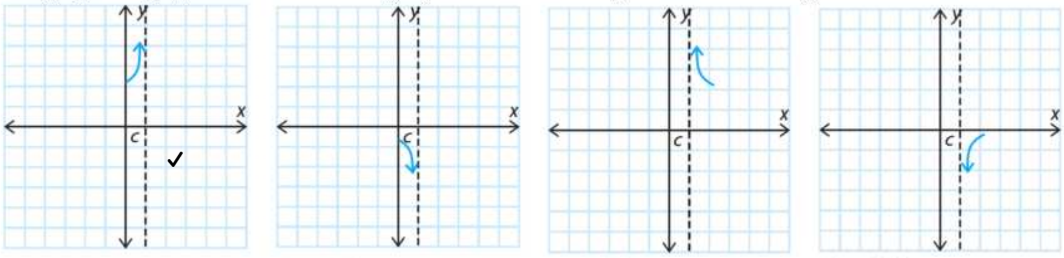
4.3: <u>Vertical, Horizontal and Oblique Asymptotes</u> but focusing on rational functions $y = \frac{g(x)}{h(x)}$ Date: ______________

Algorithm for Curve Sketching (so far...)

1. <u>Check for Asymptotes and Discontinuities</u>

i. Are there any vertical asymptotes?

Note: The graph of $f(x)$ has a vertical asymptote $x = c$ if any of the following is true:



$\lim_{x\to c^-} f(x) = +\infty$ ✓ $\lim_{x\to c^-} f(x) = -\infty$ $\lim_{x\to c^+} f(x) = +\infty$ $\lim_{x\to c^+} f(x) = -\infty$

ii. Is there a horizontal or oblique asymptote?

Test end behaviour by considering $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

IF the limit exists $\left[\text{ i.e. } \lim_{x\to+\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L\right]$,

THEN there is a horizontal asymptote at $y = L$.

For rational functions $f(x) = \frac{g(x)}{h(x)}$,

Ex: $y = \frac{10x^3+3x^2+7000}{x^4+1}$

HA @ $y=0$

- there will be a horizontal asymptote at $y = 0$ when $deg[g(x)] < deg[h(x)]$
- there will be a horizontal asymptote at $y = \frac{a}{b}$ when $deg[g(x)] = deg[h(x)]$
  - where a and b are the coefficients of the highest degree terms of g and h respectively
  - we can find this limit by factoring the highest degree terms from g and h

$y = \frac{10x^3+3x^2+7000}{2x^3+1}$

HA @ $y=5$

- there will be an oblique asymptote when $deg[g(x)] = deg[h(x)] + 1$

  We can find this oblique asymptote by dividing by $h(x)$ (long division or synthetic division)

  so that $f(x) = \frac{g(x)}{h(x)} = (mx + b) + \frac{r(x)}{h(x)}$ where $r(x)$ is the remainder.

  Then $y = mx + b$ is the equation of the oblique asymptote.

$y = \frac{10x^3+3x^2+7000}{2x^2+1}$

OA @ $y = 5x + ?$

iii. Are there any holes? Can you tell an asymptote from a hole in the graph?

2. <u>First Derivative Analysis</u>

Determine when the function is increasing, decreasing, horizontal, vertical, or a cusp. Identify all local maximums and minimums.

3. <u>Find Points and Graph</u>

i. Find and use all critical points.

ii. Find and use intercepts if the intercepts are easy to find and/or if they're necessary.

iii. Find and use any other points if necessary.

Ex1. Sketch the graph of each function.

a. $f(x) = \dfrac{3x-1}{x+5}$

VA @ $x=-5$

HA @ $y=3$

x	y
0	$-\frac{1}{5}$
$\frac{1}{3}$	0

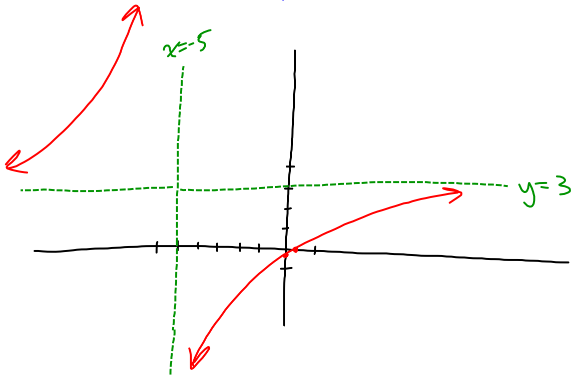
$$f'(x) = \frac{3(x+5)-(3x-1)(1)}{(x+5)^2}$$

$$= \frac{16}{(x+5)^2}$$

$f'(x) > 0$

$\therefore f(x)$ is always increasing

$\therefore$ NO C.P.'s



b. $f(x) = \dfrac{x^2+3x-2}{(x-1)^2}$

$$= \frac{x^2+3x-2}{x^2-2x+1}$$

HA @ $y=1$

VA @ $x=1$

x	y
0	-2
?	0

later??

$$f'(x) = \frac{(2x+3)(x-1)^2-(x^2+3x-2)(2(x-1)'(1))}{(x-1)^4}$$

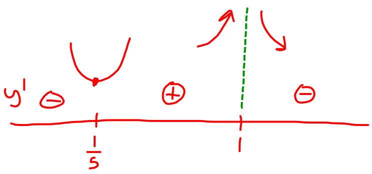
$$= \frac{(2x+3)(x-1)-2(x^2+3x-2)}{(x-1)^3}$$

$$= \frac{2x^2+x-3-2x^2-6x+4}{*}$$

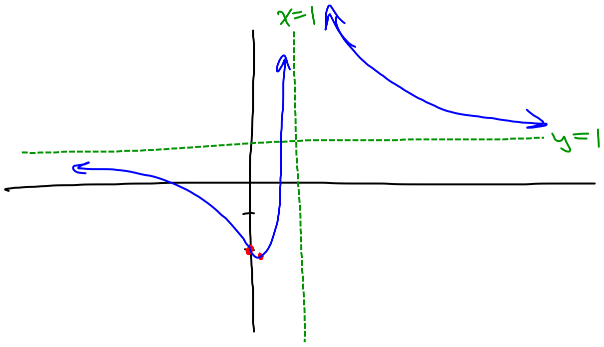
$$= \frac{-5x+1}{(x-1)^3}$$

$$y' = \frac{-(5x-1)}{(x-1)^3}$$

C.N.'s: $x = \frac{1}{5}$



min @ $\sim(0.2, -2.1)$



c. $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$

$$= \frac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

$$= \frac{x+3}{x+2}, \quad x \neq 2$$

HA @ $y=1$

VA @ $x=-2$

HOLE @ $x=2$

x	y
0	$\frac{3}{2}$
-3	0

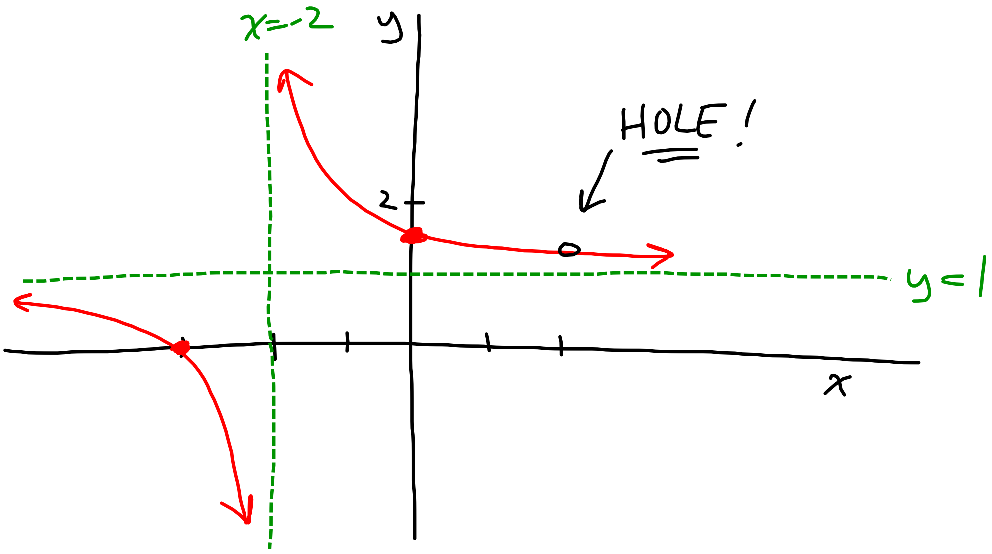
$$f'(x) = \frac{(1)(x+2)-(x+3)(1)}{(x+2)^2}, \quad x \neq 2$$

$$= \frac{-1}{(x+2)^2}, \quad x \neq 2$$

$f'(x) < 0$

$\therefore f(x)$ is decreasing

i.e. NO c.P.'s !!



d. $f(x) = \dfrac{2x^2 - 3x + 2}{x - 2} = 2x + 1 + \dfrac{4}{x-2}$

2	2	−3	2
		4	2
	2	1	4

$\therefore$ OA @ $y = 2x + 1$

VA @ $x = 2$

x	y
0	−1
?	0

later?

$$f'(x) = \frac{(4x-3)(x-2) - (2x^2-3x+2)(1)}{(x-2)^2}$$

$$= \frac{4x^2 - 11x + 6 - 2x^2 + 3x - 2}{(x-2)^2}$$

$$= \frac{2x^2 - 8x + 4}{(x-2)^2}$$

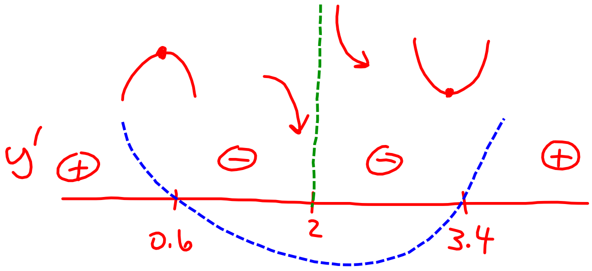
$$f'(x) = \frac{2(x^2 - 4x + 2)}{(x-2)^2}$$

$f'(x) = 0$ when

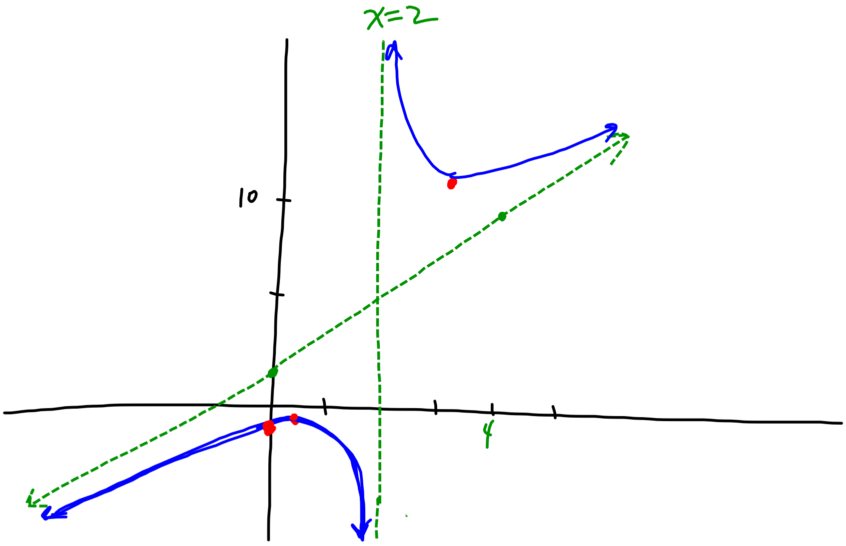
$$x = \frac{4 \pm \sqrt{8}}{2}$$

$$= 2 \pm \sqrt{2}$$

$$\doteq 0.6,\ 3.4$$



x	y	
0.6	−0.7	max
3.4	10.7	min



Ex2. Find the equation of the oblique asymptote.

$$f(x) = \frac{2x^3 + x^2 + 2}{x^2 - 4} = 2x + 1 + \frac{8x+6}{x^2-4}$$

As $x \to \infty$

$f(x) \to (2x+1)^+$

$x \to -\infty$

$f(x) \to (2x+1)^-$

$$\begin{array}{r|l}
 & 2x + 1 \\
\hline
x^2 + 0x - 4 & 2x^3 + x^2 + 0x + 2 \\
 & \underline{2x^3 + 0x^2 - 8x} \quad \downarrow \\
 & \qquad x^2 + 8x + 2 \\
 & \qquad \underline{x^2 + 0x - 4} \\
 & \qquad\qquad 8x + 6
\end{array}$$

Homefun: Page 193 # 1, (5, 7[Ans: 7a) $y = 3x + 7$], 8)(horizontal and oblique asymptote practice, if needed), (6, 10)(make a rough sketch of graph of each function), 14