

Curso de integrais duplas e triplas

César Antônio de Magalhães
ceanma@gmail.com

24 de agosto de 2016

Sumário

I	Integrais duplas	4
1	Invertendo os limites de integração - Aula 1	4
2	Determinação da região de integração - Aula 2	5
3	Cálculo de volume - Aula 3	9

Lista de Figuras

1	Integrais duplas - Aula 1 - Exercício I e II	4
2	Integrais duplas - Aula 2 - Exercício I	5
3	Integrais duplas - Aula 2 - Exercício II	6
4	Integrais duplas - Aula 2 - Exercício III	7
5	Integrais duplas - Aula 2 - Exercício IV	8
6	Integrais duplas - Aula 2 - Exercício V	9
7	Integrais duplas - Aula 3 - Exercício I	10
8	Integrais duplas - Aula 3 - Exercício II	11

Parte I

Integrais duplas

1 Invertendo os limites de integração - Aula 1

1. Exercício

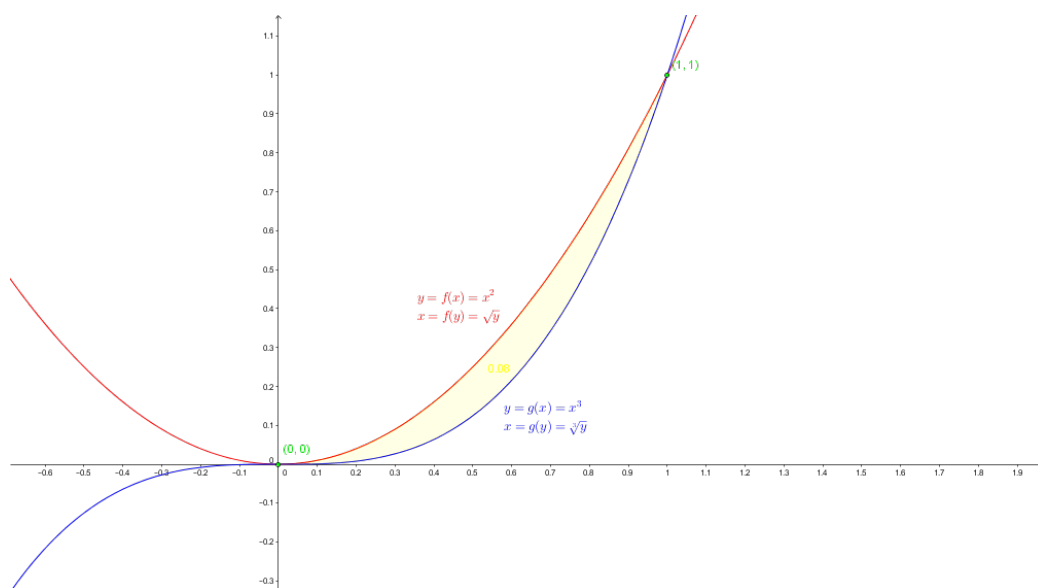


Figura 1: Integrais duplas - Aula 1 - Exercício I e II

$$\begin{aligned} f(x) &= x^2; \quad g(x) = x^3 \\ x = 0 &\Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3 \\ x = 1 &\Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3 \end{aligned}$$

$$\begin{aligned} \int_0^1 dx \int_{g(x)}^{f(x)} dy &= \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx [y]_{x^3}^{x^2} = \int_0^1 dx [x^2 - x^3] = \\ \int_0^1 x^2 dx - \int_0^1 x^3 dx &= \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left[\frac{4x^3 - 3x^4}{12} \right]_0^1 = \frac{1}{12} [4x^3 - 3x^4]_0^1 = \\ \frac{1}{12} [x^2(4x - 3)]_0^1 &= \frac{1}{12} [1^2(4 \cdot 1 - 3) - 0^2(4 \cdot 0 - 3)] = \frac{1}{12} = 0,08\bar{3} \end{aligned}$$

2. Exercício

$$f(x) = x^2 \Rightarrow f(y) = \sqrt{y}; g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$$

$$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$$

$$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$$

$$\int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy [\sqrt[3]{y} - \sqrt{y}] =$$

$$\int_0^1 \sqrt[3]{y} dy - \int_0^1 \sqrt{y} dy = \int_0^1 y^{\frac{1}{3}} dy - \int_0^1 y^{\frac{1}{2}} dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^1 =$$

$$\left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3} \right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12} \right]_0^1 = \frac{1}{12} [9\sqrt[3]{y^4} - 8\sqrt{y^3}]_0^1 =$$

$$\frac{1}{12} [(9\sqrt[3]{1^4} - 8\sqrt{1^3}) - (9\sqrt[3]{0^4} - 8\sqrt{0^3})] = \frac{1}{12}(9 - 8) = \frac{1}{12} = 0,08\bar{3}$$

2 Determinação da região de integração - Aula 2

1. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 6\}$$

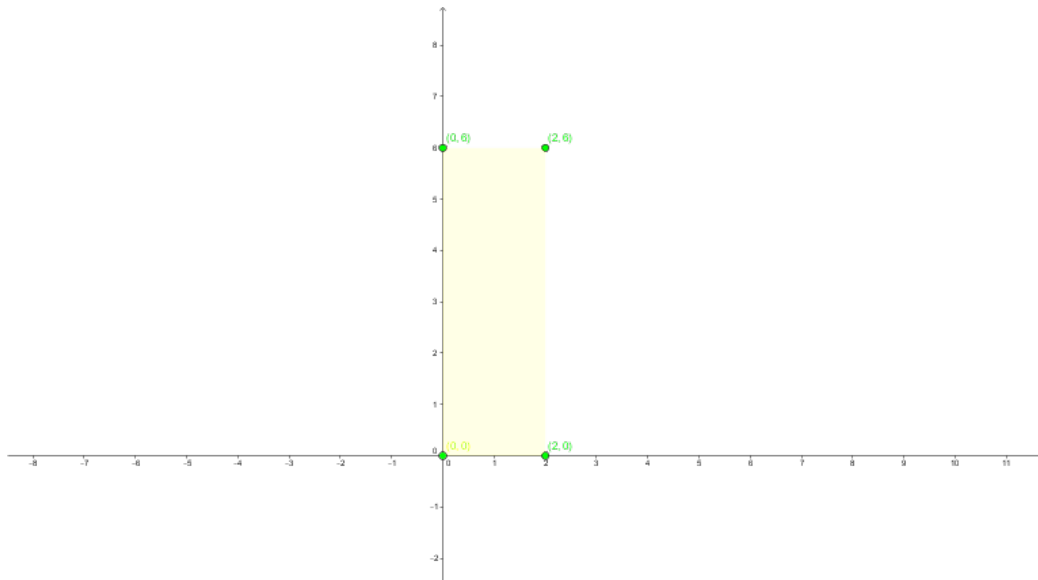


Figura 2: Integrais duplas - Aula 2 - Exercício I

$$\int_0^2 dx \int_0^6 dy = \int_0^2 dx [y]_0^6 = \int_0^2 dx [6 - 0] = 6 \int_0^2 dx = 6[x]_0^2 = 6[2 - 0] = 6 \cdot 2 = 12$$

2. Exercício

$$R = \{(x, y) \in \mathbb{R} \mid 0 \leq x \leq 1, x \leq y \leq 2x\}$$

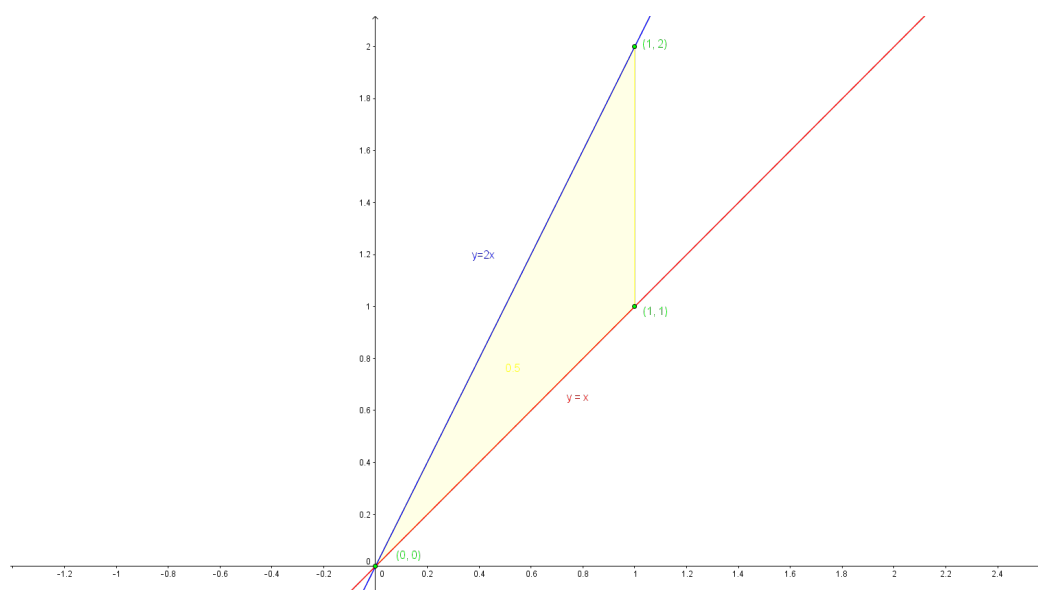


Figura 3: Integrais duplas - Aula 2 - Exercício II

$$\begin{aligned} \int_0^1 dx \int_x^{2x} dy &= \int_0^1 dx [y]_x^{2x} = \int_0^1 dx [2x - x] = 2 \int_0^1 x dx - \int_0^1 x dx = \\ &= \left[\frac{x^2}{2} - \frac{x^2}{2} \right]_0^1 = \left[\frac{2x^2 - x^2}{2} \right]_0^1 = \frac{1}{2} [x^2]_0^1 = \frac{1}{2} [1^2 - 0^2] = \frac{1}{2} = 0,5 \end{aligned}$$

3. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, 0 \leq x \leq \sqrt{1 - y^2}\}$$

$$y = 0, y = 1$$

$$x = 0, x = \sqrt{1 - y^2} \Rightarrow x^2 = 1 - y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow$$

$$y = \sqrt{1 - x^2}$$

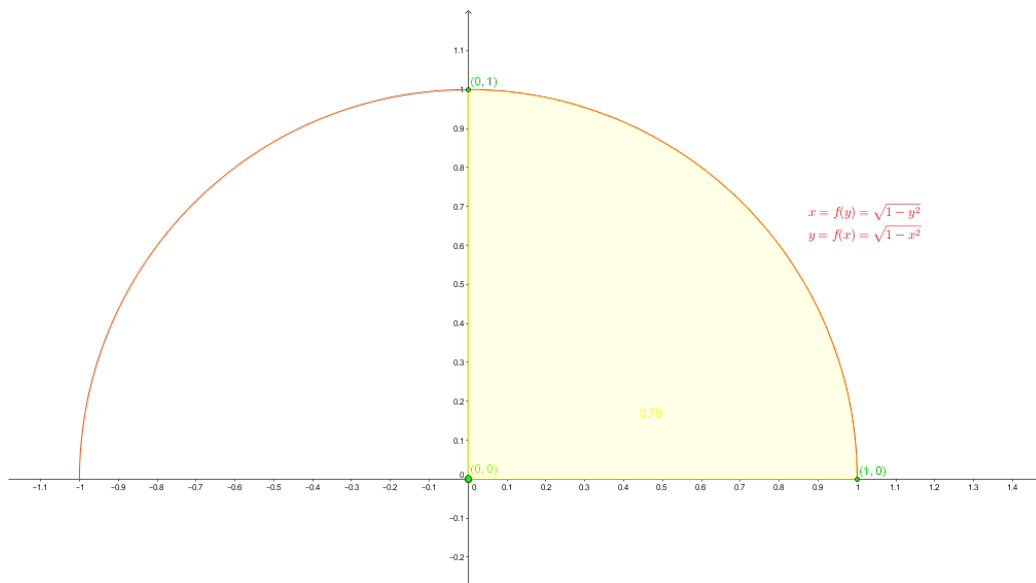


Figura 4: Integrais duplas - Aula 2 - Exercício III

$$\begin{aligned}
 \int_0^1 dy \int_0^{f(y)} dx &= \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy [\sqrt{1-y^2} - 0] = \\
 \int_0^1 \sqrt{1-y^2} dy &= \int_0^1 \sqrt{1-\sin^2(t)} \cos(t) dt = \int_0^1 \sqrt{\cos^2(t)} \cos(t) dt = \\
 \int_0^1 \cos(t) \cos(t) dt &= \int_0^1 \cos^2(t) dt = \int_0^1 \frac{1+\cos(2t)}{2} dt = \frac{1}{2} \int_0^1 [1+\cos(2t)] dt = \\
 \frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_0^1 \cos(2t) dt &= \frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_0^1 \cos(u) \frac{du}{2} = \frac{1}{2} \int_0^1 dt + \\
 \frac{1}{4} \int_0^1 \cos(u) du &= \left[\frac{1}{2} t + \frac{1}{4} \sin(u) \right]_0^1 = \left[\frac{t}{2} + \frac{\sin(2t)}{4} \right]_0^1 = \left[\frac{t}{2} + \frac{2 \sin(t) \cos(t)}{4 \cdot 2} \right]_0^1 = \\
 \left[\frac{t + \sin(t) \cos(t)}{2} \right]_0^1 &= \frac{1}{2} [\arcsen(y) + y \sqrt{1-y^2}]_0^1 = \\
 \frac{1}{2} [(\arcsen(1) + 1 \cdot \sqrt{1-1^2}) - (\arcsen(0) + 0 \cdot \sqrt{1-0^2})] &= \frac{1}{2} \left[\frac{\pi}{2} - 0 \right] = \\
 \frac{\pi}{4} &= 0,785
 \end{aligned}$$

$$y = \sin(t) \Rightarrow dy = \cos(t) dt$$

$$u = 2t \Rightarrow \frac{du}{2} = dt$$

$$\sin(t) = \frac{co}{h} = \frac{y}{1} = y$$

$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1-y^2}$$

$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1-y^2}}{1} = \sqrt{1-y^2}$$

$$y = \text{sen}(t) \Rightarrow t = \arcsen(y)$$

4. Exercício

$$y = x^2 + 1, y = -x^2 - 1; x = 1, x = -1$$

$$R = \{(x, y) \in \mathbb{R} \mid -1 \leq x \leq 1, -x^2 - 1 \leq y \leq x^2 + 1\}$$

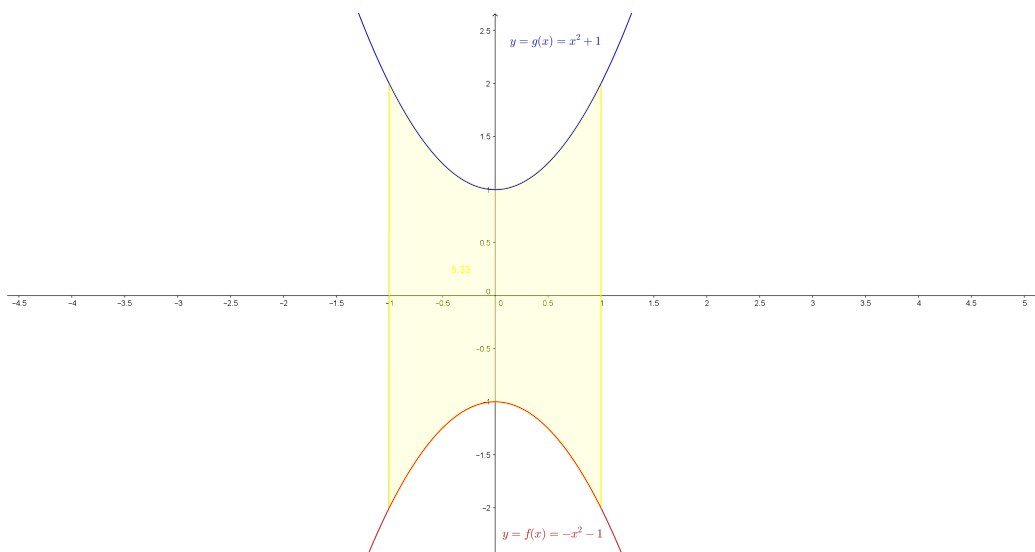


Figura 5: Integrais duplas - Aula 2 - Exercício IV

$$\begin{aligned} \int_{-1}^1 dx \int_{f(x)}^{g(x)} dy &= \int_{-1}^1 dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^1 dx [y]_{-x^2-1}^{x^2+1} = \int_{-1}^1 dx [x^2 + 1 - (-x^2 - 1)] = \\ &= \int_{-1}^1 dx [x^2 + 1 + x^2 + 1] = \int_{-1}^1 dx [2x^2 + 2] = 2 \int_{-1}^1 x^2 dx + 2 \int_{-1}^1 dx = \\ &= \left[\frac{2x^3}{3} + 2x \right]_{-1}^1 = \left[2 \left(\frac{x^3 + 3x}{3} \right) \right]_{-1}^1 = \frac{2}{3} [x(x^2 + 3)]_{-1}^1 = \\ &= \frac{2}{3} [1 \cdot (1^2 + 3) - (-1) \cdot ((-1)^2 + 3)] = \frac{2}{3} (4 + 4) = \frac{2}{3} 8 = \frac{16}{3} = 5, \bar{3} \end{aligned}$$

5. Exercício

$$R = \{(x, y) \in \mathbb{R} \mid 0 \leq y \leq 2, -y \leq x \leq y\}$$

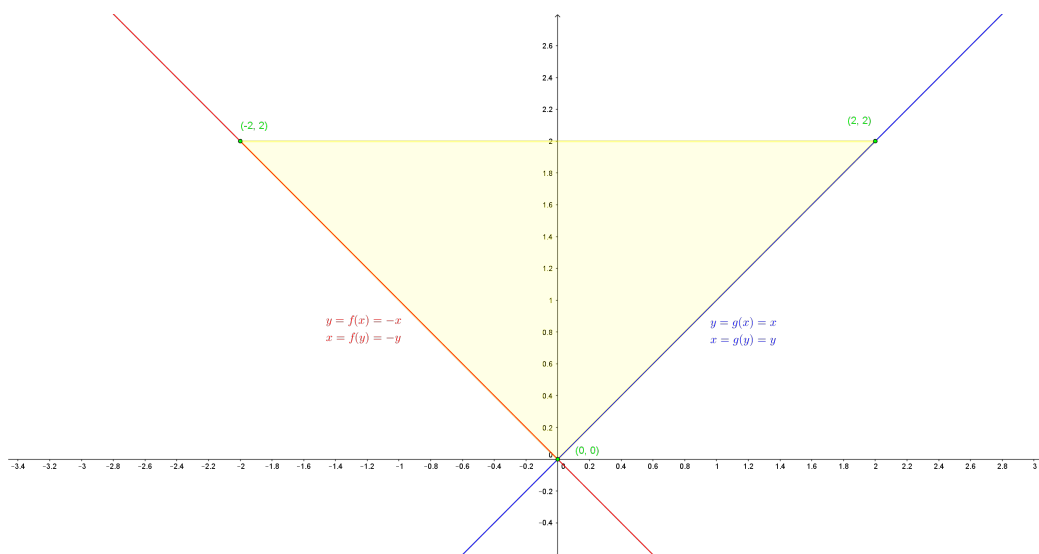


Figura 6: Integrais duplas - Aula 2 - Exercício V

$$\begin{aligned} \int_0^2 dy \int_{f(y)}^{g(y)} dx &= \int_0^2 dy \int_{-y}^y dx = \int_0^2 dy [x]_{-y}^y = \int_0^2 dy [y - (-y)] = \\ \int_0^2 dy [2y] &= 2 \int_0^2 y dy = \left[2 \frac{y^2}{2} \right]_0^2 = 2^2 - 0^2 = 4 \end{aligned}$$

3 Cálculo de volume - Aula 3

1. Exercício

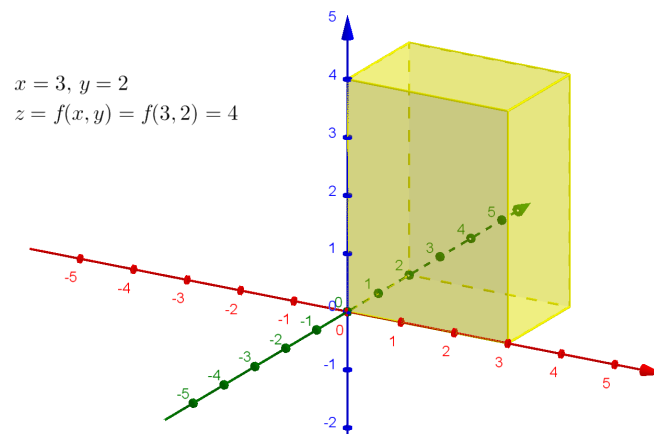


Figura 7: Integrais duplas - Aula 3 - Exercício I

$$z = 4; \quad dz = dx dy$$

$$\begin{aligned}
 v &= \int_0^3 \int_0^2 z \, dz = \int_0^3 \int_0^2 4 \, dy dx = 4 \int_0^3 dx \int_0^2 dy = 4 \int_0^3 dx [y]_0^2 = \\
 &4 \int_0^3 dx [2 - 0] = 8 \int_0^3 dx = 8[x]_0^3 = 8[3 - 0] = 8 \cdot 3 = 24
 \end{aligned}$$

2. Exercício

$$R = [0, 3] \times [0, 4]$$

$$\int \int_R (8 - 2y) da$$

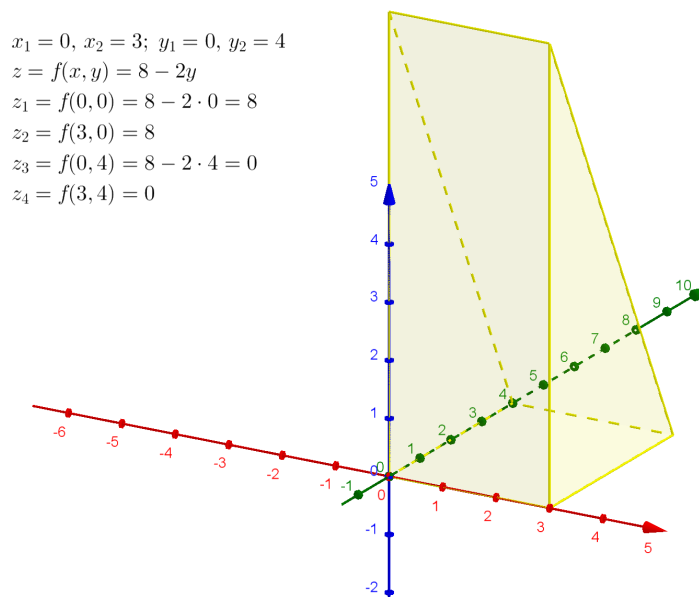


Figura 8: Integrais duplas - Aula 3 - Exercício II

$$z = 8 - 2y; da = dz = dxdy$$

$$\begin{aligned}
 v &= \int_0^3 \int_0^4 z dz = \int_0^3 \int_0^4 (8 - 2y) dxdy = \int_0^3 dx \int_0^4 (8 - 2y) dy = \\
 &\int_0^3 dx \left(8 \int_0^4 dy - 2 \int_0^4 y dy \right) = \int_0^3 dx 2 \left(4 \int_0^4 dy - \int_0^4 y dy \right) = 2 \int_0^3 dx \left[4y - \frac{y^2}{2} \right]_0^4 = \\
 &2 \int_0^3 dx \left[\frac{8y - y^2}{2} \right]_0^4 = 2 \int_0^3 dx \frac{1}{2} [y(8-y)]_0^4 = \int_0^3 dx [4(8-4) - 0(8-0)] = \\
 &16 \int_0^3 dx = 16[x]_0^3 = 16[3 - 0] = 48
 \end{aligned}$$