MATH 273 Assignment 1

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1 For each true statement, give a proof. For each false statement, write out its negation and prove that.

a For all sets A, B and C, if $A \setminus B = C$ then $A = B \cup C$.

This statement is False. Its negation: $\exists$ sets A, B, C, such that $A \setminus B = C$ and $A \neq B \cup C$. Proof of negation by example: Choose $A = C = \varnothing, B = \{1\}$, where A, B, C are sets. Then $A \setminus B = \varnothing \setminus \{1\} = \varnothing = C$. And $A = \varnothing \neq \{1\} = \{1\} \cup \varnothing = B \cup C$. Therefore the negation is true and the statement is false.

b For all sets A, B and C, if $A \setminus (B \cap C) = \varnothing$ then $A \setminus C \subseteq B$.

This statement is True. Proof: Let A, B, C be sets, such that $A \setminus (B \cap C) = \varnothing$. This means that if $x \in A$ then $x \in B$ and $x \in C$. Thus $A \subseteq B$ and $A \subseteq C$. So $A \setminus C = \varnothing \subseteq B$. Therefore the statement is true.

c For all sets A, B and C, if $A \setminus C \subseteq B$ then $A \setminus (B \cap C) = \varnothing$.

This statement is False. Its negation: $\exists$ sets A, B, C, such that $A \setminus C \subseteq B$ and $A \setminus (B \cap C) \neq \varnothing$. Proof of negation by example: Choose $A = B = \{1\}, C = \varnothing$, where A, B, C are sets. Then $A \setminus C = \{1\} \setminus \varnothing = \{1\} \subseteq \{1\} = B$. And $A \setminus (B \cap C) = \{1\} \setminus (\{1\} \cap \varnothing) = \{1\} \setminus \varnothing = \{1\} \neq \varnothing$. Therefore the negation is true and the statement is false.

2 Let $f : A \to B$ and $g : B \to C$ be functions. Prove or disprove each of the following.

a If f and g are one-to-one then $g \circ f$ is one-to-one.

This statement is True. Proof: Let $f : A \to B$ and $g : B \to C$ be functions. Let f and g be one-to-one. Suppose $a, b \in A$, and $c, d \in B$. This means, if $f(a) = f(b)$, then $a = b$. And, if $g(c) = g(d)$, then $c = d$. I'm trying to prove that $g \circ f$ is one-to-one. Suppose $g \circ f(a) = g \circ f(b)$.

$$\begin{aligned} g(f(a)) &= g(f(b)) && \\ f(a) &= f(b) && \text{because g is one-to-one} \\ a &= b && \text{because f is one-to-one} \end{aligned}$$

Therefore $g \circ f$ is one-to-one, and the statement is true.

b If $g \circ f$ is one-to-one then f is one-to-one.

This statement is True. Proof: Let $f : A \to B$ and $g : B \to C$ be functions. Let $g \circ f$ be one-to-one. Suppose $a, b \in A$, such that $f(a) = f(b)$. Because $f(a) = f(b), g \circ f(a) = g \circ f(b)$. Because $g \circ f$ is one-to-one, $a = b$. Therefore f is one-to-one, and the statement is true.

c If $g \circ f$ is one-to-one then g is one-to-one.

This statement is False. Its negation: $\exists$ functions $f : A \to B$ and $g : B \to C$ such that, $g \circ f$ is one-to-one, and g is not one-to-one. Proof of negation by example: Let $f : A \to B$ and $g : B \to C$ be functions. Choose $A = C = \{1\}, B = \{1, 2\}$. Choose $f = \{(1, 1)\}, g = \{(1, 1), (2, 1)\}$. So $g \circ f = \{(1, 1)\}$, and it is one-to-one. Choose $a = 1, b = 2$. $g(a) = g(b)$ and $a \neq b$, thus g is not one-to-one. Therefore the negation is true and the statement is false.

d If $g \circ f$ is one-to-one and f is onto then g is one-to-one.

This statement is True. Proof: Let $f : A \to B$ and $g : B \to C$ be functions. Let $g \circ f$ be one-to-one, and f be onto. Suppose $a, b \in B$, and $g(a) = g(b)$. Because f is onto, for every $y \in B, \exists x \in A$, such that $y = f(x)$. Hence $g(a) = g(b)$ can be rewritten as $g(f(c)) = g(f(d))$, where $c, d \in A$ such that, $a = f(c), b = f(d)$. Because $g \circ f$ is one-to-one, $c = d$. Becasue $c = d$, $f(c) = f(d)$, which means $a = b$. Therefore g is one-to-one, and the statement is true.

3 Let $A = \{1, 2, 3, 4\}$. Prove or disprove each of the following statements.

a For all functions $f : A \to A$, there exists a function $g : A \to A$ so that $g \circ f(1) = 2$.

This statement is True. Proof: Let $f : A \to A$. Choose $g : A \to A$, such that $\forall x \in A, g(x) = 2$. Since $f(1) \in A, g \circ f(1) = 2$. Therefore the statement is true.

b There exists a function $g : A \to A$ so that for all functions $f : A \to A, g \circ f(1) = 2$.

This statement is True. Proof: Choose $g = \{(1, 2), (2, 2), (3, 2), (4, 2)\}$. So that $\forall x \in A, g(x) = 2$. Let $f : A \to A$ be a function. Since $1 \in A$, and $f(1) \in A$, this means $g \circ f(1) = 2, \forall f : A \to A$. Therefore the statement is true.

c For all functions $f : A \to A$, there is a function $g : A \to A$ so that $g \circ f(1) = 2$ and $g \circ f(2) = 1$.

This statement is False. Its negation: $\exists f : A \to A$, such that $\forall g : A \to A$, $g \circ f(1) \neq 2$, or $g \circ f(2) \neq 1$. Proof of negation by contradiction: Choose $f = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. Let $g : A \to A$. Assume the negation to be false, so $g \circ f(1) = 2$ and $g \circ f(2) = 1$. It's true that $f(2) = f(1)$ as $1 = 1$. So $g \circ f(2) = g \circ f(1)$. However from assumption, $g \circ f(2) = 1 \neq 2 = g \circ f(1)$ leading to a contradiction. Therefore the negation can't be false, and the statement is false.

d There exists a function $g : A \to A$ so that for all functions $f : A \to A, g \circ f(1) = 2$ and $g \circ f(2) = 1$.

This statment is False. Its negation: $\forall g : A \to A$, $\exists f : A \to A$, such that, $g \circ f(1) \neq 2$, or $g \circ f(2) \neq 1$. Proof of negation by contridiction: Let $g : A \to A$. Choose $f : A \to A$ such that, $f(1) = f(2)$. Assume the negation to be false, so $g \circ f(1) = 2$ and $g \circ f(2) = 1$. Because $f(1) = f(2)$, so $g \circ f(1) = g \circ f(2)$. However from assumption, $g \circ f(1) = 2 \neq 1 = g \circ f(2)$, which is a contridiction. Thus the negation can not be false, and the statement is false.