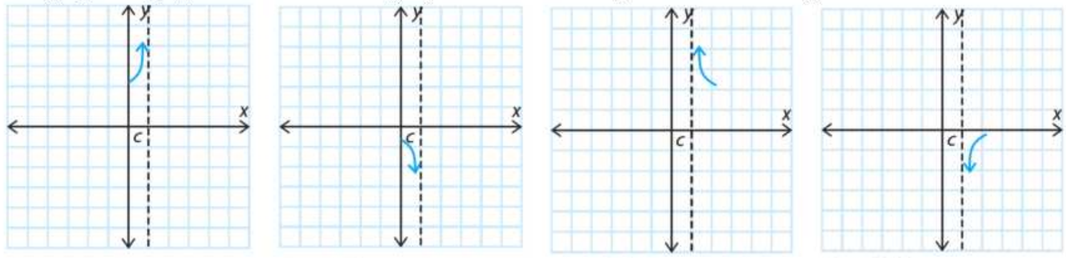
4.3: <u>Vertical, Horizontal and Oblique Asymptotes</u> but focusing on rational functions $y = \frac{g(x)}{h(x)}$ Date: ________________

Algorithm for Curve Sketching (so far...)

1. <u>Check for Asymptotes and Discontinuities</u>

i. Are there any vertical asymptotes?

Note: The graph of $f(x)$ has a vertical asymptote $x = c$ if any of the following is true:



$\lim_{x \to c^-} f(x) = +\infty$ $\lim_{x \to c^-} f(x) = -\infty$ $\lim_{x \to c^+} f(x) = +\infty$ $\lim_{x \to c^+} f(x) = -\infty$

ii. Is there a horizontal or oblique asymptote?

Test end behaviour by considering $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

IF the limit exists $\left[\text{ i.e. } \lim_{x \to +\infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L \right]$,

THEN there is a horizontal asymptote at $y = L$.

Ex: $y = \frac{10x^3+3x^2+7000}{x^4+1}$

For rational functions $f(x) = \frac{g(x)}{h(x)}$,

HA @ $y = 0$

- there will be a horizontal asymptote at $y = 0$ when $deg[g(x)]$ $<$ $deg[h(x)]$ --------------
- there will be a horizontal asymptote at $y = \frac{a}{b}$ when $deg[g(x)]$ $=$ $deg[h(x)]$ $y = \frac{10x^3+3x^2+7000}{2x^3+1}$
  - where a and b are the coefficients of the highest degree terms of g and h respectively
  - we can find this limit by factoring the highest degree terms from g and h
- there will be an oblique asymptote when $deg[g(x)]$ $=$ $deg[h(x)] + 1$ HA @ $y = 5$

  We can find this oblique asymptote by dividing by $h(x)$ (long division or synthetic division)

  so that $f(x) = \frac{g(x)}{h(x)} = (mx + b) + \frac{r(x)}{h(x)}$ where $r(x)$ is the remainder.

  $y = \frac{10x^3+3x^2+7000}{2x^2+1}$

  Then $y = mx + b$ is the equation of the oblique asymptote.

iii. Are there any holes? Can you tell an asymptote from a hole in the graph?

OA @ $y = 5x + ?$

2. <u>First Derivative Analysis</u>

Determine when the function is increasing, decreasing, horizontal, vertical, or a cusp. Identify all local maximums and minimums.

3. <u>Find Points and Graph</u>

i. Find and use all critical points.

ii. Find and use intercepts if the intercepts are easy to find and/or if they're necessary.

iii. Find and use any other points if necessary.

Ex1. Sketch the graph of each function.

a. $f(x) = \dfrac{3x - 1}{x + 5}$

VA @ $x = -5$

HA @ $y = 3$

x	y
0	$-\frac{1}{5}$
$\frac{1}{3}$	0

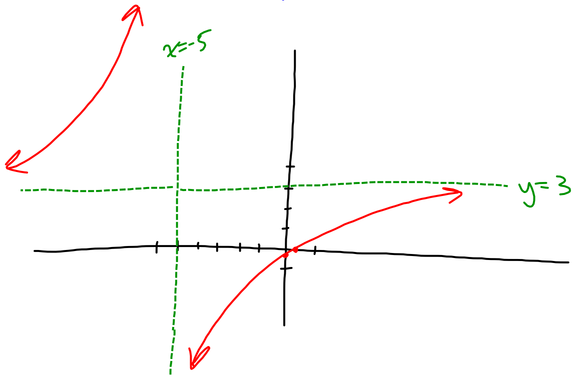
$$f'(x) = \frac{3(x+5) - (3x-1)(1)}{(x+5)^2}$$

$$= \frac{16}{(x+5)^2}$$

$f'(x) > 0$

$\therefore f(x)$ is always increasing

$\therefore$ NO C.P.'s



b. $f(x) = \dfrac{x^2 + 3x - 2}{(x - 1)^2}$

$$= \frac{x^2+3x-2}{x^2-2x+1}$$

HA @ $y = 1$

VA @ $x = 1$

x	y
0	-2
?	0

later ??

$$f'(x) = \frac{(2x+3)(x-1)^2 - (x^2+3x-2)(2(x-1)'(1))}{(x-1)^4}$$

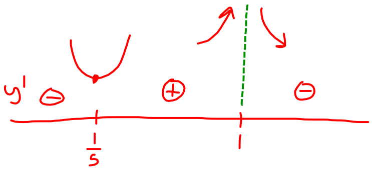
$$= \frac{(2x+3)(x-1) - 2(x^2+3x-2)}{(x-1)^3}$$

$$= \frac{2x^2+x-3-2x^2-6x+4}{*}$$

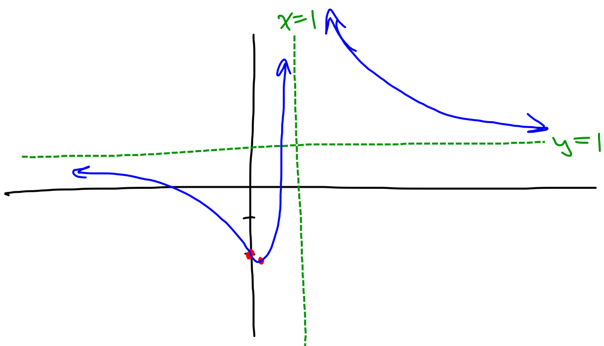
$$= \frac{-5x+1}{(x-1)^3}$$

$$y' = \frac{-(5x-1)}{(x-1)^3}$$

C.N.'s : $x = \frac{1}{5}$



min @ ~(0.2, −2.1)



c. $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$

$\therefore$

$$= \frac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

$$= \frac{x+3}{x+2}, \quad x \neq 2$$

HA @ $y = 1$

VA @ $x = -2$

HOLE @ $x = 2$

x	y
0	$\frac{3}{2}$
-3	0

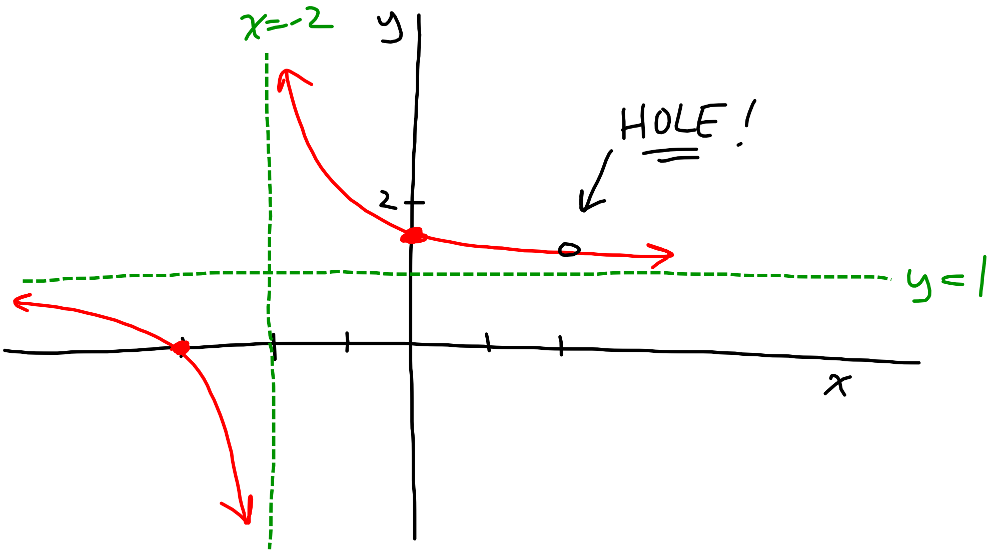
$$f'(x) = \frac{(1)(x+2) - (x+3)(1)}{(x+2)^2}, \quad x \neq 2$$

$$= \frac{-1}{(x+2)^2}, \quad x \neq 2$$

$f'(x) < 0$

$\therefore$ $f(x)$ is decreasing

i.e. NO C.P.'s !!



d. $f(x) = \dfrac{2x^2 - 3x + 2}{x - 2} = 2x+1 + \dfrac{4}{x-2}$

$$\begin{array}{r|rrr} 2 & 2 & -3 & 2 \\ & & 4 & 2 \\ \hline & 2 & 1 & 4 \end{array}$$

$\therefore$ OA @ $y = 2x+1$

VA @ $x = 2$

x	y
0	-1
?	0 later?

$$f'(x) = \frac{(4x-3)(x-2) - (2x^2-3x+2)(1)}{(x-2)^2}$$

$$= \frac{4x^2 - 11x + 6 - 2x^2 + 3x - 2}{(x-2)^2}$$

$$= \frac{2x^2 - 8x + 4}{(x-2)^2}$$

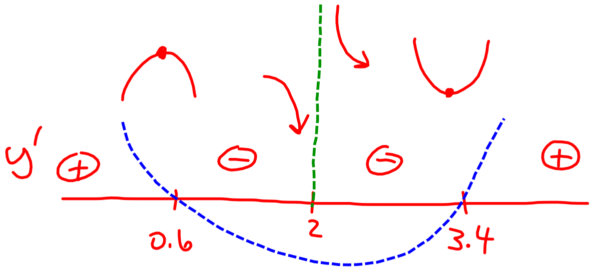
$$f'(x) = \frac{2(x^2-4x+2)}{(x-2)^2}$$

$f'(x) = 0$ when

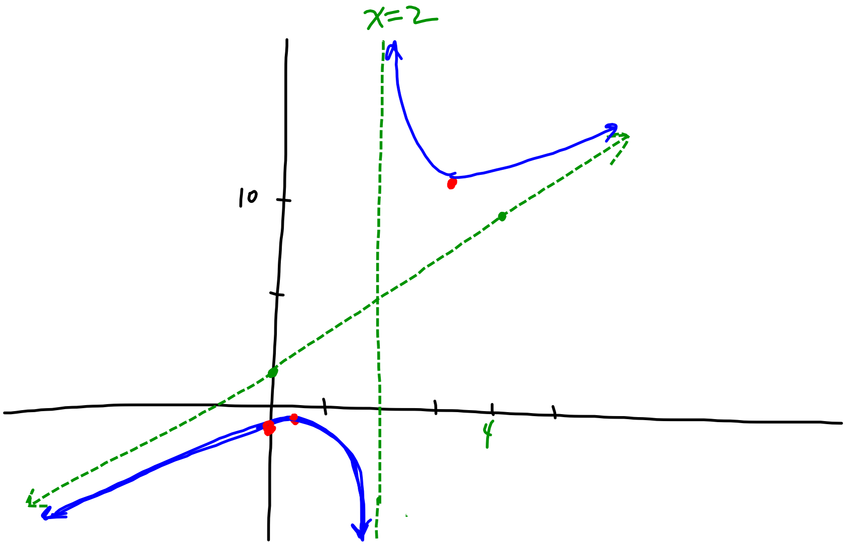
$$x = \frac{4 \pm \sqrt{8}}{2}$$

$$= 2 \pm \sqrt{2}$$

$$\doteq 0.6,\ 3.4$$



x	y	
0.6	-0.7	max
3.4	10.7	min



Ex2. Find the equation of the oblique asymptote.

$$f(x) = \frac{2x^3 + x^2 + 2}{x^2 - 4} = 2x + 1 + \frac{8x+6}{x^2-4}$$

As $x \to \infty$

$f(x) \to (2x+1)^+$

$x \to -\infty$

$f(x) \to (2x+1)^-$

$$\begin{array}{r|l} & 2x + 1 \\ \hline x^2 + 0x - 4 & 2x^3 + x^2 + 0x + 2 \\ & \underline{2x^3 + 0x^2 - 8x} \downarrow \\ & \quad x^2 + 8x + 2 \\ & \quad \underline{x^2 + 0x - 4} \\ & \qquad 8x + 6 \end{array}$$

Homefun: Page 193 # 1, (5, 7[Ans: 7a) $y = 3x + 7$], 8)(horizontal and oblique asymptote practice, if needed), (6, 10)(make a rough sketch of graph of each function), 14