Math 355 Assignment 3

Instructor: Dr Ryan Hamilton
Name: Yifeng Pan
UCID: 30063828

Fall 2019

1 Suppose $S \subseteq \mathbb{R}$.

a Show that if $S \neq \mathbb{R}$ and $S \neq \varnothing$, then $\mathbf{bd}(S) \neq \varnothing$.

We prove the contrapositive: If $\mathrm{bd}(S) = \varnothing$, then $S = \mathbb{R}$ or $S = \varnothing$. Suppose $\mathrm{bd}(S) = \varnothing$. If $S = \varnothing$, we are done. Let $S \neq \varnothing$. We prove $S = \mathbb{R}$. Since $\mathrm{bd}(S) = \varnothing$, $S = \mathrm{int}(S)$. Since $S \neq \varnothing$, let $x_1 \in S$. Since x_1 is an interior point of S, $\exists \epsilon_1$ such that $(x_1 - \epsilon_1, x_1 + \epsilon_1) \subseteq S$. Let $x_2 = x_1 + \epsilon_1$. If $x_2 \notin S$, then x_2 would be a boundary point of S, which would be a contradiction. Therefore $x_2 \in S$. We repeat to construct the sequence $\{x_n\}$, and $\{\epsilon_n\}$, such that $x_{n+1} > x_n$, and if $x_n < y < x_{n+1}$, then $y \in N_{\epsilon_n}(x_n) \subseteq S$. Therefore, $\forall n, [x_1, x_n) \subseteq S$. As $n \to \infty$, if $\{x_n\}$ is convergent to L, then L is a boundary point of S, which is a contradiction. Therefore $\{x_n\}$ is divergent to infinity, as it's increasing. Therefore $[x_1, \infty) \subseteq S$. Simularly, we construct the sequence in the negative direction to prove $(-\infty, x_1] \subseteq S$. Therefore $S = \mathbb{R}$.

b Show that $\mathbf{bd}(S) = \overline{S} \cap \overline{\mathbb{R} \setminus S}$.

We know $\mathrm{bd}(S) = \mathrm{bd}(\mathbb{R} \setminus S)$, As they have the same definition.

Suppose $x \in \mathrm{bd}(S)$. Since $\mathrm{bd}S \subseteq \overline{S}$, $x \in \overline{S}$. Since $x \in \mathrm{bd}(S) = \mathrm{bd}(\mathbb{R} \setminus S)$, $x \in \overline{\mathbb{R} \setminus S}$. Therefore $x \in \overline{S} \cap \overline{\mathbb{R} \setminus S}$.

Suppose $x \in \overline{S} \cap \overline{\mathbb{R} \setminus S} = (S \cup \mathrm{bd}(S)) \cap ((\mathbb{R} \setminus S \cup \mathrm{bd}(\mathbb{R} \setminus S))$. Therefore $x \notin S \to x \in \mathrm{bd}(S)$ and $x \in S \to x \notin \mathbb{R} \setminus S \to x \in \mathrm{bd}(\mathbb{R} \setminus S) = \mathrm{bd}(S)$. Therefore $x \in \mathrm{bd}(S)$ is both cases.

Therefore $\mathrm{bd}(S) = \overline{S} \cap \overline{\mathbb{R} \setminus S}$.

2 Using ϵ-δ arguments, directly prove the following limits:

a

$$\lim_{x \to 4} \frac{2x - 3}{\sqrt{x - 3}} = 5$$

Let $\epsilon > 0$. Choose $\delta = \min\{\frac{1}{2}, \epsilon \left(\frac{18+21+\frac{148}{0.5}}{4\sqrt{0.5}+2.5}\right)^{-1}\}$. Suppose $0 < |x - 4| < \delta$. So, $|x - 4| < \frac{1}{2}$, $3.5 < x < 4.5$. Now,

$$\begin{aligned}
|f(x) - 5| &= \left|\frac{2x - 3}{\sqrt{x - 3}} - 5\right| = \left|\frac{2x - 3 - 5\sqrt{x - 3}}{\sqrt{x - 3}}\right| \\
&= \left|\frac{(2x - 3)^2 - (5\sqrt{x - 3})^2}{(2x - 3 + 5\sqrt{x - 3})\sqrt{x - 3}}\right| = \left|\frac{4x^2 - 12x + 9 - 25(x - 3)}{(2x - 3 + 5\sqrt{x - 3})\sqrt{x - 3}}\right| = \left|\frac{4x^2 - 37x - 64}{(2x - 3 + 5\sqrt{x - 3})\sqrt{x - 3}}\right| \\
&= \left|\frac{(4x - 21 - \frac{148}{x-4})(x - 4)}{(2x - 3 + 5\sqrt{x - 3})\sqrt{x - 3}}\right| = \frac{\left|4x - 21 - \frac{148}{x-4}\right| |x - 4|}{\left|(2x - 3 + 5\sqrt{x - 3})\sqrt{x - 3}\right|} \\
&< |x - 4| \frac{\left|4(4.5) - 21 - \frac{148}{x-4}\right|}{\left|(2(3.5) - 3 + 5\sqrt{3.5 - 3})\sqrt{3.5 - 3}\right|} = |x - 4| \frac{\left|4(4.5) - 21 - \frac{148}{x-4}\right|}{4\sqrt{0.5} + 2.5} \\
&< |x - 4| \frac{18 + 21 + \left|\frac{148}{x-4}\right|}{4\sqrt{0.5} + 2.5} < |x - 4| \frac{18 + 21 + \frac{148}{0.5}}{4\sqrt{0.5} + 2.5} \\
&< \epsilon \left(\frac{18 + 21 + \frac{148}{0.5}}{4\sqrt{0.5} + 2.5}\right)^{-1} \frac{18 + 21 + \frac{148}{0.5}}{4\sqrt{0.5} + 2.5} = \epsilon
\end{aligned}$$

b

$$\lim_{x \to -1^-} \frac{1}{x^2 - 1} = \infty$$

Let $M > 0$. Choose $\delta = \min\{1, \frac{1}{3M}\}$. Suppose $0 < -1-x < \delta$. So, $0 < -1-x < 1$, $1 < -x < 2$, $-1 > x > -2$. Now,

$$\begin{aligned} f(x) &= \frac{1}{x^2-1} \\ &= \frac{1}{(x+1)(x-1)} \\ &> \frac{1}{(x+1)(-2-1)} = \frac{1}{-(-x-1)(-3)} \\ &> \frac{1}{-\frac{1}{3M}(-3)} = M \end{aligned}$$

3 Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous, monotone and bounded. Show that f is uniformly continuous. (Hint: We know that $f([0, \infty))$ is an interval.)

TODO

Since f is monotone and bounded, $f(n) = L$ as $n \to \infty$.

Let $a = \min\{f(0), L\}, b = \max\{f(0), L\}$.

From the Intermediate Value Theorem, we know $\forall t \in (a, b), \exists c \in [0, \infty)$ such that $f(c) = t$.

We define $g : (a, b) \to [0, \infty)$. The range of $g \subseteq [0, \infty)$. So $\forall y \in (a, b), f \circ g(y) = y$.

Let $\epsilon > 0$.

Let $h : (a, b - \epsilon) \to (0, \infty), h(y) = |g(y) - g(y + \epsilon)|$. h is strictly positive.

Let δ be the absolute minimum of the range of h.

Let $x, y \in [0, \infty), |x - y| < \delta$.

So $\forall z \in (a, b - \epsilon), |x - y| < |g(z) - g(z + \epsilon)|$.

4 Suppose $f : I \to \mathbb{R}$ where I is an open interval containing a. Suppose further that f is n times differentiable at a and let p_n be the nth Taylor polynomial for f at a. Apply L'Hôpital's rule $n - 1$ times to show that

$$\lim_{x \to a} \frac{f(x) - p_n(x)}{(x - a)^n} = 0$$

Let $g(x) = (f(x) - p_n(x))$. Let $h(x) = (x - a)^n$. As $\forall k, 1 < k \leq n, h^{(k)}(x)$ is non-zero for all deleted nhds of a, we can apply L'Hôpital's rule n times. So $g^{(n)} = f^{(n)}(x) - p_n^{(n)}(x)$, and $h^{(n)} = n!$.

$$\begin{aligned} \lim_{x \to a} \frac{f(x) - p_n(x)}{(x - a)^n} &= \lim_{x \to a} \frac{f^{(n)}(x) - p_n^{(n)}(x)}{n!} \\ &= \lim_{x \to a} \frac{f^{(n)}(a) - p_n^{(n)}(a)}{n!} \text{ (TODO, are } f^{(n)}, p_n^{(n)} \text{ are continuous?)} \\ &= \lim_{x \to a} \frac{0}{n!} \\ &= 0 \end{aligned}$$

5 Use the inverse function theorem to verify that $f(x) = \arcsin(x)$ is differentiable on $(-1, 1)$ and that $arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$.

We know $g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1), g(x) = \sin(x)$ is diff., and $g'(x) = \cos(x)$ is strictly positive for $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, by the Inverse Function Theorem:

1. $g^{-1} : (-1, 1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ exists. Let $f(x) = g^{-1}(x) = \arcsin(x)$.
2. g^{-1} is differentiable.
3. $(g^{-1})'(\sin(x)) = \frac{1}{\cos(x)}$.

Therefore $(g^{-1})'(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$.

6 Suppose that f, g are bounded and integrable functions on $[a, b]$.

a Show that f^2 is integrable on $[a, b]$.

(6a) is a corollary of (6b).

b Show that fg is integrable on $[a, b]$.

Draft 1:

Let $\epsilon > 0$.

We split the interval $[a, b]$ into 4 parts, $[a, b] = A \cup B \cup C \cup D$, where

$$\begin{aligned} A &= \{x | x \in [a, b], f(x) \geq 0, g(x) \geq 0\} \\ B &= \{x | x \in [a, b], f(x) \geq 0, g(x) < 0\} \\ C &= \{x | x \in [a, b], f(x) < 0, g(x) \geq 0\} \\ D &= \{x | x \in [a, b], f(x) < 0, g(x) < 0\} \end{aligned}$$

Case A:

Let $F = \int_A f, G = \int_A g$, where F and G are both non-negative from construction of A.
Let $\delta = \min\{\frac{\epsilon}{2(F+G+1)}, \frac{\sqrt{\epsilon}}{2}\}$. We know $\exists$ partition P on A [1] so that $U(f, P) - L(f, P) < \delta$, and $U(g, P) - L(g, P) < \delta$. So,

$$\begin{aligned} U(fg, P) &= \sup\{f(x)g(x) | x \in [x_{i-1}, x_i]\} \\ &= \sup\{f(x) | x \in [x_{i-1}, x_i]\} \sup\{g(x) | x \in [x_{i-1}, x_i]\} \\ &= U(f, P)U(g, P) \end{aligned}$$

$$\begin{aligned} L(fg, P) &= \inf\{f(x)g(x) | x \in [x_{i-1}, x_i]\} \\ &= \inf\{f(x) | x \in [x_{i-1}, x_i]\} \inf\{g(x) | x \in [x_{i-1}, x_i]\} \\ &= L(f, P)L(g, P) \end{aligned}$$

$$\begin{aligned} U(fg, P) &= U(f, P)U(g, P) \\ &< (L(f, P) + \delta)(L(g, P) + \delta) \\ &= L(f, P)L(g, P) + (L(f, P) + L(g, P))\delta + \delta^2 \\ &= L(fg, P) + (L(f, P) + L(g, P))\delta + \delta^2 \end{aligned}$$

[1] A might not be an interval, but the definition of partition can be extended in this case as $A \subseteq [a, b]$. I can't prove this.

$$
\begin{aligned}
U(fg,P) - L(fg,P) &< (L(f,P) + L(g,P))\delta + \delta^2 \\
&\leq (F+G)\delta + \delta^2 \\
&\leq (F+G)\frac{\epsilon}{2(F+G+1)} + \frac{\sqrt{\epsilon}^2}{2^2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon
\end{aligned}
$$

Case B:

Let $F = \int_B f, G = \int_B g$, where $F \geq 0$ and $G < 0$ from construction of B.
Let $\delta = \min\{\frac{\epsilon}{2|G|}, \frac{\sqrt{\epsilon}}{2}\}$. We know $\exists$ partition P on B so that $U(f,P) - L(f,P) < \delta$, and $U(g,P) - L(g,P) < \delta$.
So,

$$
\begin{aligned}
0 \geq U(fg,P) &= \sup\{f(x)g(x) | x \in [x_{i-1}, x_i]\} \\
&= \inf\{f(x) | x \in [x_{i-1}, x_i]\} \sup\{g(x) | x \in [x_{i-1}, x_i]\} \\
&= L(f,P)U(g,P)
\end{aligned}
$$

$$
\begin{aligned}
0 \geq L(fg,P) &= \inf\{f(x)g(x) | x \in [x_{i-1}, x_i]\} \\
&= \sup\{f(x) | x \in [x_{i-1}, x_i]\} \inf\{g(x) | x \in [x_{i-1}, x_i]\} \\
&= U(f,P)L(g,P)
\end{aligned}
$$

$$
\begin{aligned}
0 \geq U(fg,P) &= L(f,P)U(g,P) \\
&< (U(f,P) - \delta)U(g,P) \\
&< (U(f,P) - \delta)L(g,P) \\
&= U(f,P)L(g,P) - \delta L(g,P) \\
&= L(fg,P) - \delta L(g,P)
\end{aligned}
$$

$$
\begin{aligned}
U(fg,P) - L(fg,P) &< -\delta L(g,P) \\
&= \delta\,|L(g,P)| \\
&< \delta\,|U(g,P) - \delta| \\
&\leq \delta\,|G - \delta| \\
&= \left|G\delta - \delta^2\right| \\
&\leq |G\delta| + \delta^2 \\
&< \left|G\frac{\epsilon}{2\,|G|}\right| + \frac{\sqrt{\epsilon}^2}{2^2} \\
&< \epsilon
\end{aligned}
$$

Case C, D are simular.

c Show that $\max(f,g)(x) := \max\{f(x), g(x)\}$ and $\min(f,g)(x) := \min\{f(x), g(x)\}$ are both integrable on $[a,b]$.

TODO. Simular to (6b), split $[a,b]$ into two sets.

7 For what real values of α is $f(x) = x^\alpha \log(x)$ uniformly continuous on $(0,\infty)$? Support your claims.

$f(x) = x^\alpha \log(x)$ is UC on $(0,\infty) \iff f'(x)$ is bounded on $(0,\infty)$.

$f'(x) = \alpha x^{\alpha-1}\log(x) + x^\alpha/x = (\alpha\log(x) + 1)x^{\alpha-1}$.

8 Suppose f is continuous on $[a, b]$ where $a < b$ and let $M = \sup_{a \le x \le b}(|f(x)|)$.

a If $M > 0$ and p is any positive constant, show that for every $\epsilon > 0$ there are constants $c < d$ so that $[c, d] \subseteq [a, b]$ and

$$(M - \epsilon)^p (d - c) \le \int_a^b |f(x)|^p \, dx \le M^p (d - a)$$

b Prove that

$$\lim_{p \to \infty} \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} = M$$