Practice Ch 5 Test

Q5 $y = \cos x$ $\therefore$ $y' = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$

$= \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h}$

$= \lim_{h \to 0} \left[\cos x \left[\frac{\cosh - 1}{h} \right] - \sin x \sinh \right]$

$= (\cos x)(0) - (\sin x)(1)$

$= -\sin x$

Ch 5 Test

1. a) $f(x) = e^x$
$f'(x) = e^x$

b) $y = \log_2 x$
$y' = \frac{1}{x \ln 2}$

c) $s(t) = \sin t$
$s'(t) = \cos t$

d) $f(x) = \tan x$ $\therefore$ $f'(x) = \sec^2 x$

2. a) $y = e^{x^2 - 2x}$

$y = e^{x^2-2x} \cdot (2x - 2)$
$= 2(x-1)e^{x^2-2x}$

b) $f(x) = \ln(\cos x)$

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$

c) $s(t) = \frac{e^t}{\sin t}$

$s'(t) = \frac{e^t(\sin t) - e^t(\cos t)}{\sin^2 t}$

$= \frac{e^t(\sin t - \cos t)}{\sin^2 t}$

3. $y = -2\sin x$, $x \in [-\pi, \pi]$

$y' = -2\cos x$

$y' = 0$ when $x = -\frac{\pi}{2}, \frac{\pi}{2}$

$\therefore$ max $= 2$
min $= -2$

check

when $x = \frac{-\pi}{2}$, $y = -2\sin(\frac{-\pi}{2})$
$= -2(-1)$
$= 2$

when $x = \frac{\pi}{2}$, $y = -2\sin(\frac{\pi}{2})$
$= -2(1)$
$= -2$

Q4 $f(x) = \dfrac{(x+4)^4\sqrt{2x+3}}{4x^3-8x}$

a) $\ln f(x) = \ln(x+4)^4 + \ln(2x+3)^{\frac{1}{2}} - \ln(4x^3-8x)$

$= 4\ln(x+4) + \frac{1}{2}\ln(2x+3) - \ln[4x(x^2-2)]$

$= 4\ln(x+4) + \frac{1}{2}\ln(2x+3) - \ln 4x + \ln(x^2-2)$

b) $\dfrac{f'(x)}{f(x)} = \dfrac{4}{x+4} + \dfrac{2}{2(2x+3)} - \dfrac{4}{x} + \dfrac{2x}{x^2-2}$

$f'(x) = f(x)\cdot\left[\dfrac{4}{x+4} + \dfrac{1}{2x+3} - \dfrac{4}{x} + \dfrac{2x}{x^2-2}\right]$

c) logarithmic differentiation

Q5

$f(x) = e^x$

$f'(x) = \lim\limits_{h\to 0} \dfrac{e^{x+h} - e^x}{h}$

$= \lim\limits_{h\to 0} e^x\left[\dfrac{e^h-1}{h}\right]$

$= e^x(1)$

$= e^x$

$g(x) = \ln x$

$g'(x) = \lim\limits_{h\to 0} \dfrac{\ln(x+h) - \ln x}{h}$

$= \lim\limits_{h\to 0} \dfrac{1}{h}\cdot\ln\left(\dfrac{x+h}{x}\right)$

$= \lim\limits_{h\to 0} \dfrac{1}{h}\cdot\ln\left(1+\dfrac{h}{x}\right)$

$= \lim\limits_{h\to 0} \dfrac{1}{xt}\cdot\ln\left(1+\dfrac{xt}{x}\right)$

$= \lim\limits_{h\to 0} \dfrac{1}{x}\left[\dfrac{1}{t}\ln(1+t)\right]$

$= \lim\limits_{h\to 0} \dfrac{1}{x}\left[\ln(1+t)^{\frac{1}{t}}\right]$

$= \lim\limits_{h\to 0} \dfrac{1}{x}[\ln(e)]$

$= \dfrac{1}{x}(1)$

$= \dfrac{1}{x}$

Note:
Let $t = \dfrac{h}{x}$ $(x>0)$

$\therefore xt = h$

as $h\to 0$
$t\to 0$

Recall,

$\lim\limits_{t\to 0}(1+t)^{\frac{1}{t}} = e$

Q5

$h(x) = \sin x$

$$h'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\sin x \left[\frac{\cosh - 1}{h} \right] + \cos x \left[\frac{\sinh}{h} \right] \right]$$

$$h'(x) = (\sin x)(0) + (\cos x)(1)$$

$$\underline{\underline{h'(x) = \cos x}}$$

Q6. $f(x) = \frac{\ln(x-1)}{e^2 x}$

a) $$f'(x) = \frac{\frac{1}{x-1} \cdot e^2 x - \ln(x-1)\left[e^2\right]}{e^4 x^2}$$

$$= \frac{e^2\left[x - \ln(x-1)\right]}{(x-1)\, e^4 x^2}$$

So, $$f'(2) = \frac{e^2\left[2 - \ln 1\right]}{(1)e^4(2)^2}$$

$$= \frac{2e^2}{4e^4}$$

$$= \frac{1}{2e^2}$$

$$f(2) = \frac{\ln(1)}{e^2(2)}$$

$$= 0$$

$m = \frac{1}{2e^2}$; $P_0 = (2,0)$

$$\frac{x-2}{2e^2} = \frac{y-0}{1}$$

$$\therefore y = \frac{1}{2e^2} x - \frac{1}{e^2}$$

(b)

Q7 $s(t) = \cos t \sin t$, $t \geq 0$

a) $s'(t) = -\sin^2 t + \cos^2 t$

$= -\sin^2 t + 1 - \sin^2 t$

$= 1 - 2\sin^2 t$

$s'(t) = 0$ when

$\sin^2 t = \frac{1}{2}$

$\sin t = \frac{1}{\sqrt{2}}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}$

Check

$s(\frac{\pi}{4}) = \cos\frac{\pi}{4} \sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$

$= \frac{1}{2}$

$s(\frac{3\pi}{4}) = \cos\frac{3\pi}{4} \sin\frac{3\pi}{4}$

$= \frac{-1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$

$= \frac{-1}{2}$

$\therefore$ max. distance away from the fixed position is $+\frac{1}{2}$ cm

Q7 b) $s'(t) = \cos^2 t - \sin^2 t$

$s'(0) = \cos^2 0 - \sin^2 0$
$= 1$ cm/s

$\therefore$ the initial velocity is 1 cm/s.

c) $a(t) = s''(t)$
$= -4(\sin t)' \cos t$
$= -4 \sin t \cos t$

Note: $s'(t) = 1 - 2\sin^2 t$
$= 1 - 2(\sin t)^2$

IF $a(t) = 0$ $\therefore$ $\sin t = 0$ or $\cos t = 0$
$t = 0, \pi$ $\quad$ $t = \frac{\pi}{2}$

So, the required velocity is $s'(0) = 1$ cm/s.

Q8 $V(t) = e(2^{-\frac{t}{12}}) + 1.6$

$V(6) = e(2^{-\frac{1}{2}}) + 1.6$
$= 3.5221155...$

So, the value of the collection in 2008 is \$352.21

$V'(t) = e\left[\ln 2 \cdot 2^{-\frac{t}{12}} \cdot -\frac{1}{12}\right]$

So, $V'(6) = -\frac{e \ln 2 \cdot 2^{-\frac{1}{2}}}{12}$
$= -0.078507...$

So, the IROC of the value of the collection in 2008 is decreasing by \$7.85/yr.

Q9 $y = A\cos Kt + B\sin Kt$

$y' = -AK\sin Kt + BK\cos Kt$

$y'' = -AK^2\cos Kt - BK^2\sin Kt$

So, $y'' + K^2 y = -AK^2\cos Kt - BK^2\sin Kt + AK^2\cos Kt + BK^2\sin Kt$
$= 0$ ■