

César Antônio de Magalhães

Curso de integrais duplas e triplas

Brasil

2016

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Curso de integrais duplas e triplas

Exercícios de integrais duplas e triplas em
conformidade com as normas ABNT.

Universidade Norte do Paraná – Unopar

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Lista de abreviaturas e siglas

ABNT	Associação Brasileira de Normas Técnicas
v	Volume
a	Área
R	Região
P	Ponto
r	Raio
co	Cateto oposto
ca	Cateto adjacente
h	Hipotenusa
sen	Seno
cos	Cosseno
tg	Tangente
sec	Secante
cossec	Cossecante
cotg	Cotangente
arcsen	Arco seno
arccos	Arco cosseno
arctg	Arco tangente
arcsec	Arco secante
arccossec	Arco cossecante
arccotg	Arco cotangente
log	Logaritmo
ln	Logaritmo natural
e	Número de Euler
lim	Limite

Lista de símbolos

\int	Integral
\iint	Integral dupla
\iiint	Integral tripla
π	Letra grega minúscula pi
α	Ângulo alfa
θ	Ângulo theta
\in	Pertence

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Introdução

Esse documento contém exercícios retirados do Youtube através do canal OMate-matico.com, acesse-o em <https://www.youtube.com/c/omatematicogrings>.

Uma lista de exercícios prontos sobre *derivadas duplas e triplas* é apresentado em Grings (2016).

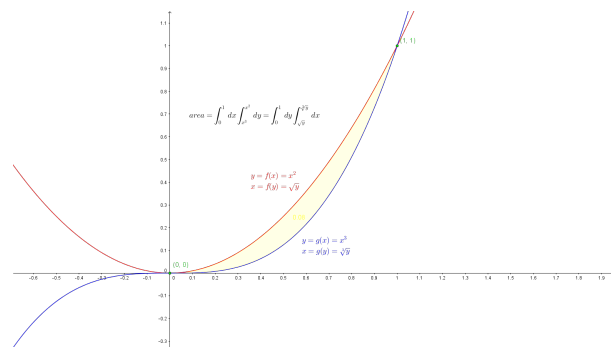
1 Integrais duplas

Cálculo de integrais duplas.

1.1 Invertendo os limites de integração - Aula 1

1. Exercício

Figura 1 – Integrais duplas - Aula 1 - Exercício I e II



$$f(x) = x^2; g(x) = x^3$$

$$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$$

$$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$$

$$\begin{aligned} a &= \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx [y]_{x^3}^{x^2} = \int_0^1 dx [x^2 - x^3] = \\ &= \int_0^1 x^2 dx - \int_0^1 x^3 dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left[\frac{4x^3 - 3x^2}{12} \right]_0^1 = \frac{1}{12} [4x^3 - 3x^2]_0^1 = \\ &= \frac{1}{12} [x^2(4x - 3)]_0^1 = \frac{1}{12} [1^2(4 \cdot 1 - 3) - 0^2(4 \cdot 0 - 3)] = \frac{1}{12} = 0,08\bar{3} \end{aligned}$$

2. Exercício

$$f(x) = x^2 \Rightarrow f(y) = \sqrt{y}; g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$$

$$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$$

$$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$$

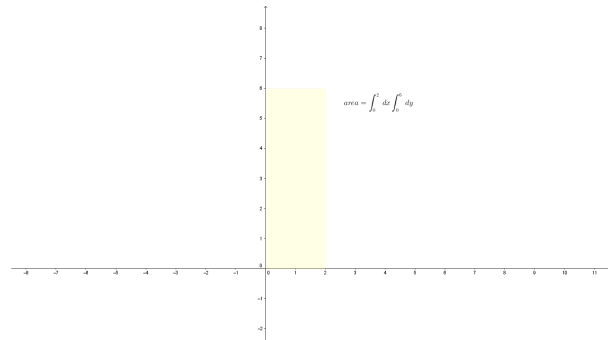
$$\begin{aligned}
a &= \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy [\sqrt[3]{y} - \sqrt{y}] = \\
&\int_0^1 \sqrt[3]{y} dy - \int_0^1 \sqrt{y} dy = \int_0^1 y^{\frac{1}{3}} dy - \int_0^1 y^{\frac{1}{2}} dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^1 = \\
&\left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3} \right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12} \right]_0^1 = \frac{1}{12} [9\sqrt[3]{y^4} - 8\sqrt{y^3}]_0^1 = \\
&\frac{1}{12} [(9\sqrt[3]{1^4} - 8\sqrt{1^3}) - (9\sqrt[3]{0^4} - 8\sqrt{0^3})] = \frac{1}{12}(9 - 8) = \frac{1}{12} = 0,08\bar{3}
\end{aligned}$$

1.2 Determinação da região de integração - Aula 2

1. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 6\}$$

Figura 2 – Integrais duplas - Aula 2 - Exercício I



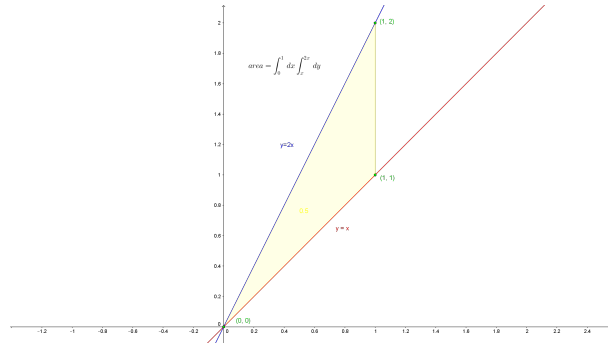
$$\begin{aligned}
a &= \int_0^2 dx \int_0^6 dy = \int_0^2 dx [y]_0^6 = \int_0^2 dx [6 - 0] = 6 \int_0^2 dx = 6[x]_0^2 = \\
&6[2 - 0] = 6 \cdot 2 = 12
\end{aligned}$$

2. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, x \leq y \leq 2x\}$$

$$\begin{aligned}
a &= \int_0^1 dx \int_x^{2x} dy = \int_0^1 dx [y]_x^{2x} = \int_0^1 dx [2x - x] = 2 \int_0^1 x dx - \int_0^1 x dx = \\
&\left[\frac{2x^2}{2} - \frac{x^2}{2} \right]_0^1 = \left[\frac{2x^2 - x^2}{2} \right]_0^1 = \frac{1}{2} [x^2]_0^1 = \frac{1}{2} [1^2 - 0^2] = \frac{1}{2} = 0,5
\end{aligned}$$

Figura 3 – Integrais duplas - Aula 2 - Exercício II



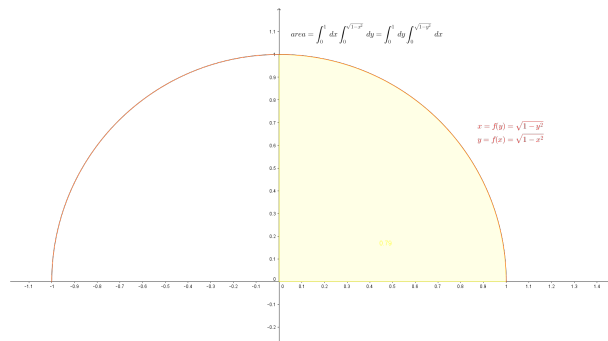
3. Exercício

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, 0 \leq x \leq \sqrt{1-y^2} \right\}$$

$$y = 0, y = 1$$

$$x = 0, x = \sqrt{1-y^2} \Rightarrow x^2 = 1-y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow y = \sqrt{1-x^2}$$

Figura 4 – Integrais duplas - Aula 2 - Exercício III



$$\begin{aligned} a &= \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy [\sqrt{1-y^2} - 0] = \\ &= \int_0^1 \sqrt{1-y^2} dy = \int_0^1 \sqrt{1-\sin^2(t)} \cos(t) dt = \int_0^1 \sqrt{\cos^2(t)} \cos(t) dt = \\ &= \int_0^1 \cos(t) \cos(t) dt = \int_0^1 \cos^2(t) dt = \int_0^1 \frac{1+\cos(2t)}{2} dt = \frac{1}{2} \int_0^1 [1+\cos(2t)] dt = \\ &= \frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_0^1 \cos(2t) dt = \frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_0^1 \cos(u) \frac{du}{2} = \frac{1}{2} \int_0^1 dt + \frac{1}{4} \int_0^1 \cos(u) du = \\ &= \left[\frac{1}{2}t + \frac{1}{4} \sin(u) \right]_0^1 = \left[\frac{t}{2} + \frac{\sin(2t)}{4} \right]_0^1 = \left[\frac{t}{2} + \frac{2\sin(t)\cos(t)}{4} \right]_0^1 = \left[\frac{t+\sin(t)\cos(t)}{2} \right]_0^1 = \\ &= \frac{1}{2} [\arcsen(y) + y\sqrt{1-y^2}]_0^1 = \frac{1}{2} [(\arcsen(1)+1\cdot\sqrt{1-1^2}) - (\arcsen(0)+0\cdot\sqrt{1-0^2})] = \\ &= \frac{1}{2} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{4} = 0,785 \end{aligned}$$

$$y = \text{sen}(t) \Rightarrow dy = \cos(t)dt$$

$$u = 2t \Rightarrow \frac{du}{2} = dt$$

$$\text{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$

$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1 - y^2}$$

$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1 - y^2}}{1} = \sqrt{1 - y^2}$$

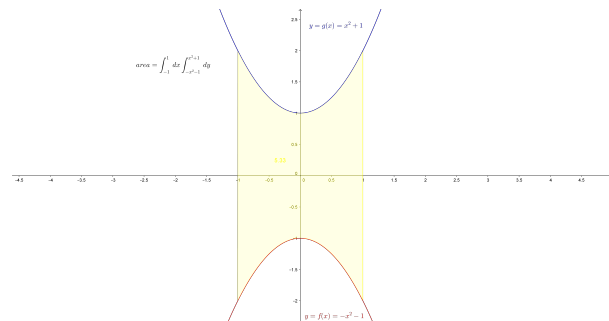
$$y = \text{sen}(t) \Rightarrow t = \arcsen(y)$$

4. Exercício

$$y = x^2 + 1, y = -x^2 - 1; x = 1, x = -1$$

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -x^2 - 1 \leq y \leq x^2 + 1\}$$

Figura 5 – Integrais duplas - Aula 2 - Exercício IV

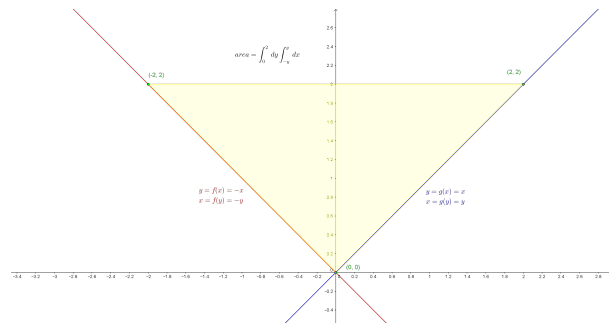


$$\begin{aligned} a &= \int_{-1}^1 dx \int_{f(x)}^{g(x)} dy = \int_{-1}^1 dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^1 dx [y]_{-x^2-1}^{x^2+1} = \\ &= \int_{-1}^1 dx [x^2 + 1 - (-x^2 - 1)] = \int_{-1}^1 dx [x^2 + 1 + x^2 + 1] = \\ &= \int_{-1}^1 dx [2x^2 + 2] = 2 \int_{-1}^1 x^2 dx + 2 \int_{-1}^1 dx = \left[2 \frac{x^3}{3} + 2x \right]_{-1}^1 = \left[2 \left(\frac{x^3 + 3x}{3} \right) \right]_{-1}^1 = \\ &= \frac{2}{3} [x(x^2 + 3)]_{-1}^1 = \frac{2}{3} [1 \cdot (1^2 + 3) - (-1) \cdot ((-1)^2 + 3)] = \frac{2}{3} (4 + 4) = \frac{2}{3} 8 = \frac{16}{3} = 5, \bar{3} \end{aligned}$$

5. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 2, -y \leq x \leq y\}$$

Figura 6 – Integrais duplas - Aula 2 - Exercício V

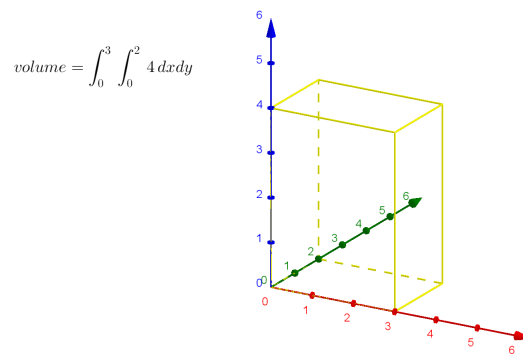


$$\begin{aligned}
 a &= \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^y dx = \int_0^2 dy [x]_{-y}^y = \int_0^2 dy [y - (-y)] = \int_0^2 dy [2y] = \\
 &2 \int_0^2 y dy = \left[2 \frac{y^2}{2} \right]_0^2 = 2^2 - 0^2 = 4
 \end{aligned}$$

1.3 Cálculo de volume - Aula 3

1. Exercício

Figura 7 – Integrais duplas - Aula 3 - Exercício I



$$z = 4; dz = dxdy$$

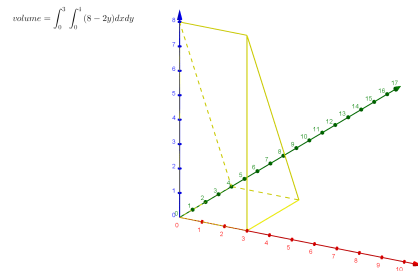
$$\begin{aligned}
 v &= \int_0^3 \int_0^2 z dz = \int_0^3 \int_0^2 4 dy dx = 4 \int_0^3 dx \int_0^2 dy = 4 \int_0^3 dx [y]_0^2 = 4 \int_0^3 dx [2 - 0] = \\
 &8 \int_0^3 dx = 8[x]_0^3 = 8[3 - 0] = 8 \cdot 3 = 24
 \end{aligned}$$

2. Exercício

$$R = [0, 3] \times [0, 4]$$

$$\iint_R (8 - 2y) da$$

Figura 8 – Integrais duplas - Aula 3 - Exercício II



$$z = 8 - 2y; \quad da = dz = dx dy$$

$$\begin{aligned} v &= \int_0^3 \int_0^4 z \, dz = \int_0^3 \int_0^4 (8 - 2y) dx dy = \int_0^3 dx \int_0^4 (8 - 2y) dy = \\ &= \int_0^3 dx \left(8 \int_0^4 dy - 2 \int_0^4 y \, dy \right) = \int_0^3 dx \, 2 \left(4 \int_0^4 dy - \int_0^4 y \, dy \right) = \\ &= 2 \int_0^3 dx \left[4y - \frac{y^2}{2} \right]_0^4 = 2 \int_0^3 dx \left[\frac{8y - y^2}{2} \right]_0^4 = 2 \int_0^3 dx \, \frac{1}{2} [y(8 - y)]_0^4 = \\ &= \int_0^3 dx [4(8 - 4) - 0(8 - 0)] = 16 \int_0^3 dx = 16[x]_0^3 = 16[3 - 0] = 48 \end{aligned}$$

1.4 Invertendo a ordem de integração - Aula 4

1. Exercício

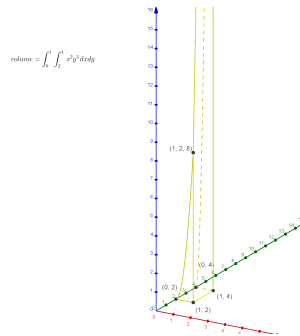
$$z = f(x, y) = y e^x; \quad dz = dx dy$$

$$\begin{aligned} v &= \int_2^4 \int_1^9 z \, dz = \int_2^4 \int_1^9 y e^x \, dy dx = \int_2^4 e^x \, dx \int_1^9 y \, dy = \int_2^4 e^x \, dx \left[\frac{y^2}{2} \right]_1^9 = \\ &= \int_2^4 e^x \, dx \frac{1}{2} [y^2]_1^9 = \frac{1}{2} \int_2^4 e^x \, dx [9^2 - 1^2] = 40 \int_2^4 e^x \, dx = 40 [e^x]_2^4 = 40 [e^4 - e^2] = \\ &= 40e^2 (e^2 - 1) \end{aligned}$$

2. Exercício

$$z = f(x, y) = x^2 y^3; \quad dz = dx dy$$

Figura 9 – Integrais duplas - Aula 4 - Exercício II



$$\begin{aligned}
 v &= \int_0^1 \int_2^4 z \, dz = \int_0^1 \int_2^4 x^2 y^3 \, dx dy = \int_0^1 x^2 \, dx \int_2^4 y^3 \, dy = \int_0^1 x^2 \, dx \left[\frac{y^4}{4} \right]_2^4 = \\
 &= \frac{1}{4} \int_0^1 x^2 \, dx [y^4]_2^4 = \frac{1}{4} \int_0^1 x^2 \, dx [4^4 - 2^4] = \frac{1}{4} \int_0^1 x^2 \, dx [2^8 - 2^4] = \\
 &= \frac{1}{4} \int_0^1 x^2 \, dx [2^4 (2^4 - 1)] = \frac{1}{4} \int_0^1 x^2 \, dx [16 \cdot 15] = 60 \int_0^1 x^2 \, dx = 60 \left[\frac{x^3}{3} \right]_0^1 = 20 [x^3]_0^1 = \\
 &= 20 [1^3 - 0^3] = 20 \cdot 1 = 20
 \end{aligned}$$

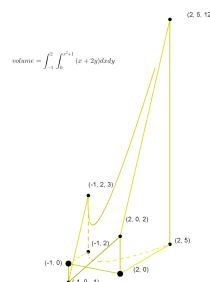
3. Exercício

$$\iint_R (x + 2y) \, da$$

R = Região limitada pela parábola $y = x^2 + 1$ e as retas $x = -1$ e $x = 2$.

$$z = f(x, y) = x + 2y; \, da = dz = dx dy$$

Figura 10 – Integrais duplas - Aula 4 - Exercício III



$$\begin{aligned}
v &= \int_{-1}^2 \int_0^{x^2+1} z \, dz = \int_{-1}^2 \int_0^{x^2+1} (x+2y) \, dx \, dy = \\
&\int_{-1}^2 dx \int_0^{x^2+1} (x+2y) \, dy = \int_{-1}^2 dx \left(x \int_0^{x^2+1} dy + 2 \int_0^{x^2+1} y \, dy \right) = \\
&\int_{-1}^2 dx \left[xy + 2 \frac{y^2}{2} \right]_0^{x^2+1} = \int_{-1}^2 dx [y(x+y)]_0^{x^2+1} = \\
&\int_{-1}^2 dx [(x^2+1) [x + (x^2+1)] - 0(x+0)] = \int_{-1}^2 dx [(x^2+1) (x^2+x+1)] = \\
&\int_{-1}^2 dx (x^4 + x^3 + 2x^2 + x + 1) = \\
&\int_{-1}^2 x^4 \, dx + \int_{-1}^2 x^3 \, dx + 2 \int_{-1}^2 x^2 \, dx + \int_{-1}^2 x \, dx + \int_{-1}^2 dx = \\
&\left[\frac{x^5}{5} + \frac{x^4}{4} + 2 \frac{x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^2 = \left[\frac{12x^5 + 15x^4 + 40x^3 + 30x^2 + 60x}{60} \right]_{-1}^2 = \\
&\frac{1}{60} [x (12x^4 + 15x^3 + 40x^2 + 30x + 60)]_{-1}^2 = \\
&\frac{1}{60} [2 (12 \cdot 2^4 + 15 \cdot 2^3 + 40 \cdot 2^2 + 30 \cdot 2 + 60) \\
&- (-1) (12(-1)^4 + 15(-1)^3 + 40(-1)^2 + 30(-1) + 60)] = \\
&\frac{1}{60} [2(192 + 120 + 160 + 60 + 60) + (12 - 15 + 40 - 30 + 60)] = \frac{1}{60} (1184 + 67) = \\
&\frac{1251}{60} = \frac{417}{20} = 20,85
\end{aligned}$$

1.5 Cálculo de integrais duplas ou iteradas

1.5.1 Aula 5

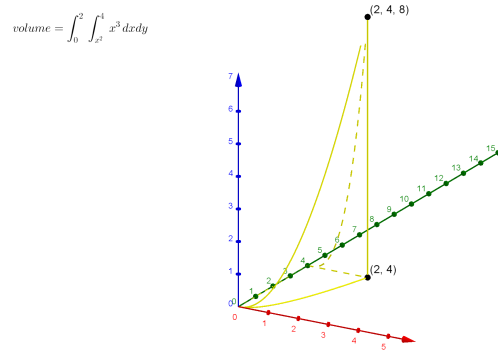
1. Exercício

$$\begin{aligned}
f(x, y) &= x^3; \quad 0 \leq x \leq 2; \quad x^2 \leq y \leq 4 \\
&\iint_R f(x, y) \, dy \, dx
\end{aligned}$$

$$\begin{aligned}
v &= \int_0^2 \int_{x^2}^4 x^3 \, dx \, dy = \int_0^2 x^3 \, dx \int_{x^2}^4 dy = \int_0^2 x^3 \, dx [y]_{x^2}^4 = \int_0^2 x^3 \, dx [4 - x^2] = \\
&4 \int_0^2 x^3 \, dx - \int_0^2 x^5 \, dx = \left[4 \frac{x^4}{4} - \frac{x^6}{6} \right]_0^2 = \left[\frac{6x^4 - x^6}{6} \right]_0^2 = \frac{1}{6} [x^4 (6 - x^2)]_0^2 = \\
&\frac{1}{6} [2^4 (6 - 2^2) - 0^4 (6 - 0^2)] = \frac{1}{6} (16 \cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2
\end{aligned}$$

2. Exercício

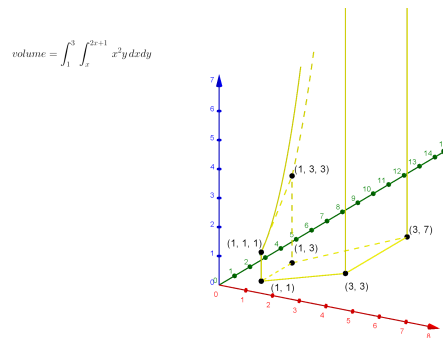
Figura 11 – Integrais duplas - Aula 5 - Exercício I



$$f(x, y) = x^2 y; \quad 1 \leq x \leq 3; \quad x \leq y \leq 2x + 1$$

$$\iint_R f(x, y) dy dx$$

Figura 12 – Integrais duplas - Aula 5 - Exercício II



$$\begin{aligned} v &= \int_1^3 \int_x^{2x+1} x^2 y dx dy = \int_1^3 x^2 dx \int_x^{2x+1} y dy = \int_1^3 x^2 dx \left[\frac{y^2}{2} \right]_x^{2x+1} = \\ &= \int_1^3 x^2 dx \frac{1}{2} [(2x+1)^2 - (x)^2] = \frac{1}{2} \int_1^3 x^2 dx (3x^2 + 4x + 1) = \\ &= \frac{3}{2} \int_1^3 x^4 dx + 2 \int_1^3 x^3 dx + \frac{1}{2} \int_1^3 x^2 dx = \left[\frac{3x^5}{2 \cdot 5} + 2 \frac{x^4}{4} + \frac{1}{2} \frac{x^3}{3} \right]_1^3 = \left[\frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6} \right]_1^3 = \\ &= \left[\frac{18x^5 + 30x^4 + 10x^3}{60} \right]_1^3 = \left[\frac{2x^3 (9x^2 + 15x + 5)}{60} \right]_1^3 = \\ &= \frac{1}{30} [x^3 (9x^2 + 15x + 5)]_1^3 = \frac{1}{30} [3^3 (9 \cdot 3^2 + 15 \cdot 3 + 5) - 1^3 (9 \cdot 1^2 + 15 \cdot 1 + 5)] = \\ &= \frac{1}{30} [27(81 + 45 + 5) - (9 + 15 + 5)] = \frac{1}{30} [27 \cdot 131 - 29] = \frac{3508}{30} = 116,9\bar{3} \end{aligned}$$

1.5.2 Aula 6

1. Exercício

$$f(x, y) = 1; 0 \leq x \leq 1; 1 \leq y \leq e^x$$

$$\iint_R f(x, y) dy dx$$

$$\begin{aligned} v &= \int_0^1 \int_1^{e^x} dy dx = \int_0^1 dx [y]_1^{e^x} = \int_0^1 dx (e^x - 1) = [e^x - x]_0^1 = \\ &= e^1 - 1 - (e^0 - 0) = e - 1 - 1 = e - 2 \end{aligned}$$

2. Exercício

$$f(x, y) = x; 0 \leq x \leq 1; 1 \leq y \leq e^{x^2}$$

$$\iint_R f(x, y) dy dx$$

$$\begin{aligned} v &= \int_0^1 \int_1^{e^{x^2}} x dx dy = \int_0^1 x dx \int_1^{e^{x^2}} dy = \int_0^1 x dx [y]_1^{e^{x^2}} = \int_0^1 x dx (e^{x^2} - 1) = \\ &= \int_0^1 x e^{x^2} dx - \int_0^1 x dx = \int_0^1 e^u \frac{du}{2} - \int_0^1 x dx = \frac{1}{2} \int_0^1 e^u du - \int_0^1 x dx = \\ &= \left[\frac{1}{2} e^u - \frac{x^2}{2} \right]_0^1 = \left[\frac{e^{x^2} - x^2}{2} \right]_0^1 = \frac{1}{2} [e^{x^2} - x^2]_0^1 = \frac{1}{2} [e^{1^2} - 1^2 - (e^{0^2} - 0^2)] = \\ &= \frac{1}{2} (e - 1 - 1) = \frac{e - 2}{2} \end{aligned}$$

$$u = x^2; \frac{du}{2} = x dx$$

3. Exercício

$$f(x, y) = 2xy; 0 \leq y \leq 1; y^2 \leq x \leq y$$

$$\iint_R f(x, y) dx dy$$

$$\begin{aligned} v &= \int_0^1 \int_{y^2}^y 2xy dx dy = 2 \int_0^1 y dy \int_{y^2}^y x dx = 2 \int_0^1 y dy \left[\frac{x^2}{2} \right]_{y^2}^y = 2 \int_0^1 y dy \frac{1}{2} [x^2]_{y^2}^y = \\ &= \int_0^1 y dy (y^2 - y^4) = \int_0^1 (y^3 - y^5) dy = \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_0^1 = \\ &= \left[\frac{6y^4 - 4y^6}{24} \right]_0^1 = \left[\frac{2y^4 (3 - 2y^2)}{24} \right]_0^1 = \frac{1}{12} [1^4 (3 - 2 \cdot 1^2) - 0^4 (3 - 2 \cdot 0^2)] = \frac{1}{12} = 0,08\bar{3} \end{aligned}$$

1.5.3 Aula 7

1. Exercício

$$f(x, y) = \frac{1}{x+y}; \quad 1 \leq y \leq e; \quad 0 \leq x \leq y$$

$$\iint_R f(x, y) dx dy$$

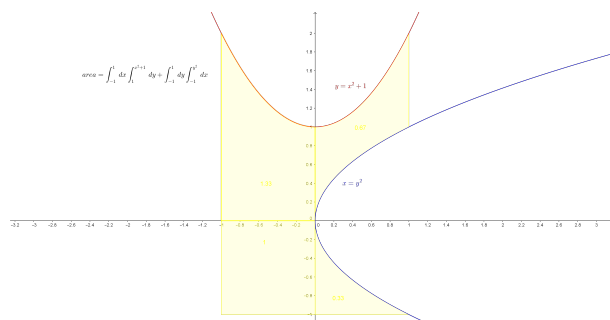
$$\begin{aligned} v &= \int_1^e \int_0^y \frac{1}{x+y} dx dy = \int_1^e dy \int_0^y (x+y)^{-1} dx = \int_1^e dy \int_0^y u^{-1} du = \\ &= \int_1^e dy \int_0^y [\ln |u|]_0^y = \int_1^e dy \int_0^y [\ln |x+y|]_0^y = \int_1^e dy \int_0^y (\ln |y+y| - \ln |0+y|) = \\ &= \int_1^e dy \int_0^y (\ln |2y| - \ln |y|) = \int_1^e dy \int_0^y (\ln |2| + \ln |y| - \ln |y|) = \ln |2| \int_1^e dy = \\ &= \ln |2| [y]_1^e = \ln |2|(e-1) \end{aligned}$$

$$u = x + y; \quad du = (1 + 0)dx = dx$$

1.6 Cálculo de área - Aula 8

1. Exercício

Figura 13 – Integrais duplas - Aula 8 - Exercício I



$$\begin{aligned}
a &= \int_{-1}^0 dx \int_0^{x^2+1} dy + \int_{-1}^0 dx \int_{-1}^0 dy + \int_0^{y^2} dx \int_{-1}^0 dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} dy = \\
&= \int_{-1}^0 dx \left(\int_0^{x^2+1} dy + \int_{-1}^0 dy \right) + \int_0^{y^2} dx \int_{-1}^0 dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} dy = \\
&= \int_{-1}^0 dx ([y]_0^{x^2+1} + [y]_{-1}^0) + \int_{-1}^0 dy [x]_0^{y^2} + \int_0^1 dx [y]_{\sqrt{x}}^{x^2+1} = \\
&= \int_{-1}^0 dx (x^2 + 1 + 1) + \int_{-1}^0 dy y^2 + \int_0^1 dx (x^2 + 1 - \sqrt{x}) = \\
&= \int_{-1}^0 (x^2 + 2) dx + \int_{-1}^0 y^2 dy + \int_0^1 (x^2 - x^{\frac{1}{2}} + 1) dx = \\
&= \left[\frac{x^3}{3} + 2x \right]_{-1}^0 + \left[\frac{y^3}{3} \right]_{-1}^0 + \left[\frac{x^3}{3} - \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + x \right]_0^1 = \\
&= \left[\frac{x^3 + 6x}{3} \right]_{-1}^0 + \frac{1}{3} [y^3]_{-1}^0 + \left[\frac{x^3}{3} - \frac{2\sqrt{x^3}}{3} + x \right]_0^1 = \\
&= \frac{1}{3} [x(x^2 + 6)]_{-1}^0 + \frac{1}{3} [\theta^3 - (-1)^3] + \left[\frac{x^3 - 2\sqrt{x^3} + 3x}{3} \right]_0^1 = \\
&= \frac{1}{3} [0(0^2 + 6) - (-1)((-1)^2 + 6)] + \frac{1}{3} + \frac{1}{3} [x^3 - 2\sqrt{x^3} + 3x]_0^1 = \\
&= \frac{7}{3} + \frac{1}{3} + \frac{1}{3} [1^3 - 2\sqrt{1^3} + 3 \cdot 1 - (0^3 - 2\sqrt{0^3} + 3 \cdot 0)] = \frac{7}{3} + \frac{1}{3} + \frac{2}{3} = \\
&= \frac{7 + 1 + 2}{3} = \frac{10}{3} = 3, \bar{3}
\end{aligned}$$

$$\begin{aligned}
a &= \int_{-1}^1 dx \int_1^{x^2+1} dy + \int_{-1}^{y^2} dx \int_{-1}^1 dy = \int_{-1}^1 dx [y]_1^{x^2+1} + \int_{-1}^1 dy [x]_{-1}^{y^2} = \\
&= \int_{-1}^1 dx (x^2 + 1 - 1) + \int_{-1}^1 dy (y^2 + 1) = \left[\frac{x^3}{3} \right]_{-1}^1 + \left[\frac{y^3}{3} + y \right]_{-1}^1 = \\
&= \frac{1}{3} [x^3]_{-1}^1 + \frac{1}{3} [y(y^2 + 3)]_{-1}^1 = \\
&= \frac{1}{3} ([1^3 - (-1)^3] + [1(1^2 + 3) - (-1)((-1)^2 + 3)]) = \frac{1}{3}(2 + 4 + 4) = \frac{10}{3} = 3, \bar{3}
\end{aligned}$$

1.7 Cálculo de volume

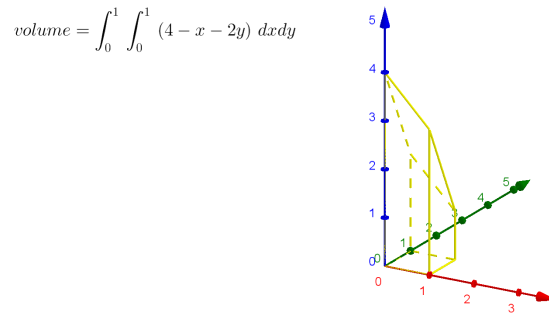
1.7.1 Aula 9

1. Exercício

Esboce a região de integração e o sólido cujo volume é dado pela integral abaixo:

$$\int_0^1 \int_0^1 (4 - x - 2y) \, dx \, dy$$

Figura 14 – Integrais duplas - Aula 9 - Exercício I



$$\begin{aligned}
 v &= \int_0^1 \int_0^1 (4 - x - 2y) \, dx dy = \int_0^1 dx \left(4 \int_0^1 dy - x \int_0^1 dy - 2 \int_0^1 y \, dy \right) = \\
 &= 4 \int_0^1 dx \int_0^1 dy - \int_0^1 x \, dx \int_0^1 dy - 2 \int_0^1 dx \int_0^1 y \, dy = \\
 &= 4[x]_0^1[y]_0^1 - \left[\frac{x^2}{2} \right]_0^1 [y]_0^1 - 2[x]_0^1 \left[\frac{y^2}{2} \right]_0^1 = 4 - \frac{1}{2} - 2 \frac{1}{2} = \frac{8 - 1 - 2}{2} = \frac{5}{2} = 2,5
 \end{aligned}$$

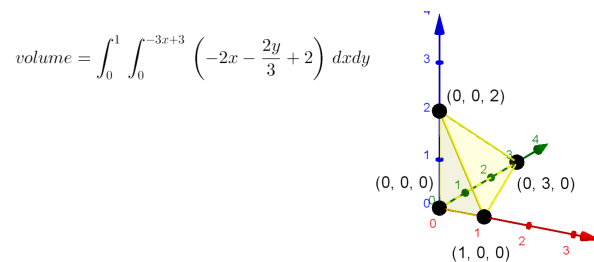
1.7.2 Aula 10

1. Exercício

Calcule o volume do sólido limitado pelos planos:

$$x = 0, y = 0, z = 0 \text{ e } 6x + 2y + 3z = 6$$

Figura 15 – Integrais duplas - Aula 10 - Exercício I



$$P_1 = (0, 0, 0)$$

$$6x = -2y - 3z + 6 \Rightarrow x = \frac{-2y - 3z + 6}{6} = \frac{-2 \cdot 0 - 3 \cdot 0 + 6}{6} = \frac{6}{6} = 1 \Rightarrow P_2 = (1, 0, 0)$$

$$2y = -6x - 3z + 6 \Rightarrow y = \frac{-6x - 3z + 6}{2} = \frac{-6 \cdot 0 - 3 \cdot 0 + 6}{2} = \frac{6}{2} = 3 \Rightarrow P_3 = (0, 3, 0)$$

$$3z = -6x - 2y + 6 \Rightarrow z = \frac{-6x - 2y + 6}{3} = \frac{-6 \cdot 0 - 2 \cdot 0 + 6}{3} = \frac{6}{3} = 2 \Rightarrow P_4 = (0, 0, 2)$$

$$x = 0, x = 1$$

$$y = 0, y = \frac{-6x - 3z + 6}{2} = \frac{-6x - 3 \cdot 0 + 6}{2} = -3x + 3$$

$$z = \frac{-6x - 2y + 6}{3} = -2x - \frac{2y}{3} + 2$$

$$\begin{aligned} v &= \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2 \right) dx dy = \int_0^1 dx \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2 \right) dy = \\ &= \int_0^1 dx \left[-2xy - \frac{2}{3} \frac{y^2}{2} + 2y \right]_0^{-3x+3} = \int_0^1 dx \frac{1}{3} [-6xy - y^2 + 6y]_0^{-3x+3} = \\ &= \frac{1}{3} \int_0^1 dx [-y(6x + y - 6)]_0^{-3x+3} = \\ &= \frac{1}{3} \int_0^1 dx [-(-3x + 3)(6x + (-3x + 3) - 6) + 0(6x + 0 - 6)] = \\ &= \frac{1}{3} \int_0^1 dx [(3x - 3)(3x - 3)] = \frac{1}{3} \int_0^1 (9x^2 - 18x + 9) dx = \frac{1}{3} \left[9 \frac{x^3}{3} - 18 \frac{x^2}{2} + 9x \right]_0^1 = \\ &= \frac{1}{3} [3x^3 - 9x^2 + 9x]_0^1 = \frac{1}{3} [3x(x^2 - 3x + 3)]_0^1 = \\ &= \frac{1}{3} [1(1^2 - 3 \cdot 1 + 3) - 0(0^2 - 3 \cdot 0 + 3)] = 1 \end{aligned}$$

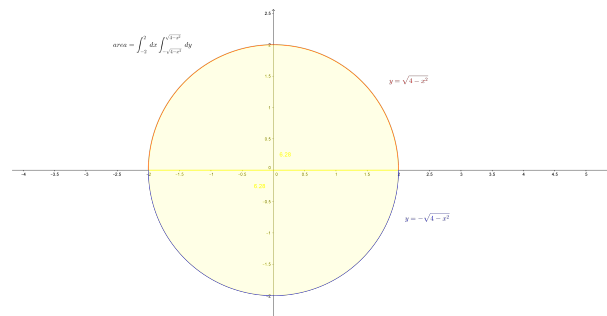
1.8 Coordenadas polares

1.8.1 Aula 1

1. Exercício

Calcule a área do círculo de raio igual a dois

Figura 16 – Coordenadas polares - Aula 01 - Exercício I



$$r = 2 \Rightarrow a = \pi r^2 = 2^2 \pi = 4\pi$$

$$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 2^2 \Rightarrow x^2 + y^2 = 4 \Rightarrow y = \pm \sqrt{4 - x^2}$$

$$R = \{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\}$$

$$\begin{aligned}
a &= \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{-2}^2 dx \left(\sqrt{4-x^2} + \sqrt{4-x^2} \right) = 2 \int_{-2}^2 \sqrt{4-x^2} dx = \\
&= 2 \int_{-2}^2 \sqrt{4 - (2 \operatorname{sen}(\alpha))^2} 2 \cos(\alpha) d\alpha = 4 \int_{-2}^2 \sqrt{4 - 4 \operatorname{sen}^2(\alpha)} \cos(\alpha) d\alpha = \\
&= 4 \int_{-2}^2 \sqrt{4 - 4(1 - \cos^2(\alpha))} \cos(\alpha) d\alpha = 4 \int_{-2}^2 \sqrt{4 - (4 - 4 \cos^2(\alpha))} \cos(\alpha) d\alpha = \\
&= 4 \int_{-2}^2 \sqrt{4 - 4 + 4 \cos^2(\alpha)} \cos(\alpha) d\alpha = 4 \int_{-2}^2 2 \cos(\alpha) \cos(\alpha) d\alpha = \\
&= 8 \int_{-2}^2 \cos^2(\alpha) d\alpha = 8 \int_{-2}^2 \left(\frac{1 + \cos(2\alpha)}{2} \right) d\alpha = 8 \int_{-2}^2 \left(\frac{1}{2} + \frac{\cos(2\alpha)}{2} \right) d\alpha = \\
&= 4 \int_{-2}^2 d\alpha + 4 \int_{-2}^2 \cos(2\alpha) d\alpha = 4 \int_{-2}^2 d\alpha + 4 \int_{-2}^2 \cos(u) \frac{du}{2} = \\
&= 4 \int_{-2}^2 d\alpha + 2 \int_{-2}^2 \cos(u) du = [4\alpha + 2 \operatorname{sen}(u)]_{-2}^2 = [4\alpha + 2 \operatorname{sen}(2\alpha)]_{-2}^2 = \\
&= [4\alpha + 4 \operatorname{sen}(\alpha) \cos(\alpha)]_{-2}^2 = [4(\alpha + \operatorname{sen}(\alpha) \cos(\alpha))]_{-2}^2 = \\
&= \left[4 \left(\operatorname{arcsen} \left(\frac{x}{2} \right) + \frac{x \sqrt{4-x^2}}{2} \right) \right]_{-2}^2 = \left[4 \left(\operatorname{arcsen} \left(\frac{x}{2} \right) + \frac{x \sqrt{4-x^2}}{4} \right) \right]_{-2}^2 = \\
&= 4 \left(\operatorname{arcsen} \left(\frac{2}{2} \right) + \frac{2 \sqrt{4-2^2}}{4} \right) - 4 \left(\operatorname{arcsen} \left(\frac{(-2)}{2} \right) + \frac{(-2) \sqrt{4-(-2)^2}}{4} \right) = \\
&= 4 \operatorname{arcsen}(1) - 4 \operatorname{arcsen}(-1) = 4(\operatorname{arcsen}(1) - \operatorname{arcsen}(-1)) = 4 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 4 \left(\frac{2\pi}{2} \right) = 4\pi
\end{aligned}$$

$$x = 2 \operatorname{sen}(\alpha); \quad dx = 2 \cos(\alpha) d\alpha$$

$$u = 2\alpha; \quad \frac{du}{2} = d\alpha$$

$$\operatorname{sen}(\alpha) = \frac{co}{h} = \frac{x}{2} \Rightarrow \alpha = \operatorname{arcsen} \left(\frac{x}{2} \right)$$

$$h^2 = co^2 + ca^2 \Rightarrow 2^2 = x^2 + ca^2 \Rightarrow ca = \sqrt{4-x^2}$$

$$\cos(\alpha) = \frac{ca}{h} = \frac{\sqrt{4-x^2}}{2}$$

$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

$$\begin{aligned}
a &= \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_0^2 \int_0^{2\pi} r dr d\theta = \int_0^2 r dr \int_0^{2\pi} d\theta = \left[\frac{r^2}{2} \right]_0^2 [\theta]_0^{2\pi} = \\
&= \frac{1}{2} [2^2 - 0^2] [2\pi - 0] = \frac{4}{2} 2\pi = 4\pi
\end{aligned}$$

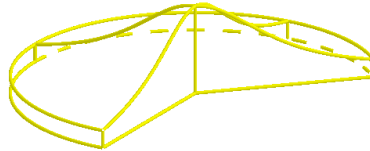
2. Exercício

$$\iint_R \frac{da}{1+x^2+y^2}$$

$$R = \left\{ (r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{2} \right\}$$

Figura 17 – Coordenadas polares - Aula 01 - Exercício II

$$volume = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r \, dr \, d\theta}{1 + r^2}$$



$$\begin{aligned} v &= \iint_R \frac{da}{1 + x^2 + y^2} = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r \, dr \, d\theta}{1 + r^2} = \int_0^2 \frac{r \, dr}{1 + r^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} d\theta = \\ &= \int_0^2 (1 + r^2)^{-1} r \, dr \left[\theta \right]_{\frac{\pi}{4}}^{\frac{3\pi}{2}} = \int_0^2 (1 + r^2)^{-1} r \, dr \left(\frac{3\pi}{2} - \frac{\pi}{4} \right) = \\ &= \int_0^2 (1 + r^2)^{-1} r \, dr \left(\frac{6\pi - \pi}{4} \right) = \frac{5\pi}{4} \int_0^2 (1 + r^2)^{-1} r \, dr = \frac{5\pi}{4} \int_0^2 u^{-1} \frac{du}{2} = \\ &= \frac{5\pi}{8} \int_0^2 u^{-1} du = \frac{5\pi}{8} [\ln|u|]_0^2 = \frac{5\pi}{8} [\ln|1 + r^2|]_0^2 = \frac{5\pi}{8} [\ln|1 + 2^2| - \ln|1 + 0^2|] = \\ &= \frac{5\pi}{8} [\ln|5| - \ln|1|] = \frac{5\pi \ln|5|}{8} \end{aligned}$$

$$u = 1 + r^2 \Rightarrow \frac{du}{2} = r \, dr$$

$$e^x = 1 = e^0 \Rightarrow x = 0$$

1.8.2 Aula 2

1. Exercício

$$\iint_R e^{x^2+y^2} \, dx \, dy$$

R , região entre as curvas abaixo:

$$x^2 + y^2 = 4$$

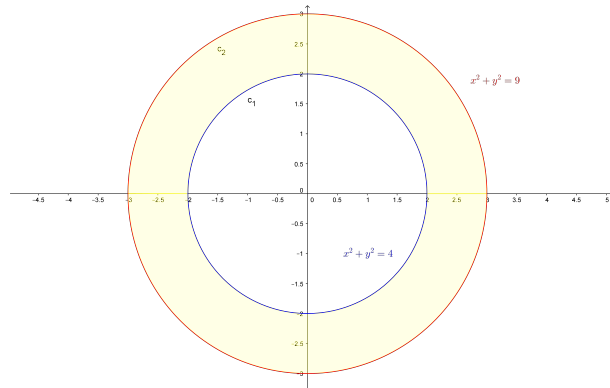
$$x^2 + y^2 = 9$$

$$x^2 + y^2 = r^2 \Rightarrow e^{x^2+y^2} = e^{r^2}$$

$$da = dx \, dy = r \, dr \, d\theta$$

$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 2 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$$

Figura 18 – Coordenadas polares - Aula 02 - Exercício I



$$v = \iint_R e^{x^2+y^2} dx dy = \int_2^3 \int_0^{2\pi} e^{r^2} r dr d\theta = \int_2^3 e^{r^2} r dr \int_0^{2\pi} d\theta = \int_2^3 e^u \frac{du}{2} \int_0^{2\pi} d\theta =$$

$$\frac{1}{2} \int_2^3 e^u du \int_0^{2\pi} d\theta = \frac{1}{2} [e^u]_2^3 [\theta]_0^{2\pi} = \frac{1}{2} [e^{r^2}]_2^3 2\pi = (e^3 - e^2) \pi = \pi (e^9 - e^4)$$

$$u = r^2 \Rightarrow \frac{du}{2} = r dr$$

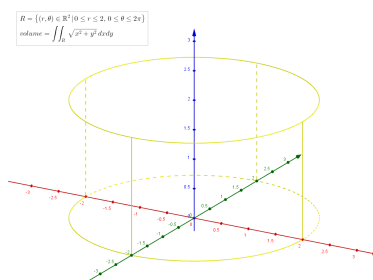
2. Exercício

$$\iint_R \sqrt{x^2 + y^2} dx dy$$

R , região cujo o contorno é:

$$x^2 + y^2 = 4$$

Figura 19 – Coordenadas polares - Aula 02 - Exercício II



$$x^2 + y^2 = r^2 \Rightarrow \sqrt{x^2 + y^2} = \sqrt{r^2} = r$$

$$da = dx dy = r dr d\theta$$

$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

$$v = \iint_R \sqrt{x^2 + y^2} dx dy = \int_0^2 \int_0^{2\pi} r^2 dr d\theta = \int_0^2 r^2 dr \int_0^{2\pi} d\theta =$$

$$\left[\frac{r^3}{3} \right]_0^2 [\theta]_0^{2\pi} = \frac{2^3}{3} 2\pi = \frac{16\pi}{3}$$

1.8.3 Aula 3

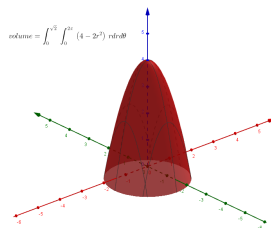
1. Exercício

Calcular o volume do sólido acima do plano xoy delimitado pela função abaixo.

xoy

$$z = 4 - 2x^2 - 2y^2$$

Figura 20 – Coordenadas polares - Aula 03 - Exercício I



$$4 - 2x^2 - 2y^2 = 0 \Rightarrow -2x^2 - 2y^2 = -4 \Rightarrow -2(x^2 + y^2) = -4 \Rightarrow$$

$$x^2 + y^2 = \frac{-4}{-2} = 2 \Rightarrow r = \sqrt{2}$$

$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi\}$$

$$z = 4 - 2x^2 - 2y^2 = 4 - 2(x^2 + y^2) = 4 - 2r^2$$

$$da = dxdy = r dr d\theta$$

$$\begin{aligned} \iint_R z da &= \iint_R (4 - 2x^2 - 2y^2) dxdy = \int_0^{\sqrt{2}} \int_0^{2\pi} (4 - 2r^2) r dr d\theta = \\ &= \int_0^{\sqrt{2}} (4r - 2r^3) dr \int_0^{2\pi} d\theta = \int_0^{\sqrt{2}} (4r - 2r^3) dr [\theta]_0^{2\pi} = 2\pi \int_0^{\sqrt{2}} (4r - 2r^3) dr = \\ &= 8\pi \int_0^{\sqrt{2}} r dr - 4\pi \int_0^{\sqrt{2}} r^3 dr = \left[\frac{8\pi r^2}{2} - \frac{4\pi r^4}{4} \right]_0^{\sqrt{2}} = [4\pi r^2 - \pi r^4]_0^{\sqrt{2}} = [\pi r^2 (4 - r^2)]_0^{\sqrt{2}} = \\ &= \pi (\sqrt{2})^2 (4 - (\sqrt{2})^2) = 2\pi(4 - 2) = 4\pi \end{aligned}$$

2 Integrais triplas

Cálculo de integrais triplas.

2.1 Introdução - Aula 1

1. Exercício

Calcule a integral tripla abaixo.

$$\iiint_R 12xy^2z^3 dv$$

$$R = \{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq x \leq 2, 0 \leq y \leq 3, 0 \leq z \leq 2\}$$

$$dv = dxdydz$$

$$\begin{aligned} \iiint_R 12xy^2z^3 dv &= \int_{-1}^2 \int_0^3 \int_0^2 12xy^2z^3 dxdydz = 12 \int_{-1}^2 x dx \int_0^3 y^2 dy \int_0^2 z^3 dz = \\ 12 \left[\frac{x^2}{2} \right]_{-1}^2 \left[\frac{y^3}{3} \right]_0^3 \left[\frac{z^4}{4} \right]_0^2 &= \frac{1}{2} [x^2]_{-1}^2 [y^3]_0^3 [z^4]_0^2 = \frac{1}{2} (2^2 - (-1)^2) 3^3 2^4 = \frac{1}{2} 3 \cdot 27 \cdot 16 = 648 \end{aligned}$$

2. Exercício

Observe a integral e preencha os retângulos abaixo.

$$\int_1^5 \int_2^4 \int_3^6 f(x, y, z) dxdzdy$$

$$[3] \leq x \leq [6]$$

$$[1] \leq y \leq [5]$$

$$[2] \leq z \leq [4]$$

3. Exercício

$$\begin{aligned}
\int_{-1}^1 \int_0^2 \int_0^1 (x^2 + y^2 + z^2) \, dx dy dz &= \int_{-1}^1 dz \int_0^2 dy \int_0^1 (x^2 + y^2 + z^2) \, dx = \\
&= \int_{-1}^1 dz \int_0^2 dy \left(\int_0^1 x^2 \, dx + y^2 \int_0^1 dx + z^2 \int_0^1 dx \right) = \\
&= \int_{-1}^1 dz \int_0^2 dy \int_0^1 x^2 \, dx + \int_{-1}^1 dz \int_0^2 y^2 \, dy \int_0^1 dx + \int_{-1}^1 z^2 \, dz \int_0^2 dy \int_0^1 dx = \\
&= [z]_{-1}^1 [y]_0^2 \left[\frac{x^3}{3} \right]_0^1 + [z]_{-1}^1 \left[\frac{y^3}{3} \right]_0^2 [x]_0^1 + \left[\frac{z^3}{3} \right]_{-1}^1 [y]_0^2 [x]_0^1 = \\
&= [z]_{-1}^1 [y]_0^2 \frac{1}{3} [x^3]_0^1 + [z]_{-1}^1 \frac{1}{3} [y^3]_0^2 [x]_0^1 + \frac{1}{3} [z^3]_{-1}^1 [y]_0^2 [x]_0^1 = \\
&= \frac{1}{3} ([1+1]2 \cdot 1^3 + [1+1]2^3 \cdot 1 + [1^3 - (-1)^3]2 \cdot 1) = \frac{1}{3} (4 + 16 + 4) = \frac{24}{3} = 8
\end{aligned}$$

4. Exercício

$$\begin{aligned}
\int_0^2 \int_{-1}^{y^2} \int_{-1}^z yz \, dx dz dy &= \int_0^2 \int_{-1}^{y^2} \left(yz \int_{-1}^z dx \right) dz dy = \int_0^2 \int_{-1}^{y^2} [yzx]_{-1}^z dz dy = \\
\int_0^2 \int_{-1}^{y^2} [yz^2 + yz] dz dy &= \int_0^2 \left(y \int_{-1}^{y^2} z^2 dz + y \int_{-1}^{y^2} z dz \right) dy = \int_0^2 \left[y \frac{z^3}{3} + y \frac{z^2}{2} \right]_{-1}^{y^2} dy = \\
\int_0^2 \left[\frac{2yz^3 + 3yz^2}{6} \right]_{-1}^{y^2} dy &= \frac{1}{6} \int_0^2 [yz^2(2z+3)]_{-1}^{y^2} dy = \\
\frac{1}{6} \int_0^2 \left[y(y^2)^2(2y^2+3) - y(-1)^2(2(-1)+3) \right] dy &= \frac{1}{6} \int_0^2 [y^5(2y^2+3) - y] dy = \\
\frac{1}{6} \int_0^2 (2y^7 + 3y^5 - y) dy &= \frac{1}{6} \left[\frac{2y^8}{8} + \frac{3y^6}{6} - \frac{y^2}{2} \right]_0^2 = \frac{1}{6} \left[\frac{y^8}{4} + \frac{y^6}{2} - \frac{y^2}{2} \right]_0^2 = \\
\frac{1}{6} \left[\frac{y^8 + 2y^6 - 2y^2}{4} \right]_0^2 &= \frac{1}{24} [y^2(y^6 + 2y^4 - 2)]_0^2 = \frac{1}{24} [2^2(2^6 + 2 \cdot 2^4 - 2)] = \\
\frac{1}{24} [4(64 + 32 - 2)] &= \frac{94}{6} = \frac{47}{3}
\end{aligned}$$

2.2 Cálculo de integrais triplas - Aula 2

1. Exercício

$$\iiint_R xy \operatorname{sen}(yz) \, dv$$

$$R = \left\{ (x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq \pi, 0 \leq y \leq 1, 0 \leq z \leq \frac{\pi}{6} \right\}$$

$$\begin{aligned}
\iiint_R xy \operatorname{sen}(yz) \, dv &= \int_0^\pi \int_0^1 \int_0^{\frac{\pi}{6}} xy \operatorname{sen}(yz) \, dz dy dx = \\
\int_0^\pi \int_0^1 \left(x \int_0^{\frac{\pi}{6}} \operatorname{sen}(yz) y \, dz \right) dy dx &= \int_0^\pi \int_0^1 \left(x \int_0^{\frac{\pi}{6}} \operatorname{sen}(u) \, du \right) dy dx = \\
\int_0^\pi \int_0^1 [-x \cos(u)]_0^{\frac{\pi}{6}} dy dx &= \int_0^\pi \int_0^1 [-x \cos(yz)]_0^{\frac{\pi}{6}} dy dx = \\
\int_0^\pi \int_0^1 \left(-x \cos\left(\frac{y\pi}{6}\right) + x \cos(0) \right) dy dx &= \int_0^\pi \int_0^1 \left(-x \cos\left(\frac{y\pi}{6}\right) + x \right) dy dx = \\
\int_0^\pi \left(-x \int_0^1 \cos\left(\frac{y\pi}{6}\right) dy + x \int_0^1 dy \right) dx &= \int_0^\pi \left(-x \int_0^1 \cos(v) \frac{6 \, dv}{\pi} + x \int_0^1 dy \right) dx = \\
\int_0^\pi \left(\frac{-6x}{\pi} \int_0^1 \cos(v) \, dv + x \int_0^1 dy \right) dx &= \int_0^\pi \left[\frac{-6x \operatorname{sen}(v)}{\pi} + xy \right]_0^1 dx = \\
\int_0^\pi \left[\frac{-6x \operatorname{sen}\left(\frac{y\pi}{6}\right) + xy\pi}{\pi} \right]_0^1 dx &= \frac{1}{\pi} \int_0^\pi \left[-x \left(6 \operatorname{sen}\left(\frac{y\pi}{6}\right) - y\pi \right) \right]_0^1 dx = \\
\frac{1}{\pi} \int_0^\pi \left[-x \left(6 \operatorname{sen}\left(\frac{\pi}{6}\right) - \pi \right) + x(6 \operatorname{sen}(0) - 0) \right] dx &= \frac{1}{\pi} \int_0^\pi \left(-6x \operatorname{sen}\left(\frac{\pi}{6}\right) + x\pi \right) dx = \\
\frac{-6 \operatorname{sen}\left(\frac{\pi}{6}\right)}{\pi} \int_0^\pi x \, dx + \pi \int_0^\pi x \, dx &= \left[\frac{-6 \operatorname{sen}\left(\frac{\pi}{6}\right)}{\pi} \frac{x^2}{2} + \frac{\pi x^2}{2} \right]_0^\pi = \\
\left[\frac{-3x^2 \operatorname{sen}\left(\frac{\pi}{6}\right)}{\pi} + \frac{\pi x^2}{2} \right]_0^\pi &= \left[\frac{-6x^2 \operatorname{sen}\left(\frac{\pi}{6}\right) + \pi^2 x^2}{2\pi} \right]_0^\pi = \frac{1}{2\pi} \left[-x^2 \left(6 \operatorname{sen}\left(\frac{\pi}{6}\right) - \pi^2 \right) \right]_0^\pi = \\
\frac{1}{2\pi} \left[-\pi^2 \left(6 \operatorname{sen}\left(\frac{\pi}{6}\right) - \pi^2 \right) \right] &= \frac{-\pi \left(6 \operatorname{sen}\left(\frac{\pi}{6}\right) - \pi^2 \right)}{2} = \frac{-\pi \left(6 \frac{1}{2} - \pi^2 \right)}{2} = \\
\frac{-\pi (3 - \pi^2)}{2} &= \frac{\pi^3 - 3\pi}{2}
\end{aligned}$$

$$u = yz \Rightarrow du = y \, dz$$

$$v = \frac{y\pi}{6} \Rightarrow \frac{6 \, dv}{\pi} = dy$$

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y z \, dx \, dz \, dy &= \int_0^1 \int_0^{\sqrt{1-y^2}} \left(z \int_0^y dx \right) dz \, dy = \int_0^1 \int_0^{\sqrt{1-y^2}} [zx]_0^y dz \, dy = \\
&\int_0^1 \int_0^{\sqrt{1-y^2}} (zy) \, dz \, dy = \int_0^1 \left(y \int_0^{\sqrt{1-y^2}} z \, dz \right) dy = \int_0^1 \left[\frac{yz^2}{2} \right]_0^{\sqrt{1-y^2}} dy = \\
&\int_0^1 \left(\frac{y(\sqrt{1-y^2})^2}{2} \right) dy = \int_0^1 \frac{y-y^3}{2} dy = \frac{1}{2} \int_0^1 (y-y^3) dy = \frac{1}{2} \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \\
&\frac{1}{2} \left[\frac{2y^2 - y^4}{4} \right]_0^1 = \frac{1}{8} [y^2(2-y^2)]_0^1 = \frac{1}{8} [1^2(2-1^2)] = \frac{1}{8}
\end{aligned}$$

3. Exercício

$$\begin{aligned}
\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^x xy \, dy \, dx \, dz &= \int_0^3 \int_0^{\sqrt{9-z^2}} \left(x \int_0^x y \, dy \right) dx \, dz = \\
&\int_0^3 \int_0^{\sqrt{9-z^2}} \left[\frac{xy^2}{2} \right]_0^x dx \, dz = \frac{1}{2} \int_0^3 \int_0^{\sqrt{9-z^2}} x^3 \, dx \, dz = \\
&\frac{1}{2} \int_0^3 \left[\frac{x^4}{4} \right]_0^{\sqrt{9-z^2}} dz = \frac{1}{2} \int_0^3 \left[\frac{(\sqrt{9-z^2})^4}{4} \right] dz = \frac{1}{8} \int_0^3 [(9-z^2)^2] dz = \\
&\frac{1}{8} \int_0^3 (81 - 18z^2 + z^4) dz = \frac{1}{8} \left[81z - \frac{18z^3}{3} + \frac{z^5}{5} \right]_0^3 = \frac{1}{8} \left[\frac{1215z - 90z^3 + 3z^5}{15} \right]_0^3 = \\
&\frac{1}{120} [3z(405 - 30z^2 + z^4)]_0^3 = \frac{1}{40} [z(405 - 30z^2 + z^4)]_0^3 = \frac{1}{40} [3(405 - 30 \cdot 3^2 + 3^4)] = \\
&\frac{1}{40} [3(405 - 270 + 81)] = \frac{648}{40} = \frac{81}{5}
\end{aligned}$$

2.3 Cálculo do volume de um sólido - Aula 3

1. Exercício

Use integral tripla para encontrar o volume do sólido no primeiro octante limitado pelos planos coordenados e pelo plano dado pela equação abaixo.

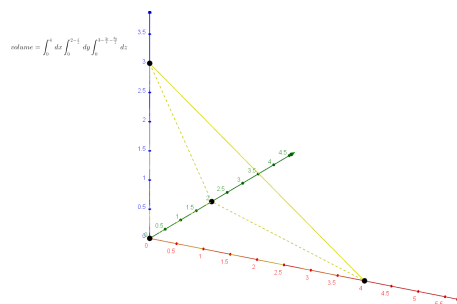
$$3x + 6y + 4z = 12$$

$$P_0(0, 0, 0)$$

$$x = 0, y = 0; 4z = 12 \Rightarrow z = \frac{12}{4} = 3; P_1(0, 0, 3)$$

$$x = 0, z = 0; 6y = 12 \Rightarrow y = \frac{12}{6} = 2; P_2(0, 2, 0)$$

Figura 21 – Integrais triplas - Aula 03 - Exercício I



$$y = 0, z = 0; 3x = 12 \Rightarrow x = \frac{12}{3} = 4; P_3(4, 0, 0)$$

$$0 \leq x \leq 4$$

$$3x + 6y = 12 \Rightarrow x + 2y = 4 \Rightarrow y = \frac{4-x}{2} = 2 - \frac{x}{2}; 0 \leq y \leq \left(2 - \frac{x}{2}\right)$$

$$3x + 6y + 4z = 12 \Rightarrow z = \frac{12 - 3x - 6y}{4} = 3 - \frac{3x}{4} - \frac{3y}{2}; 0 \leq z \leq \left(3 - \frac{3x}{4} - \frac{3y}{2}\right)$$

$$\begin{aligned}
& \int_0^4 dx \int_0^{2-\frac{x}{2}} dy \int_0^{3-\frac{3x}{4}-\frac{3y}{2}} dz = \int_0^4 dx \int_0^{2-\frac{x}{2}} dy [z]_0^{3-\frac{3x}{4}-\frac{3y}{2}} = \\
& \int_0^4 dx \int_0^{2-\frac{x}{2}} \left(3 - \frac{3x}{4} - \frac{3y}{2}\right) dy = \int_0^4 dx \left[3y - \frac{3xy}{4} - \frac{3y^2}{4}\right]_0^{2-\frac{x}{2}} = \\
& \int_0^4 \left(3\left(2 - \frac{x}{2}\right) - \frac{3x\left(2 - \frac{x}{2}\right)}{4} - \frac{3\left(2 - \frac{x}{2}\right)^2}{4}\right) dx = \\
& \int_0^4 \left(6 - \frac{3x}{2} - \frac{\left(6x - \frac{3x^2}{2}\right)}{4} - \frac{3\left(4 - 2x + \frac{x^2}{4}\right)}{4}\right) dx = \\
& \int_0^4 \left(6 - \frac{3x}{2} - \frac{\left(\frac{12x - 3x^2}{2}\right)}{4} - \frac{\left(12 - 6x + \frac{3x^2}{4}\right)}{4}\right) dx = \\
& \int_0^4 \left(6 - \frac{3x}{2} - \left(\frac{12x - 3x^2}{8}\right) - \frac{\left(48 - 24x + 3x^2\right)}{4}\right) dx = \\
& \int_0^4 \left(6 - \frac{3x}{2} - \left(\frac{3x}{2} - \frac{3x^2}{8}\right) - \left(\frac{48 - 24x + 3x^2}{16}\right)\right) dx = \\
& \int_0^4 \left(6 - \frac{3x}{2} - \frac{3x}{2} + \frac{3x^2}{8} - \left(3 - \frac{3x}{2} + \frac{3x^2}{16}\right)\right) dx = \\
& \int_0^4 \left(6 - \frac{3x}{2} - \frac{3x}{2} + \frac{3x^2}{8} - 3 + \frac{3x}{2} - \frac{3x^2}{16}\right) dx = \int_0^4 \left(3 - \frac{3x}{2} + \frac{3x^2}{16}\right) dx = \\
& \left[3x - \frac{3x^2}{4} + \frac{3x^3}{48}\right]_0^4 = 3 \cdot 4 - \frac{3 \cdot 4^2}{4} + \frac{3 \cdot 4^3}{48} = 12 - 12 + 4 = 4
\end{aligned}$$

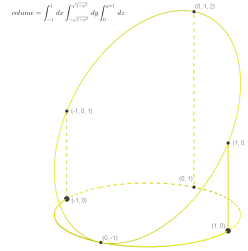
2.4 Esboço de um sólido - Aula 4

1. Exercício

Faça o esboço do sólido cujo volume é dado pela integral abaixo.

$$v = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{y+1} dz dy dx = \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_0^{y+1} dz$$

Figura 22 – Integrais triplas - Aula 04 - Exercício I



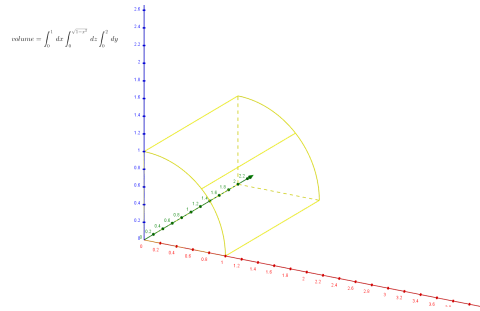
$$\begin{aligned}
 v &= \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_0^{y+1} dz = \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy [z]_0^{y+1} = \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y+1) dy = \\
 &= \int_{-1}^1 dx \left[\frac{y^2}{2} + y \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \\
 &= \int_{-1}^1 \left[\frac{(\sqrt{1-x^2})^2}{2} + \sqrt{1-x^2} - \left(\frac{(-\sqrt{1-x^2})^2}{2} - \sqrt{1-x^2} \right) \right] dx = \\
 &= \int_{-1}^1 \left(\frac{1-x^2}{2} + \sqrt{1-x^2} - \frac{1-x^2}{2} + \sqrt{1-x^2} \right) dx = \\
 &= \int_{-1}^1 \left(\frac{1}{2} - \frac{x^2}{2} + \sqrt{1-x^2} - \frac{1}{2} + \frac{x^2}{2} + \sqrt{1-x^2} \right) dx = \int_{-1}^1 (2\sqrt{1-x^2}) dx = \\
 &= 2 \int_{-1}^1 \sqrt{1-\sin^2(\theta)} \cos(\theta) d\theta = 2 \int_{-1}^1 \sqrt{1-(1-\cos^2(\theta))} \cos(\theta) d\theta = \\
 &= 2 \int_{-1}^1 \sqrt{\cos^2(\theta)} \cos(\theta) d\theta = 2 \int_{-1}^1 \cos^2(\theta) d\theta = 2 \int_{-1}^1 \left(\frac{1+\cos(2\theta)}{2} \right) d\theta = \\
 &= 2 \int_{-1}^1 \left(\frac{1}{2} + \frac{\cos(2\theta)}{2} \right) d\theta = \int_{-1}^1 d\theta + \int_{-1}^1 \cos(2\theta) d\theta = \int_{-1}^1 d\theta + \int_{-1}^1 \cos(u) \frac{du}{2} = \\
 &= \int_{-1}^1 d\theta + \frac{1}{2} \int_{-1}^1 \cos(u) du = \left[\theta + \frac{\sin(u)}{2} \right]_{-1}^1 = \left[\theta + \frac{\sin(2\theta)}{2} \right]_{-1}^1 = \\
 &= \left[\theta + \frac{2\sin(\theta)\cos(\theta)}{2} \right]_{-1}^1 = [\theta + \sin(\theta)\cos(\theta)]_{-1}^1 = [\arcsen(x) + x\sqrt{1-x^2}]_{-1}^1 = \\
 &= [\arcsen(1) + 1\sqrt{1-1^2} - (\arcsen(-1) + (-1)\sqrt{1-(-1)^2})] = [\arcsen(1) - \arcsen(-1)] = \\
 &= \frac{\pi}{2} + \frac{\pi}{2} = \frac{\pi + \pi}{2} = \frac{2\pi}{2} = \pi \\
 &= x = \sin(\theta) \Rightarrow dx = \cos(\theta) d\theta \\
 &= u = 2\theta \Rightarrow \frac{du}{2} = d\theta \\
 &= \sin(\theta) = \frac{co}{h} = \frac{x}{1} = x; \theta = \arcsen(x) \\
 &= 1 = x^2 + ca \Rightarrow ca = \sqrt{1-x^2} \\
 &= \cos(\theta) = \frac{ca}{h} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}
 \end{aligned}$$

2. Exercício

Faça o esboço do sólido cujo volume é dado pela integral abaixo.

$$v = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^2 dy dz dx = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dz \int_0^2 dy$$

Figura 23 – Integrais triplas - Aula 04 - Exercício II



$$\begin{aligned} v &= \int_0^1 dx \int_0^{\sqrt{1-x^2}} dz [y]_0^2 = 2 \int_0^1 dx \int_0^{\sqrt{1-x^2}} dz = 2 \int_0^1 dx [z]_0^{\sqrt{1-x^2}} = 2 \int_0^1 \sqrt{1-x^2} dx = \\ &= 2 \int_0^1 \sqrt{1-\sin^2(\theta)} \cos(\theta) d\theta = 2 \int_0^1 \sqrt{1-(1-\cos^2(\theta))} \cos(\theta) d\theta = \\ &= 2 \int_0^1 \sqrt{\cos^2(\theta)} \cos(\theta) d\theta = 2 \int_0^1 \cos^2(\theta) d\theta = 2 \int_0^1 \left(\frac{1+\cos(2\theta)}{2} \right) d\theta = \\ &= 2 \int_0^1 \left(\frac{1}{2} + \frac{\cos(2\theta)}{2} \right) d\theta = \int_0^1 d\theta + \int_0^1 \cos(2\theta) d\theta = \int_0^1 d\theta + \int_0^1 \cos(u) \frac{du}{2} = \\ &= \int_0^1 d\theta + \frac{1}{2} \int_0^1 \cos(u) du = \left[\theta + \frac{\sin(u)}{2} \right]_0^1 = \left[\theta + \frac{\sin(2\theta)}{2} \right]_0^1 = \left[\theta + \frac{2\sin(\theta)\cos(\theta)}{2} \right]_0^1 = \\ &= [\theta + \sin(\theta)\cos(\theta)]_0^1 = [\arcsen(x) + x\sqrt{1-x^2}]_0^1 = \\ &= \arcsen(1) + 1\sqrt{1-1^2} - (\arcsen(0) + 0\sqrt{1-0^2}) = \arcsen(1) - \arcsen(0) = \frac{\pi}{2} \end{aligned}$$

$$x = \sin(\theta) \Rightarrow dx = \cos(\theta) d\theta$$

$$u = 2\theta \Rightarrow \frac{du}{2} = d\theta$$

$$\sin(\theta) = \frac{co}{h} = \frac{x}{1} = x; \theta = \arcsen(x)$$

$$1 = x^2 + ca \Rightarrow ca = \sqrt{1-x^2}$$

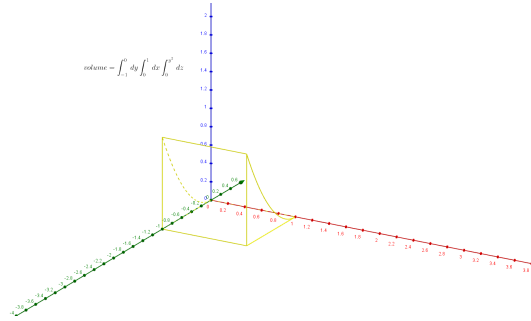
$$\cos(\theta) = \frac{ca}{h} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

3. Exercício

Faça o esboço do sólido cujo volume é dado pela integral abaixo.

$$v = \int_{-1}^0 \int_0^1 \int_0^{y^2} dz dx dy = \int_{-1}^0 dy \int_0^1 dx \int_0^{y^2} dz$$

Figura 24 – Integrais triplas - Aula 04 - Exercício III



$$v = \int_{-1}^0 dy \int_0^1 dx \int_0^{y^2} dz = \int_{-1}^0 dy \int_0^1 dx [z]_0^{y^2} = \int_{-1}^0 y^2 dy \int_0^1 dx = \int_{-1}^0 y^2 dy [x]_0^1 = \int_{-1}^0 y^2 dy = \left[\frac{y^3}{3} \right]_{-1}^0 = \frac{0^3}{3} - \frac{(-1)^3}{3} = \frac{1}{3}$$

2.5 Coordenadas esféricas

2.5.1 Aula 1

1. Exercício

Use coordenadas esféricas para calcular

$$\iiint_R (x^2 + y^2 + z^2) dv$$

, onde R é a bola unitária $x^2 + y^2 + z^2 \leq 1$.

$$x^2 + y^2 + z^2 = r^2$$

$$dv = dx dy dz = r^2 \sin(\varphi) dr d\varphi d\theta$$

$$0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$$

$$\begin{aligned} \iiint_R (x^2 + y^2 + z^2) dv &= \int_0^1 \int_0^{2\pi} \int_0^\pi r^4 \sin(\varphi) d\varphi d\theta dr = \\ &= \int_0^1 \int_0^{2\pi} \left(r^4 \int_0^\pi \sin(\varphi) d\varphi \right) d\theta dr = \int_0^1 \int_0^{2\pi} \left(r^4 [-\cos(\varphi)]_0^\pi \right) d\theta dr = \\ &= \int_0^1 \int_0^{2\pi} \left(r^4 [-\cos(\pi) + \cos(0)] \right) d\theta dr = \int_0^1 \int_0^{2\pi} 2r^4 d\theta dr = \int_0^1 \left(2r^4 \int_0^{2\pi} d\theta \right) dr = \\ &= \int_0^1 \left(2r^4 [\theta]_0^{2\pi} \right) dr = \int_0^1 4\pi r^4 dr = 4\pi \int_0^1 r^4 dr = 4\pi \left[\frac{r^5}{5} \right]_0^1 = \frac{4\pi}{5} \end{aligned}$$

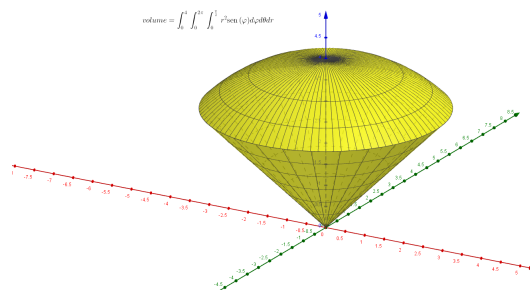
2.5.2 Aula 2

1. Exercício

$$x^2 + y^2 + z^2 = 16$$

$$z = \sqrt{x^2 + y^2}$$

Figura 25 – Coordenadas esféricas - Aula 02 - Exercício I



$$\begin{aligned} z = \sqrt{x^2 + y^2} &\Rightarrow r \cos(\varphi) = \sqrt{(r \sin(\varphi) \cos(\theta))^2 + (r \sin(\varphi) \sin(\theta))^2} = \\ &= \sqrt{r^2 \sin^2(\varphi) \cos^2(\theta) + r^2 \sin^2(\varphi) \sin^2(\theta)} = \sqrt{r^2 \sin^2(\varphi) (\cos^2(\theta) + \sin^2(\theta))} = \\ &= r \sin(\varphi) \Rightarrow \frac{r \cos(\varphi)}{r \sin(\varphi)} = \frac{r \sin(\varphi)}{r \cos(\varphi)} \Rightarrow 1 = \operatorname{tg}(\varphi) \Rightarrow \varphi = \arctg(1) = \frac{\pi}{4} \end{aligned}$$

$$0 \leq r \leq 4, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{4}$$

$$\begin{aligned} v &= \int_0^4 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} r^2 \sin(\varphi) d\varphi d\theta dr = \int_0^4 \int_0^{2\pi} \left(r^2 \int_0^{\frac{\pi}{4}} \sin(\varphi) d\varphi \right) d\theta dr = \\ &= \int_0^4 \int_0^{2\pi} \left(r^2 [-\cos(\varphi)]_0^{\frac{\pi}{4}} \right) d\theta dr = \int_0^4 \int_0^{2\pi} \left(r^2 \left[-\cos\left(\frac{\pi}{4}\right) + \cos(0) \right] \right) d\theta dr = \\ &= \int_0^4 \int_0^{2\pi} \left(r^2 \left[-\frac{\sqrt{2}}{2} + 1 \right] \right) d\theta dr = \int_0^4 \int_0^{2\pi} \frac{r^2 (2 - \sqrt{2})}{2} d\theta dr = \\ &= \int_0^4 \left(\frac{r^2 (2 - \sqrt{2})}{2} [\theta]_0^{2\pi} \right) dr = \int_0^4 \frac{r^2 (2 - \sqrt{2})}{2} 2\pi dr = \int_0^4 \pi (2 - \sqrt{2}) r^2 dr = \\ &= \pi (2 - \sqrt{2}) \left[\frac{r^3}{3} \right]_0^4 = \frac{64\pi (2 - \sqrt{2})}{3} \end{aligned}$$

2.5.3 Aula 3

1. Exercício

$$x^2 + y^2 + z^2 = r^2 = 4^2 = 16$$

$$0 \leq r \leq 4, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$$

$$\begin{aligned} v &= \int_0^\pi \int_0^{2\pi} \int_0^4 r^2 \operatorname{sen}(\varphi) dr d\theta d\varphi = \int_0^\pi \int_0^{2\pi} \left(\operatorname{sen}(\varphi) \left[\frac{r^3}{3} \right]_0^4 \right) d\theta d\varphi = \\ &= \int_0^\pi \int_0^{2\pi} \left(\operatorname{sen}(\varphi) \frac{64}{3} \right) d\theta d\varphi = \frac{64}{3} \int_0^\pi \int_0^{2\pi} \operatorname{sen}(\varphi) d\theta d\varphi = \frac{64}{3} \int_0^\pi \left(\operatorname{sen}(\varphi) [\theta]_0^{2\pi} \right) d\varphi = \\ &= \frac{64}{3} \int_0^\pi \operatorname{sen}(\varphi) 2\pi d\varphi = \frac{128\pi}{3} \int_0^\pi \operatorname{sen}(\varphi) d\varphi = \frac{128\pi}{3} [-\cos(\varphi)]_0^\pi = \\ &= \frac{128\pi}{3} (-\cos(\pi) + \cos(0)) = \frac{128\pi}{3} (1 + 1) = \frac{256\pi}{3} \end{aligned}$$

2.5.4 Aula 4

1. Exercício

$$\iiint_R e^{\sqrt{(x^2+y^2+z^2)^3}} dv$$

$$R = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

$$x^2 + y^2 + z^2 = r^2 \Rightarrow e^{\sqrt{(x^2+y^2+z^2)^3}} = e^{\sqrt{(r^2)^3}} = e^{r^3}$$

$$dv = dx dy dz = r^2 \operatorname{sen}(\varphi) dr d\theta d\varphi$$

$$x^2 + y^2 + z^2 = r^2 = 1^2 = 1$$

$$0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$$

$$\begin{aligned} &\int_0^1 \int_0^{2\pi} \int_0^\pi e^{r^3} r^2 \operatorname{sen}(\varphi) d\varphi d\theta dr = \int_0^1 e^{r^3} r^2 dr \int_0^{2\pi} d\theta \int_0^\pi \operatorname{sen}(\varphi) d\varphi = \\ &= \int_0^1 e^u \frac{du}{3} [\theta]_0^{2\pi} [-\cos(\varphi)]_0^\pi = \frac{1}{3} [e^{r^3}]_0^1 2\pi (-\cos(\pi) + \cos(0)) = \frac{2\pi}{3} (e^{1^3} - e^{0^3}) (1 + 1) = \\ &= \frac{4\pi}{3} (e - 1) \end{aligned}$$

$$u = r^3 \Rightarrow \frac{du}{3} = r^2 dr$$

2.5.5 Cálculo de massa com coordenadas esféricas - Aula 5

1. Exercício retirado da página 899 de (ROGAWSKI, 2009)

Encontre a massa de uma esfera S de raio 4 centrada na origem com densidade de massa dado abaixo.

$$\mu = f(x, y, z) = x^2 + y^2$$

$$\begin{aligned}\mu &= \frac{m}{v} \Rightarrow m = \mu v \Rightarrow \int dm = \int \mu dv \Rightarrow \\ m &= \int \mu dv = \iiint_R \mu dv = \iiint_S (x^2 + y^2) dv\end{aligned}$$

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 16\}$$

$$\begin{aligned}x^2 + y^2 &= (r \operatorname{sen}(\varphi) \cos(\theta))^2 + (r \operatorname{sen}(\varphi) \operatorname{sen}(\theta))^2 = \\ r^2 \operatorname{sen}^2(\varphi) \cos^2(\theta) + r^2 \operatorname{sen}^2(\varphi) \operatorname{sen}^2(\theta) &= r^2 \operatorname{sen}^2(\varphi) (\cos^2(\theta) + \operatorname{sen}^2(\theta)) = \\ r^2 \operatorname{sen}^2(\varphi)\end{aligned}$$

$$dv = dx dy dz = r^2 \operatorname{sen}(\varphi) dr d\theta d\varphi$$

$$0 \leq r \leq 4, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$$

$$\begin{aligned}m &= \iiint_S (x^2 + y^2) dv = \int_0^4 \int_0^{2\pi} \int_0^\pi (r^2 \operatorname{sen}^2(\varphi)) r^2 \operatorname{sen}(\varphi) d\varphi d\theta dr = \\ &= \int_0^4 \int_0^{2\pi} \int_0^\pi r^4 \operatorname{sen}^3(\varphi) d\varphi d\theta dr = \int_0^4 r^4 dr \int_0^{2\pi} d\theta \int_0^\pi \operatorname{sen}^2 \operatorname{sen}(\varphi) d\varphi = \\ &= \left[\frac{r^5}{5} \right]_0^4 [\theta]_0^{2\pi} \int_0^\pi (1 - \cos^2(\varphi)) \operatorname{sen}(\varphi) d\varphi = \\ &= \frac{1024}{5} 2\pi \left(\int_0^\pi \operatorname{sen}(\varphi) d\varphi - \int_0^\pi \cos^2(\varphi) \operatorname{sen}(\varphi) d\varphi \right) = \\ &= \frac{2048\pi}{5} \left(\int_0^\pi \operatorname{sen}(\varphi) d\varphi + \int_0^\pi u^2 du \right) = \frac{2048\pi}{5} \left[-\cos(\varphi) + \frac{u^3}{3} \right]_0^\pi = \\ &= \frac{2048\pi}{5} \left[\frac{-3\cos(\varphi) + \cos^3(\varphi)}{3} \right]_0^\pi = \frac{2048\pi}{15} \left[-\cos(\varphi) (3 - \cos^2(\varphi)) \right]_0^\pi = \\ &= \frac{2048\pi}{15} \left[-\cos(\pi) (3 - \cos^2(\pi)) + \cos(0) (3 - \cos^2(0)) \right] = \frac{2048\pi}{15} [(3 - 1) + (3 - 1)] = \\ &= \frac{2048\pi}{15} (2 + 2) = \frac{8192\pi}{15}\end{aligned}$$

$$u = \cos(\varphi) \Rightarrow -du = \operatorname{sen}(\varphi) d\varphi$$

2.5.6 Aula 6

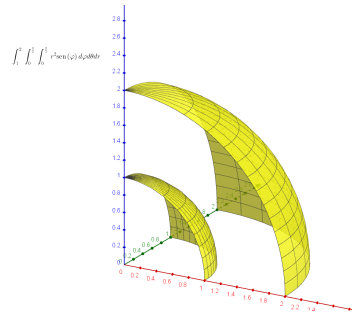
1. Exercício retirado da página 1023 de (STEWART, 2002)

Calcule a integral abaixo, onde E está contido entre as esferas dadas abaixo no 1º octante.

$$\iiint_E z \, dv$$

$$E = \{(x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 4\}$$

Figura 26 – Coordenadas esféricas - Aula 06 - Exercício I



$$z = r \cos(\varphi)$$

$$dv = dx dy dz = r^2 \sin(\varphi) dr d\theta d\varphi$$

$$1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq \frac{\pi}{2}$$

$$\begin{aligned} \iiint_E z \, dv &= \int_1^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (r \cos(\varphi)) r^2 \sin(\varphi) d\varphi d\theta dr = \\ \int_1^2 r^3 dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin(\varphi) \cos(\varphi) d\varphi &= \left[\frac{r^4}{4} \right]_1^2 [\theta]_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} u du = \left(\frac{16}{4} - \frac{1}{4} \right) \frac{\pi}{2} \left[\frac{u^2}{2} \right]_0^{\frac{\pi}{2}} = \\ \frac{16-1}{4} \frac{\pi}{2} \left[\frac{\sin^2(\varphi)}{2} \right]_0^{\frac{\pi}{2}} &= \frac{15\pi}{16} \left[\sin^2\left(\frac{\pi}{2}\right) - \sin^2(0) \right] = \frac{15\pi}{16} \end{aligned}$$

$$u = \sin(\varphi) \Rightarrow du = \cos(\varphi) d\varphi$$

2.5.7 Aula 7

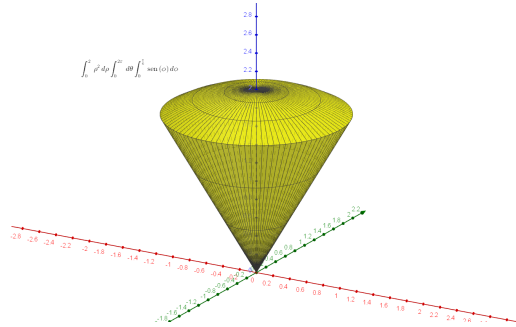
1. Exercício retirado da página 1023 de (STEWART, 2002)

Calcule

$$\iiint_E \sqrt{x^2 + y^2 + z^2} \, dv$$

, onde E é limitado abaixo pelo cone $\phi = \frac{\pi}{6}$ e acima pela esfera $\rho = 2$.

Figura 27 – Coordenadas esféricas - Aula 07 - Exercício I



$$0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{6}$$

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2} = \rho$$

$$dv = \rho^2 \sin(\phi) d\rho d\theta d\phi$$

$$\begin{aligned} \iiint_E \sqrt{x^2 + y^2 + z^2} dv &= \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{6}} (\rho) \rho^2 \sin(\phi) d\phi d\theta d\rho = \\ &= \int_0^2 \rho^3 d\rho \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} \sin(\phi) d\phi = \left[\frac{\rho^4}{4} \right]_0^2 [\theta]_0^{2\pi} [-\cos(\phi)]_0^{\frac{\pi}{6}} = \\ &= \frac{16}{4} 2\pi \left(-\cos\left(\frac{\pi}{6}\right) + \cos(0) \right) = 8\pi \left(\frac{-\sqrt{3}}{2} + 1 \right) = 8\pi \frac{-\sqrt{3} + 2}{2} = 4\pi (2 - \sqrt{3}) \end{aligned}$$

2.6 Coordenadas cilíndricas

2.6.1 Aula 1

1. Exercício

Encontre o volume do sólido limitado pelas funções abaixo.

$$z = x^2 + y^2$$

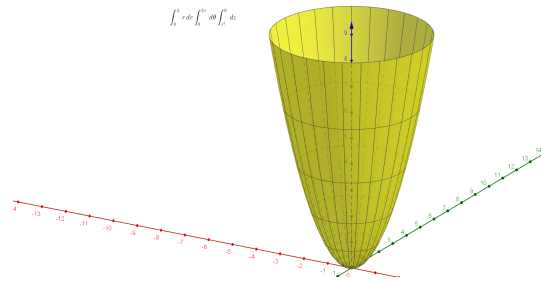
$$z = 9$$

$$z = x^2 + y^2 = r^2$$

$$x^2 + y^2 = r^2 = 3^2 = 9$$

$$0 \leq r \leq 3, 0 \leq \theta \leq 2\pi, r^2 \leq z \leq 9$$

Figura 28 – Coordenadas cilíndricas - Aula 01 - Exercício I



$$\begin{aligned}
 \int_0^3 \int_0^{2\pi} \int_{r^2}^9 r \, dz \, d\theta \, dr &= \int_0^3 r \, dr \int_0^{2\pi} d\theta \int_{r^2}^9 dz = \int_0^3 r \, dr \int_0^{2\pi} d\theta [z]_{r^2}^9 = \\
 \int_0^3 r (9 - r^2) \, dr \int_0^{2\pi} d\theta &= \int_0^3 (9r - r^3) \, dr \int_0^{2\pi} d\theta = \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_0^3 [\theta]_0^{2\pi} = \\
 \left[\frac{18r^2 - r^4}{4} \right]_0^3 2\pi &= \frac{\pi}{2} [r^2 (18 - r^2)]_0^3 = \frac{\pi}{2} 9 (18 - 9) = \frac{81\pi}{2}
 \end{aligned}$$

Referências

GRINGS, F. *Curso de Integrais Duplas e Triplas*. [S.l.], 2016. Disponível em: <https://www.youtube.com/playlist?list=PL82B9E5FF3F2B3BD3>. Citado na página 13.

ROGAWSKI, J. *Cálculo Vol.II*. [S.l.]: Bookman, 2009. Acesso em: 02 sep 2016. Citado na página 44.

STEWART, J. *Cálculo Vol.II*. 4. ed. [S.l.]: Pioneira Thomson Learning Ltda, 2002. Acesso em: 02 sep 2016. Citado na página 45.

Anexos

ANEXO A – Derivadas

A.1 Derivadas simples

Tabela 1 – Derivadas simples

$y = c$	$\Rightarrow y' = 0$
$y = x$	$\Rightarrow y' = 1$
$y = x^c$	$\Rightarrow y' = cx^{c-1}$
$y = e^x$	$\Rightarrow y' = e^x$
$y = \ln x $	$\Rightarrow y' = \frac{1}{x}$
$y = uv$	$\Rightarrow y' = u'v + uv'$
$y = \frac{u}{v}$	$\Rightarrow y' = \frac{u'v - uv'}{v^2}$
$y = u^c$	$\Rightarrow y' = cu^{c-1}u'$
$y = e^u$	$\Rightarrow y' = e^u u'$
$y = c^u$	$\Rightarrow y' = c^u u' \ln c $
$y = \ln u $	$\Rightarrow y' = \frac{u'}{u}$
$y = \log_c u $	$\Rightarrow y' = \frac{u'}{u} \log_c e $

A.2 Derivadas trigonométricas

Tabela 2 – Derivadas trigonométricas

$y = \text{sen}(x)$	$\Rightarrow y' = \cos(x)$
$y = \cos(x)$	$\Rightarrow y' = -\text{sen}(x)$
$y = \text{tg}(x)$	$\Rightarrow y' = \sec^2(x)$
$y = \text{cotg}(x)$	$\Rightarrow y' = -\text{cossec}^2(x)$
$y = \sec(x)$	$\Rightarrow y' = \sec(x) \text{tg}(x)$
$y = \text{cossec}(x)$	$\Rightarrow y' = -\text{cossec}(x) \text{cotg}(x)$
$y = \arcsen(x)$	$\Rightarrow y' = \frac{1}{\sqrt{1-x^2}}$
$y = \arccos(x)$	$\Rightarrow y' = \frac{-1}{\sqrt{1-x^2}}$
$y = \arctg(x)$	$\Rightarrow y' = \frac{1}{1+x^2}$
$y = \text{arccotg}(x)$	$\Rightarrow y' = \frac{-1}{1+x^2}$
$y = \text{arcsec}(x)$	$\Rightarrow y' = \frac{1}{ x \sqrt{x^2-1}}$
$y = \text{arccossec}(x)$	$\Rightarrow y' = \frac{-1}{ x \sqrt{x^2-1}}$

ANEXO B – Integrais

B.1 Integrais simples

Tabela 3 – Integrais simples

$\int dx$	$=$	$x + c$
$\int x^p dx$	$=$	$\frac{x^{p+1}}{p+1} + c \rightarrow p \neq -1$
$\int e^x dx$	$=$	$e^x + c$
$\int \frac{dx}{x}$	$=$	$\ln x + c$
$\int u^p du$	$=$	$\frac{u^{p+1}}{p+1} + c \rightarrow p \neq -1$
$\int e^u du$	$=$	$e^u + c$
$\int \frac{du}{u}$	$=$	$\ln u + c$
$\int p^u du$	$=$	$\frac{p^u}{\ln p } + c$

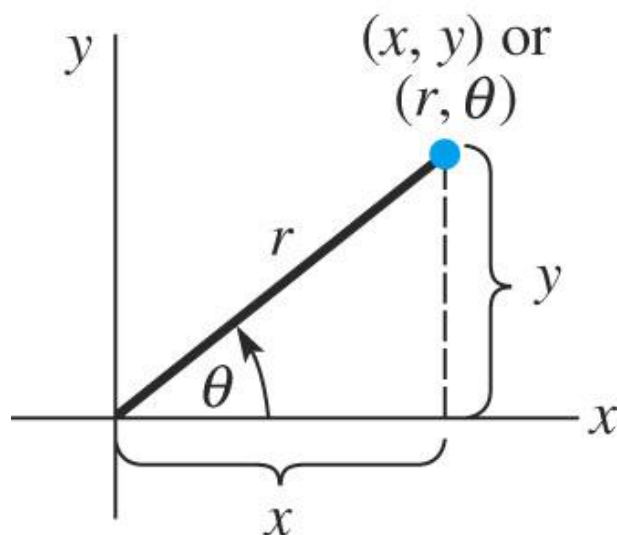
B.2 Integrais trigonométricas

Tabela 4 – Integrais trigonométricas

$\int \text{sen}(u) du$	$= -\cos(u) + c$
$\int \cos(u) du$	$= \text{sen}(u) + c$
$\int \text{tg}(u) du$	$= \ln \sec(u) + c$
$\int \text{cotg}(u) du$	$= \ln \text{sen}(u) + c$
$\int \sec(u) du$	$= \ln \sec(u) + \text{tg}(u) + c$
$\int \text{cossec}(u) du$	$= \ln \text{cossec}(u) - \text{cotg}(u) + c$
$\int \sec^2(u) du$	$= \text{tg}(u) + c$
$\int \text{cossec}^2(u) du$	$= -\text{cotg}(u) + c$
$\int \sec(u) \text{tg}(u) du$	$= \sec(u) + c$
$\int \text{cossec}(u) \text{cotg}(u) du$	$= -\text{cossec}(u) + c$
$\int \frac{du}{\sqrt{1-x^2}}$	$= \arcsen(x) + c$
$-\int \frac{du}{\sqrt{1-x^2}}$	$= \arccos(x) + c$
$\int \frac{du}{1+x^2}$	$= \text{arctg}(x) + c$
$-\int \frac{du}{1+x^2}$	$= \text{arccotg}(x) + c$
$\int \frac{du}{ x \sqrt{x^2-1}}$	$= \text{arcsec}(x) + c$
$-\int \frac{du}{ x \sqrt{x^2-1}}$	$= \text{arccossec}(x) + c$

B.3 Coordenadas polares

Figura 29 – Coordenadas polares



$$r \in [0, \infty), \theta \in [0, 2\pi]$$

Tabela 5 – Transformação de coordenadas cartesianas em polares

$$\begin{cases} y = r \operatorname{sen}(\theta) \\ x = r \operatorname{cos}(\theta) \end{cases}$$

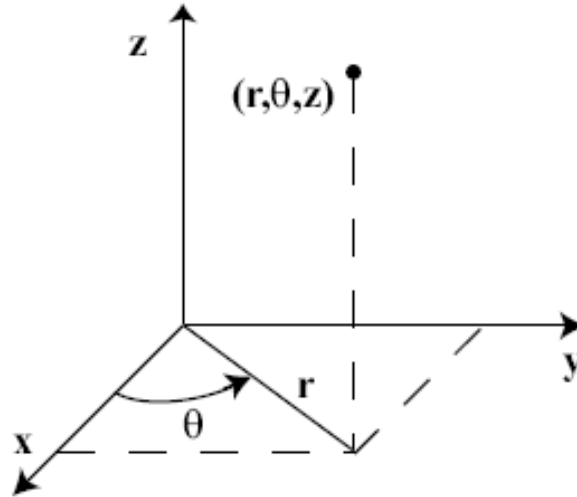
Tabela 6 – Coordenadas polares a partir das suas correspondentes cartesianas

$$\begin{cases} r^2 = x^2 + y^2 \\ \theta = \operatorname{arcsen}\left(\frac{y}{r}\right) \\ \theta = \operatorname{arccos}\left(\frac{x}{r}\right) \\ \theta = \operatorname{arctg}\left(\frac{y}{x}\right) \end{cases}$$

$$\iint_R z \, da = \iint_R f(x, y) \, dx dy = \iint_R f(r \operatorname{cos}(\theta), r \operatorname{sen}(\theta)) r \, dr d\theta$$

B.4 Coordenadas cilíndricas

Figura 30 – Coordenadas cilíndricas



$$r \in [0, \infty), \theta \in [0, 2\pi]$$

Tabela 7 – Transformação de coordenadas cartesianas em cilíndricas

$$\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \\ z = z \end{cases}$$

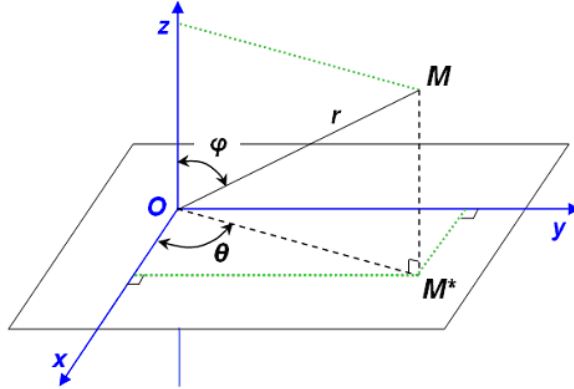
Tabela 8 – Coordenadas cilíndricas a partir das suas correspondentes cartesianas

$$\begin{cases} r^2 = x^2 + y^2 \\ \theta = \arctg\left(\frac{y}{x}\right) \end{cases}$$

$$\iiint_R f(x, y, z) dv = \iiint_R f(x, y, z) dx dy dz = \iiint_R f(r \cos(\theta), r \sin(\theta), z) r dr d\theta dz$$

B.5 Coordenadas esféricas

Figura 31 – Coordenadas esféricas



$$r \in [0, \infty), \theta \in [0, 2\pi], \varphi \in [0, \pi]$$

Tabela 9 – Transformação de coordenadas cartesianas em esféricas

$$\begin{cases} x = r \operatorname{sen}(\varphi) \cos(\theta) \\ y = r \operatorname{sen}(\varphi) \operatorname{sen}(\theta) \\ z = r \cos(\varphi) \end{cases}$$

Tabela 10 – Coordenadas esféricas a partir das suas correspondentes cartesianas

$$\begin{cases} r^2 = x^2 + y^2 + z^2 \\ \theta = \operatorname{arctg}\left(\frac{y}{x}\right) \\ \varphi = \operatorname{arctg}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \end{cases}$$

$$\begin{aligned} \iiint_R f(x, y, z) \, dv &= \iiint_R f(x, y, z) \, dx dy dz = \\ \iiint_R f(r \operatorname{sen}(\varphi) \cos(\theta), r \operatorname{sen}(\varphi) \operatorname{sen}(\theta), r \cos(\varphi)) \, r^2 \operatorname{sen}(\varphi) \, dr d\varphi d\theta \end{aligned}$$

ANEXO C – Funções trigonométricas

C.1 Determinação do seno, cosseno e tangente

Figura 32 – Determinação do seno, cosseno e tangente

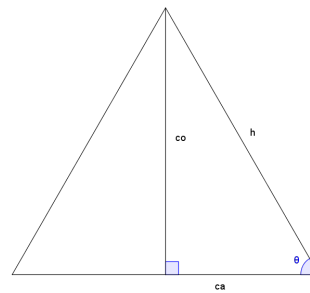
θ = ângulo
 h = hipotenusa
 co = cateto oposto
 ca = cateto adjacente

$$h^2 = co^2 + ca^2$$

$$\text{sen } \theta = \frac{co}{h}$$

$$\text{cos } \theta = \frac{ca}{h}$$

$$\text{tg } \theta = \frac{co}{ca}$$



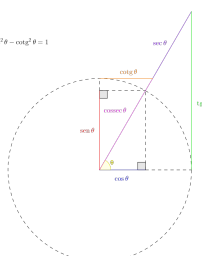
C.2 Círculo trigonométrico

Figura 33 – Círculo trigonométrico

$$\text{tg } \theta = \frac{\text{sen } \theta}{\text{cos } \theta} \quad \text{cotg } \theta = \frac{\text{cos } \theta}{\text{sen } \theta}$$

$$\text{sec } \theta = \frac{1}{\text{cos } \theta} \quad \text{cosec } \theta = \frac{1}{\text{sen } \theta}$$

$$\text{sen}^2 \theta + \text{cos}^2 \theta = 1 \quad \text{sec}^2 \theta - \text{tg}^2 \theta = 1 \quad \text{cosec}^2 \theta - \text{cotg}^2 \theta = 1$$



C.3 Identidades trigonométricas

Tabela 11 – Identidades trigonométricas

$\operatorname{tg}(x)$	$=$	$\frac{\operatorname{sen}(x)}{\cos(x)}$
$\operatorname{cotg}(x)$	$=$	$\frac{\cos(x)}{\operatorname{sen}(x)}$
$\sec(x)$	$=$	$\frac{1}{\cos(x)}$
$\operatorname{cosec}(x)$	$=$	$\frac{1}{\operatorname{sen}(x)}$
$\operatorname{sen}^2(x) + \cos^2(x)$	$=$	1
$\sec^2(x) - \operatorname{tg}^2(x)$	$=$	1
$\operatorname{cosec}^2(x) - \operatorname{cotg}^2(x)$	$=$	1
$\operatorname{sen}^2(x)$	$=$	$\frac{1 - \cos(2x)}{2}$
$\cos^2(x)$	$=$	$\frac{1 + \cos(2x)}{2}$
$\operatorname{sen}(2x)$	$=$	$2 \operatorname{sen}(x) \cos(x)$
$\cos(2x)$	$=$	$\cos^2(x) - \operatorname{sen}^2(x)$

C.4 Relação entre trigonométricas e inversas

Tabela 12 – Relação entre trigonométricas e inversas

$\operatorname{sen}(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arcsen}(x)$
$\cos(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arccos}(x)$
$\operatorname{tg}(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arctg}(x)$
$\operatorname{cosec}(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arccosec}(x)$
$\sec(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arcsec}(x)$
$\operatorname{cotg}(\theta)$	$=$	x	\Rightarrow	θ	$=$	$\operatorname{arccotg}(x)$

C.5 Substituição trigonométrica

Tabela 13 – Substituição trigonométrica

$\sqrt{a^2 - x^2}$	\Rightarrow	x	$=$	$a \operatorname{sen}(\theta)$
$\sqrt{a^2 + x^2}$	\Rightarrow	x	$=$	$a \operatorname{tg}(\theta)$
$\sqrt{x^2 - a^2}$	\Rightarrow	x	$=$	$a \sec(\theta)$

C.6 Ângulos notáveis

Tabela 14 – Ângulos notáveis

ângulo	$0^\circ (0)$	$30^\circ \left(\frac{\pi}{6}\right)$	$45^\circ \left(\frac{\pi}{4}\right)$	$60^\circ \left(\frac{\pi}{3}\right)$	$90^\circ \left(\frac{\pi}{2}\right)$
sen	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	\nexists