MATH 273 Assignment 4

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1 Prove the following using the definition of limit.

a If $\lim_{n\to\infty}(a_n) = 2$ and $\lim_{n\to\infty}(b_n) = 3$ then $\lim_{n\to\infty}(a_n + b_n) = 5$.

Let $\lim_{n\to\infty}(a_n) = 2, \lim_{n\to\infty}(b_n) = 3$. Let $n \in \mathbb{Z}, \epsilon \in \mathbb{R}, \epsilon > 0$. Because $\lim_{n\to\infty}(a_n) = 2, \exists N_1 \in \mathbb{R}$ so that $\forall n > N_1, |a_n - 2| < \frac{\epsilon}{2}$. Because $\lim_{n\to\infty}(b_n) = 3, \exists N_2 \in \mathbb{R}$ so that $\forall n > N_2, |b_n - 3| < \frac{\epsilon}{2}$. Let $N = \max(N_1, N_2), n > N$. Now:

$$\begin{aligned}
|(a_n + b_n) - 5| &= |(a_n - 2) + (b_n - 3)| \\
&\leq |a_n - 2| + |b_n - 3| \text{ (because of triangular inequlity)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ (because } |a_n - 2| < \frac{\epsilon}{2} \text{ and } |b_n - 3| < \frac{\epsilon}{2}) \\
&= \epsilon
\end{aligned}$$

Therefore $\lim_{n\to\infty}(a_n + b_n) = 5$.

b If $\lim_{n\to\infty}(a_n) = 2$ and $\lim_{n\to\infty}(b_n) = 3$ then $\lim_{n\to\infty}(a_n b_n) = 6$.

Let $\lim_{n\to\infty}(a_n) = 2, \lim_{n\to\infty}(b_n) = 3$. Let $n \in \mathbb{Z}, \epsilon \in \mathbb{R}, \epsilon > 0$. Because $\lim_{n\to\infty}(a_n) = 2, \exists N_1 \in \mathbb{R}$ so that $\forall n > N_1, |a_n - 2| < \frac{\epsilon}{8}$. Because $\lim_{n\to\infty}(b_n) = 3, \exists N_2 \in \mathbb{R}$ so that $\forall n > N_2, |b_n - 3| < 1$, and $\exists N_3 \in \mathbb{R}$ so that $\forall n > N_3, |b_n - 3| < \frac{\epsilon}{4}$. Because $|b_n - 3| < 1$,

$$-1 < (b_n - 3) < 1$$

$$\text{or } 2 < b_n < 4.$$

Let $N = \max(N_1, N_2, N_3), n > N$. Now:

$$\begin{aligned}
|a_n b_n - 6| &= |a_n b_n - 2b_n + 2b_n - 6| \\
&= |b_n(a_n - 2) + 2(b_n - 3)| \\
&\leq |b_n|\,|a_n - 2| + |2|\,|b_n - 3| \text{ (because of triangular inequlity)} \\
&< b_n \frac{\epsilon}{8} + 2\frac{\epsilon}{4} \text{ (because } b_n > 2 \text{ and } |a_n - 2| < \frac{\epsilon}{8} \text{ and } |b_n - 3| < \frac{\epsilon}{4}) \\
&< 4\frac{\epsilon}{8} + 2\frac{\epsilon}{4} \text{ (because } b_n < 4) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Therefore $\lim_{n\to\infty}(a_n b_n) = 6$.

c If $\lim_{n\to\infty}(a_n) = 2$ and $\lim_{n\to\infty}(b_n) = 3$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{3}$.

Let $\lim_{n\to\infty}(a_n) = 2, \lim_{n\to\infty}(b_n) = 3$. Let $n \in \mathbb{Z}, \epsilon \in \mathbb{R}, \epsilon > 0$. Because $\lim_{n\to\infty}(a_n) = 2, \exists N_1 \in \mathbb{R}$ so that $\forall n > N_1, |a_n - 2| < \epsilon$. Because $\lim_{n\to\infty}(b_n) = 3, \exists N_2 \in \mathbb{R}$ so that $\forall n > N_2, |b_n - 3| < \epsilon$, and $\exists N_3 \in \mathbb{R}$ so that $\forall n > N_3, |b_n - 3| < 1$. Because $|b_n - 3| < 1$,

$$-1 < (b_n - 3) < 1$$

$$\text{or } 2 < b_n < 4.$$

Let $N = \max(N_1, N_2, N_3), n > N$. Now:

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{2}{3}\right| &= \left|\frac{3a_n - 2b_n}{3b_n}\right| \\
&= \left|\frac{3a_n - 6 + 6 - 2b_n}{3b_n}\right| \\
&= \left|\frac{3(a_n - 2) + 2(3 - b_n)}{3b_n}\right| \\
&\leq \left|\frac{3(a_n - 2)}{3b_n}\right| + \left|\frac{2(3 - b_n)}{3b_n}\right| \text{ (because of triangular inequlity)} \\
&= \frac{|a_n - 2|}{|b_n|} + \frac{|2(b_n - 3)|}{|3b_n|} \\
&< \frac{|a_n - 2|}{2} + \frac{2\,|(b_n - 3)|}{3 \times 2} \text{ (because } b_n > 2) \\
&< \frac{\epsilon}{2} + \frac{2\epsilon}{3 \times 2} \text{ (because } |a_n - 2| < \epsilon \text{ and } |b_n - 3| < \epsilon) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{3} \\
&= \frac{5\epsilon}{6} \\
&< \epsilon
\end{aligned}$$

Therefore $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{2}{3}$.

2 Prove the following using the definition of limit.

a If $\lim_{n\to\infty}(a_n) = 2$ then $\lim_{n\to\infty}(a_n^3) = 8$.

Let $\lim_{n\to\infty}(a_n) = 2$. Let $n \in \mathbb{Z}, \epsilon \in \mathbb{R}, \epsilon > 0$. Because $\lim_{n\to\infty}(a_n) = 2, \exists N_1 \in \mathbb{R}$ so that $\forall n > N_1, |a_n - 2| < \frac{\epsilon}{100}$, and $\exists N_2 \in \mathbb{R}$ so that $\forall n > N_2, |a_n - 2| < 1$. Because $|a_n - 2| < 1$.

$$-1 < (a_n - 2) < 1$$

$$\text{or } 1 < a_n < 3.$$

Let $N = \max(N_1, N_2), n > N$. Now:

$$\begin{aligned}
\left|a_n^3 - 8\right| &= \left|(a_n - 2)(a_n^2 + 2a_n + 4)\right| \\
&= |(a_n - 2)| \times \left|(a_n^2 + 2a_n + 4)\right| \\
&< \frac{\epsilon}{100}\left|(a_n^2 + 2a_n + 4)\right| \text{ (because } |a_n - 2| < \frac{\epsilon}{100}) \\
&= \frac{\epsilon}{100}((a_n - 2)^2 + 6a_n) \\
&< \frac{\epsilon}{100}(1^2 + 6 \times 3) \text{ (because } a_n < 3 \text{ and } a_n - 2 < 1) \\
&= \frac{18\epsilon}{100} \\
&< \epsilon
\end{aligned}$$

Therefore $\lim_{n\to\infty}(a_n^3) = 8$.

b If $\lim_{n\to\infty}(a_n) = 4$ then $\lim_{n\to\infty}(\sqrt{a_n}) = 2$.

Let $\lim_{n\to\infty}(a_n) = 4$. Let $n \in \mathbb{Z}, \epsilon \in \mathbb{R}, \epsilon > 0$. Because $\lim_{n\to\infty}(a_n) = 2, \exists N_1 \in \mathbb{R}$ so that $\forall n > N_1, |a_n - 4| < \epsilon$, and $\exists N_2 \in \mathbb{R}$ so that $\forall n > N_2, |a_n - 4| < 3$. Because $|a_n - 4| < 3$.

$$-3 < (a_n - 4) < 3$$

$$\text{or } 1 < a_n < 7.$$

Let $N = \max(N_1, N_2), n > N$. Now:

$$\begin{aligned}
|\sqrt{a_n} - 2| &= \left|\frac{(\sqrt{a_n} - 2)(\sqrt{a_n} + 2)}{\sqrt{a_n} + 2}\right| \\
&= \frac{|a_n - 4|}{|\sqrt{a_n} + 2|} \\
&< \frac{|a_n - 4|}{|1 + 2|} \text{ (because } a_n > 1 \text{ so } \sqrt{a_n} > 1) \\
&< \frac{\epsilon}{3} \text{ (because } |a_n - 4| < \epsilon) \\
&< \epsilon
\end{aligned}$$

Therefore $\lim_{n\to\infty}(\sqrt{a_n}) = 2$

c If $\lim_{n\to\infty}(a_n) = 1$ then $\lim_{n\to\infty}(a_n^{1/3}) = 1$.

Let $\lim_{n\to\infty}(a_n) = 1$. Let $n \in \mathbb{Z}, \epsilon \in \mathbb{R}, \epsilon > 0$. Because $\lim_{n\to\infty}(a_n) = 1, \exists N_1 \in \mathbb{R}$ so that $\forall n > N_1, |a_n - 1| < \epsilon$, and $\exists N_2 \in \mathbb{R}$ so that $\forall n > N_2, |a_n - 1| < 1$. Because $|a_n - 1| < 1$.

$$-1 < (a_n - 1) < 1$$

$$\text{or } 0 < a_n < 2.$$

Let $N = \max(N_1, N_2), n > N$. Now:

$$\begin{aligned}
\left|a_n^{1/3} - 1\right| &= \left|\frac{(a_n^{1/3} - 1)(a_n^{2/3} + a_n^{1/3} + 1)}{(a_n^{2/3} + a_n^{1/3} + 1)}\right| \\
&= \left|\frac{(a_n - 1)}{(a_n^{2/3} + a_n^{1/3} + 1)}\right| \\
&< \frac{|a_n - 1|}{|0 + 0 + 1|} \text{ (because } a_n > 0\text{, so } a_n^{1/3} > 0 \text{ and } a_n^{2/3} > 0) \\
&= |a_n - 1| \\
&< \epsilon \text{ (because } |a_n - 1| < \epsilon)
\end{aligned}$$

Therefore $\lim_{n\to\infty}(a_n^{1/3}) = 1$

3 Prove the following using the definition of limit.

a Let s be a positive real number. Prove by induction on n that $(1+s)^n > 1+ns$ for all integers $n \geq 2$.

Let $s \in \mathbb{R}, s > 0$, and $n \in \mathbb{Z}, n \geq 2$. Proof of $(1+s)^n > 1+ns$ through induction on n: Base case: $n = 2$. Then,

$$\begin{aligned}(1+s)^n &= (1+s)^2 \\ &= s^2 + 2s + 1 \\ &> 2s + 1 \text{ (because } s > 0\text{, so } s^2 > 0) \\ &= 1 + ns\end{aligned}$$

Therefore $(1+s)^n > 1+ns$ for $n = 2$. Inductive step: Assume $(1+s)^n > 1+ns$ (IH). Proof that $(1+s)^{(n+1)} > 1+(n+1)s$:

$$\begin{aligned}(1+s)^{(n+1)} &= (1+s)(1+s)^n \\ &> (1+s)(1+ns) \text{ (from (IH))} \\ &= ns^2 + ns + s + 1 \\ &> ns + s + 1 \text{ (because } n > 0 \text{ and } s > 0\text{, so } ns^2 > 0) \\ &= 1 + (n+1)s\end{aligned}$$

Therefore $(1+s)^{(n+1)} > 1+(n+1)s$ if $(1+s)^n > 1+ns$. Therefore $(1+s)^n > 1+ns$ for $n >= 2$.

b Let a be a real number so that $a > 1$. Prove that $\lim_{n\to\infty}(a^n) = \infty$.

Let $a \in \mathbb{R}, a > 1$, and $M \in \mathbb{R}, M > 0, N = \log_a M$, and $n \in Z, n > N$. Then:

$$\begin{aligned}a^n &> a^N \text{ (because } n > N\text{, and } a > 1) \\ &= a^{\log_a M} \\ &= M\end{aligned}$$

Therefore $\lim_{n\to\infty}(a^n) = \infty$ for $a > 1$.

c Let a be a real number so that $|a| < 1$. Prove that $\lim_{n\to\infty}(a^n) = 0$.

Let $a \in \mathbb{R}, |a| < 1$,and $\epsilon \in \mathbb{R}, e > 0$. Case 1: $a > 0$. Let $N = \log_a \epsilon, n \in \mathbb{Z}, n > N$. Then:

$$\begin{aligned} |a^n - 0| &= a^n \\ &< a^N \text{ (because } n > N\text{, and } a < 1) \\ &= a^{\log_a \epsilon} \\ &= \epsilon \end{aligned}$$

Therefore $\lim_{n\to\infty}(a^n) = 0$ for $0 < a < 1$. Case 2: $a = 0$. Let $N = 1, n \in \mathbb{Z}, n > N$. Then:

$$\begin{aligned} |a^n - 0| &= 0^n \\ &= 0 \text{ (because } n > N = 1\text{, so } n \neq 0) \\ &< \epsilon \end{aligned}$$

Therefore $\lim_{n\to\infty}(a^n) = 0$ for $a = 0$. Case 3: $a < 0$. Let $s = -a$, and $N = \log_c \epsilon, n \in \mathbb{Z}, n > N$. Then:

$$\begin{aligned} |a^n - 0| &= |a^n| \\ &= |a|^n \text{ (because } n \in \mathbb{Z}) \\ &= c^n \\ &< c^N \text{ (because } n > N\text{, and } 0 < c < 1) \\ &= c^{\log_c \epsilon} \\ &= \epsilon \end{aligned}$$

Therefore $\lim_{n\to\infty}(a^n) = 0$ for $-1 < a < 0$. Therefore $\lim_{n\to\infty}(a^n) = 0$ for $|a| < 1$.