Curso de integrais duplas e triplas

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Resumo

Exercícios retirados do canal do Youtube, O Matematico [1].

Parte I
Integrais duplas

1 Invertendo os limites de integração - Aula 1

1. Exercício

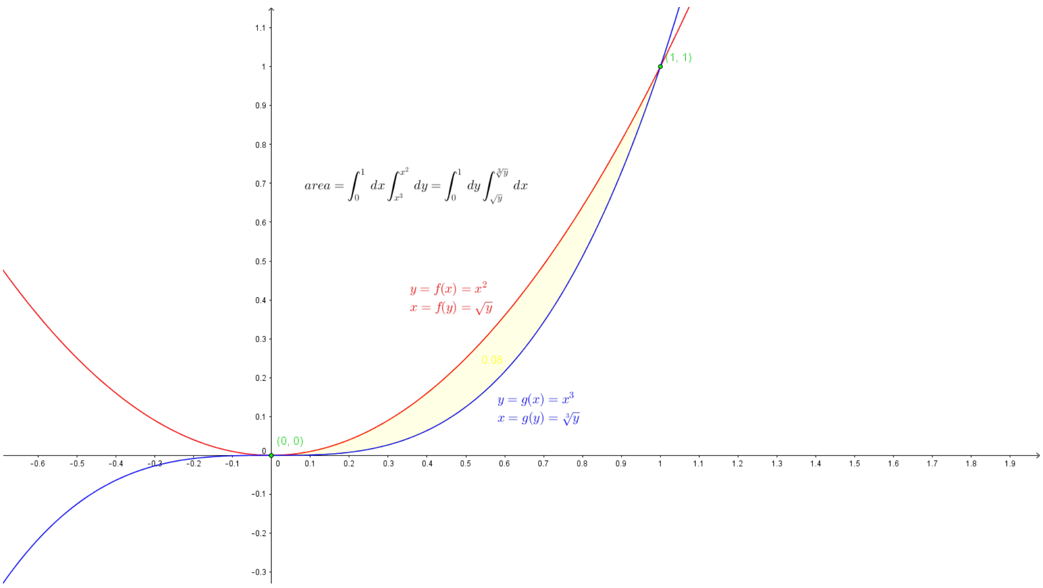


Figura 1: Integrais duplas - Aula 1 - Exercício I e II

$f(x) = x^2;\ g(x) = x^3$
$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$
$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$

$$a = \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx\, [y]_{x^3}^{x^2} = \int_0^1 dx \left[x^2 - x^3\right] =$$
$$\int_0^1 x^2\, dx - \int_0^1 x^3\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{4x^3 - 3x^2}{12}\right]_0^1 = \frac{1}{12}\left[4x^3 - 3x^2\right]_0^1 =$$
$$\frac{1}{12}\left[x^2(4x-3)\right]_0^1 = \frac{1}{12}\left[1^2(4\cdot 1 - 3) \cancel{- 0^2(4\cdot 0 - 3)}\right] = \frac{1}{12} = 0,08\overline{3}$$

2. Exercício

$f(x) = x^2 \Rightarrow f(y) = \sqrt{y};\ g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$
$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$
$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$

$$a = \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy\, [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy\, [\sqrt[3]{y} - \sqrt{y}] =$$
$$\int_0^1 \sqrt[3]{y}\, dy - \int_0^1 \sqrt{y}\, dy = \int_0^1 y^{\frac{1}{3}}\, dy - \int_0^1 y^{\frac{1}{2}}\, dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 =$$
$$\left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3}\right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12}\right]_0^1 = \frac{1}{12}\left[9\sqrt[3]{y^4} - 8\sqrt{y^3}\right]_0^1 =$$
$$\frac{1}{12}\left[\left(9\sqrt[3]{1^4} - 8\sqrt{1^3}\right) \cancel{- \left(9\sqrt[3]{0^4} - 8\sqrt{0^3}\right)}\right] = \frac{1}{12}(9-8) = \frac{1}{12} = 0,08\overline{3}$$

2 Determinação da região de integração - Aula 2

1. Exercício

$R = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 2\,,\ 0 \leq y \leq 6\}$

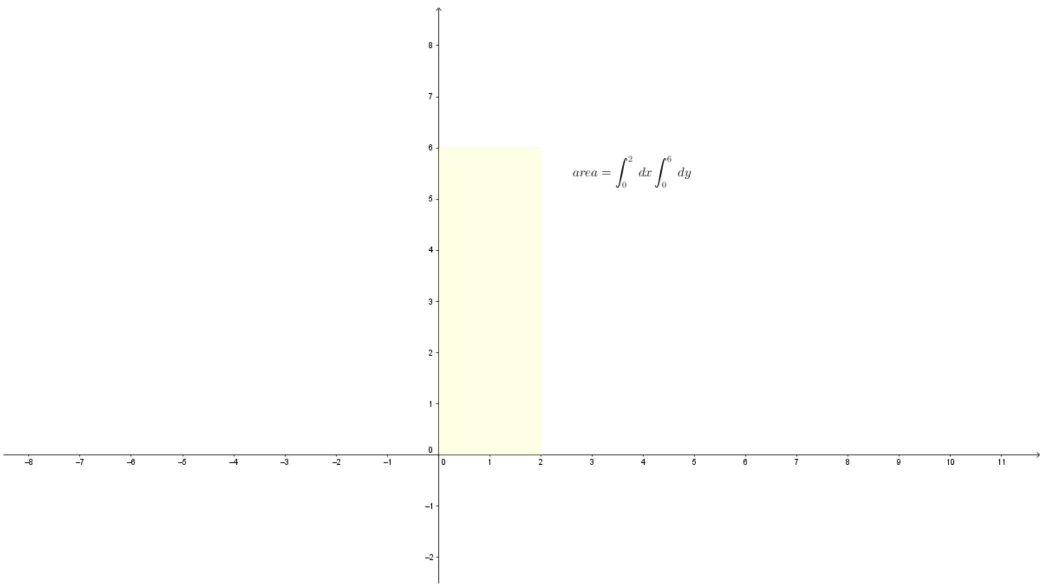


Figura 2: Integrais duplas - Aula 2 - Exercício I

$$a = \int_0^2 dx \int_0^6 dy = \int_0^2 dx\,[y]_0^6 = \int_0^2 dx\,[6-0] = 6\int_0^2 dx = 6[x]_0^2 = 6[2-0] = 6 \cdot 2 = 12$$

2. Exercício

   $R = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1\,,\; x \leq y \leq 2x\}$

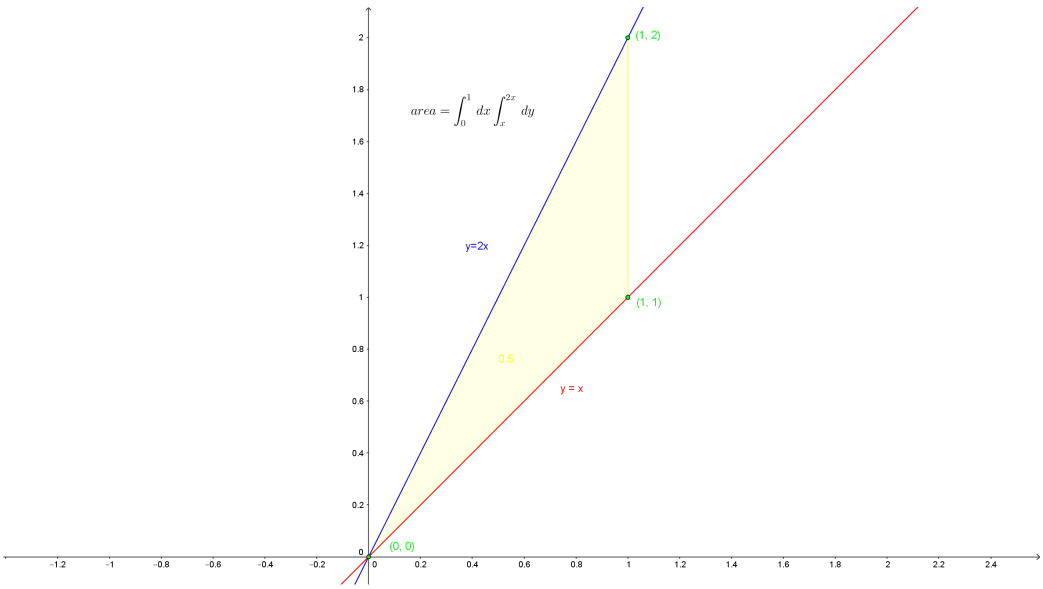


Figura 3: Integrais duplas - Aula 2 - Exercício II

$a = \int_0^1 dx \int_x^{2x} dy = \int_0^1 dx\, [y]_x^{2x} = \int_0^1 dx\, [2x - x] = 2\int_0^1 x\, dx - \int_0^1 x\, dx = \left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^2}{2}\right]_0^1 = \left[\frac{2x^2 - x^2}{2}\right]_0^1 = \frac{1}{2}\left[x^2\right]_0^1 = \frac{1}{2}\left[1^2 \cancel{- 0^2}\right] = \frac{1}{2} = 0,5$

3. Exercício

   $R = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 \,,\, 0 \leq x \leq \sqrt{1 - y^2}\right\}$

   $y = 0,\ y = 1$

   $x = 0,\ x = \sqrt{1 - y^2} \Rightarrow x^2 = 1 - y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow y = \sqrt{1 - x^2}$

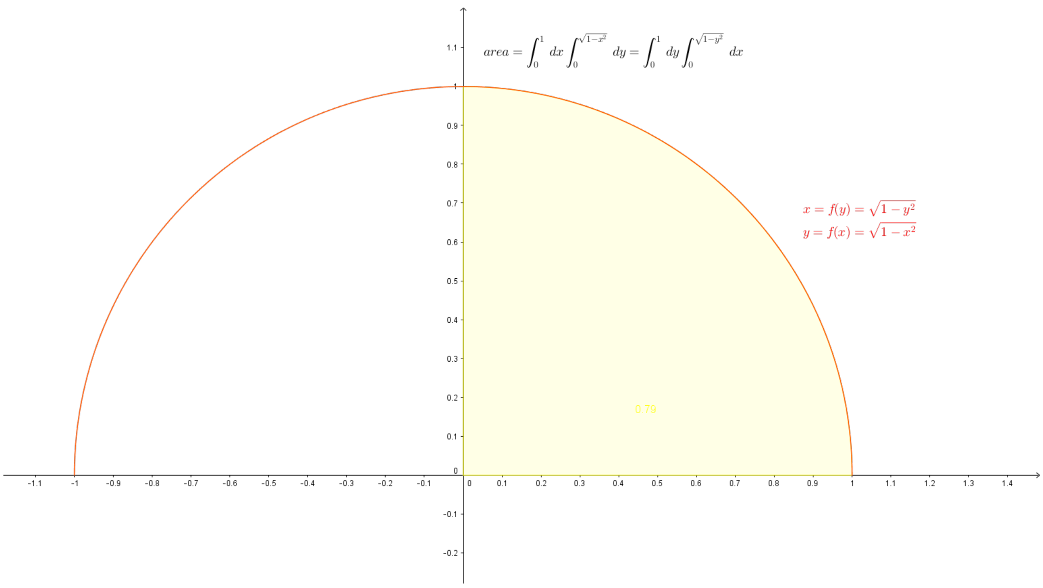


Figura 4: Integrais duplas - Aula 2 - Exercício III

$$a = \int_0^1 dy \int_0^{f(y)} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy \, [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy \left[\sqrt{1-y^2} - 0\right] =$$
$$\int_0^1 \sqrt{1-y^2}\, dy = \int_0^1 \sqrt{1-\operatorname{sen}^2(t)}\, \cos(t) dt = \int_0^1 \sqrt{\cos^2(t)}\, \cos(t) dt =$$
$$\int_0^1 \cos(t)\cos(t) dt = \int_0^1 \cos^2(t) dt = \int_0^1 \frac{1+\cos(2t)}{2} dt = \frac{1}{2}\int_0^1 [1+\cos(2t)]\, dt =$$
$$\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(2t) dt = \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(u)\frac{du}{2} = \frac{1}{2}\int_0^1 dt +$$
$$\frac{1}{4}\int_0^1 \cos(u)\, du = \left[\frac{1}{2}t + \frac{1}{4}\operatorname{sen}(u)\right]_0^1 = \left[\frac{t}{2} + \frac{\operatorname{sen}(2t)}{4}\right]_0^1 = \left[\frac{t}{2} + \frac{2\operatorname{sen}(t)\cos(t)}{4}\right]_0^1 =$$
$$\left[\frac{t+\operatorname{sen}(t)\cos(t)}{2}\right]_0^1 = \frac{1}{2}\left[\operatorname{arcsen}(y) + y\sqrt{1-y^2}\right]_0^1 =$$
$$\frac{1}{2}\left[\left(\operatorname{arcsen}(1) \cancel{+1\cdot\sqrt{1-1^2}}\right) - \left(\operatorname{arcsen}(0) \cancel{+0\cdot\sqrt{1-0^2}}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] =$$
$$\frac{\pi}{4} = 0,785$$

$$y = \operatorname{sen}(t) \Rightarrow dy = \cos(t) dt$$
$$u = 2t \Rightarrow \frac{du}{2} = dt$$

$$\operatorname{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$
$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1-y^2}$$

$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1-y^2}}{1} = \sqrt{1-y^2}$$
$$y = \operatorname{sen}(t) \Rightarrow t = \operatorname{arcsen}(y)$$

4. Exercício

$$y = x^2+1,\ y = -x^2-1;\ x = 1,\ x = -1$$
$$R = \{(x,y) \in \mathbb{R}^2 \,|\, -1 \le x \le 1\,,\ -x^2-1 \le y \le x^2+1\}$$

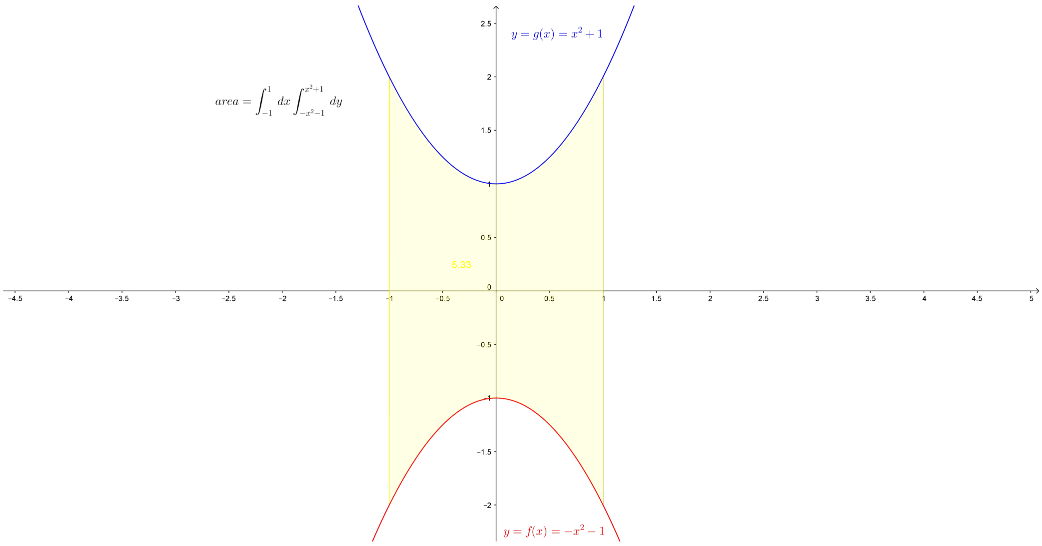


Figura 5: Integrais duplas - Aula 2 - Exercício IV

$$a = \int_{-1}^{1} dx \int_{f(x)}^{g(x)} dy = \int_{-1}^{1} dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^{1} dx\, [y]_{-x^2-1}^{x^2+1} = \int_{-1}^{1} dx\, \left[x^2+1-\left(-x^2-1\right)\right] =$$
$$\int_{-1}^{1} dx\, \left[x^2+1+x^2+1\right] = \int_{-1}^{1} dx\, \left[2x^2+2\right] = 2\int_{-1}^{1} x^2\,dx + 2\int_{-1}^{1} dx =$$
$$\left[2\frac{x^3}{3}+2x\right]_{-1}^{1} = \left[2\left(\frac{x^3+3x}{3}\right)\right]_{-1}^{1} = \frac{2}{3}\left[x\left(x^2+3\right)\right]_{-1}^{1} =$$
$$\frac{2}{3}\left[1\cdot\left(1^2+3\right)-(-1)\left((-1)^2+3\right)\right] = \frac{2}{3}(4+4) = \frac{2}{3}8 = \frac{16}{3} = 5,\overline{3}$$

5. Exercício

$$R = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le y \le 2\,,\ -y \le x \le y\}$$

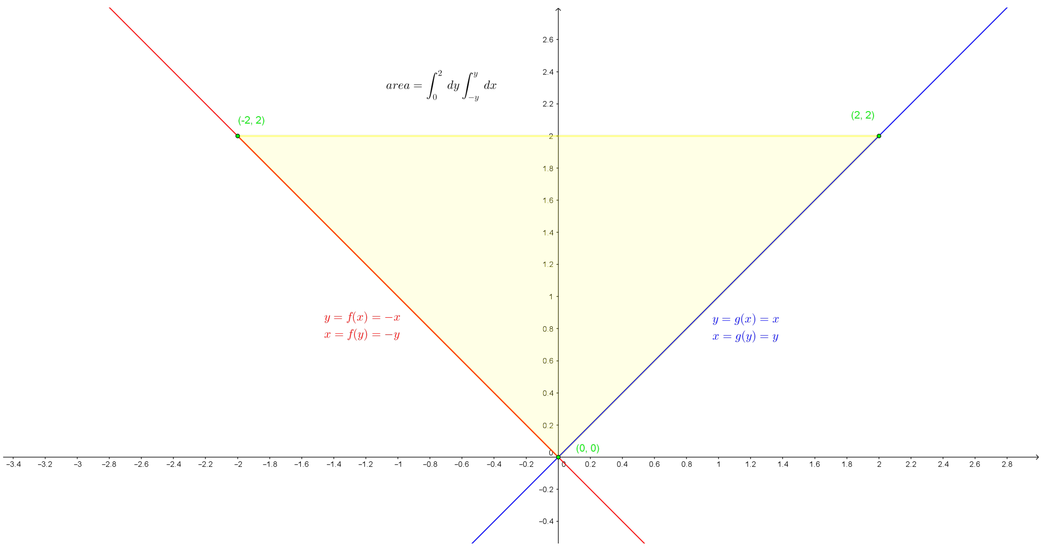


Figura 6: Integrais duplas - Aula 2 - Exercício V

$$a = \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^{y} dx = \int_0^2 dy\, [x]_{-y}^{y} = \int_0^2 dy\, [y-(-y)] = \int_0^2 dy\, [2y] = 2\int_0^2 y\, dy = \left[\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^2 = 2^2 - 0^2 = 4$$

3 Cálculo de volume - Aula 3

1. Exercício

$$volume = \int_0^3 \int_0^2 4\, dxdy$$

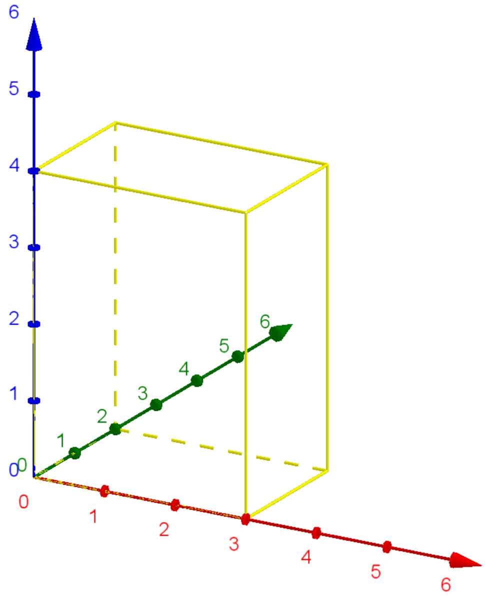


Figura 7: Integrais duplas - Aula 3 - Exercício I

$z = 4;\ dz = dxdy$

$$v = \int_0^3 \int_0^2 z\, dz = \int_0^3 \int_0^2 4\, dydx = 4 \int_0^3 dx \int_0^2 dy = 4 \int_0^3 dx\, [y]_0^2 = 4 \int_0^3 dx\, [2-0] = 8 \int_0^3 dx = 8[x]_0^3 = 8[3-0] = 8 \cdot 3 = 24$$

2. Exercício

   $R = [0,3] \times [0,4]$

   $$\iint_R (8-2y) da$$

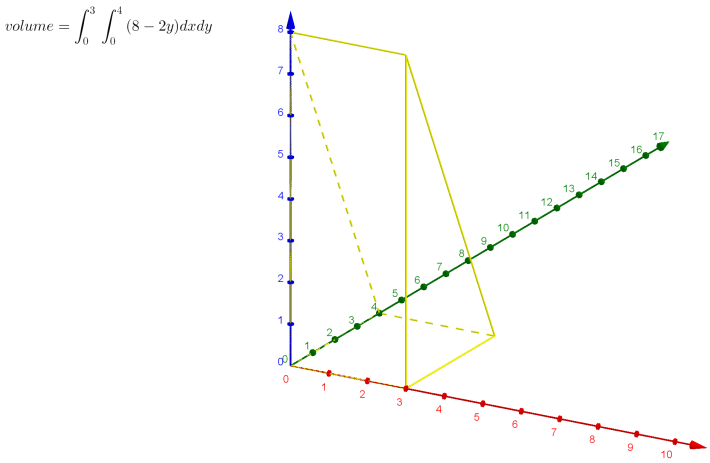


Figura 8: Integrais duplas - Aula 3 - Exercício II

$z = 8 - 2y;\ da = dz = dxdy$

$v = \int_0^3 \int_0^4 z\,dz = \int_0^3 \int_0^4 (8 - 2y)dxdy = \int_0^3 dx \int_0^4 (8 - 2y)dy = \int_0^3 dx \left(8 \int_0^4 dy - 2 \int_0^4 y\,dy\right) = \int_0^3 dx\, 2 \left(4 \int_0^4 dy - \int_0^4 y\,dy\right) = 2 \int_0^3 dx \left[4y - \frac{y^2}{2}\right]_0^4 = 2 \int_0^3 dx \left[\frac{8y - y^2}{2}\right]_0^4 = \cancel{2} \int_0^3 dx\, \frac{1}{\cancel{2}}[y(8-y)]_0^4 = \int_0^3 dx[4(8-4)\cancel{-0(8-0)}] = 16 \int_0^3 dx = 16[x]_0^3 = 16[3-0] = 48$

4 Invertendo a ordem de integração - Aula 4

1. Exercício

   $z = f(x, y) = y\,\mathrm{e}^x;\ dz = dxdy$

   $v = \int_2^4 \int_1^9 z\,dz = \int_2^4 \int_1^9 y\,\mathrm{e}^x\ dydx = \int_2^4 \mathrm{e}^x\ dx \int_1^9 y\,dy = \int_2^4 \mathrm{e}^x\ dx \left[\frac{y^2}{2}\right]_1^9 = \int_2^4 \mathrm{e}^x\ dx \frac{1}{2}\left[y^2\right]_1^9 = \frac{1}{2} \int_2^4 \mathrm{e}^x\ dx \left[9^2 - 1^2\right] = 40 \int_2^4 \mathrm{e}^x\ dx = 40\left[\mathrm{e}^x\right]_2^4 =$

$40\left[e^4 - e^2\right] = 40e^2\left(e^2 - 1\right)$

2. Exercício

   $z = f(x, y) = x^2y^3;\ dz = dxdy$

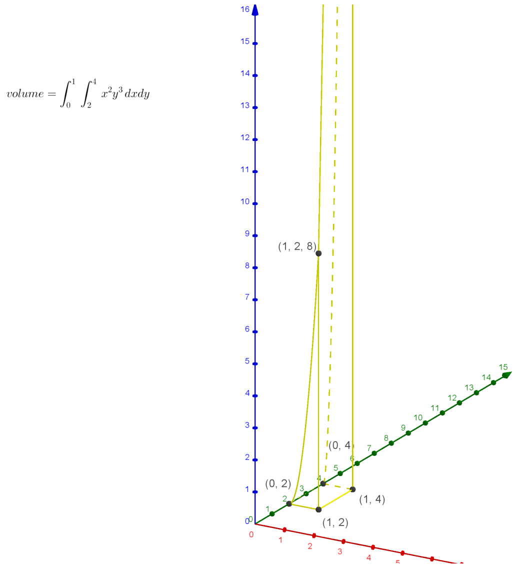
   

   Figura 9: Integrais duplas - Aula 4 - Exercício II

   $v = \int_0^1 \int_2^4 z\,dz = \int_0^1 \int_2^4 x^2y^3\,dxdy = \int_0^1 x^2\,dx \int_2^4 y^3\,dy = \int_0^1 x^2\,dx \left[\frac{y^4}{4}\right]_2^4 =$
   $\frac{1}{4}\int_0^1 x^2\,dx\left[y^4\right]_2^4 = \frac{1}{4}\int_0^1 x^2\,dx\left[4^4 - 2^4\right] = \frac{1}{4}\int_0^1 x^2\,dx\left[2^8 - 2^4\right] =$
   $\frac{1}{4}\int_0^1 x^2\,dx\left[2^4\left(2^4 - 1\right)\right] = \frac{1}{4}\int_0^1 x^2\,dx\,[16 \cdot 15] = 60\int_0^1 x^2\,dx = 60\left[\frac{x^3}{3}\right]_0^1 =$
   $20\left[x^3\right]_0^1 = 20\left[1^3 - 0^3\right] = 20 \cdot 1 = 20$

3. Exercício

   $\iint_R (x + 2y)da$

   R = Região limitada pela parábola $y = x^2 + 1$ e as retas $x = -1$ e $x = 2$.

   $z = f(x, y) = x + 2y;\ da = dz = dxdy$

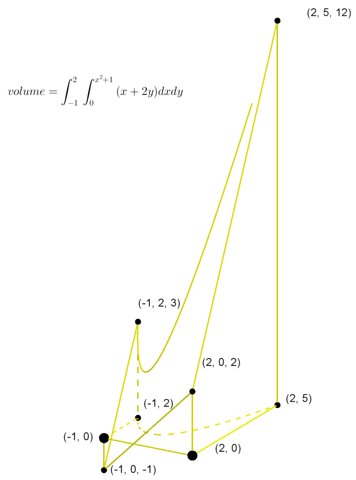


Figura 10: Integrais duplas - Aula 4 - Exercício III

$$v = \int_{-1}^{2}\int_{0}^{x^2+1} z\,dz = \int_{-1}^{2}\int_{0}^{x^2+1}(x+2y)dxdy = \int_{-1}^{2}dx\int_{0}^{x^2+1}(x+2y)dy = \int_{-1}^{2}dx\left(x\int_{0}^{x^2+1}dy + 2\int_{0}^{x^2+1}y\,dy\right) = \int_{-1}^{2}dx\left[xy+\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^{x^2+1} =$$

$$\int_{-1}^{2}dx\,[y(x+y)]_0^{x^2+1} = \int_{-1}^{2}dx\left[\left(x^2+1\right)\left[x+\left(x^2+1\right)\right]\cancel{-0(x+0)}\right] =$$

$$\int_{-1}^{2}dx\left[\left(x^2+1\right)\left(x^2+x+1\right)\right] = \int_{-1}^{2}dx\left(x^4+x^3+2x^2+x+1\right) = \int_{-1}^{2}x^4\,dx+$$

$$\int_{-1}^{2}x^3\,dx+2\int_{-1}^{2}x^2\,dx+\int_{-1}^{2}x\,dx+\int_{-1}^{2}dx = \left[\frac{x^5}{5}+\frac{x^4}{4}+2\frac{x^3}{3}+\frac{x^2}{2}+x\right]_{-1}^{2} =$$

$$\left[\frac{12x^5+15x^4+40x^3+30x^2+60x}{60}\right]_{-1}^{2} = \frac{1}{60}\left[x\left(12x^4+15x^3+40x^2+30x+60\right)\right]_{-1}^{2} =$$

$$\frac{1}{60}\left[2\left(12\cdot 2^4+15\cdot 2^3+40\cdot 2^2+30\cdot 2+60\right)\right.$$
$$\left.-(-1)\left(12(-1)^4+15(-1)^3+40(-1)^2+30(-1)+60\right)\right] =$$

$$\frac{1}{60}[2(192+120+160+60+60)+(12-15+40-30+60)] = \frac{1}{60}(1184+67) =$$

$$\frac{1251}{60} = \frac{417}{20} = 20,85$$

5 Cálculo de integrais duplas ou iteradas

5.1 Aula 5

1. Exercício

   $f(x,y) = x^3;\ 0 \leq x \leq 2;\ x^2 \leq y \leq 4$

   $$\iint_R f(x,y)dydx$$

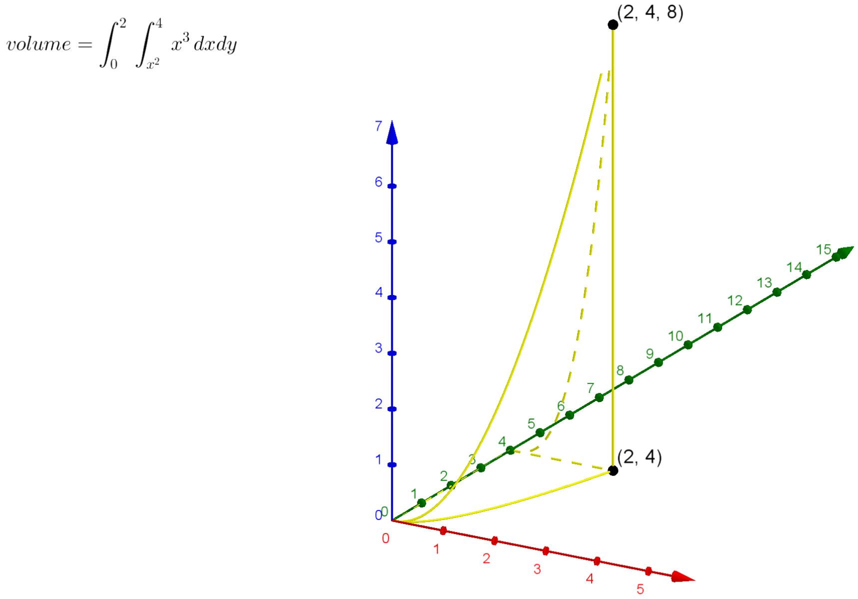


Figura 11: Integrais duplas - Aula 5 - Exercício I

$$v = \int_0^2 \int_{x^2}^4 x^3\, dxdy = \int_0^2 x^3\, dx \int_{x^2}^4 dy = \int_0^2 x^3\, dx\, [y]_{x^2}^4 = \int_0^2 x^3\, dx\, \left[4 - x^2\right] =$$
$$4\int_0^2 x^3\, dx - \int_0^2 x^5\, dx = \left[\cancel{4}\frac{x^4}{\cancel{4}} - \frac{x^6}{6}\right]_0^2 = \left[\frac{6x^4 - x^6}{6}\right]_0^2 = \frac{1}{6}\left[x^4\left(6 - x^2\right)\right]_0^2 =$$
$$\frac{1}{6}\left[2^4\left(6 - 2^2\right) - \cancel{0^4\left(6 - 0^2\right)}\right] = \frac{1}{6}(16 \cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2$$

2. Exercício

   $f(x,y) = x^2y;\ 1 \leq x \leq 3;\ x \leq y \leq 2x + 1$

   $$\iint_R f(x,y)dydx$$

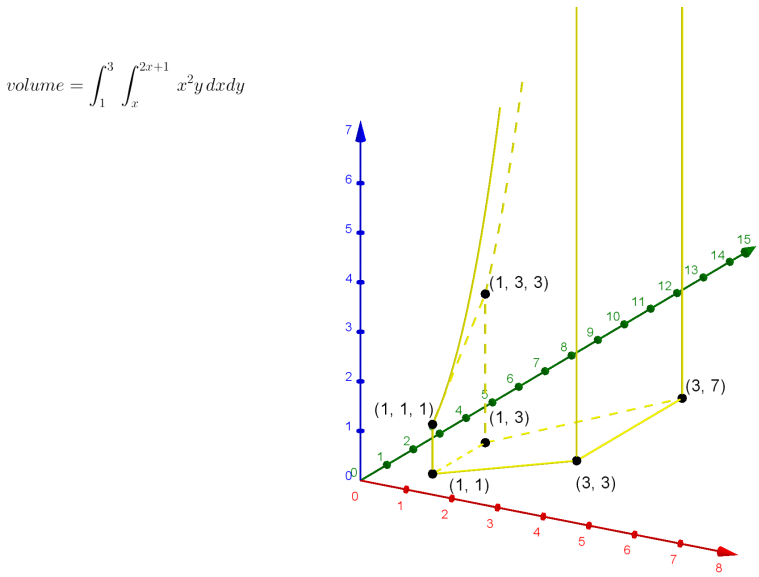


Figura 12: Integrais duplas - Aula 5 - Exercício II

$$v = \int_1^3 \int_x^{2x+1} x^2 y\, dx dy = \int_1^3 x^2\, dx \int_x^{2x+1} y\, dy = \int_1^3 x^2\, dx \left[\frac{y^2}{2}\right]_x^{2x+1} =$$
$$\int_1^3 x^2\, dx \frac{1}{2}\left[(2x+1)^2 - (x)^2\right] = \frac{1}{2}\int_1^3 x^2\, dx \left(3x^2 + 4x + 1\right) = \frac{3}{2}\int_1^3 x^4\, dx +$$
$$2\int_1^3 x^3\, dx + \frac{1}{2}\int_1^3 x^2\, dx = \left[\frac{3}{2}\frac{x^5}{5} + 2\frac{x^4}{4} + \frac{1}{2}\frac{x^3}{3}\right]_1^3 = \left[\frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6}\right]_1^3 =$$
$$\left[\frac{18x^5 + 30x^4 + 10x^3}{60}\right]_1^3 = \left[\frac{2x^3\left(9x^2 + 15x + 5\right)}{60}\right]_1^3 = \frac{1}{30}\left[x^3\left(9x^2 + 15x + 5\right)\right]_1^3 =$$
$$\frac{1}{30}\left[3^3\left(9\cdot 3^2 + 15\cdot 3 + 5\right) - 1^3\left(9\cdot 1^2 + 15\cdot 1 + 5\right)\right] =$$
$$\frac{1}{30}\left[27(81 + 45 + 5) - (9 + 15 + 5)\right] = \frac{1}{30}\left[27\cdot 131 - 29\right] = \frac{3508}{30} = 116,9\overline{3}$$

5.2 Aula 6

1. Exercício

$f(x,y) = 1;\ 0 \le x \le 1;\ 1 \le y \le \mathrm{e}^x$

$\iint_R f(x,y) dy dx$

$$v = \int_0^1 \int_1^{\mathrm{e}^x} dy dx = \int_0^1 dx\, [y]_1^{\mathrm{e}^x} = \int_0^1 dx\, (\mathrm{e}^x - 1) = [\mathrm{e}^x - x]_0^1 = \mathrm{e}^1 - 1 -$$

$\left(e^0 - 0\right) = e - 1 - 1 = e - 2$

2. Exercício

   $f(x,y) = x;\ 0 \leq x \leq 1;\ 1 \leq y \leq e^{x^2}$

   $\iint_R f(x,y)dydx$

   $v = \int_0^1 \int_1^{e^{x^2}} x\,dxdy = \int_0^1 x\,dx \int_1^{e^{x^2}} dy = \int_0^1 x\,dx\,[y]_1^{e^{x^2}} = \int_0^1 x\,dx\,\left(e^{x^2} - 1\right) =$
   $\int_0^1 x\,e^{x^2}\,dx - \int_0^1 x\,dx = \int_0^1 e^u\,\frac{du}{2} - \int_0^1 x\,dx = \frac{1}{2}\int_0^1 e^u\,du - \int_0^1 x\,dx =$
   $\left[\frac{1}{2}e^u - \frac{x^2}{2}\right]_0^1 = \left[\frac{e^{x^2} - x^2}{2}\right]_0^1 = \frac{1}{2}\left[e^{x^2} - x^2\right]_0^1 = \frac{1}{2}\left[e^{1^2} - 1^2 - \left(e^{0^2} - 0^2\right)\right] =$
   $\frac{1}{2}(e - 1 - 1) = \frac{e - 2}{2}$

   $u = x^2;\ \frac{du}{2} = x\,dx$

3. Exercício

   $f(x,y) = 2xy;\ 0 \leq y \leq 1;\ y^2 \leq x \leq y$

   $\iint_R f(x,y)dxdy$

   $v = \int_0^1 \int_{y^2}^{y} 2xy\,dxdy = 2\int_0^1 y\,dy \int_{y^2}^{y} x\,dx = 2\int_0^1 y\,dy\,\left[\frac{x^2}{2}\right]_{y^2}^{y} = \cancel{2}\int_0^1 y\,dy\,\frac{1}{\cancel{2}}\left[x^2\right]_{y^2}^{y} =$
   $\int_0^1 y\,dy\,\left(y^2 - y^4\right) = \int_0^1 \left(y^3 - y^5\right)dy = \left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1 = \left[\frac{6y^4 - 4y^6}{24}\right]_0^1 =$
   $\left[\frac{2y^4\left(3 - 2y^2\right)}{24}\right]_0^1 = \frac{1}{12}\left[1^4\left(3 - 2\cdot 1^2\right) \cancel{- 0^4\left(3 - 2\cdot 0^2\right)}\right] = \frac{1}{12} = 0{,}08\overline{3}$

5.3 Aula 7

1. Exercício

   $f(x,y) = \frac{1}{x+y};\ 1 \leq y \leq e;\ 0 \leq x \leq y$

   $\iint_R f(x,y)dxdy$

   $v = \int_1^e \int_0^y \frac{1}{x+y}\,dxdy = \int_1^e dy \int_0^y (x+y)^{-1}\,dx = \int_1^e dy \int_0^y u^{-1}\,du =$

$$\int_1^{\mathrm{e}} dy \int_0^y [\ln|u|]_0^y = \int_1^{\mathrm{e}} dy \int_0^y [\ln|x+y|]_0^y = \int_1^{\mathrm{e}} dy \int_0^y (\ln|y+y| - \ln|0+y|) = \int_1^{\mathrm{e}} dy \int_0^y (\ln|2y| - \ln|y|) = \int_1^{\mathrm{e}} dy \int_0^y (\ln|2|\cancel{+\ln|y| - \ln|y|}) = \ln|2| \int_1^{\mathrm{e}} dy = \ln|2|[y]_1^{\mathrm{e}} = \ln|2|(\mathrm{e}-1)$$

$$u = x+y;\ du = (1+0)dx = dx$$

6 Cálculo de área - Aula 8

1. Exercício

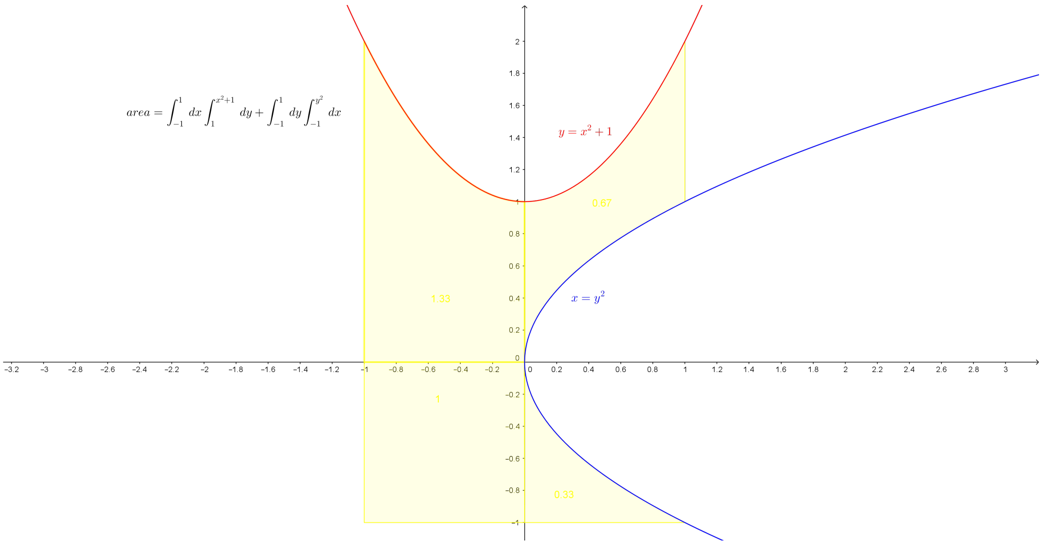


Figura 13: Integrais duplas - Aula 8 - Exercício I

$$a = \int_{-1}^{0} dx \int_0^{x^2+1} dy + \int_{-1}^{0} dx \int_{-1}^{0} dy + \int_0^{y^2} dx \int_{-1}^{0} dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} = \int_{-1}^{0} dx \left(\int_0^{x^2+1} dy + \int_{-1}^{0} dy \right) + \int_0^{y^2} dx \int_{-1}^{0} dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} = \int_{-1}^{0} dx \left([y]_0^{x^2+1} + [y]_{-1}^{0} \right) + \int_{-1}^{0} dy\, [x]_0^{y^2} + \int_0^1 dx\, [y]_{\sqrt{x}}^{x^2+1} = \int_{-1}^{0} dx\ (x^2+1+1) + \int_{-1}^{0} dy\, y^2 + \int_0^1 dx\ (x^2+1-\sqrt{x}) = \int_{-1}^{0} (x^2+2)\, dx + \int_{-1}^{0} y^2\, dy + \int_0^1 \left(x^2 - x^{\frac{1}{2}} + 1\right) dx = \left[\frac{x^3}{3} + 2x\right]_{-1}^{0} +$$

$$\left[\frac{y^3}{3}\right]_{-1}^{0}+\left[\frac{x^3}{3}-\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+x\right]_{0}^{1}=\left[\frac{x^3+6x}{3}\right]_{-1}^{0}+\frac{1}{3}\left[y^3\right]_{-1}^{0}+\left[\frac{x^3}{3}-\frac{2\sqrt{x^3}}{3}+x\right]_{0}^{1}=$$
$$\frac{1}{3}\left[x\left(x^2+6\right)\right]_{-1}^{0}+\frac{1}{3}\left[\cancel{0^3}-(-1)^3\right]+\left[\frac{x^3-2\sqrt{x^3}+3x}{3}\right]_{0}^{1}=$$
$$\frac{1}{3}\left[\cancel{0\left(0^2+6\right)}-(-1)\left((-1)^2+6\right)\right]+\frac{1}{3}+\frac{1}{3}\left[x^3-2\sqrt{x^3}+3x\right]_{0}^{1}=\frac{7}{3}+\frac{1}{3}+$$
$$\frac{1}{3}\left[1^3-2\sqrt{1^3}+3\cdot 1\cancel{-\left(0^3-2\sqrt{0^3}+3\cdot 0\right)}\right]=\frac{7}{3}+\frac{1}{3}+\frac{2}{3}=\frac{7+1+2}{3}=$$
$$\frac{10}{3}=3,\overline{3}$$

$$a=\int_{-1}^{1}dx\int_{1}^{x^2+1}dy+\int_{-1}^{y^2}dx\int_{-1}^{1}dy=\int_{-1}^{1}dx\,[y]_{1}^{x^2+1}+\int_{-1}^{1}dy\,[x]_{-1}^{y^2}=$$
$$\int_{-1}^{1}dx\,\left(x^2\cancel{+1-1}\right)+\int_{-1}^{1}dy\,\left(y^2+1\right)=\left[\frac{x^3}{3}\right]_{-1}^{1}+\left[\frac{y^3}{3}+y\right]_{-1}^{1}=\frac{1}{3}\left[x^3\right]_{-1}^{1}+$$
$$\frac{1}{3}\left[y\left(y^2+3\right)\right]_{-1}^{1}=\frac{1}{3}\left(\left[1^3-(-1)^3\right]+\left[1\left(1^2+3\right)-(-1)\left((-1)^2+3\right)\right]\right)\frac{1}{3}(2+$$
$$4+4)=\frac{10}{3}=3,\overline{3}$$

7 Cálculo de volume

7.1 Aula 9

1. Exercício

   Esboçe a região de integração e o sólido cujo volume é dado pela integral abaixo:

$$\int_{0}^{1}\int_{0}^{1}(4-x-2y)\;dxdy$$

$$volume = \int_0^1 \int_0^1 (4 - x - 2y)\ dxdy$$

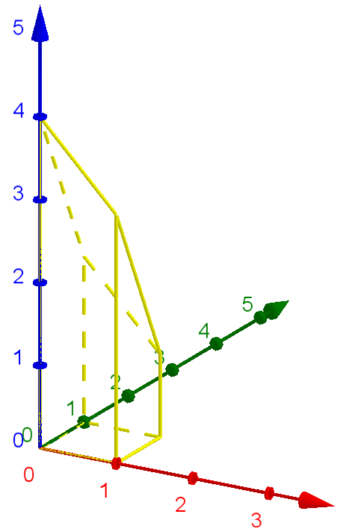


Figura 14: Integrais duplas - Aula 9 - Exercício I

$$v = \int_0^1 \int_0^1 (4 - x - 2y)\ dxdy = \int_0^1 dx \left(4 \int_0^1 dy - x \int_0^1 dy - 2 \int_0^1 y\, dy \right) =$$
$$4 \int_0^1 dx \int_0^1 dy - \int_0^1 x\, dx \int_0^1 dy - 2 \int_0^1 dx \int_0^1 y\, dy = 4[x]_0^1 [y]_0^1 - \left[\frac{x^2}{2} \right]_0^1 [y]_0^1 -$$
$$2[x]_0^1 \left[\frac{y^2}{2} \right]_0^1 = 4 - \frac{1}{2} - \cancel{2}\frac{1}{\cancel{2}} = \frac{8 - 1 - 2}{2} = \frac{5}{2} = 2,5$$

7.2 Aula 10

1. Exercício

   Calcule o volume do sólido limitado pelos planos:

$$x = 0,\ y = 0,\ z = 0 \text{ e } 6x + 2y + 3z = 6$$

$$volume = \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dxdy$$

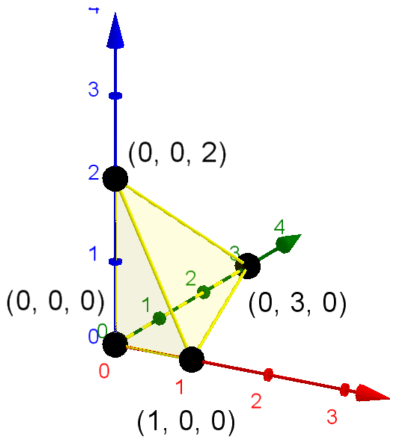


Figura 15: Integrais duplas - Aula 10 - Exercício I

$P_1 = (0,0,0)$
$6x = -2y - 3z + 6 \Rightarrow x = \frac{-2y-3z+6}{6} = \frac{-2\cdot 0 - 3 \cdot 0 + 6}{6} = \frac{6}{6} = 1 \Rightarrow P_2 = (1,0,0)$
$2y = -6x - 3z + 6 \Rightarrow y = \frac{-6x-3z+6}{2} = \frac{-6\cdot 0 - 3 \cdot 0 + 6}{2} = \frac{6}{2} = 3 \Rightarrow P_3 = (0,3,0)$
$3z = -6x - 2y + 6 \Rightarrow z = \frac{-6x-2y+6}{3} = \frac{-6\cdot 0 - 2 \cdot 0 + 6}{3} = \frac{6}{3} = 2 \Rightarrow P_4 = (0,0,2)$

$x = 0, x = 1$
$y = 0,\ y = \frac{-6x-3z+6}{2} = \frac{-6x - 3\cdot 0 + 6}{2} = -3x+3$
$z = \frac{-6x-2y+6}{3} = -2x - \frac{2y}{3} + 2$

$$v = \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dxdy = \int_0^1 dx \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dy = \int_0^1 dx \left[-2xy - \frac{\cancel{2}}{3}\frac{y^2}{\cancel{2}} + 2y\right]_0^{-3x+3} = \int_0^1 dx\, \frac{1}{3}\left[-6xy - y^2 + 6y\right]_0^{-3x+3} =$$

$$\frac{1}{3}\int_0^1 dx\, \left[-y(6x+y-6)\right]_0^{-3x+3} =$$

$$\frac{1}{3}\int_0^1 dx\, \left[-(-3x+3)(6x+(-3x+3)-6)\cancel{+0(6x+0-6)}\right] = \frac{1}{3}\int_0^1 dx\, \left[(3x-3)(3x-3)\right] =$$

$$\frac{1}{3}\int_0^1 \left(9x^2 - 18x + 9\right) dx = \frac{1}{3}\left[9\frac{x^3}{3} - 18\frac{x^2}{2} + 9x\right]_0^1 = \frac{1}{3}\left[3x^3 - 9x^2 + 9x\right]_0^1 =$$

$$\frac{1}{\cancel{3}}\left[\cancel{3}x\left(x^2-3x+3\right)\right]_0^1=\left[1\left(1^2\cancel{-3\cdot 1+3}\right)\cancel{-0\left(0^2-3\cdot 0+3\right)}\right]=1$$

8 Coordenadas polares

8.1 Aula 1

1. Exercício

   Calcule a área do circulo de raio igual a dois

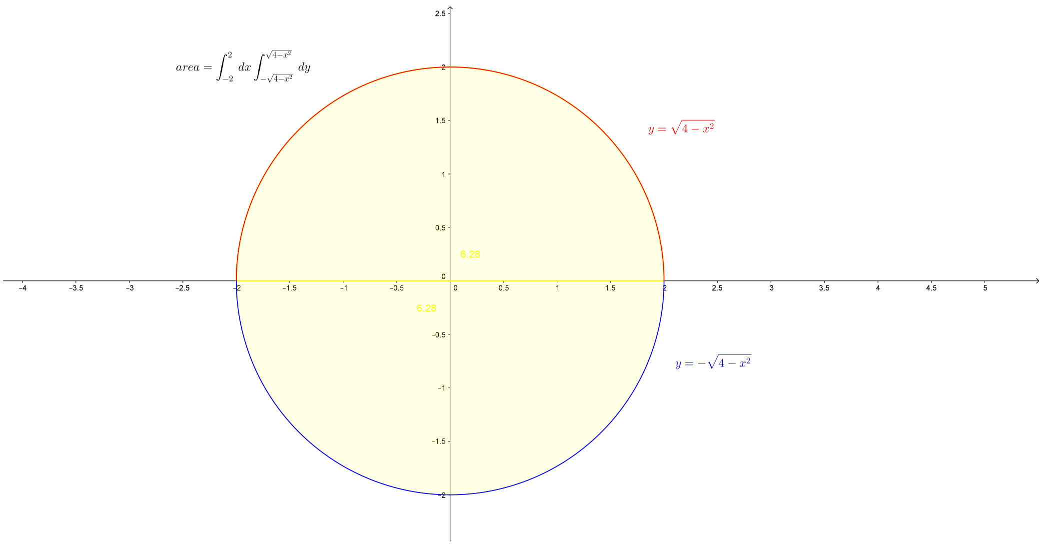


Figura 16: Coordenadas polares - Aula 01 - Exercício I

$$r=2\Rightarrow a=\pi r^2=2^2\pi=4\pi$$

$$x^2+y^2=r^2\Rightarrow x^2+y^2=2^2\Rightarrow x^2+y^2=4\Rightarrow y=\pm\sqrt{4-x^2}$$
$$R=\left\{(x,y)\in\mathbb{R}^2\,|\,-2\le x\le 2,\,-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\right\}$$

$$a=\int_{-2}^{2}dx\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dy=\int_{-2}^{2}dx\left(\sqrt{4-x^2}+\sqrt{4-x^2}\right)=2\int_{-2}^{2}\sqrt{4-x^2}\,dx=$$
$$2\int_{-2}^{2}\sqrt{4-(2\operatorname{sen}(\alpha))^2}\,2\cos(\alpha)\,d\alpha=4\int_{-2}^{2}\sqrt{4-4\operatorname{sen}^2(\alpha)}\,\cos(\alpha)\,d\alpha=$$
$$4\int_{-2}^{2}\sqrt{4-4\left(1-\cos^2(\alpha)\right)}\,\cos(\alpha)\,d\alpha=4\int_{-2}^{2}\sqrt{4-\left(4-4\cos^2(\alpha)\right)}\,\cos(\alpha)\,d\alpha=$$
$$4\int_{-2}^{2}\sqrt{\cancel{4-4}+4\cos^2(\alpha)}\,\cos(\alpha)\,d\alpha=4\int_{-2}^{2}2\cos(\alpha)\cos(\alpha)\,d\alpha=8\int_{-2}^{2}\cos^2(\alpha)\,d\alpha=$$

$$8\int_{-2}^{2}\left(\frac{1+\cos(2\alpha)}{2}\right)\,d\alpha = 8\int_{-2}^{2}\left(\frac{1}{2}+\frac{\cos(2\alpha)}{2}\right)\,d\alpha = 4\int_{-2}^{2}d\alpha + 4\int_{-2}^{2}\cos(2\alpha)\,d\alpha =$$
$$4\int_{-2}^{2}d\alpha + 4\int_{-2}^{2}\cos(u)\,\frac{du}{2} = 4\int_{-2}^{2}d\alpha + 2\int_{-2}^{2}\cos(u)\,du = \left[4\alpha + 2\,\mathrm{sen}(u)\right]_{-2}^{2} =$$
$$\left[4\alpha + 2\,\mathrm{sen}(2\alpha)\right]_{-2}^{2} = \left[4\alpha + 4\,\mathrm{sen}(\alpha)\cos(\alpha)\right]_{-2}^{2} = \left[4\left(\alpha + \mathrm{sen}(\alpha)\cos(\alpha)\right)\right]_{-2}^{2} =$$
$$\left[4\left(\mathrm{arcsen}\left(\frac{x}{2}\right) + \frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right)\right]_{-2}^{2} = \left[4\left(\mathrm{arcsen}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4}\right)\right]_{-2}^{2} =$$
$$4\left(\mathrm{arcsen}\left(\frac{2}{2}\right) + \frac{2\sqrt{4-2^2}}{4}\right) - 4\left(\mathrm{arcsen}\left(\frac{(-2)}{2}\right) + \frac{(-2)\sqrt{4-(-2)^2}}{4}\right) =$$
$$4\,\mathrm{arcsen}(1) - 4\,\mathrm{arcsen}(-1) = 4(\mathrm{arcsen}(1) - \mathrm{arcsen}(-1)) = 4\left(\frac{\pi}{2}+\frac{\pi}{2}\right) =$$
$$4\left(\frac{\cancel{2}\pi}{\cancel{2}}\right) = 4\pi$$

$x = 2\,\mathrm{sen}(\alpha);\ dx = 2\cos(\alpha)\,d\alpha$

$u = 2\alpha;\ \dfrac{du}{2} = d\alpha$

$sen(\alpha) = \dfrac{co}{h} = \dfrac{x}{2} \Rightarrow \alpha = \mathrm{arcsen}\left(\dfrac{x}{2}\right)$

$h^2 = co^2 + ca^2 \Rightarrow 2^2 = x^2 + ca^2 \Rightarrow ca = \sqrt{4-x^2}$

$cos(\alpha) = \dfrac{ca}{h} = \dfrac{\sqrt{4-x^2}}{2}$

$R = \{(r,\theta) \in \mathbb{R}^2 \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$$a = \int_{-2}^{2}dx\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dy = \int_{0}^{2}\int_{0}^{2\pi} r\,drd\theta = \int_{0}^{2} r\,dr\int_{0}^{2\pi}d\theta = \left[\frac{r^2}{2}\right]_{0}^{2}[\theta]_{0}^{2\pi} =$$
$$\frac{1}{2}\left[2^2-0^2\right][2\pi-0] = \frac{4}{\cancel{2}}\cancel{2}\pi = 4\pi$$

2. Exercício

$$\iint_R \frac{da}{1+x^2+y^2}$$

$$volume = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\, dr d\theta}{1+r^2}$$

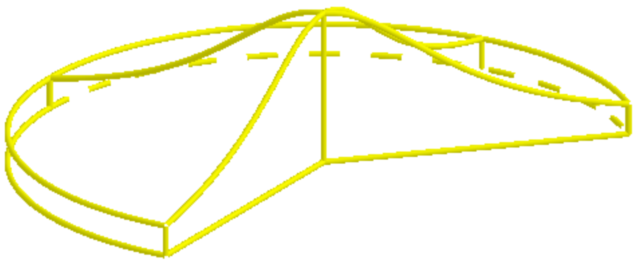


Figura 17: Coordenadas polares - Aula 01 - Exercício II

$$R = \left\{ (r,\theta) \in \mathbb{R}^2 \,|\, 0 \leq r \leq 2,\ \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{2} \right\}$$

$$v = \iint_R \frac{da}{1+x^2+y^2} = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\, dr d\theta}{1+r^2} = \int_0^2 \frac{r\, dr}{1+r^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} d\theta = \int_0^2 \left(1+r^2\right)^{-1} r\, dr\, [\theta]_{\frac{\pi}{4}}^{\frac{3\pi}{2}} =$$

$$\int_0^2 \left(1+r^2\right)^{-1} r\, dr \left(\frac{3\pi}{2} - \frac{\pi}{4}\right) = \int_0^2 \left(1+r^2\right)^{-1} r\, dr \left(\frac{6\pi - \pi}{4}\right) = \frac{5\pi}{4} \int_0^2 \left(1+r^2\right)^{-1} r\, dr =$$

$$\frac{5\pi}{4} \int_0^2 u^{-1} \frac{du}{2} = \frac{5\pi}{8} \int_0^2 u^{-1} du = \frac{5\pi}{8} \left[ln|u|\right]_0^2 = \frac{5\pi}{8} \left[ln|1+r^2|\right]_0^2 =$$

$$\frac{5\pi}{8} \left[ln|1+2^2| - ln|1+0^2|\right] = \frac{5\pi}{8} \left[ln|5| - ln|1|\right] = \frac{5\pi ln|5|}{8}$$

$$u = 1 + r^2 \Rightarrow \frac{du}{2} = r\, dr$$

$$\mathrm{e}^x = 1 = \mathrm{e}^0 \Rightarrow x = 0$$

A Derivadas

A.1 Derivadas simples

$$
\left|
\begin{array}{lllllll}
y & = & c & \Rightarrow & y' & = & 0 \\
y & = & x & \Rightarrow & y' & = & 1 \\
y & = & x^c & \Rightarrow & y' & = & cx^{c-1} \\
y & = & \mathrm{e}^x & \Rightarrow & y' & = & \mathrm{e}^x \\
y & = & \ln|x| & \Rightarrow & y' & = & \dfrac{1}{x} \\
y & = & uv & \Rightarrow & y' & = & u'v + uv' \\
y & = & \dfrac{u}{v} & \Rightarrow & y' & = & \dfrac{u'v - uv'}{v^2} \\
y & = & u^c & \Rightarrow & y' & = & cu^{c-1}u' \\
y & = & \mathrm{e}^u & \Rightarrow & y' & = & \mathrm{e}^u\, u' \\
y & = & c^u & \Rightarrow & y' & = & c^u u' ln|c| \\
y & = & \ln|u| & \Rightarrow & y' & = & \dfrac{u'}{u} \\
y & = & \log_c|u| & \Rightarrow & y' & = & \dfrac{u'}{u}\log_c|\,\mathrm{e}\,|
\end{array}
\right|
$$

Tabela 1: Derivadas simples

A.2 Derivadas trigonométricas

$$\left|\begin{array}{lllllll}
y & = & \operatorname{sen}(x) & \Rightarrow & y' & = & \cos(x) \\
y & = & \cos(x) & \Rightarrow & y' & = & -\operatorname{sen}(x) \\
y & = & \operatorname{tg}(x) & \Rightarrow & y' & = & \sec^2(x) \\
y & = & \operatorname{cotg}(x) & \Rightarrow & y' & = & -\operatorname{cossec}^2(x) \\
y & = & \sec(x) & \Rightarrow & y' & = & \sec(x)\operatorname{tg}(x) \\
y & = & \operatorname{cossec}(x) & \Rightarrow & y' & = & -\operatorname{cossec}(x)\operatorname{cotg}(x) \\
y & = & \operatorname{arcsen}(x) & \Rightarrow & y' & = & \dfrac{1}{\sqrt{1-x^2}} \\
y & = & \arccos(x) & \Rightarrow & y' & = & \dfrac{-1}{\sqrt{1-x^2}} \\
y & = & \operatorname{arctg}(x) & \Rightarrow & y' & = & \dfrac{1}{1+x^2} \\
y & = & \operatorname{arccotg}(x) & \Rightarrow & y' & = & \dfrac{-1}{1+x^2} \\
y & = & \operatorname{arcsec}(x) & \Rightarrow & y' & = & \dfrac{1}{|x|\sqrt{x^2-1}} \\
y & = & \operatorname{arccossec}(x) & \Rightarrow & y' & = & \dfrac{-1}{|x|\sqrt{x^2-1}}
\end{array}\right|$$

Tabela 2: Derivadas trigonométricas

B Integrais

B.1 Integrais simples

$$
\left|
\begin{array}{lcll}
\displaystyle\int dx & = & x + c & \\
\displaystyle\int x^p \, dx & = & \dfrac{x^{p+1}}{p+1} + c & \rightarrow \quad p \neq -1 \\
\displaystyle\int \mathrm{e}^x \; dx & = & \mathrm{e}^x + c & \\
\displaystyle\int \dfrac{dx}{x} & = & \ln |x| + c & \\
\displaystyle\int u^p \, du & = & \dfrac{u^{p+1}}{p+1} + c & \rightarrow \quad p \neq -1 \\
\displaystyle\int \mathrm{e}^u \; du & = & \mathrm{e}^u + c & \\
\displaystyle\int \dfrac{du}{u} & = & \ln |u| + c & \\
\displaystyle\int p^u \, du & = & \dfrac{p^u}{\ln |p|} + c &
\end{array}
\right|
$$

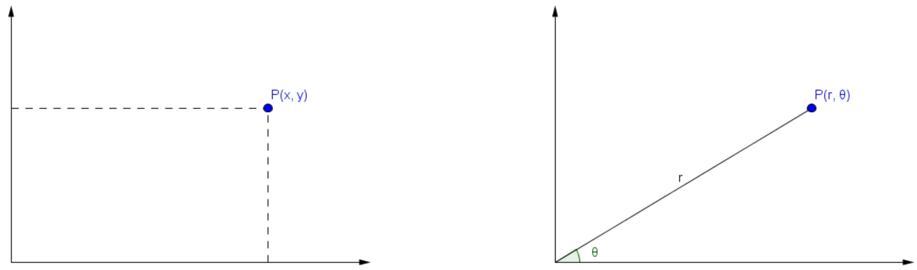
Tabela 3: Integrais simples

B.2 Integrais trigonométricas

$$
\begin{array}{|lcl|}
\int \operatorname{sen}(u)du & = & -\cos(u) + c \\
\int \cos(u)du & = & \operatorname{sen}(u) + c \\
\int \operatorname{tg}(u)du & = & \ln|\sec(u)| + c \\
\int \operatorname{cotg}(u)du & = & \ln|\operatorname{sen}(u)| + c \\
\int \sec(u)du & = & \ln|\sec(u) + \operatorname{tg}(u)| + c \\
\int \operatorname{cossec}(u)du & = & \ln|\operatorname{cossec}(u) - \operatorname{cotg}(u)| + c \\
\int \sec^2(u)du & = & \operatorname{tg}(u) + c \\
\int \operatorname{cossec}^2(u)du & = & -\operatorname{cotg}(u) + c \\
\int \sec(u)\operatorname{tg}(u)du & = & \sec(u) + c \\
\int \operatorname{cossec}(u)\operatorname{cotg}(u)du & = & -\operatorname{cossec}(u) + c \\
\int \frac{du}{\sqrt{1-x^2}} & = & \operatorname{arcsen}(x) + c \\
-\int \frac{du}{\sqrt{1-x^2}} & = & \arccos(x) + c \\
\int \frac{du}{1+x^2} & = & \operatorname{arctg}(x) + c \\
-\int \frac{du}{1+x^2} & = & \operatorname{arccotg}(x) + c \\
\int \frac{du}{|x|\sqrt{x^2-1}} & = & \operatorname{arcsec}(x) + c \\
-\int \frac{du}{|x|\sqrt{x^2-1}} & = & \operatorname{arccossec}(x) + c
\end{array}
$$

Tabela 4: Integrais trigonométricas

B.3 Relação entre coordenada cartesina e polar



(a) Coordenada cartesiana ou retangular (b) Coordenada polar

Figura 18: Coordenada cartesina e polar

$$P(x, y) \rightarrow P(r, \theta)$$

$$\left| \begin{array}{lllll} & & x & = & r\,\cos\theta \\ & & y & = & r\,\operatorname{sen}\theta \\ & & x^2 + y^2 & = & r^2 \\ da & = & dxdy & = & r\,drd\theta \\ v & = & \displaystyle\iint_{R(x,y)} f(x,y)\,dxdy & = & \displaystyle\iint_{R(r,\theta)} f(r\,\cos\theta, r\,\operatorname{sen}\theta) r\,drd\theta \end{array} \right|$$

Tabela 5: Relação entre coordenada cartesina e polar

C Funções trigonométricas

C.1 Determinação do seno, cosseno e tangente

$\theta =$ ângulo
$h =$ hipotenusa
$co =$ cateto oposto
$ca =$ cateto adjacente

$$h^2 = co^2 + ca^2$$
$$sen\,\theta = \frac{co}{h}$$
$$cos\,\theta = \frac{ca}{h}$$
$$tg\,\theta = \frac{co}{ca}$$

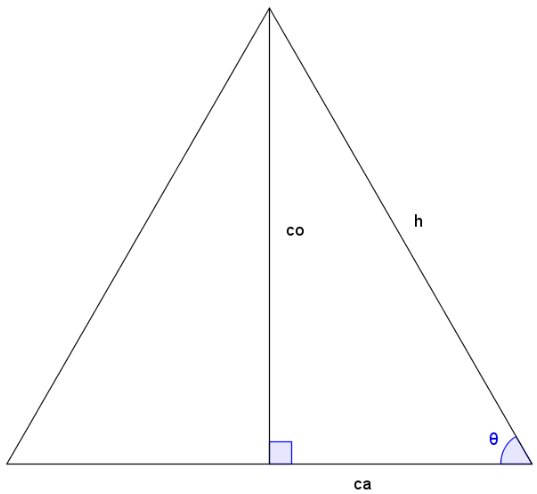


Figura 19: Determinação do seno, cosseno e tangente

C.2 Círculo trigonométrico

$$\operatorname{tg}\theta = \frac{\operatorname{sen}\theta}{\cos\theta}\,;\ \operatorname{cotg}\theta = \frac{\cos\theta}{\operatorname{sen}\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}\,;\ \operatorname{cossec}\theta = \frac{1}{\operatorname{sen}\theta}$$
$$\operatorname{sen}^2\theta + \cos^2\theta = 1;\ \sec^2\theta - \operatorname{tg}^2\theta = 1; \operatorname{cossec}^2\theta - \operatorname{cotg}^2\theta = 1$$

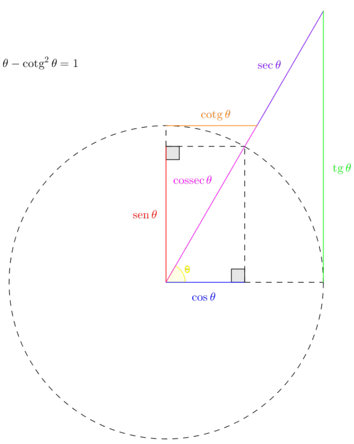


Figura 20: Círculo trigonométrico

D Identidades trigonométricas

$$
\begin{array}{|lcl|}
\mathrm{tg}(x) & = & \dfrac{\mathrm{sen}(x)}{\cos(x)} \\
\mathrm{cotg}(x) & = & \dfrac{\cos(x)}{\mathrm{sen}(x)} \\
\sec(x) & = & \dfrac{1}{\cos(x)} \\
\mathrm{cossec}(x) & = & \dfrac{1}{\mathrm{sen}(x)} \\
\mathrm{sen}^2(x) + \cos^2(x) & = & 1 \\
\sec^2(x) - \mathrm{tg}^2(x) & = & 1 \\
\mathrm{cossec}^2(x) - \mathrm{cotg}^2(x) & = & 1 \\
\mathrm{sen}^2(x) & = & \dfrac{1 - \cos(2x)}{2} \\
\cos^2(x) & = & \dfrac{1 + \cos(2x)}{2} \\
\mathrm{sen}(2x) & = & 2\,\mathrm{sen}(x)\cos(x) \\
\cos(2x) & = & \cos^2(x) - \mathrm{sen}^2(x)
\end{array}
$$

Tabela 6: Identidades trigonométricas

D.1 Relação entre trigonométricas e inversas

$$
\begin{array}{|lcccccl|}
\mathrm{sen}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arcsen}(x) \\
\cos(\theta) & = & x & \Rightarrow & \theta & = & \arccos(x) \\
\mathrm{tg}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arctg}(x) \\
\mathrm{cossec}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arccossec}(x) \\
\sec(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arcsec}(x) \\
\mathrm{cotg}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arccotg}(x)
\end{array}
$$

Tabela 7: Relação entre trigonométricas e inversas

D.2 Substituição trigonométrica

$$
\begin{array}{|lcccl|}
\sqrt{a^2 - x^2} & \Rightarrow & x & = & a\,\mathrm{sen}(\theta) \\
\sqrt{a^2 + x^2} & \Rightarrow & x & = & a\,\mathrm{tg}(\theta) \\
\sqrt{x^2 - a^2} & \Rightarrow & x & = & a\sec(\theta)
\end{array}
$$

Tabela 8: Substituição trigonométrica

D.3 Ângulos notáveis

ângulo	$0°$ (0)	$30°$ $\left(\frac{\pi}{6}\right)$	$45°$ $\left(\frac{\pi}{4}\right)$	$60°$ $\left(\frac{\pi}{3}\right)$	$90°$ $\left(\frac{\pi}{2}\right)$
sen	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\nexists$

Tabela 9: Ângulos notáveis

Referências

[1] Fernando Grings. Curso de integrais duplas e triplas. `https://www.youtube.com/playlist?list=PL82B9E5FF3F2B3BD3`, 2016.