# Curso de integrais duplas e triplas

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# Sumário

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**Parte I**

# Integrais duplas

## 1 Invertendo os limites de integração - Aula 1

1. Exercício

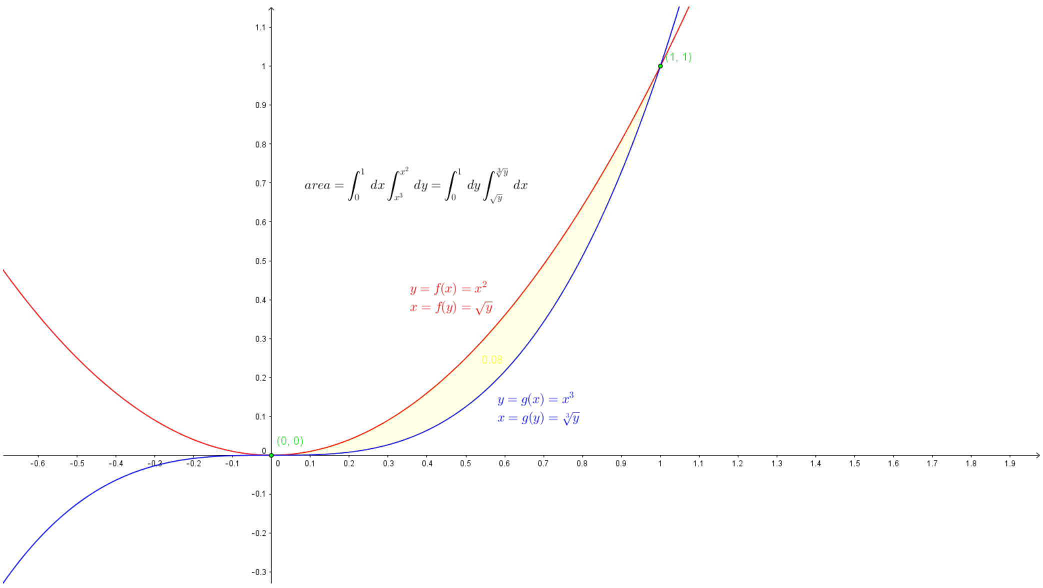



Figura 1: Integrais duplas - Aula 1 - Exercício I e II

$f(x) = x^2;\ g(x) = x^3$
$x = 0 \Rightarrow f(0) = g(0) \Rightarrow 0^2 = 0^3$
$x = 1 \Rightarrow f(1) = g(1) \Rightarrow 1^2 = 1^3$

$$a = \int_0^1 dx \int_{g(x)}^{f(x)} dy = \int_0^1 dx \int_{x^3}^{x^2} dy = \int_0^1 dx\, [y]_{x^3}^{x^2} = \int_0^1 dx \left[x^2 - x^3\right] = \int_0^1 x^2\,dx - \int_0^1 x^3\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{4x^3 - 3x^2}{12}\right]_0^1 = \frac{1}{12}\left[4x^3 - 3x^2\right]_0^1 = \frac{1}{12}\left[x^2(4x-3)\right]_0^1 = \frac{1}{12}\left[1^2(4\cdot 1 - 3) \cancel{- 0^2(4\cdot 0 - 3)}\right] = \frac{1}{12} = 0,08\overline{3}$$

2. Exercício

$f(x) = x^2 \Rightarrow f(y) = \sqrt{y};\ g(x) = x^3 \Rightarrow g(y) = \sqrt[3]{y}$
$y = 0 \Rightarrow f(0) = g(0) \Rightarrow \sqrt{0} = \sqrt[3]{0}$
$y = 1 \Rightarrow f(1) = g(1) \Rightarrow \sqrt{1} = \sqrt[3]{1}$

$$a = \int_0^1 dy \int_{f(y)}^{g(y)} dx = \int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} dx = \int_0^1 dy\, [x]_{\sqrt{y}}^{\sqrt[3]{y}} = \int_0^1 dy\, [\sqrt[3]{y} - \sqrt{y}] =$$

$$\int_0^1 \sqrt[3]{y}\, dy - \int_0^1 \sqrt{y}\, dy = \int_0^1 y^{\frac{1}{3}}\, dy - \int_0^1 y^{\frac{1}{2}}\, dy = \left[ \frac{y^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^1 =$$

$$\left[ \frac{3\sqrt[3]{y^4}}{4} - \frac{2\sqrt{y^3}}{3} \right]_0^1 = \left[ \frac{9\sqrt[3]{y^4} - 8\sqrt{y^3}}{12} \right]_0^1 = \frac{1}{12}\left[ 9\sqrt[3]{y^4} - 8\sqrt{y^3} \right]_0^1 =$$

$$\frac{1}{12}\left[ \left( 9\sqrt[3]{1^4} - 8\sqrt{1^3} \right) \cancel{- \left( 9\sqrt[3]{0^4} - 8\sqrt{0^3} \right)} \right] = \frac{1}{12}(9-8) = \frac{1}{12} = 0,08\overline{3}$$

# 2 Determinação da região de integração - Aula 2

1. Exercício

   $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2\,,\ 0 \le y \le 6\}$

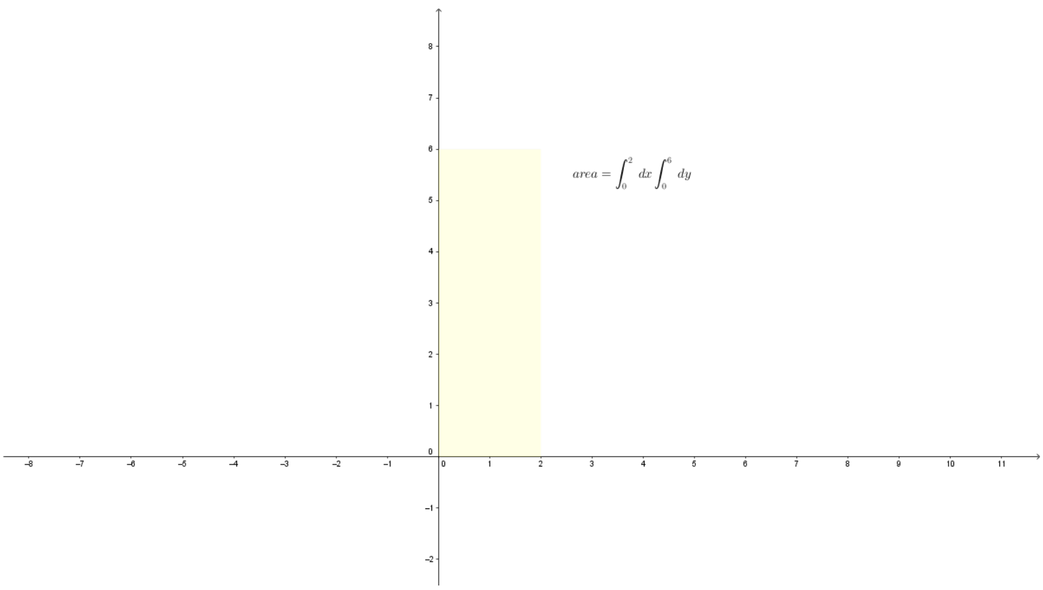



Figura 2: Integrais duplas - Aula 2 - Exercício I

$$a = \int_0^2 dx \int_0^6 dy = \int_0^2 dx \, [y]_0^6 = \int_0^2 dx \, [6-0] = 6\int_0^2 dx = 6[x]_0^2 = 6[2-0] = 6 \cdot 2 = 12$$

2. Exercício

   $R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1 \,,\, x \le y \le 2x\}$

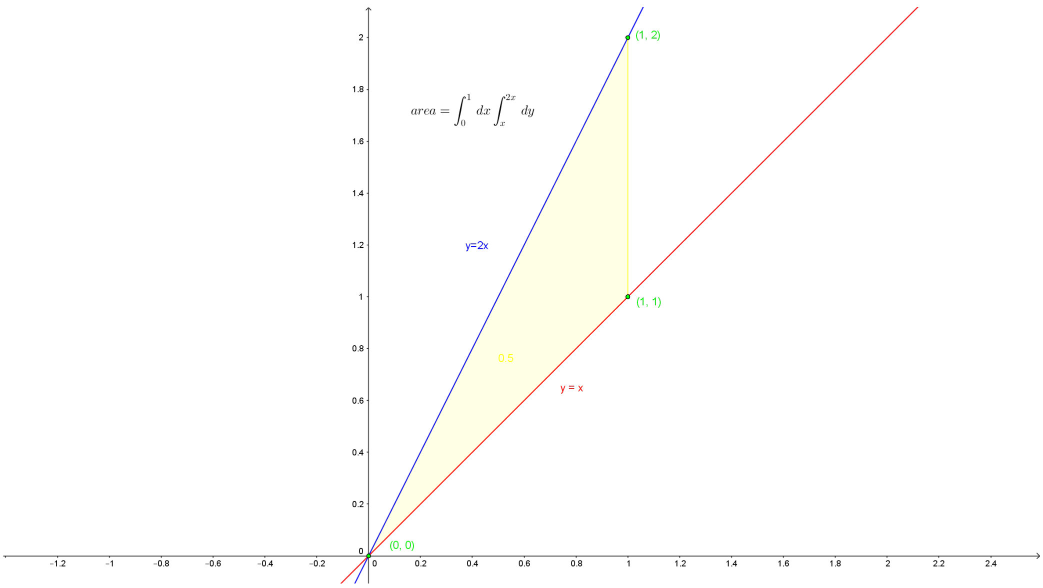



Figura 3: Integrais duplas - Aula 2 - Exercício II

$$a = \int_0^1 dx \int_x^{2x} dy = \int_0^1 dx \, [y]_x^{2x} = \int_0^1 dx \, [2x - x] = 2\int_0^1 x \, dx - \int_0^1 x \, dx = \left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^2}{2}\right]_0^1 = \left[\frac{2x^2 - x^2}{2}\right]_0^1 = \frac{1}{2}\left[x^2\right]_0^1 = \frac{1}{2}\left[1^2 \cancel{- 0^2}\right] = \frac{1}{2} = 0,5$$

3. Exercício

   $R = \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 \le y \le 1 \,,\, 0 \le x \le \sqrt{1 - y^2}\right\}$

   $y = 0, \; y = 1$

   $x = 0, \; x = \sqrt{1 - y^2} \Rightarrow x^2 = 1 - y^2 \Rightarrow x^2 - 1 = -y^2 \Rightarrow y^2 = -x^2 + 1 \Rightarrow y = \sqrt{1 - x^2}$

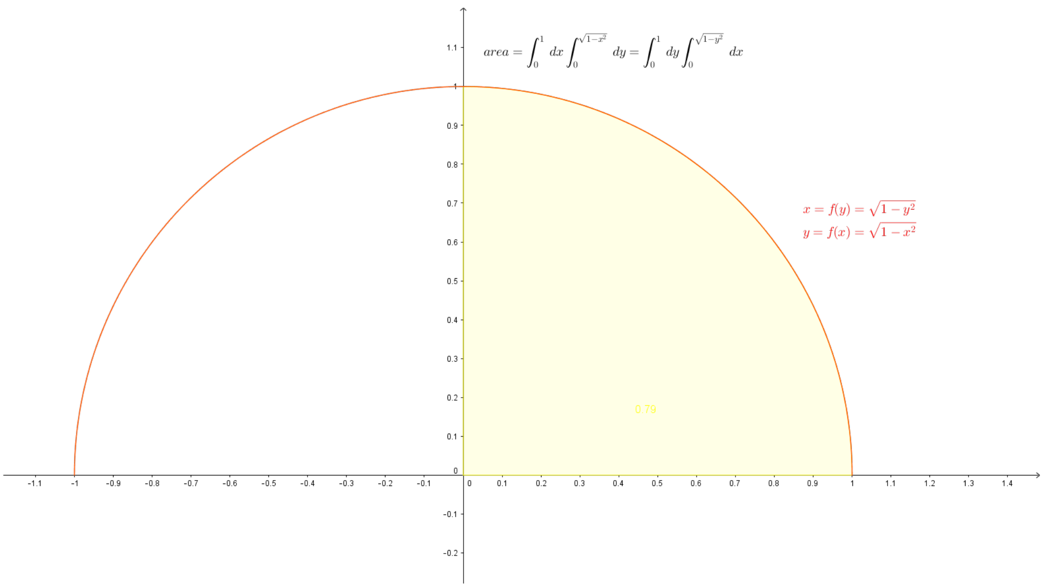



Figura 4: Integrais duplas - Aula 2 - Exercício III

$$a = \int_0^1 dy \int_0^{f(y)} dx = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy\, [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy \left[\sqrt{1-y^2} - 0\right] =$$
$$\int_0^1 \sqrt{1-y^2}\, dy = \int_0^1 \sqrt{1-\operatorname{sen}^2(t)}\, \cos(t)dt = \int_0^1 \sqrt{\cos^2(t)}\, \cos(t)dt =$$
$$\int_0^1 \cos(t)\cos(t)dt = \int_0^1 \cos^2(t)dt = \int_0^1 \frac{1+\cos(2t)}{2}dt = \frac{1}{2}\int_0^1 [1+\cos(2t)]\, dt =$$
$$\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(2t)dt = \frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_0^1 \cos(u)\frac{du}{2} = \frac{1}{2}\int_0^1 dt +$$
$$\frac{1}{4}\int_0^1 \cos(u)\, du = \left[\frac{1}{2}t + \frac{1}{4}\operatorname{sen}(u)\right]_0^1 = \left[\frac{t}{2} + \frac{\operatorname{sen}(2t)}{4}\right]_0^1 = \left[\frac{t}{2} + \frac{2\operatorname{sen}(t)\cos(t)}{4}\right]_0^1 =$$
$$\left[\frac{t+\operatorname{sen}(t)\cos(t)}{2}\right]_0^1 = \frac{1}{2}\left[\operatorname{arcsen}(y) + y\sqrt{1-y^2}\right]_0^1 =$$
$$\frac{1}{2}\left[\left(\operatorname{arcsen}(1)\cancel{+1\cdot\sqrt{1-1^2}}\right) - \left(\operatorname{arcsen}(0)\cancel{+0\cdot\sqrt{1-0^2}}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] =$$
$$\frac{\pi}{4} = 0,785$$

$$y = \operatorname{sen}(t) \Rightarrow dy = \cos(t)dt$$
$$u = 2t \Rightarrow \frac{du}{2} = dt$$
$$\operatorname{sen}(t) = \frac{co}{h} = \frac{y}{1} = y$$
$$h^2 = co^2 + ca^2 \Rightarrow 1 = y^2 + ca^2 \Rightarrow ca = \sqrt{1-y^2}$$

$$\cos(t) = \frac{ca}{h} = \frac{\sqrt{1-y^2}}{1} = \sqrt{1-y^2}$$
$$y = \operatorname{sen}(t) \Rightarrow t = \operatorname{arcsen}(y)$$

4. Exercício

$$y = x^2+1,\ y = -x^2-1;\ x = 1,\ x = -1$$
$$R = \{(x,y) \in \mathbb{R}^2 \,|\, -1 \leq x \leq 1\,,\ -x^2-1 \leq y \leq x^2+1\}$$

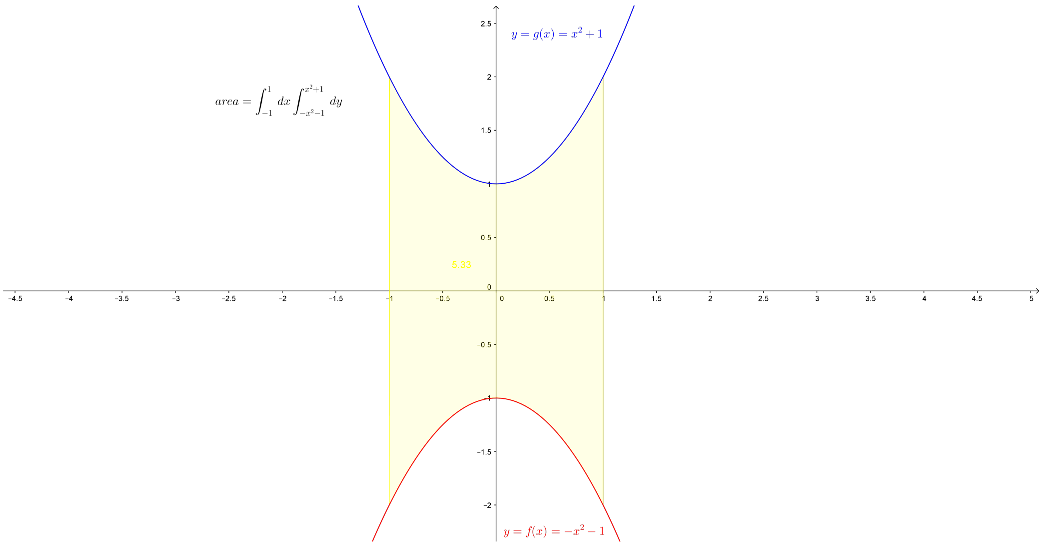



Figura 5: Integrais duplas - Aula 2 - Exercício IV

$$a = \int_{-1}^{1} dx \int_{f(x)}^{g(x)} dy = \int_{-1}^{1} dx \int_{-x^2-1}^{x^2+1} dy = \int_{-1}^{1} dx\, [y]_{-x^2-1}^{x^2+1} = \int_{-1}^{1} dx\, \left[x^2+1-\left(-x^2-1\right)\right] =$$
$$\int_{-1}^{1} dx\, \left[x^2+1+x^2+1\right] = \int_{-1}^{1} dx\, \left[2x^2+2\right] = 2\int_{-1}^{1} x^2\, dx + 2\int_{-1}^{1} dx =$$
$$\left[2\frac{x^3}{3}+2x\right]_{-1}^{1} = \left[2\left(\frac{x^3+3x}{3}\right)\right]_{-1}^{1} = \frac{2}{3}\left[x\left(x^2+3\right)\right]_{-1}^{1} =$$
$$\frac{2}{3}\left[1\cdot\left(1^2+3\right)-(-1)\left((-1)^2+3\right)\right] = \frac{2}{3}(4+4) = \frac{2}{3}8 = \frac{16}{3} = 5,\overline{3}$$

5. Exercício

$$R = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 2\,,\ -y \leq x \leq y\}$$

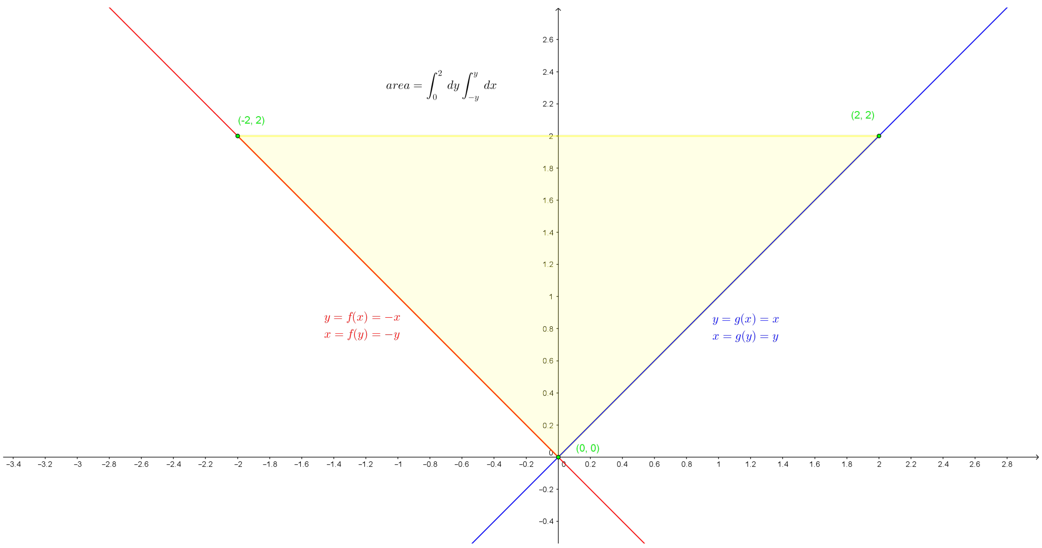



Figura 6: Integrais duplas - Aula 2 - Exercício V

$$a = \int_0^2 dy \int_{f(y)}^{g(y)} dx = \int_0^2 dy \int_{-y}^{y} dx = \int_0^2 dy\,[x]_{-y}^{y} = \int_0^2 dy\,[y-(-y)] = \int_0^2 dy\,[2y] = 2\int_0^2 y\,dy = \left[\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^2 = 2^2 - 0^2 = 4$$

# 3 Cálculo de volume - Aula 3

1. Exercício

$$volume = \int_0^3 \int_0^2 4\,dxdy$$

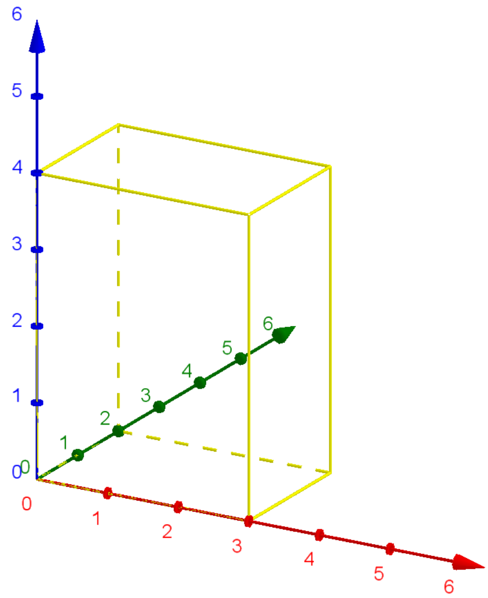



Figura 7: Integrais duplas - Aula 3 - Exercício I

$z = 4;\ dz = dxdy$

$v = \int_0^3 \int_0^2 z\,dz = \int_0^3 \int_0^2 4\,dydx = 4 \int_0^3 dx \int_0^2 dy = 4 \int_0^3 dx\,[y]_0^2 = 4 \int_0^3 dx\,[2-0] = 8 \int_0^3 dx = 8[x]_0^3 = 8[3-0] = 8 \cdot 3 = 24$

2. Exercício
$R = [0,3] \times [0,4]$
$\iint_R (8-2y)da$

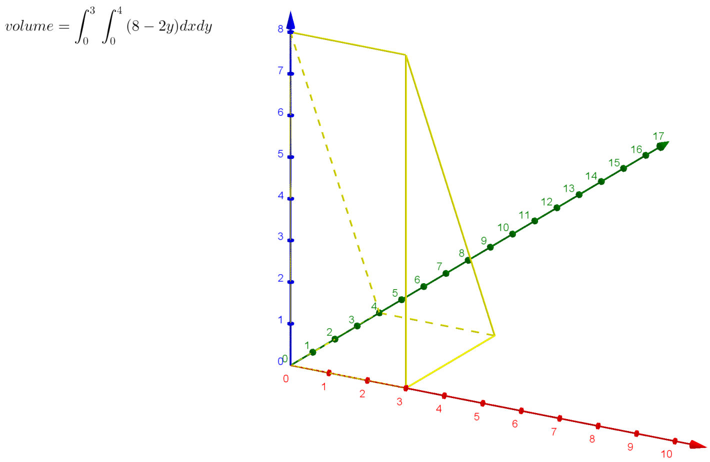



Figura 8: Integrais duplas - Aula 3 - Exercício II

$z = 8 - 2y;\ da = dz = dxdy$

$$v = \int_0^3 \int_0^4 z\,dz = \int_0^3 \int_0^4 (8-2y)dxdy = \int_0^3 dx \int_0^4 (8-2y)dy = \int_0^3 dx \left(8\int_0^4 dy - 2\int_0^4 y\,dy\right) = \int_0^3 dx\,2\left(4\int_0^4 dy - \int_0^4 y\,dy\right) = 2\int_0^3 dx\left[4y - \frac{y^2}{2}\right]_0^4 = 2\int_0^3 dx\left[\frac{8y-y^2}{2}\right]_0^4 = \cancel{2}\int_0^3 dx\,\frac{1}{\cancel{2}}[y(8-y)]_0^4 = \int_0^3 dx[4(8-4)\cancel{-0(8-0)}] = 16\int_0^3 dx = 16[x]_0^3 = 16[3-0] = 48$$

# 4 Invertendo a ordem de integração - Aula 4

1. Exercício

   $z = f(x,y) = y\,\mathrm{e}^x;\ dz = dxdy$

   $$v = \int_2^4 \int_1^9 z\,dz = \int_2^4 \int_1^9 y\,\mathrm{e}^x\ dydx = \int_2^4 \mathrm{e}^x\ dx \int_1^9 y\,dy = \int_2^4 \mathrm{e}^x\ dx\left[\frac{y^2}{2}\right]_1^9 = \int_2^4 \mathrm{e}^x\ dx\frac{1}{2}\left[y^2\right]_1^9 = \frac{1}{2}\int_2^4 \mathrm{e}^x\ dx\left[9^2 - 1^2\right] = 40\int_2^4 \mathrm{e}^x\ dx = 40\left[\mathrm{e}^x\right]_2^4 =$$

$40\left[\mathrm{e}^4-\mathrm{e}^2\right]=40e^2\left(e^2-1\right)$

2. Exercício

   $z=f(x,y)=x^2y^3;\ dz=dxdy$

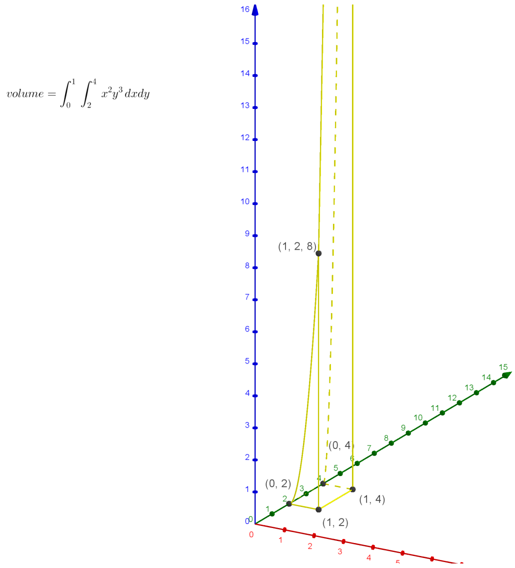



Figura 9: Integrais duplas - Aula 4 - Exercício II

   $v=\int_0^1\int_2^4 z\,dz=\int_0^1\int_2^4 x^2y^3\,dxdy=\int_0^1 x^2\,dx\int_2^4 y^3\,dy=\int_0^1 x^2\,dx\left[\frac{y^4}{4}\right]_2^4=$
   $\frac{1}{4}\int_0^1 x^2\,dx\left[y^4\right]_2^4=\frac{1}{4}\int_0^1 x^2\,dx\left[4^4-2^4\right]=\frac{1}{4}\int_0^1 x^2\,dx\left[2^8-2^4\right]=$
   $\frac{1}{4}\int_0^1 x^2\,dx\left[2^4\left(2^4-1\right)\right]=\frac{1}{4}\int_0^1 x^2\,dx\,[16\cdot 15]=60\int_0^1 x^2\,dx=60\left[\frac{x^3}{3}\right]_0^1=$
   $20\left[x^3\right]_0^1=20\left[1^3-0^3\right]=20\cdot 1=20$

3. Exercício

   $\iint_R (x+2y)da$

   R = Região limitada pela parábola $y=x^2+1$ e as retas $x=-1$ e $x=2$.

   $z=f(x,y)=x+2y;\ da=dz=dxdy$

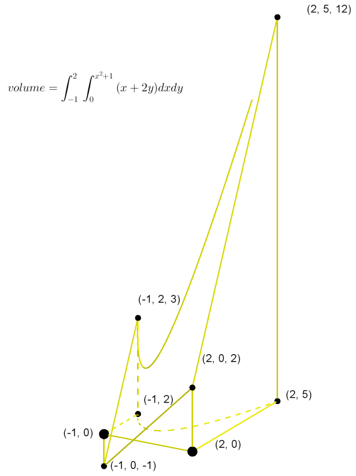



Figura 10: Integrais duplas - Aula 4 - Exercício III

$$v = \int_{-1}^{2}\int_{0}^{x^2+1} z\,dz = \int_{-1}^{2}\int_{0}^{x^2+1}(x+2y)dxdy = \int_{-1}^{2}dx\int_{0}^{x^2+1}(x+2y)dy = \int_{-1}^{2}dx\left(x\int_{0}^{x^2+1}dy + 2\int_{0}^{x^2+1}y\,dy\right) = \int_{-1}^{2}dx\left[xy+\cancel{2}\frac{y^2}{\cancel{2}}\right]_{0}^{x^2+1} =$$

$$\int_{-1}^{2}dx\,[y(x+y)]_{0}^{x^2+1} = \int_{-1}^{2}dx\left[\left(x^2+1\right)\left[x+\left(x^2+1\right)\right]\cancel{-0(x+0)}\right] =$$

$$\int_{-1}^{2}dx\left[\left(x^2+1\right)\left(x^2+x+1\right)\right] = \int_{-1}^{2}dx\left(x^4+x^3+2x^2+x+1\right) = \int_{-1}^{2}x^4\,dx+$$

$$\int_{-1}^{2}x^3\,dx+2\int_{-1}^{2}x^2\,dx+\int_{-1}^{2}x\,dx+\int_{-1}^{2}dx = \left[\frac{x^5}{5}+\frac{x^4}{4}+2\frac{x^3}{3}+\frac{x^2}{2}+x\right]_{-1}^{2} =$$

$$\left[\frac{12x^5+15x^4+40x^3+30x^2+60x}{60}\right]_{-1}^{2} = \frac{1}{60}\left[x\left(12x^4+15x^3+40x^2+30x+60\right)\right]_{-1}^{2} =$$

$$\frac{1}{60}\left[2\left(12\cdot 2^4+15\cdot 2^3+40\cdot 2^2+30\cdot 2+60\right)\right.$$
$$\left.-(-1)\left(12(-1)^4+15(-1)^3+40(-1)^2+30(-1)+60\right)\right] =$$

$$\frac{1}{60}[2(192+120+160+60+60)+(12-15+40-30+60)] = \frac{1}{60}(1184+67) =$$

$$\frac{1251}{60} = \frac{417}{20} = 20,85$$

# 5 Cálculo de integrais duplas ou iteradas

## 5.1 Aula 5

1. Exercício

   $f(x, y) = x^3;\ 0 \leq x \leq 2;\ x^2 \leq y \leq 4$

   $\iint_R f(x, y)dydx$

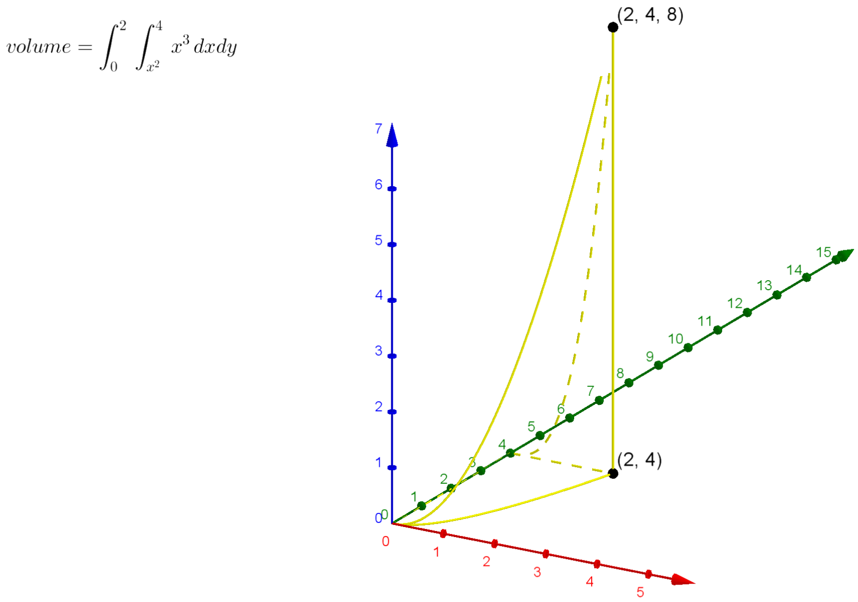



Figura 11: Integrais duplas - Aula 5 - Exercício I

$$v = \int_0^2 \int_{x^2}^4 x^3\, dxdy = \int_0^2 x^3\, dx \int_{x^2}^4 dy = \int_0^2 x^3\, dx\, [y]_{x^2}^4 = \int_0^2 x^3\, dx\, \left[4 - x^2\right] =$$
$$4\int_0^2 x^3\, dx - \int_0^2 x^5\, dx = \left[\cancel{4}\frac{x^4}{\cancel{4}} - \frac{x^6}{6}\right]_0^2 = \left[\frac{6x^4 - x^6}{6}\right]_0^2 = \frac{1}{6}\left[x^4\left(6 - x^2\right)\right]_0^2 =$$
$$\frac{1}{6}\left[2^4\left(6 - 2^2\right) - \cancel{0^4\left(6 - 0^2\right)}\right] = \frac{1}{6}(16 \cdot 2) = \frac{32}{6} = \frac{16}{3} = 5,2$$

2. Exercício

   $f(x, y) = x^2y;\ 1 \leq x \leq 3;\ x \leq y \leq 2x + 1$

   $\iint_R f(x, y)dydx$

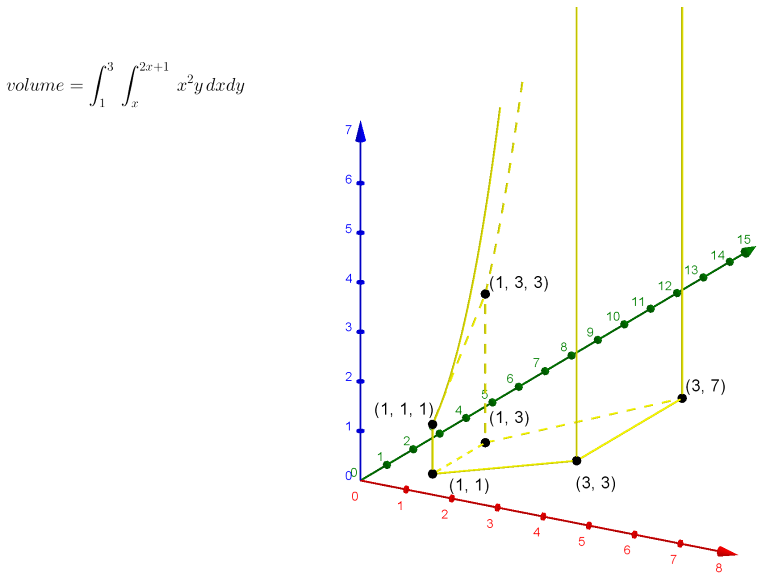



Figura 12: Integrais duplas - Aula 5 - Exercício II

$$v = \int_1^3 \int_x^{2x+1} x^2 y\,dxdy = \int_1^3 x^2\,dx \int_x^{2x+1} y\,dy = \int_1^3 x^2\,dx \left[\frac{y^2}{2}\right]_x^{2x+1} =$$
$$\int_1^3 x^2\,dx \frac{1}{2}\left[(2x+1)^2 - (x)^2\right] = \frac{1}{2}\int_1^3 x^2\,dx\left(3x^2+4x+1\right) = \frac{3}{2}\int_1^3 x^4\,dx +$$
$$2\int_1^3 x^3\,dx + \frac{1}{2}\int_1^3 x^2\,dx = \left[\frac{3}{2}\frac{x^5}{5} + 2\frac{x^4}{4} + \frac{1}{2}\frac{x^3}{3}\right]_1^3 = \left[\frac{3x^5}{10} + \frac{x^4}{2} + \frac{x^3}{6}\right]_1^3 =$$
$$\left[\frac{18x^5 + 30x^4 + 10x^3}{60}\right]_1^3 = \left[\frac{2x^3\left(9x^2+15x+5\right)}{60}\right]_1^3 = \frac{1}{30}\left[x^3\left(9x^2+15x+5\right)\right]_1^3 =$$
$$\frac{1}{30}\left[3^3\left(9\cdot 3^2 + 15\cdot 3 + 5\right) - 1^3\left(9\cdot 1^2 + 15\cdot 1 + 5\right)\right] =$$
$$\frac{1}{30}\left[27(81+45+5) - (9+15+5)\right] = \frac{1}{30}\left[27\cdot 131 - 29\right] = \frac{3508}{30} = 116,9\overline{3}$$

## 5.2 Aula 6

1. Exercício

$$f(x,y) = 1;\ 0 \le x \le 1;\ 1 \le y \le \mathrm{e}^x$$
$$\iint_R f(x,y)dydx$$
$$v = \int_0^1 \int_1^{\mathrm{e}^x} dydx = \int_0^1 dx\,[y]_1^{\mathrm{e}^x} = \int_0^1 dx\,\left(\mathrm{e}^x - 1\right) = \left[\mathrm{e}^x - x\right]_0^1 = \mathrm{e}^1 - 1 -$$

$\left(e^0 - 0\right) = e - 1 - 1 = e - 2$

2. Exercício

   $f(x,y) = x;\ 0 \leq x \leq 1;\ 1 \leq y \leq e^{x^2}$

   $\iint_R f(x,y) dy dx$

   $v = \int_0^1 \int_1^{e^{x^2}} x\, dx dy = \int_0^1 x\, dx \int_1^{e^{x^2}} dy = \int_0^1 x\, dx\, [y]_1^{e^{x^2}} = \int_0^1 x\, dx\, \left(e^{x^2} - 1\right) =$
   $\int_0^1 x\, e^{x^2}\, dx - \int_0^1 x\, dx = \int_0^1 e^u\, \frac{du}{2} - \int_0^1 x\, dx = \frac{1}{2} \int_0^1 e^u\, du - \int_0^1 x\, dx =$
   $\left[\frac{1}{2} e^u - \frac{x^2}{2}\right]_0^1 = \left[\frac{e^{x^2} - x^2}{2}\right]_0^1 = \frac{1}{2}\left[e^{x^2} - x^2\right]_0^1 = \frac{1}{2}\left[e^{1^2} - 1^2 - \left(e^{0^2} - 0^2\right)\right] =$
   $\frac{1}{2}(e - 1 - 1) = \frac{e - 2}{2}$

   $u = x^2;\ \frac{du}{2} = x\, dx$

3. Exercício

   $f(x,y) = 2xy;\ 0 \leq y \leq 1;\ y^2 \leq x \leq y$

   $\iint_R f(x,y) dx dy$

   $v = \int_0^1 \int_{y^2}^{y} 2xy\, dx dy = 2 \int_0^1 y\, dy \int_{y^2}^{y} x\, dx = 2 \int_0^1 y\, dy \left[\frac{x^2}{2}\right]_{y^2}^{y} = \cancel{2} \int_0^1 y\, dy\, \frac{1}{\cancel{2}} \left[x^2\right]_{y^2}^{y} =$
   $\int_0^1 y\, dy\, \left(y^2 - y^4\right) = \int_0^1 \left(y^3 - y^5\right) dy = \left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1 = \left[\frac{6y^4 - 4y^6}{24}\right]_0^1 =$
   $\left[\frac{2y^4\left(3 - 2y^2\right)}{24}\right]_0^1 = \frac{1}{12}\left[1^4\left(3 - 2 \cdot 1^2\right) \cancel{- 0^4\left(3 - 2 \cdot 0^2\right)}\right] = \frac{1}{12} = 0{,}08\overline{3}$

## 5.3 Aula 7

1. Exercício

   $f(x,y) = \frac{1}{x+y};\ 1 \leq y \leq e;\ 0 \leq x \leq y$

   $\iint_R f(x,y) dx dy$

   $v = \int_1^e \int_0^y \frac{1}{x+y}\, dx dy = \int_1^e dy \int_0^y (x+y)^{-1}\, dx = \int_1^e dy \int_0^y u^{-1}\, du =$

$$\int_1^{\mathrm{e}} dy \int_0^y [\ln|u|]_0^y = \int_1^{\mathrm{e}} dy \int_0^y [\ln|x+y|]_0^y = \int_1^{\mathrm{e}} dy \int_0^y (\ln|y+y| - \ln|0+y|) =$$
$$\int_1^{\mathrm{e}} dy \int_0^y (\ln|2y| - \ln|y|) = \int_1^{\mathrm{e}} dy \int_0^y (\ln|2|\cancel{+\ln|y| - \ln|y|}) = \ln|2| \int_1^{\mathrm{e}} dy =$$
$$\ln|2|[y]_1^{\mathrm{e}} = \ln|2|(\mathrm{e}-1)$$

$u = x + y;\ du = (1+0)dx = dx$

# 6 Cálculo de área - Aula 8

1. Exercício

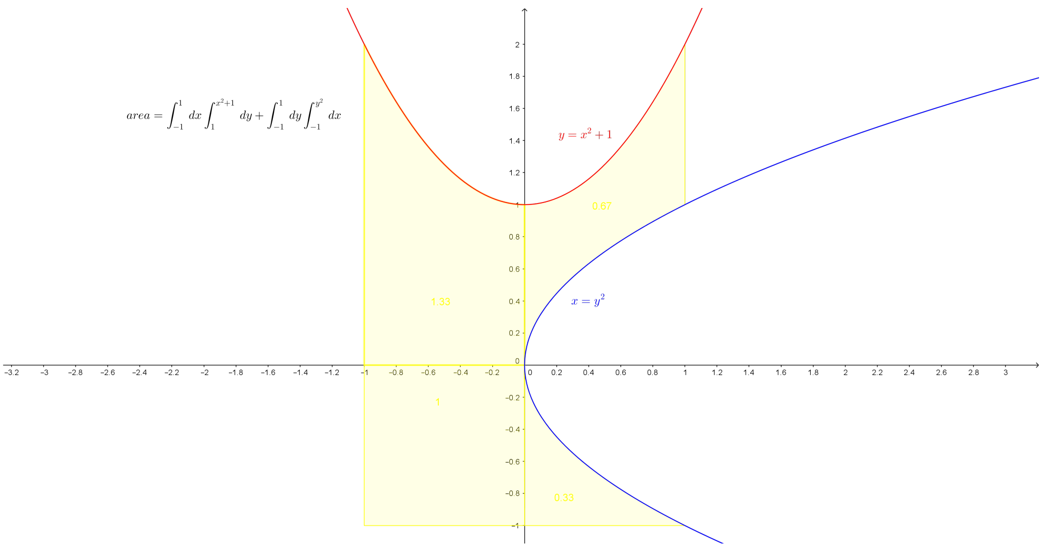



Figura 13: Integrais duplas - Aula 8 - Exercício I

$$a = \int_{-1}^0 dx \int_0^{x^2+1} dy + \int_{-1}^0 dx \int_{-1}^0 dy + \int_0^{y^2} dx \int_{-1}^0 dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} =$$
$$\int_{-1}^0 dx \left( \int_0^{x^2+1} dy + \int_{-1}^0 dy \right) + \int_0^{y^2} dx \int_{-1}^0 dy + \int_0^1 dx \int_{\sqrt{x}}^{x^2+1} = \int_{-1}^0 dx \left( [y]_0^{x^2+1} + [y]_{-1}^0 \right) +$$
$$\int_{-1}^0 dy\, [x]_0^{y^2} + \int_0^1 dx\, [y]_{\sqrt{x}}^{x^2+1} = \int_{-1}^0 dx\ (x^2+1+1) + \int_{-1}^0 dy\, y^2 + \int_0^1 dx\ (x^2+1-\sqrt{x}) =$$
$$\int_{-1}^0 (x^2+2)\, dx + \int_{-1}^0 y^2\, dy + \int_0^1 \left( x^2 - x^{\frac{1}{2}} + 1 \right) dx = \left[ \frac{x^3}{3} + 2x \right]_{-1}^0 +$$

$$\left[\frac{y^3}{3}\right]_{-1}^{0}+\left[\frac{x^3}{3}-\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+x\right]_{0}^{1}=\left[\frac{x^3+6x}{3}\right]_{-1}^{0}+\frac{1}{3}\left[y^3\right]_{-1}^{0}+\left[\frac{x^3}{3}-\frac{2\sqrt{x^3}}{3}+x\right]_{0}^{1}=$$
$$\frac{1}{3}\left[x\left(x^2+6\right)\right]_{-1}^{0}+\frac{1}{3}\left[\cancel{0^3}-(-1)^3\right]+\left[\frac{x^3-2\sqrt{x^3}+3x}{3}\right]_{0}^{1}=$$
$$\frac{1}{3}\left[\cancel{0\left(0^2+6\right)}-(-1)\left((-1)^2+6\right)\right]+\frac{1}{3}+\frac{1}{3}\left[x^3-2\sqrt{x^3}+3x\right]_{0}^{1}=\frac{7}{3}+\frac{1}{3}+$$
$$\frac{1}{3}\left[1^3-2\sqrt{1^3}+3\cdot 1\cancel{-\left(0^3-2\sqrt{0^3}+3\cdot 0\right)}\right]=\frac{7}{3}+\frac{1}{3}+\frac{2}{3}=\frac{7+1+2}{3}=$$
$$\frac{10}{3}=3,\overline{3}$$

$$a=\int_{-1}^{1}dx\int_{1}^{x^2+1}dy+\int_{-1}^{y^2}dx\int_{-1}^{1}dy=\int_{-1}^{1}dx\,[y]_{1}^{x^2+1}+\int_{-1}^{1}dy\,[x]_{-1}^{y^2}=$$
$$\int_{-1}^{1}dx\,\left(x^2\cancel{+1-1}\right)+\int_{-1}^{1}dy\,\left(y^2+1\right)=\left[\frac{x^3}{3}\right]_{-1}^{1}+\left[\frac{y^3}{3}+y\right]_{-1}^{1}=\frac{1}{3}\left[x^3\right]_{-1}^{1}+$$
$$\frac{1}{3}\left[y\left(y^2+3\right)\right]_{-1}^{1}=\frac{1}{3}\left(\left[1^3-(-1)^3\right]+\left[1\left(1^2+3\right)-(-1)\left((-1)^2+3\right)\right]\right)\frac{1}{3}(2+$$
$$4+4)=\frac{10}{3}=3,\overline{3}$$

# 7 Cálculo de volume

## 7.1 Aula 9

1. Exercício

   Esboçe a região de integração e o sólido cujo volume é dado pela integral abaixo:

$$\int_{0}^{1}\int_{0}^{1}(4-x-2y)\ dxdy$$

$$volume = \int_0^1 \int_0^1 (4 - x - 2y)\ dxdy$$

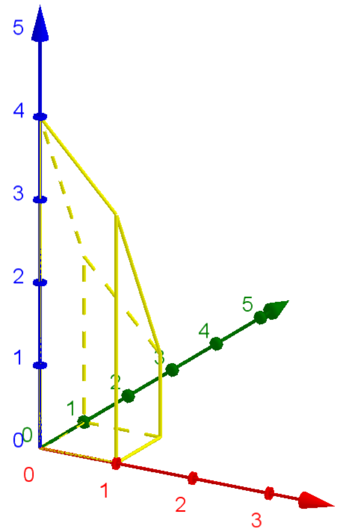


Figura 14: Integrais duplas - Aula 9 - Exercício I

$$v = \int_0^1 \int_0^1 (4 - x - 2y)\ dxdy = \int_0^1 dx \left( 4 \int_0^1 dy - x \int_0^1 dy - 2 \int_0^1 y\, dy \right) =$$
$$4 \int_0^1 dx \int_0^1 dy - \int_0^1 x\, dx \int_0^1 dy - 2 \int_0^1 dx \int_0^1 y\, dy = 4[x]_0^1[y]_0^1 - \left[\frac{x^2}{2}\right]_0^1 [y]_0^1 -$$
$$2[x]_0^1 \left[\frac{y^2}{2}\right]_0^1 = 4 - \frac{1}{2} - \cancel{2}\frac{1}{\cancel{2}} = \frac{8 - 1 - 2}{2} = \frac{5}{2} = 2,5$$

## 7.2 Aula 10

1. Exercício

   Calcule o volume do sólido limitado pelos planos:

   $$x = 0,\ y = 0,\ z = 0 \text{ e } 6x + 2y + 3z = 6$$

$$volume = \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dxdy$$

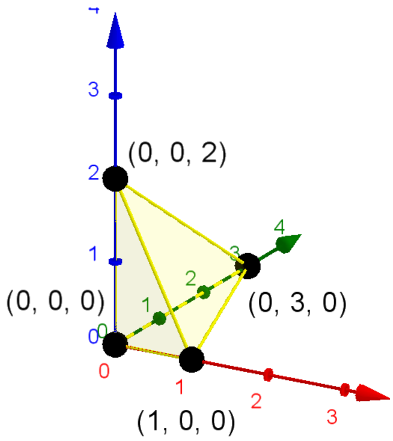



Figura 15: Integrais duplas - Aula 10 - Exercício I

$P_1 = (0,0,0)$

$6x = -2y - 3z + 6 \Rightarrow x = \dfrac{-2y-3z+6}{6} = \dfrac{-2\cdot 0 - 3\cdot 0 + 6}{6} = \dfrac{6}{6} = 1 \Rightarrow P_2 = (1,0,0)$

$2y = -6x - 3z + 6 \Rightarrow y = \dfrac{-6x-3z+6}{2} = \dfrac{-6\cdot 0 - 3\cdot 0 + 6}{2} = \dfrac{6}{2} = 3 \Rightarrow P_3 = (0,3,0)$

$3z = -6x - 2y + 6 \Rightarrow z = \dfrac{-6x-2y+6}{3} = \dfrac{-6\cdot 0 - 2\cdot 0 + 6}{3} = \dfrac{6}{3} = 2 \Rightarrow P_4 = (0,0,2)$

$x = 0, x = 1$

$y = 0,\ y = \dfrac{-6x-3z+6}{2} = \dfrac{-6x - 3\cdot 0 + 6}{2} = -3x + 3$

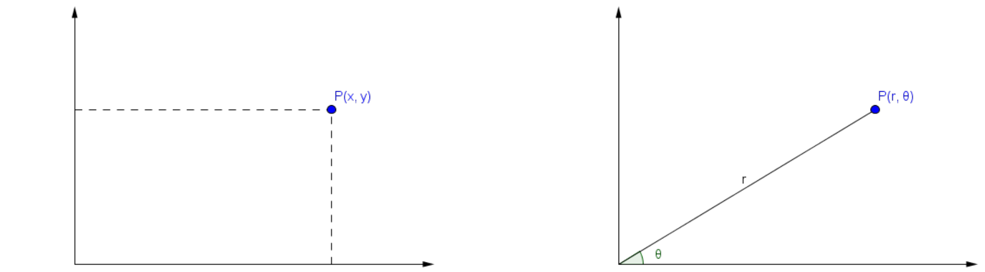
$z = \dfrac{-6x-2y+6}{3} = -2x - \dfrac{2y}{3} + 2$

$$v = \int_0^1 \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dxdy = \int_0^1 dx \int_0^{-3x+3} \left(-2x - \frac{2y}{3} + 2\right) dy =$$
$$\int_0^1 dx \left[-2xy - \frac{\cancel{2}}{3}\frac{y^2}{\cancel{2}} + 2y\right]_0^{-3x+3} = \int_0^1 dx\, \frac{1}{3}\left[-6xy - y^2 + 6y\right]_0^{-3x+3} =$$
$$\frac{1}{3}\int_0^1 dx\, \left[-y(6x+y-6)\right]_0^{-3x+3} =$$
$$\frac{1}{3}\int_0^1 dx\, \left[-(-3x+3)(6x+(-3x+3)-6)\cancel{+0(6x+0-6)}\right] = \frac{1}{3}\int_0^1 dx\, \left[(3x-3)(3x-3)\right] =$$
$$\frac{1}{3}\int_0^1 \left(9x^2 - 18x + 9\right) dx = \frac{1}{3}\left[9\frac{x^3}{3} - 18\frac{x^2}{2} + 9x\right]_0^1 = \frac{1}{3}\left[3x^3 - 9x^2 + 9x\right]_0^1 =$$

$$\frac{1}{\cancel{3}}\left[\cancel{3}x\left(x^2-3x+3\right)\right]_0^1=\left[1\left(1^2\cancel{-3\cdot 1+3}\right)\cancel{-0\left(0^2-3\cdot 0+3\right)}\right]=1$$

# 8 Coordenadas polares

## 8.1 Aula 1




(a) Coordenada cartesiana ou retangular (b) Coordenada polar

Figura 16: Coordenada cartesina e polar

$$P(x,y)\rightarrow P(r,\theta)$$

$$\boxed{\begin{array}{lllll} & & x & = & r\,\cos\theta \\ & & y & = & r\,\operatorname{sen}\theta \\ & & x^2+y^2 & = & r^2 \\ da & = & dxdy & = & r\,drd\theta \\ v & = & \displaystyle\iint_{R(x,y)} f(x,y)\,dxdy & = & \displaystyle\iint_{R(r,\theta)} f(r\,\cos\theta, r\,\operatorname{sen}\theta)r\,drd\theta \end{array}}$$

Tabela 1: Relação entre coordenada cartesina e polar

| | $0°$ $(0)$ | $30°$ $\left(\frac{\pi}{6}\right)$ | $45°$ $\left(\frac{\pi}{4}\right)$ | $60°$ $\left(\frac{\pi}{3}\right)$ | $90°$ $\left(\frac{\pi}{2}\right)$ |
|---|---|---|---|---|---|
| sen | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| cos | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| tg | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | $\infty$ |

Tabela 2: Seno, cosseno e tangente dos principais ângulos

1. Exercício

   Calcule a área do circulo de raio igual a dois

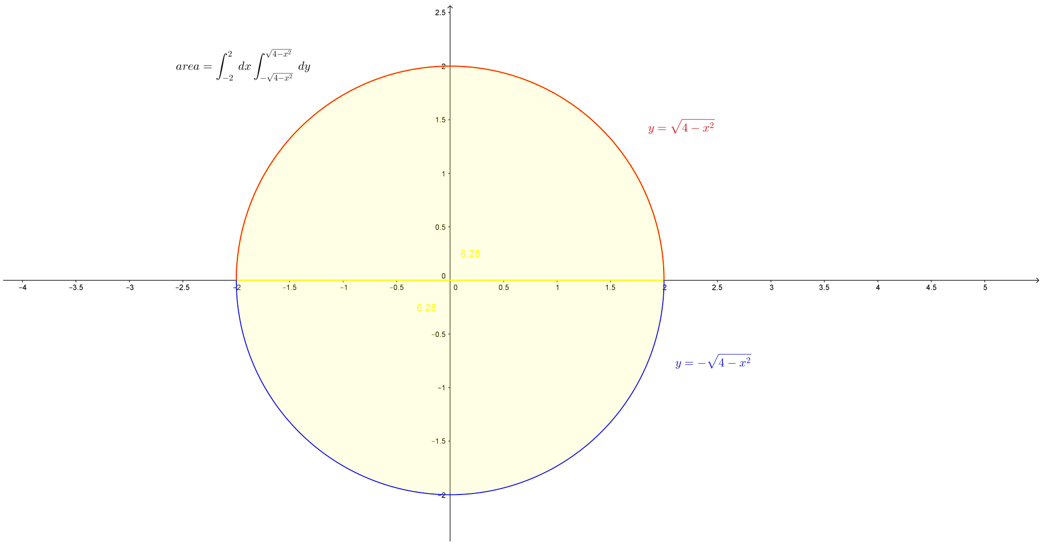



Figura 17: Coordenadas polares - Aula 01 - Exercício I

$r = 2 \Rightarrow a = \pi r^2 = 2^2 \pi = 4\pi$

$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 2^2 \Rightarrow x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$
$R = \left\{(x, y) \in \mathbb{R}^2 \,|\, -2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\right\}$

$a = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy = \int_{-2}^{2} dx \left(\sqrt{4 - x^2} + \sqrt{4 - x^2}\right) = 2 \int_{-2}^{2} \sqrt{4 - x^2}\, dx =$
$2 \int_{-2}^{2} \sqrt{4 - (2\,\mathrm{sen}(\alpha))^2}\, 2\cos(\alpha)\, d\alpha = 4 \int_{-2}^{2} \sqrt{4 - 4\,\mathrm{sen}^2(\alpha)}\, \cos(\alpha)\, d\alpha =$
$4 \int_{-2}^{2} \sqrt{4 - 4\left(1 - \cos^2(\alpha)\right)}\, \cos(\alpha)\, d\alpha = 4 \int_{-2}^{2} \sqrt{4 - \left(4 - 4\cos^2(\alpha)\right)}\, \cos(\alpha)\, d\alpha =$
$4 \int_{-2}^{2} \sqrt{\cancel{4 - 4} + 4\cos^2(\alpha)}\, \cos(\alpha)\, d\alpha = 4 \int_{-2}^{2} 2\cos(\alpha)\cos(\alpha)\, d\alpha = 8 \int_{-2}^{2} \cos^2(\alpha)\, d\alpha =$
$8 \int_{-2}^{2} \left(\frac{1 + \cos(2\alpha)}{2}\right) d\alpha = 8 \int_{-2}^{2} \left(\frac{1}{2} + \frac{\cos(2\alpha)}{2}\right) d\alpha = 4 \int_{-2}^{2} d\alpha + 4 \int_{-2}^{2} \cos(2\alpha)\, d\alpha =$
$4 \int_{-2}^{2} d\alpha + 4 \int_{-2}^{2} \cos(u)\, \frac{du}{2} = 4 \int_{-2}^{2} d\alpha + 2 \int_{-2}^{2} \cos(u)\, du = \left[4\alpha + 2\,\mathrm{sen}(u)\right]_{-2}^{2} =$
$\left[4\alpha + 2\,\mathrm{sen}(2\alpha)\right]_{-2}^{2} = \left[4\alpha + 4\,\mathrm{sen}(\alpha)\cos(\alpha)\right]_{-2}^{2} = \left[4\left(\alpha + \mathrm{sen}(\alpha)\cos(\alpha)\right)\right]_{-2}^{2} =$

$$\left[4\left(\text{arcsen}\left(\frac{x}{2}\right)+\frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right)\right]_{-2}^{2}=\left[4\left(\text{arcsen}\left(\frac{x}{2}\right)+\frac{x\sqrt{4-x^2}}{4}\right)\right]_{-2}^{2}=$$
$$4\left(\text{arcsen}\left(\frac{2}{2}\right)+\frac{2\sqrt{4-2^2}}{4}\right)-4\left(\text{arcsen}\left(\frac{(-2)}{2}\right)+\frac{(-2)\sqrt{4-(-2)^2}}{4}\right)=$$
$$4\,\text{arcsen}(1)-4\,\text{arcsen}(-1)=4(\text{arcsen}(1)-\text{arcsen}(-1))=4\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=$$
$$4\left(\frac{\cancel{2}\pi}{\cancel{2}}\right)=4\pi$$

$x=2\,\text{sen}(\alpha);\ dx=2\cos(\alpha)\,d\alpha$

$u=2\alpha;\ \frac{du}{2}=d\alpha$

$sen(\alpha)=\frac{co}{h}=\frac{x}{2}\Rightarrow\alpha=\text{arcsen}\left(\frac{x}{2}\right)$

$h^2=co^2+ca^2\Rightarrow 2^2=x^2+ca^2\Rightarrow ca=\sqrt{4-x^2}$

$cos(\alpha)=\frac{ca}{h}=\frac{\sqrt{4-x^2}}{2}$

$R=\{(r,\theta)\in\mathbb{R}^2\,|\,0\leq r\leq 2,\ 0\leq\theta\leq 2\pi\}$

$$a=\int_{-2}^{2}dx\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dy=\int_0^2\int_0^{2\pi}r\,drd\theta=\int_0^2 r\,dr\int_0^{2\pi}d\theta=\left[\frac{r^2}{2}\right]_0^2[\theta]_0^{2\pi}=$$
$$\frac{1}{2}\left[2^2-0^2\right][2\pi-0]=\frac{4}{\cancel{2}}\cancel{2}\pi=4\pi$$

2. Exercício

$$\iint_R\frac{da}{1+x^2+y^2}$$

$$volume = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\, drd\theta}{1+r^2}$$

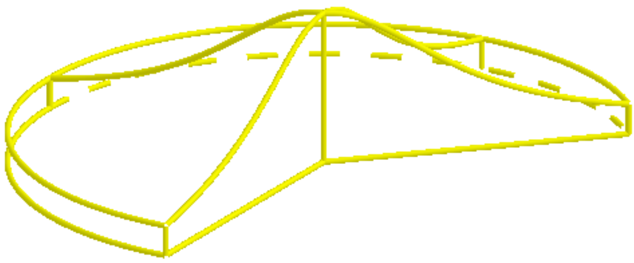


Figura 18: Coordenadas polares - Aula 01 - Exercício II

$R = \left\{ (r,\theta) \in \mathbb{R}^2 \,|\, 0 \leq r \leq 2,\ \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{2} \right\}$

$v = \iint_R \frac{da}{1+x^2+y^2} = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{r\, drd\theta}{1+r^2} = \int_0^2 \frac{r\, dr}{1+r^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} d\theta = \int_0^2 \left(1+r^2\right)^{-1} r\, dr\, [\theta]_{\frac{\pi}{4}}^{\frac{3\pi}{2}} =$
$\int_0^2 \left(1+r^2\right)^{-1} r\, dr \left(\frac{3\pi}{2} - \frac{\pi}{4}\right) = \int_0^2 \left(1+r^2\right)^{-1} r\, dr \left(\frac{6\pi-\pi}{4}\right) = \frac{5\pi}{4} \int_0^2 \left(1+r^2\right)^{-1} r\, dr =$
$\frac{5\pi}{4} \int_0^2 u^{-1} \frac{du}{2} = \frac{5\pi}{8} \int_0^2 u^{-1} du = \frac{5\pi}{8} \left[ln|u|\right]_0^2 = \frac{5\pi}{8} \left[ln|1+r^2|\right]_0^2 =$
$\frac{5\pi}{8} \left[ln|1+2^2| - ln|1+0^2|\right] = \frac{5\pi}{8} \left[ln|5| - ln|1|\right] = \frac{5\pi ln|5|}{8}$

$u = 1 + r^2 \Rightarrow \frac{du}{2} = r\, dr$
$\mathrm{e}^x = 1 = \mathrm{e}^0 \Rightarrow x = 0$