1. Sets

1.1 Sets

The study of sets and their properties is the object of set theory.

Definition 1.1.1 – Set. A **set** A set is a finite or infinite collection of objects in which order has no significance, and multiplicity is generally also ignored (*unlike a list*).

Members of a set are often referred to as elements and the notation $a \in A$ is used to denote that a is an element of a set A.

$$\underbrace{S}_{\text{Set Name}} = \{\underbrace{a, b, c}_{\text{elements}}\}$$

R We are intentionally using a non-specific definition of a set.

Definition 1.1.2 – Cardinality. The number of distinct elements of a set S is called its **cardinality**, which is represented as $|S|$

Definition 1.1.3 – Subset. Suppose that A and B are sets. We say that A is a **subset** of B if whenever $x \in A$, then $x \in B$.

$$A \subseteq B$$

We could also say that all the elements of set A are also elements of set B.

Definition 1.1.4 – Proper Subset. Suppose that A and B are sets. We say that A is a ***proper* subset** of B if whenever $x \in A$, then $x \in B$, but $A \neq B$

$$A \subset B$$

R Some textbooks makes a point of differentiating between a subset and a proper subset. This convention may later cause trouble, in terms of notation, as different writers may use the notation $A \subset B$ to denote a subsets. Making matters worse $A \subset B$ can sometimes be used to describe a proper subset as does the textbook.

Definition 1.1.5 – Equality of Sets. Suppose that both $A \subseteq B$ and $B \subseteq A$, we then say A and B are equal and write

$$A = B$$

Definition 1.1.6 – Empty Set. The empty set has no elements. It is written $\emptyset$ and called the **empty set** or **null set**.

$$\emptyset = \{\ \}$$

R $\emptyset$ and $\{\emptyset\}$ are not the same thing. By definition $\emptyset$ has no elements; however, $\{\emptyset\}$ is a set with one element, which happens to be the empty set.

1.2 Well-Known Sets

- Naturals: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Rationals: $\mathbb{Q} = \{a/b \mid a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$

1.2.1 Rationals

Definition 1.2.1 – Field. Field A **field** F is a set of elements that contains at least the elements 0 and 1, given the operations of addition and multiplication, and satisfies the field axioms.

Field Axioms

addition is commutative $a + b = b + a \ \forall a, b, c \in F$
addition is associative $(a + b) + c = a + (b + c) \ \forall a, b \in F$
additive identity $\exists 0 \in F$ such that $a + 0 = a \ \forall a \in F$
additive inverse $\forall a \in F \ \exists - a \in F$ such that $-a + a = 0$
multiplication is commutative $a \times b = b \times a \ \forall a, b \in F$
multiplication is associative $(a \times b) \times c = a \times (b \times c) \ \forall a, b, c \in F$
multiplicative identity $\exists 1 \in F$ such that $a \times 1 = a \ \forall a \in F$
multiplicative inverse $\forall a \in F \setminus \{0\} \ \exists a^{-1} \in F$ such that $a^{-1} \times a = 1$
distributive law $a \times (b + c) = a \times b + a \times c \ \forall a, b, c \in F$

The usual priority to $\times$ over $+$ to reduce the number of delimiters.

1.3 Set Operations

Definition 1.3.1 – Union.

$$A \cup B = \{x \mid x \in a \text{ or } x \in B\}$$

R The word *or* is not exclusive meaning that we have $x \in A \cup B$ even if x is an element of both set A and set B.

Definition 1.3.2 – Intersection.

$$A \cap B = \{x \mid x \in a \text{ and } x \in B\}$$

Definition 1.3.3 Disjoint Sets We say sets A and B are disjoint when

$$A \cap B = \emptyset$$

for all sets A and B

Property 1.3.1 – Commutative Law of Union.

$$A \cup B = B \cup A$$

for all sets A and B

Property 1.3.2 – Commutative Law of Intersection.

$$A \cap B = B \cap A$$

for all sets A and B

Property 1.3.3 – Distributive Law of Intersection over Union.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Property 1.3.4 – Distributive Law of Intersection over Union.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Property 1.3.5 – Union of a Set and the Empty Set.

$$A \cup \emptyset = A$$

For set A

Property 1.3.6 – Intersection of a Set and the Empty Set.

$$A \cap \emptyset = \emptyset$$

For set A

Definition 1.3.4 – Difference of two sets.

$$A \setminus B \{x \mid x \in A, x \notin B\}$$

For all sets A and B

Definition 1.3.5 – Universe Set. The universe set U is a given fixed set from which all subsets are discussed from.

Definition 1.3.6 – Compliment Set. If A is a set such that $A \subseteq B$, then

$$A' = U \setminus A$$

or

$$A' = \{x \mid x \notin A\}$$

where A' is called the **compliment** of A.

R Complimentation has no meaning unless there is a universe U.

Property 1.3.7 Double Compliment

$$A'' = A$$

Property 1.3.8

$$A \cup A' = U$$

Property 1.3.9

$$A \cap A' = \emptyset$$