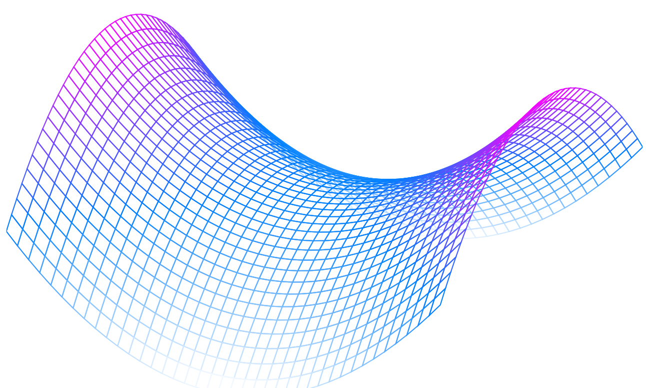
Noah's Guide to Calculus

Noah Stockwell

Summer 2016



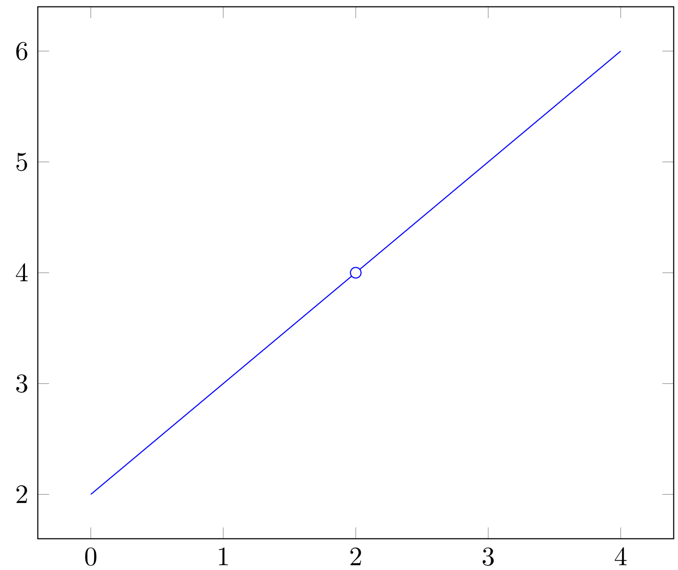
Introduction

Most Calculus textbooks are long and boring with seemingly endless and sometimes meaningless examples. This text is an attempt to solve that problem with comprehensive yet concise explanations for all of the topics covered in the AP Calculus AB exam for the 2016-17 school year. As the text does not provide many examples, most of the topics are presented in an abstract way and the author recommends this text be supplementary to another textbook.

Contents

1 Limits

Limits are a way of skirting the normal rules of math. Without the knowledge of limits, whenever a function divides by 0 or involves ∞ in any way, calculations become impossible. Limits take the rules of math a little less seriously and can be used to calculate what a value "should be". A simple example of where limits come in handy is when there is a "hole" in a graph:



$$f(x) = \frac{x^2 - 4}{x - 2}$$

Because $f(x)$ divides by 0 when $x = 2$, there can be no answer here. However, we can tell that $f(2)$ should be 4 ignoring the division by zero. We can tell this because as x becomes greater and nearer to 2 (approaching $x = 2$ from the left), the value of $f(x)$ approaches 4. Similarly, when x decreases and becomes nearer to $x = 2$ (approaching $x = 2$ from the right), the value of $f(x)$ approaches 4. Therefore, as both sides of $x = 2$ become closer and closer, they converge upon a single point: $f(2) = 4$.

1.1 Types of Limits

Limits are just a way of finding what a graph should be. Looking at the previous example, we can trace the graph as it comes closer to $x = 2$ from the left to make an educated guess and say that $f(2)$ should be about 4. We can write this in special limit notation:

$$\lim_{x \to 2^-} f(x) = 4$$

In English, that means *the limit as x approaches* 2 *from the left is* 4. The - sign is appended to the 2 to show that we are using how the graph is to the left of $x = 2$ to approximate $x = 2$. If we were using the graph to the right to approximate $x = 2$, we would write:

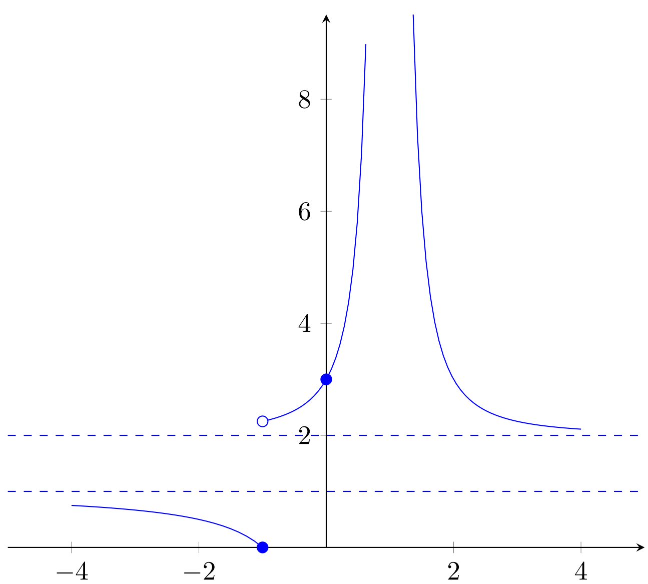
$$\lim_{x \to 2^+} f(x) = 4$$

These are called **one-sided limits**. The one-sided limits from both sides of the graph do not necessarily need to be the same.

The Limit We define $\lim\limits_{x \to c}$ (the value that the graph approaches)to be:

$$\lim_{x \to c} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

which should be read as: *the limit of $f(x)$ as x approaches c* **is equal to** *the limit of $f(x)$ as x approaches c from the right side* **and equal to** *the limit of $f(x)$ as x approaches c from the left side*. The limit from the left side is simply the value that the graph of $f(x)$ approaches as x increases to the limit point. The same is true for the right side: the limit is what the graph approaches as x becomes smaller towards the the limit point. If $\lim\limits_{x \to c^+} f(x) \neq \lim\limits_{x \to c^-} f(x)$, we say that the limit of $f(x)$ at c must not exist.



In the above graph, $\lim\limits_{x \to -1} f(x)$ doesn't exist because $\lim\limits_{x \to -1^+} f(x) = \sqrt{2} + 1$ and $\lim\limits_{x \to -1^-} f(x) = 0$. The graph doesn't need to have a hole in it to have a limit. In fact, **any continuous part of a graph has a limit**. For example, the limit as x approaches 0 exists. The limit is equal to 2. Limits can also be infinite. Because $\lim\limits_{x \to 1^+} f(x)$ and $\lim\limits_{x \to 1^-} f(x)$ are both equal to ∞, $\lim\limits_{x \to 1} f(x) = \infty$. Similarly, we can take the limit at infinity. Because they are asymptotes. $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to\ -\infty} f(x) = 1$. Because we can only approach limits at infinity from one side, we omit the + and - following the value as they are implied. However, this means that **all asymptotes are limits taken at infinity**.

1.2 Directly Calculating Limits

Unfortunately, there's no simple and easy way to calculate limits. The simplest way is just to "plug in" to the function but at some points like holes in the graph or ∞, we don't have that luxury. Instead, we can reduce or rewrite equations and also apply some general common sense. For example, $\lim_{x\to\infty} \frac{1}{x}$ must be 0 because as x gets larger, $\frac{1}{x}$ gets smaller to some point at which it must be 0. Using reduction and logic, we may progress on to more complex ideas involving limits where direct substitution fails.

Rules There are simple rules for limits:

$$\lim_{x\to c} f(x) * g(x) = \lim_{x\to c} f(x) * \lim_{x\to c} g(x)$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

$$\lim_{x\to c} f(x)^n = (\lim_{x\to c} f(x))^n$$

The same is true for function addition, subtraction, etc.

"Plugging In" In the end, all limit problems will need to have their value inserted at some time. For example, the limit $\lim_{x\to 0} x^2 + 1 = 1$. No tricks here, plugging in 0 does return 1. We do not have to deal with any division by zero or infinities, so there is no need to manipulate the problem.

Reduction Reduction is relatively simple. If we go back to our example of

$$f(x) = \frac{x^2 - 4}{x - 2}$$

we can see that this problem can factor into

$$f(x) = \frac{(x-2)(x+2)}{x-2}$$

which cancels and gives us

$$f(x) = x + 2$$

We now see that there does exist a limit at $x = 2$, $f(2) = 2 + 2 = 4$. Previously, directly plugging in 2 would not return a real value.

Sense at ∞ ∞ is a hard concept to grasp and work with in mathematics. Limits can make this easier because while we cannot *directly compute* ∞, we can approximate it exactly. For example, in the case of $f(x) = \frac{1}{x^2}$, we see that the limit at ∞ must be zero. As x increases, $f(x)$ drops ever closer to 0:

$$f(10) = 0.01$$

$$f(100) = 0.0001$$
$$f(1000) = 0.000001$$
$$etc.$$

Similarly, as x approaches ∞,

$$g(x) = \frac{3x}{4x+1}$$

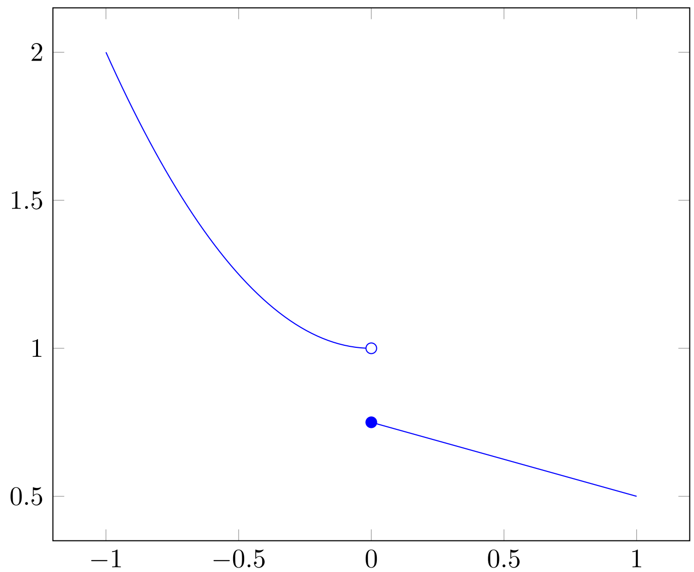
approaches $\frac{3}{4}$:

$$g(10) = \frac{30}{41}$$
$$g(100) = \frac{300}{401}$$
$$g(1000) = \frac{3000}{4001}$$
$$etc.$$

1.3 When Limits Don't Exist

Limits don't exist when:

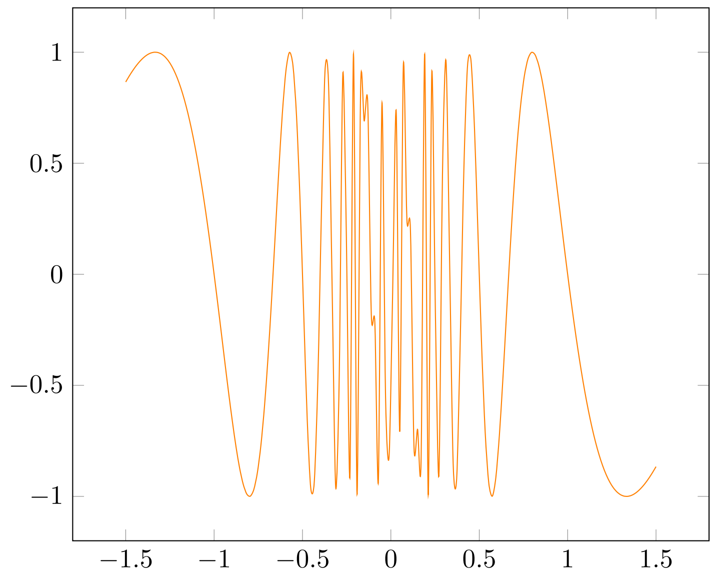
1. $\lim\limits_{x \to c^+} \neq \lim\limits_{x \to c^-}$



$\lim_{x \to 0} f(x)$ Does Not Exist

The reason why the limit cannot exist at 0 here is that when $f(x)$ approaches 0 from the right side (0^-), the limit is 1. As $f(x)$ approaches 0 from the left side (0^+), the limit is 0.75. The limit is one point and because $0.75 \neq 1$, there is no solution.

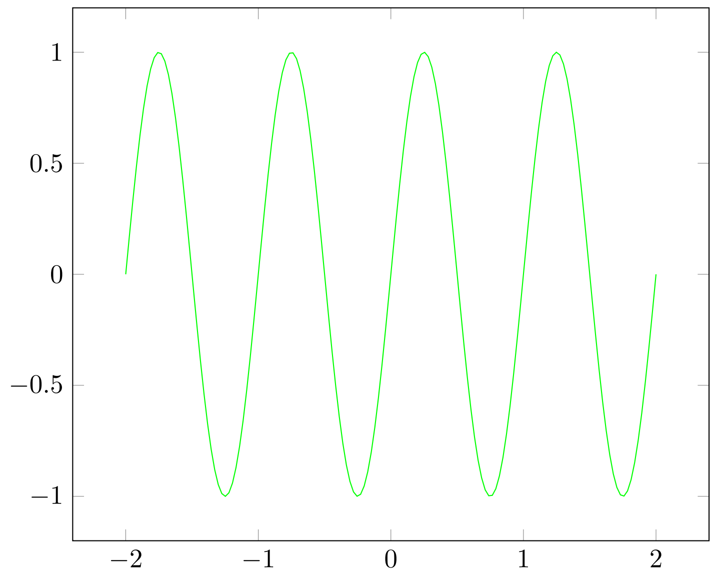
2. The graph oscillates uncontrollably:



$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ Does Not Exist}$$

The graphing utility doesn't even have any idea what's going on. We can't blame it because the graph starts oscillating faster and faster closer to 0. There is no point the graph can approach. Therefore, the limit must not exist here.

3. The graph oscillates at ∞



$\lim_{x \to \infty} \sin(x)$ Does Not Exist

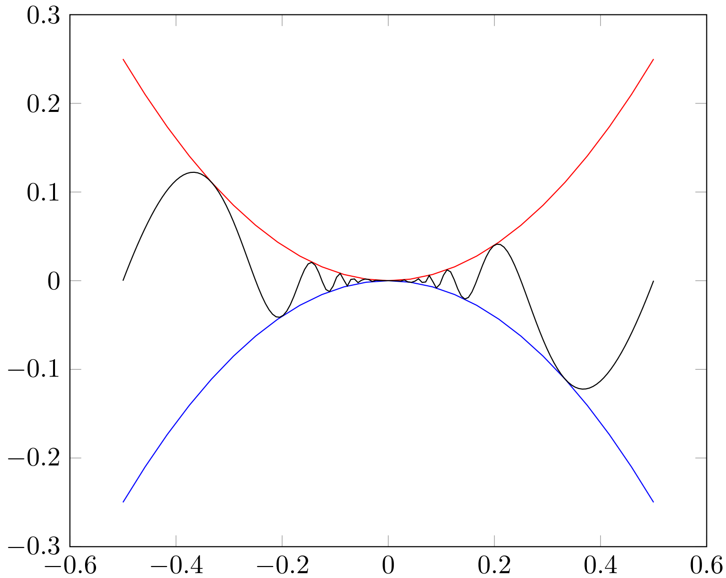
> Is ∞ an integer? Or where does it fall exactly? If you define ∞ to be a value, something can always be added to that value. Therefore, there is no single value that this graph will approach. In fact, it will approach all numbers in the interval [-1,1]. For a periodic graph like $\sin(x)$, there can be no limit because the graph will never stop oscillating.

1.4 The Squeeze Theorem

Sometimes none of the aforementioned ways for calculating limits works. One way to calculate possibly impossible limits is the squeeze theorem:

> On an interval containing a point c, if $f(x) \leq g(x) \leq h(x)$ for all of the interval, and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = a$, then $\lim_{x \to c} g(x) = a$.

The easiest way to see this is a graph:



Here we can see the graph $g(x) = x^2 * \sin(\frac{4}{x})$ being squeezed between $f(x) = x^2$ and $h(x) = -x^2$. $g(x)$ has no limit we can compute directly at 0 because $\frac{1}{x}$ has no real limit. However, it is a simple proof to show that $f(x) \leq g(x) \leq h(x)$ and $f(x)$ and $g(x)$ both have the same, real limit for $x = 0$, so

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = \lim_{x\to 0} g(x) = 0$$

1.5 Relative Magnitudes

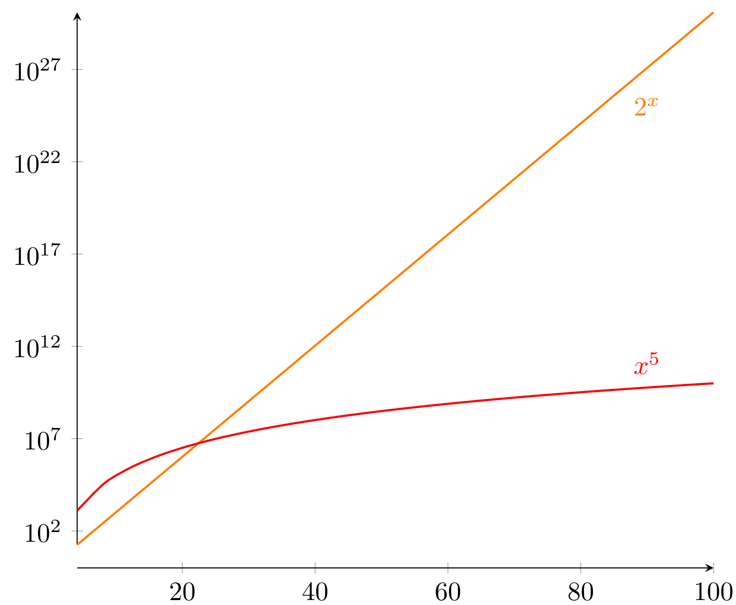
It is beneficial in some cases to calculate the limit at ∞ for some functions. Obviously, in some cases, the limit itself doesn't exist because it is ∞. For example:

$$\lim_{x\to\infty} x^2 = \infty$$

In other cases, the limit is 0. For example:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

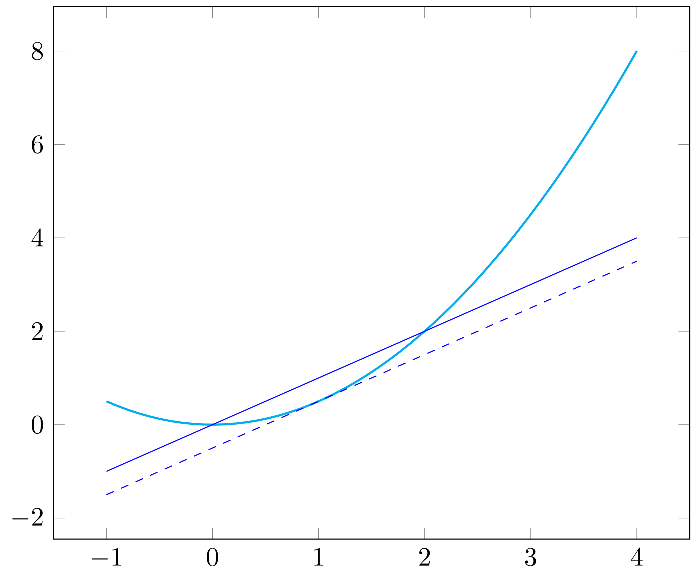
Sometimes it is not so obvious. For fractions, we can compare the rates of increase of functions (or their magnitude) of their numerator and denominator. If the denominator has a greater magnitude, it will become larger faster than the numerator. Therefore, the limit will be zero. If the numerator has a larger magnitude, it will increase faster and the limit will be ∞.



Functions like x^5 will always increase slower than functions like 2^x towards ∞.

2 Derivatives

Slopes It is useful and easy to find the rate of change for any point on a line. It is difficult to approximate how a curved function changes, however. Functions that aren't lines don't have defined slopes and we have to use a tangent line to find the slope at a point. It is difficult to find the tangent line at a point, so we must resort to finding secant approximations for the tangent line.



The secant line $y = x$ approximates the dashed line tangent to the graph $f(x) = \frac{1}{2}x^2$ for the point $x = 1$.

2.1 Difference Quotient

The Slope of a Secant Line The slope of a secant line through a point x and another point h units away will be the rise ($\triangle y$) divided by the run ($\triangle x$). Plugging in for the points $(x, f(x))$ and $(x + h, f(x + h))$, we get

$$\frac{f(x+h) - f(x)}{x + h - x}$$

which solves to

$$\frac{f(x+h) - f(x)}{h}$$

2.2 The Derivative

The derivative is a tool that can find the slope of a tangent line at (almost) any point on a graph. While we can't find the slope between one point and itself because there is no difference, we can find the slope between some point and another that is **really** close to that point. We can minimize the difference between these two points by using a limit! We will find the difference between a point and a point whose position is almost zero units away:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is the derivative! If you were to plug in 0 for h immediately, you would divide by zero. Therefore, you must use the reduction technique for solving equations of this type. The derivative is often abbreviated $\frac{d}{dt}f(t)$ with respect to some variable t and some function f; or simply $f'(x)$. If you take n more derivatives of the same function past the first one, we call it the n^{th} order derivative. Derivatives beyond the first order are labeled $\frac{d^2}{dt^2}f(t)$ or $f''(t)$ for the second order; $\frac{d^3}{dt^3}f(t)$ or $f'''(t)$ for the third order; etc. We say the derivative is taken "with respect to" something else. In a pretty standard Calculus case, the objective is to find the change in *y with respect to x*. We use the notation to describe this relationship as $\frac{dy}{dx}$. Extending this, the second order derivative of *y with respect to x* is denoted $\frac{d^2y}{dx^2}$. When the derivative of a variable is taken with respect to itself, i.e. $\frac{dx}{dx}$, we do not write it as it is equal to 1.

Example For $f(x) = x^2$, the derivative is:

$$\begin{aligned}
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x + 0 \\
&= 2x
\end{aligned} \tag{1}$$

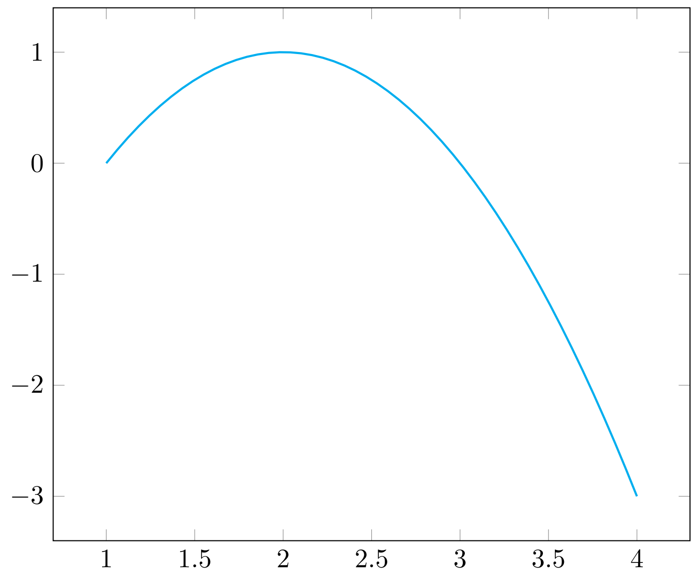
So for any point x on the graph of $f(x) = x^2$, the slope of the tangent line will be $y = 2x$.

2.3 Estimating Derivatives

Estimating derivatives is not alltogether too difficult. Remember that the derivative itself is just the limit of the difference quotient. If one needs to find the derivative, all one has to do is take the difference quotient at two known points to create a secant line. Because it is an estimate, one doesn't need to minimize h.

$$\frac{f(x+h)-f(x)}{h}$$

It is also possible to “eyeball” a graph to estimate the derivative. For example:



For the point (3,f(3)), the tangent line obviously has a negative slope. It is not too drastic so by a guesstimate, the slope must be greater than -3. The slope is also not too flat so it should be less than -1. Therefore, the derivative can be estimated to something around -2.

2.4 Rules

The derivative has patterns, or “forms” as they will be called here. You are expected to know these forms backwards and forwards. Many textbooks include proofs for these but, as it is unnecessary, this text will not. The reader is strongly encouraged to prove these forms as exercises. Some of the most important forms that Calculus students are expected to know are below. The derivative of:

1. c is 0
2. u^n is $(nu^n - 1)\ du$
3. cu is $c * (du)$
4. e^u is $e^u\ du$
5. $\ln(u)$ is $\frac{1}{u}\ du$
6. $\sin(u)$ is $\cos(u)\ du$

7. $\cos(u)$ is $-\sin(u)\ du$

Where u is some equation, c is a constant, and du is the derivative of u. Most other forms can be derived from these.

2.5 Special Rules

There are some more patterns outside of the traditional forms that the reader should know for Calculus. These have to do with composite and combined equations.

The Product Rule Given two equations $f(x)$ and $g(x)$, the derivative of the function $h(x) = f(x) * g(x)$ is:

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

For example, the derivative of $h(x) = x^2 sin(x)$ is:

$$\begin{aligned} h(x) &= f(x) * g(x) \\ h'(x) &= f'(x)g(x) + g'(x)f(x) \\ h'(x) &= 2x * \sin(x) + \cos(x) * x^2 \end{aligned} \tag{2}$$

The Quotient Rule *Avoid this at all costs.* The quotient rule is helpful but it is very, needlessly complex. For some function $h(x)$ that is the quotient of two functions $f(x)$ and $g(x)$: $h(x) = \dfrac{f(x)}{g(x)}$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(f(x))^2}$$

An easy way to remember this god-awful formula is a little song to the tune of *Low Rider* by War:

> Low...d...High...minus High d Low...all...o-ver...the square of what's below
>
> (All...my...friends...know the low rider...the...low...rid-er...is a little higher)

All jokes aside, this is one of the worst things in Calculus. Avoid it at all costs. For example, you could rewrite $\dfrac{3-x}{x^2}$ as $3x^{-2} - x^{-1}$, thereby eliminating all need for the quotient rule.

The Chain Rule The name is confusing. No, this rule doesn't have anything to do chains in the normal sense. Instead, it gives a form for a composite function $h(x) = f(g(x))$. This is required for functions like $\sin(x^2)$ where $\sin(x) = f(x)$ and $x^2 = g(x)$. The chain rule says the derivative of $h(x) = f(g(x))$ is:

$$h'(x) = f'(g(x)) * g'(x)$$

In our previous example, the derivative of $\sin(x^2)$ is

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x)) * g'(x) \\ h'(x) &= \cos(x^2) * 2x \\ h'(x) &= 2x * \cos(x^2) \end{aligned} \tag{3}$$

2.6 Implicit Differentiation

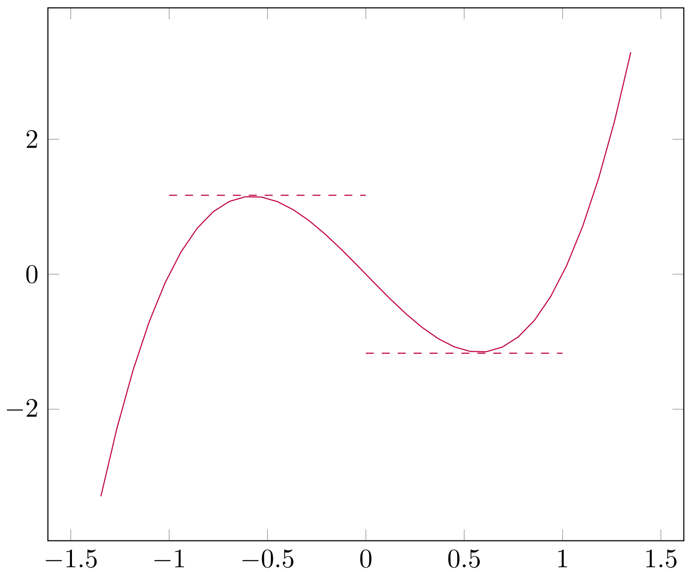
In some cases, it does not make sense to find the derivative with respect to one specific variable in the equation. Perhaps there is another equation that may go with the one you are working with and has different variables. There may also be an advantage if you take the derivative with respect to some parameter, usually t. I like to think that t is sort of like time; you can increase it or decrease it and see how x and y change with respect to this parameter. It is quite simple: all one has to do is take the derivative with respect to this parameter: $\frac{d}{dt}$. For example:

$$\begin{aligned} &\frac{d}{dt}(y = x^2 - 5x + 1) \\ &\frac{dy}{dt} = 2x\frac{dx}{dt} - 5\frac{dx}{dt} \\ &\frac{dy}{dt} = (2x - 5)\frac{dx}{dt} \end{aligned} \tag{4}$$

Note: while the derivative with respect to x normally ends up being $\frac{dx}{dx} = 1$, this is not the case with implicit differentiation.

2.7 Maxima and Minima

For any graph, the local minima and maxima will occur when the graph levels off. To put this in Calculus words, the derivative at the point must be 0 because the tangent line is horizontal.

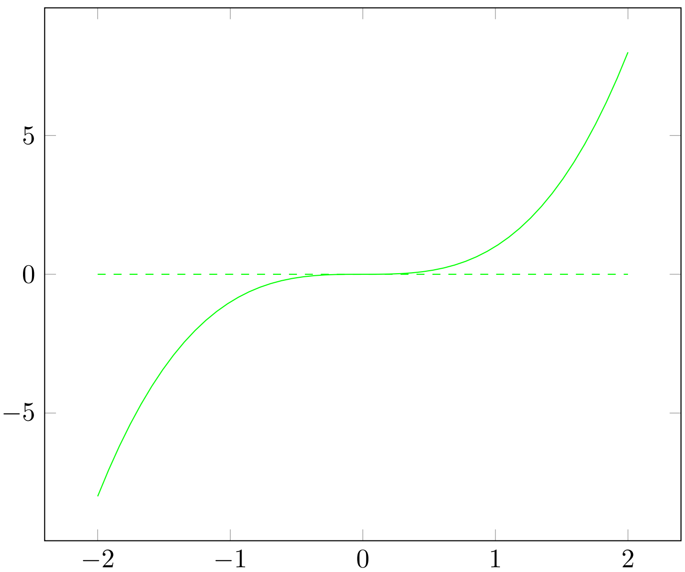


This graph has local maxima and minima where the tangent line is horizontal.

Armed with the knowledge of the derivative, we can exactly calculate at what point(s) any function has a local maximum or minimum. It is actually very simple, all one has to do is to find at what points the derivative is 0. We call this *The First Derivative Test*. The first derivative test gives us *critical numbers*. For example, let's find the points at which $f(x) = \dfrac{x^3}{3} - x$ has a local maximum or mimimum:

$$\begin{aligned} f'(x) &= 3\frac{x^2}{3} - x^0 \\ f'(x) &= x^2 - 1 \\ 0 &= x^2 - 1 \\ 0 &= (x-1)(x+1) \\ x &= \pm 1 \end{aligned} \tag{5}$$

I lied. When the derivative is zero, all it means is the line tangent to the graph is horizontal. For example, the graph of $f(x) = x^3$ has a horizontal tangent when $x = 0$:



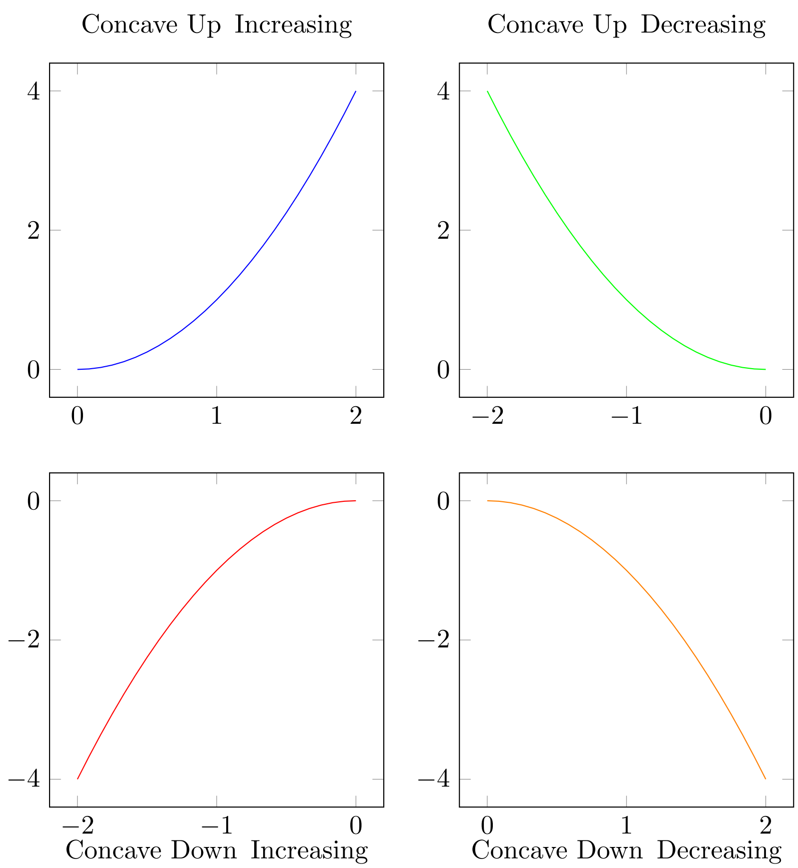
There must be some way to verify if the critical numbers we get from the first derivative test are truly maxima and minima, right? Yes! If we look at the graph of $f(x) = x^3$ above, we can tell that the tangent line has a positive slope moving towards $x = 0$ and also moving away. In calculus-speak: the first order derivative does not change signs from positive to negative. If we look at the graph of $f(x) = x^2$, we can see that the slope of the tangent is negative moving towards $x = 0$ and positive moving away. This means the first order derivative changes signs. This makes logical sense because for a function to have a local mimimum, it must decrease, hit the lowest point, and increase. We can flip this around, too: for a function to have a local maximum, it must increase, hit the highest point, and come down again. The first order derivative (the slope of the tangent line) must change signs. If it changes from positive to negative, there is a local maximum. If it changes from negative to positive, there is a local mimimum. We can write up this in a table where we have the intervals from the beginning of the values we are checking to the first critical number, then from the first to second critical number, second to third critical number,$\cdots$, and from the last critical number to the end of the values we are testing. "Values we are testing" means the ends of the interval we are working with. Normally the ends are $-\infty$ and ∞, but sometimes you might be constrained to [0,5] or something similar. For example, to find the relative minima and maxima of the function whose derivative is $f'(x) = (x-1)(x+1)$, this is the table one would create:

(-∞, -1)	-1	(-1, 1)	1	(1, ∞)
+	0	-	0	+

...and this is the true first derivative test. Because it changes signs from positive to negative at $x = -1$, that point is a local maximum, and because it changes from negative to positive at $x = 1$, that must be a local minimum.

2.8 Concavity

Recall that the second derivative gives the rate of change of the first derivative. Because the first derivative gives the slope of the tangent line, the second derivative will give the change in slope. If the second derivative is positive, the tangent line must be increasing and therefore moving up (counterclockwise). If the second derivative is negative, the tangent line must be decreasing and therefore moving down (clockwise). When the slope is changed, a curve is created. We use two words to quantify the curve that we see: **concave up** and **concave down**. Concave up is, as you may have guessed, when the graph is in a bucket shape facing upwards (think of the $y = x^2$ graph). Concave down is the opposite, where the graph looks more like a hat facing downwards (think of $y = -x^2$). We also append *increasing* or *decreasing* to the description of the function at a point to describe the sign of the first derivative, in other words whether the points are going up or going down moving right.

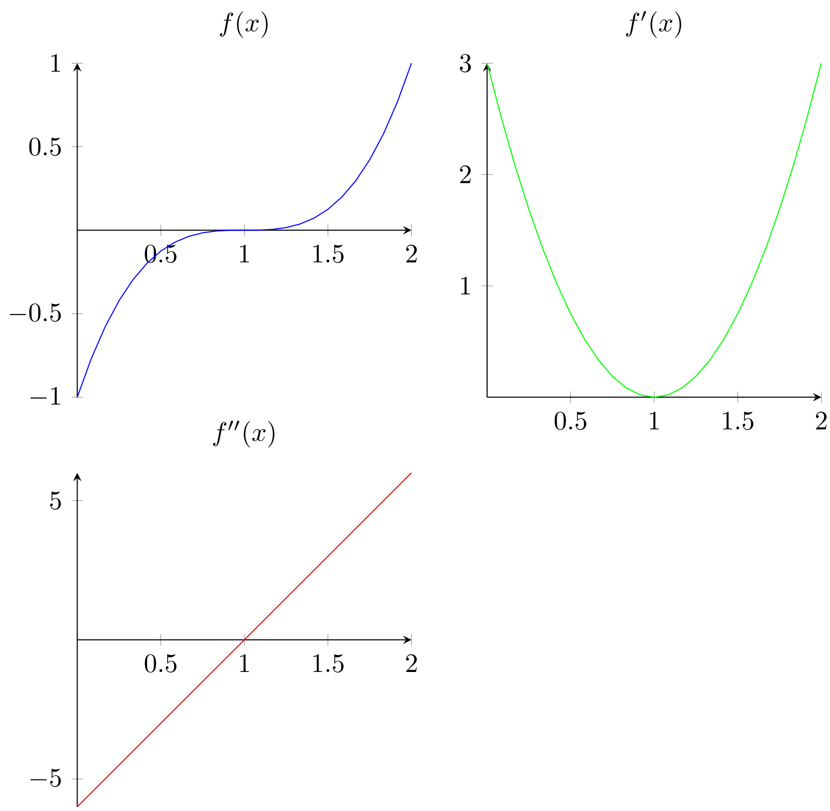


2.9 Graphical Relations

Functions and their derivatives are connected. If the derivative is positive, the function is increasing. If the derivative is negative, the function is decreasing. If the second derivative is positive, the function is accelerating in the positive direction. One of the many things a Calculus student is expected to do is identify the relation between the graphs of the derivatives of a function and the function itself. Let $f(x) = (x-1)^3$:

$$
\begin{aligned}
f(x) &= (x^2 - 2x + 1)(x - 1) \\
f(x) &= x^3 - 2x^2 + x - x^2 + 2x - 1 \\
f(x) &= x^3 - 3x^2 + 3x - 1 \\
f'(x) &= 3x^2 - 6x + 3 \\
f''(x) &= 6x - 6
\end{aligned} \tag{6}
$$

Graphing the functions, we get:



$f'(x)$ has a zero at $x = 1$ but because it doesn't change signs, $f(x)$ cannot have a local maximum or minimum there. Because $f'(x)$ is always positive, the function is always increasing. $f''(x)$ is negative before $x = 1$ and positive after, so at that point, $f(x)$ switches from concave down to concave up. This sums up the relationship between derivatives and their graphs.

2.10 Relationship with Continuity

Continuity does not guarantee that the derivative will always exist. Even though a function like

$$f(x) = \begin{cases} x & x \leq 1 \\ x^2 & x > 1 \end{cases}$$

is continuous everywhere, its derivative is not. The derivative is a limit, so for it to exist, it must be the same on both sides. If the limit of the difference quotient (the derivative) is calculated at $x = 1$, the results are:

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} 1 = 1$$

$$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} 2x = 2$$

which are not equal. The derivative must not exist for all values of x. However, if the derivative of a function is continuous everywhere, that means that the function itself must be differentiable everywhere. Therefore, functions that do not result in a piecewise derivative and have a domain of $(-\infty, \infty)$ must be differentiable everywhere. Functions that are differentiable and expected to be calculated by Calculus students are::

1. Polynomials $(x^n + x^{n-1} + ...)$
2. Power functions (ax^b)
3. Sin(x)
4. Cos(x)
5. Exponential functions (ba^x)
6. Logarithmic functions

3 Applications of Derivatives

3.1 Units

Functions can be used to model real-world phenomena and find use in everything from quantum physics to music[1]. However, in the real world, there are units of measure that are significant and go hand-in-hand with values. Think of *miles*: a unit of distance, and *hours*: a unit of time. We can combine these together to show a relationship between the two with *miles per hour*: a unit of speed. Because derivatives give rates of change, the units no longer stay the same. For example, if we wanted to find the rate of change of speed, we would look for how the speed changes with relation to time. Our units then would be $\frac{miles}{hour} \div hour$ or miles per hour per hour. We rewrite this as $\frac{miles}{hour^2}$. This gives us the rate of change of speed, or acceleration. If we wanted to find how the area of a circle in square meters changes when we change the radius in meters, the units would be $\frac{meters^2}{meters} = meters$.

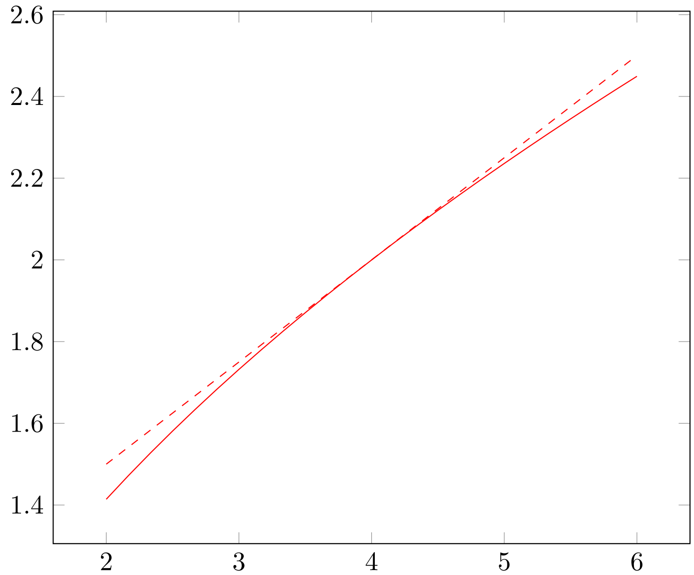
3.2 Instantaneous Rate of Change

This is perhaps the most classic Calculus derivative problem. When you are riding in a car, the speedometer tells you how fast you are going at any point in time. This is your instantaneous rate of change, how fast you are moving at one point in time. You could be travelling at 40 miles per hour but then have to stop at a stoplight for a little bit. Your average speed would be less than 40 because of that time you stopped, but you know that at that certain time you measured it, you were travelling 40 miles per hour. Problems will ask you, given a function, how fast is something changing at an exact point in time. All it is asking is for the slope of the tangent line (derivative) and not the average slope at that point. Simply calculate the derivative.

3.3 Tangent Line Approximations

The tangent line can be used as an approximations for points near a point on a graph. Take this example of the graph $f(x) = \sqrt{x}$:

[1] Just ask Bach...



The tangent line for $f(4)$ is $y - 2 = .25(x - 4)$ and approximates the graph pretty well for the near area. Because $\sqrt{5}$ would be incredibly difficult to find by hand, we can use the tangent line to approximate the value.

$$f(5) \approx .25(5 - 4) + 2$$

which is 2.25. The actual value is about 2.236. The approximation worked pretty well in this case with an error of about 0.014, or less than 1% of the actual value.

3.4 Physics

It has come up a bit before, but now it is time to formalize some theory. Because the derivative finds the rate of change for something at a certain point, it is useful in physics. For example, the derivative of a position function gives the rate of change of position of that point, or *velocity*. Similarily, the change in velocity at a point is *acceleration*. The change in acceleration at a point does get a special name in physics, too, but it is not used as much: *the jerk*. Given $x(t)$ is position, $v(t)$ is velocity, and $a(t)$ is acceleration with respect to time, the following are true:

$$x'(t) = v(t)$$
$$x''(t) = v'(t) = a(t)$$

I did not include the jerk in here because it is so very uncommon.

3.5 Related Rates

Derivatives find the rate of change of something with respect to something else. Two or more derivatives can be combined, as a result, to find how one value affects another. For example, if we know the the radius of a circle at any given time is $\ln(t)$, we know the rate of change musb be $\frac{1}{t}$, which we can rewrite as $\frac{dr}{dt} = \frac{1}{t}$. If we wanted to find how the area of a circle changes with respect to the radius, all we need to do is take the derivative.

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

substituting for r and $\frac{dr}{dt}$, we get

$$\frac{dA}{dt} = 2\pi \ln(t) * \frac{1}{t}$$

which means that for any given time t, the rate of change of the area A must be

$$\frac{2\ln(t)}{t}$$

Is it not super cool that the derivative of the area of a circle is the circumfrence?

3.6 Optimization

Using knowledge of how maxima and minima work, it becomes easy to "optimize" functions. If there is a formula for something like cash input and return, you can "optimize" the function by finding a maximum or minimum. This may be combined with the ideas behind related rates, too. Given two constraints, what is the maximum value of some function? It is simple if you use the derivative. Find the points at which the derivative of your target function is 0 and evaluate the function at those points **and** the ends of the interval you are working with.

3.7 Differential Equations

The reader should be familiar with $\frac{dy}{dx}$ at this point. Remember that what that means is the instantaneous change in y with respect to the instantaneous change in x. This can be rewritten and solved for dy as a function of dx. It's just as simple as getting $\frac{dy}{dx}$ on one side and multiplying by dx so the equation looks like:

$$\frac{dy}{dx} = f'(x)$$

$$dy = f'(x)dx$$

Given the equation $x^2 + y^2 = 1$, the differential equation will be:

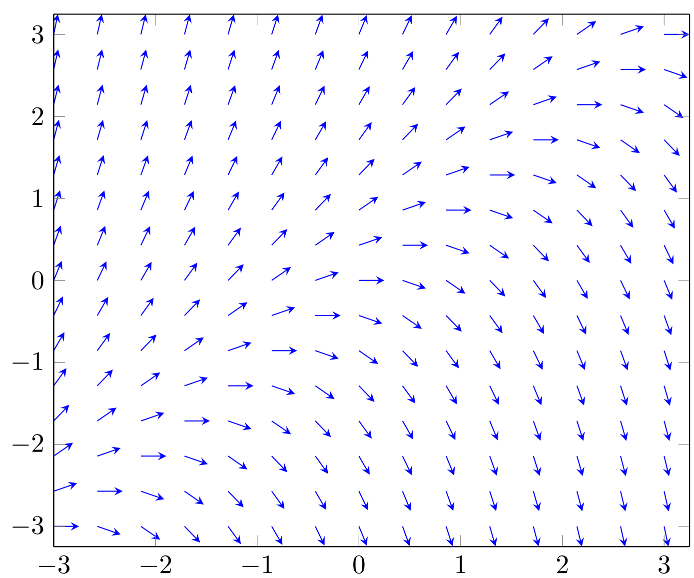
$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$dy = -\frac{x}{y}dx$$

3.8 Slope Fields

Slope fields are (in Noah's opinion) one of the most fun things in Calculus. Given a graph, the differential equation $\frac{dy}{dx}$ is solved for the (x, y) coordinate. The resulting solution is the slope of the graph **should** the graph actually pass through that point. This step is repeated for a certain number of points on the graph. All the work that has to be done is plug in (x, y) to the equation and draw a small line with the same slope at that point. Here's an example slope field for $y^2 + 2y = x^2$:



Note: arrow heads are normally not drawn.

3.9 Mean Value Theorem

The mean value theorem is relatively simple. If the reader would like to find a proof for the theorem, they are readily available on the internet.

> Given a closed interval $[a,b]$, if a function $f(x)$ is always continuous, there exists a point $a \leq c \leq b$ such that the derivative at c is equal to the slope of the secant line through a and b.

In other words, there exists some c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

3.10 L'hôpital's Rule

Taking the limit of a function with a fraction sometimes ends up with a confusing result like $\frac{\infty}{\infty}$ or $\frac{0}{0}$. Neither of these actually have a real value. L'hôpital's rule is:

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

if and only if $\lim f(x) = \lim g(x)$ and $\lim f(x)$ is either $\pm\infty$ or 0. What this means is if, when taking the limit, you get one of these "indeterminate forms", you may take the derivative of the top and the bottom **separately** and then re-evaluate the limit. You may do this as many times as you need until you no longer reach an indeterminate form.

4 Antiderivatives and Definite Integrals

4.1 Antiderivatives

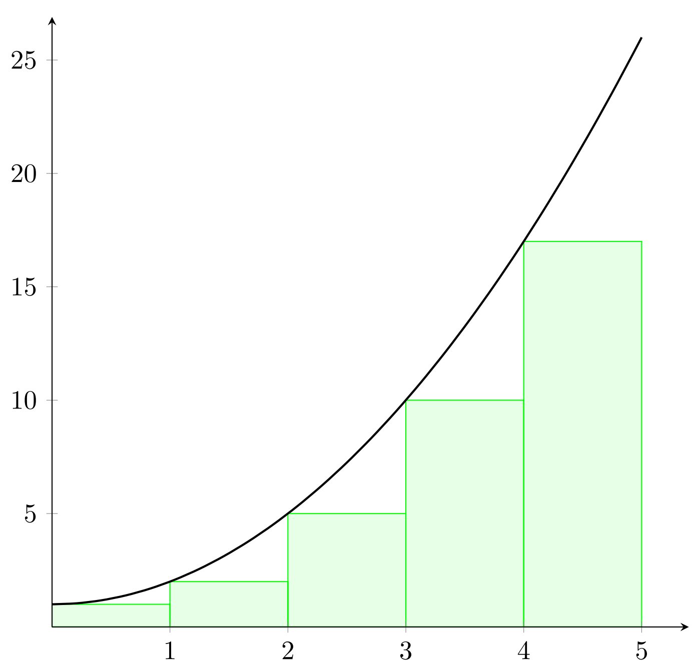
So far we have worked with derivatives. Because the derivative is a function, it should have an inverse. It is obvious now that the derivative of x^2 is $2x$, so we should say the inverse of the derivative (or antiderivative) of $2x$ is x^2. This doesn't exactly work out as x^2, $x^2 - 1$, and $x^2 + \pi$ all share $2x$ as their derivative. Instead, we say that the antiderivative of $2x$ is $x^2 + C$ where C is any constant. It should be noted that a function which is created by a derivative will have an infinite number of antiderivatives but only one derivative itself. Later in this section, you will read about ways to solve specific antiderivative problems.

4.2 Riemann Sums

Moving onto a related topic, it can be beneficial to calculate the area underneath a graph of some function. For example, the area underneath a graph between two points a and b can be used to find the average value of a graph. We can't exactly find the area underneath most graphs because there are no formulas for the area of an area bounded by, say, $f(x) = 2x\sin(x^2) + 5$. However, we can approximate the area using different methods. Four that will be covered on the AP test are:

1. **Left Riemann Sum**

$f(x) = x^2 + 1$



The Left Riemann sum divides an interval from a to b into n subintervals and adds together rectangles stretching up from the x-axis until the *left* corner touches the graph. In this case, [0,5] is divided up into 5 subintervals with width 1 each. The heights for the rectangles are $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$ because they are the left endpoints. The sum of n rectangles is:

$$(width_1 * height_1) + (width_2 * height_2) + \cdots + (width_n * height_n)$$

In Riemann sums, the widths are all equal, so the sum siplifies to:

$$width * (height_1 + height_2 + \cdots + height_n)$$

which in this case is:

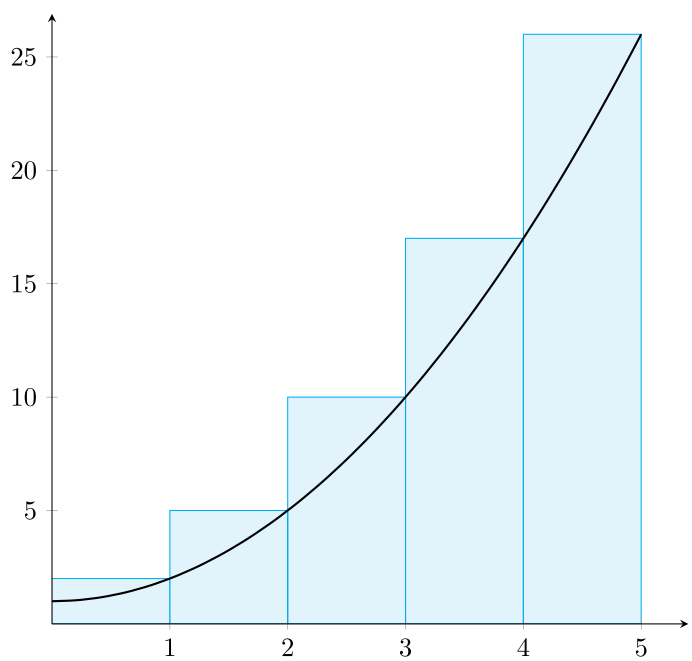
$$1 * (f(0) + f(1) + f(2) + f(3) + f(4))$$

$$= 1 + 2 + 5 + 10 + 17$$

$$= 35$$

2. **Right Riemann Sum**

$$f(x) = x^2 + 1$$

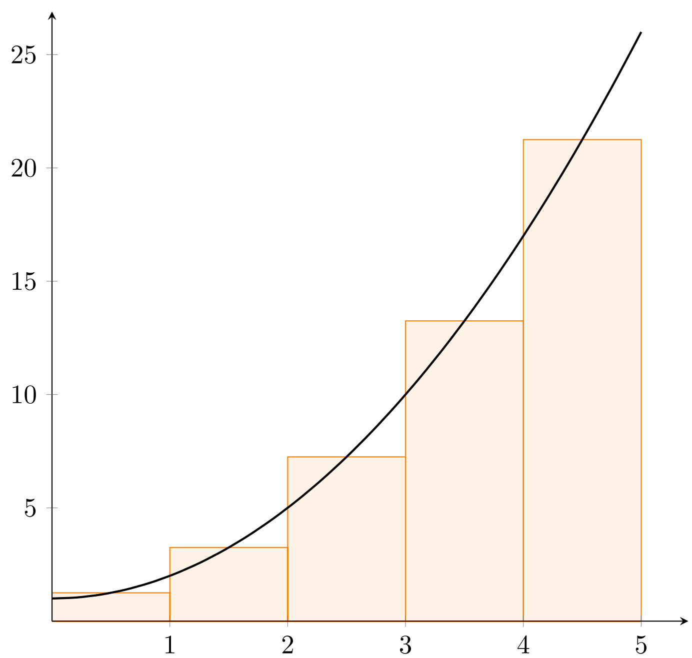


The Right Riemann sum is identical to the Left Riemann Sum except that it evaluates the function at the *right* side. In this case, the approximation is:

$$1 * (f(1) + f(2) + f(3) + f(4) + f(5))$$
$$= 2 + 5 + 10 + 17 + 26$$
$$= 60$$

3. **Midpoint Riemann Sum**

$$f(x) = x^2 + 1$$



Just like the other two, the only difference is that the Midpoint Riemann Sum calculates at the midpoint of the subintervals, so the approximation is:

$$1 * (f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5))$$
$$= 1.125 + 3.25 + 7.25 + 13.25 + 21.25$$
$$= 46.125$$

4. **Trapezoidal Sum** The trapezoid sum has the potential to have the least error of any of the previous 3 methods. Instead of summing rectangles, it sums trapezoids that look identical to the rectangles but instead of being a rectangle on the top, the top is bounded by a line that starts at $(a, f(a))$ and ends at $(b, f(b))$ instead of being a horizontal line at one of

those points. The area of a trapezoid is $\frac{h}{2}(base_1 + base_2)$. In this case, the bases are vertical and the width is the total interval divided into n subintervals. Therefore, the total area for one trapezoid from $x = a$ to $x = b$ would be:

$$\frac{width}{2}(f(a) + f(b))$$

If we were to add up all of these rectangles over an interval [c,d] divided into n subintervals, the area would be:

$$\frac{\frac{d-c}{n}}{2}(f(c)+f(c+\frac{d-c}{n})+\frac{\frac{d-c}{n}}{2}(f(\frac{d-c}{n})+f(c+2\frac{d-c}{n})+\cdots+\frac{\frac{d-c}{n}}{2}(f(d-\frac{d-c}{n})+f(d)$$

$$= \frac{d-c}{2n}\left(f(c) + f(c+2\frac{d-c}{n}) + 2f(c+2\frac{d-c}{2}) + \cdots + 2f(d - \frac{d-c}{n}) + f(d)\right)$$

If we were to abbreviate the width $\frac{d-c}{n}$ as i, it simplifies down to:

$$\frac{i}{2}\left(f(c) + 2f(c+i) + 2f(c+2i) + \cdots + 2f(d-i) + f(d)\right)$$

The trapezoid approximation is somewhat complex yet elegant at the same time.

4.3 Limits of Riemann Sums

To reduce the amount of error in our Riemann Sums, one could increase the total number of rectangles in the interval. 2,000 rectangles should be a closer approximation than 5 (generally). As we increase the number of rectangles, the width is reduced and the rectangles become closer to the graph. One way to minimize the total error is to maximize the number of rectangles. We can't directly compute ∞ rectangles but we can take the limit at ∞ subintervals. Turning the additions of the heights into a summation, we can express this as:

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + i\frac{b-a}{n})$$

Because we are dividing the interval [a,b] into n subintervals and multiplying the height (function at each point) by the width (which is constant for all rectangles) and adding all of the rectangles together. The function $f(x)$ is evaluated at the first point $x =$ starting point + (how many rectangles we have already counted) * width, or $x = a + i\frac{b-a}{n}$. We end up needing to find this sum quite often, and as it is bulky to read, we use shorthand. We write a fancy s that stands for *sum* like this: $\int$ and place our starting and ending points on the s like this: $\int_a^b$. We also abbreviate the width $\frac{b-a}{n}$ as dx usually. It is *d-some variable* always, and the variable we are moving along is usually x because we are dividing up the $x-$axis. We can write our sum then as height * width or:

$$\int_a^b f(x) * dx$$

which is more commonly written as

$$\int_a^b f(x)dx$$

We call this **the integral**. We read the last equation as *the definite integral from a to b of f(x) dx.* The reason why we choose dx to represent our width is because dx is the infintesimal change in x, similar to the infintesimal change in x of the derivative. Because the derivative uses dx to represent this change, we use the same notation to represent the change in our integral.

4.4 Rules of Integrals

One could try to find the integral the long summation way, however it is just simpler and time-reducing to memorize rules similar to the derivative rules.

1. The integral of $f(x) + g(x)$ is equal to the integral of $f(x)$ + the integral of $g(x)$ evaluated from a to b
2. The integral of $c * f(x)$ for some constant c is equal to c * the integral of $f(x)$ evaluated from a to b
3. The integral of dx is $b - a$
4. The integral of x^n is $\dfrac{x^n + 1}{n + 1}$ evaluated from a to b
5. The integral of e^u is e^u evaluated from a to b
6. The integral of $\dfrac{1}{x}$ is $\ln(x)$ evaluated from a to b
7. The integral of $\sin(x)$ is $-\cos(x)$ evaluated from a to b
8. The integral of $\cos(x)$ is $\sin(x)$ evaluated from a to b
9. The integral of $f(x)$ from a to b is equal to the integral of $-f(x)$ from b to a

A function $f(x)$ evaluated from a to b is the same as $f(b) - f(a)$. The integral has some curious properties. However, it only returns a real number and not a function like the derivative because all it does is calculate the area. The properties are eerily familiar though...

4.5 Average Value

This topic is simple enough. The integral calculates the total area between points a and b. The arithmetic mean (average) of anything is $\frac{\text{total amount of things}}{\text{total number of things}}$. In this case, the total amount of things is the total area (the integral) and the total number of things is $b - a$, so the average value of the graph of $f(x)$ is:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

4.6 Integrals Generating Functions

The integral finds the area underneath a graph between two points. Those two points do not necessarily have to be constants, and one can find this integral easily:

$$\int_0^t \cos(x)dx$$

where t is a variable. The answer is $\sin(t) - \sin(0) = \sin(t)$. This is interesting, especially because for any point t on the graph, the area underneath the curve of $\cos(x)$ between $x = 0$ and $x = t$ will always be $\sin(t)$.

4.7 Calculating Area

We have already established that the integral finds area. It can be used for more than just $f(x)$ graphs, though. For example, if we wanted to find the area of the graph of $f(t) = 2\pi t$ (the circumfrence of a circle with radius t) for some radius r, the integral evaluates to:

$$\int_0^r 2\pi t dt$$

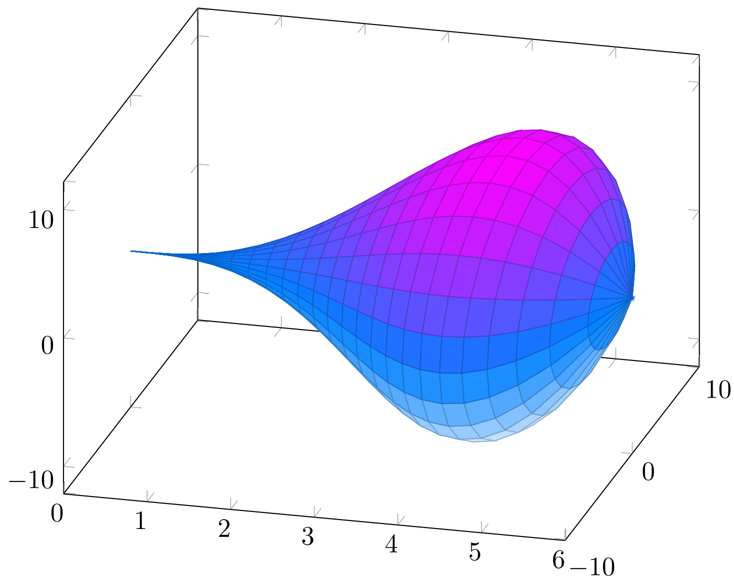
$$= 2\pi \int_0^r t dt$$

$$= \pi r^2$$

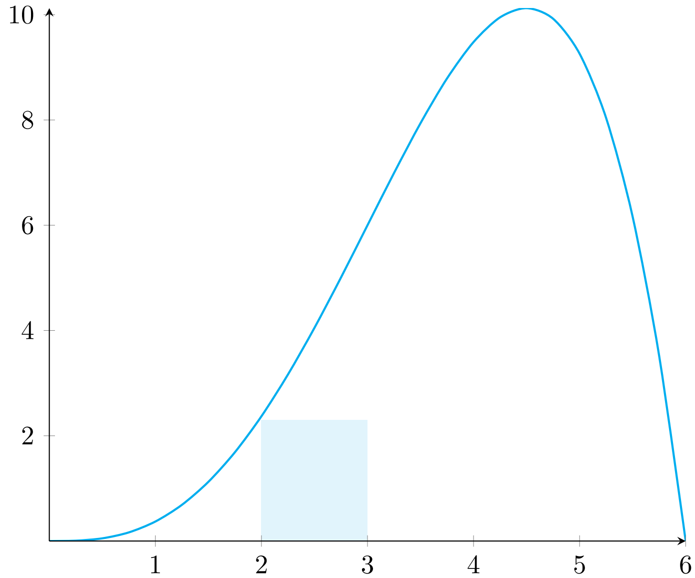
Interesting, right? Anyways, you may ask why we start at 0. The answer is that it's convenient. We want to find the difference in area between some t and, in this case, a circle with no area. That is why we use 0. It may be adventageous in other calculations to not start at 0, to find the area underneath the graph between some variable and some known value.

4.8 Disc Method

The disc method is a way to calculate the volume (yes volume, not area) of a cylindrical object. We do not know the volume formula for an odd graph like this:



But we can try and approximate it. Just like how we can use rectangles to approximate 2-d area, we can use cylinders to approximate cylindrical shapes. If we can find the two-dimensional cross-section of a revolved shape, we can calculate the volume of the revolved shape by taking an infinite sum of infinitely small discs knowing their radius (height in 2-dimensions). This means that we can find a Riemann-style rectangle in a cross-section like this:



...and revolve it around the x-axis to create a cylinder. We can revolve infinitely small cylinders over the axis just like taking the integral. In fact, the volume of a cylinder is just $r^2 * \pi *$ height. Because we have the rectangle, we know $r = f(x)$ and height$=dx$, so we end up with a formula for revolving a cross-section about the x-axis:

$$\pi \int_a^b f(x)^2 dx$$

And if it is hollow (there is a smaller function $g(x)$ that forms the inside of the cross section, the area is

$$\pi \int_a^b f(x)^2 dx - \pi \int_a^b g(x)^2 dx$$

$$= \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

An easier way to remember this is big radius R minus small radius r:

$$\pi \int_a^b \left(R^2 - r^2 \right) dx$$

4.9 Shell Method

5 The Fundamental Theorem

5.1 Introduction

5.2 Consequences

6 Applications of the Funadamental Theorem

6.1 Differential Equations

6.2 Accumulation and Net Change

6.3 Physics