MATH 355 Assignment 2

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1 Using formal ϵ arguments, prove the following

a $\lim_{n\to\infty} \frac{2n^3-1}{-n^3+1} = -2$.

Let $\epsilon \in \mathbb{R}, \epsilon > 0$. Let $N > \sqrt[3]{1/\epsilon + 1}$. Let $n \in \mathbb{N}, n \geq N$. Now:

$$\begin{aligned}|a_n - L| = \left|\frac{2n^3-1}{-n^3+1} + 2\right| &= \left|\frac{2n^3-1}{-n^3+1} + \frac{-2n^3+2}{-n^3+1}\right| \\ &= \left|\frac{1}{-n^3+1}\right| = \frac{1}{|1-n^3|} \\ &= \frac{1}{n^3-1} \text{ (as } n \geq 1) \\ &\leq \frac{1}{N^3-1} < \frac{1}{\sqrt[3]{1/\epsilon+1}^3 - 1} = \frac{1}{1/\epsilon} = \epsilon\end{aligned}$$

b $\lim_{n\to\infty} \sqrt{9n^2-n} - 3n = -\frac{1}{6}$.

Let $\epsilon \in \mathbb{R}, \epsilon > 0$. Let $N > \max(\frac{1/(6\epsilon)+1}{18+6\sqrt{8}}, 1)$. Let $n \in \mathbb{N}, n \geq N$. Now:

$$\begin{aligned}|a_n - L| = \left|\sqrt{9n^2-n} - 3n + \frac{1}{6}\right| &= \left|\frac{-n}{\sqrt{9n^2-n}+3n} + \frac{1}{6}\right| = \left|\frac{\sqrt{9n^2-n}+3n-6n}{6(\sqrt{9n^2-n}+3n)}\right| \\ &= \left|\frac{\sqrt{9n^2-n}-3n}{6(\sqrt{9n^2-n}+3n)}\right| = \left|\frac{-n}{6(\sqrt{9n^2-n}+3n)^2}\right| \\ &= \frac{n}{6(\sqrt{9n^2-n}+3n)^2} = \frac{n}{6(9n^2-n+9n^2+6n\sqrt{9n^2-n})} = \frac{1}{6(18n+6\sqrt{9n^2-n}-1)} \\ &< \frac{1}{6(18n+6\sqrt{9n^2-n^2}-1)} \text{ (as } n > 1) = \frac{1}{6(18n+6n\sqrt{8}-1)} = \frac{1}{6(n(18+6\sqrt{8})-1)} \\ &< 1/(6(\frac{1/(6\epsilon)+1}{18+6\sqrt{8}}(18+6\sqrt{8})-1)) = 1/(6(1/(6\epsilon)+1-1)) = \frac{1}{6\frac{1}{6\epsilon}} = \epsilon\end{aligned}$$

2 Find a divergent sequence a_n with the property that $\lim_{n\to\infty}(a_{n+p} - a_n) = 0$ for every natural number p

Let $a_n = \ln(n)$. Let $\epsilon \in \mathbb{R}, \epsilon > 0$. Let $p \in \mathbb{N}$. Let $N \geq p/(e^\epsilon - 1)$. Let $n \in \mathbb{N}, n > N$. Now:

$$\begin{aligned}|a_{n+p} - a_n - L| = |\ln(n+p) - \ln(n)| &= \left|\ln(\frac{n+p}{n})\right| \\ &= \ln(1 + \frac{p}{n}) \\ &< \ln(1 + \frac{p}{p/(e^\epsilon - 1)}) \\ &= \ln(1 + (e^\epsilon - 1)) \\ &= \epsilon\end{aligned}$$

Where a_n itself is divergent.

3

a Show that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = L$

Suppose $\lim_{n\to\infty} a_n = L$.

Since a_n is convergent, we know it's bounded. So $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, |a_n| \leq M$.

Let $\epsilon \in \mathbb{R}, \epsilon > 0$.

Since a_n is convergent to L, we can choose $S \in \mathbb{N}$ such that $\forall s \in \mathbb{N}, s \geq S, |a_n - L| < \frac{\epsilon}{2}$.

Let $N > \max(\frac{2S(M+|L|)}{\epsilon}, S)$. [1] Let $n \in \mathbb{N}, n \geq N$. So $1 \leq S < N \leq n$.

$$\begin{aligned}
\left|\frac{a_1+a_2+\cdots+a_n}{n} - L\right| &= \left|\frac{a_1+a_2+\cdots+a_n - nL}{n}\right| \\
&= \left|\frac{(a_1-L)+(a_2-L)+\cdots+(a_n-L)}{n}\right| \\
&= \left|\frac{\sum_{k=1}^{n}(a_k-L)}{n}\right| \\
&= \frac{|\sum_{k=1}^{n}(a_k-L)|}{n} \\
&\leq \frac{\sum_{k=1}^{n}|a_k-L|}{n} \\
&= \frac{\sum_{k=1}^{S}|a_k-L|}{n} + \frac{\sum_{k=S+1}^{n}|a_k-L|}{n} \\
&\leq \frac{\sum_{k=1}^{S}|a_k|+|L|}{n} + \frac{\sum_{k=S+1}^{n}|a_k-L|}{n} \\
&\leq \frac{S(M+|L|)}{n} + \frac{\sum_{k=S+1}^{n}|a_k-L|}{n} \\
&< \frac{S(M+|L|)}{n} + \frac{\sum_{k=S+1}^{n}\epsilon/2}{n} \\
&= \frac{S(M+|L|)}{n} + \frac{(n-S)}{n}\frac{\epsilon}{2} \\
&< \frac{S(M+|L|)}{n} + \frac{\epsilon}{2} \\
&< \frac{S(M+|L|)}{2S(M+|L|)/\epsilon} + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

b Is the converse to the above statement true? Justify your reasoning.

False. Suppose $a_n = (-1)^n$. Then $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = 0$ by the Squeeze Theorem. [2] But a_n itself is divergent. [3]

[1] In case ϵ is really big.

[2] Let $c_n = \left|\sum_{k=1}^{n} a_n\right|, b_n = -c_n$. $-1 = b_n \leq \sum_{k=1}^{n} a_n \leq c_n = 1$ for all $n \in \mathbb{N}$. Divide everything by n, then apply Squeeze.

[3] $\mathrm{sub}(a_n) = \{0, 1\}$, therefore a_n not convergent.

4 Suppose $x > 1$ and define a sequence $\{y_i\}$ by $y_1 = x$, $y_{k+1} = \frac{1}{2}(y_k + \frac{x}{y_k})$ for $k \geq 1$.

a Show that $y_k - y_{k+1} = \frac{y_k^2 - x}{2y_k}$ and $y_{k+1}^2 - x = \frac{(y_k^2-x)^2}{4y_k^2}$.

Proposition 1. $y_k - y_{k+1} = \frac{y_k^2 - x}{2y_k}$.

Proof by induction: Base case $k = 1$:

$$\begin{aligned} y_k - y_{k+1} &= x - \frac{1}{2}(x + \frac{x}{x}) = x - \frac{x+1}{2} \\ &= \frac{2x - x - 1}{2} = \frac{x-1}{2} \\ &= \frac{x^2 - x}{2x} = \frac{y_k^2 - x}{2y_k} \end{aligned}$$

Now, suppose $y_k - y_{k+1} = \frac{y_k^2 - x}{2y_k}$ for $k \geq 1$. [4]

$$\begin{aligned} y_{k+1} - y_{k+2} &= \frac{1}{2}(y_k + \frac{x}{y_k}) - \frac{1}{2}(y_{k+1} + \frac{x}{y_{k+1}}) \\ &= \frac{1}{2}(y_k + \frac{x}{y_k} - y_{k+1} - \frac{x}{y_{k+1}}) \\ &= \frac{1}{2}(\frac{y_k^2 - x}{2y_k} + \frac{x}{y_k} - \frac{x}{y_{k+1}}) \\ &= \frac{1}{2}(\frac{y_k^2}{2y_k} + \frac{x}{y_k}) - \frac{x}{4y_k} - \frac{x}{2y_{k+1}} \\ &= \frac{1}{2}(\frac{y_k}{2} + \frac{x}{y_k}) - \frac{x}{4y_k} - \frac{x}{2y_{k+1}} \\ &= \frac{1}{2}(y_k + \frac{x}{y_k}) - \frac{y_k}{4} - \frac{x}{4y_k} - \frac{x}{2y_{k+1}} \\ &= y_{k+1} - \frac{y_k}{4} - \frac{x}{4y_k} - \frac{x}{2y_{k+1}} \\ &= \frac{2y_{k+1}^2}{2y_{k+1}} - \frac{x}{2y_{k+1}} - \frac{y_k}{4} - \frac{x}{4y_k} \\ &= \frac{2y_{k+1}^2 - x}{2y_{k+1}} - \frac{1}{4}(y_k + \frac{x}{y_k}) \\ &= \frac{2y_{k+1}^2 - x}{2y_{k+1}} - \frac{y_{k+1}}{2} \\ &= \frac{2y_{k+1}^2 - x}{2y_{k+1}} - \frac{y_{k+1}^2}{2y_{k+1}} \\ &= \frac{y_{k+1}^2 - x}{2y_{k+1}} \quad \square \end{aligned}$$

Proposition 2. $y_{k+1}^2 - x = \frac{(y_k^2 - x)^2}{4y_k^2}$.

Proof:

$$\begin{aligned} y_{k+1}^2 - x &= (\frac{1}{2}(y_k + \frac{x}{y_k}))^2 - x \\ &= \frac{1}{4}(y_k^2 + (\frac{x}{y_k})^2 + 2y_k\frac{x}{y_k}) - x \\ &= \frac{1}{4}(y_k^2 + \frac{x^2}{y_k^2} + 2x) - x \\ &= \frac{y_k^2 + \frac{x^2}{y_k^2} + 2x - 4x}{4} \\ &= \frac{y_k^4 + x^2 - 2xy_k^2}{4y_k^2} \\ &= \frac{(y_k^2 - x)^2}{4y_k^2} \quad \square \end{aligned}$$

[4] Not concise. Apparently I did'nt need induction, but I can't be bothered to redo it.

b Show that $y_k \geq 1$ and $y_k^2 \geq x$ for each $k \geq 1$.

Lemma 1. $y_k > 0, \forall k \in \mathbb{N}$.

Proof by induction: Base case: $y_1 = x > 1 > 0$. Now suppose $y_k > 0$ for $k \geq 1$. So $y_{k+1} = \frac{y_k + x/y_k}{2} > \frac{x/y_k}{2} > 0$, as $x > 0$ and $y_k > 0$. □

Lemma 2. $\{y_k\}$ is decreasing.

Proof by induction: We need to prove $y_k - y_{k+1} \geq 0, \forall k \in \mathbb{N}$. Base case: $y_1 - y_2 = x - \frac{x+1}{2} = \frac{2x-x-1}{2} = \frac{x-1}{2} \geq 0$ (as $x \geq 1$). Now, suppose $y_k - y_{k+1} \geq 0$ for $k \geq 1$. From (Proposition 1, 2), we know $y_k - y_{k+1} = \frac{y_k^2 - x}{2y_k}$ and $y_{k+1}^2 - x = \frac{(y_k^2-x)^2}{4y_k^2}$.

$$
\begin{aligned}
y_{k+1} - y_{k+2} = \frac{y_{k+1}^2 - x}{2y_{k+1}} = \frac{\frac{(y_k^2-x)^2}{4y_k^2}}{2y_{k+1}} = \frac{(y_k^2 - x)^2}{8y_k^2 y_{k+1}} \\
(y_k^2 - x)^2 \geq 0 \\
8y_k^2 > 0 \text{ (Lemma 1)} \\
y_{k+1} > 0 \text{ (Lemma 1)} \\
\therefore y_{k+1} - y_{k+2} = \frac{(y_k^2 - x)^2}{8y_k^2 y_{k+1}} \geq 0
\end{aligned}
$$

□

Corollary 1. $y_k \leq x, \forall k \in \mathbb{N}$.

Proof: We know $y_1 = x$ and $\{y_k\}$ is decreasing (Lemma 2), therefore $y_k \leq x, \forall k \in \mathbb{N}$. □

Proposition 3. $y_k \geq 1, \forall k \in \mathbb{N}$.

Proof by induction: Base case: $y_1 = x \geq 1$. Now, suppose $y_k \geq 1$ for $k \geq 1$. So

$$
\begin{aligned}
y_{k+1} &= \frac{1}{2}(y_k + \frac{x}{y_k}) \\
&= \frac{1}{2}y_k + \frac{1}{2}\frac{x}{y_k} \\
&\geq \frac{1}{2} + \frac{1}{2}\frac{x}{y_k} \\
&\geq \frac{1}{2} + \frac{1}{2}\frac{x}{x} \text{ (Corollary 1: } y_k \leq x) \\
&= 1
\end{aligned}
$$

□

Proposition 4. $y_k^2 \geq x, \forall k \in \mathbb{N}$.

Proof by induction: Base case: $y_k^2 = x^2 > x$ as $x > 1$. Now, suppose $y_k^2 \geq x$ for $k \geq 1$. From (Proposition 2), we know $y_{k+1}^2 - x = \frac{(y_k^2-x)^2}{4y_k^2}$. So,

$$
\begin{aligned}
y_{k+1}^2 = \frac{(y_k^2 - x)^2}{4y_k^2} + x \\
(y_k^2 - x)^2 \geq 0 \\
4y_k^2 \geq 4x > 4 > 0 \\
\therefore \frac{(y_k^2 - x)^2}{4y_k^2} \geq 0 \\
\therefore y_{k+1}^2 \geq 0 + x = x
\end{aligned}
$$

□

c By applying the Monotone Convergence Theorem, prove that y_k converges and find its limit.

Since y_k is decreasing (Lemma 2) and 1 is a lower bound of y_k (Proposition 3), y_k is convergent by MCT.

To find the limit:

$$\begin{aligned} L &= \frac{1}{2}(L + \frac{x}{L}) \\ 2L &= L + \frac{x}{L} \\ L^2 &= x \\ L &= +\sqrt{x} \text{ (as 1 is a lower bound)} \end{aligned}$$

5 For the following sets of real numbers, calculate all interior points, boundary points, accumulation points and isolated points. Are they open, closed or compact (or several or none)?

a $S = \mathbb{Q} \cap (0, 1)$.

$\text{int}(S) = \varnothing$. Proof: Suppose $x \in \text{int}(S)$. Then $\exists \epsilon > 0$ so that $N_\epsilon(x) \subseteq S$. Now, the interval $(x - \epsilon, x + \epsilon)$ is uncountable. Therefore $(x-\epsilon, x+\epsilon) = N_\epsilon(x) \nsubseteq \mathbb{Q}$, so $N_\epsilon(x) \nsubseteq S \subseteq \mathbb{Q}$. Contradiction. Therefore there exists no such x, and $\text{int}(S) = \varnothing$. □

$S' = [0, 1]$. Proof: If $x < 0$ or $x > 1$, let $\epsilon = \frac{\min(|x-1|,|x-0|)}{2} > 0$. It's easy to see that $N^*_\epsilon(x) \cap S = \varnothing$. Therefore $S' \subseteq [0, 1]$. Now, suppose $x \in [0, 1)$. Let $\epsilon > 0$. Since $x, x + \epsilon, 1 \in \mathbb{R}$, and $x < \min(1, x + \epsilon)$, $\exists q \in \mathbb{Q}$ such that $x < q < \min(1, x + \epsilon)$ [5]. Since $q \in \mathbb{Q}, q < 1, q > x \geq 0$, so $q \in S$. Therefore $q \in ((x - \epsilon, x + \epsilon) \setminus \{x\}) \cap S = N^*_\epsilon(x) \cap S$. Therefore $[0, 1) \subseteq S'$. Now, suppose $x = 1$. Similarly, $\exists p \in \mathbb{Q}$ such that $\max(0, 1 - \epsilon) < p < 1$. So $\{1\} \subseteq S'$. Therefore $[0, 1] = [0, 1) \cup \{1\} \subseteq S'$. Therefore $S' = [0, 1]$. □

$\text{bd}(S) = [0, 1]$. Proof: Since $\overline{S} = S \cup S' = S'$, so $S' = \overline{S} = S \cup \text{bd}(S)$, so $\text{bd}(S) \subseteq S' = [0, 1]$. Now, let $\epsilon > 0$. Suppose $x \in [0, 1]$. Since $(x - \epsilon, x + \epsilon)$ is uncountable, so $N_\epsilon(x) \nsubseteq \mathbb{Q}$ which is countable. Since $S \subseteq \mathbb{Q}$, so $N_\epsilon(x) \nsubseteq S$. Therefore $N_\epsilon(x) \cap S \neq N_\epsilon(x)$. Therefore $N_\epsilon(x) \cap (\mathbb{R} \setminus S) \neq \varnothing$. Now, since $x \in [0, 1] = S'$, so $N^*_\epsilon(x) \cap S \neq \varnothing$. Since $N^*_\epsilon(x) \subseteq N_\epsilon(x)$, so $N_\epsilon(x) \cap S \neq \varnothing$. Therefore $[0, 1] \subseteq \text{bd}(S)$. Therefore $\text{bd}(S) = [0, 1]$. □

Isolated points: $S \setminus S' = (\mathbb{Q} \cap (0, 1)) \setminus [0, 1] = \varnothing$.

S is not open, as $S \neq \varnothing = \text{int}(S)$.

S is not closed, as $S \neq [0, 1] = S'$. [6]

S is not compact, as S is not closed: (Heine-Borel)

b $\{x \in \mathbb{Q} | x = \frac{k}{2^n}$ where $n, k \in \mathbb{N} \cup \{0\}$ and $0 \leq k \leq 2^n\}$.

Let S be the above set. It's easy to see that $S \neq \varnothing$.

Lemma 3. $S \subsetneq [0, 1]$,or equivalently $S \neq [0, 1] \wedge S \subseteq [0, 1]$.

Proof: Let $x \in S$. Since $x = k/2^n$ and $0 \leq k \leq 2^n$, so $0 \leq x \leq 1$. Therefore $S \subseteq [0, 1]$. Since $[0, 1]$ is uncountable, and $S \subseteq \mathbb{Q}$ is countable, so $S \neq [0, 1]$. Therefore $S \subsetneq [0, 1]$. □

$\text{int}(S) = \varnothing$. Proof is word-for-word identical to the interior proof in (5a). □

$S' = [0, 1]$. Proof: If $x < 0$ or $x > 1$, let $\epsilon = \frac{\min(|x-1|,|x-0|)}{2} > 0$. It's easy to see that $N^*_\epsilon(x) \cap S = \varnothing$. Therefore $S' \subseteq [0, 1]$. Now, suppose $x \in [0, 1)$. Let $\epsilon > 0$. Choose $n \in \mathbb{N} \cup \{0\}$ such that $0 < \frac{1}{2^n} < \epsilon$ [7], where $\bigcup_{k=0}^{2^n-1} [\frac{k}{2^n}, \frac{k+1}{2^n}) = [0, 1)$, and each $[\frac{k}{2^n}, \frac{k+1}{2^n})$ is disjoint from each other. Since $x \in [0, 1)$, find the unique j such that $x \in [\frac{j}{2^n}, \frac{j+1}{2^n})$, where $j \in \mathbb{N} \cup \{0\}, 0 \leq j \leq 2^n - 1$. Therefore $\frac{j+1}{2^n} \in S$. Now, $|\frac{j+1}{2^n} - x| = \frac{j+1}{2^n} - x \leq \frac{j+1}{2^n} - \frac{j}{2^n} = \frac{1}{2^n} < \epsilon$, where $\frac{j+1}{2^n} \neq x$. Therefore $[0, 1) \subseteq S'$. Similarly we can prove $(0, 1] \subseteq S'$. Therefore $[0, 1] = [0, 1) \cup (0, 1] \subseteq S' \cup S' = S'$. Therefore $S' = [0, 1]$. □

$\text{bd}(S) = [0, 1]$. Proof is word-for-word identical to the boundary proof in (5a) □

Isolated points: $S \setminus S' = \varnothing$, as $S \subseteq [0, 1] = S'$.

S is not open, as $S \neq \varnothing = \text{int}(S)$.

S is not closed, as $S \neq [0, 1] = S'$.

S is not compact, as S is not closed: (Heine-Borel)

[5] From: Assignment 1, Problem (5a)
[6] Property of closeness.
[7] A corollary of the Archimedean Property

6 Construct a sequence a_n so that the set of subsequential limits S is the integers $\mathbb{Z}$.

Let

$$\begin{aligned} a_n = (&0, \\ &0, 1, \\ &0, 1, -1, \\ &0, 1, -1, 2, \\ &0, 1, -1, 2, -2 \\ &\ldots) \end{aligned}$$

where every "column" [8] is a subsequence of $a_{n_k} = c, \forall n_k$ where $c \in \mathbb{Z}$. Therefore $\mathbb{Z} \subseteq \mathrm{sub}(a_n)$. If a_{n_k} contains infinite elements from > 1 "columns" of a_n, then a_{n_k} won't be cauchy (set $\epsilon = 1/2$), therefore not convergent. Therefore there are no other elements in $\mathrm{sub}(a_n)$ besides $\mathbb{Z}$. Therefore $\mathrm{sub}(a_n) = \mathbb{Z}$.

7 Suppose S is closed set of real numbers with no isolated points. Show that S is uncountable.

Suppose $S \neq \varnothing$ [9] is closed with no isolated points.

Lemma 4. $S = S'$

Proof: Since S is closed, we know $S = \overline{S} = S \cup S' \to S' \subseteq S$. Since S has no isolated points, we know $S \setminus S' = \varnothing \to S \subseteq S'$. Therefore $S = S'$. ☐

Lemma 5. S is not finite.

Proof: Choose some $\epsilon_1 > 0$. Since $x_1 \in S'$, $\exists x_2 \in S$ such that $0 < |x_1 - x_2| < \epsilon_1$. Now let $\epsilon_2 = \frac{|x_1 - x_2|}{4} > 0$. Since $x_2 \in S'$, $\exists x_3 \in S$ such that $0 < |x_2 - x_3| < \epsilon_2$. We repeate to construct $\{x_n\} \subseteq S$ and $\{\epsilon_n\} \subseteq \mathbb{R}$ for $n \in \mathbb{N}$, where $\epsilon_n = \frac{|x_{n-1} - x_n|}{n^2}$ and x_n is chosen such that $|x_{n-1} - x_n| < \epsilon_{n-1}$.

Theorem 1. S is uncountable.

Proof: [10] Suppose S is denumerable. Then let $\{s_n\}$ be some denumeration of S where $n \in \mathbb{N}$. So $\{s_n\} = S = S'$. Now, we define $\{K_n\}, n \in \mathbb{N}$ recursively: Let $\epsilon_1 > 0$. Let $K_1 = N_{\epsilon_1}(s_1)$ where $s_1 \in K_1 \cap S \neq \varnothing$. Now, for $n \geq 2$: Since $S = S'$ and $K_{n-1} \cap S \neq \varnothing$, $\exists \epsilon_n \in \mathbb{R}, \epsilon_n < \epsilon_{n-1}$ and $x \in K_{n-1} \cap S'$ [11] [12] such that $K_n = N_{\epsilon_n}(x)$ has the properties: $\overline{K_n} \subseteq K_{n-1}$, [13] and $s_{n-1} \notin \overline{K_n}$. [14] [15] Since $x \in K_n \cap S$, $K_n \cap S \neq \varnothing$, Since $\forall n, \overline{K_n}$ is some bounded and closed interval, $\overline{K_n}$ is compact.

Now, we define $\{T_n\}$ such that $T_n = \overline{K_n} \cap \{s_n\}$ where $n \in \mathbb{N}$. Since $\overline{K_n}$ is compact and $\{s_n\} = S$ is closed, T_n is compact. Since $K_n \subseteq \overline{K_n}$ and $K_n \cap S \neq \varnothing$, we know $T_n \neq \varnothing$. Since $\overline{K_n} \supseteq K_n \supseteq \overline{K_{n+1}}$, we know $T_n \supseteq T_{n+1}$. Since $s_n \notin \overline{K_{+1}}$, we know $s_n \notin T_{n+1}$. Since $\forall s_n \in \{s_n\}, s_n \notin T_{n+1}$, $\bigcap_{n=1}^{\infty} T_n = \varnothing$. This violates Cantor's Intersection Theorem. Therefore S is not denumerable. Therefore S is uncountable. ☐

Citations

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Proofread by Devin Kwok (UCID: 10016484).

[8] Columns can be defined explicitly using Triangle Numbers.
[9] If $S = \varnothing$, then S is closed with no isolated points and finite, therefore countable.
[10] I got the idea for this proof from Baby Rudin, Theorem 2.43.
[11] x doesn't necessarily equal to s_n. If $s_n \notin K_{n-1}$, then choose some other element as $S \cap K_{n-1} \neq \varnothing$.
[12] In other words, we can choose x to making K_n smaller regardless of where s_n is.
[13] $x \in K_{n-1} \cap S'$, K_{n-1} is an open interval, and ϵ_n can be arbitrarily small.
[14] s_{n-1} is an accumulation point. Choose some point near s_{n-1}, then choose ϵ_n to be much smaller than the difference of the two.
[15] If $s_{n-1} \notin K_{n-1}$ then this case is trival.