1. Equations

1.1 Equality

Property 1.1.1 – Reflexive Property of Eqaulity (RPE).

$$a = a \tag{1.1a}$$

Property 1.1.2 – Substitution Property of Equality (SPE).

Given $a = b$, then

$$E(a) = E(b) \tag{1.2}$$

$E(x)$ represents any expression.

Property 1.1.3 – Symmetric Property of Equality (SyPE).

$$a = b \quad \text{then} \quad b = a \tag{1.3a}$$

Property 1.1.4 – Transitive Property of Equality (TPE).

$$\text{if} \quad a = b \quad \text{and} \quad b = c \quad \text{then} \quad a = c \tag{1.4a}$$

1.2 Solving Linear Equations

Example 1.1 – id:20141206-102142.

Solve the equation $x + a = b$ for x

S

Solution:

$$\begin{aligned}
[x + a] + \neg a &= [b] + \neg a && \text{SPE(1.2)} + \text{AI(\textbf{??})}\\
x + (a + \neg a) &= b + \neg a && \text{APA(\textbf{??})}\\
x + 0 &= b + \neg a && \text{OOA(\textbf{??})}\\
x &= b + \neg a && \text{AId(\textbf{??})}\\
x &= b - a && \text{DOS(\textbf{??})}
\end{aligned}$$

Example 1.2 – id:20141111-222931.

Solve the equations $x + 8 = 0$

S

Solution:

$$\begin{aligned}
[x + 8] + \neg 8 &= [0] + \neg 8 && \text{SPE(1.2)} + \text{AI(\textbf{??})}\\
x + (8 + \neg 8) &= 0 + \neg 8 && \text{APA(\textbf{??})}\\
x + 0 &= \neg 8 && \text{OOA(\textbf{??})}\\
x &= \neg 8 && \text{AId(\textbf{??})}\\
x &= -8 && \text{ONeg(\textbf{??})}
\end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned}
[x + 8] + \neg 8 &= [0] + \neg 8 && \text{SPE(1.2)} + \text{AI(\textbf{??})}\\
x &= -8 && \text{OOA(\textbf{??})}
\end{aligned}$$

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Dependencies:
example **??**-20141111-190212

Example 1.3 – id:20141206-101632.

Solve the equation $x + 4 = 7$

S

Solution:

$$\begin{aligned} [x + 4] + \neg 4 &= [7] + \neg 4 && \text{SPE(1.2) + AI(\textbf{??})} \\ x + (4 + \neg 4) &= 7 + \neg 4 && \text{APA(\textbf{??})} \\ x + 0 &= 3 && \text{OOA(\textbf{??})} \\ x &= 3 && \text{AId(\textbf{??})} \end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned} [x] + 4 + \neg 4 &= [7] + \neg 4 && \text{SPE(1.2) + AI(\textbf{??})} \\ x &= 3 && \text{OOA(\textbf{??})} \end{aligned}$$

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Example 1.4 – id:20141206-101107.

Solve the equation $x - 8 = 15$ for x

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Solution:

$$\begin{aligned} x + \neg 8 &= 15 && \text{DOS(\textbf{??})} \\ [x + \neg 8] + 8 &= [15] + 8 && \text{SPE(1.2) + AI(\textbf{??})} \\ x + (\neg 8 + 8) &= 15 + 8 && \text{APA(\textbf{??})} \\ x + 0 &= 23 && \text{OOA(\textbf{??})} \\ x &= 23 && \text{AId(\textbf{??})} \end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned} [x + \neg 8] + 8 &= [15] + 8 && \text{SPE(1.2) + AI(\textbf{??})} \\ x &= 23 && \text{OOA(\textbf{??})} \end{aligned}$$

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Example 1.5 – id:20141206-102404.

Solve the equation $5x = 9$ for x.

S

Solution:

$$
\begin{aligned}
\frac{1}{5}[5x] &= \frac{1}{5}[9] && \text{SPE(1.2) + MI(??)} \\
\frac{1}{5}\cdot[5\cdot x] &= \frac{1}{5}\cdot 9 && \text{JTC(??)} \\
\left(\frac{1}{5}\cdot 5\right)\cdot x &= \frac{1}{5}\cdot 9 && \text{APM(??)} \\
1\cdot x &= \frac{9}{5} && \text{OOM(??)} \\
x &= \frac{9}{5} && \text{MId(??)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
\frac{1}{5}[5x] &= \frac{1}{5}[9] && \text{SPE(1.2) + MI(??)} \\
x &= \frac{9}{5} && \text{OOM(??)}
\end{aligned}
$$

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Example 1.6 – id:20141206-104404.

Solve the equation $ax = b$ for x.

S

Solution:

$$
\begin{aligned}
\frac{1}{a}[ax] &= \frac{1}{a}[b] && \text{SPE(1.2) + MI(??)} \\
\frac{1}{a}\cdot(a\cdot x) &= \frac{1}{a}\cdot b && \text{JTC(??)} \\
\left(\frac{1}{a}\cdot a\right)\cdot x &= \frac{1}{a}\cdot b && \text{APM(??)} \\
1\cdot x &= \frac{b}{a} && \text{OOM(??)} \\
x &= \frac{b}{a} && \text{MId(??)}
\end{aligned}
$$

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Example 1.7 – id:20141206-102723.

Solve the equation $-2x = 7$ for x

S

Solution:

$$\begin{aligned}
\neg 2x &= 7 && \text{ONeg(??)} \\
\neg\frac{1}{2}[\neg 2x] &= \neg\frac{1}{2}[7] && \text{SPE(1.2)} + \text{MI(??)} \\
\neg\frac{1}{2}\cdot(\neg 2\cdot x) &= \neg\frac{1}{2}\cdot 7 && \text{JTC(??)} \\
\left(\neg\frac{1}{2}\cdot\neg 2\right)\cdot x &= \neg\frac{1}{2}\cdot 7 && \text{APM(??)} \\
1\cdot x &= \neg\frac{7}{2} && \text{OOM(??)} \\
1\cdot x &= -\frac{7}{2} && \text{ONeg(??)} \\
x &= -\frac{7}{2} && \text{MId(??)}
\end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned}
\neg\frac{1}{2}[\neg 2x] &= \neg\frac{1}{2}[7] && \text{SPE(1.2)} + \text{MI(??)} \\
x &= -\frac{7}{2} && \text{OOM(??)}
\end{aligned}$$

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Example 1.8 – id:20141111-215726.

Solve the equation $2x + 5 = 0$ for x

S

Solution:

$$
\begin{aligned}
[2x+5] + \neg 5 &= [0] + \neg 5 && \text{SPE(1.2) + AI(??)} \\
2x + (5 + \neg 5) &= 0 + \neg 5 && \text{APA(??)} \\
2x + 0 &= \neg 5 && \text{OOA(??)} \\
2x &= \neg 5 && \text{AId(??)} \\
\frac{1}{2}[2x] &= \frac{1}{2}[\neg 5] && \text{SPE(1.2) + MI(??)} \\
\frac{1}{2} \cdot 2 \cdot x &= \frac{1}{2} \cdot \neg 5 && \text{JTC(??)} \\
\left(\frac{1}{2} \cdot 2\right) \cdot x &= \frac{1}{2} \cdot \neg 5 && \text{APM(??)} \\
1 \cdot x &= \frac{\neg 5}{2} && \text{OOM(??)} \\
1x &= -\frac{5}{2} && \text{ONeg(??)} \\
x &= -\frac{5}{2} && \text{MId(??)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
[2x+5] + \neg 5 &= [0] + \neg 5 && \text{SPE(1.2) + AI(??)} && (1.5) \\
2x &= \neg 5 && \text{OOA(??)} && (1.6) \\
\frac{1}{2}[2x] &= \frac{1}{2}[\neg 5] && \text{SPE(1.2) + MI(??)} && (1.7) \\
x &= -\frac{5}{2} && \text{OOM(??)} && (1.8)
\end{aligned}
$$

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Dependencies:
example **??**-20141111-192213

1.3 Solving Quadratic Equations

Example 1.9 – id:20141107-131748.

Solve the equation $2 - x^2 = 0$ for x

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Solution:

$$
\begin{aligned}
2 - 1x^2 &= 0 && \text{MId(??)}\\
2 + \neg 1x^2 &= 0 && \text{DOS(??)}\\
\left[2 + \neg 1x^2\right] + 1x^2 &= [0] + 1x^2 && \text{SPE(1.2) + AI(??)}\\
2 + \left(\neg 1x^2 + 1x^2\right) &= 0 + 1x^2 && \text{APA(??)}\\
2 + 0 &= 0 + 1x^2 && \text{OOA(??)}\\
2 &= 1x^2 && \text{AId(??)}\\
2 &= x^2 && \text{MId(??)}\\
\pm [2]^{\frac{1}{2}} &= \left[x^2\right]^{\frac{1}{2}} &&\\
\pm 2^{\frac{1}{2}} &= x && \text{PoPo(??)}\\
\pm\sqrt{2} &= x && \text{PoTR(??)}\\
x &= \pm\sqrt{2} && \text{SyPE(1.3a)}
\end{aligned}
$$

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1.3.1 Completing the Square

Completing the square is an algebraic algorithm used to find the solutions of quadratic equations of the form, $ax^2 + bx + c = 0$. Essentially, we want to manipulate this equation such that $x =$ some values. To gain some understanding of how this algorithm works, we will consider each step individually. Let's begin with a quadratic equation in the general form: $ax^2 + bx + c = 0$.

Since we are trying to manipulate the equation $ax^2+bx+c = 0$ such that $x =$ some value, we first want the coefficient factor a to be equal to 1. This is done by multiplying both expressions by the multiplicative inverse of a (step 1) followed by simplifying both expressions.

$$
\begin{aligned}
\frac{1}{a}\left[ax^2 + bx + c\right] &= \frac{1}{a}[0] && \text{SPE(1.2) + MI(??)}\\
\frac{1}{a}\cdot ax^2 + \frac{1}{a}\cdot bx + \frac{1}{a}\cdot c &= \frac{1}{a}[0] && \text{DPE(??)}\\
\frac{1}{a}\cdot a\cdot x^2 + \frac{1}{a}\cdot b\cdot x + \frac{1}{a}\cdot c &= \frac{1}{a}[0] && \text{JTC(??)}\\
x^2 + \frac{b}{a}\cdot x + \frac{c}{a} &= 0 && \text{OOM(??)}\\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \text{CTJ(??)}
\end{aligned}
$$

We now have three terms in the left hand expression where the first two terms have at least one variable factor, x. The problem is that the third term is a constant and we want $x =$ some value. This text step is to get rid of the $\frac{c}{a}$ term, which can be done by using the additive inverse followed by simplifying both expressions.

$$\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] + \neg\frac{c}{a} = [0] + \neg\frac{c}{a} \qquad \text{SPE(1.2) + AI(\textbf{??})}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} + \neg\frac{c}{a} = 0 + \neg\frac{c}{a} \qquad \text{APA(\textbf{??})}$$

$$x^2 + \frac{b}{a}x = \neg\frac{c}{a} \qquad \text{OOA(\textbf{??})}$$

The next step is called completing the square. The idea is to add a *NeW* constant, k, to the left-hand expression, $x^2 + \frac{b}{a}x + k$, such that the quadratic expression can then be factored as two identical factors, $(x + m)(x + m) = (x + m)^2$, where $k = m \cdot m$. Notice that since we are adding a constant term, k, to the left-hand expression, then we must also add this constant, k, to the right-hand expression, $x^2 + \frac{b}{a}x + k = \neg\frac{c}{a} + k$. To determine the values of both m and k we should refer to the the organization of the two factors that make up the product of a quadratic expression, $x^2 + \frac{b}{a}x + k = (x + m)^2$.

Since both factors of this new quadratic expression are the same, both terms that make up the middle term, must also be the same. We know that $mx + mx = \frac{b}{a}x$, so we should be able to determine the value of m from this equation. If we can determine the value of m, then we can determine the value of k.

$$\begin{aligned} \frac{b}{a}x &= mx + mx \\ &= 2mx && \text{OOA(\textbf{??})} \end{aligned}$$

Solving for m,

$$\begin{aligned} 2mx &= \frac{b}{a}x \\ 2 \cdot m \cdot x &= \frac{b}{a} \cdot x && \text{JTC(\textbf{??})} \\ \frac{1}{2}[2 \cdot m \cdot x] &= \frac{1}{2}\left[\frac{b}{a} \cdot x\right] && \text{SPE(1.2) + MI(\textbf{??})} \\ m \cdot x &= \frac{b}{2a} \cdot x && \text{OOM(\textbf{??})} \\ [m \cdot x]\frac{1}{x} &= \left[\frac{b}{2a} \cdot x\right]\frac{1}{x} && \text{SPE(1.2) + MI(\textbf{??})} \\ m &= \frac{b}{2a} && \text{OOM(\textbf{??})} \end{aligned}$$

$$
\begin{aligned}
\left[x^2 + \frac{b}{a}x\right] + \left(\frac{b}{2a}\right)^2 &= \left[\neg\frac{c}{a}\right] + \left(\frac{b}{2a}\right)^2 && \text{SPE(1.2)}\\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= \neg\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \text{APA(}\textbf{??}\text{)}\\
\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) &= \neg\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \text{DPF(}\textbf{??}\text{)}\\
\left(x + \frac{b}{2a}\right)^2 &= \neg\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \text{PoTF(}\textbf{??}\text{)}\\
\left(x + \frac{b}{2a}\right)^2 &= \neg\frac{c}{a} + \frac{(b)^2}{(2a)^2} && \text{PoQPo(}\textbf{??}\text{)}\\
\left(x + \frac{b}{2a}\right)^2 &= \neg\frac{c}{a} + \frac{b^2}{2^2a^2} && \text{PoPrPo}\\
\left(x + \frac{b}{2a}\right)^2 &= \neg\frac{c}{a} + \frac{b^2}{4a^2} && \text{OOE}\\
\left(x + \frac{b}{2a}\right)^2 &= \neg\frac{c}{a} \cdot \frac{4a}{4a} + \frac{b^2}{4a^2} && \text{MId}\\
\left(x + \frac{b}{2a}\right)^2 &= \neg\frac{c \cdot 4 \cdot a}{a \cdot 4 \cdot a} + \frac{b^2}{4a^2} && \text{JTC}\\
\left(x + \frac{b}{2a}\right)^2 &= \neg\frac{4 \cdot a \cdot c}{4 \cdot a \cdot a} + \frac{b^2}{4a^2} && \text{CPM}\\
\left(x + \frac{b}{2a}\right)^2 &= \neg\frac{4 \cdot a \cdot c}{4 \cdot a^2} + \frac{b^2}{4a^2} && \text{PrCBPo}\\
\left(x + \frac{b}{2a}\right)^2 &= \neg\frac{4ac}{4a^2} + \frac{b^2}{4a^2} && \text{CTJ}\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{\neg 4ac + b^2}{4a^2} && \text{CD}\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 + \neg 4ac}{4a^2} && \text{CPA}\\
\left[\left(x + \frac{b}{2a}\right)^2\right]^{\frac{1}{2}} &= \pm\left[\frac{b^2 + \neg 4ac}{4a^2}\right]^{\frac{1}{2}} && \text{PoI}
\end{aligned}
$$

$$x + \frac{b}{2a} = \pm \left[\frac{b^2 + \neg 4ac}{4a^2}\right]^{\frac{1}{2}} \qquad \text{PoPrPo}$$

$$x + \frac{b}{2a} = \pm \frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{[4a^2]^{\frac{1}{2}}} \qquad \text{PoQPo}$$

$$x + \frac{b}{2a} = \pm \frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{4^{\frac{1}{2}}a} \qquad \text{PoPrPo}$$

$$x + \frac{b}{2a} = \pm \frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{OOE}$$

$$\left[x + \frac{b}{2a}\right] + \neg \frac{b}{2a} = \left[\pm \frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a}\right] + \neg \frac{b}{2a} \qquad \text{AI}$$

$$x + \frac{b}{2a} + \neg \frac{b}{2a} = \pm \frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} + \neg \frac{b}{2a} \qquad \text{APA}$$

$$x + \frac{b}{2a} + \neg \frac{b}{2a} = \neg \frac{b}{2a} \pm \frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{CPA}$$

$$x = \neg \frac{b}{2a} \pm \frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{OOA}$$

$$x = \frac{\neg b \pm \left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{CD}$$

$$x = \frac{\neg b \pm \left[b^2 - 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{DOS}$$

$$x = \frac{\neg b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{ETR}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{ONeg}$$