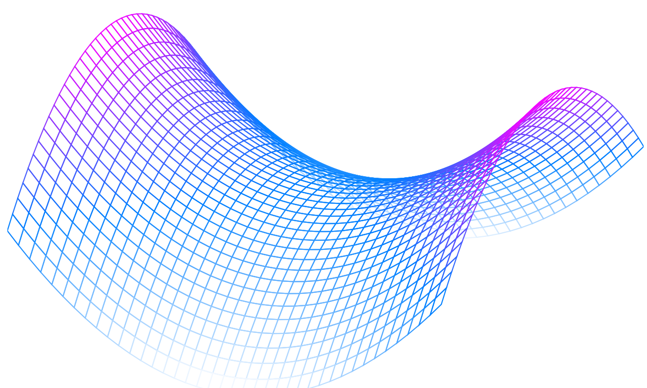
Noah's Guide to Calculus

Noah Stockwell

2016-2017



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I would like to thank my first- and second-semester Calculus teacher for teaching me the foundations of Calculus and my third-semester Calculus professor for showing me new ways to use math.

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This textbook was created using LaTeX in TeXstudio.

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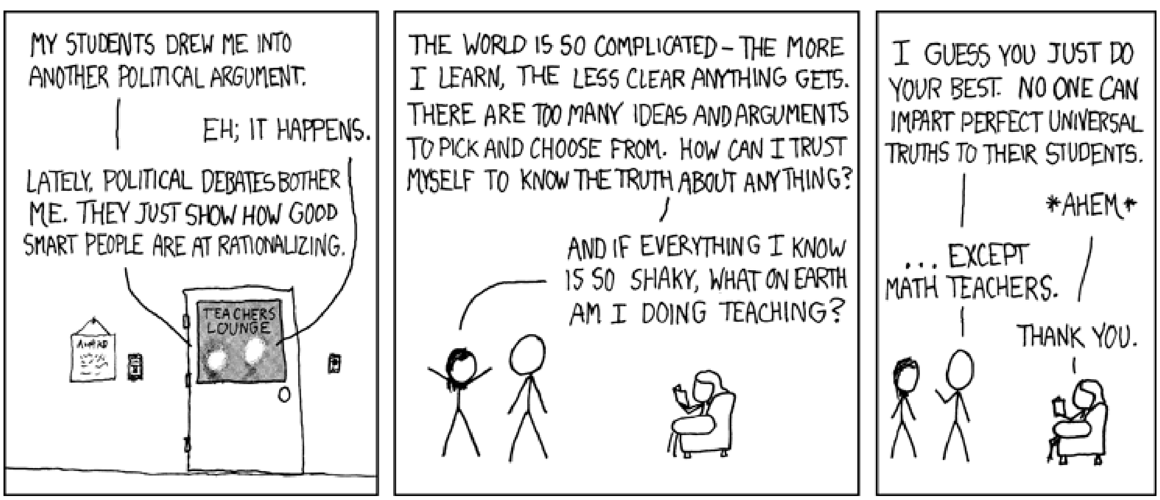
Introduction

Most Calculus textbooks are long and boring with seemingly endless and sometimes meaningless examples. This text is an attempt to solve that problem with comprehensive yet concise explanations for all of the topics covered in the AP Calculus AB exam for the 2016-17 school year. As the text does not provide many examples, most of the topics are presented in an abstract way and I strongly recommend this text be supplementary to another textbook.

All trigonometry is in radians.

This textbook will have example problems throughout but exercises will be used as more of a checkpoint for the reader rather than homework.

If you have any comments or suggestions about the textbook or math in general, feel free to email me! You can reach me at noah.stockwell@edgewoodhs.org.

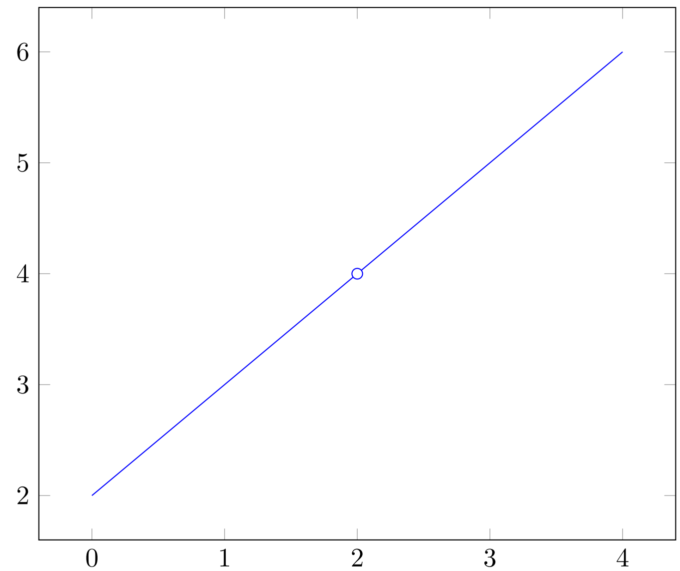


Contents

Limits

1 Limits

Limits are a way of skirting the normal rules of math. Without the knowledge of limits, whenever a function divides by 0 or involves ∞ in any way, calculations become impossible. Limits take the rules of math a little less seriously and can be used to calculate what a value "should be". A simple example of where limits come in handy is when there is a "hole" in a graph:



$$f(x) = \frac{x^2 - 4}{x - 2}$$

Because $f(x)$ divides by 0 when $x = 2$, there can be no answer here. However, we can tell that $f(2)$ should be 4 ignoring the division by zero. We can tell this because as x becomes greater and nearer to 2 (approaching $x = 2$ from the left), the value of $f(x)$ approaches 4. Similarly, when x decreases and becomes nearer to $x = 2$ (approaching $x = 2$ from the right), the value of $f(x)$ approaches 4. Therefore, as both sides of $x = 2$ become closer and closer, they converge upon a single point: $f(2) = 4$.

1.1 Types of Limits

Limits are just a way of finding what a graph should be. Looking at the previous example, we can trace the graph as it comes closer to $x = 2$ from the left to make an educated guess and say that $f(2)$ should be about 4. We can write this in special limit notation:

$$\lim_{x \to 2^-} f(x) = 4$$

In English, that means *the limit as x approaches 2 from the left is 4*. The - sign is appended to the 2 to show that we are using how the graph is to the left of $x = 2$ to approximate $x = 2$. If we were using the graph to the right to approximate $x = 2$, we would write:

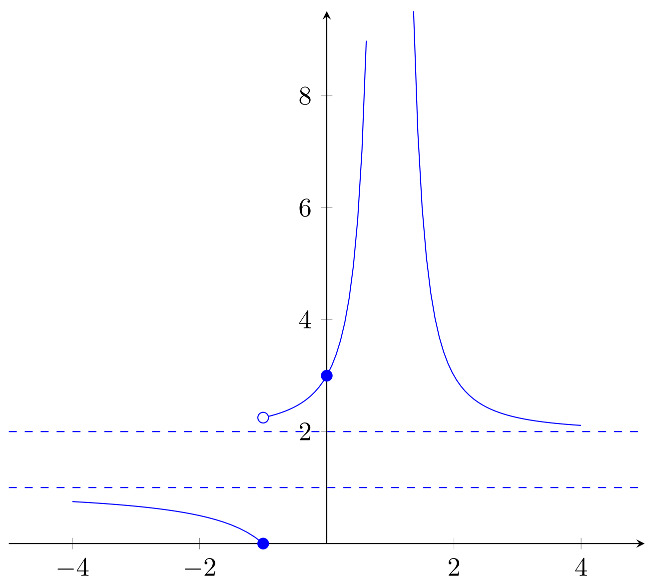
$$\lim_{x \to 2^+} f(x) = 4$$

These are called **one-sided limits**. The one-sided limits from both sides of the graph do not necessarily need to be the same.

The Limit We define $\lim\limits_{x \to c}$ (the value that the graph approaches)to be:

$$\lim_{x \to c} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

which should be read as: *the limit of $f(x)$ as x approaches c* **is equal to** *the limit of $f(x)$ as x approaches c from the right side* **and equal to** *the limit of $f(x)$ as x approaches c from the left side.* The limit from the left side is simply the value that the graph of $f(x)$ approaches as x increases to the limit point. The same is true for the right side: the limit is what the graph approaches as x becomes smaller towards the the limit point. If $\lim\limits_{x \to c^+} f(x) \neq \lim\limits_{x \to c^-} f(x)$, we say that the limit of $f(x)$ at c must not exist.



In the above graph, $\lim\limits_{x \to -1} f(x)$ doesn't exist because $\lim\limits_{x \to -1^+} f(x) = \sqrt{2}+1$ and $\lim\limits_{x \to -1^-} f(x) = 0$. The graph doesn't need to have a hole in it to have a limit. In fact, **any continuous part of a graph has a limit**. For example, the limit as x approaches 0 exists. The limit is equal to 2. Limits can also be infinite. Because $\lim\limits_{x \to 1^+} f(x)$ and $\lim\limits_{x \to 1^-} f(x)$ are both equal to ∞, $\lim\limits_{x \to 1} f(x) = \infty$. Similarly, we can take the limit at infinity. Because they are asymptotes. $\lim\limits_{x \to \infty} f(x) = 2$ and $\lim\limits_{x \to -\infty} f(x) = 1$. Because we can only approach limits at infinity from one side, we omit the + and - following the value as they are implied. However, this means that all asymptotes are limits taken at infinity.

1.2 Directly Calculating Limits

Unfortunately, there's no simple and easy way to calculate limits. The simplest way is just to "plug in" to the function but at some points like holes in the graph or ∞, we don't have that luxury. Instead, we can reduce or rewrite equations and also apply some general common sense. For example, $\lim_{x\to\infty} \frac{1}{x}$ must be 0 because as x gets larger, $\frac{1}{x}$ gets smaller to some point at which it must be 0. Using reduction and logic, we may progress on to more complex ideas involving limits where direct substitution fails.

Rules There are simple rules for limits:

$$\lim_{x\to c} f(x) * g(x) = \lim_{x\to c} f(x) * \lim_{x\to c} g(x)$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

$$\lim_{x\to c} f(x)^n = \left(\lim_{x\to c} f(x)\right)^n$$

The same is true for function addition, subtraction, etc.

"Plugging In" In the end, all limit problems will need to have their value inserted at some time. For example, the limit $\lim_{x\to 0} x^2 + 1 = 1$. No tricks here, plugging in 0 does return 1. We do not have to deal with any division by zero or infinities, so there is no need to manipulate the problem.

Reduction Reduction is relatively simple. If we go back to our example of

$$f(x) = \frac{x^2 - 4}{x - 2}$$

we can see that this problem can factor into

$$f(x) = \frac{(x-2)(x+2)}{x-2}$$

which cancels and gives us

$$f(x) = x + 2$$

We now see that there does exist a limit at $x = 2$, $f(2) = 2 + 2 = 4$. Previously, directly plugging in 2 would not return a real value.

Sense at ∞ ∞ is a hard concept to grasp and work with in mathematics. Limits can make this easier because while we cannot *directly compute* ∞, we can approximate it exactly. For example, in the case of $f(x) = \dfrac{1}{x^2}$, we see that the limit at ∞ must be zero. As x increases, $f(x)$ drops ever closer to 0:

$$f(10) = 0.01$$

$$f(100) = 0.0001$$
$$f(1000) = 0.000001$$
$$etc.$$

Similarly, as x approaches ∞,

$$g(x) = \frac{3x}{4x+1}$$

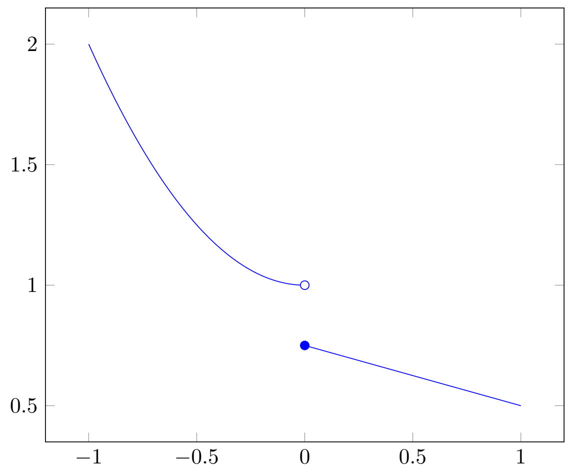
approaches $\frac{3}{4}$:

$$g(10) = \frac{30}{41}$$
$$g(100) = \frac{300}{401}$$
$$g(1000) = \frac{3000}{4001}$$
$$etc.$$

1.3 When Limits Don't Exist

Limits don't exist when:

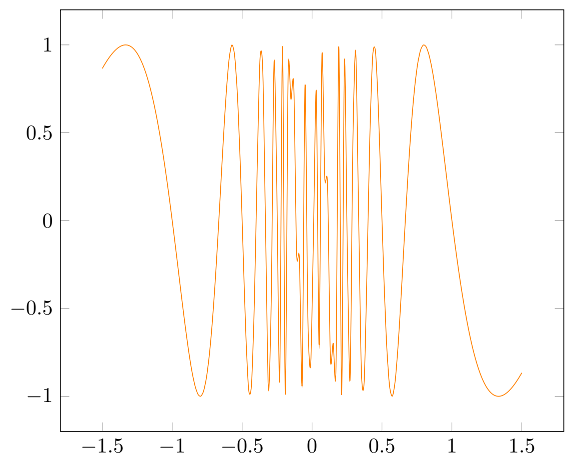
1. $\lim\limits_{x \to c^+} \neq \lim\limits_{x \to c^-}$



$$\lim_{x \to 0} f(x) \text{ Does Not Exist}$$

The reason why the limit cannot exist at 0 here is that when $f(x)$ approaches 0 from the right side (0^-), the limit is 1. As $f(x)$ approaches 0 from the left side (0^+), the limit is 0.75. The limit is one point and because $0.75 \neq 1$, there is no solution.

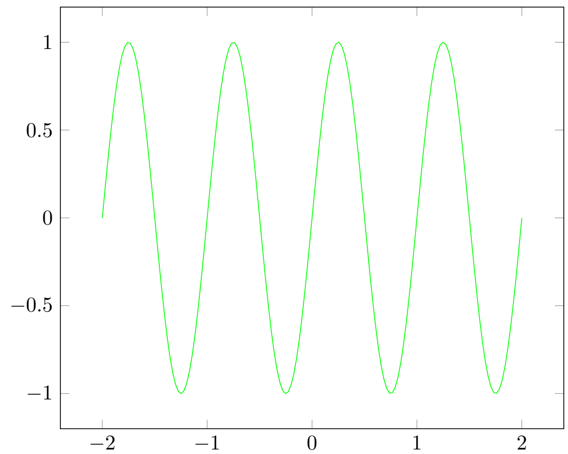
2. **The graph oscillates uncontrollably:**



$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ Does Not Exist}$$

The graphing utility doesn't even have any idea what's going on. We can't blame it because the graph starts oscillating faster and faster closer to 0. There is no point the graph can approach. Therefore, the limit must not exist here.

3. **The graph oscillates at ∞**



$$\lim_{x\to\infty} \sin(x) \text{ Does Not Exist}$$

Is ∞ an integer? Or where does it fall exactly? If you define ∞ to be a value, something can always be added to that value. Therefore, there is no single number that this graph will approach. In fact, it will approach all numbers in the interval [-1,1]. For a periodic graph like $\sin(x)$, there can be no limit because the graph will never stop oscillating.

Example 1:

$$\lim_{q\to 5} \frac{q^2}{q+1}$$

To find the limit, we must find the limit from the left-hand and right-hand side. For the left-hand side:

$$\lim_{q\to 5^-} \frac{q^2}{q+1} = \frac{25}{6}$$

For the right-hand side:

$$\lim_{q\to 5^+} \frac{q^2}{q+1} = \frac{25}{6}$$

The two limits are equal, therefore:

1. The limit $\lim_{q\to 5} \frac{q^2}{q+1}$ must exist, and
2. The limit must have the value $\frac{25}{6}$.

Example 2:

$$\lim_{x\to 3} \frac{x^2-9}{x^4-81}$$

Again, find the limit from the left-hand and right-hand sides:

$$\lim_{x\to 3^-} \frac{x^2-9}{x^4-81} = \frac{0}{0}$$

$\frac{0}{0}$ is a bad number. There is no defined value for $\frac{0}{0}$, so we should try and work around it to see if we can get an actual answer:

$$\frac{x^2-9}{x^4-81} = \frac{x^2-9}{(x^2-9)(x^2+9)}$$
$$\implies \lim_{x\to 3^-} \frac{x^2-9}{x^4-81} = \lim_{x\to 3^-} \frac{1}{x^2+9} = \frac{1}{18}$$

Similarly for $\lim_{x\to 3^+}$, we run into the same issue and must reduce:

$$\lim_{x\to 3^+} \frac{x^2-9}{x^4-81} = \frac{0}{0}$$
$$\implies \frac{x^2-9}{x^4-81} = \frac{x^2-9}{(x^2-9)(x^2+9)}$$
$$\implies \lim_{x\to 3^+} \frac{x^2-9}{x^4-81} = \lim_{x\to 3^+} \frac{1}{x^2+9} = \frac{1}{18}$$

Therefore, the limit must exist and its value is $\frac{1}{18}$.

Example 3:

$$\lim_{y\to\infty} \sin\left(\frac{1}{y}\right)$$

We only need to calculate the left-handed limit because we can only approach ∞ from the left side

$$\begin{aligned}
&\lim_{y\to\infty} \sin\left(\frac{1}{y}\right) \\
=&\sin\left(\lim_{y\to\infty} \frac{1}{y}\right) \\
=&\sin(0) \\
=&0
\end{aligned}$$

Example 4:

$$\lim_{t\to\infty} \cos\left(\frac{e^t}{2t+4}\right)$$

This has no solution because, unlike Example 3:

$$\lim_{t\to\infty} \frac{e^t}{2t+4}$$

does not exist and neither can the limit with cosine. This problem does not have a solution.

Exercises

1. **Find the limit if it exists:**
   (a) $\lim\limits_{x\to 4^+} \sqrt{x+5}$
   (b) $\lim\limits_{t\to 0^-} \dfrac{t}{|t|}$
   (c) $\lim\limits_{y\to 2} \dfrac{y-2}{y^2-4}$
   (d) $\lim\limits_{t\to \pi} \tan\left(\dfrac{t}{2}\right)$
   (e) $\lim\limits_{x\to 9} \log_x(y) = 4$
   (f) $\lim\limits_{x\to 0} \ln(x)$
2. **Find the limit at ∞ if it exists:**
   (a) $\lim\limits_{q\to\infty} \dfrac{q^2}{(q+2)(q-1)}$
   (b) $\lim\limits_{t\to -\infty} e^t$
   (c) $\lim\limits_{x\to\infty} (-1)^x$
   (d) $\lim\limits_{t\to -\infty} \cos\left(\dfrac{1}{t}\right)$
   (e) $\lim\limits_{n\to\infty} f(n)$ where $f(n) = \dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^n}$

1.4 ϵ-δ Formation of the Limit

The *epsilon* (ϵ) - *delta* (δ) formation of the limit actually explains how the limit works. Here is what this limiting process is defined as:

> For a function $f(x)$, a point $x = c$, the limit at that point $f(c) = L$, and a positive ϵ error, I can find you a δ window ($c - \delta$ to $c + \delta$) where the difference between all $f(k)$ with $c - \delta \leq k \leq c + \delta$ and the limit L is less than ϵ

Essentially, if you have a function $f(x)$ and you want to know a point $f(x) = L$ with a maximum error of ϵ, I can find you an δ window where if you plug in any point in the window, you will be less than your ϵ error away from your point $f(x) = L$.
Let's say you have $f(x) = x^2$ and you want to know 5.1^2 with an *epsilon*-error of 3, the window where all of the acceptable points lie can be $\sqrt{25.01 - 3} \leq x \leq \sqrt{25.01 + 3}$

That sounds good, but how does this show us that the limit exists? The answer is that if you can show this for any ϵ in general, you can make it as small as you want, and as your ϵ error gets smaller, your δ window closes in around a point $x = c$. If this is the case, we say that $f(c)$ has a limit L. If you cannot show this, then $f(c)$ must not have a limit L.

When we want to find a limit, we need a guess for what L actually is. Let's take the function we started out our discussion of limits with: $f(x) = \dfrac{x^2 - 4}{x - 2}$ when $x = 2$. If we try to evaluate $f(2)$, we do not get a real answer. Since it appears to us like the limit does exist at that point, let's use our new definition of the limit to evaluate it:

First, let's write the difference between some $f(x)$ and the limit L:

$$|f(x) - L|$$

Step two is to require our error to be bounded by ϵ:

$$|f(x) - L| \leq \epsilon$$

Third, let's write in our equations:

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| \leq \epsilon$$

We are trying to find a δ-window around $x = 2$, so plugging that in, we get:

$$\left| \frac{(2 + \delta)^2 - 4}{(2 + \delta) - 2} - 4 \right| \leq \epsilon$$

$$\left| \frac{\delta^2 + 4\delta}{\delta} - 4 \right| \leq \epsilon$$

$$|4 + \delta - 4| \leq \epsilon$$

$$|\delta| \leq \epsilon$$

$$\delta \leq \epsilon$$

Where $\epsilon \geq 0$ - this is guaranteed by the fact that ϵ is positive (see definition).

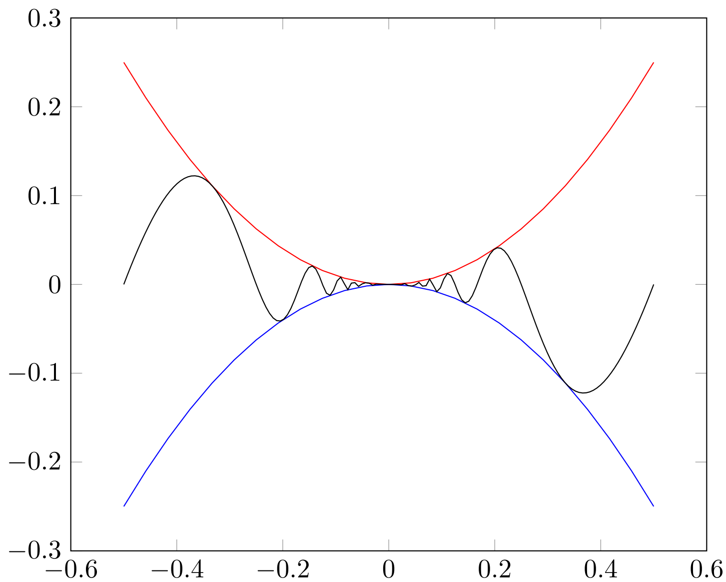
So in fact, if we want to be ϵ away from the limit L, all we have to do is choose a δ such that $\delta \leq \epsilon$. This should make intuitive sense because if you simplify the equation, the slope is 1. An ϵ change in y will result in a ϵ change in x.

1.5 The Squeeze Theorem

Sometimes none of the aforementioned ways for calculating limits works. One way to calculate possibly impossible limits is the squeeze theorem:

> On an interval containing a point c, if $f(x) \leq g(x) \leq h(x)$ for all of the interval, and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = a$, then $\lim_{x\to c} g(x) = a$.

The easiest way to see this is a graph:



Here we can see the graph $g(x) = x^2 * \sin(\frac{4}{x})$ being squeezed between $f(x) = -x^2$ and $h(x) = x^2$. $g(x)$ has no limit we can compute directly at 0 because $\frac{1}{x}$ has no real limit. However, it is a simple proof to show that $f(x) \leq g(x) \leq h(x)$ and $f(x)$ and $g(x)$ both have the same, real limit for $x = 0$, so

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = \lim_{x\to 0} g(x) = 0$$

Example 1: The squeeze theorem is uncommon and only applies (to Calculus) in a small number of situations. As a result, we will only prove the previous problem and then move on. For the squeeze theorem to work, we need to show that two *intersecting* graphs will be the maximum and minimum boundaries for a third function. In this case:

$$\begin{aligned} f(x) &\leq g(x) \\ -x^2 &\leq x^2 * \sin\left(\frac{4}{x}\right) \\ -1 &\leq \sin\left(\frac{4}{x}\right) \end{aligned}$$

That statement is true. The smallest value that $\sin(t)$ for some t can be is -1. Going the other way:

$$\begin{aligned} h(x) &\geq g(x) \\ x^2 &\geq x^2 * \sin\left(\frac{4}{x}\right) \\ 1 &\geq \sin\left(\frac{4}{x}\right) \end{aligned}$$

Again, a true statement. Because $f(x)$ and $h(x)$ intersect at $x = 0$, $g(x)$ must be between $f(0) = 0$ and $h(0) = 0$ where the only solution is $\lim_{x \to 0} g(x) = 0$.

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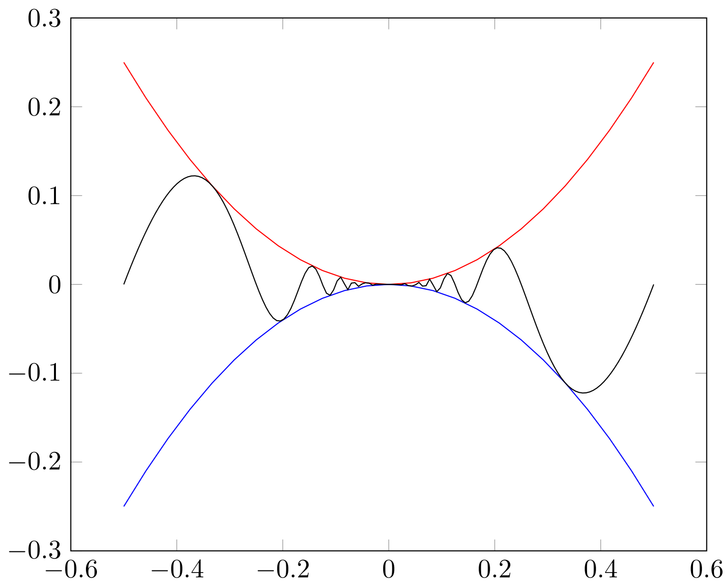

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1.7 Relative Magnitudes

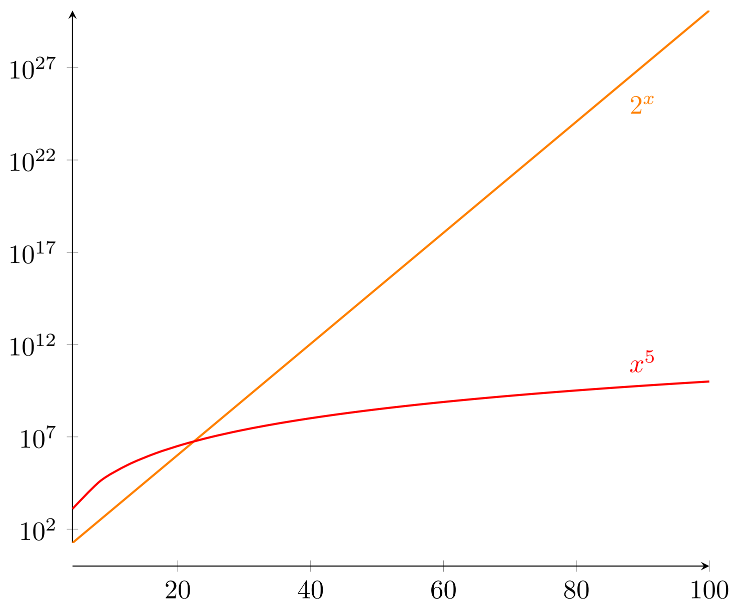
It is beneficial in some cases to calculate the limit at ∞ for some functions. Obviously, in some cases, the limit doesn't exist because it itself is ∞. For example:

$$\lim_{x \to \infty} x^2 = \infty$$

In other cases, the limit is 0. For example:

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Sometimes it is not so obvious. For fractions, we can compare the rates of increase of functions (or their magnitude) of their numerator and denominator. If the denominator has a greater magnitude, it will become larger faster than the numerator. Therefore, the limit will be zero. If the numerator has a larger magnitude, it will increase faster and the limit will be ∞.

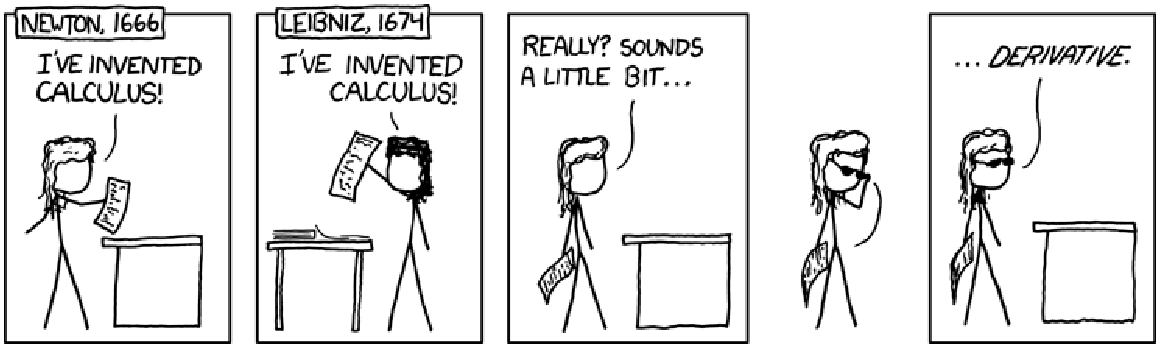


Polynomial functions like x^5 will always increase slower than exponential functions like 2^x towards ∞, therefore:

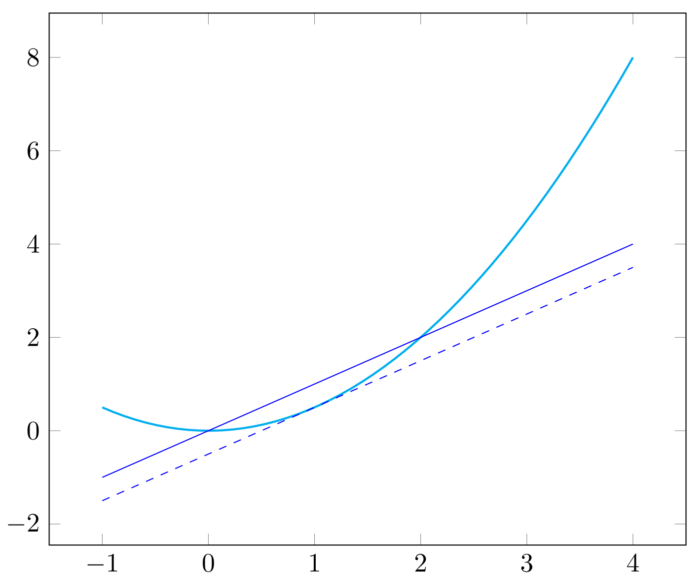
$$\lim_{x\to\infty} \frac{x^5}{2^x} = 0$$

Derivatives

2 Derivatives



Slopes It is useful and easy to find the rate of change for any point on a line. It is difficult to approximate how a curved function changes, however. Functions that aren't lines don't have defined slopes and we have to use a tangent line to find the slope at a point. It is difficult to find the tangent line at a point, so we must resort to finding secant approximations for the tangent line.



The secant line $y = x$ approximates the dashed line tangent to the graph $f(x) = \frac{1}{2}x^2$ for the point $x = 1$.

2.1 The Difference Quotient

The Slope of a Secant Line The slope of a secant line through a point x and another point h units away will be the rise ($\triangle y$) divided by the run ($\triangle x$). Plugging in for the points $(x, f(x))$ and $(x + h, f(x + h))$, we get

$$\frac{f(x+h) - f(x)}{x + h - x}$$

which solves to

$$\frac{f(x+h)-f(x)}{h}$$

2.2 The Derivative

The derivative is a tool that can find the slope of a tangent line at (almost) any point on a graph. While we can't find the slope between one point and itself because there is no difference, we can find the slope between some point and another that is **really** close to that point. We can minimize the difference between these two points by using a limit! We will find the difference between a point and a point whose position is almost zero units away:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

This is the derivative! If you were to plug in 0 for h immediately, you would divide by zero. Therefore, you must use the reduction technique for solving equations of this type. The derivative is often abbreviated $\frac{d}{dt}f(t)$ or sometimes simply $f'(x)$. If you take n more derivatives of the same function past the first one, we call it the n^{th} order derivative. Derivatives beyond the first order are labeled $\frac{d^2}{dt^2}f(t)$ or $f''(t)$ for the second order; $\frac{d^3}{dt^3}f(t)$ or $f'''(t)$ for the third order; etc. We say the derivative is taken "with respect to" something else. In a pretty standard Calculus case, the objective is to find the change in *y with respect to x*. We use the notation to describe this relationship as $\frac{dy}{dx}$. Extending this, the second order derivative of *y with respect to x* is denoted $\frac{d^2y}{dx^2}$. When the derivative of a variable is taken with respect to itself, i.e. $\frac{dx}{dx}$, we do not write it as it is equal to 1.

Example 1: We already know how to find the slope of a linear equation. We can prove the difference quotient gives the slope for any linear equation. The general slope-intercept form for linear equations is $f(x) = mx + b$ where m is the slope and b is the y-intercept.

$$\begin{aligned}
m &\stackrel{?}{=} \lim_{h\to 0} \frac{(m(x+h)+b)-(mx+b)}{h} \\
m &\stackrel{?}{=} \lim_{h\to 0} \frac{mx+mh+b-mx-b}{h} \\
m &\stackrel{?}{=} \lim_{h\to 0} \frac{mh}{h} \\
m &\stackrel{?}{=} \lim_{h\to 0} m \\
m &= m \\
&\checkmark
\end{aligned}$$

Example 2: For $f(x) = x^2$, the derivative is:

$$\begin{aligned}
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2}{h} \\
&= \lim_{h\to 0} 2x+h \\
&= 2x+0 \\
&= 2x
\end{aligned}$$

So for any point x on the graph of $f(x) = x^2$, the slope of the tangent line will be $\frac{\Delta y}{\Delta x} = 2x$. Be familiar with the limit of the difference quotient as the definition of the derivative. You will prove more of these in the exercises but the math required is pre-Calculus[1] and generally unneccesary in the understanding of Calculus.

Example 3: Working off of the previous problem, we can calculate the slope of the tangent line at any point on the graph. For example, the slope at the point (3.5,12.25) is:

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= 2x \\
\frac{\Delta y}{\Delta x} &= 2(3.5) \\
\frac{\Delta y}{\Delta x} &= 7
\end{aligned}$$

Knowing the slope, we can calculate the equation itself. In Calculus, it is often the easiest when finding a tangent line equation to use the point-slope form of a line which is:

$$(y - y_0) = m(x - x_0)$$

[1] All math before Calculus will be referred to as pre-Calculus in this textbook

where m is the slope at a point (x_0, y_0). The tangent line to the graph of $f(x) = x^2$ at the point (3.5,12.5) is therefore

$$y - 12.25 = 7(x - 3.5)$$

or

$$y = 7(x - 3.5) + 12.25$$

See the Tips about the AP Test section for more information.

Exercises

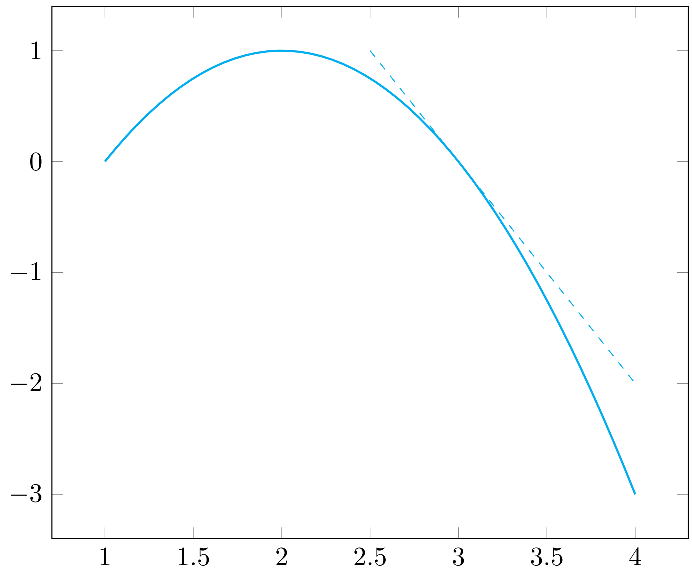
1. **Write, but do not solve, the difference quotient for:**
   (a) $f(x) = \sin(x)$
   (b) $g(x) = \ln(2x)$
   (c) $h(x) = \sqrt{x}$
   (d) $q(x) = e^x$
2. **Calculate the difference quotient for the following equations:**
   (a) 1
   (b) x
   (c) $5x - 1$
   (d) $(x - 2)^2$
   (e) $5x^3$
3. **Use the difference quotient to calculate the slope of the tangent line at the point given:**
   (a) $f(x) = 2x - 1, x = 0$
   (b) $g(x) = 3x^2 + 5x - 1, x = 3$
   (c) $h(x) = x^3 - 5x, x = 1$

2.3 Estimating Derivatives

Estimating derivatives is not altogether too difficult. Remember that the derivative itself is just the limit of the difference quotient. If one needs to find the derivative, all one has to do is take the difference quotient at two known points to create a secant line. Because it is an estimate, one doesn't need to minimize h.

$$\frac{f(x+h) - f(x)}{h}$$

It is also possible to "eyeball" a graph to estimate the derivative. For example:



For the point (3,f(3)), the tangent line obviously has a negative slope. It is not too drastic so by a guesstimate, the slope must be greater than -3. The slope is also not too flat so it should be less than -1. Therefore, the derivative can be estimated to something around -2.

2.4 Rules of Derivatives

The derivative has patterns, or "forms" as they will be called here. You are expected to know these forms backwards and forwards. Many textbooks include proofs for these but, as it is unnecessary, this text will not. The reader is strongly encouraged to prove these forms as exercises. Some of the most important forms that Calculus students are expected to know are below. The derivative of:

1. c is 0
2. u^n is $(nu^{n-1})\ du$
3. cu is $c * (du)$
4. e^u is $e^u\ du$
5. ln(u) is $\frac{1}{u}\ du$
6. sin(u) is cos(u) du
7. cos(u) is -sin(u) du

Where u is some equation, c is a constant, and du is the derivative of u. Most other forms can be derived from these.

2.5 Special Rules

There are some more patterns outside of the traditional forms that the reader should know for Calculus. These have to do with composite and combined equations.

The Product Rule Given two equations $f(x)$ and $g(x)$, the derivative of the function $h(x) = f(x) * g(x)$ is:

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

For example, the derivative of $h(x) = x^2 \sin(x)$ is:

$$\begin{aligned} h(x) &= f(x) * g(x) \\ h'(x) &= f'(x)g(x) + g'(x)f(x) \\ h'(x) &= 2x * \sin(x) + \cos(x) * x^2 \end{aligned}$$

The Quotient Rule *Avoid this at all costs.* The quotient rule is helpful but it is very, needlessly complex. For some function $h(x)$ that is the quotient of two functions $f(x)$ and $g(x)$: $h(x) = \dfrac{f(x)}{g(x)}$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(f(x))^2}$$

An easy way to remember this god-awful formula that was taught to me is a little song to the tune of *Low Rider* by War:

> Low...d...High...minus High d Low...all...o-ver...the square of what's below
>
> (All...my...friends...know the low rider...the...low...rid-er...is a little higher)

All jokes aside, this is one of the worst things in Calculus. Avoid it at all costs. For example, you could rewrite $\dfrac{3-x}{x^2}$ as $3x^{-2} - x^{-1}$, thereby eliminating all need for the quotient rule.

The Chain Rule The name is confusing. No, this rule doesn't have anything to do chains in the normal sense. Instead, it gives a form for a composite function $h(x) = f(g(x))$. This is required for functions like $\sin(x^2)$ where $\sin(x) = f(x)$ and $x^2 = g(x)$. The chain rule says the derivative of $h(x) = f(g(x))$ is:

$$h'(x) = f'(g(x)) * g'(x)$$

In our previous example, the derivative of $\sin(x^2)$ is

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x)) * g'(x) \\ h'(x) &= \cos(x^2) * 2x \\ h'(x) &= 2x * \cos(x^2) \end{aligned}$$

Example 1: Find the derivative of $f(x) = \sqrt{x^3}$
This can be rewritten as

$$f(x) = x^{1.5}$$

which fits into derivative form #2:

$$\begin{aligned}
f(x) &= x^{1.5} \\
f'(x) &= 1.5 * x^{1.5-1}\frac{dx}{dx} \\
f'(x) &= 1.5x^{.5} \\
f'(x) &= 1.5\sqrt{x}
\end{aligned}$$

Example 2: Find the derivative of $f(x) = \ln(x^2 + 2x + 1)$
This can be rewritten using rules of logarithms as:

$$\begin{aligned}
f(x) &= \ln(x^2 + 2x + 1) \\
f(x) &= \ln\left((x+1)^2\right) \\
f(x) &= 2\ln(x+1) \\
f'(x) &= 2 * \frac{1}{x+1}\frac{dx}{dx} \\
f'(x) &= \frac{2}{x+1}
\end{aligned}$$

Example 3: Find the derivative of $f(x) = \sin\left(\frac{\cos(x)}{x^2+1}\right)$
This will require the chain rule (twice) and the product rule. First, let's do the chain rule:

$$\begin{aligned}
f(x) = &\sin\left(\frac{\cos(x)}{x^2+1}\right) \\
&\cos\left(\frac{\cos(x)}{x^2+1}\right) * \frac{d}{dx}\left(\frac{\cos(x)}{x^2+1}\right)
\end{aligned}$$

Now we have to find the still unfinished part:

$$\begin{aligned}
&\cos\left(\frac{\cos(x)}{x^2+1}\right) * \frac{d}{dx}\left(\frac{\cos(x)}{x^2+1}\right) \\
&\cos\left(\frac{\cos(x)}{x^2+1}\right) * \frac{d}{dx}\left(\cos(x)(x^2+1)^{-1}\right) \\
&\cos\left(\frac{\cos(x)}{x^2+1}\right) * \left(\cos(x) * \frac{d}{dx}\left((x^2+1)^{-1}\right) + (-\sin(x)) * (x^2+1)^{-1}\right) \\
&\cos\left(\frac{\cos(x)}{x^2+1}\right) * \left(\cos(x) * -(x^2+1)^{-2} * 2x) - \sin(x) * (x^2+1)^{-1}\right) \\
&\cos\left(\frac{\cos(x)}{x^2+1}\right) * \left(\frac{2x\cos(x)}{(x^2+1)^2} - \frac{\sin(x)}{x^2+1}\right) \\
&\cos\left(\frac{\cos(x)}{x^2+1}\right) * \left(\frac{2x\cos(x)}{(x^2+1)^2} - \frac{\sin(x)(x^2+1)}{(x^2+1)^2}\right) \\
&\cos\left(\frac{\cos(x)}{x^2+1}\right) * \left(\frac{2x\cos(x) - \sin(x)(x^2+1)}{(x^2+1)^2}\right) \\
f'(x) = &\cos\left(\frac{\cos(x)}{x^2+1}\right) * \frac{2x\cos(x) - \sin(x)(x^2+1)}{(x^2+1)^2}
\end{aligned}$$

It's not pretty, but Calculus has this nasty habit of making things complex sometimes.

Exercises

1. **Find the derivative:**
   (a) $2x^5$
   (b) $\ln(2x^5)$
   (c) e^{x-4}
   (d) e^{x^2}
   (e) $\tan(x)$
2. **Find the derivative using the product rule:**
   (a) $x\sin(x)$
   (b) $\ln(x^{2x})$
   (c) xe^x
   (d) $\cos(x)(x+5)^2$
3. **Find the derivative using the quotient rule:**
   (a) $\dfrac{x^2}{\sin(x)}$
   (b) $\dfrac{e^5x}{x^2+2x-9}$
   (c) $\cot(x)$
   (d) $\dfrac{\ln(x)}{xe^{2x}}$
4. **Find the derivative using the chain rule:**
   (a) $\cos(3x^2)$
   (b) $\sin(\ln(x))$
   (c) $\ln(\sec(x))$
   (d) $\sin(x^2+2x)$

2.6 Implicit Differentiation

In some cases, it does not make sense to find the derivative with respect to one specific variable in the equation. Perhaps there is another equation that may go with the one you are working with and has different variables. There may also be an advantage if you take the derivative with respect to some parameter, usually t. I like to think that t is sort of like time; you can increase it or decrease it and see how x and y change with respect to this parameter. It is quite simple: all one has to do is take the derivative with respect to this parameter: $\frac{d}{dt}$. For example:

$$\begin{aligned} \frac{d}{dt}&(y = x^2 - 5x + 1) \\ \frac{dy}{dt} &= 2x\frac{dx}{dt} - 5\frac{dx}{dt} \\ \frac{dy}{dt} &= (2x-5)\frac{dx}{dt} \end{aligned}$$

Note: while the derivative with respect to x normally ends up being $\frac{dx}{dx} = 1$, this is not the case with implicit differentiation.

Example 1: Using implicit differentiation, find the derivative of the circle $(x-4)^2 + (y+3)^2 = 16$

$$\begin{aligned} (x-4)^2 + (y+3)^2 &= 16 \\ 2(x-4)^1\frac{dx}{dt} + 2(y+3)^1\frac{dy}{dt} &= 0 \\ (2x-8)\frac{dx}{dt} + (2y+6)\frac{dy}{dt} &= 0 \end{aligned}$$

Pretty simple.

Example 2: Using implicit differentiation, find the derivative of the function $y^2 = 2xy + x^2$

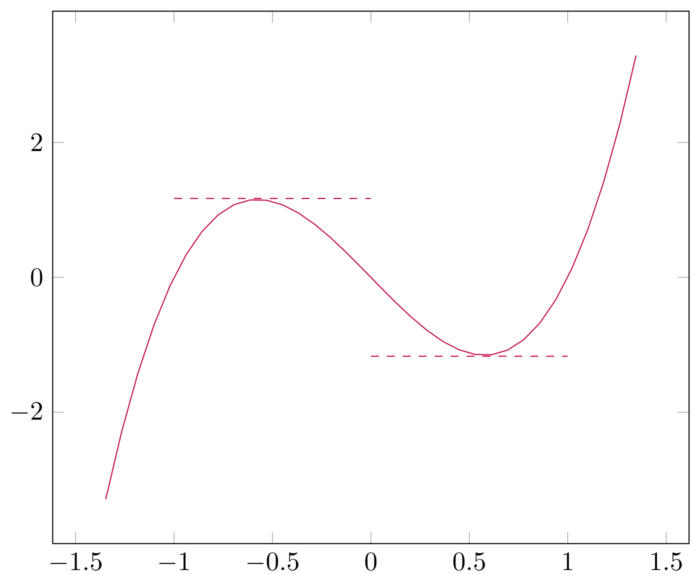
$$\begin{aligned} y^2 &= 2xy + x^2 \\ 2y\frac{dy}{dt} &= 2x\frac{dy}{dt} + 2y\frac{dx}{dt} + 2x\frac{dx}{dt} \\ -2y\frac{dy}{dt} + 2y\frac{dy}{dt} &= 2x\frac{dx}{dt} + 2x\frac{dx}{dt} \\ 0 &= 4x\frac{dx}{dt} \end{aligned}$$

Exercises

1. **Find the derivative using implicit differentiation:**
   (a) $y = \sin(x)$
   (b) $y = \ln(2x^5)$
   (c) $y = \dfrac{x+1}{\sqrt{x^2-9}}$
2. **Find the derivative using implicit differentiation:**
   (a) $6 = \sqrt[5]{(x+2y)^2}$
   (b) $y + 2 = \dfrac{\tan(x)}{xy}$
   (c) $1 = \sin(xy) + (y - x^{-2})^3$

2.7 Maxima and Minima

For any graph, the local minima and maxima will occur when the graph levels off. To put this in Calculus words, the derivative at the point must be 0 because the tangent line is horizontal.

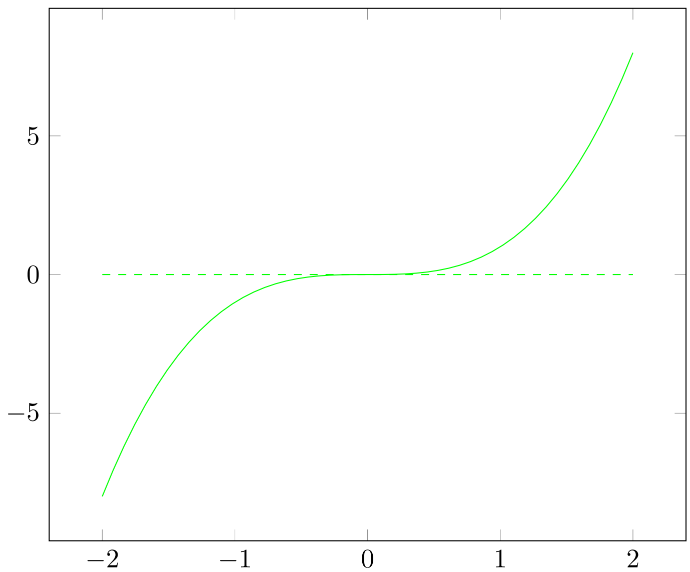


This graph has local maxima and minima where the tangent line is horizontal.

Armed with the knowledge of the derivative, we can exactly calculate at what point(s) any function has a local maximum or minimum. It is actually very simple, all one has to do is to find at what points the derivative is 0. We call this *The First Derivative Test.* The first derivative test gives us *critical numbers.* For example, let's find the points at which $f(x) = \dfrac{x^3}{3} - x$ has a local maximum or mimimum:

$$\begin{aligned}
f'(x) &= 3\frac{x^2}{3} - x^0 \\
f'(x) &= x^2 - 1 \\
0 &= x^2 - 1 \\
0 &= (x-1)(x+1) \\
x &= \pm 1
\end{aligned}$$

I lied. When the derivative is zero, all it means is the line tangent to the graph is horizontal. For example, the graph of $f(x) = x^3$ has a horizontal tangent when $x = 0$:



There must be some way to verify if the critical numbers we get from the first derivative test are truly maxima and minima, right? Yes! If we look at the graph of $f(x) = x^3$ above, we can tell that the tangent line has a positive slope moving towards $x = 0$ and also moving away. In calculus-speak: the first order derivative does not change signs from positive to negative. If we look at the graph of $f(x) = x^2$, we can see that the slope of the tangent is negative moving towards $x = 0$ and positive moving away. This means the first order derivative changes signs. This makes logical sense because for a function to have a local mimimum, it must decrease, hit the lowest point, and increase. We can flip this around, too: for a function to have a local maximum, it must increase, hit the highest point, and come down again. The first order derivative (the slope of the tangent line) must change signs. If it changes from positive to negative, there is a local maximum. If it changes from negative to positive, there is a local mimimum. We can write up this in a table where we have the intervals from the beginning of the values we are checking to the first critical number, then from the first to second critical number, second to third critical number,$\cdots$, and from the last critical number to the end of the values we are testing. "Values we are testing" means the ends of the interval we are working with. Normally the ends are $-\infty$ and ∞, but sometimes you might be constrained to [0,5] or something similar. For example, to find the relative minima and maxima of the function whose derivative is $f'(x) = (x-1)(x+1)$, this is the table one would create:

(-∞, -1)	-1	(-1, 1)	1	(1, ∞)
+	0	-	0	+

...and this is the true first derivative test. Because it changes signs from positive to negative at $x = -1$, that point is a local maximum, and because it changes from negative to positive at $x = 1$, that must be a local minimum.

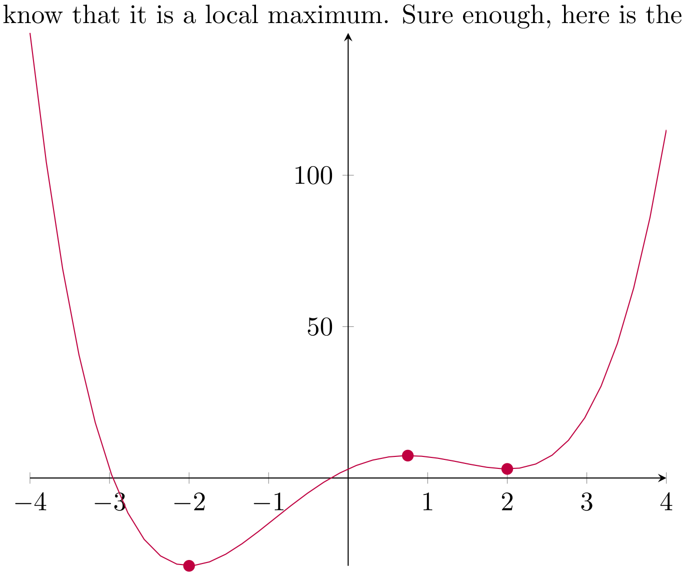
Example 1: Find the relative maxima and minima of the function $f(x) = x^4 - x^3 - 8x^2 + 12x + 3$. Since we have no bounds, we will evaluate this over $(-\infty, \infty)$

$$f(x) = x^4 - x^3 - 8x^2 + 12x + 3$$
$$f'(x) = 4x^3 - 3x^2 - 16x + 12$$
$$f'(x) = (4x - 3)(x^2) - (4x - 3)(4)$$
$$f'(x) = (4x - 3)(x^2 - 4)$$
$$f'(x) = (4x - 3)(x - 2)(x + 2)$$
$$f'(x) \text{ must be zero at } x = \frac{3}{4}, \pm 2$$

Writing out our first derivative test table, we get:

(-∞, -2)	-2	(-2, 0.75)	0.75	(0.75,2)	2	(2, ∞)
-	0	+	0	-	0	+

Because when $x = \pm 2$ the derivative changes signs from negative to positive, we know that these two points are relative minimums. Because at $x = 0.75$ the derivative changes signs from positive to negative, we know that it is a local maximum. Sure enough, here is the graph for $f(x)$:



Example 2: Find the absolute maximum and minimum of the function $f(x) = \sin(\pi x)$ over the interval [0,0.75]

$$\begin{aligned} f(x) &= \sin(\pi x) \\ f'(x) &= \pi\cos(\pi x) \\ 0 &= \pi\cos(\pi x) \quad \{0 \le x \le 0.75\} \\ x &= 0.5 \end{aligned}$$

[0,0.5)	0.5	(0.5,0.75]
+	0	-

So $x = 0.5$ must be an absolute maximum. However, we are also constrained to an interval, so we must check the bounds as well:

$$f(0) = 0$$

$$f(0.5) = 1$$

$$f(0.75) = \frac{\sqrt{2}}{2}$$

From this, we can see that $x = 0.5$ is indeed our absolute maximum but $x = 0$ is our absolute minimum on this interval as well.

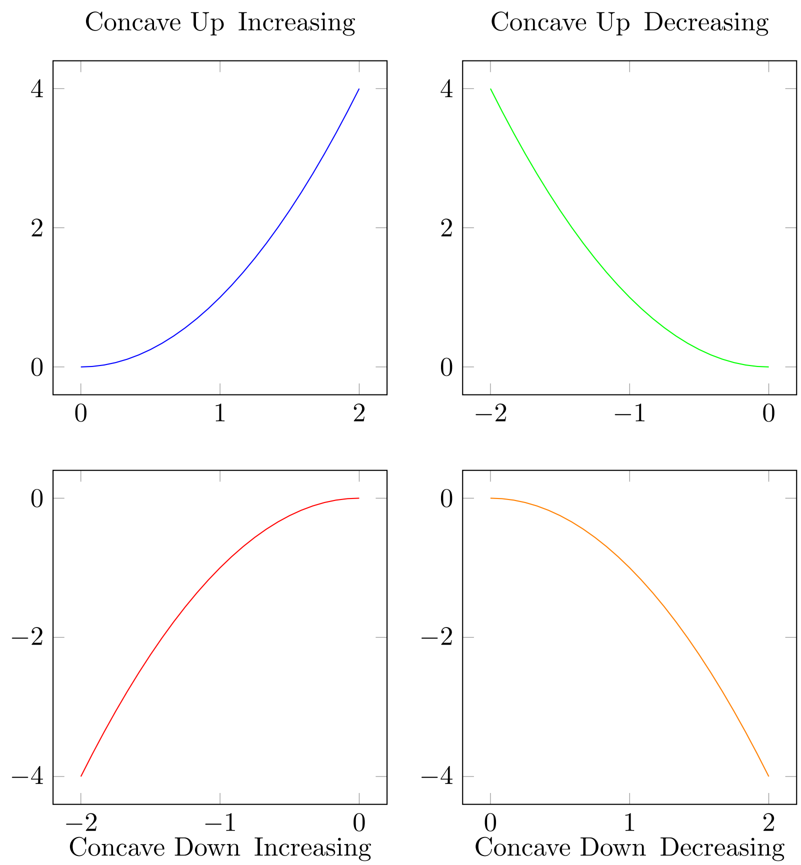
Exercises

1. **Find the local maxima and minima of the function:**
   (a) $f(x) = x(x-3)(x+3)$
   (b) $g(x) = \dfrac{5}{x^2+5}$
   (c) $h(x) = \sin(\ln(x)) \quad \{0.1 \le x\}$
2. **Find the absolute maximum and minimum of the function on the given interval:**
   (a) $f(x) = x^3 - x \quad \{-5 \le x \le 1.5\}$
   (b) $g(x) = x^2\sin(x) \quad \{-9 \le x \le 0\}$

2.8 Concavity

Recall that the second derivative gives the rate of change of the first derivative. Because the first derivative gives the slope of the tangent line, the second derivative will give the change in slope. If the second derivative is positive, the tangent line must be increasing and therefore moving up (counterclockwise). If the second derivative is negative, the tangent line must be decreasing and therefore moving down (clockwise). When the slope is changed, a curve is created. We use two words to quantify the curve that we see: **concave up** and **concave down**. Concave up is, as you may have guessed, when the graph is in a bucket shape facing upwards (think of the $y = x^2$ graph).

Concave down is the opposite, where the graph looks more like a hat facing downwards (think of $y = -x^2$). We also append *increasing* or *decreasing* to the description of the function at a point to describe the sign of the first derivative, in other words whether the points are going up or going down moving right.



Example 1: Find when the function $f(x) = 1/4x^4 + x^3 - 1/2x^2 - 3x + 17$ is: Concave up increasing, concave down increasing, concave up decreasing, and concave down decreasing.

$$f(x) = 1/4x^4 + x^3 - 1/2x^2 - 3x + 17$$
$$f'(x) = x^3 + 3x^2 - x - 3$$
$$f''(x) = 3x^2 + 6x - 1$$

First, we will find when the first derivative is positive or negative to determine where $f(x)$ is increasing or decreasing.

$$\begin{aligned}
f'(x) &= x^3 + 3x^2 - x - 3 \\
f'(x) &= x^3 - x + 3x^2 - 3 \\
f'(x) &= x(x^2 - 1) + 3(x^2 - 1) \\
f'(x) &= (x + 3)(x^2 - 1) \\
f'(x) &= (x + 3)(x - 1)(x + 1) \\
0 &= (x + 3)(x - 1)(x + 1) \\
x &= -3, \pm 1
\end{aligned}$$

$(-\infty, -3)$	$(-3, -1)$	$(-1, 1)$	$(1, \infty)$
-	+	-	+

So $f(x)$ is increasing over $(-3, -1)$ and $(1, \infty)$, and decreasing over $(-\infty, -3)$ and $(-1, 1)$. Now, we will find where the function is concave up and down:

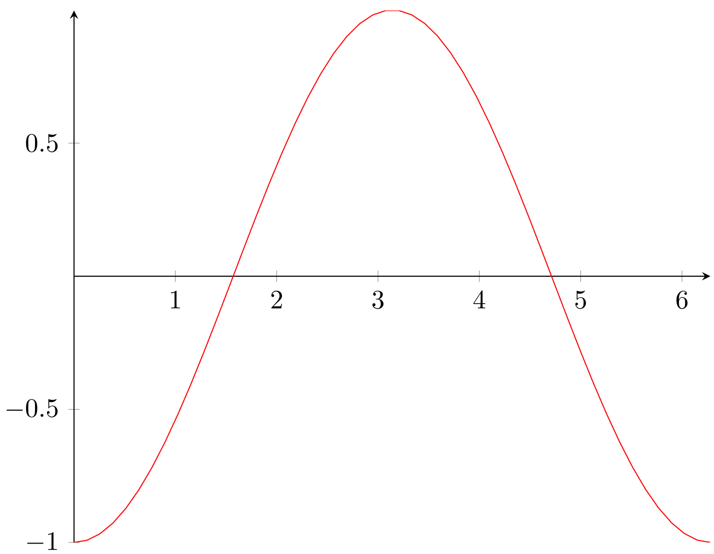
$$\begin{aligned}
f''(x) &= 3x^2 + 6x - 1 \\
0 &= 3x^2 + 6x - 1 \\
x &= \frac{-6 \pm \sqrt{6^2 - 4 * 3 * -1}}{2 * 3} \\
x &= \frac{-6 \pm \sqrt{48}}{6} \\
x &= \frac{-3 \pm 2\sqrt{3}}{3} \\
x &\approx -2.155, 0.155
\end{aligned}$$

$(-\infty, -2.155)$	$(-2.155, -0.155)$	$(-0.155, \infty)$
+	-	+

So $f(x)$ is concave up over $(-\infty, -2.155)$ and $(-0.155, \infty)$, and concave down over $(-2.155, -0.155)$.

Example 2: Find where $\sin(x - \frac{\pi}{2})$ is concave up and down:

$$\begin{aligned}
f(x) &= \sin(x - \frac{\pi}{2}) \\
f'(x) &= \cos(x - \frac{\pi}{2}) \\
f''(x) &= -\sin(x - \frac{\pi}{2})
\end{aligned}$$



$\sin(x-1)$

Interestingly, the concavity is just the inverse of the function itself. $\sin(x-\frac{\pi}{2})$ is concave up when it is negative, written out mathematically: $(-\frac{\pi}{2}, \frac{\pi}{2}) + n * 2\pi$, where n is any integer.

Exercises

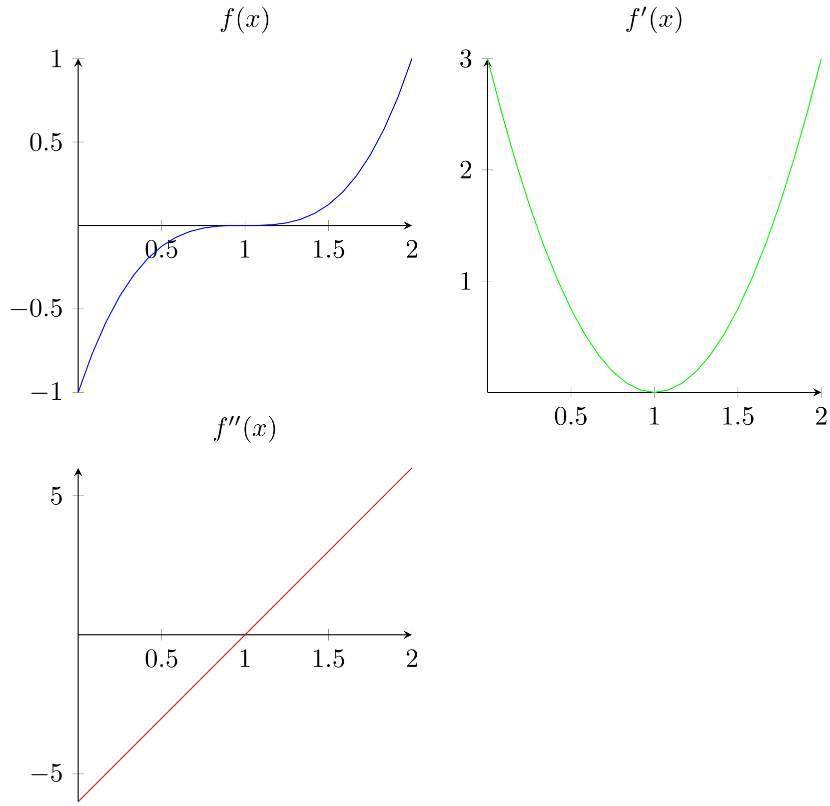
1. **Find where the functions are concave up and down:**
   (a) $f(x) = -1/3x^3 - x^2 + x$
   (b) $g(x) = \tan(x)$
   (c) $h(x) = \dfrac{(x-1)}{(x+2)(x-5)}$

2.9 Graphical Relations

Functions and their derivatives are connected. If the derivative is positive, the function is increasing. If the derivative is negative, the function is decreasing. If the second derivative is positive, the function is accelerating in the positive direction. One of the many things a Calculus student is expected to do is identify the relation between the graphs of the derivatives of a function and the function itself. Let $f(x) = (x-1)^3$:

$$f(x) = (x^2 - 2x + 1)(x - 1)$$
$$f(x) = x^3 - 2x^2 + x - x^2 + 2x - 1$$
$$f(x) = x^3 - 3x^2 + 3x - 1$$
$$f'(x) = 3x^2 - 6x + 3$$
$$f''(x) = 6x - 6$$

Graphing the functions, we get:



$f'(x)$ has a zero at $x = 1$ but because it doesn't change signs, $f(x)$ cannot have a local maximum or minimum there. Because $f'(x)$ is always positive, the function is always increasing. $f''(x)$ is negative before $x = 1$ and positive after, so at that point, $f(x)$ switches from concave down to concave up. This sums up the relationship between derivatives and their graphs.

2.10 Relationship with Continuity

Continuity does not guarantee that the derivative will always exist. Even though a function like

$$f(x) = \begin{cases} x & x \leq 1 \\ x^2 & x > 1 \end{cases}$$

is continuous everywhere, its derivative is not. The derivative is a limit, so for it to exist, it must be the same on both sides. If the limit of the difference quotient (the derivative) is calculated at $x = 1$, the results are:

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} 1 = 1$$

$$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} 2x = 2$$

which are not equal. The derivative must not exist for all values of x. However, if the derivative of a function is continuous everywhere, that means that the function itself must be differentiable everywhere. Therefore, functions that do not result in a piecewise derivative and have a domain of $(-\infty, \infty)$ must be differentiable everywhere. Functions that are differentiable and expected to be calculated by Calculus students are::

1. Polynomials ($x^n + x^{n-1} + ...$)
2. Power functions (ax^b)
3. sin(x)
4. cos(x)
5. Exponential functions (ba^x)
6. Logarithmic functions

3 Applications of Derivatives

3.1 Units

Functions can be used to model real-world phenomena and find use in everything from quantum physics to music. However, in the real world, there are units of measure that are significant and go hand-in-hand with values. Think of *miles*: a unit of distance, and *hours*: a unit of time. We can combine these together to show a relationship between the two with *miles per hour*: a unit of speed. Because derivatives give rates of change, the units no longer stay the same. For example, if we wanted to find the rate of change of speed, we would look for how the speed changes with relation to time. Our units then would be *miles per hour per hour*: $\frac{miles}{hour} \div hour$. We rewrite this as $\frac{miles}{hour^2}$. This gives us the rate of change of speed, or acceleration. If we wanted to find how the area of a circle in *square meters* changes when we change the radius in *meters*, the units would be $\frac{meters^2}{meters} = meters$.

Exercises

1. **What would be the unit of:**
   (a) $f'(x)$ where $y = f(x)$ is in y inches and x seconds
   (b) The second derivative of the area of a circle: $A = \pi r^2$ where A is in square meters and r is in linear meters
   (c) The rate of change of acceleration of the function $x(t) = 15t^2 + 15t + 5$ where $x(t)$ is in meters and t is in seconds

2. **Find the value including correct units:**
   (a) $V''(5)$ where $V(r) = \frac{4}{3}\pi r^3$ with $V(r)$ in cubic meters and r in linear meters
   (b) The acceleration and veolcity of a model rocket at time $t = 2$ seconds with the rocket's position is given by $x(t) = -4.9t^2 + 100t - 2$ in meters
   (c) The second derivative of $g(s) = s^2 * \sin(s)$ where $g(s)$ is in dollars and s is in quality units (abbreviated q)

3. **Solve the problems and give correct units:**
   (a) Mr. Regina is a NASCAR driver and he recently got a new car, the *Eyremobile*. Powered by high-quality literature, the *Eyremobile* performs better than his previous racecar and its position, if run at full power, is given by the function:

$$x(t) = \frac{t}{20} - \frac{3}{10}\ln(t + 6)$$

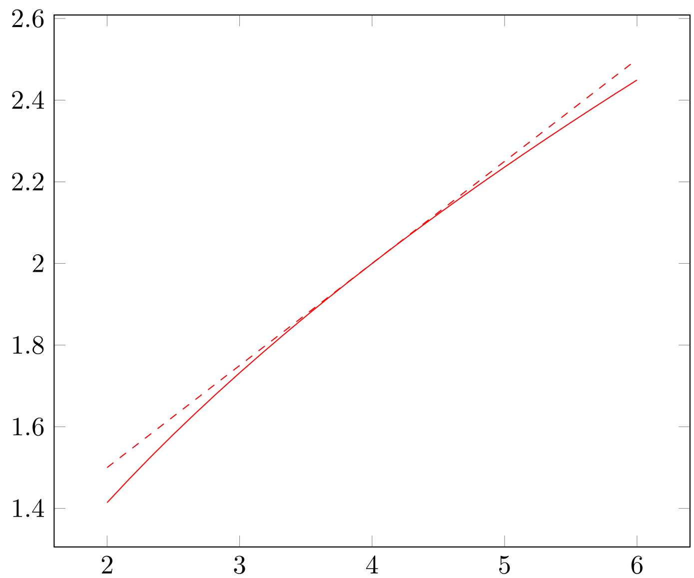
   where $x(t)$ is measured in miles and t is time in seconds.
   i. Find the velocity equation for the *Eyremobile* running at full power.
   ii. If Mr. Regina exceeds 210 miles per hour on a curve, he will fly off the track. Will Mr. Regina be safe if he runs the *Eyremobile* at full power over the 30 minute race? Give a sound explanation with evidence.

3.2 Instantaneous Rate of Change

This is perhaps the most classic Calculus derivative problem. When you are riding in a car, the speedometer tells you how fast you are going at any point in time. This is your instantaneous rate of change, how fast you are moving at one point in time. You could be travelling at 40 miles per hour but then have to stop at a stoplight for a little bit. Your average speed would be less than 40 because of that time you stopped, but you know that at that certain time you measured it, you were travelling 40 miles per hour. Problems will ask you, given a function, how fast is something changing at an exact point in time. All it is asking is for the slope of the tangent line (derivative) and not the average slope at that point. Simply calculate the derivative.

3.3 Tangent Line Approximations

The tangent line can be used as an approximations for points near a point on a graph. Take this example of the graph $f(x) = \sqrt{x}$:



The tangent line for $f(4)$ is $y - 2 = .25(x - 4)$ and approximates the graph pretty well for the near area. Because $\sqrt{5}$ would be incredibly difficult to find by hand, we can use the tangent line to approximate the value.

$$f(5) \approx .25(5 - 4) + 2$$

which is 2.25. The actual value is about 2.236. The approximation worked pretty well in this case with an error of about 0.014, or less than 1% of the actual value.

Example 1: Estimate the value of $f(8.125)$ for $f(x) = -2\sqrt[3]{x^2}$

$$\begin{aligned}
f(x) &= -2\sqrt[3]{x^2} \\
f(x) &= -2x^{\frac{2}{3}} \\
\frac{df}{dx} &= -2x^{-\frac{1}{3}}\left(\frac{2}{3}\right) \\
\frac{df}{dx} &= -\frac{4}{3} * \frac{1}{\sqrt[3]{x}} \\
\frac{df}{dx}(8) &= -\frac{4}{3} * \frac{1}{2} \\
\frac{df}{dx}(8) &= -\frac{2}{3}
\end{aligned}$$

Now that we know the derivative at 8, we can plug that into a point-slope linear equation:

$$\begin{aligned}
y - y_0 &= m(x - x_0) \\
y - f(8) &= -\frac{2}{3}(x - 8) \\
y + 8 &= -\frac{2}{3}(x - 8)
\end{aligned}$$

We can now use that to estimate $f(8.125)$

$$\begin{aligned}
y + 8 &= -\frac{2}{3}(8.125 - 8) \\
y &= -\frac{2}{3} * \frac{1}{8} - 8 \\
y &= -\frac{1}{12} - 8 \\
y &= -8\,\frac{1}{12}
\end{aligned}$$

A good approximation for $f(8.125)$ is therefore $-8\,\frac{1}{12}$.

Exercises

1. **Find the point-slope form of the tangent line for the equation at the given point:**
   (a) $y = (x-3)(x+5)$ when $x = 4$
   (b) $y = \sin(x)$ when $x = \frac{\pi}{6}$
   (c) $y = \dfrac{x}{\sqrt{x^2 - 2x - 12}}$ when $x = 5$
2. **Approximate the value at the given point:**
   (a) $y = \tan(x)$ when $x = \dfrac{3\pi}{16}$
   (b) $y = 2\ln(x)$ when $x = 3$ (hint: $e \approx 2.713$)

(c) $y = \dfrac{1}{1+e^x}$ when $x = 0.5$

3.4 Physics

It has come up a bit before, but now it is time to formalize some theory. Because the derivative finds the rate of change for something at a certain point, it is useful in physics. For example, the derivative of a position function gives the rate of change of position of that point, or *velocity.* Similarily, the change in velocity at a point is *acceleration.* The change in acceleration at a point does get a special name in physics, too, but it is not used as much: *the jerk.* Given $x(t)$ is position, $v(t)$ is velocity, and $a(t)$ is acceleration with respect to time, the following are true:

$$x'(t) = v(t)$$
$$x''(t) = v'(t) = a(t)$$
$$etc.$$

3.5 Related Rates

Derivatives find the rate of change of something with respect to something else. Two or more derivatives can be combined, as a result, to find how one value affects another. For example, if we know the the radius of a circle at any given time is $\ln(t)$, we know the rate of change must be $\frac{1}{t}$, which we can rewrite as $\frac{dr}{dt} = \frac{1}{t}$. If we wanted to find how the area of a circle changes with respect to the radius, all we need to do is take the derivative.

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

substituting for r and $\frac{dr}{dt}$, we get

$$\frac{dA}{dt} = 2\pi \ln(t) * \frac{1}{t}$$

which means that for any given time t, the rate of change of the area A must be

$$\frac{2\ln(t)}{t}$$

Is it not super cool that the derivative of the area of a circle is the circumference?

Example 1: Find the rate of change in the volume of a sphere if the radius is $\sin^2(t)$ at time t. We will start with finding the derivative of the volume equation, implicitly:

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = \frac{4}{3}\,\pi\, 3\, r^2\, \frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\,\pi\, r^2\, \frac{dr}{dt}$$

Next, we will find the derivative of the radius equation:

$$r = \sin^2(t)$$
$$r = \sin(t) * \sin(t)$$
$$\frac{dr}{dt} = \sin(t) * \cos(t) + \cos(t) * \sin(t)$$
$$\frac{dr}{dt} = 2\sin(t)\cos(t)$$

Knowing these two equations and the equation for r, we can substitute:

$$\frac{dV}{dt} = 4\pi r^2 * 2\sin(t)\cos(t)$$
$$\frac{dV}{dt} = 8\pi(\sin^2(t))^2\sin(t)\cos(t)$$
$$\frac{dV}{dt} = 8\pi\sin^5(t)\cos(t)$$

Exercises

1. **Find the equation for the rate:**
   (a) The change in the surface area of a cube with respect to the change of its side length
   (b) The change in the area of a right triangle with respect to the change of one of the non-right angles
2. Coach Lou is filling a form with concrete to create a prop for the musical. The form's inside is an equilateral triangular prism with side lengths of 4ft and a depth of 15ft. He is filling it at a rate of 1.5 cubic feet per minute. Find the equation for the height of the concrete at time t if one of the rectangular sides is flush with the ground.

3.6 Optimization

Using knowledge of how maxima and minima work, it becomes easy to "optimize" functions. If there is a formula for something like cash input and return, you can "optimize" the function by finding a maximum or minimum. Given constraints, what is the maximum value of some function? Usually, you are given one function to maximize and one constraint function.

Example 1: As part of a startup that sells Jell-O® through the internet, you are tasked with finding the dimensions for a new box for shipping. You are told that the length must be three times the width. The bottom and top parts of the box cost 2 per square foot and the sides cost 5 per square foot. You must design the cheapest box that can hold 30 cubic feet of Jell-O®.

First, let's define the function that we want to optimize. The objective is to reduce cost, so we want to find the minimum of the cost function. Given length ℓ, width w, and height h, our cost function C is:

$$C = 2 * (w * h) * 5 + 2 * (\ell * h) * 5 + 2 * (w * \ell) * 2$$

Some other information we have: $\ell = 3w$, and $\ell * w * h = 30$. Rewriting the cost function, we have:

$$C = 10 * (w * h) + 10 * (3w * h) + 4 * (w * 3w)$$

$$C = 40wh + 12w^2$$

We can also solve our volume function:

$$3w * w * h = 30$$

$$h = \frac{10}{w^2}$$

And substitute that back into our cost function:

$$C = \frac{400}{w} + 10w^2$$

So now we have a function that combines the cost and constraint information that we were given! Our goal is to minimize cost, so we will take the derivative and find the critical numbers:

$$C(w) = 10w^2 + 400w^{-1}$$
$$C'(w) = 20w - 400w^{-2}$$
$$0 = 20w - 400w^{-2}$$
$$0 = w - 20w^{-2}$$
$$20w^{-2} = w$$
$$20 = w^3$$
$$w = 0?, \sqrt[3]{20}$$

A box with width 0 would not only be silly but impossible without infinite material. We can throw that one out immediately. As good practice, though, we should make sure $\sqrt[3]{20}$ is in fact a minimum. We can use the second derivative test to do so:

$$C'(w) = 20w - 400w^{-2}$$
$$C''(w) = 20 + 800w^{-3}$$

0	$(0, \sqrt[3]{20})$	$\sqrt[3]{20}$	$(\sqrt[3]{20}, \infty)$
∞	+	60	+

So because there is no change in sign at the second derivative at $w = \sqrt[3]{20}$, it must not be a point of inflection, and because it's positive, it must be concave up. This means that our function must have a minimum at this point. We now know w but we also need h and ℓ, so we're not entirely done yet.

$$\ell = 3w$$
$$\ell = 3\sqrt[3]{20}$$
$$\ell * w * h = 30$$
$$3 * \sqrt[3]{20} * \sqrt[3]{20} * h = 30$$
$$h = \frac{30}{3\sqrt[3]{400}}$$
$$h = \frac{10}{\sqrt[3]{400}}$$

And now we have our dimensions!

Exercises

1. Minimize the surface area of a cylinder with a volume of 6 meters.
2. Mr. Young and Dr. Bob are working on building a new rocket launcher. The maximum possible height capability of a launcher is determined by the total pressure of air passing through the

outlet a in psi and the diameter of the outlet s in inches with the equation $\left(\dfrac{99.5\sqrt[4]{a^3}}{s}\right) + s$. Additionally, a must be 175 times greater than s. Optimize the function to find the best (most efficient) height to aim for. Round to the nearest integer.

3.7 Differential Equations

The reader should be familiar with $\dfrac{dy}{dx}$ at this point. Remember that what that means is the instantaneous change in y with respect to the instantaneous change in x. This can be rewritten and solved for dy as a function of dx. It's just as simple as getting $\dfrac{dy}{dx}$ on one side and multiplying by dx so the equation looks like:

$$\frac{dy}{dx} = f'(x)$$

$$dy = f'(x)dx$$

Given the equation $x^2 + y^2 = 1$, the differential equation will be:

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

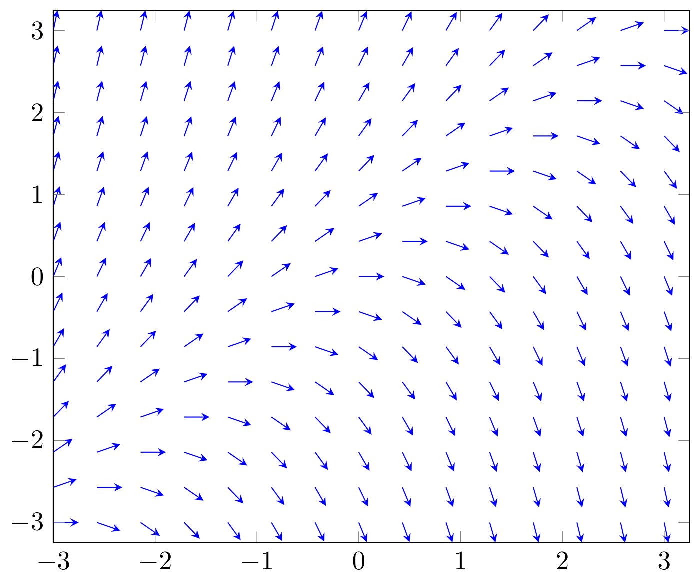
$$2y\frac{dy}{dx} = -2x$$

$$dy = -\frac{x}{y}dx$$

3.8 Slope Fields

Slope fields are (in my opinion) one of the most fun things in Calculus. Given a graph, the differential equation $\dfrac{dy}{dx}$ is solved for the (x, y) coordinate. The resulting solution is the slope of the graph **should** the graph actually pass through that point. This step is repeated for a certain number of points on the graph. All the work that has to be done is plug in (x, y) to the equation and draw a small line with the same slope at that point.

Example 1: Draw the slope field for $y^2 + 2y = x^2$ $\left(\text{the differential equation is } \dfrac{dy}{dx} = \dfrac{x}{y+1}\right)$:



Note: arrow heads are normally not drawn. Software limitation.

Exercises

1. **Find the differential equation for the function:**
   (a) $y = \sin(x)$
   (b) $y = (x-2)^4$
   (c) $y = \ln(x^2+1)$
2. **Draw the slope field for these functions at the points: (-1,-1), (0,-1), (1,-1), (-1,0), (0,0), (1,0), (-1,1), (0,1), and (1,1)**
   (a) $y = x^2$
   (b) $y = \dfrac{(x+2)}{(x-1)(2x+1)}$
   (c) $y = \sec(2x)$

3.9 Mean Value Theorem

The mean value theorem is relatively simple. If the reader would like to find a proof for the theorem, they are readily available on the internet.

> Given a closed interval $[a,b]$, if a function $f(x)$ is always continuous, there exists a point $a \leq c \leq b$ such that the derivative at c is equal to the slope of the secant line through a and b.

In other words, there exists some c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

You will see this on the AP test. The test will ask you which one of the given options is guaranteed by the mean value theorem. Simply find the slope of the secant line. Do note: this only works for continuous functions.

3.10 L'hôpital's Rule

Taking the limit of a function with a fraction sometimes ends up with a confusing result like $\frac{\infty}{\infty}$ or $\frac{0}{0}$. Neither of these actually have a real value. L'hôpital's rule is:

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

if and only if $\lim f(x) = \lim g(x)$ and $\lim f(x)$ is either $\pm\infty$ or 0. What this means is if, when taking the limit, you get one of these "indeterminate forms", you may take the derivative of the top and the bottom **separately** and then re-evaluate the limit. You may do this as many times as you need until you no longer reach an indeterminate form. The limit must exist at ∞; that is a requirement for this rule to work.

Example 1: Find $\lim\limits_{x\to\infty} \frac{x^3-x+1}{4x^3+2x^2+x-1}$. A method for solving this particular type of problem is shown before Calculus, but here is the reason why it works:

$$\lim_{x\to\infty} \frac{x^3 - x + 1}{4x^3 + 2x^2 + x - 1} = \frac{\infty}{\infty}$$

This is an indeterminate form so we can apply L'hôpital's rule:

$$\lim_{x\to\infty} \frac{x^3 - x + 1}{4x^3 + 2x^2 + x - 1} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{x^3 - x + 1}{4x^3 + 2x^2 + x - 1} = \lim_{x\to\infty} \frac{3x^2 - 1}{12x^2 + 4x + 1}$$

$$\lim_{x\to\infty} \frac{3x^2 - 1}{12x^2 + 4x + 1} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{3x^2 - 1}{12x^2 + 4x + 1} = \lim_{x\to\infty} \frac{6x}{24x + 4}$$

$$\lim_{x\to\infty} \frac{6x}{24x + 4} = \lim_{x\to\infty} \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{6x}{24x + 4} = \frac{6}{24}$$

$$\lim_{x\to\infty} \frac{6}{24} = \frac{6}{24} = \frac{1}{4}$$

Exercises

1. **Find the limit if it exists:**
   (a) $\lim_{x\to\infty} \frac{\sin(x)}{15x^3}$
   (b) $\lim_{x\to\infty} \frac{2x^3-1}{x^2+5}$
   (c) $\lim_{x\to\infty} \frac{xe^x}{x^2-1}$
   (d) $\lim_{x\to\infty} \frac{2\ln(x)}{\sqrt{x-5}}$

Integrals

4 Antiderivatives and Definite Integrals

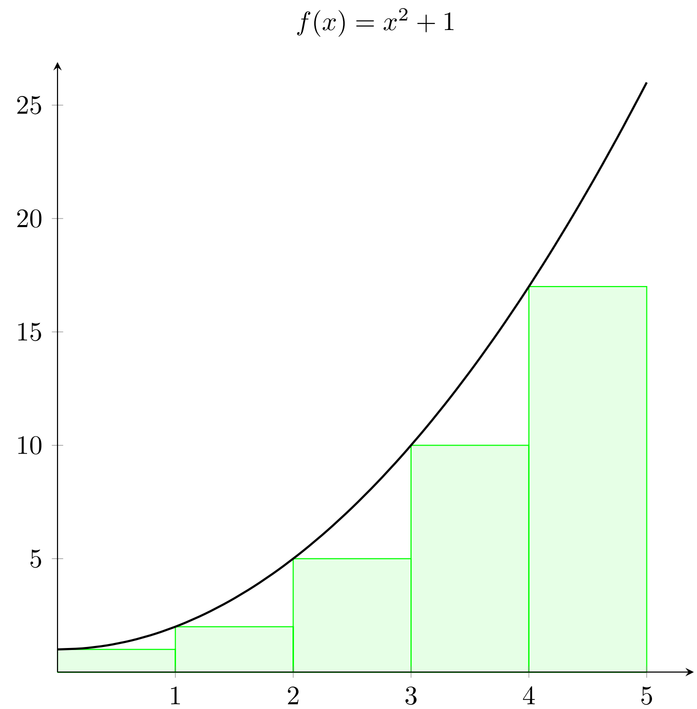
4.1 Introduction to Antiderivatives

So far we have worked with derivatives. The derivative is a function that eats a function and spits out its rate of change. Because math is full of inverse operations (addition and subtraction, multiplication and division, etc.), shouldn't the derivative have an inverse? It is obvious now that the derivative of x^2 is $2x$, so the inverse of the derivative (or antiderivative) of $2x$ is x^2. This statement isn't entirely true as x^2, $x^2 - 1$, and $x^2 + \pi$ all share $2x$ as their derivative. Instead, we can say that the antiderivative of $2x$ is $x^2 + C$ where C is any constant. It should be noted that a function which is created by a derivative will have an infinite number of antiderivatives but only one derivative itself. Later in this section, you will read about ways to solve specific antiderivative problems.

4.2 Riemann Sums

Moving onto a related topic, it can be useful to calculate the area underneath a graph of some function. For example, the area underneath a graph between two points a and b can be used to find the average value of a graph. We can't exactly find the area underneath most graphs because there are no formulas for the area of an area bounded by, say, $f(x) = 2x\sin(x^2) + 5$. However, we can approximate the area using different methods. Four that will be covered on the AP test are:

1. **Left Riemann Sum**



The Left Riemann sum divides an interval from a to b into n subintervals and adds together rectangles stretching up from the x-axis until the *left* corner touches the graph. In this case, [0,5] is divided up into 5 subintervals with width 1 each. The heights for the rectangles are $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$ because they are the left endpoints. The sum of n rectangles is:

$$(width_1 * height_1) + (width_2 * height_2) + \cdots + (width_n * height_n)$$

In Riemann sums, the widths are all equal, so the sum siplifies to:

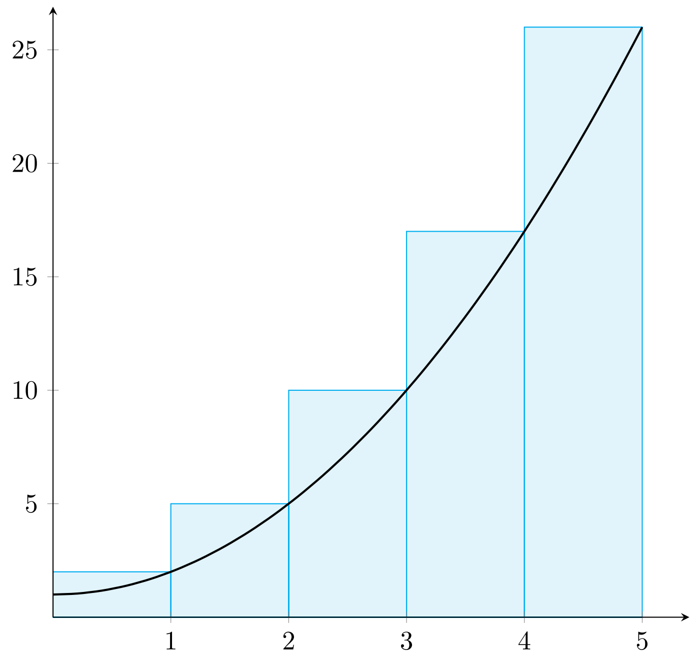
$$width * (height_1 + height_2 + \cdots + height_n)$$

which in this case is:

$$\begin{gathered} 1 * (f(0) + f(1) + f(2) + f(3) + f(4)) \\ = 1 + 2 + 5 + 10 + 17 \\ = 35 \end{gathered}$$

2. **Right Riemann Sum**

$f(x) = x^2 + 1$



The Right Riemann sum is identical to the Left Riemann Sum except that it evaluates the function at the *right* side. In this case, the approximation is:

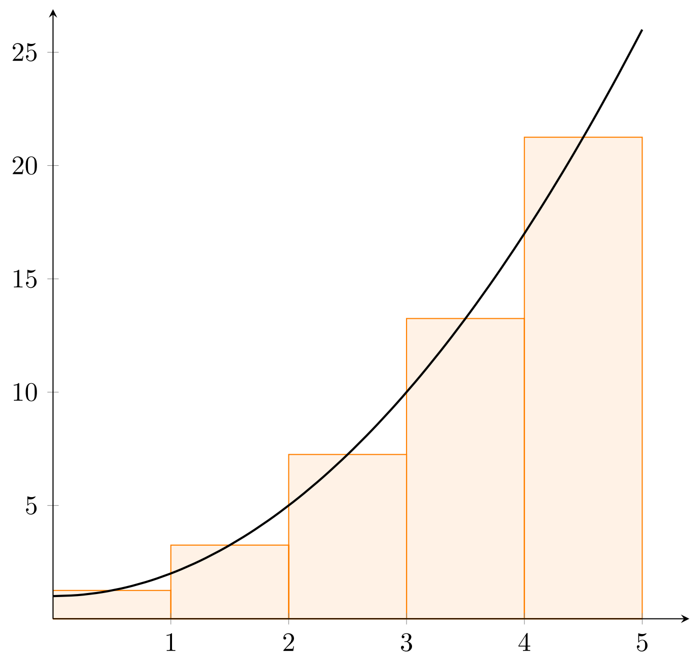
$$1 * (f(1) + f(2) + f(3) + f(4) + f(5))$$

$$= 2 + 5 + 10 + 17 + 26$$

$$= 60$$

3. **Midpoint Riemann Sum**

$f(x) = x^2 + 1$



Just like the other two, the only difference is that the Midpoint Riemann Sum calculates at the midpoint of the subintervals, so the approximation is:

$$1 * (f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5))$$

$$= 1.125 + 3.25 + 7.25 + 13.25 + 21.25$$

$$= 46.125$$

4. **Trapezoidal Sum** The trapezoid sum has the potential to have the least error of any of the previous 3 methods. Instead of summing rectangles, it sums trapezoids that look identical to the rectangle except the top is bounded by a line that starts at $(a, f(a))$ and ends at $(b, f(b))$ instead of being a horizontal line at one of those points. The area of a trapezoid is $\frac{h}{2}(base_1 + base_2)$. In this case, the bases are vertical and the width is the total interval divided into n subintervals. Therefore, the total area for one trapezoid from $x = a$ to $x = b$ would be:

$$\frac{width}{2}(f(a) + f(b))$$

If we were to add up all of these rectangles over an interval [c,d] divided into n subintervals, the area would be:

$$\frac{\frac{d-c}{n}}{2}\left(f(c) + f\left(c + \frac{d-c}{n}\right)\right) + \frac{\frac{d-c}{n}}{2}\left(f\left(\frac{d-c}{n}\right) + f\left(c + 2\frac{d-c}{n}\right)\right) + \cdots + \frac{\frac{d-c}{n}}{2}\left(f\left(d - \frac{d-c}{n}\right) + f(d)\right)$$

$$= \frac{d-c}{2n}\left(f(c) + f\left(c + 2\frac{d-c}{n}\right) + 2f\left(c + 2\frac{d-c}{2}\right) + \cdots + 2f\left(d - \frac{d-c}{n}\right) + f(d)\right)$$

If we were to abbreviate the width $\frac{d-c}{n}$ as i, it simplifies down to:

$$\frac{i}{2}\left(f(c) + 2f(c+i) + 2f(c+2i) + \cdots + 2f(d-i) + f(d)\right)$$

The trapezoid approximation is somewhat complex yet elegant at the same time.

Exercises

1. **Calculate the Left Riemann Sum using 4 rectangles over the interval specified:**
   (a) $2x + 2$ over $(1, 5)$
   (b) $\cos(x)$ over $(0, \frac{\pi}{2})$
   (c)
   $$\frac{e^x}{x-1}\text{over}(2,3)$$
2. **Calculate the Right Riemann Sum using 4 rectangles over the interval specified:**
   (a) $2x + 2$ over $(1, 5)$
   (b) $\cos(x)$ over $(0, \frac{\pi}{2})$
   (c)
   $$\frac{e^x}{x-1}\text{over}(2,3)$$
3. **Calculate the Midpoint Riemann Sum using 4 rectangles over the interval specified:**
   (a) $2x + 2$ over $(1, 5)$
   (b) $\cos(x)$ over $(0, \frac{\pi}{2})$

(c)

$$\frac{e^x}{x-1}\text{over}(2,3)$$

4. **Calculate the Trapezoidal Sum using 6 trapezoids over the interval specified:**
   (a) $2x+2$ over $(1,4)$
   (b) $x\sin(x)$ over $(0,\pi)$
   (c) $.5x^3-2x^2+1$ over $(-2,4)$

4.3 Limits of Riemann Sums

To reduce the amount of error in our Riemann Sums, one could increase the total number of rectangles in the interval. 2,000 rectangles should be a closer approximation than 5 (generally). As we increase the number of rectangles, the width is reduced and the rectangles become closer to the graph. One way to minimize the total error is to maximize the number of rectangles. We can't directly compute ∞ rectangles but we can take the limit at ∞ subintervals. Turning the additions of the heights into a summation, we can express this as:

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}f\left(a+i\frac{b-a}{n}\right)$$

because we are dividing the interval $[a,b]$ into n subintervals and multiplying the height (function at each point) by the width (which is constant for all rectangles) and adding all of the rectangles together. The function $f(x)$ is evaluated at the first point $x=$ starting point + (how many rectangles we have already counted) * width, or $x=a+i\frac{b-a}{n}$. We end up needing to find this sum quite often, and as it is bulky to read, we use shorthand. We write a fancy s that stands for *sum* like this: $\int$ and place our starting and ending points on the s like this: $\int_a^b$. We also abbreviate the width $\frac{b-a}{n}$ as dx usually. It is *d-some variable* always, and the variable we are moving along is usually x because we are dividing up the $x-$axis. We can write our sum then as height * width or:

$$\int_a^b f(x)*dx$$

which is more commonly written as

$$\int_a^b f(x)dx$$

We call this **the integral**. We read the last equation as *the definite integral from a to b of f(x) dx*. The reason why we choose dx to represent our width is because dx is the infintesimal change in x, similar to the infintesimal change in x of the derivative. Because the derivative uses dx to represent this change, we use the same notation to represent the change in our integral.

4.4 Rules of Integrals

One could try to find the integral the long summation way, however it is just simpler and time-reducing to memorize rules similar to the derivative rules.

1. The integral of $f(x) + g(x)$ is equal to the integral of $f(x)$ + the integral of $g(x)$ evaluated from a to b
2. The integral of $c * f(x)$ for some constant c is equal to c * the integral of $f(x)$ evaluated from a to b
3. The integral of dx is $b - a$
4. The integral of x^n is $\dfrac{x^n + 1}{n + 1}$ evaluated from a to b
5. The integral of x^{-1} is $\ln(x)$
6. The integral of e^u is e^u evaluated from a to b
7. The integral of $\dfrac{1}{x}$ is $\ln(x)$ evaluated from a to b
8. The integral of $\sin(x)$ is $-\cos(x)$ evaluated from a to b
9. The integral of $\cos(x)$ is $\sin(x)$ evaluated from a to b
10. The integral of $f(x)$ from a to b is equal to the opposite (-) of the integral of $f(x)$ evaluated from b to a

A function $f(x)$ evaluated from a to b is the same as $f(b) - f(a)$. We usually represent it as $f(x)\Big|_a^b$ The integral has some curious properties. However, it only returns a real number and not a function like the derivative because all it does is calculate the area. The properties are eerily familiar though...

Example 1: Find the area underneath the graph of $g(x) = \dfrac{3}{x} + \sin(x)$ between $x = 1$ and $x = \dfrac{3\pi}{4}$

$$
\begin{aligned}
g(x) &= \frac{3}{x} + \sin(x) \\
\int_1^{\frac{3\pi}{4}} g(x)\,dx &= \int_1^{\frac{3\pi}{4}} \quad dx \\
\int_1^{\frac{3\pi}{4}} g(x)\,dx &= \int_1^{\frac{3\pi}{4}} \frac{3}{x}\,dx + \int_1^{\frac{3\pi}{4}} \sin(x)dx \\
\int_1^{\frac{3\pi}{4}} g(x)\,dx &= 3\int_1^{\frac{3\pi}{4}} \frac{1}{x}\,dx + \int_1^{\frac{3\pi}{4}} \sin(x)dx \\
\int_1^{\frac{3\pi}{4}} g(x)\,dx &= 3\ln(x)\Big|_1^{\frac{3\pi}{4}} - \cos(x)\Big|_1^{\frac{3\pi}{4}} \\
\int_1^{\frac{3\pi}{4}} g(x)\,dx &= 3\left(\frac{3\pi}{4}\right) - 3\ln(1) - \cos\left(\frac{3\pi}{4}\right) + \cos(1) \\
\int_1^{\frac{3\pi}{4}} g(x)\,dx &= 3\ln\left(\frac{3\pi}{4}\right) + \frac{\sqrt{2}}{2} + \cos(1)
\end{aligned}
$$

It is not a fully simplified answer, but there is no need to fully evaluate it as there is nothing but pre-calculus left. This answer is good enough.

Example 2: Find the area underneath the graph of $f(x) = 3x^2 - 2\cos(x)$ between $x = 2$ and $x = 5$

$$f(x) = 3x^2 - 2\cos(x)$$
$$\int_2^5 f(x)\,dx = \int_2^5 3x^2 - 2\cos(x)\,dx$$
$$\int_2^5 f(x)\,dx = \int_2^5 3x^2\,dx - \int_2^5 2\cos(x)\,dx$$
$$\int_2^5 f(x)\,dx = \int_2^5 3x^2\,dx - 2\int_2^5 \cos(x)\,dx$$
$$\int_2^5 f(x)\,dx = \left(\frac{1}{3} * 3x^{2+1}\right)\Big|_2^5 - 2(\sin(x))\Big|_2^5$$
$$\int_2^5 f(x)\,dx = x^3\Big|_2^5 - 2\sin(x)\Big|_2^5$$
$$\int_2^5 f(x)\,dx = (125 - 8) - 2\sin(5) + 2\sin(2)$$
$$\int_2^5 f(x)\,dx = 117 + 2\sin(2) - 2\sin(5)$$

Exercises

1. Calculate the area underneath the curve between $x = 1$ and $x = 3$:
   (a) x^2
   (b) $\cos(x) - \sqrt{x}$
   (c) $2.5e^x - \frac{1}{\sqrt{x}}$

4.5 u Substitution

u-Substitution is a very important tool to have in your integral toolbox. It works a bit like this:

> One of the most basic elements in math is the variable. Variables can represent *anything*, from a simple constant $r = 2$ to a complex system of other variables like $y = \ln(2x)$. These statements are true simply because we say they are. The only thing that we're doing when we assign a variable is essentially re-naming a phrase. In the previous example, we take $\ln(2x)$ and give it the name y.

Let's look at this equation:

$$f(x) = \int_0^2 x\cos(x^2 + 1)dx$$

Here, we are using x as our variable of choice. Unfortunately, there is no integral form that this falls under. We can change x to a different variable and the equation would still be true as long as the equality is kept:

$$x = t$$
$$f(t) = \int_0^2 x\cos(t^2+1)dt$$

Although this change doesn't change the value of the equality, it's not too helpful. We can assign an entire phrase to a variable though, and we will call that phrase stand-in u. In this case, let's say

$$u = x^2 + 1$$

That helps clear up the bottom. Our equation is now

$$f(x) = \int_0^2 t\cos(u)dx$$

That wasn't helpful at all. We now have one integral and two variables. A mess! That being said, we can start putting things in terms of u:

$$x = 0 \mapsto u = 1$$
$$x = 2 \mapsto u = 5$$

So we can rewrite the equation as

$$f(u) = \int_1^5 x\cos(u)dx$$

Again, unhelpful. One more thing, though. dx is the infinitesimal value of x that we're using in this integral. Because u is just x changed slightly, we can find du in terms of dx!

$$u = x^2 + 1$$
$$du = 2x\,dx$$

Which can be re-written as

$$dx = \frac{du}{2x}$$

which we can substitute back into the integral:

$$f(u) = \int_1^5 x\cos(u)\frac{du}{2x}$$

which simplifies down to

$$f(u) = \frac{1}{2}\int_1^5 \cos(u)du$$

which is an easy integral! $f(u) = \sin(5) - \sin(1)$.

Example 1: Find the integral of $h(x) = x^2\sqrt{x^3+1}$ over [0,1] First, let's find a good phrase to set for u. $x^3 + 1$ looks good:

$$x = 0 \mapsto u = 1$$
$$x = 1 \mapsto u = 2$$

Next, let's find du:

$$u = x^3 + 1$$
$$du = x^2\,dx$$

Plugging in:

$$\int_1^2 \sqrt{u}\,du$$

Which is simple from here on out.

Exercises

1. **Find the integral using u-substitution:**
   (a) $\frac{\sin(\ln(x))}{x}$ over (1,e)
   (b) $\sin(x)\cos(x)$ over (0,$\frac{\pi}{2}$)
   (c) $\frac{2x+1}{x^2+x-5}$ over (2,4)

4.6 Trigonometric Substitution

Don't worry! Trig sub is easier than it may seem at first. Abstraction is a big thing in mathematics. We like to take something and weed it down to the exact core of how it works. In this section, we will take trigonometry, find out exactly how it works, and use our newfound understanding to help us with Calculus!

Up until now, trigonometry has always meant angles, triangles, the unit circle, all that fun stuff. Let's go back to the very beginning and look at what trigonometry actually is: a set of tools for calculating ratios. Nothing more. You might say that angles are a part of this, too, but remember that angles are determined by ratios!

Trigonometry is just a way to *represent* ratios. Let's take a look at an example:

$$\int \frac{1}{\sqrt{16-x^2}}\,dx$$

This is an impossible integral. Or at least it is in its current form. Again, the great thing about math is that we can do whatever we want as long as the equality is preserved. Using a combination of the Pythagorean theorem and our knowledge of trigonometry, we can simplify that down.

We know this problem is a ratio, so we're going to use a triangle to model the pieces. Let's look at $\sqrt{16-x^2}$. The square root should be a hint that we can use the Pythagorean theorem. From this, it should be clear that this square root could easily also be representative of a leg of a triangle. Specifically a triangle where the other leg is x and the hypotenuse is 4. Because we're using a right triangle, we get to use trigonometry to rewrite. We're going to choose one of the two non-right angles and call it θ. For the purpose of this problem, we're going to choose θ so $\sqrt{16-x^2}$ is related to cosine. Cosine is adjacent over hypotenuse, so in this case:

$$\cos(\theta) = \frac{\sqrt{16-x^2}}{4}$$

The only reason why we did this is to change the problem into something easier. There is no special secret, nothing like that. Moving on, we can rewrite the previous equation as:

$$4\cos(\theta) = \sqrt{16-x^2}$$

We have now found a different way to represent our denominator! Success! Our integral now can be rewritten as:

$$\int \frac{1}{4\cos(\theta)}\,dx$$

But we're still left with this dx. We need to find some way to change that to $d\theta$. We have six ways to do so: sine, cosine, secant, etc.! We get to have our pick because they all relate x in some way to θ. What we're going to do is take the easiest possible route. In this triangle, the easiest way is $\sin(\theta)$ which is simply $\frac{x}{4}$. Our next job is to find $d\theta$ in terms of dx:

$$\sin(\theta) = \frac{x}{4}$$
$$4\sin(\theta) = x$$
$$4\cos(\theta)\,d\theta = dx$$

So plugging back into our original integral:

$$\int \frac{1}{4\cos(\theta)} 4\cos(\theta)d\theta$$

Which is easy to calculate!

$$\int \frac{1}{4\cos(\theta)} 4\cos(\theta)\,d\theta$$
$$\int d\theta$$
$$\theta + C$$

We should convert θ back to x now, and the easiest way to do that is just to calculate the arcsine in this case.

$$\sin(\theta) = \frac{x}{4}$$
$$\theta = \arcsin\left(\frac{x}{4}\right)$$

So our answer is

$$\arcsin\left(\frac{x}{4}\right) + C$$

Why did we choose the trigonometric functions in the order that we did? It took me almost an entire year of trying to figure out exactly why we were able to do trigonometric substitution. The easiest way to put it is *we do it because we can.* There is no secret that you're missing out on. We're just changing how we represent some values. Why didn't we use, say, tangent, or something other than sine or cosine? The answer is that in this specific problem, those were the easiest. If we were to use tangent, we would complicate the problem even more. After a lot of work, we should end up at the same answer. It will just take much longer.

4.7 Average Value

This topic is simple enough. The integral calculates the total area between points a and b. The arithmetic mean (average) of anything is $\frac{\text{total amount of things}}{\text{total number of things}}$. In this case, the total amount of things is the total area (the integral) and the total number of things is $b - a$, so the average value of the graph of $f(x)$ is:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Example 1: Find the average value of $\sin(x)$ over $(0, \frac{\pi}{4})$

$$\frac{1}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \sin(x)dx$$
$$\frac{4}{\pi}(-\cos(x))\Big|_0^{\frac{\pi}{4}}$$
$$\frac{4}{\pi}\left(\frac{-\sqrt{2}}{2}+1\right)$$
$$\frac{2-2\sqrt{2}}{2\pi}$$

Example 2: Find the average value of $x^3 - 2x + 1$ over $(5, 9)$

$$\frac{1}{9-5}\int_5^9 x^3 - 2x + 1dx$$
$$\frac{1}{4}\left(\frac{1}{4}x^4 - x^2 + x\right)\Big|_5^9$$
$$\frac{1}{4}(1568.25 - 136.25)$$

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Exercises

1. **Find the average value over the interval:**
   (a) $2\sin(x) - x^2$ over (0,2)
   (b) $x^2 - x + \frac{1}{2x}$ over (3,2)
   (c) $\dfrac{2x}{x^2+1}$ over $(-10000, 10000)$

4.8 Integrals Generating Functions

The integral finds the area underneath a graph between two points. Those two points do not necessarily have to be constants, and one can find this integral easily:

$$\int_0^t \cos(x)dx$$

where t is a variable. The answer is $\sin(t) - \sin(0) = \sin(t)$. This is interesting, especially because for any point t on the graph, the area underneath the curve of $\cos(x)$ between $x = 0$ and $x = t$ will always be $\sin(t)$.

Example 1: Compute the integral $\int_0^{2x} t\cos(t^2)dt$

$$u = t^2$$
$$t = 0 \mapsto u = 0$$
$$t = 2x \mapsto u = 4x^2$$
$$.5\int_0^{4x^2} \cos(u)du$$
$$.5\sin(4x^2) - .5\sin(1)$$

Example 2: Compute the integral $\int_1^r x^2 + 1 - \frac{3}{x}dx$

$$\int_1^r x^2dx + \int_1^r dx - 3\int_1^r \frac{1}{x}dx$$
$$\left(\frac{r^3}{3} - \frac{0}{3}\right) + (r-1) - (\ln(r) - \ln(3))$$
$$\frac{r^3-1}{3} + r - 1 - \ln(r) - \ln(3)$$

Exercises

1. Find the function generated by the integral:

   (a)
   $$\int_0^n 17x - 3\cos(x)$$

   (b)
   $$\int_p^2 t + 5 + \frac{7}{2t}dt$$

   (c)
   $$int_1^x \ln(2t)dt$$

4.9 Calculating Area

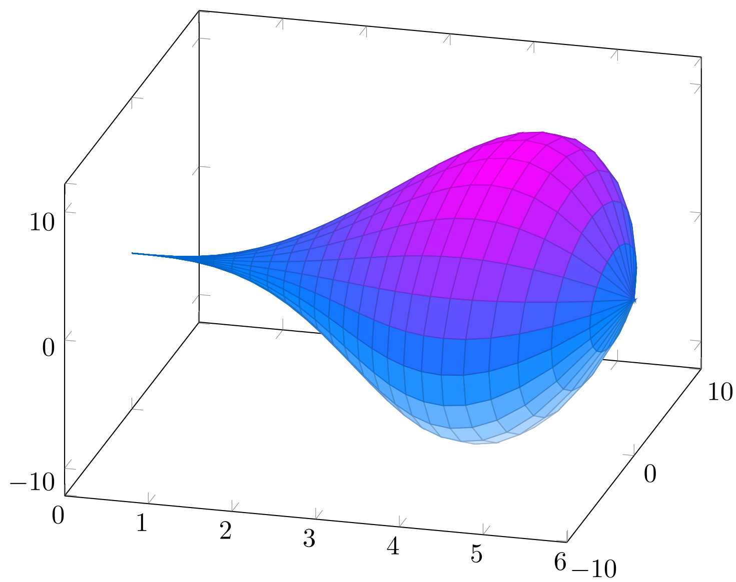
We have already established that the integral finds area. It can be used for more than just $f(x)$ graphs, though. For example, if we wanted to find the area of the graph of $f(t) = 2\pi t$ (the circumfrence of a circle with radius t) for some radius r, the integral evaluates to:

$$\int_0^r 2\pi t dt$$
$$= 2\pi \int_0^r t dt$$
$$= \pi r^2$$

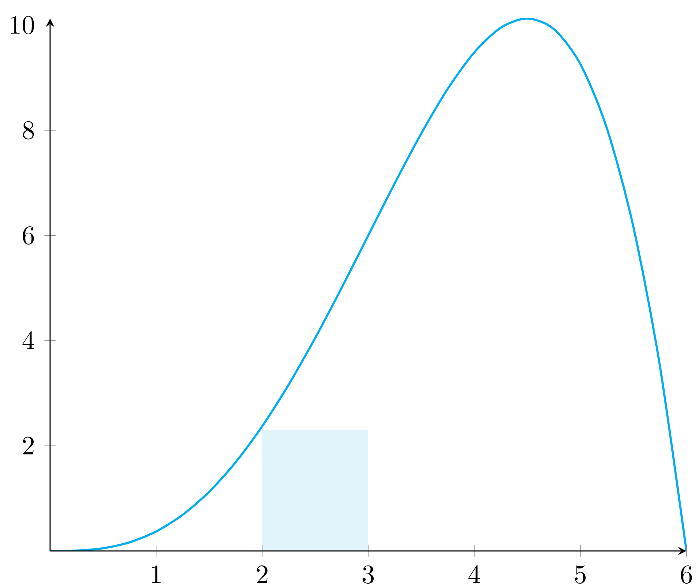
Interesting, right? Anyways, you may ask why we start at 0. The answer is that it's convenient. We want to find the difference in area between some t and, in this case, a circle with no area. That is why we use 0. It may be adventageous in other calculations to not start at 0, to find the area underneath the graph between some variable and some known value.

4.10 Disc Method

The disc method is a way to calculate the volume (yes volume, not area) of a cylindrical object. We do not know the volume formula for an odd graph like this:



But we can try and approximate it. Just like how we can use rectangles to approximate 2-d area, we can use cylinders to approximate cylindrical shapes. If we can find the two-dimensional cross-section of a revolved shape, we can calculate the volume of the revolved shape by taking an infinite sum of infinitely small discs knowing their radius (height in 2-dimensions). This means that we can find a Riemann-style rectangle in a cross-section like this:



...and revolve it around the x-axis to create a cylinder. We can revolve infinitely small cylinders over the axis just like taking the integral. In fact, the volume of a cylinder is just $r^2 * \pi *$ height. Because we have the rectangle, we know $r = f(x)$ and height$=dx$, so we end up with a formula for revolving a cross-section about the x-axis:

$$\pi \int_a^b f(x)^2 dx$$

And if it is hollow (there is a smaller function $g(x)$ that forms the inside of the cross section), the area is

$$\pi \int_a^b f(x)^2 dx - \pi \int_a^b g(x)^2 dx$$

$$= \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

An easier way to remember this is big radius R minus small radius r:

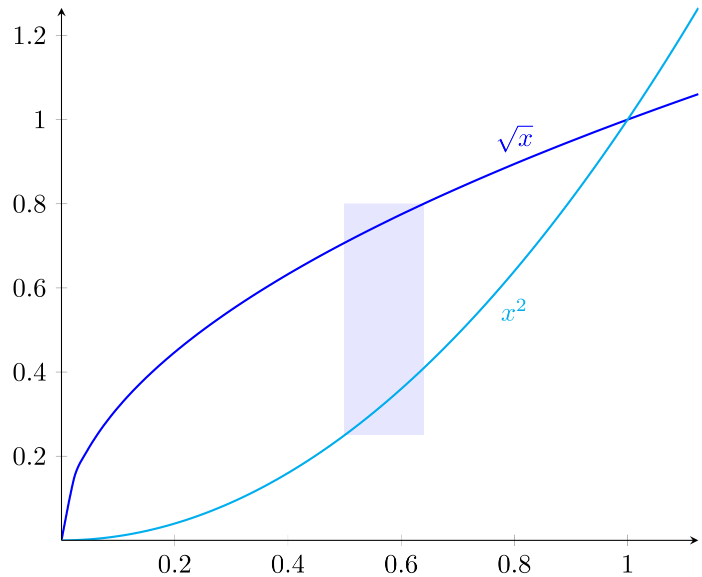
$$\pi \int_a^b \left(R^2 - r^2 \right) dx$$

or, in this case,

$$\pi \int_0^6 f(x)^2 dx$$

4.11 Shell Method

The shell method is the opposite way to calculate revolved solids of the disc method. If one is given a cross section like this:



An area bounded by two functions revolved around the y-axis, it can be easier to calculate shells instead of discs. We can say that we have approximations for a thin “shell” that is just a circle's circumference multiplied by some width. This is not exactly the formula for a shell-like object but because the width is so small, it's practically 2-dimensional and the approximation is close enough. The error becomes closer to 0 as the width of the shells closes in on 0 (a limit!). We end up with the formula for each shell being:

$$2 * \pi * r * height * width$$

The radius in this case is x because every point will be x units away from the y-axis, the height will be the top function minus the bottom ($\sqrt{x} - x^2$). The width is an infinitely small amount of x, or dx. Therefore the formula is:

$$2\pi \int_0^1 x(\sqrt{x} - x^2)dx$$

in this case.

A simple way I was taught to remember this is that the she***ll*** method is useful when the axis of revolution and cross-section are para***ll***el.

Example 1: Find the volume of the solid bounded by $y = x^2$, $y = \sin\left(\pi x^2\right) + x$, x=0, and x=1 revolved around the y-axis.
The only way to calculate this is by finding the difference between x^2 and $\sin\left(\pi x^2\right) + x$, so we must use the shell method. Remember, the shell of a cylinder is $2 * \pi * r * height * width$. r is the distance away from the axis of revolution, which in this case is $x - 0 = x$. $height$ will be the height of the rectangle, or $\sin\left(\pi x^2\right) + x - x^2$. $width$ is going to be infinitely small, or dx. That leaves us with the volume being:

$$dV = 2\pi * x * \left(\sin\left(\pi x^2\right) + x - x^2\right) * dx$$

which simplifies down to

$$dV = 2\pi\left(x\sin\left(\pi x^2\right) + x^2 - x^3\right)dx$$

Now, we will add up all of these volumes between $x = 0$ and $x = 1$:

$$V = \int_0^1 2\pi\left(x\sin(\pi x^2) + x^2 - x^3\right)dx$$
$$V = 2\pi\left(\int_0^1 x\sin(\pi x^2)dx + \int_0^1 x^2dx - \int_0^1 x^3dx\right)$$
$$u = x^2$$
$$du = 2x\,dx$$
$$x = 0 \mapsto u = 0$$
$$x = 1 \mapsto u = 1$$
$$V = 2\pi\left(.5\int_0^1 \sin(\pi u)du + \int_0^1 x^2dx - \int_0^1 x^3dx\right)$$
$$V = 2\pi\left(\frac{1}{2\pi}(-\cos(\pi * 1) + \cos(\pi * 0)\right) + \frac{1}{3} - \frac{0}{3} - \frac{1}{4} + \frac{0}{4}$$
$$V = 2\pi\left(\frac{1}{2\pi}(1+1) + \frac{1}{3} - \frac{1}{4}\right)$$
$$V = 2 + \frac{\pi}{6}$$

Example 2: Find the volume of the solid bounded by $y = x^2$, $y = 4$, and $x = 0$ revolved around the y-*axis*.
It's probably best to approach this problem using the washer method. First, we have to find everything in terms of y:

$$y = x^2$$
$$\sqrt{y} = x$$

Now for the boundary points $x = 0$ and $x = 1$

$$x = 0 \mapsto y = 0$$
$$x = 2 \mapsto y = 4$$

Here's what we have so far: the boundaries are $y = 0$ and $y = 4$, the function is $x = \sqrt{y}$, and we are revolving around $x = 0$. Our infinitesimal volume is $\pi * r^2 * height$ which is:

$$dV = \pi\left(\sqrt{y}\right)^2 * dy$$

Therefore, our total sum of all of the volumes is:

$$V = \int_0^4 \pi * (x)^2 * dy$$
$$V = \pi \int_0^4 (\sqrt{y})^2 dy$$
$$V = \pi \int_0^4 y\, dy$$
$$V = \pi \left(.5(4)^2 - .5(0)^2\right)$$
$$V = 8\pi$$

Exercises

1. **Find the volume of the solid**
   (a) x^3 and $\sqrt{x}$ revolved around the x-axis
   (b) $\frac{1}{x}$ bounded by $y = 0$, $x = 1$, and $x = 5$ revolved around $y = -1$
   (c) $5 - (x-2)^2$ bounded by $y = 1$, $x = 0$, and $x = 4$ revolved around the y-axis

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The Fundamental Theorem

5 The Fundamental Theorem

5.1 Introduction

The Fundamental Theorem of Calculus ties together derivatives, definite integrals, and antiderivatives. We have already seen that the integral can act like an antiderivative in some ways from time to time. The Fundamental Theorem of Calculus formalizes this:

1. $\displaystyle\int_a^b f'(x)dx = f(b) - f(a)$

2. $\displaystyle\frac{d}{dx}\int f(x)dx = f(x)$

5.2 Consequences

We can use the integral as an antiderivative! It is common to find the antiderivative with $\int_c^x$ for some constant c, so when finding an antiderivative with the integral, we drop writing the c and the x: $\int$. The reason we can drop the bounds is because the lower bound will return a constant. We don't care about what that constant is, we can always find it later.

Without bounds, we say that it is an **indefinite integral**. As seen at the beginning of §4, a function created by a derivative has an infinite amount of antiderivatives. Therefore, we write the resulting antiderivative as the indefinite integral$+C$, where C is the **constant of integration**. C is just the placeholder for some constant.

For any funciton created by a derivative[2], the indefinite integral can calculate the antiderivative. For example, for $f'(x)$:

$$f(x) = x^2 + 5x - 1$$

$$f'(x) = 2x + 5$$

we can use the integral to reverse and find our original equation $f(x)$

$$\int f'(x)dx$$

$$\int 2x + 5\,dx$$

$$x^2 + 5x + C$$

to which $f(x)$ is a valid solution when $C = -1$.

Okay, let's slow down a bit. Here's what we just found out:

1. The derivative of an indefinite integral of $f(x)$ is $f(x)$ itself
2. The integral from a to b of $f(x)$ is $f(b) - f(a)$

This shouldn't be *too* surprising. There's also something else we learned:

> We can now calculate the (change in) area under a curve without needing to know the bounds!

[2]I keep saying created by a derivative because there are some functions that do not have an antiderivative; they cannot be created with a derivative.

As you will see later, this is very handy. Note that we can't actually find the exact value of the integral given only the derivative because of the $+C$.
This is really cool! The limit of the difference quotient of an *infinite* sum of a function is the function itself! It may seem obvious knowing the rules of derivatives and integrals, but think about it this way:

$$\lim_{h\to 0} \frac{\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n} + h\right) - \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)}{h} = f(x)$$

How cool is that?! Really cool. Math rocks.

Exercises

1. **Are these statements always true? If the statement is false, correct it.**
   (a) $\int f'(x)\,dx = f(x)$
   (b) $\frac{d}{dx}\int 3\,dx = 3$
   (c) Given a derivative $g'(x)$, you can find the value of any point on $g(x)$
2. **Find the function given by the indefinite integral and the point:**
   (a) $f'(x) = x\cos(x^2),\ f(0) = 2$
   (b) $h'(x) = \frac{3}{x}$, $h(e) = 1$
   (c) $\frac{dV}{dt} = 2\sin(t) + t^2 - 3t$, $V(0) = -2$

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6 Applications of the Fundamental Theorem

6.1 Solving Differential Equations

Knowing the integral is the antiderivative, we can solve differential equations now. All one has to do is separate the variables so all y's and dy are on one side and all x's and dx are on the other of the equality. For example, we can calculate the classic example of the slope of a graph being dependent upon the $y-$coordinate:

$$\begin{aligned}
\frac{dy}{dx} &= yk \\
\frac{1}{y}dy &= k * dx \\
\int \frac{1}{y}dy &= k \int dx \\
\ln(y) &= kx + C \\
y &= e^{kx+C} \\
y &= e^{kx} * e^{C} \\
y &= e^{kx} * P \\
y &= Pe^{kx}
\end{aligned}$$

for some rate k and some initial value (principal) P. P is the intial value because when $x = 0$, $y = Pe^0 = P$.

Example 1: Solve the differential equation $dy = \frac{-x}{y}dx$ which passes through the point $(0, 2)$

$$\begin{aligned}
y\,dy &= -x\,dx \\
\int y\,dy &= -\int x\,dx \\
.5y^2 + D &= -x^2 + E
\end{aligned}$$

We don't know either the value of D or E, so we can just subtact D from both sides. The benefit of this is we only have one constant now, $E - D$ on the right side. Because $E - D$ is unknown, we can just write it as another constant $.5 * C$. We only do this to make the problem a little simpler.

$$\begin{aligned}
.5y^2 &= -.5x^2 + .5C \\
y^2 &= -x^2 + C \\
y^2 + x^2 &= C \\
2^2 + 0^2 &= C \\
4 &= C
\end{aligned}$$

So our final answer is a circle centered at the origin that has a radius of 2: $y^2 + x^2 = 4$.

Exercises

1. **Solve the differential equation:**
   (a) $dy = 2xy\,dx$
   (b) $dy = \frac{2x}{y^2}\,dx$
   (c) $dy = x\sec(y)dx$
2. **Solve the differential equation that passes through the given point:**
   (a) $dy = 3y\,dx$ through (1,100e)
   (b) $dy = 2y(100 - y)$ through (0,25)

6.2 Accumulation and Net Change

Because the integral is the antiderivative, the antiderivative of a function will return another function that can be used to find the area under the graph or net accumulation of the original function. This was touched upon in §4.6. In any case, the antiderivative finds many uses and can be used to calculate the average value of a function and also how much value the graph has underneath it. The most important takeaway here is (in many cases):

$$\text{The Antiderivative} = \text{The Integral}$$

Calculating the integral of the rate of change over an interval will give you the total (net) change.

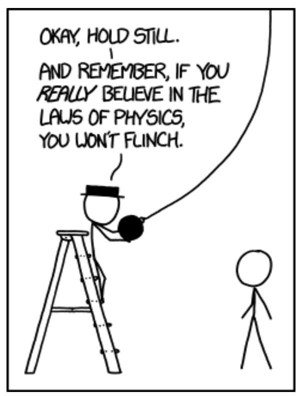
Example 1: The change in time Mr. Hylkema must spend grading Calculus homework each day t is modeled by the function $h\left(t\right) = \sin\left(\frac{x\pi}{4}\right) + \cos\left(\frac{x\pi}{2}\right)$ where $h\left(t\right)$ is in hours. By how many hours has Mr. Hylkema's homework load changed between day 0 and day 20?

$$h\left(t\right) = \sin\left(\frac{x\pi}{4}\right) + \cos\left(\frac{x\pi}{2}\right)$$
$$\int_0^{20} h\left(t\right)dt = \int_0^{20} \sin\left(\frac{x\pi}{4}\right) + \cos\left(\frac{x\pi}{2}\right)dt$$
$$\int_0^{20} h\left(t\right)dt = \int_0^{20} \sin\left(\frac{x\pi}{4}\right)dt + \int_0^{20} \cos\left(\frac{x\pi}{2}\right)dt$$
$$H\left(t\right) = \frac{-4}{\pi}\cos\left(\frac{x\pi}{4}\right)\Big|_0^{20} + \frac{2}{\pi}\sin\left(\frac{x\pi}{2}\right)\Big|_0^{20}$$
$$H\left(t\right) \approx 2.546$$

Exercises

1. **Find the net change:**
   (a) A factory's rate of output over a 10-hour day is modeled by $P(t) = \dfrac{1}{.0625t^2 - .5t + 2} + .5$ units in thousands. How much has the factory created between $t = 2$ hours and the end of the day?
   (b) A sample of Unobtanium-1050 radioactively decays over a period of time modeled by the differential equation $dr = \dfrac{-1}{4r}dt$ where r is the amount of radioactive particles given off and t is in decades. What percentage of the sample is still emitting radiation after 2 centuries?

6.3 Physics



The antiderivative is very helpful in physics problems. We already know that $x(t)$, $v(t)$, and $a(t)$ are all related through their rates (derivatives), so they must be related with their antiderivatives as well. There is no single antiderivative for a function, so instead of saying

$$\int v(t)dt = x(t)$$

which is false, we can say

$$\int v(t)dt = \text{change in } x(t)$$

or

$$\int v(t)dt = x(t) + x_0$$

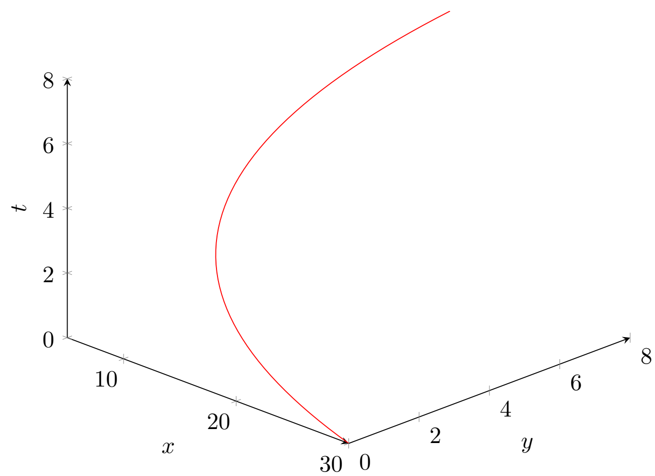
At any point t, the change in $x(t)$ will be $\int v(t)dt$. This works for all relations as well:

$$\int \left(\int a(t)dt \right) dt = \int (v(t) + v_0)dt = x(t) + v_0 t + x_0$$

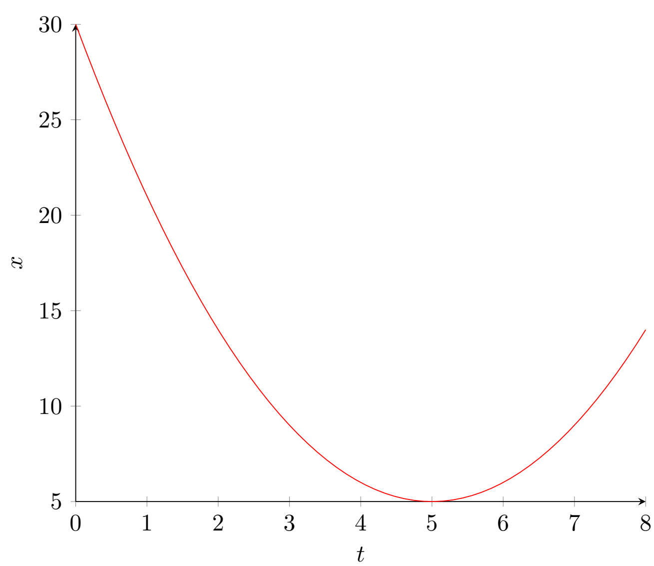
We are not using C as our constant of integration because the y-intercept has defined meaning in physics. We are instead using x_0 and v_0 as our constants to show initial conditions ($time = 0$).

Example 1: The velocity of a particle moving along the x-axis is $v(t) = t^2 - 10t + 30$. What is the change in position from time $t = 0$ to $t = 7$?

The AP test has multiple problems where a particle is moving along the x-axis dependent on time t. Don't think of this as a 3-dimensional problem even though it seems like it. This is *not* how you should think about it:



Because they're only asking for the change in the x-coordinate with relation to time t. This is how you should approach the problem:



Now that the confusing wording is out of the way, the problem is pretty simple. All we need to do is find the net change in position given the velocity:

$$v(t) = t^2 - 10t + 30$$
$$\int_0^7 v(t)\,dt = \int_0^7 t^2 - 10t + 30\,dt$$
$$\int_0^7 v(t)\,dt = \frac{t^3}{3} - 5t^2 + 30t\,\Big|_0^7$$
$$\int_0^7 v(t)\,dt = \frac{343}{3} - 245 + 210$$
$$79\frac{1}{3}$$

Exercises

1. A particle travels along the x-axis with acceleration $a(t)$ given by $a(t) = \cos(t) + 3t$. How far has the particle traveled from its initial point after time $t = \frac{3\pi}{4}$ seconds if the particle had an initial velocity of 0?

2. Dr. Bob is driving at 70 miles per hour when he suddenly realizes that he has to turn off at the next exit in 500 feet. He applies the brake and the car slows down at an acceleration of $a(t) = -0.5 * t$ miles per square hours. The maximum exit ramp velocity is 45 miles per hour. Will Dr. Bob make it? If so, what will his velocity be? If not, how much more distance would it take Dr. Bob to slow down to 45 miles per hour?

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7 Tips on the Course and AP Test

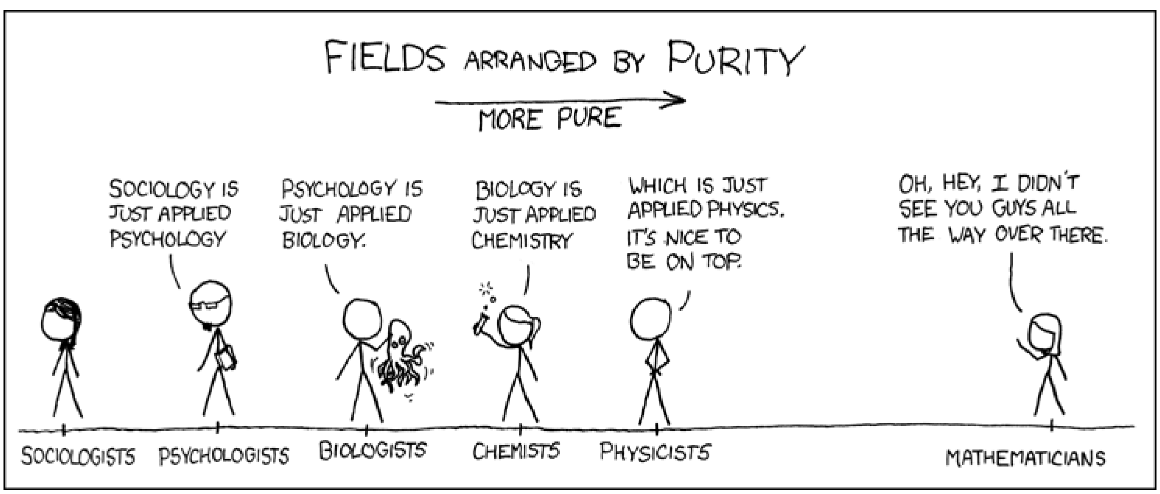
1. As with almost everything, the best way to succeed on the test is to study.
2. Calculus builds off of itself. If you are lost or unsure about something, seek help.
3. Do not give up. Calculus is hard for everybody.
4. The College Board *loves* to make the test confusing. Take as many practice tests as possible.
5. When selecting answers, check to see if there are different ways to write your answer. For example, $\sqrt{x}$ is the same as $x^{.5}$. Check to make sure you have fully simplified, too.
6. If you have extra time, check your work. If you have extra time beyond that, redo whatever problems you are not 100% sure about.
7. Doing homework is more important than ever. You should never, ever copy any one else's homework. A good grade on a homework assignment is not worth a bad score on quizzes, tests, or the AP test.
8. If you are having extreme difficulty on a problem and cannot figure out how to complete it even after reviewing your notes and the textook(s), do not fake your way through it. Leave it unfinished and ask for help.
9. Ask for help! Do not be afraid to ask. Your teacher is an exceptional (and most likely the best) resource. Other great resources include:
   (a) **Khan Academy -** good resource, can be a little slow moving at times
   (b) **Wolfram MathWorld -** the best resource on the web, explanations can be too advanced or abstract occasionally
   (c) **Stack Exchange -** often has the best explanations of problems but not all topics are asked about

   Some resources you may want to double-check are:

   (a) People who have passed AP Calculus
   (b) Other online forums
   (c) Other students in your class

8 Endnotes

This is a short and abstract book. If any topics need to be explained further or new topics need to be added, feel free to contact Noah Stockwell or Edgewood High School with instructions to be forwarded to Noah Stockwell. I'd be happy to answer any questions and welcome input on this book!



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