Curso de integrais duplas e triplas

César Antônio de Magalhães
ceanma@gmail.com

26 de agosto de 2016

Sumário

Lista de Figuras

Parte I
Integrais duplas

1 Invertendo os limites de integração - Aula 1

1. Exercício

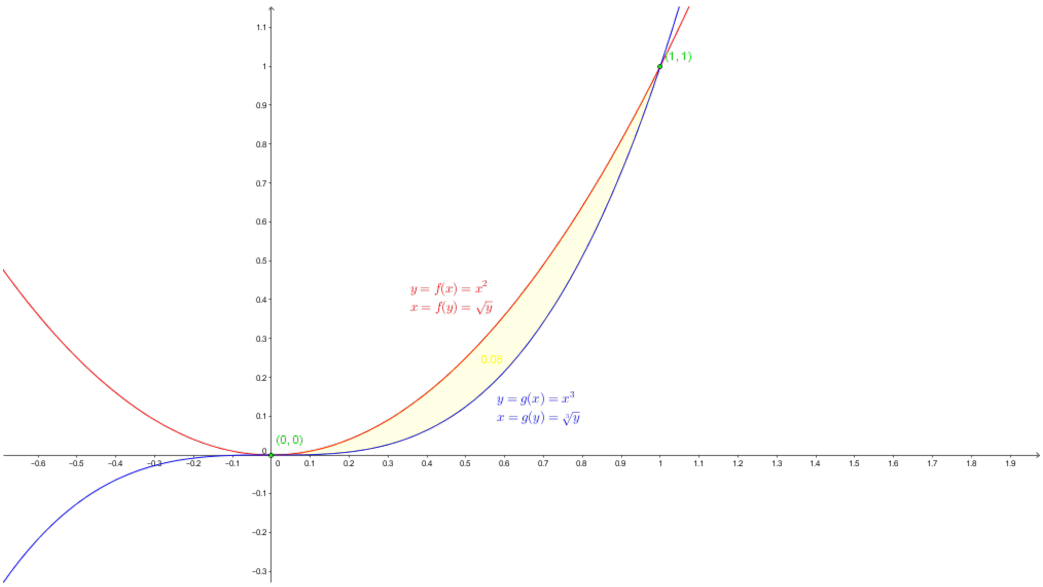

Figura 1: Integrais duplas - Aula 1 - Exercício I e II

$f(x) = x^2;\ g(x) = x^3$
$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$
$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$

$$a = \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx\, [y]_{x^3}^{x^2} = \int_0^1 dx \left[x^2 - x^3\right] = \int_0^1 x^2\,dx - \int_0^1 x^3\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{4x^3 - 3x^2}{12}\right]_0^1 = \frac{1}{12}\left[4x^3 - 3x^2\right]_0^1 = \frac{1}{12}\left[x^2\,(4x-3)\right]_0^1 = \frac{1}{12}\left[1^2\,(4\cdot 1 - 3)\cancel{-0^2\,(4\cdot 0 - 3)}\right] = \frac{1}{12} = 0,08\overline{3}$$

2. Exercício

$f(x) = x^2 \Rightarrow f(y) = \sqrt{y};\ g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$
$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$
$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$

$$a = \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy\, [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy\, [\sqrt[3]{y} - \sqrt{y}] =$$

$$\int_0^1 \sqrt[3]{y}\, dy - \int_0^1 \sqrt{y}\, dy = \int_0^1 y^{\frac{1}{3}}\, dy - \int_0^1 y^{\frac{1}{2}}\, dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^1 =$$

$$\left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3} \right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12} \right]_0^1 = \frac{1}{12}\left[9\sqrt[3]{y^4} - 8\sqrt{y^3} \right]_0^1 =$$

$$\frac{1}{12}\left[\left(9\sqrt[3]{1^4} - 8\sqrt{1^3}\right) \cancel{- \left(9\sqrt[3]{0^4} - 8\sqrt{0^3}\right)} \right] = \frac{1}{12}(9-8) = \frac{1}{12} = 0,08\overline{3}$$

2 Determinação da região de integração - Aula 2

1. Exercício

   $R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 2\,,\, 0 \leq y \leq 6\}$

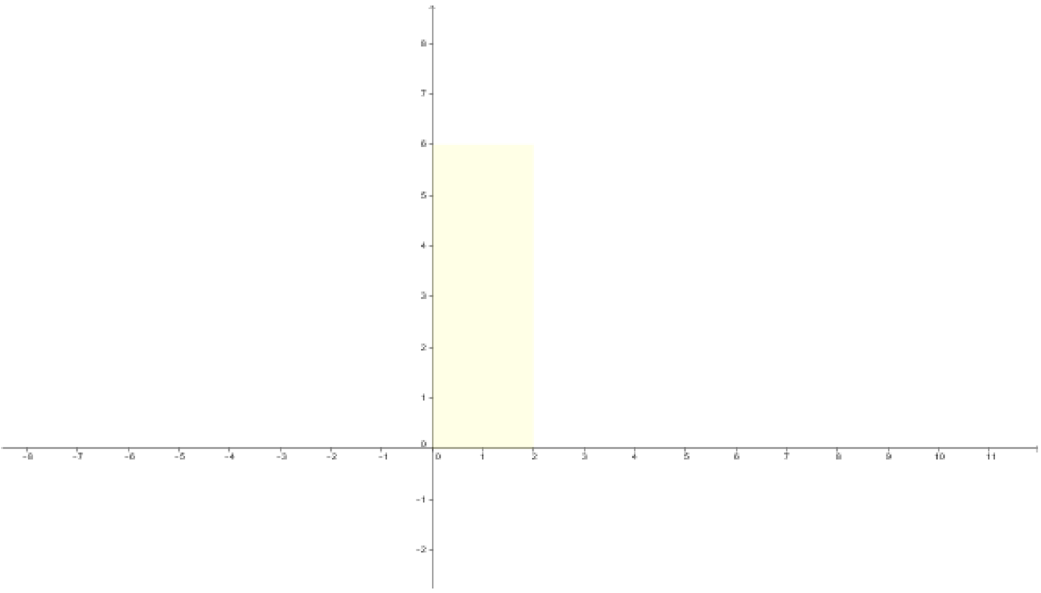

Figura 2: Integrais duplas - Aula 2 - Exercício I

$$a = \int_0^2 dx \int_0^6 dy = \int_0^2 dx\,[y]_0^6 = \int_0^2 dx\,[6-0] = 6\int_0^2 dx = 6[x]_0^2 = 6[2-0] = 6 \cdot 2 = 12$$

2. Exercício

   $R = \{(x,y) \in \mathbb{R} \,|\, 0 \leq x \leq 1\,,\; x \leq y \leq 2x\}$

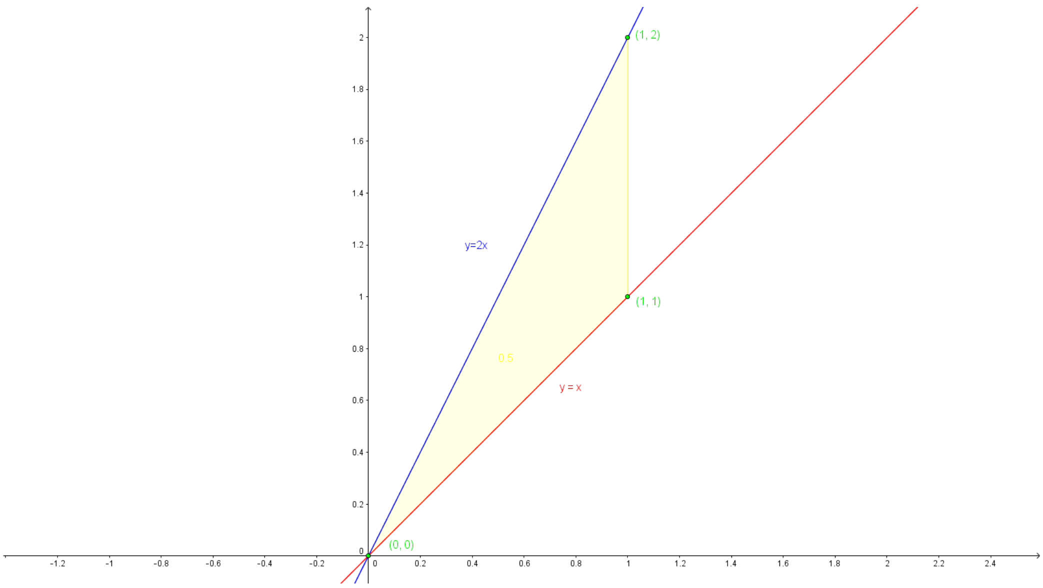

Figura 3: Integrais duplas - Aula 2 - Exercício II

$$a = \int_0^1 dx \int_x^{2x} dy = \int_0^1 dx\,[y]_x^{2x} = \int_0^1 dx\,[2x - x] = 2\int_0^1 x\,dx - \int_0^1 x\,dx = \left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^2}{2}\right]_0^1 = \left[\frac{2x^2 - x^2}{2}\right]_0^1 = \frac{1}{2}\left[x^2\right]_0^1 = \frac{1}{2}\left[1^2 \cancel{-0^2}\right] = \frac{1}{2} = 0,5$$

3. Exercício

   $R = \left\{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 1\,,\; 0 \leq x \leq \sqrt{1-y^2}\right\}$

   $y = 0,\ y = 1$

   $x = 0,\ x = \sqrt{1-y^2} \Rightarrow x^2 = 1 - y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow y = \sqrt{1-x^2}$

Figura 4: Integrais duplas - Aula 2 - Exercício III

$$a = \int_0^1 dy \int_0^{f(y)} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy \, [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy \left[\sqrt{1-y^2} - 0\right] =$$
$$\int_0^1 \sqrt{1-y^2}\, dy = \int_0^1 \sqrt{1-\operatorname{sen}^2(t)}\, \cos(t) dt = \int_0^1 \sqrt{\cos^2(t)}\, \cos(t) dt =$$
$$\int_0^1 \cos(t)\cos(t) dt = \int_0^1 \cos^2(t) dt = \int_0^1 \frac{1+\cos(2t)}{2} dt = \frac{1}{2}\int_0^1 \left[1+\cos(2t)\right] dt =$$
$$\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(2t) dt = \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(u)\frac{du}{2} = \frac{1}{2}\int_0^1 dt +$$
$$\frac{1}{4}\int_0^1 \cos(u)\, du = \left[\frac{1}{2}t + \frac{1}{4}\operatorname{sen}(u)\right]_0^1 = \left[\frac{t}{2} + \frac{\operatorname{sen}(2t)}{4}\right]_0^1 = \left[\frac{t}{2} + \frac{2\operatorname{sen}(t)\cos(t)}{4}\right]_0^1 =$$
$$\left[\frac{t + \operatorname{sen}(t)\cos(t)}{2}\right]_0^1 = \frac{1}{2}\left[\operatorname{arcsen}(y) + y\sqrt{1-y^2}\right]_0^1 =$$
$$\frac{1}{2}\left[\left(\operatorname{arcsen}(1)\cancel{+1\cdot\sqrt{1-1^2}}\right) - \left(\operatorname{arcsen}(0)\cancel{+0\cdot\sqrt{1-0^2}}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] =$$
$$\frac{\pi}{4} = 0,785$$

$$y = \operatorname{sen}(t) \Rightarrow dy = \cos(t) dt$$
$$u = 2t \Rightarrow \frac{du}{2} = dt$$

$$\operatorname{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$
$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1-y^2}$$

$\cos(t) = \dfrac{ca}{h} = \dfrac{\sqrt{1-y^2}}{1} = \sqrt{1-y^2}$
$y = \text{sen}(t) \Rightarrow t = \text{arcsen}(y)$

4. Exercício

   $y = x^2+1,\ y = -x^2-1;\ x = 1,\ x = -1$
   $R = \{(x,y) \in \mathbb{R} \,|\, -1 \leq x \leq 1\,,\ -x^2-1 \leq y \leq x^2+1\}$

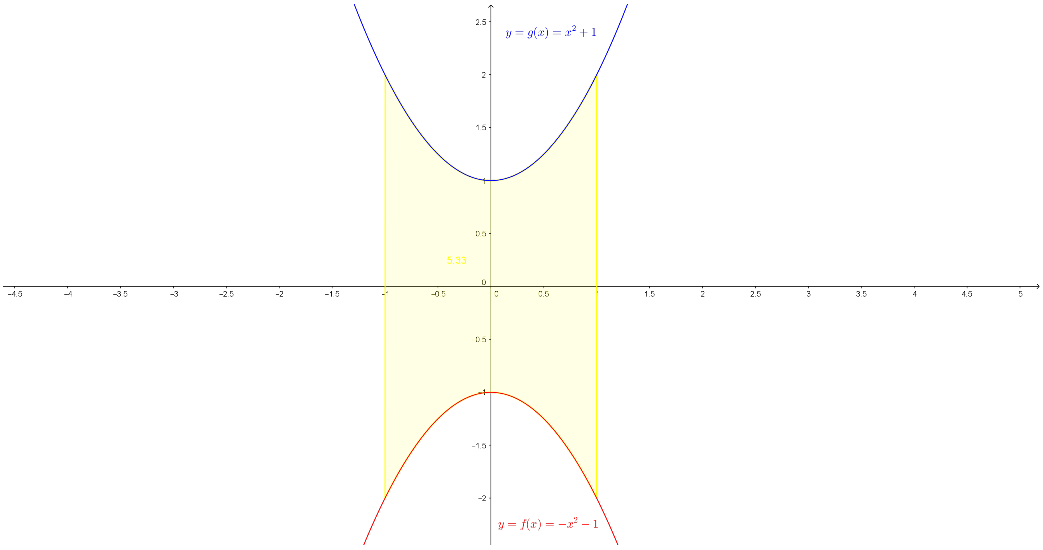

Figura 5: Integrais duplas - Aula 2 - Exercício IV

$$a = \int_{-1}^{1} dx \int_{f(x)}^{g(x)} dy = \int_{-1}^{1} dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^{1} dx\, [y]_{-x^2-1}^{x^2+1} = \int_{-1}^{1} dx\ \left[x^2+1-\left(-x^2-1\right)\right] =$$
$$\int_{-1}^{1} dx\ \left[x^2+1+x^2+1\right] = \int_{-1}^{1} dx\ \left[2x^2+2\right] = 2\int_{-1}^{1} x^2\,dx + 2\int_{-1}^{1} dx =$$
$$\left[2\frac{x^3}{3}+2x\right]_{-1}^{1} = \left[2\left(\frac{x^3+3x}{3}\right)\right]_{-1}^{1} = \frac{2}{3}\left[x\left(x^2+3\right)\right]_{-1}^{1} =$$
$$\frac{2}{3}\left[1\cdot\left(1^2+3\right)-(-1)\left((-1)^2+3\right)\right] = \frac{2}{3}(4+4) = \frac{2}{3}8 = \frac{16}{3} = 5,\overline{3}$$

5. Exercício

   $R = \{(x,y) \in \mathbb{R} \,|\, 0 \leq y \leq 2\,,\ -y \leq x \leq y\}$

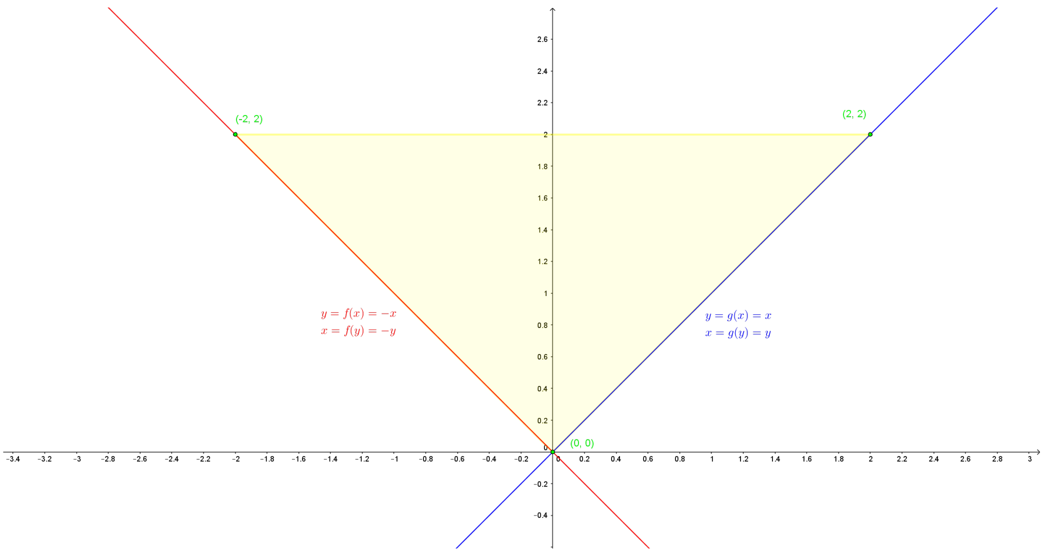

Figura 6: Integrais duplas - Aula 2 - Exercício V

$$a = \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^{y} dx = \int_0^2 dy\, [x]_{-y}^{y} = \int_0^2 dy\, [y-(-y)] = \int_0^2 dy\, [2y] = 2\int_0^2 y\, dy = \left[\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^2 = 2^2 - 0^2 = 4$$

3 Cálculo de volume - Aula 3

1. Exercício

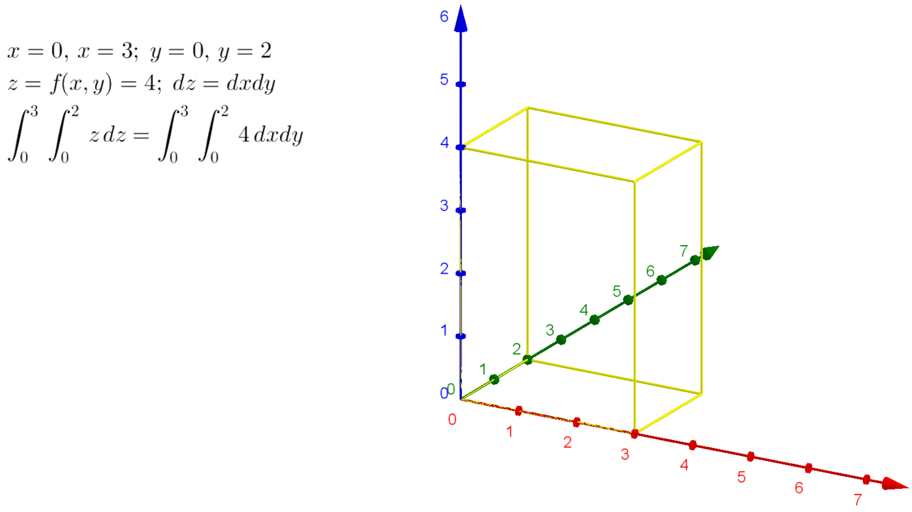

Figura 7: Integrais duplas - Aula 3 - Exercício I

$z = 4;\ dz = dxdy$

$v = \int_0^3 \int_0^2 z\,dz = \int_0^3 \int_0^2 4\,dydx = 4\int_0^3 dx \int_0^2 dy = 4\int_0^3 dx\,[y]_0^2 =$
$4\int_0^3 dx\,[2-0] = 8\int_0^3 dx = 8[x]_0^3 = 8[3-0] = 8 \cdot 3 = 24$

2. Exercício

$R = [0, 3] \times [0, 4]$
$\iint_R (8 - 2y)da$

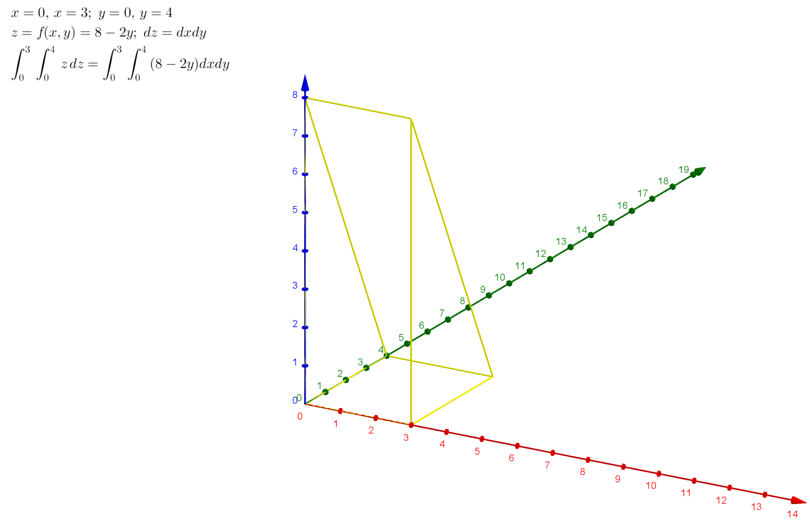

Figura 8: Integrais duplas - Aula 3 - Exercício II

$z = 8 - 2y;\ da = dz = dxdy$

$$v = \int_0^3 \int_0^4 z\,dz = \int_0^3 \int_0^4 (8-2y)dxdy = \int_0^3 dx \int_0^4 (8-2y)dy = \int_0^3 dx \left(8\int_0^4 dy - 2\int_0^4 y\,dy\right) = \int_0^3 dx\, 2\left(4\int_0^4 dy - \int_0^4 y\,dy\right) = 2\int_0^3 dx \left[4y - \frac{y^2}{2}\right]_0^4 = 2\int_0^3 dx \left[\frac{8y - y^2}{2}\right]_0^4 = \cancel{2}\int_0^3 dx\, \frac{1}{\cancel{2}}[y(8-y)]_0^4 = \int_0^3 dx[4(8-4)\cancel{-0(8-0)}] = 16\int_0^3 dx = 16[x]_0^3 = 16[3-0] = 48$$

4 Invertendo a ordem de integração - Aula 4

1. Exercício

   $z = f(x, y) = y\,\mathrm{e}^x;\ dz = dxdy$

   $$v = \int_2^4 \int_1^9 z\,dz = \int_2^4 \int_1^9 y\,\mathrm{e}^x\ dydx = \int_2^4 \mathrm{e}^x\ dx \int_1^9 y\,dy = \int_2^4 \mathrm{e}^x\ dx \left[\frac{y^2}{2}\right]_1^9 = \int_2^4 \mathrm{e}^x\ dx \frac{1}{2}\left[y^2\right]_1^9 = \frac{1}{2}\int_2^4 \mathrm{e}^x\ dx \left[9^2 - 1^2\right] = 40\int_2^4 \mathrm{e}^x\ dx = 40\left[\mathrm{e}^x\right]_2^4 = 40\left[\mathrm{e}^4 - \mathrm{e}^2\right] = 40e^2\left(e^2 - 1\right)$$

2. Exercício

   $z = f(x, y) = x^2y^3;\ dz = dxdy$

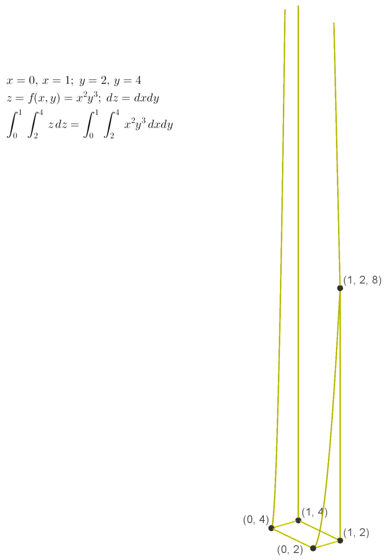

Figura 9: Integrais duplas - Aula 4 - Exercício II

$$v = \int_0^1 \int_2^4 z\, dz = \int_0^1 \int_2^4 x^2y^3\, dxdy = \int_0^1 x^2\, dx \int_2^4 y^3\, dy = \int_0^1 x^2\, dx \left[\frac{y^4}{4}\right]_2^4 =$$
$$\frac{1}{4}\int_0^1 x^2\, dx \left[y^4\right]_2^4 = \frac{1}{4}\int_0^1 x^2\, dx \left[4^4 - 2^4\right] = \frac{1}{4}\int_0^1 x^2\, dx \left[2^8 - 2^4\right] =$$
$$\frac{1}{4}\int_0^1 x^2\, dx \left[2^4\left(2^4 - 1\right)\right] = \frac{1}{4}\int_0^1 x^2\, dx\, [16 \cdot 15] = 60\int_0^1 x^2\, dx = 60\left[\frac{x^3}{3}\right]_0^1 =$$
$$20\left[x^3\right]_0^1 = 20\left[1^3 - 0^3\right] = 20 \cdot 1 = 20$$

3. Exercício

   $$\iint_R (x + 2y) da$$

   R = Região limitada pela parábola $y = x^2 + 1$ e as retas $x = -1$ e $x = 2$.

   $z = f(x, y) = x + 2y;\ da = dz = dxdy$

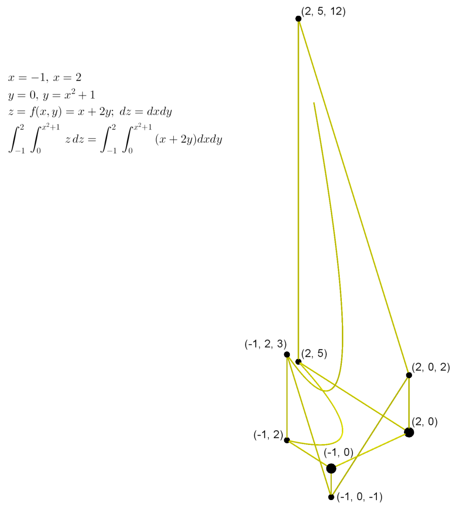

Figura 10: Integrais duplas - Aula 4 - Exercício III

$$v = \int_{-1}^{2}\int_{0}^{x^2+1} z\,dz = \int_{-1}^{2}\int_{0}^{x^2+1}(x+2y)dxdy = \int_{-1}^{2} dx \int_{0}^{x^2+1}(x+2y)dy = \int_{-1}^{2} dx \left(x\int_{0}^{x^2+1} dy + 2\int_{0}^{x^2+1} y\,dy\right) = \int_{-1}^{2} dx \left[xy + \cancel{2}\frac{y^2}{\cancel{2}}\right]_0^{x^2+1} =$$

$$\int_{-1}^{2} dx\,[y(x+y)]_0^{x^2+1} = \int_{-1}^{2} dx\left[\left(x^2+1\right)\left[x+\left(x^2+1\right)\right]\cancel{-0(x+0)}\right] =$$

$$\int_{-1}^{2} dx\left[\left(x^2+1\right)\left(x^2+x+1\right)\right] = \int_{-1}^{2} dx\left(x^4+x^3+2x^2+x+1\right) = \int_{-1}^{2} x^4\,dx +$$

$$\int_{-1}^{2} x^3\,dx + 2\int_{-1}^{2} x^2\,dx + \int_{-1}^{2} x\,dx + \int_{-1}^{2} dx = \left[\frac{x^5}{5}+\frac{x^4}{4}+2\frac{x^3}{3}+\frac{x^2}{2}+x\right]_{-1}^{2} =$$

$$\left[\frac{12x^5+15x^4+40x^3+30x^2+60x}{60}\right]_{-1}^{2} = \frac{1}{60}\left[x\left(12x^4+15x^3+40x^2+30x+60\right)\right]_{-1}^{2} =$$

$$\frac{1}{60}\left[2\left(12\cdot 2^4+15\cdot 2^3+40\cdot 2^2+30\cdot 2+60\right)\right.$$
$$\left.-(-1)\left(12(-1)^4+15(-1)^3+40(-1)^2+30(-1)+60\right)\right] =$$

$$\frac{1}{60}[2(192+120+160+60+60)+(12-15+40-30+60)] = \frac{1}{60}(1184+67) =$$

$$\frac{1251}{60} = \frac{417}{20} = 20,85$$

5 Cálculo de integrais duplas ou iteradas

5.1 Aula 5

1. Exercício

   $f(x,y) = x^3;\ 0 \le x \le 2;\ x^2 \le y \le 4$
   $$\iint_R f(x,y)dydx$$

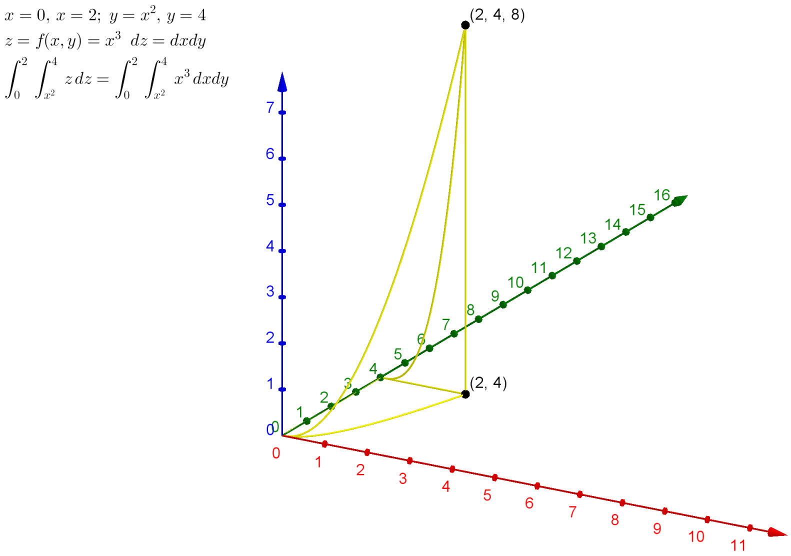

Figura 11: Integrais duplas - Aula 5 - Exercício I

$$v = \int_0^2 \int_{x^2}^4 x^3\,dxdy = \int_0^2 x^3\,dx \int_{x^2}^4 dy = \int_0^2 x^3\,dx\,[y]_{x^2}^4 = \int_0^2 x^3\,dx\,\left[4 - x^2\right] =$$
$$4\int_0^2 x^3\,dx - \int_0^2 x^5\,dx = \left[\cancel{4}\frac{x^4}{\cancel{4}} - \frac{x^6}{6}\right]_0^2 = \left[\frac{6x^4 - x^6}{6}\right]_0^2 = \frac{1}{6}\left[x^4\left(6 - x^2\right)\right]_0^2 =$$
$$\frac{1}{6}\left[2^4\left(6 - 2^2\right) - \cancel{0^4\left(6 - 0^2\right)}\right] = \frac{1}{6}(16 \cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2$$

2. Exercício

   $f(x,y) = x^2y;\ 1 \le x \le 3;\ x \le y \le 2x+1$
   $$\iint_R f(x,y)dydx$$

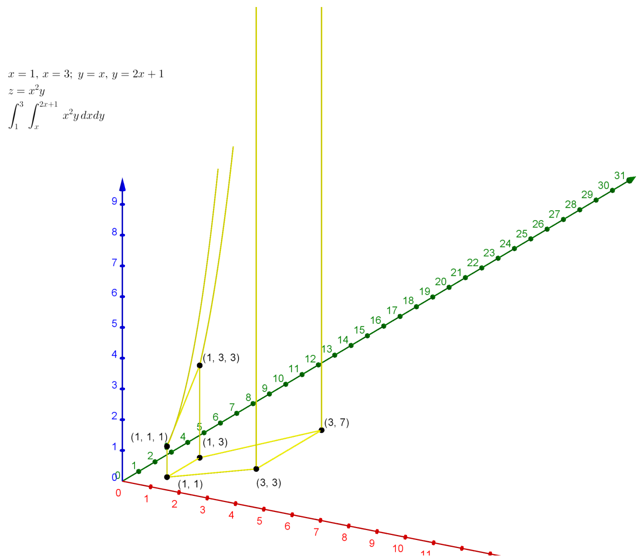

Figura 12: Integrais duplas - Aula 5 - Exercício II

$$v = \int_1^3 \int_x^{2x+1} x^2 y\, dxdy = \int_1^3 x^2\, dx \int_x^{2x+1} y\, dy = \int_1^3 x^2\, dx \left[\frac{y^2}{2}\right]_x^{2x+1} =$$
$$\int_1^3 x^2\, dx \frac{1}{2}\left[(2x+1)^2 - (x)^2\right] = \frac{1}{2}\int_1^3 x^2\, dx \left(3x^2 + 4x + 1\right) = \frac{3}{2}\int_1^3 x^4\, dx +$$
$$2\int_1^3 x^3\, dx + \frac{1}{2}\int_1^3 x^2\, dx = \left[\frac{3}{2}\frac{x^5}{5} + 2\frac{x^4}{4} + \frac{1}{2}\frac{x^3}{3}\right]_1^3 = \left[\frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6}\right]_1^3 =$$
$$\left[\frac{18x^5 + 30x^4 + 10x^3}{60}\right]_1^3 = \left[\frac{2x^3\left(9x^2 + 15x + 5\right)}{60}\right]_1^3 = \frac{1}{30}\left[x^3\left(9x^2 + 15x + 5\right)\right]_1^3 =$$
$$\frac{1}{30}\left[3^3\left(9 \cdot 3^2 + 15 \cdot 3 + 5\right) - 1^3\left(9 \cdot 1^2 + 15 \cdot 1 + 5\right)\right] =$$
$$\frac{1}{30}\left[27(81 + 45 + 5) - (9 + 15 + 5)\right] = \frac{1}{30}\left[27 \cdot 131 - 29\right] = \frac{3508}{30} = 116,9\overline{3}$$

5.2 Aula 6

1. Exercício

$$f(x, y) = 1;\ 0 \leq x \leq 1;\ 1 \leq y \leq \mathrm{e}^x$$
$$\iint_R f(x, y) dydx$$
$$v = \int_0^1 \int_1^{\mathrm{e}^x} dydx = \int_0^1 dx\, [y]_1^{\mathrm{e}^x} = \int_0^1 dx\, \left(\mathrm{e}^x - 1\right) = \left[\mathrm{e}^x - x\right]_0^1 = \mathrm{e}^1 - 1 -$$
$$\left(\mathrm{e}^0 - 0\right) = \mathrm{e} - 1 - 1 = \mathrm{e} - 2$$

2. Exercício

$f(x,y) = x;\ 0 \leq x \leq 1;\ 1 \leq y \leq \mathrm{e}^{x^2}$

$\iint_R f(x,y)dydx$

$v = \int_0^1 \int_1^{\mathrm{e}^{x^2}} x\,dxdy = \int_0^1 x\,dx \int_1^{\mathrm{e}^{x^2}} dy = \int_0^1 x\,dx\,[y]_1^{\mathrm{e}^{x^2}} = \int_0^1 x\,dx\,\left(\mathrm{e}^{x^2} - 1\right) =$
$\int_0^1 x\,\mathrm{e}^{x^2}\,dx - \int_0^1 x\,dx = \int_0^1 \mathrm{e}^u\,\frac{du}{2} - \int_0^1 x\,dx = \frac{1}{2}\int_0^1 \mathrm{e}^u\,du - \int_0^1 x\,dx =$
$\left[\frac{1}{2}e^u - \frac{x^2}{2}\right]_0^1 = \left[\frac{e^{x^2} - x^2}{2}\right]_0^1 = \frac{1}{2}\left[e^{x^2} - x^2\right]_0^1 = \frac{1}{2}\left[e^{1^2} - 1^2 - \left(e^{0^2} - 0^2\right)\right] =$
$\frac{1}{2}(\mathrm{e} - 1 - 1) = \frac{\mathrm{e} - 2}{2}$

$u = x^2;\ \frac{du}{2} = x\,dx$

3. Exercício

$f(x,y) = 2xy;\ 0 \leq y \leq 1;\ y^2 \leq x \leq y$

$\iint_R f(x,y)dxdy$

$v = \int_0^1 \int_{y^2}^{y} 2xy\,dxdy = 2\int_0^1 y\,dy \int_{y^2}^{y} x\,dx = 2\int_0^1 y\,dy\,\left[\frac{x^2}{2}\right]_{y^2}^{y} = \cancel{2}\int_0^1 y\,dy\,\frac{1}{\cancel{2}}\left[x^2\right]_{y^2}^{y} =$
$\int_0^1 y\,dy\,\left(y^2 - y^4\right) = \int_0^1 \left(y^3 - y^5\right)dy = \left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1 = \left[\frac{6y^4 - 4y^6}{24}\right]_0^1 =$
$\left[\frac{2y^4\left(3 - 2y^2\right)}{24}\right]_0^1 = \frac{1}{12}\left[1^4\left(3 - 2\cdot 1^2\right)\cancel{- 0^4\left(3 - 2\cdot 0^2\right)}\right] = \frac{1}{12} = 0,08\overline{3}$

5.3 Aula 7

1. Exercício

$f(x,y) = \frac{1}{x+y};\ 1 \leq y \leq \mathrm{e};\ 0 \leq x \leq y$

$\iint_R f(x,y)dxdy$

$v = \int_1^{\mathrm{e}} \int_0^y \frac{1}{x+y}\,dxdy = \int_1^{\mathrm{e}} dy \int_0^y (x+y)^{-1}\,dx = \int_1^{\mathrm{e}} dy \int_0^y u^{-1}\,du =$
$\int_1^{\mathrm{e}} dy \int_0^y \left[\ln|u|\right]_0^y = \int_1^{\mathrm{e}} dy \int_0^y \left[\ln|x+y|\right]_0^y = \int_1^{\mathrm{e}} dy \int_0^y \left(\ln|y+y| - \ln|0+y|\right) =$
$\int_1^{\mathrm{e}} dy \int_0^y \left(\ln|2y| - \ln|y|\right) = \int_1^{\mathrm{e}} dy \int_0^y \left(\ln|2| \cancel{+ \ln|y| - \ln|y|}\right) = \ln|2| \int_1^{\mathrm{e}} dy =$

$\ln|2|[y]_1^{\mathrm{e}} = \ln|2|(\mathrm{e}-1)$

$u = x + y;\ du = (1+0)dx = dx$