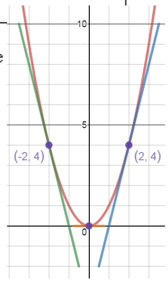
The Definition of the Derivative Function

The derivative of $f(x)$ with respect to x is the function $f'(x)$, where $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided this limit exists.

So, the derivative of f at the number a is $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, or $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

From Chapter 1, $f'(x)$ is therefore the instantaneous rate of $f(x)$ with respect to x.
That is, $f'(x)$ is a function which gives the slope of the tangent for any x on the function $f(x)$.



Ex1. Determine the derivative of $f(x) = x^2$ and use it to find the slopes of the tangents to the parabola when $x = -2$, $x = 0$, and $x = 2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h \to 0} (2x+h)$$
$$= 2x$$

$$\boxed{\begin{aligned} f'(2) &= 2(2) \\ &= 4 \\ f'(0) &= 2(0) \\ &= 0 \\ f'(-2) &= 2(-2) \\ &= -4 \end{aligned}}$$

$$f(x+h) = (x+h)^2$$
$$= x^2 + 2xh + h^2$$
$$f(x) = x^2$$

Ex2. Determine the derivative with respect to x of each of the following functions. What pattern do you see developing? Use the pattern to predict the derivative of $p(x) = x^{39}$.

a. $f(x) = x^3$ b. $g(x) = x^4$ c. $h(x) = x^5$

a) $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$
$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$
$$= 3x^2$$

$$f(x+h)$$
$$= (x+h)^3$$
$$= x^3 + 3x^2h + 3xh^2 + h^3$$
$$f(x) = x^3$$

b) $g(x) = x^4$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$
$$= \lim_{h \to 0} \frac{4x^3h + h^2(\sim)}{h}$$
$$= \lim_{h \to 0} [4x^3 + h(\sim)]$$
$$= 4x^3$$

$$g(x+h)$$
$$= (x+h)^4$$
$$= x^4 + 4x^3h + h^2(\sim)$$
$$g(x) = x^4$$

(c) $h(x) = x^5$

$h'(x) = 5x^4$

$p(x) = x^{39}$

$p'(x) = 39x^{38}$

Ex3. Determine the derivative from first principles for each of the following functions.

a. $f(t) = \sqrt{2t-1}$

b. $g(x) = \frac{1}{x}$

$f(t+h) = \sqrt{2(t+h)-1}$

$= \sqrt{2t+2h-1}$

$f(t) = \sqrt{2t-1}$

a) $f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$

$= \lim_{h \to 0} \frac{\sqrt{2t+2h-1} - \sqrt{2t-1}}{h} \cdot \frac{\sqrt{\ } + \sqrt{\ }}{\sqrt{\ } + \sqrt{\ }}$

$= \lim_{h \to 0} \frac{2t+2h-1-2t+1}{h(\qquad)}$

$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2t+2h-1} + \sqrt{2t-1}\right)}$

$= \frac{2}{\sqrt{2t-1} + \sqrt{2t-1}}$

$= \frac{2}{2\sqrt{2t-1}}$

$= \frac{1}{\sqrt{2t-1}}$

$g(x) = \frac{1}{x}$

$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$g'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$g(x) = \lim_{h \to 0} \frac{1}{\cancel{h}} \cdot \frac{\cancel{x} - \cancel{x} - \cancel{h}}{x(x+h)}$

$g'(x) = \lim_{h \to 0} \frac{-1}{x(x+h)}$

So $g'(x) = \frac{-1}{x^2}$

Ex4. Determine an equation of the normal to the graph of $g(x) = \frac{1}{x}$ at $x = 2$.

Note: The **NORMAL** to the graph of f at point P is the line that is **PERPENDICULAR TO THE TANGENT** at P.

$g(2) = \frac{1}{2} \longrightarrow$ So $P_0 = (2, \frac{1}{2})$

$g'(2) = \frac{-1}{2^2} \longrightarrow$ Since $m = -\frac{1}{4}$

$= -\frac{1}{4}$ $\quad \therefore n = 4$

Thus, $\frac{x-2}{1} = \frac{y - \frac{1}{2}}{4}$

$4x - 8 = y - \frac{1}{2}$

$8x - 16 = 2y - 1$

$8x - 2y - 15 = 0$ ITRE

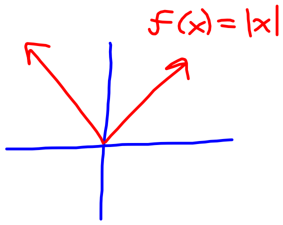
Ex5. Use the definition of a derivative to show that the absolute value function $f(x) = |x|$ is not differentiable at $x = 0$.

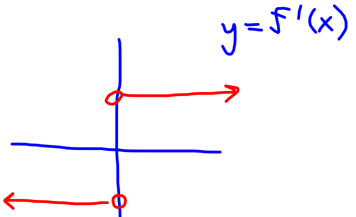
$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

Well... $\displaystyle\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = +1$ and $\displaystyle\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$

Thus $\displaystyle\lim_{h\to 0} \frac{|h|}{h}$ DNE i.e. $f'(0)$ DNE !!

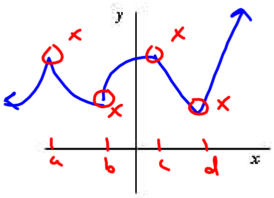




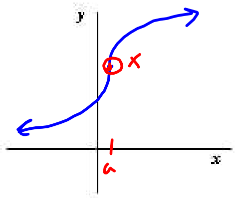
$f(0)=0$ $\quad \displaystyle\lim_{x\to 0} f(x) = 0$ $\qquad f'(0)$ DNE b/c $\displaystyle\lim_{h\to 0} \frac{|h|}{h}$ DNE.

The Existence of Derivatives

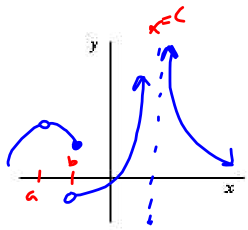
A function f is said to be differentiable at a if $f'(a)$ exists. At points where f is not differentiable, we say that the derivative does not exist. Three common ways for a derivative to fail to exist are shown.



A Cusp or Point



Vertical Tangent



Discontinuity

Ex6. Answer true or false.

a. If a function is continuous at a point, then it is differentiable at this point. F

b. If a function is not differentiable at a point, then it is not continuous at this point. F

c. If a function is not continuous at a point, then it is also not differentiable at this point. T

d. If a function is differentiable at a point, then it is also continuous at this point. T

Other notations to note: $f'(x) = y' = \dfrac{dy}{dx} = \dfrac{d}{dx}f(x)$. For example, $\dfrac{d}{dx}(x^2) = 2x$.

Homefun: Page 73 # 1, 8, 6acd, 7bc, 14, 13, 15, 16(consider graphing the function and corresponding tangent too), 17