Inquiry Based Vector Calculus

About the Document

This document was originally designed in the spring of 2015 to guide students through an eleven week Linear Algebra course (Math 211, Linear Algebra for Scientists) at the University of Victoria. The order of topics closely follows that in *Linear Algebra for Science and Engineering* second edition by Daniel Norman and Dan Wolczuk.

A typical class day using the problem-sets:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

4. **Repeat step 2.**

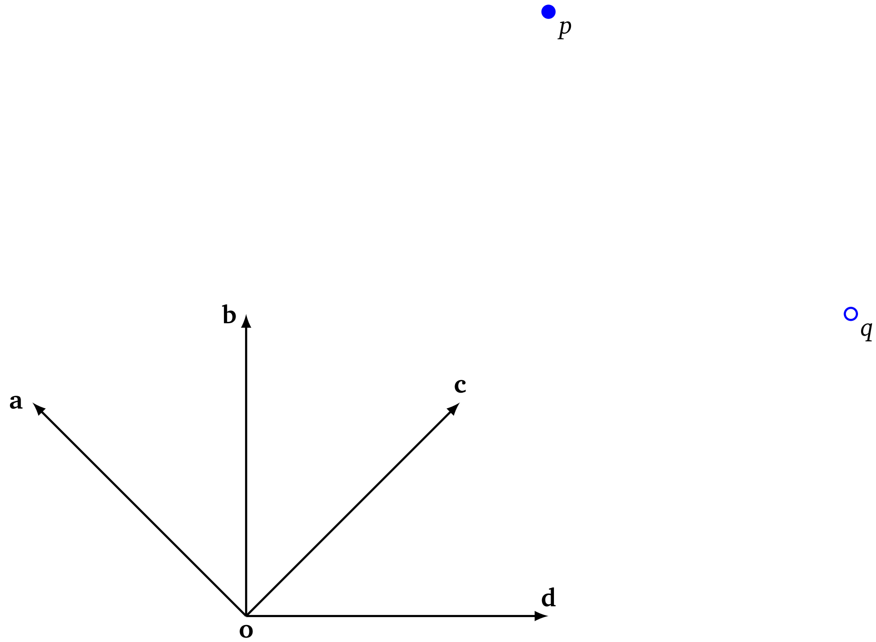
Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

These problem-sets strike a balance between concepts and computation, leaning towards the conceptual side. Two algorithms, *row reduction* and *determinant of a matrix by cofactors*, are not introduced in these problem-sets. Instead, students are expected to learn them on their own and be prepared to apply them on problems in the problem-set (these topics were left up to the students because of time constraints). Further, not ever Linear Algebra definition is given in these problem-sets, since these notes are not intended to replace a Linear Algebra textbook, but most definitions are given to expedite the transition to new topics in the middle of class time.

If you modify this document, you may add your name to the copyright list. Also, if you think your contributions would be helpful to others, consider making a pull requestion, or opening an *issue* at https://github.com/siefkenj/IBLLinearAlgebra

Vectors

1



Notice that all arrows in this diagram are the same length. We will call this length a *unit*.

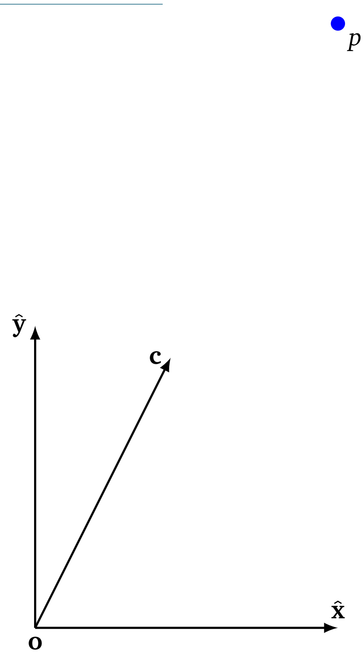
1.1 Give directions from $\mathbf{o}$ to p of the form “Walk ____units in the direction of arrow ____, then walk ____units in the direction of arrow ____.”

1.2 Can you give directions with the two arrows you haven’t used? Give such directions, or explain why it cannot be done.

1.3 Give directions from $\mathbf{o}$ to q.

1.4 Can you give directions from $\mathbf{o}$ to q using $\mathbf{c}$ and $\mathbf{a}$? Give such directions, or explain why it cannot be done.

2



We are going to start using a more mathematical notation for giving directions. Our directions will now look like

$$p = \_\_\_\_\hat{\mathbf{x}} + \_\_\_\_\hat{\mathbf{y}}$$

which is read as “To get to p (=) go ____units in the direction $\hat{\mathbf{x}}$ then (+) go ____units in the direction $\hat{\mathbf{y}}$.”

2.1 What is the difference between $p = \_\_\_\_\hat{\mathbf{x}} + \_\_\_\_\hat{\mathbf{y}}$ and $p = \_\_\_\_\hat{\mathbf{y}} + \_\_\_\_\hat{\mathbf{x}}$? Can they both give valid directions?

2.2 (a) Give directions to p using the new notation.
(b) Give directions to p using $\mathbf{c}$.
(c) What is the distance from $\mathbf{o}$ to p in units?

2.3 (a) $r = 1\mathbf{c}$. Give directions from $\mathbf{o}$ to r using $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.
(b) What is the distance from $\mathbf{o}$ to r?

2.4 (a) $q = -2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$; find the exact distance from $\mathbf{o}$ to q.
(b) $s = 2\hat{\mathbf{x}} + \mathbf{c}$; find the exact distance from $\mathbf{o}$ to s.

The vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are called the *standard basis vectors* for $\mathbb{R}^2$ (the plane).

Column Vector Notation

We previously wrote $q = -2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$. In column vector notation we write

$$q = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

We may call q either a *vector* or a *point*. If we call q a vector, we are emphasizing that q gives direction of some sort. If we call q a point, we emphasize that q is some absolute location in space. (What's the philosophical difference between a location in space and directions from the origin to said location?)

3

$r = 1\mathbf{c}$ and $s = 2\hat{\mathbf{x}} + \mathbf{c}$ where $\mathbf{c}$ is the vector from before.

3.1 Write r and s in column vector form.

Sets and Set Notation

Def

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1, 2, 3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X.

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

Def

Some common sets are

$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}$.

$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}$.

$\mathbb{R} = \{\text{real numbers}\}$.

$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}$.

4

4.1 Which of the following are true?

(a) $3 \in \{1, 2, 3\}$.
(b) $4 \in \{1, 2, 3\}$.
(c) "b"$\in \{x : x \text{ is an English letter}\}$.
(d) "ò"$\in \{x : x \text{ is an English letter}\}$.
(e) $\{1, 2\} \subseteq \{1, 2, 3\}$.
(f) For some $a \in \{1, 2, 3\}$, $a \geq 3$.
(g) For any $a \in \{1, 2, 3\}$, $a \geq 3$.
(h) $1 \subseteq \{1, 2, 3\}$.
(i) $\{1, 2, 3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$.
(j) $\{1, 2, 3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$.

5 Write the following in set-builder notation

5.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

5.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

Def Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)}\quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)}\quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

6 Let $X = \{1,2,3\}$ and $Y = \{2,3,4,5\}$ and $Z = \{4,5,6\}$. Compute

6.1 $X \cup Y$

6.2 $X \cap Y$

6.3 $X \cup Y \cup Z$

6.4 $X \cap Y \cap Z$

7 Draw the following subsets of $\mathbb{R}^2$.

7.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

7.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

7.3 $J = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

7.4 $V \cup H$.

7.5 $V \cap H$.

7.6 Does $V \cup H = \mathbb{R}^2$?

Dot Product

Def If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

8

Let $\vec{a} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}3\\2\end{bmatrix}$, and $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$.

8.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

8.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

8.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

8.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

Def The ***norm*** of a vector $\vec{v} \in \mathbb{R}^n$, denoted $\|\vec{v}\|$ is its length and is given by the formula

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}.$$

9

9.1 Let $\vec{a} = \begin{bmatrix}3\\2\end{bmatrix}$. Find $\|\vec{a}\|$ using the Pythagorean theorem and using the formula from the definition of the norm. How do these quantities relate?

9.2 Let $\vec{b} = \begin{bmatrix}1\\1\\-2\\2\end{bmatrix}$, and find $\|\vec{b}\|$. Did you know how to find 4-d lengths before?

9.3 Suppose $\vec{u} = \begin{bmatrix}x\\y\end{bmatrix}$ for some $x, y \in \mathbb{R}$. Could $\vec{u} \cdot \vec{u}$ be negative? Compute $\vec{u} \cdot \vec{u}$ algebraically and use this to justify your answer.

Def The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} - \vec{v}\|$.

Def A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\| = 1$.

10

Let $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\1\\3\end{bmatrix}$.

10.1 Find the distance between $\vec{u}$ and $\vec{v}$.

10.2 Find a unit vector in the direction of $\vec{u}$.

10.3 Does there exists a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

10.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

Def Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u}\cdot\vec{v}=0$. The word orthogonal is synonymous with the word perpendicular.

11

11.1 Find two vectors orthogonal to $\vec{a}=\begin{bmatrix}1\\-3\end{bmatrix}$. Can you find two such vectors that are not parallel?

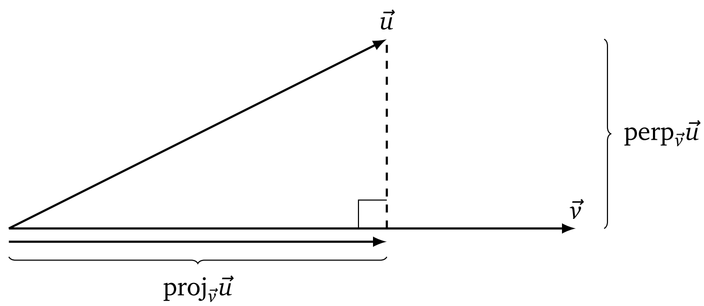
11.2 Find two vectors orthogonal to $\vec{b}=\begin{bmatrix}1\\-3\\4\end{bmatrix}$. Can you find two such vectors that are not parallel?

11.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\|=5$ and $\|\vec{y}\|=3$. What is the distance between $\vec{x}$ and $\vec{y}$?

Projections

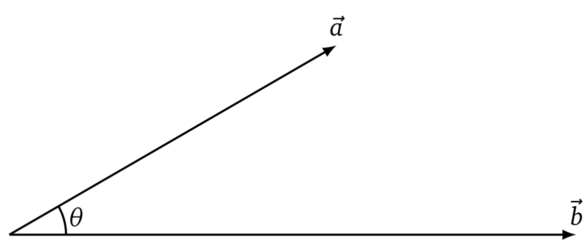
Projections (sometimes called orthogonal projections) are a way to measure how much one vector points in the direction of another.

Def



The ***projection*** of $\vec{u}$ onto $\vec{v}$ is written $\mathrm{proj}_{\vec{v}}\vec{u}$ and is the vector in the direction of $\vec{v}$ such that $\vec{u}-\mathrm{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$. The vector $\vec{u}-\mathrm{proj}_{\vec{v}}$ is called the ***perpendicular component*** of $\vec{u}$ with respect to $\vec{v}$ and is notated as $\mathrm{perp}_{\vec{v}}\vec{u}$.

12



In this picture $\|\vec{a}\|=4$, $\theta=\pi/6$, and $\vec{b}=\begin{bmatrix}6\\0\end{bmatrix}$.

12.1 Write $\vec{a}$ in column vector form.

12.2 Find $\|\mathrm{proj}_{\vec{b}}\vec{a}\|$ and $\|\mathrm{perp}_{\vec{b}}\vec{a}\|$.

12.3 Write down $\mathrm{proj}_{\vec{b}}\vec{a}$ and $\mathrm{perp}_{\vec{b}}\vec{a}$ in column vector form.

12.4 If $\vec{c}=\begin{bmatrix}-4\\0\end{bmatrix}$, write down $\mathrm{proj}_{\vec{c}}\vec{a}$ and $\mathrm{perp}_{\vec{c}}\vec{a}$ in column vector form.

12.5 If $\vec{d}=\begin{bmatrix}3\\1\end{bmatrix}$, write down $\mathrm{proj}_{\vec{d}}\vec{a}$ and $\mathrm{perp}_{\vec{d}}\vec{a}$ in column vector form. (You may need to use your knowledge of how dot products and angles relate to answer this one.)

12.6 Consider $\vec{d}=\begin{bmatrix}3\\1\end{bmatrix}$. Compute $\mathrm{proj}_{\hat{\mathbf{x}}}\vec{d}$ and $\mathrm{proj}_{\hat{\mathbf{y}}}\vec{d}$. How do these projections relate to the coordinates of $\vec{d}$? What can you say in general about projections onto $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$?

Lines, Planes, Normals, and Equations

13

13.1 Draw $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and *all* vectors perpendicular to it.

13.2 If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{x}$ is perpendicular to $\vec{u}$, what is $\vec{x} \cdot \vec{u}$?

13.3 Expand the dot product $\vec{u} \cdot \vec{x}$ to get an equation for a line. This equation is called the *scalar equation* representing the line.

Def A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to it.

13.4 Rewrite the line $\vec{u} \cdot \vec{x} = 0$ in $y = mx + b$ form and verify it matches the line you drew above.

14

We can also write a line in *parametric form* by introducing a parameter that traces out the line as the parameter runs over all real numbers.

14.1 Draw the line L with x, y coordinates given by

$$\begin{aligned} x &= t \\ y &= 2t \end{aligned}$$

as t ranges over $\mathbb{R}$.

14.2 Write the line $\vec{u} \cdot \vec{x} = 0$ (where $\vec{u}$ is the same as before) in parametric form.

15

Vector form is the same as parametric form but written in vector notation. For example, the line L from earlier could be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix}$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

15.1 Write the line $\vec{u} \cdot \vec{x} = 0$ in vector form. That is, find a vector $\vec{v}$ so the line $\vec{u} \cdot \vec{x} = 0$ can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\vec{v}$$

as t ranges over $\mathbb{R}$.

15.2 What is $\vec{v} \cdot \vec{u}$? Why? Will this always happen?

Moving to Planes

16 16.1 Write down three solutions $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ to

$$2x + y - z = 0. \tag{1}$$

16.2 Find $\vec{n} \in \mathbb{R}^3$ so that equation (1) is equivalent to $\vec{n} \cdot \vec{x} = 0$ where $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

16.3 What do you notice about the angle between solutions to equation (1) and $\vec{n}$?

When writing down solutions to equation (1), you got to choose two coordinates before the remaining coordinate became determined. This means the solutions have two parameters (and consequently form a two dimensional space).

16.4 Write down parametric form of a line of solutions to equation (1).

16.5 Write down parametric form of a different line of solutions to equation (1).

16.6 Write down all solutions to equation (1) in parametric form. That is, find $a_x, a_y, a_z, b_x, b_y, b_z$ so that

$$\begin{aligned} x &= a_x t + b_x s \\ y &= a_y t + b_y s \\ z &= a_z t + b_z s \end{aligned}$$

gives all solutions as t, s vary over all of $\mathbb{R}$.

16.7 Write all solutions to equation (1) in vector form.

Arbitrary Lines and Planes

So far, all of our lines and planes have passed through the origin. To produce the equation of an arbitrary line/plane, we first make one of same "slope" that passes through the origin, then we translate it to the appropriate place.

17 We'd like to write the equation of a line L with normal vector $\vec{n} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ that passes through the point $p = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

17.1 Give a scalar equation of the line L_2 which is parallel to L but passes through the origin.

17.2 Draw a picture of L and L_2, and find two points that lie on L. Call these points p_1 and p_2.

17.3 Verify the vector $\overline{p_1 p_2}$ is orthogonal to $\vec{n}$.

17.4 What is $\vec{n} \cdot p_1$, $\vec{n} \cdot p_2$, $\vec{n} \cdot p$? Should these values be zero, equal, or different? Explain (think about projections).

17.5 How does the equation $\vec{n} \cdot (\vec{x} - p) = 0$ relate to L?

18 W is the plane with normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and that passes through the point $p = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

18.1 Write normal form of W.

18.2 Write vector form of W.

Arc Length

19 The parameterized curve

$$\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix}$$

describes the position of a particle at time t.

19.1 Describe the path and motion of this particle in words.

19.2 Compute the displacement of the particle between $t = 0$ and $t = \Delta t$ and call the resulting vector $\Delta\vec{r}$. (Assume Δt is small.)

19.3 Approximate the length of $\Delta\vec{r}$. You may use the fact that

$$\sin x \approx x \qquad \text{and} \qquad \cos x \approx -\tfrac{1}{2}x^2 + 1$$

when $x \approx 0$.

19.4 Use a limit to compute the velocity of the particle at $t = 0$. Call this vector $\vec{v}_0$.

19.5 Use a limit to compute the speed at $t = 0$. Call this value s_0.

19.6 How do $\|\vec{v}_0\|$ and s_0 relate? Why?

20 A particle's path is parameterized by

$$\vec{m}(t) = (f(t), g(t), h(t))$$

where t represents time.

20.1 Derive (with explanation) a formula for the velocity of the particle at time $t = t_0$.

20.2 Derive (with explanation) a formula for the speed of the particle at time $t = t_0$.

21 Recall the particle whose path is given by $\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix}$ where t represents time.

21.1 Use the fact that

$$\text{distance traveled} = \int \textit{speed}\ d\textit{time}$$

to produce a formula for how far the particle has traveled from $t = 0$ to $t = t_0$.

21.2 Use geometry to do the same thing.

21.3 Derive an expression (with explanation) for the arc length of $\vec{m}(t) = (f(t), g(t), h(t))$ from $t = 0$ to $t = t_0$.

Arc Length Parameterization

An *arc length parameterization* of a curve C is a function $\vec{s}: \mathbb{R} \to \mathbb{R}^n$ whose image is C with the added property that the arc length of $\vec{s}(t)$ from $t = 0$ to $t = t_0$ is t_0 for all valid choices of t_0. I.e., the distance traveled by the parameter along $\mathbb{R}$ is the same as the distance traveled by the point $\vec{s}(t)$ in $\mathbb{R}^n$.

22

22.1 Produce an arc length parameterization of the curve parameterized by $\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix}$.

22.2 Produce an arc length parameterization of the curve parameterized by $\vec{q}(t) = \begin{bmatrix} t \\ t^{3/2} \end{bmatrix}$.

An arc length parameterization of a curve can also be thought of as a parameterization where a particle always moves at unit speed (if you interpret a parameterized curve as describing the motion of a particle).

By reparameterizing, we can describe the motion of a particle along a path at any speed.

23 A particle moves along a path C, which is a circle in $\mathbb{R}^2$ of radius 3, centered at the origin, and oriented counter-clockwise.

23.1 Parameterize C so that the speed is 2.5.

23.2 Parameterize C so that the speed of the particle starts and ends at 0.

23.3 Parameterize C so that the speed of the particle starts at 0 and ends at 4.

23.4 Parameterize C so that the speed of the particle is 0 at six points along the curve.

24 A particle's motion is described by the function $\vec{h}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$, which is a parameterization of the curve H. The arc length of H from $t = 0$ to $t = t_0$ using this parameterization is is given by the function $s(t_0) = t_0^2$.

24.1 Write an expression for the speed of the particle at time t.

24.2 Give a formula for the arc length parameterization of H.

Tangents, Normals, and Acceleration

Def Suppose $\vec{r}(t)$ describes the motion of a particle. The ***velocity*** of the particle is defined as

$$\vec{v}(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

The ***acceleration*** of the particle is defined as

$$\vec{a}(t) = \lim_{h\to 0} \frac{\vec{v}(t+h) - \vec{v}(t)}{h}.$$

Both $\vec{v}$ and $\vec{a}$ are vector-valued derivatives.

25 Let $\vec{r}(t) = \begin{bmatrix} t+2 \\ \sin t \\ t^3 \end{bmatrix}$ represent the position of a particle at time t.

25.1 Find the velocity of the particle at time t.

25.2 Find the acceleration of the particle at time t.

26 Let $\vec{r}_\ell(t) = \begin{bmatrix} \frac{t^2}{2} \\ \frac{t^2}{2} \end{bmatrix}$ and $\vec{r}_c(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ represent the position of the particles r_ℓ and r_c at time t.

26.1 Describe the paths of each of these particles. What should their accelerations be? Why?

26.2 Compute the accelerations of r_ℓ and r_c at time t.

26.3 Compute the tangent vectors for the functions $\vec{r}_\ell$ and $\vec{r}_c$ at time t. How does the direction of the tangent vectors and the direction of the acceleration vectors compare?

If $\vec{r}(t)$ describes the position of a particle at time t and $\vec{a}(t)$ its acceleration, $\vec{a}(t)$ can be decomposed into a *tangential* component, $\vec{a}_T(t)$, and a *normal* component $\vec{a}_N(t)$ so that

$$\vec{a}(t) = \vec{a}_T(t) + \vec{a}_N(t).$$

27

Let $\vec{r}_{c\ell}(t) = \begin{bmatrix} \frac{t^2}{2} + \cos t \\ \frac{t^2}{2} + \sin t \end{bmatrix}$ represent the position of the particle $r_{c\ell}$ at time t.

27.1 Compute the tangential and normal components of the acceleration of $r_{c\ell}$.

27.2 By changing the speed of $r_{c\ell}$ (but not its path), is it possible to make the tangential component of its acceleration zero?

27.3 By changing the speed of $r_{c\ell}$ (but not its path), is it possible to make the normal component of its acceleration zero?

We're going to develop some tools to mathematically verify our intuition about the acceleration vector.

28

28.1 Suppose $\vec{p}(t)$ and $\vec{q}(t)$ are both vector-valued functions. Expand $\vec{p}(t) \cdot \vec{q}(t)$ using components. Now, come up with an expression for $\Big(\vec{p}(t) \cdot \vec{q}(t)\Big)'$. Look at your result and rewrite it as an expression involving dot products. You've just discovered a product rule for the dot product!

28.2 Using similar methods to the computation of $\Big(\vec{p}(t) \cdot \vec{q}(t)\Big)'$, find an expression for $\|\vec{p}(t)\|'$.

28.3 If a particle whose path is given by $\vec{r}(t)$ is moving at unit speed, then $\|\vec{v}(t)\| = 1$. Take derivatives of both sides of this expression to show that $\vec{v}(t) \cdot \vec{a}(t) = 0$.

Visualizing Surfaces

As we've already seen, being able to picture multi-dimensional objects is invaluable when intuiting solutions and when coming up with mathematical justifications for intuition. There are two main ways we visualize 2D surfaces: perspective drawings and level-curves.

29

Consider the surface defined by $z = f(x, y)$ where $f(x, y) = x^2 + y^2$.

29.1 On a single xy-plane, draw the set of points satisfying the equations $0 = f(x, y)$, $1 = f(x, y)$, $2 = f(x, y)$, $3 = f(x, y)$, and $4 = f(x, y)$. What is the set of points satisfying $-1 = f(x, y)$?

The curves you just drew are called *level curves*.

29.2 Plot in 3D the function $z = f(x, y)$ for $z = 0, 1, 2, 3, 4$. Are you able to see what this surface looks like?

29.3 We can plot more and more contours of this surface until we have a good idea of what it looks like. Add the set of points $z = f(0, y)$ and $z = f(x, 0)$ to your plot.

30

Level curves are contours produced by fixing z and graphing the result. Other useful contours come about by fixing x or y. Use any contour-based method to plot the following surfaces.

30.1 $z = x^2 - y^2$

30.2 $z^2 = 4x^2 + y^2$

30.3 $z^2 = 9 - (4x^2 + y^2)$

30.4 $z = (\sin x)(\sin y)$

Directional Derivatives

31 It is winter in Chicago. Imagine a bug crawling along the floor of a room that is 7×7 meters. Figure 1 shows the contour plot of the heat of the floor in Fahrenheit.[1]

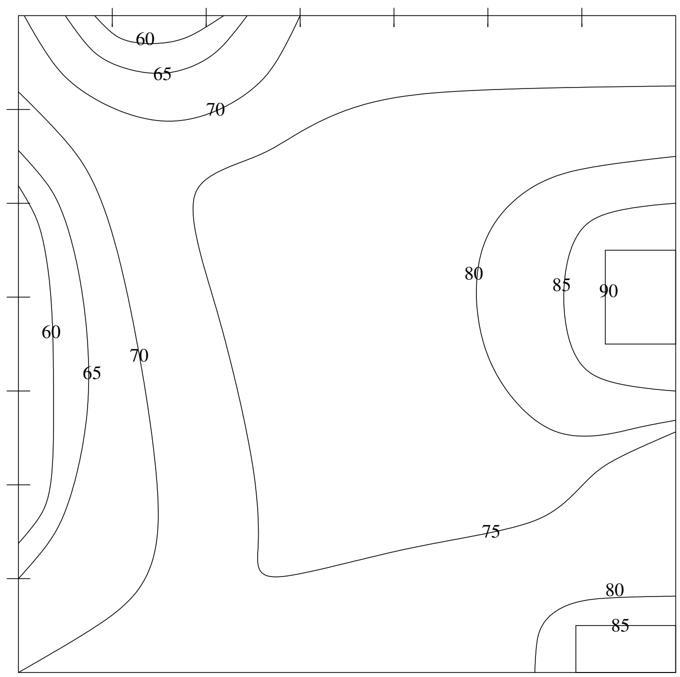


Figure 1: Heat contours for a room.

31.1 Where is the door to the outside? Where is the window? Where is the heater? Why? What might be in the lower right corner of the room?

31.2 The bug would like to crawl from the window to the right wall along a path that minimizes the change in temperature. Draw two such paths.

31.3 The bug makes a circuit around the entire room by walking along the walls clockwise. Draw a graph of distance verses temperature for this part of the bug's journey. How do the starting values and ending values of your graph compare? Is the accuracy of your graph limited?

32 Figure 2 shows the path the bug decided to follow.

32.1 Find the average change in temperature from P to Q for the bugs journey.

32.2 Estimate the instantaneous rate of change of temperature at the point P.

32.3 Suppose $T(x, y)$ gives the temperature of the room at the coordinates (x, y). Write a limit expression to compute the exact rate of change of temperature at the beginning of the bug's journey. Does the rate of change depend on time?

32.4 Write a limit expression to compute the rate of change of temperature *with respect to distance* at the start of the bug's journey.

32.5 If the bug had decided to travel in a different direction, would the rate of change of temperature be the same? Explain.

[1]The bug questions are due to Mairead Greene, Amy Ksir, and Christine von Renesse.

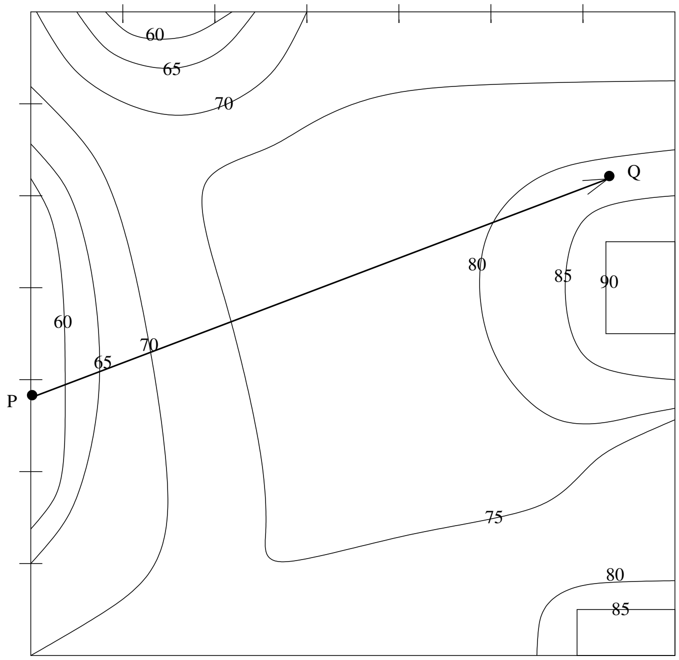


Figure 2: Heat contours for a room.

33 Let $f(x, y) = 2(x-3)^2 + y$

33.1 Calculate the rate of change of f at the origin in the directions $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\vec{a} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

33.2 Calculate the rate of change of f at $p = (1, -1)$ in the directions $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\vec{a} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

A New Perspective on Derivatives

In single-variable calculus, we deal with functions $g : \mathbb{R} \to \mathbb{R}$. The derivative of this function at a point a is the rate of change of g at the point a.

34 Let $g(x) = x^2$.

34.1 What is the rate of change of g at the points $x = 0, 1$, and 2? How about at $x = x_0$?

34.2 Use your knowledge of the rate of change of g to *approximate* $g(x)$ where $x = 0 + \varepsilon, 1 + \varepsilon$, and $2 - \varepsilon$ where $\varepsilon > 0$ is a tiny number.

34.3 Write down three functions: G_0, G_1, and G_2 so that if ε is tiny, $g(a + \varepsilon) \approx g(a) + G_a(\varepsilon)$.

35 Recall $f(x, y) = 2(x-3)^2 + y$ from before.

35.1 Write down a function $F_{(0,0)} : \mathbb{R}^2 \to \mathbb{R}$ such that if $\vec{u}$ is a vector and $||\vec{u}||$ is tiny,

$$f(\vec{0} + \vec{u}) \approx f(\vec{0}) + F_{(0,0)}(\vec{u}).$$

Notice, $F_{(0,0)}$ is somehow representative of the rate of change of f at $(0, 0)$. It may help to compute $F_{(0,0)}$ for several example directions before coming up with a formula.

35.2 Write down a function $F_{(1,-1)}$ so that if $\vec{u}$ is a vector and $||\vec{u}||$ is tiny, $f\Big((1, -1) + \vec{u}\Big) \approx f(1, -1) + F_{(1,-1)}(\vec{u})$.

35.3 Can a single number represent the rate of change of f at the point $(0, 0)$? Why or why not? If a number cannot, can you think of another object that can?

Def The ***directional derivative*** of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\vec{a}$ in the direction $\vec{u}$ is

$$D_{\vec{a}} f(\vec{u}) = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}.$$

I.e., it is the rate of change of f at $\vec{a}$ heading in the direction of $\vec{u}$ with speed $\|\vec{u}\|$.

36 Let $f(x, y) = (x+1)^2 + 2y$.

36.1 Give a parameterization of all two dimensional unit vectors.

36.2 Use the definition of the directional derivative to compute the directional derivative of f at the point $(0,0)$ in the direction of each of your parameterized unit vectors.

36.3 Find the direction of the maximal directional derivative. You may find the identity $\sin t + \cos t = \sqrt{2}\sin(t + \frac{\pi}{4})$ helpful.

Def The ***gradient*** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

where $x_1, \dots, x_n$ are the parameters of f.

37 Let $f(x, y) = (x+1)^2 + 2y$ as before.

37.1 Compute $\nabla f(0,0)$ (the gradient of f at the point $(0,0)$).

37.2 Compute $\nabla f(0,0)\cdot\vec{u}$ for each of your parameterized unit vectors $\vec{u}$. What do you notice?

37.3 Make a conjecture about how the gradient of a function can be used to compute directional derivatives.

38 Sticking with $f(x, y) = (x+1)^2 + 2y$, consider the surface $S = \{(x, y, f(x, y)) : x, y \in \mathbb{R}^2\}$.

38.1 Describe S. If $a \in S$, what should *the tangent plane to S at a* mean?

38.2 Compute the tangent plane, $\mathcal{P}$, to S at the point $(0, 0, f(0,0))$ in vector form.

38.3 Find standard/normal form of the tangent plane and solve for z. That is, find an equation for the tangent plane in the form $z = ax + by + c$. Do the values a, b, and c look familiar?

38.4 Consider the function $g = f - \mathcal{P}$. That is, $g(x, y) = f(x, y) - z(x, y)$ where $z(x, y)$ is the equation of the tangent plane. What should the directional derivative of g at $(0,0)$ be? Does it depend on direction?

So far, the gradient has been appearing in directional derivatives and tangent planes. One might even argue that the gradient *is* the derivative. Unfortunately, the gradient of a function existing is not sufficient for a function to be differentiable.

Def The function $f : \mathbb{R}^n \to \mathbb{R}$ is ***differentiable*** at $\vec{a} = (a_1, \dots, a_n)$ if there exists a tangent plane at $\vec{a}$. That is, there exists some function $p(x_1, \dots, x_n) = c + \sum \alpha_i(x_i - a_i) = c + \vec{\alpha}\cdot(\vec{x} - \vec{a})$ so that

$$\lim_{\|\vec{u}\|\to 0} \frac{f(\vec{a}+\vec{u}) - p(\vec{a}+\vec{u})}{\|\vec{u}\|} = 0.$$

There's just one trouble with this definition. How do we take a limit as $\|\vec{u}\| \to 0$?

Def If $f : \mathbb{R}^n \to \mathbb{R}$, then the ***limit of f at $\vec{a}$*** exists and is equal to L, written $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$0 < \|\vec{x} - \vec{a}\| < \delta \qquad \text{implies} \qquad |f(\vec{x}) - L| < \varepsilon.$$

If no such L exists, we say the limit does not exist.

When first considered, the definition of limit can be confusing, so let's play with some functions that we're familiar with.

39 Let $f(x) = 3x$.

39.1 Draw the graph of f and use your intuition to guess $\lim_{x\to 0} f(x)$. Call your guess L.

39.2 If $\varepsilon = 1/2$, can you find a δ so that if $|x - 0| < \delta$ we can be assured $|f(x) - L| < \varepsilon$?

39.3 If $\varepsilon = 1/4$, can you find a δ so that if $|x - 0| < \delta$ we can be assured $|f(x) - L| < \varepsilon$?

39.4 Come up with a rule for how to choose an appropriate δ for any ε.

40 Let $g(x) = \frac{x}{|x|}$, and let L be a guess for $\lim_{x\to 0} \frac{x}{|x|}$.

40.1 Suppose we guess $L = 1$. Picking $\varepsilon = 3/2$, can you find an appropriate δ? What if $L = -1$? For $\varepsilon = 3/2$, can you find an appropriate δ? What if $L = 0$? What values can we rule out as possibilities for L?

40.2 Suppose we guess $L = 0$. Picking $\varepsilon = 3/2$, can you find an appropriate δ? Is 0 the limit? Explain.

40.3 Find an ε that shows that g has no limit at 0.

41 Let $h(x, y) = \frac{xy^2}{x^2+y^2}$.

41.1 Give a parameterization of all vectors in $\mathbb{R}^2$ with length r.

41.2 If $\vec{u} \in \mathbb{R}^2$ satisfies $\|\vec{u}\| = r$, can you give upper and lower bounds on $h(\vec{u})$?

41.3 Explain why the limit $\lim_{\|\vec{u}\|\to 0} h(\vec{u})$ does or doesn't exist.

42 Let $h(x, y) = \frac{xy}{x^2+y^2}$.

42.1 Give a parameterization of all vectors in $\mathbb{R}^2$ with length r.

42.2 If $\vec{u} \in \mathbb{R}^2$ satisfies $\|\vec{u}\| = r$, can you give upper and lower bounds on $h(\vec{u})$?

42.3 Explain why the limit $\lim_{\|\vec{u}\|\to 0} h(\vec{u})$ does or doesn't exist.

Lagrange Multipliers

43 We know that $\nabla f(\vec{a})\cdot\vec{u}$ gives the directional derivative of f at the point $\vec{a}$ in the direction $\vec{u}$. Suppose $\nabla f(\vec{a}) = (1,3)$.

43.1 What is the direction of the biggest rate of change of f at $\vec{a}$? Why?

43.2 In what direction is the least change in f at $\vec{a}$?

44 Suppose C is a level curve of f. For $\vec{a}\in C$, suppose $\vec{T}_a$ is a tangent vector to C at $\vec{a}$.

44.1 What is the rate of change of the function f at $\vec{a}$ in the direction $\vec{T}_a$?

44.2 What is $\nabla f(\vec{a})\cdot\vec{T}_a$?

44.3 If $f(x,y) = x^3 + y^2$, find a tangent vector to the curve $C = \{(x,y) : f(x,y) = 2\}$ at $(1,1)$.

45 Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a curve C (not a level curve). Let $\vec{r}(t)$ be a parameterization of C. We are interested in the minimum and maximum values for $f\circ\vec{r}$ (i.e., the min/max values of f on the curve C).

45.1 What will $(f\circ\vec{r})'$ be at a local maximum or minimum?

45.2 Explain how to interpret the quantity $(f\circ\vec{r})'(t)$ as a directional derivative.

45.3 Use your knowledge of how the gradient allows you to compute directional derivatives to come up with a "chain rule" for the expression $(f\circ\vec{r})'(t)$.

45.4 Suppose $f(x,y) = 2x + y$ and C is a circle of radius 1 centered at $\vec{0}$. Parameterize C and attempt to find the maximum value that f takes on the curve C. Are these computations easy?

46 Let $f(x,y) = 2x + y$ and let C be a circle of radius 1 centered at $\vec{0}$.

46.1 Find a function $g : \mathbb{R}^2 \to \mathbb{R}$ so that C is a level curve of g. Can you pick g so that $C = \{(x,y) : g(x,y) = 0\}$?

46.2 What is the relationship between a tangent vector to C at $\vec{a}$ and $\nabla g(\vec{a})$?

46.3 Suppose that on the curve C, f attains a maximum at $\vec{a}\in C$. What then is the relationship between $\nabla f(\vec{a})$ and $\nabla g(\vec{a})$? Can you describe this relationship with a formula?

46.4 Use your formula from the previous part along with the constraint $g(x,y) = 0$ to find the maximum value the function f attains on the curve C. Congratulations, you've just discovered *Lagrange Multipliers*!

47 We will use Lagrange Multipliers to find the dimensions of the largest box (the one with the most volume) given that the surface area is 32.

47.1 Write down the function we wish to optimize, call it f, and rephrase our constraint on surface area so that we may interpret it as the level curve of some function g.

47.2 Write down a relationship between ∇f and ∇g at the maximum.

47.3 Write down all equations obtained thus far, and attempt to solve for the dimensions of the box. Do we need to solve for every variable?

Iterated Integrals

Integrals add things up, and iterated integrals are no different; it's just that instead of adding things up in one dimension, we add things up in multiple dimensions.

48 To get our bearings, let's revisit integrating with respect of $\mathrm{d}x$ and $\mathrm{d}y$ when finding area. Let T be the triangle with vertices $(0,0)$, $(4,0)$, and $(4,3)$.

48.1 Draw a picture representing how a Riemann sum could be used to compute the area of T.

48.2 Draw a picture representing a different way to compute the area of T using a Riemann sum.

48.3 Compute the area of T using an integral with respect to x; now using an integral with respect to y; now using the formula $\frac{1}{2}base \times height$.

49 Let T be the triangle with vertices $(0,0)$, $(4,0)$, and $(4,3)$, and imagine that T outlines a triangular strip of foil. The density of this foil at any given point (in the first quadrant) is $\rho(x,y) = x^2 + y$.

49.1 Using the fact that for this foil, $mass = density \times area$, write down two different integrals that would compute the total mass of T, and draw their corresponding Riemann-sum pictures.

49.2 Evaluate your two integrals. Is one easier to evaluate than the other?

49.3 Can $(x^2 + y)\mathrm{d}x\mathrm{d}y$ be interpreted on its own?

50 Let the region $R \subset \mathbb{R}^2$ be the area between the parabola $x = y^2$ and the line $x = 1$. Let $f(x,y) = 3x+y^2$.

50.1 Write down two different iterated integrals to compute $\displaystyle\iint_R f(x,y)\mathrm{d}x\mathrm{d}y$.

50.2 Is one of your integrals easier to evaluate than the other?

50.3 Describe and draw a region A so that $\displaystyle\iint_A f(x,y)\mathrm{d}x\mathrm{d}y = \int_{x=0}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} f(x,y)\mathrm{d}y\mathrm{d}x$.

The Volume Form

We've been using integrals to find the area or volume (or hypervolume) under a curve. Abstractly, this process has been the same every time: chop the region into little pieces, and add up each piece's contribution with an integral to find the total. We're continuing the same process, but we're going to start chopping regions up into (sometimes) non-rectangular chunks.

51

51.1 Draw the lines $x = x_0$, $x = x_0 + \Delta x$, $y = y_0$ and $y = y_0 + \Delta y$ where Δx and Δy are assume to be small. Write down a formula for the area enclosed by those lines. Does your formula depend on x_0 and y_0?

51.2 Draw the curves, specified in polar coordinates by $r = r_0$, $r = r_0 + \Delta r$, $\theta = \theta_0$, and $\theta = \theta_0 + \Delta\theta$ where Δr and $\Delta\theta$ are assumed to be small. Write down a formula for the area enclosed by those curves. Does your formula depend on r_0 and θ_0?

51.3 Using the idea that a limit as $\Delta ? \to 0$ of a Riemann sum results in an integral, write down a double-integral expression for the area of a semicircle of radius 1 in both rectangular and polar coordinates.

51.4 Let $V(r_0,\theta_0) = r_0\Delta r\Delta\theta$, and let $\Delta V(r_0,\theta_0)$ be the exact area of the annular (ring-shaped) section you found earlier. What is $\displaystyle\lim_{\Delta r,\Delta\theta\to 0}\frac{\Delta V(r_0,\theta_0)}{V(r_0,\theta_0)}$? Do we need the $(\Delta r)^2\Delta\theta$ term at infinitesimal scales?

Def

> Suppose $\mathcal{F}$ is a coordinate system for $\mathbb{R}^2$ with a relationship to rectangular coordinates given by $x = f_1(a,b)$ and $y = f_2(a,b)$. The ***pre-volume form*** associated with $\mathcal{F}$ at the point (a,b) is written $\Delta V(a,b)$ and is the area between the curves $a = a_0$, $a = a_0 + \Delta a$, $b = b_0$, and $b = b_0 + \Delta b$.
>
> The ***volume form*** associated with $\mathcal{F}$ at (a,b) is the infinitesimal $\mathrm{d}V(a,b) = V(a,b)\mathrm{d}a\mathrm{d}b$ where V is the unique function satisfying
> $$\lim_{\Delta a,\Delta b\to 0} \frac{\Delta V(a,b)}{V(a,b)\Delta a\Delta b} = 1$$
> and can obtained by replacing Δw with $\mathrm{d}w$ and $(\Delta w)^2$ with 0 in the pre-volume form (where w stands for any variable).

For coordinate systems in $\mathbb{R}^3$ (and $\mathbb{R}^n$), the volume form is defined in an analogous way.

52

52.1 Write down the pre-volume and volume forms for polar coordinates. What are the functions f_1 and f_2 for polar coordinates?

52.2 Write down the pre-volume and volume forms for rectangular coordinates. What are the functions f_1 and f_2?

52.3 Let $\mathcal{S}$ be a stretched coordinate system. That is, the point (a,b) in $\mathcal{S}$-coordinates corresponds to the point $(a/2, b/3)$ in rectangular coordinates. Write down the pre-volume and volume forms for $\mathcal{S}$-coordinates.

52.4 Consider the set $X = \{(a,b) : 0 \le b \le a^2 \text{ and } 0 \le a \le 1\}$ specified in $\mathcal{S}$ coordinates. Draw X on the $\mathcal{S}$-coordinate plane and in standard coordinate plane. Then, compute the area of X by setting up an integral using the volume form for $\mathcal{S}$ coordinates.

53

Let $f(x,y) = x^2 + y^2$ and let D be a disk of radius 2 centered at the origin. We would like to find $A = \displaystyle\int_D f\,\mathrm{d}V$.

53.1 Write down the bounds of D in rectangular coordinates and set up (but don't evaluate) an integral for A.

53.2 Write down the bounds of D in polar coordinates and set up (but don't evaluate) an an integral for A.

53.3 Find A by evaluating whichever integral you want.

54

Cylindrical coordinates for $\mathbb{R}^3$ take the form (r,θ,z) where the xy-plane is specified in polar coordinates by the first two components and the height along the z-axis is given by the third component.

54.1 Compute the volume form for cylindrical coordinates.

54.2 A discuss is constructed from gluing two cones' bases together. The radius of the base of each cone is 4 and the height of each cone is 1. Describe the surface of the discuss using cylindrical coordinates (you may need to describe the top and bottom separately)

54.3 The discuss has a density given by $\rho(x,y,z) = x^2 + y^2 + 2|z|$. Find the mass of the discuss. (You may exploit symmetry.)

Surface Integrals

We've done line integrals, which were adding up the values of a function along a curve. Now we're going to do the same thing with surfaces.

55

55.1 Find the area of a parallelogram with adjacent sides given by $\vec{a} = (0, 1)$ and $\vec{b} = (4, 0)$.

55.2 Find the area of a parallelogram with adjacent sides given by $\vec{a} = (0, 1, 0)$ and $\vec{b} = (4, 0, 4)$.

55.3 Find the area of a parallelogram with adjacent sides given by $\vec{a} = (1, 2, 3)$ and $\vec{b} = (2, 1, -1)$.

56

Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface and let $f : \mathcal{S} \to \mathbb{R}$ be a function.

56.1 Think of two examples of surfaces $\mathcal{S}$ and functions f that correspond to the world you live in.

56.2 We'd like to add up (integrate) the values of f over the surface $\mathcal{S}$. Explain how a Riemann sum can help us do this.

56.3 Suppose that $\vec{r}(a, b)$ for $0 \le a \le 2$ and $-1 \le b \le 1$ is an isometric parameterization of $\mathcal{S}$ (i.e., $\vec{r}$ preserves area). Write down an iterated integral to compute $\displaystyle\int_{\mathcal{S}} f$ with respect to surface area.

56.4 Suppose that $\vec{q}(\alpha, \beta)$ for $-3 \le \alpha \le 0$ and $5 \le \beta \le 6$ is a non-isometric parameterization of $\mathcal{S}$. However, you also know that if $R \subset \mathbb{R}^2$ is a rectangle with side lengths $\Delta\alpha$ and $\Delta\beta$ and lower-left corner (α, β), then $\vec{q}(R)$ has surface area $Q(\alpha, \beta)\Delta\alpha\Delta\beta$. Set up an iterated integral to compute $\displaystyle\int_{\mathcal{S}} f$ with respect to surface area using $\vec{q}$ and Q. Justify your intuition with an explanation in terms of Riemann sums.

In the previous problem, Q looked a lot like a volume form, but finding it from first principles is hard! Instead, we'll develop intuition and a formula for Q using tangent planes.

57

Suppose $\mathcal{P} \subset \mathbb{R}^3$ is a plane with parameterization $\vec{p}(t, s) = t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

57.1 Suppose $R_{(t_0, s_0)} \subset \mathbb{R}^2$ is a rectangle with lower left corner at (t_0, s_0) and sides of length Δt and Δs. Find a function Q so that $Q(t, s) =$ area of $\vec{p}(R_{(t_0, s_0)})$. Does Q depend on t_0 and s_0? Does this make sense?

57.2 Consider the new parameterization $\tilde{p}$ given by $\tilde{p}(t, s) = \vec{p}(t^2, s)$. What does $\tilde{p}$ parameterize? How does it relate to $\mathcal{P}$?

57.3 Let $R_{(t_0, s_0)}$ be a rectangle as before. Find a function $\tilde{Q}$ so that $\tilde{Q}(t, s) =$ area of $\tilde{p}(R_{(t_0, s_0)})$. Does $\tilde{Q}$ depend on (t_0, s_0)?

57.4 Find a function $V(t, s)$ so that $\displaystyle\lim_{\Delta t, \Delta s \to 0} \frac{\tilde{Q}(t, s)}{V(t, s)\Delta t \Delta s} = 1$. Does V remind you of anything?

Given a surface $\mathcal{S}$ parameterized by $r(t, s)$, there is a canonical way to write vector form of the tangent plane to $\mathcal{S}$ at the point $(t, s, r(t, s))$ by using the directional derivatives of r in the t direction and the s direction as two direction vectors for your plane.

58

Consider the surface $\mathcal{S}$ parameterized by $(t, s, r(t, s))$ where $r(t, s) = t^2 + s^2$ and $0 \le t \le 1$ and $0 \le s \le 1$.

58.1 Find the canonical representation of the tangent plane to $\mathcal{S}$ at the point (t_0, s_0).

58.2 Let $R_{(t_0, s_0)}$ be a rectangle with lower left corner (t_0, s_0) and sides of very short lengths Δt and Δs. Describe the set $I_{(t_0, s_0)} = \{(a, b, r(a, b)) : (a, b) \in R_{(t_0, s_0)}\}$.

58.3 Give an estimation of the surface area of $I_{(t_0, s_0)}$. What assumptions are you making?

58.4 Write down an integral to compute the surface area $\mathcal{S}$.

Vector Fields

Def A ***scalar field*** is a function $f : \mathbb{R}^n \to \mathbb{R}$. A ***vector field*** is a function $g : \mathbb{R}^n \to \mathbb{R}^m$.

We've already worked with scalar fields—when we found the area above a curve and below a surface, we were integrating a scalar field (namely the height of the function at a given point). We'll now explore vector fields, which assign a vector to each point in space. Since vectors can be used to represent forces, and velocities, and accelerations, vector fields are a key object of study in physics.

59 Sketch the following vector fields.

59.1 $\vec{f}(x,y) = x^2\hat{\mathbf{x}} + \hat{\mathbf{y}}$.

59.2 $\vec{g}(x,y) = (x,-y)$

59.3 $\nabla h(x,y)$ where $h(x,y) = -(x^2+y^2)$.

We already have experience with some vector fields. Namely, the gradient of a function is a vector field. We're going to explore some special properties of vector fields arising from gradients.

60 Let $f(x,y) = x^2 + y$, and let $A = (0,0)$ and $B = (1,1)$.

60.1 Let $\vec{x} = (x,y)$, and compute $\nabla f(\vec{x})$. If $\vec{u}$ is a vector with small magnitude, how can $\nabla f(\vec{x})$ be used to estimate the change in f from $\vec{x}$ to $\vec{x}+\vec{u}$?

60.2 Parameterize a straight line segment from A to B. If we imagine $\nabla f(\vec{x})$ as representing a force at position $\vec{x}$, how much work is done moving along a straight line from A to B? (This is a line integral that you've done before!)

60.3 How much work is done moving from A to B if you first move horizontally and then move vertically?

60.4 What does the amount of work represent? Will the work done depend on your path from A to B?

Def A vector field $\vec{f}$ is called ***conservative*** if $\vec{f} = \nabla g$ for some g.

61 61.1 Give two examples of conservative vector fields in $\mathbb{R}^2$.

61.2 Is the vector field $\vec{f}(x,y) = (x,-y)$ conservative? Why or why not?

61.3 Give two examples of non-conservative vector fields in $\mathbb{R}^2$.

Let's get phisicsical

Vector fields merely assign a vector to every point in space. We've seen how it can be fruitful to interpret a vector field as describing the force at a particular point. It's also useful to think of vector fields as describing the velocity at particular points of some fluid.

Interpreting a vector field as describing the velocity of a fluid brings us to the idea of flux. Given a vector field $\vec{f}$, the *flux* of $\vec{f}$ through the surface $\mathcal{S}$ is the volume of fluid that passes through $\mathcal{S}$ in one time unit (assuming the fluid has density one). If $\mathcal{S}$ has an orientation, the flux is positive if the fluid moves "out" of $\mathcal{S}$ and negative if the fluid moves "in" $\mathcal{S}$.

62 Let $\vec{f}(x,y,z) = (0,0,2)$. Let $P(\vec{u},\vec{v})$ be the function that outputs the parallelogram with adjacent sides given by $\vec{u}$ and $\vec{v}$ and whose orientation is given by the right hand rule.

62.1 Describe $P(\hat{y},\hat{x})$. What is its orientation?

62.2 Compute the flux of $\vec{f}$ through the surface $R_1 = P(\hat{y},\hat{x})$.

62.3 Compute the flux of $\vec{f}$ through the surface $R_2 = P\Big((\sqrt{2}/2, 0, \sqrt{2}/2), (0,1,0)\Big)$

62.4 Compute the flux through a box with side-lengths of one which is parallel to the xz-plane and meets the xy-plane at an angle of $\pi/4$ and whose faces are oriented outwards.

62.5 If $\mathcal{S}$ is a closed surface (i.e., $\mathcal{S}$ encloses a region and has no holes), what is the flux of $\vec{f}$ through $\mathcal{S}$? Would your answer be the same for the flux of $\vec{g}(x,y,z) = (0,0,2z)$ through $\vec{S}$? Why or why not?

63 Let $\vec{f}(x,y,z) = (0,0,-z)$ and let $\mathcal{S}$ be the surface parameterized by $\vec{s}(u,v) = (u,v,u^2+v^2)$ for $u^2+v^2 \leq 4$ oriented downwards.

63.1 Find the flux of $\vec{f}$ through $\mathcal{S}$ using the parameterization $\vec{s}$.

63.2 Let $\vec{p}(r,\theta)$ be a parameterization of $\mathcal{S}$ in cylindrical coordinates. Find $\vec{p}$ and use $\vec{p}$ to find the flux of $\vec{f}$ through $\mathcal{S}$.

63.3 Does flux depend on the choice of parameterization?

We're going to play the approximation game again (just like we did when we discovered that tangent planes approximate a surface near their point of tangency). But, this time, we're going to be approximating flux.

64 Let $C_{\Delta x,\Delta y,\Delta z}$ be the box with side lengths Δx, Δy, Δz, parallel to the xy and xz planes and with lower left corner at $\vec{0}$. Assume each face of the box is oriented outwards. Let $\vec{f}:\mathbb{R}^3 \to \mathbb{R}^3$ be a vector field.

64.1 Find the normal vector and the surface area for each face of $C_{\Delta x,\Delta y,\Delta z}$.

64.2 Let B be the bottom face of $C_{\Delta x,\Delta y,\Delta z}$. Assuming Δx and Δy are very small, what might be a reasonable approximation for the flux of $\vec{f}$ through B?

64.3 Compute the approximate flux through $C_{\Delta x,\Delta y,\Delta z}$.

Def Let $C_{\Delta x,\Delta y,\Delta z}(x,y,z)$ be a rectangular prism with side lengths Δx, Δy, and Δz, lower left corner at the point (x,y,z) and faces oriented outwards. For a vector field $\vec{f}:\mathbb{R}^3 \to \mathbb{R}^3$, the ***divergence*** of $\vec{f}$ at the point (x,y,z) is

$$\nabla\cdot\vec{f}(x,y,z) = \lim_{\Delta x,\Delta y,\Delta z\to 0} \frac{\text{flux of } \vec{f} \text{ through } C_{\Delta x,\Delta y,\Delta z}(x,y,z)}{\text{volume of } C_{\Delta x,\Delta y,\Delta z}(x,y,z)}.$$

In other words, the divergence of $\vec{f}$ at a point is the amount of outward flux of $\vec{f}$ per unit volume at that point.

64.4 Compute the divergence of $\vec{f}$ at $(0,0,0)$.

64.5 Compute the divergence of $\vec{f}$ at (x,y,z). Is $\nabla\cdot\vec{f}(x,y,z)$ a reasonable notation for this?

Divergence works in 2d just like it does in 3d (but 2d is easier to draw pictures of).

65 Plot each vector field and estimate whether the divergence is positive, negative, or zero. Then, check your answer with a computation.

65.1 $\vec{f}(x,y) = (x,y)$.

65.2 $\vec{g}(x,y) = (|x|, 0)$.

65.3 $\vec{h}(x,y) = (-y,x)$

65.4 $\vec{l}(x,y) = \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix}$.

Divergence

Def Given a volume or area V, the ***boundary*** of V is denoted ∂V and is assumed to have an outwards or counter clockwise orientation.

Def The ***divergence theorem*** (also called ***Gauss's theorem*** or ***Ostrogradsky's theorem***) states that for a vector field $\vec{f}$ and a region R, the flux of $\vec{f}$ through ∂R equals the integral of the divergence of $\vec{f}$ over R. In symbols,

$$\iint_{\partial R} \vec{f}\cdot\hat{n}\,\mathrm{d}V_{\partial R} = \iiint_R \nabla\cdot\vec{f}\,\mathrm{d}V_R,$$

where $\mathrm{d}V_R$ is the volume form for R, $\mathrm{d}V_{\partial R}$ is the volume form for ∂R, and $\hat{n}$ is a unit normal vector to ∂R.

If we interpret $\vec{f}$ as the velocity of a fluid, the divergence of $\vec{f}$ can be thought of as how much fluid is being created or destroyed at a point. The divergence theorem then says, the amount of fluid leaving R (the flux through ∂R) is equal to the sum of all the fluid being created or destroyed in R (the integral of $\nabla\cdot\vec{f}$ over R).

We will produce an intuition for the divergence theorem in 2 dimensions, since it's easier to draw.

66 Let $\vec{f}(x,y)=(x,y)$ be a vector field and let $R=[0,3]\times[0,3]$.

66.1 Draw $\vec{f}$ and R.

66.2 Compute the flux of $\vec{f}$ through ∂R.

66.3 Divide R up into tiny subsquares of equal size. What is the sum of the flux of $\vec{f}$ through all the subsquares? Why?

66.4 Explain how a Riemann sum and the definition of divergence can be used to motivate the divergence theorem.

Curl

Curl is the compliment to divergence, and as the name suggests it has something to do with rotations.

Imagine a vector field $\vec{f}:\mathbb{R}^3\to\mathbb{R}^3$. If you stuck a small sphere at the point $\vec{x}\in\mathbb{R}^3$ and interpreted $\vec{f}$ as the velocity of a fluid, the sphere might start spinning, and this spinning is exactly what curl measures.

Def An isometric parameterization $\vec{p}:\mathbb{R}^3\to\mathbb{R}^3$ is called an ***isometry***. An isometry $\vec{r}:\mathbb{R}^3\to\mathbb{R}^3$ is called a ***rotation*** if $\vec{r}(0,0,0)=(0,0,0)$ and if $\vec{r}$ is orientation preserving. That is, $\vec{r}(1,0,0)\times\vec{r}(0,1,0)=\vec{r}(0,0,1)$.

If $\vec{r}:\mathbb{R}^3\to\mathbb{R}^3$ is a rotation, an ***axis of rotation*** for $\vec{r}$ is a line $\ell=\{t\vec{d}:t\in\mathbb{R}\}$ such that $\vec{r}(\ell)=\ell$.

Amazingly (though maybe you're used to this) any non-trivial rotation has exactly one axis of rotation! This is something very special about 3D space (and it's not true in 4D and higher)! Further, a non-trivial rotation $\vec{r}$ can be described by its axis of rotation, the magnitude of the rotation, and the attribute of clockwise or counterclockwise.

Another amazing coincidence of 3D space (following from the first coincidence) is that we can encode all of this information in a single vector. Let $\vec{r}:\mathbb{R}^3\to\mathbb{R}^3$ be a rotation and let $\vec{u}$ be a unit vector in the direction of $\vec{r}$'s axis of rotation. We can describe $\vec{r}$ by the vector $\alpha\vec{u}$ where $|\alpha|$ is the magnitude of the rotation (i.e., the number of radians $\vec{r}$ rotates by), and the sign of α is positive if the rotation is counterclockwise and negative if it's clockwise (as judged by looking at how the plane $\vec{u}\cdot\vec{x}=0$ rotates from the vantage point of $\vec{u}$).

67 For each of the following rotations in $\mathbb{R}^3$, write down its representation as a vector.

67.1 Rotation of the xy plane counterclockwise by $\pi/3$.

67.2 Rotation of the xy plane clockwise by $\pi/3$.

67.3 The rotation that sends $\hat{\mathbf{x}}$ to $\vec{v}=(\sqrt{2}/2,0,\sqrt{2}/2)$.

68 Let $\vec{f}:\mathbb{R}^3\to\mathbb{R}^3$ be a vector field and let $R_{\Delta x,\Delta y}$ be the rectangle parallel to the xy-plane with lower left corner at $\vec{0}$ and side lengths Δx and Δy. Give $\partial R_{\Delta x,\Delta y}$ a counterclockwise orientation.

68.1 Let B be the bottom of $\partial R_{\Delta x,\Delta y}$. That is, $B=\{(x,0,0):0\le x\le \Delta x\}$ with the orientation inherited from $\partial R_{\Delta x,\Delta y}$. If Δx is very small, what is a reasonable estimate for the amount of work done as a point moves through $\vec{f}$ along B?

68.2 Estimate the amount of work done traversing $\partial R_{\Delta x,\Delta y}$, and call this quantity $C_{\Delta x,\Delta y}$.

68.3 Consider the limit

$$\lim_{\Delta x,\Delta y\to 0}\frac{C_{\Delta x,\Delta y}}{\Delta x\Delta y}.$$

Can you express this limit in terms of quantities we're already familiar with (directional/partial derivatives maybe)?

68.4 You've just computed the $\hat{z}$ component of the curl of $\vec{f}$! Now, suppose that Δx and Δy are large. Subdivide $R_{\Delta x,\Delta y}$ into small rectangles. Using a Riemann sum idea, come up with a conjecture about a theorem of similar form to the divergence theorem but for curl.

Def Let $R^{\vec{v}}(x,y,z)$ be a rectangle with side lengths Δa and Δb, corner through (x,y,z), and with normal vector $\vec{v}$.
Let $\vec{f}:\mathbb{R}^3\to\mathbb{R}^3$ be a vector field. The $\vec{v}$ component of the ***curl*** of $\vec{f}$ at the point (x,y,z) is

$$curl_{\vec{v}}(\vec{f})=\lim_{\Delta a,\Delta b\to 0}\frac{\text{work done by }\vec{f}\text{ on a particle traversing }\partial R^{\vec{v}}(x,y,z)}{\Delta a\Delta b}.$$

The ***curl*** of $\vec{f}$ at the point (x,y,z) is the vector, notated $\nabla\times\vec{f}(x,y,z)$, so that $(\nabla\times\vec{f}(x,y,z))\cdot\frac{\vec{v}}{\|\vec{v}\|}=curl_{\vec{v}}(\vec{f})$.

Computing curl from the definition each time can be cumbersome, so let's see if there's a formula.

69 Let $\vec{f}:\mathbb{R}^3\to\mathbb{R}^3$ be a vector field.

69.1 Write down the definition for the $\hat{x}$, $\hat{y}$, and $\hat{z}$ components of the curl of $\vec{f}$. Do you see anything that looks like a derivative?

69.2 Does the notation $\nabla\times\vec{f}(x,y,z)$ make sense for the curl of $\vec{f}$ at a point? Explain.

70 Let $\vec{f}:\mathbb{R}^3\to\mathbb{R}^3$ be a vector field and let $g:\mathbb{R}^3\to\mathbb{R}$ be a scalar field.

70.1 What is the curl of the gradient of g?

70.2 What is the divergence of the curl of $\vec{f}$? Does it make sense to take the curl of the divergence of $\vec{f}$?

70.3 Could the vector field $\vec{h}(x,y,z)=(x,y,z)$ be the curl of $\vec{f}$?

Def Let $\vec{f}:\mathbb{R}^3\to\mathbb{R}^3$ be a vector field, and let $\mathcal{S}$ be a surface. ***Stokes's theorem*** says that the work done by $\vec{f}$ on a particle traversing $\partial\mathcal{S}$ is equal to the flux of $\nabla\times\vec{f}$ through $\mathcal{S}$. That is,

$$\int_{\partial\mathcal{S}}\vec{f}\cdot\hat{r}\,\mathrm{d}V_{\partial\mathcal{S}}=\iint_{\mathcal{S}}(\nabla\times\vec{f})\cdot\hat{n}\,\mathrm{d}V_{\mathcal{S}},$$

where $\hat{r}$ is a unit tangent vector to $\partial\mathcal{S}$, $\hat{n}$ is a unit normal vector to $\mathcal{S}$, $\mathrm{d}V_{\partial\mathcal{S}}$ is the volume form for $\partial\mathcal{S}$ (in this case, it's the arc-length form), and $\mathrm{d}V_{\mathcal{S}}$ is the volume form for $\mathcal{S}$.

Systems of Linear Equations

Linear equations are equations only involving variables, multiplication by constants, and addition/subtraction. *Systems* of equations are sets of equations that share common variables.

71 Consider the system

$$\begin{aligned} x - y &= 2 \\ 2x + y &= 1 \end{aligned} \tag{2}$$

71.1 Draw the lines in (2) on the same coordinate plane.

71.2 Algebraically solve the system (2). What does this solution represent on your graph?

72 Let L be the line given by $x - y = 2$.

72.1 Write an equation of a line that doesn't intersect L.

72.2 Write an equation of a line that intersects L in

(a) one place.
(b) infinitely many places
(c) exactly two places

or explain why no such equation exists.

72.3 For each equation you came up with, solve the system algebraically. How can you tell algebraically how many solutions there are?

The Row Reduction Algorithm

73 73.1 Solve the system

$$\begin{aligned} x - y - 2z &= -5 \\ 2x + 3y + z &= 5 \\ 0x + 2y + 3z &= 8 \end{aligned} \tag{3}$$

any way you like.

73.2 Use an augmented matrix to solve the system (3).

The system (3) can be interpreted in two ways (and switching between these interpretations when appropriate is one of the most powerful tools of Linear Algebra). We can think of solutions to (3) as the intersection of three planes, or we can interpret the solution as coefficients of a linear combination.

73.3 Rewrite (3) as a vector equation of the form

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{p}$$

where x, y, z are interpreted as scalar quantities.

73.4 If (x, y, z) is a solution to (3), explain how to get from the origin to $\vec{p}$ using only $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

73.5 If (x, y, z) is a solution to (3), is $\vec{p} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

74 Consider the augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -7 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

74.1 Write the system of equations corresponding to A.

74.2 Solve the system of equations corresponding to A.

Infinite Solutions

75 Consider the system

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 6 \end{aligned} \tag{4}$$

75.1 How many solutions does (4) have?

75.2 Write the solutions to (4) in vector form.

75.3 What happens when you use an augmented matrix to solve (4)?

Free Variables

76 Suppose the row-reduced augmented matrix corresponding to a system is

$$B = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

After reducing, we have 1 equation and 2 unknowns, so we can make $2 - 1 = 1$ choices when writing a solution. Let's make the choice $y = t$.

76.1 With the added equation $y = t$, solve the system represented by B.

77 Consider the system given by the augmented matrix

$$C = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

and call the variables in this system x_1, x_2, x_3, x_4, x_5.

77.1 Write the system of equations represented by C.

77.2 Identify how many choices you can make when writing down a solution corresponding to C.

77.3 Add one equation (of the form $x_i = t$ or $x_j = s$, etc.) for each choice you must make when solving the system.

77.4 Write in vector form all solutions to C.

78 78.1 An unknown system U is represented by an augmented matrix with 4 rows and 6 columns. What is the minimum number of free variables solutions to U will have?

78.2 An unknown system V is represented by an augmented matrix with 6 rows and 4 columns. What is the minimum number of free variables solutions to V will have?

79

Def A system is called ***homogeneous*** if all equations equal 0.

Let A be an unknown system of 3 equations and 3 variables and suppose $(x, y, z) = (1, 2, 1)$ and $(x, y, z) = (-1, 1, 1)$ are solutions to A.

79.1 Can you produce another solution to the system?

79.2 Can you produce a solution to the homogeneous version of A (the version of A where every equation equals 0)?

79.3 Suppose when you use an augmented matrix to solve the system A, you only have one free variable. Could A be homogeneous? Can you produce all solutions to the system A?

Rank

Def The ***rank*** of the matrix A is the number of leading ones in the reduced row echelon form of A.

80

80.1 Determine the rank of (a) $\begin{bmatrix}1&1\\2&2\end{bmatrix}$ (b) $\begin{bmatrix}1&2\\3&4\end{bmatrix}$ (c) $\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$ (d) $\begin{bmatrix}3\\3\\2\end{bmatrix}$ (e) $\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$.

81

Consider the homogeneous system

$$\begin{array}{llll} x & +2y & +z & =0\\ x & +2y & +3z & =0\\ -x & -2y & +z & =0\end{array} \tag{5}$$

and the non-augmented matrix of coefficients $A=\begin{bmatrix}1&2&1\\1&2&3\\-1&-2&1\end{bmatrix}$.

81.1 What is rank(A)?

81.2 Give the general solution to (5).

81.3 Are the column vectors of A linearly independent?

81.4 Give a non-homogeneous system with the same coefficients as (5) that has

(a) infinitely many solutions

(b) no solutions.

82

82.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

82.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Span Again

83

Consider the system

$$\begin{array}{llll} x & -y & -z & =0\\ 0x & +1y & +2z & =0\\ 3x & -3y & +3z & =0\end{array} \tag{6}$$

which has the unique solution $(x,y,z)=(0,0,0)$.

83.1 Give vectors $\vec{u},\vec{v},\vec{w}$ so that the system (6) corresponds to the vector equation $x\vec{u}+y\vec{v}+z\vec{w}=\vec{0}$.

83.2 Is $\vec{w}\in\operatorname{span}\{\vec{u},\vec{v}\}$? If so, write it as a linear combination of $\vec{u}$ and $\vec{v}$.

The matrix M is the non-augmented matrix corresponding to a homogeneous system of linear equations. M also corresponds to the vector equation $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$. Further, we know

$$\operatorname{rref}(M)=\begin{bmatrix}1&0&1\\0&1&-2\\0&0&0\end{bmatrix}.$$

83.3 Give a solution to the vector equation $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$.

83.4 Is $\vec{c}\in\operatorname{span}\{\vec{a},\vec{b}\}$? If so, write it as a linear combination of $\vec{a}$ and $\vec{b}$.

83.5 Do you have enough information to tell if $\{\vec{a},\vec{b}\}$ is linearly independent? Why or why not?

Finding Linearly Independent Subsets

84 Suppose when you use an augmented matrix to solve $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$ you have no free variables.

84.1 Is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

Suppose when you use an augmented matrix to solve $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$, the second column corresponds to a free variable.

84.2 Is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

84.3 Is $\{\vec{u},\vec{w}\}$ linearly independent?

84.4 Is $\{\vec{u},\vec{v}\}$ linearly independent?

Def Given a set of vectors X, a ***maximal linearly independent subset*** of X is a linearly independent subset $V\subseteq X$ with the most possible vectors in it (i.e., if you took any subset of X with more vectors, it would be linearly dependent).

85

85.1 Give a maximal linearly independent subset, T, of $\left\{\begin{bmatrix}a\\b\\c\end{bmatrix} : a,b,c\in\mathbb{R}\right\}$.

85.2 What is the size of T?

86 Consider the vectors

$$\vec{v}_1=\begin{bmatrix}1\\2\\1\end{bmatrix}\qquad \vec{v}_2=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}\qquad \vec{v}_3=\begin{bmatrix}0\\1\\0\end{bmatrix}\qquad \vec{v}_4=\begin{bmatrix}-1\\2\\0\end{bmatrix}\qquad \vec{v}_5=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

and the matrices

$$A=\begin{bmatrix}1&-1&0&-1&1\\2&-1&1&2&-1\\1&-1&0&0&1\end{bmatrix}\qquad \operatorname{rref}(A)=\begin{bmatrix}1&0&1&0&-2\\0&1&1&0&-3\\0&0&0&1&0\end{bmatrix}.$$

(Notice that the columns of A are the vectors $\vec{v}_1,\ldots,\vec{v}_5$)

86.1 Is $V=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4,\vec{v}_5\}$ linearly independent?

86.2 Pick a maximal linearly independent subset of V.

86.3 Pick another (different) maximal linearly independent subset of V.

86.4 Give a basis for $\operatorname{span}(V)$.

86.5 What is the dimension of $\operatorname{span}(V)$?

Matrices

87

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

87.1 Write the shape of the matrices A, B, C (i.e., for each one, write the dimensions in $m \times n$ form).

87.2 List *all* products between the matrices A, B, C that are defined. (Your list will be some subset of AB, AC, BA, CA, BC, CB.)

87.3 Compute AC and CA.

88

88.1 If the matrices X and Y are both square $n \times n$ matrices, does $XY = YX$? Explain.

88.2 If the matrices X and Y are both square $n \times n$ matrices, does $X + Y = Y + X$? Explain.

89

Consider the system

$$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned} \tag{7}$$

89.1 Find values of a, b, c, d, e, f so that the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

represents the same system as (7).

Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

89.2 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

89.3 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

90

The entries of a matrix are specified by (row,column) pairs of integers. If a_{ij} is the (i, j) entry of a matrix A, we may write $A = [a_{ij}]$.

90.1 Write the 2×2 matrix A with entries $a_{11} = 4$, $a_{12} = 3$, $a_{21} = 7$ and $a_{22} = 9$.

90.2 Let $B = [b_{ij}]$ be the 3×3 matrix where $b_{ij} = i + j$. Write B.

90.3 Let $C = [c_{ij}]$ be the 3×4 matrix where $c_{ij} = 0$ if $i = j$ and $c_{ij} = 1$ if $i \neq j$.

91

Def The ***transpose*** of a matrix $A = [a_{ij}]$ is the matrix $A^T = [a_{ji}]$.

Visually, the transpose of a matrix swaps rows and columns.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

91.1 What is the shape of A and A^T?

91.2 Write down A^T.

B and D are 4×6 matrices and C is a 6×4 matrix.

91.3 Does $(BC)^T = B^T C^T$? Explain.

91.4 Does $(B + D)^T = B^T + D^T$? Explain.

91.5 Compute AA^T and $A^T A$ (where A is the matrix defined earlier). What do you notice?

92

Def A matrix X is called ***symmetric*** if $X = X^T$.

Symmetric matrices have many useful properties, and have deep connections with orthogonality and eigenvectors (which we will get to later on).

92.1 Prove that if W is a square matrix, then $V = W^T W + W + W^T$ is a symmetric matrix.

93

Def A ***zero matrix*** is a matrix whose entries are all zeros. An ***identity matrix*** is a square matrix whose diagonal entries are 1 and non-diagonal entries are 0.

We write the $m \times n$ zero matrix as $0_{m\times n}$ or just 0 if the shape is determined by context. The $n \times n$ identity matrix is notated $I_{n\times n}$ or just I if the shape is determined by context.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

93.1 Write down the 3×3 identity matrix and the 3×3 zero matrix.

93.2 Compute $I_{3\times 3}A$, $AI_{3\times 3}$, $0_{3\times 3}A$, and $A0_{3\times 3}$.

93.3 If we were to think of matrices as numbers, what numbers would the zero matrix and the identity matrix correspond to?

94

94.1 Solve the matrix equation

$$I_{4\times 4} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$

Linear Transformations

95 $\mathcal{R}:\mathbb{R}^2\to\mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

95.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

95.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

95.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

95.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter clockwise by 90°.

96 $\mathcal{S}:\mathbb{R}^3\to\mathbb{R}^3$ stretches in the $\hat{\mathbf{z}}$ direction by a factor of 2 and contracts in the $\hat{\mathbf{y}}$ direction by a factor of 3.

96.1 Write a matrix representation of $\mathcal{S}$.

Def A ***linear transformation*** is a function that takes in vectors and outputs vectors and which distributes with respect to vector addition and scalar multiplication. That is $T:\mathbb{R}^n\to\mathbb{R}^m$ is a linear transformation if

$$T(\vec{u}+\vec{v})=T\vec{u}+T\vec{v}\qquad\text{and}\qquad T(a\vec{v})=aT\vec{v}$$

for all scalars a.

97 97.1 Classify the following as linear transformation or not

(a) $\mathcal{R}$ from above.

(b) $\mathcal{S}$ from above.

(c) $W:\mathbb{R}^2\to\mathbb{R}^2$ where $W\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x^2\\y\end{bmatrix}$.

(d) $T:\mathbb{R}^2\to\mathbb{R}^2$ where $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+2\\y\end{bmatrix}$.

(e) $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix}x\\y\end{bmatrix}=\mathrm{proj}_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

It turns out every linear transformation can be written as a matrix (in fact this is why matrix multiplication was invented).

98 Define $\mathcal{P}$ to be projection onto $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

98.1 Write down a matrix for $\mathcal{P}$.

98.2 What is the rank of the matrix corresponding to $\mathcal{P}$?

Matrix multiplication was designed to exactly model composition of linear transformations.

98.3 Write down a matrix for $\mathcal{P}$ and for $\mathcal{R}$, the counter-clockwise rotation by 90°.

98.4 Write down matrices for $\mathcal{P}\circ\mathcal{R}$ and $\mathcal{R}\circ\mathcal{P}$.

Def The ***range*** (or ***image***) of a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ is the set of vectors that T can output. That is,

$$\mathrm{range}(T)=\{\vec{y}\in\mathbb{R}^m:\vec{y}=T\vec{x}\text{ for some }\vec{x}\in\mathbb{R}^n\}.$$

Def The ***null space*** (or ***kernel***) of a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ is the set of vectors that get mapped to zero under T. That is,

$$\mathrm{null}(T)=\{\vec{x}\in\mathbb{R}^n:T\vec{x}=\vec{0}\}.$$

99 Let $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ be projection onto the vector $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$ (like before).

99.1 What is the range of $\mathcal{P}$?

99.2 What is the null space of $\mathcal{P}$?

100 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

100.1 Show that the null space of T is a subspace.

100.2 Show that the range of T is a subspace.

Def Associated with any matrix M are three fundamental subspaces: the ***row space*** of M is the span of the rows of M; the ***column space*** of M is the span of the columns of M; and the ***null space*** of M is the set of solutions to $M\vec{x} = \vec{0}$.

The ***nullity*** of M is the dimension of the null space of M.

101 Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

101.1 Describe the row space of A.

101.2 Describe the column space of A.

101.3 Is the row space of A the same as the column space of A?

101.4 Describe the set of all vectors perpendicular to the rows of A.

101.5 Describe the null space of A.

102

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

102.1 How does the row space of B relate to the row space of C?

102.2 How does the null space of B relate to the null space of C?

102.3 Compute the null space of B.

103

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \mathrm{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

103.1 How does the column space of P relate to the column space of Q?

103.2 Describe the columns space of P and the column space of Q.

Def The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem states

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = \#\text{ of columns in } A.$$

104 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

104.1 What is the rank and nullity of A^T?

104.2 What is the rank and nullity of A?

Matrix Inverses

105

105.1 Apply the row operation $R_3 \to R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

105.2 Apply the row operation $R_3 \to R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

Def An ***elementary matrix*** is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

105.3 Compute E_1A and E_2A. How do the resulting matrices relate to row operations?

105.4 Without computing, what should the result of applying the row operation $R_3 \to R_3 - 2R_1$ to E_1 be? Compute and verify.

105.5 Without computing, what should E_1E_2 be? What about E_2E_1? Now compute and verify.

Def The ***inverse*** of an $n \times n$ matrix A is an $n \times n$ matrix B such that $AB = I_{n\times n} = BA$. In this case, B is called the inverse of A and is notated as A^{-1}.

106

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

106.1 Which pairs of matrices above are inverses of each other?

107

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

107.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

107.2 What is E_2E_1B?

107.3 Find B^{-1}.

107.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

108

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

108.1 What is $A^{-1}A$?

108.2 What is rref(A)?

108.3 What is rref(C)? (Hint, there is no need to actually do row reduction!)

108.4 Solve the system $A\vec{x} = \vec{b}$.

109

109.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

109.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

Change of Basis

110 Let $\vec{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, $\vec{c} = \begin{bmatrix}4\\0\end{bmatrix}$, and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$.

110.1 Is $\mathcal{B}$ a basis for $\mathbb{R}^2$?

110.2 Find coefficients α_1 and α_2 so that $\vec{c} = \alpha_1\vec{b}_1 + \alpha_2\vec{b}_2$.

We call the vector $\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}$ the representation of $\vec{c}$ in the $\mathcal{B}$ basis and notate this by $[\vec{c}]_{\mathcal{B}} = \begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}$.

110.3 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

Let $X = [\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$.

110.4 Compute $X[\vec{c}]_{\mathcal{B}}$. What do you notice?

111 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X = [\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{S}}.$$

111.1 Should X^{-1} exist? Explain.

111.2 Consider the equation

$$X^{-1}[\vec{v}]_? = [\vec{v}]_?.$$

Can you fill in the "?" symbols so that the equation makes sense?

111.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

112 Let $\vec{c}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, and $A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

112.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

112.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}$ and $T\begin{bmatrix}5\\3\end{bmatrix}$.

112.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

112.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

112.5 Find a matrix for T in the $\mathcal{C}$ basis.

112.6 Find a matrix for T in the standard basis.

Def A matrix A and a matrix B are ***similar matrices***, denoted $A \sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A \sim B$ if there is an invertible matrix X so that

$$A = XBX^{-1}.$$

Determinants

Def The unit n-cube is the n-dimensional cube with side length 1 and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The volume of the unit n-cube is always 1.

113 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).

113.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}, T\begin{bmatrix}0\\1\end{bmatrix}, T\begin{bmatrix}1\\1\end{bmatrix}$?

113.2 Write down a matrix for T.

113.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may need to use trigonometry.

Def The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors or the row vectors.

114 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

114.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

114.2 Compute the area of $A(C_2)$.

114.3 Compute $\det(A)$.

115 Suppose R is a rotation counterclockwise by 30°.

115.1 Draw C_2 and $R(C_2)$.

115.2 Compute the area of $R(C_2)$.

115.3 Compute $\det(R)$.

116 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \quad \text{and} \quad F\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}.$$

116.1 What is $\det(F)$?

117

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1 \to R_1 + 2R_2$ applied.

117.1 What is $\det(E_f)$?

117.2 What is $\det(E_m)$?

117.3 What is $\det(E_a)$?

117.4 What is $\det(E_f E_m)$?

117.5 What is $\det(4I_{3\times 3})$?

117.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

118

$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

118.1 What is $\det(U)$?

When you row reduce the square matrix V, there is a row of zeros.

118.2 What is $\det(V)$?

P is projection onto the vector $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

118.3 What is $\det(P)$?

119

Suppose you know $\det(X) = 4$.

119.1 What is $\det(X^{-1})$?

119.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

119.3 Suppose Y is not invertible. What is $\det(Y)$?

After all this work with determinants, we see that (like dot products) there is a geometric and an algebraic way of thinking about them, and they *determine* if a matrix is invertible.

Eigenvectors

Def For a transformation X, an ***eigenvector*** for X is a vector that doesn't change directions when X is applied. That is, $\vec{v}$ is an eigenvector for X if
$$X\vec{v} = \lambda\vec{v}$$
for some $\lambda \in \mathbb{R}$. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

120 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).

120.1 Give an eigenvector for T. What is the eigenvalue?

120.2 Can you give another?

121 For some matrix A,
$$A\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B = A - \tfrac{2}{3}I.$$

121.1 Give an eigenvector and a corresponding eigenvalue for A.

121.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

121.3 What is the dimension of $\mathrm{null}(B)$?

121.4 What is $\det(B)$?

122 Let $C = \begin{bmatrix}-1 & 2\\ 1 & 0\end{bmatrix}$ and $E_\lambda = C - \lambda I$.

122.1 For what values of λ does E_λ have a non-trivial null space?

122.2 What are the eigenvalues of C?

122.3 Find the eigenvectors of C.

Def For a matrix A, the ***characteristic polynomial*** of A is
$$\mathrm{char}(A) = \det(A - \lambda I).$$

123 Let $D = \begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$.

123.1 Compute $\mathrm{char}(D)$.

123.2 Find the eigenvalues of D.

124 Suppose $\mathrm{char}(E) = \lambda(\lambda-2)(\lambda+3)$ for some unknown 3×3 matrix E.

124.1 What are the eigenvalues of E?

124.2 Is E invertible?

124.3 What is $\mathrm{nullity}(E)$, $\mathrm{nullity}(E-3I)$, $\mathrm{nullity}(E+3I)$?

125 Define
$$A = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix}1\\1\\-2\end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$$
and notice that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors for A.

125.1 Find the eigenvalues of A.

125.2 Find the characteristic polynomial of A.

125.3 Compute $A\vec{w}$ where $w = 2\vec{v}_1 - \vec{v}_2$.

125.4 Compute $A\vec{u}$ where $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for unknown scalar coefficients a, b, c.

Notice that $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

125.5 If $\vec{x} = \begin{bmatrix}1\\3\\4\end{bmatrix}_{\mathcal{V}}$ is $\vec{x}$ written in the $\mathcal{V}$ basis, compute $A\vec{x}$ in the $\mathcal{V}$ basis.

126 The transformation P^{-1} takes vectors in the standard basis and outputs vectors in their $\mathcal{V}$-basis representation (where $\mathcal{V}$ is from above).

126.1 Describe in words what P does.

126.2 Describe how you can use P and P^{-1} to easily compute $A\vec{y}$ for any $\vec{y} \in \mathbb{R}^3$.

126.3 Can you find a matrix D so that

$$PDP^{-1} = A?$$

126.4 $\vec{x} = \begin{bmatrix}1\\3\\4\end{bmatrix}_{\mathcal{V}}$. Compute $A^{100}\vec{x}$.

127 For an $n \times n$ matrix T, suppose its eigenvectors $\{\vec{v}_1, \dots \vec{v}_n\}$ form a basis for $\mathbb{R}^n$. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues.

127.1 Is T diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

127.2 What if one of the eigenvalues of T is zero? Is T diagonalizable?

127.3 What if the eigenvectors of T did not form a basis for $\mathbb{R}^n$. Would T be diagonalizable?

Def Let A be a matrix with eigenvalues $\{\lambda_1, \dots, \lambda_m\}$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the eigenspace corresponding to λ_i. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

128 Define $F = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$.

128.1 Is F diagonalizable? Why or why not?

128.2 What is the geometric and algebraic multiplicity of each eigenvalue of F?

128.3 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?

Orthogonality

Def A set of vectors is ***orthogonal*** if every pair of vectors in the set is orthogonal.

Def A set of vectors is ***orthonormal*** if the set is orthogonal and every vector is a unit vector.

129

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} \qquad \vec{b}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

The matrix $A = [\vec{b}_1|\vec{b}_2]$ takes vectors in the $\mathcal{B}$ basis and rewrites them in the standard basis.

129.1 What does A^{-1} do?

129.2 Find a matrix B that takes vectors in the standard basis and rewrites them in the $\mathcal{B}$ basis.

129.3 Write $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_S$ in the $\mathcal{B}$ basis.

129.4 What is the relationship between A and B?

Def An ***orthogonal*** matrix is a square matrix whose columns are orthonormal (Yes, a better name would be orthonormal matrix, but that is not the term the rest of the world uses).

130 Suppose $X = [\vec{x}_1|\vec{x}_2|\vec{x}_3|\vec{x}_4]$ is an orthogonal matrix.

130.1 What is the shape of X (i.e., it is a what×what matrix)?

130.2 Compute X^TX.

130.3 What is X^{-1}?

131

$$Y = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

131.1 Is Y an orthogonal matrix?

131.2 Fix Y so it is an orthogonal matrix. Call the new matrix X.

131.3 Compute X^{-1}.

131.4 Compute Y^{-1}.

131.5 Compute $|\det(X)|$ and $|\det(Y)|$ (the absolute value of the determinant of X and Y).

Matrix equations involving orthogonal matrices are easy to solve because the inverse of an orthogonal matrix is so easy to compute!

132 Let $A = [\vec{a}_1|\vec{a}_2|\vec{a}_3|\vec{a}_4]$ be an orthogonal matrix.

132.1 Explain why $\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{b} \\ \vec{a}_2 \cdot \vec{b} \\ \vec{a}_3 \cdot \vec{b} \\ \vec{a}_4 \cdot \vec{b} \end{bmatrix}$ is a solution to $A\vec{x} = \vec{b}$.

132.2 Find scalars a, b, c, d so $\vec{b} = a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3 + d\vec{a}_4$ (your answers will have variables in them).

Orthogonal matrices also allow us to compute projections quite easily.

Def If V is a subspace of $\mathbb{R}^n$, the ***projection*** (sometimes called the orthogonal projection) of $\vec{x}$ onto V is the closest point in V to $\vec{x}$. We notate the projection of $\vec{x}$ onto V as $\text{proj}_V \vec{x}$.

Projections are normally hard to compute and a priori might require some sort of calculus-style optimization to find. However, from geometry we know that if we travel from $\text{proj}_V \vec{x}$ to $\vec{x}$, we should always trace out a path perpendicular to V. Otherwise, we could find a point in V that was slightly closer to $\vec{x}$, violating the definition of $\text{proj}_V \vec{x}$. Thus, orthogonality will be our savior.

133 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis.

133.1 If $\vec{x} = 1\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$, find the projection of $\vec{x}$ onto the xy-plane.

Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

133.2 If $\vec{y} = 3\vec{b}_1 - 2\vec{b}_2 + 2\vec{b}_3$, find the projection of $\vec{y}$ onto $\text{span}\{\vec{b}_1, \vec{b}_3\}$.

Suppose $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ is a basis for $\mathbb{R}^3$ with

$$\|\vec{c}_1\| = \|\vec{c}_2\| = \|\vec{c}_3\| = 1 \qquad \vec{c}_1 \cdot \vec{c}_2 = 0 \qquad \vec{c}_1 \cdot \vec{c}_3 = 0 \qquad \vec{c}_2 \cdot \vec{c}_3 = \sqrt{2}/2.$$

133.3 If $\vec{z} = 5\vec{c}_1 + 2\vec{c}_2 - \vec{c}_3$, find the projection of $\vec{z}$ onto $\text{span}\,\{\vec{c}_1, \vec{c}_2\}$.

134 Let's put this all together. $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Let $\mathcal{P}$ be the plane defined by

$$0x + y - z = 0.$$

134.1 Write $\mathcal{P}$ in vector form (Hint: think about the vectors listed in the $\mathcal{B}$ basis).

134.2 Find an orthonormal basis $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ for $\mathbb{R}^3$ so $\mathcal{P} = \text{span}\,\{\vec{c}_1, \vec{c}_2\}$.

134.3 Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\text{proj}_{\mathcal{P}} \vec{x}$.

Gram-Schmidt Orthogonalization

We've seen how useful orthonormal bases are. The incredible thing is that we can turn any basis into an orthonormal basis through a process called Gram-Schmidt orthogonalization.

135 Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

135.1 Draw $\vec{a}$ and $\vec{b}$ and find $\vec{w} = \text{proj}_{\vec{b}} \vec{a}$.

135.2 Add $\vec{c} = \vec{a} - \vec{w}$ to your drawing. What is the angle between $\vec{c}$ and $\vec{b}$.

135.3 Can you write $\vec{a}$ as the sum of two vectors, one in the direction of $\vec{b}$ and one orthogonal to $\vec{b}$? If so, do it.

136 Let $\vec{a}=\begin{bmatrix}1\\2\\6\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$.

136.1 Write $\vec{a}=\vec{u}+\vec{v}$ where $\vec{u}$ is parallel to $\vec{b}$ and $\vec{v}$ is orthogonal to $\vec{b}$.

136.2 Find an orthonormal basis for $\operatorname{span}\{\vec{a},\vec{b}\}$.

With two vectors, making an orthonormal set without changing the span is quite easy. With more vectors, it is only slightly harder.

Def The ***Gram-Schmidt*** orthogonalization procedure takes in a set of vectors and outputs a set of orthonormal vectors with the same span. The idea is to iteratively produce a set of vectors where each new vector you produce is orthogonal to the previous vectors.

The algorithm is as follows: Let $\{v_1,\ldots,v_n\}$ be a set of vectors. Produce a set $\{v_2',\ldots,v_n'\}$ that is orthogonal to v_1 by subtracting off the respective projections of $v_2,\ldots,v_n$ onto v_1. Next, produce a set $\{v_3'',\ldots,v_n''\}$ orthogonal to both v_1 and v_2' by subtracting off the respective projections onto v_2'. Continue this process until you have a set $V=\{v_1,v_2',v_3'',v_4''',\ldots\}$ that is orthogonal. Finally, normalize V so all vectors have unit length.

137 Let $\vec{x}_1=\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$, $\vec{x}_2=\begin{bmatrix}2\\1\\0\\1\end{bmatrix}$, and $\vec{x}_3=\begin{bmatrix}2\\2\\1\\2\end{bmatrix}$.

137.1 Use the Gram-Schmidt procedure to find an orthonormal basis for $\operatorname{span}\{\vec{x}_1,\vec{x}_2,\vec{x}_3\}$.

137.2 Find an orthonormal basis $\mathcal{V}=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4\}$ for $\mathbb{R}^4$ so that $\operatorname{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}=\operatorname{span}\{\vec{x}_1,\vec{x}_2,\vec{x}_3\}$.

Let $R=\begin{bmatrix}1&-1&-1&1\\2&1&0&1\\2&2&1&2\end{bmatrix}$.

137.3 Find an orthonormal basis for the row space of R.

137.4 Find the null space of R (Hint, you've already done the work, so there is no need to row reduce).

138 Let

$$\vec{y}_1=\begin{bmatrix}1\\1\\2\end{bmatrix}\qquad \vec{y}_2=\begin{bmatrix}-1\\-1\\2\end{bmatrix}\qquad \vec{y}_3=\begin{bmatrix}1\\1\\6\end{bmatrix}.$$

138.1 Find an orthonormal basis $\mathcal{W}$ so that $\operatorname{span}\mathcal{W}=\operatorname{span}\{\vec{y}_1,\vec{y}_2,\vec{y}_3\}$.

Def The ***orthogonal complement*** of a subspace V is written $V^\perp$ and defined as

$$V^\perp=\{\vec{x}:\vec{x}\text{ is orthogonal to }V\}.$$

138.2 Find the orthogonal complement of $\operatorname{span}\mathcal{W}$.

138.3 Write $\vec{v}=\begin{bmatrix}1\\0\\1\end{bmatrix}$ in the form $\vec{v}=\vec{r}+\vec{n}$ where $\vec{r}\in\operatorname{span}\mathcal{W}$ and $\vec{n}\in(\operatorname{span}\mathcal{W})^\perp$.

QR Decomposition

Def For a matrix A, we can rewrite $A = QR$ where Q is an orthogonal matrix and R is an upper triangular matrix. Writing A as QR is called the ***QR decomposition*** of A.

139 Suppose A, B, C are square matrices and $C = AB$.

139.1 How do the column spaces of A and C relate?

139.2 How do the column spaces of B and C relate?

140 $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms a basis for $\mathbb{R}^3$. When we apply the Gram-Schmidt process to $\mathcal{V}$, we get

$$\begin{aligned} q_1' &= \vec{v} \\ q_2' &= \vec{v}_2 - \tfrac{1}{2}\vec{v}_2 \\ q_3' &= \vec{v}_3 - \vec{v}_1 + 2\vec{v}_2 \end{aligned}$$

form an orthogonal set. Normalizing we get

$$\begin{aligned} \vec{q}_1 &= 2q_1' \\ \vec{q}_2 &= 3q_2' \\ \vec{q}_3 &= \tfrac{1}{2}q_3' \end{aligned}$$

form an orthonormal set.

140.1 Write $\vec{v}_1$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

140.2 Write $\vec{v}_2$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

140.3 Write $\vec{v}_3$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

Define $A = [\vec{v}_1|\vec{v}_2|\vec{v}_2]$ and $Q = [\vec{q}_1|\vec{q}_2|\vec{q}_3]$.

140.4 Find a matrix R so that $A = QR$.

We've just discovered one process to find the QR decomposition of a matrix. It's really as simple as doing Gram-Schmidt and keeping track of your coefficients. Now, we have another way to the matrix equation $A\vec{x} = \vec{b}$. If we do a QR decomposition and exploit the fact that $Q^{-1} = Q^T$, we have

$$A\vec{x} = QR\vec{x} = \vec{b} \qquad \Longrightarrow \qquad R\vec{x} = Q^T\vec{b}$$

and R is a triangular matrix, so we can just do back substitution! (It turns out that if you solve systems this way, there is less rounding error than if you use row reduction.)

Symmetric Matrices

When you're new to Linear Algebra, learning lots of new concepts and algorithms, it's sometimes hard to grasp the significance of certain properties of a matrix.

Symmetric matrices are easy to forget at first, but they have many profound properties (not to mention they are one of the key concepts of Quantum Mechanics).

141 Let A be a symmetric matrix and let $\vec{v}$ be an eigenvector with eigenvalue 3 and $\vec{w}$ be an eigenvector with eigenvalue 4. Note, for this problem, we are thinking of $\vec{v}$ and $\vec{w}$ as column vectors.

141.1 Write $A\vec{v}$, $\vec{v}^TA^T$, $\vec{v}^TA$, $A\vec{w}$, $\vec{w}^TA^T$, and $\vec{w}^TA$ in terms of $\vec{v}$, $\vec{w}$ and scalars.

141.2 How do $\vec{v}^T\vec{w}$ and $\vec{w}^T\vec{v}$ relate?

141.3 What should $\vec{v}^T A\vec{w}$ be in terms of $\vec{v}^T$ and $\vec{w}$? (Note, you could compute $(\vec{v}^T A)\vec{w}$ or $\vec{v}^T(A\vec{w})$. Better do both to be safe).

141.4 What can you conclude about $\vec{v}^T\vec{w}$? How about $\vec{v}\cdot\vec{w}$?

We've just deduced that all eigenspaces of a symmetric matrix are orthogonal! On top of that, symmetric matrices always have a basis of eigenvectors. That means that not only can you always diagonalize a symmetric matrix, but you can *orthogonally* diagonalize a symmetric matrix. (i.e. if A is symmetric, then $A = QDQ^T$ where Q is orthogonal and D is diagonal). This is like the best of all worlds in one!