AN INTRODUCTION TO BUSINESS MATHEMATICS

Lecture notes for the Bachelor degree programmes
IB/IMC/IMA/ITM/IEVM/ACM/IEM/IMM
at Karlshochschule International University

Module

0.1.1 IMQM: Introduction to Management and its Quantitative Methods

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Abstract

These lecture notes provide a self-contained introduction to the mathematical methods required in a Bachelor degree programme in Business, Economics, or Management. In particular, the topics covered comprise real-valued vector and matrix algebra, systems of linear algebraic equations, Leontief's stationary input–output matrix model, linear programming, elementary financial mathematics, as well as differential and integral calculus of real-valued functions of one real variable. A special focus is set on applications in quantitative economical modelling.

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Contents

Qualification objectives of the module (excerpt)

The qualification objectives shall be reached by an integrative approach.

A broad instructive range is aspired. The students shall acquire a 360 degree orientation concerning the task- and personnel-related tasks and roles of a manager and supporting tools and methods and be able to describe the coherence in an integrative way. The knowledge concerning the tasks and the understanding of methods and tools shall be strengthened by a constructivist approach based on case studies and exercises.

Students who have successfully participated in this module will be able to

- ...,
- solve problems in Linear Algebra and Analysis and apply such mathematical methods to quantitative problems in management.
- apply and challenge the knowledge critically on current issues and selected case studies.

Introduction

These lecture notes contain the entire material of the quantitative methods part of the first semester module **0.1.1 IMQM: Introduction to Management and its Quantitative Methods** at Karlshochschule International University. The aim is to provide a selection of tried-and-tested mathematical tools that proved efficient in actual practical problems of **Economics** and **Management**. These tools constitute the foundation for a systematic treatment of the typical kinds of quantitative problems one is confronted with in a Bachelor degree programme. Nevertheless, they provide a sufficient amount of points of contact with a quantitatively oriented subsequent Master degree programme in **Economics**, **Management**, or the **Social Sciences**.

The prerequisites for a proper understanding of these lecture notes are modest, as they do not go much beyond the basic A-levels standards in **Mathematics**. Besides the four fundamental arithmetical operations of addition, subtraction, multiplication and division of real numbers, you should be familiar, e.g., with manipulating fractions, dealing with powers of real numbers, the binomial formulae, determining the point of intersection for two straight lines in the Euclidian plane, solving a quadratic algebraic equation, and the rules of differentiation of real-valued functions of one variable.

It might be useful for the reader to have available a modern **graphic display calculator (GDC)** for dealing with some of the calculations that necessarily arise along the way, when confronted with specific quantitative problems. Some current models used in public schools and in undergraduate studies are, amongst others,

- Texas Instruments *TI–84 plus*,
- Casio *CFX–9850GB PLUS*.

However, the reader is strongly encouraged to think about resorting, as an alternative, to a **spreadsheet programme** such as EXCEL or OpenOffice to handle the calculations one encounters in one's quantitative work.

The central theme of these lecture notes is the acquisition and application of a number of effective mathematical methods in a business oriented environment. In particular, we hereby focus on **quantitative processes** of the sort

$$\text{INPUT} \to \text{OUTPUT} \ ,$$

for which different kinds of **functional relationships** between some numerical **INPUT quantities** and some numerical **OUTPUT quantities** are being considered. Of special interest in this context will be **ratios** of the structure

$$\frac{\text{OUTPUT}}{\text{INPUT}} .$$

In this respect, it is a general objective in **Economics** to look for ways to optimise the value of such ratios (in favour of some **economic agent**), either by seeking to increase the OUTPUT when the INPUT is confined to be fixed, or by seeking to decrease the INPUT when the OUTPUT is confined to be fixed. Consequently, most of the subsequent considerations in these lecture notes will therefore deal with issues of **optimisation** of given **functional relationships** between some **variables**, which manifest themselves either in **minimisation** or in **maximisation** procedures.

The structure of these lecture notes is the following. Part I presents selected mathematical methods from **Linear Algebra**, which are discussed in Chs. 1 to 5. Applications of these methods focus on the quantitative aspects of flows of goods in simple economic models, as well as on problems in linear programming. In Part II, which is limited to Ch. 6, we turn to discuss elementary aspects of **Financial Mathematics**. Fundamental principles of **Analysis**, comprising differential and integral calculus for real-valued functions of one real variable, and their application to quantitative economic problems, are reviewed in Part III; this extends across Chs. 7 and 8.

We emphasise the fact that there are *no* explicit examples nor exercises included in these lecture notes. These are reserved exclusively for the lectures given throughout term time.

Recommended textbooks accompanying the lectures are the works by Asano (2013) [2], Dowling (2009) [11], Dowling (1990) [10], Bauer *et al* (2008) [3], Bosch (2003) [6], and Hülsmann *et al* (2005) [16]. Some standard references of **Applied Mathematics** are, e.g., Bronstein *et al* (2005) [7] and Arens *et al* (2008) [1]. Should the reader feel inspired by the aesthetics, beauty, ellegance and efficiency of the mathematical methods presented, and, hence, would like to know more about their background and relevance, as well as being introduced to further mathematical techniques of interest, she/he is recommended to take a look at the brilliant books by Penrose (2004) [21], Singh (1997) [23], Gleick(1987) [13] and Smith (2007) [24]. Note that most of the textbooks and monographs mentioned in this Introduction are available from the library at Karlshochschule International University.

Finally, we draw the reader's attention to the fact that the *.pdf version of these lecture notes contains interactive features such as fully hyperlinked references to original publications at the websites `dx.doi.org` and `jstor.org`, as well as active links to biographical information on scientists that have been influential in the historical development of **Mathematics**, hosted by the websites The MacTutor History of Mathematics archive (`www-history.mcs.st-and.ac.uk`) and `en.wikipedia.org`.

Chapter 1

Vector algebra in Euclidian space $\mathbb{R}^n$

Let us begin our elementary considerations of **vector algebra** with the introduction of a special class of mathematical objects. These will be useful at a later stage, when we turn to formulate certain problems of a quantitative nature in a compact and elegant way. Besides introducing these mathematical objects, we also need to define which kinds of mathematical operations they can be subjected to, and what computational rules we have to take care of.

1.1 Basic concepts

Given be a set V of mathematical objects $\boldsymbol{a}$ which, for now, we want to consider merely as a collection of n arbitrary real numbers $a_1, \ldots, a_i, \ldots, a_n$. In explicit terms,

$$V = \left\{ \boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} \mid a_i \in \mathbb{R},\ i = 1, \ldots, n \right\} . \tag{1.1}$$

Formally the n real numbers considered can either be assembled in an ordered pattern as a column or a row. We define

<u>**Def.:**</u> Real-valued **column vector** with n components

$$\boxed{\boldsymbol{a} := \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} , \qquad a_i \in \mathbb{R}, \quad i = 1, \ldots, n \ ,} \tag{1.2}$$

Notation: $\boldsymbol{a} \in \mathbb{R}^{n\times 1}$,

and

<u>Def.:</u> Real-valued **row vector** with n components

$$\boxed{\boldsymbol{a}^T := (a_1, \ldots, a_i, \ldots, a_n) \ , \qquad a_i \in \mathbb{R}, \quad i = 1, \ldots, n \ ,} \tag{1.3}$$

Notation: $\boldsymbol{a}^T \in \mathbb{R}^{1\times n}$.

Correspondingly, we define the n-component objects

$$\mathbf{0} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{0}^T := (0, \ldots, 0, \ldots, 0) \tag{1.4}$$

to constitute related **zero vectors**.

Next we define for like objects in the set V, i.e., either for n-component column vectors or for n-component row vectors, two simple computational operations. These are

<u>Def.:</u> **Addition** of vectors

$$\boxed{\boldsymbol{a} + \boldsymbol{b} := \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_i + b_i \\ \vdots \\ a_n + b_n \end{pmatrix} \ , \qquad a_i, b_i \in \mathbb{R} \ ,} \tag{1.5}$$

and

<u>Def.:</u> **Rescaling** of vectors

$$\boxed{\lambda\boldsymbol{a} := \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_i \\ \vdots \\ \lambda a_n \end{pmatrix} \ , \qquad \lambda, a_i \in \mathbb{R} \ .} \tag{1.6}$$

The rescaling of a vector $\boldsymbol{a}$ with an arbitrary non-zero real number λ has the following effects:

- $|\lambda| > 1$ — stretching of the length of $\boldsymbol{a}$
- $0 < |\lambda| < 1$ — shrinking of the length of $\boldsymbol{a}$
- $\lambda < 0$ — directional reversal of $\boldsymbol{a}$.

The notion of the length of a vector $\boldsymbol{a}$ will be made precise shortly.

The addition and the rescaling of n-component vectors satisfy the following addition and multiplication laws:

Computational rules for addition and rescaling of vectors
For vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$:

1. $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ (**commutative addition**)
2. $\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$ (**associative addition**)
3. $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$ (**addition identity element**)
4. For every $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, there exists exactly one $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{a} + \boldsymbol{x} = \boldsymbol{b}$ (**invertibility of addition**)
5. $(\lambda\mu)\boldsymbol{a} = \lambda(\mu\boldsymbol{a})$ with $\lambda \in \mathbb{R}$ (**associative rescaling**)
6. $1\boldsymbol{a} = \boldsymbol{a}$ (**rescaling identity element**)
7. $\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$;
   $(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \mu\boldsymbol{a}$ with $\lambda, \mu \in \mathbb{R}$ (**distributive rescaling**).

In conclusion of this section, we remark that every set of mathematical objects V constructed in line with Eq. (1.1), with an addition and a rescaling defined according to Eqs. (1.5) and (1.6), and satisfying the laws stated above, constitutes a **linear vector space over Euclidian space** $\mathbb{R}^n$.[1]

1.2 Dimension and basis of $\mathbb{R}^n$

Let there be given m n-component vectors[2] $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$, as well as m real numbers $\lambda_1, \ldots, \lambda_i, \ldots, \lambda_m \in \mathbb{R}$. The new n-component vector $\boldsymbol{b}$ resulting from the addition of arbitrarily rescaled versions of these m vectors according to

$$\boxed{\boldsymbol{b} = \lambda_1\boldsymbol{a}_1 + \ldots + \lambda_i\boldsymbol{a}_i + \ldots + \lambda_m\boldsymbol{a}_m =: \sum_{i=1}^{m} \lambda_i\boldsymbol{a}_i \in \mathbb{R}^n} \tag{1.7}$$

is referred to as a **linear combination** of the m vectors $\boldsymbol{a}_i$, $i = 1, \ldots, m$.

<u>**Def.:**</u> A set of m vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$ is called **linearly independent** when the condition

$$\boldsymbol{0} \stackrel{!}{=} \lambda_1\boldsymbol{a}_1 + \ldots + \lambda_i\boldsymbol{a}_i + \ldots + \lambda_m\boldsymbol{a}_m = \sum_{i=1}^{m} \lambda_i\boldsymbol{a}_i \ , \tag{1.8}$$

[1]This is named after the ancient greek mathematician Euclid of Alexandria (about 325 BC–265 BC).

[2]A slightly s horter notation for n-component column vectors $\boldsymbol{a} \in \mathbb{R}^{n\times 1}$ is given by $\boldsymbol{a} \in \mathbb{R}^n$; likewise $\boldsymbol{a}^T \in \mathbb{R}^n$ for n-component row vectors $\boldsymbol{a}^T \in \mathbb{R}^{1\times n}$.

i.e., the problem of forming the **zero vector** $\mathbf{0} \in \mathbb{R}^n$ from a linear combination of the m vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$, can *only* be solved trivially, namely by $0 = \lambda_1 = \ldots = \lambda_i = \ldots = \lambda_m$. When, however, this condition can be solved non-trivially, with some $\lambda_i \neq 0$, then the set of m vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$ is called **linearly dependent**.

In Euclidian space $\mathbb{R}^n$, there is a maximum number n (!) of vectors which can be linearly independent. This maximum number is referred to as the **dimension of Euclidian space** $\mathbb{R}^n$. Every set of n linearly independent vectors in Euclidian space $\mathbb{R}^n$ constitutes a possible **basis of Euclidian space** $\mathbb{R}^n$. If the set $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_n\}$ constitutes a basis of $\mathbb{R}^n$, then every other vector $\boldsymbol{b} \in \mathbb{R}^n$ can be expressed in terms of these basis vectors by

$$\boldsymbol{b} = \beta_1 \boldsymbol{a}_1 + \ldots + \beta_i \boldsymbol{a}_i + \ldots + \beta_n \boldsymbol{a}_n = \sum_{i=1}^{n} \beta_i \boldsymbol{a}_i \ . \tag{1.9}$$

The rescaling factors $\beta_i \in \mathbb{R}$ of the $\boldsymbol{a}_i \in \mathbb{R}^n$ are called the **components of vector $\boldsymbol{b}$ with respect to the basis** $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_n\}$.

Remark: The n **unit vectors**

$$\boldsymbol{e}_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ , \quad \boldsymbol{e}_2 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \ , \quad \ldots \ , \quad \boldsymbol{e}_n := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \ , \tag{1.10}$$

constitute the so-called **canonical basis of Euclidian space** $\mathbb{R}^n$. With respect to this basis, all vectors $\boldsymbol{b} \in \mathbb{R}^n$ can be represented as a linear combinationen

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = b_1 \boldsymbol{e}_1 + b_2 \boldsymbol{e}_2 + \cdots + b_n \boldsymbol{e}_n = \sum_{i=1}^{n} b_i \boldsymbol{e}_i \ . \tag{1.11}$$

1.3 Euclidian scalar product

Finally, to conclude this section, we introduce a third mathematical operation defined for vectors on $\mathbb{R}^n$.

Def.: For an n-component row vector $\boldsymbol{a}^T \in \mathbb{R}^{1\times n}$ and an n-component column vector $\boldsymbol{b} \in \mathbb{R}^{n\times 1}$, the **Euclidian scalar product**

$$\boxed{\boldsymbol{a}^T \cdot \boldsymbol{b} := (a_1, \ldots, a_i, \ldots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \ldots + a_i b_i \ldots + a_n b_n =: \sum_{i=1}^{n} a_i b_i} \tag{1.12}$$

defines a mapping $f : \mathbb{R}^{1\times n} \times \mathbb{R}^{n\times 1} \to \mathbb{R}$ from the product set of n-component row and column vectors to the set of real numbers. Note that, in contrast to the addition and the rescaling of n-component vectors, the outcome of forming a Euclidian scalar product between two n-component vectors is a *single real number*.

In the context of the Euclidian scalar product, two non-zero vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ (wit $\boldsymbol{a} \neq \boldsymbol{0} \neq \boldsymbol{b}$) are referred to as **mutually orthogonal** when they exhibit the property that $0 = \boldsymbol{a}^T \cdot \boldsymbol{b} = \boldsymbol{b}^T \cdot \boldsymbol{a}$.

Computational rules for Euclidian scalar product of vectors
For vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$:

1. $(\boldsymbol{a} + \boldsymbol{b})^T \cdot \boldsymbol{c} = \boldsymbol{a}^T \cdot \boldsymbol{c} + \boldsymbol{b}^T \cdot \boldsymbol{c}$ (**distributive scalar product**)
2. $\boldsymbol{a}^T \cdot \boldsymbol{b} = \boldsymbol{b}^T \cdot \boldsymbol{a}$ (**commutative scalar product**)
3. $(\lambda \boldsymbol{a}^T) \cdot \boldsymbol{b} = \lambda(\boldsymbol{a}^T \cdot \boldsymbol{b})$ with $\lambda \in \mathbb{R}$ (**homogeneous scalar product**)
4. $\boldsymbol{a}^T \cdot \boldsymbol{a} > 0$ for all $\boldsymbol{a} \neq \boldsymbol{0}$ (**positive definite scalar product**).

Now we turn to introduce the notion of the length of an n-component vector.

Def.: The **length** of a vector $\boldsymbol{a} \in \mathbb{R}^n$ is defined via the Euclidian scalar product as

$$\boxed{|\boldsymbol{a}| := \sqrt{\boldsymbol{a}^T \cdot \boldsymbol{a}} = \sqrt{a_1^2 + \ldots + a_i^2 + \ldots + a_n^2} =: \sqrt{\sum_{i=1}^{n} a_i^2}\ .} \tag{1.13}$$

Technically one refers to the non-negative real number $|\boldsymbol{a}|$ as the **absolute value** or the **Euclidian norm** of the vector $\boldsymbol{a} \in \mathbb{R}^n$. The length of $\boldsymbol{a} \in \mathbb{R}^n$ has the following properties:

- $|\boldsymbol{a}| \geq 0$, and $|\boldsymbol{a}| = 0 \Leftrightarrow \boldsymbol{a} = \boldsymbol{0}$;
- $|\lambda \boldsymbol{a}| = |\lambda||\boldsymbol{a}|$ for $\lambda \in \mathbb{R}$;
- $|\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$ (**triangle inequality**).

Every non-zero vector $\boldsymbol{a} \in \mathbb{R}^n$, i.e., $|\boldsymbol{a}| > 0$, can be rescaled by the reciprocal of its length. This procedure defines the

Def.: **Normalisation** of a vector $\boldsymbol{a} \in \mathbb{R}^n$;

$$\hat{\boldsymbol{a}} := \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \qquad \Rightarrow \qquad |\hat{\boldsymbol{a}}| = 1\ . \tag{1.14}$$

By this method one generates a vector of length 1, i.e., a **unit vector** $\hat{\boldsymbol{a}}$. To denote unit vectors we will employ the “hat” symbol.

Lastly, also by means of the Euclidian scalar product, we introduce the angle enclosed between two non-zero vectors.

Def.: **Angle** enclosed between $\boldsymbol{a}, \boldsymbol{b} \neq \boldsymbol{0} \in \mathbb{R}^n$

$$\boxed{\cos[\varphi(\boldsymbol{a}, \boldsymbol{b})] = \frac{\boldsymbol{a}^T}{|\boldsymbol{a}|} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|} = \hat{\boldsymbol{a}}^T \cdot \hat{\boldsymbol{b}} \quad \Rightarrow \quad \varphi(\boldsymbol{a}, \boldsymbol{b}) = \cos^{-1}(\hat{\boldsymbol{a}}^T \cdot \hat{\boldsymbol{b}})\ .} \tag{1.15}$$

Remark: The inverse cosine function[3] $\cos^{-1}(\ldots)$ is available on every standard GDC or spreadsheet.

[3]The notion of on inverse function will be discussed later in Ch. 7.

Chapter 2

Matrices

In this chapter, we introduce a second class of mathematical objects that are more general than vectors. For these objects, we will also define certain mathematical operations, and a set of computational rules that apply in this context.

2.1 Matrices as linear mappings

Consider given a collection of $m \times n$ arbitrary real numbers a_{11}, a_{12} $\ldots$, a_{ij}, $\ldots$, a_{mn}, which we arrange systematically in a particular kind of array.

<u>**Def.:**</u> A real-valued $\boldsymbol{(m \times n)}$**-matrix** is formally defined to constitute an array of real numbers according to

$$\mathbf{A} := \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{pmatrix} , \tag{2.1}$$

where $a_{ij} \in \mathbb{R}$, $i = 1, \ldots, m; j = 1, \ldots, n$.
Notation: $\mathbf{A} \in \mathbb{R}^{m \times n}$.

Characteristic features of this array of real numbers are:

- m denotes the number of **rows** of $\mathbf{A}$, n the number of **columns** of $\mathbf{A}$.

- a_{ij} represents the **elements** of $\mathbf{A}$; a_{ij} is located at the point of intersection of the ith row and the jth column of $\mathbf{A}$.

- elements of the ith row constitute the **row vector** $(a_{i1}, a_{i2}, \ldots, a_{ij}, \ldots, a_{in})$, elements of the

jth column the **column vector** $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix}$.

Formally column vectors need to be viewed as $(n \times 1)$-matrices, row vectors as $(1 \times n)$-matrices. An $\boldsymbol{(m \times n)}$**-zero matrix**, denoted by $\mathbf{0}$, has all its elements equal to zero, i.e.,

$$\mathbf{0} := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} . \tag{2.2}$$

Matrices which have an *equal* number of rows and columns, i.e. $m = n$, are referred to as **quadratic matrices**. In particular, the $\boldsymbol{(n \times n)}$**-unit matrix** (or identity matrix)

$$\mathbf{1} := \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \tag{2.3}$$

holds a special status in the family of $(n \times n)$-matrices.

Now we make explicit in what sense we will comprehend $(m \times n)$-matrices as mathematical objects.

Def.: A real-valued matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ defines by the computational operation

$$\begin{aligned} \mathbf{A}\boldsymbol{x} \quad &:= \quad \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} \\ &:= \quad \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1j}x_j + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2j}x_j + \dots + a_{2n}x_n \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{ij}x_j + \dots + a_{in}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mj}x_j + \dots + a_{mn}x_n \end{pmatrix} =: \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{pmatrix} = \boldsymbol{y} \end{aligned} \tag{2.4}$$

a **mapping** $\mathbf{A} : \mathbb{R}^{n\times 1} \to \mathbb{R}^{m\times 1}$, i.e. a mapping from the set of real-valued n-component column vectors (here: $\boldsymbol{x}$) to the set of real-valued m-component column vectors (here: $\boldsymbol{y}$).

In loose analogy to the photographic process, $\boldsymbol{x}$ can be viewed as representing an "object," $\mathbf{A}$ a "camera," and $\boldsymbol{y}$ the resultant "image."

Since for real-valued vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^{n\times 1}$ and real numbers $\lambda \in \mathbb{R}$, mappings defined by real-valued matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ exhibit the two special properties

$$\boxed{\begin{array}{c}\mathbf{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = (\mathbf{A}\boldsymbol{x}_1) + (\mathbf{A}\boldsymbol{x}_2) \\ \\ \mathbf{A}(\lambda\boldsymbol{x}_1) = \lambda(\mathbf{A}\boldsymbol{x}_1)\ ,\end{array}} \tag{2.5}$$

they constitute **linear mappings**.[1]

We now turn to discuss the most important mathematical operations defined for $(m \times n)$-matrices, as well as the computational rules that obtain.

2.2 Basic concepts

<u>Def.:</u> Transpose of a matrix
For $\mathbf{A} \in \mathbb{R}^{m\times n}$, we define the process of transposing $\mathbf{A}$ by

$$\boxed{\mathbf{A}^T: \quad a^T_{ij} := a_{ji}\ ,} \tag{2.6}$$

where $i = 1, \ldots, m$ und $j = 1, \ldots, n$. Note that it holds that $\mathbf{A}^T \in \mathbb{R}^{n\times m}$.

When transposing an $(m\times n)$-matrix, one simply has to exchange the matrix' rows with its columns (and vice versa): the elements of the first row become the elements of the first column, etc. It follows that, in particular,

$$(\mathbf{A}^T)^T = \mathbf{A} \tag{2.7}$$

applies.

Two special cases may occur for quadratic matrices (where $m = n$):

- When $\mathbf{A}^T = \mathbf{A}$, one refers to $\mathbf{A}$ as a **symmetric matrix**.
- When $\mathbf{A}^T = -\mathbf{A}$, one refers to $\mathbf{A}$ as an **antisymmetric matrix**.

<u>Def.:</u> Addition of matrices
For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$, the sum is given by

$$\boxed{\mathbf{A} + \mathbf{B} =: \mathbf{C}: \quad a_{ij} + b_{ij} =: c_{ij}\ ,} \tag{2.8}$$

with $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Note that an addition can be performed meaningfully only for matrices of the *same format*.

[1]It is important to note at this point that many advanced mathematical models designed to describe quantitative aspects of some natural and economic phenomena do *not* satisfy the conditions (2.5), as they employ *non-linear mappings* for this purpose. However, in such contexts, linear mappings often provide useful first approximations.

Def.: **Rescaling** of matrices
For $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\lambda \in \mathbb{R}\backslash\{0\}$, let

$$\boxed{\lambda\mathbf{A} =: \mathbf{C}: \quad \lambda a_{ij} =: c_{ij} \ ,} \tag{2.9}$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

When rescaling a matrix, all its elements simply have to be multiplied by the same non-zero real number λ.

Computational rules for addition and rescaling of matrices
For matrices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{m \times n}$:

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ **(commutative addition)**
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ **(associative addition)**
3. $\mathbf{A} + \mathbf{0} = \mathbf{A}$ **(addition identity element)**
4. For every $\mathbf{A}$ and $\mathbf{B}$, there exists exactly one $\mathbf{Z}$ such that $\mathbf{A} + \mathbf{Z} = \mathbf{B}$. **(invertibility of addition)**
5. $(\lambda\mu)\mathbf{A} = \lambda(\mu\mathbf{A})$ with $\lambda, \mu \in \mathbb{R}\backslash\{0\}$ **(associative rescaling)**
6. $1\mathbf{A} = \mathbf{A}$ **(rescaling identity element)**
7. $\lambda(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}$;
   $(\lambda + \mu)\mathbf{A} = \lambda\mathbf{A} + \mu\mathbf{A}$ with $\lambda, \mu \in \mathbb{R}\backslash\{0\}$ **(distributive rescaling)**
8. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ **(transposition rule 1)**
9. $(\lambda\mathbf{A})^T = \lambda\mathbf{A}^T$ with $\lambda \in \mathbb{R}\backslash\{0\}$. **(transposition rule 2)**

Next we introduce a particularly useful mathematical operation for matrices.

2.3 Matrix multiplication

Def.: For a real-valued $(m \times n)$-matrix $\mathbf{A}$ and a real-valued $(n \times r)$-matrix $\mathbf{B}$, a **matrix multiplication** is defined by

$$\boxed{\begin{array}{c} \mathbf{A}\mathbf{B} =: \mathbf{C} \\ \\ a_{i1}b_{1j} + \ldots + a_{ik}b_{kj} + \ldots + a_{in}b_{nj} =: \sum_{k=1}^{n} a_{ik}b_{kj} =: c_{ij} \ , \end{array}} \tag{2.10}$$

with $i = 1, \ldots, m$ and $j = 1, \ldots, r$, thus yielding as an outcome a real-valued $(m \times r)$-matrix $\mathbf{C}$.

The element of $\mathbf{C}$ at the intersection of the ith row and the jth column is determined by the computational rule

$$c_{ij} = \text{Euclidian scalar product of } i\text{th row vector of } \mathbf{A} \text{ and } j\text{th column vector of } \mathbf{B}\ . \tag{2.11}$$

It is important to realise that the definition of a matrix multiplication just provided depends in an essential way on the fact that *matrix* $\mathbf{A}$ *on the left in the product needs to have as many (!) columns as matrix* $\mathbf{B}$ *on the right rows*. Otherwise, a matrix multiplication *cannot* be defined in a meaningful way.

<u>GDC:</u> For matrices [A] and [B] edited beforehand, of matching formats, their matrix multiplication can be evaluated in mode `MATRIX` $\rightarrow$ `NAMES` by [A] $*$ [B].

Computational rules for matrix multiplication
For $\mathbf{A}, \mathbf{B}, \mathbf{C}$ real-valued matrices of correspondingly matching formats we have:

1. $\mathbf{AB} = \mathbf{0}$ is possible with $\mathbf{A} \neq \mathbf{0}, \mathbf{B} \neq \mathbf{0}$. (**zero divisor**)
2. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (**associative matrix multiplication**)
3. $\mathbf{A} \underbrace{\mathbf{1}}_{\in \mathbb{R}^{n\times n}} = \underbrace{\mathbf{1}}_{\in \mathbb{R}^{m\times m}} \mathbf{A} = \mathbf{A}$ (**multiplicative identity element**)
4. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
   $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$ (**distributive matrix multiplication**)
5. $\mathbf{A}(\lambda\mathbf{B}) = (\lambda\mathbf{A})\mathbf{B} = \lambda(\mathbf{AB})$ with $\lambda \in \mathbb{R}$ (**homogeneous matrix multiplication**)
6. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ (**transposition rule**).

Chapter 3

Systems of linear algebraic equations

In this chapter, we turn to address a particular field of application of the notions of matrices and vectors, or of linear mappings in general.

3.1 Basic concepts

Let us begin with a system of $m \in \mathbb{N}$ *linear* algebraic equations, wherein every single equation can be understood to constitute a **constraint** on the range of values of $n \in \mathbb{N}$ variables $x_1, \ldots, x_n \in \mathbb{R}$. The objective is to determine all possible values of $x_1, \ldots, x_n \in \mathbb{R}$ which satisfy these constraints simultaneously. Problems of this kind, namely **systems of linear algebraic equations**, are often represented in the form

• Representation 1:

$$
\begin{aligned}
a_{11}x_1 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n &= b_1 \\
&\vdots \\
a_{i1}x_1 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n &= b_i \\
&\vdots \\
a_{m1}x_1 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n &= b_m \ .
\end{aligned}
\tag{3.1}
$$

Depending on how the natural numbers m and n relate to one another, systems of linear algebraic equations can be classified as follows:

- $m < n$: fewer equations than variables; the linear system is **under-determined**,
- $m = n$: same number of equations as variables; the linear system is **well-determined**,
- $m > n$: more equations than variables; the linear system is **over-determined**.

A more compact representation of a linear system of format $(m \times n)$ is given by

• Representation 2:

$$\boxed{\mathbf{A}\boldsymbol{x} = \left(\begin{array}{ccccc} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{array}\right)\left(\begin{array}{c} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{array}\right) = \left(\begin{array}{c} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{array}\right) = \boldsymbol{b}\ .} \tag{3.2}$$

The mathematical objects employed in this variant of a linear system are as follows: $\mathbf{A}$ takes the central role of the **coefficient matrix** of the linear system, of format $(m \times n)$, $\boldsymbol{x}$ is its **variable vector**, of format $(n \times 1)$, and, lastly, $\boldsymbol{b}$ is its **image vector**, of format $(m \times 1)$.

When dealing with systems of linear algebraic equations in the form of Representation 2, i.e. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, the main question to be answered is:

Question: For given **coefficient matrix** $\mathbf{A}$ and **image vector** $\boldsymbol{b}$, can we find a **variable vector** $\boldsymbol{x}$ that $\mathbf{A}$ maps onto $\boldsymbol{b}$?

In a sense this describes the inversion of the photographic process we had previously referred to: we *have* given the camera and we already *know* the image, but we have yet to find a matching object. Remarkably, to address this issue, we can fall back on a simple algorithmic method due to the German mathematician and astronomer Carl Friedrich Gauß (1777–1855).

3.2 Gaußian elimination

The algorithmic solution technique developed by Gauß is based on the insight that the solution set of a **linear system** of m algebraic equations for n real-valued variables, i.e.

$$\boxed{\mathbf{A}\boldsymbol{x} = \boldsymbol{b}\ ,} \tag{3.3}$$

remains unchanged under the following algebraic **equivalence transformations** of the linear system:

1. changing the order amongst the equations,
2. multiplication of any equation by a non-zero real number $c \neq 0$,
3. addition of a multiple of one equation to another equation,
4. changing the order amongst the equations.

Specifically, this implies that we may manipulate a given linear system by means of these four different kinds of equivalence transformations without ever changing its identity. In concrete cases, however, one should not apply these equivalence transformations at random but rather follow a target oriented strategy. This is what Gaußian elimination can provide.

Target: To cast the **augmented coefficient matrix** $(\mathbf{A}|\boldsymbol{b})$, i.e., the array

$$\begin{array}{ccccc|c} a_{11} & \dots & a_{1j} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} & b_i \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} & b_m \end{array} \;, \tag{3.4}$$

when possible, into **upper triangular form**

$$\begin{array}{ccccc|c} 1 & \dots & \tilde{a}_{1j} & \dots & \tilde{a}_{1n} & \tilde{b}_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \tilde{a}_{ij} & \dots & \tilde{a}_{in} & \tilde{b}_i \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & \tilde{a}_{mn} & \tilde{b}_m \end{array} \;, \tag{3.5}$$

by means of the four kinds of equivalence transformations such that the resultant simpler final linear system may easily be solved using **backward substitution**.

Three exclusive cases of possible **solution content** for a given system of linear algebraic equations do exist. The linear system may possess either

1. *no solution* at all, or
2. a *unique solution*, or
3. *multiple solutions*.

Remark: Linear systems that are under-determined, i.e., when $m < n$, can *never* be solved uniquely due to the fact that in such a case there not exist enough equations to constrain the values of *all* of the n variables.

GDC: For a stored augmented coefficient matrix [A] of format $(m \times n+1)$, associated with a given $(m \times n)$ linear system, select mode `MATRIX` $\rightarrow$ `MATH` and then call the function `rref([A])`. It is possible that backward substitution needs to be employed to obtain the final solution.

For completeness, we want to turn briefly to the issue of solvability of a system of linear algebraic equations. To this end, we need to introduce the notion of the rank of a matrix.

3.3 Rank of a matrix

Def.: A real-valued matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ possesses the **rank**

$$\boxed{\text{rank}(\mathbf{A}) = r\;, \qquad r \leq \min\{m, n\}} \tag{3.6}$$

if and only if r is the **maximum number** of row resp. column vectors of $\mathbf{A}$ which are linearly independent. Clearly, r can only be as large as the smaller of the numbers m and n that determine the format of $\mathbf{A}$.

For **quadratic matrices** $\mathbf{A} \in \mathbb{R}^{n\times n}$, there is available a more elegant measure to determine its rank. This (in the present case real-valued) measure is referred to as the **determinant** of matrix $\mathbf{A}$, $\det(\mathbf{A})$, and is defined as follows.

Def.:

(i) When $\mathbf{A} \in \mathbb{R}^{2\times 2}$, its **determinant** is given by

$$\det(\mathbf{A}) := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := a_{11}a_{22} - a_{12}a_{21} \ , \tag{3.7}$$

i.e. the difference between the products of $\mathbf{A}$'s on-diagonal elements and $\mathbf{A}$'s off-diagonal elements.

(ii) When $\mathbf{A} \in \mathbb{R}^{3\times 3}$, the definition of $\mathbf{A}$'s **determinant** is more complex. In that case it is given by

$$\begin{aligned} \det(\mathbf{A}) &:= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &:= a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{21}(a_{32}a_{13} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \end{aligned} \tag{3.8}$$

Observe, term by term, the cyclic permutation of the first index of the elements a_{ij} according to the rule $1 \to 2 \to 3 \to 1$.

(iii) Finally, for the (slightly involved) definition of the **determinant** of a higher-dimensional matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, please refer to the literature; e.g. Bronstein *et al* (2005) [7, p 267].

To determine the rank of a given quadtratic matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, one now installs the following criteria: rank$(\mathbf{A}) = r = n$, if $\det(\mathbf{A}) \neq 0$, and rank$(\mathbf{A}) = r < n$, if $\det(\mathbf{A}) = 0$. In the first case, $\mathbf{A}$ is referred to as **regular**, in the second as **singular**. For quadratic matrices $\mathbf{A}$ that are singular, rank$(\mathbf{A}) = r$ (with $r < n$) is given by the number r of rows (or columns) of the largest possible non-zero subdeterminant of $\mathbf{A}$.

GDC: For a stored quadratic matrix [A], select mode `MATRIX` $\to$ `MATH` and obtain its determinant by calling the function `det([A])`.

3.4 Criteria for solving systems of linear algebraic equations

Making use of the concept of the **rank** of a real-valued matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we can now summarise the solution content of a specific system of linear algebraic equations of format $(m \times n)$ in a table. For given linear system

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b} \ ,$$

with coefficient matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, variable vector $\boldsymbol{x} \in \mathbb{R}^{n\times 1}$ and image vector $\boldsymbol{b} \in \mathbb{R}^{m\times 1}$, there exist(s)

	$\boldsymbol{b} \neq \mathbf{0}$	$\boldsymbol{b} = \mathbf{0}$
1. rank$(\mathbf{A}) \neq$ rank$(\mathbf{A}\vert\boldsymbol{b})$	no solution	———
2. rank$(\mathbf{A}) =$ rank$(\mathbf{A}\vert\boldsymbol{b}) = r$		
(a) $r = n$	a unique solution	$\boldsymbol{x} = \mathbf{0}$
(b) $r < n$	multiple solutions: $n - r$ free parameters	multiple solutions: $n - r$ free parameters

$(\mathbf{A}|\boldsymbol{b})$ here denotes the augmented coefficient matrix.

Next we discuss a particularly useful property of *regular* quadratic matrices.

3.5 Inverse of a regular $(n \times n)$-matrix

Def.: Let a real-valued quadratic matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ be **regular**, i.e., $\det(\mathbf{A}) \in \mathbb{R}\backslash\{0\}$. Then there exists an **inverse matrix** $\mathbf{A}^{-1}$ to $\mathbf{A}$ defined by the characterising properties

$$\boxed{\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{1}\ .} \tag{3.9}$$

Here $\mathbf{1}$ denotes the $(\boldsymbol{n} \times \boldsymbol{n})$**-unit matrix** [cf. Eq. (2.3)].

When a computational device is not at hand, the inverse matrix $\mathbf{A}^{-1}$ of a regular quadratic matrix $\mathbf{A}$ can be obtained by solving the matrix-valued linear system

$$\mathbf{A}\mathbf{X} \stackrel{!}{=} \mathbf{1} \tag{3.10}$$

for the unknown matrix $\mathbf{X}$ by means of **simultaneous Gaußian elimination**.

GDC: For a stored quadratic matrix $[\mathtt{A}]$, its inverse matrix can be simply obtained as $[\mathtt{A}]^{-1}$, where the x^{-1} function key needs to be used.

Computational rules for the inverse operation
For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, with $\det(\mathbf{A}) \neq 0 \neq \det(\mathbf{B})$, it holds that

1. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
2. $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
3. $(\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$
4. $(\lambda\mathbf{A})^{-1} = \dfrac{1}{\lambda}\,\mathbf{A}^{-1}$.

The special interest in applications in the concept of **inverse matrices** arises for the following reason. Consider given a well-determined linear system

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}\,,$$

with *regular* quadratic coefficient matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, i.e., $\det(\mathbf{A}) \neq 0$. Then, for $\mathbf{A}$, there exists an inverse matrix $\mathbf{A}^{-1}$. Matrix-multiplying both sides of the equation above *from the left (!)* by the inverse $\mathbf{A}^{-1}$, results in

$$\underbrace{\mathbf{A}^{-1}(\mathbf{A}\boldsymbol{x}) = (\mathbf{A}^{-1}\mathbf{A})\boldsymbol{x} = \mathbf{1}\boldsymbol{x} = \boldsymbol{x}}_{\text{left-hand side}} = \underbrace{\mathbf{A}^{-1}\boldsymbol{b}}_{\text{right-hand side}}\quad. \tag{3.11}$$

In this case, the *unique solution (!)* $\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b}$ of the linear system arises simply from matrix multiplication of the image vector $\boldsymbol{b}$ by the inverse matrix of $\mathbf{A}$. (Of course, it might actually require a bit of computational work to determine $\mathbf{A}^{-1}$.)

3.6 Outlook

There are a number of exciting advanced topics in **Linear Algebra**. Amongst them one finds the concept of the characteristic **eigenvalues** and associated **eigenvectors** of **quadratic matrices**, which has particularly high relevance in practical applications. The question to be answered here is the following: for given real-valued quadratic matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, do there exist real numbers $\lambda_n \in \mathbb{R}$ and real-valued vectors $\boldsymbol{v}_n \in \mathbb{R}^{n\times 1}$ which satisfy the condition

$$\mathbf{A}\boldsymbol{v}_n \stackrel{!}{=} \lambda_n\boldsymbol{v}_n\ ? \tag{3.12}$$

Put differently: for which vectors $\boldsymbol{v}_n \in \mathbb{R}^{n\times 1}$ does their mapping by a quadratic matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ amount to simple rescalings by real numbers $\lambda_n \in \mathbb{R}$?

By re-arranging, Eq. (3.12) can be recast into the form

$$\mathbf{0} \stackrel{!}{=} (\mathbf{A} - \lambda_n\mathbf{1})\,\boldsymbol{v}_n\ , \tag{3.13}$$

with $\mathbf{1}$ an $(n \times n)$-unit matrix [cf. Eq. (2.3)] and $\mathbf{0}$ an n-component zero vector. This condition corresponds to a homogeneous system of linear algebraic equations of format $(n \times n)$. Non-trivial solutions $\boldsymbol{v}_n \neq \mathbf{0}$ to this system exist provided that the so-called **characteristic equation**

$$0 \stackrel{!}{=} \det\left(\mathbf{A} - \lambda_n \mathbf{1}\right) \;, \tag{3.14}$$

a polynomial of degree n (cf. Sec. 7.1.1), allows for real-valued roots $\lambda_n \in \mathbb{R}$. Note that *symmetric* quadratic matrices (cf. Sec. 2.2) possess exclusively real-valued eigenvalues λ_n. When these eigenvalues turn out to be all *different*, then the associated eigenvectors $\boldsymbol{v}_n$ prove to be mutually orthogonal.

Knowledge of the spectrum of **eigenvalues** $\lambda_n \in \mathbb{R}$ and associated **eigenvectors** $\boldsymbol{v}_n \in \mathbb{R}^{n\times 1}$ of a real-valued matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ provides the basis of a transformation of $\mathbf{A}$ to its **diagonal form** $\mathbf{A}_{\lambda_n}$, thus yielding a diagonal matrix which features the eigenvalues λ_n as its on-diagonal elements; cf. Leon (2009) [19].

Amongst other examples, the concept of eigenvalues and eigenvectors of quadratic real-valued matrices plays a special role in **Statistics**, in the context of exploratory **principal component analyses** of multivariate data sets, where the objective is to identify dominant intrinsic structures; cf. Hair *et al* (2010) [14, Ch. 3] and Ref. [12, App. A].

Chapter 4

Leontief's stationary input–output matrix model

We now turn to discuss some specific applications of **Linear Algebra** in economic theory. To begin with, let us consider quantitative aspects of the exchange of goods between a certain number of **economic agents**. We here aim at a simplified abstract description of real economic processes.

4.1 General considerations

The quantitative model to be described in the following is due to the Russian economist Wassily Wassilyovich Leontief (1905–1999), cf. Leontief (1936) [20], for which, besides other important contributions, he was awarded the 1973 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel.

Suppose given an economic system consisting of $n \in \mathbb{N}$ **interdependent economic agents** exchanging between them the goods they produce. For simplicity we want to *assume* that every one of these **economic agents** represents the production of a *single* good only. Presently we intend to monitor the flow of goods in this simple economic system during a specified **reference period of time**. The total numbers of the n goods leaving the production sector of this model constitute the **OUTPUT quantities**. The **INPUT quantities** to the production sector are twofold. On the one hand, there are **exogenous** INPUT quantities which we take to be given by $m \in \mathbb{N}$ different kinds of external **resources** needed in differing proportions to produce the n goods. On the other hand, due to their mutual interdependence, some of the **economic agents** require **goods made by their neighbours** to be able to produce their own goods; these constitute the **endogenous** INPUT quantities of the system. Likewise, the production sector's total OUTPUT during the chosen reference period of the n goods can be viewed to flow through one of *two* separate channels: (i) the **exogenous** channel linking the production sector to **external consumers** representing an open market, and (ii) the **endogenous** channel linking the **economic agents** to their **neighbours** (thus respresenting their interdependencies). It is supposed that momentum is injected into the economic system, triggering the flow of goods between the different actors, by the prospect of **increasing**

the value of the INPUT quantities, in line with the notion of the economic **value chain**.

Leontief's model is based on the following three elementary

Assumptions:

1. For all goods involved the functional relationship between INPUT and OUTPUT quantities be of a **linear nature** [cf. Eq. (2.5)].

2. The proportions of "INPUT quantities to OUTPUT quantities" be **constant** over the reference period of time considered; the flows of goods are thus considered to be **stationary**.

3. **Economic equilibrium** obtains during the reference period of time: the numbers of goods then supplied equal the numbers of goods then demanded.

The mathematical formulation of Leontief's quantitative model employs the following

Vector- and matrix-valued quantities:

1. $\boldsymbol{q}$ — **total output vector** $\in \mathbb{R}^{n\times 1}$, components $q_i \geq 0$ units (dim: units)

2. $\boldsymbol{y}$ — **final demand vector** $\in \mathbb{R}^{n\times 1}$, components $y_i \geq 0$ units (dim: units)

3. $\mathbf{P}$ — **input–output matrix** $\in \mathbb{R}^{n\times n}$, components $P_{ij} \geq 0$ (dim: 1)

4. $(\mathbf{1} - \mathbf{P})$ — **technology matrix** $\in \mathbb{R}^{n\times n}$, regular, hence, invertible (dim: 1)

5. $(\mathbf{1} - \mathbf{P})^{-1}$ — **total demand matrix** $\in \mathbb{R}^{n\times n}$ (dim: 1)

6. $\boldsymbol{v}$ — **resource vector** $\in \mathbb{R}^{m\times 1}$, components $v_i \geq 0$ units (dim: units)

7. $\mathbf{R}$ — **resource consumption matrix** $\in \mathbb{R}^{m\times n}$, components $R_{ij} \geq 0$, (dim: 1)

where $\mathbf{1}$ denotes the $(\boldsymbol{n} \times \boldsymbol{n})$**-unit matrix** [cf. Eq. (2.3)]. Note that the components of all the vectors involved, as well as of the input–output matrix and of the resource consumption matrix, can assume *non-negative values (!)* only.

4.2 Input–output matrix and resource consumption matrix

We now turn to provide the definition of the two central matrix-valued quantities in Leontief's model. We will also highlight their main characteristic features.

4.2.1 Input–output matrix

Suppose the **reference period of time** has ended for the economic system in question, i.e. the stationary **flows of goods** have stopped eventually. We now want to take stock of the **numbers of goods** that have been delivered by each of the n **economic agents** in the system. Say that during the reference period considered, agent 1 delivered of their good the number n_{11} to themselves, the number n_{12} to agent 2, the number n_{13} to agent 3, and so on, and, lastly, the number n_{1n} to agent n.

The number delivered by agent 1 to external consumers shall be denoted by y_1. Since in this model a good produced *cannot* all of a sudden disappear again, and since by Assumption 3 above the number of goods supplied must be equal to the number of goods demanded, we find that for the total output of agent 1 it holds that $q_1 := n_{11} + \ldots + n_{1j} + \ldots + n_{1n} + y_1$. Analogous relations hold for the total output q_2, q_3, ..., q_n of each of the remaining $n-1$ agents. We thus obtain the intermediate result

$$\begin{aligned} q_1 &= n_{11} + \ldots + n_{1j} + \ldots + n_{1n} + y_1 > 0 \quad (4.1) \\ &\vdots \\ q_i &= n_{i1} + \ldots + n_{ij} + \ldots + n_{in} + y_i > 0 \quad (4.2) \\ &\vdots \\ q_n &= n_{n1} + \ldots + n_{nj} + \ldots + n_{nn} + y_n > 0 \; . \quad (4.3) \end{aligned}$$

This simple system of **balance equations** can be summarised in terms of a standard **input–output table** as follows:

[Values in units]	agent 1	$\cdots$	agent j	$\cdots$	agent n	external consumers	Σ: total output
agent 1	n_{11}	$\ldots$	n_{1j}	$\ldots$	n_{1n}	y_1	q_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
agent i	n_{i1}	$\ldots$	n_{ij}	$\ldots$	n_{in}	y_i	q_i
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
agent n	n_{n1}	$\ldots$	n_{nj}	$\ldots$	n_{nn}	y_n	q_n

The first column of this table lists all the n different **sources of flows of goods** (or suppliers of goods), while its first row shows the $n+1$ different **sinks of flows of goods** (or consumers of goods). The last column contains the total output of each of the n agents in the **reference period of time**.

Next we compute for each of the n agents the respective values of the *non-negative ratios*

$$P_{ij} := \frac{\text{INPUT (in units) of agent } i \text{ for agent } j \text{ (during reference period)}}{\text{OUTPUT (in units) of agent } j \text{ (during reference period)}} \; , \quad (4.4)$$

or, employing a compact and economical index notation,[1]

$$\boxed{P_{ij} := \frac{n_{ij}}{q_j} \; ,} \quad (4.5)$$

[1]Note that the normalisation quantities in these ratios P_{ij} are given by the total output q_j of the receiving agent j and *not* by the total output q_i of the supplying agent i. In the latter case the P_{ij} would represent percentages of the total output q_i.

with $i, j = 1, \dots, n$. These $n \times n = n^2$ different ratios may be naturally viewed as the elements of a quadratic matrix $\mathbf{P}$ of format $(n \times n)$. In general, this matrix is given by

$$\boxed{\mathbf{P} = \left(\begin{array}{ccccc} \frac{n_{11}}{n_{11}+\ldots+n_{1j}+\ldots+n_{1n}+y_1} & \cdots & \frac{n_{1j}}{n_{j1}+\ldots+n_{jj}+\ldots+n_{jn}+y_j} & \cdots & \frac{n_{1n}}{n_{n1}+\ldots+n_{nj}+\ldots+n_{nn}+y_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{n_{i1}}{n_{11}+\ldots+n_{1j}+\ldots+n_{1n}+y_1} & \cdots & \frac{n_{ij}}{n_{j1}+\ldots+n_{jj}+\ldots+n_{jn}+y_j} & \cdots & \frac{n_{in}}{n_{n1}+\ldots+n_{nj}+\ldots+n_{nn}+y_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{n_{n1}}{n_{11}+\ldots+n_{1j}+\ldots+n_{1n}+y_1} & \cdots & \frac{n_{nj}}{n_{j1}+\ldots+n_{jj}+\ldots+n_{jn}+y_j} & \cdots & \frac{n_{nn}}{n_{n1}+\ldots+n_{nj}+\ldots+n_{nn}+y_n} \end{array}\right)} , \tag{4.6}$$

and is referred to as Leontief's **input–output matrix** of the stationary economic system under investigation.

For the very simple case with just $n = 3$ producing agents, the input–output matrix reduces to

$$\mathbf{P} = \left(\begin{array}{ccc} \frac{n_{11}}{n_{11}+n_{12}+n_{13}+y_1} & \frac{n_{12}}{n_{21}+n_{22}+n_{23}+y_2} & \frac{n_{13}}{n_{31}+n_{32}+n_{33}+y_3} \\ \frac{n_{21}}{n_{11}+n_{12}+n_{13}+y_1} & \frac{n_{22}}{n_{21}+n_{22}+n_{23}+y_2} & \frac{n_{23}}{n_{31}+n_{32}+n_{33}+y_3} \\ \frac{n_{31}}{n_{11}+n_{12}+n_{13}+y_1} & \frac{n_{32}}{n_{21}+n_{22}+n_{23}+y_2} & \frac{n_{33}}{n_{31}+n_{32}+n_{33}+y_3} \end{array}\right) .$$

It is important to realise that for an actual economic system the input–output matrix $\mathbf{P}$ can be determined only once *the reference period of time chosen has come to an end*.

The utility of Leontief's stationary input–output matrix model is in its application for the purpose of **forecasting**. This is done on the basis of an **extrapolation**, namely by *assuming* that an input–output matrix $\mathbf{P}_{\text{reference period}}$ obtained from data taken during a specific reference period also is valid (to an acceptable degree of accuracy) during a subsequent period, i.e.,

$$\boxed{\mathbf{P}_{\text{subsequent period}} \approx \mathbf{P}_{\text{reference period}} ,} \tag{4.7}$$

or, in component form,

$$P_{ij}\big|_{\text{subsequent period}} = \left.\frac{n_{ij}}{q_j}\right|_{\text{subsequent period}} \approx P_{ij}\big|_{\text{reference period}} = \left.\frac{n_{ij}}{q_j}\right|_{\text{reference period}} . \tag{4.8}$$

In this way it becomes possible to compute for a given (idealised) economic system approximate numbers of **INPUT quantities** required during a near future production period from the known numbers of **OUTPUT quantities** of the most recent production period. Long-term empirical experience has shown that this method generally leads to useful results to a reasonable approximation. All of these calculations are grounded on linear relationships describing the quantitative aspects of stationary flows of goods, as we will soon elucidate.

4.2.2 Resource consumption matrix

The second matrix-valued quantity central to Leontief's stationary model is the **resource consumption matrix** $\mathbf{R}$. This may be interpreted as providing a recipe for the amounts of the m different kinds of external resources (the exogenous **INPUT quantities**) that are needed in the production of the n goods (the **OUTPUT quantities**). Its elements are defined as the ratios

$$\boxed{R_{ij} := \text{amounts (in units) required of resource } i \text{ for the production of one unit of good } j ,} \tag{4.9}$$

with $i = 1, \ldots, m$ und $j = 1, \ldots, n$. The rows of matrix $\mathbf{R}$ thus contain information concerning the m resources, the columns information concerning the n goods. Note that in general the $(m \times n)$ **resource consumption matrix** $\mathbf{R}$ is *not (!)* a quadratic matrix and, therefore, in general *not* invertible.

4.3 Stationary linear flows of goods

4.3.1 Flows of goods: endogenous INPUT to total OUTPUT

We now turn to a quantitative description of the stationary **flows of goods** that are associated with the **total output** $\boldsymbol{q}$ during a specific period of time considered. According to Leontief's Assumption 1, there exists a *linear* functional relationship between the endogenous vector-valued **INPUT quantity** $\boldsymbol{q} - \boldsymbol{y}$ and the vector-valued **OUTPUT quantity** $\boldsymbol{q}$. This may be represented in terms of a matrix-valued relationship as

$$\boxed{\boldsymbol{q} - \boldsymbol{y} = \mathbf{P}\boldsymbol{q} \quad \Leftrightarrow \quad q_i - y_i = \sum_{j=1}^{n} P_{ij} q_j \ ,} \tag{4.10}$$

with $i = 1, \ldots, n$, in which the **input–output matrix** $\mathbf{P}$ takes the role of mediating a mapping between either of these vector-valued quantities. According to Assumption 2, the elements of the **input–output matrix** $\mathbf{P}$ remain *constant* for the period of time considered, i.e. the corresponding flows of goods are assumed to be **stationary**.

Relation (4.10) may also be motivated from an alternative perspective that takes the **physical sciences** as a guidline. Namely, the total numbers $\boldsymbol{q}$ of the n goods produced during the period of time considered which, by Assumption 3, are equal to the numbers supplied of the n goods satisfy a **conservation law**: "whatever has been produced of the n goods during the period of time considered *cannot* get lost in this period." In quantitative terms this simple relationship may be cast into the form

$$\underbrace{\boldsymbol{q}}_{\text{total output}} = \underbrace{\boldsymbol{y}}_{\text{final demand (exogenous)}} + \underbrace{\mathbf{P}\boldsymbol{q}}_{\text{deliveries to production sector (endogenous)}} \ .$$

For computational purposes this central stationary flow of goods relation (4.10) may be rearranged as is convenient. In this context it is helpful to make use of the matrix identity $\boldsymbol{q} = \mathbf{1}\boldsymbol{q}$, where $\mathbf{1}$ denotes the $(\boldsymbol{n} \times \boldsymbol{n})$**-unit matrix** [cf. Eq. (2.3)].

Examples:

(i) given/known: $\mathbf{P}$, $\boldsymbol{q}$

Then it applies that

$$\boldsymbol{y} = (\mathbf{1} - \mathbf{P})\boldsymbol{q} \quad \Leftrightarrow \quad y_i = \sum_{j=1}^{n} (\delta_{ij} - P_{ij}) q_j \ , \tag{4.11}$$

with $i = 1, \ldots, n$; $(\mathbf{1} - \mathbf{P})$ represents the invertible **technology matrix** of the economic system regarded.

(ii) given/known: $\mathbf{P}$, $\boldsymbol{y}$

Then it holds that

$$\boldsymbol{q} = (\mathbf{1} - \mathbf{P})^{-1}\boldsymbol{y} \quad \Leftrightarrow \quad q_i = \sum_{j=1}^{n}(\delta_{ij} - P_{ij})^{-1}y_j \ , \tag{4.12}$$

with $i = 1, \ldots, n$; $(\mathbf{1} - \mathbf{P})^{-1}$ here denotes the **total demand matrix**, i.e., the inverse of the technology matrix.

4.3.2 Flows of goods: exogenous INPUT to total OUTPUT

Likewise, by Assumption 1, a *linear* functional relationship is supposed to exist between the exogenous vector-valued **INPUT quantity** $\boldsymbol{v}$ and the vector-valued **OUTPUT quantity** $\boldsymbol{q}$. In matrix language this can be expressed by

$$\boxed{\boldsymbol{v} = \mathbf{R}\boldsymbol{q} \quad \Leftrightarrow \quad v_i = \sum_{j=1}^{n} R_{ij}q_j \ ,} \tag{4.13}$$

with $i = 1, \ldots, m$. By Assumption 2, the elements of the **resource consumption matrix** $\mathbf{R}$ remain *constant* during the period of time considered, i.e., the corresponding resource flows are supposed to be **stationary**.

By combination of Eqs. (4.13) and (4.12), it is possible to compute the numbers $\boldsymbol{v}$ of resources required (during the period of time considered) for the production of the n goods for given final demand $\boldsymbol{y}$. It applies that

$$\boldsymbol{v} = \mathbf{R}\boldsymbol{q} = \mathbf{R}(\mathbf{1} - \mathbf{P})^{-1}\boldsymbol{y} \quad \Leftrightarrow \quad v_i = \sum_{j=1}^{n}\sum_{k=1}^{n} R_{ij}(\delta_{jk} - P_{jk})^{-1}y_k \ , \tag{4.14}$$

with $i = 1, \ldots, m$.

GDC: For problems with $n \leq 5$, and known matrices $\mathbf{P}$ and $\mathbf{R}$, Eqs. (4.11), (4.12) and (4.14) can be immediately used to calculate the quantities $\boldsymbol{q}$ from given quantities $\boldsymbol{y}$, or vice versa.

4.4 Outlook

Leontief's input–output matrix model may be extended in a straightforward fashion to include more advanced considerations of **economic theory**. Supposing a closed though not necessarily stationary economic system G comprising n interdependent **economic agents** producing n different goods, one may assign **monetary values** to the **INPUT quantity** $\boldsymbol{v}$ as well as to the **OUTPUT quantities** $\boldsymbol{q}$ and $\boldsymbol{y}$ of the system. Besides the numbers of goods produced and the associated flows of goods one may monitor with respect to G for a given period of time, one can in addition analyse in time and space the **amount of money** coupled to the different goods, and the corresponding **flows of money**. However, contrary to the number of goods, in general there does *not* exist a **conservation law** for the amount of money with respect to G. This may render the analysis of flows of money more difficult, because, in the sense of an **increase in value**, *money can either be generated inside G during the period of time considered or it can likewise be annihilated*; it is *not* just limited to either flowing into respectively flowing out of G. Central to considerations of this kind is a **balance equation** for the amount of money contained in G during a given period of time, which is an *additive* quantity. Such balance equations constitute familiar tools in **Physics** (cf. Herrmann (2003) [15, p 7ff]). Its structure in the present case is given by[2]

$$\left(\begin{array}{c}\textbf{rate of change in time}\\ \text{of the }\textbf{amount of money}\\ \text{in } G \text{ [in CU/TU]}\end{array}\right) = \left(\begin{array}{c}\textbf{flux of money}\\ \text{into } G \text{ [in CU/TU]}\end{array}\right) + \left(\begin{array}{c}\textbf{rate of generation of money}\\ \text{in } G \text{ [in CU/TU]}\end{array}\right) .$$

Note that, with respect to G, both fluxes of money and rates of generation of money can in principle possess either sign, positive or negative. To deal with these quantitative issues properly, one requires the technical tools of the **differential and integral calculus** which we will discuss at an elementary level in Chs. 7 and 8. We make contact here with the interdisciplinary science of **Econophysics** (cf., e.g., Bouchaud and Potters (2003) [5]), a very interesting and challenging subject which, however, is beyond the scope of these lecture notes.

Leontief's input–output matrix model, and its possible extension as outlined here, provide the quantitative basis for considerations of economical ratios of the kind

$$\frac{\text{OUTPUT [in units]}}{\text{INPUT [in units]}} ,$$

as mentioned in the Introduction. In addition, *dimensionless* (scale-invariant) ratios of the form

$$\frac{\text{REVENUE [in CU]}}{\text{COSTS [in CU]}} ,$$

referred to as **economic efficiency**, can be computed for and compared between different economic systems and their underlying production sectors. In Ch. 7 we will briefly reconsider this issue.

[2] Here the symbols CU and TU denote "currency units" and "time units," respectively.

Chapter 5

Linear programming

On the backdrop of the **economic principle**, we discuss in this chapter a special class of quantitative problems that frequently arise in specific practical applications in **Business** and **Management**. Generally one distinguishes between two variants of the **economic principle**: either (i) to draw maximum utility from limited resources, or (ii) to reach a specific target with minimum effort (costs). With regard to the ratio $(\text{OUTPUT})/(\text{INPUT})$ put into focus in the Introduction, the issue is to find an **optimal value** for this ratio under given **boundary conditions**. This aim can be realised either (i) by increasing the (positive) value of the numerator for fixed (positive) value of the denominator, or (ii) by decreasing the (positive) value of the denominator for fixed (positive) value of the numerator. The class of quantitative problems to be looked at in some detail in this chapter typically relate to boundary conditions according to case (i).

5.1 Exposition of a quantitative problem

To be maximised is a (non-negative) real-valued quantity z, which depends in a *linear functional fashion* on a fixed number of n (non-negative) real-valued variables $x_1, \ldots, x_n$. We suppose that the n variables $x_1, \ldots, x_n$ in turn are constrained by a fixed number m of algebraic conditions, which also are assumed to depend on $x_1, \ldots, x_n$ in a *linear fashion*. These m constraints, or restrictions, shall have the character of imposing upper limits on m different kinds of resources.

Def.: Consider a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, a vector $\boldsymbol{b} \in \mathbb{R}^{m\times 1}$, two vectors $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^{n\times 1}$, and a constant $d \in \mathbb{R}$. A quantitative problem of the form

$$\boxed{\max\left\{z = \boldsymbol{c}^T \cdot \boldsymbol{x} + d\,|\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\right\}\ ,} \tag{5.1}$$

or, expressed in terms of a component notation,

$$
\begin{aligned}
\text{max } z(x_1,\dots,x_n) = c_1x_1 + \dots + c_nx_n + d & & & \quad (5.2)\\
a_{11}x_1 + \dots + a_{1n}x_n &\leq& b_1 & \quad (5.3)\\
&\vdots& &\\
a_{m1}x_1 + \dots + a_{mn}x_n &\leq& b_m & \quad (5.4)\\
x_1 &\geq& 0 & \quad (5.5)\\
&\vdots& &\\
x_n &\geq& 0\,, & \quad (5.6)
\end{aligned}
$$

is referred to as a **standard maximum problem of linear programming** with n real-valued variables. The different quantities and relations appearing in this definition are called

- $z(x_1,\dots,x_n)$ — **linear objective function**, the dependent variable,
- $x_1,\dots,x_n$ — n **independent variables**,
- $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$ — m **restrictions**,
- $\boldsymbol{x} \geq \mathbf{0}$ — n **non-negativity constraints**.

<u>**Remark:**</u> In an analogous fashion one may also formulate a **standard minimum problem of linear programming**, which can be cast into the form

$$\text{min}\left\{z = \boldsymbol{c}^T\cdot\boldsymbol{x} + d\,|\,\mathbf{A}\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\right\} \ .$$

In this case, the components of the vector $\boldsymbol{b}$ need to be interpreted as lower limits on certain capacities.

For given linear objective function $z(x_1,\dots,x_n)$, the set of points $\boldsymbol{x} = (x_1,\dots,x_n)^T$ satisfying the condition

$$\boxed{z(x_1,\dots,x_n) = C = \text{constant} \in \mathbb{R}\ ,} \quad (5.7)$$

for fixed value of C, is referred to as an **isoquant** of z. **Isoquants** of linear objective functions of $n = 2$ independent variables constitute straight lines, of $n = 3$ independent variables Euclidian planes, of $n = 4$ independent variables Euclidian 3-spaces (or hyperplanes), and of $n \geq 5$ independent variables Euclidian $(n-1)$-spaces (or hyperplanes).

In the simplest cases of **linear programming**, the linear **objective function** z depends on $n = 2$ **variables** x_1 and x_2 only. An illustrative and efficient method of solving problems of this kind will be looked at in the following section.

5.2 Graphical method for solving problems with two independent variables

The systematic graphical solution method of standard maximum problems of **linear programming** with $n = 2$ independent variables comprises the following steps:

1. Derivation of the **linear objective function**

$$z(x_1, x_2) = c_1x_1 + c_2x_2 + d$$

   in dependence on the **variables** x_1 and x_2.

2. Identification in the x_1, x_2–plane of the **feasible region** D of z which is determined by the m restrictions imposed on x_1 and x_2. Specifically, D constitutes the domain of z (cf. Ch. 7).

3. Plotting in the x_1, x_2–plane of the projection of the **isoquant** of the linear objective function z which intersects the origin ($0 = x_1 = x_2$). When $c_2 \neq 0$, this projection is described by the equation

$$x_2 = -(c_1/c_2)x_1 \ .$$

4. Erecting in the origin of the x_1, x_2–plane the **direction of optimisation** for z which is determined by the constant z-gradient

$$(\boldsymbol{\nabla} z)^T = \begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \frac{\partial z}{\partial x_2} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \ .$$

5. **Parallel displacement** in the x_1, x_2–plane of the projection of the $(0, 0)$-isoquant of z along the direction of optimisation $(\boldsymbol{\nabla} z)^T$ across the feasible region D out to a distance where the projected isoquant just about touches D.

6. Determination of the **optimal solution** $(x_{1\text{O}}, x_{2\text{O}})$ as the point resp. set of points of intersection between the displaced projection of the $(0, 0)$-isoquant of z and the *far* boundary of D.

7. Computation of the **optimal value** of the linear objective function $z_\text{O} = z(x_{1\text{O}}, x_{2\text{O}})$ from the optimal solution $(x_{1\text{O}}, x_{2\text{O}})$.

8. Specification of potential **remaining resources** by substitution of the optimal solution $(x_{1\text{O}}, x_{2\text{O}})$ into the m restrictions.

In general one finds that for a linear **objective function** z with $n = 2$ **independent variables** x_1 and x_2, the feasible region D, when *non-empty and bounded*, constitutes an area in the x_1, x_2–plane with straight edges and a certain number of vertices. In these cases, the **optimal values** of the linear objective function z are always to be found either at the vertices or on the edges of the feasible region D. When D is an empty set, then there exists no solution to the corresponding linear programming problem. When D is unbounded, again there may not exist a solution to the linear programming problem, but this then depends on the specific circumstances that apply.

Remark: To solve a **standard minimum problem of linear programming** with $n = 2$ independent variables by means of the graphical method, one needs to parallelly displace in the x_1, x_2–plane the projection of the $(0, 0)$-isoquant of z along the direction of optimisation $(\boldsymbol{\nabla} z)^T$ until contact is made with the feasible region D for the first time. The optimal solution is then given by the point resp. set of points of intersection between the displaced projection of the $(0, 0)$-isoquant of z and the *near* boundary of D.

5.3 Dantzig's simplex algorithm

The main disadvantage of the graphical solution method is its limitation to problems with only $n = 2$ independent variables. In actual practice, however, one is often concerned with **linear programming problems** that depend on *more* than two **independent variables**. To deal with these more complex problems in a systematic fashion, the US-American mathematician George Bernard Dantzig (1914–2005) has devised during the 1940ies an efficient algorithm which can be programmed on a computer in a fairly straightforward fashion; cf. Dantzig (1949,1955) [8, 9].

In mathematics, **simplex** is an alternative name used to refer to a convex polyhedron, i.e., a body of finite (hyper-)volume in two or more dimensions bounded by linear (hyper-)surfaces which intersect in linear edges and vertices. In general the feasible regions of linear programming problems constitute such simplexes. Since the **optimal solutions** for the **independent variables** of **linear programming problems**, when they exist, are always to be found at a vertex or along an edge of simplex feasible regions, Dantzig developed his so-called **simplex algorithm** such that it systematically scans the edges and vertices of a feasible region to identify the **optimal solution** (when it exists) in as few steps as possible.

The starting point be a **standard maximum problem of linear programming** with n **independent variables** in the form of relations (5.2)–(5.6). First, by introducing m non-negative **slack variables** $s_1, \ldots, s_m$, one transforms the m linear **restrictions** (inequalities) into an equivalent set of m linear equations. In this way, potential differences between the left-hand and the right-hand sides of the m inequalities are represented by the slack variables. In combination with the defining equation of the linear **objective function** z, one thus is confronted with a system of $1 + m$ linear algebraic equations for the $1 + n + m$ variables $z, x_1, \ldots, x_n, s_1, \ldots, s_m$, given by

Maximum problem of linear programming in canonical form

$$
\begin{aligned}
z - c_1x_1 - c_2x_2 - \ldots - c_nx_n &= d && (5.8)\\
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 &= b_1 && (5.9)\\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + s_2 &= b_2 && (5.10)\\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + s_m &= b_m\,. && (5.11)
\end{aligned}
$$

As discussed previously in Ch. 3, a system of linear algebraic equations of format $(1 + m) \times (1 + n + m)$ is *under-determined* and so, at most, allows for *multiple solutions*. The general

$(1 + n + m)$-dimensional solution vector

$$\boldsymbol{x}_L = (z_L, x_{1,L}, \ldots, x_{n,L}, s_{1,L}, \ldots, s_{m,L})^T \tag{5.12}$$

thus contains n variables the values of which can be chosen *arbitrarily*. It is very important to be aware of this fact. It implies that, given the linear system is solvable in the first place, one has a *choice* amongst different solutions, and so one can pick the solution which proves **optimal** for the given problem at hand. **Dantzig's simplex algorithm** constitues a tool for determining such an **optimal solution** in a systematic way.

Let us begin by transferring the coefficients and right-hand sides (RHS) of the under-determined linear system introduced above into a particular kind of **simplex tableau**.

Initial simplex tableau

$$\begin{array}{c|cccc|cccc|c}
z & x_1 & x_2 & \ldots & x_n & s_1 & s_2 & \ldots & s_m & \text{RHS} \\
\hline
1 & -c_1 & -c_2 & \ldots & -c_n & 0 & 0 & \ldots & 0 & d \\
\hline
0 & a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 & b_1 \\
0 & a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 & b_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a_{m1} & a_{m2} & \ldots & a_{mn} & 0 & 0 & \ldots & 1 & b_m
\end{array} \tag{5.13}$$

In such a **simplex tableau** one distinguishes so-called **basis variables** from **non-basis variables**. Basis variables are those that contain in their respective columns in the number tableau a $(1+m)$-component canonical unit vector [cf. Eq. (1.10)]; in total the **simplex tableau** contains $1 + m$ of these. Non-basis variables are the remaining ones that do *not* contain a canonical basis vector in their respective columns; there exist n of this kind. The complete basis can thus be perceived as spanning a $(1 + m)$-dimensional Euclidian space $\mathbb{R}^{1+m}$. Initially, always z and the m slack variables $s_1, \ldots, s_m$ constitute the basis variables, while the n independent variables $x_1, \ldots, x_n$ classify as non-basis variables [cf. the initial tableau (5.13)]. The corresponding so-called (first) **basis solution** has the general appearance

$$\boldsymbol{x}_{B_1} = (z_{B_1}, x_{1,B_1}, \ldots, x_{n,B_1}, s_{1,B_1}, \ldots, s_{m,B_1})^T = (d, 0, \ldots, 0, b_1, \ldots, b_m)^T \ ,$$

since, for simplicity, each of the n arbitrarily specifiable non-basis variables may be assigned the special value zero. In this respect basis solutions will always be *special solutions* (as opposed to general ones) of the under-determined system (5.8)–(5.11) — the maximum problem of linear programming in canonical form.

Central aim of the **simplex algorithm** is to bring as many of the n **independent variables** $x_1, \ldots, x_n$ as possible into the $(1 + m)$-dimensional basis, at the expense of one of the m **slack variables** $s_1, \ldots, s_m$, one at a time, in order to construct successively more favourable special vector-valued solutions to the optimisation problem at hand. Ultimately, the **simplex algorithm** needs to be viewed as a special variant of Gaußian elimination as discussed in Ch. 3, with a set of systematic instructions concerning allowable equivalence transformations of the underlying under-determined linear system (5.8)–(5.11), resp. the initial **simplex tableau** (5.13). This set of systematic algebraic simplex operations can be summarised as follows:

Simplex operations

S1: Does the current simplex tableau show $-c_j \geq 0$ for all $j \in \{1, \ldots, n\}$? If so, then the corresponding basis solution is **optimal**. *END*. Otherwise goto S2.

S2: Choose a **pivot column index** $j^* \in \{1, \ldots, n\}$ such that $-c_{j^*} := \min\{-c_j | j \in \{1, \ldots, n\}\} < 0$.

S3: Is there a row index $i^* \in \{1, \ldots, m\}$ such that $a_{i^*j^*} > 0$? If not, the objective function z is unbounded from above. *END*. Otherwise goto S4.

S4: Choose a **pivot row index** i^* such that $a_{i^*j^*} > 0$ and $b_{i^*}/a_{i^*j^*} := \min\{b_i/a_{i^*j^*} | a_{i^*j^*} > 0, i \in \{1, \ldots, m\}\}$. Perform a **pivot operation** with the **pivot element** $a_{i^*j^*}$. Goto S1.

When the final **simplex tableau** has been arrived at, one again assigns the non-basis variables the value zero. The values of the final basis variables corresponding to the **optimal solution** of the given **linear programming problem** are then to be determined from the final **simplex tableau** by backward substitution, beginning at the bottom row. Note that slack variables with positive values belonging to the basis variables in the **optimal solution** provide immediate information on existing remaining capacities in the problem at hand.

Chapter 6

Elementary financial mathematics

In this chapter we want to provide a brief introduction into some basic concepts of **financial mathematics**. As we will try to emphasise, many applications of these concepts (that have immediate practical relevance) are founded on only two simple and easily accessible mathematical structures: the so-called arithmetical and geometrical real-valued sequences and their associated finite series.

6.1 Arithmetical and geometrical sequences and series

6.1.1 Arithmetical sequence and series

An **arithmetical sequence** of $n \in \mathbb{N}$ real numbers $a_n \in \mathbb{R}$,

$$(a_n)_{n\in\mathbb{N}} \ ,$$

is defined by the property that the **difference** d between neighbouring elements in the sequence be *constant*, i.e., for $n > 1$

$$\boxed{a_n - a_{n-1} =: d = \text{constant} \neq 0 \ ,} \tag{6.1}$$

with $a_n, a_{n-1}, d \in \mathbb{R}$. Given this recursive formation rule, one may infer the **explicit representation** of an **arithmetical sequence** as

$$a_n = a_1 + (n-1)d \quad \text{with} \quad n \in \mathbb{N} \ . \tag{6.2}$$

Note that any **arithmetical sequence** is *uniquely determined* by the two free parameters a_1 and d, the starting value of the sequence and the constant difference between neighbours in the sequence, respectively. Equation (6.2) shows that the elements a_n in a non-trivial **arithmetical sequence** exhibit either **linear** growth or **linear** decay with n.

When one calculates for an **arithmetical sequence** of $n + 1$ real numbers the **arithmetical mean** of the immediate neighbours of any particular element a_n (with $n \geq 2$), one finds that

$$\frac{1}{2}\left(a_{n-1} + a_{n+1}\right) = \frac{1}{2}\left(a_1 + (n-2)d + a_1 + nd\right) = a_1 + (n-1)d = a_n \ . \tag{6.3}$$

Summation of the first n elements of an arbitrary **arithmetical sequence** of real numbers leads to a **finite arithmetical series**,

$$S_n := a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} [a_1 + (k-1)d] = na_1 + \frac{d}{2}(n-1)n \; . \tag{6.4}$$

In the last algebraic step use was made of the Gaußian **identity**[1] (cf., e.g., Bosch (2003) [6, p 21])

$$\boxed{\sum_{k=1}^{n-1} k \equiv \frac{1}{2}(n-1)n \; .} \tag{6.5}$$

6.1.2 Geometrical sequence and series

A **geometrical sequence** of $n \in \mathbb{N}$ real numbers $a_n \in \mathbb{R}$,

$$(a_n)_{n\in\mathbb{N}} \; ,$$

is defined by the property that the **quotient** q between neighbouring elements in the sequence be *constant*, i.e., for $n > 1$

$$\boxed{\frac{a_n}{a_{n-1}} =: q = \text{constant} \neq 0 \; ,} \tag{6.6}$$

with $a_n, a_{n-1} \in \mathbb{R}$ and $q \in \mathbb{R}\backslash\{0, 1\}$. Given this recursive formation rule, one may infer the **explicit representation** of a **geometrical sequence** as

$$a_n = a_1 q^{n-1} \quad \text{with} \quad n \in \mathbb{N}. \tag{6.7}$$

Note that any **geometrical sequence** is *uniquely determined* by the two free parameters a_1 and q, the starting value of the sequence and the constant quotient between neighbours in the sequence, respectively. Equation (6.7) shows that the elements a_n in a non-trivial **geometrical sequence** exhibit either **exponential** growth or **exponential** decay with n (cf. Sec. 7.1.4).

When one calculates for a **geometrical sequence** of $n+1$ real numbers the **geometrical mean** of the immediate neighbours of any particular element a_n (with $n \geq 2$), one finds that

$$\sqrt{a_{n-1} \cdot a_{n+1}} = \sqrt{a_1 q^{n-2} \cdot a_1 q^n} = a_1 q^{n-1} = a_n \; . \tag{6.8}$$

Summation of the first n elements of an arbitrary **geometrical sequence** of real numbers leads to a **finite geometrical series**,

$$S_n := a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left[a_1 q^{k-1}\right] = a_1 \sum_{k=0}^{n-1} q^k = a_1 \frac{q^n - 1}{q - 1} \; . \tag{6.9}$$

In the last algebraic step use was made of the **identity** (cf., e.g., Bosch (2003) [6, p 27])

$$\boxed{\sum_{k=0}^{n-1} q^k \equiv \frac{q^n - 1}{q - 1} \quad \text{for} \quad q \in \mathbb{R}\backslash\{0, 1\} \; .} \tag{6.10}$$

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[1] Analogously, the modified Gaußian identity $\sum_{k=1}^{n}(2k-1) \equiv n^2$ applies.

6.2 Interest and compound interest

Let us consider a first rather simple interest model. Suppose given an **initial capital** of positive value $K_0 > 0$ CU paid into a bank account at some initial instant, and a time interval consisting of $n \in \mathbb{N}$ **periods** of equal lengths. At the end of each period, the money in this bank account shall earn a service fee corresponding to an **interest rate** of $p > 0$ percent. Introducing the dimensionless **interest factor**[2]

$$q := 1 + \frac{p}{100} > 1 \;, \tag{6.11}$$

one finds that by the end of the first interest period a total capital of value (in CU)

$$K_1 = K_0 + K_0 \cdot \frac{p}{100} = K_0 \left(1 + \frac{p}{100}\right) = K_0 q$$

will have accumulated. When the entire time interval of n interest periods has ended, a **final capital** worth of (in CU)

$$\boxed{\text{recursively: } K_n = K_{n-1} q \;, \quad n \in \mathbb{N} \;,} \tag{6.12}$$

will have accumulated, where K_{n-1} denotes the capital (in CU) accumulated by the end of $n-1$ interest periods. This recursive representation of the growth of the initial capital K_0 due to a total of n interest payments and the effect of **compound interest** makes explicit the direct link with the mathematical structure of a **geometrical sequence** of real numbers (6.6).

It is a straightforward exercise to show that in this simple interest model the final capital K_n is related to the initial capital K_0 by

$$\boxed{\text{explicitly: } K_n = K_0 q^n \;, \quad n \in \mathbb{N} \;.} \tag{6.13}$$

Note that this equation links the four non-negative quantities K_n, K_0, q and n to one another. Hence, knowing the values of three of these quantities, one may solve Eq. (6.13) to obtain the value of the fourth. For example, solving Eq. (6.13) for K_0 yields

$$K_0 = \frac{K_n}{q^n} =: B_0 \;. \tag{6.14}$$

In this particular variant, K_0 is referred to as the **present value** B_0 of the final capital K_n; this is obtained from K_n by an n-fold division with the interest factor q.

Further possibilities of re-arranging Eq. (6.13) are:

(i) Solving for the **interest factor** q:

$$q = \sqrt[n]{\frac{K_n}{K_0}} \;, \tag{6.15}$$

(ii) Solving for the **contract period** n:

$$n = \frac{\ln\left(K_n/K_0\right)}{\ln(q)} \;. \tag{6.16}$$

[2]Inverting this defining relation for q leads to $p = 100 \cdot (q-1)$.

From now on, $n \in \mathbb{N}$ shall denote the number of full years that have passed in a specific interest model.

Now we turn to discuss a second, more refined interest model. Let us suppose that an **initial capital** $K_0 > 0$ CU earns interest during one full year $m \in \mathbb{N}$ times at the mth part of a **nominal annual interest rate** $p_{\text{nom}} > 0$. At the end of the first out of m periods of equal length $1/m$, the initial capital K_0 will thus have increased to an amount

$$K_{1/m} = K_0 + K_0 \cdot \frac{p_{\text{nom}}}{m \cdot 100} = K_0 \left(1 + \frac{p_{\text{nom}}}{m \cdot 100}\right) .$$

By the end of the kth ($k \leq m$) out of m periods the **account balance** will have become

$$K_{k/m} = K_0 \left(1 + \frac{p_{\text{nom}}}{m \cdot 100}\right)^k ;$$

the interest factor $\left(1 + \frac{p_{\text{nom}}}{m \cdot 100}\right)$ will then have been applied k times to K_0. At the end of the full year, K_0 in this interest model will have increased to

$$K_1 = K_{m/m} = K_0 \left(1 + \frac{p_{\text{nom}}}{m \cdot 100}\right)^m , \quad m \in \mathbb{N} .$$

This relation defines an **effective interest factor**

$$q_{\text{eff}} := \left(1 + \frac{p_{\text{nom}}}{m \cdot 100}\right)^m , \tag{6.17}$$

with associated **effective annual interest rate**

$$\boxed{p_{\text{eff}} = 100 \cdot \left[\left(1 + \frac{p_{\text{nom}}}{m \cdot 100}\right)^m - 1\right] , \quad m \in \mathbb{N} ,} \tag{6.18}$$

obtained from re-arranging $q_{\text{eff}} = 1 + \frac{p_{\text{eff}}}{100}$.

When, ultimately, $n \in \mathbb{N}$ full years will have passed in the second interest model, the initial capital K_0 will have been transformed into a final capital of value

$$\boxed{K_n = K_0 \left(1 + \frac{p_{\text{nom}}}{m \cdot 100}\right)^{n \cdot m} = K_0 q_{\text{eff}}^n , \quad n, m \in \mathbb{N} .} \tag{6.19}$$

The **present value** B_0 of K_n is thus given by

$$B_0 = \frac{K_n}{q_{\text{eff}}^n} = K_0 . \tag{6.20}$$

Finally, as a third interest model relevant to applications in **Finance**, we turn to consider the concept of **installment savings**. For simplicity, let us restrict our discussion to the case when $n \in \mathbb{N}$ equal **installments** of *constant* value $E > 0$ CU are paid into an account that earns $p > 0$ percent annual interest (i.e., $q > 1$) at the beginning of each of n full years. The initial account balance be $K_0 = 0$ CU. At the end of a first full year in this interest model, the account balance will have increased to

$$K_1 = E + E \cdot \frac{p}{100} = E\left(1 + \frac{p}{100}\right) = Eq\ .$$

At the end of two full years one finds, substituting for K_1,

$$K_2 = (K_1 + E)q = (Eq + E)q = E(q^2 + q) = Eq(q + 1)\ .$$

At the end of n full years we have, recursively substituting for K_{n-1}, K_{n-2}, etc.,

$$K_n = (K_{n-1} + E)q = \cdots = E(q^n + \ldots + q^2 + q) = Eq(q^{n-1} + \ldots + q + 1) = Eq\sum_{k=0}^{n-1} q^k\ .$$

Using the identity (6.10), since presently $q > 1$, the **account balance** at the end of n full years can be reduced to the expression

$$\boxed{K_n = Eq\,\frac{q^n - 1}{q - 1}\ , \quad q \in \mathbb{R}_{>1}\ , \quad n \in \mathbb{N}\ .} \tag{6.21}$$

The **present value** B_0 associated with K_n is obtained by n-fold division of K_n with the interest factor q:

$$B_0 := \frac{K_n}{q^n} \overbrace{=}^{\text{Eq. } 6.21} = \frac{E(q^n - 1)}{q^{n-1}(q - 1)}\ . \tag{6.22}$$

This gives the value of an initial capital B_0 which will grow to the *same* final value K_n after n annual interest periods with constant interest factor $q > 1$.

Lastly, re-arranging Eq. (6.21) to solve for the **contract period** n yields.

$$n = \frac{\ln\left[1 + (q - 1)(K_n/Eq)\right]}{\ln(q)}\ . \tag{6.23}$$

6.3 Redemption payments in constant annuities

The starting point of the next discussion be a **mortgage loan** of amount $R_0 > 0$ CU that an **economic agent** borrowed from a bank at the obligation of annual service payments of $p > 0$ percent (i.e., $q > 1$) on the **remaining debt**. We suppose that the contract between the agent and the bank fixes the following conditions:

(i) the first **redemption payment** T_1 amount to $t > 0$ percent of the mortgage R_0,

(ii) the remaining debt shall be paid back to the bank in *constant* **annuities** of value $A > 0$ CU at the end of each full year that has passed.

The **annuity** A is defined as the *sum* of the variable nth **interest payment** $Z_n > 0$ CU and the variable nth **redemption payment** $T_n > 0$ CU. In the present model we impose on the annuity the condition that it be *constant* across full years,

$$A = Z_n + T_n \stackrel{!}{=} \text{constant}\ . \tag{6.24}$$

For $n = 1$, for example, we thus obtain

$$A = Z_1 + T_1 = R_0 \cdot \frac{p}{100} + R_0 \cdot \frac{t}{100} = R_0 \left(\frac{p+t}{100}\right) = R_0 \left[(q-1) + \frac{t}{100}\right] \stackrel{!}{=} \text{constant} \ . \quad (6.25)$$

For the first full year of a running mortgage contract, the interest payment, the redemption payment, and, following the payment of a first annuity, the remaining debt take the values

$$\begin{aligned} Z_1 &= R_0 \cdot \frac{p}{100} \ = \ R_0(q-1) \\ T_1 &= A - Z_1 \\ R_1 &= R_0 + Z_1 - A \overbrace{=}^{\text{substitute for } Z_1} R_0 + R_0 \cdot \frac{p}{100} - A \ = \ R_0 q - A \ . \end{aligned}$$

By the end of a second full year, these become

$$\begin{aligned} Z_2 &= R_1(q-1) \\ T_2 &= A - Z_2 \\ R_2 &= R_1 + Z_2 - A \overbrace{=}^{\text{substitute for } Z_2} R_1 q - A \overbrace{=}^{\text{substitute for } R_1} R_0 q^2 - A(q+1) \ . \end{aligned}$$

At this stage, it has become clear according to which patterns the different quantities involved in the redemption payment model need to be formed. The **interest payment** for the nth full year in a mortgage contract of constant anuities amounts to (recursively)

$$Z_n = R_{n-1}(q-1) \ , \quad n \in \mathbb{N} \ , \qquad (6.26)$$

where R_{n-1} denotes the remaining debt at the end of the previous full year. The **redemption payment** for full year n is then given by (recursively)

$$T_n = A - Z_n \ , \quad n \in \mathbb{N} \ . \qquad (6.27)$$

The **remaining debt** at the end of the nth full year then is (in CU)

$$\boxed{\text{recursively:} \quad R_n = R_{n-1} + Z_n - A \ = \ R_{n-1} q - A \ , \quad n \in \mathbb{N} \ .} \qquad (6.28)$$

By successive backward substitution for R_{n-1}, R_{n-2}, etc., R_n can be re-expressed as

$$R_n = R_0 q^n - A(q^{n-1} + \ldots + q + 1) = R_0 q^n - A \sum_{k=0}^{n-1} q^k \ .$$

Now employing the identity (6.10), we finally obtain (since $q > 1$)

$$\boxed{\text{explicitly:} \quad R_n = R_0 q^n - A \, \frac{q^n - 1}{q - 1} \ , \quad n \in \mathbb{N} \ .} \qquad (6.29)$$

All the formulae we have now derived for computing the values of the quantities $\{n, Z_n, T_n, R_n\}$ form the basis of a formal **redemption payment plan**, given by

n	Z_n [CU]	T_n [CU]	R_n [CU]
0	–	–	R_0
1	Z_1	T_1	R_1
2	Z_2	T_2	R_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

,

a standard scheme that banks must make available to their mortgage customers for the purpose of financial orientation.

Remark: For known values of the free parameters $R_0 > 0$ CU, $q > 1$ and $A > 0$ CU, the simple recursive formulae (6.26), (6.27) and (6.28) can be used to implement a redemption payment plan in a modern spreadsheet programme such as EXCEL or OpenOffice.

We emphasise the following observation concerning Eq. (6.29): since the constant annuity A contains implicitly a factor $(q-1)$ [cf. Eq. (6.25)], the two competing terms in this relation each grow exponentially with n. For the redemption payments to eventually terminate, it is thus essential to fix the free parameter t (for known $p > 0 \Leftrightarrow q > 1$) in such a way that the second term on the right-hand side of Eq. (6.29) is given the possibility to catch up with the first as n progresses (the latter of which has a head start of $R_0 > 0$ CU at $n = 0$). The necessary condition following from the requirement that $R_n \stackrel{!}{\leq} R_{n-1}$ is thus $t > 0$.

Equation (6.29) links the five non-negative quantities R_n, R_0, q, n and A to one another. Given one knows the values of four of these, one can solve for the fifth. For example:

(i) Calculation of the **contract period** n of a mortgage contract, knowing the mortgage R_0, the interest factor q and the annuity A. Solving the condition $R_n \stackrel{!}{=} 0$ imposed on R_n for n yields (after a few algebraic steps)

$$n = \frac{\ln\left(1 + \frac{p}{t}\right)}{\ln(q)} \; ; \tag{6.30}$$

the contract period is thus independent of the value of the mortgage loan, R_0.

(ii) Evaluation of the **annuity** A, knowing the contract period n, the mortgage loan R_0, and the interest factor q. Solving the condition $R_n \stackrel{!}{=} 0$ imposed on R_n for A immediately yields

$$A = \frac{q^n(q-1)}{q^n - 1}\, R_0 \; . \tag{6.31}$$

Now equating the two expressions (6.31) and (6.25) for the annuity A, one finds in addition that

$$\frac{t}{100} = \frac{q-1}{q^n - 1} \; . \tag{6.32}$$

6.4 Pension calculations

Quantitative models for **pension calculations** assume given an **initial capital** $K_0 > 0$ CU that was paid into a bank account at a particular moment in time. The issue is to monitor the subsequent evolution in **discrete time** n of the **account balance** K_n (in CU), which is subjected to two competing influences: on the one-hand side, the bank account earns interest at an **annual interest rate** of $p > 0$ percent (i.e., $q > 1$), on the other, it is supposed that throughout one full year a total of $m \in \mathbb{N}$ pension payments of the *constant* **amount** a are made from this bank account, always at the beginning of each of m intervals of equal duration per year.

Let us begin by evaluating the amount of interest earned per year by the bank account. An important point in this respect is the fact that throughout one full year there is a total of m deductions of value a from the bank account, i.e., in general the account balance does *not* stay constant throughout that year but rather decreases in discrete steps. For this reason, the account is credited by the bank with interest only at the mth part of $p > 0$ percent for each interval (out of the total of m) that has passed, with *no* compound interest effect. Hence, at the end of the first out of m intervals per year the bank account has earned interest worth of (in CU)

$$Z_{1/m} = (K_0 - a) \cdot \frac{p}{m \cdot 100} = (K_0 - a)\,\frac{(q-1)}{m}\ .$$

The interest earned for the kth interval (out of m; $k \leq m$) is then given by

$$Z_{k/m} = (K_0 - ka)\,\frac{(q-1)}{m}\ .$$

Summation over the contributions of each of the m intervals to the interest earned then yields for the entire interest earned during the first full year (in CU)

$$Z_1 = \sum_{k=1}^{m} Z_{k/m} = \sum_{k=1}^{m} (K_0 - ka)\,\frac{(q-1)}{m} = \frac{(q-1)}{m}\left[mK_0 - a\sum_{k=1}^{m} k\right]\ .$$

By means of substitution from the identity (6.5), this result can be recast into the equivalent form

$$Z_1 = \left[K_0 - \frac{1}{2}\,(m+1)a\right](q-1)\ . \tag{6.33}$$

Note that this quantity decreases linearly with the number of deductions m made per year resp. with the pension payment amount a.

One now finds that the account balance at the end of the first full year that has passed is given by

$$K_1 = K_0 - ma + Z_1 \overset{\text{Eq. }(6.33)}{\overbrace{=}} K_0 q - \left[m + \frac{1}{2}\,(m+1)(q-1)\right] a\ .$$

At the end of a second full year of the pension payment contract the interest earned is

$$Z_2 = \left[K_1 - \frac{1}{2}\,(m+1)a\right](q-1)\ ,$$

while the account balance amounts to

$$K_2 = K_1 - ma + Z_2 \overbrace{=}^{\text{substitute for } K_1 \text{ and } Z_2} K_0 q^2 - \left[m + \frac{1}{2}(m+1)(q-1)\right] a(q+1) \ .$$

At this stage, certain fairly simple patterns for the **interest earned** during full year n, and the **account balance** after n full years, reveal themselves. For Z_n we have

$$Z_n = \left[R_{n-1} - \frac{1}{2}(m+1)a\right](q-1) \ , \tag{6.34}$$

and for K_n one obtains

$$K_n = K_{n-1} - ma + Z_n \overbrace{=}^{\text{substitute for } K_{n-1} \text{ and } Z_n} K_0 q^n - \left[m + \frac{1}{2}(m+1)(q-1)\right] a \sum_{k=0}^{n-1} q^k \ .$$

The latter result can be re-expressed upon substitution from the identity (6.10). Thus, K_n can finally be given by

$$\boxed{\text{explicitly:} \quad K_n = K_0 q^n - \left[m + \frac{1}{2}(m+1)(q-1)\right] a \, \frac{q^n - 1}{q-1} \ , \quad n, m \in \mathbb{N} \ .} \tag{6.35}$$

In a fashion practically identical to our discussion of the redemption payment model in Sec. 6.3, the two competing terms on the right-hand side of Eq. (6.35) likewise exhibit exponential growth with the number n of full years passed. Specifically, it depends on the values of the parameters $K_0 > 0$ CU, $q > 1$, $a > 0$ CU, as well as $m \geq 1$, whether the second term eventually manages to catch up with the first as n progresses (the latter of which, in this model, is given a head start of value $K_0 > 0$ CU at $n = 0$).

We remark that Eq. (6.35), again, may be algebraically re-arranged at one's convenience (as long as division by zero is avoided). For example:

(i) The **duration** n (in full years) of a particular pension contract is obtained from solving the condition $K_n \stackrel{!}{=} 0$ accordingly. Given that $[\ldots]a - K_0(q-1) > 0$, one thus finds[3]

$$n = \frac{\ln\left(\frac{[\ldots]a}{[\ldots]a - K_0(q-1)}\right)}{\ln(q)} \ . \tag{6.36}$$

(ii) The **present value** B_0 of a pension scheme results from the following consideration: for fixed interest factor $q > 1$, which initial capital $K_0 > 0$ CU must be paid into a bank account such that for a duration of n full years one can receive payments of constant amount a at the beginning of each of m intervals (of equal length) per year? The value of $B_0 = K_0$ is again obtained from imposing on Eq. (6.35) the condition $K_n \stackrel{!}{=} 0$ and solving for K_0. This yields

$$B_0 = K_0 = \left[m + \frac{1}{2}(m+1)(q-1)\right] a \, \frac{q^n - 1}{q^n(q-1)} \ . \tag{6.37}$$

[3] To avoid notational overload, the brackets $[\ldots]$ here represent the term $\left[m + \frac{1}{2}(m+1)(q-1)\right]$.

(iii) The idea of so-called **everlasting pension payments** of amount $a_{\text{ever}} > 0$ CU is based on the strategy to consume only the annual interest earned by an initial capital $K_0 > 0$ CU residing in a bank account with interest factor $q > 1$. Imposing now on Eq. (6.35) the condition $K_n \stackrel{!}{=} K_0$ to hold for all values of n, and then solving for a, yields the result

$$a_{\text{ever}} = \frac{q-1}{m + \frac{1}{2}(m+1)(q+1)}\, K_0 \; ; \tag{6.38}$$

Note that, naturally, a_{ever} is directly proportional to the initial capital K_0!

6.5 Linear and declining-balance depreciation methods

Attempts at the quantitative description of the process of declining material value of industrial goods, properties or other assets, of **initial value** $K_0 > 0$ CU, are referred to as **depreciation**. International tax laws generally provide investors with a choice between two particular mathematical methods of calculating **depreciation**. We will discuss these options in turn.

6.5.1 Linear depreciation method

When the **initial value** $K_0 > 0$ CU is supposed to decline to 0 CU in the space of N full years by equal annual amounts, the **remaining value** R_n (in CU) at the end of n full years is described by

$$\boxed{R_n = K_0 - n\left(\frac{K_0}{N}\right) \; , \quad n = 1, \ldots, N \; .} \tag{6.39}$$

Note that for the difference of remaining values for years adjacent one obtains $R_n - R_{n-1} = -\left(\frac{K_0}{N}\right) =: d < 0$. The underlying mathematical structure of the **straight line depreciation method** is thus an **arithmetical sequence** of real numbers, with constant *negative* difference d between neighbouring elements (cf. Sec. 6.1.1).

6.5.2 Declining-balance depreciation method

The foundation of the second depreciation method to be described here, for an industrial good of **initial value** $K_0 > 0$ CU, is the idea that per year the value declines by a certain **percentage rate** $p > 0$ of the value of the good during the previous year. Introducing a dimensionless **depreciation factor** by

$$q := 1 - \frac{p}{100} < 1 \; , \tag{6.40}$$

the **remaining value** R_n (in CU) after n full years amounts to

$$\boxed{\text{recursively:} \quad R_n = R_{n-1}q \; , \quad R_0 \equiv K_0 \; , \quad n \in \mathbb{N} \; .} \tag{6.41}$$

The underlying mathematical structure of the **declining balance depreciation method** is thus a **geometrical sequence** of real numbers, with constant ratio $0 < q < 1$ between neighbouring elements (cf. Sec. 6.1.2). With increasing n the values of these elements become ever smaller. By means of successive backward substitution expression (6.41) can be transformed to

$$\boxed{\text{explicitly:} \quad R_n = K_0 q^n \,, \quad 0 < q < 1 \,, \quad n \in \mathbb{N} \,.} \tag{6.42}$$

From Eq. (6.42), one may derive results concerning the following questions of a quantitative nature:

(i) Suppose given a depreciation factor q and a projected remaining value R_n for some industrial good. After which **depreciation period** n will this value be attained? One finds

$$n = \frac{\ln\left(R_n/K_0\right)}{\ln(q)} \,. \tag{6.43}$$

(ii) Knowing a projected depreciation period n and corresponding remaining value R_n, at which **percentage rate** $p > 0$ must the depreciation method be operated? This yields

$$q = \sqrt[n]{\frac{R_n}{K_0}} \quad \Rightarrow \quad p = 100 \cdot \left(1 - \sqrt[n]{\frac{R_n}{K_0}}\right) \,. \tag{6.44}$$

6.6 Summarising formula

To conclude this chapter, let us summarise the results on **elementary financial mathematics** that we derived along the way. Remarkably, these can be condensed in a single formula which contains the different concepts discussed as special cases. This formula, in which n represents the number of full years that have passed, is given by (cf. Zeh–Marschke (2010) [26]):

$$\boxed{K_n = K_0 q^n + R\,\frac{q^n - 1}{q - 1} \,, \quad q \in \mathbb{R}_{>0} \backslash \{1\} \,, \quad n \in \mathbb{N} \,.} \tag{6.45}$$

The different special cases contained therein are:

(i) **Compound interest** for an initial capital $K_0 > 0$ CU: with $R = 0$ and $q > 1$, Eq. (6.45) reduces to Eq. (6.13).

(ii) **Installment savings** with constant installments $E > 0$ CU: with $K_0 = 0$ CU, $q > 1$ and $R = Eq$, Eq. (6.45) reduces to Eq. (6.19).

(iii) **Redemption payments in constant annuities**: with $K_0 = -R_0 < 0$ CU, $q > 1$ and $R = A > 0$ CU, Eq. (6.45) reduces to the *negative (!)* of Eq. (6.29). In this dual formulation, remaining debt $K_n = -R_n$ is (meaningfully) expressed as a negative account balance.

(iv) **Pension payments** of constant amount $a > 0$ CU: with $q > 1$ and $R = -\left[m + \frac{1}{2}\,(m+1)(q-1)\right] a$, Eq. (6.45) transforms to Eq. (6.35).

(v) **Declining balance depreciation** of an asset of initial value $K_0 > 0$ CU: with $R = 0$ and $0 < q < 1$, Eq. (6.45) converts to Eq. (6.42) for the remaining value $K_n = R_n$.

Chapter 7

Differential calculus of real-valued functions of one real variable

In Chs. 1 to 5 of these lecture notes, we confined our considerations to functional relationships between **INPUT quantities** and **OUTPUT quantities** of a *linear* nature. In this chapter now, we turn to discuss characteristic properties of truly **non-linear functional relationships** between one **INPUT quantity** and one **OUTPUT quantity**.

7.1 Real-valued functions of one real variable

Let us begin by introducing the concept of a **real-valued function of one real variable**. This constitutes a special kind of a **mapping**[1] that needs to satisfy the following simple but very strict rule:

a mapping f that assigns to *every element* x from a subset D of the real numbers $\mathbb{R}$ (i.e., $D \subseteq \mathbb{R}$) *one and only one element* y from a second subset W of the real numbers $\mathbb{R}$ (i.e., $W \subseteq \mathbb{R}$).

Def.: A **unique** mapping f of a subset $D \subseteq \mathbb{R}$ of the real numbers onto a subset $W \subseteq \mathbb{R}$ of the real numbers,

$$\boxed{f\colon D \to W\ , \qquad x \mapsto y = f(x)} \tag{7.1}$$

is referred to as a **real-valued function of one real variable**.

We now fix some terminology concerning the concept of a real-valued function of one real variable:

- D: **domain** of f,
- W: **target space** of f,
- $x \in D$: **independent variable** of f, also referred to as the *argument* of f,
- $y \in W$: **dependent variable** of f,

[1] Cf. our introduction in Ch. 2 of matrices as a particular class of mathematical objects.

- $f(x)$: **mapping prescription**,
- **graph** of f: the set of pairs of values $G = \{(x, f(x))|x \in D\} \subseteq \mathbb{R}^2$.

For later analysis of the mathematical properties of real-valued functions of one real variable, we need to address a few more technical issues.

Def.: Given a mapping f that is **one-to-one and onto**, with domain $D(f) \subseteq \mathbb{R}$ and target space $W(f) \subseteq \mathbb{R}$, not only is every $x \in D(f)$ assigned to one and only one $y \in W(f)$, but also every $y \in W(f)$ is assigned to one and only one $x \in D(f)$. In this case, there exists an associated mapping f^{-1}, with $D(f^{-1}) = W(f)$ and $W(f^{-1}) = D(f)$, which is referred to as the **inverse function** of f.

Def.: A real-valued function f of one real variable x is **continuous** at some value $x \in D(f)$ when for $\Delta x \in \mathbb{R}_{>0}$ the condition

$$\lim_{\Delta x \to 0} f(x - \Delta x) = \lim_{\Delta x \to 0} f(x + \Delta x) = f(x) \tag{7.2}$$

obtains, i.e., when at x the left and right limits of the function f coincide and are equal to the value $f(x)$. A real-valued function f as such is **continuous** when f is continuous *for all* $x \in D(f)$.

Def.: When a real-valued function f of one real variable x satisfies the condition

$$f(a) < f(b) \qquad \textit{for all} \quad a, b \in D(f) \text{ with } a < b\ , \tag{7.3}$$

then f is called **strictly monotonously increasing**. When, however, f satisfies the condition

$$f(a) > f(b) \qquad \textit{for all} \quad a, b \in D(f) \text{ with } a < b\ , \tag{7.4}$$

then f is called **strictly monotonously decreasing**.

Note, in particular, that real-valued functions of one real variable that are strictly monotonous and continous are always one-to-one and onto and, therefore, are invertible.

In the following, we briefly review five elementary classes of real-valued functions of one real variable that find frequent application in the modelling of quantitative problems in **economic theory**.

7.1.1 Polynomials of degree n

Polynomials of degree n are real-valued functions of one real variable of the form

$$\boxed{\begin{array}{c} y = f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_i x^i + \ldots + a_2 x^2 + a_1 x + a_0 \\[1em] \text{with } a_i \in \mathbb{R},\ i = 1, \ldots, n,\ \ n \in \mathbb{N},\ a_n \neq 0\ . \end{array}} \tag{7.5}$$

Their domain comprises the entire set of real numbers, i.e., $D(f) = \mathbb{R}$. The extent of their target space depends specifically on the values of the real constant **coefficients** $a_i \in \mathbb{R}$. Functions in this class possess a maximum of n real **roots**.

7.1.2 Rational functions

Rational functions are constructed by forming the **ratio of two polynomials** of degrees m resp. n, i.e.,

$$\boxed{\begin{array}{c} y = f(x) = \dfrac{p_m(x)}{q_n(x)} = \dfrac{a_m x^m + \ldots + a_1 x + a_0}{b_n x^n + \ldots + b_1 x + b_0} \\ \text{with } a_i, b_j \in \mathbb{R},\ i = 1, \ldots, m,\ j = 1, \ldots, n,\ \ m, n \in \mathbb{N},\ a_m, b_n \neq 0\ . \end{array}} \tag{7.6}$$

Their domain is given by $D(f) = \mathbb{R} \backslash \{x | q_n(x) = 0\}$. When for the degrees m and n of the polynomials $p_m(x)$ and $q_n(x)$ we have

(i) $m < n$, then f is referred to as a **proper rational function**, or

(ii) $m \geq n$, then f is referred to as an **improper rational function**.

In the latter case, application of *polynomial division* leads to a separation of f into a purely polynomial part and a proper rational part. The **roots** of f always correspond to those roots of the numerator polynomial $p_m(x)$ for which simultaneously $q_n(x) \neq 0$ applies. The roots of the denominator polynomial $q_n(x)$ constitute **poles** of f. Proper rational functions always tend for very small (i.e., $x \to -\infty$) and for very large (i.e., $x \to +\infty$) values of their argument to zero.

7.1.3 Power-law functions

Power-law functions exhibit the specific structure given by

$$\boxed{y = f(x) = x^\alpha \quad \text{with } \alpha \in \mathbb{R}\ .} \tag{7.7}$$

We here confine ourselves to cases with domains $D(f) = \mathbb{R}_{>0}$, such that for the coresponding target spaces we have $W(f) = \mathbb{R}_{>0}$. Under these conditions, power-law functions are strictly monotonously increasing when $\alpha > 0$, and strictly monotonously decreasing when $\alpha < 0$. Hence, they are inverted by $y = \sqrt[\alpha]{x} = x^{1/\alpha}$. There do *not* exist any roots under the conditions stated here.

7.1.4 Exponential functions

Exponential functions have the general form

$$\boxed{y = f(x) = a^x \quad \text{with } a \in \mathbb{R}_{>0} \backslash \{1\}\ .} \tag{7.8}$$

Their domain is $D(f) = \mathbb{R}$, while their target space is $W(f) = \mathbb{R}_{>0}$. They exhibit strict monotonous increase for $a > 1$, and strict monotonous decrease for $0 < a < 1$. Hence, they too are invertible. Their y-intercept is generally located at $y = 1$. For $a > 1$, exponential functions are also known as **growth functions**.

Special case: When the *constant (!)* base number is chosen to be $a = e$, where e denotes the irrational **Euler's number** (according to the Swiss mathematician Leonhard Euler, 1707–1783) defined by the infinite series

$$e := \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ ,$$

one obtains the **natural exponential function**

$$y = f(x) = e^x =: \exp(x) \ . \tag{7.9}$$

In analogy to the definition of e, the relation

$$e^x = \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

applies.

7.1.5 Logarithmic functions

Logarithmic functions, denoted by

$$\boxed{y = f(x) = \log_a(x) \quad \text{with } a \in \mathbb{R}_{>0} \backslash \{1\} \ ,} \tag{7.10}$$

are defined as *inverse functions* of the strictly monotonous exponential functions $y = f(x) = a^x$ — and vice versa. Correspondingly, $D(f) = \mathbb{R}_{>0}$ and $W(f) = \mathbb{R}$ apply. Strictly monotonously increasing behaviour is given when $a > 1$, strictly monotonously decreasing behaviour when $0 < a < 1$. In general, the x-intercept is located at $x = 1$.

Special case: The **natural logarithmic function** (lat.: logarithmus naturalis) obtains when the constant basis number is set to $a = e$. This yields

$$y = f(x) = \log_e(x) := \ln(x) \ . \tag{7.11}$$

7.1.6 Concatenations of real-valued functions

Real-valued functions from all five categories considered in the previous sections may be combined arbitrarily (respecting relevant computational rules), either via the four **fundamental arithmetical operations**, or via **concatenations**.

Theorem: Let real-valued functions f and g be continuous on domains $D(f)$ resp. $D(g)$. Then the combined real-valued functions

1. **sum/difference** $f \pm g$, where $(f \pm g)(x) := f(x) \pm g(x)$ with $D(f) \cap D(g)$,
2. **product** $f \cdot g$, where $(f \cdot g)(x) := f(x)g(x)$ with $D(f) \cap D(g)$,
3. **quotient** $\dfrac{f}{g}$, where $\left(\dfrac{f}{g}\right)(x) := \dfrac{f(x)}{g(x)}$ with $g(x) \neq 0$ and $D(f) \cap D(g) \backslash \{x|g(x) = 0\}$,
4. **concatenation** $f \circ g$, where $(f \circ g)(x) := f(g(x))$ mit $\{x \in D(g)|g(x) \in D(f)\}$,

are also continuous on the respective domains.

7.2 Derivation of differentiable real-valued functions

The central theme of this chapter is the mathematical description of the **local variability** of *continuous* real-valued function of one real variable, $f : D \subseteq \mathbb{R} \to W \subseteq \mathbb{R}$. To this end, let us consider the effect on f of a small change of its argument x. Supposing we affect a change $x \to x + \Delta x$, with $\Delta x \in \mathbb{R}$, what are the resultant consequences for f? We immediately find that $y \to y + \Delta y = f(x + \Delta x)$, with $\Delta y \in \mathbb{R}$, obtains. Hence, a prescribed change of the argument x by a (small) value Δx triggers in f a change by the amount $\Delta y = f(x + \Delta x) - f(x)$. It is of general quantitative interest to compare the **sizes** of these two changes. This is accomplished by forming the respective **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \ .$$

It is then natural, for given f, to investigate the limit behaviour of this difference quotient as the change Δx of the argument of f is made successively smaller.

Def.: A continuous real-valued function f of one real variable is called **differentiable at $\boldsymbol{x \in D(f)}$**, when for arbitrary $\Delta x \in \mathbb{R}$ the limit

$$\boxed{f'(x) := \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}} \tag{7.12}$$

exists and is unique. When f is differentiable *for all* $x \in D(f)$, then f as such is referred to as being **differentiable**.

The existence of this limit in a point $(x, f(x))$ for a real-valued function f requires that the latter exhibits neither "jumps" nor "kinks," i.e., that at $(x, f(x))$ the function is sufficiently "smooth." The quantity $f'(x)$ is referred to as the **first derivative** of the (differentiable) function f at position x. It provides a quantitative measure for the **local rate of change** of the function f in the point $(x, f(x))$. In general one interprets the first derivative $f'(x)$ as follows: an increase of the argument x of a differentiable real-valued function f by 1 (one) unit leads to a change in the value of f by approximately $f'(x) \cdot 1$ units.

Alternative notation for the first derivative of f:

$$f'(x) \equiv \frac{\mathrm{d}f(x)}{\mathrm{d}x} \ .$$

The differential calculus was developed in parallel with the integral calculus (see Ch. 7) during the second half of the 17^{th} Century, independent of one another by the English physicist, mathematiccian, astronomer and philosopher Sir Isaac Newton (1643–1727) and the German philosopher, mathematician and physicist Gottfried Wilhelm Leibniz (1646–1716).

Via the first derivative of a differentiable function f at an argument $x_0 \in D(f)$, i.e., $f'(x_0)$, one defines the so-called **linearisation** of f in a neighbourhood of x_0. The equation describing the associated **tangent** to f in the point $(x_0, f(x_0))$ is given by

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0) \ .} \tag{7.13}$$

GDC: Local values $f'(x_0)$ of first derivatives can be computed for given function f in mode `CALC` using the interactive routine `dy/dx`.

The following rules of differentiation apply for the five families of elementary real-valued functions discussed in Sec. 7.1, as well as concatenations thereof:

Rules of differentiation

1. $(c)' = 0$ for $c = \text{constant} \in \mathbb{R}$ (**constants**)
2. $(x)' = 1$ (**linear function**)
3. $(x^n)' = nx^{n-1}$ for $n \in \mathbb{N}$ (**natural power-law functions**)
4. $(x^\alpha)' = \alpha x^{\alpha-1}$ for $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}_{>0}$ (**general power-law functions**)
5. $(a^x)' = \ln(a)a^x$ for $a \in \mathbb{R}_{>0}\backslash\{1\}$ (**exponential functions**)
6. $(e^{ax})' = ae^{ax}$ for $a \in \mathbb{R}$ (**natural exponential functions**)
7. $(\log_a(x))' = \dfrac{1}{x\ln(a)}$ for $a \in \mathbb{R}_{>0}\backslash\{1\}$ and $x \in \mathbb{R}_{>0}$ (**logarithmic functions**)
8. $(\ln(x))' = \dfrac{1}{x}$ for $x \in \mathbb{R}_{>0}$ (**natural logarithmic function**).

For differentiable real-valued functions f and g it holds that:

1. $(cf(x))' = cf'(x)$ for $c = \text{constant} \in \mathbb{R}$
2. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$ (**summation rule**)
3. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ (**product rule**)
4. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ (**quotient rule**)
5. $((f \circ g)(x))' = \left. f'(g)\right|_{g=g(x)} \cdot g'(x)$ (**chain rule**)
6. $(\ln(f(x)))' = \dfrac{f'(x)}{f(x)}$ for $f(x) > 0$ (**logarithmic differentiation**)
7. $(f^{-1}(x))' = \left.\dfrac{1}{f'(y)}\right|_{y=f^{-1}(x)}$, if f is one-to-one and onto.

   (**differentiation of inverse functions**).

The methods of differential calculus just introduced shall now be employed to describe the local change behaviour of a few simple examples of functions in **economic theory**, and also to determine their local extremal values. The following section provides an overview of such frequently occurring **economic functions**.

7.3 Common functions in economic theory

1. **total cost function** $K(x) \geq 0$ (dim: CU)
   argument: level of physical output $x \geq 0$ (dim: units)

2. **marginal cost function** $K'(x) > 0$ (dim: CU/unit)
   argument: level of physical output $x \geq 0$ (dim: units)

3. **average cost function** $K(x)/x > 0$ (dim: CU/unit)
   argument: level of physical output $x > 0$ (dim: units)

4. **unit price function** $p(x) \geq 0$ (dim: CU/unit)
   argument: level of physical output $x > 0$ (dim: units)

5. **total revenue function** $E(x) := xp(x) \geq 0$ (dim: CU)
   argument: level of physical output $x > 0$ (dim: units)

6. **marginal revenue function** $E'(x) = xp'(x) + p(x)$ (dim: CU/unit)
   argument: level of physical output $x > 0$ (dim: units)

7. **profit function** $G(x) := E(x) - K(x)$ (dim: CU)
   argument: level of physical output $x > 0$ (dim: units)

8. **marginal profit function** $G'(x) := E'(x) - K'(x) = xp'(x) + p(x) - K'(x)$ (dim: CU/unit)
   argument: level of physical output $x > 0$ (dim: units)

9. **utility function** $U(x)$ (dim: case dependent)
   argument: material wealth, opportunity, action x (dim: case dependent)

   The fundamental concept of a utility function as a means to capture in quantitative terms the psychological value (happiness) assigned by an economic agent to a certain amount of money, or to owning a specific good, was introduced to **economic theory** in 1738 by the Swiss mathematician and physicist Daniel Bernoulli FRS (1700–1782); cf. Bernoulli (1738) [4]. The utility function is part of the folklore of the theory, and often taken to be a piecewise differentiable, right-handedly curved (concave) function, i.e., $U''(x) < 0$, on the grounds of the *assumption* of diminishing marginal utility (happiness) with increasing material wealth.

10. **economic efficiency** $W(x) := E(x)/K(x) \geq 0$ (dim: 1)
    argument: level of physical output $x > 0$ (dim: units)

11. **demand function** $N(p) \geq 0$, monotonously decreasing (dim: units)
    argument: unit price p, ($0 \leq p \leq p_{\max}$) (dim: CU/unit)

12. **supply function** $A(p) \geq 0$, monotonously increasing (dim: units)
    argument: unit price p, ($p_{\min} \leq p$) (dim: CU/units).

A particularly prominent example of a real-valued economic function of one real variable constitutes the **psychological value function**, devised by the Israeli–US-American experimental psychologists Daniel Kahneman and Amos Tversky (1937–1996) in the context of their **Prospect Theory** (a pillar of **Behavioural Economics**), which was later awarded a Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel in 2002 (cf. Kahneman and Tversky (1979) [18, p 279], and Kahneman (2011) [17, p 282f]). A possible representation of this function is given by the piecewise description

$$v(x) = \begin{cases} a\log_{10}(1+x) & \text{for} \quad x \in \mathbb{R}_{\geq 0} \\ \\ -2a\log_{10}(1-x) & \text{for} \quad x \in \mathbb{R}_{<0} \end{cases}, \tag{7.14}$$

with parameter $a \in \mathbb{R}_{>0}$. Overcoming a conceptual problem of Bernoulli's utility function, here, in contrast, the argument x quantifies a *change in wealth (or welfare)* with respect to some given reference point (rather than a specific value of wealth itself).

7.4 Curve sketching

Before we turn to discuss applications of **differential calculus** to simple quantitative problems in **economic theory**, we briefly summarize the main steps of **curve sketching** for a real-valued function of one real variable, also referred to as **analysis** of the properties of differentiability of a real-valued function.

1. **domain**: $D(f) = \{x \in \mathbb{R} | f(x) \text{ is regular}\}$
2. **symmetries**: for all $x \in D(f)$, is
   (i) $f(-x) = f(x)$, i.e., is f **even**, or
   (ii) $f(-x) = -f(x)$, i.e., is f **odd**, or
   (iii) $f(-x) \neq f(x) \neq -f(x)$, i.e., f exhibits **no symmetries**?
3. **roots**: identify all $x_N \in D(f)$ that satisfy the condition $f(x) \stackrel{!}{=} 0$.
4. **local extremal values**:
   (i) **local minima** of f exist at all $x_E \in D(f)$, for which the
   necessary condition $f'(x) \stackrel{!}{=} 0$, and the
   sufficient condition $f''(x) \stackrel{!}{>} 0$ are satisfied simultaneously.
   (ii) **local maxima** of f exist at all $x_E \in D(f)$, for which the
   necessary condition $f'(x) \stackrel{!}{=} 0$, and the
   sufficient condition $f''(x) \stackrel{!}{<} 0$ are satisfied simultaneously.

5. **points of inflection**: find all $x_W \in D(f)$, for which the

   necessary condition $f''(x) \stackrel{!}{=} 0$, and the

   sufficient condition $f'''(x) \stackrel{!}{\neq} 0$ are satisfied simultaneously.

6. **monotonous behaviour**:

   (i) f is **monotonously increasing** for all $x \in D(f)$ with $f'(x) > 0$

   (ii) f is **monotonously decreasing** for all $x \in D(f)$ with $f'(x) < 0$

7. **local curvature**:

   (i) f behaves **left-handedly curved** for $x \in D(f)$ with $f''(x) > 0$

   (ii) f behaves **right-handedly curved** for $x \in D(f)$ with $f''(x) < 0$

8. **asymptotic behaviour**:

   asymptotes to f are constituted by

   (i) straight lines $y = ax + b$ with the property $\lim_{x\to+\infty}[f(x) - ax - b] = 0$ or $\lim_{x\to-\infty}[f(x) - ax - b] = 0$

   (ii) straight lines $x = x_0$ at poles $x_0 \notin D(f)$

9. **range**: $W(f) = \{y \in \mathbb{R} | y = f(x)\}$.

7.5 Analytic investigations of economic functions

7.5.1 Total cost functions according to Turgot and von Thünen

According to the **law of diminishing returns**, which was introduced to **economic theory** by the French economist and statesman Anne Robert Jacques Turgot (1727–1781) and also by the German economist Johann Heinrich von Thünen (1783–1850), it is meaningful to model non-negative **total cost functions** $K(x)$ (in CU) relating to typical production processes, with argument **level of physical output** $x \geq 0$ units, as a mathematical mapping in terms of a special *polynomial of degree 3* [cf. Eq. (7.5)], which is given by

$$\boxed{\begin{aligned} K(x) &= \underbrace{a_3x^3 + a_2x^2 + a_1x}_{=K_v(x)} + \underbrace{a_0}_{=K_f} \\ &\text{with } a_3, a_1 > 0,\ a_2 < 0,\ a_0 \geq 0,\ a_2^2 - 3a_3a_1 < 0\ . \end{aligned}} \tag{7.15}$$

The model thus contains a total of four free parameters. It is the outcome of a systematic **regression analysis** of agricultural quantitative–empirical data with the aim to describe an inherently **non-linear functional relationship** between a few economic variables. As such, the functional relationship for $K(x)$ expressed in Eq. (7.15) was derived from a practical consideration. It is a reflection of the following observed features:

(i) for levels of physical output $x \geq 0$ units, the total costs relating to typical production processes exhibit strictly monotonously increasing behaviour; thus

(ii) for the total costs there do *not* exist neither roots nor local extremal values;[2] however,

(iii) the total costs display *exactly one* point of inflection.

The continuous curve for $K(x)$ resulting from these considerations exhibits the characteristic shape of an inverted capital letter "S": beginning at a positive value corresponding to fixed costs, the total costs first increase degressively up to a point of inflection, whereafter they continue to increase, but in a progressive fashion.

In broad terms, the functional expression given in Eq. (7.15) to model totals costs in dependence of the level of physical output is the sum of two contributions, the **variable costs** $K_v(x)$ and the **fixed costs** $K_f = a_0$, viz.

$$K(x) = K_v(x) + K_f \ . \tag{7.16}$$

In **economic theory**, it is commonplace to partition **total cost functions** in the diminishing returns picture into *four phases*, the boundaries of which are designated by special values of the level of physical output of a production process:

- **phase I** (interval 0 units $\leq x \leq x_W$):

  the total costs $K(x)$ possess at a level of physical output $x_W = -a_2/(3a_3) > 0$ units a **point of inflection**. For values of x smaller than x_W, one obtains $K''(x) < 0$ CU/unit^2, i.e., $K(x)$ increases in a degressive fashion. For values of x larger than x_W, the opposite applies, $K''(x) > 0$ CU/unit^2, i.e., $K(x)$ increases in a progressive fashion. The **marginal costs**, given by

  $$K'(x) = 3a_3x^2 + 2a_2x + a_1 > 0 \text{ CU/unit} \qquad \text{for all} \quad x \geq 0 \text{ units} \ , \tag{7.17}$$

  attain a **minimum** at the same level of physical output, $x_W = -a_2/(3a_3)$.

- **phase II** (interval $x_W < x \leq x_{g_1}$):

  the **variable average costs**

  $$\frac{K_v(x)}{x} = a_3x^2 + a_2x + a_1 \ , \qquad x > 0 \text{ units} \tag{7.18}$$

  become **minimal** at a level of physical output $x_{g_1} = -a_2/(2a_3) > 0$ units. At this value of x, *equality of variable average costs and marginal costs* applies, i.e.,

  $$\frac{K_v(x)}{x} = K'(x) \ , \tag{7.19}$$

[2] The last condition in Eq. (7.15) ensures a first derivative of $K(x)$ that does *not* possess any roots; cf. the case of a quadratic algebraic equation $0 \stackrel{!}{=} ax^2 + bx + c$, with discriminant $b^2 - 4ac < 0$.

which follows by the quotient rule of differentiation from the necessary condition for an extremum of the variable average costs,

$$0 \stackrel{!}{=} \left(\frac{K_v(x)}{x}\right)' = \frac{(K(x) - K_f)' \cdot x - K_v(x) \cdot 1}{x^2} \ ,$$

and the fact that $K_f' = 0$ CU/unit. Taking care of the equality (7.19), one finds for the tangent to $K(x)$ in the point $(x_{g_1}, K(x_{g_1}))$ the equation [cf. Eq. (7.13)]

$$T(x) = K(x_{g_1}) + K'(x_{g_1})(x - x_{g_1}) = K_v(x_{g_1}) + K_f + \frac{K_v(x_{g_1})}{x_{g_1}}(x - x_{g_1}) = K_f + \frac{K_v(x_{g_1})}{x_{g_1}} x \ .$$

Its intercept with the K-axis is at K_f.

- **phase III** (interval $x_{g_1} < x \leq x_{g_2}$):

  The **average costs**

$$\frac{K(x)}{x} = a_3x^2 + a_2x + a_1 + \frac{a_0}{x} \ , \qquad x > 0 \text{ units} \tag{7.20}$$

attain a **minimum** at a level of physical output $x_{g_2} > 0$ units, the defining equation of which is given by 0 CU $\stackrel{!}{=} 2a_3x_{g_2}^3 + a_2x_{g_2}^2 - a_0$. At this value of x, *equality of average costs and marginal costs* obtains, viz.

$$\frac{K(x)}{x} = K'(x) \ , \tag{7.21}$$

which follows by the quotient rule of differentiation from the necessary condition for an extremum of the average costs,

$$0 \stackrel{!}{=} \left(\frac{K(x)}{x}\right)' = \frac{K'(x) \cdot x - K(x) \cdot 1}{x^2} \ .$$

Since a quotient can be zero only when its numerator vanishes (and its denominator remains non-zero), one finds from re-arranging the numerator expression equated to zero the property

$$\frac{K'(x)}{K(x)/x} = x\,\frac{K'(x)}{K(x)} = 1 \qquad \text{for} \quad x = x_{g_2} \ . \tag{7.22}$$

The corresponding extremal value pair $(x_{g_2}, K(x_{g_2}))$ is referred to in **economic theory** as the **minimum efficient scale (MES)**. From a business economics perspective, at a level of physical output $x = x_{g_2}$ the (compared to our remarks in the Introduction inverted) ratio "INPUT over OUTPUT," i.e., $\dfrac{K(x)}{x}$, becomes most favourable. By respecting the property (7.21), the equation for the tangent to $K(x)$ in this point [cf. Eq. (7.13)] becomes

$$T(x) = K(x_{g_2}) + K'(x_{g_2})(x - x_{g_2}) = K(x_{g_2}) + \frac{K(x_{g_2})}{x_{g_2}}(x - x_{g_2}) = \frac{K(x_{g_2})}{x_{g_2}} x \ .$$

Its intercept with the K-axis is thus at 0 CU.

- **phase IV** (half-interval $x > x_{g_2}$):

  In this phase $K'(x)/K(x)/x > 1$ obtains; the costs associated with the production of one additional unit of a good, approximately the marginal costs $K'(x)$, now exceed the average costs, $K(x)/x$. This situation is considered unfavourable from a business economics perspective.

7.5.2 Profit functions in the diminishing returns picture

In this section, we confine our considerations, for reasons of *simplicity*, to a market sitution with only a single supplier of a good in demand. The price policy that this single supplier may thus inact defines a state of **monopoly**. Moreover, in addition we want to *assume* that for the market situation considered **economic equilibrium** obtains. This manifests itself in equality of **supply** and **demand**, viz.

$$x(p) = N(p) \, , \tag{7.23}$$

wherein x denotes a non-negative **supply function** (in units) (which is synonymous with the supplier's level of physical output) and N a non-negative **demand function** (in units), both of which are taken to depend on the positive **unit price** p (in CU/unit) of the good in question. The **supply function**, and with it the **unit price**, can, of course, be prescribed by the monopolistic supplier in an arbitrary fashion. In a specific quantitative economic model, for instance, the **demand function** $x(p)$ (recall that by Eq. (7.23) $x(p) = N(p)$ obtains) could be assumed to be either a linear or a quadratic function of p. In any case, in order for $x(p)$ to realistically describe an actual demand–unit price relationship, it should be chosen as a strictly monotonously decreasing function, and as such it is *invertible*. The non-negative **demand function** $x(p)$ features two characteristic points, signified by its intercepts with the x- and the p-axes. The **prohibitive price** p_{proh} is to be determined from the condition $x(p_{\text{proh}}) \stackrel{!}{=} 0$ units; therefore, it constitutes a root of $x(p)$. The **saturation quantity** x_{sat}, on the other hand, is defined by $x_{\text{sat}} := x(0\ \text{CU/unit})$.

The inverse function associated with the strictly monotonously decreasing non-negative **demand function** $x(p)$, the **unit price function** $p(x)$ (in CU/unit), is likewise strictly monotonously decreasing. Via $p(x)$, one calculates, in dependence on a known amount x of units supplied/demanded (i.e., sold), the **total revenue** (in CU) made by a monopolist according to (cf. Sec. 7.3)

$$E(x) = xp(x) \, . \tag{7.24}$$

Under the *assumption* that the non-negative **total costs** $K(x)$ (in CU) underlying the production process of the good in demand can be modelled according to the diminishing returns picture of Turgot and von Thünen, the **profit function** (in CU) of the monopolist in dependence on the level of physical output takes the form

$$G(x) = E(x) - K(x) = \underbrace{x\ \overbrace{p(x)}^{\text{unit price}}}_{\text{total revenue}} - \underbrace{\left[a_3x^3 + a_2x^2 + a_1x + a_0\right]}_{\text{total costs}} \, . \tag{7.25}$$

The first two derivatives of $G(x)$ with respect to its argument x are given by

$$\begin{aligned} G'(x) = E'(x) - K'(x) &= xp'(x) + p(x) - \left[3a_3x^2 + 2a_2x + a_1\right] && (7.26) \\ G''(x) = E''(x) - K''(x) &= xp''(x) + 2p'(x) - \left[6a_3x + 2a_2\right] \ . && (7.27) \end{aligned}$$

Employing the principles of curve sketching set out in Sec. 7.4, the following characteristic values of $G(x)$ can thus be identified:

- **break-even point**

  $x_S > 0$ units, as the unique solution to the conditions

  $$G(x) \stackrel{!}{=} 0 \text{ CU} \qquad \text{(necessary condition)} \tag{7.28}$$

  and

  $$G'(x) \stackrel{!}{>} 0 \text{ CU/unit} \qquad \text{(sufficient condition)} \ , \tag{7.29}$$

- **end of the profitable zone**

  $x_G > 0$ units, as the unique solution to the conditions

  $$G(x) \stackrel{!}{=} 0 \text{ CU} \qquad \text{(necessary condition)} \tag{7.30}$$

  and

  $$G'(x) \stackrel{!}{<} 0 \text{ CU/unit} \qquad \text{(sufficient condition)} \ , \tag{7.31}$$

- **maximum profit**

  $x_M > 0$ CU, as the unique solution to the conditions

  $$G'(x) \stackrel{!}{=} 0 \text{ CU/unit} \qquad \text{(necessary condition)} \tag{7.32}$$

  and

  $$G''(x) \stackrel{!}{<} 0 \text{ CU/unit}^2 \qquad \text{(sufficient condition)} \ . \tag{7.33}$$

At this point, we like to draw the reader's attention to a special geometric property of the quantitative model for **profit** that we just have outlined: at maximum profit, the **total revenue function** $E(x)$ and the **total cost function** $K(x)$ always possess *parallel tangents*. This is due to the fact that by the necessary condition for an extremum to exist, one finds that

$$0 \text{ CU/unit} \stackrel{!}{=} G'(x) = E'(x) - K'(x) \qquad \Leftrightarrow \qquad E'(x) \stackrel{!}{=} K'(x) \ . \tag{7.34}$$

<u>**GDC:**</u> Roots and local maxima resp. minima can be easily determined for a given stored function in mode `CALC` by employing the interactive routines `zero` and `maximum` resp. `minimum`.

To conclude these considerations, we briefly turn to elucidate the technical term **Cournot's point**, which frequently arises in quantitative discussions in **economic theory**; this is named after the French mathematician and economist Antoine–Augustin Cournot (1801–1877). **Cournot's point** simply labels the profit-optimal combination of the level of physical output and the associated unit price, $(x_M, p(x_M))$, for the **unit price function** $p(x)$ of a good in a monopolistic market situation. Note that for this specific combination of optimal values the **Amoroso–Robinson formula** applies, which was developed by the Italian mathematician and economist Luigi Amoroso (1886–1965) and the British economist Joan Violet Robinson (1903–1983). This states that

$$p(x_M) = \frac{K'(x_M)}{1+\varepsilon_p(x_M)} , \tag{7.35}$$

with $K'(x_M)$ the value of the marginal costs at x_M, and $\varepsilon_p(x_M)$ the value of the **elasticity** of the unit price function at x_M (see the following Sec. 7.6). Starting from the defining equation of the **total revenue** $E(x) = xp(x)$, the **Amoroso–Robinson formula** is derived by evaluating the first derivative of $E(x)$ at x_M, so

$$E'(x_M) = p(x_M) + x_M p'(x_M) = p(x_M)\left[1 + x_M\,\frac{p'(x_M)}{p(x_M)}\right] \overbrace{=}^{\text{Sec. 7.6}} = p(x_M)\left[1+\varepsilon_p(x_M)\right] ,$$

and then re-arranging to solve for $p(x_M)$, using the fact that $E'(x_M) = K'(x_M)$.

Remark: In a market situation where **perfect competition** applies, one *assumes* that the **unit price function** has settled to a *constant* value $p(x) = p =$ constant > 0 CU/unit (and, hence, $p'(x) = 0$ CU/unit^2 obtains).

7.5.3 Extremal values of rational economic functions

Now we want to address the determination of extremal values of economic functions that constitute ratios in the sense of the construction

$$\frac{\text{OUTPUT}}{\text{INPUT}} ,$$

a topic raised in the Introduction.

Let us consider two examples for determining **local maxima** of ratios of this kind.

(i) We begin with the **average profit** in dependence on the level of physical output $x \geq 0$ units,

$$\frac{G(x)}{x} . \tag{7.36}$$

The conditions that determine a local maximum are $[G(x)/x]' \stackrel{!}{=} 0$ CU/unit^2 and $[G(x)/x]'' \stackrel{!}{<} 0$ CU/unit^3. Respecting the quotient rule of differentiation (cf. Sec. 7.2), the first condition yields

$$\frac{G'(x)x - G(x)}{x^2} = 0 \text{ GE/ME}^2 . \tag{7.37}$$

Since a quotient can only be zero when its numerator vanishes while its denominator remains non-zero, it immediately follows that

$$G'(x)x - G(x) = 0 \text{ CU} \quad \Rightarrow \quad x\,\frac{G'(x)}{G(x)} = 1 . \tag{7.38}$$

The task at hand now is to find a (unique) value of the level of physical output x which satisfies this last condition, and for which the second derivative of the average profit becomes negative.

(ii) To compare the performance of two companies over a given period of time in a meaningful way, it is recommended to adhere only to measures that are *dimensionless ratios*, and so independent of **scale**. An example of such a dimensionless ratio is the measure referred to as **economic efficiency**,

$$W(x) = \frac{E(x)}{K(x)} , \tag{7.39}$$

which expresses the **total revenue** (in CU) of a company for a given period as a multiple of the **total costs** (in CU) it had to endure during this period, both as functions of the **level of physical output**. In analogy to our discussion in (i), the conditions for the existence of a local maximum amount to $[E(x)/K(x)]' \stackrel{!}{=} 0 \times 1/\text{unit}$ and $[E(x)/K(x)]'' \stackrel{!}{<} 0 \times 1/\text{unit}^2$. By the quotient rule of differentiation (see Sec. 7.2), the first condition leads to

$$\frac{E'(x)K(x) - E(x)K'(x)}{K^2(x)} = 0 \times 1/\text{unit} , \tag{7.40}$$

i.e., for $K(x) > 0$ CU,

$$E'(x)K(x) - E(x)K'(x) = 0 \text{ CU}^2/\text{unit} . \tag{7.41}$$

By re-arranging and multiplication with $x > 0$ unit, this can be cast into the particular form

$$x\,\frac{E'(x)}{E(x)} = x\,\frac{K'(x)}{K(x)} . \tag{7.42}$$

The reason for this special kind of representation of the necessary condition for a local maximum to exist [and also for Eq. (7.38)] will be clarified in the subsequent section. Again, a value of the level of physical output which satisfies Eq. (7.42) must in addition lead to a negative second derivative of the **economic efficiency** in order to satisfy the sufficient condition for a local maximum to exist.

7.6 Elasticities

Finally, we pick up once more the discussion on quantifying the **local variability** of differentiable real-valued functions of one real variable, $f : D \subseteq \mathbb{R} \to W \subseteq \mathbb{R}$, though from a slightly different perspective. For reasons to be elucidated shortly, we confine ourselves to considerations of regimes of f with *positive* values of the argument x and also *positive* values $y = f(x) > 0$ of the function itself.

As before in Sec. 7.2, we want to assume a small change of the value of the argument x and evaluate its resultant effect on the value $y = f(x)$. This yields

$$x \stackrel{\Delta x \in \mathbb{R}}{\longrightarrow} x + \Delta x \qquad \Longrightarrow \qquad y = f(x) \stackrel{\Delta y \in \mathbb{R}}{\longrightarrow} y + \Delta y = f(x + \Delta x) . \tag{7.43}$$

We remark in passing that **relative changes** of non-negative quantities are defined by the quotient

$$\frac{\text{new value} - \text{old value}}{\text{old value}}$$

under the prerequisite that "old value > 0" applies. It follows from this specific construction that the minimum value a relative change can possibly attain amounts to "-1" (corresponding to a decrease of the "old value" by 100%).

Related to this consideration we identify the following terms:

- a prescribed **absolute change** of the independent variable x: Δx ,
- the resultant **absolute change** of the function f: $\Delta y = f(x + \Delta x) - f(x)$,
- the associated **relative change** of the independent variable x: $\dfrac{\Delta x}{x}$,
- the associated resultant **relative change** of the function f: $\dfrac{\Delta y}{y} = \dfrac{f(x + \Delta x) - f(x)}{f(x)}$.

Now let us compare the **order-of-magnitudes** of the two relative changes just envisaged, $\dfrac{\Delta x}{x}$ and $\dfrac{\Delta y}{y}$. This is realised by considering the value of their quotient, "resultant relative change of f divided by the prescribed relative change of x"':

$$\frac{\dfrac{\Delta y}{y}}{\dfrac{\Delta x}{x}} = \frac{\dfrac{f(x + \Delta x) - f(x)}{f(x)}}{\dfrac{\Delta x}{x}} \ .$$

Since we assumed f to be differentiable, it is possible to investigate the behaviour of this quotient of relative changes in the limit of increasingly smaller prescribed relative changes $\dfrac{\Delta x}{x} \to 0 \Rightarrow \Delta x \to 0$ near some $x > 0$. One thus defines:

Def.: For a differentiable real-valued function f of one real variable x, the *dimensionless* (i.e., units-independent) quantity

$$\boxed{\varepsilon_f(x) := \lim_{\Delta x \to 0} \frac{\dfrac{\Delta y}{y}}{\dfrac{\Delta x}{x}} = \lim_{\Delta x \to 0} \frac{\dfrac{f(x + \Delta x) - f(x)}{f(x)}}{\dfrac{\Delta x}{x}} = x \, \frac{f'(x)}{f(x)}} \tag{7.44}$$

is referred to as the **elasticity** of the function f at position x.

The elasticity of f quantifies its resultant relative change in response to a prescribed infinitesimally small relative change of its argument x, starting from some positive initial value $x > 0$. As such it constitutes a measure for the **relative local rate of change** of a function f in a point $(x, f(x))$. In

economic theory, in particular, one adheres to the following interpretation of the elasticity $\varepsilon_f(x)$: when the postive argument x of some positive differentiable real-valued function f is increased by $1\ \%$, then in consequence f will change approximately by $\varepsilon_f(x) \times 1\ \%$.

In the scientific literature one often finds the elasticity of a positive differentiable function f of a positive argument x expressed in terms of logarithmic differentiation. That is,

$$\varepsilon_f(x) := \frac{\mathrm{d}\ln[f(x)]}{\mathrm{d}\ln(x)} \qquad \text{for } x > 0 \text{ and } f(x) > 0\ ,$$

since by the chain rule of differentiation it holds that

$$\frac{\mathrm{d}\ln[f(x)]}{\mathrm{d}\ln(x)} = \frac{\dfrac{\mathrm{d}f(x)}{f(x)}}{\dfrac{\mathrm{d}x}{x}} = x\,\frac{\dfrac{\mathrm{d}f(x)}{\mathrm{d}x}}{f(x)} = x\,\frac{f'(x)}{f(x)}\ .$$

The logarithmic representation of the elasticity of a differentiable function f immediately explains why, at the beginning, we confined our considerations to positive differentiable functions of positive arguments only.[3] A brief look at the list of standard economic functions provided in Sec. 7.3 reveals that most of these (though not all) are positive functions of non-negative arguments.

For the elementary classes of real-valued functions of one real variable discussed in Sec. 7.1 one finds:

Standard elasticities

1. $f(x) = x^n$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}_{>0}$ $\Rightarrow$ $\varepsilon_f(x) = n$ (**natural power-law functions**)
2. $f(x) = x^\alpha$ for $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}_{>0}$ $\Rightarrow$ $\varepsilon_f(x) = \alpha$ (**general power-law functions**)
3. $f(x) = a^x$ for $a \in \mathbb{R}_{>0}\backslash\{1\}$ and $x \in \mathbb{R}_{>0}$ $\Rightarrow$ $\varepsilon_f(x) = \ln(a)x$ (**exponential functions**)
4. $f(x) = e^{ax}$ for $a \in \mathbb{R}$ and $x \in \mathbb{R}_{>0}$ $\Rightarrow$ $\varepsilon_f(x) = ax$ (**natural exponential functions**)
5. $f(x) = \log_a(x)$ for $a \in \mathbb{R}_{>0}\backslash\{1\}$ and $x \in \mathbb{R}_{>0}$
$$\Rightarrow \quad \varepsilon_f(x) = \frac{1}{\ln(a)\log_a(x)}$$
(**logarithmic functions**)
6. $f(x) = \ln(x)$ for $x \in \mathbb{R}_{>0}$ $\Rightarrow$ $\varepsilon_f(x) = \dfrac{1}{\ln(x)}$ (**natural logarithmic function**).

In view of these results, we would like to emphasise the fact that for the entire family of **general power-law functions** the elasticity $\varepsilon_f(x)$ has a *constant value*, independent of the value of the argument x. It is this very property which classifies **general power-law functions** as **scale-invariant**. When **scale-invariance** obtains, dimensionless ratios, i.e., quotients of variables of the

[3]To extend the regime of applicability of the measure ε_f, one may consider working in terms of absolute values $|x|$ and $|f(x)|$. Then one has to distinguish between four cases, which need to be looked at separately: (i) $x > 0$, $f(x) > 0$, (ii) $x < 0$, $f(x) > 0$, (iii) $x < 0$, $f(x) < 0$ and (iv) $x > 0$, $f(x) < 0$.

same physical dimension, reduce to *constants*. In this context, we would like to remark that scale-invariant (fractal) power-law functions of the form $f(x) = Kx^{\alpha}$, with $K > 0$ and $\alpha \in \mathbb{R}_{<0}\backslash\{-1\}$, are frequently employed in **Economics** and the **Social Sciences** for modelling **uncertainty** of **economic agents** in **decision-making processes**, or for describing probability distributions of **rare event phenomena**; see, e.g., Taleb (2007) [25, p 326ff] or Gleick (1987) [13, Chs. 5 and 6]. This is due, in part, to the curious property that for certain values of the exponent α general power-law probability distributions attain unbounded variance; cf. Ref. [12, Sec. 8.9].

Practical applications in **economic theory** of the concept of an elasticty as a measure of relative change of a differentiable real-valued function f of one real variable x are generally based on the following *linear (!)* approximation: beginning at $x_0 > 0$, for small prescribed percentage changes of the argument x in the interval $0\ \% < \dfrac{\Delta x}{x_0} \leq 5\ \%$, the resultant percentage changes of f amount approximately to

$$(\text{percentage change of } f) \approx (\text{elasticity of } f \text{ at } x_0) \times (\text{percentage change of } x)\ , \tag{7.45}$$

or, in terms of a mathematical formula, to

$$\frac{f(x_0 + \Delta x) - f(x_0)}{f(x_0)} \approx \varepsilon_f(x_0)\frac{\Delta x}{x_0}\ . \tag{7.46}$$

We now draw the reader's attention to a special kind of terminology developed in **economic theory** to describe the **relative local change behaviour** of economic functions in qualitative terms. For $x \in D(f)$, the relative local change behaviour of a function f is called

- **inelastic**, whenever $|\varepsilon_f(x)| < 1$,
- **unit elastic**, when $|\varepsilon_f(x)| = 1$, and
- **elastic**, whenever $|\varepsilon_f(x)| > 1$.

For example, a total cost function $K(x)$ in the diminishing returns picture exhibits unit elastic behaviour at the minimum efficient scale $x = x_{g_2}$ where, by Eq. (7.22), $\varepsilon_K(x_{g_2}) = 1$. Also, at the local maximum of an average profit function $G(x)/x$, the property $\varepsilon_G(x) = 1$ applies; cf. Eq. (7.38).

Next, we review the computational rules one needs to adhere to when calculating elasticities for combinations of two real-valued functions of one real variable in the sense of Sec. 7.1.6:

Computational rules for elasticities
If f and g are differentiable real-valued functions of one real variable, with elasticities ε_f and ε_g, it holds that:

1. **product** $f \cdot g$: $\quad \varepsilon_{f\cdot g}(x) = \varepsilon_f(x) + \varepsilon_g(x)$,
2. **quotient** $\dfrac{f}{g}$, $g \neq 0$: $\quad \varepsilon_{f/g}(x) = \varepsilon_f(x) - \varepsilon_g(x)$,
3. **concatenation** $f \circ g$: $\quad \varepsilon_{f\circ g}(x) = \varepsilon_f(g(x)) \cdot \varepsilon_g(x)$,

4. **inverse function** f^{-1}: $$\varepsilon_{f^{-1}}(x) = \left.\frac{1}{\varepsilon_f(y)}\right|_{y=f^{-1}(x)} .$$

To end this chapter, we remark that for a positive differentiable real-valued function f of one positive real variable x, a second elasticity may be defined according to

$$\varepsilon_f\left[\varepsilon_f(x)\right] := x\,\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x}{f(x)}\,\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right] . \tag{7.47}$$

Of course, by analogy this procedure may be generalised to higher derivatives of f still.

Chapter 8

Integral calculus of real-valued functions of one real variable

In the final chapter of these lecture notes we give a brief overview of the main definitions and laws of the **integral calculus** of real-valued functions of one variable. Subsequently we consider a simple application of this tool in **economic theory**.

8.1 Indefinite integrals

Def.: Let f be a continuous real-valued function of one real variable and F a differentiable real-valued function of the same real variable, with $D(f) = D(F)$. Given that f and F are related according to

$$\boxed{F'(x) = f(x) \qquad \textit{for all} \quad x \in D(f)\ ,} \tag{8.1}$$

then F is referred to as a **primitive** of f.

Remark: The primitive of a given continuous real-valued function f *cannot* be unique. By the rules of differentiation discussed in Sec. 7.2, besides F also $F + c$, with $c \in \mathbb{R}$ a real-valued constant, constitutes a primitive of f since $(c)' = 0$.

Def.: If F is a primitive of a continuous real-valued function f of one real variable, then

$$\boxed{\int f(x)\,\mathrm{d}x = F(x) + c\ , \quad c \;=\; \text{constant} \;\in\; \mathbb{R}\ , \quad \text{with } F'(x) = f(x)} \tag{8.2}$$

defines the **indefinite integral** of the function f. The following names are used to refer to the different ingredients in this expression:

- x — the **integration variable**,
- $f(x)$ — the **integrand**,
- $\mathrm{d}x$ — the **differential**, and, lastly,

- c — the **constant of integration**.

For the elementary, continuous real-valued functions of one variable introduced in Sec. 7.1, the following rules of indefinite integration apply:

Rules of indefinite integration

1. $\int \alpha \, \mathrm{d}x = \alpha x + c$ with $\alpha = \text{constant} \in \mathbb{R}$ (**constants**)
2. $\displaystyle\int x \, \mathrm{d}x = \frac{x^2}{2} + c$ (**linear functions**)
3. $\displaystyle\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ for $n \in \mathbb{N}$ (**natural power-law functions**)
4. $\displaystyle\int x^\alpha \, \mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + c$ for $\alpha \in \mathbb{R}\backslash\{-1\}$ and $x \in \mathbb{R}_{>0}$ (**general power-law functions**)
5. $\displaystyle\int a^x \, \mathrm{d}x = \frac{a^x}{\ln(a)} + c$ for $a \in \mathbb{R}_{>0}\backslash\{1\}$ (**exponential functions**)
6. $\displaystyle\int e^{ax} \, \mathrm{d}x = \frac{e^{ax}}{a} + c$ for $a \in \mathbb{R}\backslash\{0\}$ (**natural exponential functions**)
7. $\int x^{-1} \, \mathrm{d}x = \ln|x| + c$ for $x \in \mathbb{R}\backslash\{0\}$.

Special methods of integration need to be employed when the integrand consists of a concatanation of elementary real-valued functions. Here we provide a list with the main tools for this purpose. For differentiable real-valued functions f and g, it holds that

1. $\int (\alpha f(x) \pm \beta g(x)) \, \mathrm{d}x = \alpha \int f(x) \, \mathrm{d}x \pm \beta \int g(x) \, \mathrm{d}x$
   with $\alpha, \beta = \text{constant} \in \mathbb{R}$ (**summation rule**)
2. $\int f(x)g'(x) \, \mathrm{d}x = f(x)g(x) - \int f'(x)g(x) \, \mathrm{d}x$ (**integration by parts**)
3. $\int f(g(x))g'(x) \, \mathrm{d}x \overbrace{=}^{u=g(x) \text{ and } \mathrm{d}u=g'(x)\mathrm{d}x} \int f(u) \, \mathrm{d}u = F(g(x)) + c$ (**substitution method**)
4. $\displaystyle\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln|f(x)| + c$ for $f(x) \neq 0$ (**logarithmic integration**).

8.2 Definite integrals

<u>**Def.:**</u> Let f be a real-valued function of one variable which is continuous on an interval $[a, b] \subset D(f)$, and let F be a primitive of f. Then the expression

$$\boxed{\int_a^b f(x) \, \mathrm{d}x = F(x)|_{x=a}^{x=b} = F(b) - F(a)} \tag{8.3}$$

defines the **definite integral** of f in the **limits of integration** a and b.

For definite integrals the following general rules apply:

1. $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$ (**identical limits of integration**)

2. $\displaystyle\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$ (**interchange of limits of integration**)

3. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$ for $c \in [a, b]$ (**split of integration interval**).

Remark: The main qualitative difference between an (i) indefinite integral and a (ii) definite integral of a continuous real-valued function of one variable reveals intself in the different kinds of outcome: while (i) yields as a result a real-valued (primitive) *function*, (ii) simply yields a single real *number*.

GDC: For a stored real-valued function, the evaluation of a definite integral can be performed in mode `CALC` with the pre-programmed function `∫f(x)dx`. The corresponding limits of integration need to be specified interactively.

As indicated in Sec. 7.6, the scale-invariant power-law functions $f(x) = x^{\alpha}$ for $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}_{>0}$ play a special role in practical applications. For $x \in [a, b] \subset \mathbb{R}_{>0}$ and $\alpha \neq -1$ it holds that

$$\int_a^b x^{\alpha}\,\mathrm{d}x = \left.\frac{x^{\alpha+1}}{\alpha+1}\right|_{x=a}^{x=b} = \frac{1}{\alpha+1}\left(b^{\alpha+1} - a^{\alpha+1}\right) \ . \tag{8.4}$$

Problematic in this context can be considerations of taking limits of the form $a \to 0$ resp. $b \to \infty$, since for either of the two cases

(i) case $\alpha < -1$:

$$\lim_{a\to 0} \int_a^b x^{\alpha}\,\mathrm{d}x \to \infty \ , \tag{8.5}$$

(ii) case $\alpha > -1$:

$$\lim_{b\to\infty} \int_a^b x^{\alpha}\,\mathrm{d}x \to \infty \ , \tag{8.6}$$

one ends up with **divergent** mathematical expressions.

8.3 Applications in economic theory

The starting point shall be a simple market situation for a single product. For this product, on the one-hand side, there be a **demand function** $N(p)$ (in units) which is monotonously decreasing on the price interval $[p_u, p_o]$; the limit values p_u and p_o denote the minimum and maximum prices per unit (in CU/u) acceptable for the product. On the other hand, the market situation be described by a **supply function** $A(p)$ (in units) which is monotonously increasing on $[p_u, p_o]$.

The **equilibrium unit price** p_M (in CU/unit) for this product is defined by assuming a state of **economic equilibrium** of the market, quantitatively expressed by the condition

$$A(p_M) = N(p_M) \ . \tag{8.7}$$

Geometrically, this condition defines common points of intersection for the functions $A(p)$ and $N(p)$ (when they exist).

GDC: Common points of intersection for stored functions f and g can be easily determined interactively in mode `CALC` employing the routine `intersect`.

In a drastically simplified fashion, we now turn to compute the revenue made on the market by the suppliers of a new product for either of three possible **strategies of market entry**.

1. **Strategy 1:** The revenue obtained by the suppliers when the new product is being sold straight at the equilibrium unit price p_M, in an amount $N(p_M)$, is simply given by

$$U_1 = U(p_M) = p_M N(p_M) \qquad \text{(in CU)}\ . \tag{8.8}$$

2. **Strategy 2:** Some consumers would be willing to purchase the product intially also at a unit price which is higher than p_M. If, hence, the suppliers decide to offer the product initially at a unit price $p_o > p_M$, and then, in order to generate further demand, to continuously[1] (!) *reduce* the unit price to the lower p_M, the revenue obtained yields the larger value

$$U_2 = U(p_M) + \int_{p_M}^{p_o} N(p)\,\mathrm{d}p\ . \tag{8.9}$$

   Since the amount of money

$$K := \int_{p_M}^{p_o} N(p)\,\mathrm{d}p \qquad \text{(in CU)} \tag{8.10}$$

   is (theoretically) safed by the consumers when the product is introduced to the market according to strategy 1, this amount is referred to in the economic literature as **consumer surplus**.

3. **Strategy 3:** Some suppliers would be willing to introduce the product to the market initially at a unit price which is lower than p_M. If, hence, the suppliers decide to offer the product initially at a unit price $p_u < p_M$, and then to continuously[2] (!) *raise* it to the higher p_M, the revenue obtaines amounts to the smaller value

$$U_3 = U(p_M) - \int_{p_u}^{p_M} A(p)\,\mathrm{d}p\ . \tag{8.11}$$

   Since the suppliers (theoretically) earn the extra amount

$$P := \int_{p_u}^{p_M} A(p)\,\mathrm{d}p \qquad \text{(in CU)} \tag{8.12}$$

   when the product is introduced to the market according to strategy 1, this amount is referred to in the economic literature as **producer surplus**.

[1] This is a strong mathematical assumption aimed at facilitating the actual calculation to follow.
[2] See previous footnote.

Appendix A

Glossary of technical terms (GB – D)

A
absolute change: absolute Änderung
absolute value: Betrag
account balance: Kontostand
addition: Addition
analysis: Analysis, Untersuchung auf Differenzierbarkeitseigenschaften
arithmetical mean: arithmetischer Mittelwert
arithmetical sequence: arithmetische Zahlenfolge
arithmetical series: arithmetische Reihe
augmented coefficient matrix: erweiterte Koeffizientenmatrix
average costs: Stückkosten
average profit: Durchschnittsgewinn, Gewinn pro Stück

B
backward substitution: rückwertige Substitution
balance equation: Bilanzgleichung
basis: Basis
basis solution: Basislösung
basis variable: Basisvariable
Behavioural Economics: Verhaltensökonomik
boundary condition: Randbedingung
break-even point: Gewinnschwelle

C
chain rule: Kettenregel
characteristic equation: charakteristische Gleichung
coefficient matrix: Koeffizientenmatrix
column: Spalte
column vector: Spaltenvektor
component: Komponente
compound interest: Zinseszins
concatenation: Verschachtelung, Verknüpfung
conservation law: Erhaltungssatz

constant of integration: Integrationskonstante
constraint: Zwangsbedingung
cost function: Kostenfunktion
consumer surplus: Konsumentenrente
continuity: Stetigkeit
contract period: Laufzeit
Cournot's point: Cournotscher Punkt
curve sketching: Kurvendiskussion

D

decision-making: Entscheidungsfindung
declining-balance depreciation method: geometrisch–degressive Abschreibung
definite integral: bestimmtes Integral
demand function: Nachfragefunktion
dependent variable: abhängige Variable
depreciation: Abschreibung
depreciation factor: Abschreibungsfaktor
derivative: Ableitung
determinant: Determinante
difference: Differenz
difference quotient: Differenzenquotient
differentiable: differenzierbar
differential: Integrationsdifferenzial
differential calculus: Differenzialrechnung
dimension: Dimension
direction of optimisation: Optimierungsrichtung
divergent: divergent, unbeschr"ankt
domain: Definitionsbereich

E

economic agent: Wirtschaftstreibende(r) (meistens ein *homo oeconomicus*)
economic efficiency: Wirtschaftlichkeit
economic equilibrium: ökonomisches Gleichgewicht
economic principle: ökonomisches Prinzip
economic theory: Wirtschaftstheorie
Econophysics: Ökonophysik
eigenvalue: Eigenwert
eigenvector: Eigenvektor
elastic: elastisch
elasticity: Elastizität
element: Element
end of profitable zone: Gewinngrenze
endogenous: endogen
equilibrium price: Marktpreis
equivalence transformation: Äquivalenztransformation

exogenous: exogen
exponential function: Exponentialfunktion
extrapolation: Extrapolation, über bekannten Gütigkeitsbereich hinaus verallgemeinern

F
feasible region: zulässiger Bereich
final capital: Endkapital
fixed costs: Fixkosten
forecasting: Vorhersagen erstellen
function: Funktion

G
Gaußian elimination: Gauß'scher Algorithmus
GDC: GTR, grafikfähiger Taschenrechner
geometrical mean: geometrischer Mittelwert
geometrical sequence: geometrische Zahlenfolge
geometrical series: geometrische Reihe
growth function: Wachstumsfunktion

H

I
identity: Identität
image vector: Absolutgliedvektor
indefinite integral: unbestimmtes Integral
independent variable: unabhängige Variable
inelastic: unelastisch
initial capital: Anfangskapital
installment: Ratenzahlung
installment savings: Ratensparen
integral calculus: Integralrechnung
integrand: Integrand
integration variable: Integrationsvariable
interest factor: Aufzinsfaktor
interest rate: Zinsfuß
inverse function: Inversfunktion, Umkehrfunktion
inverse matrix: inverse Matrix, Umkehrmatrix
isoquant: Isoquante

J

K

L
law of diminishing returns: Ertragsgesetz

length: Länge
level of physical output: Ausbringungsmenge
limits of integration: Integrationsgrenzen
linear combination: Linearkombination
linearisation: Linearisierung
linear programming: lineare Optimierung
local rate of change: lokale Änderungsrate
logarithmic function: Logarithmusfunktion

M

mapping: Abbildung
marginal costs: Grenzkosten
maximisation: Maximierung
minimisation: Minimierung
minimum efficient scale: Betriebsoptimum
monetary value: Geldwert
monopoly: Monopol
monotonicity: Monotonie
mortgage loan: Darlehen

N

non-basis variable: Nichtbasisvariable
non-negativity constraints: Nichtnegativitätsbedingungen
non-linear functional relationship: nichtlineare Funktionalbeziehung

O

objective function: Zielfunktion
one-to-one and onto: eineindeutig
optimal solution: optimalen Lösung
optimal value: optimaler Wert
optimisation: Optimierung
order-of-magnitude: Größenordnung
orthogonal: orthogonal
over-determined: überbestimmt

P

parallel displacement: Parallelverschiebung
pension calculations: Rentenrechnung
percentage rate: Prozentsatz
perfect competition: totale Konkurrenz
period: Periode
pivot column index: Pivotspaltenindex
pivot element: Pivotelement
pivot operation: Pivotschritt
pivot row index: Pivotzeilenindex
pole: Polstelle, Singularität
polynomial division: Polynomdivision

polynomial of degree n: Polynom vom Grad n
power-law function: Potenzfunktion
present value: Barwert
primitive: Stammfunktion
principal component analysis: Hauptkomponentenanalyse
producer surplus: Produzentenrente
product rule: Produktregel
profit function: Gewinnfunktion
prohibitive price: Pohibitivpreis
Prospect Theory: Neue Erwartungstheorie
psychological value function: psychologische Wertfunktion

Q
quadratic matrix: quadratische Matrix
quotient: Quotient
quotient rule: Quotientenregel

R
range: Wertespektrum
rank: Rang
rare event: seltenes Ereignis
rational function: gebrochen rationale Funktion
real-valued function: reellwertige Funktion
reference period: Referenzzeitraum
regression analysis: Regressionsanalyse
regular: regulär
relative change: relative Änderung
remaining debt: Restschuld
remaining resources: Restkapazitäten
remaining value: Restwert
rescaling: Skalierung
resources: Rohstoffe
resource consumption matrix: Rohstoffverbrauchsmatrix
restrictions: Restriktionen
root: Nullstelle
row: Reihe
row vector: Zeilenvektor

S
saturation quantity: Sättigungsmenge
scale: Skala, Größenordnung
scale-invariant: skaleninvariant
simplex: Simplex, konvexer Polyeder
simplex tableau: Simplextabelle
singular: singulär
sink: Senke

slack variable: Schlupfvariable
source: Quelle
stationary: stationär, konstant in der Zeit
straight line depreciation method: lineare Abschreibung
strictly monotonously decreasing: streng monoton fallend
strictly monotonously increasing: streng monoton steigend
summation rule: Summationsregel
supply function: Angebotsfunktion

T

tangent: Tangente
target space: Wertebereich
technology matrix: Technologiematrix
total demand matrix: Gesamtbedarfsmatrix
total revenue: Ertrag
transpose: Transponierte

U

uncertainty: Unsicherheit
under-determined: unterbestimmt
uniqueness: Eindeutigkeit
unit elastic: proportional elastisch
unit matrix: Einheitsmatrix
unit price: Stückpreis
unit vector: Einheitsvektor
utility function: Nutzenfunktion

V

value chain: Wertschöpfungskette
variability: Änderungsverhalten, Variabilität
variable average costs: variable Stückkosten
variable costs: variable Kosten
variable vector: Variablenvektor
vector: Vektor
vector algebra: Vektoralgebra

W

well-determined: wohlbestimmt

Z

zero matrix: Nullmatrix
zero vector: Nullvektor

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