Curso de integrais duplas e triplas

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Sumário

Lista de Figuras

Parte I
Integrais duplas

1 Invertendo os limites de integração - Aula 1

1. Exercício

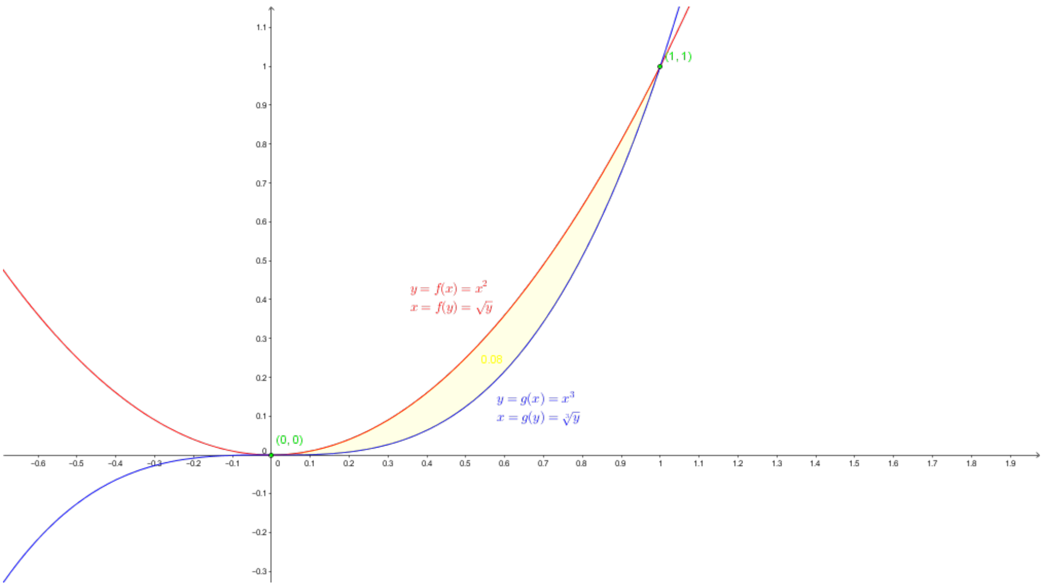

Figura 1: Integrais duplas - Aula 1 - Exercício I e II

$f(x) = x^2;\ g(x) = x^3$
$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$
$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$

$$\int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx\, [y]_{x^3}^{x^2} = \int_0^1 dx \left[x^2 - x^3\right] = \int_0^1 x^2\, dx - \int_0^1 x^3\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{4x^3 - 3x^2}{12}\right]_0^1 = \frac{1}{12}\left[4x^3 - 3x^2\right]_0^1 = \frac{1}{12}\left[x^2\,(4x-3)\right]_0^1 = \frac{1}{12}\left[1^2\,(4\cdot 1 - 3) \cancel{- 0^2\,(4\cdot 0 - 3)}\right] = \frac{1}{12} = 0,08\overline{3}$$

2. Exercício

$$f(x) = x^2 \Rightarrow f(y) = \sqrt{y};\ g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$$
$$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$$
$$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$$

$$\int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy\, [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy\, [\sqrt[3]{y} - \sqrt{y}] =$$
$$\int_0^1 \sqrt[3]{y}\, dy - \int_0^1 \sqrt{y}\, dy = \int_0^1 y^{\frac{1}{3}}\, dy - \int_0^1 y^{\frac{1}{2}}\, dy = \left[\frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^1 =$$
$$\left[\frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3} \right]_0^1 = \left[\frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12} \right]_0^1 = \frac{1}{12}\left[9\sqrt[3]{y^4} - 8\sqrt{y^3} \right]_0^1 =$$
$$\frac{1}{12}\left[\left(9\sqrt[3]{1^4} - 8\sqrt{1^3}\right) \cancel{- \left(9\sqrt[3]{0^4} - 8\sqrt{0^3}\right)} \right] = \frac{1}{12}(9-8) = \frac{1}{12} = 0,08\overline{3}$$

2 Determinação da região de integração - Aula 2

1. Exercício

   $R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 2\,,\ 0 \leq y \leq 6\}$

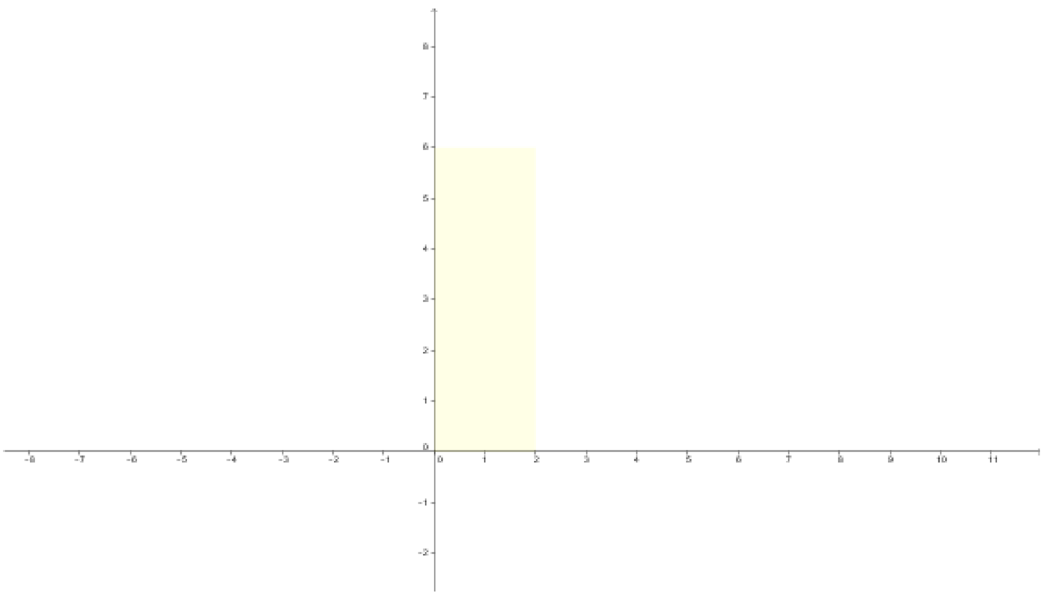

Figura 2: Integrais duplas - Aula 2 - Exercício I

$$\int_0^2 dx \int_0^6 dy = \int_0^2 dx\,[y]_0^6 = \int_0^2 dx\,[6-0] = 6\int_0^2 dx = 6[x]_0^2 = 6[2-0] = 6\cdot 2 = 12$$

2. Exercício

   $R = \{(x,y) \in \mathbb{R} \,|\, 0 \leq x \leq 1\,,\ x \leq y \leq 2x\}$

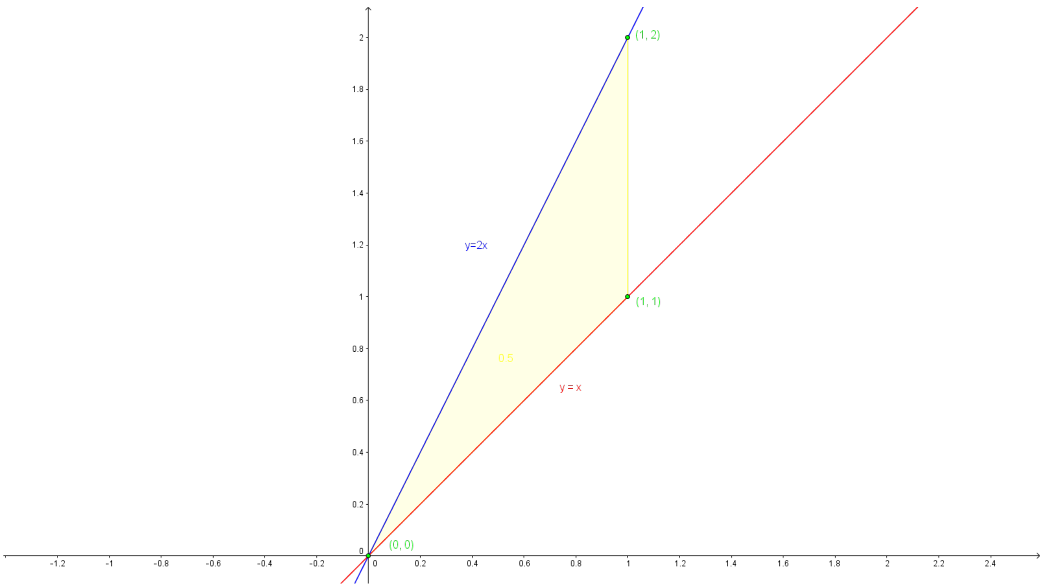

Figura 3: Integrais duplas - Aula 2 - Exercício II

$$\int_0^1 dx \int_x^{2x} dy = \int_0^1 dx\,[y]_x^{2x} = \int_0^1 dx\,[2x - x] = 2\int_0^1 x\,dx - \int_0^1 x\,dx = \left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^2}{2}\right]_0^1 = \left[\frac{2x^2 - x^2}{2}\right]_0^1 = \frac{1}{2}\left[x^2\right]_0^1 = \frac{1}{2}\left[1^2 \cancel{-0^2}\right] = \frac{1}{2} = 0,5$$

3. Exercício

   $R = \left\{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 1\,,\ 0 \leq x \leq \sqrt{1-y^2}\right\}$

   $y = 0,\ y = 1$

   $x = 0,\ x = \sqrt{1-y^2} \Rightarrow x^2 = 1 - y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow y = \sqrt{1-x^2}$

Figura 4: Integrais duplas - Aula 2 - Exercício III

$$\int_0^1 dy \int_0^{f(y)} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy\, [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy \left[\sqrt{1-y^2} - 0\right] =$$
$$\int_0^1 \sqrt{1-y^2}\, dy = \int_0^1 \sqrt{1-\operatorname{sen}^2(t)}\, \cos(t)dt = \int_0^1 \sqrt{\cos^2(t)}\, \cos(t)dt =$$
$$\int_0^1 \cos(t)\cos(t)dt = \int_0^1 \cos^2(t)dt = \int_0^1 \frac{1+\cos(2t)}{2}dt = \frac{1}{2}\int_0^1 [1+\cos(2t)]\, dt =$$
$$\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(2t)dt = \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(u)\frac{du}{2} = \frac{1}{2}\int_0^1 dt +$$
$$\frac{1}{4}\int_0^1 \cos(u)\, du = \left[\frac{1}{2}t + \frac{1}{4}\operatorname{sen}(u)\right]_0^1 = \left[\frac{t}{2} + \frac{\operatorname{sen}(2t)}{4}\right]_0^1 = \left[\frac{t}{2} + \frac{\cancel{2}\operatorname{sen}(t)\cos(t)}{\cancel{4}\,2}\right]_0^1 =$$
$$\left[\frac{t+\operatorname{sen}(t)\cos(t)}{2}\right]_0^1 = \frac{1}{2}\left[\operatorname{arcsen}(y) + y\sqrt{1-y^2}\right]_0^1 =$$
$$\frac{1}{2}\left[\left(\operatorname{arcsen}(1) + \cancel{1\cdot\sqrt{1-1^2}}\right) - \left(\operatorname{arcsen}(0) \cancel{+0\cdot\sqrt{1-0^2}}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] =$$
$$\frac{\pi}{4} = 0,785$$

$$y = \operatorname{sen}(t) \Rightarrow dy = \cos(t)dt$$
$$u = 2t \Rightarrow \frac{du}{2} = dt$$

$$\operatorname{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$
$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1-y^2}$$

$\cos(t) = \dfrac{ca}{h} = \dfrac{\sqrt{1-y^2}}{1} = \sqrt{1-y^2}$
$y = \text{sen}(t) \Rightarrow t = \text{arcsen}(y)$

4. Exercício

$y = x^2 + 1$, $y = -x^2 - 1$; $x = 1$, $x = -1$
$R = \{(x, y) \in \mathbb{R} \,|\, -1 \leq x \leq 1\,,\, -x^2 - 1 \leq y \leq x^2 + 1\}$

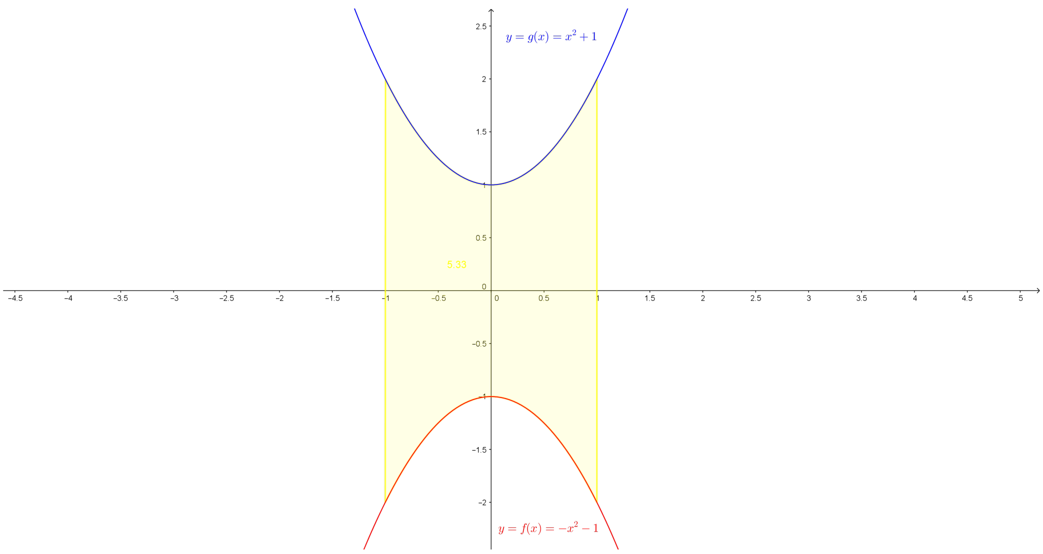

Figura 5: Integrais duplas - Aula 2 - Exercício IV

$$\int_{-1}^{1} dx \int_{f(x)}^{g(x)} dy = \int_{-1}^{1} dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^{1} dx\, [y]_{-x^2-1}^{x^2+1} = \int_{-1}^{1} dx\, \left[x^2 + 1 - \left(-x^2 - 1\right)\right] =$$
$$\int_{-1}^{1} dx\, \left[x^2 + 1 + x^2 + 1\right] = \int_{-1}^{1} dx\, \left[2x^2 + 2\right] = 2\int_{-1}^{1} x^2\, dx + 2\int_{-1}^{1} dx =$$
$$\left[2\frac{x^3}{3} + 2x\right]_{-1}^{1} = \left[2\left(\frac{x^3 + 3x}{3}\right)\right]_{-1}^{1} = \frac{2}{3}\left[x\left(x^2 + 3\right)\right]_{-1}^{1} =$$
$$\frac{2}{3}\left[1 \cdot \left(1^2 + 3\right) - (-1)\left((-1)^2 + 3\right)\right] = \frac{2}{3}(4 + 4) = \frac{2}{3}8 = \frac{16}{3} = 5,\overline{3}$$

5. Exercício

$R = \{(x, y) \in \mathbb{R} \,|\, 0 \leq y \leq 2\,,\, -y \leq x \leq y\}$

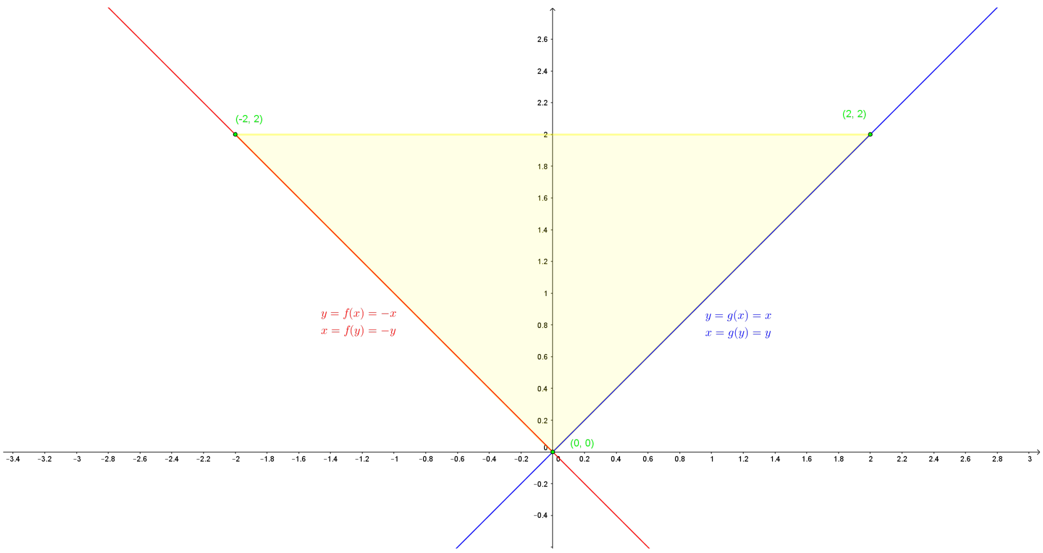

Figura 6: Integrais duplas - Aula 2 - Exercício V

$$\int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^{y} dx = \int_0^2 dy\, [x]_{-y}^{y} = \int_0^2 dy\, [y-(-y)] = \int_0^2 dy\, [2y] = 2\int_0^2 y\, dy = \left[\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^2 = 2^2 - 0^2 = 4$$

3 Cálculo de volume - Aula 3

1. Exercício

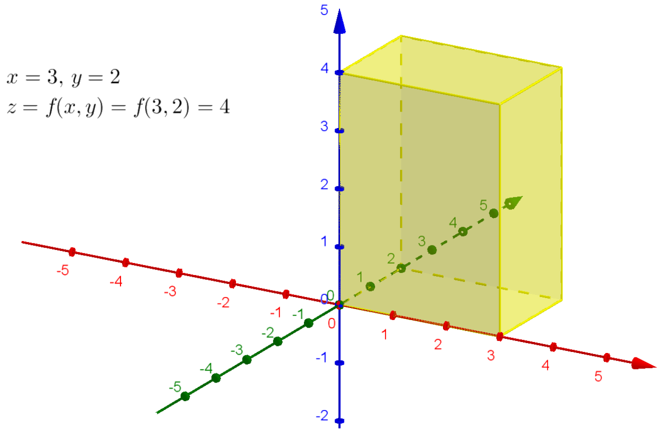

Figura 7: Integrais duplas - Aula 3 - Exercício I

$z = 4;\ dz = dxdy$

$v \;=\; \int_0^3 \int_0^2 z\,dz \;=\; \int_0^3 \int_0^2 4\,dydx \;=\; 4\int_0^3 dx \int_0^2 dy \;=\; 4\int_0^3 dx\,[y]_0^2 \;=\; 4\int_0^3 dx\,[2-0] = 8\int_0^3 dx = 8[x]_0^3 = 8[3-0] = 8 \cdot 3 = 24$

2. Exercício

   $R = [0,3] \times [0,4]$

   $\int\int_R (8-2y)da$

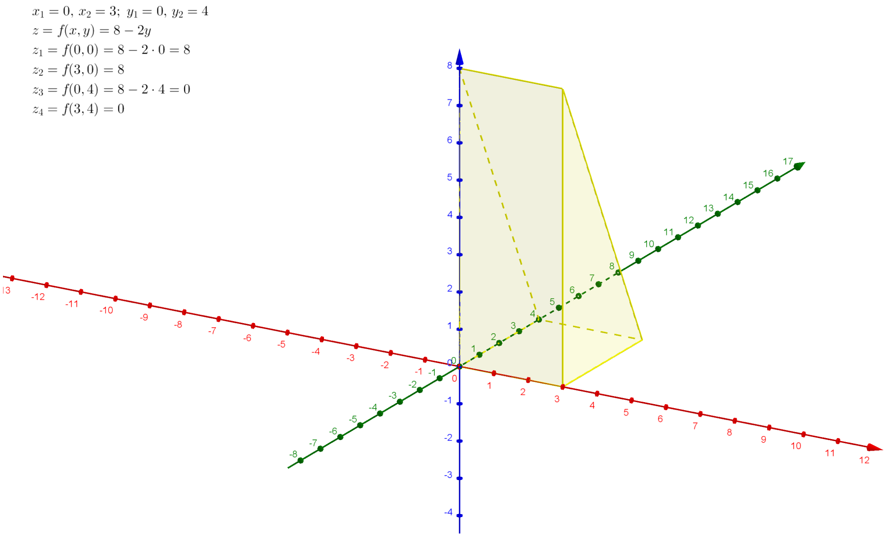

Figura 8: Integrais duplas - Aula 3 - Exercício II

$$z = 8 - 2y;\ da = dz = dxdy$$

$$v = \int_0^3 \int_0^4 z\,dz = \int_0^3 \int_0^4 (8 - 2y)dxdy = \int_0^3 dx \int_0^4 (8 - 2y)dy = \int_0^3 dx \left(8 \int_0^4 dy - 2\int_0^4 y\,dy\right) = \int_0^3 dx\,2\left(4\int_0^4 dy - \int_0^4 y\,dy\right) = 2\int_0^3 dx\left[4y - \frac{y^2}{2}\right]_0^4 = 2\int_0^3 dx\left[\frac{8y - y^2}{2}\right]_0^4 = \cancel{2}\int_0^3 dx\,\frac{1}{\cancel{2}}[y(8-y)]_0^4 = \int_0^3 dx[4(8-4)\cancel{-0(8-0)}] = 16\int_0^3 dx = 16[x]_0^3 = 16[3-0] = 48$$

4 Invertendo a ordem de integração - Aula 4

1. Exercício

$$z = f(x,y) = y\,\mathrm{e}^x;\ dz = dxdy$$

$$v = \int_2^4 \int_1^9 z\,dz = \int_2^4 \int_1^9 y\,\mathrm{e}^x\ dydx = \int_2^4 \mathrm{e}^x\ dx \int_1^9 y\,dy = \int_2^4 \mathrm{e}^x\ dx\left[\frac{y^2}{2}\right]_1^9 = \int_2^4 \mathrm{e}^x\ dx\frac{1}{2}\left[y^2\right]_1^9 = \frac{1}{2}\int_2^4 \mathrm{e}^x\ dx\left[9^2 - 1^2\right] = 40\int_2^4 \mathrm{e}^x\ dx = 40\left[\mathrm{e}^x\right]_2^4 = 40\left[\mathrm{e}^4 - \mathrm{e}^2\right] = 40e^2\left(e^2 - 1\right)$$

2. Exercício

$z = f(x,y) = x^2y^3;\ dz = dxdy$

$$v = \int_0^1 \int_2^4 z\,dz = \int_0^1 \int_2^4 x^2y^3\,dxdy = \int_0^1 x^2\,dx \int_2^4 y^3\,dy = \int_0^1 x^2\,dx \left[\frac{y^4}{4}\right]_2^4 =$$
$$\frac{1}{4}\int_0^1 x^2\,dx \left[y^4\right]_2^4 = \frac{1}{4}\int_0^1 x^2\,dx \left[4^4 - 2^4\right] = \frac{1}{4}\int_0^1 x^2\,dx \left[2^8 - 2^4\right] = \frac{1}{4}\int_0^1 x^2\,dx \left[2^4\left(2^4 - 1\right)\right] =$$
$$\frac{1}{4}\int_0^1 x^2\,dx\,[16 \cdot 15] = 60\int_0^1 x^2\,dx = 60\left[\frac{x^3}{3}\right]_0^1 = 20\left[x^3\right]_0^1 = 20\left[1^3 - 0^3\right] =$$
$$20 \cdot 1 = 20$$

3. Exercício

$$\int\int_D (x+2y)da$$

D = Região limitada pela parábola $y = x^2 + 1$ e as retas $x = -1$ e $x = 2$.

$z = f(x,y) = x + 2y;\ da = dz = dxdy$

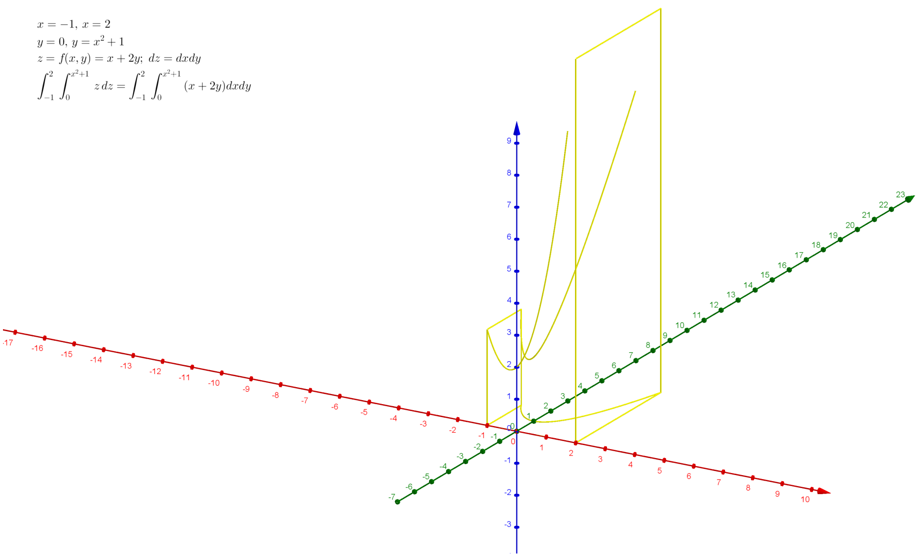

Figura 9: Integrais duplas - Aula 4 - Exercício III

$$\int_{-1}^{2}\int_0^{x^2+1} z\,dz = \int_{-1}^{2}\int_0^{x^2+1} (x+2y)dxdy = \int_{-1}^{2} dx \int_0^{x^2+1} (x+2y)dy =$$

$$\int_{-1}^{2} dx \left(x \int_{0}^{x^2+1} dy + 2 \int_{0}^{x^2+1} y\, dy \right) = \int_{-1}^{2} dx \left[xy + \cancel{2} \frac{y^2}{\cancel{2}} \right]_0^{x^2+1} = \int_{-1}^{2} dx \left[y(x+2y) \right]_0^{x^2+1} =$$

$$\int_{-1}^{2} dx \left[\left(x^2+1\right) \left[x + 2\left(x^2+1\right) \right] \cancel{-0(x+2\cdot 0)} \right] = \int_{-1}^{2} dx \left[\left(x^2+1\right) \left(2x^2+x+2\right) \right] =$$

$$\int_{-1}^{2} dx \left(2x^4 + x^3 + 4x^2 + x + 2\right) = 2\int_{-1}^{2} x^4\, dx + \int_{-1}^{2} x^3\, dx + 4\int_{-1}^{2} x^2\, dx +$$

$$\int_{-1}^{2} x\, dx + 2\int_{-1}^{2} dx = \left[2\frac{x^5}{5} + \frac{x^4}{4} + 4\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^{2} =$$

$$\left[\frac{24x^5 + 15x^4 + 80x^3 + 30x^2 + 120x}{60} \right]_{-1}^{2} = \frac{1}{60} \left[x\left(24x^4 + 15x^3 + 80x^2 + 30x + 120\right) \right]_{-1}^{2} =$$

$$\frac{1}{60} \left[2\left(24\cdot 2^4 + 15\cdot 2^3 + 80\cdot 2^2 + 30\cdot 2 + 120\right) \right.$$

$$\left. -(-1)\left(24(-1)^4 + 15(-1)^3 + 80(-1)^2 + 30(-1) + 120\right) \right] =$$

$$\frac{1}{60}[2(384 + 120 + 320 + 60 + 120) + (24 - 15 + 80 - 30 + 120)] =$$

$$\frac{1}{60}(2008 + 179) = \frac{2187}{60} = \frac{729}{20} = 36,45$$