

Limites de Funções de duas Variáveis – Aula 1

Exercício I

$$\lim_{\substack{(x,y) \rightarrow (-1,2) \\ x \rightarrow -1 \\ y \rightarrow 2}} \left[\frac{xy}{x^2+y^2} \right] = \lim_{\substack{x \rightarrow -1 \\ y \rightarrow 2}} \left[\frac{xy}{x^2+y^2} \right] = \frac{(-1)2}{(-1)^2+2^2} = \frac{-2}{1+4} = -\frac{2}{5} \quad (1)$$

Exercício II

$$\lim_{(x,y) \rightarrow (0,0)} \left[\frac{5}{x^2+3y^2} \right] = \frac{5}{(0)^2+3(0)^2} = \frac{5}{0^+} = +\infty \quad (2)$$

Exercício III

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,-2)} \left[\frac{4x^2-y^2}{2x+y} \right] &= \frac{4(1)^2-(-2)^2}{2(1)+(-2)} = \frac{4-4}{2-2} = \frac{0}{0} \\ \frac{(2x)^2-y^2}{2x+y} &= \frac{(2x-y)(2x+y)}{2x+y} = 2x-y \\ \lim_{(x,y) \rightarrow (1,-2)} [2x-y] &= 2 \cdot 1 - (-2) = 2+2 = 4 \end{aligned} \quad (3)$$

Exercício IV

$$\begin{aligned} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{2xy}{x^2+y^2} \right] &= \frac{2(0)(0)}{(0)^2+(0)^2} = \frac{0}{0} \\ \lim_{\substack{x \rightarrow 0 \\ y=0}} \left[\frac{2x(0)}{x^2+(0)^2} \right] &= \lim_{x \rightarrow 0} \left[\frac{0}{x^2} \right] = \lim_{x \rightarrow 0} [0] = 0 \\ \lim_{\substack{x=0 \\ y \rightarrow 0}} \left[\frac{2(0)y}{(0)^2+y^2} \right] &= \lim_{y \rightarrow 0} \left[\frac{0}{y^2} \right] = \lim_{y \rightarrow 0} [0] = 0 \\ \lim_{\substack{x \rightarrow 0 \\ y=x}} \left[\frac{2x(x)}{x^2+(x)^2} \right] &= \lim_{x \rightarrow 0} \left[\frac{2x^2}{2x^2} \right] = \lim_{x \rightarrow 0} [1] = 1 \\ \lim_{\substack{x \rightarrow 0 \\ y=x^2}} \left[\frac{2xy}{x^2+y^2} \right] &= \lim_{x \rightarrow 0} \left[\frac{2x(x^2)}{x^2+(x^2)^2} \right] = \lim_{x \rightarrow 0} \left[\frac{2x^3}{x^2+x^4} \right] = \lim_{x \rightarrow 0} \left[\frac{2x^3}{x^2(1+x^2)} \right] = \lim_{x \rightarrow 0} \left[\frac{2x}{1+x^2} \right] = \frac{2(0)}{1+(0)^2} = 0 \\ \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{2xy}{x^2+y^2} \right] &= \nexists \end{aligned} \quad (4)$$

Limites de Funções de duas Variáveis – [Aula 2](#)

Exercício I

$$\begin{aligned}
 \lim_{(x,y) \rightarrow (0,0)} \left[\frac{x^2 - y^2}{x^2 + y^2} \right] &= \frac{(0)^2 - (0)^2}{(0)^2 + (0)^2} = \frac{0}{0} \\
 \lim_{\substack{x \rightarrow 0 \\ y=0}} \left[\frac{x^2 - (0)^2}{x^2 + (0)^2} \right] &= \lim_{x \rightarrow 0} \left[\frac{x^2}{x^2} \right] = \lim_{x \rightarrow 0} [1] = \mathbf{1} \\
 \lim_{\substack{x=0 \\ y \rightarrow 0}} \left[\frac{(0)^2 - y^2}{(0)^2 + y^2} \right] &= \lim_{y \rightarrow 0} \left[\frac{-y^2}{y^2} \right] = \lim_{y \rightarrow 0} [-1] = \mathbf{-1} \\
 \lim_{(x,y) \rightarrow (0,0)} \left[\frac{x^2 - y^2}{x^2 + y^2} \right] &= \nexists
 \end{aligned} \tag{5}$$

Exercício II

$$\begin{aligned}
 \lim_{(x,y) \rightarrow (0,0)} \left[\frac{xy^2}{x^2 + y^4} \right] &= \frac{(0)(0)^2}{(0)^2 + (0)^4} = \frac{0}{0} \\
 \lim_{\substack{x \rightarrow 0 \\ y=0}} \left[\frac{x(0)^2}{x^2 + (0)^4} \right] &= \lim_{x \rightarrow 0} \left[\frac{0}{x^2} \right] = \lim_{x \rightarrow 0} [0] = \mathbf{0} \\
 \lim_{\substack{x=0 \\ y \rightarrow 0}} \left[\frac{(0)y^2}{(0)^2 + y^4} \right] &= \lim_{y \rightarrow 0} \left[\frac{0}{y^4} \right] = \lim_{y \rightarrow 0} [0] = \mathbf{0} \\
 \lim_{\substack{x \rightarrow 0 \\ y=x^2}} \left[\frac{x(x^2)^2}{x^2 + (x^2)^4} \right] &= \lim_{x \rightarrow 0} \left[\frac{x \cdot x^4}{x^2 + x^8} \right] = \lim_{x \rightarrow 0} \left[\frac{x^5}{x^2(1+x^6)} \right] = \lim_{x \rightarrow 0} \left[\frac{x^3}{1+x^6} \right] = \frac{(0)^3}{1+(0)^6} = \frac{0}{1} = \mathbf{0} \\
 \lim_{\substack{x=y^2 \\ y \rightarrow 0}} \left[\frac{(y^2)y^2}{(y^2)^2 + y^4} \right] &= \lim_{y \rightarrow 0} \left[\frac{y^4}{y^4 + y^4} \right] = \lim_{y \rightarrow 0} \left[\frac{y^4}{2y^4} \right] = \lim_{y \rightarrow 0} \left[\frac{1}{2} \right] = \mathbf{\frac{1}{2}} \\
 \lim_{(x,y) \rightarrow (0,0)} \left[\frac{xy^2}{x^2 + y^4} \right] &= \nexists
 \end{aligned} \tag{6}$$

Limites de Funções de duas Variáveis – [Aula 3](#)

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = 0$$

$$|g(x, y)| \leq M \text{ (função limitada)}$$

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} [f(x, y) \cdot g(x, y)] = 0$$

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}}$$

Exercício I

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} [x \cdot \cos(y)] = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} [0] \cdot \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} [\cos(y)] = 0 \cdot [-1:1] = 0 \quad (7)$$

$$0 \leq |\cos(0)| \leq 1$$

Exercício II

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[x \cdot \sin\left(\frac{1+y}{y}\right) \right] = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} [x] \cdot \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\sin\left(\frac{1+y}{y}\right) \right] = 0 \cdot [-1:1] = 0 \quad (8)$$

$$0 \leq \left| \sin\left(\frac{1}{0}\right) \right| \leq 1$$

Exercício III

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{3x^2y}{x^2+y^2} \right] = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[3y \left(\frac{x^2}{x^2+y^2} \right) \right] = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} [3y] \cdot \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{x^2}{x^2+y^2} \right] = 0 \cdot [0:1] = 0 \quad (9)$$

$$\frac{x^2}{x^2+y^2} = \frac{(0)^2}{(0)^2+y^2} = \frac{0}{y^2} = 0$$

$$\frac{x^2}{x^2+y^2} = \frac{x^2}{x^2+(0)^2} = \frac{x^2}{x^2} = 1$$

Exercício IV

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{3x^3+4y^3}{x^2+y^2} \right] = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{3x^3}{x^2+y^2} \right] \cdot \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{4y^3}{x^2+y^2} \right] = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[3x \left(\frac{x^2}{x^2+y^2} \right) \right] \cdot \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[4y \left(\frac{y^2}{x^2+y^2} \right) \right] =$$

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} [3x] \cdot \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{x^2}{x^2+y^2} \right] \cdot \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} [4y] \cdot \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{y^2}{x^2+y^2} \right] = 0 \cdot [0:1] \cdot 0 \cdot [0:1] = 0$$

$$\frac{x^2}{x^2+y^2} = \frac{(0)^2}{(0)^2+y^2} = \frac{0}{y^2} = 0 \quad (10)$$

$$\frac{x^2}{x^2+y^2} = \frac{x^2}{x^2+(0)^2} = \frac{x^2}{x^2} = 1$$

$$\frac{y^2}{x^2+y^2} = \frac{(0)^2}{x^2+(0)^2} = \frac{0}{x^2} = 0$$

$$\frac{y^2}{x^2+y^2} = \frac{y^2}{(0)^2+y^2} = \frac{y^2}{y^2} = 1$$