Chapter 1

Conditional Probability

1.1 Conditional Probability

The conditional probability of an event is the probability that an event A occurs given that another event B has already occurred. This type of probability is calculated by restricting the sample space that were working with to only the set B.

The formula for conditional probability can be rewritten using some basic algebra. Instead of the formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

1.2 Conditional Probability

Suppose B is an event in a sample space S with $P(B) > 0$. The probability that an event A occurs once Bhas occurred or, specifically, the conditional probability of A given B (written $P(A|B)$), is defined as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- This can be expressed as a multiplication theorem

$$P(A \cap B) = P(A|B) \times P(B)$$

- The symbol | is a vertical line and does not imply division.
- Also $P(A|B)$ is not the same as $P(B|A)$.

Remark: The Prosecutor's Fallacy , with reference to the O.J. Simpson trial.

What is the probability of one event given that another event occurs? For example, what is the probability of a mouse finding the end of the maze, given that it finds the room before the end of the maze?

This is represented as:

$$P[A|B]$$

or "the probability of A given B."

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If A and B are independent of one another, such as with coin tosses or child births, then:

$$P[A|B] = P[A]$$

Thus, "what is the probability that the next child a family bears will be a boy, given that the last child is a boy."

This can also be stacked where the probability of A with several "givens."

$$P[A|B_1, B_2, B_3]$$

or "the probability of A given that B1, B2, and B3 are true?"

- pairwise disjoint sets
- The addition principle

1.3 Conditional Probability

$$P(X|Y) = \frac{P(X \text{ and } Y)}{P(Y)}$$

Re-arranging

$$P(X \text{ and } Y) = P(X|Y) \times P(Y)$$

Therefore we can say

$$P(F \text{ and } A) = P(F|A) \times P(A)$$

$$P(F \text{ and } B) = P(F|B) \times P(B)$$

$$P(F \text{ and } A) = P(F|A) \times P(A)$$

$$P(F \text{ and } B) = P(F|B) \times P(B)$$

$$P(F \text{ and } A) = P(F|A) \times P(A) = 0.80 \times 0.01$$

$$P(F \text{ and } A) = 0.008$$

$$P(F \text{ and } B) = P(F|B) \times P(B) = 0.20 \times 0.03$$

$$P(F \text{ and } B) = 0.006$$

Recall:

$$P(F) = P(F \text{ and } A) + P(F \text{ and } B)$$

$$P[A|B]$$

or "the probability of A given B."

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If A and B are independent of one another, such as with coin tosses or child births, then:

$$P[A|B] = P[A]$$

Thus, "what is the probability that the next child a family bears will be a boy, given that the last child is a boy."

This can also be stacked where the probability of A with several "givens."

$$P[A|B_1, B_2, B_3]$$

or "the probability of A given that B1, B2, and B3 are true?"

1.4 Algebaic Results

1.4.1 Multiplication Rule

The multiplication rule is a result used to determine the probability that two events, A and B, both occur. The multiplication rule follows from the definition of conditional probability.

The result is often written as follows, using set notation:

$$P(A|B) \times P(B) = P(B|A) \times P(A) \qquad (= P(A \cap B))$$

Recall that for independent events, that is events which have no influence on one another, the rule simplifies to:

$$P(A \cap B) = P(A) \times P(B)$$

From the first year intake example, check that

$$P(E|F) \times P(F) = P(F|E) \times P(E)$$

- $P(E|F) \times P(F) = 0.58 \times 0.38 = 0.22$
- $P(F|E) \times P(E) = 0.55 \times 0.40 = 0.22$

1.4.2 Law of Total Probability

The law of total probability is a fundamental rule relating marginal probabilities to conditional probabilities. The result is often written as follows, using set notation:

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

where $P(A \cap B^c)$ is probability that event A occurs and B does not.

Using the multiplication rule, this can be expressed as

$$P(A) = P(A|B) \times P(B) + P(A|B^c) \times P(B^c)$$

From the first year intake example , check that

$$P(E) = P(E \cap M) + P(E \cap F)$$

with $P(E) = 0.40$, $P(E \cap M) = 0.18$ and $P(E \cap F) = 0.22$

$$0.40 = 0.18 + 0.22$$

Remark: M and F are complement events.

1.5 Bayes' Theorem

Bayes' Theorem is a result that allows new information to be used to update the conditional probability of an event.

Recall the definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Using the multiplication rule, gives Bayes' Theorem in its simplest form:

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

Using the multiplication rule, gives Bayes' Theorem in its simplest form:

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

The general form of Bayes theorem is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Equivalently Bayes' Theorem can be expressed as

$$P(B|A) = \frac{P(A|B) \times P(B)}{P(A)}.$$

1.6 Conditional Probabilities

Bayes Theorem

$$P(A|B) = \frac{P(A\text{and}B)}{P(B)}$$