Fourier Series and Boundary Value Problems

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1 A peek ahead

Consider a cylinder solid with perfectly insulated sides, from $x = 0$ to $x = c$. Let $u(x,t)$ be the temperature at position x and time t. Under appropriate assumptions, a mathematical model is the heat equation PDE

$$u_t(x,t) = ku_{xx}(x,t), \tag{1}$$

where $0 < x < c$ and $t > 0$. The boundary conditions are $u_x(0,t) = 0$ and $u_x(c,t) = 0$ for $t > 0$. For the initial conditions we have $u(x,0) = f(x)$ for $0 < x < c$. We shall assume the thermal diffusivity $k > 0$. This is derived in section 22 of the textbook. We will consider a solution method from section 36 in the textbook.

Idea: Look for solutions of the PDE (1) and the boundary conditions. There are infinitely many of them. Since (1) and the BC are linear and homogeneous, linear combinations are also solutions by the superposition principle. Then we try to determine a particular linear combination which also satisfies the initial conditions.

Idea: Separation of variables: Seek solutions of the form $u(x,t) = X(x)T(t)$, where $X, T \neq 0$. Plug in to (1) and BC to get

$$X(x)T'(t) = kX''(x)T(t) \tag{2}$$

with $0 < x < c$ and $t > 0$. Let's separate the variables to get

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}. \tag{3}$$

Since LHS is a function of only t and RHS is a function of only x, so both must be constant. $\frac{T'(t)}{kT(t)} = -\lambda = \frac{X''(x)}{X(x)}$, where $\lambda \in \mathbb{R}$. This gives two separate ODEs.

Note. According to the boundary conditions we have $X'(0)T(t) = 0$ and $X'(c)T(t) = 0$.

So we end up with

$$X''(x) + \lambda X(x) = 0, \qquad X'(0) = 0, \qquad X'(c) = 0, \tag{4}$$

which is the Sturm-Liouville BVP and

$$T'(t) + k\lambda T(t) = 0. \tag{5}$$

Equation (5) has a solution for any λ, i.e. $T(t) = e^{-k\lambda t}$. Equation (4) has solutions only for certain values of λ. Next we investigate (4) for different λ's.

When $\lambda = 0$, $X''(x) = 0$, $X'(0) = 0$, $X'(c) = 0$. Then $X(x) = Ax + B$ and $X'(x) = A$, so $A = 0$. So $X(x) = 1$ and any constant multiple.

When $\lambda > 0$, lets assume that we can write $\lambda = \alpha^2$, where $\alpha > 0$. Then $X''(x) + \alpha^2 X(x) = 0$. The general solution is $X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$. Then $X'(x) = -\alpha c_1 \sin\alpha x + \alpha c_2 \cos\alpha x$. $X'(0) = 0$ implies $0 = \alpha c_2 \cos 0$, so $c_2 = 0$ and $X(x) = c_1 \cos\alpha x$. $X'(c) = 0$ implies $0 = \alpha c_1 \sin\alpha c$, where $c_1 \neq 0$. Therefore $\sin\alpha c = 0$. So $\alpha = n\pi/c$, for $n \in \mathbb{N}$. $X(x) = \cos\left(\frac{n\pi}{c}x\right), n \in \mathbb{N}$ and any constant multiple.

When $\lambda < 0$ set $\lambda = -\alpha^2$ for $\alpha > 0$. $X''(x) - \alpha^2 X(x) = 0$. The general solution is $X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. Then $X'(x) = \alpha c_1 e^{\alpha x} - \alpha c_2 e^{-\alpha x}$. With $X'(0) = 0$ we get $\alpha c_1 - \alpha c_2 = 0$ so $c_1 = c_2$. That means that $X'(x) = \alpha c_1 e^{\alpha x} - \alpha c_1 e^{-\alpha x}$ with $X'(c) = 0$ implying $0 = \alpha c_1(e^{\alpha c} - e^{-\alpha c})$. This implies $c_1 = 0$, so we only get trivial solution $X(x) = 0$.

Set $\lambda_0 = 0$, then $\lambda_n = \left(\frac{n\pi}{c}\right)^2$ for $n \in \mathbb{N}$. $X_0(x) = 1$, $T_0(t) = 1$, then $X_n(x) = \cos\frac{n\pi}{c}x$ and $T_n(t) = e^{-k\frac{n^2\pi^2}{c^2}t}$. Then $u_0(x,t) = 1$ and $u_n(x,t) = \cos\left(\frac{n\pi}{c}x\right)e^{-k\frac{n^2\pi^2}{c^2}t}$ solve (1) and BCs. By superposition, $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{c}x\right)e^{-k\frac{n^2\pi^2}{c^2}t}$ is a solution. Can such a solution also satisfy the initial conditions? Can we determine $A_0, A_1, \ldots$ so that $A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{c}x = f(x)$?

HW: Sec 5 p.13: 2,3,4,6,8

2 Fourier Series

Consider the finite interval (a,b).

Definition 1. $f(x)$ is piecewise continuous (PWC) if f is continuous for all except a finite set of points on (a,b), and if those points and at endpoints the one-sided limits exist. $C_p(a,b)$ is the set of all functions that are PWC on the interval (a,b).

Example 1. $y = \tan x$ is <u>not</u> PWC on $(-\pi/2, \pi/2)$ because $\lim_{x\to\pi/2^-} f(x)$ does not exist.

Example 2. $f(x) = \begin{cases} 0 & \text{for } -1 < x < 0 \\ 1 & \text{for } 0 \le x < 1 \end{cases}$: then f is PWC.

Example 3. $f(x) = \sin(1/x)$ on $(0,1)$ is not PWC.

Note. If $f \in C_p(a,b)$, then $\int_a^b f(x)\,dx$ exists. If $f, g \in C_p(a,b)$, then $c_1 f(x) + c_2 g(x) \in C_p(a,b)$. Thus $C_p(a,b)$ is a vector space. It is an example of <u>function spaces</u>. If $f, g \in C_p(a,b)$, then $f \cdot g \in C_p(a,b)$.

We can define an inner product $(f,g) = \int_a^b f(x)g(x)\,dx$. Further, we can define orthogonal functions as $f \perp g \iff (f,g) = 0$. We can also define norm (length) as $||f|| = \sqrt{(f,f)} = \left(\int_a^b f^2(x)\,dx\right)^{1/2}$. Thus $C_p(a,b)$ is an infinite dimensional normed vector space.

Let $f \in C_p(0,\pi)$. <u>Assume</u> f has a Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{6}$$

for $0 < x < \pi$. Can we determine the the coefficients a_i? Yes, if we assume also we can integrate term by term. $\int_0^\pi f(x)\,dx = \int_0^\pi \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty}\int_0^\pi a_n \cos nx\,dx$. $\int_0^\pi f(x)\,dx = \frac{\pi}{2}a_0$, so $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$ because $\int_0^\pi a_n \cos nx\,dx = \frac{1}{n}\sin nx\Big|_0^\pi = 0$. In HW: $\int_0^\pi \cos mx \cdot \cos nx\,dx = \begin{cases} 0 & \text{for } m \ne n \\ \pi/2 & \text{for } m = n \end{cases}$. Rewrite f as $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mx$. Multiply by $\cos nx$ and integrate:

$$\int_0^\pi f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^\pi \cos nx\,dx + \sum_{m=1}^{\infty} a_m \int_0^\pi \cos mx \cdot \cos nx\,dx$$

$\int_0^\pi f(x)\cos nx\,dx = a_n\frac{\pi}{2}$, so $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$. We write: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$ for $n \in \mathbb{N}$.

Example 4. Consider $f(x) = x$ on $(0,\pi)$, $f \in C_p(0,\pi)$. Then

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^\pi = \pi$$

Set $x = u$ and $\cos nx = dv$ so that

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx = \frac{2}{\pi}x\frac{1}{n}\sin nx\Big|_0^\pi - \frac{2}{\pi}\frac{1}{n}\int_0^\pi \sin nx\,dx = \frac{2}{\pi}\frac{1}{n^2}\cos nx\Big|_0^\pi = \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

So $x \sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}((-1)^n - 1)\cos nx$, where even index terms are 0. So we re-index $n = 2k-1$ with

$$x \sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{-2}{(2k-1)^2}\cos(2k-1)x.$$

<u>Assume</u> f has a Fourier sine series
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$
for $0 < x < \pi$. We can determine a formula for b_n just like for the cosine series. In HW you show $\int_0^\pi \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n \end{cases}$. Write $f(x) = \sum_{m=1}^{\infty} b_m \sin mx$. Multiply by $\sin nx$ and integrate
$$\int_0^\pi f(x) \sin nx \, dx = \sum_{m=1}^{\infty} b_m \int_0^\pi \sin mx \sin nx \, dx = b_n \frac{\pi}{2},$$
so $b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx$.

Example 5. Calculate Fourier sine series for $f(x) = x$ on $(0, \pi)$
$$b_n = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx = \frac{2}{\pi} \left(x \frac{-1}{n} \cos nx \Big|_0^\pi + \frac{1}{n} \int_0^\pi \cos nx \, dx \right) = \frac{2}{\pi} \frac{-\pi}{n} (-1)^n = \frac{2}{n} (-1)^{n+1}$$
So $x \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

Note. Observe, even though $f(x)$ is only defined on $(0, \pi)$, the Fourier sine or cosine series are defined for all $x \in (-\infty, \infty)$. So on $(0, \pi)$, the series is 'equal' to $f(x)$. What does the series look like outside the interval?

Note. Note that every term in cosine series is an even function and every term in sine series is odd.

Example 6. $f(x) = x$ on $(0, \pi)$, $f(x) =$ cosine series. So on $(-\pi, \pi)$, the cosine series will be the even extension of f. It can also be extended periodically on the entire x-axis.

Example 7. Assuming on $[0, \pi)$ that the Fourier sine series converges. Then on $(-\infty, \infty)$.

Last time: Fourier cosine + sine series on $0 < x < \pi$.
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad a_n = \frac{2}{\pi} \int_0^\pi f(s) \cos ns \, ds \tag{7}$$
$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n \frac{2}{\pi} \int_0^\pi f(s) \sin ns \, ds \tag{8}$$
with even, odd, periodic extensions.

Thus, we can construct a Fourier series on $-\pi < x < \pi$ for any even or odd function.

What about other functions? Let $f \in C_p(-\pi, \pi)$. Observe $f(x) = g(x) + h(x)$, where $g(x) = \frac{f(x)+f(-x)}{2}$ and $h(x) = \frac{f(x)-f(-x)}{2}$. Then g is even and h is odd. Thus, $g(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ and $h(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$. If the series converge on $0 < x < \pi$, then they also converge on $-\pi < x < \pi$. So
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{9}$$
on $-\pi < x < \pi$. Set $x = -s$. Observe that
$$\begin{aligned} a_n = \frac{2}{\pi} \int_0^\pi g(x) \cos nx \, dx &= \frac{1}{\pi} \left[\int_0^\pi f(x) \cos nx \, dx + \int_0^\pi f(-s) \cos ns \, ds \right] \\ &= \frac{1}{\pi} \left[\int_0^\pi f(x) \cos nx \, dx + \int_{-\pi}^0 f(x) \cos nx dx \right] = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \cos nx \, dx. \end{aligned}$$
Similarly,
$$b_n = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin nx \, dx.$$

Equation (9) is called the Fourier series for f on $-\pi < x < \pi$. If f is even, it reduces to Fourier cosine series on $0 < x < \pi$. If f is odd it reduces to Fourier sine series on $0 < x < \pi$.

Recall. $\cos(A-B) = \cos A\cos B + \sin A \sin B$.

From equation (9),

$$\begin{aligned} f(x) &\sim \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\cos nx \int_{-\pi}^{\pi} f(s)\cos ns\,ds + \sin nx \int_{-\pi}^{\pi} f(s)\sin ns\,ds\right] \\ &\sim \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(s)\left[\cos ns\cos nx + \sin ns \sin nx\right]\,ds \\ &\sim \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(s)\cos n(s-x)\,ds. \end{aligned}$$

Will be useful later in convergence proof.

Example 8. $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi & 0 < x < \pi \end{cases}$. Then

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \pi\,dx = \pi, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{\pi}{\pi}\int_{0}^{\pi}\cos nx\,dx = 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -\frac{1}{n}\cos nx\Big|_0^{\pi} = \begin{cases} \frac{2}{n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}, \\ f(x) &\sim \frac{\pi}{2} + \sum_{n=1}^{\infty} b_n \sin nx = \frac{\pi}{2} + 2\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin(2k-1)x. \end{aligned}$$

Note. $f(x) - \frac{\pi}{2}$ is odd, only requiring terms from the Fourier sine series.

More general intervals of form $-c < x < c$. Let $f \in C_p(-c,c)$. Change of variables to interval $-\pi < s < \pi$. Define $g(s) = f(\frac{cs}{\pi})$, so $-c < \frac{cs}{\pi} < c$. Use $x = \frac{cs}{\pi}$, $s = \frac{\pi x}{s}$. $f(x) = f(\frac{cs}{\pi}) = g(s) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos ns + b_n \sin ns]$ on $-\pi < s < \pi$, where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{cs}{\pi})\cos ns\,ds$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{cs}{\pi})\sin ns\,ds$. By change of variables,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right]$$

where $a_n = \frac{1}{\pi}\frac{\pi}{c}\int_{-c}^{c} f(x)\cos n\pi x c\,dx$ and $b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$.

Homework: Sec. 7 p.18: 2,3,4; Sec. 8 p.22: 3,4 (Tuesday)

3 Convergence of Fourier Series

Goal is to prove a Fourier theorem- conditions on $f(x)$ which guarantee convergence of Fourier series for f. Also analyze convergence behavior. Develop theory for $-\pi < x < \pi$ then extend to $-c < x < c$.

Recall. $f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ provided the limit exists.

Notation for one-sided limits: $g(x_0+) = \lim_{x\to x_0^+} g(x)$ and $g(x_0-) = \lim_{x\to x_0^-} g(x)$.

Definition 2. Suppose $f(x_0+)$ exists. Then the right hand derivative of f at x_0 is

$$f'_+(x_0) = \lim_{x\to x_0^+}\frac{f(x)-f(x_0+)}{x-x_0} \tag{10}$$

provided the limit exists. Similarly if $f(x_0-)$ exists, then the left hand derivative is

$$f'_-(x_0) = \lim_{x\to x_0^-}\frac{f(x)-f(x_0-)}{x-x_0} \tag{11}$$

if it exists.

Note. $f(x_0)$ need not be defined. Other texts do require $f(x_0)$ to be defined.
Note. If ordinary derivative exists at x_0, then f is continuous at x_0 and $f'_+(x_0) = f'_-(x_0) = f'(x_0)$.

Converse not necessarily true: $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$, so $f'_+(0) = 0$ and $f'_-(0) = 0$ but $f'(0)$ does not exist and f is not continuous at $x = 0$.
Note. Usual derivative rules apply to right + left hand derivatives.

Definition 3. $C_p^1(a,b) = \{f \in C_p(a,b) : f' \in C_p(a,b)\} = \{f : f \text{ is piecewise smooth (PWS)}\}$.

Theorem 1. *If $f \in C_p^1(a,b)$, then at each point $x_0 \in (a,b)$, the one-sided derivatives exist and*

$$f'_+(x_0) = f'(x_0+), \qquad f'_-(x_0) = f'(x_0-). \tag{12}$$

Proof. If f is PWS, then f and f' are continuous on the interiors of subintervals. It is sufficient to prove this at the end points assuming f, f' are continuous on (a,b). We will show $f'_+(a)$ exists and equal to $f'(a+) = \lim_{x\to a^+} f'(x)$. Let $s \in (a,b)$. Since f' is continuous on (a,b), by mean value theorem (MVT) $\frac{f(s)-f(a+)}{s-a} = f'(c)$ for some $c \in (a,s)$. As $s \to a^+$, then $c \to c^+$. This tells us $f'_+(a) = \lim_{s\to a+} \frac{f(s)-f(a+)}{s-a} = \lim_{s\to a+} f'(c) = \lim_{c\to a+} f'(c) = f'(a+)$. Similarly $f'_-(b) = f'(b-)$. ☐

Example 9. $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Note. $f(0+) = 0 = f(0-)$, so f is continuous on $\mathbb{R}$ for all x.

For $x \neq 0$, f is differentiable and $f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$. One sided limits of f' do not exist at $x = 0$, so $f'(0+)$ does not exist but $f'_+(0) = \lim_{x\to 0+} \frac{f(x)-f(0+)}{x-0} = \lim_{x\to 0+} x \sin\frac{1}{x} = 0$ as with $f'_-(0) = \cdots = 0$. In other words, $f \notin C_p^1(a,b)$ if $0 \in [a,b]$.
Recall. We have numerical evidence that Fourier coefficients satisfy $a_n \to 0$ and $b_n \to 0$.

Geometric argument: If $f(x) = c$ is constant, then $a_n = \frac{2}{\pi} c \int_0^\pi \cos nx\, dx = 0$. When f is not constant, it is "almost" constant over small subintervals.

Let $f \in C_p(a,b)$ and $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\, dx$ for $n = 0, 1, \ldots$ with the partial sum S_N defined as $S_N(x) = \frac{a_0}{2} + \sum_{n=1}^N a_n \cos nx$. Then $\int_0^\pi [f(x) - S_N(x)]^2\, dx = \int_0^\pi [f(x)]^2\, dx + \int_0^\pi [S_N(x)]^2\, dx - 2\int_0^\pi f(x) S_N(x)\, dx$.

$$I_N = \int_0^\pi f(x) S_N(x)\, dx = \frac{a_0}{2}\int_0^\pi f(x)\, dx + \sum_{n=1}^N a_n \int_0^\pi f(x)\cos nx\, dx = \frac{\pi}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^N a_n^2\right]$$

For J_N: $\int_0^\pi S_N(x)\cdot 1\, dx = \int_0^\pi \left[\frac{a_0}{2} + \sum_{n=1}^N a_n \cos nx\right] dx = \frac{a_0}{2}$ and

$$\int_0^\pi S_N(x)\cdot \cos nx\, dx = \int_0^\pi \left[\frac{a_0}{2} + \sum_{m=1}^N a_m \cos mx\right]\cos nx\, dx = \frac{\pi}{2} a_n.$$

So $J_N = \int_0^\pi [S_N(x)]^2\, dx = \int_0^\pi S_N(x) S_N(x)\, dx$ and

$$J_N = \frac{a_0}{2}\int_0^\pi S_N(x)\, dx + \sum_{n=1}^N a_n \int_0^\pi S_N(x)\cos nx\, dx = \frac{\pi}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^N a_n^2\right].$$

So $0 \leq \int_0^\pi [f(x) - S_N(x)]^2\, dx = \int_0^\pi [f(x)]^2\, dx - \frac{\pi}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^N a_n^2\right]$. So

$$0 \leq \frac{a_0^2}{2} + \sum_{n=1}^N a_n^2 \leq \frac{2}{\pi}\int_0^\pi [f(x)]^2\, dx \tag{13}$$

for all $N = 1, 2, \ldots$. This is known as Bessel's inequality. The partial sums form a non-decreasing sequence bounded above, hence converges. Hence, series (sequence of partial sums) converges. $\frac{a_0^2}{2} + \sum_{n=1}^\infty a_n^2 < \infty$. Thus $a_n^2 \to 0$, so $a_n \to 0$.

HW Sect. 11, p.33: 1,2,4

4 Fourier Theorem

Last time: one-sided derivatives, Bessel's inequality: $\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty} a_n^2 \leq \dfrac{2}{\pi}\int_0^\pi [f(x)]^2\,dx$, this implies $\lim_{n\to\infty} a_n = 0$, similarly for Fourier sine coefficients b_n.

Lemma (Riemann-Lebesgue). If $G(u)$ is PWC on $0 < u < \pi$ then $\displaystyle\lim_{N\to\infty}\int_0^\pi G(u)\sin\left(\frac{u}{2}+Nu\right)\,du = 0.$

Proof. Observe $\sin\left(\dfrac{u}{2}+Nu\right) = \sin\dfrac{u}{2}\cos Nu + \cos\dfrac{u}{2}\sin Nu$. Hence

$$\begin{aligned}\int_0^\pi G(u)\sin\left(\frac{u}{2}+Nu\right)\,du &= \frac{\pi}{2}\cdot\frac{2}{\pi}\int_0^\pi \left[G(u)\sin\frac{u}{2}\right]\cos Nu\,du + \frac{\pi}{2}\cdot\frac{2}{\pi}\int_0^\pi\left[G(u)\cos\frac{u}{2}\right]\sin Nu\,du\\ &= \frac{\pi}{2}a_n + \frac{\pi}{2}b_n,\end{aligned}$$

where a_n is Fourier cosine coefficient for $G(u)\sin\frac{u}{2}$ and b_n is Fourier sine coefficient for $G(u)\cos\frac{u}{2}$. By previous result, $a_n \to 0$ and $b_n \to 0$ as $N\to\infty$. □

Definition 4 (Dirichlet Kernel)**.** $D_N(u) = \dfrac{1}{2}+\displaystyle\sum_{n=1}^{N}\cos nu$. Observe

1. $D_N(u)$ is continuous, even, and has period 2π.
2. $\int_0^\pi D_N(u)\,du = \frac{\pi}{2}$.
3. $D_N(u) = \dfrac{\sin\left(\frac{u}{2}+Nu\right)}{2\sin\frac{u}{2}}$, $u\neq 2n\pi$, $n = 0, \pm1, \pm2, \ldots$.

To see this: $\sin(A+B) = [\sin A\cos B + \cos A\sin B]$ and $\sin(A-B) = [\sin A\cos B - \cos A\sin B]$ and $2\sin A\cos B = [\sin(A+B)+\sin(A-B)]$. So

$$\begin{aligned}2\sin\frac{u}{2}D_N(u) &= \sin\frac{u}{2} + 2\sum_{n=1}^{N}\sin\frac{u}{2}\cos nu = \sin\frac{u}{2}+\sum_{n=1}^{N}\left[\sin\left(\frac{u}{2}+nu\right)+\sin\left(\frac{u}{2}-nu\right)\right]\\ &= \sin\frac{u}{2}+\left[\left(\sin\frac{3}{2}u-\sin\frac{u}{2}\right)+\left(\sin\frac{5}{2}u-\sin\frac{3}{2}u\right)+\cdots+\left(\sin\frac{2N+1}{2}u-\sin\frac{2N-1}{2}u\right)\right]\\ &= \sin\left(\frac{u}{2}+Nu\right),\end{aligned}$$

which is a telescoping sum.

Lemma. Suppose $g(u)$ is PWC on $(0,\pi)$ and $g'_+(0)$ exists. Then $\lim_{N\to\infty}\int_0^\pi g(u)D_N(u)\,du = \frac{\pi}{2}g(0+)$.

Proof. Since $g(u) = g(u) - g(0+) + g(0+)$, define I_N and J_N by

$$\int_0^\pi g(u)D_N(u)\,du = \int_0^\pi [g(u)-g(0+)]D_N(u)\,du + \int_0^\pi g(0+)D_N(u)\,du = I_N + J_N.$$

Then $I_N = \displaystyle\int_0^\pi [g(u)-g(0+)]\frac{\sin\left(\frac{u}{2}+Nu\right)}{2\sin\frac{u}{2}}\,du = \int_0^\pi G(u)\sin\left(\frac{u}{2}+Nu\right)du$, where $G(u) = \dfrac{g(u)-g(0+)}{2\sin\frac{u}{2}}$. From the Riemann-Lebesgue lemma, we know that $\lim_{N\to\infty} = 0$ as long as $G(u)$ is PWC on $(0,\pi)$. One possible problem is at $u=0$, where denominator is 0. $G(u)$ is PWC on $(0,\pi)$ as long as $G(0+) = \lim_{u\to0+} G(u)$ exists. $G(0+) = \displaystyle\lim_{u\to0+}\frac{g(u)-g(0+)}{u}\cdot\frac{u}{2\sin\frac{u}{2}}$. But $\displaystyle\lim_{u\to0+}\frac{g(u)-g(0+)}{u}$ exists because it is $g'_+(0)$ which is assumed to exist. And $\displaystyle\lim_{u\to0+}\frac{u}{2\sin\frac{u}{2}} = 1$. Also $J_N = g(0+)\int_0^\pi D_N(u)\,du = \frac{\pi}{2}g(0+)$. □

Given $f \in C_p(-\pi,\pi)$, use notation $S(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$, where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ for $n = 0, 1, \ldots$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$. Also define partial sum $S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}[a_n \cos nx + b_n \sin nx]$. Observe:

$$\begin{aligned}
S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{N}\left[\cos nx \int_{-\pi}^{\pi} f(s)\cos ns\,ds + \sum_{n=1}^{N} f(s)\sin ns\,ds\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi} f(s)[\cos nx\cos ns + \sin nx \sin ns]\,ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi} f(s)\cos n(s-x)\,ds.
\end{aligned}$$

Theorem 2. *Let f be PWC on $(-\pi,\pi)$, and is periodic with period 2π on $\mathbb{R}$. The Fourier series for f converges to the mean value*

$$\frac{f(x+) - f(x-)}{2} \tag{14}$$

at each $x \in (-\infty,\infty)$ where both one sided derivatives $f'_+(x)$ and $f'_-(x)$ exist.

Note. When f is continuous at x, then equation (14) $= f(x)$, so Fourier series converges to $f(x)$.

Proof. We want to show $S_N(x) \to \frac{f(x+)+f(x-)}{2}$. We will set $x \pm \pi$ because f, D_N are period 2π:

$$\begin{aligned}
S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi} f(s)\cos n(s-x)\,ds = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\left[\frac{1}{2} + \sum_{n=1}^{N}\cos n(s-x)\right]dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)D_N(s-x)\,ds = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(s)D_N(s-x)\,ds = \frac{1}{\pi}[I_N + J_N],
\end{aligned}$$

where $I_N = \int_x^{x+\pi} f(s)D_N(s-x)\,ds$ and $J_N = \int_{x-\pi}^{x} f(s)D_N(s-x)\,ds$. Consider I_N: Let $u = s - x$, so $I_N = \displaystyle\int_0^{\pi} f(x+u)D_N(u)\,du = \int_0^{\pi} g(u)D_N(u)\,du$, where $g(u) = f(x+u)$. Observe g is PWC on $(0,\pi)$. Also, $g'_+(0) = \displaystyle\lim_{u\to 0+}\frac{g(u)-g(0+)}{u} = \lim_{u\to 0+}\frac{f(x+u)-f(x+)}{u} = \lim_{v\to x+}\frac{f(v)-f(x+)}{v-x} = f'_+(x)$, which exists. We can apply the previous theorem, which says $\displaystyle\lim_{N\to\infty} I_N = \lim_{N\to\infty}\int_0^{\pi} g(u)D_N(u)\,du = \frac{\pi}{2}g(0+) = \frac{\pi}{2}f(x+)$. Similarly, for J_N we get $\lim\limits_{N\to\infty} J_N = \frac{\pi}{2}f(x-)$. $S_N(x) = \frac{1}{\pi}[I_N + J_N]$, so $S_N(x) \to \frac{1}{2}f(x+) + \frac{1}{2}f(x-)$. □

Last time: Proved a Fourier theorem, conditions on $f(x)$ which guarantee Fourier series converges.

Corollary 1. Let f be PWS on $(-\pi,\pi)$ and let $F(x)$ be 2π-periodic extension to $(-\infty,\infty)$. At each $x \in (-\infty,\infty)$, the Fourier series (for $f(x)$ on $(-\pi,\pi)$) converges to $\frac{F(x+)+F(x-)}{2}$.

Example 10. $f(x) = x$ on $(-\pi,\pi)$. f is PWS on $(-\pi,\pi)$. Let $F(x)$ be 2π-periodic extension. The Fourier series for $f(x)$ is $S(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$ (Example 5.1). At $x = \frac{\pi}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}\right)$, where all even terms are 0. $\frac{\pi}{4} = \sum_{k=1}^{\infty}\frac{(-1)^{2k}}{2k-1}\sin\frac{(2k-1)\pi}{2} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots$.

Example 11. $f(x) = \begin{cases} 0 & -\pi < x \le 0 \\ x & 0 < x < \pi \end{cases}$. We know from Example 7.1 that the Fourier series is

$$S(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right].$$

f is PWS on $(-\pi,\pi)$ (f' not exist at 0). At $x = \pi$, $S(\pi) = \frac{\pi}{2}$. So $\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n-1}{\pi n^2}\cos n\pi + 0$. $\frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = \sum_{k=1}^{\infty}\frac{2}{(2k-1)^2}$ and $\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$.

Other Intervals. Given $f(x)$ on $-c < x < c$, we use $g(s) = f\left(\frac{cs}{\pi}\right)$ using $x = \frac{cs}{\pi}$. g defined on $-\pi < s < \pi$. Apply results to g, then translate them back to f.

Theorem 3. *Suppose f is PWS on $(-c, c)$. Let $F(x)$ be the $2c$-periodic extension of $f(x)$. Def. Fourier series $S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)\right]$ where $a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$, $b_n = \frac{1}{c}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{c}\,dx$. Then the Fourier series converges to $\frac{F(x+)-F(x-)}{2}$ for all $x \in (-\infty, \infty)$.*

Next: further investigation of convergence "behavior".

Example 12. $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$. Fourier series is $S(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$ (use problem 7.2).

Example 13. $f(x) = x^2$ on $-\pi < x < \pi$

Note. $F(x)$ is continuous.

Last time: Fourier theorem on other intervals, numerical investigation of convergence, Gibb's phenomenon, "nonuniform" convergence.

Next we consider further the convergence behavior of Fourier series. Since a Fourier series is a sequence of partial sums of functions, that is, a sequence of functions, we begin with convergence for sequences of functions. Let $\{g_n(x)\}_{n=1}^{\infty}$ be a sequence of functions on domain E and let $g(x)$ also be a function on E.

Definition 5. We say $\{g_n\}$ converges point-wise to $g(x)$ on E if $\lim_{n\to\infty} g_n(x) = g(x)$ for each fixed $x \in E$. This means for each $x \in E$ and any $\epsilon > 0$, there exists $N = N(\epsilon, x)$ such that if $n \geq N$, then $|g_n(x) - g(x)| < \epsilon$.

Definition 6. We say $\{g_n\}$ converges uniformly to g on E if for any $\epsilon > 0$ there exists $N(\epsilon)$ such that if $n \geq N$, then $|g_n(x) - g(x)| < \epsilon$ for all $x \in E$.

Note. N does not depend on x.

Example 14. $g_n(x) = \begin{cases} 1 - nx & 0 \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$. $\{g_n\}$ converges point-wise to $g(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases}$ but not uniformly.

Note. If each $g_n(x)$ is continuous on E and $g_n \to g(x)$ uniformly on E, then g is continuous on E.

Let's apply ideas to series of functions. Consider sequence $\{f_n(x)\}_{n=1}^{\infty}$ on $a \leq x \leq b$. Define partial sums $S_N(x) = \sum_{n=1}^{N} f_n(x)$. The series $S(x) = \sum_{n=1}^{\infty} f_n(x)$ converges if the sequence $S_N(x)$ converges point-wise to $S(x)$.

Note. Our Fourier theorem showed point-wise convergence.

The series converges uniformly on $a \leq x \leq b$ if for any $\epsilon > 0$ there exists N such that if $n \geq N$, then $|\sum_{k=1}^{n} f_k(x) - S(x)| < \epsilon$.

To show this type of convergence it is often useful to use Weierstrass M-test: If there is a convergent series (of positive real numbers) $\sum_{n=1}^{\infty} M_n < \infty$ such that $|f_n(x)| \leq M_n$ for each n and all $x \in [a, b]$, then $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[a, b]$.

Theorem 4. *Let f be a function satisfying:*

1. *f is continuous on $-\pi \leq x \leq \pi$*
2. *$f(-\pi) = f(\pi)$*
3. *f' is PWC on $-\pi < x < \pi$.*

The Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$, converges uniformly to $f(x)$ on $-\pi \leq x \leq \pi$.

Proof. Observe that $|a_n \cos nx + b_n \sin nx| \leq |a_n \cos nx| + |b_n \sin nx| \leq |a_n| + |b_n| \leq 2\sqrt{a_n^2 + b_n^2}$. the result follows from Weierstrass M-test once we verify that $\sum_{n=1}^{\infty} 2\sqrt{a_n^2 + b_n^2} < \infty$ (next). $\square$

Definition 7 (Cauchy-Schwarz inequality)**.** $|\langle p,q\rangle| \le ||p||\cdot||q||$ on $\mathbb{R}^N$ with Euclidean norm $\vec{p} = \begin{bmatrix} p_1 & \cdots & p_N \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} q_1 & \cdots & q_N \end{bmatrix}$. Then

$$\left|\sum_{n=1}^N p_n q_n\right| \le \sqrt{\sum_{n=1}^N p_n^2}\sqrt{\sum_{n=1}^N q_n^2}, \qquad \left|\sum_{n=1}^N p_n q_n\right|^2 \le \sum_{n=1}^N p_n^2 \sum_{n=1}^N q_n^2.$$

Lemma. Assume hypotheses on f from previous theorem. Then $\sum_{n=1}^\infty \sqrt{a_n^2+b_n^2} < \infty$.

Proof. Since f' is PWC, its Fourier series exists with coefficients $\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx$ and $\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx$. Observe: $\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\,dx = \frac{1}{\pi}[f(\pi)-f(-\pi)] = 0$. For $n = 1, 2, \ldots$

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx = \frac{1}{\pi}\left[f(x)\cos nx\Big|_0^{\pi} + n\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right] = nb_n,$$
$$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx = \frac{1}{\pi}\left[f(x)\sin nx\Big|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} f'(x)\cos nx\,dx\right] = -na_n.$$

We have $T_N = \sum_{n=1}^N \sqrt{a_n^2+b_n^2} = \sum_{n=1}^N \frac{1}{n}\sqrt{\alpha_n^2+\beta_n^2}$. Then

$$T_N^2 = \left(\sum_{n=1}^N \frac{1}{n}\sqrt{\alpha_n^2+\beta_n^2}\right)^2 \le \left(\sum_{n=1}^N \frac{1}{n^2}\right)\left(\sum_{n=1}^N (\alpha_n^2+\beta_n^2)\right).$$

Converges to $\zeta(2)$. By Bessel's inequality $\sum_{n=1}^N (\alpha_n^2+\beta_n^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi}[f'(x)]^2\,dx$. Thus T_N^2 are increasing and bounded above, so convergent. Thus T_N is convergent. So $\sum_{n=1}^\infty \sqrt{a_n^2+b_n^2} < \infty$. □

HW Sec 14. p41, 1,2,4,5,6 Sec 15, p45, 1,3.

Last time: conditions under which Fourier series converges uniformly. We next investigate differentiating Fourier series.

Theorem 5. *Let f be a function which satisfies:*

a. f is continuous on $-\pi \le x \le \pi$

b. $f(-\pi) = f(\pi)$

c. f' is PWC on $-\pi < x < \pi$

Then the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty [a_n \cos nx + b_n \sin nx]$, where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$, is differentiable at each x in $-\pi < x < \pi$ where $f''(x)$ exists; and in this case $f'(x) = \sum_{n=1}^\infty [-na_n \sin nx + nb_n \cos nx]$.

Proof. Fix x such that $f''(x)$ exists. Thus f' is continuous at x. Apply Fourier Theorem to f'. For fixed x: $f'(x) = \frac{\alpha_0}{2} + \sum_{n=1}^\infty [\alpha_n \cos ns + \beta_n \sin nx]$, where $\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx$ and $\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx$. From last time: $\alpha_0 = 0$, $\alpha_n = nb_n$, $\beta_n = -na_n$, $n = 1, 2, \ldots$. The result follows. □

Example 15. $f(x) = x$ on $0 < x < \pi$. Sine series: $S(x) = 2\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sin nx$. Calculate derivative term by term. $S'(x) = 2\sum_{n=1}^\infty (-1)^{n+1}\cos nx$. This series doesn't even converge for any x. Cosine series: $C(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^\infty \frac{\cos(2n-1)x}{(2n-1)^2}$. Calculate derivative term by term: $C'(x) = \frac{4}{\pi}\sum_{n=1}^\infty \frac{\sin(2n-1)x}{2n-1}$

Note. The odd extension of $f(x) = x$ to $(-\pi, \pi)$ is (plot), $f(-\pi) \neq f(\pi)$. $S(x)$ is the full Fourier series for this function. The even extension $\tilde{f}$ is (plot), satisfies $\tilde{f}(-\pi) = \tilde{f}(\pi)$. $C(x)$ is the full Fourier series for this function. (HW)

We can verify that $C'(x)$ is the actual Fourier series for $\tilde{f}'(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$.

Example 16. Suppose we want to compute Fourier series for $f(x) = x^2$ on $-\pi < x < \pi$. We know it has the form $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx$. Rather than compute the A_n directly, we know the series is differentiable and $2x = f'(x) = \sum_{n=1}^{\infty} -nA_n \sin nx$. Also we know $2x = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$. So $A_n = \frac{4}{n^2}(-1)^n$. Must compute A_0 directly: $A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}$. So $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$.

Theorem 6. *Let f be PWC on $-\pi < x < \pi$ with Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$. Then term-by-term integration is valid. That is, for any $x \in [-\pi, \pi]$,*

$$\int_{-\pi}^{\pi} f(s)\,ds = \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \sin nx - b_n\left(\cos nx + (-1)^{n+1}\right)\right].$$

Note. This may no longer be a Fourier series.

HW: Sec.20 p.58: 1,2,3,5

Example 17 (Quiz 1, Problem 1). $f(x) = 1$ on $0 < x < \pi$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx = \frac{2}{\pi}\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\pi} = \frac{-2}{n\pi}\left[(-1)^n - 1\right] = \frac{2}{\pi}[1-(-1)^n] = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

So $f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$ and set $n = 2k-1$ to get $f(x) = \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(2k-1)x$.

The Fourier sine series on $(0,\pi)$ represents the odd extension of f on $(-\pi,\pi)$ and the 2π-periodic extension (of the odd extension) on $(-\infty,\infty)$.

Example 18 (Quiz 2, Problem 1). $f(x) = \frac{\pi}{2} - x$ on $0 < x < \pi$. Given Fourier cosine series $f(x) \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x$. The even extension to $(-\pi,\pi)$ is *graph*. Use $S(0) = \frac{\pi}{2}$ to get $\frac{\pi}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Example 19 (Quiz 2, Problem 2). $S(x)$ is the Fourier series for $f(x) = x^3$ on $-1 < x < 1$. Find $S(\pi)$. $S(x)$ "converges" to periodic extension of period 2 of $F(x) = x^3$. *graph*. So $S(\pi) = (\pi - 4)^3$.

Theorem 7. *Let f be PWC on $-\pi < x < \pi$, with Fourier series $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. Then term by term integration is valid. That is, for any $x \in [-\pi, \pi]$,*

$$\int_{-\pi}^{\pi} f(s)\,ds = \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \sin nx + b_n\left(\cos nx + (-1)^{n+1}\right)\right].$$

Proof. Define $F(x) = \int_{-\pi}^{x} f(s)\,ds - \frac{a_0}{2}x$. $F(x)$ is continuous on $-\pi \le x \le \pi$. If f is continuous at x, then F is differentiable at x and $F'(x) = f(x) - \frac{a_0}{2}$. Thus F is PWS on $(-\pi,\pi)$, since f is PWC there. Also, $F(-\pi) = \frac{a_0\pi}{2}$ and $F(\pi) = \int_{-\pi}^{\pi} f(s)\,ds - \frac{a_0\pi}{2} = \pi a_0 - \frac{\pi a_0}{2} = \frac{\pi a_0}{2}$. So $F(-\pi) = F(\pi)$. We can apply our Fourier theorem to $F(x)$.

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx) \tag{15}$$

on $-\pi \le x \le \pi$, where $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$ and $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx$. For $n = 1, 2, \ldots$, set $u = F(x)$ and $dv = \cos nx\,dx$ so that

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx = \frac{1}{\pi}F(x)\left(\frac{1}{n}\right)\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}\left[f(x) - \frac{a_0}{2}\right]\sin nx\,dx \\ &= -\frac{1}{n\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -\frac{b_n}{n} \end{aligned}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\left[-F(x)\frac{1}{n}\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\left(f(x) - \frac{a_0}{2}\right)\cos nx\,dx\right] = \frac{1}{n}a_n$$

$$A_0 : \frac{\pi a_0}{2} = F(\pi) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n \cos \pi n + B_n \sin \pi n] = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(-1)^n \implies \frac{A_0}{2} = \frac{\pi a_0}{2} + \sum_{n=1}^{\infty} A_n(-1)^n$$

Plug in A_n, B_n to F(x), check. $\square$

5 Derivation of heat equation

Three types of models of heat transfer.

1. Conduction: due to molecular activity.
    - energy transfer from more active to less active particles
    - Fourier's law
2. Convection: due to bulk transfer/motion of mass.
    - Newton's law of cooling
3. Radiation: electromagnetic waves, i.e. sun heating Earth.

We consider heat transfer in a solid body, where conduction is the appropriate model. Let $u(x, y, z, t) =$ temperature at location (x, y, z) and time t. Given a surface S through the point $P(x, y, z)$ with unit normal $\vec{n}$. Let $\Phi(x, y, z, t) =$ flux of heat across S in the direction $\vec{n} =$ quantity of heat per unit of area per unit of time conducted across S in direction $\vec{n}$.

Definition 8. Fourier's Law: magnitude of flux is proportional to magnitude of directional derivative $\frac{du}{d\vec{n}}$. In other words, $\Phi(x, y, z, t) = -K\frac{du}{dn}$, for $K > 0$. $K =$ thermal conductivity.

Example 20 (Sec. 20, Problem 5). $f(s) = s$ has Fourier series $f(s) \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin ns$. Integrate the series from 0 to x + sketch graph. From the theorem, $\int_0^x s\,ds = 2\sum_{n=1}^{\infty} \int_0^x \frac{(-1)^{n+1}}{n} \sin ns\,ds$. Then $\frac{x^2}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}(-1)\cos ns\big|_0^x = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[\cos nx - 1] = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$.

Last time: Began derivation of heat equation $u(x, y, z, t) =$ temperature at (x, y, z) at time t. $\Phi(x, y, z, t) =$ flux .... Fourier's law: $\Phi(x, y, z, t) = -K\frac{du}{dn}$.

Definition 9. $\sigma =$ specific heat $=$ energy required to raise temperature of one unit of mass one degree. $\delta =$ mass density.

Note. In general, K, σ, δ may not be constant and may depend on x, y, z or even t or u. We usually assume constant.

Definition 10 (One-dimensional heat equation)**.** Assume:

- the solid is a circular cylinder with a constant cross sectional area A in yz-plane.
- heat flows only parallel to x-axis. Thus: $\Phi = \Phi(x, t)$ and $u(x, t)$.
- K, σ, δ, A are constant.
- temperature is constant over a cross-section.
- perfect insulation, so no heat escapes through the side of cylinder.
- no heat generated or lost inside cylinder (no sources or sinks).

We derive a model by considering conservation of thermal energy in a small segment of width Δx. WLOG (without loss of generality) assume thermal energy flows left to right. Conservation law: (1) net rate of heat accumulation = (2) rate of heat entering - rate of heat leaving. From definition of specific heat: (1) rate of heat change per unit time $\approx \sigma \cdot m \cdot u_t(x^*, t)$ on $x < x^* < x + \Delta x$. But $m = \delta A \Delta x$, $A =$ cross sectional area (in text, $A = \Delta y \Delta z$). So (1) $\approx \sigma\delta A\Delta x u_t(x^*, t)$. Also

$$(2) \approx \Phi(x, t) \cdot A - \Phi(x + \Delta x, t) \cdot A = -Ku_x(x, t) \cdot A + Ku_x(x + \Delta x, t) \cdot A.$$

Set equal: $\sigma\delta A\Delta x u_t(x^*, t) = K \cdot A[u_x(x + \Delta x, t) - u_x(x, t)]$. So $u_t(x^*, t) = \frac{1}{\sigma\delta}K\left[\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}\right]$. Take lim as $\Delta x \to 0$, so $x^* \to x$, $u_t(x, t) = ku_{xx}(x, t)$, where $k = \frac{K}{\sigma\delta} > 0$ is the thermal diffusivity.

HW: derive the 1-d heat equation when $K = K(x)$.

Note. In this 1-d model, thermal energy can only enter or leave through boundaries at left + right end. The full mathematical model consists of

1. The PDE $u_t = ku_{xx}$ on $0 \leq x \leq c$.
2. Initial temperature distribution (IC) $u(x,0) = f(x)$ on $0 < x < c$.
3. Two boundary conditions at $x = 0$ and $x = c$. For example: Dirichlet: $u(0,t) = 0$ and $u(c,t) = 0$. or Neumann: $u_x(0,t) = 0$ and $u_x(c,t) = 0$. Third: $\Phi|_{x=0,c} = H(T - u|_{x=0,c})$.

Example 21. Let $u(x)$ denote steady state temperature in a cylinder whose faces at $x = 0$ and $x = c$ are kept at constant temperature $u = 0$ and $u = u_0 > 0$. Since it's steady state, $u_t = 0$. So $ku_{xx}(x,t) = 0$. Can write $u(x,t) = u(x)$, so $u''(x) = 0$, $u(0) = 0$, and $u(c) = u_0$. So $u(x) = ax + b$ and $u(0) = 0$ implies $b = 0$, so $u(x) = ax$. To determine constant a, use other BC: $u(c) = u_0$ implies $u_0 = ac$, where $a = \frac{u_0}{c}$. So $u(x) = \frac{u_0}{c}x$.

TEST: verify PWC, left right derivative, Convergence theorem, calculate fourier series, quizzes, heat equation

Example 22 (Q3.1). $f(x) = \begin{cases} 1 - x & 0 < x \leq 1 \\ 0 & 1 < x \leq 2 \end{cases}$, $C(s)$ = cosine series, $S(s)$ = sine series. $C(s)$ is full Fourier series of even extension to $(-2,2)$. $S(s)$ is full Fourier series of odd extension of f to $(-2,2)$. (graph).

Example 23 (Q3.2). $S(x)$ is the Fourier series for $f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ \sin x & 0 \leq x \leq \pi \end{cases}$. So $S\left(\frac{5\pi}{2}\right) = 1$ and $S(3\pi) = 0$. Can differentiate term by term to get series for $S'(x)$. Note, $f'(0)$ not defined. So $S'\left(\frac{\pi}{2}\right)$ and $S'(\pi) = -\frac{1}{2}$.

Last time: Heat equation.

6 Model of vibrating elastic string

Definition 11. Consider a tightly stretched elastic string. Assume:

- motion consists only of vertical displacements, which are small. $y(x,t)$ = vertical displacement at time t of the point whose equilibrium position is $(x,0)$ in xy-plane.
- string is perfectly flexible, so there is an elastic restoring force, but no resistance to bending.
- horizontal component of tensile force is constant, $H > 0$.

$V(x,t)$ = vertical component of tensile force, exerted by left part of string on the right part of string at (x,y). At (x,y), if sloped down, then V is negative (downward) and $y_x(x,t)$ is positive. Then $y_x(x,t) = \frac{-V(x,t)}{H}$. On the other hand, if at (x,y) the slope is downward, then V is positive (upward) and $y_x(x,t)$ is negative. Again, $y_x(x,t) = \frac{-V(x,t)}{H}$. So $V(x,t) = -Hy_x(x,t)$ and $V(x+\Delta x,t) = Hy_x(x+\Delta x,t)$. Apply Newton's law $F = ma$. δ = density = mass per unit length. So $\delta\Delta x$ = mass of piece of string. Then $\delta\Delta x y_{tt}(\bar{x},t) = Hy_x(x+\Delta x,t) - Hy_x(x,t)$. Then the one dimensional wave equation can be derived as $y_{tt}(x,t) = a^2 y_{xx}(x,t)$, where $a^2 = \frac{H}{\delta}$. Can apply separation of variables to wave equation, just like we did previously for the heat equation.

HW: Sec. 27: 2,3. Sec. 29, 1.

Example 24 (Sec. 27, problem 2). At $x = 0$ flux into cylinder is constant Φ_0. At $x = c$ temperature is held constant to value 0. Steaty state means $u_t(x,t) = 0$. So $u(x,t) = u(x)$ and PDE becomes ODE $u''(x) = 0$. BC are: At $x = c$: $u(c) = 0$. At $x = 0$: $\Phi_0 = -Ku'(0)$. The general solution is $u(x) = Ax + B$.

Example 25 (HW Sec. 20, 1). $f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ \sin x & 0 \leq x \leq \pi \end{cases}$ continuous on $[-\pi,\pi]$, PWS, $f(-\pi) = 0 = f(\pi)$. By Theorem in §17 get uniform convergence of Fourier series $S(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$. Observe $\left|\frac{\cos 2nx}{4n^2-1}\right| \leq \frac{1}{4n^2-1} \leq \frac{1}{3n^2} = M_n$ and $\sum M_n < \infty$. Theorem in §19 implies $S(x)$ is differentiable at all x except

$x = 0$ in $(-\pi, \pi)$ because $f''(x)$ exists except at $x = 0$. Graph $S'(x)$. Use fact that $S'(x)$ is Fourier series for $f'(x) = \begin{cases} 0 & -\pi < x < 0 \\ \cos x & 0 < x < \pi \end{cases}$.

Example 26 (HW §20, 2). $f(x) = x \sim S(x) =$ Fourier cosine series on $(0, \pi)$. $S(x)$ is the Fourier series for $g(x) = |x|$ on $(-\pi, \pi)$. g is continuous on $[-\pi, \pi]$, satisfies $g(-\pi) = g(\pi)$, and g is PWS. By Theorem, $S'(x)$ is Fourier series for $f'(x)$ and converges for all $x \neq 0$, so all $x \in (0, \pi)$ since $f''(x)$ exists for all x except $x = 0$.

Example 27 (HW §20, #3). Theorem: Let f be a function satisfying

i) f is continuous on $0 \leq x \leq \pi$

ii) $f(0) = 0 = f(\pi)$

iii) f' is PWC on $0 < x < \pi$.

Then the Fourier sine series for f is differentiable at each point where $f''(x)$ exists.

Example 28 (HW §14, #1). $f(x) = x \sim S(x) =$ Fourier sine series on $(0, \pi)$. $S(x)$ is the Fourier series for odd extension to $(-\pi, \pi)$, which is continuous on $(-\pi, \pi)$. Theorem says $S(x)$ converges to mean value of 2π-periodic extension of $g(x)$.

Example 29 (HW §14, #4). $f(x) = \begin{cases} 0 \\ \sin x \end{cases}$

7 The Fourier Method

(Previously did §36) Some ideas aboud linear combinations.

Definition 12 (Function space)**.** A vector space whose elements are functions

Example 30. $C_p(a, b)$, $C(a, b)$, $L^2(a, b)$.

Definition 13 (Linear Operator)**.** A linear operator L on a function space V is a map from V to V with the properties: For all $u_1, u_2 \in V$ and scalars c_1, c_2 we have $L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$.

Example 31. Differential operator: $L(u) = \frac{du}{dx}$.

Example 32. Multiplication operator: $L(u) = f(x) \cdot u(x)$.

Definition 14 (Superposition Principle)**.** If L is a linear operator and $L(u_1) = 0$ and $L(u_2) = 0$, then $L(c_1, u_1 + c_2u_2) = 0$. This extends to any finite linear combination $L(c_1u_1 + c_2u_2 + \cdots + c_nu_n) = 0$.

Example 33. $L(u) = u_t - ku_{xx} = 0$, with function space defined by the BVP. Under suitable assumptions we can extend to infinite linear combinations, which we need for Fourier method (separation of variables). We did one example (§36) for a heat equation.

Example 34 (A wave equation example). $y_{tt}(x, t) = a^2y_{xx}(x, t)$ on $0 < x < c$ and $0 < t$ with BC $y(0, t) = 0$ and $y(c, t) = 0$. IC is $y(x, 0) = f(x)$ and $y_t(x, 0) = 0$. Assume solution has the form $y(x, t) = X(x)T(t)$. Plug in: $\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda \implies \underline{X''(x) + \lambda X(x) = 0,\ X(0) = 0, \text{ and } X(c) = 0}$. Also $T''(t) + \lambda a^2T(t) = 0$ and $T'(t) = 0$.

$\underline{\lambda = 0}$: $X''(x) = 0 \implies X(x) = Ax + B$ but BC imply $X(x) = 0$.

$\underline{\lambda < 0}$: $\lambda = -\alpha^2$, $X''(x) - \alpha^2X(x) = 0$. Then $X(x) = c_1e^{\alpha x} + c_2e^{-\alpha x}$. At $X(0) = 0$: $c_1 + c_2 = 0$, so $c_1 = -c_2$. At $X(c) = 0$: $c_1e^{\alpha c} - c_1e^{-\alpha c} = 0 = c_1\left[e^{\alpha c} - e^{-\alpha c}\right]$, which implies $c_1 = 0$, so $c_2 = 0$.

$\underline{\lambda > 0}$: $\lambda = \alpha^2$, $X''(x) + \alpha^2X(x) = 0$ and $X(x) = c_1\cos\alpha x + c_2\sin\alpha x$. At $X(0) = 0$: $c_1 + 0 = 0$, so $c_1 = 0$. So $X(x) = c_2\sin\alpha x$. At $X(c) = 0$: $\sin(\alpha c) = 0$, so $\alpha = \pm\frac{n\pi}{c}$, $n = 1, 2, \ldots.$ So $\lambda_n = \frac{n^2\pi^2}{c^2}$, $n = 1, 2, \ldots.$ with corresponding solutions (eigenfunctions) $X_n(x) = \sin\frac{n\pi}{c}x$. Use λ_n to get corresponding $T_n(t)$: $T''(t) + \frac{n^2\pi^2a^2}{c^2}T(t) = 0$, $T_n(t) = c_1\cos\frac{n\pi a}{c}t + c_2\sin\frac{n\pi a}{c}t$. So $T_n(t) = \cos\frac{n\pi a}{c}t$.

So $y_n(x,t) = \sin\frac{n\pi}{c}x\cos\frac{n\pi a}{c}t$ for $n = 1, 2, \dots$ is a solution of PDE and BC and $y_t(x,0) = 0$. Thus the series (Infinite linear combination) also satisfies these. $y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{c}x\cos\frac{n\pi a}{c}t$ satisfies PDE, BC, and $y_t(x,0) = 0$. Can we pick B_n so $y(x,t)$ also satisfies $y(x,0) = f(x)$? That is, $f(x) \stackrel{?}{=} \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{c}x$? Yes, if $f(x)$ has a Fourier series.

HW: §34: 2,4,5,6. §37 1,3.

Previously, we used Fourier method (separation of variables) to solve two different problems:

1. Heat Equation with insulated BC. PDE: $u_t(x,t) = ku_{xx}(x,t)$, BC: $u_x(0,t) = 0 = u_x(c,t)$, IC: $u(x,0) = f(x)$. This leads to the BVP $X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $X'(c) = 0$. Eigenvalues: $\lambda_0 = 0$, $\lambda_n = \frac{n^2\pi^2}{c^2}$, $n = 1, 2, \dots$. Eigenfunctions: $X_0(x) = 1$, $X_n(x) = \cos\frac{n\pi}{c}x$.

2. Wave equation. PDE: $y_{tt}(x,t) = a^2 y_{xx}(x,t)$. BC: $y(0,t) = 0 = y(c,t)$, IC: $y(x,0) = f(x)$, $y_t(x,0) = 0$. This leads to the BVP: $X''(x) + \lambda X(x) = 0$, $X(0) = 0$, $X(c) = 0$. Eigenvalues $\lambda_n = \frac{n^2\pi^2}{c^2}$ for $n = 1, 2, \dots$. Eigenfunctions: $X_n(x) = \sin\frac{n\pi}{c}x$. We will study other BC's and BVP's later (Sturm-Liouville).

Example 35. Heat Equation with zero temperature BC $u_t(x,t) = ku_{xx}(x,t)$, $u(0,t) = 0 = u(\pi,t)$, $u(x,0) = f(x)$. Apply Fourier Method, $X''(x) + \lambda X(x) = 0$, $X(0) = 0$, $X(\pi) = 0$, and $T'(t) + \lambda k T(t) = 0$. So $\lambda_n = n^2$, $X_n(x) = \sin(nx)$. $T_n(t) = e^{-n^2kt}$. So $u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)e^{-n^2kt}$. From IC: $f(x) = \sum_{n=1}^{\infty} B_n \sin(nx)$. So $B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$.

Example 36. Heat equation with nonzero temperature BC: $u_t(x,t) = ku_{xx}(x,t)$, $u(0,t) = 0$, $u(\pi,t) = u_0$, $u(x,0) = 0$. Note: BC is nonhomogeneous, so Fourier method doesn't apply directly (because Superposition Principle only applies to linear, homogeneous problems). Look for solution of form $u(x,t) = U(x,t) + \Phi(x) \rightarrow$ Plug in: $U_t(x,t) = u_t(x,t) = ku_{xx}(x,t) = kU_{xx}(x,t) + k\Phi''(x)$. At boundary conditions: $0 = u(0,t) = U(0,t) + \Phi(0)$ and $u_0 = u(\pi,t) = U(\pi,t) + \Phi(\pi)$. Suppose we select $\Phi(x)$ to satisfy $\Phi''(x) = 0$, $\Phi(0) = 0$, $\Phi(\pi) = u_0$. If such a $\Phi(x)$ is used, then $U(x,t)$ must satisfy $U_t(x,t) = kU_{xx}(x,t)$, $U(0,t) = 0$, $U(\pi,t) = 0$, $U(x,0) = -\Phi(x)$. We just solved this for $U(x,t)$. $\Phi''(x) = 0$ implies $\Phi(x) = Ax + B$ or $\Phi(x) = \frac{u_0}{\pi}x$. So $U(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)e^{-n^2kt}$, where $\sum_{n=1}^{\infty} B_n \sin(nx) = -\frac{u_0}{\pi}x$. We know $x = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}\sin nx$, so $B_n = \frac{-u_0}{\pi}\cdot\frac{2(-1)^{n+1}}{n} = \frac{2u_0}{\pi n}(-1)^n$. So

$$u(x,t) = \frac{u_0}{\pi}x + \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)e^{-kn^2t}.$$

Example 37. Two numerical examples:

1. $u_t = ku_{xx}$, $u_x(x,t) = 0 = u_x(1,t)$, $u(x,0) = x$.
   Solution $u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2kt}\cos(2n-1)\pi x$

2. $u_t = ku_{xx}$, $u(0,t) = 0 = u(\pi,t)$, $u(x,0) = x(\pi - x)$.
   Solution $u(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)^2kt}\sin(2n-1)x$.

HW §39: 1, 2

Example 38. Heat equation with a heat source: $u_t(x,t) = ku_{xx}(x,t) + q(t)$, $u(0,t) = 0$, $u(\pi,t) = 0$, $u(x,0) = f(x)$. We generalize the "variation of parameters" idea from ODE's and seek a solution of the form $u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin nx$. Plug in to PDE: $\sum_{n=1}^{\infty} B_n'(t)\sin nx = k\sum_{n=1}^{\infty} -n^2 B_n(t)\sin(nx) + q(t)\cdot\sum_{n=1}^{\infty}\frac{2[1-(-1)^n]}{n\pi}\sin nx$, where we used $1 = \sum_{n=1}^{\infty}\frac{2[1-(-1)^n]}{n\pi}\sin nx$ on $0 < x < \pi$. Thus

$$B_n'(t) + n^2kB_n(t) = \frac{2[1-(-1)^n]}{n\pi}q(t), \tag{16}$$

where $n = 1, 2, \dots$. Also $u(x,0) = f(x)$ implies $\sum_{n=1}^{\infty} B_n(0)\sin(nx) = f(x)$, so

$$B_n(0) = b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx).$$

We can solve equation (16) using integrating factor $e^{n^2kt}B_n'(t) + n^2ke^{n^2kt}B_n(t) = \frac{2[1-(-1)^n]}{n\pi}e^{n^2kt}q(t)$, where the LHS is $\frac{d}{dt}\left[e^{n^2kt}B_n(t)\right] = \frac{2[1-(-1)^n]}{n\pi}e^{n^2kt}q(t)$. Integrate 0 to t: $e^{n^2kt}B_n(t) - b_n = \frac{2[1-(01)^n]}{n\pi}\int_0^t e^{n^2k\tau}q(\tau)\,d\tau$. So $B_n(t) = b_ne^{-n^2kt} + \frac{2[1-(-1)^n]}{n\pi}\int_0^t e^{-n^2k(t-\tau)}q(\tau)\,d\tau$. So the solution

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n(t)\sin nx = \sum_{n=1}^{\infty} b_ne^{-n^2kt}\sin nx + \sum_{n=1}^{\infty}\frac{2[1-(-1)^n]}{n\pi}\left[\int_0^t e^{-n^2k(t-\tau)}q(\tau)\,d\tau\right]\sin nx \\ &= \sum_{n=1}^{\infty} b_ne^{-n^2kt}\sin nx + \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\sin(2j-1)x}{2j-1}\int_0^t e^{-(2j-1)^2k(t-\tau)}q(\tau)\,d\tau \end{aligned}$$

HW: §42 : 1, 2. Extra Credit: Use separation of variables to find a solution of $u_t(x,t) = ku_{xx}(x,t)$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = \pi x$. Put in all the details to the derivation.

We continue to use the two basic BVP's. We solved several heat equation examples.

Example 39 (already solved). $y_{tt}(x,t) = a^2y_{xx}(x,t)$ on $0 < x < c$ with $y(0,t) = 0$ and $y(c,t) = 0$; $y(x,0) = f(x)$ and $y_t(x,0) = 0$; where f is continuous on $[0,c]$ and $f(0) = 0 = f(c)$. We used separation of variables to get the solution $y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{c}\cos\frac{n\pi at}{c}$, where $B_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$. We observe the solution can be written in a different form, not involving series. Observe:

$$\sin\frac{n\pi x}{c}\cos\frac{n\pi at}{c} = \frac{1}{2}\left[\sin\frac{n\pi}{c}(x+at) + \sin\frac{n\pi}{c}(x-at)\right],$$

where we used $2\sin E\cos F = \sin(E+F) + \sin(E-F)$. Then

$$y(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{c}(x+at) + \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{c}(x-at)\right].$$

Also, $f(x) \sim \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{c}x$. Let $F(x)$ be the $2c$ periodic extension of the odd extension of $f(x)$ to $(-c,c)$. Then $F(x) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{c}$ for all $x \in (-\infty,\infty)$. Thus

$$y(x,t) = \frac{1}{2}\left[F(x+at) + F(x-at)\right].$$

Let's check that this is a solution. $y_x(x,t) = \frac{1}{2}\left[F'(x+at) + F'(x-at)\right]$, $y_{xx}(x,t) = \frac{1}{2}[F''(x+at) + F''(x-at)]$, $y_{tt}(x,t) = \frac{1}{2}[a^2F''(x+at) + a^2F''(x-at)] = a^2y_{xx}(x,t)$. BC: $y(0,t) = \frac{1}{2}[F(at) + F(-at)] = \frac{1}{2}[F(at) - F(at)] = 0$ because F is odd; $y(c,t) = \frac{1}{2}[F(c+at) + F(c-at)] = 0$ since F is odd and $2c$ periodic. IC: $y(x,0) = F(x) = f(x)$ on $(0,c)$; $y_t(x,0) = \frac{1}{2}[aF'(x) - aF'(x)] = 0$. Thus $y(x,t)$ is a solution to the given wave equation.

What about other IC? Consider $y_{tt}(x,t) = a^2y_{xx}(x,t)$ on $0 < x < c$ with $y(0,t)$ and $y(c,t) = 0$ using IC $y(x,0) = 0$ and $y_t(x,0) = g(x)$. Separation of variables: $y(x,t) = X(x)T(t)$ gives $X'' + \lambda X = 0$, $X(0) = 0$, and $X(c) = 0$; and $T''(t) + \lambda a^2T(t) = 0$, $T(0) = 0$. Eigenvalues: $\lambda_n = \frac{n^2\pi^2}{c^2}$, eigenfunctions $X_n(x) = \sin\frac{n\pi}{c}x$, $T_n(t) = \sin\frac{n\pi a}{c}t$. Then

$$y(x,t) = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{c}\cos\frac{n\pi at}{c}. \tag{17}$$

This satisfies PDE, BC, and $y(x,0) = 0$. To get the other IC, $y_t(x,t) = \sum_{n=1}^{\infty}\frac{n\pi a}{c}C_n\sin\frac{n\pi x}{c}\cos\frac{n\pi at}{c}$. From $y_t(x,0) = g(x) = \sum_{n=1}^{\infty}\frac{n\pi a}{c}\sin\frac{n\pi x}{c}$. So $\frac{n\pi a}{c}C_n$ are the Fourier sine coefficients of $g(x)$, i.e. $\frac{n\pi a}{c}C_n = \frac{2}{c}\int_0^c g(x)\sin\frac{n\pi x}{c}\,dx$ or $C_n = \frac{2}{n\pi a}\int_0^c g(x)\sin\frac{n\pi x}{c}\,dx$. From equation (17), like last example, $y_t(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{n\pi a}{c}C_n\sin\frac{n\pi}{c}(x+at) + \sum_{n=1}^{\infty}\frac{n\pi a}{c}C_n\sin\frac{n\pi}{c}(x-at)\right]$. So if $G(x)$ is the odd, $2c$ periodic extension of $g(x)$, then $y_t(x,t) = \frac{1}{2}[G(x+at) + G(x-at)]$. Hence, $y(x,t) = \frac{1}{2}\left[\int_0^t G(x+a\tau)\,d\tau + \int_0^t G(x-a\tau)\,d\tau\right]$. Substitute: $s = x + a\tau$ and $s = x - a\tau \to y(x,t) = \frac{1}{2a}\left[\int_x^{x+at} G(s)\,ds - \int_x^{x-at} G(s)\,ds\right] = \frac{1}{2a}\int_{x-at}^{x+at} G(s)\,ds$.

For general case: $y_{tt} = a^2y_{xx}$; BC: $y(0,t) = 0$, $y(c,t) = 0$; IC: $y(x,0) = f(x)$, $y_t(x,0) = g(x)$. Solution is

$$y(x,t) = \frac{1}{2}[F(x+at) + F(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} G(s)\,ds$$

HW §45; 1,2,3,4

8 Orthonormal Sets

We will generalize geometry of vectors in 3-space to more general vector spaces whose elements (vectors) are functions. In 3-space, vectors are defined as directed line segments. Define the following geometrically:

- Length of vector
- Angle between vectors
- Add two vectors (parallelogram)
- Multiply by a real number a: get new vector in same direction if $a > 0$, opposite direction if $a < 0$, and length is the original length multiplied by $|a|$.
- Given a vector and plane in 3-space, can define the projection of vector onto the plane as follows: move to equivalent vector with tail in plane, then drop a perpendicular line from head to plane. Then connect those two points in the plane.

  Note. The length of the projection is always less than or equal to length of original vector and only equal if original vector is in the plane.

All these geometric properties and definitions can be given algebraically. This is important because it is the algebraic definitions which can extend to higher dimensions and more general vector spaces.

Definition 15. Given a 'geometric' vector, take an equivalent vector with tail at origin $(0,0,0)$. Define 'algebraic' vector to have the coordinates of the point at the head: $x = (x_1, x_2, x_3)$.

- Define addition of two vectors and multiplication by a real number a, component-wise. $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$, $ax = (ax_1, ax_2, ax_3)$. This is consistent with the geometric definition.
- Define algebraic length by Euclidean norm: $||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. This is consistent with the geometric definition (Pythagorean Theorem).
- Define the dot product: $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$. This has important geometric interpretations. If θ is the 'geometric' angle between vectors x and y, then $\cos\theta = \frac{x\cdot y}{||x||\cdot||y||}$ (Law of cosines). Two vectors are orthogonal if and only if $x \cdot y = 0$.

  Note. $||x|| = \sqrt{x \cdot x}$.

Recall the Euclidean basis vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$. Observe following properties: $||e_i|| = 1$, $i = 1,2,3$; $e_i \cdot e_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$. Let $x = (x_1, x_2, x_3)$. Then $x = \displaystyle\sum_{i=1}^{3} x_ie_i = \sum_{i=1}^{3}(x \cdot e_i)e_i$. Also $||x||^2 = \sum_{i=1}^3 |x_i|^2 = \sum_{i=1}^3 (x\cdot e_i)^2$. Also, if $y \cdot e_i = 0$, $i = 1,2,3$, then $y = (0,0,0)$. Thus $\{e_1, e_2, e_3\}$ forms an orthonormal basis of $\mathbb{R}^3$.

Example 40. This works for any orthonormal basis in any vector space. Consider in $\mathbb{R}^3$: $u_1 = \frac{1}{\sqrt{6}}(1,2,1)$, $u_2 = \frac{1}{\sqrt{21}}(2,1,-4)$, $u_3 = \frac{1}{\sqrt{4}}(3,-2,1)$. Can check $\{u_1,u_2,u_3\}$ is orthonormal set. Define $W = \operatorname{span}\{u_1,u_2\}$. Let $x \in \mathbb{R}^3$. Define $\hat{x} = \sum_{i=1}^2 (x \cdot u_i)u_i = (x\cdot u_1)u_1 + (x \cdot u_2)u_2$. Then $\hat{x}$ is the orthogonal projection of x onto W. Actually, $x = \sum_{i=1}^3 (x \cdot e_i)e_i$.

Example 41. Recall our first class: $u_t(x,t) = ku_{xx}(x,t)$, $u_x(0,t) = 0$, $u_x(c,t) = 0$, and $u(x,0) = f(x)$. Apply Fourier Method and seek solutions of form $u(x,t) = X(x)T(t)$. Plug in to PDE: $X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $X'(c) = 0$, and $T'(t) + k\lambda T(t) = 0$. For $\lambda = 0$, $X_(x) = 1$. For $\lambda = \left(\frac{n\pi}{c}\right)^2$, $X_n(x) = \cos\frac{n\pi}{c}x$. Also, $T_0(t) = 1$ and $T_n(t) = e^{-\frac{n^2\pi^2}{c^2}kt}$. Then get $u_0(x,t) = 1$ and $u_n(x,t) = \cos\frac{n\pi}{c}x \cdot e^{-\frac{n^2\pi^2}{c^2}kt}$ for $n = 1,2,\ldots$. Each is a solution of PDE + BC, so $u(x,t) = A_0 \cdot 1 + \sum_{n=1}^\infty A_n \cos\frac{n\pi}{c}x \cdot e^{-\frac{n^2\pi^2}{c^2}kt}$ is also a solution. Next step is to determine coefficients $A_0, A_1, \ldots$ so $u(x,t)$ also satisfies $u(x,0) = f(x)$.

Example 42. For 37.1, seek solution of form $u(x,t) = X(x)Y(y)$ for $u_{xx}(x,y) + u_{yy}(x,y) = 0$ on $0 < x < \pi$ and $0 < y < 2$ with $u_x(0,y) = 0$, $u_x(\pi,y) = 0$, $u(x,0) = 0$, and $u(x,2) = f(x)$. Plug in to PDE: $X''(x) + \lambda X(x) = 0$ with $X'(0) = 0$ and $X'(\pi) = 0$. So $\lambda_0 = 0$, $X_0(x) = 1$, $\lambda_n = n^2$, and $X_n(x) = \cos nx$. Do on your own, get $Y'' - \lambda_nY = 0$, $Y(0) = 0$, etc.

Last time: geometry of dot product in $\mathbb{R}^3$, to extend to general function spaces.

Recall. $C_p(a,b) = \{f : f \text{ is PWC on } [a,b]\}$. This is a vector space.

Definition 16. Define an inner product on $C_p(a,b)$ by $(f,g) = \int_a^b f(x)\overline{g(x)}\,dx$.

Recall. Recall in $\mathbb{R}^3$, $u = \begin{bmatrix} a_1, a_2, a_3 \end{bmatrix}^T$, $u \cdot u = \sum a_i^2 = ||u||^2$.

So $C_p(a,b)$ is an inner product space (infinite dimensional). Compatible norm:

$$||f|| = \sqrt{(f,f)} = \left(\int_a^b [f(x)]^2\,dx \right)^{1/2}.$$

Definition 17. Functions f and g are <u>orthogonal</u> if $(f,g) = 0$, that is $\int_a^b f(x)\overline{g(x)}\,dx = 0$

A set of functions $\{\psi_n\}_{n=1}^{\infty}$ is <u>orthogonal</u> if $(\psi_m, \psi_n) = 0$ when $m \neq n$. If $\{\psi_n\}_{n=1}^{\infty}$ is an orthogonal set, then define $\phi_n = \frac{1}{||\psi_n||}\psi_n$. Then $||\phi_n|| = 1$ and $(\phi_n, \phi_m) = 0$ if $m \neq n$, so $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal set. This means that $(\phi_n, \phi_m) = \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$.

Example 43. $\psi_n(x) = \sin nx$, $n = 1, 2, \dots$ on $0 \leq x \leq \pi$. Then $(\psi_m, \psi_n) = \int_0^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases}$. So $\{\psi_n\}_1^{\infty}$ is orthogonal. Define $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$. Then $||\phi_n|| = 1$ for $n = 1, 2, \dots$, so $\{\phi_n\}_1^{\infty}$ is an orthonormal set.

Example 44. $\phi_n = \sqrt{\frac{2}{\pi}} \cos nx$ for $n = 1, 2, \dots$ and $\phi_0(x) = \frac{1}{\sqrt{\pi}}$. Then $\{\phi_n\}_{n=0}^{\infty}$ is an orthonormal set.

Example 45. $\phi_0(x) = \frac{1}{\sqrt{2\pi}}$, $\phi_{2n-1}(x) = \frac{1}{\sqrt{\pi}} \cos nx$, $\phi_{2n}(x) = \frac{1}{\sqrt{\pi}} \sin nx$ for $n = 1, 2, \dots$. So $\{\phi_n\}_{n=0}^{\infty}$ is an orthonormal set.

Example 46. $\{1, x, x^2, \dots\}$, then use Gram-Schmidt process to make the functions orthonormal.

<u>*Question*</u>. Given an orthonormal set $\{\phi_n\}_{n=1}^{\infty}$ in $C_p(a,b)$, is it possible to represent every $f \in C_p(a,b)$ as <u>a linear</u> combination (infinite series) of the ϕ_n's. That is, can we write $f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x)$? We write $f(x) \sim \sum_{n=1}^{\infty} C_n \phi_n(x)$ if they are equal for all but a finite number of x values on (a,b).

Suppose the answer is yes. Then $f(x) = \sum_{m=1}^{\infty} C_m \phi_m(x)$. Then

$$(f(x), \phi_n(x)) = \left(\sum_{m=1}^{\infty} C_m \phi_m(x), \phi_n(x) \right) = \sum_{m=1}^{\infty} C_m (\phi_m(x), \phi_n(x)) = \sum_{m=1}^{\infty} C_m \delta_{mn} = C_n.$$

So $C_n = (f, \phi_n)$.

HW Sect 61, 1,2,4,5

<u>Last time</u>: inner product spaces, orthonormal sets of functions, $C_p(a,b)$. Given an orthonormal set $\{\phi_n(x)\}_{n=1}^{\infty}$ in $C_p(a,b)$, can we represent any function f (in $C_p(a,b)$ or in a subspace W of $C_p(a,b)$), as $\sum_{n=1}^{\infty} C_n \phi_n(x)$? If yes, then $C_n = (f(x), \phi_n(x)) = \int_a^b f(x)\phi_n(x)\,dx$. The constants C_n are called Fourier constants or coefficients.

Definition 18. A sequence of functions $\{S_N(x)\}_{N=1}^{\infty}$ in $C_p(a,b)$ is said to <u>converge in the mean</u> to the function $f(x)$ in $C_p(a,b)$ if $E_N = ||f - S_N||^2 = \int_a^b |f(x) - S_N(x)|^2\,dx$ satisfies $\lim\limits_{N \to \infty} E_n = 0$, or equivalently $\lim\limits_{N \to \infty} ||f - S_N|| = 0$.

Note. not the same as pointwise or uniform convergence.

Definition 19. An orthonormal set $\{\phi_n\}_{n=1}^{\infty}$ is <u>complete</u> in a subspace W of $C_p(a,b)$ if, for every $f \in W$, the partial sums of the generalized Fourier series $S_N(x) = \sum_{n=1}^{N} C_n \phi_n(x)$, $C_n = (f, \phi_n)$ converge in the mean to f. $||f - S_n|| \to 0$ as $N \to \infty$.

Remark. Consider $C[a,b]$ is a vector space and is a subspace of $L_2(a,b)$. Define $||f||_\infty = \max_{a\leq x\leq b} |f(x)|$. Then $C[a,b]$ with $||\cdot||_\infty$ is a complete normed space. But $C[a,b]$ with our norm is not complete. $||f||_2 = \sqrt{\int_a^b |f(x)|^2\,dx}$. But $L_2(a,b)$ is complete with $||\cdot||_2$.

Consider $\mathbb{R}^3$. Given vetor $\vec{v}\in\mathbb{R}^3$, which vector in $W = \text{span}\{\vec{i},\vec{j}\} = xy$-plane is closest to $\vec{v}$? $\vec{v} = (x,y,z)$, then closest vector in xy-plane is $(x,y,0)$, that is $(\vec{v}\cdot\vec{i})\vec{i} + (\vec{v}\cdot\vec{j})\vec{j}$. So closest vector to $\vec{v}$ is the Fourier sum.

Let's extend to general inner product space. Let $\{\phi_n\}_{n=1}^\infty$ be an orthonormal set in $C_p(a,b)$, and let $f\in C_p(a,b)$. Which vector in span$\{\phi_N\}_{n=1}^\infty$ is closest to f? Let $\Phi_n(x) = \sum_{n=1}^N \gamma_n\phi_n(x)$. $\rightarrow$ which choice of $\gamma_1,\dots,\gamma_N$ minimizes $||f-\Phi_N||$? It is equivalent to minimize $E = ||f-\Phi_N||^2$.

$$\begin{aligned}
E = ||f-\Phi_n||^2 &= \left(f - \sum_{n=1}^N \gamma_n\phi_n, f - \sum_{n=1}^N \gamma_n\phi_n\right) = (f,f) + \left(\sum_{n=1}^N \gamma_n\phi_n, \sum_{n=1}^N \gamma_n\phi_n\right) - 2\left(f, \sum_{n=1}^N \gamma_n\phi_n\right)\\
&= ||f||^2 + \sum_{n=1}^N \gamma_n^2 - 2\sum_{n=1}^N \gamma_n C_n + \sum_{n=1}^N C_n^2 - \sum_{n=1}^N C_n^2 = ||f||^2 + \sum_{n=1}^N (\gamma_n - C_n)^2 - \sum_{n=1}^N C_n^2
\end{aligned}$$

The "best approxmation" which minimizes E is obtained by choosing $\gamma_n = C_n$: In this case, since $E\geq 0$, $\sum_{n=1}^N C_n^2 \leq ||f||^2$, called Bessel's Inequality.

Theorem 8. *If $\{C_n\}_{n=1}^\infty$ are the Fourier constants for $f\in C_p(a,b)$ (or any vector in any inner product space) with respect to some orthonormal set $\{\phi_n\}_{n=1}^\infty$, then* $\lim_{n\to\infty} C_n = 0$

Proof. $\sum_{n=1}^\infty C_n^2 \leq ||f||^2$. So $C_n^2\to 0$, so $C_n \to 0$. □

Suppose $\{\phi_n\}_{n=1}^\infty$ is orthonormal set in $W\subset C_p(a,b)$ and $f\in W$. Set $C_n = (f,\phi_n)$. Let $S_N(x) = \sum_{n=1}^N C_n\phi_n(x)$. So $||f - S_n(x)||^2 = ||f||^2 - \sum_{n=1}^N C_n^2$. If $\{\phi_n\}_{n=1}^\infty$ is complete, then $||f(x) - S_N(x)|| \to 0$, $\rightarrow$ so $\sum_{n=1}^\infty C_n^2 = ||f||^2$, which is Parseval's equation.

Remark. Whether an orthonormal set is complete is equivalent to Parseval's equation.

Last time: orthonormal sets, complete, Parseval's equation.

Recall. Two fundamental BVP's:

1. $X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $X'(c) = 0$. Eigenvalues: $\lambda_0 = 0$, $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ for $n = 1,2,\dots$. Eigenfunctions: $\psi_0(x) = 1$, $\psi_n(x) = \cos\frac{n\pi}{c}x$

2. $X''(x) + \lambda X(x) = 0$, $X(0) = 0$, $X(c) = 0$. Eigenvalues: $\lambda_n = \left(\frac{n\pi}{c}\right)^2$ for $n = 1,2,\dots$. Eigenfunctions $\psi_n(x) = \sin\frac{n\pi}{c}x$ for $n = 1,2,\dots$.

These arise by applying separation of variables to certain PDE's. For other more general PDE's or BC's, the method of separation of variables leads to more general BVP, which we now consider.

Consider the Sturm-Liouville problem (SL) on $a < x < b$:

$$(r(x)X'(x))' + [q(x) + \lambda p(x)]\,X(x) = 0.$$

with 'separated BC's' $a_1X(a) + a_2X'(a) = 0$ and $b_1X(b) + b_2X'(b) = 0$, where a_1, a_2 not both 0 and b_1, b_2 not both 0. The functions p, q, r and parameters a_1, a_2, b_1, b_2 are real and also independent of λ.

Definition 20. The SL problem is regular if interval (a,b) is bounded and

(i) p, q, r, r' are continuous on $[a,b]$.

(ii) $p(x) > 0$ and $r(x) > 0$ on $[a,b]$.

Otherwise SL problem is singular.

Definition 21. λ is an eigenvalue (possibly complex) of (SL) if for that value of λ there is a nontrivial solution $X(x)$, which is called an eigenfunction. The spectrum of (SL) is the set of all eigenvalues.

Fact 1 (which will be assumed without proof). A regular SL problem has countably infinite many eigenvalues, $\lambda_1, \lambda_2, \ldots$.

Next we prove that eigenvalues and eigenfunctions of regular SL problems have many similar properties to these of problems (1) and (2).

Recall. For $f, g \in C_p(a,b)$, $(f,g) = \int_a^b f(x)g(x)\,dx$. f is orthogonal to $g \iff 0 = (f,g)$. Also $||f||\sqrt{(f,f)} = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$.

Definition 22. Let $p \in C_p(a,b)$ satisfy $p(x) > 0$ and $f, g \in C_p(a,b)$. f and g are orthogonal with respect to weight function p if $(f,g) = \int_a^b p(x)f(x)g(x)\,dx = 0$. We define weighted norm $||f|| = \left(\int_a^b p(x)|f(x)|^2\,dx\right)^{1/2}$.

Consider the SL problem: (1) $[r(x)X'(x)]' + [g(x) + \lambda p(x)]X(x) = 0$ and (2) $a_1X(a) + a_2X'(a) = 0$, $b_1X(b) + b_2X'(b) = 0$, where

- (i) p, r, r' are continuous on $[a,b]$ and q is continuous on (a,b).
- (ii) $p(x) > 0$ and $r(x) > 0$ and $a < x < b$.

This includes all regular SL problems, plus some singular.

Theorem 9. *If λ_m and λ_n are distinct eigenvalues of (1)-(2), then the corresponding eigenfunction $X_m(x)$ and $X_n(x)$ are orthogonal with respect to $p(x)$.*

Last time: (1) $[r(x)X'(x)]' + [q(x) + \lambda p(x)]X(x) = 0$, (2) $a_1X(a) + a_2X'(a) = 0$ and $b_1X(b) + b_2X'(b) = 0$, where

- (i) p, r, r' continuous on $[a,b]$, q continuous on (a,b)
- (ii) $p(x) > 0$, $r(x) > 0$ on (a,b)

Recall (Theorem). If λ_m, λ_n are distinct eigenvalues of (1)-(2), then the corresponding eigenfunctions $X_m(x)$ and $X_n(x)$ are orthogonal with respect to weight function $p(x)$. This is also true when:

- a. $r(a) = 0$ and left BC is dropped
- b. $r(b) = 0$ and right BC is dropped
- c. $r(a) = r(b)$ and (2) is replaced with $X(a) = X(b)$ and $X'(a) = X'(b)$

Proof. We have $[rX_m']' + [q + \lambda_m p]X_m = 0$ and $[rX_n']' + [q + \lambda_n p]X_n = 0$. Multiply by X_n and X_m respectively and subtract the equations to get

$$\begin{aligned}(\lambda_m - \lambda_n)pX_mX_n &= X_m(rX_n')' - X_n(rX_m')' = [X_m(rX_n')' + X_m' rX_n'] - [X_n(rX_m')' + X_m' rX_n'] \\ &= \frac{d}{dx}[X_m(rX_n') - X_n(rX_m')]\end{aligned}$$

Integrate from a to b $(\lambda_m - \lambda_n)\int_a^b pX_mX_n\,dx = r(x)\left[X_m(x)X_n'(x) - X_m'(x)X_n(x)\right]_a^b = r(b)\Delta(b) - r(a)\Delta(a)$, where

$$\Delta(x) = \left|\begin{bmatrix} X_m(x) & X_n(x) \\ X_m'(x) & X_n'(x)\end{bmatrix}\right|.$$

Left BC implies $a_xX_m(a) + a_2X_m'(a) = 0$ and $a_1X_n(a) + a_2X_n'(x) = 0$ is

$$\begin{bmatrix} X_m(a) & X_m'(a) \\ X_m'(a) & X_n'(a)\end{bmatrix}\begin{bmatrix} a_1 \\ a_2\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix}.$$

This implies $\Delta(a) = 0$. Similarly right BC implies $\Delta(b) = 0$. Thus $(\lambda_m - \lambda_n)\int_a^b p(x)X_m(x)X_n(x)\,dx = r(a)\Delta(a) - r(b)\Delta(b) = 0$. Since $\lambda_m - \lambda_n \neq 0$, so $\int_a^b pX_mX_n\,dx = 0$. If $r(a) = r(b)$ and BC become $X(a) = X(b)$ and $X'(a) = X'(b)$, then BC imply $\Delta(a) = \Delta(b)$, so again $r(a)\Delta(a) - r(b)\Delta(b) = 0$. ☐

Theorem 10. *If λ is an eigenvalue for SL problem in previous theorem, then λ is real.*

Proof. Suppose λ is an eigenvalue with corresponding eigenfunction $X(x)$. We write $\lambda = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$ and $X(x) = u(x) = iv(x)$, $u, v \in \mathbb{R}$. Take conjugate of (1)-(2), use fact that $p, q, r, a_1, a_2, b_1, b_2$ are real. So $[r(x)\overline{X'}(x)]' + [q(x) + \overline{\lambda}p(x)]\overline{X}(x) = 0$, $a_1\overline{X}(a) + a_2\overline{X'}(a) = 0$, and $b_1\overline{X}(b) + b_2\overline{X'}(b) = 0$. Thus $\overline{\lambda}$ is also an eigenvalue, with corresponding eigenfunction $\overline{X}(x)$. By way of contradiction, suppose λ not real, so $\beta \neq 0$. Then $\lambda \neq \overline{\lambda}$, so $X(x)$ and $\overline{X}(x)$ are orthogonal with respect to weight function p. Thus $0 = \int_a^b p(x)X(x)\overline{X}(x)\,dx = \int_a^b p(x)|X(x)|^2\,dx = \int_a^b p(x)(u(x)^2 + v(x)^2)\,dx$. This implies $u(x) \equiv 0$ and $v(x) \equiv 0$, so $X(x) \equiv 0$. Contradiction since $X(x)$ is an eigenfunction. $\square$

Theorem 11. *If λ is an eigenvalue of regular SL problem, then it has a real eigenfunction. If X, Y are eigenfunctions for same eigenvalue λ, then $Y(x) = cX(x)$, $c \neq 0$ $(*)$.*

Proof. Prove 2nd statement first. Suppose $X(x), Y(x)$ are eigenfunctions for $\lambda \in \mathbb{R}$. Define (linear combination) $Z(x) = Y'(a)X(x) - X'(a)Y(x)$. Thus $Z(x)$ satisfies: $[rZ']' + [q + \lambda p]Z = 0$ and $Z'(a) = 0$. Also: $a_1X(a) + a_2X'(a) = 0$ and $a_1Y(a) + a_2Y'(a) = 0$. This implies that $Z(a) = 0$, $\begin{bmatrix} X(a) & X'(a) \\ Y(a) & Y'(a) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. By Existence and Uniqueness Theorem for 2nd order linear IVP, since 0 function is also a solution, $Z(x) \equiv 0$. Thus $Y'(a)X(x) - X'(a)Y(x) \equiv 0$. Must have either $Y'(a)$ and $X'(a)$ both zero or both nonzero. If $Y'(a) \neq 0$ and $X'(a) \neq 0$, then $(*)$ holds. If $Y'(a) = 0$ and $X'(a) = 0$, then $Y(a) \neq 0$ and $X(a) \neq 0$. In this case, define $W(x) = Y(a)X(x) - X(a)Y(x)$. Follow argument similar to $Z(x)$, conclude $W(x) \equiv 0$, so $Y(x) = \frac{Y(a)}{X(a)}X(x)$, so $(*)$ holds. Let $X(x) = U(x) + iV(x)$ be eigenfunction for λ, where $U(x), V(x)$ are real. Plug in to DE and see that $U(x)$ and $V(x)$ are also eigenfunctions for λ. Thus $V(x) = \beta U(x)$, so β is real. Thus $X(x) = (1 + i\beta)U(x)$. $\square$

HW Sect. 69: 1,2

Example 47 (§69.1). $[xX'(x)]' + \frac{\lambda}{x}X(x) = 0$. Let $x = e^s$ so $s = \ln x$. $X'(x) = \frac{dX}{dx} = \frac{dX}{ds} \cdot \frac{ds}{dx}$ or $\frac{d}{dx}[e^sX'(x)]$. $X'(x) = \frac{d}{dx}X(x) = \frac{dX}{dx} \cdot \frac{ds}{dx} = e^{-s}\frac{dX}{ds}$.

Recall. Regular SL problems $[r(x)X'(x)]' + [q(x) + \lambda p(x)]X(x) = 0$ with $a_1X(a) + a_2X'(a) = 0$, $b_1X(b) + b_2X'(b) = 0$, where p, q, r, r' continuous on $[a, b)$, $p > 0$, $r > 0$ on $[a, b]$. We know

- countably infinitely many real eigenvalues,
- each eigenvalue has a real eigenfunction (dimension of eigenspace is 1),
- eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to $p(x)$.

Theorem 12. *Let λ be an eigenvalue of a regular SL problem. If $q(x) \leq 0$ on $[a, b]$ and $a_1a_2 \leq 0$, $b_1b_2 \geq 0$, then $\lambda \geq 0$.*

Proof. Let $X(x)$ be a real eigenfunction for eigenvalue λ. So $[rX']' + [q + \lambda p]X = 0$. Multiply by X and integrate:

$$\begin{aligned}
\lambda\int_a^b p(x)X^2(x)\,dx &= -\int_a^b [rX']'X\,dx + \int_a^b (-q)X^2\,dx \\
&= -r(x)X'(x)X(x)\big|_a^b + \int_a^b r(x)[X'(x)]^2\,dx + \int_a^b (-q)X^2\,dx \\
&= r(a)X(a)X'(a) - r(b)X(b)X'(b) + \int_a^b r(X')^2\,dx + \int_a^b (-q)X^2\,dx,
\end{aligned}$$

using integration by parts. Consider $r(a)X(a)X'(a)$: If either $a_1 = 0$ or $a_2 = 0$, then $r(a)X(a)X'(a) = 0$. If $a_1 \neq 0$ and $a_2 \neq 0$, $r(a)X(a)X'(a) = -r(a)X(a)\left(\frac{a_1}{a_2}\right)X(a) = r(a)[X(a)]^2\left(-\frac{a_1}{a_2}\right) \geq 0$ because $-\frac{a_1}{a_2} > 0$. Similar argument shows $-r(b)X(b)X'(b) \geq 0$, so $\lambda\int_a^b p(x)[X(x)]^2\,dx \geq 0$, so $\lambda \geq 0$. $\square$

Example 48. $X''(x) + \lambda X(x) = 0$ with $X'(0) = 0$ and $hX(c) + X'(c) = 0$, $h > 0$. This is a regular SL problem, $r(x) \equiv 1$, $p(x) \equiv 1$, $q(x) \equiv 0$, $a_1 = 0$, $a_2 = 1$, $b_1 = h$ and $b_2 = 1$.

Case $\lambda = 0$: Then $X''(x) = 0$. Genereal solution is $X(x) = Ax + B$. $X'(0)$ implies $A = 0$, so $X(x) = B$. Then $hX(c) + X'(c) = 0$ implies $hB = 0$, so $B = 0$. So $X(x) = 0$, $\lambda = 0$ not an eigenvalue.

Consider $\lambda > 0$, say $\lambda = \alpha^2$, where $\alpha > 0$. Thus $X''(x) + \alpha^2 X(x) = 0$. General solution is $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$. $X'(0) = 0$ implies $\alpha c_2 \cos(0) = 0$, so $c_2 = 0$. So $X(x) = c_1 \cos \alpha x$. $hX(c) + X'(c) = 0$ implies $c_1 h \cos(\alpha c) - \alpha c_1 \sin(\alpha c) = 0 = h \cos(\alpha c) - \alpha \sin(\alpha c)$, so $\frac{h}{\alpha} = \tan(\alpha c)$ or $\frac{hc}{\alpha c} = \tan(\alpha c)$. $\to$ so $\frac{hc}{x} = \tan x$ for $x = \alpha c$. If x is a root, then $\alpha = \frac{x}{c}$ and $\lambda = \alpha^2$ is an eigenvalue. Plot $y = \tan x$ and $y = \frac{hc}{x}$: geometrically there are countably infinite number of solutions at the intersection points. As $n \to \infty$, $x_n \sim (n-1)\pi$. Then $\alpha_n = \frac{x_n}{c} \sim \frac{(n-1)\pi}{c}$ as $n \to \infty$. So $\lambda_n = \alpha_n^2$. Corresponding eigenfunction $X_n(x) = \cos \alpha_n x$, for $n = 1, 2, \ldots$. Also $\{\cos \alpha_n x\}_{n=1}^{\infty}$ is orthogonal. Note: $||X_n(x)||^2 = \int_0^c \cos^2 \alpha_n x \, dx = \cdots = \frac{ch + \sin^2(\alpha_n c)}{2h}$. So $\phi_n(x) = \frac{1}{||X_n||} X_n(x) = \sqrt{\frac{2h}{ch + \sin^2(\alpha_n c)}} \cos(\alpha_n x)$, where $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal set.

HW Section 72: 1,2,6,9

Exam: Chapter 3, Starting with §26 we did other boundary conditions, §28. §32-34, all of Chapter 4. Chapter 5 the 2 basic boundary value problems, §39, 40, 42, 45. All of Chapter 7: Chapter 8: §67-73.

Example 49 (Done in HW). $(xX'(x))' + \frac{1}{x}\lambda X(x) = 0$ with $X(1) = 0$ and $X(b) = 0$. In HW you show: $\lambda_n = \alpha_n^2$, $\alpha_n = \frac{n\pi}{\ln b}$, $n = 1, 2, \ldots$ and $\phi_n(x) = \sqrt{\frac{2}{\ln b}} \sin(\alpha_n \ln x)$. $\{\phi_n\}_{n=1}^{\infty}$ is orthonormal with respect to weight function $p(x) = \frac{1}{x}$. Let us represent $f(x) = 1$ with these eigenfunctions $1 = \sum_{n=1}^{\infty} c_n \phi_n(x)$. If this is possible, and we assume it is, we know the Fourier coefficients:

$$\begin{aligned} c_n = (f(x), \phi_n(x)) &= \int_1^b \frac{1}{x} \cdot (1) \phi_n(x) \, dx = \sqrt{\frac{2}{\ln b}} \int_1^b \frac{1}{x} \sin(\alpha_n \ln x) \, dx = \sqrt{\frac{2}{\ln b}} \left(-\frac{1}{\alpha_n}\right) \cos(\alpha_n \ln x) \Big|_1^b \\ &= \sqrt{\frac{2}{\ln b}} \left(-\frac{1}{\alpha_n}\right) [\cos(n\pi) - 1] = \sqrt{\frac{2}{\ln b}} \left(\frac{1}{\alpha_n}\right) (1 - (-1)^n) \end{aligned}$$

So $1 = \displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{2}{\ln b}} \cdot \frac{1 - (-1)^n}{\alpha_n} \sqrt{\frac{2}{\ln b}} \sin(\alpha_n \ln b) = \frac{4}{\ln b} \sum_{n=1}^{\infty} \frac{1}{\alpha_{2n-1}} \sin(\alpha_{2n-1} \ln x)$ on $1 < x < b$. If we make the substitution $s = \frac{\pi}{\ln b} \ln x$, we translate from x in $[1, b]$ to s in $[0, \pi]$, and $(2n-1)s = \alpha_{2n-1} \ln x$, so we get:

$$1 = \frac{4}{\ln b} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi / \ln b} \sin(2n-1)s = \frac{4}{\pi} \sum_{n=1} \frac{1}{2n-1} \sin[(2n-1)s]$$

on $0 < s < \pi$, which we proven to be valid on $0 < s < \pi$.

Example 50. Consider temperature in cylinder on $0 < x < 1$ with perfect insulation at $x = 0$ and at $x = 1$, surface heat transfer into a medium with temperature 0. Initial temperature distribution $f(x)$. The model: $u_t(x,t) = ku_{xx}(x,t)$, $u_x(0,t) = 0$, $u_x(1,t) = -hu(1,t)$, $u(x,0) = f(x)$. Newton's law of cooling $h[T - u(x,t)]$, $h > 0$. Do separation of variables: $u(x,t) = X(x)T(t)$. Get: $X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $hX(1) + X'(1) = 0$. We solved this last time: $(c = 1)$ $X_n(x) = \cos \alpha_n x$, $n = 1, 2, \ldots$, $\lambda_n = \alpha_n^2$, α_n's are roots of $\tan \alpha_n = \frac{h}{\alpha_n}$. Normalized eigenfunctions $\phi_n(x) = \sqrt{\frac{2h}{h + \sin^2 \alpha_n}} \cos \alpha_n x$. We also get $T_n(t) = e^{-\lambda_n kt} = e^{-\alpha_n^2 kt}$. So solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha_n^2 kt} \sqrt{\frac{2h}{h + \sin^2 \alpha_n}} \cos \alpha_n x.$$

Using IC: $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$, so $c_n = (f(x), \phi_n(x)) = \int_0^1 f(x)\phi_n(x) \, dx$.

HW 73: 1,2,4. Next: (not in text) Vibrations of solid elastic beam.

Transverse vibrations of solid elastic beam. Consider elastic beam of length l (much larger than cross sectional are). Let $w(x,t)$ = transverse displacement at time t and position x. Under reasonable assumptions, can derive the Euler-Bernoulli beam equation:

$$m(x)w_{tt}(x,t) + \frac{\partial^2}{\partial x^2}[EI(x)w_{xx}(x,t)] = 0.$$

Assume all parameters are constant, we get $w_{tt}(x,t) + \beta w_{xxxx}(x,t) = 0$, $\beta = \frac{EI}{m} > 0$. Boundary Conditions:

Clamped: $w(\bar{x},t) = 0$, $w_x(\bar{x},t) = 0$ for $\bar{x} = 0$ or l.

Hinged: $w(\bar{x}) = 0$, $w_{xx}(\bar{x},t) = 0$ for $\bar{x} = 0$ or l.

Free: $w_{xx}(\bar{x},t) = 0$, $w_{xxx}(\bar{x},t) = 0$ for $\bar{x} = 0$ or l.

Example 51. Consider Euler-Bernoulli beam hinge at both ends. $w_{tt}(x,t) + \beta w_{xxxx}(x,t) = 0$. BC $w(0,t) = 0 = w_x(0,t)$, $w(l,t) = 0 = w_{xx}(l,t)$. IC $w(x,0) = f(x)$. Apply Fourier method: $w(x,t) = X(x)T(t)$, so $T''(t) + \lambda T(t) = 0$ and $X'''' - \frac{\lambda}{\beta}X = 0$ with $X(0) = 0 = X''(0)$ and $X(l) = 0 = X''(l)$. Get $\lambda > 0$, assume $\frac{\lambda}{\beta} = \mu^4$ so that $X'''' - \mu^4 X = 0$. Characteristic euqation: $r^4 - \mu^4 = 0 = (r^2 - \mu^2)(r^2 + \mu^2) = (r - \mu)(r + \mu)(r^2 + \mu^2)$, with roots $r = \mu, -\mu, i\mu, -i\mu$. General solution: $X(x,t) = c_1 \sin \mu x + c_2 \cos \mu x + c_3 \sinh \mu x + c_4 \cosh \mu x$. BC: $X(0) = 0$: $c_2 + c_4 = 0$. $X''(0) = 0$: $-c_2 + c_4 = 0$. $\implies c_2 = 0 = c_4$. $X(l) = 0$: $c_1 \sin \mu l + c_2 \cos \mu l + c_3 \sinh \mu l + c_4 \cosh \mu l = 0$. $X''(l) = 0$: $-c_1 \sin \mu l - c_2 \cos \mu l + c_3 \sinh \mu l + c_4 \cosh \mu l = 0$.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ \sin \mu l & \cos \mu l & \sinh \mu l & \cosh \mu l \\ -sin \mu l & -\cos \mu l & -\sinh \mu l & -\cosh \mu l \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$c_2 = 0 = c_4$, so $c_1 \sin \mu l + c_3 \sinh \mu l = 0$ and $-c_1 \sin \mu l + c_3 \sinh \mu l = 0$. Then $2c_3 \sinh \mu l = 0 \implies c_3 = 0$. Thus $c_1 \sin \mu l = 0$, $c_1 \neq 0$, so $\sin \mu l = 0$. So $\mu l = n\pi$, $n = 1, 2, \ldots$, or $\mu_n = \frac{n\pi}{l}$ and $\lambda_n = \beta \left(\frac{n\pi}{l}\right)^4$. Therefore $X_n(x) = \sin \frac{n\pi}{l} x$.

HW: Find eigenvalues + eigenfunctions for cantilever beam (clamped at $x = 0$ and free at $x = l$).

Recall. Beam example $w_{tt}(x,t) + \beta w_{xxxx}(x,t) = 0$. Separate variables $x''''(x) + \frac{\lambda}{\beta}x(x) = 0$, where $\frac{\lambda}{\beta} = \mu^4$ with $X(0) = 0 = X'(0)$ and $X''(l) = 0 = X'''(l)$. Then $X'''' + \mu^4 X = 0$. $X(x) = c_1 \sin \mu x + c_2 \cos \mu x + c_3 \sinh \mu x + c_4 \cosh \mu x$.

Modifications for certain non-homogeneous problems (section 77, similar to section 39)

Example 52. $u_t(x,t) = ku_{xx}(x,t)$ with $u(0,t) = 0$ and $Ku_x(1,t) = A$, $A > 0$; $u(x,0) = 0$. Non homogeneous BC. Similar to idea in section 39, suppose $u(x,t) = U(x,t) + \Phi(x)$. Try to select $\Phi(x)$ so that $U(x,t)$ is a solution to homogeneous PDE with homogeneous BC. If possible, then can apply Fourier Method to determine $U(x,t)$, and recover $u(x,t)$. Then $U(x,t) = u(x,t) - \Phi(x)$. So $U_t(x,t) = u_t(x,t) = ku_{xx}(x,t)$ and $U_t(x,t) = kU_{xx}(x,t) + k\Phi''(x)$. Then $U(0,t) = u(0,t) - \Phi(0) = -\Phi(0)$ and $U_x(1,t) = u_x(1,t) - \Phi'(1) = \frac{A}{K} - \Phi'(1)$. To make PDE and BC for $U(x,t)$ homogeneous, we want $\Phi(x)$ to satisfy:

i) $k\Phi''(x) = 0$

ii) $\Phi(0) = 0$

iii) $\frac{A}{K} - \Phi'(1) = 0$

So i) implies $\Phi(x) = Bx + C$. ii) implies $0 = B(0) + C$, so $C = 0$ so $\Phi(x) = Bx$. Now iii) implies $\Phi'(1) = \frac{A}{K}$ so $B = \frac{A}{K}$. So $\Phi(x) = \frac{A}{K}x$. Then $U(x,t)$ satisfies: $U_t(x,t) = kU_{xx}(x,t)$, $U(0,t) = 0$, $U_x(1,t) = 0$, and $U(x,0) = u(x,0) - \Phi(x) = -\frac{A}{K}x$. Next apply Fourier method to determine $U(x,t)$. $U(x,t) = X(x)T(t)$. $X''(x) + \lambda X(x) = 0$, $X(0) = 0$, and $X'(1) = 0$. By problem 72.1, $\lambda_n = \alpha_n^2$; $\alpha_n = \frac{(2n-1)\pi}{2}$ for $n = 1, 2, \ldots$; $X_n(x) = \phi_n(x) = \sqrt{2}\sin(\alpha_n x)$. Then $T_n(t) = e^{-\alpha_n^2 kt}$. So $U(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha_n^2 kt}\phi_n(x)$. Then determine c_n so that $-\frac{A}{K}x = \sum_{n=1}^{\infty} c_n \phi_n(x)$, $c_n = \sqrt{2}\frac{A}{K}\frac{(-1)^n}{\alpha_n^2}$. Thus $u(x,t) = \frac{A}{K}x + \frac{2A}{K}\sum_{n=1}^{\infty} e^{-\alpha_n^2 kt}\frac{(-1)^n}{\alpha_n^2}\sin(\alpha_n x)$.

HW: §77: 2,3. Not due.

Example 53 (Test #4). $u_t(x,t) = u_{xx}(x,t)$, $u_x(0,t) = 0$, $u(1,t) = 4$, $u(x,0) = 0$. $u(x,t) = U(x,t) + \Phi(x)$. $U(x,t) = u(x,t) - \Phi(x)$. $U_t(x,t) = u_t(x,t) = u_{xx}(x,t) = U_{xx}(x,t) + \Phi''(x)$. $U_x(0,t) = u_x(0,t) - \Phi'(0) = -\Phi'(0)$. $U(1,t) = u(1,t) - \Phi(1) = 4 - \Phi(1)$. Select $\Phi(x)$ so that i) $\Phi''(x) = 0 \implies \Phi(x) = Ax + B$; ii) $\Phi'(0) = 0 \implies A = 0, \Phi(x) = B$; iii) $\Phi(1) = 4 \implies B = 4$, so $\Phi(x) = 4$. So $U(x,t)$ solves: $U_t(x,t) = U_{xx}(x,t)$, $U_x(0,t) = 0$, $U(1,t) = 0$, $U(x,0) = -4$.

Example 54 (Test #5). $X''(x) + \lambda X(x) = 0$, $X(0) = 0$, $X'(1) + 2X(1) = 0$. Know $\lambda \geq 0$. Case $\lambda = 0$: $\to$ only trivial solution, so not an eigenvalue. Case $\lambda > 0$: Assume $\lambda = \alpha^2$, $\alpha > 0$. $X(x) = c_1 \sin \alpha x + c_2 \cos \alpha x$. $X(0) = 0$ implies $0 = c_2$, so $X(x) = c_1 \sin \alpha x$, $X'(x) = c_1 \alpha \cos \alpha x$. $X'(1) + 2X(1) = 0$ implies $c_1 \alpha \cos \alpha + 2c_1 \sin \alpha = 0$. Want $c_1 \neq 0$, $\alpha \cos \alpha + 2 \sin \alpha = 0$ means $\tan \alpha = -\frac{\alpha}{2}$. By considering graphs of $y = \tan x$ and $y = -\frac{x}{2}$, we see there are countable infinite number of solutions α_n, $n = 1, 2 \ldots$. Then $\lambda_n = \alpha_n^2$, $X_n(x) = \sin(\alpha_n x)$. To get normalized eigenfunction $\phi_n(x) = \frac{1}{||X_n(x)||} X_n(x)$. $||X_n(x)||^2 = \int_0^1 \sin^2(\alpha_n x)\, dx = \int_0^1 \frac{1}{2}[1 - \cos 2\alpha_n x]\, dx = \frac{1}{2}x + \frac{1}{4\alpha_n} \sin 2\alpha_n x \Big|_0^1 = \frac{1}{2} - \frac{1}{4\alpha_n} \sin 2\alpha_n = \frac{1}{2} - \frac{1}{2\alpha_n} \sin \alpha_n \cos \alpha_n = \frac{1}{2} - \frac{1}{2}\left(-\frac{1}{2}\cos \alpha_n\right) \cos \alpha_n = \frac{1}{2} + \frac{1}{4}\cos^2 \alpha_n = \frac{2 + \cos^2 \alpha_n}{4}$. So $\phi_n(x) = \frac{2}{\sqrt{2 + \cos^2 \alpha_n}} \sin(\alpha_n x)$.

Example 55 (Beam Equation). Last time got to $X'''' - \frac{\lambda}{\beta} X = 0$, where $\frac{\lambda}{\beta} = \mu^4$. Then $X'''' - \mu^4 X = 0$ with $X(0) = 0 = X'(0)$, $X''(l) = X'''(l) = 0$, and $X(x) = c_1 \sin \mu x + c_2 \cos \mu x + c_3 \sinh \mu x + c_4 \cosh \mu x$. Then $X'(x) = \mu[c_1 \cos \mu x - c_2 \sin \mu x + c_3 \cosh \mu x + c_4 \sinh \mu x]$, $X''(x) = \mu^2[-c_1 \sin \mu x - c_2 \cos \mu x + c_3 \sinh \mu x + c_4 \cosh \mu x]$, and $X'''(x) = \mu^3[-c_1 \cos \mu x + c_2 \sin \mu x + c_3 \cosh \mu x + c_4 \sinh \mu x]$. $X(0) = 0$ implies $c_2 = -c_4$. $X'(0) = 0$ implies $c_1 = -c_3$. So $X''(l) = 0$ implies $c_3[\sin \mu l + \sinh \mu l] + c_2[\cos \mu l + \cosh \mu l] = 0$ and $X'''(l) = 0$ implies $-c_1[\cos \mu l + \cosh \mu l] + c_2[\sin \mu l - \sinh \mu l] = 0$.

$$\begin{bmatrix} \sin \mu l + \sinh \mu l & \cos \mu l + \cosh \mu l \\ -\cos \mu l - \cosh \mu l & \sin \mu l - \sinh \mu l \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This has a nontrivial solution only if determinant of matrix is zero. Then

$$(\sin \mu l + \sinh \mu l)(\sin \mu l - \sinh \mu l) + (\cos \mu l + \cosh \mu l)^2 = 0$$
$$\sin^2 \mu l - \sinh^2 \mu l + \cos^2 \mu l + \cosh^2 \mu l + 2 \cos \mu l \cosh \mu l = 0 = 2 + 2 \cos \mu l \cosh \mu l = 0$$

So $\cos \mu l \cosh \mu l = -1$.