Curso de integrais duplas e triplas

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Sumário

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Resumo

Exercícios retirados do canal do Youtube, O Matematico [1].

Parte I
Integrais duplas

1 Invertendo os limites de integração - Aula 1

1. Exercício

$$f(x) = x^2;\ g(x) = x^3$$

Figura 1: Integrais duplas - Aula 1 - Exercício I e II

img/v01_a01_e01.png

$$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$$
$$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$$

$$a = \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx \, [y]_{x^3}^{x^2} = \int_0^1 dx \left[x^2 - x^3\right] =$$
$$\int_0^1 x^2 \, dx - \int_0^1 x^3 \, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{4x^3 - 3x^2}{12}\right]_0^1 = \frac{1}{12}\left[4x^3 - 3x^2\right]_0^1 =$$
$$\frac{1}{12}\left[x^2(4x-3)\right]_0^1 = \frac{1}{12}\left[1^2(4 \cdot 1 - 3) \cancel{- 0^2(4 \cdot 0 - 3)}\right] = \frac{1}{12} = 0,08\overline{3}$$

2. Exercício

$$f(x) = x^2 \Rightarrow f(y) = \sqrt{y}; \; g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$$
$$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$$
$$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$$

$$a = \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy \, [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy \, [\sqrt[3]{y} - \sqrt{y}] =$$
$$\int_0^1 \sqrt[3]{y} \, dy - \int_0^1 \sqrt{y} \, dy = \int_0^1 y^{\frac{1}{3}} \, dy - \int_0^1 y^{\frac{1}{2}} \, dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 =$$
$$\left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3}\right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12}\right]_0^1 = \frac{1}{12}\left[9\sqrt[3]{y^4} - 8\sqrt{y^3}\right]_0^1 =$$
$$\frac{1}{12}\left[\left(9\sqrt[3]{1^4} - 8\sqrt{1^3}\right) \cancel{- \left(9\sqrt[3]{0^4} - 8\sqrt{0^3}\right)}\right] = \frac{1}{12}(9-8) = \frac{1}{12} = 0,08\overline{3}$$

2 Determinação da região de integração - Aula 2

1. Exercício

$$R = \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 2\,,\, 0 \leq y \leq 6\right\}$$

Figura 2: Integrais duplas - Aula 2 - Exercício I

img/v02_a02_e01.png

$$a = \int_0^2 dx \int_0^6 dy = \int_0^2 dx\, [y]_0^6 = \int_0^2 dx\, [6-0] = 6 \int_0^2 dx = 6[x]_0^2 = 6[2-0] = 6 \cdot 2 = 12$$

2. Exercício

$$R = \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1\,,\, x \leq y \leq 2x\right\}$$

Figura 3: Integrais duplas - Aula 2 - Exercício II

img/v02_a02_e02.png

$$a = \int_0^1 dx \int_x^{2x} dy = \int_0^1 dx\, [y]_x^{2x} = \int_0^1 dx\, [2x - x] = 2\int_0^1 x\, dx - \int_0^1 x\, dx =$$

$$\left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^2}{2}\right]_0^1 = \left[\frac{2x^2 - x^2}{2}\right]_0^1 = \frac{1}{2}\left[x^2\right]_0^1 = \frac{1}{2}\left[1^2 \cancel{- 0^2}\right] = \frac{1}{2} = 0,5$$

3. Exercício

$$R = \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 1 \,,\, 0 \leq x \leq \sqrt{1 - y^2}\right\}$$

$$y = 0,\ y = 1$$

$$x = 0,\ x = \sqrt{1 - y^2} \Rightarrow x^2 = 1 - y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow y = \sqrt{1 - x^2}$$

Figura 4: Integrais duplas - Aula 2 - Exercício III

img/v02_a02_e03.png

$$a = \int_0^1 dy \int_0^{f(y)} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy\, [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy\, \left[\sqrt{1 - y^2} - 0\right] =$$

$$\int_0^1 \sqrt{1 - y^2}\, dy = \int_0^1 \sqrt{1 - \operatorname{sen}^2(t)}\, \cos(t)dt = \int_0^1 \sqrt{\cos^2(t)}\, \cos(t)dt =$$

$$\int_0^1 \cos(t)\cos(t)dt = \int_0^1 \cos^2(t)dt = \int_0^1 \frac{1 + \cos(2t)}{2}dt = \frac{1}{2}\int_0^1 [1 + \cos(2t)]\, dt =$$

$$\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(2t)dt = \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(u)\frac{du}{2} = \frac{1}{2}\int_0^1 dt + \frac{1}{4}\int_0^1 \cos(u)\, du =$$

$$\left[\frac{1}{2}t + \frac{1}{4}\operatorname{sen}(u)\right]_0^1 = \left[\frac{t}{2} + \frac{\operatorname{sen}(2t)}{4}\right]_0^1 = \left[\frac{t}{2} + \frac{2\operatorname{sen}(t)\cos(t)}{4}\right]_0^1 = \left[\frac{t + \operatorname{sen}(t)\cos(t)}{2}\right]_0^1 =$$

$$\frac{1}{2}\left[\operatorname{arcsen}(y) + y\sqrt{1 - y^2}\right]_0^1 = \frac{1}{2}\left[\left(\operatorname{arcsen}(1) \cancel{+ 1 \cdot \sqrt{1 - 1^2}}\right) - \left(\operatorname{arcsen}(0) \cancel{+ 0 \cdot \sqrt{1 - 0^2}}\right)\right] =$$

$$\frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4} = 0,785$$

$$y = \text{sen}(t) \Rightarrow dy = \cos(t)dt$$
$$u = 2t \Rightarrow \frac{du}{2} = dt$$
$$\text{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$
$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1-y^2}$$
$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1-y^2}}{1} = \sqrt{1-y^2}$$
$$y = \text{sen}(t) \Rightarrow t = \text{arcsen}(y)$$

4. Exercício

$$y = x^2 + 1,\ y = -x^2 - 1;\ x = 1,\ x = -1$$
$$R = \left\{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1\,,\ -x^2 - 1 \leq y \leq x^2 + 1\right\}$$

Figura 5: Integrais duplas - Aula 2 - Exercício IV

`img/v02_a02_e04.png`

$$a = \int_{-1}^{1} dx \int_{f(x)}^{g(x)} dy = \int_{-1}^{1} dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^{1} dx\, [y]_{-x^2-1}^{x^2+1} =$$
$$\int_{-1}^{1} dx\ \left[x^2 + 1 - \left(-x^2 - 1\right)\right] = \int_{-1}^{1} dx\ \left[x^2 + 1 + x^2 + 1\right] =$$
$$\int_{-1}^{1} dx\ \left[2x^2 + 2\right] = 2\int_{-1}^{1} x^2\, dx + 2\int_{-1}^{1} dx = \left[2\frac{x^3}{3} + 2x\right]_{-1}^{1} = \left[2\left(\frac{x^3 + 3x}{3}\right)\right]_{-1}^{1} =$$
$$\frac{2}{3}\left[x\left(x^2 + 3\right)\right]_{-1}^{1} = \frac{2}{3}\left[1 \cdot \left(1^2 + 3\right) - (-1)\left((-1)^2 + 3\right)\right] = \frac{2}{3}(4 + 4) = \frac{2}{3}8 = \frac{16}{3} = 5,\overline{3}$$

5. Exercício

$$R = \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 2\,,\; -y \leq x \leq y\right\}$$

Figura 6: Integrais duplas - Aula 2 - Exercício V

img/v02_a02_e05.png

$$a = \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^{y} dx = \int_0^2 dy\, [x]_{-y}^{y} = \int_0^2 dy\, [y - (-y)] = \int_0^2 dy\, [2y] =$$

$$2\int_0^2 y\, dy = \left[\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^2 = 2^2 - 0^2 = 4$$

3 Cálculo de volume - Aula 3

1. Exercício

Figura 7: Integrais duplas - Aula 3 - Exercício I

img/v03_a03_e01.png

$$z = 4;\; dz = dxdy$$

$$v = \int_0^3 \int_0^2 z\,dz = \int_0^3 \int_0^2 4\,dydx = 4\int_0^3 dx \int_0^2 dy = 4\int_0^3 dx\,[y]_0^2 = 4\int_0^3 dx\,[2-0] =$$
$$8\int_0^3 dx = 8[x]_0^3 = 8[3-0] = 8\cdot 3 = 24$$

2. Exercício

$$R = [0,3]\times[0,4]$$
$$\iint_R (8-2y)da$$

Figura 8: Integrais duplas - Aula 3 - Exercício II

`img/v03_a03_e02.png`

$$z = 8-2y;\ da = dz = dxdy$$

$$v = \int_0^3 \int_0^4 z\,dz = \int_0^3 \int_0^4 (8-2y)dxdy = \int_0^3 dx \int_0^4 (8-2y)dy =$$
$$\int_0^3 dx \left(8\int_0^4 dy - 2\int_0^4 y\,dy\right) = \int_0^3 dx\,2\left(4\int_0^4 dy - \int_0^4 y\,dy\right) =$$
$$2\int_0^3 dx\left[4y - \frac{y^2}{2}\right]_0^4 = 2\int_0^3 dx\left[\frac{8y-y^2}{2}\right]_0^4 = \cancel{2}\int_0^3 dx\,\frac{1}{\cancel{2}}[y(8-y)]_0^4 =$$
$$\int_0^3 dx[4(8-4)\cancel{-0(8-0)}] = 16\int_0^3 dx = 16[x]_0^3 = 16[3-0] = 48$$

4 Invertendo a ordem de integração - Aula 4

1. Exercício

$$z = f(x,y) = y\,\mathrm{e}^x;\ dz = dxdy$$

$$v = \int_2^4 \int_1^9 z\,dz = \int_2^4 \int_1^9 y\,\mathrm{e}^x\ dydx = \int_2^4 \mathrm{e}^x\ dx \int_1^9 y\,dy = \int_2^4 \mathrm{e}^x\ dx \left[\frac{y^2}{2}\right]_1^9 =$$
$$\int_2^4 \mathrm{e}^x\ dx \frac{1}{2}\left[y^2\right]_1^9 = \frac{1}{2}\int_2^4 \mathrm{e}^x\ dx \left[9^2 - 1^2\right] = 40\int_2^4 \mathrm{e}^x\ dx = 40\left[\mathrm{e}^x\right]_2^4 = 40\left[\mathrm{e}^4 - \mathrm{e}^2\right] =$$
$$40e^2\left(e^2 - 1\right)$$

2. Exercício

$$z = f(x,y) = x^2y^3;\ dz = dxdy$$

Figura 9: Integrais duplas - Aula 4 - Exercício II

img/v04_a04_e02.png

$$v = \int_0^1 \int_2^4 z\,dz = \int_0^1 \int_2^4 x^2y^3\,dxdy = \int_0^1 x^2\,dx \int_2^4 y^3\,dy = \int_0^1 x^2\,dx \left[\frac{y^4}{4}\right]_2^4 =$$
$$\frac{1}{4}\int_0^1 x^2\,dx \left[y^4\right]_2^4 = \frac{1}{4}\int_0^1 x^2\,dx \left[4^4 - 2^4\right] = \frac{1}{4}\int_0^1 x^2\,dx \left[2^8 - 2^4\right] =$$
$$\frac{1}{4}\int_0^1 x^2\,dx \left[2^4\left(2^4 - 1\right)\right] = \frac{1}{4}\int_0^1 x^2\,dx\,[16\cdot 15] = 60\int_0^1 x^2\,dx = 60\left[\frac{x^3}{3}\right]_0^1 = 20\left[x^3\right]_0^1 =$$
$$20\left[1^3 - 0^3\right] = 20\cdot 1 = 20$$

3. Exercício

$$\iint_R (x+2y)da$$

R = Região limitada pela parábola $y = x^2 + 1$ e as retas $x = -1$ e $x = 2$.

$$z = f(x,y) = x + 2y;\ da = dz = dxdy$$

Figura 10: Integrais duplas - Aula 4 - Exercício III

img/v04_a04_e03.png

$$v = \int_{-1}^{2}\int_{0}^{x^2+1} z\,dz = \int_{-1}^{2}\int_{0}^{x^2+1}(x+2y)dxdy =$$

$$\int_{-1}^{2} dx \int_{0}^{x^2+1}(x+2y)dy = \int_{-1}^{2} dx \left(x\int_{0}^{x^2+1} dy + 2\int_{0}^{x^2+1} y\,dy \right) =$$

$$\int_{-1}^{2} dx \left[xy + \cancel{2}\frac{y^2}{\cancel{2}} \right]_{0}^{x^2+1} = \int_{-1}^{2} dx \left[y(x+y) \right]_{0}^{x^2+1} =$$

$$\int_{-1}^{2} dx \left[\left(x^2+1\right)\left[x+\left(x^2+1\right)\right] \cancel{-0(x+0)} \right] = \int_{-1}^{2} dx \left[\left(x^2+1\right)\left(x^2+x+1\right) \right] =$$

$$\int_{-1}^{2} dx \left(x^4+x^3+2x^2+x+1\right) =$$

$$\int_{-1}^{2} x^4\,dx + \int_{-1}^{2} x^3\,dx + 2\int_{-1}^{2} x^2\,dx + \int_{-1}^{2} x\,dx + \int_{-1}^{2} dx =$$

$$\left[\frac{x^5}{5} + \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^{2} = \left[\frac{12x^5+15x^4+40x^3+30x^2+60x}{60} \right]_{-1}^{2} =$$

$$\frac{1}{60}\left[x\left(12x^4+15x^3+40x^2+30x+60\right) \right]_{-1}^{2} =$$

$$\frac{1}{60}\left[2\left(12\cdot 2^4+15\cdot 2^3+40\cdot 2^2+30\cdot 2+60\right) \right.$$

$$\left. -(-1)\left(12(-1)^4+15(-1)^3+40(-1)^2+30(-1)+60\right) \right] =$$

$$\frac{1}{60}[2(192+120+160+60+60)+(12-15+40-30+60)] = \frac{1}{60}(1184+67) =$$

$$\frac{1251}{60} = \frac{417}{20} = 20,85$$

5 Cálculo de integrais duplas ou iteradas

5.1 Aula 5

1. Exercício

$$f(x,y) = x^3;\ 0 \leq x \leq 2;\ x^2 \leq y \leq 4$$

$$\iint_R f(x,y)dydx$$

Figura 11: Integrais duplas - Aula 5 - Exercício I

`img/v05_a05_e01.png`

$$v = \int_0^2 \int_{x^2}^4 x^3\, dxdy = \int_0^2 x^3\, dx \int_{x^2}^4 dy = \int_0^2 x^3\, dx\, [y]_{x^2}^4 = \int_0^2 x^3\, dx\, \left[4 - x^2\right] =$$

$$4\int_0^2 x^3\, dx - \int_0^2 x^5\, dx = \left[\cancel{4}\frac{x^4}{\cancel{4}} - \frac{x^6}{6}\right]_0^2 = \left[\frac{6x^4 - x^6}{6}\right]_0^2 = \frac{1}{6}\left[x^4\left(6 - x^2\right)\right]_0^2 =$$

$$\frac{1}{6}\left[2^4\left(6 - 2^2\right) - \cancel{0^4\left(6 - 0^2\right)}\right] = \frac{1}{6}(16 \cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2$$

2. Exercício

$$f(x,y) = x^2 y;\ 1 \leq x \leq 3;\ x \leq y \leq 2x + 1$$

$$\iint_R f(x,y)dydx$$

Figura 12: Integrais duplas - Aula 5 - Exercício II

img/v05_a05_e02.png

$$
v = \int_1^3 \int_x^{2x+1} x^2 y \, dx dy = \int_1^3 x^2 \, dx \int_x^{2x+1} y \, dy = \int_1^3 x^2 \, dx \left[\frac{y^2}{2} \right]_x^{2x+1} =
$$

$$
\int_1^3 x^2 \, dx \frac{1}{2} \left[(2x+1)^2 - (x)^2 \right] = \frac{1}{2} \int_1^3 x^2 \, dx \left(3x^2 + 4x + 1 \right) =
$$

$$
\frac{3}{2} \int_1^3 x^4 \, dx + 2 \int_1^3 x^3 \, dx + \frac{1}{2} \int_1^3 x^2 \, dx = \left[\frac{3}{2} \frac{x^5}{5} + 2 \frac{x^4}{4} + \frac{1}{2} \frac{x^3}{3} \right]_1^3 = \left[\frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6} \right]_1^3 =
$$

$$
\left[\frac{18x^5 + 30x^4 + 10x^3}{60} \right]_1^3 = \left[\frac{2x^3 \left(9x^2 + 15x + 5 \right)}{60} \right]_1^3 =
$$

$$
\frac{1}{30} \left[x^3 \left(9x^2 + 15x + 5 \right) \right]_1^3 = \frac{1}{30} \left[3^3 \left(9 \cdot 3^2 + 15 \cdot 3 + 5 \right) - 1^3 \left(9 \cdot 1^2 + 15 \cdot 1 + 5 \right) \right] =
$$

$$
\frac{1}{30} \left[27(81 + 45 + 5) - (9 + 15 + 5) \right] = \frac{1}{30} \left[27 \cdot 131 - 29 \right] = \frac{3508}{30} = 116,9\overline{3}
$$

5.2 Aula 6

1. Exercício

$$
f(x,y) = 1; \; 0 \leq x \leq 1; \; 1 \leq y \leq \mathrm{e}^x
$$

$$
\iint_R f(x,y) dy dx
$$

$$
v = \int_0^1 \int_1^{\mathrm{e}^x} dy dx = \int_0^1 dx \left[y \right]_1^{\mathrm{e}^x} = \int_0^1 dx \, \left(\mathrm{e}^x - 1 \right) = \left[\mathrm{e}^x - x \right]_0^1 =
$$

$$
\mathrm{e}^1 - 1 - \left(\mathrm{e}^0 - 0 \right) = \mathrm{e} - 1 - 1 = \mathrm{e} - 2
$$

2. Exercício

$$f(x,y) = x;\ 0 \leq x \leq 1;\ 1 \leq y \leq \mathrm{e}^{x^2}$$
$$\iint_R f(x,y)dydx$$

$$v = \int_0^1 \int_1^{\mathrm{e}^{x^2}} x\,dxdy = \int_0^1 x\,dx \int_1^{\mathrm{e}^{x^2}} dy = \int_0^1 x\,dx\,[y]_1^{\mathrm{e}^{x^2}} = \int_0^1 x\,dx\,\left(\mathrm{e}^{x^2} - 1\right) =$$
$$\int_0^1 x\,\mathrm{e}^{x^2}\,dx - \int_0^1 x\,dx = \int_0^1 \mathrm{e}^u\,\frac{du}{2} - \int_0^1 x\,dx = \frac{1}{2}\int_0^1 \mathrm{e}^u\,du - \int_0^1 x\,dx =$$
$$\left[\frac{1}{2}e^u - \frac{x^2}{2}\right]_0^1 = \left[\frac{e^{x^2} - x^2}{2}\right]_0^1 = \frac{1}{2}\left[e^{x^2} - x^2\right]_0^1 = \frac{1}{2}\left[e^{1^2} - 1^2 - \left(e^{0^2} - 0^2\right)\right] =$$
$$\frac{1}{2}(\mathrm{e} - 1 - 1) = \frac{\mathrm{e} - 2}{2}$$

$$u = x^2;\ \frac{du}{2} = x\,dx$$

3. Exercício

$$f(x,y) = 2xy;\ 0 \leq y \leq 1;\ y^2 \leq x \leq y$$
$$\iint_R f(x,y)dxdy$$

$$v = \int_0^1 \int_{y^2}^{y} 2xy\,dxdy = 2\int_0^1 y\,dy \int_{y^2}^{y} x\,dx = 2\int_0^1 y\,dy\,\left[\frac{x^2}{2}\right]_{y^2}^{y} = \cancel{2}\int_0^1 y\,dy\,\frac{1}{\cancel{2}}\left[x^2\right]_{y^2}^{y} =$$
$$\int_0^1 y\,dy\,\left(y^2 - y^4\right) = \int_0^1 \left(y^3 - y^5\right)dy = \left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1 =$$
$$\left[\frac{6y^4 - 4y^6}{24}\right]_0^1 = \left[\frac{2y^4\left(3 - 2y^2\right)}{24}\right]_0^1 = \frac{1}{12}\left[1^4\left(3 - 2\cdot 1^2\right) \cancel{- 0^4\left(3 - 2\cdot 0^2\right)}\right] = \frac{1}{12} = 0,08\overline{3}$$

5.3 Aula 7

1. Exercício

$$f(x,y) = \frac{1}{x+y};\ 1 \leq y \leq \mathrm{e};\ 0 \leq x \leq y$$
$$\iint_R f(x,y)dxdy$$

$$v = \int_1^{\mathrm{e}} \int_0^y \frac{1}{x+y}\, dxdy = \int_1^{\mathrm{e}} dy \int_0^y (x+y)^{-1}\, dx = \int_1^{\mathrm{e}} dy \int_0^y u^{-1}\, du =$$
$$\int_1^{\mathrm{e}} dy \int_0^y \left[\ln|u|\right]_0^y = \int_1^{\mathrm{e}} dy \int_0^y \left[\ln|x+y|\right]_0^y = \int_1^{\mathrm{e}} dy \int_0^y (\ln|y+y| - \ln|0+y|) =$$
$$\int_1^{\mathrm{e}} dy \int_0^y (\ln|2y| - \ln|y|) = \int_1^{\mathrm{e}} dy \int_0^y (\ln|2|\cancel{+\ln|y| - \ln|y|}) = \ln|2| \int_1^{\mathrm{e}} dy =$$
$$\ln|2|[y]_1^{\mathrm{e}} = \ln|2|(\mathrm{e}-1)$$

$$u = x+y;\ du = (1+0)dx = dx$$

6 Cálculo de área - Aula 8

1. Exercício

Figura 13: Integrais duplas - Aula 8 - Exercício I

img/v08_a08_e01.png

$$
a=\int_{-1}^{0}dx\int_{0}^{x^2+1}dy+\int_{-1}^{0}dx\int_{-1}^{0}dy+\int_{0}^{y^2}dx\int_{-1}^{0}dy+\int_{0}^{1}dx\int_{\sqrt{x}}^{x^2+1}=
$$

$$
\int_{-1}^{0}dx\left(\int_{0}^{x^2+1}dy+\int_{-1}^{0}dy\right)+\int_{0}^{y^2}dx\int_{-1}^{0}dy+\int_{0}^{1}dx\int_{\sqrt{x}}^{x^2+1}=
$$

$$
\int_{-1}^{0}dx\left([y]_0^{x^2+1}+[y]_{-1}^{0}\right)+\int_{-1}^{0}dy\,[x]_0^{y^2}+\int_{0}^{1}dx\,[y]_{\sqrt{x}}^{x^2+1}=
$$

$$
\int_{-1}^{0}dx\,\left(x^2+1+1\right)+\int_{-1}^{0}dy\,y^2+\int_{0}^{1}dx\,\left(x^2+1-\sqrt{x}\right)=
$$

$$
\int_{-1}^{0}\left(x^2+2\right)dx+\int_{-1}^{0}y^2\,dy+\int_{0}^{1}\left(x^2-x^{\frac{1}{2}}+1\right)dx=
$$

$$
\left[\frac{x^3}{3}+2x\right]_{-1}^{0}+\left[\frac{y^3}{3}\right]_{-1}^{0}+\left[\frac{x^3}{3}-\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+x\right]_{0}^{1}=
$$

$$
\left[\frac{x^3+6x}{3}\right]_{-1}^{0}+\frac{1}{3}\left[y^3\right]_{-1}^{0}+\left[\frac{x^3}{3}-\frac{2\sqrt{x^3}}{3}+x\right]_{0}^{1}=
$$

$$
\frac{1}{3}\left[x\left(x^2+6\right)\right]_{-1}^{0}+\frac{1}{3}\left[\cancel{0^3}-(-1)^3\right]+\left[\frac{x^3-2\sqrt{x^3}+3x}{3}\right]_{0}^{1}=
$$

$$
\frac{1}{3}\left[\cancel{0\left(0^2+6\right)}-(-1)\left((-1)^2+6\right)\right]+\frac{1}{3}+\frac{1}{3}\left[x^3-2\sqrt{x^3}+3x\right]_{0}^{1}=
$$

$$
\frac{7}{3}+\frac{1}{3}+\frac{1}{3}\left[1^3-2\sqrt{1^3}+3\cdot 1\cancel{-\left(0^3-2\sqrt{0^3}+3\cdot 0\right)}\right]=\frac{7}{3}+\frac{1}{3}+\frac{2}{3}=
$$

$$
\frac{7+1+2}{3}=\frac{10}{3}=3,\overline{3}
$$

$$
a=\int_{-1}^{1}dx\int_{1}^{x^2+1}dy+\int_{-1}^{y^2}dx\int_{-1}^{1}dy=\int_{-1}^{1}dx\,[y]_1^{x^2+1}+\int_{-1}^{1}dy\,[x]_{-1}^{y^2}=
$$

$$
\int_{-1}^{1}dx\,\left(x^2\cancel{+1-1}\right)+\int_{-1}^{1}dy\,\left(y^2+1\right)=\left[\frac{x^3}{3}\right]_{-1}^{1}+\left[\frac{y^3}{3}+y\right]_{-1}^{1}=
$$

$$
\frac{1}{3}\left[x^3\right]_{-1}^{1}+\frac{1}{3}\left[y\left(y^2+3\right)\right]_{-1}^{1}=
$$

$$
\frac{1}{3}\left(\left[1^3-(-1)^3\right]+\left[1\left(1^2+3\right)-(-1)\left((-1)^2+3\right)\right]\right)\frac{1}{3}(2+4+4)=\frac{10}{3}=3,\overline{3}
$$

7 Cálculo de volume

7.1 Aula 9

1. Exercício

   Esboçe a região de integração e o sólido cujo volume é dado pela integral abaixo:

$$\int_0^1 \int_0^1 (4 - x - 2y)\ dxdy$$

Figura 14: Integrais duplas - Aula 9 - Exercício I

img/v09_a09_e01.png

$$v = \int_0^1 \int_0^1 (4 - x - 2y)\ dxdy = \int_0^1 dx \left(4 \int_0^1 dy - x \int_0^1 dy - 2 \int_0^1 y\, dy \right) =$$

$$4 \int_0^1 dx \int_0^1 dy - \int_0^1 x\, dx \int_0^1 dy - 2 \int_0^1 dx \int_0^1 y\, dy =$$

$$4[x]_0^1[y]_0^1 - \left[\frac{x^2}{2}\right]_0^1 [y]_0^1 - 2[x]_0^1 \left[\frac{y^2}{2}\right]_0^1 = 4 - \frac{1}{2} - \cancel{2}\frac{1}{\cancel{2}} = \frac{8 - 1 - 2}{2} = \frac{5}{2} = 2,5$$

7.2 Aula 10

1. Exercício

   Calcule o volume do sólido limitado pelos planos:

$$x = 0,\ y = 0,\ z = 0 \text{ e } 6x + 2y + 3z = 6$$

$$P_1 = (0, 0, 0)$$

$$6x = -2y - 3z + 6 \Rightarrow x = \frac{-2y - 3z + 6}{6} = \frac{-2 \cdot 0 - 3 \cdot 0 + 6}{6} = \frac{6}{6} = 1 \Rightarrow P_2 = (1, 0, 0)$$

Figura 15: Integrais duplas - Aula 10 - Exercício I

img/v10_a10_e01.png

$$2y = -6x - 3z + 6 \Rightarrow y = \frac{-6x-3z+6}{2} = \frac{-6\cdot 0 - 3\cdot 0 + 6}{2} = \frac{6}{2} = 3 \Rightarrow P_3 = (0,3,0)$$
$$3z = -6x - 2y + 6 \Rightarrow z = \frac{-6x-2y+6}{3} = \frac{-6\cdot 0 - 2\cdot 0 + 6}{3} = \frac{6}{3} = 2 \Rightarrow P_4 = (0,0,2)$$
$$x = 0, x = 1$$
$$y = 0,\ y = \frac{-6x-3z+6}{2} = \frac{-6x-3\cdot 0+6}{2} = -3x+3$$
$$z = \frac{-6x-2y+6}{3} = -2x - \frac{2y}{3} + 2$$

$$v = \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dxdy = \int_0^1 dx \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dy =$$
$$\int_0^1 dx \left[-2xy - \frac{\cancel{2}}{3}\frac{y^2}{\cancel{2}} + 2y\right]_0^{-3x+3} = \int_0^1 dx\, \frac{1}{3}\left[-6xy - y^2 + 6y\right]_0^{-3x+3} =$$
$$\frac{1}{3}\int_0^1 dx\ \left[-y(6x+y-6)\right]_0^{-3x+3} =$$
$$\frac{1}{3}\int_0^1 dx\ \left[-(-3x+3)(6x+(-3x+3)-6)\cancel{+0(6x+0-6)}\right] =$$
$$\frac{1}{3}\int_0^1 dx\ \left[(3x-3)(3x-3)\right] = \frac{1}{3}\int_0^1 \left(9x^2 - 18x + 9\right) dx = \frac{1}{3}\left[9\frac{x^3}{3} - 18\frac{x^2}{2} + 9x\right]_0^1 =$$
$$\frac{1}{3}\left[3x^3 - 9x^2 + 9x\right]_0^1 = \frac{1}{\cancel{3}}\left[\cancel{3}x\left(x^2 - 3x + 3\right)\right]_0^1 =$$
$$\left[1\left(1^2 \cancel{- 3\cdot 1 + 3}\right) \cancel{- 0\left(0^2 - 3\cdot 0 + 3\right)}\right] = 1$$

8 Coordenadas polares

8.1 Aula 1

1. Exercício

Figura 16: Coordenadas polares - Aula 01 - Exercício I

img/v11_a01_e01.png

Calcule a área do circulo de raio igual a dois

$$r = 2 \Rightarrow a = \pi r^2 = 2^2\pi = 4\pi$$
$$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 2^2 \Rightarrow x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$$
$$R = \left\{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2,\ -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\right\}$$

$$a = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{-2}^{2} dx \left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) = 2\int_{-2}^{2} \sqrt{4-x^2}\, dx =$$

$$2\int_{-2}^{2} \sqrt{4-(2\operatorname{sen}(\alpha))^2}\, 2\cos(\alpha)\, d\alpha = 4\int_{-2}^{2} \sqrt{4-4\operatorname{sen}^2(\alpha)}\, \cos(\alpha)\, d\alpha =$$

$$4\int_{-2}^{2} \sqrt{4-4\left(1-\cos^2(\alpha)\right)}\, \cos(\alpha)\, d\alpha = 4\int_{-2}^{2} \sqrt{4-\left(4-4\cos^2(\alpha)\right)}\, \cos(\alpha)\, d\alpha =$$

$$4\int_{-2}^{2} \sqrt{\cancel{4-4}+4\cos^2(\alpha)}\, \cos(\alpha)\, d\alpha = 4\int_{-2}^{2} 2\cos(\alpha)\cos(\alpha)\, d\alpha =$$

$$8\int_{-2}^{2} \cos^2(\alpha)\, d\alpha = 8\int_{-2}^{2} \left(\frac{1+\cos(2\alpha)}{2}\right) d\alpha = 8\int_{-2}^{2} \left(\frac{1}{2} + \frac{\cos(2\alpha)}{2}\right) d\alpha =$$

$$4\int_{-2}^{2} d\alpha + 4\int_{-2}^{2} \cos(2\alpha)\, d\alpha = 4\int_{-2}^{2} d\alpha + 4\int_{-2}^{2} \cos(u)\, \frac{du}{2} =$$

$$4\int_{-2}^{2} d\alpha + 2\int_{-2}^{2} \cos(u)\, du = \left[4\alpha + 2\operatorname{sen}(u)\right]_{-2}^{2} = \left[4\alpha + 2\operatorname{sen}(2\alpha)\right]_{-2}^{2} =$$

$$\left[4\alpha + 4\operatorname{sen}(\alpha)\cos(\alpha)\right]_{-2}^{2} = \left[4\left(\alpha + \operatorname{sen}(\alpha)\cos(\alpha)\right)\right]_{-2}^{2} =$$

$$\left[4\left(\operatorname{arcsen}\left(\frac{x}{2}\right) + \frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right)\right]_{-2}^{2} = \left[4\left(\operatorname{arcsen}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4}\right)\right]_{-2}^{2} =$$

$$4\left(\operatorname{arcsen}\left(\frac{2}{2}\right) + \frac{2\sqrt{4-2^2}}{4}\right) - 4\left(\operatorname{arcsen}\left(\frac{(-2)}{2}\right) + \frac{(-2)\sqrt{4-(-2)^2}}{4}\right) =$$

$$4\operatorname{arcsen}(1) - 4\operatorname{arcsen}(-1) = 4(\operatorname{arcsen}(1) - \operatorname{arcsen}(-1)) = 4\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 4\left(\frac{\cancel{2}\pi}{\cancel{2}}\right) = 4\pi$$

$$x = 2\operatorname{sen}(\alpha);\ dx = 2\cos(\alpha)\, d\alpha$$

$$u = 2\alpha;\ \frac{du}{2} = d\alpha$$

$$\operatorname{sen}(\alpha) = \frac{co}{h} = \frac{x}{2} \Rightarrow \alpha = \operatorname{arcsen}\left(\frac{x}{2}\right)$$

$$h^2 = co^2 + ca^2 \Rightarrow 2^2 = x^2 + ca^2 \Rightarrow ca = \sqrt{4-x^2}$$

$$\cos(\alpha) = \frac{ca}{h} = \frac{\sqrt{4-x^2}}{2}$$

$$R = \left\{(r,\theta) \in \mathbb{R}^2 \,|\, 0 \le r \le 2,\ 0 \le \theta \le 2\pi\right\}$$

$$a = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{0}^{2}\int_{0}^{2\pi} r\, dr d\theta = \int_{0}^{2} r\, dr \int_{0}^{2\pi} d\theta = \left[\frac{r^2}{2}\right]_{0}^{2} [\theta]_{0}^{2\pi} =$$

$$\frac{1}{2}\left[2^2 - 0^2\right][2\pi - 0] = \frac{4}{\cancel{2}}\cancel{2}\pi = 4\pi$$

2. Exercício

$$\iint_R \frac{da}{1+x^2+y^2}$$

Figura 17: Coordenadas polares - Aula 01 - Exercício II

img/v11_a01_e02.png

$$R = \left\{ (r,\theta) \in \mathbb{R}^2 \,|\, 0 \leq r \leq 2,\, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{2} \right\}$$

$$v = \iint_R \frac{da}{1+x^2+y^2} = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\,dr d\theta}{1+r^2} = \int_0^2 \frac{r\,dr}{1+r^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} d\theta =$$
$$\int_0^2 \left(1+r^2\right)^{-1} r\,dr \left[\theta\right]_{\frac{\pi}{4}}^{\frac{3\pi}{2}} = \int_0^2 \left(1+r^2\right)^{-1} r\,dr \left(\frac{3\pi}{2} - \frac{\pi}{4}\right) =$$
$$\int_0^2 \left(1+r^2\right)^{-1} r\,dr \left(\frac{6\pi - \pi}{4}\right) = \frac{5\pi}{4} \int_0^2 \left(1+r^2\right)^{-1} r\,dr = \frac{5\pi}{4} \int_0^2 u^{-1} \frac{du}{2} =$$
$$\frac{5\pi}{8} \int_0^2 u^{-1} du = \frac{5\pi}{8} \left[ln|u|\right]_0^2 = \frac{5\pi}{8} \left[ln|1+r^2|\right]_0^2 = \frac{5\pi}{8} \left[ln|1+2^2| - ln|1+0^2|\right] =$$
$$\frac{5\pi}{8} \left[ln|5| - ln|1|\right] = \frac{5\pi ln|5|}{8}$$

$$u = 1 + r^2 \Rightarrow \frac{du}{2} = r\,dr$$
$$\mathrm{e}^x = 1 = \mathrm{e}^0 \Rightarrow x = 0$$

8.2 Aula 2

1. Exercício

$$\iint_R \mathrm{e}^{x^2+y^2}\, dxdy$$

R, região entre as curvas abaixo:

$$x^2 + y^2 = 4$$

$$x^2 + y^2 = 9$$

Figura 18: Coordenadas polares - Aula 02 - Exercício I

`img/v12_a02_e01.png`

$$x^2 + y^2 = r^2 \Rightarrow \mathrm{e}^{x^2+y^2} = \mathrm{e}^{r^2}$$
$$da = dxdy = r\, drd\theta$$
$$R = \left\{(r,\theta) \in \mathbb{R}^2 \,|\, 2 \le r \le 3,\ 0 \le \theta \le 2\pi\right\}$$

$$v = \iint_R \mathrm{e}^{x^2+y^2}\, dxdy = \int_2^3 \int_0^{2\pi} \mathrm{e}^{r^2}\, r\, drd\theta = \int_2^3 \mathrm{e}^{r^2}\, r\, dr \int_0^{2\pi} d\theta = \int_2^3 \mathrm{e}^{u}\, \frac{du}{2} \int_0^{2\pi} d\theta =$$
$$\frac{1}{2}\int_2^3 \mathrm{e}^u\, du \int_0^{2\pi} d\theta = \frac{1}{2}\left[\mathrm{e}^u\right]_2^3 \left[\theta\right]_0^{2\pi} = \frac{1}{\cancel{2}}\left[\mathrm{e}^{r^2}\right]_2^3 \cancel{2}\pi = \left(\mathrm{e}^{3^2} - \mathrm{e}^{2^2}\right)\pi = \pi\left(\mathrm{e}^9 - \mathrm{e}^4\right)$$

$$u = r^2 \Rightarrow \frac{du}{2} = r\, dr$$

2. Exercício

$$\iint_R \sqrt{x^2+y^2}\, dxdy$$

Figura 19: Coordenadas polares - Aula 02 - Exercício II

img/v12_a02_e02.png

R, região cujo o contorno é:

$$x^2 + y^2 = 4$$

$$x^2 + y^2 = r^2 \Rightarrow \sqrt{x^2 + y^2} = \sqrt{r^2} = r$$
$$da = dxdy = r\,drd\theta$$
$$R = \left\{(r, \theta) \in \mathbb{R}^2 \,|\, 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\right\}$$

$$v = \iint_R \sqrt{x^2 + y^2}\,dxdy = \int_0^2 \int_0^{2\pi} r^2\,drd\theta = \int_0^2 r^2\,dr \int_0^{2\pi} d\theta =$$
$$\left[\frac{r^3}{3}\right]_0^2 [\theta]_0^{2\pi} = \frac{2^3}{3} 2\pi = \frac{16\pi}{3}$$

8.3 Aula 3

1. Exercício

   Calcular o volume do sólido acima do plano xoy delimitado pela função abaixo.

$$xoy$$
$$z = 4 - 2x^2 - 2y^2$$

$$4 - 2x^2 - 2y^2 = 0 \Rightarrow -2x^2 - 2y^2 = -4 \Rightarrow -2\left(x^2 + y^2\right) = -4 \Rightarrow$$
$$x^2 + y^2 = \frac{-4}{-2} = 2 \Rightarrow r = \sqrt{2}$$

$$R = \left\{(r, \theta) \in \mathbb{R}^2 \,|\, 0 \leq r \leq \sqrt{2},\ 0 \leq \theta \leq 2\pi\right\}$$

Figura 20: Coordenadas polares - Aula 03 - Exercício I

img/v13_a03_e01.png

$$z = 4 - 2x^2 - 2y^2 = 4 - 2\left(x^2 + y^2\right) = 4 - 2r^2$$
$$da = dxdy = r\,drd\theta$$

$$\iint_R z\,da = \iint_R \left(4 - 2x^2 - 2y^2\right)\,dxdy = \int_0^{\sqrt{2}} \int_0^{2\pi} \left(4 - 2r^2\right) r\,drd\theta =$$
$$\int_0^{\sqrt{2}} \left(4r - 2r^3\right)\,dr \int_0^{2\pi} d\theta = \int_0^{\sqrt{2}} \left(4r - 2r^3\right)\,dr[\theta]_0^{2\pi} = 2\pi \int_0^{\sqrt{2}} \left(4r - 2r^3\right)\,dr =$$
$$8\pi \int_0^{\sqrt{2}} r\,dr - 4\pi \int_0^{\sqrt{2}} r^3\,dr = \left[\frac{8\pi r^2}{2} - \frac{\cancel{4}\pi r^4}{\cancel{4}}\right]_0^{\sqrt{2}} = \left[4\pi r^2 - \pi r^4\right]_0^{\sqrt{2}} = \left[\pi r^2\left(4 - r^2\right)\right]_0^{\sqrt{2}} =$$
$$\pi\left(\sqrt{2}\right)^2\left(4 - \left(\sqrt{2}\right)^2\right) = 2\pi(4-2) = 4\pi$$

Parte II
Integrais triplas

9 Introdução - Aula 1

1. Exercício

   Calcule a integral tripla abaixo.

$$\iiint_R 12xy^2z^3\,dv$$
$$R = \left\{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq x \leq 2,\ 0 \leq y \leq 3,\ 0 \leq z \leq 2\right\}$$

$$dv = dxdydz$$

$$\iiint_R 12xy^2z^3\,dv = \int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12xy^2z^3\,dxdydz = 12\int_{-1}^{2} x\,dx \int_{0}^{3} y^2\,dy \int_{0}^{2} z^3\,dz =$$
$$\cancel{12}\left[\frac{x^2}{2}\right]_{-1}^{2}\left[\frac{y^3}{\cancel{3}}\right]_{0}^{3}\left[\frac{z^4}{\cancel{4}}\right]_{0}^{2} = \frac{1}{2}\left[x^2\right]_{-1}^{2}\left[y^3\right]_{0}^{3}\left[z^4\right]_{0}^{2} = \frac{1}{2}\left(2^2-(-1)^2\right)3^3 2^4 = \frac{1}{2}3\cdot 27\cdot 16 = 648$$

2. Exercício

   Observe a integral e preencha os retângulos abaixo.

$$\int_{1}^{5}\int_{2}^{4}\int_{3}^{6} f(x,y,z)\,dxdzdy$$
$$[3] \leq x \leq [6]$$
$$[1] \leq y \leq [5]$$
$$[2] \leq z \leq [4]$$

3. Exercício

$$\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}\left(x^2+y^2+z^2\right)\,dxdydz = \int_{-1}^{1} dz \int_{0}^{2} dy \int_{0}^{1}\left(x^2+y^2+z^2\right)\,dx =$$
$$\int_{-1}^{1} dz \int_{0}^{2} dy \left(\int_{0}^{1} x^2\,dx + y^2\int_{0}^{1} dx + z^2\int_{0}^{1} dx\right) =$$
$$\int_{-1}^{1} dz \int_{0}^{2} dy \int_{0}^{1} x^2\,dx + \int_{-1}^{1} dz \int_{0}^{2} y^2\,dy \int_{0}^{1} dx + \int_{-1}^{1} z^2\,dz \int_{0}^{2} dy \int_{0}^{1} dx =$$
$$[z]_{-1}^{1}[y]_{0}^{2}\left[\frac{x^3}{3}\right]_{0}^{1} + [z]_{-1}^{1}\left[\frac{y^3}{3}\right]_{0}^{2}[x]_{0}^{1} + \left[\frac{z^3}{3}\right]_{-1}^{1}[y]_{0}^{2}[x]_{0}^{1} =$$
$$[z]_{-1}^{1}[y]_{0}^{2}\frac{1}{3}\left[x^3\right]_{0}^{1} + [z]_{-1}^{1}\frac{1}{3}\left[y^3\right]_{0}^{2}[x]_{0}^{1} + \frac{1}{3}\left[z^3\right]_{-1}^{1}[y]_{0}^{2}[x]_{0}^{1} =$$
$$\frac{1}{3}\left([1+1]2\cdot 1^3 + [1+1]2^3\cdot 1 + \left[1^3-(-1)^3\right]2\cdot 1\right) = \frac{1}{3}(4+16+4) = \frac{24}{3} = 8$$

4. Exercício

$$\int_0^2\int_{-1}^{y^2}\int_{-1}^{z} yz\,dxdzdy = \int_0^2\int_{-1}^{y^2}\left(yz\int_{-1}^{z} dx\right)dzdy = \int_0^2\int_{-1}^{y^2}[yzx]_{-1}^{z}\,dzdy =$$
$$\int_0^2\int_{-1}^{y^2}[yz^2+yz]\,dzdy = \int_0^2\left(y\int_{-1}^{y^2} z^2\,dz + y\int_{-1}^{y^2} z\,dz\right)dy = \int_0^2\left[y\frac{z^3}{3}+y\frac{z^2}{2}\right]_{-1}^{y^2}dy =$$
$$\int_0^2\left[\frac{2yz^3+3yz^2}{6}\right]_{-1}^{y^2}dy = \frac{1}{6}\int_0^2\left[yz^2\,(2z+3)\right]_{-1}^{y^2}dy =$$
$$\frac{1}{6}\int_0^2\left[y\left(y^2\right)^2\left(2y^2+3\right)-y(-1)^2\,(2(-1)+3)\right]dy = \frac{1}{6}\int_0^2\left[y^5\left(2y^2+3\right)-y\right]dy =$$
$$\frac{1}{6}\int_0^2\left(2y^7+3y^5-y\right)dy = \frac{1}{6}\left[\frac{2y^8}{8}+\frac{3y^6}{6}-\frac{y^2}{2}\right]_0^2 = \frac{1}{6}\left[\frac{y^8}{4}+\frac{y^6}{2}-\frac{y^2}{2}\right]_0^2 =$$
$$\frac{1}{6}\left[\frac{y^8+2y^6-2y^2}{4}\right]_0^2 = \frac{1}{24}\left[y^2\left(y^6+2y^4-2\right)\right]_0^2 = \frac{1}{24}\left[2^2\left(2^6+2\cdot 2^4-2\right)\right] =$$
$$\frac{1}{24}\left[4\,(64+32-2)\right] = \frac{94}{6} = \frac{47}{3}$$

10 Cálculo de integrais triplas - Aula 2

1. Exercício

$$\iiint_R xy\,\mathrm{sen}(yz)\,dv$$
$$R = \left\{(x,y,z)\in\mathbb{R}^3 \,|\, 0\le x\le\pi,\ 0\le y\le 1,\ 0\le z\le\frac{\pi}{6}\right\}$$

$$\iiint_R xy\,\mathrm{sen}(yz)\,dv = \int_0^{\pi}\int_0^1\int_0^{\frac{\pi}{6}} xy\,\mathrm{sen}(yz)\,dzdydx =$$

$$\int_0^{\pi}\int_0^1\left(x\int_0^{\frac{\pi}{6}}\mathrm{sen}(yz)y\,dz\right)dydx = \int_0^{\pi}\int_0^1\left(x\int_0^{\frac{\pi}{6}}\mathrm{sen}(u)\,du\right)dydx =$$

$$\int_0^{\pi}\int_0^1\left[-x\cos(u)\right]_0^{\frac{\pi}{6}}\,dydx = \int_0^{\pi}\int_0^1\left[-x\cos(yz)\right]_0^{\frac{\pi}{6}}\,dydx =$$

$$\int_0^{\pi}\int_0^1\left(-x\cos\left(\frac{y\pi}{6}\right)+x\cos(0)\right)dydx = \int_0^{\pi}\int_0^1\left(-x\cos\left(\frac{y\pi}{6}\right)+x\right)dydx =$$

$$\int_0^{\pi}\left(-x\int_0^1\cos\left(\frac{y\pi}{6}\right)dy + x\int_0^1 dy\right)dx = \int_0^{\pi}\left(-x\int_0^1\cos(v)\frac{6\,dv}{\pi}+x\int_0^1 dy\right)dx =$$

$$\int_0^{\pi}\left(\frac{-6x}{\pi}\int_0^1\cos(v)\,dv + x\int_0^1 dy\right)dx = \int_0^{\pi}\left[\frac{-6x\,\mathrm{sen}(v)}{\pi}+xy\right]_0^1 dx =$$

$$\int_0^{\pi}\left[\frac{-6x\,\mathrm{sen}\left(\frac{y\pi}{6}\right)+xy\pi}{\pi}\right]_0^1 dx = \frac{1}{\pi}\int_0^{\pi}\left[-x\left(6\,\mathrm{sen}\left(\frac{y\pi}{6}\right)-y\pi\right)\right]_0^1 dx =$$

$$\frac{1}{\pi}\int_0^{\pi}\left[-x\left(6\,\mathrm{sen}\left(\frac{\pi}{6}\right)-\pi\right)\cancel{+x(6\,\mathrm{sen}(0)-0)}\right]dx = \frac{1}{\pi}\int_0^{\pi}\left(-6x\,\mathrm{sen}\left(\frac{\pi}{6}\right)+x\pi\right)dx =$$

$$\frac{-6\,\mathrm{sen}\left(\frac{\pi}{6}\right)}{\pi}\int_0^{\pi}x\,dx+\pi\int_0^{\pi}x\,dx = \left[\frac{-6\,\mathrm{sen}\left(\frac{\pi}{6}\right)}{\pi}\frac{x^2}{2}+\frac{\pi x^2}{2}\right]_0^{\pi} =$$

$$\left[\frac{-3x^2\,\mathrm{sen}\left(\frac{\pi}{6}\right)}{\pi}+\frac{\pi x^2}{2}\right]_0^{\pi} = \left[\frac{-6x^2\,\mathrm{sen}\left(\frac{\pi}{6}\right)+\pi^2x^2}{2\pi}\right]_0^{\pi} = \frac{1}{2\pi}\left[-x^2\left(6\,\mathrm{sen}\left(\frac{\pi}{6}\right)-\pi^2\right)\right]_0^{\pi} =$$

$$\frac{1}{2\pi}\left[-\pi^2\left(6\,\mathrm{sen}\left(\frac{\pi}{6}\right)-\pi^2\right)\right] = \frac{-\pi\left(6\,\mathrm{sen}\left(\frac{\pi}{6}\right)-\pi^2\right)}{2} = \frac{-\pi\left(6\frac{1}{2}-\pi^2\right)}{2} =$$

$$\frac{-\pi\left(3-\pi^2\right)}{2} = \frac{\pi^3-3\pi}{2}$$

$$u = yz \Rightarrow du = y\,dz$$

$$v = \frac{y\pi}{6} \Rightarrow \frac{6\,dv}{\pi} = dy$$

2. Exercício

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y z\, dxdzdy = \int_0^1 \int_0^{\sqrt{1-y^2}} \left(z \int_0^y dx \right) dzdy = \int_0^1 \int_0^{\sqrt{1-y^2}} [zx]_0^y \, dzdy =$$

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (zy)\ dzdy = \int_0^1 \left(y \int_0^{\sqrt{1-y^2}} z\, dz \right) dy = \int_0^1 \left[\frac{yz^2}{2} \right]_0^{\sqrt{1-y^2}} dy =$$

$$\int_0^1 \left(\frac{y\left(\sqrt{1-y^2}\right)^2}{2} \right) dy = \int_0^1 \frac{y-y^3}{2}\, dy = \frac{1}{2}\int_0^1 \left(y - y^3\right)\, dy = \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 =$$

$$\frac{1}{2}\left[\frac{2y^2 - y^4}{4}\right]_0^1 = \frac{1}{8}\left[y^2\left(2-y^2\right)\right]_0^1 = \frac{1}{8}\left[1^2\left(2-1^2\right)\right] = \frac{1}{8}$$

3. Exercício

$$\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^x xy\, dydxdz = \int_0^3 \int_0^{\sqrt{9-z^2}} \left(x \int_0^x y\, dy \right) dxdz =$$

$$\int_0^3 \int_0^{\sqrt{9-z^2}} \left[\frac{xy^2}{2}\right]_0^x = \frac{1}{2}\int_0^3 \int_0^{\sqrt{9-z^2}} x^3\, dxdz =$$

$$\frac{1}{2}\int_0^3 \left[\frac{x^4}{4}\right]_0^{\sqrt{9-z^2}} dz = \frac{1}{2}\int_0^3 \left[\frac{\left(\sqrt{9-z^2}\right)^4}{4}\right] dz = \frac{1}{8}\int_0^3 \left[\left(9-z^2\right)^2\right] dz =$$

$$\frac{1}{8}\int_0^3 \left(81 - 18z^2 + z^4\right)\, dz = \frac{1}{8}\left[81z - \frac{18z^3}{3} + \frac{z^5}{5}\right]_0^3 = \frac{1}{8}\left[\frac{1215z - 90z^3 + 3z^5}{15}\right]_0^3 =$$

$$\frac{1}{120}\left[3z\left(405 - 30z^2 + z^4\right)\right]_0^3 = \frac{1}{40}\left[z\left(405 - 30z^2 + z^4\right)\right]_0^3 = \frac{1}{40}\left[3\left(405 - 30\cdot 3^2 + 3^4\right)\right] =$$

$$\frac{1}{40}\left[3\left(405 - 270 + 81\right)\right] = \frac{648}{40} = \frac{81}{5}$$

11 Cálculo do volume de um sólido - Aula 3

1. Exercício

   Use integral tripla para encontrar o volume do sólido no primeiro octante limitado pelos planos coordenados e pelo plano dado pela equação abaixo.

$$3x + 6y + 4z = 12$$

Figura 21: Integrais triplas - Aula 03 - Exercício I

img/v16_a03_e01.png

$$P_0(0,0,0)$$
$$x=0,\ y=0;\ 4z=12 \Rightarrow z=\frac{12}{4}=3;\ P_1(0,0,3)$$
$$x=0,\ z=0;\ 6y=12 \Rightarrow y=\frac{12}{6}=2;\ P_2(0,2,0)$$
$$y=0,\ z=0;\ 3x=12 \Rightarrow x=\frac{12}{3}=4;\ P_3(4,0,0)$$

$$0 \le x \le 4$$
$$3x+6y=12 \Rightarrow x+2y=4 \Rightarrow y=\frac{4-x}{2}=2-\frac{x}{2};\ 0 \le y \le \left(2-\frac{x}{2}\right)$$
$$3x+6y+4z=12 \Rightarrow z=\frac{12-3x-6y}{4}=3-\frac{3x}{4}-\frac{3y}{2};\ 0 \le z \le \left(3-\frac{3x}{4}-\frac{3y}{2}\right)$$

$$\int_0^4 dx \int_0^{2-\frac{x}{2}} dy \int_0^{3-\frac{3x}{4}-\frac{3y}{2}} dz = \int_0^4 dx \int_0^{2-\frac{x}{2}} dy \, [z]_0^{3-\frac{3x}{4}-\frac{3y}{2}} =$$

$$\int_0^4 dx \int_0^{2-\frac{x}{2}} \left(3 - \frac{3x}{4} - \frac{3y}{2}\right) dy = \int_0^4 dx \left[3y - \frac{3xy}{4} - \frac{3y^2}{4}\right]_0^{2-\frac{x}{2}} =$$

$$\int_0^4 \left(3\left(2-\frac{x}{2}\right) - \frac{3x\left(2-\frac{x}{2}\right)}{4} - \frac{3\left(2-\frac{x}{2}\right)^2}{4}\right) dx =$$

$$\int_0^4 \left(6 - \frac{3x}{2} - \frac{\left(6x - \frac{3x^2}{2}\right)}{4} - \frac{3\left(4 - 2x + \frac{x^2}{4}\right)}{4}\right) dx =$$

$$\int_0^4 \left(6 - \frac{3x}{2} - \frac{\left(\frac{12x - 3x^2}{2}\right)}{4} - \frac{\left(12 - 6x + \frac{3x^2}{4}\right)}{4}\right) dx =$$

$$\int_0^4 \left(6 - \frac{3x}{2} - \left(\frac{12x - 3x^2}{8}\right) - \frac{\left(\frac{48 - 24x + 3x^2}{4}\right)}{4}\right) dx =$$

$$\int_0^4 \left(6 - \frac{3x}{2} - \left(\frac{3x}{2} - \frac{3x^2}{8}\right) - \left(\frac{48 - 24x + 3x^2}{16}\right)\right) dx =$$

$$\int_0^4 \left(6 - \frac{3x}{2} - \frac{3x}{2} + \frac{3x^2}{8} - \left(3 - \frac{3x}{2} + \frac{3x^2}{16}\right)\right) dx =$$

$$\int_0^4 \left(6 - \cancel{\frac{3x}{2}} - \frac{3x}{2} + \frac{3x^2}{8} - 3 + \cancel{\frac{3x}{2}} - \frac{3x^2}{16}\right) dx = \int_0^4 \left(3 - \frac{3x}{2} + \frac{3x^2}{16}\right) dx =$$

$$\left[3x - \frac{3x^2}{4} + \frac{3x^3}{48}\right]_0^4 = 3 \cdot 4 - \frac{3 \cdot 4^2}{4} + \frac{3 \cdot 4^3}{48} = \cancel{12 - 12} + 4 = 4$$

12 Esboço de um sólido - Aula 4

1. Exercício

   Faça o esboço do sólido cujo volume é dado pela integral abaixo.

$$v = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{y+1} dzdydx = \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_0^{y+1} dz$$

Figura 22: Integrais triplas - Aula 04 - Exercício I

`img/v17_a04_e01.png`

$$v = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{0}^{y+1} dz = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, [z]_0^{y+1} = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y+1)\, dy =$$

$$\int_{-1}^{1} dx \left[\frac{y^2}{2} + y\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} =$$

$$\int_{-1}^{1} \left[\frac{\left(\sqrt{1-x^2}\right)^2}{2} + \sqrt{1-x^2} - \left(\frac{\left(-\sqrt{1-x^2}\right)^2}{2} - \sqrt{1-x^2}\right)\right] dx =$$

$$\int_{-1}^{1} \left(\frac{1-x^2}{2} + \sqrt{1-x^2} - \frac{(1-x^2)}{2} + \sqrt{1-x^2}\right) dx =$$

$$\int_{-1}^{1} \left(\cancel{\frac{1}{2}} - \cancel{\frac{x^2}{2}} + \sqrt{1-x^2} - \cancel{\frac{1}{2}} + \cancel{\frac{x^2}{2}} + \sqrt{1-x^2}\right) dx = \int_{-1}^{1} \left(2\sqrt{1-x^2}\right) dx =$$

$$2\int_{-1}^{1} \sqrt{1-\operatorname{sen}^2(\theta)}\cos(\theta)\, d\theta = 2\int_{-1}^{1} \sqrt{1-(1-\cos^2(\theta))}\cos(\theta)\, d\theta =$$

$$2\int_{-1}^{1} \sqrt{\cos^2(\theta)}\cos(\theta)\, d\theta = 2\int_{-1}^{1} \cos^2(\theta)\, d\theta = 2\int_{-1}^{1} \left(\frac{1+\cos(2\theta)}{2}\right) d\theta =$$

$$2\int_{-1}^{1} \left(\frac{1}{2} + \frac{\cos(2\theta)}{2}\right) d\theta = \int_{-1}^{1} d\theta + \int_{-1}^{1} \cos(2\theta)\, d\theta = \int_{-1}^{1} d\theta + \int_{-1}^{1} \cos(u)\frac{du}{2} =$$

$$\int_{-1}^{1} d\theta + \frac{1}{2}\int_{-1}^{1} \cos(u)\, du = \left[\theta + \frac{\operatorname{sen}(u)}{2}\right]_{-1}^{1} = \left[\theta + \frac{\operatorname{sen}(2\theta)}{2}\right]_{-1}^{1} =$$

$$\left[\theta + \frac{\cancel{2}\operatorname{sen}(\theta)\cos(\theta)}{\cancel{2}}\right]_{-1}^{1} = [\theta + \operatorname{sen}(\theta)\cos(\theta)]_{-1}^{1} = \left[\operatorname{arcsen}(x) + x\sqrt{1-x^2}\right]_{-1}^{1} =$$

$$\left[\operatorname{arcsen}(1)\cancel{+1\sqrt{1-1^2}} - \left(\operatorname{arcsen}(-1)\cancel{+(-1)\sqrt{1-(-1)^2}}\right)\right] = [\operatorname{arcsen}(1) - \operatorname{arcsen}(-1)] =$$

$$\frac{\pi}{2} + \frac{\pi}{2} = \frac{\pi+\pi}{2} = \frac{2\pi}{2} = \pi$$

$$x = \operatorname{sen}(\theta) \Rightarrow dx = \cos(\theta)\, d\theta$$

$$u = 2\theta \Rightarrow \frac{du}{2} = d\theta$$

$$\text{sen}(\theta) = \frac{co}{h} = \frac{x}{1} = x;\ \theta = \text{arcsen}(x)$$

$$1 = x^2 + ca \Rightarrow ca = \sqrt{1-x^2}$$

$$\cos(\theta) = \frac{ca}{h} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

2. Exercício

   Faça o esboço do sólido cujo volume é dado pela integral abaixo.

$$v = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^2 dydzdx = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dz \int_0^2 dy$$

Figura 23: Integrais triplas - Aula 04 - Exercício II

img/v17_a04_e02.png

$$v = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dz\, [y]_0^2 = 2\int_0^1 dx \int_0^{\sqrt{1-x^2}} dz = 2\int_0^1 dx\, [z]_0^{\sqrt{1-x^2}} = 2\int_0^1 \sqrt{1-x^2}\, dx =$$
$$2\int_0^1 \sqrt{1-\text{sen}^2(\theta)}\cos(\theta)\, d\theta = 2\int_0^1 \sqrt{1-(1-\cos^2(\theta))}\cos(\theta)\, d\theta =$$
$$2\int_0^1 \sqrt{\cos^2(\theta)}\cos(\theta)\, d\theta = 2\int_0^1 \cos^2(\theta)\, d\theta = 2\int_0^1 \left(\frac{1+\cos(2\theta)}{2}\right) d\theta =$$
$$2\int_0^1 \left(\frac{1}{2} + \frac{\cos(2\theta)}{2}\right) d\theta = \int_0^1 d\theta + \int_0^1 \cos(2\theta)\, d\theta = \int_0^1 d\theta + \int_0^1 \cos(u)\frac{du}{2} =$$
$$\int_0^1 d\theta + \frac{1}{2}\int_0^1 \cos(u)\, du = \left[\theta + \frac{\text{sen}(u)}{2}\right]_0^1 = \left[\theta + \frac{\text{sen}(2\theta)}{2}\right]_0^1 = \left[\theta + \frac{\cancel{2}\,\text{sen}(\theta)\cos(\theta)}{\cancel{2}}\right]_0^1 =$$
$$[\theta + \text{sen}(\theta)\cos(\theta)]_0^1 = \left[\text{arcsen}(x) + x\sqrt{1-x^2}\right]_0^1 =$$
$$\text{arcsen}(1)\cancel{+1\sqrt{1-1^2}} - \left(\text{arcsen}(0)\cancel{+0\sqrt{1-0^2}}\right) = \text{arcsen}(1)\cancel{-\text{arcsen}(0)} = \frac{\pi}{2}$$

$$x = \text{sen}(\theta) \Rightarrow dx = \cos(\theta)\, d\theta$$
$$u = 2\theta \Rightarrow \frac{du}{2} = d\theta$$
$$\text{sen}(\theta) = \frac{co}{h} = \frac{x}{1} = x;\ \theta = \text{arcsen}(x)$$
$$1 = x^2 + ca \Rightarrow ca = \sqrt{1 - x^2}$$
$$\cos(\theta) = \frac{ca}{h} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

3. Exercício

   Faça o esboço do sólido cujo volume é dado pela integral abaixo.

$$v = \int_{-1}^{0} \int_{0}^{1} \int_{0}^{y^2} dzdxdy = \int_{-1}^{0} dy \int_{0}^{1} dx \int_{0}^{y^2} dz$$

Figura 24: Integrais triplas - Aula 04 - Exercício III

img/v17_a04_e03.png

$$v = \int_{-1}^{0} dy \int_{0}^{1} dx \int_{0}^{y^2} dz = \int_{-1}^{0} dy \int_{0}^{1} dx\, [z]_0^{y^2} = \int_{-1}^{0} y^2 dy \int_{0}^{1} dx = \int_{-1}^{0} y^2 dy\, [x]_0^1 =$$
$$\int_{-1}^{0} y^2 dy = \left[\frac{y^3}{3}\right]_{-1}^{0} = \frac{\cancel{0^3}}{\cancel{3}} - \frac{(-1)^3}{3} = \frac{1}{3}$$

13 Coordenadas esféricas

13.1 Aula 1

1. Exercício

   Use coordenadas esféricas para calcular

$$\iiint_R \left(x^2 + y^2 + z^2\right) dv$$

, onde R é a bola unitária $x^2 + y^2 + z^2 \leq 1$.

$$x^2 + y^2 + z^2 = r^2$$
$$dv = dxdydz = r^2 \operatorname{sen}(\varphi)\, dr d\varphi d\theta$$
$$0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi$$

$$\iiint_R \left(x^2 + y^2 + z^2\right) dv = \int_0^1 \int_0^{2\pi} \int_0^{\pi} r^4 \operatorname{sen}(\varphi)\, d\varphi d\theta dr =$$
$$\int_0^1 \int_0^{2\pi} \left(r^4 \int_0^{\pi} \operatorname{sen}(\varphi)\, d\varphi \right) d\theta dr = \int_0^1 \int_0^{2\pi} \left(r^4 \left[-\cos(\varphi)\right]_0^{\pi} \right) d\theta dr =$$
$$\int_0^1 \int_0^{2\pi} \left(r^4 \left[-\cos(\pi) + \cos(0)\right] \right) d\theta dr = \int_0^1 \int_0^{2\pi} 2r^4\, d\theta dr = \int_0^1 \left(2r^4 \int_0^{2\pi} d\theta \right) dr =$$
$$\int_0^1 \left(2r^4 \left[\theta\right]_0^{2\pi} \right) dr = \int_0^1 4\pi r^4 dr = 4\pi \int_0^1 r^4 dr = 4\pi \left[\frac{r^5}{5}\right]_0^1 = \frac{4\pi}{5}$$

13.2 Aula 2

1. Exercício

$$x^2 + y^2 + z^2 = 16$$
$$z = \sqrt{x^2 + y^2}$$

Figura 25: Coordenadas esféricas - Aula 02 - Exercício I

img/v19_a02_e01.png

$$z = \sqrt{x^2 + y^2} \Rightarrow r\cos(\varphi) = \sqrt{\left(r\operatorname{sen}(\varphi)\cos(\theta)\right)^2 + \left(r\operatorname{sen}(\varphi)\operatorname{sen}(\theta)\right)^2} =$$
$$\sqrt{r^2\operatorname{sen}^2(\varphi)\cos^2(\theta) + r^2\operatorname{sen}^2(\varphi)\operatorname{sen}^2(\theta)} = \sqrt{r^2\operatorname{sen}^2(\varphi)\left(\cos^2(\theta) + \operatorname{sen}^2(\theta)\right)} =$$
$$r\operatorname{sen}(\varphi) \Rightarrow \frac{r\cos(\varphi)}{r\cos(\varphi)} = \frac{r\operatorname{sen}(\varphi)}{r\cos(\varphi)} \Rightarrow 1 = \operatorname{tg}(\varphi) \Rightarrow \varphi = \operatorname{arctg}(1) = \frac{\pi}{4}$$

$$0 \leq r \leq 4,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \frac{\pi}{4}$$

$$v = \int_0^4 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} r^2 \operatorname{sen}(\varphi) d\varphi d\theta dr = \int_0^4 \int_0^{2\pi} \left(r^2 \int_0^{\frac{\pi}{4}} \operatorname{sen}(\varphi) d\varphi \right) d\theta dr =$$
$$\int_0^4 \int_0^{2\pi} \left(r^2 \left[-\cos(\varphi) \right]_0^{\frac{\pi}{4}} \right) d\theta dr = \int_0^4 \int_0^{2\pi} \left(r^2 \left[-\cos\left(\frac{\pi}{4}\right) + \cos(0) \right] \right) d\theta dr =$$
$$\int_0^4 \int_0^{2\pi} \left(r^2 \left[-\frac{\sqrt{2}}{2} + 1 \right] \right) d\theta dr = \int_0^4 \int_0^{2\pi} \frac{r^2 \left(2 - \sqrt{2}\right)}{2} d\theta dr =$$
$$\int_0^4 \left(\frac{r^2 \left(2 - \sqrt{2}\right)}{2} \left[\theta\right]_0^{2\pi} \right) dr = \int_0^4 \frac{r^2 \left(2 - \sqrt{2}\right)}{\cancel{2}} \cancel{2}\pi \, dr = \int_0^4 \pi \left(2 - \sqrt{2}\right) r^2 \, dr =$$
$$\pi \left(2 - \sqrt{2}\right) \left[\frac{r^3}{3} \right]_0^4 = \frac{64\pi \left(2 - \sqrt{2}\right)}{3}$$

13.3 Aula 3

1. Exercício

$$x^2 + y^2 + z^2 = r^2 = 4^2 = 16$$

$$0 \leq r \leq 4,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi$$

$$v = \int_0^{\pi} \int_0^{2\pi} \int_0^4 r^2 \operatorname{sen}(\varphi) \, dr d\theta d\varphi = \int_0^{\pi} \int_0^{2\pi} \left(\operatorname{sen}(\varphi) \left[\frac{r^3}{3} \right]_0^4 \right) d\theta d\varphi =$$
$$\int_0^{\pi} \int_0^{2\pi} \left(\operatorname{sen}(\varphi) \frac{64}{3} \right) d\theta d\varphi = \frac{64}{3} \int_0^{\pi} \int_0^{2\pi} \operatorname{sen}(\varphi) \, d\theta d\varphi = \frac{64}{3} \int_0^{\pi} \left(\operatorname{sen}(\varphi) \left[\theta\right]_0^{2\pi} \right) d\varphi =$$
$$\frac{64}{3} \int_0^{\pi} \operatorname{sen}(\varphi) 2\pi \, d\varphi = \frac{128\pi}{3} \int_0^{\pi} \operatorname{sen}(\varphi) \, d\varphi = \frac{128\pi}{3} \left[-\cos(\varphi) \right]_0^{\pi} =$$
$$\frac{128\pi}{3} (-\cos(\pi) + \cos(0)) = \frac{128\pi}{3} (1 + 1) = \frac{256\pi}{3}$$

13.4 Aula 4

1. Exercício

$$\iiint_R \mathrm{e}^{\sqrt{\left(x^2+y^2+z^2\right)^3}} \, dv$$

$$R = \{(x,y,z)\,|\,x^2+y^2+z^2 \le 1\}$$

$$x^2+y^2+z^2 = r^2 \Rightarrow \mathrm{e}^{\sqrt{(x^2+y^2+z^2)^3}} = \mathrm{e}^{\sqrt{(r^2)^3}} = \mathrm{e}^{r^3}$$
$$dv = dxdydz = r^2\operatorname{sen}(\varphi)\,drd\theta d\varphi$$
$$x^2+y^2+z^2 = r^2 = 1^2 = 1$$
$$0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi$$

$$\int_0^1\int_0^{2\pi}\int_0^{\pi} e^{r^3} r^2\operatorname{sen}(\varphi)\,d\varphi d\theta dr = \int_0^1 e^{r^3} r^2\,dr \int_0^{2\pi} d\theta \int_0^{\pi}\operatorname{sen}(\varphi)\,d\varphi =$$
$$\int_0^1 e^u \frac{du}{3}\,[\theta]_0^{2\pi}\,[-\cos(\varphi)]_0^{\pi} = \frac{1}{3}\left[e^{r^3}\right]_0^1 2\pi\left(-\cos(\pi)+\cos(0)\right) = \frac{2\pi}{3}\left(e^{1^3}-e^{0^3}\right)(1+1) =$$
$$\frac{4\pi}{3}(e-1)$$

$$u = r^3 \Rightarrow \frac{du}{3} = r^2\,dr$$

13.5 Cálculo de massa com coordenadas esféricas - Aula 5

1. Exercício retirado da página 899 de [2]

   Encontre a massa de uma esfera S de raio 4 centrada na origem com densidade de massa dado abaixo.

$$\mu = f(x,y,z) = x^2+y^2$$

$$\mu = \frac{m}{v} \Rightarrow m = \mu v \Rightarrow \int dm = \int \mu dv \Rightarrow$$
$$m = \int \mu dv = \iiint_R \mu\,dv = \iiint_S \left(x^2+y^2\right)\,dv$$
$$S = \{(x,y,z)\,|\,x^2+y^2+z^2 \le 16\}$$

$$x^2+y^2 = (r\operatorname{sen}(\varphi)\cos(\theta))^2 + (r\operatorname{sen}(\varphi)\operatorname{sen}(\theta))^2 =$$
$$r^2\operatorname{sen}^2(\varphi)\cos^2(\theta) + r^2\operatorname{sen}^2(\varphi)\operatorname{sen}^2(\theta) = r^2\operatorname{sen}^2(\varphi)\left(\cos^2(\theta)+\operatorname{sen}^2(\theta)\right) =$$
$$r^2\operatorname{sen}^2(\varphi)$$

$$dv = dxdydz = r^2 \operatorname{sen}(\varphi)\, drd\theta d\varphi$$

$$0 \leq r \leq 4,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi$$

$$m = \iiint_S \left(x^2 + y^2\right)\, dv = \int_0^4 \int_0^{2\pi} \int_0^{\pi} \left(r^2 \operatorname{sen}^2(\varphi)\right) r^2 \operatorname{sen}(\varphi)\, d\varphi d\theta dr =$$

$$\int_0^4 \int_0^{2\pi} \int_0^{\pi} r^4 \operatorname{sen}^3(\varphi)\, d\varphi d\theta dr = \int_0^4 r^4\, dr \int_0^{2\pi} d\theta \int_0^{\pi} \operatorname{sen}^2 \operatorname{sen}(\varphi)\, d\varphi =$$

$$\left[\frac{r^5}{5}\right]_0^4 [\theta]_0^{2\pi} \int_0^{\pi} \left(1 - \cos^2(\varphi)\right) \operatorname{sen}(\varphi)\, d\varphi =$$

$$\frac{1024}{5} 2\pi \left(\int_0^{\pi} \operatorname{sen}(\varphi)\, d\varphi - \int_0^{\pi} \cos^2(\varphi) \operatorname{sen}(\varphi)\, d\varphi\right) =$$

$$\frac{2048\pi}{5} \left(\int_0^{\pi} \operatorname{sen}(\varphi)\, d\varphi + \int_0^{\pi} u^2\, du\right) = \frac{2048\pi}{5} \left[-\cos(\varphi) + \frac{u^3}{3}\right]_0^{\pi} =$$

$$\frac{2048\pi}{5} \left[\frac{-3\cos(\varphi) + \cos^3(\varphi)}{3}\right]_0^{\pi} = \frac{2048\pi}{15} \left[-\cos(\varphi) \left(3 - \cos^2(\varphi)\right)\right]_0^{\pi} =$$

$$\frac{2048\pi}{15} \left[-\cos(\pi) \left(3 - \cos^2(\pi)\right) + \cos(0) \left(3 - \cos^2(0)\right)\right] = \frac{2048\pi}{15} \left[(3-1) + (3-1)\right] =$$

$$\frac{2048\pi}{15} (2+2) = \frac{8192\pi}{15}$$

$$u = \cos(\varphi) \Rightarrow -du = \operatorname{sen}(\varphi)\, d\varphi$$

13.6 Aula 6

1. Exercício retirado da página 1023 de [3]

   Calcule a integral abaixo, onde E está contido entre as esféras dadas abaixo no 1º octante.

$$\iiint_E z\, dv$$

$$E = \{(x, y, z) \,|\, 1 \leq x^2 + y^2 + z^2 \leq 4\}$$

$$z = r\cos(\varphi)$$

$$dv = dxdydz = r^2 \operatorname{sen}(\varphi)\, drd\theta d\varphi$$

$$1 \leq r \leq 2,\ 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq \varphi \leq \frac{\pi}{2}$$

Figura 26: Coordenadas esféricas - Aula 06 - Exercício I

img/v23_a06_e01.png

$$\iiint_E z\, dv = \int_1^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (r\cos(\varphi))\, r^2 \operatorname{sen}(\varphi)\, d\varphi d\theta dr =$$

$$\int_1^2 r^3\, dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \operatorname{sen}(\varphi)\cos(\varphi)\, d\varphi = \left[\frac{r^4}{4}\right]_1^2 [\theta]_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} u\, du = \left(\frac{16}{4} - \frac{1}{4}\right)\frac{\pi}{2}\left[\frac{u^2}{2}\right]_0^{\frac{\pi}{2}} =$$

$$\frac{16-1}{4}\frac{\pi}{2}\left[\frac{\operatorname{sen}^2(\varphi)}{2}\right]_0^{\frac{\pi}{2}} = \frac{15\pi}{16}\left[\operatorname{sen}^2\left(\frac{\pi}{2}\right) \cancel{- \operatorname{sen}(0)}\right] = \frac{15\pi}{16}$$

$$u = \operatorname{sen}(\varphi) \Rightarrow du = \cos(\varphi)\, d\varphi$$

13.7 Aula 7

1. Exercício retirado da página 1023 de [3]

   Calcule

   $$\iiint_E \sqrt{x^2+y^2+z^2}\, dv$$

   , onde E é limitado abaixo pelo cone $\phi = \frac{\pi}{6}$ e acima pela esfera $\rho = 2$.

Figura 27: Coordenadas esféricas - Aula 07 - Exercício I

img/v24_a07_e01.png

$$0 \leq \rho \leq 2,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{6}$$

$$\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$$
$$dv = \rho^2 \operatorname{sen}(\phi)\, d\rho d\theta d\phi$$

$$\iiint_E \sqrt{x^2+y^2+z^2}\, dv = \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{6}} (\rho)\rho^2 \operatorname{sen}(\phi)\, d\phi d\theta d\rho =$$
$$\int_0^2 \rho^3\, d\rho \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} \operatorname{sen}(\phi)\, d\phi = \left[\frac{\rho^4}{4}\right]_0^2 [\theta]_0^{2\pi} \left[-\cos(\phi)\right]_0^{\frac{\pi}{6}} =$$
$$\frac{16}{4} 2\pi \left(-\cos\left(\frac{\pi}{6}\right) + \cos(0)\right) = 8\pi \left(\frac{-\sqrt{3}}{2} + 1\right) = 8\pi \frac{-\sqrt{3}+2}{2} = 4\pi\left(2-\sqrt{3}\right)$$

14 Coordenadas cilíndricas

14.1 Aula 1

1. Exercício

   Encontre o volume do sólido limitado pelas funções abaixo.

$$z = x^2 + y^2$$
$$z = 9$$

Figura 28: Coordenadas cilíndricas - Aula 01 - Exercício I

img/v25_a01_e01.png

$$z = x^2 + y^2 = r^2$$

$$x^2 + y^2 = r^2 = 3^2 = 9$$

$$0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi,\ r^2 \leq z \leq 9$$

$$\int_0^3 \int_0^{2\pi} \int_{r^2}^9 r\, dz d\theta dr = \int_0^3 r\, dr \int_0^{2\pi} d\theta \int_{r^2}^9 dz = \int_0^3 r\, dr \int_0^{2\pi} d\theta\, [z]_{r^2}^9 =$$
$$\int_0^3 r\left(9 - r^2\right)\, dr \int_0^{2\pi} d\theta = \int_0^3 \left(9r - r^3\right)\, dr \int_0^{2\pi} d\theta = \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_0^3 [\theta]_0^{2\pi} =$$
$$\left[\frac{18r^2 - r^4}{4}\right]_0^3 2\pi = \frac{\pi}{2}\left[r^2\left(18 - r^2\right)\right]_0^3 = \frac{\pi}{2} 9\,(18 - 9) = \frac{81\pi}{2}$$

A Derivadas

B Derivadas simples

Tabela 1: Derivadas simples

y	$=$	c	$\Rightarrow$	y'	$=$	0
y	$=$	x	$\Rightarrow$	y'	$=$	1
y	$=$	x^c	$\Rightarrow$	y'	$=$	cx^{c-1}
y	$=$	e^x	$\Rightarrow$	y'	$=$	e^x
y	$=$	$\ln\|x\|$	$\Rightarrow$	y'	$=$	$\frac{1}{x}$
y	$=$	uv	$\Rightarrow$	y'	$=$	$u'v + uv'$
y	$=$	$\frac{u}{v}$	$\Rightarrow$	y'	$=$	$\frac{u'v - uv'}{v^2}$
y	$=$	u^c	$\Rightarrow$	y'	$=$	$cu^{c-1}u'$
y	$=$	e^u	$\Rightarrow$	y'	$=$	$\mathrm{e}^u\, u'$
y	$=$	c^u	$\Rightarrow$	y'	$=$	$c^u u' ln\|c\|$
y	$=$	$\ln\|u\|$	$\Rightarrow$	y'	$=$	$\frac{u'}{u}$
y	$=$	$\log_c\|u\|$	$\Rightarrow$	y'	$=$	$\frac{u'}{u}\log_c\|\,\mathrm{e}\,\|$

C Derivadas trigonométricas

Tabela 2: Derivadas trigonométricas

y	$=$	$\operatorname{sen}(x)$	$\Rightarrow$	y'	$=$	$\cos(x)$
y	$=$	$\cos(x)$	$\Rightarrow$	y'	$=$	$-\operatorname{sen}(x)$
y	$=$	$\operatorname{tg}(x)$	$\Rightarrow$	y'	$=$	$\sec^2(x)$
y	$=$	$\operatorname{cotg}(x)$	$\Rightarrow$	y'	$=$	$-\operatorname{cossec}^2(x)$
y	$=$	$\sec(x)$	$\Rightarrow$	y'	$=$	$\sec(x)\operatorname{tg}(x)$
y	$=$	$\operatorname{cossec}(x)$	$\Rightarrow$	y'	$=$	$-\operatorname{cossec}(x)\operatorname{cotg}(x)$
y	$=$	$\operatorname{arcsen}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{1}{\sqrt{1-x^2}}$
y	$=$	$\arccos(x)$	$\Rightarrow$	y'	$=$	$\dfrac{-1}{\sqrt{1-x^2}}$
y	$=$	$\operatorname{arctg}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{1}{1+x^2}$
y	$=$	$\operatorname{arccotg}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{-1}{1+x^2}$
y	$=$	$\operatorname{arcsec}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$
y	$=$	$\operatorname{arccossec}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

D Integrais

E Integrais simples

Tabela 3: Integrais simples

$$
\begin{array}{|lclcl|}
\displaystyle\int dx & = & x + c & & \\
\displaystyle\int x^p \, dx & = & \dfrac{x^{p+1}}{p+1} + c & \rightarrow & p \neq -1 \\
\displaystyle\int \mathrm{e}^x \; dx & = & \mathrm{e}^x + c & & \\
\displaystyle\int \dfrac{dx}{x} & = & \ln|x| + c & & \\
\displaystyle\int u^p \, du & = & \dfrac{u^{p+1}}{p+1} + c & \rightarrow & p \neq -1 \\
\displaystyle\int \mathrm{e}^u \; du & = & \mathrm{e}^u + c & & \\
\displaystyle\int \dfrac{du}{u} & = & \ln|u| + c & & \\
\displaystyle\int p^u \, du & = & \dfrac{p^u}{\ln|p|} + c & &
\end{array}
$$

F Integrais trigonométricas

Tabela 4: Integrais trigonométricas

$$
\begin{array}{|lcl|}
\int \operatorname{sen}(u)du & = & -\cos(u)+c \\
\int \cos(u)du & = & \operatorname{sen}(u)+c \\
\int \operatorname{tg}(u)du & = & \ln|\sec(u)|+c \\
\int \operatorname{cotg}(u)du & = & \ln|\operatorname{sen}(u)|+c \\
\int \sec(u)du & = & \ln|\sec(u)+\operatorname{tg}(u)|+c \\
\int \operatorname{cossec}(u)du & = & \ln|\operatorname{cossec}(u)-\operatorname{cotg}(u)|+c \\
\int \sec^2(u)du & = & \operatorname{tg}(u)+c \\
\int \operatorname{cossec}^2(u)du & = & -\operatorname{cotg}(u)+c \\
\int \sec(u)\operatorname{tg}(u)du & = & \sec(u)+c \\
\int \operatorname{cossec}(u)\operatorname{cotg}(u)du & = & -\operatorname{cossec}(u)+c \\
\int \frac{du}{\sqrt{1-x^2}} & = & \operatorname{arcsen}(x)+c \\
-\int \frac{du}{\sqrt{1-x^2}} & = & \arccos(x)+c \\
\int \frac{du}{1+x^2} & = & \operatorname{arctg}(x)+c \\
-\int \frac{du}{1+x^2} & = & \operatorname{arccotg}(x)+c \\
\int \frac{du}{|x|\sqrt{x^2-1}} & = & \operatorname{arcsec}(x)+c \\
-\int \frac{du}{|x|\sqrt{x^2-1}} & = & \operatorname{arccossec}(x)+c
\end{array}
$$

G Coordenadas polares

Figura 29: Coordenadas polares

img/coordenadas_polares.jpg

$$r \in [0, \infty),\ \theta \in [0,\ 2\pi]$$

Tabela 5: Transformação de coordenadas cartesinas em polares

$$\left| \begin{array}{lcl} y & = & r\ \mathrm{sen}(\theta) \\ x & = & r\ \cos(\theta) \end{array} \right|$$

Tabela 6: Coordenadas polares a partir das suas correspondentes cartesianas

$$\left| \begin{array}{lcl} r^2 & = & x^2 + y^2 \\ \theta & = & \mathrm{arcsen}\left(\frac{y}{r}\right) \\ \theta & = & \arccos\left(\frac{x}{r}\right) \\ \theta & = & \mathrm{arctg}\left(\frac{y}{x}\right) \end{array} \right|$$

$$\iint_R z\, da = \iint_R f(x, y)\, dxdy = \iint_R f(r\ \cos(\theta),\ r\ \mathrm{sen}(\theta)) r\, drd\theta$$

H Coordenadas cilíndricas

Figura 30: Coordenadas cilíndricas

img/coordenadas_cilindricas.png

$$r \in [0, \infty),\ \theta \in [0,\ 2\pi]$$

Tabela 7: Transformação de coordenadas cartesinas em cilíndricas

$$\left|\begin{array}{lcl} x & = & r\ \cos(\theta) \\ y & = & r\ \mathrm{sen}(\theta) \\ z & = & z \end{array}\right|$$

Tabela 8: Coordenadas cilíndricas a partir das suas correspondentes cartesianas

$$\left|\begin{array}{lcl} r^2 & = & x^2 + y^2 \\ \theta & = & \mathrm{arctg}\left(\dfrac{y}{x}\right) \end{array}\right|$$

$$\iiint_R f(x,y,z)\, dv = \iiint_R f(x,y,z)\, dxdydz = \iiint_R f(r\ \cos(\theta),\ r\ \mathrm{sen}(\theta), z) r\, drd\theta dz$$

I Coordenadas esféricas

Figura 31: Coordenadas esféricas

img/coordenadas_esfericas.png

$$r \in [0, \infty),\ \theta \in [0,\ 2\pi],\ \varphi \in [0,\ \pi]$$

Tabela 9: Transformação de coordenadas cartesinas em esféricas

$$\left| \begin{array}{lcl} x & = & r\ \mathrm{sen}(\varphi)\cos(\theta) \\ y & = & r\ \mathrm{sen}(\varphi)\,\mathrm{sen}(\theta) \\ z & = & r\ \cos(\varphi) \end{array} \right|$$

Tabela 10: Coordenadas esféricas a partir das suas correspondentes cartesianas

$$\left| \begin{array}{lcl} r^2 & = & x^2 + y^2 + z^2 \\ \theta & = & \mathrm{arctg}\left(\dfrac{y}{x}\right) \\ \varphi & = & \mathrm{arctg}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) \end{array} \right|$$

$$\iiint_R f(x, y, z)\, dv = \iiint_R f(x, y, z)\, dxdydz =$$
$$\iiint_R f(r\ \mathrm{sen}(\varphi)\cos(\theta),\ r\ \mathrm{sen}(\varphi)\,\mathrm{sen}(\theta),\ r\ \cos(\varphi))\, r^2\,\mathrm{sen}(\varphi)\, drd\varphi d\theta$$

J Funções trigonométricas

K Determinação do seno, cosseno e tangente

Figura 32: Determinação do seno, cosseno e tangente

img/triangulo_retangulo.png

L Círculo trigonométrico

Figura 33: Círculo trigonométrico

img/circulo_trigonometrico.png

M Identidades trigonométricas

Tabela 11: Identidades trigonométricas

$$
\begin{array}{|lcl|}
\mathrm{tg}(x) & = & \dfrac{\mathrm{sen}(x)}{\cos(x)} \\
\mathrm{cotg}(x) & = & \dfrac{\cos(x)}{\mathrm{sen}(x)} \\
\sec(x) & = & \dfrac{1}{\cos(x)} \\
\mathrm{cossec}(x) & = & \dfrac{1}{\mathrm{sen}(x)} \\
\mathrm{sen}^2(x) + \cos^2(x) & = & 1 \\
\sec^2(x) - \mathrm{tg}^2(x) & = & 1 \\
\mathrm{cossec}^2(x) - \mathrm{cotg}^2(x) & = & 1 \\
\mathrm{sen}^2(x) & = & \dfrac{1 - \cos(2x)}{2} \\
\cos^2(x) & = & \dfrac{1 + \cos(2x)}{2} \\
\mathrm{sen}(2x) & = & 2\,\mathrm{sen}(x)\cos(x) \\
\cos(2x) & = & \cos^2(x) - \mathrm{sen}^2(x)
\end{array}
$$

N Relação entre trigonométricas e inversas

Tabela 12: Relação entre trigonométricas e inversas

$$
\begin{array}{|lcccccl|}
\mathrm{sen}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arcsen}(x) \\
\cos(\theta) & = & x & \Rightarrow & \theta & = & \arccos(x) \\
\mathrm{tg}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arctg}(x) \\
\mathrm{cossec}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arccossec}(x) \\
\sec(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arcsec}(x) \\
\mathrm{cotg}(\theta) & = & x & \Rightarrow & \theta & = & \mathrm{arccotg}(x)
\end{array}
$$

O Substituição trigonométrica

Tabela 13: Substituição trigonométrica

$$\left|\begin{array}{cccc} \sqrt{a^2-x^2} & \Rightarrow & x & = & a\,\mathrm{sen}(\theta) \\ \sqrt{a^2+x^2} & \Rightarrow & x & = & a\,\mathrm{tg}(\theta) \\ \sqrt{x^2-a^2} & \Rightarrow & x & = & a\sec(\theta) \end{array}\right|$$

P Ângulos notáveis

Tabela 14: Ângulos notáveis

ângulo	0° (0)	30° $\left(\frac{\pi}{6}\right)$	45° $\left(\frac{\pi}{4}\right)$	60° $\left(\frac{\pi}{3}\right)$	90° $\left(\frac{\pi}{2}\right)$
sen	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\nexists$

Referências

[1] Fernando Grings. *Curso de Integrais Duplas e Triplas*. Youtube, 2016.

[2] Jon Rogawski. *Cálculo Vol.II*. Bookman, 2009.

[3] James Stewart. *Cálculo Vol.II*. Pioneira Thomson Learning Ltda, 4 edition, 2002.