f(x) = x^2 \Rightarrow f(y) = \sqrt{y}; \quad g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$$

$$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$$

$$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$$

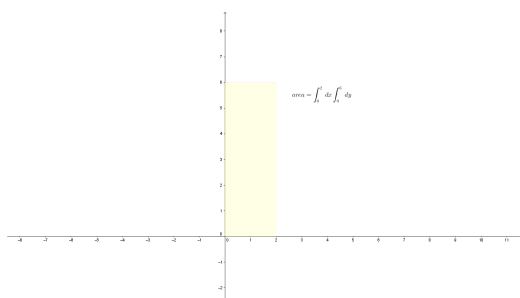
$$\begin{aligned}
 a &= \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy [\sqrt[3]{y} - \sqrt{y}] = \\
 &\int_0^1 \sqrt[3]{y} dy - \int_0^1 \sqrt{y} dy = \int_0^1 y^{\frac{1}{3}} dy - \int_0^1 y^{\frac{1}{2}} dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^1 = \\
 &\left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3} \right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12} \right]_0^1 = \frac{1}{12} [9\sqrt[3]{y^4} - 8\sqrt{y^3}]_0^1 = \\
 &\frac{1}{12} [(9\sqrt[3]{1^4} - 8\sqrt{1^3}) - (9\sqrt[3]{0^4} - 8\sqrt{0^3})] = \frac{1}{12} (9 - 8) = \frac{1}{12} = 0,08\bar{3}
 \end{aligned}$$

2 Determinação da região de integração - Aula 2

1. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 6\}$$

Figura 2: Integrais duplas - Aula 2 - Exercício I

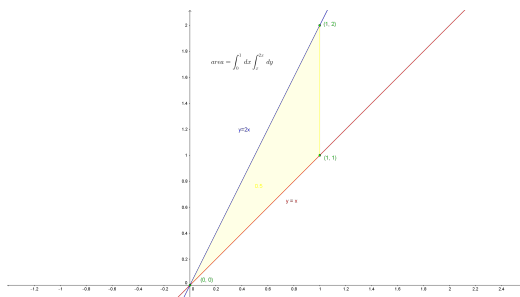


$$\begin{aligned} a &= \int_0^2 dx \int_0^6 dy = \int_0^2 dx [y]_0^6 = \int_0^2 dx [6 - 0] = 6 \int_0^2 dx = 6[x]_0^2 = \\ &6[2 - 0] = 6 \cdot 2 = 12 \end{aligned}$$

2. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, x \leq y \leq 2x\}$$

Figura 3: Integrais duplas - Aula 2 - Exercício II



$$a = \int_0^1 dx \int_x^{2x} dy = \int_0^1 dx [y]_x^{2x} = \int_0^1 dx [2x - x] = 2 \int_0^1 x dx - \int_0^1 x dx =$$

$$\left[\frac{x^2}{2} - \frac{x^2}{2} \right]_0^1 = \left[\frac{2x^2 - x^2}{2} \right]_0^1 = \frac{1}{2} [x^2]_0^1 = \frac{1}{2} [1^2 - 0^2] = \frac{1}{2} = 0,5$$

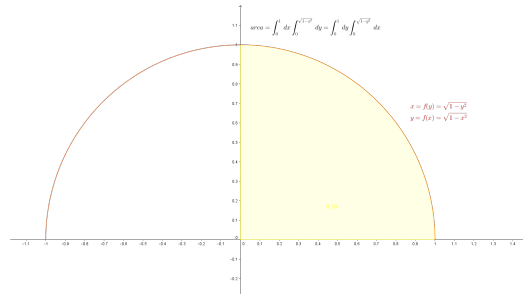
3. Exercício

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, 0 \leq x \leq \sqrt{1-y^2} \right\}$$

$$y = 0, y = 1$$

$$x = 0, x = \sqrt{1-y^2} \Rightarrow x^2 = 1-y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow y = \sqrt{1-x^2}$$

Figura 4: Integrais duplas - Aula 2 - Exercício III



$$a = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy [\sqrt{1-y^2} - 0] =$$

$$\int_0^1 \sqrt{1-y^2} dy = \int_0^1 \sqrt{1-\sin^2(t)} \cos(t) dt = \int_0^1 \sqrt{\cos^2(t)} \cos(t) dt =$$

$$\int_0^1 \cos(t) \cos(t) dt = \int_0^1 \cos^2(t) dt = \int_0^1 \frac{1+\cos(2t)}{2} dt = \frac{1}{2} \int_0^1 [1+\cos(2t)] dt =$$

$$\frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_0^1 \cos(2t) dt = \frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_0^1 \cos(u) \frac{du}{2} = \frac{1}{2} \int_0^1 dt + \frac{1}{4} \int_0^1 \cos(u) du =$$

$$\left[\frac{1}{2}t + \frac{1}{4} \sin(u) \right]_0^1 = \left[\frac{t}{2} + \frac{\sin(2t)}{4} \right]_0^1 = \left[\frac{t}{2} + \frac{2\sin(t)\cos(t)}{4} \right]_0^1 = \left[\frac{t + \sin(t)\cos(t)}{2} \right]_0^1 =$$

$$\frac{1}{2} [\arcsen(y) + y\sqrt{1-y^2}]_0^1 = \frac{1}{2} [(\arcsen(1) + 1 \cdot \sqrt{1-1^2}) - (\arcsen(0) + 0 \cdot \sqrt{1-0^2})] =$$

$$\frac{1}{2} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{4} = 0,785$$

$$y = \text{sen}(t) \Rightarrow dy = \cos(t)dt$$

$$u = 2t \Rightarrow \frac{du}{2} = dt$$

$$\text{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$

$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1 - y^2}$$

$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1 - y^2}}{1} = \sqrt{1 - y^2}$$

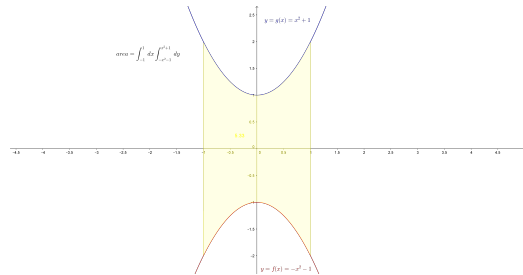
$$y = \text{sen}(t) \Rightarrow t = \arcsen(y)$$

4. Exercício

$$y = x^2 + 1, y = -x^2 - 1; x = 1, x = -1$$

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -x^2 - 1 \leq y \leq x^2 + 1\}$$

Figura 5: Integrais duplas - Aula 2 - Exercício IV

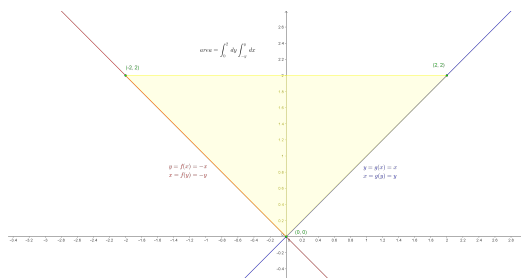


$$\begin{aligned} a &= \int_{-1}^1 dx \int_{f(x)}^{g(x)} dy = \int_{-1}^1 dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^1 dx [y]_{-x^2-1}^{x^2+1} = \\ &= \int_{-1}^1 dx [x^2 + 1 - (-x^2 - 1)] = \int_{-1}^1 dx [x^2 + 1 + x^2 + 1] = \\ &= \int_{-1}^1 dx [2x^2 + 2] = 2 \int_{-1}^1 x^2 dx + 2 \int_{-1}^1 dx = \left[2\frac{x^3}{3} + 2x \right]_{-1}^1 = \left[2\left(\frac{x^3 + 3x}{3}\right) \right]_{-1}^1 = \\ &= \frac{2}{3} [x(x^2 + 3)]_{-1}^1 = \frac{2}{3} [1 \cdot (1^2 + 3) - (-1)((-1)^2 + 3)] = \frac{2}{3}(4 + 4) = \frac{2}{3}8 = \frac{16}{3} = 5, \bar{3} \end{aligned}$$

5. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 2, -y \leq x \leq y\}$$

Figura 6: Integrais duplas - Aula 2 - Exercício V

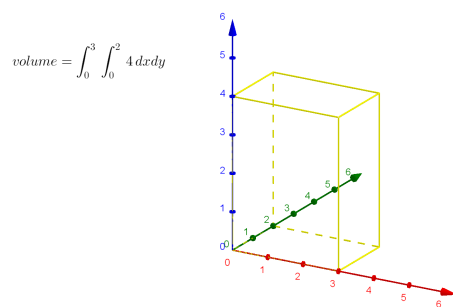


$$\begin{aligned} a &= \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^y dx = \int_0^2 dy [x]_{-y}^y = \int_0^2 dy [y - (-y)] = \int_0^2 dy [2y] = \\ &2 \int_0^2 y dy = \left[2 \frac{y^2}{2} \right]_0^2 = 2^2 - 0^2 = 4 \end{aligned}$$

3 Cálculo de volume - Aula 3

1. Exercício

Figura 7: Integrais duplas - Aula 3 - Exercício I



$$z = 4; dz = dx dy$$

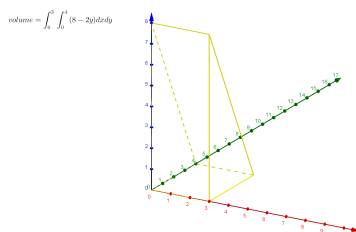
$$v = \int_0^3 \int_0^2 z \, dz = \int_0^3 \int_0^2 4 \, dy \, dx = 4 \int_0^3 dx \int_0^2 dy = 4 \int_0^3 dx [y]_0^2 = 4 \int_0^3 dx [2 - 0] = 8 \int_0^3 dx = 8[x]_0^3 = 8[3 - 0] = 8 \cdot 3 = 24$$

2. Exercício

$$R = [0, 3] \times [0, 4]$$

$$\iint_R (8 - 2y) \, da$$

Figura 8: Integrais duplas - Aula 3 - Exercício II



$$z = 8 - 2y; \, da = dz = dx \, dy$$

$$\begin{aligned} v &= \int_0^3 \int_0^4 z \, dz = \int_0^3 \int_0^4 (8 - 2y) \, dx \, dy = \int_0^3 dx \int_0^4 (8 - 2y) \, dy = \\ &= \int_0^3 dx \left(8 \int_0^4 dy - 2 \int_0^4 y \, dy \right) = \int_0^3 dx \, 2 \left(4 \int_0^4 dy - \int_0^4 y \, dy \right) = \\ &= 2 \int_0^3 dx \left[4y - \frac{y^2}{2} \right]_0^4 = 2 \int_0^3 dx \left[\frac{8y - y^2}{2} \right]_0^4 = 2 \int_0^3 dx \frac{1}{2} [y(8 - y)]_0^4 = \\ &= \int_0^3 dx [4(8 - 4) - 0(8 - 0)] = 16 \int_0^3 dx = 16[x]_0^3 = 16[3 - 0] = 48 \end{aligned}$$

4 Invertendo a ordem de integração - Aula 4

1. Exercício

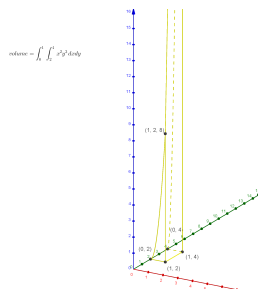
$$z = f(x, y) = y e^x; \, dz = dx \, dy$$

$$\begin{aligned}
 v &= \int_2^4 \int_1^9 z \, dz = \int_2^4 \int_1^9 y e^x \, dy dx = \int_2^4 e^x \, dx \int_1^9 y \, dy = \int_2^4 e^x \, dx \left[\frac{y^2}{2} \right]_1^9 = \\
 &= \int_2^4 e^x \, dx \frac{1}{2} [y^2]_1^9 = \frac{1}{2} \int_2^4 e^x \, dx [9^2 - 1^2] = 40 \int_2^4 e^x \, dx = 40 [e^x]_2^4 = 40 [e^4 - e^2] = \\
 &= 40e^2 (e^2 - 1)
 \end{aligned}$$

2. Exercício

$$z = f(x, y) = x^2 y^3; \, dz = dx dy$$

Figura 9: Integrais duplas - Aula 4 - Exercício II



$$\begin{aligned}
 v &= \int_0^1 \int_2^4 z \, dz = \int_0^1 \int_2^4 x^2 y^3 \, dx dy = \int_0^1 x^2 \, dx \int_2^4 y^3 \, dy = \int_0^1 x^2 \, dx \left[\frac{y^4}{4} \right]_2^4 = \\
 &= \frac{1}{4} \int_0^1 x^2 \, dx [y^4]_2^4 = \frac{1}{4} \int_0^1 x^2 \, dx [4^4 - 2^4] = \frac{1}{4} \int_0^1 x^2 \, dx [2^8 - 2^4] = \\
 &= \frac{1}{4} \int_0^1 x^2 \, dx [2^4 (2^4 - 1)] = \frac{1}{4} \int_0^1 x^2 \, dx [16 \cdot 15] = 60 \int_0^1 x^2 \, dx = 60 \left[\frac{x^3}{3} \right]_0^1 = 20 [x^3]_0^1 = \\
 &= 20 [1^3 - 0^3] = 20 \cdot 1 = 20
 \end{aligned}$$

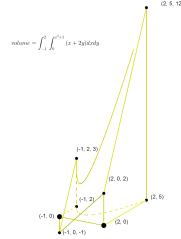
3. Exercício

$$\iint_R (x + 2y) da$$

R = Região limitada pela parábola $y = x^2 + 1$ e as retas $x = -1$ e $x = 2$.

$$z = f(x, y) = x + 2y; \, da = dz = dx dy$$

Figura 10: Integrais duplas - Aula 4 - Exercício III



$$\begin{aligned}
 v &= \int_{-1}^2 \int_0^{x^2+1} z \, dz = \int_{-1}^2 \int_0^{x^2+1} (x + 2y) \, dx \, dy = \\
 &\int_{-1}^2 dx \int_0^{x^2+1} (x + 2y) \, dy = \int_{-1}^2 dx \left(x \int_0^{x^2+1} dy + 2 \int_0^{x^2+1} y \, dy \right) = \\
 &\int_{-1}^2 dx \left[xy + \frac{y^2}{2} \right]_0^{x^2+1} = \int_{-1}^2 dx [y(x + y)]_0^{x^2+1} = \\
 &\int_{-1}^2 dx [(x^2 + 1) [x + (x^2 + 1)] - 0(x + 0)] = \int_{-1}^2 dx [(x^2 + 1) (x^2 + x + 1)] = \\
 &\int_{-1}^2 dx (x^4 + x^3 + 2x^2 + x + 1) = \\
 &\int_{-1}^2 x^4 \, dx + \int_{-1}^2 x^3 \, dx + 2 \int_{-1}^2 x^2 \, dx + \int_{-1}^2 x \, dx + \int_{-1}^2 dx = \\
 &\left[\frac{x^5}{5} + \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^2 = \left[\frac{12x^5 + 15x^4 + 40x^3 + 30x^2 + 60x}{60} \right]_{-1}^2 = \\
 &\frac{1}{60} [x (12x^4 + 15x^3 + 40x^2 + 30x + 60)]_{-1}^2 = \\
 &\frac{1}{60} [2 (12 \cdot 2^4 + 15 \cdot 2^3 + 40 \cdot 2^2 + 30 \cdot 2 + 60) \\
 &\quad - (-1) (12(-1)^4 + 15(-1)^3 + 40(-1)^2 + 30(-1) + 60)] = \\
 &\frac{1}{60} [2(192 + 120 + 160 + 60 + 60) + (12 - 15 + 40 - 30 + 60)] = \frac{1}{60} (1184 + 67) = \\
 &\frac{1251}{60} = \frac{417}{20} = 20,85
 \end{aligned}$$

5 Cálculo de integrais duplas ou iteradas

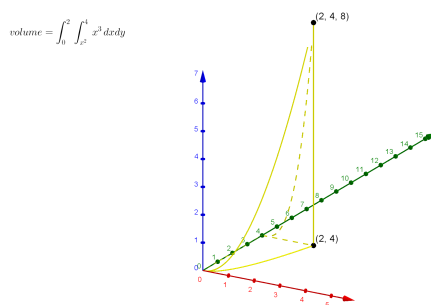
5.1 Aula 5

1. Exercício

$$f(x, y) = x^3; 0 \leq x \leq 2; x^2 \leq y \leq 4$$

$$\iint_R f(x, y) dy dx$$

Figura 11: Integrais duplas - Aula 5 - Exercício I



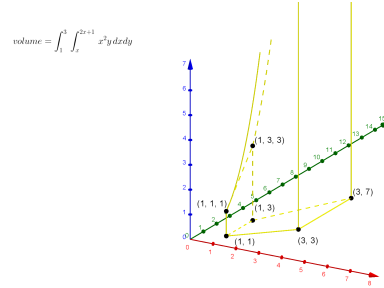
$$\begin{aligned} v &= \int_0^2 \int_{x^2}^4 x^3 dx dy = \int_0^2 x^3 dx \int_{x^2}^4 dy = \int_0^2 x^3 dx [y]_{x^2}^4 = \int_0^2 x^3 dx [4 - x^2] = \\ &= 4 \int_0^2 x^3 dx - \int_0^2 x^5 dx = \left[4 \frac{x^4}{4} - \frac{x^6}{6} \right]_0^2 = \left[\frac{6x^4 - x^6}{6} \right]_0^2 = \frac{1}{6} [x^4 (6 - x^2)]_0^2 = \\ &= \frac{1}{6} [2^4 (6 - 2^2) - 0^4 (6 - 0^2)] = \frac{1}{6} (16 \cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2 \end{aligned}$$

2. Exercício

$$f(x, y) = x^2 y; 1 \leq x \leq 3; x \leq y \leq 2x + 1$$

$$\iint_R f(x, y) dy dx$$

Figura 12: Integrais duplas - Aula 5 - Exercício II



$$\begin{aligned}
 v &= \int_1^3 \int_x^{2x+1} x^2 y \, dx \, dy = \int_1^3 x^2 \, dx \int_x^{2x+1} y \, dy = \int_1^3 x^2 \, dx \left[\frac{y^2}{2} \right]_x^{2x+1} = \\
 &= \int_1^3 x^2 \, dx \frac{1}{2} [(2x+1)^2 - (x)^2] = \frac{1}{2} \int_1^3 x^2 \, dx (3x^2 + 4x + 1) = \\
 \frac{3}{2} \int_1^3 x^4 \, dx + 2 \int_1^3 x^3 \, dx + \frac{1}{2} \int_1^3 x^2 \, dx &= \left[\frac{3}{2} \frac{x^5}{5} + 2 \frac{x^4}{4} + \frac{1}{2} \frac{x^3}{3} \right]_1^3 = \left[\frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6} \right]_1^3 = \\
 &= \left[\frac{18x^5 + 30x^4 + 10x^3}{60} \right]_1^3 = \left[\frac{2x^3 (9x^2 + 15x + 5)}{60} \right]_1^3 = \\
 \frac{1}{30} [x^3 (9x^2 + 15x + 5)]_1^3 &= \frac{1}{30} [3^3 (9 \cdot 3^2 + 15 \cdot 3 + 5) - 1^3 (9 \cdot 1^2 + 15 \cdot 1 + 5)] = \\
 \frac{1}{30} [27(81 + 45 + 5) - (9 + 15 + 5)] &= \frac{1}{30} [27 \cdot 131 - 29] = \frac{3508}{30} = 116,9\bar{3}
 \end{aligned}$$

5.2 Aula 6

1. Exercício

$$f(x, y) = 1; \quad 0 \leq x \leq 1; \quad 1 \leq y \leq e^x$$

$$\iint_R f(x, y) \, dy \, dx$$

$$\begin{aligned}
 v &= \int_0^1 \int_1^{e^x} dy \, dx = \int_0^1 dx [y]_1^{e^x} = \int_0^1 dx (e^x - 1) = [e^x - x]_0^1 = \\
 &= e^1 - 1 - (e^0 - 0) = e - 1 - 1 = e - 2
 \end{aligned}$$

2. Exercício

$$f(x, y) = x; 0 \leq x \leq 1; 1 \leq y \leq e^{x^2}$$

$$\iint_R f(x, y) dy dx$$

$$\begin{aligned} v &= \int_0^1 \int_1^{e^{x^2}} x dx dy = \int_0^1 x dx \int_1^{e^{x^2}} dy = \int_0^1 x dx [y]_1^{e^{x^2}} = \int_0^1 x dx (e^{x^2} - 1) = \\ &= \int_0^1 x e^{x^2} dx - \int_0^1 x dx = \int_0^1 e^u \frac{du}{2} - \int_0^1 x dx = \frac{1}{2} \int_0^1 e^u du - \int_0^1 x dx = \\ &= \left[\frac{1}{2} e^u - \frac{x^2}{2} \right]_0^1 = \left[\frac{e^{x^2} - x^2}{2} \right]_0^1 = \frac{1}{2} [e^{x^2} - x^2]_0^1 = \frac{1}{2} [e^{1^2} - 1^2 - (e^{0^2} - 0^2)] = \\ &= \frac{1}{2} (e - 1 - 1) = \frac{e-2}{2} \end{aligned}$$

$$u = x^2; \frac{du}{2} = x dx$$

3. Exercício

$$f(x, y) = 2xy; 0 \leq y \leq 1; y^2 \leq x \leq y$$

$$\iint_R f(x, y) dx dy$$

$$\begin{aligned} v &= \int_0^1 \int_{y^2}^y 2xy dx dy = 2 \int_0^1 y dy \int_{y^2}^y x dx = 2 \int_0^1 y dy \left[\frac{x^2}{2} \right]_{y^2}^y = 2 \int_0^1 y dy \frac{1}{2} [x^2]_{y^2}^y = \\ &= \int_0^1 y dy (y^2 - y^4) = \int_0^1 (y^3 - y^5) dy = \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_0^1 = \\ &= \left[\frac{6y^4 - 4y^6}{24} \right]_0^1 = \left[\frac{2y^4 (3 - 2y^2)}{24} \right]_0^1 = \frac{1}{12} [1^4 (3 - 2 \cdot 1^2) - 0^4 (3 - 2 \cdot 0^2)] = \frac{1}{12} = 0,08\bar{3} \end{aligned}$$

5.3 Aula 7

1. Exercício

$$f(x, y) = \frac{1}{x+y}; 1 \leq y \leq e; 0 \leq x \leq y$$

$$\iint_R f(x, y) dx dy$$

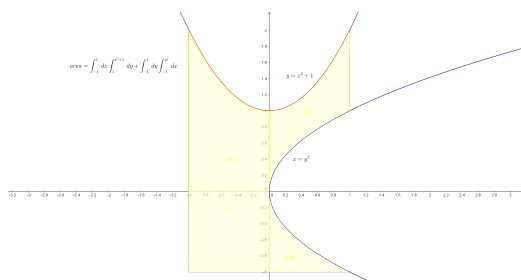
$$\begin{aligned}
v &= \int_1^e \int_0^y \frac{1}{x+y} dx dy = \int_1^e dy \int_0^y (x+y)^{-1} dx = \int_1^e dy \int_0^y u^{-1} du = \\
&\int_1^e dy \int_0^y [\ln |u|]_0^y = \int_1^e dy \int_0^y [\ln |x+y|]_0^y = \int_1^e dy \int_0^y (\ln |y+y| - \ln |0+y|) = \\
&\int_1^e dy \int_0^y (\ln |2y| - \ln |y|) = \int_1^e dy \int_0^y (\ln |2| + \ln |y| - \ln |y|) = \ln |2| \int_1^e dy = \\
&\ln |2| [y]_1^e = \ln |2|(e-1)
\end{aligned}$$

$$u = x + y; du = (1 + 0)dx = dx$$

6 Cálculo de área - Aula 8

1. Exercício

Figura 13: Integrais duplas - Aula 8 - Exercício I



$$\begin{aligned}
a &= \int_{-1}^0 dx \int_0^{x^2+1} dy + \int_{-1}^0 dx \int_{-1}^0 dy + \int_0^{y^2} dx \int_{-1}^0 dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} dy = \\
&= \int_{-1}^0 dx \left(\int_0^{x^2+1} dy + \int_{-1}^0 dy \right) + \int_0^{y^2} dx \int_{-1}^0 dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} dy = \\
&= \int_{-1}^0 dx \left([y]_0^{x^2+1} + [y]_{-1}^0 \right) + \int_{-1}^0 dy [x]_0^{y^2} + \int_0^1 dx [y]_{\sqrt{x}}^{x^2+1} = \\
&= \int_{-1}^0 dx (x^2 + 1 + 1) + \int_{-1}^0 dy y^2 + \int_0^1 dx (x^2 + 1 - \sqrt{x}) = \\
&= \int_{-1}^0 (x^2 + 2) dx + \int_{-1}^0 y^2 dy + \int_0^1 \left(x^2 - x^{\frac{1}{2}} + 1 \right) dx = \\
&= \left[\frac{x^3}{3} + 2x \right]_{-1}^0 + \left[\frac{y^3}{3} \right]_{-1}^0 + \left[\frac{x^3}{3} - \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + x \right]_0^1 = \\
&= \left[\frac{x^3 + 6x}{3} \right]_{-1}^0 + \frac{1}{3} [y^3]_{-1}^0 + \left[\frac{x^3}{3} - \frac{2\sqrt{x^3}}{3} + x \right]_0^1 = \\
&= \frac{1}{3} [x(x^2 + 6)]_{-1}^0 + \frac{1}{3} [\theta^3 - (-1)^3] + \left[\frac{x^3 - 2\sqrt{x^3} + 3x}{3} \right]_0^1 = \\
&= \frac{1}{3} [0(0^2 + 6) - (-1)((-1)^2 + 6)] + \frac{1}{3} + \frac{1}{3} [x^3 - 2\sqrt{x^3} + 3x]_0^1 = \\
&= \frac{7}{3} + \frac{1}{3} + \frac{1}{3} [1^3 - 2\sqrt{1^3} + 3 \cdot 1 - (0^3 - 2\sqrt{0^3} + 3 \cdot 0)] = \frac{7}{3} + \frac{1}{3} + \frac{2}{3} = \\
&= \frac{7+1+2}{3} = \frac{10}{3} = 3, \bar{3}
\end{aligned}$$

$$\begin{aligned}
a &= \int_{-1}^1 dx \int_1^{x^2+1} dy + \int_{-1}^{y^2} dx \int_{-1}^1 dy = \int_{-1}^1 dx [y]_1^{x^2+1} + \int_{-1}^1 dy [x]_{-1}^{y^2} = \\
&= \int_{-1}^1 dx (x^2 + 1 - 1) + \int_{-1}^1 dy (y^2 + 1) = \left[\frac{x^3}{3} \right]_{-1}^1 + \left[\frac{y^3}{3} + y \right]_{-1}^1 = \\
&= \frac{1}{3} [x^3]_{-1}^1 + \frac{1}{3} [y(y^2 + 3)]_{-1}^1 = \\
&= \frac{1}{3} ([1^3 - (-1)^3] + [1(1^2 + 3) - (-1)((-1)^2 + 3)]) = \frac{1}{3}(2 + 4 + 4) = \frac{10}{3} = 3, \bar{3}
\end{aligned}$$

7 Cálculo de volume

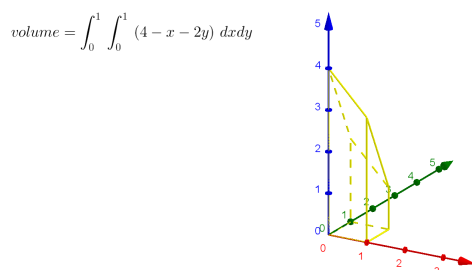
7.1 Aula 9

1. Exercício

Esboce a região de integração e o sólido cujo volume é dado pela integral abaixo:

$$\int_0^1 \int_0^1 (4 - x - 2y) \, dx dy$$

Figura 14: Integrais duplas - Aula 9 - Exercício I



$$\begin{aligned} v &= \int_0^1 \int_0^1 (4 - x - 2y) \, dx dy = \int_0^1 dx \left(4 \int_0^1 dy - x \int_0^1 dy - 2 \int_0^1 y \, dy \right) = \\ &= 4 \int_0^1 dx \int_0^1 dy - \int_0^1 x \, dx \int_0^1 dy - 2 \int_0^1 dx \int_0^1 y \, dy = \\ &= 4[x]_0^1 [y]_0^1 - \left[\frac{x^2}{2} \right]_0^1 [y]_0^1 - 2[x]_0^1 \left[\frac{y^2}{2} \right]_0^1 = 4 - \frac{1}{2} - 2 \cdot \frac{1}{2} = \frac{8 - 1 - 2}{2} = \frac{5}{2} = 2,5 \end{aligned}$$

7.2 Aula 10

1. Exercício

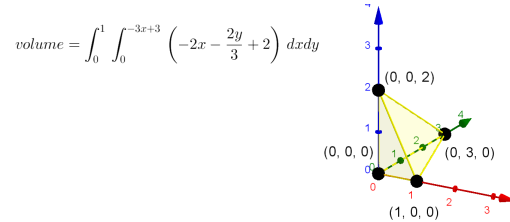
Calcule o volume do sólido limitado pelos planos:

$$x = 0, y = 0, z = 0 \text{ e } 6x + 2y + 3z = 6$$

$$P_1 = (0, 0, 0)$$

$$6x = -2y - 3z + 6 \Rightarrow x = \frac{-2y - 3z + 6}{6} = \frac{-2 \cdot 0 - 3 \cdot 0 + 6}{6} = \frac{6}{6} = 1 \Rightarrow P_2 = (1, 0, 0)$$

Figura 15: Integrais duplas - Aula 10 - Exercício I



$$2y = -6x - 3z + 6 \Rightarrow y = \frac{-6x - 3z + 6}{2} = \frac{-6 \cdot 0 - 3 \cdot 0 + 6}{2} = \frac{6}{2} = 3 \Rightarrow P_3 = (0, 3, 0)$$

$$3z = -6x - 2y + 6 \Rightarrow z = \frac{-6x - 2y + 6}{3} = \frac{-6 \cdot 0 - 2 \cdot 0 + 6}{3} = \frac{6}{3} = 2 \Rightarrow P_4 = (0, 0, 2)$$

$$x = 0, x = 1$$

$$y = 0, y = \frac{-6x - 3z + 6}{2} = \frac{-6x - 3 \cdot 0 + 6}{2} = -3x + 3$$

$$z = \frac{-6x - 2y + 6}{3} = -2x - \frac{2y}{3} + 2$$

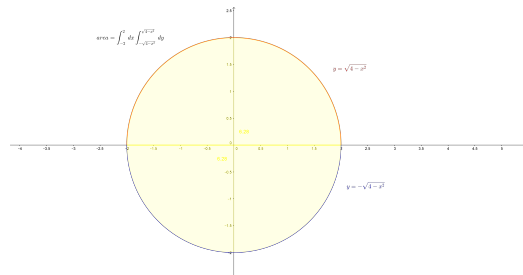
$$\begin{aligned} v &= \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2 \right) dx dy = \int_0^1 dx \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2 \right) dy = \\ &= \int_0^1 dx \left[-2xy - \frac{2}{3} \frac{y^2}{2} + 2y \right]_0^{-3x+3} = \int_0^1 dx \frac{1}{3} [-6xy - y^2 + 6y]_0^{-3x+3} = \\ &= \frac{1}{3} \int_0^1 dx [-y(6x + y - 6)]_0^{-3x+3} = \\ &= \frac{1}{3} \int_0^1 dx [-(-3x+3)(6x + (-3x+3) - 6) + 0(6x + 0 - 6)] = \\ &= \frac{1}{3} \int_0^1 dx [(3x-3)(3x-3)] = \frac{1}{3} \int_0^1 (9x^2 - 18x + 9) dx = \frac{1}{3} \left[9 \frac{x^3}{3} - 18 \frac{x^2}{2} + 9x \right]_0^1 = \\ &= \frac{1}{3} [3x^3 - 9x^2 + 9x]_0^1 = \frac{1}{3} [3x(x^2 - 3x + 3)]_0^1 = \\ &= [1(1^2 - 3 \cdot 1 + 3) - 0(0^2 - 3 \cdot 0 + 3)] = 1 \end{aligned}$$

8 Coordenadas polares

8.1 Aula 1

1. Exercício

Figura 16: Coordenadas polares - Aula 01 - Exercício I



Calcule a área do círculo de raio igual a dois

$$r = 2 \Rightarrow A = \pi r^2 = 2^2 \pi = 4\pi$$

$$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 2^2 \Rightarrow x^2 + y^2 = 4 \Rightarrow y = \pm \sqrt{4 - x^2}$$

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2} \right\}$$

$$\begin{aligned}
a &= \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{-2}^2 dx \left(\sqrt{4-x^2} + \sqrt{4-x^2} \right) = 2 \int_{-2}^2 \sqrt{4-x^2} dx = \\
&= 2 \int_{-2}^2 \sqrt{4 - (2 \sin(\alpha))^2} 2 \cos(\alpha) d\alpha = 4 \int_{-2}^2 \sqrt{4 - 4 \sin^2(\alpha)} \cos(\alpha) d\alpha = \\
&= 4 \int_{-2}^2 \sqrt{4 - 4(1 - \cos^2(\alpha))} \cos(\alpha) d\alpha = 4 \int_{-2}^2 \sqrt{4 - (4 - 4 \cos^2(\alpha))} \cos(\alpha) d\alpha = \\
&= 4 \int_{-2}^2 \sqrt{4 - 4 + 4 \cos^2(\alpha)} \cos(\alpha) d\alpha = 4 \int_{-2}^2 2 \cos(\alpha) \cos(\alpha) d\alpha = \\
&= 8 \int_{-2}^2 \cos^2(\alpha) d\alpha = 8 \int_{-2}^2 \left(\frac{1 + \cos(2\alpha)}{2} \right) d\alpha = 8 \int_{-2}^2 \left(\frac{1}{2} + \frac{\cos(2\alpha)}{2} \right) d\alpha = \\
&= 4 \int_{-2}^2 d\alpha + 4 \int_{-2}^2 \cos(2\alpha) d\alpha = 4 \int_{-2}^2 d\alpha + 4 \int_{-2}^2 \cos(u) \frac{du}{2} = \\
&= 4 \int_{-2}^2 d\alpha + 2 \int_{-2}^2 \cos(u) du = [4\alpha + 2 \sin(u)]_{-2}^2 = [4\alpha + 2 \sin(2\alpha)]_{-2}^2 = \\
&= [4\alpha + 4 \sin(\alpha) \cos(\alpha)]_{-2}^2 = [4(\alpha + \sin(\alpha) \cos(\alpha))]_{-2}^2 = \\
&= \left[4 \left(\arcsin\left(\frac{x}{2}\right) + \frac{x}{2} \frac{\sqrt{4-x^2}}{2} \right) \right]_{-2}^2 = \left[4 \left(\arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4} \right) \right]_{-2}^2 = \\
&= 4 \left(\arcsin\left(\frac{2}{2}\right) + \frac{2\sqrt{4-2^2}}{4} \right) - 4 \left(\arcsin\left(\frac{-2}{2}\right) + \frac{(-2)\sqrt{4-(-2)^2}}{4} \right) = \\
&= 4 \arcsin(1) - 4 \arcsin(-1) = 4(\arcsin(1) - \arcsin(-1)) = 4 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 4 \left(\frac{2\pi}{2} \right) = 4\pi
\end{aligned}$$

$$x = 2 \sin(\alpha); dx = 2 \cos(\alpha) d\alpha$$

$$u = 2\alpha; \frac{du}{2} = d\alpha$$

$$\sin(\alpha) = \frac{co}{h} = \frac{x}{2} \Rightarrow \alpha = \arcsin\left(\frac{x}{2}\right)$$

$$h^2 = co^2 + ca^2 \Rightarrow 2^2 = x^2 + ca^2 \Rightarrow ca = \sqrt{4-x^2}$$

$$\cos(\alpha) = \frac{ca}{h} = \frac{\sqrt{4-x^2}}{2}$$

$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

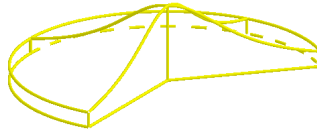
$$\begin{aligned}
a &= \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_0^2 \int_0^{2\pi} r dr d\theta = \int_0^2 r dr \int_0^{2\pi} d\theta = \left[\frac{r^2}{2} \right]_0^2 [\theta]_0^{2\pi} = \\
&= \frac{1}{2} [2^2 - 0^2] [2\pi - 0] = \frac{4}{2} 2\pi = 4\pi
\end{aligned}$$

2. Exercício

$$\iint_R \frac{da}{1+x^2+y^2}$$

Figura 17: Coordenadas polares - Aula 01 - Exercício II

$$volume = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r dr d\theta}{1+r^2}$$



$$R = \left\{ (r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{2} \right\}$$

$$\begin{aligned} v &= \iint_R \frac{da}{1+x^2+y^2} = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r dr d\theta}{1+r^2} = \int_0^2 \frac{r dr}{1+r^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} d\theta = \\ &= \int_0^2 (1+r^2)^{-1} r dr \left[\theta \right]_{\frac{\pi}{4}}^{\frac{3\pi}{2}} = \int_0^2 (1+r^2)^{-1} r dr \left(\frac{3\pi}{2} - \frac{\pi}{4} \right) = \\ &= \int_0^2 (1+r^2)^{-1} r dr \left(\frac{6\pi - \pi}{4} \right) = \frac{5\pi}{4} \int_0^2 (1+r^2)^{-1} r dr = \frac{5\pi}{4} \int_0^2 u^{-1} \frac{du}{2} = \\ &= \frac{5\pi}{8} \int_0^2 u^{-1} du = \frac{5\pi}{8} [ln|u|]_0^2 = \frac{5\pi}{8} [ln|1+r^2|]_0^2 = \frac{5\pi}{8} [ln|1+2^2| - ln|1+0^2|] = \\ &= \frac{5\pi}{8} [ln|5| - ln|1|] = \frac{5\pi ln|5|}{8} \end{aligned}$$

$$u = 1 + r^2 \Rightarrow \frac{du}{2} = r dr$$

$$e^x = 1 = e^0 \Rightarrow x = 0$$

A Derivadas

B Derivadas simples

Tabela 1: Derivadas simples

$y = c$	\Rightarrow	$y' = 0$
$y = x$	\Rightarrow	$y' = 1$
$y = x^c$	\Rightarrow	$y' = cx^{c-1}$
$y = e^x$	\Rightarrow	$y' = e^x$
$y = \ln x $	\Rightarrow	$y' = \frac{1}{x}$
$y = uv$	\Rightarrow	$y' = u'v + uv'$
$y = \frac{u}{v}$	\Rightarrow	$y' = \frac{u'v - uv'}{v^2}$
$y = u^c$	\Rightarrow	$y' = cu^{c-1}u'$
$y = e^u$	\Rightarrow	$y' = e^u u'$
$y = c^u$	\Rightarrow	$y' = c^u u' \ln c $
$y = \ln u $	\Rightarrow	$y' = \frac{u'}{u}$
$y = \log_c u $	\Rightarrow	$y' = \frac{u'}{u} \log_c e $

C Derivadas trigonométricas

D Integrais

E Integrais simples

F Integrais trigonométricas

G Relação entre coordenada cartesina e polar

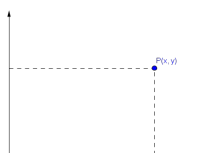
$$P(x, y) \rightarrow P(r, \theta)$$

Tabela 2: Derivadas trigonométricas

$y = \text{sen}(x)$	$\Rightarrow y' = \cos(x)$
$y = \cos(x)$	$\Rightarrow y' = -\text{sen}(x)$
$y = \text{tg}(x)$	$\Rightarrow y' = \sec^2(x)$
$y = \text{cotg}(x)$	$\Rightarrow y' = -\text{cossec}^2(x)$
$y = \sec(x)$	$\Rightarrow y' = \sec(x) \text{tg}(x)$
$y = \text{cossec}(x)$	$\Rightarrow y' = -\text{cossec}(x) \text{cotg}(x)$
$y = \arcsen(x)$	$\Rightarrow y' = \frac{1}{\sqrt{1-x^2}}$
$y = \arccos(x)$	$\Rightarrow y' = \frac{-1}{\sqrt{1-x^2}}$
$y = \arctg(x)$	$\Rightarrow y' = \frac{1}{1+x^2}$
$y = \text{arccotg}(x)$	$\Rightarrow y' = \frac{-1}{1+x^2}$
$y = \text{arcsec}(x)$	$\Rightarrow y' = \frac{1}{ x \sqrt{x^2-1}}$
$y = \text{arccossec}(x)$	$\Rightarrow y' = \frac{-1}{ x \sqrt{x^2-1}}$

Figura 18: Coordenada cartesina e polar

(a) Coordenada cartesiana ou retangular



(b) Coordenada polar

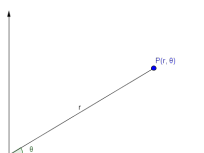
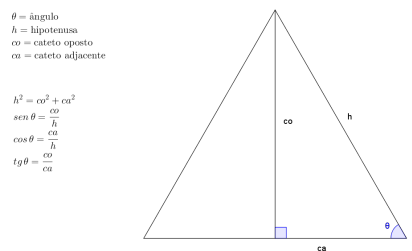


Tabela 3: Integrais simples

$$\left| \begin{array}{lcl} \int dx & = & x + c \\ \int x^p dx & = & \frac{x^{p+1}}{p+1} + c \rightarrow p \neq -1 \\ \int e^x dx & = & e^x + c \\ \int \frac{dx}{x} & = & \ln |x| + c \\ \int u^p du & = & \frac{u^{p+1}}{p+1} + c \rightarrow p \neq -1 \\ \int e^u du & = & e^u + c \\ \int \frac{du}{u} & = & \ln |u| + c \\ \int p^u du & = & \frac{p^u}{\ln |p|} + c \end{array} \right|$$

Figura 19: Determinação do seno, cosseno e tangente



H Funções trigonométricas

I Determinação do seno, cosseno e tangente

J Círculo trigonométrico

K Identidades trigonométricas

L Relação entre trigonométricas e inversas

M Substituição trigonométrica

N Ângulos notáveis

Referências

- [1] Fernando Grings. *Curso de Integrais Duplas e Triplas*. Youtube, 2016.

Figura 20: Círculo trigonométrico

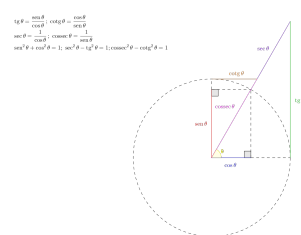


Tabela 4: Integrais trigonométricas

$\int \text{sen}(u) du$	$= -\cos(u) + c$
$\int \cos(u) du$	$= \text{sen}(u) + c$
$\int \text{tg}(u) du$	$= \ln \sec(u) + c$
$\int \text{cotg}(u) du$	$= \ln \text{sen}(u) + c$
$\int \sec(u) du$	$= \ln \sec(u) + \text{tg}(u) + c$
$\int \text{cossec}(u) du$	$= \ln \text{cossec}(u) - \text{cotg}(u) + c$
$\int \sec^2(u) du$	$= \text{tg}(u) + c$
$\int \text{cossec}^2(u) du$	$= -\text{cotg}(u) + c$
$\int \sec(u) \text{tg}(u) du$	$= \sec(u) + c$
$\int \text{cossec}(u) \text{cotg}(u) du$	$= -\text{cossec}(u) + c$
$\int \frac{du}{\sqrt{1-x^2}}$	$= \arcsen(x) + c$
$-\int \frac{du}{\sqrt{1-x^2}}$	$= \arccos(x) + c$
$\int \frac{du}{1+x^2}$	$= \arctg(x) + c$
$-\int \frac{du}{1+x^2}$	$= \text{arccotg}(x) + c$
$\int \frac{du}{ x \sqrt{x^2-1}}$	$= \text{arcsec}(x) + c$
$-\int \frac{du}{ x \sqrt{x^2-1}}$	$= \text{arccossec}(x) + c$

Tabela 5: Relação entre coordenada cartesina e polar

x	$= r \cos \theta$
y	$= r \text{sen } \theta$
$x^2 + y^2$	$= r^2$
$da = dx dy$	$= r dr d\theta$
$v = \iint_{R(x,y)} f(x, y) dx dy$	$= \iint_{R(r,\theta)} f(r \cos \theta, r \text{sen } \theta) r dr d\theta$

Tabela 6: Identidades trigonométricas

$\operatorname{tg}(x)$	$=$	$\frac{\operatorname{sen}(x)}{\cos(x)}$
$\operatorname{cotg}(x)$	$=$	$\frac{\cos(x)}{\operatorname{sen}(x)}$
$\sec(x)$	$=$	$\frac{1}{\cos(x)}$
$\operatorname{cosec}(x)$	$=$	$\frac{1}{\operatorname{sen}(x)}$
$\operatorname{sen}^2(x) + \cos^2(x)$	$=$	1
$\sec^2(x) - \operatorname{tg}^2(x)$	$=$	1
$\operatorname{cosec}^2(x) - \operatorname{cotg}^2(x)$	$=$	1
$\operatorname{sen}^2(x)$	$=$	$\frac{1 - \cos(2x)}{2}$
$\cos^2(x)$	$=$	$\frac{1 + \cos(2x)}{2}$
$\operatorname{sen}(2x)$	$=$	$2 \operatorname{sen}(x) \cos(x)$
$\cos(2x)$	$=$	$\cos^2(x) - \operatorname{sen}^2(x)$

Tabela 7: Relação entre trigonométricas e inversas

$\operatorname{sen}(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arcsen}(x)$
$\cos(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arccos}(x)$
$\operatorname{tg}(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arctg}(x)$
$\operatorname{cosec}(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arccosec}(x)$
$\sec(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arcsec}(x)$
$\operatorname{cotg}(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arccotg}(x)$

Tabela 8: Substituição trigonométrica

$\sqrt{a^2 - x^2}$	\Rightarrow	x	$=$	$a \operatorname{sen}(\theta)$
$\sqrt{a^2 + x^2}$	\Rightarrow	x	$=$	$a \operatorname{tg}(\theta)$
$\sqrt{x^2 - a^2}$	\Rightarrow	x	$=$	$a \sec(\theta)$

Tabela 9: Ângulos notáveis

ângulo	$0^\circ (0)$	$30^\circ (\frac{\pi}{6})$	$45^\circ (\frac{\pi}{4})$	$60^\circ (\frac{\pi}{3})$	$90^\circ (\frac{\pi}{2})$
sen	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	\nexists