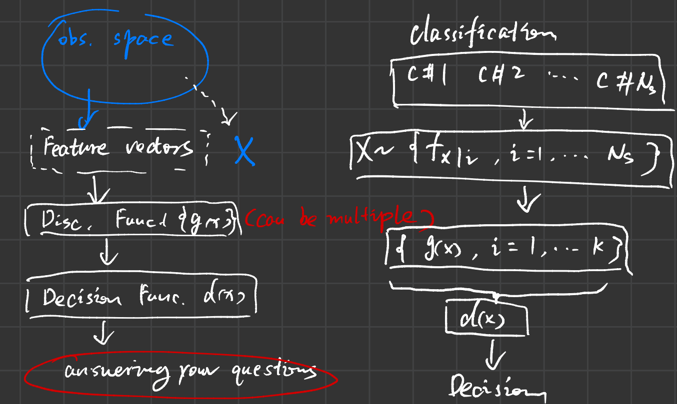
Tutorial 5



d f

- New instance of g(x), HMM
  i.e. $p(\underline{x} \mid \lambda)$, where $\lambda = \{\{q, A\}, B\}$
- Forward/Backward, Viterbi
  inference on $p(\underline{x} \mid \lambda)$, HMM

P5.1 page 121

$$A = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \rightleftharpoons \text{draw state graph}$$

a). about definition HMMs

By obs. A, state i can not transfere to state j for $i > j$. $i, j = \{1, 2, 3\}$

Not ergodic

b). In your project Ass 3, you need implement Forward.

$\alpha_{j,t} = P[x_1 \cdots x_t, S_t = j \mid \lambda]$ $\qquad \alpha_{j,t}^{tmp}$

$\hat{\alpha}_{j,t} = P[S_t = j \mid x_1 \cdots x_t, \lambda]$ $\qquad = P[X_t = x_t, S_t = j \mid x_1 \cdots x_{t-1}, \lambda]$

$c_t = P[x_t \mid x_1 \cdots x_{t-1}, \lambda]$

For $t=1$, initialization step

$$\begin{cases} \alpha_{j,1} = P[Z_t = z_1, S_1 = j \mid \lambda] = q_j b_j(z_1) \\ c_1 = \sum_{k=1}^{N} \alpha_{k,1}, \quad \hat{\alpha}_{j,1} = \dfrac{\alpha_{j,1}}{c_1} \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_{1,1} = q_1 b_1(1) = 1 \\ \alpha_{2,1} = q_2 b_2(1) = 0 \\ \alpha_{3,1} = q_3 b_3(1) = 0 \end{cases} \quad \begin{array}{l} c_1 = \sum_{k=1}^{3} \alpha_{k,1} \\ = 1 \end{array} \quad \begin{array}{l} \hat{\alpha}_{1,1} = P[S_1 = 1 \mid z_1, \lambda] = 1 \\ \hat{\alpha}_{2,1} = \cdots = 0 \\ \hat{\alpha}_{3,1} = \quad = 0 \end{array}$$

For the $t=1$, $\alpha_{j,1}^{tmp} = \alpha_{j,1}$

Loop to compute all $t \geq 2$

From this step onwards,

For $t=2$

$$\alpha_{1,2}^{tmp} = b_1(2) \sum_{i=1}^{N} \hat{\alpha}_{i,1} a_{i1} = 0 \qquad c_2 = \sum_{k=1}^{N} \alpha_{k,2}^{tmp} = 0.35$$

$$\alpha_{2,2}^{tmp} = b_2(2) \sum_{i=1}^{N} \hat{\alpha}_{i,1} a_{i2} = 0.35 \qquad \hat{\alpha}_{1,2} = \frac{0}{0.35} = 0$$

$$\alpha_{3,2}^{tmp} = b_3(2) \sum_{i=1}^{N} \hat{\alpha}_{i,1} a_{i3} = 0 \qquad \hat{\alpha}_{2,2} = 1, \quad \hat{\alpha}_{3,2} = 0$$

$t=3$,

$\alpha_{1,3}^{tmp} = b_1(4)\sum_{i=1}^{N}\hat{\alpha}_{i,2}a_{i,3} = 0$, $\alpha_{2,3}^{tmp} = b_2(4)\sum_{i=1}^{N}\hat{\alpha}_{i,2}a_{i,3} = 0.05$

$\alpha_{3,3}^{tmp} = b_3(4)\sum_{i=1}^{N}\hat{\alpha}_{i,2}\,a_{i,3} = 0.3$,

$C_3 = \sum_{k=1}^{3}\alpha_{k,3}^{tmp} = 0.35$

$\hat{\alpha}_{1,3} = 0$, $\hat{\alpha}_{2,3} = \frac{1}{7}$ $\hat{\alpha}_{3,3} = \frac{6}{7}$

$t=4$,

$\alpha_{1,4}^{tmp} = b_1(4)\sum_{i=1}^{3}\hat{\alpha}_{i,2}a_{i1} = 0$, $\alpha_{2,4}^{tmp} = b_2(4)\sum_{i=1}^{3}\hat{\alpha}_{i,3}\,a_{i2} = \frac{1}{140}$

$\alpha_{3,4}^{tmp} = b_3(4)\sum_{i=1}^{3}\hat{\alpha}_{i,3}\,a_{i2} = \frac{78}{140}$

$C_4 = \frac{79}{140}$

$\hat{\alpha}_{1,4} = 0$, $\hat{\alpha}_{2,4} = \frac{1}{79}$, $\hat{\alpha}_{3,4} = \frac{78}{79}$

$t=5$

$\alpha_{1,5}^{tmp} = b_1(1)\sum_{i=1}^{3}\hat{\alpha}_{i,4}a_{i1} = 0$, $\alpha_{2,5}^{tmp} = 0$, $\alpha_{3,5}^{tmp} \simeq 0.0994$

$C_5 = 0.0994$

$\hat{\alpha}_{1,5} = \hat{\alpha}_{2,5} = 0$, $\hat{\alpha}_{3,5} = 1$

We summarize the results into the table below:

t	1	2	3	4	5
$\hat{\alpha}_{1,t}$	1	0	0	0	0
$\hat{\alpha}_{2,t}$	0	1	1/7	1/79	0
$\hat{\alpha}_{3,t}$	0	0	6/7	78/79	1
C_t	1	0.35	0.35	79/140	0.0994

Then the answer to (b) is

$P(\underline{Z} = \underline{z} \mid x) = \prod_{i=1}^{5} C_i \approx 0.0069$

(c) $P[S_1 = i_1, \cdots S_T = i_T \mid x_1 \cdots x_T\, X]$

$P(A)$, $A \in \{1, 2, 3, 4\}$

toy random variable A is corresponding $\underline{S}$

(c). number of events $\underline{S}$

(d). $\underset{S_1 \cdots S_T}{\text{argmax}}$

enum. $S_1 \cdots S_T$

① no transition at all. 1

② transfer only once to state 2 at $t=5$, 1

③ transition to state 2 at $t=4$; then at $t=5$, either stay at 2, or go to 3, 2

④ . . . .

. . .

$1+1+2+3+4 = 1$

(d) Viterbi algorithm

① $t=1$. initialization step

$$x_{j,1} = q_j\, b_j(z_1), \quad z_1 = 1$$

$$\begin{cases} x_{1,1} = q_1 b_1(z_1) = 1\cdot 1 = 1 \\ x_{2,1} = q_2 b_2(z_1) = 0 \\ x_{3,1} = q_3 b_3(z_1) = 0 \end{cases}$$

For $t \geq 2$, $\begin{cases} x_{j,t} = b_j(z_t) \max\limits_i x_{i,t-1} a_{ij} \\ \xi_{j,t} = \arg\max\limits_i x_{i,t-1} a_{ij} \end{cases}$

again, for $t \geq 2$, you can use loop implentation to do all the rest computations.

② $t=2$

$$x_{2,2} = b_2(z_2) \max_i x_{i,1} a_{i2} = 0.5 \cdot \max\{1\cdot 0.7, 0, 0\} = 0.35$$

$$x_{1,2} = b_1(z_2) \max_i x_{i,1} a_{i1} = 0$$

$$x_{3,2} = b_3(z_2) \max_i x_{i,1} a_{i3} = 0$$

$$\xi_{1,2} = \arg\max_i x_{i,1} a_{i1} = 1$$

$$\xi_{2,2} = \arg\max_i x_{i,1} a_{i2} = 1$$

$$\xi_{3,2} = \arg\max_i x_{i,1} a_{i3} = 0$$

③ $t=3$

$x_{1,3} = 0$, $x_{2,3} = 0.0175$, $x_{3,3} = 0.105$

$\xi_{2,3} = \xi_{3,3} = 2$

④ $t=4$,

$x_{2,4} = 8.75\times 10^{-4}$, $x_{3,4} = 0.063$, $x_{1,4} = 0$

$\xi_{2,4} = 2$, $\xi_{3,4} = 3$

⑤ $t=5$

$x_{1,5} = 0$, $x_{2,5} = 0$, $x_{3,5} = 0.0063$,

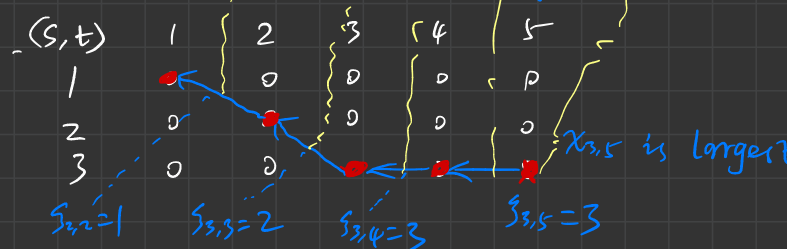
$\xi_{2,4} = 2$. $\xi_{3,4} = 3$

we summarize into table

t	1	2	3	4	5
$x_{1,t}$	1	0	0	0	0
$x_{2,t}$	0	0.35	0.0175	8.75×10^{-4}	0
$x_{3,t}$	0	0	0.105	0.063	0.0063
$\xi_{1,t}$		1			
$\xi_{2,t}$		1	2	2	2
$\xi_{3,t}$			2	3	3

Decode path

start



5.3

(a) $P(S_{19}=j \mid x_1 x_2 \cdots x_{18} x_{19})$

$$= \frac{P(S_{19}=j, x_{19}=x_{19} \mid x_1 \cdots x_{18})}{P(X_{19}=x_{19} \mid x_1 \cdots x_{18})}$$

where

$$P(X_{19}=x_{19} \mid x_1 \cdots x_{18}) = \sum_j P(S_{19}=j, X_{19}=x_{19} \mid X_1 \cdots x_{18})$$

and $P(S_{19}=j, X_{19}=x_{19} \mid x_1 \cdots x_{18})$

$$= \underline{P(X_{19}=x_{19} \mid S_{19}=j, x_1 \cdots} x_{18}) P(S_{19}=j \mid x_1 \cdots x_{18})$$

$$= \underline{P(X_{19}=x_{19} \mid S_{19}=j)} P(S_{19}=j \mid x_1 \cdots x_{18})$$

$$= b_j(x_{19}) P(S_{19}=j \mid x_1 \cdots x_{18}) \qquad \# \ x_{19}=3$$

$$= b_j(3) P(S_{19}=j \mid x_1 \cdots x_{18})$$

$P(S_{19}=j \mid x_1 \cdots x_{18})$

$$= \sum_i P(S_{19}=j \mid S_{18}=i, x_1 \cdots x_{18}) P(S_{18}=i \mid x_1 \cdots x_{18})$$

$$= \sum_i P(S_{19}=j \mid S_{18}=i) P(S_{18}=i \mid x_1 \cdots x_{18})$$

$$= \sum_i a_{ij} \hat{\alpha}_{i,18}$$

Therefore $P(S_{19}=j, x_{19}=x_{19} \mid x_1 x_2 \cdots x_{18}) = b_j(3) \sum_{i=1}^{2} a_{ij} \hat{\alpha}_{i,18}$

for $j=1$

$P(S_{19}=j, x_{19}=x_{19} \mid x_1 x_2 \cdots x_{18})$

$$= 0.2 \cdot [0.9, 0.2] \cdot [0.3 \quad 0.7]^T = 0.082$$

For $j=2$

$P(S_{19}=2, X_{19}=x_{19} \mid x_1 x_2 \cdots x_{18})$

$$= 0.3 \cdot [0.1 \quad 0.8] \cdot [0.3 \quad 0.7]^T$$

$$= 0.177$$

Therefore $P(S_{19}=j \mid x_1 \cdots x_{19}) = \begin{cases} \frac{0.082}{0.082+0.177}, & j=1 \\ \frac{0.177}{0.082+0.177}, & j=2 \end{cases}$

(b) $P(S_{19}=j \mid x_1 \cdots x_{19} x_{20})$

$$= \frac{P(S_{19}=j, x_{20} \mid x_1 \cdots x_{19})}{P(x_{20} \mid x_1 \cdots x_{19})}$$

$P(S_{19}=j, x_{20} \mid x_1 \cdots x_{19}, \lambda)$

$$= \sum_k P(S_{19}=j, S_{20}=k, x_{20} \mid x_1 \cdots x_{19})$$

$$= \sum_k a_{jk} b_k(4) P(S_{19}=j \mid x_1 \cdots x_{19})$$

$$\approx \begin{cases} [0.9 \quad 0.1] \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix} \cdot 0.3166, & j=1 \\ [0.2, \quad 0.8] \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix} \cdot 0.6834, & j=2 \end{cases}$$

$$P(x_{20} \mid x_1 \cdots x_{19}) = \sum_{j=1}^{2} P(S_{19}=j, x_{20} \mid x_1 \cdots x_{19})$$

$$\approx 0.273$$