The **AVERAGE RATE OF CHANGE** in y with respect to x over the interval from $x = a$ to $x = a + h$

is the slope of the secant with endpoints $(a, f(a))$ and $(a+h, f(a+h))$

To calculate this average rate of change (slope), we can use the **DIFFERENCE QUOTIENT** (slope formula).

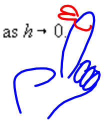
So, the *slope of the secant from $(a, f(a))$ to $(a+h, f(a+h))$* $= \dfrac{\Delta y}{\Delta x} = \dfrac{f(a+h) - f(a)}{(a+h) - a} = \dfrac{f(a+h) - f(a)}{h}$.

The **INSTANTANEOUS RATE OF CHANGE** in y with respect to x at $x = a$

is the slope of the tangent at the point $(a, f(a))$.

So, to calculate this slope, we can use the limiting value of the average rate of change as $h \to 0$.

That is, the *slope of the tangent at $(a, f(a))$* $= \lim_{h \to 0} \dfrac{\Delta y}{\Delta x} = \boxed{\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}}$.



Or, we can use an alternate, but equivalent form to find the slope where $\lim_{x \to a} \dfrac{\Delta y}{\Delta x} = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

Ex1. Ferd is cleaning the outside of the patio windows at his auntie's apartment, which is located 45 m above the ground. Ferd accidently trips and falls over the edge of the balcony. His height, in metres, after t seconds is given by the position function, $s(t) = 45 - 5t^2$.

a. Calculate the average velocity during the first, second and third seconds.

t	$s(t)$
0	45
1	40
2	25
3	0

$+1$, $+1$, $+1$; -5, -15, -25

(i) AROC $= -5$ m/s

(ii) AROC $= -15$ m/s

(iii) AROC $= -25$ m/s

b. Calculate the average velocity over the first 3 seconds.

$$\text{AROC} = \frac{-45\text{ m}}{3\text{ s}} = -15 \text{ m/s}$$

c. Calculate the average velocity $\boxed{\text{at exactly 3 seconds}}$. Use both forms of the limit.

$$\text{IROC} = \lim_{h \to 0} \frac{s(3+h) - s(3)}{h}$$

$$= \lim_{h \to 0} \frac{-30h - 5h^2}{h}$$

$$= \lim_{h \to 0} (-30 - 5h)$$

$$= -30 \text{ m/s}$$

$s(3+h)$
$= 45 - 5(3+h)^2$
$= 45 - 5(9 + 6h + h^2)$
$= 45 - 45 - 30h - 5h^2$
$= -30h - 5h^2$

$s(3) = 0$

$$\text{IROC} = \lim_{x \to 3} \frac{s(x) - s(3)}{x - 3}$$

$$= \lim_{x \to 3} \frac{45 - 5x^2}{x - 3}$$

$$= \lim_{x \to 3} \frac{-5(x^2 - 9)}{x - 3}$$

$$\text{IROC} = \lim_{x \to 3} \frac{-5\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$$

$$= \lim_{x \to 3} -5(x+3)$$

$$= -5(3+3)$$

$$= -30 \text{ m/s}$$

Ex2. The height of a soccer ball in metres t seconds after being hit with a knee is given by the function $H(t) = 1 + 3.5t - 5t^2$.

a. Find the velocity of the soccer ball at $t = 0.5$ s.

$$IROC = \lim_{h \to 0} \frac{H(0.5+h) - H(0.5)}{h}$$

$$H(0.5+h) = 1 + 3.5(0.5+h) - 5(0.5+h)^2$$
$$= 1 + 1.75 + 3.5h - 5(0.25 + h + h^2)$$
$$= 2.75 + 3.5h - 1.25 - 5h - 5h^2$$
$$= -5h^2 - 1.5h + 1.5$$

$$H(0.5) = 1 + 3.5(0.5) - 5(0.5)^2$$
$$= 1.5$$

$$\therefore IROC = \lim_{h \to 0} \frac{-5h^2 - 1.5h}{h} = \lim_{h \to 0} (-5h - 1.5) = -1.5 \text{ m/s}$$

b. When does the ball momentarily stop? What is the height of the ball at this time?

$$\lim_{h \to 0} \frac{H(t+h) - H(t)}{h} = 0$$

$$H(t+h) = 1 + 3.5(t+h) - 5(t+h)^2$$
$$= 1 + 3.5t + 3.5h - 5(t^2 + 2ht + h^2)$$
$$= 1 + 3.5t + 3.5h - 5t^2 - 10ht - 5h^2$$

$$H(t) = 1 + 3.5t - 5t^2$$

$$\therefore \lim_{h \to 0} \frac{3.5h - 10ht - 5h^2}{h} = 0$$

$$\therefore \lim_{h \to 0} (3.5 - 10t - 5h) = 0$$

So $3.5 - 10t = 0$

$10t = 3.5$

$t = 0.35$ s

$H(0.35) =$ ______ m

Ex3. The total cost, in dollars, of manufacturing x synthetic hair balls for ceramic cats is given by the function $C(x) = 50\sqrt{2x} + 1500$.

a. What is the total cost of manufacturing 2 synthetic hair balls? 50 synthetic hair balls? Compare the corresponding unit costs.

$$C(2) = 50\sqrt{4} + 1500 = \$1600$$

$$AROC = \frac{\$1600}{2 \text{ balls}} = \$800/\text{ball}$$

$$C(50) = 50\sqrt{100} + 1500 = \$2000$$

$$AROC = \frac{\$2000}{50 \text{ balls}} = \$40/\text{ball}$$

b. What is the rate of change in the total cost with respect to the number of synthetic hair balls, x, being produced when $x = 50$.

$$C(50+h) = 50\sqrt{2(50+h)} + 1500 = 50\sqrt{100+2h} + 1500$$

$$C(50) = 2000$$

$$IROC = \lim_{h\to 0} \frac{C(50+h) - C(50)}{h}$$

$$= \lim_{h\to 0} \frac{50\sqrt{100+2h} + 1500 - 2000}{h}$$

$$= \lim_{h\to 0} \frac{50\sqrt{100+2h} - 500}{h} \times \frac{50\sqrt{} + 500}{50\sqrt{} + 500}$$

$$= \lim_{h\to 0} \frac{2500(100+2h) - 250000}{h(\qquad)}$$

$$= \lim_{h\to 0} \frac{\cancel{250000} + 5000\cancel{h} - \cancel{250000}}{\cancel{h}(50\sqrt{100+2h} + 500)}$$

$$= \frac{5000}{50\sqrt{100} + 500}$$

$$= \frac{5000}{1000}$$

$$= \$5/\text{ball}$$