

# Curso de integrais duplas e triplas

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## Resumo

Exercícios retirados do canal do Youtube, O Matematico [1].

## Parte I

# Integrais duplas

## 1 Invertendo os limites de integração - Aula 1

### 1. Exercício

Figura 1: Integrais duplas - Aula 1 - Exercício I e II

img/v01\_a01\_e01.png

$$\begin{aligned}
 f(x) &= x^2; \quad g(x) = x^3 \\
 x = 0 &\Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3 \\
 x = 1 &\Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3 \\
 a &= \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx [y]_{x^3}^{x^2} = \int_0^1 dx [x^2 - x^3] = \\
 &\int_0^1 x^2 dx - \int_0^1 x^3 dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left[ \frac{4x^3 - 3x^2}{12} \right]_0^1 = \frac{1}{12} [4x^3 - 3x^2]_0^1 = \\
 &\frac{1}{12} [x^2 (4x - 3)]_0^1 = \frac{1}{12} [1^2 (4 \cdot 1 - 3) - 0^2 (4 \cdot 0 - 3)] = \frac{1}{12} = 0,08\bar{3}
 \end{aligned}$$

## 2. Exercício

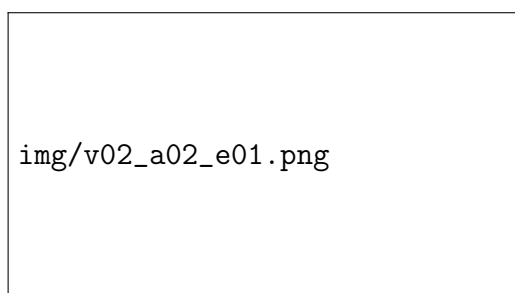
$$\begin{aligned}
 f(x) &= x^2 \Rightarrow f(y) = \sqrt{y}; \quad g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y} \\
 y = 0 &\Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0} \\
 y = 1 &\Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1} \\
 a &= \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy [\sqrt[3]{y} - \sqrt{y}] = \\
 &\int_0^1 \sqrt[3]{y} dy - \int_0^1 \sqrt{y} dy = \int_0^1 y^{\frac{1}{3}} dy - \int_0^1 y^{\frac{1}{2}} dy = \left[ \frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^1 = \\
 &\left[ \frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3} \right]_0^1 = \left[ \frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12} \right]_0^1 = \frac{1}{12} [9\sqrt[3]{y^4} - 8\sqrt{y^3}]_0^1 = \\
 &\frac{1}{12} [(9\sqrt[3]{1^4} - 8\sqrt{1^3}) - (9\sqrt[3]{0^4} - 8\sqrt{0^3})] = \frac{1}{12} (9 - 8) = \frac{1}{12} = 0,08\bar{3}
 \end{aligned}$$

## 2 Determinação da região de integração - Aula 2

### 1. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 6\}$$

Figura 2: Integrais duplas - Aula 2 - Exercício I

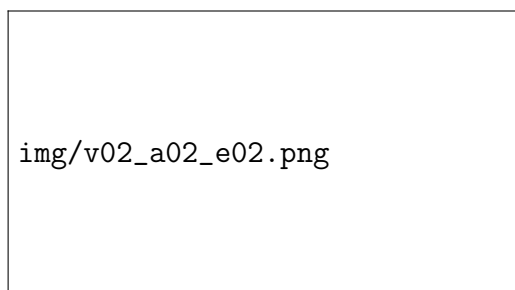


$$\begin{aligned} a &= \int_0^2 dx \int_0^6 dy = \int_0^2 dx [y]_0^6 = \int_0^2 dx [6 - 0] = 6 \int_0^2 dx = 6[x]_0^2 = \\ &6[2 - 0] = 6 \cdot 2 = 12 \end{aligned}$$

### 2. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, x \leq y \leq 2x\}$$

Figura 3: Integrais duplas - Aula 2 - Exercício II



$$a = \int_0^1 dx \int_x^{2x} dy = \int_0^1 dx [y]_x^{2x} = \int_0^1 dx [2x - x] = 2 \int_0^1 x dx - \int_0^1 x dx =$$

$$\left[ \frac{x^2}{2} - \frac{x^2}{2} \right]_0^1 = \left[ \frac{2x^2 - x^2}{2} \right]_0^1 = \frac{1}{2} [x^2]_0^1 = \frac{1}{2} [1^2 - 0^2] = \frac{1}{2} = 0,5$$

### 3. Exercício

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, 0 \leq x \leq \sqrt{1-y^2} \right\}$$

$$y = 0, y = 1$$

$$x = 0, x = \sqrt{1-y^2} \Rightarrow x^2 = 1-y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow y = \sqrt{1-x^2}$$

Figura 4: Integrais duplas - Aula 2 - Exercício III

img/v02\_a02\_e03.png

$$a = \int_0^1 dy \int_0^{f(y)} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy [\sqrt{1-y^2} - 0] =$$

$$\int_0^1 \sqrt{1-y^2} dy = \int_0^1 \sqrt{1-\sin^2(t)} \cos(t) dt = \int_0^1 \sqrt{\cos^2(t)} \cos(t) dt =$$

$$\int_0^1 \cos(t) \cos(t) dt = \int_0^1 \cos^2(t) dt = \int_0^1 \frac{1 + \cos(2t)}{2} dt = \frac{1}{2} \int_0^1 [1 + \cos(2t)] dt =$$

$$\frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_0^1 \cos(2t) dt = \frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_0^1 \cos(u) \frac{du}{2} = \frac{1}{2} \int_0^1 dt + \frac{1}{4} \int_0^1 \cos(u) du =$$

$$\left[ \frac{1}{2} t + \frac{1}{4} \sin(u) \right]_0^1 = \left[ \frac{t}{2} + \frac{\sin(2t)}{4} \right]_0^1 = \left[ \frac{t}{2} + \frac{2 \sin(t) \cos(t)}{4} \right]_0^1 = \left[ \frac{t + \sin(t) \cos(t)}{2} \right]_0^1 =$$

$$\frac{1}{2} [\arcsen(y) + y\sqrt{1-y^2}]_0^1 = \frac{1}{2} [(\arcsen(1) + 1 \cdot \sqrt{1-1^2}) - (\arcsen(0) + 0 \cdot \sqrt{1-0^2})] =$$

$$\frac{1}{2} \left[ \frac{\pi}{2} - 0 \right] = \frac{\pi}{4} = 0,785$$

$$y = \text{sen}(t) \Rightarrow dy = \cos(t)dt$$

$$u = 2t \Rightarrow \frac{du}{2} = dt$$

$$\text{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$

$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1 - y^2}$$

$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1 - y^2}}{1} = \sqrt{1 - y^2}$$

$$y = \text{sen}(t) \Rightarrow t = \arcsen(y)$$

#### 4. Exercício

$$y = x^2 + 1, y = -x^2 - 1; x = 1, x = -1$$

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -x^2 - 1 \leq y \leq x^2 + 1\}$$

Figura 5: Integrais duplas - Aula 2 - Exercício IV

img/v02\_a02\_e04.png

$$\begin{aligned} a &= \int_{-1}^1 dx \int_{f(x)}^{g(x)} dy = \int_{-1}^1 dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^1 dx [y]_{-x^2-1}^{x^2+1} = \\ &= \int_{-1}^1 dx [x^2 + 1 - (-x^2 - 1)] = \int_{-1}^1 dx [x^2 + 1 + x^2 + 1] = \\ &= \int_{-1}^1 dx [2x^2 + 2] = 2 \int_{-1}^1 x^2 dx + 2 \int_{-1}^1 dx = \left[ 2\frac{x^3}{3} + 2x \right]_{-1}^1 = \left[ 2\left(\frac{x^3 + 3x}{3}\right) \right]_{-1}^1 = \\ &= \frac{2}{3} [x(x^2 + 3)]_{-1}^1 = \frac{2}{3} [1 \cdot (1^2 + 3) - (-1)((-1)^2 + 3)] = \frac{2}{3}(4 + 4) = \frac{2}{3}8 = \frac{16}{3} = 5, \bar{3} \end{aligned}$$

### 5. Exercício

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 2, -y \leq x \leq y\}$$

Figura 6: Integrais duplas - Aula 2 - Exercício V

img/v02\_a02\_e05.png

$$\begin{aligned} a &= \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^y dx = \int_0^2 dy [x]_{-y}^y = \int_0^2 dy [y - (-y)] = \int_0^2 dy [2y] = \\ &2 \int_0^2 y dy = \left[ 2 \frac{y^2}{2} \right]_0^2 = 2^2 - 0^2 = 4 \end{aligned}$$

## 3 Cálculo de volume - Aula 3

### 1. Exercício

Figura 7: Integrais duplas - Aula 3 - Exercício I

img/v03\_a03\_e01.png

$$z = 4; dz = dx dy$$



$$v = \int_0^3 \int_0^2 z \, dz = \int_0^3 \int_0^2 4 \, dy \, dx = 4 \int_0^3 dx \int_0^2 dy = 4 \int_0^3 dx [y]_0^2 = 4 \int_0^3 dx [2 - 0] = 8 \int_0^3 dx = 8[x]_0^3 = 8[3 - 0] = 8 \cdot 3 = 24$$

## 2. Exercício

$$R = [0, 3] \times [0, 4]$$

$$\iint_R (8 - 2y) \, da$$

Figura 8: Integrais duplas - Aula 3 - Exercício II

img/v03\_a03\_e02.png

$$z = 8 - 2y; \, da = dz = dx \, dy$$

$$\begin{aligned} v &= \int_0^3 \int_0^4 z \, dz = \int_0^3 \int_0^4 (8 - 2y) \, dx \, dy = \int_0^3 dx \int_0^4 (8 - 2y) \, dy = \\ &= \int_0^3 dx \left( 8 \int_0^4 dy - 2 \int_0^4 y \, dy \right) = \int_0^3 dx \, 2 \left( 4 \int_0^4 dy - \int_0^4 y \, dy \right) = \\ &= 2 \int_0^3 dx \left[ 4y - \frac{y^2}{2} \right]_0^4 = 2 \int_0^3 dx \left[ \frac{8y - y^2}{2} \right]_0^4 = 2 \int_0^3 dx \frac{1}{2} [y(8 - y)]_0^4 = \\ &= \int_0^3 dx [4(8 - 4) - 0(8 - 0)] = 16 \int_0^3 dx = 16[x]_0^3 = 16[3 - 0] = 48 \end{aligned}$$

## 4 Invertendo a ordem de integração - Aula 4

### 1. Exercício

$$z = f(x, y) = y e^x; \, dz = dx \, dy$$

$$\begin{aligned}
 v &= \int_2^4 \int_1^9 z \, dz = \int_2^4 \int_1^9 y e^x \, dy dx = \int_2^4 e^x \, dx \int_1^9 y \, dy = \int_2^4 e^x \, dx \left[ \frac{y^2}{2} \right]_1^9 = \\
 &= \int_2^4 e^x \, dx \frac{1}{2} [y^2]_1^9 = \frac{1}{2} \int_2^4 e^x \, dx [9^2 - 1^2] = 40 \int_2^4 e^x \, dx = 40 [e^x]_2^4 = 40 [e^4 - e^2] = \\
 &= 40e^2 (e^2 - 1)
 \end{aligned}$$

## 2. Exercício

$$z = f(x, y) = x^2 y^3; \, dz = dx dy$$

Figura 9: Integrais duplas - Aula 4 - Exercício II

img/v04\_a04\_e02.png

$$\begin{aligned}
 v &= \int_0^1 \int_2^4 z \, dz = \int_0^1 \int_2^4 x^2 y^3 \, dx dy = \int_0^1 x^2 \, dx \int_2^4 y^3 \, dy = \int_0^1 x^2 \, dx \left[ \frac{y^4}{4} \right]_2^4 = \\
 &= \frac{1}{4} \int_0^1 x^2 \, dx [y^4]_2^4 = \frac{1}{4} \int_0^1 x^2 \, dx [4^4 - 2^4] = \frac{1}{4} \int_0^1 x^2 \, dx [2^8 - 2^4] = \\
 &= \frac{1}{4} \int_0^1 x^2 \, dx [2^4 (2^4 - 1)] = \frac{1}{4} \int_0^1 x^2 \, dx [16 \cdot 15] = 60 \int_0^1 x^2 \, dx = 60 \left[ \frac{x^3}{3} \right]_0^1 = 20 [x^3]_0^1 = \\
 &= 20 [1^3 - 0^3] = 20 \cdot 1 = 20
 \end{aligned}$$

## 3. Exercício

$$\iint_R (x + 2y) da$$

$R$  = Região limitada pela parábola  $y = x^2 + 1$  e as retas  $x = -1$  e  $x = 2$ .

$$z = f(x, y) = x + 2y; \, da = dz = dx dy$$

Figura 10: Integrais duplas - Aula 4 - Exercício III

img/v04\_a04\_e03.png

$$\begin{aligned}
 v &= \int_{-1}^2 \int_0^{x^2+1} z \, dz = \int_{-1}^2 \int_0^{x^2+1} (x+2y) \, dx \, dy = \\
 &= \int_{-1}^2 dx \int_0^{x^2+1} (x+2y) \, dy = \int_{-1}^2 dx \left( x \int_0^{x^2+1} dy + 2 \int_0^{x^2+1} y \, dy \right) = \\
 &= \int_{-1}^2 dx \left[ xy + 2 \frac{y^2}{2} \right]_0^{x^2+1} = \int_{-1}^2 dx [y(x+y)]_0^{x^2+1} = \\
 &= \int_{-1}^2 dx [(x^2+1) [x + (x^2+1)] - 0] = \int_{-1}^2 dx [(x^2+1)(x^2+x+1)] = \\
 &= \int_{-1}^2 dx (x^4 + x^3 + 2x^2 + x + 1) = \\
 &= \int_{-1}^2 x^4 \, dx + \int_{-1}^2 x^3 \, dx + 2 \int_{-1}^2 x^2 \, dx + \int_{-1}^2 x \, dx + \int_{-1}^2 dx = \\
 &= \left[ \frac{x^5}{5} + \frac{x^4}{4} + 2 \frac{x^3}{3} + \frac{x^2}{2} + x \right]_{-1}^2 = \left[ \frac{12x^5 + 15x^4 + 40x^3 + 30x^2 + 60x}{60} \right]_{-1}^2 = \\
 &= \frac{1}{60} [x(12x^4 + 15x^3 + 40x^2 + 30x + 60)]_{-1}^2 = \\
 &= \frac{1}{60} [2(12 \cdot 2^4 + 15 \cdot 2^3 + 40 \cdot 2^2 + 30 \cdot 2 + 60) \\
 &\quad - (-1)(12(-1)^4 + 15(-1)^3 + 40(-1)^2 + 30(-1) + 60)] = \\
 &= \frac{1}{60} [2(192 + 120 + 160 + 60 + 60) + (12 - 15 + 40 - 30 + 60)] = \frac{1}{60} (1184 + 67) = \\
 &= \frac{1251}{60} = \frac{417}{20} = 20,85
 \end{aligned}$$

## 5 Cálculo de integrais duplas ou iteradas

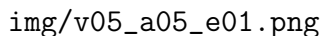
### 5.1 Aula 5

#### 1. Exercício

$$f(x, y) = x^3; 0 \leq x \leq 2; x^2 \leq y \leq 4$$

$$\iint_R f(x, y) dy dx$$

Figura 11: Integrais duplas - Aula 5 - Exercício I



$$\begin{aligned} v &= \int_0^2 \int_{x^2}^4 x^3 dx dy = \int_0^2 x^3 dx \int_{x^2}^4 dy = \int_0^2 x^3 dx [y]_{x^2}^4 = \int_0^2 x^3 dx [4 - x^2] = \\ &= 4 \int_0^2 x^3 dx - \int_0^2 x^5 dx = \left[ 4 \frac{x^4}{4} - \frac{x^6}{6} \right]_0^2 = \left[ \frac{6x^4 - x^6}{6} \right]_0^2 = \frac{1}{6} [x^4 (6 - x^2)]_0^2 = \\ &= \frac{1}{6} [2^4 (6 - 2^2) - 0] = \frac{1}{6} (16 \cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2 \end{aligned}$$

#### 2. Exercício

$$f(x, y) = x^2 y; 1 \leq x \leq 3; x \leq y \leq 2x + 1$$

$$\iint_R f(x, y) dy dx$$

Figura 12: Integrais duplas - Aula 5 - Exercício II

img/v05\_a05\_e02.png

$$\begin{aligned}
 v &= \int_1^3 \int_x^{2x+1} x^2 y \, dx dy = \int_1^3 x^2 \, dx \int_x^{2x+1} y \, dy = \int_1^3 x^2 \, dx \left[ \frac{y^2}{2} \right]_x^{2x+1} = \\
 &\int_1^3 x^2 \, dx \frac{1}{2} [(2x+1)^2 - (x)^2] = \frac{1}{2} \int_1^3 x^2 \, dx (3x^2 + 4x + 1) = \\
 \frac{3}{2} \int_1^3 x^4 \, dx + 2 \int_1^3 x^3 \, dx + \frac{1}{2} \int_1^3 x^2 \, dx &= \left[ \frac{3}{2} \frac{x^5}{5} + 2 \frac{x^4}{4} + \frac{1}{2} \frac{x^3}{3} \right]_1^3 = \left[ \frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6} \right]_1^3 = \\
 &\left[ \frac{18x^5 + 30x^4 + 10x^3}{60} \right]_1^3 = \left[ \frac{2x^3(9x^2 + 15x + 5)}{60} \right]_1^3 = \\
 \frac{1}{30} [x^3(9x^2 + 15x + 5)]_1^3 &= \frac{1}{30} [3^3(9 \cdot 3^2 + 15 \cdot 3 + 5) - 1^3(9 \cdot 1^2 + 15 \cdot 1 + 5)] = \\
 \frac{1}{30} [27(81 + 45 + 5) - (9 + 15 + 5)] &= \frac{1}{30} [27 \cdot 131 - 29] = \frac{3508}{30} = 116,9\bar{3}
 \end{aligned}$$

## 5.2 Aula 6

### 1. Exercício

$$f(x, y) = 1; \quad 0 \leq x \leq 1; \quad 1 \leq y \leq e^x$$

$$\iint_R f(x, y) \, dy dx$$

$$\begin{aligned}
 v &= \int_0^1 \int_1^{e^x} dy dx = \int_0^1 dx [y]_1^{e^x} = \int_0^1 dx (e^x - 1) = [e^x - x]_0^1 = \\
 &e^1 - 1 - (e^0 - 0) = e - 1 - 1 = e - 2
 \end{aligned}$$

### 2. Exercício

$$f(x, y) = x; 0 \leq x \leq 1; 1 \leq y \leq e^{x^2}$$

$$\iint_R f(x, y) dy dx$$

$$\begin{aligned} v &= \int_0^1 \int_1^{e^{x^2}} x dx dy = \int_0^1 x dx \int_1^{e^{x^2}} dy = \int_0^1 x dx [y]_1^{e^{x^2}} = \int_0^1 x dx (e^{x^2} - 1) = \\ &= \int_0^1 x e^{x^2} dx - \int_0^1 x dx = \int_0^1 e^u \frac{du}{2} - \int_0^1 x dx = \frac{1}{2} \int_0^1 e^u du - \int_0^1 x dx = \\ &= \left[ \frac{1}{2} e^u - \frac{x^2}{2} \right]_0^1 = \left[ \frac{e^{x^2} - x^2}{2} \right]_0^1 = \frac{1}{2} [e^{x^2} - x^2]_0^1 = \frac{1}{2} [e^{1^2} - 1^2 - (e^{0^2} - 0^2)] = \\ &= \frac{1}{2} (e - 1 - 1) = \frac{e - 2}{2} \end{aligned}$$

$$u = x^2; \frac{du}{2} = x dx$$

### 3. Exercício

$$f(x, y) = 2xy; 0 \leq y \leq 1; y^2 \leq x \leq y$$

$$\iint_R f(x, y) dx dy$$

$$\begin{aligned} v &= \int_0^1 \int_{y^2}^y 2xy dx dy = 2 \int_0^1 y dy \int_{y^2}^y x dx = 2 \int_0^1 y dy \left[ \frac{x^2}{2} \right]_{y^2}^y = 2 \int_0^1 y dy \frac{1}{2} [x^2]_{y^2}^y = \\ &= \int_0^1 y dy (y^2 - y^4) = \int_0^1 (y^3 - y^5) dy = \left[ \frac{y^4}{4} - \frac{y^6}{6} \right]_0^1 = \\ &= \left[ \frac{6y^4 - 4y^6}{24} \right]_0^1 = \left[ \frac{2y^4(3 - 2y^2)}{24} \right]_0^1 = \frac{1}{12} [1^4(3 - 2 \cdot 1^2) - 0^4(3 - 2 \cdot 0^2)] = \frac{1}{12} = 0,08\bar{3} \end{aligned}$$

## 5.3 Aula 7

### 1. Exercício

$$f(x, y) = \frac{1}{x + y}; 1 \leq y \leq e; 0 \leq x \leq y$$

$$\iint_R f(x, y) dx dy$$

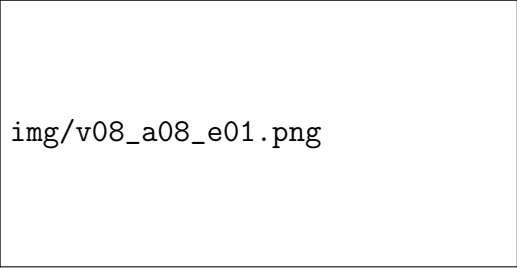
$$\begin{aligned}
v &= \int_1^e \int_0^y \frac{1}{x+y} dx dy = \int_1^e dy \int_0^y (x+y)^{-1} dx = \int_1^e dy \int_0^y u^{-1} du = \\
&\int_1^e dy \int_0^y [\ln |u|]_0^y = \int_1^e dy \int_0^y [\ln |x+y|]_0^y = \int_1^e dy \int_0^y (\ln |y+y| - \ln |0+y|) = \\
&\int_1^e dy \int_0^y (\ln |2y| - \ln |y|) = \int_1^e dy \int_0^y (\ln |2| + \ln |y| - \ln |y|) = \ln |2| \int_1^e dy = \\
&\ln |2| [y]_1^e = \ln |2|(e-1)
\end{aligned}$$

$$u = x + y; \quad du = (1 + 0)dx = dx$$

## 6 Cálculo de área - Aula 8

### 1. Exercício

Figura 13: Integrais duplas - Aula 8 - Exercício I



img/v08\_a08\_e01.png

$$\begin{aligned}
a &= \int_{-1}^0 dx \int_0^{x^2+1} dy + \int_{-1}^0 dx \int_{-1}^0 dy + \int_0^{y^2} dx \int_{-1}^0 dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} dy = \\
&= \int_{-1}^0 dx \left( \int_0^{x^2+1} dy + \int_{-1}^0 dy \right) + \int_0^{y^2} dx \int_{-1}^0 dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} dy = \\
&= \int_{-1}^0 dx \left( [y]_0^{x^2+1} + [y]_{-1}^0 \right) + \int_{-1}^0 dy [x]_0^{y^2} + \int_0^1 dx [y]_{\sqrt{x}}^{x^2+1} = \\
&= \int_{-1}^0 dx (x^2 + 1 + 1) + \int_{-1}^0 dy y^2 + \int_0^1 dx (x^2 + 1 - \sqrt{x}) = \\
&= \int_{-1}^0 (x^2 + 2) dx + \int_{-1}^0 y^2 dy + \int_0^1 \left( x^2 - x^{\frac{1}{2}} + 1 \right) dx = \\
&= \left[ \frac{x^3}{3} + 2x \right]_{-1}^0 + \left[ \frac{y^3}{3} \right]_{-1}^0 + \left[ \frac{x^3}{3} - \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + x \right]_0^1 = \\
&= \left[ \frac{x^3 + 6x}{3} \right]_{-1}^0 + \frac{1}{3} [y^3]_{-1}^0 + \left[ \frac{x^3}{3} - \frac{2\sqrt{x^3}}{3} + x \right]_0^1 = \\
&= \frac{1}{3} [x(x^2 + 6)]_{-1}^0 + \frac{1}{3} [\theta^3 - (-1)^3] + \left[ \frac{x^3 - 2\sqrt{x^3} + 3x}{3} \right]_0^1 = \\
&= \frac{1}{3} [0(0^2 + 6) - (-1)((-1)^2 + 6)] + \frac{1}{3} + \frac{1}{3} [x^3 - 2\sqrt{x^3} + 3x]_0^1 = \\
&= \frac{7}{3} + \frac{1}{3} + \frac{1}{3} [1^3 - 2\sqrt{1^3} + 3 \cdot 1 - (0^3 - 2\sqrt{0^3} + 3 \cdot 0)] = \frac{7}{3} + \frac{1}{3} + \frac{2}{3} = \\
&= \frac{7+1+2}{3} = \frac{10}{3} = 3, \bar{3}
\end{aligned}$$

$$\begin{aligned}
a &= \int_{-1}^1 dx \int_1^{x^2+1} dy + \int_{-1}^{y^2} dx \int_{-1}^1 dy = \int_{-1}^1 dx [y]_1^{x^2+1} + \int_{-1}^1 dy [x]_{-1}^{y^2} = \\
&= \int_{-1}^1 dx (x^2 + 1 - 1) + \int_{-1}^1 dy (y^2 + 1) = \left[ \frac{x^3}{3} \right]_{-1}^1 + \left[ \frac{y^3}{3} + y \right]_{-1}^1 = \\
&= \frac{1}{3} [x^3]_{-1}^1 + \frac{1}{3} [y(y^2 + 3)]_{-1}^1 = \\
&= \frac{1}{3} ([1^3 - (-1)^3] + [1(1^2 + 3) - (-1)((-1)^2 + 3)]) = \frac{1}{3}(2 + 4 + 4) = \frac{10}{3} = 3, \bar{3}
\end{aligned}$$



## 7 Cálculo de volume

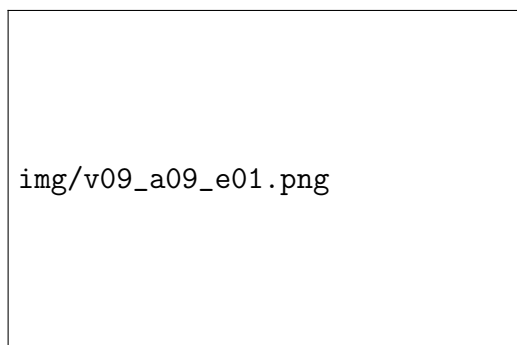
### 7.1 Aula 9

#### 1. Exercício

Esboce a região de integração e o sólido cujo volume é dado pela integral abaixo:

$$\int_0^1 \int_0^1 (4 - x - 2y) \, dx dy$$

Figura 14: Integrais duplas - Aula 9 - Exercício I



$$\begin{aligned} v &= \int_0^1 \int_0^1 (4 - x - 2y) \, dx dy = \int_0^1 dx \left( 4 \int_0^1 dy - x \int_0^1 dy - 2 \int_0^1 y \, dy \right) = \\ &= 4 \int_0^1 dx \int_0^1 dy - \int_0^1 x \, dx \int_0^1 dy - 2 \int_0^1 dx \int_0^1 y \, dy = \\ &= 4[x]_0^1 [y]_0^1 - \left[ \frac{x^2}{2} \right]_0^1 [y]_0^1 - 2[x]_0^1 \left[ \frac{y^2}{2} \right]_0^1 = 4 - \frac{1}{2} - 2 \cdot \frac{1}{2} = \frac{8 - 1 - 2}{2} = \frac{5}{2} = 2,5 \end{aligned}$$

### 7.2 Aula 10

#### 1. Exercício

Calcule o volume do sólido limitado pelos planos:

$$x = 0, y = 0, z = 0 \text{ e } 6x + 2y + 3z = 6$$

$$\begin{aligned} P_1 &= (0, 0, 0) \\ 6x &= -2y - 3z + 6 \Rightarrow x = \frac{-2y - 3z + 6}{6} = \frac{-2 \cdot 0 - 3 \cdot 0 + 6}{6} = \frac{6}{6} = 1 \Rightarrow P_2 = (1, 0, 0) \end{aligned}$$

Figura 15: Integrais duplas - Aula 10 - Exercício I

img/v10\_a10\_e01.png

$$2y = -6x - 3z + 6 \Rightarrow y = \frac{-6x - 3z + 6}{2} = \frac{-6 \cdot 0 - 3 \cdot 0 + 6}{2} = \frac{6}{2} = 3 \Rightarrow P_3 = (0, 3, 0)$$

$$3z = -6x - 2y + 6 \Rightarrow z = \frac{-6x - 2y + 6}{3} = \frac{-6 \cdot 0 - 2 \cdot 0 + 6}{3} = \frac{6}{3} = 2 \Rightarrow P_4 = (0, 0, 2)$$

$$x = 0, x = 1$$

$$y = 0, y = \frac{-6x - 3z + 6}{2} = \frac{-6x - 3 \cdot 0 + 6}{2} = -3x + 3$$

$$z = \frac{-6x - 2y + 6}{3} = -2x - \frac{2y}{3} + 2$$

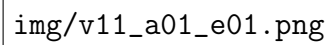
$$\begin{aligned} v &= \int_0^1 \int_0^{-3x+3} \left( -2x - \frac{2y}{3} + 2 \right) dx dy = \int_0^1 dx \int_0^{-3x+3} \left( -2x - \frac{2y}{3} + 2 \right) dy = \\ &= \int_0^1 dx \left[ -2xy - \frac{2}{3} \frac{y^2}{2} + 2y \right]_0^{-3x+3} = \int_0^1 dx \frac{1}{3} [-6xy - y^2 + 6y]_0^{-3x+3} = \\ &= \frac{1}{3} \int_0^1 dx [-y(6x + y - 6)]_0^{-3x+3} = \\ &= \frac{1}{3} \int_0^1 dx [ -(-3x + 3)(6x + (-3x + 3) - 6) + 0(6x + 0 - 6) ] = \\ &= \frac{1}{3} \int_0^1 dx [(3x - 3)(3x - 3)] = \frac{1}{3} \int_0^1 (9x^2 - 18x + 9) dx = \frac{1}{3} \left[ 9 \frac{x^3}{3} - 18 \frac{x^2}{2} + 9x \right]_0^1 = \\ &= \frac{1}{3} [3x^3 - 9x^2 + 9x]_0^1 = \frac{1}{3} [3x(x^2 - 3x + 3)]_0^1 = \\ &= [1(1^2 - 3 \cdot 1 + 3) - 0(0^2 - 3 \cdot 0 + 3)] = 1 \end{aligned}$$

## 8 Coordenadas polares

### 8.1 Aula 1

#### 1. Exercício

Figura 16: Coordenadas polares - Aula 01 - Exercício I



Calcule a área do círculo de raio igual a dois

$$r = 2 \Rightarrow a = \pi r^2 = 2^2 \pi = 4\pi$$

$$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 2^2 \Rightarrow x^2 + y^2 = 4 \Rightarrow y = \pm \sqrt{4 - x^2}$$

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2} \right\}$$

$$\begin{aligned}
a &= \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{-2}^2 dx \left( \sqrt{4-x^2} + \sqrt{4-x^2} \right) = 2 \int_{-2}^2 \sqrt{4-x^2} dx = \\
&= 2 \int_{-2}^2 \sqrt{4 - (2 \sin(\alpha))^2} 2 \cos(\alpha) d\alpha = 4 \int_{-2}^2 \sqrt{4 - 4 \sin^2(\alpha)} \cos(\alpha) d\alpha = \\
&= 4 \int_{-2}^2 \sqrt{4 - 4(1 - \cos^2(\alpha))} \cos(\alpha) d\alpha = 4 \int_{-2}^2 \sqrt{4 - (4 - 4 \cos^2(\alpha))} \cos(\alpha) d\alpha = \\
&= 4 \int_{-2}^2 \sqrt{4 - 4 + 4 \cos^2(\alpha)} \cos(\alpha) d\alpha = 4 \int_{-2}^2 2 \cos(\alpha) \cos(\alpha) d\alpha = \\
&= 8 \int_{-2}^2 \cos^2(\alpha) d\alpha = 8 \int_{-2}^2 \left( \frac{1 + \cos(2\alpha)}{2} \right) d\alpha = 8 \int_{-2}^2 \left( \frac{1}{2} + \frac{\cos(2\alpha)}{2} \right) d\alpha = \\
&= 4 \int_{-2}^2 d\alpha + 4 \int_{-2}^2 \cos(2\alpha) d\alpha = 4 \int_{-2}^2 d\alpha + 4 \int_{-2}^2 \cos(u) \frac{du}{2} = \\
&= 4 \int_{-2}^2 d\alpha + 2 \int_{-2}^2 \cos(u) du = [4\alpha + 2 \sin(u)]_{-2}^2 = [4\alpha + 2 \sin(2\alpha)]_{-2}^2 = \\
&= [4\alpha + 4 \sin(\alpha) \cos(\alpha)]_{-2}^2 = [4(\alpha + \sin(\alpha) \cos(\alpha))]_{-2}^2 = \\
&= \left[ 4 \left( \arcsin\left(\frac{x}{2}\right) + \frac{x}{2} \frac{\sqrt{4-x^2}}{2} \right) \right]_{-2}^2 = \left[ 4 \left( \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4} \right) \right]_{-2}^2 = \\
&= 4 \left( \arcsin\left(\frac{2}{2}\right) + \frac{2\sqrt{4-2^2}}{4} \right) - 4 \left( \arcsin\left(\frac{-2}{2}\right) + \frac{(-2)\sqrt{4-(-2)^2}}{4} \right) = \\
&= 4 \arcsin(1) - 4 \arcsin(-1) = 4(\arcsin(1) - \arcsin(-1)) = 4 \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = 4 \left( \frac{2\pi}{2} \right) = 4\pi
\end{aligned}$$

$$x = 2 \sin(\alpha); dx = 2 \cos(\alpha) d\alpha$$

$$u = 2\alpha; \frac{du}{2} = d\alpha$$

$$\sin(\alpha) = \frac{co}{h} = \frac{x}{2} \Rightarrow \alpha = \arcsin\left(\frac{x}{2}\right)$$

$$h^2 = co^2 + ca^2 \Rightarrow 2^2 = x^2 + ca^2 \Rightarrow ca = \sqrt{4-x^2}$$

$$\cos(\alpha) = \frac{ca}{h} = \frac{\sqrt{4-x^2}}{2}$$

$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

$$\begin{aligned}
a &= \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_0^2 \int_0^{2\pi} r dr d\theta = \int_0^2 r dr \int_0^{2\pi} d\theta = \left[ \frac{r^2}{2} \right]_0^2 [\theta]_0^{2\pi} = \\
&= \frac{1}{2} [2^2 - 0^2] [2\pi - 0] = \frac{4}{2} 2\pi = 4\pi
\end{aligned}$$

## 2. Exercício

$$\iint_R \frac{da}{1+x^2+y^2}$$

Figura 17: Coordenadas polares - Aula 01 - Exercício II

img/v11\_a01\_e02.png

$$R = \left\{ (r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{2} \right\}$$

$$\begin{aligned} v &= \iint_R \frac{da}{1+x^2+y^2} = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r \, dr \, d\theta}{1+r^2} = \int_0^2 \frac{r \, dr}{1+r^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} d\theta = \\ &= \int_0^2 (1+r^2)^{-1} r \, dr \left[ \theta \right]_{\frac{\pi}{4}}^{\frac{3\pi}{2}} = \int_0^2 (1+r^2)^{-1} r \, dr \left( \frac{3\pi}{2} - \frac{\pi}{4} \right) = \\ &= \int_0^2 (1+r^2)^{-1} r \, dr \left( \frac{6\pi - \pi}{4} \right) = \frac{5\pi}{4} \int_0^2 (1+r^2)^{-1} r \, dr = \frac{5\pi}{4} \int_0^2 u^{-1} \frac{du}{2} = \\ &= \frac{5\pi}{8} \int_0^2 u^{-1} du = \frac{5\pi}{8} [\ln|u|]_0^2 = \frac{5\pi}{8} [\ln|1+r^2|]_0^2 = \frac{5\pi}{8} [\ln|1+2^2| - \ln|1+0^2|] = \\ &= \frac{5\pi}{8} [\ln|5| - \ln|1|] = \frac{5\pi \ln|5|}{8} \end{aligned}$$

$$u = 1 + r^2 \Rightarrow \frac{du}{2} = r \, dr$$

$$e^x = 1 = e^0 \Rightarrow x = 0$$

## 8.2 Aula 2

### 1. Exercício

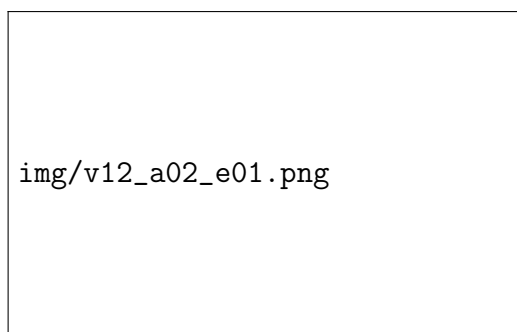
$$\iint_R e^{x^2+y^2} dx dy$$

$R$ , região entre as curvas abaixo:

$$x^2 + y^2 = 4$$

$$x^2 + y^2 = 9$$

Figura 18: Coordenadas polares - Aula 02 - Exercício I



$$x^2 + y^2 = r^2 \Rightarrow e^{x^2+y^2} = e^{r^2}$$

$$da = dx dy = r dr d\theta$$

$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 2 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$$

$$\begin{aligned} v &= \iint_R e^{x^2+y^2} dx dy = \int_2^3 \int_0^{2\pi} e^{r^2} r dr d\theta = \int_2^3 e^{r^2} r dr \int_0^{2\pi} d\theta = \int_2^3 e^u \frac{du}{2} \int_0^{2\pi} d\theta = \\ &= \frac{1}{2} \int_2^3 e^u du \int_0^{2\pi} d\theta = \frac{1}{2} [e^u]_2^3 [\theta]_0^{2\pi} = \frac{1}{2} [e^{r^2}]_2^3 2\pi = (e^{3^2} - e^{2^2}) \pi = \pi (e^9 - e^4) \end{aligned}$$

$$u = r^2 \Rightarrow \frac{du}{2} = r dr$$

### 2. Exercício

$$\iint_R \sqrt{x^2 + y^2} dx dy$$

$R$ , região cujo o contorno é:

$$x^2 + y^2 = 4$$

$$x^2 + y^2 = r^2 \Rightarrow \sqrt{x^2 + y^2} = \sqrt{r^2} = r$$

$$da = dxdy = r drd\theta$$

$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

$$\begin{aligned} v = \iint_R \sqrt{x^2 + y^2} dxdy &= \int_0^2 \int_0^{2\pi} r^2 drd\theta = \int_0^2 r^2 dr \int_0^{2\pi} d\theta = \\ &= \left[ \frac{r^3}{3} \right]_0^2 [\theta]_0^{2\pi} = \frac{2^3}{3} 2\pi = \frac{16\pi}{3} \end{aligned}$$

### 8.3 Aula 3

#### 1. Exercício

Calcular o volume do sólido acima do plano  $xoy$  delimitado pela função abaixo.

$xoy$

$$z = 4 - 2x^2 - 2y^2$$

$$4 - 2x^2 - 2y^2 = 0 \Rightarrow -2x^2 - 2y^2 = -4 \Rightarrow -2(x^2 + y^2) = -4 \Rightarrow$$

$$x^2 + y^2 = \frac{-4}{-2} = 2 \Rightarrow r = \sqrt{2}$$

$$R = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi\}$$

$$z = 4 - 2x^2 - 2y^2 = 4 - 2(x^2 + y^2) = 4 - 2r^2$$

$$da = dxdy = r drd\theta$$

$$\begin{aligned} \iint_R z da &= \iint_R (4 - 2x^2 - 2y^2) dxdy = \int_0^{\sqrt{2}} \int_0^{2\pi} (4 - 2r^2) r drd\theta = \\ &= \int_0^{\sqrt{2}} (4r - 2r^3) dr \int_0^{2\pi} d\theta = \int_0^{\sqrt{2}} (4r - 2r^3) dr [\theta]_0^{2\pi} = 2\pi \int_0^{\sqrt{2}} (4r - 2r^3) dr = \\ &= 8\pi \int_0^{\sqrt{2}} r dr - 4\pi \int_0^{\sqrt{2}} r^3 dr = \left[ \frac{8\pi r^2}{2} - \frac{4\pi r^4}{4} \right]_0^{\sqrt{2}} = [4\pi r^2 - \pi r^4]_0^{\sqrt{2}} = [\pi r^2 (4 - r^2)]_0^{\sqrt{2}} = \\ &= \pi (\sqrt{2})^2 (4 - (\sqrt{2})^2) = 2\pi(4 - 2) = 4\pi \end{aligned}$$

## Parte II

# Integrais triplas

## 9 Introdução - Aula 1

### 1. Exercício

Calcule a integral tripla abaixo.

$$\iiint_R 12xy^2z^3 dv$$
$$R = \{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq x \leq 2, 0 \leq y \leq 3, 0 \leq z \leq 2\}$$

$$dv = dxdydz$$

$$\begin{aligned} \iiint_R 12xy^2z^3 dv &= \int_{-1}^2 \int_0^3 \int_0^2 12xy^2z^3 dxdydz = 12 \int_{-1}^2 x dx \int_0^3 y^2 dy \int_0^2 z^3 dz = \\ &= 12 \left[ \frac{x^2}{2} \right]_{-1}^2 \left[ \frac{y^3}{3} \right]_0^3 \left[ \frac{z^4}{4} \right]_0^2 = \frac{1}{2} [x^2]_{-1}^2 [y^3]_0^3 [z^4]_0^2 = \frac{1}{2} (2^2 - (-1)^2) 3^3 2^4 = \frac{1}{2} 3 \cdot 27 \cdot 16 = 648 \end{aligned}$$

### 2. Exercício

Observe a integral e preencha os retângulos abaixo.

$$\int_1^5 \int_2^4 \int_3^6 f(x, y, z) dxdzdy$$

$$[3] \leq x \leq [6]$$

$$[1] \leq y \leq [5]$$

$$[2] \leq z \leq [4]$$

### 3. Exercício



$$\begin{aligned}
\int_{-1}^1 \int_0^2 \int_0^1 (x^2 + y^2 + z^2) \, dx dy dz &= \int_{-1}^1 dz \int_0^2 dy \int_0^1 (x^2 + y^2 + z^2) \, dx = \\
&= \int_{-1}^1 dz \int_0^2 dy \left( \int_0^1 x^2 \, dx + y^2 \int_0^1 dx + z^2 \int_0^1 dx \right) = \\
&= \int_{-1}^1 dz \int_0^2 dy \int_0^1 x^2 \, dx + \int_{-1}^1 dz \int_0^2 y^2 \, dy \int_0^1 dx + \int_{-1}^1 z^2 \, dz \int_0^2 dy \int_0^1 dx = \\
&= [z]_{-1}^1 [y]_0^2 \left[ \frac{x^3}{3} \right]_0^1 + [z]_{-1}^1 \left[ \frac{y^3}{3} \right]_0^2 [x]_0^1 + \left[ \frac{z^3}{3} \right]_{-1}^1 [y]_0^2 [x]_0^1 = \\
&= [z]_{-1}^1 [y]_0^2 \frac{1}{3} [x^3]_0^1 + [z]_{-1}^1 \frac{1}{3} [y^3]_0^2 [x]_0^1 + \frac{1}{3} [z^3]_{-1}^1 [y]_0^2 [x]_0^1 = \\
&= \frac{1}{3} ([1+1]2 \cdot 1^3 + [1+1]2^3 \cdot 1 + [1^3 - (-1)^3] 2 \cdot 1) = \frac{1}{3} (4 + 16 + 4) = \frac{24}{3} = 8
\end{aligned}$$

#### 4. Exercício

$$\begin{aligned}
\int_0^2 \int_{-1}^{y^2} \int_{-1}^z yz \, dx dz dy &= \int_0^2 \int_{-1}^{y^2} \left( yz \int_{-1}^z dx \right) dz dy = \int_0^2 \int_{-1}^{y^2} [yzx]_{-1}^z dz dy = \\
\int_0^2 \int_{-1}^{y^2} [yz^2 + yz] dz dy &= \int_0^2 \left( y \int_{-1}^{y^2} z^2 dz + y \int_{-1}^{y^2} z dz \right) dy = \int_0^2 \left[ y \frac{z^3}{3} + y \frac{z^2}{2} \right]_{-1}^{y^2} dy = \\
&= \int_0^2 \left[ \frac{2yz^3 + 3yz^2}{6} \right]_{-1}^{y^2} dy = \frac{1}{6} \int_0^2 [yz^2(2z+3)]_{-1}^{y^2} dy = \\
\frac{1}{6} \int_0^2 [y(y^2)^2(2y^2+3) - y(-1)^2(2(-1)+3)] dy &= \frac{1}{6} \int_0^2 [y^5(2y^2+3) - y] dy = \\
\frac{1}{6} \int_0^2 (2y^7 + 3y^5 - y) dy &= \frac{1}{6} \left[ \frac{2y^8}{8} + \frac{3y^6}{6} - \frac{y^2}{2} \right]_0^2 = \frac{1}{6} \left[ \frac{y^8}{4} + \frac{y^6}{2} - \frac{y^2}{2} \right]_0^2 = \\
\frac{1}{6} \left[ \frac{y^8 + 2y^6 - 2y^2}{4} \right]_0^2 &= \frac{1}{24} [y^2(y^6 + 2y^4 - 2)]_0^2 = \frac{1}{24} [2^2(2^6 + 2 \cdot 2^4 - 2)] = \\
&= \frac{1}{24} [4(64 + 32 - 2)] = \frac{94}{6} = \frac{47}{3}
\end{aligned}$$

## 10 Cálculo de integrais triplas - Aula 2

#### 1. Exercício

$$\iiint_R xy \operatorname{sen}(yz) dv$$

$$R = \left\{ (x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq \pi, 0 \leq y \leq 1, 0 \leq z \leq \frac{\pi}{6} \right\}$$

$$\begin{aligned} \iiint_R xy \operatorname{sen}(yz) dv &= \int_0^\pi \int_0^1 \int_0^{\frac{\pi}{6}} xy \operatorname{sen}(yz) dz dy dx = \\ &= \int_0^\pi \int_0^1 \left( x \int_0^{\frac{\pi}{6}} \operatorname{sen}(yz) y dz \right) dy dx = \int_0^\pi \int_0^1 \left( x \int_0^{\frac{\pi}{6}} \operatorname{sen}(u) du \right) dy dx = \\ &= \int_0^\pi \int_0^1 [-x \cos(u)]_0^{\frac{\pi}{6}} dy dx = \int_0^\pi \int_0^1 [-x \cos(yz)]_0^{\frac{\pi}{6}} dy dx = \\ &= \int_0^\pi \int_0^1 \left( -x \cos\left(\frac{y\pi}{6}\right) + x \cos(0) \right) dy dx = \int_0^\pi \int_0^1 \left( -x \cos\left(\frac{y\pi}{6}\right) + x \right) dy dx = \\ &= \int_0^\pi \left( -x \int_0^1 \cos\left(\frac{y\pi}{6}\right) dy + x \int_0^1 dy \right) dx = \int_0^\pi \left( -x \int_0^1 \cos(v) \frac{6 dv}{\pi} + x \int_0^1 dy \right) dx = \\ &= \int_0^\pi \left( \frac{-6x}{\pi} \int_0^1 \cos(v) dv + x \int_0^1 dy \right) dx = \int_0^\pi \left[ \frac{-6x \operatorname{sen}(v)}{\pi} + xy \right]_0^1 dx = \\ &= \int_0^\pi \left[ \frac{-6x \operatorname{sen}\left(\frac{y\pi}{6}\right) + xy\pi}{\pi} \right]_0^1 dx = \frac{1}{\pi} \int_0^\pi \left[ -x \left( 6 \operatorname{sen}\left(\frac{y\pi}{6}\right) - y\pi \right) \right]_0^1 dx = \\ &= \frac{1}{\pi} \int_0^\pi \left[ -x \left( 6 \operatorname{sen}\left(\frac{\pi}{6}\right) - \pi \right) + x(6 \operatorname{sen}(0) - 0) \right] dx = \frac{1}{\pi} \int_0^\pi \left( -6x \operatorname{sen}\left(\frac{\pi}{6}\right) + x\pi \right) dx = \\ &= \frac{-6 \operatorname{sen}\left(\frac{\pi}{6}\right)}{\pi} \int_0^\pi x dx + \pi \int_0^\pi x dx = \left[ \frac{-6 \operatorname{sen}\left(\frac{\pi}{6}\right)}{\pi} \frac{x^2}{2} + \frac{\pi x^2}{2} \right]_0^\pi = \\ &= \left[ \frac{-3x^2 \operatorname{sen}\left(\frac{\pi}{6}\right)}{\pi} + \frac{\pi x^2}{2} \right]_0^\pi = \left[ \frac{-6x^2 \operatorname{sen}\left(\frac{\pi}{6}\right) + \pi^2 x^2}{2\pi} \right]_0^\pi = \frac{1}{2\pi} \left[ -x^2 \left( 6 \operatorname{sen}\left(\frac{\pi}{6}\right) - \pi^2 \right) \right]_0^\pi = \\ &= \frac{1}{2\pi} \left[ -\pi^2 \left( 6 \operatorname{sen}\left(\frac{\pi}{6}\right) - \pi^2 \right) \right] = \frac{-\pi \left( 6 \operatorname{sen}\left(\frac{\pi}{6}\right) - \pi^2 \right)}{2} = \frac{-\pi \left( 6 \frac{1}{2} - \pi^2 \right)}{2} = \\ &= \frac{-\pi (3 - \pi^2)}{2} = \frac{\pi^3 - 3\pi}{2} \end{aligned}$$

$$u = yz \Rightarrow du = y dz$$

$$v = \frac{y\pi}{6} \Rightarrow \frac{6 dv}{\pi} = dy$$

## 2. Exercício

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y z dx dz dy &= \int_0^1 \int_0^{\sqrt{1-y^2}} \left( z \int_0^y dx \right) dz dy = \int_0^1 \int_0^{\sqrt{1-y^2}} [zx]_0^y dz dy = \\ &= \int_0^1 \int_0^{\sqrt{1-y^2}} (zy) dz dy = \int_0^1 \left( y \int_0^{\sqrt{1-y^2}} z dz \right) dy = \int_0^1 \left[ \frac{yz^2}{2} \right]_0^{\sqrt{1-y^2}} dy = \\ &= \int_0^1 \left( \frac{y \left( \sqrt{1-y^2} \right)^2}{2} \right) dy = \int_0^1 \frac{y - y^3}{2} dy = \frac{1}{2} \int_0^1 (y - y^3) dy = \frac{1}{2} \left[ \frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \\ &= \frac{1}{2} \left[ \frac{2y^2 - y^4}{4} \right]_0^1 = \frac{1}{8} [y^2 (2 - y^2)]_0^1 = \frac{1}{8} [1^2 (2 - 1^2)] = \frac{1}{8} \end{aligned}$$

## 3. Exercício

$$\begin{aligned} \int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^x xy dy dx dz &= \int_0^3 \int_0^{\sqrt{9-z^2}} \left( x \int_0^x y dy \right) dx dz = \\ &= \int_0^3 \int_0^{\sqrt{9-z^2}} \left[ \frac{xy^2}{2} \right]_0^x dx dz = \frac{1}{2} \int_0^3 \int_0^{\sqrt{9-z^2}} x^3 dx dz = \\ &= \frac{1}{2} \int_0^3 \left[ \frac{x^4}{4} \right]_0^{\sqrt{9-z^2}} dz = \frac{1}{2} \int_0^3 \left[ \frac{(\sqrt{9-z^2})^4}{4} \right] dz = \frac{1}{8} \int_0^3 [(9-z^2)^2] dz = \\ &= \frac{1}{8} \int_0^3 (81 - 18z^2 + z^4) dz = \frac{1}{8} \left[ 81z - \frac{18z^3}{3} + \frac{z^5}{5} \right]_0^3 = \frac{1}{8} \left[ \frac{1215z - 90z^3 + 3z^5}{15} \right]_0^3 = \\ &= \frac{1}{120} [3z (405 - 30z^2 + z^4)]_0^3 = \frac{1}{40} [z (405 - 30z^2 + z^4)]_0^3 = \frac{1}{40} [3 (405 - 30 \cdot 3^2 + 3^4)] = \\ &= \frac{1}{40} [3 (405 - 270 + 81)] = \frac{648}{40} = \frac{81}{5} \end{aligned}$$

Tabela 1: Derivadas simples

$y = c$	$\Rightarrow y' = 0$
$y = x$	$\Rightarrow y' = 1$
$y = x^c$	$\Rightarrow y' = cx^{c-1}$
$y = e^x$	$\Rightarrow y' = e^x$
$y = \ln  x $	$\Rightarrow y' = \frac{1}{x}$
$y = uv$	$\Rightarrow y' = u'v + uv'$
$y = \frac{u}{v}$	$\Rightarrow y' = \frac{u'v - uv'}{v^2}$
$y = u^c$	$\Rightarrow y' = cu^{c-1}u'$
$y = e^u$	$\Rightarrow y' = e^u u'$
$y = c^u$	$\Rightarrow y' = c^u u' \ln  c $
$y = \ln  u $	$\Rightarrow y' = \frac{u'}{u}$
$y = \log_c  u $	$\Rightarrow y' = \frac{u'}{u} \log_c  e $

**A Derivadas**

**B Derivadas simples**

**C Derivadas trigonométricas**

**D Integrais**

**E Integrais simples**

**F Integrais trigonométricas**

**G Relação entre coordenada cartesina e polar**

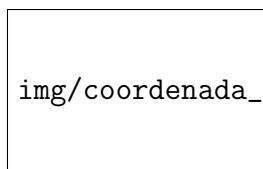
$$P(x, y) \rightarrow P(r, \theta)$$

Tabela 2: Derivadas trigonométricas

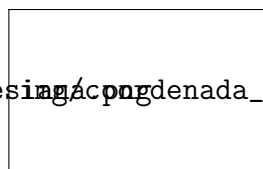
$y = \text{sen}(x)$	$\Rightarrow y' = \cos(x)$
$y = \cos(x)$	$\Rightarrow y' = -\text{sen}(x)$
$y = \text{tg}(x)$	$\Rightarrow y' = \sec^2(x)$
$y = \text{cotg}(x)$	$\Rightarrow y' = -\text{cossec}^2(x)$
$y = \sec(x)$	$\Rightarrow y' = \sec(x) \text{tg}(x)$
$y = \text{cossec}(x)$	$\Rightarrow y' = -\text{cossec}(x) \text{cotg}(x)$
$y = \arcsen(x)$	$\Rightarrow y' = \frac{1}{\sqrt{1-x^2}}$
$y = \arccos(x)$	$\Rightarrow y' = \frac{-1}{\sqrt{1-x^2}}$
$y = \arctg(x)$	$\Rightarrow y' = \frac{1}{1+x^2}$
$y = \text{arccotg}(x)$	$\Rightarrow y' = \frac{-1}{1+x^2}$
$y = \text{arcsec}(x)$	$\Rightarrow y' = \frac{1}{ x \sqrt{x^2-1}}$
$y = \text{arccossec}(x)$	$\Rightarrow y' = \frac{-1}{ x \sqrt{x^2-1}}$

Figura 19: Coordenada cartesina e polar

(a) Coordenada cartesiana ou retangular



(b) Coordenada polar



img/coordenada\_cartesiana.png img/coordenada\_polar.png

Tabela 3: Integrais simples

$$\begin{array}{lcl}
 \int dx & = & x + c \\
 \int x^p dx & = & \frac{x^{p+1}}{p+1} + c \rightarrow p \neq -1 \\
 \int e^x dx & = & e^x + c \\
 \int \frac{dx}{x} & = & \ln |x| + c \\
 \int u^p du & = & \frac{u^{p+1}}{p+1} + c \rightarrow p \neq -1 \\
 \int e^u du & = & e^u + c \\
 \int \frac{du}{u} & = & \ln |u| + c \\
 \int p^u du & = & \frac{p^u}{\ln |p|} + c
 \end{array}$$

Figura 20: Determinação do seno, cosseno e tangente

img/triangulo\_retangulo.png

**H Funções trigonométricas**

**I Determinação do seno, cosseno e tangente**

**J Círculo trigonométrico**

**K Identidades trigonométricas**

**L Relação entre trigonométricas e inversas**

**M Substituição trigonométrica**

**N Ângulos notáveis**

**Referências**

Figura 21: Círculo trigonométrico

img/circulo\_trigonometrico.png

Tabela 4: Integrais trigonométricas

$\int \text{sen}(u) du$	$= -\cos(u) + c$
$\int \cos(u) du$	$= \text{sen}(u) + c$
$\int \text{tg}(u) du$	$= \ln  \sec(u)  + c$
$\int \text{cotg}(u) du$	$= \ln  \text{sen}(u)  + c$
$\int \sec(u) du$	$= \ln  \sec(u) + \text{tg}(u)  + c$
$\int \text{cossec}(u) du$	$= \ln  \text{cossec}(u) - \text{cotg}(u)  + c$
$\int \sec^2(u) du$	$= \text{tg}(u) + c$
$\int \text{cossec}^2(u) du$	$= -\text{cotg}(u) + c$
$\int \sec(u) \text{tg}(u) du$	$= \sec(u) + c$
$\int \text{cossec}(u) \text{cotg}(u) du$	$= -\text{cossec}(u) + c$
$\int \frac{du}{\sqrt{1-x^2}}$	$= \arcsen(x) + c$
$-\int \frac{du}{\sqrt{1-x^2}}$	$= \arccos(x) + c$
$\int \frac{du}{1+x^2}$	$= \arctg(x) + c$
$-\int \frac{du}{1+x^2}$	$= \text{arccotg}(x) + c$
$\int \frac{du}{ x \sqrt{x^2-1}}$	$= \text{arcsec}(x) + c$
$-\int \frac{du}{ x \sqrt{x^2-1}}$	$= \text{arccossec}(x) + c$

Tabela 5: Relação entre coordenada cartesina e polar

$x$	$= r \cos \theta$
$y$	$= r \text{sen } \theta$
$x^2 + y^2$	$= r^2$
$da = dx dy$	$= r dr d\theta$
$v = \iint_{R(x,y)} f(x, y) dx dy$	$= \iint_{R(r,\theta)} f(r \cos \theta, r \text{sen } \theta) r dr d\theta$



Tabela 6: Identidades trigonométricas

$\operatorname{tg}(x)$	$=$	$\frac{\operatorname{sen}(x)}{\cos(x)}$
$\operatorname{cotg}(x)$	$=$	$\frac{\cos(x)}{\operatorname{sen}(x)}$
$\sec(x)$	$=$	$\frac{1}{\cos(x)}$
$\operatorname{cosec}(x)$	$=$	$\frac{1}{\operatorname{sen}(x)}$
$\operatorname{sen}^2(x) + \cos^2(x)$	$=$	1
$\sec^2(x) - \operatorname{tg}^2(x)$	$=$	1
$\operatorname{cosec}^2(x) - \operatorname{cotg}^2(x)$	$=$	1
$\operatorname{sen}^2(x)$	$=$	$\frac{1 - \cos(2x)}{2}$
$\cos^2(x)$	$=$	$\frac{1 + \cos(2x)}{2}$
$\operatorname{sen}(2x)$	$=$	$2 \operatorname{sen}(x) \cos(x)$
$\cos(2x)$	$=$	$\cos^2(x) - \operatorname{sen}^2(x)$

Tabela 7: Relação entre trigonométricas e inversas

$\operatorname{sen}(\theta)$	$=$	$x$	$\Rightarrow$	$\theta$	$=$	$\operatorname{arcsen}(x)$
$\cos(\theta)$	$=$	$x$	$\Rightarrow$	$\theta$	$=$	$\operatorname{arccos}(x)$
$\operatorname{tg}(\theta)$	$=$	$x$	$\Rightarrow$	$\theta$	$=$	$\operatorname{arctg}(x)$
$\operatorname{cosec}(\theta)$	$=$	$x$	$\Rightarrow$	$\theta$	$=$	$\operatorname{arccosec}(x)$
$\sec(\theta)$	$=$	$x$	$\Rightarrow$	$\theta$	$=$	$\operatorname{arcsec}(x)$
$\operatorname{cotg}(\theta)$	$=$	$x$	$\Rightarrow$	$\theta$	$=$	$\operatorname{arccotg}(x)$

Tabela 8: Substituição trigonométrica

$$\left| \begin{array}{l} \sqrt{a^2 - x^2} \Rightarrow x = a \operatorname{sen}(\theta) \\ \sqrt{a^2 + x^2} \Rightarrow x = a \operatorname{tg}(\theta) \\ \sqrt{x^2 - a^2} \Rightarrow x = a \sec(\theta) \end{array} \right|$$

Tabela 9: Ângulos notáveis

ângulo	$0^\circ (0)$	$30^\circ (\frac{\pi}{6})$	$45^\circ (\frac{\pi}{4})$	$60^\circ (\frac{\pi}{3})$	$90^\circ (\frac{\pi}{2})$
sen	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\nexists$