César Antônio de Magalhães

Curso de integrais duplas e triplas

Brasil

2016

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Curso de integrais duplas e triplas

Exercícios de integrais duplas e triplas em conformidade com as normas ABNT.

Universidade Norte do Paraná – Unopar

Brasil

2016

Lista de ilustrações

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ABNT Associação Brasileira de Normas Técnicas

$f(x)$, $g(x)$, $f(y)$, $g(y)$, $f(x,y)$, ... Função

dx, dy, $d\theta$, ... Derivada

v Volume

a Área

R Região

P Ponto

r Raio

co Cateto oposto

ca Cateto adjacente

h Hipotenusa

sen Seno

cos Cosseno

tg Tangente

sec Secante

cossec Cossecante

cotg Cotangente

arcsen Arco seno

arccos Arco cosseno

arctg Arco tangente

arcsec Arco secante

arccossec Arco cossecante

arccotg Arco cotangente

log	Logaritmo
ln	Logaritmo natural
e	Número de Euler
lim	Limite

Lista de símbolos

$\int$	Integral
$\iint$	Integral dupla
π	Letra grega minúscula pi
α	Ângulo alfa
θ	Ângulo theta
$\in$	Pertence

Sumário

Introdução

Esse documento contém exercícios retirados do Youtube através do canal OMatematico.com, acesse-o em <https://www.youtube.com/c/omatematicogrings>.

Uma lista de exercícios prontos sobre *derivadas duplas e triplas* é apresentado em Grings (2016).

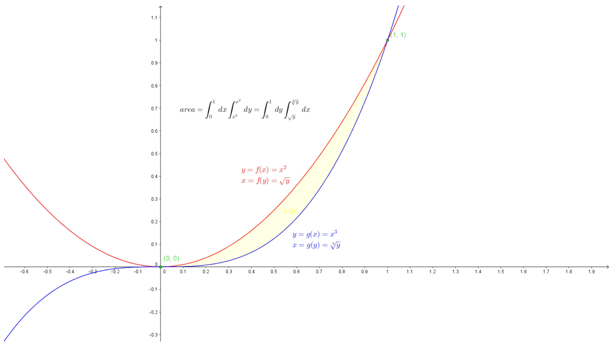
1 Integrais duplas

Cálculo de integrais duplas.

1.1 Invertendo os limites de integração - Aula 1

1. Exercício

Figura 1 – Integrais duplas - Aula 1 - Exercício I e II



$$f(x) = x^2;\ g(x) = x^3$$

$$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$$

$$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$$

$$a = \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx\, [y]_{x^3}^{x^2} = \int_0^1 dx \left[x^2 - x^3\right] =$$
$$\int_0^1 x^2\, dx - \int_0^1 x^3\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{4x^3 - 3x^2}{12}\right]_0^1 = \frac{1}{12}\left[4x^3 - 3x^2\right]_0^1 =$$
$$\frac{1}{12}\left[x^2(4x-3)\right]_0^1 = \frac{1}{12}\left[1^2(4\cdot 1 - 3) \cancel{- 0^2(4\cdot 0 - 3)}\right] = \frac{1}{12} = 0,08\overline{3}$$

2. Exercício

$$f(x) = x^2 \Rightarrow f(y) = \sqrt{y};\ g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$$

$$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$$

$$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$$

$$a = \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy\, [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy\, [\sqrt[3]{y} - \sqrt{y}] =$$

$$\int_0^1 \sqrt[3]{y}\, dy - \int_0^1 \sqrt{y}\, dy = \int_0^1 y^{\frac{1}{3}}\, dy - \int_0^1 y^{\frac{1}{2}}\, dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^1 =$$

$$\left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3} \right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12} \right]_0^1 = \frac{1}{12} \left[9\sqrt[3]{y^4} - 8\sqrt{y^3} \right]_0^1 =$$

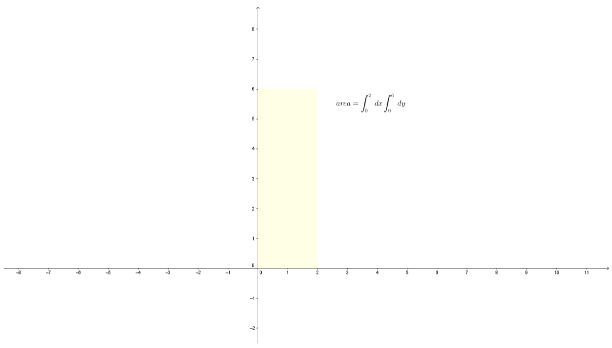
$$\frac{1}{12} \left[\left(9\sqrt[3]{1^4} - 8\sqrt{1^3}\right) \cancel{- \left(9\sqrt[3]{0^4} - 8\sqrt{0^3}\right)} \right] = \frac{1}{12}(9-8) = \frac{1}{12} = 0,08\overline{3}$$

1.2 Determinação da região de integração - Aula 2

1. Exercício

$$R = \left\{ (x,y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 2 \,,\, 0 \leq y \leq 6 \right\}$$

Figura 2 – Integrais duplas - Aula 2 - Exercício I



$$a = \int_0^2 dx \int_0^6 dy = \int_0^2 dx\, [y]_0^6 = \int_0^2 dx\, [6-0] = 6\int_0^2 dx = 6[x]_0^2 =$$
$$6[2-0] = 6 \cdot 2 = 12$$

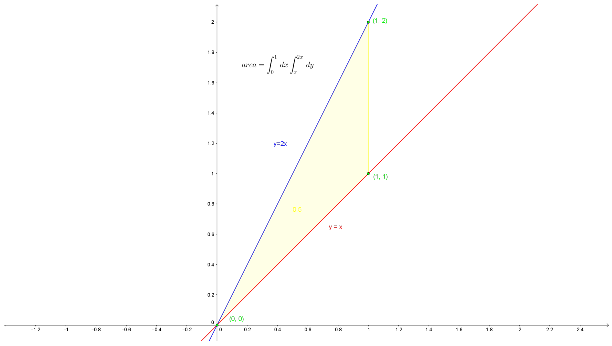
2. Exercício

$$R = \left\{ (x,y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1 \,,\, x \leq y \leq 2x \right\}$$

$$a = \int_0^1 dx \int_x^{2x} dy = \int_0^1 dx\, [y]_x^{2x} = \int_0^1 dx\, [2x - x] = 2\int_0^1 x\, dx - \int_0^1 x\, dx =$$

$$\left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^2}{2} \right]_0^1 = \left[\frac{2x^2 - x^2}{2} \right]_0^1 = \frac{1}{2}\left[x^2\right]_0^1 = \frac{1}{2}\left[1^2 \cancel{- 0^2}\right] = \frac{1}{2} = 0,5$$

Figura 3 – Integrais duplas - Aula 2 - Exercício II



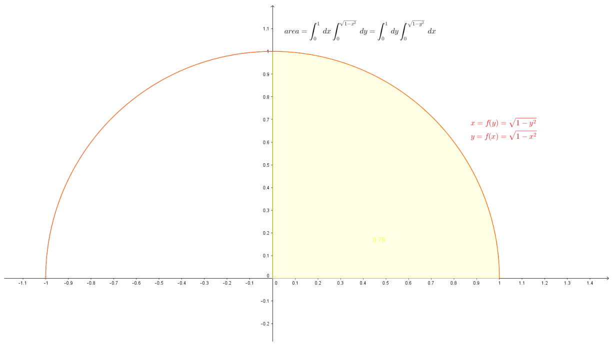
3. Exercício

$$R = \left\{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 1\,,\, 0 \leq x \leq \sqrt{1-y^2}\right\}$$

$$y = 0,\ y = 1$$

$$x = 0,\ x = \sqrt{1-y^2} \Rightarrow x^2 = 1-y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2+1 \Rightarrow y = \sqrt{1-x^2}$$

Figura 4 – Integrais duplas - Aula 2 - Exercício III



$$a = \int_0^1 dy \int_0^{f(y)} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy\, [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy\, \left[\sqrt{1-y^2} - 0\right] =$$

$$\int_0^1 \sqrt{1-y^2}\, dy = \int_0^1 \sqrt{1-\operatorname{sen}^2(t)}\, \cos(t)dt = \int_0^1 \sqrt{\cos^2(t)}\, \cos(t)dt =$$

$$\int_0^1 \cos(t)\cos(t)dt = \int_0^1 \cos^2(t)dt = \int_0^1 \frac{1+\cos(2t)}{2}dt = \frac{1}{2}\int_0^1 [1+\cos(2t)]\, dt =$$

$$\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(2t)dt = \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(u)\frac{du}{2} = \frac{1}{2}\int_0^1 dt + \frac{1}{4}\int_0^1 \cos(u)\, du =$$

$$\left[\frac{1}{2}t + \frac{1}{4}\operatorname{sen}(u)\right]_0^1 = \left[\frac{t}{2} + \frac{\operatorname{sen}(2t)}{4}\right]_0^1 = \left[\frac{t}{2} + \frac{2\operatorname{sen}(t)\cos(t)}{4}\right]_0^1 = \left[\frac{t+\operatorname{sen}(t)\cos(t)}{2}\right]_0^1 =$$

$$\frac{1}{2}\left[\operatorname{arcsen}(y) + y\sqrt{1-y^2}\right]_0^1 = \frac{1}{2}\left[\left(\operatorname{arcsen}(1)\cancel{+1\cdot\sqrt{1-1^2}}\right) - \left(\operatorname{arcsen}(0)\cancel{+0\cdot\sqrt{1-0^2}}\right)\right] =$$

$$\frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4} = 0,785$$

$$y = \operatorname{sen}(t) \Rightarrow dy = \cos(t)dt$$

$$u = 2t \Rightarrow \frac{du}{2} = dt$$

$$\operatorname{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$

$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1 - y^2}$$

$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1-y^2}}{1} = \sqrt{1 - y^2}$$

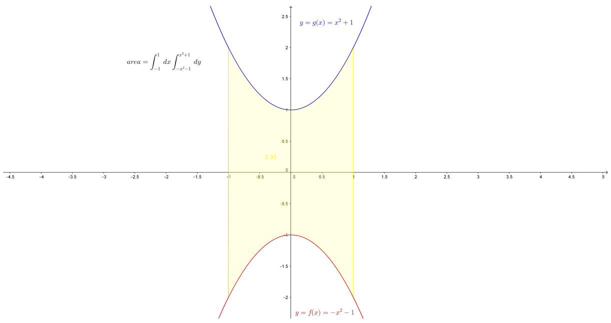
$$y = \operatorname{sen}(t) \Rightarrow t = \operatorname{arcsen}(y)$$

4. Exercício

$$y = x^2 + 1,\ y = -x^2 - 1;\ x = 1,\ x = -1$$

$$R = \Big\{(x, y) \in \mathbb{R}^2 \,|\, -1 \leq x \leq 1\,,\ -x^2 - 1 \leq y \leq x^2 + 1\Big\}$$

Figura 5 – Integrais duplas - Aula 2 - Exercício IV



$$a = \int_{-1}^{1} dx \int_{f(x)}^{g(x)} dy = \int_{-1}^{1} dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^{1} dx\, [y]_{-x^2-1}^{x^2+1} =$$

$$\int_{-1}^{1} dx \left[x^2 + 1 - \left(-x^2 - 1\right)\right] = \int_{-1}^{1} dx \left[x^2 + 1 + x^2 + 1\right] =$$

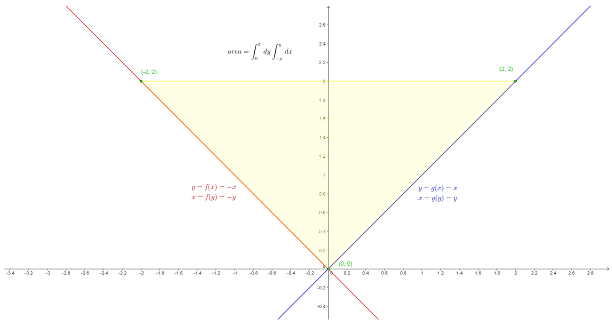
$$\int_{-1}^{1} dx \left[2x^2 + 2\right] = 2\int_{-1}^{1} x^2\, dx + 2\int_{-1}^{1} dx = \left[2\frac{x^3}{3} + 2x\right]_{-1}^{1} = \left[2\left(\frac{x^3 + 3x}{3}\right)\right]_{-1}^{1} =$$

$$\frac{2}{3}\left[x\left(x^2 + 3\right)\right]_{-1}^{1} = \frac{2}{3}\left[1 \cdot \left(1^2 + 3\right) - (-1)\left((-1)^2 + 3\right)\right] = \frac{2}{3}(4 + 4) = \frac{2}{3}8 = \frac{16}{3} = 5{,}\overline{3}$$

5. Exercício

$$R = \Big\{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 2\,,\ -y \leq x \leq y\Big\}$$

Figura 6 – Integrais duplas - Aula 2 - Exercício V

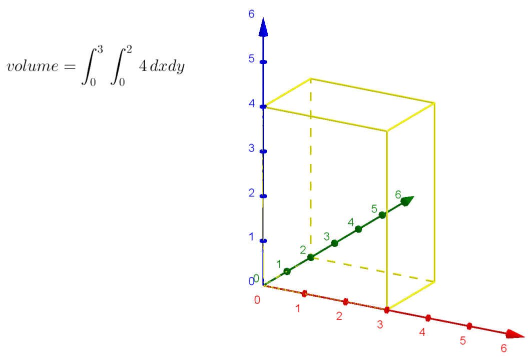


$$a = \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^{y} dx = \int_0^2 dy\, [x]_{-y}^{y} = \int_0^2 dy\, [y-(-y)] = \int_0^2 dy\, [2y] =$$
$$2\int_0^2 y\, dy = \left[\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^2 = 2^2 - 0^2 = 4$$

1.3 Cálculo de volume - Aula 3

1. Exercício

Figura 7 – Integrais duplas - Aula 3 - Exercício I



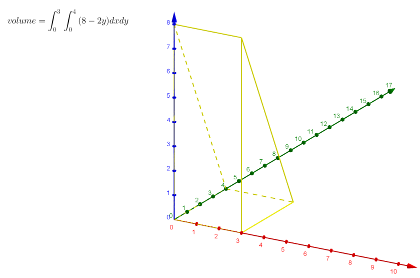
$$z = 4;\ dz = dxdy$$

$$v = \int_0^3 \int_0^2 z\, dz = \int_0^3 \int_0^2 4\, dydx = 4\int_0^3 dx \int_0^2 dy = 4\int_0^3 dx\, [y]_0^2 = 4\int_0^3 dx\, [2-0] =$$
$$8\int_0^3 dx = 8[x]_0^3 = 8[3-0] = 8 \cdot 3 = 24$$

2. Exercício

$$R = [0,3] \times [0,4]$$
$$\iint_R (8-2y) da$$

Figura 8 – Integrais duplas - Aula 3 - Exercício II



$$z = 8 - 2y;\ da = dz = dxdy$$

$$v = \int_0^3 \int_0^4 z\,dz = \int_0^3 \int_0^4 (8-2y)dxdy = \int_0^3 dx \int_0^4 (8-2y)dy =$$
$$\int_0^3 dx \left(8\int_0^4 dy - 2\int_0^4 y\,dy\right) = \int_0^3 dx\, 2\left(4\int_0^4 dy - \int_0^4 y\,dy\right) =$$
$$2\int_0^3 dx \left[4y - \frac{y^2}{2}\right]_0^4 = 2\int_0^3 dx \left[\frac{8y-y^2}{2}\right]_0^4 = \cancel{2}\int_0^3 dx\,\frac{1}{\cancel{2}}[y(8-y)]_0^4 =$$
$$\int_0^3 dx[4(8-4)\cancel{-0(8-0)}] = 16\int_0^3 dx = 16[x]_0^3 = 16[3-0] = 48$$

1.4 Invertendo a ordem de integração - Aula 4

1. Exercício

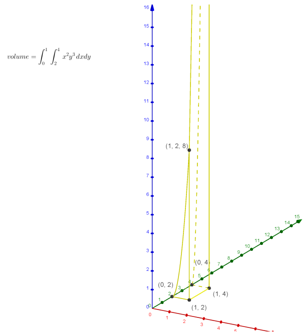
$$z = f(x,y) = y\,\mathrm{e}^x;\ dz = dxdy$$

$$v = \int_2^4 \int_1^9 z\,dz = \int_2^4 \int_1^9 y\,\mathrm{e}^x\ dydx = \int_2^4 \mathrm{e}^x\ dx \int_1^9 y\,dy = \int_2^4 \mathrm{e}^x\ dx \left[\frac{y^2}{2}\right]_1^9 =$$
$$\int_2^4 \mathrm{e}^x\ dx \frac{1}{2}\left[y^2\right]_1^9 = \frac{1}{2}\int_2^4 \mathrm{e}^x\ dx\left[9^2 - 1^2\right] = 40\int_2^4 \mathrm{e}^x\ dx = 40\left[\mathrm{e}^x\right]_2^4 = 40\left[\mathrm{e}^4 - \mathrm{e}^2\right] =$$
$$40e^2\left(e^2 - 1\right)$$

2. Exercício

$$z = f(x,y) = x^2y^3;\ dz = dxdy$$

Figura 9 – Integrais duplas - Aula 4 - Exercício II



$$v = \int_0^1 \int_2^4 z\,dz = \int_0^1 \int_2^4 x^2y^3\,dxdy = \int_0^1 x^2\,dx \int_2^4 y^3\,dy = \int_0^1 x^2\,dx \left[\frac{y^4}{4}\right]_2^4 =$$
$$\frac{1}{4}\int_0^1 x^2\,dx \left[y^4\right]_2^4 = \frac{1}{4}\int_0^1 x^2\,dx \left[4^4 - 2^4\right] = \frac{1}{4}\int_0^1 x^2\,dx \left[2^8 - 2^4\right] =$$
$$\frac{1}{4}\int_0^1 x^2\,dx \left[2^4\left(2^4 - 1\right)\right] = \frac{1}{4}\int_0^1 x^2\,dx \left[16 \cdot 15\right] = 60\int_0^1 x^2\,dx = 60\left[\frac{x^3}{3}\right]_0^1 = 20\left[x^3\right]_0^1 =$$
$$20\left[1^3 - 0^3\right] = 20 \cdot 1 = 20$$

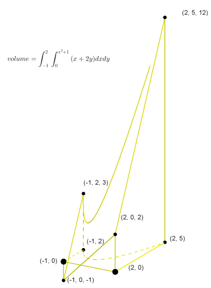
3. Exercício

$$\iint_R (x + 2y)da$$

$R =$ Região limitada pela parábola $y = x^2 + 1$ e as retas $x = -1$ e $x = 2$.

$$z = f(x, y) = x + 2y;\ da = dz = dxdy$$

Figura 10 – Integrais duplas - Aula 4 - Exercício III



$$v = \int_{-1}^{2}\int_{0}^{x^2+1} z\,dz = \int_{-1}^{2}\int_{0}^{x^2+1}(x+2y)dxdy =$$
$$\int_{-1}^{2} dx\int_{0}^{x^2+1}(x+2y)dy = \int_{-1}^{2} dx\left(x\int_{0}^{x^2+1} dy + 2\int_{0}^{x^2+1} y\,dy\right) =$$
$$\int_{-1}^{2} dx\left[xy+\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^{x^2+1} = \int_{-1}^{2} dx\,[y(x+y)]_0^{x^2+1} =$$
$$\int_{-1}^{2} dx\left[\left(x^2+1\right)\left[x+\left(x^2+1\right)\right]\cancel{-0(x+0)}\right] = \int_{-1}^{2} dx\left[\left(x^2+1\right)\left(x^2+x+1\right)\right] =$$
$$\int_{-1}^{2} dx\left(x^4+x^3+2x^2+x+1\right) =$$
$$\int_{-1}^{2} x^4\,dx + \int_{-1}^{2} x^3\,dx + 2\int_{-1}^{2} x^2\,dx + \int_{-1}^{2} x\,dx + \int_{-1}^{2} dx =$$
$$\left[\frac{x^5}{5}+\frac{x^4}{4}+2\frac{x^3}{3}+\frac{x^2}{2}+x\right]_{-1}^{2} = \left[\frac{12x^5+15x^4+40x^3+30x^2+60x}{60}\right]_{-1}^{2} =$$
$$\frac{1}{60}\left[x\left(12x^4+15x^3+40x^2+30x+60\right)\right]_{-1}^{2} =$$
$$\frac{1}{60}\left[2\left(12\cdot 2^4+15\cdot 2^3+40\cdot 2^2+30\cdot 2+60\right)\right.$$
$$\left.-(-1)\left(12(-1)^4+15(-1)^3+40(-1)^2+30(-1)+60\right)\right] =$$
$$\frac{1}{60}[2(192+120+160+60+60)+(12-15+40-30+60)] = \frac{1}{60}(1184+67) =$$
$$\frac{1251}{60} = \frac{417}{20} = 20,85$$

1.5 Cálculo de integrais duplas ou iteradas

1.5.1 Aula 5

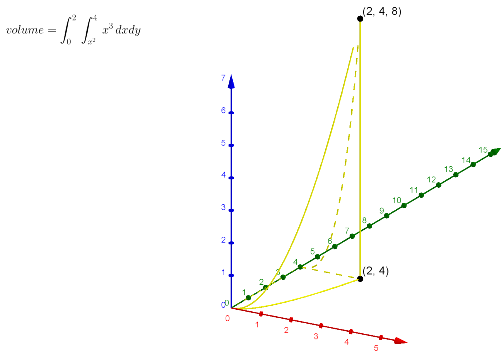
1. Exercício

$$f(x,y) = x^3;\ 0 \le x \le 2;\ x^2 \le y \le 4$$
$$\iint_R f(x,y)dydx$$

$$v = \int_0^2\int_{x^2}^4 x^3\,dxdy = \int_0^2 x^3\,dx\int_{x^2}^4 dy = \int_0^2 x^3\,dx\,[y]_{x^2}^4 = \int_0^2 x^3\,dx\left[4-x^2\right] =$$
$$4\int_0^2 x^3\,dx - \int_0^2 x^5\,dx = \left[\cancel{4}\frac{x^4}{\cancel{4}}-\frac{x^6}{6}\right]_0^2 = \left[\frac{6x^4-x^6}{6}\right]_0^2 = \frac{1}{6}\left[x^4\left(6-x^2\right)\right]_0^2 =$$
$$\frac{1}{6}\left[2^4\left(6-2^2\right)-\cancel{0^4\left(6-0^2\right)}\right] = \frac{1}{6}(16\cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2$$

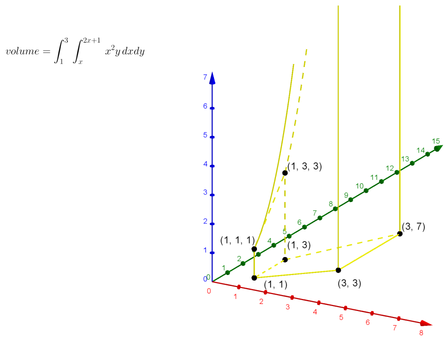
2. Exercício

Figura 11 – Integrais duplas - Aula 5 - Exercício I



$$f(x,y) = x^2y;\ 1 \leq x \leq 3;\ x \leq y \leq 2x+1$$
$$\iint_R f(x,y)dydx$$

Figura 12 – Integrais duplas - Aula 5 - Exercício II



$$v = \int_1^3 \int_x^{2x+1} x^2 y\, dxdy = \int_1^3 x^2\, dx \int_x^{2x+1} y\, dy = \int_1^3 x^2\, dx \left[\frac{y^2}{2}\right]_x^{2x+1} =$$
$$\int_1^3 x^2\, dx \frac{1}{2}\left[(2x+1)^2 - (x)^2\right] = \frac{1}{2}\int_1^3 x^2\, dx \left(3x^2 + 4x + 1\right) =$$
$$\frac{3}{2}\int_1^3 x^4\, dx + 2\int_1^3 x^3\, dx + \frac{1}{2}\int_1^3 x^2\, dx = \left[\frac{3}{2}\frac{x^5}{5} + 2\frac{x^4}{4} + \frac{1}{2}\frac{x^3}{3}\right]_1^3 = \left[\frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6}\right]_1^3 =$$
$$\left[\frac{18x^5 + 30x^4 + 10x^3}{60}\right]_1^3 = \left[\frac{2x^3\left(9x^2 + 15x + 5\right)}{60}\right]_1^3 =$$
$$\frac{1}{30}\left[x^3\left(9x^2 + 15x + 5\right)\right]_1^3 = \frac{1}{30}\left[3^3\left(9 \cdot 3^2 + 15 \cdot 3 + 5\right) - 1^3\left(9 \cdot 1^2 + 15 \cdot 1 + 5\right)\right] =$$
$$\frac{1}{30}\left[27(81 + 45 + 5) - (9 + 15 + 5)\right] = \frac{1}{30}\left[27 \cdot 131 - 29\right] = \frac{3508}{30} = 116{,}9\overline{3}$$

1.5.2 Aula 6

1. Exercício

$$f(x,y)=1;\ 0\le x\le 1;\ 1\le y\le \mathrm{e}^x$$

$$\iint_R f(x,y)dydx$$

$$v=\int_0^1\int_1^{\mathrm{e}^x} dydx=\int_0^1 dx\,[y]_1^{\mathrm{e}^x}=\int_0^1 dx\ (\mathrm{e}^x-1)=[\mathrm{e}^x-x]_0^1=$$
$$\mathrm{e}^1-1-\left(\mathrm{e}^0-0\right)=\mathrm{e}-1-1=\mathrm{e}-2$$

2. Exercício

$$f(x,y)=x;\ 0\le x\le 1;\ 1\le y\le \mathrm{e}^{x^2}$$

$$\iint_R f(x,y)dydx$$

$$v=\int_0^1\int_1^{\mathrm{e}^{x^2}} x\,dxdy=\int_0^1 x\,dx\int_1^{\mathrm{e}^{x^2}} dy=\int_0^1 x\,dx\,[y]_1^{\mathrm{e}^{x^2}}=\int_0^1 x\,dx\,\left(\mathrm{e}^{x^2}-1\right)=$$
$$\int_0^1 x\,\mathrm{e}^{x^2}\,dx-\int_0^1 x\,dx=\int_0^1 \mathrm{e}^u\,\frac{du}{2}-\int_0^1 x\,dx=\frac{1}{2}\int_0^1 \mathrm{e}^u\,du-\int_0^1 x\,dx=$$
$$\left[\frac{1}{2}e^u-\frac{x^2}{2}\right]_0^1=\left[\frac{e^{x^2}-x^2}{2}\right]_0^1=\frac{1}{2}\left[e^{x^2}-x^2\right]_0^1=\frac{1}{2}\left[e^{1^2}-1^2-\left(e^{0^2}-0^2\right)\right]=$$
$$\frac{1}{2}(\mathrm{e}-1-1)=\frac{\mathrm{e}-2}{2}$$

$$u=x^2;\ \frac{du}{2}=x\,dx$$

3. Exercício

$$f(x,y)=2xy;\ 0\le y\le 1;\ y^2\le x\le y$$

$$\iint_R f(x,y)dxdy$$

$$v=\int_0^1\int_{y^2}^{y} 2xy\,dxdy=2\int_0^1 y\,dy\int_{y^2}^{y} x\,dx=2\int_0^1 y\,dy\,\left[\frac{x^2}{2}\right]_{y^2}^{y}=\cancel{2}\int_0^1 y\,dy\,\frac{1}{\cancel{2}}\left[x^2\right]_{y^2}^{y}=$$
$$\int_0^1 y\,dy\,\left(y^2-y^4\right)=\int_0^1\left(y^3-y^5\right)dy=\left[\frac{y^4}{4}-\frac{y^6}{6}\right]_0^1=$$
$$\left[\frac{6y^4-4y^6}{24}\right]_0^1=\left[\frac{2y^4\left(3-2y^2\right)}{24}\right]_0^1=\frac{1}{12}\left[1^4\left(3-2\cdot 1^2\right)\cancel{-0^4\left(3-2\cdot 0^2\right)}\right]=\frac{1}{12}=0,08\overline{3}$$

1.5.3 Aula 7

1. Exercício

$$f(x,y) = \frac{1}{x+y};\ 1 \leq y \leq \mathrm{e};\ 0 \leq x \leq y$$

$$\iint_R f(x,y)dxdy$$

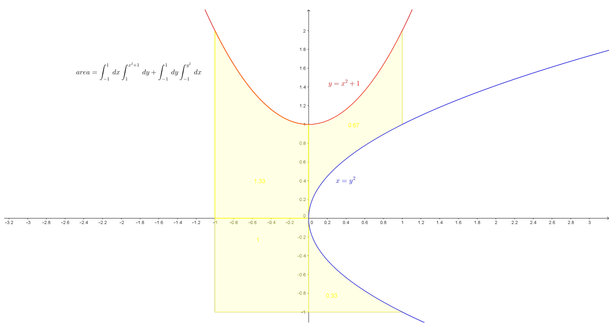
$$v = \int_1^{\mathrm{e}} \int_0^y \frac{1}{x+y}\, dxdy = \int_1^{\mathrm{e}} dy \int_0^y (x+y)^{-1}\, dx = \int_1^{\mathrm{e}} dy \int_0^y u^{-1}\, du =$$
$$\int_1^{\mathrm{e}} dy \int_0^y\ [\ln|u|]_0^y = \int_1^{\mathrm{e}} dy \int_0^y\ [\ln|x+y|]_0^y = \int_1^{\mathrm{e}} dy \int_0^y\ (\ln|y+y| - \ln|0+y|) =$$
$$\int_1^{\mathrm{e}} dy \int_0^y\ (\ln|2y| - \ln|y|) = \int_1^{\mathrm{e}} dy \int_0^y\ (\ln|2| \cancel{+ \ln|y| - \ln|y|}) = \ln|2| \int_1^{\mathrm{e}} dy =$$
$$\ln|2|[y]_1^{\mathrm{e}} = \ln|2|(\mathrm{e}-1)$$

$$u = x + y;\ du = (1+0)dx = dx$$

1.6 Cálculo de área - Aula 8

1. Exercício

Figura 13 – Integrais duplas - Aula 8 - Exercício I



$$
\begin{aligned}
a = \int_{-1}^{0} dx \int_{0}^{x^2+1} dy + \int_{-1}^{0} dx \int_{-1}^{0} dy + \int_{0}^{y^2} dx \int_{-1}^{0} dy + \int_{0}^{1} dx \int_{\sqrt{x}}^{x^2+1} &= \\
\int_{-1}^{0} dx \left(\int_{0}^{x^2+1} dy + \int_{-1}^{0} dy \right) + \int_{0}^{y^2} dx \int_{-1}^{0} dy + \int_{0}^{1} dx \int_{\sqrt{x}}^{x^2+1} &= \\
\int_{-1}^{0} dx \left([y]_0^{x^2+1} + [y]_{-1}^{0} \right) + \int_{-1}^{0} dy\, [x]_0^{y^2} + \int_{0}^{1} dx\, [y]_{\sqrt{x}}^{x^2+1} &= \\
\int_{-1}^{0} dx \left(x^2 + 1 + 1 \right) + \int_{-1}^{0} dy\, y^2 + \int_{0}^{1} dx \left(x^2 + 1 - \sqrt{x} \right) &= \\
\int_{-1}^{0} \left(x^2 + 2 \right) dx + \int_{-1}^{0} y^2\, dy + \int_{0}^{1} \left(x^2 - x^{\frac{1}{2}} + 1 \right) dx &= \\
\left[\frac{x^3}{3} + 2x \right]_{-1}^{0} + \left[\frac{y^3}{3} \right]_{-1}^{0} + \left[\frac{x^3}{3} - \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + x \right]_{0}^{1} &= \\
\left[\frac{x^3 + 6x}{3} \right]_{-1}^{0} + \frac{1}{3} \left[y^3 \right]_{-1}^{0} + \left[\frac{x^3}{3} - \frac{2\sqrt{x^3}}{3} + x \right]_{0}^{1} &= \\
\frac{1}{3} \left[x \left(x^2 + 6 \right) \right]_{-1}^{0} + \frac{1}{3} \left[\cancel{0^3} - (-1)^3 \right] + \left[\frac{x^3 - 2\sqrt{x^3} + 3x}{3} \right]_{0}^{1} &= \\
\frac{1}{3} \left[\cancel{0 \left(0^2 + 6 \right)} - (-1) \left((-1)^2 + 6 \right) \right] + \frac{1}{3} + \frac{1}{3} \left[x^3 - 2\sqrt{x^3} + 3x \right]_{0}^{1} &= \\
\frac{7}{3} + \frac{1}{3} + \frac{1}{3} \left[1^3 - 2\sqrt{1^3} + 3 \cdot 1 \cancel{- \left(0^3 - 2\sqrt{0^3} + 3 \cdot 0 \right)} \right] = \frac{7}{3} + \frac{1}{3} + \frac{2}{3} &= \\
\frac{7+1+2}{3} = \frac{10}{3} &= 3,\overline{3}
\end{aligned}
$$

$$
\begin{aligned}
a = \int_{-1}^{1} dx \int_{1}^{x^2+1} dy + \int_{-1}^{y^2} dx \int_{-1}^{1} dy = \int_{-1}^{1} dx\, [y]_1^{x^2+1} + \int_{-1}^{1} dy\, [x]_{-1}^{y^2} &= \\
\int_{-1}^{1} dx \left(x^2 \cancel{+1-1} \right) + \int_{-1}^{1} dy \left(y^2 + 1 \right) = \left[\frac{x^3}{3} \right]_{-1}^{1} + \left[\frac{y^3}{3} + y \right]_{-1}^{1} &= \\
\frac{1}{3} \left[x^3 \right]_{-1}^{1} + \frac{1}{3} \left[y \left(y^2 + 3 \right) \right]_{-1}^{1} &= \\
\frac{1}{3} \left(\left[1^3 - (-1)^3 \right] + \left[1 \left(1^2 + 3 \right) - (-1) \left((-1)^2 + 3 \right) \right] \right) \frac{1}{3} (2 + 4 + 4) = \frac{10}{3} &= 3,\overline{3}
\end{aligned}
$$

1.7 Cálculo de volume

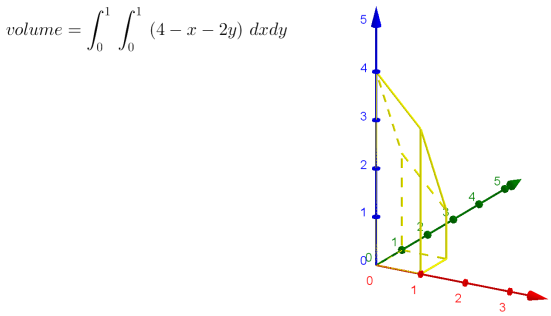
1.7.1 Aula 9

1. Exercício

   Esboçe a região de integração e o sólido cujo volume é dado pela integral abaixo:

$$\int_{0}^{1} \int_{0}^{1} (4 - x - 2y)\; dxdy$$

Figura 14 – Integrais duplas - Aula 9 - Exercício I



$$v = \int_0^1 \int_0^1 (4 - x - 2y)\, dxdy = \int_0^1 dx \left(4 \int_0^1 dy - x \int_0^1 dy - 2 \int_0^1 y\, dy\right) =$$
$$4 \int_0^1 dx \int_0^1 dy - \int_0^1 x\, dx \int_0^1 dy - 2 \int_0^1 dx \int_0^1 y\, dy =$$
$$4[x]_0^1[y]_0^1 - \left[\frac{x^2}{2}\right]_0^1 [y]_0^1 - 2[x]_0^1 \left[\frac{y^2}{2}\right]_0^1 = 4 - \frac{1}{2} - \cancel{2}\frac{1}{\cancel{2}} = \frac{8 - 1 - 2}{2} = \frac{5}{2} = 2,5$$

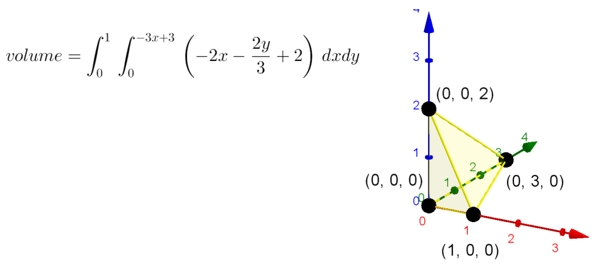
1.7.2 Aula 10

1. Exercício

   Calcule o volume do sólido limitado pelos planos:

$$x = 0,\ y = 0,\ z = 0 \text{ e } 6x + 2y + 3z = 6$$

Figura 15 – Integrais duplas - Aula 10 - Exercício I



$$P_1 = (0, 0, 0)$$
$$6x = -2y - 3z + 6 \Rightarrow x = \frac{-2y - 3z + 6}{6} = \frac{-2 \cdot 0 - 3 \cdot 0 + 6}{6} = \frac{6}{6} = 1 \Rightarrow P_2 = (1, 0, 0)$$
$$2y = -6x - 3z + 6 \Rightarrow y = \frac{-6x - 3z + 6}{2} = \frac{-6 \cdot 0 - 3 \cdot 0 + 6}{2} = \frac{6}{2} = 3 \Rightarrow P_3 = (0, 3, 0)$$
$$3z = -6x - 2y + 6 \Rightarrow z = \frac{-6x - 2y + 6}{3} = \frac{-6 \cdot 0 - 2 \cdot 0 + 6}{3} = \frac{6}{3} = 2 \Rightarrow P_4 = (0, 0, 2)$$

$$x = 0, x = 1$$

$$y = 0,\ y = \frac{-6x - 3z + 6}{2} = \frac{-6x - 3 \cdot 0 + 6}{2} = -3x + 3$$

$$z = \frac{-6x - 2y + 6}{3} = -2x - \frac{2y}{3} + 2$$

$$v = \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dxdy = \int_0^1 dx \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dy =$$

$$\int_0^1 dx \left[-2xy - \frac{\cancel{2}}{3}\frac{y^2}{\cancel{2}} + 2y\right]_0^{-3x+3} = \int_0^1 dx\, \frac{1}{3}\left[-6xy - y^2 + 6y\right]_0^{-3x+3} =$$

$$\frac{1}{3}\int_0^1 dx\ [-y(6x + y - 6)]_0^{-3x+3} =$$

$$\frac{1}{3}\int_0^1 dx\ [-(-3x+3)(6x + (-3x+3) - 6)\cancel{+0(6x+0-6)}] =$$

$$\frac{1}{3}\int_0^1 dx\ [(3x-3)(3x-3)] = \frac{1}{3}\int_0^1 \left(9x^2 - 18x + 9\right) dx = \frac{1}{3}\left[9\frac{x^3}{3} - 18\frac{x^2}{2} + 9x\right]_0^1 =$$

$$\frac{1}{3}\left[3x^3 - 9x^2 + 9x\right]_0^1 = \frac{1}{\cancel{3}}\left[\cancel{3}x\left(x^2 - 3x + 3\right)\right]_0^1 =$$

$$\left[1\left(1^2 \cancel{- 3 \cdot 1 + 3}\right) \cancel{- 0\,(0^2 - 3 \cdot 0 + 3)}\right] = 1$$

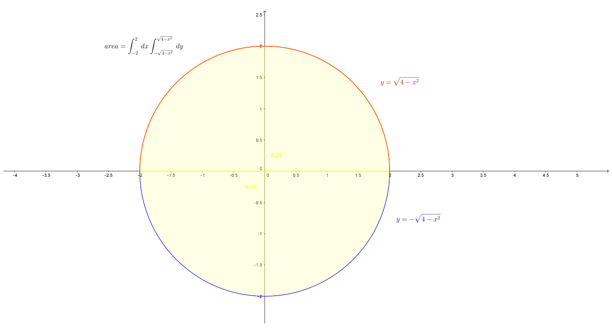
1.8 Coordenadas polares

1.8.1 Aula 1

1. Exercício

   Calcule a área do circulo de raio igual a dois

Figura 16 – Coordenadas polares - Aula 01 - Exercício I



$$r = 2 \Rightarrow a = \pi r^2 = 2^2\pi = 4\pi$$

$$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 2^2 \Rightarrow x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$$

$$R = \left\{(x, y) \in \mathbb{R}^2 \,|\, -2 \le x \le 2,\ -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}\right\}$$

$$a = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{-2}^{2} dx \left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) = 2\int_{-2}^{2} \sqrt{4-x^2}\, dx =$$
$$2\int_{-2}^{2} \sqrt{4-(2\operatorname{sen}(\alpha))^2}\, 2\cos(\alpha)\, d\alpha = 4\int_{-2}^{2} \sqrt{4-4\operatorname{sen}^2(\alpha)}\, \cos(\alpha)\, d\alpha =$$
$$4\int_{-2}^{2} \sqrt{4-4\left(1-\cos^2(\alpha)\right)}\, \cos(\alpha)\, d\alpha = 4\int_{-2}^{2} \sqrt{4-\left(4-4\cos^2(\alpha)\right)}\, \cos(\alpha)\, d\alpha =$$
$$4\int_{-2}^{2} \sqrt{\cancel{4-4}+4\cos^2(\alpha)}\, \cos(\alpha)\, d\alpha = 4\int_{-2}^{2} 2\cos(\alpha)\cos(\alpha)\, d\alpha =$$
$$8\int_{-2}^{2} \cos^2(\alpha)\, d\alpha = 8\int_{-2}^{2} \left(\frac{1+\cos(2\alpha)}{2}\right) d\alpha = 8\int_{-2}^{2} \left(\frac{1}{2} + \frac{\cos(2\alpha)}{2}\right) d\alpha =$$
$$4\int_{-2}^{2} d\alpha + 4\int_{-2}^{2} \cos(2\alpha)\, d\alpha = 4\int_{-2}^{2} d\alpha + 4\int_{-2}^{2} \cos(u)\, \frac{du}{2} =$$
$$4\int_{-2}^{2} d\alpha + 2\int_{-2}^{2} \cos(u)\, du = \left[4\alpha + 2\operatorname{sen}(u)\right]_{-2}^{2} = \left[4\alpha + 2\operatorname{sen}(2\alpha)\right]_{-2}^{2} =$$
$$\left[4\alpha + 4\operatorname{sen}(\alpha)\cos(\alpha)\right]_{-2}^{2} = \left[4\left(\alpha + \operatorname{sen}(\alpha)\cos(\alpha)\right)\right]_{-2}^{2} =$$
$$\left[4\left(\operatorname{arcsen}\left(\frac{x}{2}\right) + \frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right)\right]_{-2}^{2} = \left[4\left(\operatorname{arcsen}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4}\right)\right]_{-2}^{2} =$$
$$4\left(\operatorname{arcsen}\left(\frac{2}{2}\right) + \frac{2\sqrt{4-2^2}}{4}\right) - 4\left(\operatorname{arcsen}\left(\frac{(-2)}{2}\right) + \frac{(-2)\sqrt{4-(-2)^2}}{4}\right) =$$
$$4\operatorname{arcsen}(1) - 4\operatorname{arcsen}(-1) = 4(\operatorname{arcsen}(1) - \operatorname{arcsen}(-1)) = 4\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 4\left(\frac{\cancel{2}\pi}{\cancel{2}}\right) = 4\pi$$

$$x = 2\operatorname{sen}(\alpha);\ dx = 2\cos(\alpha)\, d\alpha$$
$$u = 2\alpha;\ \frac{du}{2} = d\alpha$$
$$\operatorname{sen}(\alpha) = \frac{co}{h} = \frac{x}{2} \Rightarrow \alpha = \operatorname{arcsen}\left(\frac{x}{2}\right)$$
$$h^2 = co^2 + ca^2 \Rightarrow 2^2 = x^2 + ca^2 \Rightarrow ca = \sqrt{4-x^2}$$
$$\cos(\alpha) = \frac{ca}{h} = \frac{\sqrt{4-x^2}}{2}$$
$$R = \left\{(r,\theta) \in \mathbb{R}^2 \,|\, 0 \le r \le 2,\ 0 \le \theta \le 2\pi\right\}$$

$$a = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{0}^{2}\int_{0}^{2\pi} r\, dr d\theta = \int_{0}^{2} r\, dr \int_{0}^{2\pi} d\theta = \left[\frac{r^2}{2}\right]_{0}^{2} [\theta]_{0}^{2\pi} =$$
$$\frac{1}{2}\left[2^2 - 0^2\right][2\pi - 0] = \frac{4}{\cancel{2}}\cancel{2}\pi = 4\pi$$

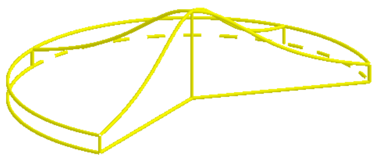
2. Exercício

$$\iint_R \frac{da}{1+x^2+y^2}$$

$$R = \left\{(r,\theta) \in \mathbb{R}^2 \,|\, 0 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{2}\right\}$$

Figura 17 – Coordenadas polares - Aula 01 - Exercício II

$$volume = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\,drd\theta}{1+r^2}$$



$$v = \iint_R \frac{da}{1+x^2+y^2} = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\,drd\theta}{1+r^2} = \int_0^2 \frac{r\,dr}{1+r^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} d\theta =$$
$$\int_0^2 \left(1+r^2\right)^{-1} r\,dr\,[\theta]_{\frac{\pi}{4}}^{\frac{3\pi}{2}} = \int_0^2 \left(1+r^2\right)^{-1} r\,dr\left(\frac{3\pi}{2}-\frac{\pi}{4}\right) =$$
$$\int_0^2 \left(1+r^2\right)^{-1} r\,dr\left(\frac{6\pi-\pi}{4}\right) = \frac{5\pi}{4}\int_0^2 \left(1+r^2\right)^{-1} r\,dr = \frac{5\pi}{4}\int_0^2 u^{-1}\frac{du}{2} =$$
$$\frac{5\pi}{8}\int_0^2 u^{-1}du = \frac{5\pi}{8}\left[ln|u|\right]_0^2 = \frac{5\pi}{8}\left[ln|1+r^2|\right]_0^2 = \frac{5\pi}{8}\left[ln|1+2^2| - ln|1+0^2|\right] =$$
$$\frac{5\pi}{8}\left[ln|5| - ln|1|\right] = \frac{5\pi ln|5|}{8}$$

$$u = 1+r^2 \Rightarrow \frac{du}{2} = r\,dr$$
$$\mathrm{e}^x = 1 = \mathrm{e}^0 \Rightarrow x = 0$$

2 Integrais triplas

Cálculo de integrais triplas.

Referências

GRINGS, F. *Curso de Integrais Duplas e Triplas*. [S.l.], 2016. Disponível em: <https://www.youtube.com/playlist?list=PL82B9E5FF3F2B3BD3>. Citado na página 13.

Anexos

ANEXO A – Derivadas

A.1 Derivadas simples

Tabela 1 – Derivadas simples

$$
\begin{array}{|lllllll|}
y & = & c & \Rightarrow & y' & = & 0 \\
y & = & x & \Rightarrow & y' & = & 1 \\
y & = & x^c & \Rightarrow & y' & = & cx^{c-1} \\
y & = & \mathrm{e}^x & \Rightarrow & y' & = & \mathrm{e}^x \\
y & = & \ln|x| & \Rightarrow & y' & = & \dfrac{1}{x} \\
y & = & uv & \Rightarrow & y' & = & u'v + uv' \\
y & = & \dfrac{u}{v} & \Rightarrow & y' & = & \dfrac{u'v - uv'}{v^2} \\
y & = & u^c & \Rightarrow & y' & = & cu^{c-1}u' \\
y & = & \mathrm{e}^u & \Rightarrow & y' & = & \mathrm{e}^u\, u' \\
y & = & c^u & \Rightarrow & y' & = & c^u u' ln|c| \\
y & = & \ln|u| & \Rightarrow & y' & = & \dfrac{u'}{u} \\
y & = & \log_c|u| & \Rightarrow & y' & = & \dfrac{u'}{u}\log_c|\,\mathrm{e}\,|
\end{array}
$$

A.2 Derivadas trigonométricas

Tabela 2 – Derivadas trigonométricas

$$
\begin{array}{|lllllll|}
y & = & \mathrm{sen}(x) & \Rightarrow & y' & = & \cos(x) \\
y & = & \cos(x) & \Rightarrow & y' & = & -\mathrm{sen}(x) \\
y & = & \mathrm{tg}(x) & \Rightarrow & y' & = & \sec^2(x) \\
y & = & \mathrm{cotg}(x) & \Rightarrow & y' & = & -\mathrm{cossec}^2(x) \\
y & = & \sec(x) & \Rightarrow & y' & = & \sec(x)\,\mathrm{tg}(x) \\
y & = & \mathrm{cossec}(x) & \Rightarrow & y' & = & -\mathrm{cossec}(x)\,\mathrm{cotg}(x) \\
y & = & \mathrm{arcsen}(x) & \Rightarrow & y' & = & \dfrac{1}{\sqrt{1-x^2}} \\
y & = & \arccos(x) & \Rightarrow & y' & = & \dfrac{-1}{\sqrt{1-x^2}} \\
y & = & \mathrm{arctg}(x) & \Rightarrow & y' & = & \dfrac{1}{1+x^2} \\
y & = & \mathrm{arccotg}(x) & \Rightarrow & y' & = & \dfrac{-1}{1+x^2} \\
y & = & \mathrm{arcsec}(x) & \Rightarrow & y' & = & \dfrac{1}{|x|\sqrt{x^2-1}} \\
y & = & \mathrm{arccossec}(x) & \Rightarrow & y' & = & \dfrac{-1}{|x|\sqrt{x^2-1}}
\end{array}
$$

ANEXO B – Integrais

B.1 Integrais simples

Tabela 3 – Integrais simples

$$
\begin{array}{|lcll|}
\int dx & = & x + c & \\
\int x^p \, dx & = & \dfrac{x^{p+1}}{p+1} + c & \rightarrow \quad p \neq -1 \\
\int \mathrm{e}^x \, dx & = & \mathrm{e}^x + c & \\
\int \dfrac{dx}{x} & = & \ln|x| + c & \\
\int u^p \, du & = & \dfrac{u^{p+1}}{p+1} + c & \rightarrow \quad p \neq -1 \\
\int \mathrm{e}^u \, du & = & \mathrm{e}^u + c & \\
\int \dfrac{du}{u} & = & \ln|u| + c & \\
\int p^u \, du & = & \dfrac{p^u}{\ln|p|} + c & \\
\end{array}
$$

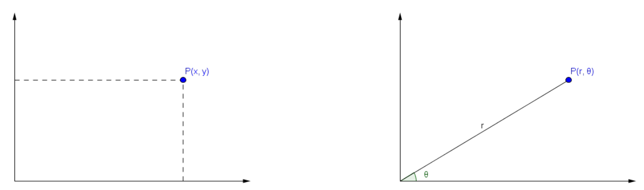
B.2 Integrais trigonométricas

B.3 Relação entre coordenada cartesina e polar

Figura 18 – Coordenada cartesina e polar

(a) Coordenada cartesiana ou retangular

(b) Coordenada polar



$$P(x, y) \rightarrow P(r, \theta)$$

Tabela 4 – Integrais trigonométricas

$$
\begin{array}{|lcl|}
\displaystyle\int \operatorname{sen}(u)du & = & -\cos(u) + c \\
\displaystyle\int \cos(u)du & = & \operatorname{sen}(u) + c \\
\displaystyle\int \operatorname{tg}(u)du & = & \ln|\sec(u)| + c \\
\displaystyle\int \operatorname{cotg}(u)du & = & \ln|\operatorname{sen}(u)| + c \\
\displaystyle\int \sec(u)du & = & \ln|\sec(u) + \operatorname{tg}(u)| + c \\
\displaystyle\int \operatorname{cossec}(u)du & = & \ln|\operatorname{cossec}(u) - \operatorname{cotg}(u)| + c \\
\displaystyle\int \sec^2(u)du & = & \operatorname{tg}(u) + c \\
\displaystyle\int \operatorname{cossec}^2(u)du & = & -\operatorname{cotg}(u) + c \\
\displaystyle\int \sec(u)\operatorname{tg}(u)du & = & \sec(u) + c \\
\displaystyle\int \operatorname{cossec}(u)\operatorname{cotg}(u)du & = & -\operatorname{cossec}(u) + c \\
\displaystyle\int \frac{du}{\sqrt{1-x^2}} & = & \operatorname{arcsen}(x) + c \\
\displaystyle-\int \frac{du}{\sqrt{1-x^2}} & = & \arccos(x) + c \\
\displaystyle\int \frac{du}{1+x^2} & = & \operatorname{arctg}(x) + c \\
\displaystyle-\int \frac{du}{1+x^2} & = & \operatorname{arccotg}(x) + c \\
\displaystyle\int \frac{du}{|x|\sqrt{x^2-1}} & = & \operatorname{arcsec}(x) + c \\
\displaystyle-\int \frac{du}{|x|\sqrt{x^2-1}} & = & \operatorname{arccossec}(x) + c
\end{array}
$$

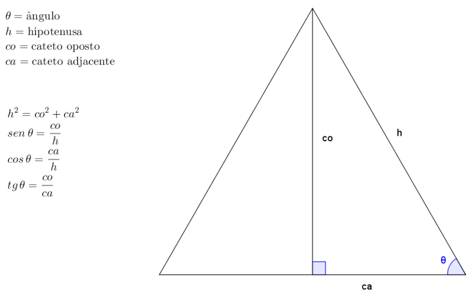
Tabela 5 – Relação entre coordenada cartesina e polar

$$
\begin{array}{|lcccl|}
 & & x & = & r\,\cos\theta \\
 & & y & = & r\,\operatorname{sen}\theta \\
 & & x^2 + y^2 & = & r^2 \\
da & = & dxdy & = & r\,drd\theta \\
v & = & \displaystyle\iint_{R(x,y)} f(x,y)\,dxdy & = & \displaystyle\iint_{R(r,\theta)} f(r\,\cos\theta, r\,\operatorname{sen}\theta)r\,drd\theta
\end{array}
$$

ANEXO C – Funções trigonométricas

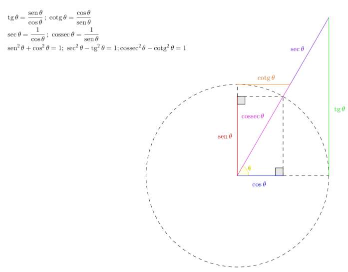
C.1 Determinação do seno, cosseno e tangente

Figura 19 – Determinação do seno, cosseno e tangente



C.2 Círculo trigonométrico

Figura 20 – Círculo trigonométrico



C.3 Identidades trigonométricas

C.4 Relação entre trigonométricas e inversas

C.5 Substituição trigonométrica

C.6 Ângulos notáveis

Tabela 6 – Identidades trigonométricas

$$
\begin{aligned}
\mathrm{tg}(x) &= \frac{\mathrm{sen}(x)}{\cos(x)} \\
\mathrm{cotg}(x) &= \frac{\cos(x)}{\mathrm{sen}(x)} \\
\sec(x) &= \frac{1}{\cos(x)} \\
\mathrm{cossec}(x) &= \frac{1}{\mathrm{sen}(x)} \\
\mathrm{sen}^2(x) + \cos^2(x) &= 1 \\
\sec^2(x) - \mathrm{tg}^2(x) &= 1 \\
\mathrm{cossec}^2(x) - \mathrm{cotg}^2(x) &= 1 \\
\mathrm{sen}^2(x) &= \frac{1 - \cos(2x)}{2} \\
\cos^2(x) &= \frac{1 + \cos(2x)}{2} \\
\mathrm{sen}(2x) &= 2\,\mathrm{sen}(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \mathrm{sen}^2(x)
\end{aligned}
$$

Tabela 7 – Relação entre trigonométricas e inversas

$$
\begin{array}{lcccccl}
\mathrm{sen}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arcsen}(x) \\
\cos(\theta) & = & x & \Rightarrow & \theta & = & \arccos(x) \\
\mathrm{tg}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arctg}(x) \\
\mathrm{cossec}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arccossec}(x) \\
\sec(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arcsec}(x) \\
\mathrm{cotg}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arccotg}(x)
\end{array}
$$

Tabela 8 – Substituição trigonométrica

$$
\begin{array}{lcccl}
\sqrt{a^2 - x^2} & \Rightarrow & x & = & a\,\mathrm{sen}(\theta) \\
\sqrt{a^2 + x^2} & \Rightarrow & x & = & a\,\mathrm{tg}(\theta) \\
\sqrt{x^2 - a^2} & \Rightarrow & x & = & a\sec(\theta)
\end{array}
$$

Tabela 9 – Ângulos notáveis

ângulo	$0°$ (0)	$30°$ $\left(\frac{\pi}{6}\right)$	$45°$ $\left(\frac{\pi}{4}\right)$	$60°$ $\left(\frac{\pi}{3}\right)$	$90°$ $\left(\frac{\pi}{2}\right)$
sen	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\nexists$