Example 1

$$\lim_{q\to 5} \frac{q^2}{q+1}$$

To find the limit, we must find the limit from the left-hand and right-hand side. For the left-hand side:

$$\lim_{q\to 5^-} \frac{q^2}{q+1} = \frac{25}{6}$$

For the right-hand side:

$$\lim_{q\to 5^+} \frac{q^2}{q+1} = \frac{25}{6}$$

The two limits are equal, therefore:

1. The limit $\lim_{q\to 5} \frac{q^2}{q+1}$ must exist, and
2. The limit must have the value $\frac{25}{6}$.

Example 2

$$\lim_{x\to 3} \frac{x^2-9}{x^4-81}$$

Again, find the limit from the left-hand and right-hand sides:

$$\lim_{x\to 3^-} \frac{x^2-9}{x^4-81} = \frac{0}{0}$$

$\frac{0}{0}$ is a bad number. There is no defined value for $\frac{0}{0}$, so we should try and work around it to see if we can get an actual answer:

$$\begin{aligned} \frac{x^2-9}{x^4-81} &= \frac{x^2-9}{(x^2-9)(x^2+9)} \\ \implies \lim_{x\to 3^-} \frac{x^2-9}{x^4-81} &= \lim_{x\to 3^-} \frac{1}{x^2+9} = \frac{1}{18} \end{aligned}$$

Similarly for $\lim_{x\to 3^+}$, we run into the same issue and must reduce:

$$\begin{aligned} \lim_{x\to 3^+} \frac{x^2-9}{x^4-81} &= \frac{0}{0} \\ \implies \frac{x^2-9}{x^4-81} &= \frac{x^2-9}{(x^2-9)(x^2+9)} \\ \implies \lim_{x\to 3^+} \frac{x^2-9}{x^4-81} &= \lim_{x\to 3^+} \frac{1}{x^2+9} = \frac{1}{18} \end{aligned}$$

Therefore, the limit must exist and its value is $\frac{1}{18}$.

Example 3

$$\lim_{y\to\infty} \sin\left(\frac{1}{y}\right)$$

We only need to calculate the left-handed limit because we can only approach ∞ from the left side

$$\begin{aligned}
&\lim_{y\to\infty} \sin\left(\frac{1}{y}\right)\\
=&\sin\left(\lim_{y\to\infty}\frac{1}{y}\right)\\
=&\sin(0)\\
=&0
\end{aligned}$$

Example 4

$$\lim_{t\to\infty} \cos\left(\frac{e^t}{2t+4}\right)$$

This has no solution because, unlike Example 3:

$$\lim_{t\to\infty}\frac{e^t}{2t+4}$$

does not exist and neither can the limit with cosine. This problem does not have a solution.