Noah's Guide to Calculus

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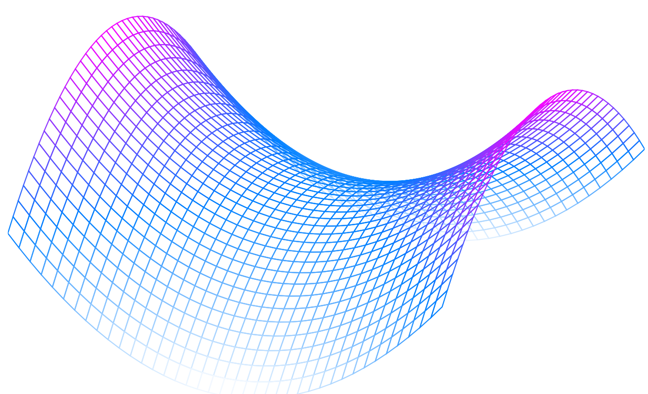

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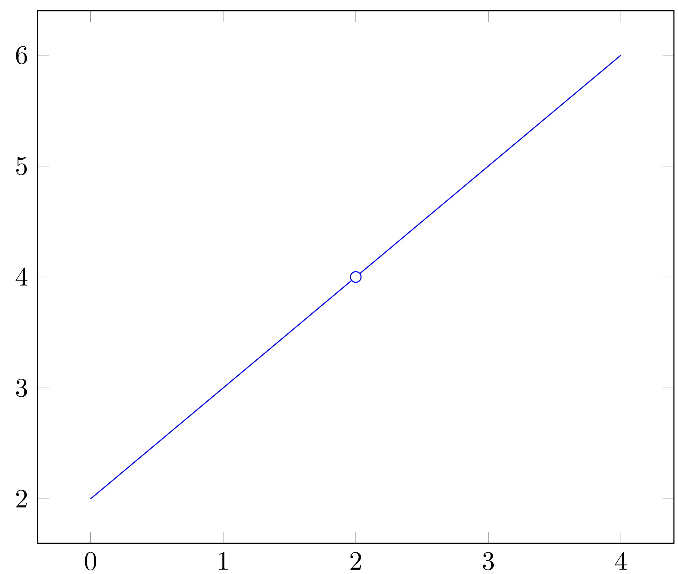
This is a simple textbook that can be used to make Calculus seem a little less confusing.

2 Limits

2.1 Introduction

Limits are a way of skirting the normal rules of math. Without the knowledge of limits, whenever a function divides by 0 or involves ∞ in any way, calculations become impossible. Limits take the rules of math a little less seriously and can be used to calculate what a value “should be”. A simple example of where limits come in handy is when there is a “hole” in a graph:



$$f(x) = \frac{x^2 - 4}{x - 2}$$

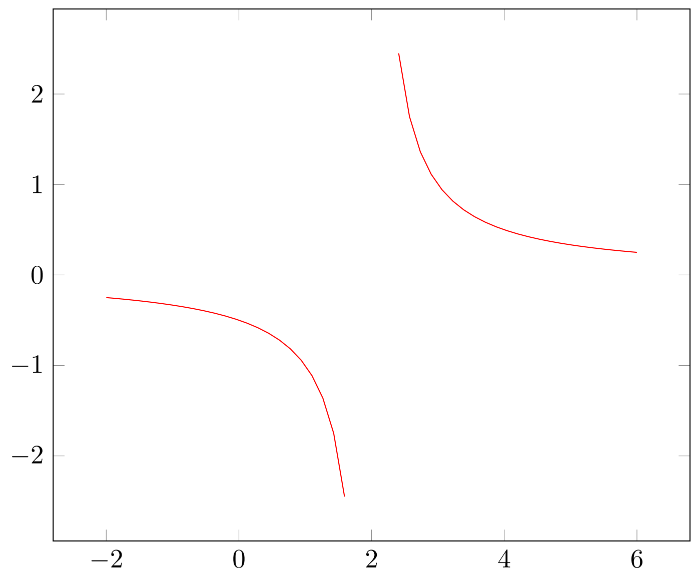
Because $f(x)$ divides by 0 when $x = 2$, there can be no answer here. However, we can tell that $f(2)$ should be 4 ignoring the division by zero. We can tell this because as x becomes greater and nearer to 2 (approaching $x = 2$ from the left), the value of $f(x)$ approaches 4. Similarly, when x decreases and becomes nearer to $x = 2$ (approaching $x = 2$ from the right), the value of $f(x)$ approaches 4. Therefore, as both sides of $x = 2$ become closer and closer, they converge upon a single point: $f(2) = 4$.

2.2 Limits

Limits can help us with the preceding problem. A limit returns the value of a function that it should be. *We define the limit to be equal to the point both sides of a graph approaches.* A simple way to write this is $\lim\limits_{x \to c^+} f(x)$ to represent the limit as x increases to the value c and $\lim\limits_{x \to c^-} f(x)$ to represent the limit as x decreases to the value c. In other words, $\lim\limits_{x \to c^+} f(x)$ is the value we approach moving to the right towards c and $\lim_{x \to c^-} f(x)$ is the value we approach moving to the left towards c. We define

$$\lim_{x \to c} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

If the limits moving to the left and moving to the right are not equal, the statement above is false and we say that the limit does not exist. For example, the limit as x approaches 2 in this graph doesn't exist:



2.3 Applications of Limits

Limits are very useful when needing to find a value that shouldn't necessarily exist. We can use limits to calculate holes in graphs from what we saw earlier, but we can also use them to approach unapproachable values like ∞. As we can only approach ∞ from one side, we write the limit as $\lim_{x \to \infty} f(x)$ for positive ∞ and $\lim_{x \to -\infty} f(x)$ for $-\infty$.

2.4 Calculating limits

Unfortunately, there's no simple and easy way to calculate limits. The simplest way is just to "plug in" to the function but at some points like holes in the graph or ∞, we don't have that luxury. Instead, we can reduce or rewrite equations and also apply some general common sense. For example, $\lim_{x \to \infty} \frac{1}{x}$ must be 0 because as x gets larger, $\frac{1}{x}$ gets smaller to some point at which it must be 0. Using reduction and logic, we may progress on to more complex ideas involving limits where direct substitution fails.

"Plugging In" In the end, all limit problems will need to have their value inserted at some time. For example, the limit $\lim_{x \to 0} x^2 + 1 = 1$. No tricks here, plugging in 0 does return 1. We do not have to deal with any division by zero or infinities, so there is no need to manipulate the problem.

Reduction Reduction is relatively simple. If we go back to our example of

$$f(x) = \frac{x^2 - 4}{x - 2}$$

we can see that this problem can factor into

$$f(x) = \frac{(x-2)(x+2)}{x-2}$$

which cancels and gives us

$$f(x) = x + 2$$

We now see that there does exist a limit at $x = 2$, $f(2) = 2 + 2 = 4$ Previously, directly plugging in 2 would not return a real value.

Sense at ∞ ∞ is a hard concept to grasp and work with in mathematics. Limits can make this easier because while we cannot *directly compute* ∞, we can approximate it exactly. For example, in the case of $f(x) = \frac{1}{x^2}$, we see that the limit at ∞ must be zero. As x increases, $f(x)$ drops ever closer to 0:

$$f(10) = 0.01$$

$$f(100) = 0.0001$$

$$f(1000) = 0.000001$$

$$etc.$$

Similarly, as x approaches ∞,

$$g(x) = \frac{3x}{4x + 1}$$

approaches $\frac{3}{4}$:

$$g(10) = \frac{30}{41}$$

$$g(100) = \frac{300}{401}$$

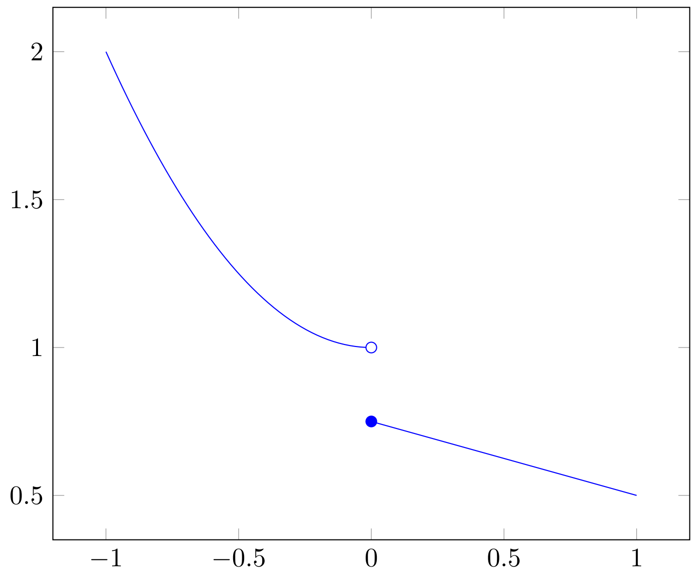
$$g(1000) = \frac{3000}{4001}$$

$$etc.$$

2.5 When Limits Don't Exist

Limits don't exist when:

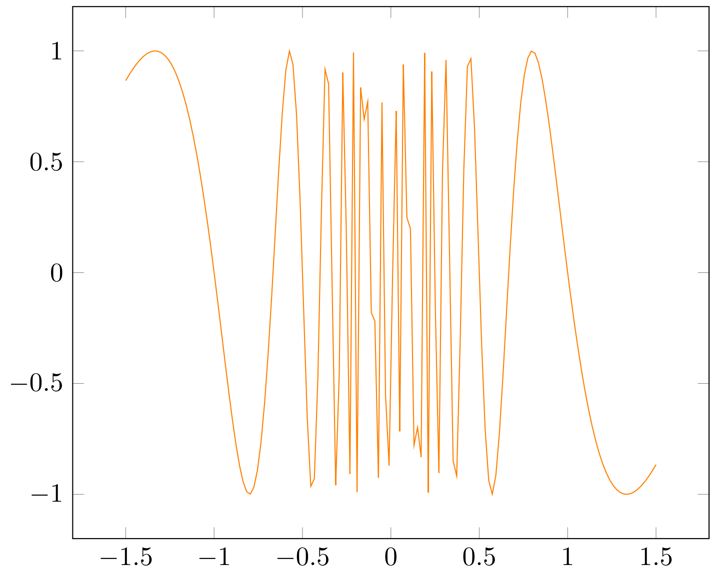
1. $\lim_{x \to c^+} \neq \lim_{x \to c^-}$



$\lim_{x \to 0} f(x)$ Does Not Exist

The reason why the limit cannot exist at 0 here is that when $f(x)$ approaches 0 from the right side (0^-), the limit is 1. As $f(x)$ approaches 0 from the left side (0^+), the limit is 0.75. The limit is one point and because $0.75 \neq 1$, there is no solution. We say that the limit *diverges*.

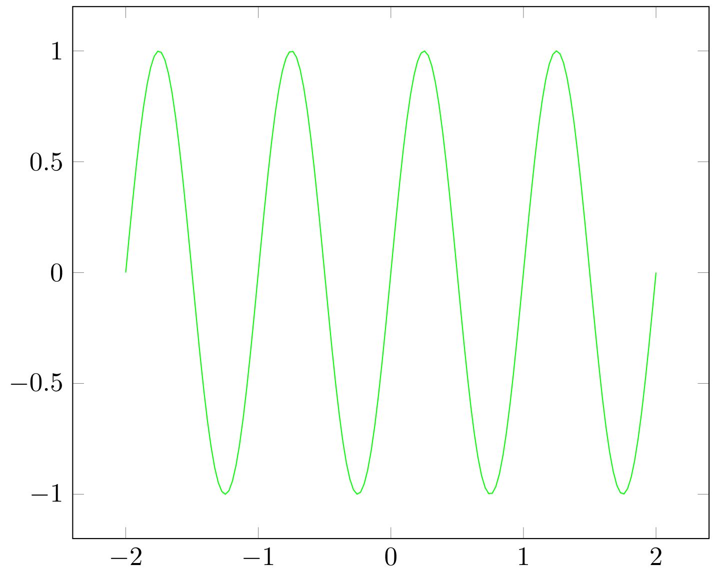
2. The graph oscillates uncontrollably:



$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ Does Not Exist}$$

The graphing utility doesn't even have any idea what's going on. We can't blame it because the graph starts oscillating faster and faster closer to 0. There is no point the graph can approach. Therefore, the limit must not exist here.

3. The graph oscillates at ∞



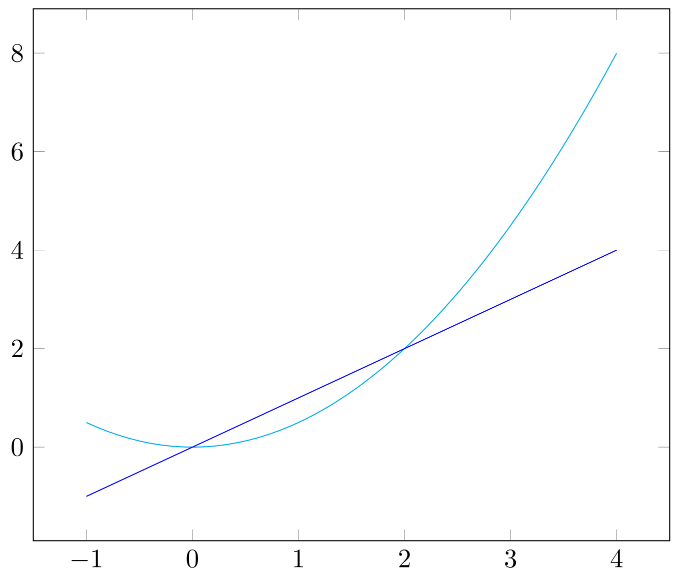
$$\lim_{x\to\infty} \sin(x) \text{ Does Not Exist}$$

Is ∞ an integer? Or where does it fall exactly? If you define ∞ to be a value, something can always be added to that value. Therefore, there is no single value that this graph will approach. In fact, it will approach all numbers in the interval [-1,1]. For a periodic graph like $\sin(x)$, there can be no limit because the graph will never stop oscillating.

3 Derivatives

3.1 Introduction

Slopes It is useful and easy to find the rate of change for any point on a line. It is difficult to approximate how a curved function changes, however. Functions that aren't lines don't have defined slopes and we have to use a tangent line to find the slope at a point. It is difficult to find the tangent line at a point, so we must resort to finding secant approximations for the tangent line.



The secant line $y = x$ approximates the line tangent to the graph $f(x) = \frac{1}{2}x^2$ for the point $x = 1$.

The Slope of a Secant Line The slope of a secant line through a point x and another point h units away will be the rise ($\triangle y$) divided by the run ($\triangle x$). Plugging in for the points $(x, f(x))$ and $(x+h, f(x+h))$, we get

$$\frac{f(x+h) - f(x)}{x + h - x}$$

which solves to

$$\frac{f(x+h) - f(x)}{h}$$

3.2 The Derivative

The derivative is a tool that can find the slope of a tangent line at (almost) any point on a graph. While we can't find the slope between one point and itself because there is no difference, we can find the slope between some point and another that is **really** close to that point. We can minimize the difference between these two points by using a limit! We will find the difference between a point and a point whose position is almost zero units away:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is the derivative! If you were to plug in 0 for h immediately, you would divide by zero. Therefore, you must use the reduction technique for solving equations of this type. The derivative is often abbreviated $\frac{d}{dt}f(t)$ with respect to some variable t and some function f; or simply $f'(x)$. If you take n more derivatives of the same function past the first one, we call it the $n^t h$ order derivative. Derivatives beyond the first order are labeled $\frac{d^2}{dt^2}f(t)$ or $f''(t)$ for the second order; $\frac{d^3}{dt^3}f(t)$ or $f'''(t)$ for the third order; etc. We say the derivative is taken "with respect to" something else. In a pretty standard Calculus case, the objective is to find the change in *y with respect to x*. We use the notation to describe this relationship as: $\frac{dy}{dx}$. Extending this, the second order derivative of *y with respect to x* is denoted $\frac{d^2y}{dx^2}$.

Example For $f(x) = x^2$, the derivative is:

$$\begin{aligned}
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x + 0 \\
&= 2x
\end{aligned} \tag{1}$$

So for any point x on the graph of $f(x) = x^2$, the slope of the tangent line will be $y = 2x$.

3.3 Forms

The derivative has patterns, or "forms" as they will be called here. You are expected to know these forms backwards and forwards. Many textbooks include proofs for these but, as it is unnecessary, this text will not. The reader is strongly encouraged to prove these forms as exercises. Some of the most important forms that Calculus students are expected to know are below. The derivative of:

1. a constant is 0

2. u^n is $(nu^n - 1)\ du$

3. cu is $c * (u')$

4. e^u is $e^u\ du$

5. $\ln(u)$ is $\frac{1}{u}\ du$

6. $\sin(u)$ is $\cos(u)\ du$

7. $\cos(u)$ is -$\sin(u)\ du$

Where u is some equation and du is the derivative of u. Most other forms can be derived from these.

3.4 Special Rules

There are some more patterns outside of the traditional forms that the reader should know for Calculus. These have to do with composite and combined equations.

The Product Rule Given two equations $f(x)$ and $g(x)$, the derivative of the function $h(x) = f(x) * g(x)$ is:

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

For example, the derivative of $h(x) = x^2 sin(x)$ is:

$$\begin{aligned} h(x) &= f(x) * g(x) \\ h'(x) &= f'(x)g(x) + g'(x)f(x) \\ h'(x) &= 2x * \sin(x) + \cos(x) * x^2 \end{aligned} \tag{2}$$

The Quotient Rule *Avoid this at all costs.* The quotient rule is helpful but it is very, needlessly complex. For some function $h(x)$ that is the quotient of two functions $f(x)$ and $g(x)$: $h(x) = \dfrac{f(x)}{g(x)}$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(f(x))^2}$$

An easy way to remember this god-awful formula is a little song to the tune of *Low Rider* by War:

> Low...d...High...minus High d Low...all...o-ver...the square of what's below
>
> (All...my...friends...know the low rider...the...low...rid-er...is a little higher)

All jokes aside, this is one of the worst things in Calculus. Avoid it at all costs. For example, you could rewrite $\dfrac{3-x}{x^2}$ as $3x^{-2} - x^{-1}$, thereby eliminating all need for the quotient rule.

The Chain Rule The name is confusing. No, this rule doesn't have anything to do chains in the normal sense. Instead, it gives a form for a composite function $h(x) = f(g(x))$. This is required for functions like $\sin(x^2)$ where $\sin(x) = f(x)$ and $x^2 = g(x)$. The chain rule says the derivative of $h(x) = f(g(x))$ is:

$$h'(x) = f'(g(x)) * g'(x)$$

In our previous example, the derivative of $\sin(x^2)$ is

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x)) * g'(x) \\ h'(x) &= \cos(x^2) * 2x \\ h'(x) &= 2x * \cos(x^2) \end{aligned} \tag{3}$$

3.5 Related Rates

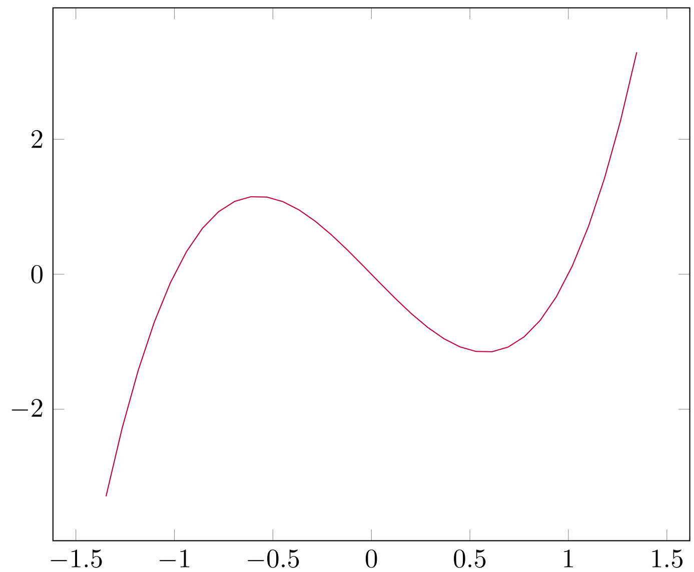
The derivative finds use in many physics and optimization problems. Because it returns the slope, it finds how one thing changes with respect to another. In physics, if you have a position equation $x(t)$, the derivative will return another equation giving the rate of change of position. This has a name in physics: the velocity. Similarly, the derivative of velocity will return the rate of change of velocity: acceleration. These can be summarized in the equations:

$$x'(t) = v(t)$$

$$x''(t) = a(t)$$

3.6 Local maxima and minima

For any graph, the local minima and maxima will occur when the graph levels off. To put this in Calculus words, the derivative at the point must be 0 because the tangent line is horizontal.

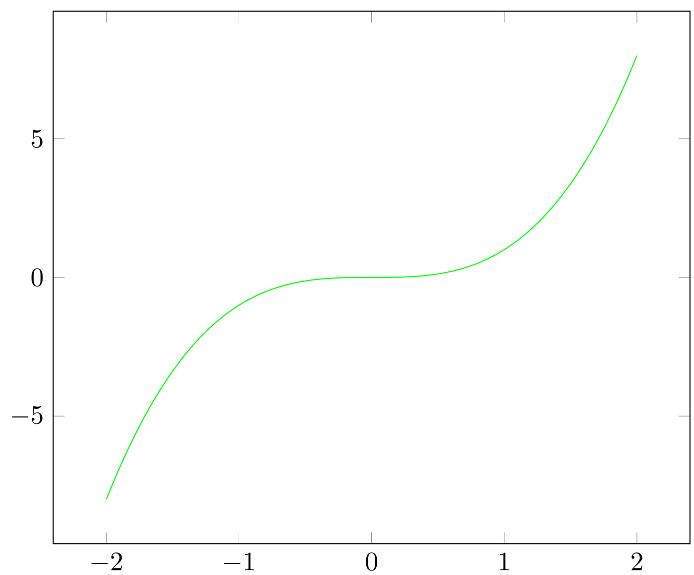


This graph has local maxima and minima where the tangent line is horizontal.

Armed with the knowledge of the derivative, we can exactly calculate at what point(s) any function has a local maximum or minimum. It is actually very simple, all one has to do is to find at what points the derivative is 0. We call this *The First Derivative Test.* The first derivative test gives us *critical numbers.* For example, let's find the points at which $f(x) = \frac{x^3}{3} - x$ has a local maximum or mimimum:

$$\begin{aligned} f'(x) &= 3\frac{x^2}{3} - x^0 \\ f'(x) &= x^2 - 1 \\ 0 &= x^2 - 1 \\ 0 &= (x-1)(x+1) \\ x &= \pm 1 \end{aligned} \tag{4}$$

I lied. When the derivative is zero, all it means is the line tangent to the graph is horizontal. For example, the graph of $f(x) = x^3$ has a horizontal tangent when $x = 0$:



There must be some way to verify if the critical numbers we get from the first derivative test are truly maxima and minima, right? Yes! If we look at the graph of $f(x) = x^3$ above, we can tell that the tangent line has a positive slope moving towards $x = 0$ and also moving away. In calculus-speak: the first order derivative does not change signs from positive to negative. If we look at the graph of $f(x) = x^2$, we can see that the slope of the tangent is negative moving towards $x = 0$ and positive moving away. This means the first order derivative changes signs. This makes logical sense because for a function to have a local mimimum, it must decrease, hit the lowest point, and increase. We can flip this around, too: for a function to have a local maximum, it must increase, hit the highest point, and come down again. The first order derivative (the slope of the tangent line) must change signs. If it changes from positive to negative, there is a local maximum. If it changes from negative to positive, there is a local mimimum. We can write up this in a table where we have the intervals from the beginning of the values we are checking to the first critical number, then from the first to second critical number, second to third critical number,$\cdots$, and from the last critical number to the end of the values we are testing. "Values we are testing" means the ends of the interval we are working with. Normally the ends are $-\infty$ and ∞, but sometimes you might be constrained to [0,5] or something similar. For example, to find the relative minima and maxima of the function whose derivative is $f'(x) = (x - 1)(x + 1)$, this is the table one would create:

(-∞, -1)	-1	(-1, 1)	1	(1, ∞)
+	0	-	0	+

...and this is the true first derivative test. Because it changes signs from positive to negative at $x = -1$, that point is a local maximum, and because it changes from negative to positive at $x = 1$, that must be a local minimum.