PHIL 377 Assignment 3

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1 For each item below, provide a Fitch-style proof of the claim.

a $\vdash \forall x A(x) \leftrightarrow \forall x(A(x) \lor A(x))$

1.	$\forall x A(x)$	
2.	$A(c)$	$\forall$ **Elim:** 1
3.	$A(c) \lor A(c)$	$\lor$ **Intro:** 2
4.	$\forall x(A(x) \lor A(x))$	$\forall$ **Intro:** 3
5.	$\forall x(A(x) \lor A(x))$	
6.	$A(c) \lor A(c)$	$\forall$ **Elim:** 5
7.	$A(c)$	
8.	$A(c) \land A(c)$	$\land$ **Intro:** 7, 7
9.	$A(c)$	$\land$ **Elim:** 8
10.	$A(c)$	$\lor$ **Elim:** 6, 7–9, 7–9
11.	$\forall x A(x)$	$\forall$ **Intro:** 10
12.	$\forall x A(x) \leftrightarrow \forall x(A(x) \lor A(x))$	$\leftrightarrow$ **Intro:** 1–4, 5–11

b $\vdash \forall x \forall y(x = y \to (G(x,y) \leftrightarrow G(y,x)))$

1.	$i = j$	
2.	$G(i,j)$	
3.	$G(i,i)$	$=$ **Elim:** 1, 2
4.	$G(j,i)$	$=$ **Elim:** 1, 3
5.	$G(j,i)$	
6.	$G(j,j)$	$=$ **Elim:** 1, 5
7.	$G(i,j)$	$=$ **Elim:** 1, 6
8.	$G(i,j) \leftrightarrow G(j,i)$	$\leftrightarrow$ **Intro:** 2–4, 5–7
9.	$i = j \to (G(i,j) \leftrightarrow G(j,i))$	$\to$ **Intro:** 1–8
10.	$\forall y(i = y \to (G(i,y) \leftrightarrow G(y,i)))$	$\forall$ **Intro:** 9
11.	$\forall x \forall y(x = y \to (G(x,y) \leftrightarrow G(y,x)))$	$\forall$ **Intro:** 10

c $\forall x(F(x) \lor G(x))$
$\vdash \exists x \neg F(x) \to \exists x G(x)$

1.	$\forall x(F(x) \lor G(x))$	
2.	$\exists x \neg F(x)$	
3.	$\neg F(c)$	
4.	$F(c) \lor G(c)$	$\forall$ **Elim:** 1
5.	$F(c)$	
6.	$\bot$	$\neg$ **Elim:** 3,5
7.	$G(c)$	**X:** 6
8.	$G(c)$	
9.	$G(c) \land G(c)$	$\land$ **Intro:** 8, 8
10.	$G(c)$	$\land$ **Elim:** 9
11.	$G(c)$	$\lor$ **Elim:** 4, 5–7, 8–10
12.	$\exists x G(x)$	$\exists$ **Intro:** 11
13.	$\exists x G(x)$	$\exists$ **Elim:** 3, 4–12
14.	$\exists x \neg F(x) \to \exists x G(x)$	$\to$ **Intro:** 2–13

d $\forall z[G(z) \to \forall y(K(y) \to H(z,y))]$,
$(K(i) \land G(j)) \land i = j$
$\vdash H(i,i)$

1.	$\forall z[G(z) \to \forall y(K(y) \to H(z,y))]$	
2.	$(K(i) \land G(j)) \land i = j$	
3.	$G(j) \to \forall y(K(y) \to H(j,y))$	$\forall$ **Elim:** 1
4.	$K(i) \land G(j)$	$\land$ **Elim:** 2
5.	$G(j)$	$\land$ **Elim:** 4
6.	$\forall y(K(y) \to H(j,y))$	$\to$ **Elim:** 3, 5
7.	$K(i) \to H(j,i)$	$\forall$ **Elim:** 6
8.	$K(i)$	$\land$ **Elim:** 4
9.	$H(j,i)$	$\to$ **Elim:** 7, 8
10.	$i = j$	$\land$ **Elim:** 2
11.	$H(i,i)$	$=$ **Elim:** 9, 10

2 For each of the following claims,

i) If it is true, provide a Fitch-style proof of the claim,

ii) If it is not true, provide an interpretation that is a counterexample to the claim.

a $\exists x F(x, a)$,
$\forall y(y = a \to y = b)$
$\vdash \exists y F(y, y)$

False. Proof by example:
Let the domain comprise of the letters a, b and c.
Suppose $F(c, a)$ and $a = b$ are true.
Then $\exists x F(x, a)$ is true.
And $\forall y(y = a \to y = b)$ is also true.
However $\exists y F(y, y)$ is false.
Therefore $\exists x F(x, a)$ and $\forall y(y = a \to y = b)$ cannot prove $\exists y F(y, y)$.

b $L(a) \leftrightarrow \forall x L(x)$
$\dashv\vdash \exists x L(x)$

False. Proof by example:
Let the domain comprise of the letters a and b.
Suppose $L(a)$ and $\neg L(b)$ are true.
Then $\exists x L(x)$ is true.
But, $L(a) \leftrightarrow \forall x L(x)$ is false,
because $L(a)$ is true while $\forall x L(x)$ is false.
Therefore $\exists x L(x)$ cannot prove $L(a) \leftrightarrow \forall x L(x)$.
Therefore they are not provably equivalent.

c $\forall x(F(x) \to \exists y(G(y, x) \land \neg G(x, y)))$,
$\exists x F(x)$
$\vdash \exists x \exists y \neg x = y$

True.

1. ∀x(F(x) → ∃y(G(y, x) ∧ ¬G(x, y)))	
2. ∃xF(x)	
3. F(c)	
4. F(c) → ∃y(G(y, c) ∧ ¬G(c, y))	**∀ Elim:** 1
5. ∃y(G(y, c) ∧ ¬G(c, y))	**→ Elim:** 4, 3
6. G(d, c) ∧ ¬G(c, d)	
7. c = d	
8. G(d, c)	**∧ Elim:** 6
9. G(c, c)	**= Elim:** 7, 8
10. ¬G(c, d)	**∧ Elim:** 6
11. ¬G(c, c)	**= Elim:** 7, 10
12. ⊥	**¬ Elim:** 9, 11
13. ¬c = d	**¬ Intro:** 7–12
14. ∃y¬c = y	**∃ Intro:** 13
15. ∃x∃y¬x = y	**∃ Intro:** 14
16. ∃x∃y¬x = y	**∃ Elim:** 5, 6–15
17. ∃x∃y¬x = y	**∃ Elim:** 5, 3–16

d $\forall x F(x, a) \leftrightarrow \neg\forall x \neg G(x, b)$,
$\exists x(F(x, a) \land \neg G(x, b))$
$\vdash \bot$

False. Proof by example:
Let the domain comprise of the letters i, j, a and b.
Suppose $F(i, a), \neg F(j, a)$ and $\neg G(i, b)$ are true.
Then $\forall x F(x, a) \leftrightarrow \neg\forall x \neg G(x, b)$ is vacuously true,
as both $\forall x F(x, a)$ and $\neg\forall x \neg G(x, b)$ are false.
Now, $\exists x(F(x, a) \land \neg G(x, b))$ is also true.
Therefore $\forall x F(x, a) \leftrightarrow \neg\forall x \neg G(x, b)$ and
$\exists x(F(x, a) \land \neg G(x, b))$ cannot prove a contradiction.

e $\forall y(H(y) \land (J(y, y) \land M(y)))$
$\vdash \exists x J(x, b) \land \forall x M(x)$

True.

1. ∀y[H(y) ∧ (J(y, y) ∧ M(y))]	
2. H(b) ∧ (J(b, b) ∧ M(b))	**∀ Elim:** 1
3. J(b, b) ∧ M(b)	**∧ Elim:** 2
4. J(b, b)	**∧ Elim:** 3
5. ∃xJ(x, b)	**∃ Intro:** 4
6. M(b)	**∧ Elim:** 3
7. ∀xM(x)	**∀ Intro:** 6
8. ∃xJ(x, b) ∧ ∀xM(x)	**∧ Intro:** 5, 7

3 Find an example of an invalid argument where only $\forall$I is used improperly because the name that is quantified away still remains in the formula, i.e.,

m.	$\mathscr{A}(\ldots a \ldots a \ldots)$	
n.	$\forall x \mathscr{A}(\ldots x \ldots a \ldots)$	$\forall$ **Intro: m**

Invalid argument in the form of a invalid Fitch-style proof.

1.	$\forall x F(x, x)$	
2.	$F(a, a)$	$\forall$ **Elim:** 1
3.	$\forall x F(x, a)$	$\forall$ **Intro:** 2 (bad)

Therefore $\forall x F(x, x) \vdash \forall x F(x, a)$.
However this is an invalid argument. Proper proof by example:
Let the domain comprise of the letters a and b.
Suppose $F(a, a), F(b, b)$ and $\neg F(b, a)$ are true.
Then $\forall x F(x, x)$ is true.
However $\forall x F(x, a)$ is false.
Therefore $\forall x F(x, x)$ cannot prove $\forall x F(x, a)$.
Therefore it's possible for the premises to be true and the conclusion to be false.
Therefore it's an invalid argument.

4 Bonus: Provide an informal proof of the following claim:

$$\forall x \forall y (R(x, y) \rightarrow R(y, x)), \forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z)), \forall x \exists y R(x, y) \vDash \forall x R(x, x)$$

Mathematical proof where R is a relation over the set S:

$\forall x \forall y (R(x, y) \rightarrow R(y, x))$ means $\forall x, y \in S$: if xRy then yRx. Therefore R is symmetric.
$\forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$ means $\forall x, y, z \in S$: if xRy and yRz then xRz. Therefore R is transitive.
And lastly, we know $\forall x \exists y R(x, y)$, this means $\forall x \in S, \exists y \in S$, such that xRy.
Now let a be an arbitrary element of S. Then we know $\exists b$ such that aRb.
Because aRb and R is symmetric: bRa.
Because aRb and bRa and R is transitive: aRa.
Therefore $\forall a \in S$: aRa.
Therefore R is reflexive, or $\forall x R(x, x)$.

Fitch-style Proof:

1.	$\forall x \forall y (R(x, y) \rightarrow R(y, x))$	
2.	$\forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$	
3.	$\forall x \exists y R(x, y)$	
4.	$\forall y (R(a, y) \rightarrow R(y, a))$	$\forall$ **Elim:** 1
5.	$R(a, b) \rightarrow R(b, a)$	$\forall$ **Elim:** 4
6.	$\forall y \forall z ((R(a, y) \wedge R(y, z)) \rightarrow R(a, z))$	$\forall$ **Elim:** 2
7.	$\forall z ((R(a, b) \wedge R(b, z)) \rightarrow R(a, z))$	$\forall$ **Elim:** 6
8.	$(R(a, b) \wedge R(b, a)) \rightarrow R(a, a)$	$\forall$ **Elim:** 7
9.	$\exists y R(a, y)$	$\forall$ **Elim:** 3
10.	$R(a, b)$	
11.	$R(b, a)$	$\rightarrow$ **Elim:** 5, 10
12.	$R(a, b) \wedge R(b, a)$	$\wedge$ **Intro:** 10, 11
13.	$R(a, a)$	$\rightarrow$ **Elim:** 8, 12
14.	$R(a, a)$	$\exists$ **Elim:** 9, 10–13
15.	$\forall x R(x, x)$	$\forall$ **Intro:** 14