Related Rates (Calculus Appendix – Page 565) Date: _______________

SOLVING RELATED RATES WITH MEASUREMENT RELATIONSHIPS

Ex1. Express the following statements in symbols.

a. The length of a rectangle is increasing at 4 cm/s.

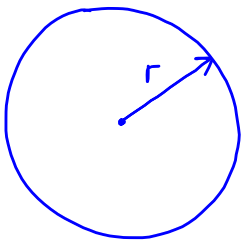
$$\frac{dl}{dt} = 4\text{ cm/s}$$

b. Water is added to a barrel at 250 cm^3/s.

$$\frac{dV}{dt} = 250\text{ cm}^3/\text{s}$$

Ex2. A ball is thrown into a pond and creates circular ripples that travel outward at 8 cm/s.

a. Determine the rate of increase of the circumference with respect to time at $t = 5$ s and $t = 10$ s.



Given:

$$\frac{dr}{dt} = 8\text{ cm/s}$$

Find

$$\frac{dC}{dt} \text{ when } t = 5\text{s}, 10\text{s}$$

Non-Related Rates Approach

$C = 2\pi r$, $r = 8t$ No!

$\therefore C = 2\pi(8t)$

$= 16\pi t$

$\therefore \frac{dC}{dt} = 16\pi$ cm/s

M1 "Chain Rule"

$$\overset{?}{\frac{dC}{dt}} = \overset{\checkmark}{\frac{dC}{dr}} \cdot \overset{\checkmark}{\frac{dr}{dt}}$$

$C = 2\pi r \quad \therefore \frac{dC}{dr} = 2\pi$

$= 2\pi \cdot 8$

$= 16\pi$ cm/s

M2 "Implicit Diff"

$$\frac{d}{dt}\left(C = 2\pi r\right)$$

$$\frac{dC}{dt} = 2\pi \cdot \frac{dr}{dt}$$

$= 2\pi \cdot 8$

b. Determine the rate of increase of the area enclosed by the ripples when the area is 576π cm^2.

Given: $\frac{dr}{dt} = 8$ cm/s Find: $\frac{dA}{dt}$ when $A = 576\pi$ cm^2

M1

$$\overset{?}{\frac{dA}{dt}} = \frac{dA}{dr} \cdot \overset{\checkmark}{\frac{dr}{dt}}$$

$A = \pi r^2 \quad \therefore \frac{dA}{dr} = 2\pi r$

$$\frac{dA}{dt} = (2\pi r)(8)$$

$$= 16\pi r$$

When $A = 576\pi$, $\pi r^2 = 576\pi$

$r^2 = 576$

$r = 24$ cm

$\therefore \frac{dA}{dt} = 16\pi(24)$

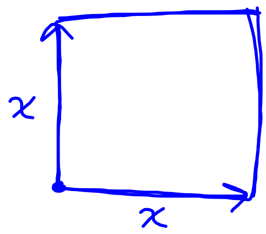
$= 384\pi$ cm^2/s.

M2 $\frac{d}{dt}\left(A = \pi r^2\right)$

$$\therefore \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

Ex3. A square is expanding so that its area increases at 10 cm²/min.

a. How fast is the side length increasing when the area is 52 cm²?



Given: $\frac{dA}{dt} = 10 \text{ cm}^2/\text{min}$

Find: $\frac{dx}{dt}$ when $A = 52 \text{ cm}^2$

M1

$$\overset{\checkmark}{\frac{dA}{dt}} = \frac{dA}{dx} \cdot \overset{?}{\frac{dx}{dt}}$$

$A = x^2$ $\quad \frac{dA}{dx} = \frac{2x}{1}$

$$10 = 2x \cdot \frac{dx}{dt}$$

$\frac{dx}{dA} = \frac{1}{2x}$

$$\frac{dx}{dt} = \frac{10}{2x} = \frac{5}{x}$$

When $A = 52$ $\quad x^2 = 52$

$x = \sqrt{52}$

$x = 2\sqrt{13}$

$\therefore \frac{dx}{dt} = \frac{5}{2\sqrt{13}}$ cm/min

$$\frac{dx}{dt} = \overset{?}{\frac{dx}{dA}} \cdot \overset{\checkmark}{\frac{dA}{dt}}$$

$$= \left(\frac{1}{2x}\right)(10)$$

$$= \frac{5}{x}$$

$x^2 = A$

$x = A^{\frac{1}{2}}$

$\frac{dx}{dA} = \frac{1}{2} A^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{A}}$

$= \frac{1}{2x}$

This is FRIGGIN' CRAY CRAY!!! ...

b. How fast is the perimeter increasing when the side length is 8 cm?

M2 $\quad \frac{d}{dt}\left(P = 4x\right)$

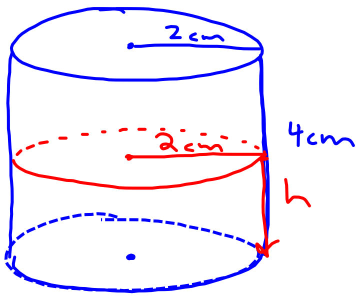
$$\frac{dP}{dt} = 4 \cdot \frac{dx}{dt} = 4\left(\frac{5}{x}\right) = \frac{20}{x}$$

So when $x = 8$,

$\frac{dP}{dt} = \frac{20}{8} = \frac{5}{2}$ cm/min.

Ex4. A cylindrical container has radius 2 cm and height 4 cm. Water drains from the base of the container at a constant rate of 6 cm³/s. How fast does the depth of the water decrease?

The radius of a partially filled cylinder will be the same as the filled cylinder.



Given: $\frac{dV}{dt} = -6 cm^3/s$

Find: $\frac{dh}{dt}$

$V = \pi r^2 h$

$V = \pi (2)^2 h$

$V = 4\pi h$

$\frac{dV}{dh} = 4\pi$

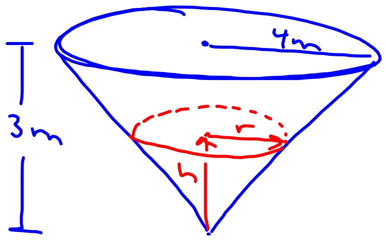
$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$-6 = 4\pi \cdot \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{-6}{4\pi} = \frac{-3}{2\pi}$ cm/s

Ex5. A conical reservoir is filling with water at a constant rate of 3 m³/min. The reservoir is 3 m deep and has a maximum diameter of 8 m. Determine the rate at which the depth of the water is increasing when the depth is 2 m.

The height and radius of a partially filled cone will be proportional to the height and radius of the filled cone.



$\frac{dV}{dt} = 3 m^3/min$

Find $\frac{dh}{dt}$ when $h = 2 m$.

✓ ?

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$3 = \frac{16\pi h^2}{9} \cdot \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{27}{16\pi h^2}$

$V = \frac{1}{3}\pi r^2 h$, $\frac{r}{h} = \frac{4}{3}$

$r = \frac{4}{3}h$

$V = \frac{1}{3}\pi \left(\frac{4}{3}h\right)^2 h$

$V = \frac{1}{3}\pi \cdot \frac{16}{9}h^2 \cdot h$

$V = \frac{16}{27}\pi \cdot h^3$

$\therefore \frac{dV}{dh} = \frac{16}{9}\pi h^2$

So when $h=2$, $\frac{dh}{dt} = \frac{27}{16\pi(4)} = \frac{27}{64\pi}$ m/min

Homefun: Page 569 #1abcd, 3→7, 12, 13[[illegible]: watch units!!], 15, 16, 20

Answers: [6a. units should be m/s]