Example 1: As part of a startup that sells Jell-O® through the internet, you are tasked with finding the dimensions for a new box for shipping. You are told that the length must be three times the width. The bottom and top parts of the box cost 2 per square foot and the sides cost 5 per square foot. You must design the cheapest box that can hold 30 cubic feet of Jell-O®.

First, let's define the function that we want to optimize. The objective is to reduce cost, so we want to find the minimum of the cost function. Given length ℓ, width w, and height h, our cost function C is:

$$C = 2 * (w * h) * 5 + 2 * (\ell * h) * 5 + 2 * (w * \ell) * 2$$

Some other information we have: $\ell = 3w$, and $\ell * w * h = 30$. Rewriting the cost function, we have:

$$C = 10 * (w * h) + 10 * (3w * h) + 4 * (w * 3w)$$

$$C = 40wh + 12w^2$$

We can also solve our volume function:

$$3w * w * h = 30$$

$$h = \frac{10}{w^2}$$

And substitute that back into our cost function:

$$C = \frac{400}{w} + 10w^2$$

So now we have a function that combines the cost and constraint information that we were given! Our goal is to minimize cost, so we will take the derivative and find the critical numbers:

$$C(w) = 10w^2 + 400w^{-1}$$

$$C'(w) = 20w - 400w^{-2}$$

$$0 = 20w - 400w^{-2}$$

$$0 = w - 20w^{-2}$$

$$20w^{-2} = w$$

$$20 = w^3$$

$$w = 0?, \sqrt[3]{20}$$

A box with width 0 would not only be silly but impossible without infinite material. We can throw that one out immediately. As good practice, though, we should make sure $\sqrt[3]{20}$ is in fact a minimum. We can use the second derivative test to do so:

$$C'(w) = 20w - 400w^{-2}$$
$$C''(w) = 20 + 800w^{-3}$$

0	$(0, \sqrt[3]{20})$	$\sqrt[3]{20}$	$(\sqrt[3]{20}, \infty)$
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So because there is no change in sign at the second derivative at $w = \sqrt[3]{20}$, it must not be a point of inflection, and because it's positive, it must be concave up. This means that our function must have a minimum at this point. We now know w but we also need h and ℓ, so we're not entirely done yet.

$$\ell = 3w$$
$$\ell = 3\sqrt[3]{20}$$
$$\ell * w * h = 30$$
$$3 * \sqrt[3]{20} * \sqrt[3]{20} * h = 30$$
$$h = \frac{30}{3\sqrt[3]{400}}$$
$$h = \frac{10}{\sqrt[3]{400}}$$

And now we have our dimensions!