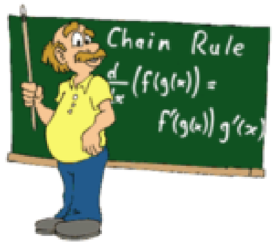
2.5: The Derivatives of Composite Functions (i.e. The Chain Rule) Date: ______________

THE CHAIN RULE

IF $h(x) = f(g(x))$, THEN $h'(x) = f'(g(x)) \cdot g'(x)$

Ex1. Find $h(x) = f(g(x))$, then use the chain rule to find $h'(x)$.



a. $f(x) = x^3$, $g(x) = 8 - 2x - x^2$

$f(x) = x^3$ $g(x) = 8 - 2x - x^2$

$f'(x) = 3x^2$ $g'(x) = -2 - 2x$

$$h(x) = f(g(x)) = f(8-2x-x^2) = (8-2x-x^2)^3$$

$$h'(x) = f'(g(x)) \cdot g'(x) = f'(8-2x-x^2) \cdot (-2-2x) = 3(8-2x-x^2)^2 \cdot (-2-2x)$$

b. $f(x) = x^{\frac{3}{2}}$, $g(x) = x^2 + x$

$f(x) = x^{\frac{3}{2}}$ $g(x) = x^2 + x$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$ $g'(x) = 2x + 1$

$$h(x) = f(g(x)) = f(x^2+x) = (x^2+x)^{\frac{3}{2}}$$

$$h'(x) = f'(g(x)) \cdot g'(x) = f'(x^2+x) \cdot (2x+1) = \frac{3}{2}(x^2+x)^{\frac{1}{2}} \cdot (2x+1)$$

Ex2. Differentiate $y = (x^2+1)^4(2x-3)^3$. Express your answer in a simplified factored form.

$$y' = 4(x^2+1)^3(2x)(2x-3)^3 + (x^2+1)^4\left(3(2x-3)^2 \cdot (2)\right)$$

$$= 8x(x^2+1)^3(2x-3)^3 + 6(x^2+1)^4(2x-3)^2$$

$$= 8x\,a^3 b^3 + 6a^4 b^2$$

$$= 2a^3 b^2(4xb + 3a)$$

$$= 2(x^2+1)^3(2x-3)^2\left[4x(2x-3) + 3(x^2+1)\right]$$

$$= 2(x^2+1)^3(2x-3)^2\left[11x^2 - 12x + 3\right]$$

Ex3. Find the equation of the tangent to the curve $y=\left(\frac{x-1}{x-8}\right)^{\frac{1}{3}}$ at $x=9$.

$$y'=\frac{1}{3}\left(\frac{x-1}{x-8}\right)^{-\frac{2}{3}}\cdot\frac{(1)(x-8)-(x-1)(1)}{(x-8)^2}$$

$$y'=\frac{1}{3}\left(\frac{x-8}{x-1}\right)^{\frac{2}{3}}\cdot\frac{-7}{(x-8)^2}$$

$x=9$

$$y'=\frac{1}{3}\cdot\left(\frac{1}{8}\right)^{\frac{2}{3}}\cdot\frac{-7}{(1)^2} \qquad y=\left(\frac{8}{1}\right)^{\frac{1}{3}}$$

$$=\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{-7}{1} \qquad y=2$$

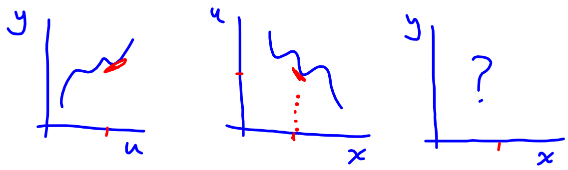
$$=\frac{-7}{12}$$

$\therefore m=\frac{-7}{12}$, $P_0=(9,2)$

$$\therefore \frac{x-9}{12}=\frac{y-2}{-7}$$

$$-7x+63=12y-24$$

$7x+12y-87=0$ ITRE.



THE CHAIN RULE A LA LEIBNIZ

IF y is a function of u and u is a function of x, THEN $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

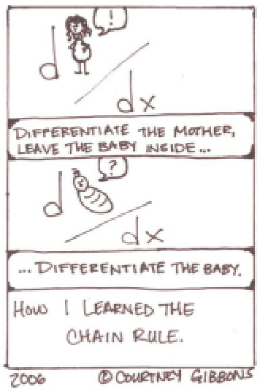
Ex4. Let $y=u^3$ where $u=\sqrt{x}$. $=x^{\frac{1}{2}}$

a. Find y in terms of x, then find y'.

$y=u^3$

$y=(x^{\frac{1}{2}})^3 \longrightarrow y'=3(x^{\frac{1}{2}})^2\cdot\frac{1}{2}x^{-\frac{1}{2}}$

$y=x^{\frac{3}{2}} \longrightarrow y'=\frac{3}{2}x^{\frac{1}{2}}$



b. Find $\frac{dy}{dx}$ using the chain rule.

$y=u^3 \qquad u=x^{\frac{1}{2}}$

$\frac{dy}{du}=3u^2 \qquad \frac{du}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=3u^2\cdot\frac{1}{2}x^{-\frac{1}{2}}$$

Ex5. Find $\frac{dy}{dx}$ at $x = 8$, if $y = 2u^2 - 5u + 3$ and $u = x^{\frac{1}{3}} - x$.

$$\frac{dy}{du} = 4u - 5 \quad , \quad \frac{du}{dx} = \frac{1}{3}x^{-\frac{2}{3}} - 1$$

when $x = 8$, $u = 8^{\frac{1}{3}} - 8$
$= 2 - 8$
$= -6$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= (4u-5) \cdot \left(\frac{1}{3}x^{-\frac{2}{3}} - 1\right)$$
$$= [4(-6)-5]\left[\frac{1}{3} \cdot (8)^{-\frac{2}{3}} - 1\right]$$
$$= (-29)\left[\frac{1}{3} \cdot \frac{1}{4} - \frac{12}{12}\right]$$
$$= (-29)\left(-\frac{11}{12}\right) \longrightarrow \therefore \frac{dy}{dx} = \frac{+319}{12}$$

Ex6. Prove the Chain Rule.

Prove: IF $h(x) = f(g(x)) \longrightarrow \therefore h'(x) = f'(g(x)) \cdot g'(x)$

$$h'(x) = \lim_{h \to 0} \frac{h(x+h) - h(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(\underline{g(x+h)}) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

Let $K = g(x+h) - g(x)$ And as $h \to 0$
$\therefore g(x+h) = g(x) + K$ $K \to 0$

$$\therefore h'(x) = \lim_{K \to 0} \frac{f(g(x)+K) - f(g(x))}{K} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(g(x)) \cdot g'(x)$$

Homefun: Page 105 # 1bde, 2b, 4→14, 17b, 18 [8a and 8e answers not completely factored: 8a. $3(x+4)(x-3)(3x+5)$ 8e. $4x(1-2x)(1+2x)(1-10x)$]

Q6, Q7, Q13, Q14 are nice because they require an understanding of the Chain Rule beyond what you learned in 2.3 with the made up Power of a Function Rule.