Curso de integrais duplas e triplas

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Sumário

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Resumo

Exercícios retirados do canal do Youtube, O Matematico [1].

Parte I
Integrais duplas

1 Invertendo os limites de integração - Aula 1

1. Exercício

$$f(x) = x^2;\ g(x) = x^3$$
$$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$$
$$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$$

Figura 1: Integrais duplas - Aula 1 - Exercício I e II

img/v01_a01_e01.png

$$a = \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx\, [y]_{x^3}^{x^2} = \int_0^1 dx\, \left[x^2 - x^3\right] =$$

$$\int_0^1 x^2\, dx - \int_0^1 x^3\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{4x^3 - 3x^2}{12}\right]_0^1 = \frac{1}{12}\left[4x^3 - 3x^2\right]_0^1 =$$

$$\frac{1}{12}\left[x^2(4x-3)\right]_0^1 = \frac{1}{12}\left[1^2(4\cdot 1 - 3) \cancel{- 0^2(4\cdot 0 - 3)}\right] = \frac{1}{12} = 0{,}08\overline{3}$$

2. Exercício

$$f(x) = x^2 \Rightarrow f(y) = \sqrt{y};\ g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$$
$$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$$
$$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$$

$$a = \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy\, [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy\, [\sqrt[3]{y} - \sqrt{y}] =$$

$$\int_0^1 \sqrt[3]{y}\, dy - \int_0^1 \sqrt{y}\, dy = \int_0^1 y^{\frac{1}{3}}\, dy - \int_0^1 y^{\frac{1}{2}}\, dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 =$$

$$\left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3}\right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12}\right]_0^1 = \frac{1}{12}\left[9\sqrt[3]{y^4} - 8\sqrt{y^3}\right]_0^1 =$$

$$\frac{1}{12}\left[\left(9\sqrt[3]{1^4} - 8\sqrt{1^3}\right) \cancel{- \left(9\sqrt[3]{0^4} - 8\sqrt{0^3}\right)}\right] = \frac{1}{12}(9-8) = \frac{1}{12} = 0{,}08\overline{3}$$

2 Determinação da região de integração - Aula 2

1. Exercício

$$R = \left\{ (x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 2 \,,\, 0 \leq y \leq 6 \right\}$$

Figura 2: Integrais duplas - Aula 2 - Exercício I

img/v02_a02_e01.png

$$a = \int_0^2 dx \int_0^6 dy = \int_0^2 dx \, [y]_0^6 = \int_0^2 dx \, [6 - 0] = 6 \int_0^2 dx = 6[x]_0^2 = 6[2 - 0] = 6 \cdot 2 = 12$$

2. Exercício

$$R = \left\{ (x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1 \,,\, x \leq y \leq 2x \right\}$$

Figura 3: Integrais duplas - Aula 2 - Exercício II

img/v02_a02_e02.png

$$a=\int_0^1 dx\int_x^{2x} dy=\int_0^1 dx\,[y]_x^{2x}=\int_0^1 dx\,[2x-x]=2\int_0^1 x\,dx-\int_0^1 x\,dx=$$

$$\left[\cancel{2}\frac{x^2}{\cancel{2}}-\frac{x^2}{2}\right]_0^1=\left[\frac{2x^2-x^2}{2}\right]_0^1=\frac{1}{2}\left[x^2\right]_0^1=\frac{1}{2}\left[1^2\cancel{-0^2}\right]=\frac{1}{2}=0,5$$

3. Exercício

$$R=\left\{(x,y)\in\mathbb{R}^2\,|\,0\leq y\leq 1\,,\,0\leq x\leq\sqrt{1-y^2}\right\}$$

$$y=0,\ y=1$$

$$x=0,\ x=\sqrt{1-y^2}\Rightarrow x^2=1-y^2\Rightarrow x^2-1=-y^2\Rightarrow y^2=-x^2+1\Rightarrow y=\sqrt{1-x^2}$$

Figura 4: Integrais duplas - Aula 2 - Exercício III

img/v02_a02_e03.png

$$a=\int_0^1 dy\int_0^{f(y)} dx=\int_0^1 dy\int_0^{\sqrt{1-y^2}} dx=\int_0^1 dy\,[x]_0^{\sqrt{1-y^2}}=\int_0^1 dy\,\left[\sqrt{1-y^2}-0\right]=$$

$$\int_0^1\sqrt{1-y^2}\,dy=\int_0^1\sqrt{1-\operatorname{sen}^2(t)}\,\cos(t)dt=\int_0^1\sqrt{\cos^2(t)}\,\cos(t)dt=$$

$$\int_0^1\cos(t)\cos(t)dt=\int_0^1\cos^2(t)dt=\int_0^1\frac{1+\cos(2t)}{2}dt=\frac{1}{2}\int_0^1\left[1+\cos(2t)\right]dt=$$

$$\frac{1}{2}\int_0^1 dt+\frac{1}{2}\int_0^1\cos(2t)dt=\frac{1}{2}\int_0^1 dt+\frac{1}{2}\int_0^1\cos(u)\frac{du}{2}=\frac{1}{2}\int_0^1 dt+\frac{1}{4}\int_0^1\cos(u)\,du=$$

$$\left[\frac{1}{2}t+\frac{1}{4}\operatorname{sen}(u)\right]_0^1=\left[\frac{t}{2}+\frac{\operatorname{sen}(2t)}{4}\right]_0^1=\left[\frac{t}{2}+\frac{2\operatorname{sen}(t)\cos(t)}{4}\right]_0^1=\left[\frac{t+\operatorname{sen}(t)\cos(t)}{2}\right]_0^1=$$

$$\frac{1}{2}\left[\operatorname{arcsen}(y)+y\sqrt{1-y^2}\right]_0^1=\frac{1}{2}\left[\left(\operatorname{arcsen}(1)\cancel{+1\cdot\sqrt{1-1^2}}\right)-\left(\operatorname{arcsen}(0)\cancel{+0\cdot\sqrt{1-0^2}}\right)\right]=$$

$$\frac{1}{2}\left[\frac{\pi}{2}-0\right]=\frac{\pi}{4}=0,785$$

$$y = \operatorname{sen}(t) \Rightarrow dy = \cos(t)dt$$
$$u = 2t \Rightarrow \frac{du}{2} = dt$$
$$\operatorname{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$
$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1-y^2}$$
$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1-y^2}}{1} = \sqrt{1-y^2}$$
$$y = \operatorname{sen}(t) \Rightarrow t = \operatorname{arcsen}(y)$$

4. Exercício

$$y = x^2 + 1,\ y = -x^2 - 1;\ x = 1,\ x = -1$$
$$R = \left\{(x, y) \in \mathbb{R}^2 \mid -1 \le x \le 1\,,\ -x^2 - 1 \le y \le x^2 + 1\right\}$$

Figura 5: Integrais duplas - Aula 2 - Exercício IV

img/v02_a02_e04.png

$$a = \int_{-1}^{1} dx \int_{f(x)}^{g(x)} dy = \int_{-1}^{1} dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^{1} dx\, [y]_{-x^2-1}^{x^2+1} =$$
$$\int_{-1}^{1} dx\, \left[x^2 + 1 - \left(-x^2 - 1\right)\right] = \int_{-1}^{1} dx\, \left[x^2 + 1 + x^2 + 1\right] =$$
$$\int_{-1}^{1} dx\, \left[2x^2 + 2\right] = 2\int_{-1}^{1} x^2\, dx + 2\int_{-1}^{1} dx = \left[2\frac{x^3}{3} + 2x\right]_{-1}^{1} = \left[2\left(\frac{x^3 + 3x}{3}\right)\right]_{-1}^{1} =$$
$$\frac{2}{3}\left[x\left(x^2 + 3\right)\right]_{-1}^{1} = \frac{2}{3}\left[1 \cdot \left(1^2 + 3\right) - (-1)\left((-1)^2 + 3\right)\right] = \frac{2}{3}(4 + 4) = \frac{2}{3}8 = \frac{16}{3} = 5,\overline{3}$$

5. Exercício

$$R = \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 2\,,\ -y \leq x \leq y\right\}$$

Figura 6: Integrais duplas - Aula 2 - Exercício V

img/v02_a02_e05.png

$$a = \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^{y} dx = \int_0^2 dy\, [x]_{-y}^{y} = \int_0^2 dy\, [y - (-y)] = \int_0^2 dy\, [2y] =$$

$$2 \int_0^2 y\, dy = \left[\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^2 = 2^2 - 0^2 = 4$$

3 Cálculo de volume - Aula 3

1. Exercício

Figura 7: Integrais duplas - Aula 3 - Exercício I

img/v03_a03_e01.png

$$z = 4;\ dz = dxdy$$

$$v = \int_0^3 \int_0^2 z\,dz = \int_0^3 \int_0^2 4\,dydx = 4 \int_0^3 dx \int_0^2 dy = 4 \int_0^3 dx\,[y]_0^2 = 4 \int_0^3 dx\,[2-0] =$$
$$8 \int_0^3 dx = 8[x]_0^3 = 8[3-0] = 8 \cdot 3 = 24$$

2. Exercício

$$R = [0,3] \times [0,4]$$
$$\iint_R (8-2y)da$$

Figura 8: Integrais duplas - Aula 3 - Exercício II

`img/v03_a03_e02.png`

$$z = 8 - 2y;\ da = dz = dxdy$$

$$v = \int_0^3 \int_0^4 z\,dz = \int_0^3 \int_0^4 (8-2y)dxdy = \int_0^3 dx \int_0^4 (8-2y)dy =$$
$$\int_0^3 dx \left(8 \int_0^4 dy - 2 \int_0^4 y\,dy\right) = \int_0^3 dx\,2\left(4 \int_0^4 dy - \int_0^4 y\,dy\right) =$$
$$2 \int_0^3 dx \left[4y - \frac{y^2}{2}\right]_0^4 = 2 \int_0^3 dx \left[\frac{8y - y^2}{2}\right]_0^4 = \cancel{2} \int_0^3 dx\,\frac{1}{\cancel{2}}[y(8-y)]_0^4 =$$
$$\int_0^3 dx[4(8-4)\cancel{-0(8-0)}] = 16 \int_0^3 dx = 16[x]_0^3 = 16[3-0] = 48$$

4 Invertendo a ordem de integração - Aula 4

1. Exercício

$$z = f(x,y) = y\,\mathrm{e}^x;\ dz = dxdy$$

$$v = \int_2^4 \int_1^9 z\,dz = \int_2^4 \int_1^9 y\,\mathrm{e}^x\ dydx = \int_2^4 \mathrm{e}^x\ dx \int_1^9 y\,dy = \int_2^4 \mathrm{e}^x\ dx \left[\frac{y^2}{2}\right]_1^9 =$$
$$\int_2^4 \mathrm{e}^x\ dx \frac{1}{2}\left[y^2\right]_1^9 = \frac{1}{2}\int_2^4 \mathrm{e}^x\ dx \left[9^2 - 1^2\right] = 40 \int_2^4 \mathrm{e}^x\ dx = 40\left[\mathrm{e}^x\right]_2^4 = 40\left[\mathrm{e}^4 - \mathrm{e}^2\right] =$$
$$40e^2\left(e^2 - 1\right)$$

2. Exercício

$$z = f(x, y) = x^2y^3;\ dz = dxdy$$

Figura 9: Integrais duplas - Aula 4 - Exercício II

img/v04_a04_e02.png

$$v = \int_0^1 \int_2^4 z\,dz = \int_0^1 \int_2^4 x^2y^3\,dxdy = \int_0^1 x^2\,dx \int_2^4 y^3\,dy = \int_0^1 x^2\,dx \left[\frac{y^4}{4}\right]_2^4 =$$
$$\frac{1}{4}\int_0^1 x^2\,dx \left[y^4\right]_2^4 = \frac{1}{4}\int_0^1 x^2\,dx \left[4^4 - 2^4\right] = \frac{1}{4}\int_0^1 x^2\,dx \left[2^8 - 2^4\right] =$$
$$\frac{1}{4}\int_0^1 x^2\,dx \left[2^4\left(2^4 - 1\right)\right] = \frac{1}{4}\int_0^1 x^2\,dx\,[16 \cdot 15] = 60\int_0^1 x^2\,dx = 60\left[\frac{x^3}{3}\right]_0^1 = 20\left[x^3\right]_0^1 =$$
$$20\left[1^3 - 0^3\right] = 20 \cdot 1 = 20$$

3. Exercício

$$\iint_R (x + 2y)da$$

R = Região limitada pela parábola $y = x^2 + 1$ e as retas $x = -1$ e $x = 2$.

$$z = f(x, y) = x + 2y;\ da = dz = dxdy$$

Figura 10: Integrais duplas - Aula 4 - Exercício III

img/v04_a04_e03.png

$$v = \int_{-1}^{2}\int_{0}^{x^2+1} z\,dz = \int_{-1}^{2}\int_{0}^{x^2+1}(x+2y)dxdy =$$

$$\int_{-1}^{2} dx \int_{0}^{x^2+1}(x+2y)dy = \int_{-1}^{2} dx\left(x\int_{0}^{x^2+1} dy + 2\int_{0}^{x^2+1} y\,dy\right) =$$

$$\int_{-1}^{2} dx\left[xy + \cancel{2}\frac{y^2}{\cancel{2}}\right]_0^{x^2+1} = \int_{-1}^{2} dx\,[y(x+y)]_0^{x^2+1} =$$

$$\int_{-1}^{2} dx\left[\left(x^2+1\right)\left[x+\left(x^2+1\right)\right]\cancel{-0(x+0)}\right] = \int_{-1}^{2} dx\left[\left(x^2+1\right)\left(x^2+x+1\right)\right] =$$

$$\int_{-1}^{2} dx\left(x^4+x^3+2x^2+x+1\right) =$$

$$\int_{-1}^{2} x^4\,dx + \int_{-1}^{2} x^3\,dx + 2\int_{-1}^{2} x^2\,dx + \int_{-1}^{2} x\,dx + \int_{-1}^{2} dx =$$

$$\left[\frac{x^5}{5}+\frac{x^4}{4}+2\frac{x^3}{3}+\frac{x^2}{2}+x\right]_{-1}^{2} = \left[\frac{12x^5+15x^4+40x^3+30x^2+60x}{60}\right]_{-1}^{2} =$$

$$\frac{1}{60}\left[x\left(12x^4+15x^3+40x^2+30x+60\right)\right]_{-1}^{2} =$$

$$\frac{1}{60}\left[2\left(12\cdot 2^4+15\cdot 2^3+40\cdot 2^2+30\cdot 2+60\right)\right.$$

$$\left.-(-1)\left(12(-1)^4+15(-1)^3+40(-1)^2+30(-1)+60\right)\right] =$$

$$\frac{1}{60}[2(192+120+160+60+60)+(12-15+40-30+60)] = \frac{1}{60}(1184+67) =$$

$$\frac{1251}{60} = \frac{417}{20} = 20,85$$

5 Cálculo de integrais duplas ou iteradas

5.1 Aula 5

1. Exercício

$$f(x,y) = x^3;\ 0 \le x \le 2;\ x^2 \le y \le 4$$
$$\iint_R f(x,y)dydx$$

Figura 11: Integrais duplas - Aula 5 - Exercício I

img/v05_a05_e01.png

$$v = \int_0^2 \int_{x^2}^4 x^3\, dxdy = \int_0^2 x^3\, dx \int_{x^2}^4 dy = \int_0^2 x^3\, dx\, [y]_{x^2}^4 = \int_0^2 x^3\, dx\ \left[4 - x^2\right] =$$
$$4\int_0^2 x^3\, dx - \int_0^2 x^5\, dx = \left[\cancel{4}\frac{x^4}{\cancel{4}} - \frac{x^6}{6}\right]_0^2 = \left[\frac{6x^4 - x^6}{6}\right]_0^2 = \frac{1}{6}\left[x^4\left(6 - x^2\right)\right]_0^2 =$$
$$\frac{1}{6}\left[2^4\left(6 - 2^2\right) - \cancel{0^4\left(6 - 0^2\right)}\right] = \frac{1}{6}(16 \cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2$$

2. Exercício

$$f(x,y) = x^2y;\ 1 \le x \le 3;\ x \le y \le 2x + 1$$
$$\iint_R f(x,y)dydx$$

Figura 12: Integrais duplas - Aula 5 - Exercício II

`img/v05_a05_e02.png`

$$v = \int_1^3 \int_x^{2x+1} x^2 y \, dx dy = \int_1^3 x^2 \, dx \int_x^{2x+1} y \, dy = \int_1^3 x^2 \, dx \left[\frac{y^2}{2} \right]_x^{2x+1} =$$

$$\int_1^3 x^2 \, dx \frac{1}{2} \left[(2x+1)^2 - (x)^2 \right] = \frac{1}{2} \int_1^3 x^2 \, dx \left(3x^2 + 4x + 1 \right) =$$

$$\frac{3}{2} \int_1^3 x^4 \, dx + 2 \int_1^3 x^3 \, dx + \frac{1}{2} \int_1^3 x^2 \, dx = \left[\frac{3}{2} \frac{x^5}{5} + 2 \frac{x^4}{4} + \frac{1}{2} \frac{x^3}{3} \right]_1^3 = \left[\frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6} \right]_1^3 =$$

$$\left[\frac{18x^5 + 30x^4 + 10x^3}{60} \right]_1^3 = \left[\frac{2x^3 \left(9x^2 + 15x + 5 \right)}{60} \right]_1^3 =$$

$$\frac{1}{30} \left[x^3 \left(9x^2 + 15x + 5 \right) \right]_1^3 = \frac{1}{30} \left[3^3 \left(9 \cdot 3^2 + 15 \cdot 3 + 5 \right) - 1^3 \left(9 \cdot 1^2 + 15 \cdot 1 + 5 \right) \right] =$$

$$\frac{1}{30} \left[27(81 + 45 + 5) - (9 + 15 + 5) \right] = \frac{1}{30} \left[27 \cdot 131 - 29 \right] = \frac{3508}{30} = 116,9\overline{3}$$

5.2 Aula 6

1. Exercício

$$f(x,y) = 1; \; 0 \le x \le 1; \; 1 \le y \le \mathrm{e}^x$$

$$\iint_R f(x,y) dy dx$$

$$v = \int_0^1 \int_1^{\mathrm{e}^x} dy dx = \int_0^1 dx \, [y]_1^{\mathrm{e}^x} = \int_0^1 dx \, \left(\mathrm{e}^x - 1 \right) = \left[\mathrm{e}^x - x \right]_0^1 =$$

$$\mathrm{e}^1 - 1 - \left(\mathrm{e}^0 - 0 \right) = \mathrm{e} - 1 - 1 = \mathrm{e} - 2$$

2. Exercício

$$f(x,y)=x;\ 0\le x\le 1;\ 1\le y\le \mathrm{e}^{x^2}$$

$$\iint_R f(x,y)dydx$$

$$v=\int_0^1\int_1^{\mathrm{e}^{x^2}} x\,dxdy=\int_0^1 x\,dx\int_1^{\mathrm{e}^{x^2}} dy=\int_0^1 x\,dx\,[y]_1^{\mathrm{e}^{x^2}}=\int_0^1 x\,dx\,\left(\mathrm{e}^{x^2}-1\right)=$$

$$\int_0^1 x\,\mathrm{e}^{x^2}\,dx-\int_0^1 x\,dx=\int_0^1 \mathrm{e}^u\,\frac{du}{2}-\int_0^1 x\,dx=\frac{1}{2}\int_0^1 \mathrm{e}^u\,du-\int_0^1 x\,dx=$$

$$\left[\frac{1}{2}e^u-\frac{x^2}{2}\right]_0^1=\left[\frac{e^{x^2}-x^2}{2}\right]_0^1=\frac{1}{2}\left[e^{x^2}-x^2\right]_0^1=\frac{1}{2}\left[e^{1^2}-1^2-\left(e^{0^2}-0^2\right)\right]=$$

$$\frac{1}{2}(\mathrm{e}-1-1)=\frac{\mathrm{e}-2}{2}$$

$$u=x^2;\ \frac{du}{2}=x\,dx$$

3. Exercício

$$f(x,y)=2xy;\ 0\le y\le 1;\ y^2\le x\le y$$

$$\iint_R f(x,y)dxdy$$

$$v=\int_0^1\int_{y^2}^{y} 2xy\,dxdy=2\int_0^1 y\,dy\int_{y^2}^{y} x\,dx=2\int_0^1 y\,dy\,\left[\frac{x^2}{2}\right]_{y^2}^{y}=\cancel{2}\int_0^1 y\,dy\,\frac{1}{\cancel{2}}\left[x^2\right]_{y^2}^{y}=$$

$$\int_0^1 y\,dy\,\left(y^2-y^4\right)=\int_0^1\left(y^3-y^5\right)dy=\left[\frac{y^4}{4}-\frac{y^6}{6}\right]_0^1=$$

$$\left[\frac{6y^4-4y^6}{24}\right]_0^1=\left[\frac{2y^4\left(3-2y^2\right)}{24}\right]_0^1=\frac{1}{12}\left[1^4\left(3-2\cdot 1^2\right)\cancel{-0^4\left(3-2\cdot 0^2\right)}\right]=\frac{1}{12}=0{,}08\overline{3}$$

5.3 Aula 7

1. Exercício

$$f(x,y)=\frac{1}{x+y};\ 1\le y\le \mathrm{e};\ 0\le x\le y$$

$$\iint_R f(x,y)dxdy$$

$$v = \int_1^{\mathrm{e}} \int_0^y \frac{1}{x+y}\, dxdy = \int_1^{\mathrm{e}} dy \int_0^y (x+y)^{-1}\, dx = \int_1^{\mathrm{e}} dy \int_0^y u^{-1}\, du =$$
$$\int_1^{\mathrm{e}} dy \int_0^y\ [\ln|u|]_0^y = \int_1^{\mathrm{e}} dy \int_0^y\ [\ln|x+y|]_0^y = \int_1^{\mathrm{e}} dy \int_0^y\ (\ln|y+y| - \ln|0+y|) =$$
$$\int_1^{\mathrm{e}} dy \int_0^y\ (\ln|2y| - \ln|y|) = \int_1^{\mathrm{e}} dy \int_0^y\ (\ln|2|\cancel{+\ln|y| - \ln|y|}) = \ln|2| \int_1^{\mathrm{e}} dy =$$
$$\ln|2|[y]_1^{\mathrm{e}} = \ln|2|(\mathrm{e}-1)$$

$$u = x + y;\ du = (1+0)dx = dx$$

6 Cálculo de área - Aula 8

1. Exercício

Figura 13: Integrais duplas - Aula 8 - Exercício I

`img/v08_a08_e01.png`

$$
\begin{aligned}
a = \int_{-1}^{0} dx \int_{0}^{x^2+1} dy + \int_{-1}^{0} dx \int_{-1}^{0} dy + \int_{0}^{y^2} dx \int_{-1}^{0} dy + \int_{0}^{1} dx \int_{\sqrt{x}}^{x^2+1} &= \\
\int_{-1}^{0} dx \left(\int_{0}^{x^2+1} dy + \int_{-1}^{0} dy \right) + \int_{0}^{y^2} dx \int_{-1}^{0} dy + \int_{0}^{1} dx \int_{\sqrt{x}}^{x^2+1} &= \\
\int_{-1}^{0} dx \left([y]_{0}^{x^2+1} + [y]_{-1}^{0} \right) + \int_{-1}^{0} dy \, [x]_{0}^{y^2} + \int_{0}^{1} dx \, [y]_{\sqrt{x}}^{x^2+1} &= \\
\int_{-1}^{0} dx \, \left(x^2 + 1 + 1 \right) + \int_{-1}^{0} dy \, y^2 + \int_{0}^{1} dx \, \left(x^2 + 1 - \sqrt{x} \right) &= \\
\int_{-1}^{0} \left(x^2 + 2 \right) dx + \int_{-1}^{0} y^2 \, dy + \int_{0}^{1} \left(x^2 - x^{\frac{1}{2}} + 1 \right) dx &= \\
\left[\frac{x^3}{3} + 2x \right]_{-1}^{0} + \left[\frac{y^3}{3} \right]_{-1}^{0} + \left[\frac{x^3}{3} - \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2} \right)} + x \right]_{0}^{1} &= \\
\left[\frac{x^3 + 6x}{3} \right]_{-1}^{0} + \frac{1}{3} \left[y^3 \right]_{-1}^{0} + \left[\frac{x^3}{3} - \frac{2\sqrt{x^3}}{3} + x \right]_{0}^{1} &= \\
\frac{1}{3} \left[x \left(x^2 + 6 \right) \right]_{-1}^{0} + \frac{1}{3} \left[\cancel{0^3} - (-1)^3 \right] + \left[\frac{x^3 - 2\sqrt{x^3} + 3x}{3} \right]_{0}^{1} &= \\
\frac{1}{3} \left[\cancel{0 \left(0^2 + 6 \right)} - (-1) \left((-1)^2 + 6 \right) \right] + \frac{1}{3} + \frac{1}{3} \left[x^3 - 2\sqrt{x^3} + 3x \right]_{0}^{1} &= \\
\frac{7}{3} + \frac{1}{3} + \frac{1}{3} \left[1^3 - 2\sqrt{1^3} + 3 \cdot 1 \cancel{- \left(0^3 - 2\sqrt{0^3} + 3 \cdot 0 \right)} \right] = \frac{7}{3} + \frac{1}{3} + \frac{2}{3} &= \\
\frac{7 + 1 + 2}{3} = \frac{10}{3} &= 3,\overline{3}
\end{aligned}
$$

$$
\begin{aligned}
a = \int_{-1}^{1} dx \int_{1}^{x^2+1} dy + \int_{-1}^{y^2} dx \int_{-1}^{1} dy = \int_{-1}^{1} dx \, [y]_{1}^{x^2+1} + \int_{-1}^{1} dy \, [x]_{-1}^{y^2} &= \\
\int_{-1}^{1} dx \, \left(x^2 \cancel{+1-1} \right) + \int_{-1}^{1} dy \, \left(y^2 + 1 \right) = \left[\frac{x^3}{3} \right]_{-1}^{1} + \left[\frac{y^3}{3} + y \right]_{-1}^{1} &= \\
\frac{1}{3} \left[x^3 \right]_{-1}^{1} + \frac{1}{3} \left[y \left(y^2 + 3 \right) \right]_{-1}^{1} &= \\
\frac{1}{3} \left(\left[1^3 - (-1)^3 \right] + \left[1 \left(1^2 + 3 \right) - (-1) \left((-1)^2 + 3 \right) \right] \right) \frac{1}{3} (2 + 4 + 4) = \frac{10}{3} &= 3,\overline{3}
\end{aligned}
$$

7 Cálculo de volume

7.1 Aula 9

1. Exercício

   Esboçe a região de integração e o sólido cujo volume é dado pela integral abaixo:

   $$\int_0^1 \int_0^1 (4 - x - 2y)\; dxdy$$

Figura 14: Integrais duplas - Aula 9 - Exercício I

img/v09_a09_e01.png

$$v = \int_0^1 \int_0^1 (4 - x - 2y)\; dxdy = \int_0^1 dx \left(4 \int_0^1 dy - x \int_0^1 dy - 2 \int_0^1 y\, dy\right) =$$

$$4 \int_0^1 dx \int_0^1 dy - \int_0^1 x\, dx \int_0^1 dy - 2 \int_0^1 dx \int_0^1 y\, dy =$$

$$4[x]_0^1[y]_0^1 - \left[\frac{x^2}{2}\right]_0^1 [y]_0^1 - 2[x]_0^1 \left[\frac{y^2}{2}\right]_0^1 = 4 - \frac{1}{2} - \cancel{2}\frac{1}{\cancel{2}} = \frac{8-1-2}{2} = \frac{5}{2} = 2,5$$

7.2 Aula 10

1. Exercício

   Calcule o volume do sólido limitado pelos planos:

   $$x = 0,\; y = 0,\; z = 0 \text{ e } 6x + 2y + 3z = 6$$

$$P_1 = (0, 0, 0)$$

$$6x = -2y - 3z + 6 \Rightarrow x = \frac{-2y - 3z + 6}{6} = \frac{-2 \cdot 0 - 3 \cdot 0 + 6}{6} = \frac{6}{6} = 1 \Rightarrow P_2 = (1, 0, 0)$$

Figura 15: Integrais duplas - Aula 10 - Exercício I

img/v10_a10_e01.png

$$2y=-6x-3z+6\Rightarrow y=\frac{-6x-3z+6}{2}=\frac{-6\cdot 0-3\cdot 0+6}{2}=\frac{6}{2}=3\Rightarrow P_3=(0,3,0)$$
$$3z=-6x-2y+6\Rightarrow z=\frac{-6x-2y+6}{3}=\frac{-6\cdot 0-2\cdot 0+6}{3}=\frac{6}{3}=2\Rightarrow P_4=(0,0,2)$$
$$x=0,x=1$$
$$y=0,\ y=\frac{-6x-3z+6}{2}=\frac{-6x-3\cdot 0+6}{2}=-3x+3$$
$$z=\frac{-6x-2y+6}{3}=-2x-\frac{2y}{3}+2$$

$$v=\int_0^1\int_0^{-3x+3}\left(-2x-\frac{2y}{3}+2\right)dxdy=\int_0^1 dx\int_0^{-3x+3}\left(-2x-\frac{2y}{3}+2\right)dy=$$
$$\int_0^1 dx\left[-2xy-\frac{\cancel{2}}{3}\frac{y^2}{\cancel{2}}+2y\right]_0^{-3x+3}=\int_0^1 dx\,\frac{1}{3}\left[-6xy-y^2+6y\right]_0^{-3x+3}=$$
$$\frac{1}{3}\int_0^1 dx\,\left[-y(6x+y-6)\right]_0^{-3x+3}=$$
$$\frac{1}{3}\int_0^1 dx\,\left[-(-3x+3)(6x+(-3x+3)-6)\cancel{+0(6x+0-6)}\right]=$$
$$\frac{1}{3}\int_0^1 dx\,\left[(3x-3)(3x-3)\right]=\frac{1}{3}\int_0^1\left(9x^2-18x+9\right)dx=\frac{1}{3}\left[9\frac{x^3}{3}-18\frac{x^2}{2}+9x\right]_0^1=$$
$$\frac{1}{3}\left[3x^3-9x^2+9x\right]_0^1=\frac{1}{\cancel{3}}\left[\cancel{3}x\left(x^2-3x+3\right)\right]_0^1=$$
$$\left[1\left(1^2\cancel{-3\cdot 1+3}\right)\cancel{-0\left(0^2-3\cdot 0+3\right)}\right]=1$$

8 Coordenadas polares

8.1 Aula 1

1. Exercício

Figura 16: Coordenadas polares - Aula 01 - Exercício I

img/v11_a01_e01.png

Calcule a área do circulo de raio igual a dois

$$r = 2 \Rightarrow a = \pi r^2 = 2^2\pi = 4\pi$$

$$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 2^2 \Rightarrow x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$$

$$R = \left\{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\right\}$$

$$
\begin{aligned}
a = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{-2}^{2} dx \left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) = 2\int_{-2}^{2} \sqrt{4-x^2}\, dx = \\
2\int_{-2}^{2} \sqrt{4-(2\operatorname{sen}(\alpha))^2}\, 2\cos(\alpha)\, d\alpha = 4\int_{-2}^{2} \sqrt{4-4\operatorname{sen}^2(\alpha)}\, \cos(\alpha)\, d\alpha = \\
4\int_{-2}^{2} \sqrt{4-4\left(1-\cos^2(\alpha)\right)}\, \cos(\alpha)\, d\alpha = 4\int_{-2}^{2} \sqrt{4-\left(4-4\cos^2(\alpha)\right)}\, \cos(\alpha)\, d\alpha = \\
4\int_{-2}^{2} \sqrt{\cancel{4-4} + 4\cos^2(\alpha)}\, \cos(\alpha)\, d\alpha = 4\int_{-2}^{2} 2\cos(\alpha)\cos(\alpha)\, d\alpha = \\
8\int_{-2}^{2} \cos^2(\alpha)\, d\alpha = 8\int_{-2}^{2} \left(\frac{1+\cos(2\alpha)}{2}\right) d\alpha = 8\int_{-2}^{2} \left(\frac{1}{2} + \frac{\cos(2\alpha)}{2}\right) d\alpha = \\
4\int_{-2}^{2} d\alpha + 4\int_{-2}^{2} \cos(2\alpha)\, d\alpha = 4\int_{-2}^{2} d\alpha + 4\int_{-2}^{2} \cos(u)\, \frac{du}{2} = \\
4\int_{-2}^{2} d\alpha + 2\int_{-2}^{2} \cos(u)\, du = \left[4\alpha + 2\operatorname{sen}(u)\right]_{-2}^{2} = \left[4\alpha + 2\operatorname{sen}(2\alpha)\right]_{-2}^{2} = \\
\left[4\alpha + 4\operatorname{sen}(\alpha)\cos(\alpha)\right]_{-2}^{2} = \left[4\left(\alpha + \operatorname{sen}(\alpha)\cos(\alpha)\right)\right]_{-2}^{2} = \\
\left[4\left(\operatorname{arcsen}\left(\frac{x}{2}\right) + \frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right)\right]_{-2}^{2} = \left[4\left(\operatorname{arcsen}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4}\right)\right]_{-2}^{2} = \\
4\left(\operatorname{arcsen}\left(\frac{2}{2}\right) + \frac{2\sqrt{4-2^2}}{4}\right) - 4\left(\operatorname{arcsen}\left(\frac{(-2)}{2}\right) + \frac{(-2)\sqrt{4-(-2)^2}}{4}\right) = \\
4\operatorname{arcsen}(1) - 4\operatorname{arcsen}(-1) = 4(\operatorname{arcsen}(1) - \operatorname{arcsen}(-1)) = 4\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 4\left(\frac{\cancel{2}\pi}{\cancel{2}}\right) = 4\pi
\end{aligned}
$$

$$x = 2\operatorname{sen}(\alpha);\; dx = 2\cos(\alpha)\, d\alpha$$

$$u = 2\alpha;\; \frac{du}{2} = d\alpha$$

$$\operatorname{sen}(\alpha) = \frac{co}{h} = \frac{x}{2} \Rightarrow \alpha = \operatorname{arcsen}\left(\frac{x}{2}\right)$$

$$h^2 = co^2 + ca^2 \Rightarrow 2^2 = x^2 + ca^2 \Rightarrow ca = \sqrt{4-x^2}$$

$$\cos(\alpha) = \frac{ca}{h} = \frac{\sqrt{4-x^2}}{2}$$

$$R = \left\{(r,\theta) \in \mathbb{R}^2 \,|\, 0 \le r \le 2,\; 0 \le \theta \le 2\pi\right\}$$

$$
\begin{aligned}
a = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{0}^{2}\int_{0}^{2\pi} r\, dr d\theta = \int_{0}^{2} r\, dr \int_{0}^{2\pi} d\theta = \left[\frac{r^2}{2}\right]_{0}^{2} [\theta]_{0}^{2\pi} = \\
\frac{1}{2}\left[2^2 - 0^2\right]\left[2\pi - 0\right] = \frac{4}{\cancel{2}}\cancel{2}\pi = 4\pi
\end{aligned}
$$

2. Exercício

$$\iint_R \frac{da}{1+x^2+y^2}$$

Figura 17: Coordenadas polares - Aula 01 - Exercício II

img/v11_a01_e02.png

$$R = \left\{ (r,\theta) \in \mathbb{R}^2 \,|\, 0 \leq r \leq 2,\, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{2} \right\}$$

$$v = \iint_R \frac{da}{1+x^2+y^2} = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\,drd\theta}{1+r^2} = \int_0^2 \frac{r\,dr}{1+r^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} d\theta =$$
$$\int_0^2 \left(1+r^2\right)^{-1} r\,dr\,[\theta]_{\frac{\pi}{4}}^{\frac{3\pi}{2}} = \int_0^2 \left(1+r^2\right)^{-1} r\,dr \left(\frac{3\pi}{2} - \frac{\pi}{4}\right) =$$
$$\int_0^2 \left(1+r^2\right)^{-1} r\,dr \left(\frac{6\pi-\pi}{4}\right) = \frac{5\pi}{4} \int_0^2 \left(1+r^2\right)^{-1} r\,dr = \frac{5\pi}{4} \int_0^2 u^{-1} \frac{du}{2} =$$
$$\frac{5\pi}{8} \int_0^2 u^{-1} du = \frac{5\pi}{8} [ln|u|]_0^2 = \frac{5\pi}{8} \left[ln|1+r^2|\right]_0^2 = \frac{5\pi}{8} \left[ln|1+2^2| - ln|1+0^2|\right] =$$
$$\frac{5\pi}{8} [ln|5| - ln|1|] = \frac{5\pi ln|5|}{8}$$

$$u = 1 + r^2 \Rightarrow \frac{du}{2} = r\,dr$$
$$\mathrm{e}^x = 1 = \mathrm{e}^0 \Rightarrow x = 0$$

8.2 Aula 2

1. Exercício

$$\iint_R \mathrm{e}^{x^2+y^2}\, dxdy$$

R, região entre as curvas abaixo:

$$x^2 + y^2 = 4$$

$$x^2 + y^2 = 9$$

Figura 18: Coordenadas polares - Aula 02 - Exercício I

img/v12_a02_e01.png

$$x^2 + y^2 = r^2 \Rightarrow \mathrm{e}^{x^2+y^2} = \mathrm{e}^{r^2}$$
$$da = dxdy = r\, drd\theta$$
$$R = \left\{(r,\theta) \in \mathbb{R}^2 \,|\, 2 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\right\}$$

$$v = \iint_R \mathrm{e}^{x^2+y^2}\, dxdy = \int_2^3 \int_0^{2\pi} \mathrm{e}^{r^2}\, r\, drd\theta = \int_2^3 \mathrm{e}^{r^2}\, r\, dr \int_0^{2\pi} d\theta = \int_2^3 \mathrm{e}^u\, \frac{du}{2} \int_0^{2\pi} d\theta =$$
$$\frac{1}{2}\int_2^3 \mathrm{e}^u\, du \int_0^{2\pi} d\theta = \frac{1}{2}\left[\mathrm{e}^u\right]_2^3 \left[\theta\right]_0^{2\pi} = \frac{1}{\cancel{2}}\left[\mathrm{e}^{r^2}\right]_2^3 \cancel{2}\pi = \left(\mathrm{e}^{3^2} - \mathrm{e}^{2^2}\right)\pi = \pi\left(\mathrm{e}^9 - \mathrm{e}^4\right)$$

$$u = r^2 \Rightarrow \frac{du}{2} = r\, dr$$

2. Exercício

$$\iint_R \sqrt{x^2 + y^2}\, dxdy$$

R, região cujo o contorno é:

$$x^2 + y^2 = 4$$

$$x^2 + y^2 = r^2 \Rightarrow \sqrt{x^2 + y^2} = \sqrt{r^2} = r$$
$$da = dxdy = r\,drd\theta$$
$$R = \left\{(r, \theta) \in \mathbb{R}^2 \,|\, 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\right\}$$

$$v = \iint_R \sqrt{x^2 + y^2}\, dxdy = \int_0^2 \int_0^{2\pi} r^2\, drd\theta = \int_0^2 r^2\, dr \int_0^{2\pi} d\theta = \left[\frac{r^3}{3}\right]_0^2 [\theta]_0^{2\pi} = \frac{2^3}{3} 2\pi = \frac{16\pi}{3}$$

A Derivadas

B Derivadas simples

Tabela 1: Derivadas simples

y	$=$	c	$\Rightarrow$	y'	$=$	0
y	$=$	x	$\Rightarrow$	y'	$=$	1
y	$=$	x^c	$\Rightarrow$	y'	$=$	cx^{c-1}
y	$=$	e^x	$\Rightarrow$	y'	$=$	e^x
y	$=$	$\ln\lvert x\rvert$	$\Rightarrow$	y'	$=$	$\frac{1}{x}$
y	$=$	uv	$\Rightarrow$	y'	$=$	$u'v + uv'$
y	$=$	$\frac{u}{v}$	$\Rightarrow$	y'	$=$	$\frac{u'v - uv'}{v^2}$
y	$=$	u^c	$\Rightarrow$	y'	$=$	$cu^{c-1}u'$
y	$=$	e^u	$\Rightarrow$	y'	$=$	$\mathrm{e}^u\, u'$
y	$=$	c^u	$\Rightarrow$	y'	$=$	$c^u u' ln\lvert c\rvert$
y	$=$	$\ln\lvert u\rvert$	$\Rightarrow$	y'	$=$	$\frac{u'}{u}$
y	$=$	$\log_c\lvert u\rvert$	$\Rightarrow$	y'	$=$	$\frac{u'}{u} \log_c \lvert\,\mathrm{e}\,\rvert$

Tabela 2: Derivadas trigonométricas

y	$=$	$\operatorname{sen}(x)$	$\Rightarrow$	y'	$=$	$\cos(x)$
y	$=$	$\cos(x)$	$\Rightarrow$	y'	$=$	$-\operatorname{sen}(x)$
y	$=$	$\operatorname{tg}(x)$	$\Rightarrow$	y'	$=$	$\sec^2(x)$
y	$=$	$\operatorname{cotg}(x)$	$\Rightarrow$	y'	$=$	$-\operatorname{cossec}^2(x)$
y	$=$	$\sec(x)$	$\Rightarrow$	y'	$=$	$\sec(x)\operatorname{tg}(x)$
y	$=$	$\operatorname{cossec}(x)$	$\Rightarrow$	y'	$=$	$-\operatorname{cossec}(x)\operatorname{cotg}(x)$
y	$=$	$\operatorname{arcsen}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{1}{\sqrt{1-x^2}}$
y	$=$	$\arccos(x)$	$\Rightarrow$	y'	$=$	$\dfrac{-1}{\sqrt{1-x^2}}$
y	$=$	$\operatorname{arctg}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{1}{1+x^2}$
y	$=$	$\operatorname{arccotg}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{-1}{1+x^2}$
y	$=$	$\operatorname{arcsec}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$
y	$=$	$\operatorname{arccossec}(x)$	$\Rightarrow$	y'	$=$	$\dfrac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

C Derivadas trigonométricas

D Integrais

E Integrais simples

F Integrais trigonométricas

G Relação entre coordenada cartesina e polar

$$P(x, y) \to P(r, \theta)$$

Tabela 3: Integrais simples

$$
\left|
\begin{array}{lcll}
\displaystyle\int dx & = & x + c & \\
\displaystyle\int x^p \, dx & = & \dfrac{x^{p+1}}{p+1} + c & \rightarrow \quad p \neq -1 \\
\displaystyle\int \mathrm{e}^x \; dx & = & \mathrm{e}^x + c & \\
\displaystyle\int \dfrac{dx}{x} & = & \ln|x| + c & \\
\displaystyle\int u^p \, du & = & \dfrac{u^{p+1}}{p+1} + c & \rightarrow \quad p \neq -1 \\
\displaystyle\int \mathrm{e}^u \; du & = & \mathrm{e}^u + c & \\
\displaystyle\int \dfrac{du}{u} & = & \ln|u| + c & \\
\displaystyle\int p^u \, du & = & \dfrac{p^u}{\ln|p|} + c &
\end{array}
\right|
$$

Figura 19: Coordenada cartesina e polar

(a) Coordenada cartesiana ou retangular

img/coordenada_cartesiana.png

(b) Coordenada polar

img/coordenada_polar.png

Figura 20: Determinação do seno, cosseno e tangente

img/triangulo_retangulo.png

Figura 21: Círculo trigonométrico

img/circulo_trigonometrico.png

H Funções trigonométricas

I Determinação do seno, cosseno e tangente

J Círculo trigonométrico

K Identidades trigonométricas

L Relação entre trigonométricas e inversas

M Substituição trigonométrica

N Ângulos notáveis

Referências

[1] Fernando Grings. *Curso de Integrais Duplas e Triplas.* Youtube, 2016.

Tabela 4: Integrais trigonométricas

$$
\begin{array}{|lcl|}
\int \operatorname{sen}(u)du & = & -\cos(u)+c \\
\int \cos(u)du & = & \operatorname{sen}(u)+c \\
\int \operatorname{tg}(u)du & = & \ln|\sec(u)|+c \\
\int \operatorname{cotg}(u)du & = & \ln|\operatorname{sen}(u)|+c \\
\int \sec(u)du & = & \ln|\sec(u)+\operatorname{tg}(u)|+c \\
\int \operatorname{cossec}(u)du & = & \ln|\operatorname{cossec}(u)-\operatorname{cotg}(u)|+c \\
\int \sec^2(u)du & = & \operatorname{tg}(u)+c \\
\int \operatorname{cossec}^2(u)du & = & -\operatorname{cotg}(u)+c \\
\int \sec(u)\operatorname{tg}(u)du & = & \sec(u)+c \\
\int \operatorname{cossec}(u)\operatorname{cotg}(u)du & = & -\operatorname{cossec}(u)+c \\
\int \frac{du}{\sqrt{1-x^2}} & = & \operatorname{arcsen}(x)+c \\
-\int \frac{du}{\sqrt{1-x^2}} & = & \arccos(x)+c \\
\int \frac{du}{1+x^2} & = & \operatorname{arctg}(x)+c \\
-\int \frac{du}{1+x^2} & = & \operatorname{arccotg}(x)+c \\
\int \frac{du}{|x|\sqrt{x^2-1}} & = & \operatorname{arcsec}(x)+c \\
-\int \frac{du}{|x|\sqrt{x^2-1}} & = & \operatorname{arccossec}(x)+c
\end{array}
$$

Tabela 5: Relação entre coordenada cartesina e polar

$$
\begin{array}{|lclcl|}
 & & x & = & r\cos\theta \\
 & & y & = & r\operatorname{sen}\theta \\
 & & x^2+y^2 & = & r^2 \\
da & = & dxdy & = & r\,drd\theta \\
v & = & \iint_{R(x,y)} f(x,y)\,dxdy & = & \iint_{R(r,\theta)} f(r\cos\theta, r\operatorname{sen}\theta)r\,drd\theta
\end{array}
$$

Tabela 6: Identidades trigonométricas

$$
\begin{array}{|lcl|}
\operatorname{tg}(x) & = & \dfrac{\operatorname{sen}(x)}{\cos(x)} \\
\operatorname{cotg}(x) & = & \dfrac{\cos(x)}{\operatorname{sen}(x)} \\
\sec(x) & = & \dfrac{1}{\cos(x)} \\
\operatorname{cossec}(x) & = & \dfrac{1}{\operatorname{sen}(x)} \\
\operatorname{sen}^2(x) + \cos^2(x) & = & 1 \\
\sec^2(x) - \operatorname{tg}^2(x) & = & 1 \\
\operatorname{cossec}^2(x) - \operatorname{cotg}^2(x) & = & 1 \\
\operatorname{sen}^2(x) & = & \dfrac{1 - \cos(2x)}{2} \\
\cos^2(x) & = & \dfrac{1 + \cos(2x)}{2} \\
\operatorname{sen}(2x) & = & 2\operatorname{sen}(x)\cos(x) \\
\cos(2x) & = & \cos^2(x) - \operatorname{sen}^2(x)
\end{array}
$$

Tabela 7: Relação entre trigonométricas e inversas

$$
\begin{array}{|lcccccl|}
\operatorname{sen}(\theta) & = & x & \Rightarrow & \theta & = & \operatorname{arcsen}(x) \\
\cos(\theta) & = & x & \Rightarrow & \theta & = & \arccos(x) \\
\operatorname{tg}(\theta) & = & x & \Rightarrow & \theta & = & \operatorname{arctg}(x) \\
\operatorname{cossec}(\theta) & = & x & \Rightarrow & \theta & = & \operatorname{arccossec}(x) \\
\sec(\theta) & = & x & \Rightarrow & \theta & = & \operatorname{arcsec}(x) \\
\operatorname{cotg}(\theta) & = & x & \Rightarrow & \theta & = & \operatorname{arccotg}(x)
\end{array}
$$

Tabela 8: Substituição trigonométrica

$$
\begin{array}{|ccccl|}
\sqrt{a^2 - x^2} & \Rightarrow & x & = & a\operatorname{sen}(\theta) \\
\sqrt{a^2 + x^2} & \Rightarrow & x & = & a\operatorname{tg}(\theta) \\
\sqrt{x^2 - a^2} & \Rightarrow & x & = & a\sec(\theta)
\end{array}
$$

Tabela 9: Ângulos notáveis

ângulo	$0°$ (0)	$30°$ $\left(\frac{\pi}{6}\right)$	$45°$ $\left(\frac{\pi}{4}\right)$	$60°$ $\left(\frac{\pi}{3}\right)$	$90°$ $\left(\frac{\pi}{2}\right)$
sen	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\nexists$