The Well-Rounded Retract and the Voronoi Polyhedron (UNCG)

Presenter: Mark McConnell, Notes by Michael Reed

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$G = GL_n(\mathbb{R}) = \{n \times n$ matrices, invertible$\}$ = space of lattice bases, by row vectors.
$GL_n(\mathbb{Z}) = \{n \times n,$ in $\mathbb{Z},$ invertible$\} \iff \det\gamma = \pm 1$. $GL_n(\mathbb{Z})\backslash G$ = space of lattices.
Consider two lattices equivalent ($\sim$) if they differ by

- orthogonal transformation (rotation/reflection). $\mathcal{O}_n(\mathbb{R})$
- scaling by positive real numbers (homotheties) $\mathbb{R}^+$.

Let $X = G/\mathbb{R}_+ \cdot \mathcal{O}_n(\mathbb{R})$ = space of lattice bases $\mod \sim$. $GL_n(\mathbb{Z})\backslash X$ = space of lattices.

Remark. Can rescale ?in?to? homothety so $g \in G$ has $\det g = \pm 1$. y to $\mathcal{O}_n(\mathbb{R})$, can get $\det g = \pm 1 \implies G = SL_n(\mathbb{R})$. $X = SL_n(\mathbb{R})/SO_n(\mathbb{R})$, both with $\det = 1$. $SL_n(\mathbb{Z})\backslash X$.

Remark. $\Gamma(N) = \{\gamma \in SL_n(\mathbb{Z}) \mid \gamma \equiv I \mod N\}$. $\Gamma_0(N,k) = \{\gamma \in SL_n(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod N\}$. [?canonincal? $\Gamma_0(N) \subseteq SL_2(\mathbb{Z})$ is $\Gamma_0(N,1)$].

Remark. Can do this for any semisimple Lie group G. G has a manifold compact subgroup K. $X = G/K$. Riemannian symmetric space. $\Gamma\backslash X$ locally symmetric space.

Let $\Gamma \subseteq SL_n(\mathbb{Z})$ any arithmetic group.

Remark. $X = \mathfrak{h}$ for SL_2.

Hecke Correspondence

1. Pick $g \in SL_n(\mathbb{R})$. lattice with basis g. Fix prime l. Fix $k = \{1, \dots, n\}$. There are only finitely many sublattices $M \subseteq L$ such that $L/M \cong (\mathbb{Z}/l\mathbb{Z})^k$.

   Definition 1. The Hecke correspondence $T(l,k)$ sending $\Gamma\backslash X \to \Gamma\backslash X$ is the one-to-many function $L \mapsto$ set of those M.

2. Let $t = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & l & & \\ & & & & \ddots & \\ & & & & & l \end{bmatrix}$. The l block is $k \times k$ and the 1 blcok is $n-k \times n-k$. Two maps:

   $\Gamma \cap \Gamma_0(l,k)\backslash X \xrightarrow{r} \Gamma\backslash X$ and $\Gamma \cap \Gamma_0(l,k)\backslash X \xrightarrow{s} \Gamma\backslash X$. $\Gamma \cap \Gamma_0(l,k) \cdot g \overset{r}{\mapsto} \Gamma \cdot g$ and $\Gamma \cap \Gamma_0(l,k) \cdot g \overset{s}{\mapsto} \Gamma \cdot tg$.

   Definition 2. The Hecke correspondence $T(l,k)$ is $\mathcal{A} \circ r^{-1}$, r^{-1} is preimage.

Why did $\Gamma_0(l,k)$ appear? Could do $X \xrightarrow{r} \Gamma\backslash X$ and $X \xrightarrow{s} \Gamma\backslash X$. Γg lifts in X to $\{g, \gamma_0 g, \gamma_1 g, \dots\}$. $s(\gamma_0 g) = s(\gamma_1 g)$ if and only if $t\gamma_0 g$ and $t\gamma_1 g$ ($\leftarrow$ right action) are in the same Γ-coset. iff $\exists \gamma_2 \in \Gamma$ so $\gamma_2 = t\gamma_0 g g^{-1}\gamma_1^{-1}t^{-1}$, where $\gamma_0 g g^{-1}\gamma_1^{-1} \in \Gamma$. iff $\gamma_0\gamma_1^{-1} \in \Gamma \cap [t^{-1}\Gamma t]$, where ?$\Gamma$ is γ_2?. iff $\gamma_0\gamma_1^{-1} \in \Gamma_0(l,k)$.

Definition 3. The Hecke operator $T(l,k)$ sending $H_*(\Gamma\backslash X) \mapsto H_*(\Gamma\backslash X)$ is $s_* \circ r^*$.

For any spaces S_1, S_2 and continuous maps $f : S_1 \to S_2$, $\exists$ natural $f_* : H_i(S_1) \to H_i(S_2)$. In general, there is no f^* back the other way. But there is if f has finitely many sheets.

Example 1. f is a covering map with finitely many sheets.

(N) is torsion-free for $N \geq 3 \implies (\Gamma(N) \cap \Gamma_0(l,k))\backslash X \to \Gamma(N)\backslash X = Y(N)$ is a covering map.

Well-Rounded Retract

This is a subset $W \subseteq X$. $\exists$ a deformation retraction $X \to W$. Furthermore, the retraction is $GL_n(\mathbb{Z})$-equivariant $\implies$?? to a retraction $\Gamma\backslash X \to \Gamma\backslash W$. $G(3)\backslash\mathfrak{h}$.

Example 2. SL_2, $Y(3)$. Graph 4 vertices, 6 edges dual to tetrahedron. $\Gamma(3)\backslash W$ is the graph $\Gamma(3)\backslash X = Y(3)$ retracts onto the graph.

How to do the WR retraction: fix g, $L =$ its lattice.

1. find the shortest non-geo retraction in L, fix the line V_1 through that vector, rescale in the $n-1$ direction $\perp V_1$.
2. eventually some linear independent vector becomes tied for shortest, fix space V_2 spanned by those two, rescale (shrink) $\perp V_2$.
3. ... etc.

STOP: a bunch of vectors all tied for shortest. <u>def</u> well-rounded

$G = SL_n(\mathbb{R}) =$ space of bases of $\mathbb{R}^n$. $X = G/K =$ space of bases of $\mathbb{R}^n \mod$ rotation. $g \in G$. $L =$ lattice spanned by $g = \{\vec{y} = \vec{y}g \mid \vec{x} \in \mathbb{Z}^n\}$.

Definition 4. The <u>arithmetic minimum</u> of L is $\min\{||\vec{y}|| \mid \vec{y} \in L, \vec{y} \neq \vec{0}\}$. The <u>minimal vectors</u> of L are the $\vec{y}$ where the arithmetic minimum is attained.

Definition 5. L is <u>well-rounded</u> if its minimal vectors span $\mathbb{R}^n$.

Definition 6. $gK \in X$ is <u>well-rounded</u> if its lattice L is well-rounded. $W \subset X$ is the set of well-rounded points.

Theorem 1 (Ash 1984)**.** *1. W is a deformation retract of X.*

2. The retraction is $GL_n(\mathbb{Z})$-equivariant $\implies$ descends to a deformation retraction $\Gamma\backslash X \to \Gamma\backslash W$.

3. W is a locally finite regular cell complex. $GL_n(\mathbb{Z})$ acts on cells. $\Gamma\backslash W$ is a finite cell complex.

A 0-cell and a 1-cell makes S^1, <u>irregular</u>. Two 0-cell's and two 1-cells also makes S^1, <u>regular</u>.

Corollary 1. Can compute $H_*(\Gamma\backslash X)$ by $\cong H_*(\Gamma\backslash W)$.

Example 3 ($n = 2$). $\vec{y} \in L$ have the form $\vec{y} = \vec{x}g$ for $\vec{x} \in \mathbb{Z}^n$. $||\vec{y}||^2 = \vec{x}gg^T\vec{x}^T$, gg^T symmetric matrix. Write $gg^T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. <u>Problem</u>: Find the lows of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are minimal vectors. (multiply by homothety so arithmetic minimum = 1.). $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \implies a = 1$. $\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a^b & \\ b & c \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \implies c = 1$. $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq 1$ is $2 + 2b \geq 1$ or $2b \geq -1$ or $-\frac{1}{2} \leq b$. $\begin{bmatrix} -1 & 1 \end{bmatrix}$ yields $b \leq \frac{1}{2}$. So $|b| \leq \frac{1}{2}$.

Lemma. When $|b| \leq \frac{1}{2}$, $\forall \begin{bmatrix} k \\ q \end{bmatrix} = \vec{x} \in \mathbb{Z}^2$, $\vec{x} \neq \vec{0}$, $\begin{bmatrix} k & q \end{bmatrix} \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} \begin{bmatrix} k \\ q \end{bmatrix} \geq 1$.

Proof. Complete the square. Use method of *Exer. 0.1.2.* $\square$

<u>Answer</u> Lows is a 1-cell $\{\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} \mid b \in [-\frac{1}{2}, \frac{1}{2}]\}$. $gg^T = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} =$ <u>Gram matrix of g</u>. (i,j) entry of Gram matrix = dot product of rows i and j of g. Up to roration, $g = \begin{bmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{bmatrix}$ and $\cos\theta = b$ and $|b| \leq \frac{1}{2}$. $60° \leq \theta \leq 120°$ are the angles of the points of radius one half from the origin of the fundamental domain.

An algorithm for Hecke Operators for SL in higher rank

Computing W as a whole is a(n infinite) linear programming (LP) problem. Fix bound r. Look at all $\begin{bmatrix} p & q \end{bmatrix} \in \mathbb{Z}^2$, $|| \begin{bmatrix} p & q \end{bmatrix} || \leq r$. Set up a LP like the previous exercise. Ask software to solve $\implies$ some bounded subset of W. Well-rounded retract for SL_n is based on work of Voronoi (1998). modern: Voronoi polyhedron.

Why computing Hecke operators with W might be hard.

Example 4. SL_2, $T(2,1)$. For $z \in \mathfrak{h}$, the Hecke correspondence $T(2,1)$ sends it to <u>three</u> points $2z$, $\frac{z}{2}$, $\frac{z+1}{2}$ or $X = \{ \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \}$. The arcs arcs are split at the center (breaking cells).

Manin had a Hecke algorithm for $n = 2$. It used <u>uni</u>modular symbols to avoid the problem of breaking cells. Ash-Rudolph generalized Manin's algorithm to SL_n $\forall n$, but only in the top non-vanishing degree of H_* or H^*. Top degree $= \frac{n(n+1)}{2} - n = \dim W$. To compute degree $(\dim W) - 1$, Gunnel's algorithm.

The Well-Tempered Complex

(MacPherson-McConnel 1994-95, 2014, 2016-present) An algorithm to compute $T(l,k)$ on $H_i(\Gamma \backslash W)$, $\forall SL_n$, $\forall i$. Fix $l, k \in \{1, \ldots, n\}$. Fix $L = g \cdot K$. Let $M \subseteq L$ be one of the sublattices where $L/M \cong (\mathbb{Z}/l\mathbb{Z})^k$.

Recall. $T(l,k)$ correspondence is the map $L \mapsto$ (set of all such M).

Definition 7. Let t be a continuously varying real parameter, $t \geq 1$. The <u>tempered length</u> of $\vec{y} \in L$ is $\begin{cases} ||\vec{y}|| & \text{if } \vec{y} \in M \\ t \cdot ||\vec{y}|| & \text{if } \vec{y} \in L, \vec{y} \notin M \end{cases}$. For a given t, let W_t be the image of the WR retraction done using the tempered length.

Remark. After $t \geq l$, all $W_t = W_l$ because $l \cdot L \subseteq M$. Stops at $t = l$.

Definition 8. The <u>well-tempered complex</u> $\tilde{W}$ is the subset of $X \times [1, l]$ that has value W_t over t in 2nd coordinate.

Theorem 2 (M-M 2016)**.**

1. *$\tilde{W}$ is a def. retract of $X \times [1, l]$*
2. *retract is $\Gamma_0(l,k)$-equivariant $\implies$ descends $(\Gamma \cap \Gamma_0(l,k)) \backslash \tilde{W}$.*
3. *$\tilde{W}$ is a loc. finite ?alg.? cell complex, computable by LP (set $\mu = \frac{1}{t^2}$, it's <u>linear</u> in $\mu, a, b, c, \ldots$).*