César Antônio de Magalhães

Curso de integrais duplas e triplas

Brasil

2016

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Introdução

Exercícios retirados do canal do Youtube, O Matematico (GRINGS, 2016).

César Magalhães

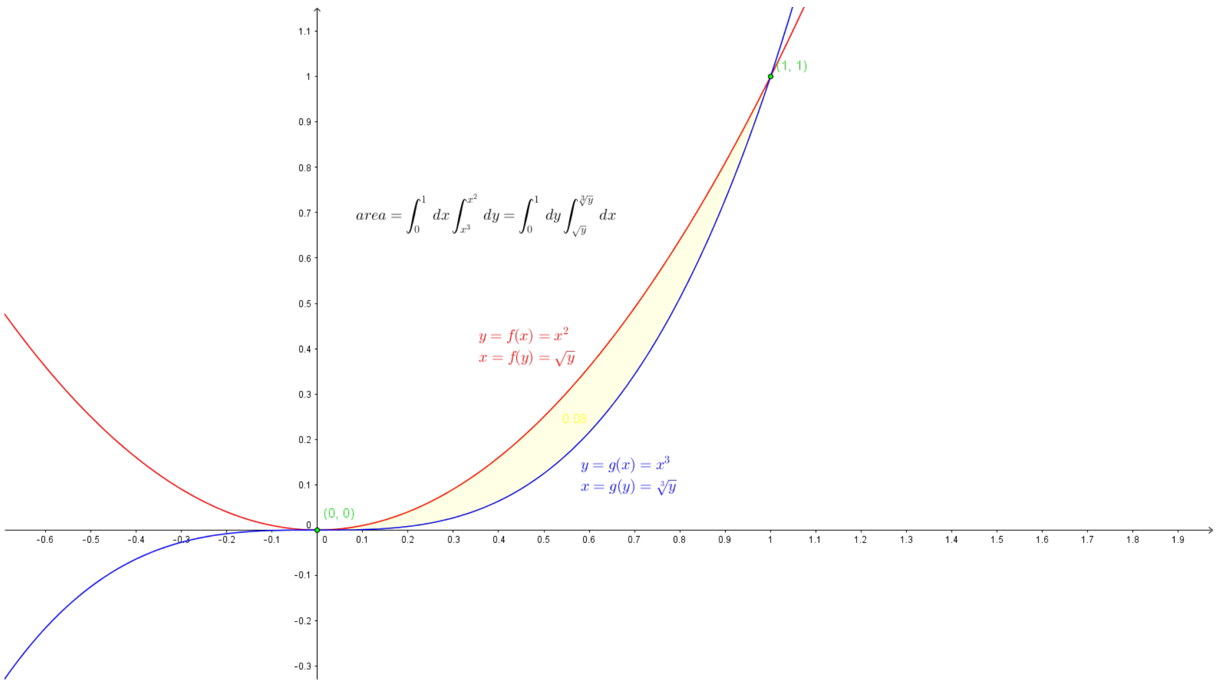
1 Integrais duplas

Cálculo de integrais duplas.

1.1 Invertendo os limites de integração - Aula 1

1. Exercício

Figura 1 – Integrais duplas - Aula 1 - Exercício I e II



$f(x) = x^2;\ g(x) = x^3$
$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$
$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$

$$a = \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx\, [y]_{x^3}^{x^2} = \int_0^1 dx \left[x^2 - x^3\right] = \int_0^1 x^2\, dx - \int_0^1 x^3\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{4x^3 - 3x^2}{12}\right]_0^1 = \frac{1}{12}\left[4x^3 - 3x^2\right]_0^1 = \frac{1}{12}\left[x^2(4x-3)\right]_0^1 = \frac{1}{12}\left[1^2(4\cdot 1 - 3) \cancel{- 0^2(4\cdot 0 - 3)}\right] = \frac{1}{12} = 0{,}08\overline{3}$$

2. Exercício

$f(x) = x^2 \Rightarrow f(y) = \sqrt{y};\ g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$
$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$
$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$

$$a = \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy\, [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy\, [\sqrt[3]{y} - \sqrt{y}] = \int_0^1 \sqrt[3]{y}\, dy -$$

$$\int_0^1 \sqrt{y}\, dy = \int_0^1 y^{\frac{1}{3}}\, dy - \int_0^1 y^{\frac{1}{2}}\, dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^1 = \left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3} \right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12} \right]_0^1 =$$

$$\frac{1}{12}\left[9\sqrt[3]{y^4} - 8\sqrt{y^3}\right]_0^1 =$$

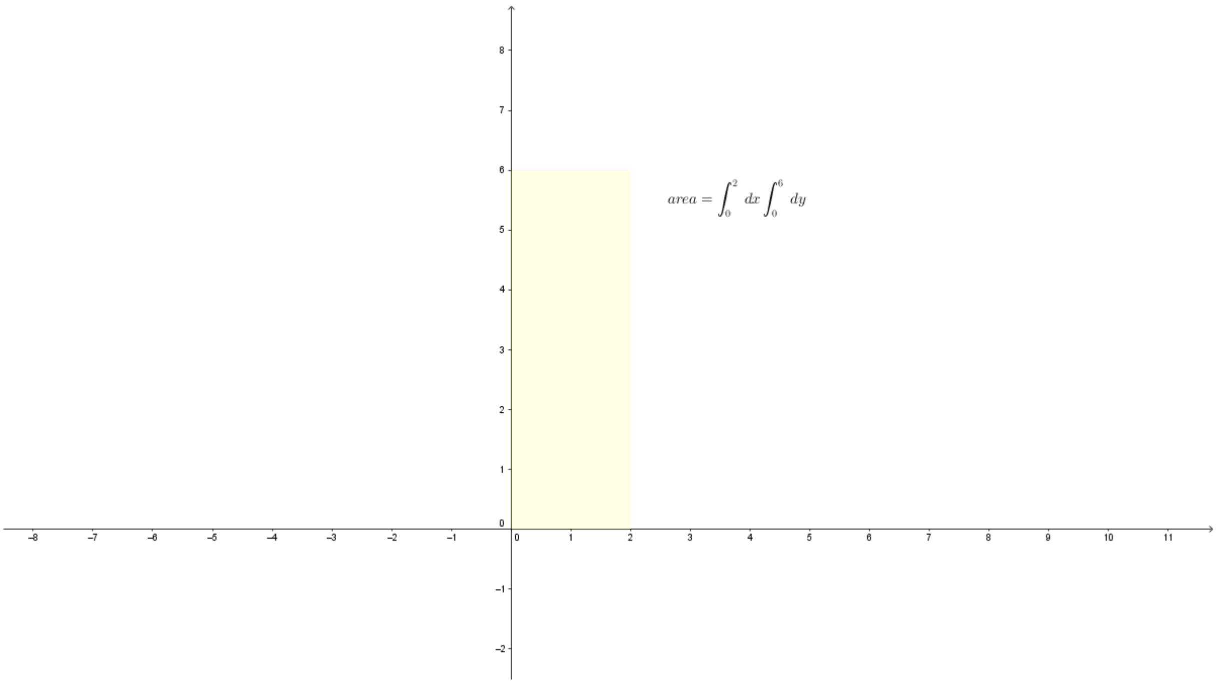
$$\frac{1}{12}\left[\left(9\sqrt[3]{1^4} - 8\sqrt{1^3}\right) \cancel{- \left(9\sqrt[3]{0^4} - 8\sqrt{0^3}\right)}\right] = \frac{1}{12}(9-8) = \frac{1}{12} = 0,08\overline{3}$$

1.2 Determinação da região de integração - Aula 2

1. Exercício

$R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 2 \,,\, 0 \leq y \leq 6\}$

Figura 2 – Integrais duplas - Aula 2 - Exercício I

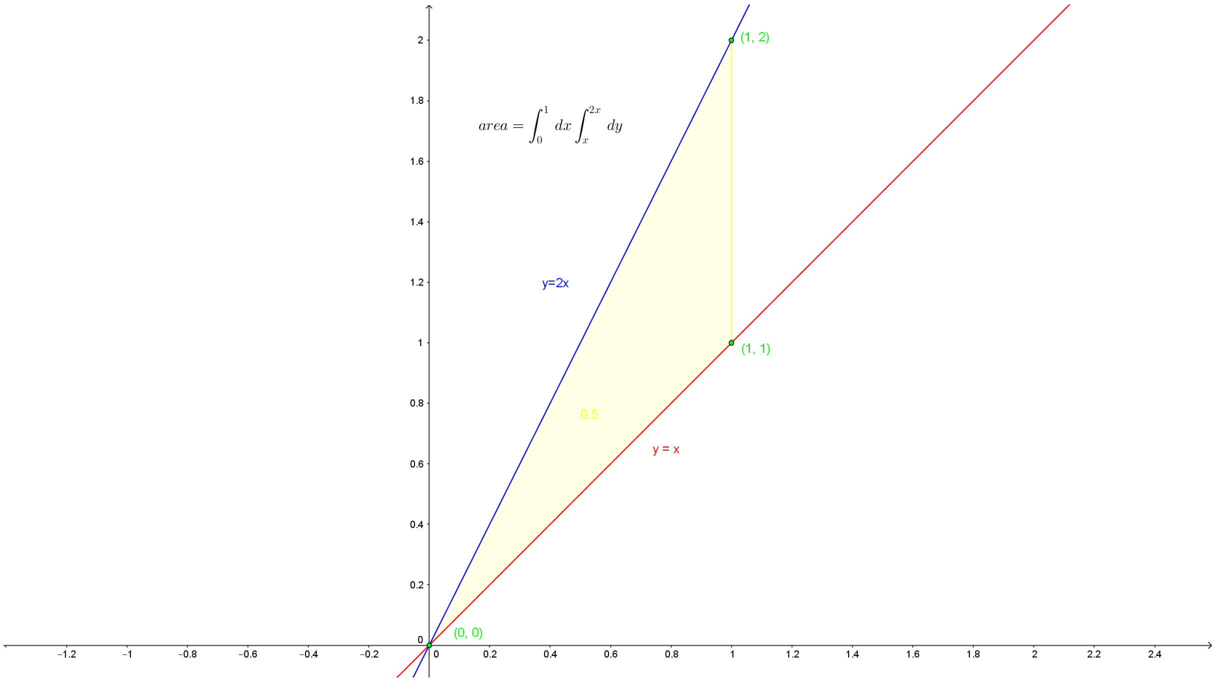


$$a = \int_0^2 dx \int_0^6 dy = \int_0^2 dx\, [y]_0^6 = \int_0^2 dx\, [6-0] = 6\int_0^2 dx = 6[x]_0^2 = 6[2-0] = 6 \cdot 2 = 12$$

2. Exercício

$R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1 \,,\, x \leq y \leq 2x\}$

Figura 3 – Integrais duplas - Aula 2 - Exercício II



$$a = \int_0^1 dx \int_x^{2x} dy = \int_0^1 dx\,[y]_x^{2x} = \int_0^1 dx\,[2x - x] = 2\int_0^1 x\,dx - \int_0^1 x\,dx = \left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^2}{2}\right]_0^1 = \left[\frac{2x^2 - x^2}{2}\right]_0^1 = \frac{1}{2}\left[x^2\right]_0^1 = \frac{1}{2}\left[1^2 \cancel{-0^2}\right] = \frac{1}{2} = 0,5$$

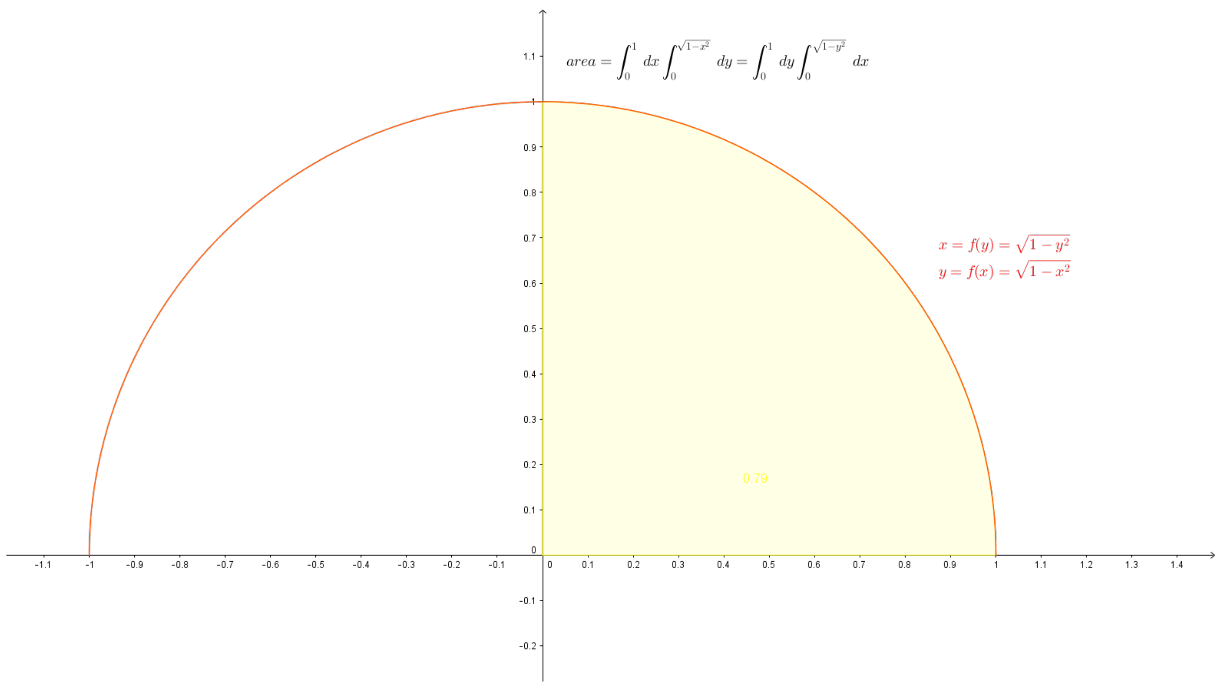
3. Exercício

$$R = \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 1\,,\; 0 \leq x \leq \sqrt{1 - y^2}\right\}$$

$y = 0,\ y = 1$

$x = 0,\ x = \sqrt{1 - y^2} \Rightarrow x^2 = 1 - y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow y = \sqrt{1 - x^2}$

Figura 4 – Integrais duplas - Aula 2 - Exercício III



$$a = \int_0^1 dy \int_0^{f(y)} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy\, [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy \left[\sqrt{1-y^2} - 0\right] =$$
$$\int_0^1 \sqrt{1-y^2}\, dy = \int_0^1 \sqrt{1-\operatorname{sen}^2(t)}\, \cos(t)dt = \int_0^1 \sqrt{\cos^2(t)}\, \cos(t)dt = \int_0^1 \cos(t)\cos(t)dt =$$
$$\int_0^1 \cos^2(t)dt = \int_0^1 \frac{1+\cos(2t)}{2}dt = \frac{1}{2}\int_0^1 [1+\cos(2t)]\, dt = \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(2t)dt =$$
$$\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(u)\frac{du}{2} = \frac{1}{2}\int_0^1 dt + \frac{1}{4}\int_0^1 \cos(u)\, du = \left[\frac{1}{2}t + \frac{1}{4}\operatorname{sen}(u)\right]_0^1 = \left[\frac{t}{2} + \frac{\operatorname{sen}(2t)}{4}\right]_0^1 =$$
$$\left[\frac{t}{2} + \frac{2\operatorname{sen}(t)\cos(t)}{4}\right]_0^1 = \left[\frac{t+\operatorname{sen}(t)\cos(t)}{2}\right]_0^1 = \frac{1}{2}\left[\operatorname{arcsen}(y) + y\sqrt{1-y^2}\right]_0^1 =$$
$$\frac{1}{2}\left[\left(\operatorname{arcsen}(1)\cancel{+1\cdot\sqrt{1-1^2}}\right) - \left(\operatorname{arcsen}(0)\cancel{+0\cdot\sqrt{1-0^2}}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4} = 0,785$$

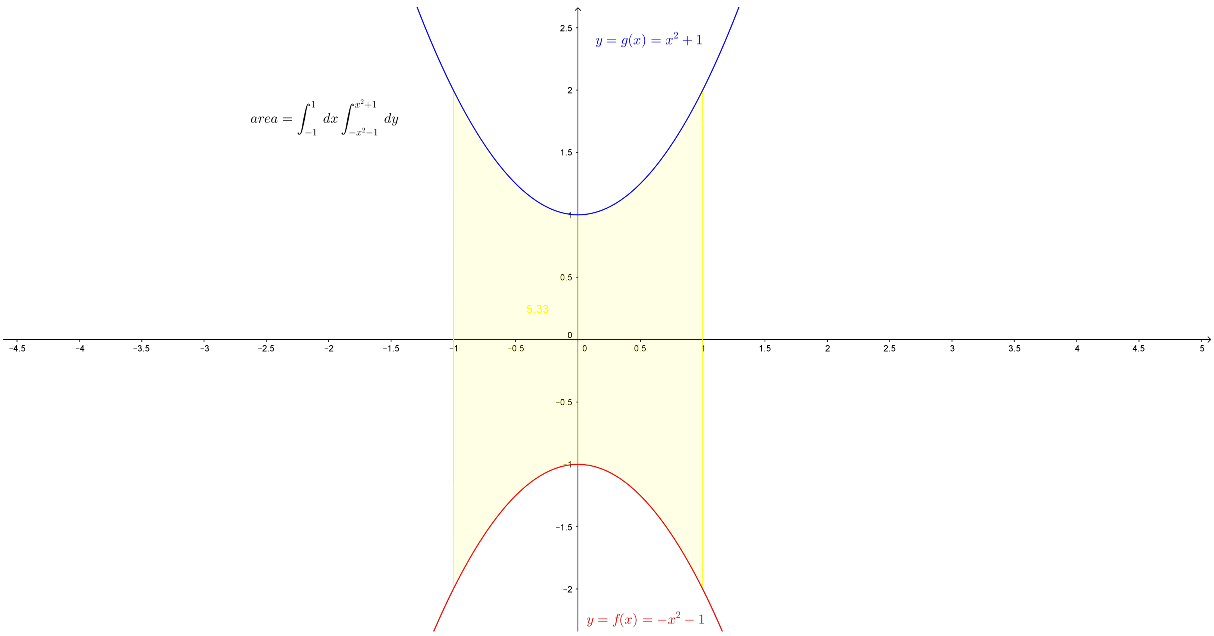
$$y = \operatorname{sen}(t) \Rightarrow dy = \cos(t)dt$$
$$u = 2t \Rightarrow \frac{du}{2} = dt$$

$$\operatorname{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$
$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1-y^2}$$
$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1-y^2}}{1} = \sqrt{1-y^2}$$
$$y = \operatorname{sen}(t) \Rightarrow t = \operatorname{arcsen}(y)$$

4. Exercício

$$y = x^2 + 1,\ y = -x^2 - 1;\ x = 1,\ x = -1$$
$$R = \{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le 1\,,\ -x^2 - 1 \le y \le x^2 + 1\}$$

Figura 5 – Integrais duplas - Aula 2 - Exercício IV

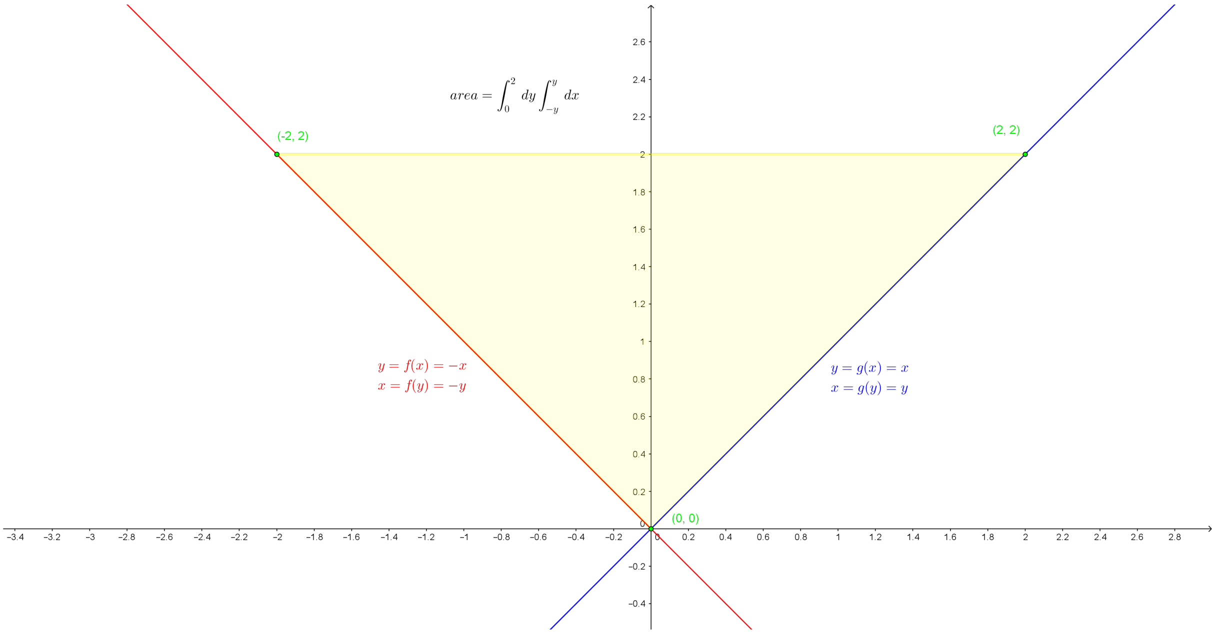


$$a = \int_{-1}^{1} dx \int_{f(x)}^{g(x)} dy = \int_{-1}^{1} dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^{1} dx \, [y]_{-x^2-1}^{x^2+1} = \int_{-1}^{1} dx \, \left[x^2 + 1 - \left(-x^2 - 1\right)\right] =$$
$$\int_{-1}^{1} dx \, \left[x^2 + 1 + x^2 + 1\right] = \int_{-1}^{1} dx \, \left[2x^2 + 2\right] = 2 \int_{-1}^{1} x^2 \, dx + 2 \int_{-1}^{1} dx = \left[2\frac{x^3}{3} + 2x\right]_{-1}^{1} =$$
$$\left[2\left(\frac{x^3 + 3x}{3}\right)\right]_{-1}^{1} = \frac{2}{3}\left[x\left(x^2 + 3\right)\right]_{-1}^{1} =$$
$$\frac{2}{3}\left[1 \cdot \left(1^2 + 3\right) - (-1)\left((-1)^2 + 3\right)\right] = \frac{2}{3}(4 + 4) = \frac{2}{3}8 = \frac{16}{3} = 5,\overline{3}$$

5. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 2 \,,\, -y \leq x \leq y\}$$

Figura 6 – Integrais duplas - Aula 2 - Exercício V



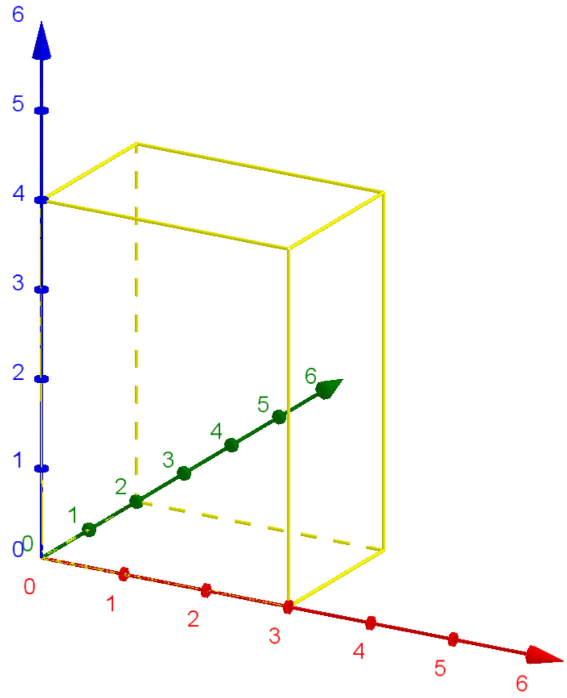
$$a = \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^{y} dx = \int_0^2 dy \, [x]_{-y}^{y} = \int_0^2 dy \, [y - (-y)] = \int_0^2 dy \, [2y] =$$
$$2 \int_0^2 y \, dy = \left[\cancel{2} \frac{y^2}{\cancel{2}} \right]_0^2 = 2^2 - 0^2 = 4$$

1.3 Cálculo de volume - Aula 3

1. Exercício

Figura 7 – Integrais duplas - Aula 3 - Exercício I

$$volume = \int_0^3 \int_0^2 4\, dxdy$$



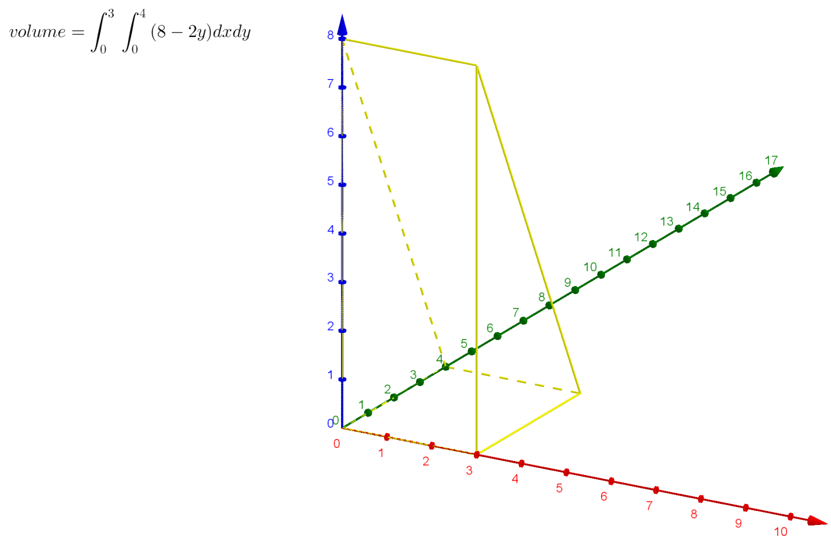
$z = 4;\ dz = dxdy$

$v = \int_0^3 \int_0^2 z\, dz = \int_0^3 \int_0^2 4\, dydx = 4\int_0^3 dx \int_0^2 dy = 4\int_0^3 dx\, [y]_0^2 = 4\int_0^3 dx\, [2-0] =$
$8\int_0^3 dx = 8[x]_0^3 = 8[3-0] = 8\cdot 3 = 24$

2. Exercício

$R = [0,3]\times[0,4]$
$\iint_R (8-2y)da$

Figura 8 – Integrais duplas - Aula 3 - Exercício II



$z = 8 - 2y;\ da = dz = dxdy$

$v = \int_0^3 \int_0^4 z\, dz = \int_0^3 \int_0^4 (8{-}2y)dxdy = \int_0^3 dx \int_0^4 (8{-}2y)dy = \int_0^3 dx \left(8\int_0^4 dy - 2\int_0^4 y\, dy\right) = \int_0^3 dx\, 2\left(4\int_0^4 dy - \int_0^4 y\, dy\right) = 2\int_0^3 dx \left[4y - \frac{y^2}{2}\right]_0^4 = 2\int_0^3 dx \left[\frac{8y-y^2}{2}\right]_0^4 = \cancel{2}\int_0^3 dx\, \frac{1}{\cancel{2}}[y(8-y)]_0^4 = \int_0^3 dx[4(8-4)\cancel{-0(8-0)}] = 16\int_0^3 dx = 16[x]_0^3 = 16[3-0] = 48$

1.4 Invertendo a ordem de integração - Aula 4

1. Exercício

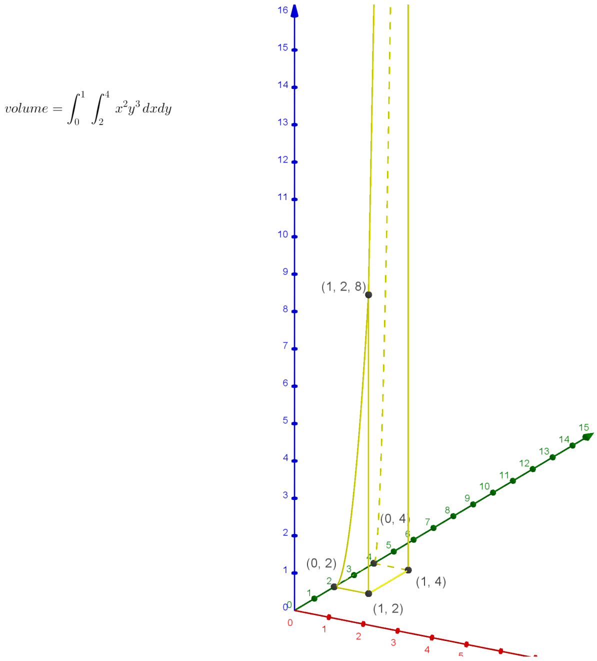
   $z = f(x,y) = y\,\mathrm{e}^x;\ dz = dxdy$

   $v = \int_2^4 \int_1^9 z\, dz = \int_2^4 \int_1^9 y\,\mathrm{e}^x\ dydx = \int_2^4 \mathrm{e}^x\ dx \int_1^9 y\, dy = \int_2^4 \mathrm{e}^x\ dx \left[\frac{y^2}{2}\right]_1^9 = \int_2^4 \mathrm{e}^x\ dx \frac{1}{2}\left[y^2\right]_1^9 = \frac{1}{2}\int_2^4 \mathrm{e}^x\ dx \left[9^2 - 1^2\right] = 40\int_2^4 \mathrm{e}^x\ dx = 40\left[\mathrm{e}^x\right]_2^4 = 40\left[\mathrm{e}^4 - \mathrm{e}^2\right] = 40e^2\left(e^2 - 1\right)$

2. Exercício

   $z = f(x,y) = x^2y^3;\ dz = dxdy$

Figura 9 – Integrais duplas - Aula 4 - Exercício II



$$v = \int_0^1 \int_2^4 z\,dz = \int_0^1 \int_2^4 x^2y^3\,dxdy = \int_0^1 x^2\,dx \int_2^4 y^3\,dy = \int_0^1 x^2\,dx \left[\frac{y^4}{4}\right]_2^4 = \frac{1}{4}\int_0^1 x^2\,dx \left[y^4\right]_2^4 =$$
$$\frac{1}{4}\int_0^1 x^2\,dx \left[4^4 - 2^4\right] = \frac{1}{4}\int_0^1 x^2\,dx \left[2^8 - 2^4\right] =$$
$$\frac{1}{4}\int_0^1 x^2\,dx \left[2^4\left(2^4 - 1\right)\right] = \frac{1}{4}\int_0^1 x^2\,dx\,[16 \cdot 15] = 60\int_0^1 x^2\,dx = 60\left[\frac{x^3}{3}\right]_0^1 = 20\left[x^3\right]_0^1 =$$
$$20\left[1^3 - 0^3\right] = 20 \cdot 1 = 20$$

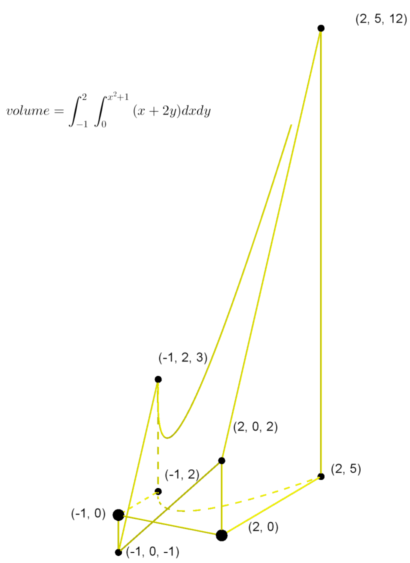
3. Exercício

$$\iint_R (x + 2y)da$$

R = Região limitada pela parábola $y = x^2 + 1$ e as retas $x = -1$ e $x = 2$.

$z = f(x, y) = x + 2y;\ da = dz = dxdy$

Figura 10 – Integrais duplas - Aula 4 - Exercício III



$$v = \int_{-1}^{2}\int_{0}^{x^2+1} z\,dz = \int_{-1}^{2}\int_{0}^{x^2+1}(x+2y)dxdy = \int_{-1}^{2} dx \int_{0}^{x^2+1}(x+2y)dy = \int_{-1}^{2} dx \left(x\int_{0}^{x^2+1} dy + 2\int_{0}^{x^2+1}\right.$$

$$\int_{-1}^{2} dx \left[xy + \cancel{2}\frac{y^2}{\cancel{2}} \right]_0^{x^2+1} = \int_{-1}^{2} dx\,[y(x+y)]_0^{x^2+1} = \int_{-1}^{2} dx \left[\left(x^2+1\right)\left[x + \left(x^2+1\right)\right] \cancel{-0(x+0)} \right] =$$

$$\int_{-1}^{2} dx \left[\left(x^2+1\right)\left(x^2+x+1\right) \right] = \int_{-1}^{2} dx \left(x^4+x^3+2x^2+x+1\right) = \int_{-1}^{2} x^4\,dx + \int_{-1}^{2} x^3\,dx +$$

$$2\int_{-1}^{2} x^2\,dx + \int_{-1}^{2} x\,dx + \int_{-1}^{2} dx = \left[\frac{x^5}{5} + \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^{2} =$$

$$\left[\frac{12x^5 + 15x^4 + 40x^3 + 30x^2 + 60x}{60} \right]_{-1}^{2} = \frac{1}{60}\left[x\left(12x^4 + 15x^3 + 40x^2 + 30x + 60\right) \right]_{-1}^{2} =$$

$$\frac{1}{60}\left[2\left(12\cdot 2^4 + 15\cdot 2^3 + 40\cdot 2^2 + 30\cdot 2 + 60\right) \right.$$

$$\left. -(-1)\left(12(-1)^4 + 15(-1)^3 + 40(-1)^2 + 30(-1) + 60\right) \right] =$$

$$\frac{1}{60}[2(192+120+160+60+60) + (12-15+40-30+60)] = \frac{1}{60}(1184+67) =$$

$$\frac{1251}{60} = \frac{417}{20} = 20,85$$

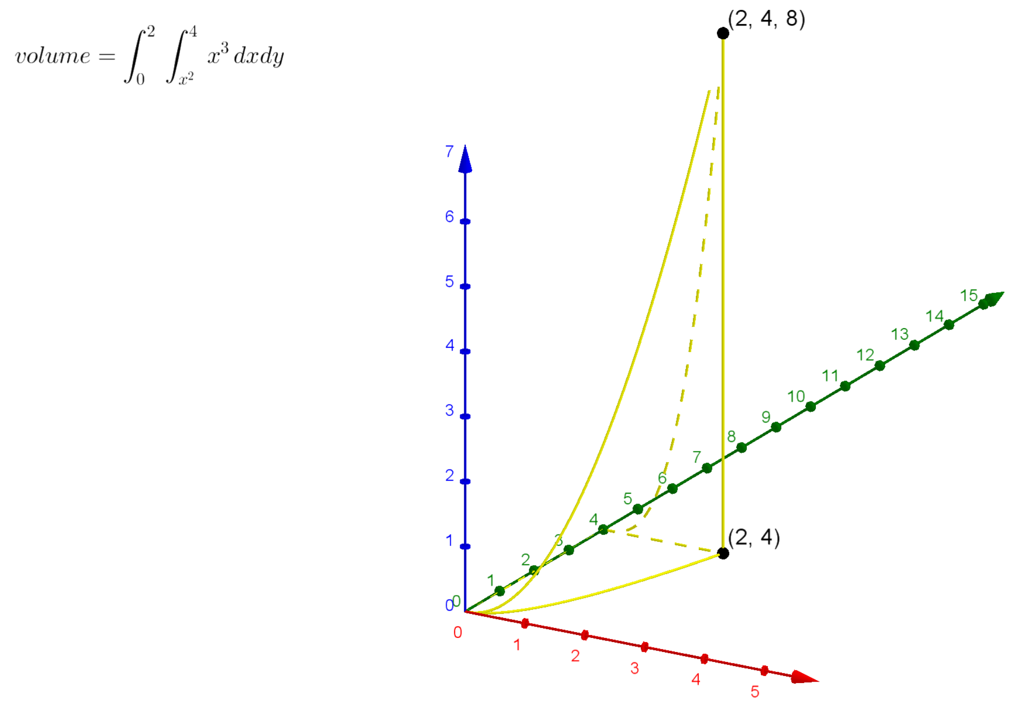
1.5 Cálculo de integrais duplas ou iteradas

1.5.1 Aula 5

1. Exercício

$$f(x,y) = x^3;\ 0 \le x \le 2;\ x^2 \le y \le 4$$
$$\iint_R f(x,y)dydx$$

Figura 11 – Integrais duplas - Aula 5 - Exercício I



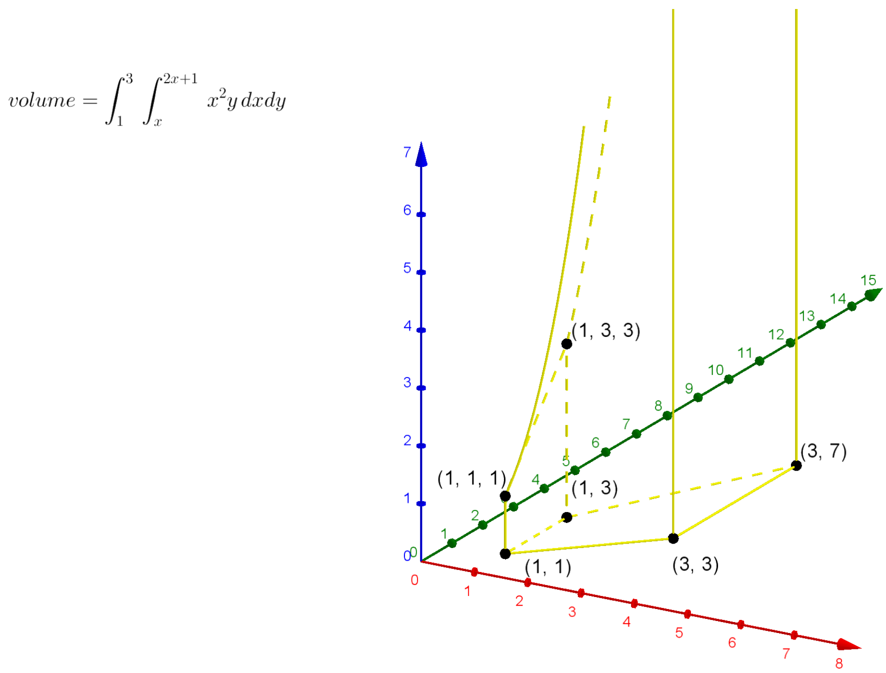
$$v = \int_0^2 \int_{x^2}^4 x^3\, dxdy = \int_0^2 x^3\, dx \int_{x^2}^4 dy = \int_0^2 x^3\, dx\, [y]_{x^2}^4 = \int_0^2 x^3\, dx\, \left[4 - x^2\right] = 4\int_0^2 x^3\, dx - \int_0^2 x^5\, dx = \left[\cancel{4}\frac{x^4}{\cancel{4}} - \frac{x^6}{6}\right]_0^2 = \left[\frac{6x^4 - x^6}{6}\right]_0^2 = \frac{1}{6}\left[x^4\left(6 - x^2\right)\right]_0^2 = \frac{1}{6}\left[2^4\left(6 - 2^2\right) - \cancel{0^4\left(6 - 0^2\right)}\right] = \frac{1}{6}(16 \cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2$$

2. Exercício

$f(x, y) = x^2 y;\ 1 \le x \le 3;\ x \le y \le 2x + 1$

$\iint_R f(x, y) dydx$

Figura 12 – Integrais duplas - Aula 5 - Exercício II



$$v = \int_1^3 \int_x^{2x+1} x^2 y \, dx dy = \int_1^3 x^2 \, dx \int_x^{2x+1} y \, dy = \int_1^3 x^2 \, dx \left[\frac{y^2}{2}\right]_x^{2x+1} =$$
$$\int_1^3 x^2 \, dx \frac{1}{2}\left[(2x+1)^2 - (x)^2\right] = \frac{1}{2}\int_1^3 x^2 \, dx \left(3x^2 + 4x + 1\right) = \frac{3}{2}\int_1^3 x^4 \, dx + 2\int_1^3 x^3 \, dx +$$
$$\frac{1}{2}\int_1^3 x^2 \, dx = \left[\frac{3}{2}\frac{x^5}{5} + 2\frac{x^4}{4} + \frac{1}{2}\frac{x^3}{3}\right]_1^3 = \left[\frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6}\right]_1^3 = \left[\frac{18x^5 + 30x^4 + 10x^3}{60}\right]_1^3 =$$
$$\left[\frac{2x^3\left(9x^2 + 15x + 5\right)}{60}\right]_1^3 = \frac{1}{30}\left[x^3\left(9x^2 + 15x + 5\right)\right]_1^3 = \frac{1}{30}\left[3^3\left(9 \cdot 3^2 + 15 \cdot 3 + 5\right) - 1^3\left(9 \cdot 1^2 + 15 \cdot 1 +\right.\right.$$
$$\frac{1}{30}\left[27(81 + 45 + 5) - (9 + 15 + 5)\right] = \frac{1}{30}\left[27 \cdot 131 - 29\right] = \frac{3508}{30} = 116{,}9\overline{3}$$

1.5.2 Aula 6

1. Exercício

   $f(x, y) = 1;\ 0 \leq x \leq 1;\ 1 \leq y \leq \mathrm{e}^x$
   $\iint_R f(x, y) dy dx$

   $$v = \int_0^1 \int_1^{\mathrm{e}^x} dy dx = \int_0^1 dx \, [y]_1^{\mathrm{e}^x} = \int_0^1 dx \, (\mathrm{e}^x - 1) = [\mathrm{e}^x - x]_0^1 = \mathrm{e}^1 - 1 - \left(\mathrm{e}^0 - 0\right) =$$
   $$\mathrm{e} - 1 - 1 = \mathrm{e} - 2$$

2. Exercício

   $f(x, y) = x;\ 0 \leq x \leq 1;\ 1 \leq y \leq \mathrm{e}^{x^2}$
   $\iint_R f(x, y) dy dx$

$$v = \int_0^1 \int_1^{e^{x^2}} x\,dxdy = \int_0^1 x\,dx \int_1^{e^{x^2}} dy = \int_0^1 x\,dx\,[y]_1^{e^{x^2}} = \int_0^1 x\,dx\left(e^{x^2} - 1\right) =$$
$$\int_0^1 x\,e^{x^2}\,dx - \int_0^1 x\,dx = \int_0^1 e^u\,\frac{du}{2} - \int_0^1 x\,dx = \frac{1}{2}\int_0^1 e^u\,du - \int_0^1 x\,dx = \left[\frac{1}{2}e^u - \frac{x^2}{2}\right]_0^1 =$$
$$\left[\frac{e^{x^2} - x^2}{2}\right]_0^1 = \frac{1}{2}\left[e^{x^2} - x^2\right]_0^1 = \frac{1}{2}\left[e^{1^2} - 1^2 - \left(e^{0^2} - 0^2\right)\right] = \frac{1}{2}(e - 1 - 1) = \frac{e - 2}{2}$$
$$u = x^2;\ \frac{du}{2} = x\,dx$$

3. Exercício

$$f(x,y) = 2xy;\ 0 \le y \le 1;\ y^2 \le x \le y$$
$$\iint_R f(x,y)dxdy$$
$$v = \int_0^1 \int_{y^2}^{y} 2xy\,dxdy = 2\int_0^1 y\,dy \int_{y^2}^{y} x\,dx = 2\int_0^1 y\,dy\left[\frac{x^2}{2}\right]_{y^2}^{y} = \cancel{2}\int_0^1 y\,dy\,\frac{1}{\cancel{2}}\left[x^2\right]_{y^2}^{y} =$$
$$\int_0^1 y\,dy\left(y^2 - y^4\right) = \int_0^1 \left(y^3 - y^5\right)dy = \left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1 = \left[\frac{6y^4 - 4y^6}{24}\right]_0^1 = \left[\frac{2y^4\left(3 - 2y^2\right)}{24}\right]_0^1 =$$
$$\frac{1}{12}\left[1^4\left(3 - 2\cdot 1^2\right)\cancel{- 0^4\left(3 - 2\cdot 0^2\right)}\right] = \frac{1}{12} = 0,08\overline{3}$$

1.5.3 Aula 7

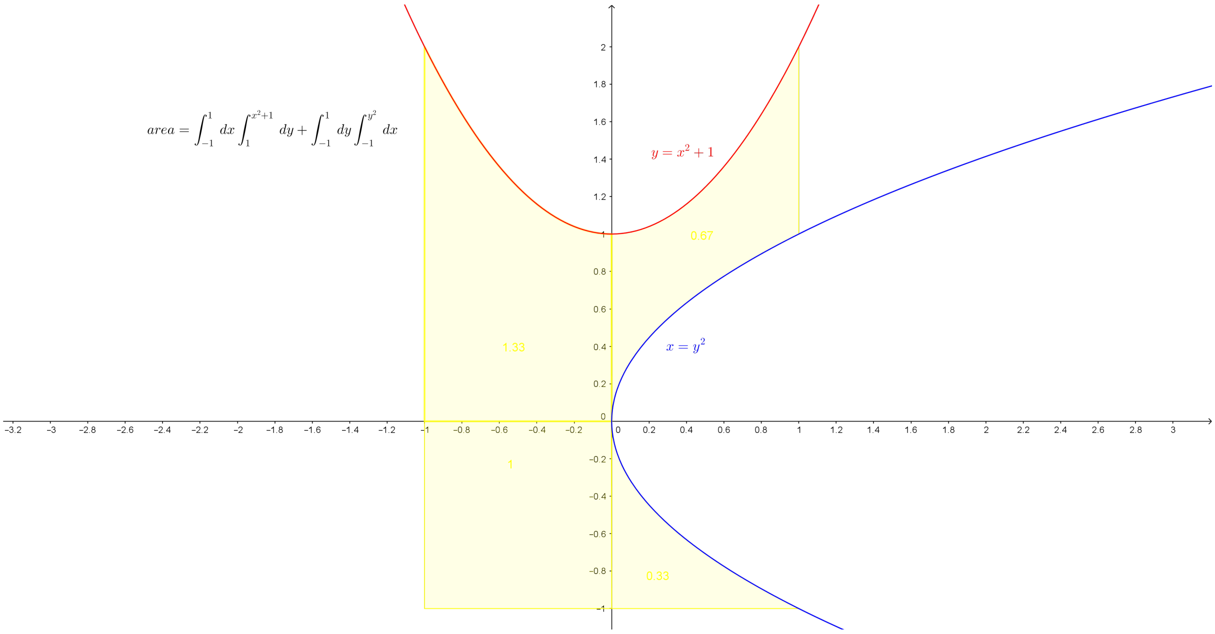
1. Exercício

$$f(x,y) = \frac{1}{x+y};\ 1 \le y \le e;\ 0 \le x \le y$$
$$\iint_R f(x,y)dxdy$$
$$v = \int_1^e \int_0^y \frac{1}{x+y}\,dxdy = \int_1^e dy \int_0^y (x+y)^{-1}\,dx = \int_1^e dy \int_0^y u^{-1}\,du = \int_1^e dy \int_0^y \left[\ln|u|\right]_0^y =$$
$$\int_1^e dy \int_0^y \left[\ln|x+y|\right]_0^y = \int_1^e dy \int_0^y \left(\ln|y+y| - \ln|0+y|\right) = \int_1^e dy \int_0^y \left(\ln|2y| - \ln|y|\right) =$$
$$\int_1^e dy \int_0^y \left(\ln|2|\cancel{+\ln|y| - \ln|y|}\right) = \ln|2|\int_1^e dy = \ln|2|[y]_1^e = \ln|2|(e - 1)$$
$$u = x + y;\ du = (1+0)dx = dx$$

1.6 Cálculo de área - Aula 8

1. Exercício

Figura 13 – Integrais duplas - Aula 8 - Exercício I



$$a = \int_{-1}^{0} dx \int_{0}^{x^2+1} dy + \int_{-1}^{0} dx \int_{-1}^{0} dy + \int_{0}^{y^2} dx \int_{-1}^{0} dy + \int_{0}^{1} dx \int_{\sqrt{x}}^{x^2+1} =$$

$$\int_{-1}^{0} dx \left(\int_{0}^{x^2+1} dy + \int_{-1}^{0} dy\right) + \int_{0}^{y^2} dx \int_{-1}^{0} dy + \int_{0}^{1} dx \int_{\sqrt{x}}^{x^2+1} = \int_{-1}^{0} dx \left([y]_0^{x^2+1} + [y]_{-1}^{0}\right) +$$

$$\int_{-1}^{0} dy\, [x]_0^{y^2} + \int_{0}^{1} dx\, [y]_{\sqrt{x}}^{x^2+1} = \int_{-1}^{0} dx\, \left(x^2 + 1 + 1\right) + \int_{-1}^{0} dy\, y^2 + \int_{0}^{1} dx\, \left(x^2 + 1 - \sqrt{x}\right) =$$

$$\int_{-1}^{0} \left(x^2 + 2\right) dx + \int_{-1}^{0} y^2\, dy + \int_{0}^{1} \left(x^2 - x^{\frac{1}{2}} + 1\right) dx = \left[\frac{x^3}{3} + 2x\right]_{-1}^{0} + \left[\frac{y^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3} - \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + x\right]_{0}^{1} =$$

$$\left[\frac{x^3 + 6x}{3}\right]_{-1}^{0} + \frac{1}{3}\left[y^3\right]_{-1}^{0} + \left[\frac{x^3}{3} - \frac{2\sqrt{x^3}}{3} + x\right]_{0}^{1} = \frac{1}{3}\left[x\left(x^2 + 6\right)\right]_{-1}^{0} + \frac{1}{3}\left[\cancel{0^3} - (-1)^3\right] +$$

$$\left[\frac{x^3 - 2\sqrt{x^3} + 3x}{3}\right]_{0}^{1} =$$

$$\frac{1}{3}\left[\cancel{0\left(0^2 + 6\right)} - (-1)\left((-1)^2 + 6\right)\right] + \frac{1}{3} + \frac{1}{3}\left[x^3 - 2\sqrt{x^3} + 3x\right]_{0}^{1} = \frac{7}{3} + \frac{1}{3} + \frac{1}{3}\left[1^3 - 2\sqrt{1^3} + 3 \cdot 1 \cancel{- \left(0^3 - 2\sqrt{}\right.}\right.$$

$$\frac{7}{3} + \frac{1}{3} + \frac{2}{3} = \frac{7 + 1 + 2}{3} = \frac{10}{3} = 3,\overline{3}$$

$$a = \int_{-1}^{1} dx \int_{1}^{x^2+1} dy + \int_{-1}^{y^2} dx \int_{-1}^{1} dy = \int_{-1}^{1} dx\, [y]_1^{x^2+1} + \int_{-1}^{1} dy\, [x]_{-1}^{y^2} = \int_{-1}^{1} dx\, \left(x^2 \cancel{+1} \cancel{-1}\right) +$$

$$\int_{-1}^{1} dy\, \left(y^2 + 1\right) = \left[\frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{y^3}{3} + y\right]_{-1}^{1} = \frac{1}{3}\left[x^3\right]_{-1}^{1} + \frac{1}{3}\left[y\left(y^2 + 3\right)\right]_{-1}^{1} = \frac{1}{3}\left(\left[1^3 - (-1)^3\right] + \left[1\left(1^2 + 3\right)\right.\right.$$

$$4 + 4) = \frac{10}{3} = 3,\overline{3}$$

1.7 Cálculo de volume

1.7.1 Aula 9

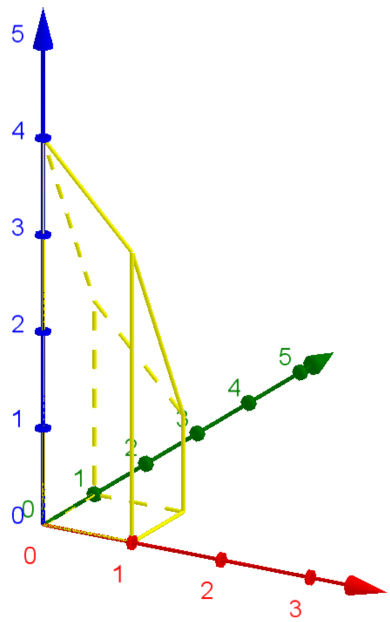
1. Exercício

   Esboçe a região de integração e o sólido cujo volume é dado pela integral abaixo:

$$\int_0^1 \int_0^1 (4 - x - 2y)\ dxdy$$

Figura 14 – Integrais duplas - Aula 9 - Exercício I

$$volume = \int_0^1 \int_0^1 (4 - x - 2y)\ dxdy$$



$v = \int_0^1 \int_0^1 (4 - x - 2y)\ dxdy = \int_0^1 dx \left(4\int_0^1 dy - x\int_0^1 dy - 2\int_0^1 y\, dy\right) = 4\int_0^1 dx \int_0^1 dy - \int_0^1 x\, dx \int_0^1 dy - 2\int_0^1 dx \int_0^1 y\, dy = 4[x]_0^1[y]_0^1 - \left[\frac{x^2}{2}\right]_0^1 [y]_0^1 - 2[x]_0^1 \left[\frac{y^2}{2}\right]_0^1 = 4 - \frac{1}{2} - \cancel{2}\frac{1}{\cancel{2}} = \frac{8-1-2}{2} = \frac{5}{2} = 2,5$

1.7.2 Aula 10

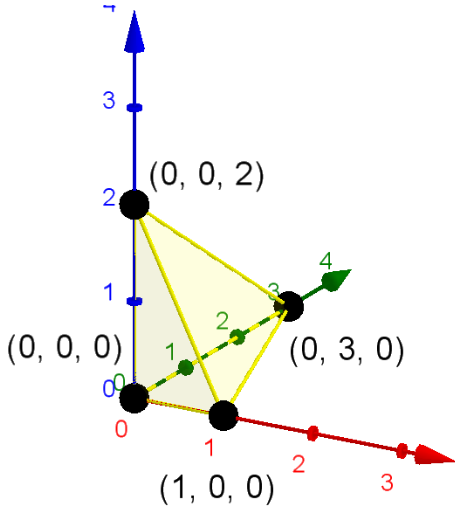
1. Exercício

   Calcule o volume do sólido limitado pelos planos:

$$x = 0,\ y = 0,\ z = 0 \text{ e } 6x + 2y + 3z = 6$$

Figura 15 – Integrais duplas - Aula 10 - Exercício I

$$volume = \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dxdy$$



$P_1 = (0,0,0)$

$6x = -2y - 3z + 6 \Rightarrow x = \dfrac{-2y-3z+6}{6} = \dfrac{-2\cdot 0 - 3\cdot 0 + 6}{6} = \dfrac{6}{6} = 1 \Rightarrow P_2 = (1,0,0)$

$2y = -6x - 3z + 6 \Rightarrow y = \dfrac{-6x-3z+6}{2} = \dfrac{-6\cdot 0 - 3\cdot 0 + 6}{2} = \dfrac{6}{2} = 3 \Rightarrow P_3 = (0,3,0)$

$3z = -6x - 2y + 6 \Rightarrow z = \dfrac{-6x-2y+6}{3} = \dfrac{-6\cdot 0 - 2\cdot 0 + 6}{3} = \dfrac{6}{3} = 2 \Rightarrow P_4 = (0,0,2)$

$x = 0, x = 1$

$y = 0,\ y = \dfrac{-6x-3z+6}{2} = \dfrac{-6x - 3\cdot 0 + 6}{2} = -3x+3$

$z = \dfrac{-6x-2y+6}{3} = -2x - \dfrac{2y}{3} + 2$

$v = \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dxdy = \int_0^1 dx \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dy = \int_0^1 dx \left[-2xy - \frac{\cancel{2}}{3}\frac{y^2}{\cancel{2}} +\right.$
$\int_0^1 dx\, \frac{1}{3}\left[-6xy - y^2 + 6y\right]_0^{-3x+3} = \frac{1}{3}\int_0^1 dx\, \left[-y(6x+y-6)\right]_0^{-3x+3} =$
$\frac{1}{3}\int_0^1 dx\, \left[-(-3x+3)(6x+(-3x+3)-6)\cancel{+0(6x+0-6)}\right] = \frac{1}{3}\int_0^1 dx\, \left[(3x-3)(3x-3)\right] =$
$\frac{1}{3}\int_0^1 \left(9x^2 - 18x + 9\right) dx = \frac{1}{3}\left[9\frac{x^3}{3} - 18\frac{x^2}{2} + 9x\right]_0^1 = \frac{1}{3}\left[3x^3 - 9x^2 + 9x\right]_0^1 = \cancel{\frac{1}{3}}\left[\cancel{3}x\left(x^2 - 3x + 3\right)\right]_0^1 =$
$\left[1\left(1^2 \cancel{-3\cdot 1 + 3}\right) \cancel{-0\left(0^2 - 3\cdot 0 + 3\right)}\right] = 1$

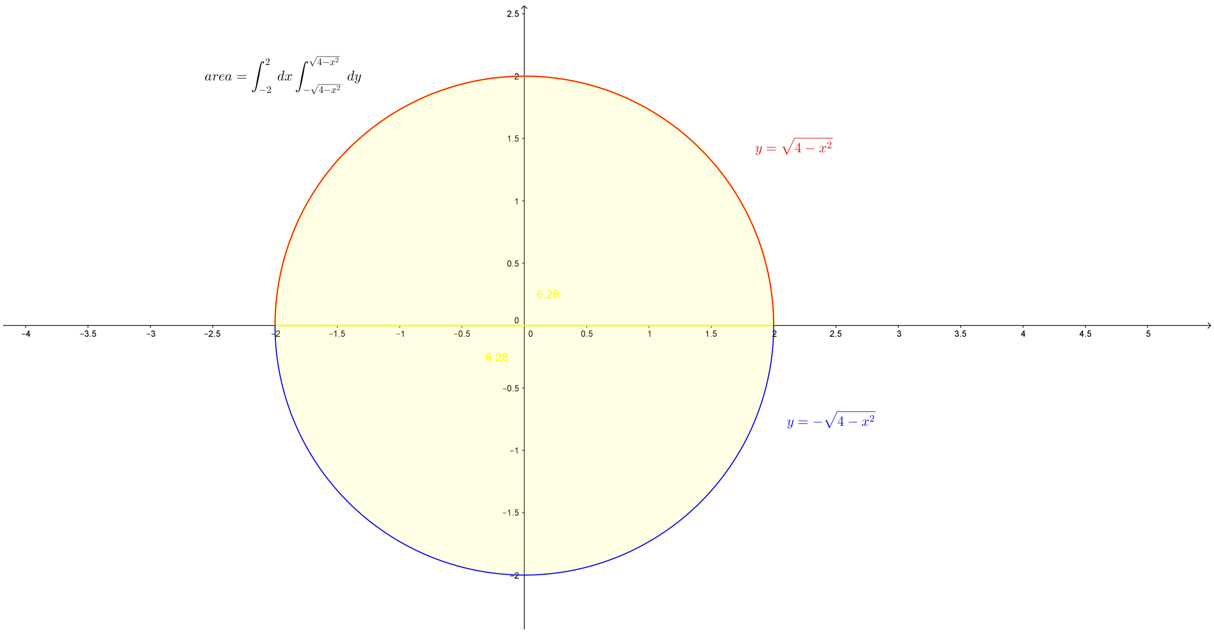
1.8 Coordenadas polares

1.8.1 Aula 1

1. Exercício

Calcule a área do circulo de raio igual a dois

Figura 16 – Coordenadas polares - Aula 01 - Exercício I



$r = 2 \Rightarrow a = \pi r^2 = 2^2\pi = 4\pi$

$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 2^2 \Rightarrow x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4-x^2}$
$R = \left\{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\right\}$

$a = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{-2}^{2} dx \left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) = 2\int_{-2}^{2} \sqrt{4-x^2}\, dx = 2\int_{-2}^{2} \sqrt{4-(2\operatorname{sen}(\alpha}$
$4\int_{-2}^{2} \sqrt{4-4\operatorname{sen}^2(\alpha)}\, \cos(\alpha)\, d\alpha = 4\int_{-2}^{2} \sqrt{4-4\left(1-\cos^2(\alpha)\right)}\, \cos(\alpha)\, d\alpha = 4\int_{-2}^{2} \sqrt{4-(4-4\cos^2(}$
$4\int_{-2}^{2} \sqrt{\cancel{4-4}+4\cos^2(\alpha)}\, \cos(\alpha)\, d\alpha = 4\int_{-2}^{2} 2\cos(\alpha)\cos(\alpha)\, d\alpha = 8\int_{-2}^{2} \cos^2(\alpha)\, d\alpha =$
$8\int_{-2}^{2} \left(\frac{1+\cos(2\alpha)}{2}\right) d\alpha = 8\int_{-2}^{2} \left(\frac{1}{2} + \frac{\cos(2\alpha)}{2}\right) d\alpha = 4\int_{-2}^{2} d\alpha + 4\int_{-2}^{2} \cos(2\alpha)\, d\alpha =$
$4\int_{-2}^{2} d\alpha + 4\int_{-2}^{2} \cos(u)\, \frac{du}{2} = 4\int_{-2}^{2} d\alpha + 2\int_{-2}^{2} \cos(u)\, du = [4\alpha + 2\operatorname{sen}(u)]_{-2}^{2} = [4\alpha + 2\operatorname{sen}(2\alpha)]_{-2}^{2} =$
$[4\alpha + 4\operatorname{sen}(\alpha)\cos(\alpha)]_{-2}^{2} = [4\left(\alpha + \operatorname{sen}(\alpha)\cos(\alpha)\right)]_{-2}^{2} = \left[4\left(\operatorname{arcsen}\left(\frac{x}{2}\right) + \frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right)\right]_{-2}^{2} =$
$\left[4\left(\operatorname{arcsen}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4}\right)\right]_{-2}^{2} = 4\left(\operatorname{arcsen}\left(\frac{2}{2}\right) + \frac{2\sqrt{4-2^2}}{4}\right) - 4\left(\operatorname{arcsen}\left(\frac{(-2)}{2}\right) + \frac{(-2)\sqrt{}}{}\right.$
$4\operatorname{arcsen}(1) - 4\operatorname{arcsen}(-1) = 4(\operatorname{arcsen}(1) - \operatorname{arcsen}(-1)) = 4\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 4\left(\frac{\cancel{2}\pi}{\cancel{2}}\right) = 4\pi$

$x = 2\operatorname{sen}(\alpha);\ dx = 2\cos(\alpha)\, d\alpha$
$u = 2\alpha;\ \frac{du}{2} = d\alpha$

$$sen(\alpha) = \frac{co}{h} = \frac{x}{2} \Rightarrow \alpha = \text{arcsen}\left(\frac{x}{2}\right)$$
$$h^2 = co^2 + ca^2 \Rightarrow 2^2 = x^2 + ca^2 \Rightarrow ca = \sqrt{4-x^2}$$
$$cos(\alpha) = \frac{ca}{h} = \frac{\sqrt{4-x^2}}{2}$$

$$R = \{(r,\theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2,\, 0 \leq \theta \leq 2\pi\}$$

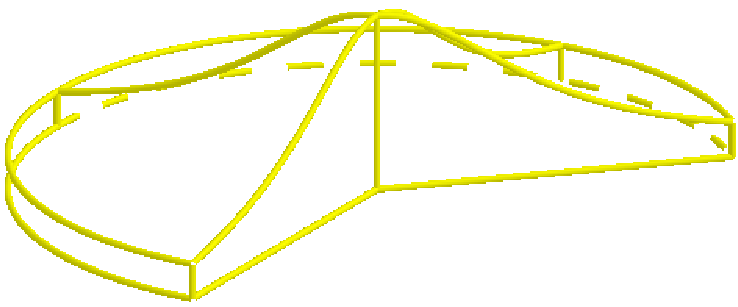
$$a = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_0^2 \int_0^{2\pi} r\, dr d\theta = \int_0^2 r\, dr \int_0^{2\pi} d\theta = \left[\frac{r^2}{2}\right]_0^2 [\theta]_0^{2\pi} = \frac{1}{2}\left[2^2 - 0^2\right][2\pi - 0] = \frac{4}{\cancel{2}}\cancel{2}\pi = 4\pi$$

2. Exercício

$$\iint_R \frac{da}{1+x^2+y^2}$$

Figura 17 – Coordenadas polares - Aula 01 - Exercício II

$$volume = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\, dr d\theta}{1+r^2}$$



$$R = \left\{(r,\theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2,\, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{2}\right\}$$
$$v = \iint_R \frac{da}{1+x^2+y^2} = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\, dr d\theta}{1+r^2} = \int_0^2 \frac{r\, dr}{1+r^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} d\theta = \int_0^2 \left(1+r^2\right)^{-1} r\, dr\, [\theta]_{\frac{\pi}{4}}^{\frac{3\pi}{2}} =$$

$$\int_0^2 \left(1+r^2\right)^{-1} r\,dr \left(\frac{3\pi}{2}-\frac{\pi}{4}\right) = \int_0^2 \left(1+r^2\right)^{-1} r\,dr \left(\frac{6\pi-\pi}{4}\right) = \frac{5\pi}{4}\int_0^2 \left(1+r^2\right)^{-1} r\,dr =$$
$$\frac{5\pi}{4}\int_0^2 u^{-1}\frac{du}{2} = \frac{5\pi}{8}\int_0^2 u^{-1}du = \frac{5\pi}{8}\left[ln|u|\right]_0^2 = \frac{5\pi}{8}\left[ln|1+r^2|\right]_0^2 =$$
$$\frac{5\pi}{8}\left[ln|1+2^2| - ln|1+0^2|\right] = \frac{5\pi}{8}\left[ln|5| - ln|1|\right] = \frac{5\pi ln|5|}{8}$$

$$u = 1+r^2 \Rightarrow \frac{du}{2} = r\,dr$$
$$\mathrm{e}^x = 1 = \mathrm{e}^0 \Rightarrow x = 0$$

2 Integrais triplas

Cálculo de integrais triplas.

Referências

GRINGS, F. *Curso de Integrais Duplas e Triplas*. 2016. <https://www.youtube.com/playlist?list=PL82B9E5FF3F2B3BD3>. Citado na página 9.

Anexos

ANEXO A – Derivadas

A.1 Derivadas simples

Tabela 1 – Derivadas simples

$$
\begin{array}{|lllllll|}
y & = & c & \Rightarrow & y' & = & 0 \\
y & = & x & \Rightarrow & y' & = & 1 \\
y & = & x^c & \Rightarrow & y' & = & cx^{c-1} \\
y & = & \mathrm{e}^x & \Rightarrow & y' & = & \mathrm{e}^x \\
y & = & \ln|x| & \Rightarrow & y' & = & \frac{1}{x} \\
y & = & uv & \Rightarrow & y' & = & u'v + uv' \\
y & = & \frac{u}{v} & \Rightarrow & y' & = & \frac{u'v - uv'}{v^2} \\
y & = & u^c & \Rightarrow & y' & = & cu^{c-1}u' \\
y & = & \mathrm{e}^u & \Rightarrow & y' & = & \mathrm{e}^u\, u' \\
y & = & c^u & \Rightarrow & y' & = & c^u u' ln|c| \\
y & = & \ln|u| & \Rightarrow & y' & = & \frac{u'}{u} \\
y & = & \log_c|u| & \Rightarrow & y' & = & \frac{u'}{u}\log_c|\,\mathrm{e}\,|
\end{array}
$$

A.2 Derivadas trigonométricas

Tabela 2 – Derivadas trigonométricas

$$
\begin{array}{|lllllll|}
y & = & \mathrm{sen}(x) & \Rightarrow & y' & = & \cos(x) \\
y & = & \cos(x) & \Rightarrow & y' & = & -\,\mathrm{sen}(x) \\
y & = & \mathrm{tg}(x) & \Rightarrow & y' & = & \sec^2(x) \\
y & = & \mathrm{cotg}(x) & \Rightarrow & y' & = & -\,\mathrm{cossec}^2(x) \\
y & = & \sec(x) & \Rightarrow & y' & = & \sec(x)\,\mathrm{tg}(x) \\
y & = & \mathrm{cossec}(x) & \Rightarrow & y' & = & -\,\mathrm{cossec}(x)\,\mathrm{cotg}(x) \\
y & = & \mathrm{arcsen}(x) & \Rightarrow & y' & = & \frac{1}{\sqrt{1-x^2}} \\
y & = & \arccos(x) & \Rightarrow & y' & = & \frac{-1}{\sqrt{1-x^2}} \\
y & = & \mathrm{arctg}(x) & \Rightarrow & y' & = & \frac{1}{1+x^2} \\
y & = & \mathrm{arccotg}(x) & \Rightarrow & y' & = & \frac{-1}{1+x^2} \\
y & = & \mathrm{arcsec}(x) & \Rightarrow & y' & = & \frac{1}{|x|\sqrt{x^2-1}} \\
y & = & \mathrm{arccossec}(x) & \Rightarrow & y' & = & \frac{-1}{|x|\sqrt{x^2-1}}
\end{array}
$$

ANEXO B – Integrais

B.1 Integrais simples

Tabela 3 – Integrais simples

$$
\begin{array}{|llll|}
\displaystyle\int dx & = & x + c & \\
\displaystyle\int x^p \, dx & = & \dfrac{x^{p+1}}{p+1} + c & \rightarrow \quad p \neq -1 \\
\displaystyle\int \mathrm{e}^x \; dx & = & \mathrm{e}^x + c & \\
\displaystyle\int \dfrac{dx}{x} & = & \ln|x| + c & \\
\displaystyle\int u^p \, du & = & \dfrac{u^{p+1}}{p+1} + c & \rightarrow \quad p \neq -1 \\
\displaystyle\int \mathrm{e}^u \; du & = & \mathrm{e}^u + c & \\
\displaystyle\int \dfrac{du}{u} & = & \ln|u| + c & \\
\displaystyle\int p^u \, du & = & \dfrac{p^u}{\ln|p|} + c &
\end{array}
$$

B.2 Integrais trigonométricas

Tabela 4 – Integrais trigonométricas

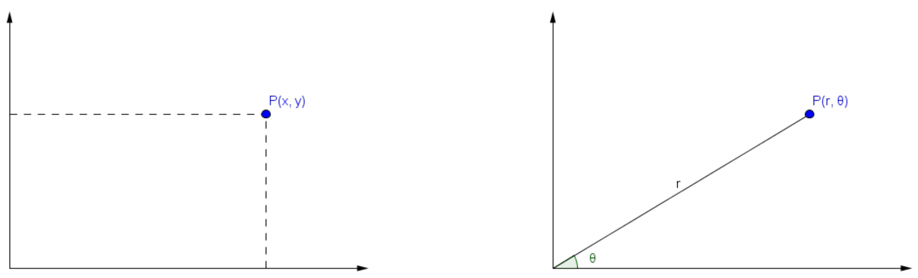
$\int \operatorname{sen}(u)du$	$=$	$-\cos(u) + c$
$\int \cos(u)du$	$=$	$\operatorname{sen}(u) + c$
$\int \operatorname{tg}(u)du$	$=$	$\ln\lvert\sec(u)\rvert + c$
$\int \operatorname{cotg}(u)du$	$=$	$\ln\lvert\operatorname{sen}(u)\rvert + c$
$\int \sec(u)du$	$=$	$\ln\lvert\sec(u) + \operatorname{tg}(u)\rvert + c$
$\int \operatorname{cossec}(u)du$	$=$	$\ln\lvert\operatorname{cossec}(u) - \operatorname{cotg}(u)\rvert + c$
$\int \sec^2(u)du$	$=$	$\operatorname{tg}(u) + c$
$\int \operatorname{cossec}^2(u)du$	$=$	$-\operatorname{cotg}(u) + c$
$\int \sec(u)\operatorname{tg}(u)du$	$=$	$\sec(u) + c$
$\int \operatorname{cossec}(u)\operatorname{cotg}(u)du$	$=$	$-\operatorname{cossec}(u) + c$
$\int \frac{du}{\sqrt{1-x^2}}$	$=$	$\operatorname{arcsen}(x) + c$
$-\int \frac{du}{\sqrt{1-x^2}}$	$=$	$\arccos(x) + c$
$\int \frac{du}{1+x^2}$	$=$	$\operatorname{arctg}(x) + c$
$-\int \frac{du}{1+x^2}$	$=$	$\operatorname{arccotg}(x) + c$
$\int \frac{du}{\lvert x\rvert\sqrt{x^2-1}}$	$=$	$\operatorname{arcsec}(x) + c$
$-\int \frac{du}{\lvert x\rvert\sqrt{x^2-1}}$	$=$	$\operatorname{arccossec}(x) + c$

B.3 Relação entre coordenada cartesina e polar

Figura 18 – Coordenada cartesina e polar

(a) Coordenada cartesiana ou retangular

(b) Coordenada polar



$$P(x, y) \to P(r, \theta)$$

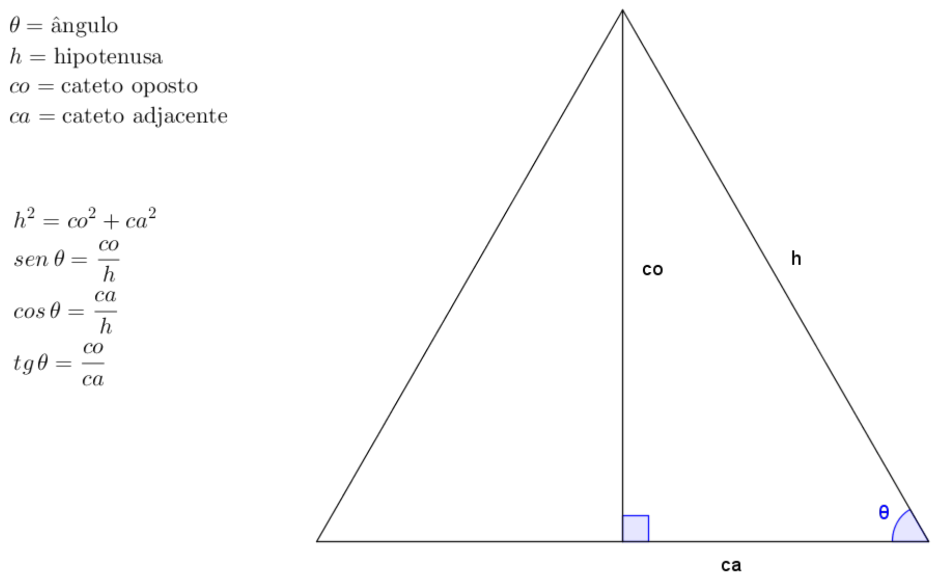
Tabela 5 – Relação entre coordenada cartesina e polar

$$\left| \begin{array}{lllll} & & x & = & r\ \cos\theta \\ & & y & = & r\ \operatorname{sen}\theta \\ & & x^2 + y^2 & = & r^2 \\ da & = & dxdy & = & r\, drd\theta \\ v & = & \displaystyle\iint_{R(x,y)} f(x, y)\, dxdy & = & \displaystyle\iint_{R(r,\theta)} f(r\ \cos\theta, r\ \operatorname{sen}\theta) r\, drd\theta \end{array} \right|$$

ANEXO C – Funções trigonométricas

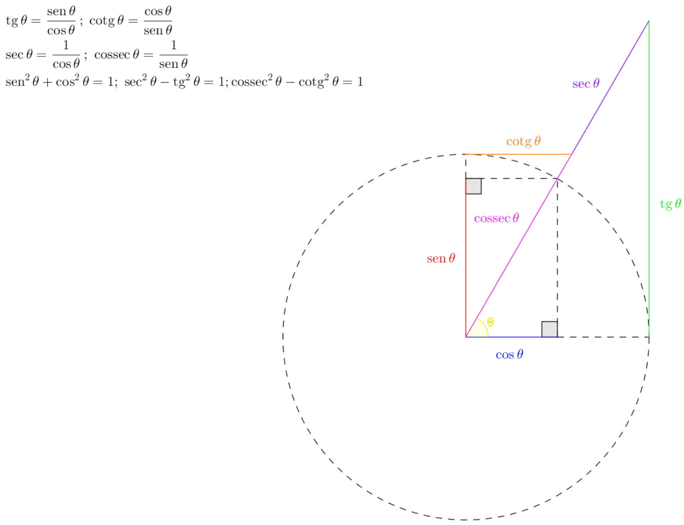
C.1 Determinação do seno, cosseno e tangente

Figura 19 – Determinação do seno, cosseno e tangente



C.2 Círculo trigonométrico

Figura 20 – Círculo trigonométrico



C.3 Identidades trigonométricas

Tabela 6 – Identidades trigonométricas

$\mathrm{tg}(x)$	$=$	$\dfrac{\mathrm{sen}(x)}{\cos(x)}$
$\mathrm{cotg}(x)$	$=$	$\dfrac{\cos(x)}{\mathrm{sen}(x)}$
$\sec(x)$	$=$	$\dfrac{1}{\cos(x)}$
$\mathrm{cossec}(x)$	$=$	$\dfrac{1}{\mathrm{sen}(x)}$
$\mathrm{sen}^2(x) + \cos^2(x)$	$=$	1
$\sec^2(x) - \mathrm{tg}^2(x)$	$=$	1
$\mathrm{cossec}^2(x) - \mathrm{cotg}^2(x)$	$=$	1
$\mathrm{sen}^2(x)$	$=$	$\dfrac{1-\cos(2x)}{2}$
$\cos^2(x)$	$=$	$\dfrac{1+\cos(2x)}{2}$
$\mathrm{sen}(2x)$	$=$	$2\,\mathrm{sen}(x)\cos(x)$
$\cos(2x)$	$=$	$\cos^2(x) - \mathrm{sen}^2(x)$

C.4 Relação entre trigonométricas e inversas

Tabela 7 – Relação entre trigonométricas e inversas

$\mathrm{sen}(\theta)$	$=$	x	$\Rightarrow$	θ	$=$	$\mathrm{arcsen}(x)$
$\cos(\theta)$	$=$	x	$\Rightarrow$	θ	$=$	$\arccos(x)$
$\mathrm{tg}(\theta)$	$=$	x	$\Rightarrow$	θ	$=$	$\mathrm{arctg}(x)$
$\mathrm{cossec}(\theta)$	$=$	x	$\Rightarrow$	θ	$=$	$\mathrm{arccossec}(x)$
$\sec(\theta)$	$=$	x	$\Rightarrow$	θ	$=$	$\mathrm{arcsec}(x)$
$\mathrm{cotg}(\theta)$	$=$	x	$\Rightarrow$	θ	$=$	$\mathrm{arccotg}(x)$

C.5 Substituição trigonométrica

Tabela 8 – Substituição trigonométrica

$\sqrt{a^2 - x^2}$	$\Rightarrow$	x	$=$	$a\,\mathrm{sen}(\theta)$
$\sqrt{a^2 + x^2}$	$\Rightarrow$	x	$=$	$a\,\mathrm{tg}(\theta)$
$\sqrt{x^2 - a^2}$	$\Rightarrow$	x	$=$	$a\sec(\theta)$

C.6 Ângulos notáveis

Tabela 9 – Ângulos notáveis

ângulo	$0°\ (0)$	$30°\ \left(\frac{\pi}{6}\right)$	$45°\ \left(\frac{\pi}{4}\right)$	$60°\ \left(\frac{\pi}{3}\right)$	$90°\ \left(\frac{\pi}{2}\right)$
sen	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\nexists$