STAT 323 Assignment 1

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1 Let X be a random variable with a density function given by

$$f(x) = \begin{cases} \frac{3}{2}x^2 & \textbf{for } -1 \leq x \leq 1 \\ 0 & \textbf{elsewhere} \end{cases}$$

a Find the density function of $Y = 3 - X$.

By method of distributon functions:
$X = 3 - Y$. So:

$$\begin{aligned} -1 \leq & X \leq 1 \\ -1 \leq & 3 - Y \leq 1 \\ -4 \leq & -Y \leq -2 \\ 2 \leq & Y \leq 4 \end{aligned}$$

$F(y) = P(Y \leq y) = P(3 - X \leq y) = P(-X \leq y - 3) = P(X \geq 3 - y)$

$$= \int_{3-y}^{\infty} \frac{3}{2}x^2 dx = \int_{3-y}^{1} \frac{3}{2}x^2 dx = \frac{x^3}{2}\Big|_{3-y}^{1} = \frac{1-(3-y)^3}{2}$$

So:

$$F(y) = \begin{cases} 0 & \text{for } y \leq 2 \\ \frac{1-(3-y)^3}{2} & \text{for } 2 \leq y \leq 4 \\ 1 & \text{for } 4 \leq y \end{cases}$$

$f(y) = \frac{d}{dy}(\frac{1-(3-y)^3}{2}) = \frac{3}{2}(3-y)^2$. Therefore: $\boxed{f(y) = \begin{cases} \frac{3}{2}(3-y)^2 & \text{for } 2 \leq y \leq 4 \\ 0 & \text{elsewhere} \end{cases}}$

b Find the density function of $Y = X^2$.

By method of distributon functions:
$-1 \leq X \leq 1$, so $0 \leq X^2 \leq 1$, and $0 \leq Y \leq 1$.
$F(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{3}{2}x^2 dx = \frac{x^3}{2}\Big|_{-\sqrt{y}}^{\sqrt{y}} = y^{3/2}$$

So:

$$F(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ y^{3/2} & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } 1 \leq y \end{cases}$$

$f(y) = \frac{d}{dy}(y^{3/2}) = \frac{3}{2}\sqrt{y}$. Therefore: $\boxed{f(y) = \begin{cases} \frac{3}{2}\sqrt{y} & \text{for } 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}}$

2 Assume that X has a beta distribution with parameters α and β. Find the density function of $Y = 1 - X$.

By method of transformations:
We know $Y = 1 - X$, and $0 \leq X \leq 1$.
It's easy to see that $0 \leq Y \leq 1$, and that Y is decreasing
$y = f(x) = 1 - x$, So $x = f^{-1}(y) = 1 - y$.

$$\begin{aligned}
F_Y(y) &= F_X(1-y) \\
f_Y(y) &= f_X(1-y)\left|\frac{d(1-y)}{dy}\right| \\
&= f_X(1-y)\left|-1\right|
\end{aligned}$$

Therefore: $\boxed{f(y) = \begin{cases} \frac{(1-y)^{\alpha-1}y^{\beta-1}}{B(\alpha,\beta)} & \text{for } 0 \leq y \leq 1, \text{ where } B(\alpha,\beta) \text{ is the Beta function.} \\ 0 & \text{elsewhere} \end{cases}}$

3 X is a uniformly-distributed random variable between 0 and 1.

a Find the probability density function of $Y = -\lambda \ln X$.

By method of transformations:
$X = \exp(\frac{Y}{-\lambda})$. Y is decreasing if λ is positive, increasing if λ is negative, and undefined PDF if $\lambda = 0$.

$$\begin{aligned}
0 \leq &X \leq 1 \\
0 \leq &\exp(\frac{Y}{-\lambda}) \leq 1 \\
\ln 0 \leq &\frac{Y}{-\lambda} \leq \ln 1 \\
&\frac{Y}{-\lambda} \leq 0 \\
0 \leq &Y
\end{aligned}$$

$$f_Y(y) = f_X(\exp(\tfrac{y}{-\lambda}))\left|\frac{d(\exp(\frac{y}{-\lambda}))}{dy}\right| = f_X(\exp(\tfrac{y}{-\lambda}))\left|\frac{\exp(\frac{y}{-\lambda})}{-\lambda}\right| = \frac{\exp(\frac{y}{-\lambda})}{|\lambda|}$$

Therefore: $\boxed{f(y) = \begin{cases} \frac{\exp(\frac{y}{-\lambda})}{|\lambda|} & \text{for } y \geq 0, \lambda \neq 0 \\ 0 & \text{elsewhere} \end{cases}}$

b Find the expected value and standard deviation of Y.

$f(y)$ is the exponenial distribution. Expected value: $\boxed{\mu = \lambda}$. Standard deviation: $\boxed{\sigma = \lambda}$.

4 The lifetime of an electronic component in an HDTV is a random variable that can be modeled by the exponential distribution with a mean lifetime β. Two components, X_1 and X_2, are randomly chosen and operated until failure. At that point, the lifetime of each component is observed. The mean lifetime of these two components is

$$\overline{x} = \frac{X_1 + X_2}{2}$$

a Find the probability density function of $\overline{x}$ using the MGF technique (the method of moment-generating functions.)

By method of moment-generating functions: $m_{\overline{x}}(t) = E(e^{t\overline{x}}) = E(\exp(t\frac{x_1+x_2}{2})) = E(\exp(t\frac{x_1}{2})\exp(t\frac{x_2}{2})) = E(\exp(t\frac{x_1}{2}))E(\exp(t\frac{x_2}{2}))$(because they are independent.) $= m_{x_1}(t/2)m_{x_2}(t/2) = (1-\beta\frac{t}{2})^{-1}(1-\beta\frac{t}{2})^{-1} = (1-\beta\frac{t}{2})^{-2}$

$(1-\beta\frac{t}{2})^{-2}$ is the MGF of the Gamma function with $\alpha' = 2, \beta' = \beta/2$.

Therefore: $\boxed{f(\overline{x}) = \begin{cases} \frac{\overline{x}^{2-1}e^{-\overline{x}/(\beta/2)}}{\Gamma(2)(\beta/2)^2} = \frac{4\overline{x}e^{-2\overline{x}/\beta}}{\beta^2} & \text{for } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}}$

b If the mean lifetime of the electronic component is two years ($\beta = 2$), what is the probability that the mean lifetime of two tested components will be more than three years? You may use R and/or additional software to calculate your answer to this question, but please show the relevant equation(s) used to obtain the final answer.

We know $f(\overline{x}) = \frac{4\overline{x}e^{-2\overline{x}/\beta}}{\beta^2}$, and $\beta = 2$. So $f(\overline{x}) = \overline{x}e^{-\overline{x}}$.

$\boxed{P(\overline{x} \geq 3) = \int_3^\infty \overline{x}e^{-\overline{x}}d\overline{x} = 4/e^3 \approx 19.915\%}$

5 Let $X_1, X_2, \ldots, X_{10}$ repensent a sample of size 10 taken from a normal distribution with $\mu = 0$ and $\sigma^2 = 1$. Define the following quantity:

$$U = X_1^2 + X_2^2 + \ldots + X_{10}^2$$

Find the distribution of U and state the mean and standard deviation of U as well.

By method of moment-generating functions: $m_u(t) = E(e^{tu}) = E(\exp(t(x_1^2 + x_2^2 + \ldots + x_{10}^2)))$
$= E(\exp(tx_1^2))E(\exp(tx_2^2)) \ldots E(\exp(tx_{10}^2))$(because they are independent.)
$= m_{x_1^2}(t)m_{x_2^2}(t) \ldots m_{x_{10}^2}(t) = (m_{x^2}(t))^{10}$(because they are the same distribution.)
Let $\sigma' = \sqrt{-\frac{1}{2t-1}}, \mu' = 0$.

$$\begin{aligned}
m_{x^2}(t) &= \int_{-\infty}^{\infty} e^{tx^2} \frac{e^{-\frac{(x-0)^2}{2(1^2)}}}{1\sqrt{2\pi}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2 - \frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x^2(t-\frac{1}{2})} dx \\
&= \frac{\sigma'}{\sigma'} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(x-0)^2 \frac{1}{2\frac{1}{2(t-\frac{1}{2})}}} dx \\
&= \sigma' \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu')^2}{2\sqrt{-\frac{1}{2t-1}}^2}}}{\sigma'\sqrt{2\pi}} dx \text{ (because imaginary numbers.)} \\
&= \sigma' \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu')^2}{2\sigma'^2}}}{\sigma'\sqrt{2\pi}} dx \\
&= \sigma' = \sqrt{-\frac{1}{2t-1}}
\end{aligned}$$

$(m_{x^2}(t))^{10} = \sqrt{-\frac{1}{2t-1}}^{10} = -\frac{1}{(2t-1)^5} = \frac{1}{(1-2t)^5} = (1-2t)^{-5}$. Which is the MGF of the Gamma distribution with $\alpha = 5, \beta = 2$.

$$\boxed{\mu = 10,\ \sigma = \sqrt{20},\ f(u) = \begin{cases} \frac{u^4 e^{-u/2}}{768} & \text{for } 0 \leq u \\ 0 & \text{elsewhere} \end{cases}}$$

6 Let $X_1, X_2, \ldots, X_n$ represent a sample of observations taken from an exponentially distributed population with parameter β. Let $Y = X_1 + X_2 + \ldots + X_n$. Assuming the observations are independent random variables, identify the distribution of the random variable defined as

$$Z = \frac{2Y}{\beta}$$

By method of moment-generating functions: $m_y(t) = E(e^{ty}) = E(\exp(t(x_1 + x_2 + \ldots + x_n)))$
$= (m_x(t))^n$ (same reasoning as question 5)
Now. $m_z(t) = E(e^{tz}) = E(e^{(t2y)/\beta}) = m_y(2t/\beta) = (m_x(2t/\beta))^n = ((1 - \frac{\beta 2t}{\beta})^{-1})^n = (1 - 2t)^{-n}$
Which is the MGF of the Chi-square distribution with $v = 2n$.

$$\boxed{\mu = 2n,\ \sigma = \sqrt{4n},\ f(z) = \begin{cases} \frac{z^{n-1}e^{-z/2}}{2^n\Gamma(n)} & \text{for } 0 \leq z^2 \\ 0 & \text{elsewhere} \end{cases}}$$