MATH 273 Assignment 3

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1 Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $T = \{1, 2, 3, 4\}$. Please explain how you get the answers.

a How many sebsets A of S are there so that $A \cap T = \varnothing$?

There are 2^5 subsets A of S so that $A \cap T = \varnothing$. To construct subset A of S, first, not choose $\{1, 2, 3, 4\}$, because if any of those elements were chosen, $A \cap T \neq \varnothing$. We may or may not include each of the next 5 elements {5,6,7,8,9}, giving us $2 \times 2 \times 2 \times 2 \times 2$ choices, or 2^5. $1 \times 2^5 = 2^5$.

b How many sebsets A of S are there so that $A \setminus T = \varnothing$?

There are 2^4 subsets A of S so that $A \setminus T = \varnothing$. To construct subset A of S, first, not choose $\{5, 6, 7, 8, 9\}$, because if any of those elements were chosen, $A \setminus T \neq \varnothing$. We may or may not include each of the first 4 elements $\{1, 2, 3, 4\}$, giving us $2 \times 2 \times 2 \times 2$ choices, or 2^4. $1 \times 2^4 = 2^4$.

c How many sebsets A of S are there so that $A \cap T \neq \varnothing$ and $A \setminus T \neq \varnothing$?

There are $(2^4 - 1) \times (2^5 - 1)$ subsets A of S so that $A \setminus T \neq \varnothing$ and $A \cap T \neq \varnothing$. We break down the counting so that $A = B \cup C$, where $B \subseteq \{1, 2, 3, 4\}$, and $C \subseteq \{5, 6, 7, 8, 9\}$. $B \cap T \neq \varnothing$ for all B, except when $B = \varnothing$. So there are $2^4 - 1$ choices for B. $C \setminus T \neq \varnothing$ for all C, except when $C = \varnothing$. So there are $2^5 - 1$ choices for C. To construct A, first, pick B, of which there are $2^4 - 1$ options. Then pick C, of which there are $2^5 - 1$ options. $A = B \cup C$. Therefore there are $(2^4 - 1) \times (2^5 - 1)$ to construct A.

2 Let n, k be positive integers.

a Prove by a combinatorial proof that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Combinatorial problem: How many subsets does the set S with n elements have? Counting method 1: For any given subset of S:
It may or may not include the 1st element of S. (2 choices)
It may or may not include the 2nd element of S. (2 choices)
It may or may not include the 3rd element of S. (2 choices)
...
It may or may not include the nth element of S. (2 choices)
Therefore there are 2^n ways to constuct a subset of S.
Counting method 2:
There are $\binom{n}{0}$ subsets of S with 0 elements.
There are $\binom{n}{1}$ subsets of S with 1 elements.
There are $\binom{n}{2}$ subsets of S with 2 elements.
...
There are $\binom{n}{n}$ subsets of S with n elements.
Therefore there are $\sum_{i=0}^{n} \binom{n}{i}$ subsets of S. Thus by combinatorial proof: $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

b Prove that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ by using the Binomial Theorem.

Binomial Theorem: $\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x+y)^n$. Let $x = y = 1$. Then $\sum_{i=0}^{n} \binom{n}{i} = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x+y)^n = (1+1)^n = 2^n$. So $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

c Prove by induction on n that $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$ for all integers $n \geq k$.

Let $n, k \in \mathbb{Z}$ such that $n \geq k$. Base case: $n = k$ $\sum_{i=k}^{n} \binom{i}{k} = \sum_{i=n}^{n} \binom{i}{n} = \binom{n}{n} = 1 = \binom{n+1}{n+1} = \binom{n+1}{k+1}$ Inductive Step: Suppose $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$. (IH) We are trying to prove $\sum_{i=k}^{n+1} \binom{i}{k} = \binom{n+2}{k+1}$.

$$\begin{aligned}
\sum_{i=k}^{n+1} \binom{i}{k} &= \sum_{i=k}^{n} \binom{i}{k} + \binom{n+1}{k} \\
&= \binom{n+1}{k+1} + \binom{n+1}{k} \text{from (IH)} \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} + \frac{(n+1)!}{(k)!(n-k+1)!} \\
&= \frac{(n+1)!(n-k+1)}{(k+1)!(n-k)!(n-k+1)} + \frac{(n+1)!(k+1)}{(k)!(n-k+1)!(k+1)} \\
&= \frac{(n+1)!(n-k+1) + (n+1)!(k+1)}{(n-k+1)!(k+1)!} \\
&= \frac{(n+1)!((n-k+1)+(k+1))}{(n-k+1)!(k+1)!} = \frac{(n+1)!(n+2)}{(n-k+1)!(k+1)!} \\
&= \frac{(n+2)!}{(n-k+1)!(k+1)!} = \binom{n+2}{k+1}
\end{aligned}$$

Therefore $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$ for all integers $n \geq k$.

3 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Please explain how you get the answers.

a How many functions $f : A \to A$ are there so that $f \circ f(1) = 2$?

There are $9^7 \times 8$ functions $f : A \to A$ so that $f \circ f(1) = 2$. f is of the form $\{(1, a), (2, b), (3, c) \dots (9, i)\}$. First pick a. If $f \circ f(1) = 2 \neq 1$ then $a \neq 1$. There are 8 choices for a. Then pick the term $(a, 2)$. There is 1 choice in this step. There are 9 choices for the remaining 7 terms, giving 9^7 choices. Therefore there are $9^7 \times 1 \times 8$ ways to construct f so that $f \circ f(1) = 2$.

b How many functions $f : A \to A$ are there so that $f \circ f(1) = 2$ and f is onto?

There are $7 \times 7!$ functions $f : A \to A$ so that $f \circ f(1) = 2$ and f is onto. f is of the form $\{(1, a), (2, b), (3, c) \dots (9, i)\}$. First we pick a. If $f \circ f(1) = 2 \neq 1$ then $a \neq 1$. If $f \circ f(1) = 2$ and f is onto, then $a \neq 2$, because $(1, 2)$ and $(2, 2)$ can't coexist. This gives us 7 choices for a. We need the term $(a, 2)$. We have 1 choice in this step. For the remaining terms, we can pick any permutation of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\} \setminus \{a, 2\}$, because f is onto. This gives us 7! ways to pick the remaining terms. Therefore there are $7 \times 1 \times 7!$ ways to construct f as so that $f \circ f(1) = 2$ and f is onto.

c How many functions $f : A \to A$ are there so that $f \circ f(1) = 2$ or f is onto?

There are $(9^7 \times 8) + (9!) - (7 \times 7!)$ functions $f : A \to A$ so that $f \circ f(1) = 2$ or f is onto. The (number of functions that are onto or $f \circ f(1) = 2$) is equal to the (number of functions that are onto) plus the (number of functions that are $f \circ f(1) = 2$) minus the (number of functions that are both onto and $f \circ f(1) = 2$). From 3.(a): There are $9^7 \times 8$ functions $f : A \to A$ so that $f \circ f(1) = 2$. From 3.(b): There are $7 \times 7!$ functions $f : A \to A$ so that $f \circ f(1) = 2$ and f is onto. The number of functions $f : A \to A$ that are onto is just the number of permutations of A. Which is equal to 9!. So there are $(9^7 \times 8) + (9!) - (7 \times 7!)$ functions $f : A \to A$ so that $f \circ f(1) = 2$ or f is onto.