1. Differentiation

1.1 Limit of the Difference Quotient

Definition 1.1.1 – Derivative. The derivative of a function $f(x)$ with respect to the variable x is defined as

$$f'(x) \equiv \lim_{\Delta x \to 0} \underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\text{Difference Quotient}} \tag{1.1}$$

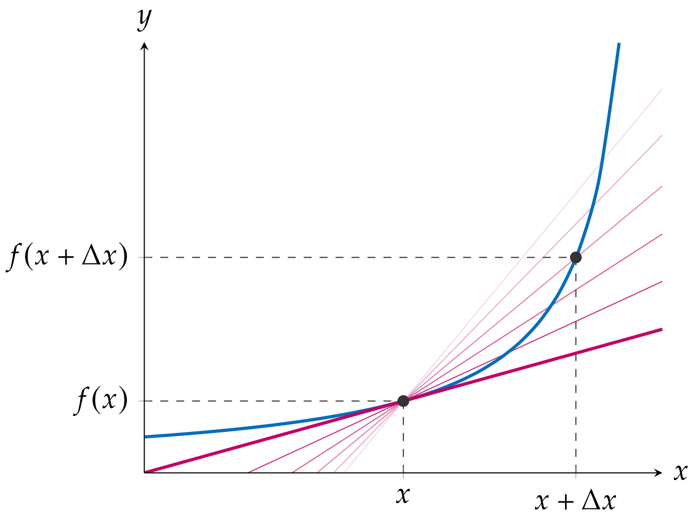

Figure 1.1: [**mooculus:textbook**]

Example 1.1 – id:20141219-212546.

Differentiate the function $f(x) = 5$

S

Solution:

$$
\begin{aligned}
f(x) &= 5x^0 && \text{PoID(??)}\\
f'(x) &= \lim_{\Delta x\to 0}\frac{5[x+\Delta x]^0 - 5[x]^0}{\Delta x} && \text{SPE(??)\&DBFP(1.1)}\\
f'(x) &= \lim_{\Delta x\to 0}\frac{5(1)-5(1)}{\Delta x} && \text{PoID(??)}\\
f'(x) &= \lim_{\Delta x\to 0} 0 && \text{OOM(??)}\\
f'(x) &= 0
\end{aligned}
$$

■

1.2 Derivative of a Monomial Functions

Rule 1.2.1 – Derivative of a Constant (DC).

$$[c]' = 0 \tag{1.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0 \tag{1.3}$$

Rule 1.2.2 – Derivative of a Constant Multiple (DCM).

$$[cf(x)]' = c\,[f(x)]' \tag{1.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = c\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] \tag{1.5}$$

Rule 1.2.3 – Derivative of a Power (DPo).

$$[x^n]' = nx^{n-1} \tag{1.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1} \tag{1.7}$$

Example 1.2 – id:20141124-153017.

Differentiate $f(x) = -3$

S

Solution:

$$
\begin{aligned}
f'(x) &= [-3]' && \text{SPE(??)}\\
f'(x) &= 0 && \text{DC(1.2)}
\end{aligned}
$$

D

Dependencies:
example 1.4-20141124-152503

Example 1.3 – id:20141124-141850.

Differentiate $f(x) = x^2$

S

Solution:

$$\begin{aligned} f'(x) &= [x^2]' && \text{SPE(??)} \\ f'(x) &= 2x^{2-1} && \text{DPo(1.6)} \\ f'(x) &= 2x^1 && \text{OOA(??)} \\ f'(x) &= 2x && \text{MId(??)} \end{aligned}$$

S

Less Steps Solution:

$$f'(x) = 2x \qquad \text{DPo(1.6)}$$

D

Dependencies:
example 1.4-20141124-152503

1.3 Derivative of Polynomial Functions

Rule 1.3.1 – Derivative of a Sum (DS).

$$[f(x) + g(x)]' = f'(x) + g'(x) \tag{1.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + \frac{\mathrm{d}}{\mathrm{d}x}[g(x)] \tag{1.9}$$

Example 1.4 – id:20141124-152503.

Differentiate $f(x) = x^2 - 3$

S

Solution:

$$\begin{aligned}
f(x) &= x^2 + \neg 3 && \text{DOS(??)}\\
f'(x) &= [x^2 + \neg 3]' && \text{SPE(??)}\\
f'(x) &= [x^2]' + [\neg 3]' && \text{DS(1.8)}\\
f'(x) &= [x^2]' + 0 && \text{DC(1.2)}\\
f'(x) &= [x^2]' && \text{AId(??)}\\
f'(x) &= 2x && \text{DPo(1.6) goto 1.3}
\end{aligned}$$

S

Less Steps Solution:

$$f(x) = 2x^2 \qquad \text{DPo(1.6)\&DC(1.6)}$$

D

Dependencies:
example 1.15-20141124-205219

Example 1.5 – id:20141128-151834.

Differentiate $f(x) = 3x^2 - 6x + 4$

S

Solution:

$$\begin{aligned}
f(x) &= 3x^2 + \neg 6x + 4 && \text{DOS(??)}\\
f'(x) &= [3x^2 + \neg 6x + 4]' && \text{SPE(??)}\\
f'(x) &= [3x^2]' + [6x]' + [4]' && \text{DS(1.8)}\\
f'(x) &= [3x^2]' + [6x]' + 0 && \text{DC(1.2)}\\
f'(x) &= [3x^2]' + [6x]' && \text{AId(??)}\\
f'(x) &= 3[x^2]' + 6[x]' && \text{DCM(1.4)}\\
f'(x) &= 3\,(2x) + 6\,(1) && \text{DPo(1.6)}\\
f'(x) &= 6x + 6 && \text{OOM(??)}
\end{aligned}$$

S

$$f'(x) = 6x + 6 \qquad \text{DS(1.8)}$$

1.3.1 Finding the vertex of a quadratic function using differentiation.

We can find the vertex of a quadratic function, $f(x)$ using differentiation by:

1. Differentiate the function: Find $f'(x)$.
2. Set the derivative equal to zero: $f'(x) = 0$.
3. Find the abscissa of the vertex by solving the equation $f'(x) = 0$ for x to find the critical x value: $x = k$.
4. Find the ordinate of the vertex by substituting the value of critical value $x = k$ into the function $f(x)$: Evaluate $f(k)$

Example 1.6 – id:20151008-110208.
Find the vertex of the quadratic function, $f(x) = ax^2 + bx + c$, using differentiation.

S

Solution:
1. Find the derivative of $f(x)$

$$\begin{aligned}
[f(x)]' &= [ax^2 + bx + c]' && \text{SPE(??)}\\
f'(x) &= [ax^2]' + [bx]' + [c]' && \text{DS(1.8)}\\
f'(x) &= a[x^2]' + b[x]' + [c]' && \text{DCM(1.4)}\\
f'(x) &= a \cdot x + b \cdot 1 + c && \text{DPo(1.6)}\\
f'(x) &= ax + b + [c]' && \text{Todo simplify}\\
f'(x) &= ax + b + 0 && \text{DC(1.2)}\\
f'(x) &= ax + b && \text{AId(??)}
\end{aligned}$$

2. Set the derivative equal to zero and solve for x.

$$\begin{aligned}
f'(x) &= 0\\
ax + b &= 0\\
x &= -\frac{b}{a} && \text{Todo Solve}
\end{aligned}$$

The abscissa of the vertex is $x = -\frac{b}{2a}$.
3. Find the ordinate of the vertex by substituting the argument $x = -\frac{b}{2a}$ into $f(x)$

Example 1.7 – id:20150923-152515.
Find the vertex of the parabola $y = x^2 - 2x - 6$ using differentiation.

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1. Differentiate the function.

$$\begin{aligned}
f(x) &= x^2 - 2x - 6 \\
f(x) &= x^2 + \neg 2x + \neg 6 && \text{DOS(??)} \\
[f(x)]' &= [x^2 + \neg 2x + \neg 6]' && \text{SPE(??)} \\
f'(x) &= [x^2]' + [\neg 2x]' + [\neg 6]' && \text{DS(1.8)} \\
f'(x) &= [x^2]' + \neg 2[x]' + [\neg 6]' && \text{DCM(1.4)} \\
f'(x) &= 2x + \neg 2 + [\neg 6]' && \text{DPo(1.6)} \\
f'(x) &= 2x + \neg 2 + 0 && \text{DC(1.2)} \\
f'(x) &= 2x + \neg 2 && \text{AId(??)} \\
f'(x) &= 2x - 2 && \text{DOS(??)}
\end{aligned}$$

2 and 3. Set the derivative equal to zero and solve for x

$$\begin{aligned}
2x - 2 &= 0 \\
x &= 1
\end{aligned}$$

4. Find the value of $f(1)$

$$\begin{aligned}
f(x) &= x^2 - 2x - 6 \\
f(1) &= [1]^2 - 2[1] - 6 && \text{SPE(??)} \\
f(1) &= -7 && \text{Evaluate}
\end{aligned}$$

The vertex of this parabola is the point $(1, -7)$ ▪

Rule 1.3.2 – Derivative of a Product (DPr).

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \tag{1.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)]g(x) + f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)] \tag{1.11}$$

Example 1.8 – id:20141209-144203.

Differentiate $f(x) = x^2(2x + 4)$

S

Solution:

$$
\begin{aligned}
f'(x) &= [x^2(2x+4)]' && \text{SPE(??)}\\
f'(x) &= [x^2]'(2x+4) + x^2[2x+4]' && \text{DPr(1.10)}\\
f'(x) &= [x^2]'(2x+4) + x^2[2x] + [4]' && \text{DS(1.8)}\\
f'(x) &= [x^2]'(2x+4) + x^2 \cdot 2[x] + [4]' && \text{DCM(1.4)}\\
f'(x) &= 2x(2x+4) + x^2 \cdot 2 \cdot 1 + [4]' && \text{DPo(1.6)}\\
f'(x) &= 2x(2x+4) + x^2 \cdot 2 \cdot 1 + 0 && \text{DC(1.2)}\\
f'(x) &= 6x^2 + 8x && \text{simplify goto ??}
\end{aligned}
$$

Example 1.9 – id:20141209-142321.

Differentiate $f(x) = x^2 \cos(x)$

S

Solution:

$$
\begin{aligned}
f'(x) &= [x^2\cos(x)]' && \text{SPE(??)}\\
f'(x) &= [x^2]'\cos(x) + x^2[\cos(x)]' && \text{DPr(1.10)}\\
f'(x) &= 2x\cos(x) + x^2[\cos(x)]' && \text{DPo(1.6)}\\
f'(x) &= 2x\cos(x) + x^2(-1\sin(x)) && \text{DCos(1.14)}\\
f'(x) &= 2x\cos(x) - x^2\sin x && \text{OOM(??)}
\end{aligned}
$$

1.4 Derivative of Trigonometric Functions

Rule 1.4.1 – Derivative of Sine (DSin).

$$[\sin(x)]' = \cos(x) \tag{1.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(x)] = \cos(x) \tag{1.13}$$

Rule 1.4.2 – Derivative of Cosine (DCos).

$$[\cos(x)]' = -\sin(x) \tag{1.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos(x)] = -\sin(x) \tag{1.15}$$

Example 1.10 – id:20150910-115935.

Differentiate $f(x) = \sin(x)\cos(x)$

S

Solution:

$$
\begin{aligned}
f'(x) &= [\sin(x)\cos(x)]' && \text{SPE(??)}\\
f'(x) &= [\sin(x)]'\cos(x) + \sin(x)[\cos(x)]' && \text{DPr(1.10)}\\
f'(x) &= \cos(x)\cos(x) + \sin(x)[\cos(x)]' && \text{DSin(1.12)}\\
f'(x) &= \cos(x)\cos(x) + \sin(x)(-\sin(x)) && \text{DCos(1.14)}\\
f'(x) &= \cos^2(x) - \sin^2(x) && \text{simplify goto ??}
\end{aligned}
$$

■

Example 1.11 – id:20141209-151354.

Differentiate $f(x) = \sin(x)\sin(x)$

S

Solution:

$$
\begin{aligned}
f'(x) &= [\sin(x)\sin(x)]' && \text{SPE(??)}\\
f'(x) &= [\sin(x)]'\sin(x) + \sin(x)[\sin(x)] && \text{DPr(1.10)}\\
f'(x) &= \cos(x)\sin(x) + \sin(x)\cos(x) && \text{DSin(1.12)}\\
f'(x) &= \cos(x)\sin(x) + \cos(x)\sin(x) && \text{CPM(??)}\\
f'(x) &= 2\cos(x)\sin(x) && \text{OOA(??)}
\end{aligned}
$$

■

1.5 Derivative of Rational Functions

Rule 1.5.1 – Derivative of a Quotient (DQ).

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \tag{1.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]\,g(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]}{[g(x)]^2} \tag{1.17}$$

1.6 Derivative of Exponential Functions

1.7 Derivative of Logarithmic Functions

Rule 1.7.1 – Derivative of a Natural Logarithm (DNL).

$$[\ln x]' = \frac{1}{x} \tag{1.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x} \tag{1.19}$$

1.8 Derivative of Composite Functions

Rule 1.8.1 – Derivative of a Composite Function (DComp).

$$[f(g(x))]' = [g(x)]'\,[f(g(x))]' \tag{1.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = \frac{\mathrm{d}}{\mathrm{d}x}[g(x)]\,\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] \tag{1.21}$$

Example 1.12 – id:20141124-203850.

Differentiate $y = \ln(3x)$

S

Solution:

After identifying that $y = \ln(3x)$ is a composite function, we let $u = 3x$ and thus we get a new function $y = \ln(u)$.

Using chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{DComp(1.21)}$$

We need to find the factors $\frac{\mathrm{d}y}{\mathrm{d}u}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$.

$$y = \ln(u) \qquad u = 3x$$

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{DComp(1.21)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{DNL(1.18)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{u} \cdot 3 \qquad \text{DPo(1.6)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3x} \cdot 3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{3x} \qquad \text{OOM(??)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

D

Dependencies:
example 1.15-20141124-205219

Example 1.13 – id:20141128-160248.

Differentiate $y = \ln(3x^2 - 6x + 4)$

S

Solution: After identifying that $y = \ln(3x^2 - 6x + 4)$ is a composite function, we let $u = 3x^2 - 6x + 4$ and thus we get a new function $y = \ln(u)$.

Using chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{DComp(1.21)}$$

We need to find the factors $\frac{\mathrm{d}y}{\mathrm{d}u}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$.

$$y = \ln(u) \qquad u = 3x^2 - 6x + 4$$

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 6x - 6 \qquad \text{goto } 1.5$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{DComp(1.21)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} \quad \text{DNL(1.18)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot (6x - 6) \quad \text{DPo(1.6)}$$
$$\frac{dy}{dx} = \frac{1}{3x^2 - 6x + 4} \cdot (6x - 6)$$
$$\frac{dy}{dx} = \frac{6x - 6}{3x^2 - 6x + 4} \quad \text{OOM(??)}$$

Example 1.14 – id:20141128-155506.

Differentiate $y = \ln(\cos x)$

S

Solution: After identifying that $y = \ln(\cos x)$ is a composite function, we let $u = \cos x$ and thus we get a new function $y = \ln(u)$.

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{DComp(1.21)}$$

We need to find the factors $\frac{dy}{du}$ and $\frac{du}{dx}$.

$$y = \ln(u) \qquad u = \cos x$$
$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{DComp(1.21)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} \quad \text{DNL(1.18)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot -\sin x \quad \text{DPo(1.6)}$$
$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot -\sin x$$
$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} \quad \text{OOM(??)}$$
$$\frac{dy}{dx} = -\tan x$$

Example 1.15 – id:20141124-205219.

Differentiate $y = \left(x^2 - 1\right) \ln(3x)$

Solution:

$$
\begin{aligned}
y' &= \left[\mathbf{x^2 - 3}\right]' \cdot \ln x + \left(x^2 - 3\right) \cdot \left[\mathbf{\ln 3x}\right]' && \text{DPr(1.10)} \\
y' &= 2x \cdot \ln x + \left(x^2 - 3\right) \cdot \left[\mathbf{\ln 3x}\right]' && \text{differentiate goto 1.4} \\
y' &= 2x \cdot \ln x + \left(x^2 - 3\right) \cdot \frac{1}{x} && \text{differentiate goto 1.12} \\
y' &= 2x \cdot \ln x + \frac{x^2 - 3}{x} && \text{OOM(\textbf{??})} \\
y' &= 2x^2 \ln x + \frac{x^2 - 3}{x} && \text{JTC(\textbf{??})} \\
y' &= \frac{2x^2 \ln x + \left(x^2 - 3\right)}{x} && \text{OOA(\textbf{??})}
\end{aligned}
$$