Functions

based on Precalculus
Version $4 - \epsilon$

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Chapter 1

Basic Concepts and Review

1.1 Basic Set Theory and Interval Notation

1.1.1 Some Basic Set Theory Notions

We begin this section with the definition of a concept that is central to all of Mathematics.

Definition 1.1. A **set** is a well-defined collection of objects which are called the elements of the set. Here, 'well-defined' means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word "smolko" is well-defined and is a set, but the collection of the worst Math teachers in the world is **not** well-defined and therefore is **not** a set.[1]

In general, there are three ways to describe sets and those methods are listed below.

Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to describe the elements the set.
2. **The Roster Method:** Begin with a left brace '$\{$', list each element of the set *only once* and then end with a right brace '$\}$'.
3. **The Set-Builder Method:** A combination of the verbal and roster methods using a "dummy variable" such as x and conditions on that variable.

Let S be the set described *verbally* as the set of letters that make up the word "smolko". A *roster* description of S is $\{s, m, o, l, k\}$. Note that we listed 'o' only once, even though it appears twice in the word "smolko". Also, the order of the elements doesn't matter, so $\{k, l, m, o, s\}$ is also a roster description of S. A *set-builder* description of S is: $\{x \mid x \text{ is a letter in the word "smolko"}\}$. The way to read this is 'The set of elements x such that x is a letter in the word "smolko".' In each of the above cases, we may use the familiar equals sign '=' and write $S = \{s, m, o, l, k\}$ or $S = \{x \mid x \text{ is a letter in the word "smolko"}\}$.

[1] For a more thought-provoking example, consider the collection of all things that do not contain themselves - this leads to the famous paradox known as Russell's Paradox.

Notice that m is in S but many other letters, such as q, are not in S. We express these ideas of set inclusion and exclusion mathematically using the symbols $m \in S$ (read 'm is in S') and $q \notin S$ (read 'q is not in S'). More precisely, we have the following.

Definition 1.2. Let A be a set.

- If x is an element of A then we write $x \in A$ which is read 'x is in A'.
- If x is *not* an element of A then we write $x \notin A$ which is read 'x is not in A'.

Now let's consider the set $C = \{x \mid x \text{ is a consonant in the word "smolko"}\}$. A roster description of C is $C = \{s, m, l, k\}$. Note that by construction, every element of C is also in S. We express this relationship by stating that the set C is a **subset** of the set S, which is written in symbols as $C \subseteq S$. The more formal definition is given at the top of the next page.

Definition 1.3. Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

In our previous example, $C \subseteq S$ yet not vice-versa since $o \in S$ but $o \notin C$. Additionally, the set of vowels $V = \{a, e, i, o, u\}$, while it does have an element in common with S, is not a subset of S. (As an added note, S is not a subset of V, either.) We could, however, *build* a set which contains both S and V as subsets by gathering all of the elements in both S and V together into a single set, say $U = \{s, m, o, l, k, a, e, i, u\}$. Then $S \subseteq U$ and $V \subseteq U$. The set U we have built is called the **union** of the sets S and V and is denoted $S \cup V$. Furthermore, S and V aren't completely *different* sets since they both contain the letter 'o.' The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of S and V is $\{o\}$, written $S \cap V = \{o\}$. We formalize these ideas below.

Definition 1.4. Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The **union** of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$

The key words in Definition 1.4 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). Please note that this mathematical use of the word 'or' differs than how we use 'or' in spoken English. In Math, we use the *inclusive or* which allows for the element to be in both sets. At a restaurant if you're asked "Do you want fries or a salad?" you must pick one and only one. This is known as the *exclusive or* and it plays a role in other Math classes. For our purposes it is good enough to say that for an element to belong to the union of two sets it must belong to *at least one* of them.

Returning to the sets C and V above, $C \cup V = \{s, m, l, k, a, e, i, o, u\}$.[2] Their intersection, however, creates a bit of notational awkwardness since C and V have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

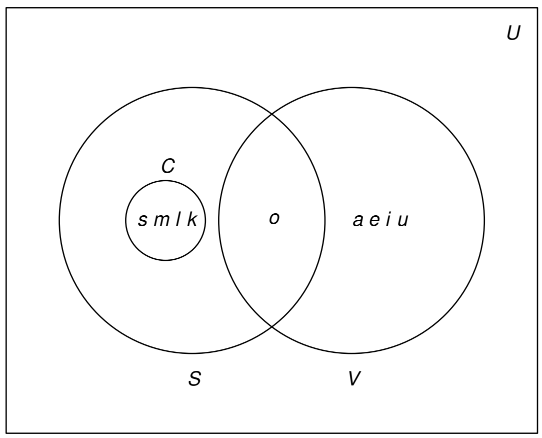
[2] Which just so happens to be the same set as $S \cup V$.

Definition 1.5. The **Empty Set** is the set which contains no elements and is denoted $\emptyset$. That is,

$$\emptyset = \{\} = \{x \mid x \neq x\}.$$

As promised, the empty set is the set containing no elements since no matter what 'x' is, '$x = x$.' Like the number '0,' the empty set plays a vital role in mathematics.[3] We introduce it here more as a symbol of convenience as opposed to a contrivance[4] because saying that $C \cap V = \emptyset$ is unambiguous whereas $\{\}$ looks like a typographical error.

A nice way to visualize the relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets S, C and V is drawn at the top of the next page.

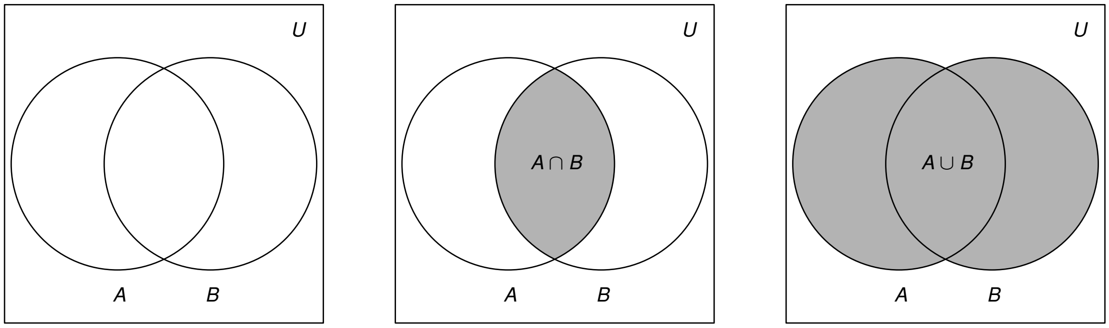


A Venn Diagram for C, S and V.

In the Venn Diagram above we have three circles - one for each of the sets C, S and V. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set C is completely inside the circle representing S. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing S and V overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that since $C \cap V = \emptyset$, the circles which represent C and V have no overlap whatsoever.

All of these circles lie in a rectangle labeled U for the 'universal' set. A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or U as the set of letters in the entire alphabet. The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no' and we refer you once again to Russell's Paradox. The usual triptych of Venn Diagrams indicating generic sets A and B along with $A \cap B$ and $A \cup B$ is given below.

[3] Sadly, the full extent of the empty set's role will not be explored in this text.
[4] Actually, the empty set can be used to generate numbers - mathematicians can create something from nothing!



Sets A and B. $A \cap B$ is shaded. $A \cup B$ is shaded.

The one major limitation of Venn Diagrams is that they become unwieldy if more than four sets need to be drawn simultaneously within the same universal set. This idea is explored in the Exercises.

1.1.2 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Much of the "real world" can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete[5] definition of a real number is given below.

Definition 1.6. A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character $\mathbb{R}$.

Certain subsets of the real numbers are worthy of note and are listed below. In fact, in more advanced texts,[6] the real numbers are *constructed* from some of these subsets.

[5]Math pun intended!

[6]See, for instance, Landau's Foundations of Analysis.

Special Subsets of Real Numbers

1. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis '...' here indicate that the natural numbers contain 1, 2, 3 'and so forth'.

2. The **Whole Numbers**: $\mathbb{W} = \{0, 1, 2, \ldots\}$.

3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.[a]

4. The **Rational Numbers**: $\mathbb{Q} = \left\{ \frac{a}{b} \,|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ where } b \neq 0 \right\}$. Rational numbers are the ratios of integers where the denominator is not zero. It turns out that another way to describe the rational numbers[b] is:

$$\mathbb{Q} = \{x \,|\, x \text{ possesses a repeating or terminating decimal representation}\}$$

5. The **Irrational Numbers**: $\mathbb{P} = \{x \,|\, x \in \mathbb{R} \text{ but } x \notin \mathbb{Q}\}$.[c] That is, an irrational number is a real number which isn't rational. Said differently,

$$\mathbb{P} = \{x \,|\, x \text{ possesses a decimal representation which neither repeats nor terminates}\}$$

[a]The symbol $\pm$ is read 'plus or minus' and it is a shorthand notation which appears throughout the text. Just remember that $x = \pm 3$ means $x = 3$ or $x = -3$.

[b]See Section **??**.

[c]Examples here include number π (See Section **??**), $\sqrt{2}$ and $0.101001000100001\ldots$.

Note that every natural number is a whole number which, in turn, is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and since every rational number is a real number[7] the sets $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are nested like Matryoshka dolls. More formally, these sets form a subset chain: $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above.

It is time to put all of this together in an example.

Example 1.1.1.

1. Write a roster description for $P = \{2^n \,|\, n \in \mathbb{N}\}$ and $E = \{2n \,|\, n \in \mathbb{Z}\}$.

2. Write a verbal description for $S = \{x^2 \,|\, x \in \mathbb{R}\}$.

3. Let $A = \{-117, \frac{4}{5}, 0.20\overline{2002}, 0.202002000200002\ldots\}$.

   (a) Which elements of A are natural numbers? Rational numbers? Real numbers?

   (b) Find $A \cap \mathbb{W}$, $A \cap \mathbb{Z}$ and $A \cap \mathbb{P}$.

4. What is another name for $\mathbb{N} \cup \mathbb{Q}$? What about $\mathbb{Q} \cup \mathbb{P}$?

[7]Thanks to long division!

Solution.

1. To find roster descriptions for each of these sets, we need to list their elements. Starting with the set $P = \{2^n \,|\, n \in \mathbb{N}\}$, we substitute natural number values n into the formula 2^n. For $n = 1$ we get $2^1 = 2$, for $n = 2$ we get $2^2 = 4$, for $n = 3$ we get $2^3 = 8$ and for $n = 4$ we get $2^4 = 16$. Hence P describes the powers of 2, so a roster description for P is $P = \{2, 4, 8, 16, ...\}$ where the '...' indicates the that pattern continues.[8]

   Proceeding in the same way, we generate elements in $E = \{2n \,|\, n \in \mathbb{Z}\}$ by plugging in integer values of n into the formula $2n$. Starting with $n = 0$ we obtain $2(0) = 0$. For $n = 1$ we get $2(1) = 2$, for $n = -1$ we get $2(-1) = -2$ for $n = 2$, we get $2(2) = 4$ and for $n = -2$ we get $2(-2) = -4$. As n moves through the integers, $2n$ produces all of the *even* integers.[9] A roster description for E is $E = \{0, \pm 2, \pm 4, ...\}$.

2. One way to verbally describe S is to say that S is the 'set of all squares of real numbers'. While this isn't incorrect, we'd like to take this opportunity to delve a little deeper.[10] What makes the set $S = \{x^2 \,|\, x \in \mathbb{R}\}$ a little trickier to wrangle than the sets P or E above is that the dummy variable here, x, runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way.[11] Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in S. For $x = -2$, $x^2 = (-2)^2 = 4$ so 4 is in S, as are $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $(\sqrt{117})^2 = 117$. Even things like $(-\pi)^2$ and $(0.101001000100001\ldots)^2$ are in S.

   So suppose $s \in S$. What can be said about s? We know there is some real number x so that $s = x^2$. Since $x^2 \geq 0$ for any real number x, we know $s \geq 0$. This tells us that everything in S is a non-negative real number.[12] This begs the question: are <u>all</u> of the non-negative real numbers in S? Suppose n is a non-negative real number, that is, $n \geq 0$. If n were in S, there would be a real number x so that $x^2 = n$. As you may recall, we can solve $x^2 = n$ by 'extracting square roots': $x = \pm\sqrt{n}$. Since $n \geq 0$, $\sqrt{n}$ is a real number.[13] Moreover, $(\sqrt{n})^2 = n$ so n is the square of a real number which means $n \in S$. Hence, S is the set of non-negative real numbers.

3. (a) The set A contains no natural numbers.[14] Clearly $\frac{4}{5}$ is a rational number as is -117 (which can be written as $\frac{-117}{1}$). It's the last two numbers listed in A, $0.20\overline{2002}$ and $0.202002000200002\ldots$, that warrant some discussion. First, recall that the 'line' over the digits 2002 in $0.20\overline{2002}$ (called the vinculum) indicates that these digits repeat, so it is a rational number.[15] As for the number $0.202002000200002\ldots$, the '...' indicates the pattern of adding an extra '0' followed by a '2' is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal

[8] This isn't the most *precise* way to describe this set - it's always dangerous to use '...' since we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

[9] This shouldn't be too surprising, since an even integer is *defined* to be an integer multiple of 2.

[10] Think of this as an opportunity to stop and smell the mathematical roses.

[11] This is a nontrivial statement. Interested readers are directed to a discussion of Cantor's Diagonal Argument.

[12] This means S is a subset of the non-negative real numbers.

[13] This is called the 'square root closed property' of the non-negative real numbers.

[14] Carl was tempted to include $0.\overline{9}$ in the set A, but thought better of it. See Section **??** for details.

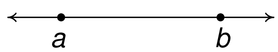
[15] So $0.20\overline{2002} = 0.20200220022002\ldots$.

is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of A are real numbers, since all of them can be expressed as decimals (remember that $\frac{4}{5} = 0.8$).

(b) The set $A \cap \mathbb{W} = \{x \,|\, x \in A \text{ and } x \in \mathbb{W}\}$ is another way of saying we are looking for the set of numbers in A which are whole numbers. Since A contains no whole numbers, $A \cap \mathbb{W} = \emptyset$. Similarly, $A \cap \mathbb{Z}$ is looking for the set of numbers in A which are integers. Since -117 is the only integer in A, $A \cap \mathbb{Z} = \{-117\}$. For the set $A \cap \mathbb{P}$, as discussed in part (a), the number $0.202002000200002\ldots$ is irrational, so $A \cap \mathbb{P} = \{0.202002000200002\ldots\}$.

4. The set $\mathbb{N} \cup \mathbb{Q} = \{x \,|\, x \in \mathbb{N} \text{ or } x \in \mathbb{Q}\}$ is the union of the set of natural numbers with the set of rational numbers. Since every natural number is a rational number, $\mathbb{N}$ doesn't contribute any new elements to $\mathbb{Q}$, so $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$.[16] For the set $\mathbb{Q} \cup \mathbb{P}$, we note that every real number is either rational or not, hence $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$, pretty much by the definition of the set $\mathbb{P}$. $\square$

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers a and b, we write $a < b$ if a is located to the left of b on the number line, as shown below.



The real number line with two numbers a and b where $a < b$.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that $\mathbb{R}$ is complete. This means that there are no 'holes' or 'gaps' in the real number line.[17] Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you'll find a solid line segment (or interval) consisting of infinitely many real numbers. The next result tells us what types of numbers we can expect to find.

Density Property of $\mathbb{Q}$ and $\mathbb{P}$ in $\mathbb{R}$

Between any two distinct real numbers, there is at least one rational number and one irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and infinitely many irrational numbers.

The root word 'dense' here communicates the idea that rationals and irrationals are 'thoroughly mixed' into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you've done that, try doing the same thing for the numbers $0.\overline{9}$ and 1. ('Try' is the operative word, here.[18])

The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers a and b, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). This property is given below.

[16] In fact, anytime $A \subseteq B$, $A \cup B = B$ and vice-versa. See the exercises.

[17] Alas, this intuitive feel for what it means to be 'complete' is as good as it gets at this level. Completeness is given a much more precise meaning later in courses like Analysis and Topology.

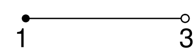
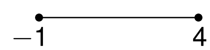
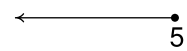
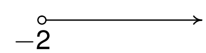
[18] Again, see Section **??** for details.

Law of Trichotomy

If a and b are real numbers then exactly one of the following statements is true:

$$a < b \qquad\qquad a > b \qquad\qquad a = b$$

Segments of the real number line are called **intervals**. They play a huge role not only in this text but also in the Calculus curriculum so we need a concise way to describe them. We start by examining a few examples of the **interval notation** associated with some specific sets of numbers.

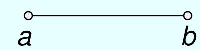
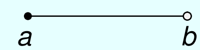
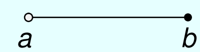
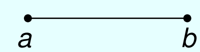
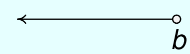
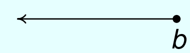
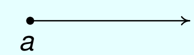
Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid 1 \leq x < 3\}$	$[1, 3)$	
$\{x \mid -1 \leq x \leq 4\}$	$[-1, 4]$	
$\{x \mid x \leq 5\}$	$(-\infty, 5]$	
$\{x \mid x > -2\}$	$(-2, \infty)$	

As you can glean from the table, for intervals with finite endpoints we start by writing 'left endpoint, right endpoint'. We use square brackets, '[' or ']', if the endpoint is included in the interval. This corresponds to a 'filled-in' or 'closed' dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, '(' or ')' that correspond to an 'open' circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions. We summarize all of the possible cases in one convenient table below.[19]

[19]The importance of understanding interval notation in this book and also in Calculus cannot be overstated so please do yourself a favor and memorize this chart.

Interval Notation

Let a and b be real numbers with $a < b$.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$	$(a, b]$	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \leq b\}$	$(-\infty, b]$	
$\{x \mid x > a\}$	(a, ∞)	
$\{x \mid x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

Intervals of the forms $(a, b), (-\infty, b)$ and (a, ∞) are said to be **open** intervals. Those of the forms $[a, b], (-\infty, b]$ and $[a, \infty)$ are said to be **closed** intervals.

Unfortunately, the words 'open' and 'closed' are not antonyms here because the empty set $\emptyset$ and the set $(-\infty, \infty)$ are simultaneously open and closed[20] while the intervals $(a, b]$ and $[a, b)$ are neither open nor closed. The inclusion or exclusion of an endpoint might seem like a terribly small thing to fuss about but these sorts of technicalities in the language become important in Calculus so we feel the need to put this material in the Precalculus book.

We close this section with an example that ties together some of the concepts presented earlier. Specifically, we demonstrate how to use interval notation along with the concepts of union and intersection to describe a variety of sets on the real number line. In many sections of the text to come you will need to be fluent with this notation so take the time to study it deeply now.

[20]You don't need to worry about that fact until you take an advanced course in Topology.

Example 1.1.2.

1. Express the following sets of numbers using interval notation.

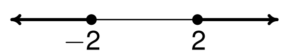
   (a) $\{x \mid x \leq -2 \text{ or } x \geq 2\}$ (b) $\{x \mid x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\}$

   (c) $\{x \mid x \neq \pm 3\}$ (d) $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$

2. Let $A = [-5, 3)$ and $B = (1, \infty)$. Find $A \cap B$ and $A \cup B$.

Solution.

1. (a) The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. The 'or' in $\{x \mid x \leq -2 \text{ or } x \geq 2\}$ tells us that we are looking for the union of these two intervals, so our answer is $(-\infty, -2] \cup [2, \infty)$.

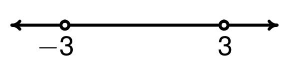


$(-\infty, -2] \cup [2, \infty)$

   (b) For the set $\{x \mid x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\}$, we need the real numbers less than (to the left of) $\sqrt{3}$ that are simultaneously greater than (to the right of) $-\frac{8}{5}$, including $-\frac{8}{5}$ but excluding $\sqrt{3}$. This yields $\{x \mid x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\} = \left[-\frac{8}{5}, \sqrt{3}\right)$.

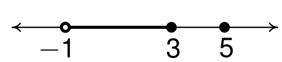
$\left[-\frac{8}{5}, \sqrt{3}\right)$

   (c) For the set $\{x \mid x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. (Refer back to page 4 for a discussion about $x = \pm 3$) This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.



$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

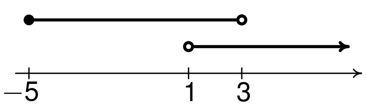
   (d) Graphing the set $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$ yields the interval $(-1, 3]$ along with the single number 5. While we *could* express this single point as $[5, 5]$, it is customary to write a single point as a 'singleton set', so in our case we have the set $\{5\}$. This means that our final answer is written $\{x \mid -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.



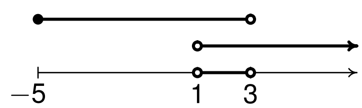
$(-1, 3] \cup \{5\}$

2. We start by graphing $A = [-5, 3)$ and $B = (1, \infty)$ on the number line. To find $A \cap B$, we need to find the numbers common to both A and B; in other words, we need to find the overlap of the two intervals. Clearly, everything between 1 and 3 is in both A and B. However, since 1 is in A but not in B, 1 is not in the intersection. Similarly, since 3 is in B but not in A, it isn't in the intersection either. Hence, $A \cap B = (1, 3)$.

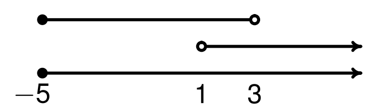
   To find $A \cup B$, we need to find the numbers in at least one of A or B. Graphically, we shade A and B along with it. Notice here that even though 1 isn't in B, it is in A, so it's in the union along with all of the other elements of A between -5 and 1. A similar argument goes for the inclusion of 3 in the union. The result of shading both A and B together gives us $A \cup B = [-5, \infty)$.



$A = [-5, 3)$, $B = (1, \infty)$



$A \cap B = (1, 3)$



$A \cup B = [-5, \infty)$ □

1.1.3 Exercises

1. Find a verbal description for $O = \{2n - 1 \mid n \in \mathbb{N}\}$

2. Find a roster description for $X = \{z^2 \mid z \in \mathbb{Z}\}$

3. Let $A = \left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \sqrt{3}, 5.2020020002\ldots, \frac{20}{10}, 117\right\}$

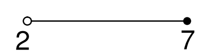
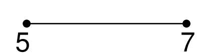
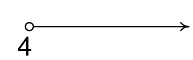
   (a) List the elements of A which are natural numbers.

   (b) List the elements of A which are irrational numbers.

   (c) Find $A \cap \mathbb{Z}$

   (d) Find $A \cap \mathbb{Q}$

4. Fill in the chart below.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0, 3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3, 3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty, 9)$	
		
$\{x \mid x \geq -3\}$		

In Exercises 5 - 10, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

5. $(-1, 5] \cap [0, 8)$

6. $(-1, 1) \cup [0, 6]$

7. $(-\infty, 4] \cap (0, \infty)$

8. $(-\infty, 0) \cap [1, 5]$

9. $(-\infty, 0) \cup [1, 5]$

10. $(-\infty, 5] \cap [5, 8)$

In Exercises 11 - 22, write the set using interval notation.

11. $\{x \mid x \neq 5\}$

12. $\{x \mid x \neq -1\}$

13. $\{x \mid x \neq -3,\ 4\}$

14. $\{x \mid x \neq 0,\ 2\}$

15. $\{x \mid x \neq 2,\ -2\}$

16. $\{x \mid x \neq 0,\ \pm 4\}$

17. $\{x \mid x \leq -1 \text{ or } x \geq 1\}$

18. $\{x \mid x < 3 \text{ and } x \geq 2\}$

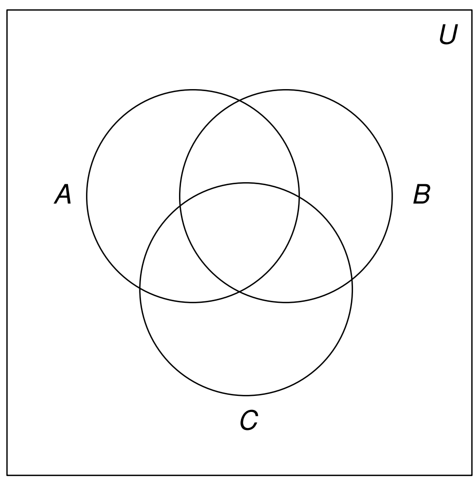
19. $\{x \mid x \leq -3 \text{ or } x > 0\}$

20. $\{x \mid x \leq 2 \text{ and } x > 3\}$

21. $\{x \mid x > 2 \text{ or } x = \pm 1\}$

22. $\{x \mid 3 < x < 13 \text{ and } x \neq 4\}$

For Exercises 23 - 28, use the blank Venn Diagram below with A, B, and C in it as a guide to help you shade the following sets.



23. $A \cup C$

24. $B \cap C$

25. $(A \cup B) \cup C$

26. $(A \cap B) \cap C$

27. $A \cap (B \cup C)$

28. $(A \cap B) \cup (A \cap C)$

29. Explain how your answers to problems 27 and 28 show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Phrased differently, this shows 'intersection *distributes* over union.' Discuss with your classmates if 'union' distributes over 'intersection.' Use a Venn Diagram to support your answer.

30. Show that $A \subseteq B$ if and only if $A \cup B = B$.

31. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8, 10\}$, $C = \{1, 6, 9\}$ and $D = \{2, 7, 10\}$. Draw one Venn Diagram that shows all four of these sets. What sort of difficulties do you encounter?

1.1.4 Answers

1. O is the odd natural numbers.

2. $X = \{0, 1, 4, 9, 16, ...\}$

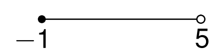
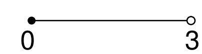
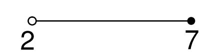
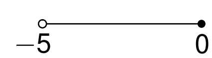
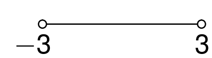
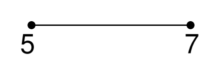
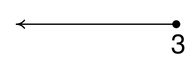
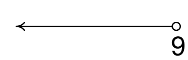
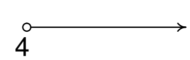
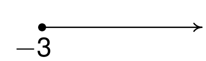
3. (a) $\frac{20}{10} = 2$ and 117

   (b) $\sqrt{3}$ and 5.2020020002

   (c) $\left\{-3, \frac{20}{10}, 117\right\}$

   (d) $\left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \frac{20}{10}, 117\right\}$

4.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$	$[-1, 5)$	
$\{x \mid 0 \leq x < 3\}$	$[0, 3)$	
$\{x \mid 2 < x \leq 7\}$	$(2, 7]$	
$\{x \mid -5 < x \leq 0\}$	$(-5, 0]$	
$\{x \mid -3 < x < 3\}$	$(-3, 3)$	
$\{x \mid 5 \leq x \leq 7\}$	$[5, 7]$	
$\{x \mid x \leq 3\}$	$(-\infty, 3]$	
$\{x \mid x < 9\}$	$(-\infty, 9)$	
$\{x \mid x > 4\}$	$(4, \infty)$	
$\{x \mid x \geq -3\}$	$[-3, \infty)$	

5. $(-1, 5] \cap [0, 8) = [0, 5]$

6. $(-1, 1) \cup [0, 6] = (-1, 6]$

7. $(-\infty, 4] \cap (0, \infty) = (0, 4]$

8. $(-\infty, 0) \cap [1, 5] = \emptyset$

9. $(-\infty, 0) \cup [1, 5] = (-\infty, 0) \cup [1, 5]$

10. $(-\infty, 5] \cap [5, 8) = \{5\}$

11. $(-\infty, 5) \cup (5, \infty)$

12. $(-\infty, -1) \cup (-1, \infty)$

13. $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

14. $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

15. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

16. $(-\infty, -4) \cup (-4, 0) \cup (0, 4) \cup (4, \infty)$

17. $(-\infty, -1] \cup [1, \infty)$

18. $[2, 3)$

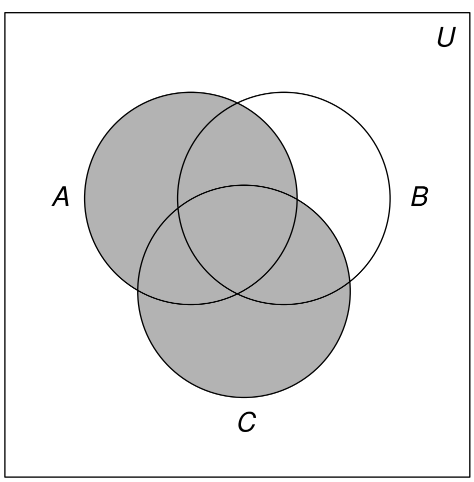
19. $(-\infty, -3] \cup (0, \infty)$

20. $\emptyset$

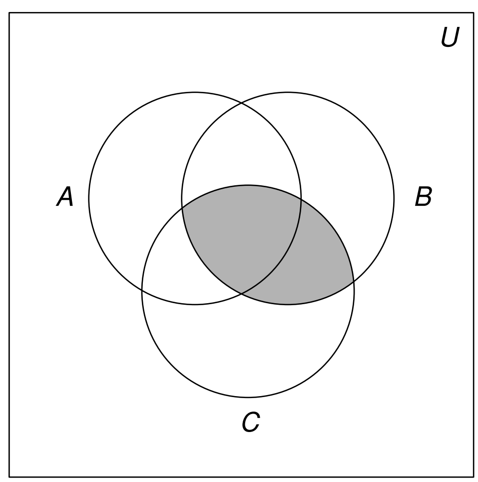
21. $\{-1\} \cup \{1\} \cup (2, \infty)$

22. $(3, 4) \cup (4, 13)$

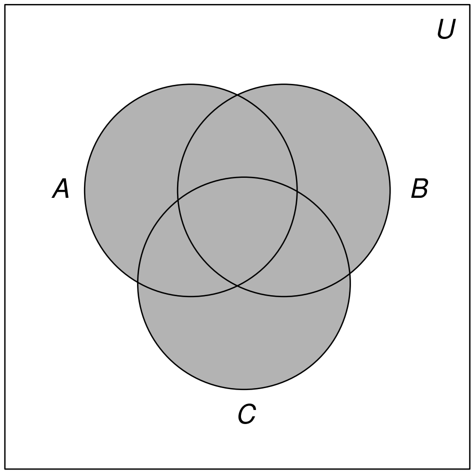
23. $A \cup C$



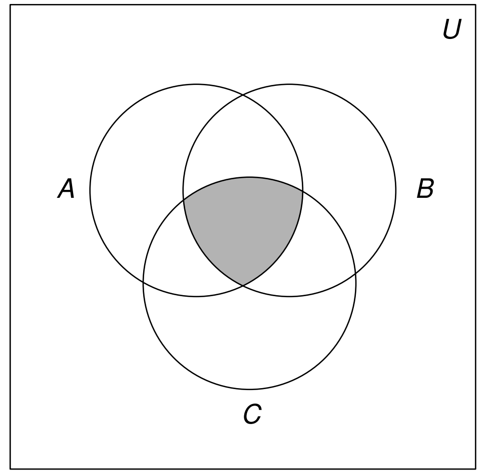
24. $B \cap C$



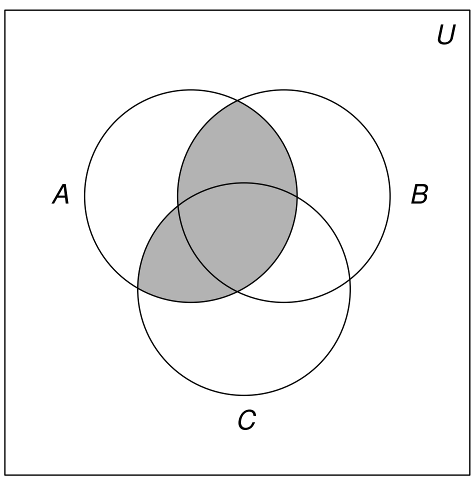
25. $(A \cup B) \cup C$



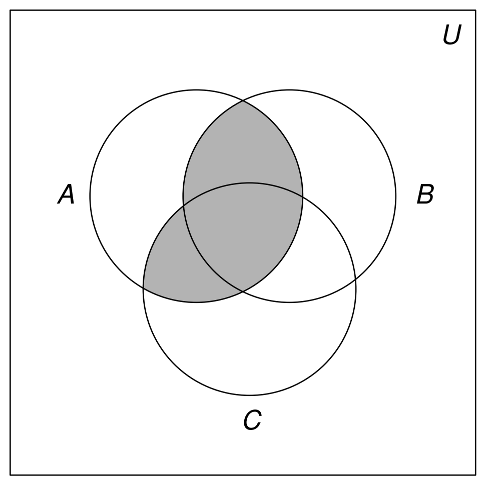
26. $(A \cap B) \cap C$



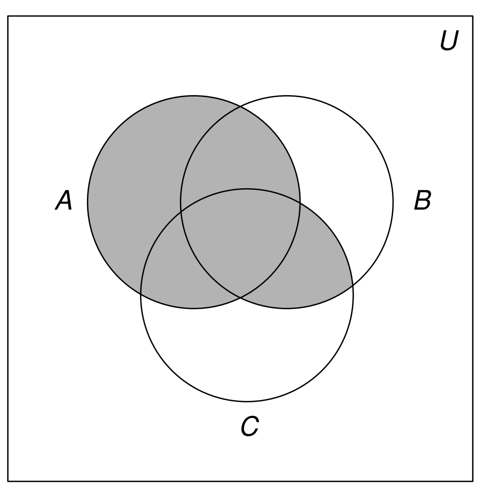
27. $A \cap (B \cup C)$



28. $(A \cap B) \cup (A \cap C)$



29. Yes, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



1.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra. There are two primary operations one can perform with real numbers: addition and multiplication. We'll start with the properties of addition.

Properties of Real Number Addition

- **Closure:** For all real numbers a and b, $a + b$ is also a real number.
- **Commutativity:** For all real numbers a and b, $a + b = b + a$.
- **Associativity:** For all real numbers a, b and c, $a + (b + c) = (a + b) + c$.
- **Identity:** There is a real number '0' so that for all real numbers a, $a + 0 = a$.
- **Inverse:** For all real numbers a, there is a real number $-a$ such that $a + (-a) = 0$.
- **Definition of Subtraction:** For all real numbers a and b, $a - b = a + (-b)$.

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers a and b a variety of ways: ab, $a \cdot b$, $a(b)$, $(a)(b)$ and so on. We'll refrain from using $a \times b$ for real number multiplication in this text with one notable exception in Definition 1.7.

Properties of Real Number Multiplication

- **Closure:** For all real numbers a and b, ab is also a real number.
- **Commutativity:** For all real numbers a and b, $ab = ba$.
- **Associativity:** For all real numbers a, b and c, $a(bc) = (ab)c$.
- **Identity:** There is a real number '1' so that for all real numbers a, $a \cdot 1 = a$.
- **Inverse:** For all real numbers $a \neq 0$, there is a real number $\dfrac{1}{a}$ such that $a\left(\dfrac{1}{a}\right) = 1$.
- **Definition of Division:** For all real numbers a and $b \neq 0$, $a \div b = \dfrac{a}{b} = a\left(\dfrac{1}{b}\right)$.

While most students and some faculty tend to skip over these properties or give them a cursory glance at best,[1] it is important to realize that the properties stated above are what drive the symbolic manipulation in all of Algebra. When listing a tally of more than two numbers, $1+2+3$ for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as $(1+2)+3$ or $1+(2+3)$. This brings up a note about 'grouping symbols'. Recall that parentheses and brackets

[1] Not unlike how Carl approached all the Elven poetry in The Lord of the Rings.

are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example, $1 + 2 \cdot 3 = 1 + 6 = 7$, but $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$. As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first: $(1 + 2) \cdot 3 = 1 \cdot 3 + 2 \cdot 3 = 3 + 6 = 9$. More generally, we have the following.

The Distributive Property and Factoring

For all real numbers a, b and c:

- **Distributive Property:** $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.
- **Factoring:**[a] $ab + ac = a(b + c)$ and $ac + bc = (a + b)c$.

[a]Or, as Carl calls it, 'reading the Distributive Property from right to left.'

It is worth pointing out that we didn't really need to list the Distributive Property both for $a(b+c)$ (distributing from the left) and $(a+b)c$ (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, 'factoring' is really the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students *see* these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression $5(2 + x)$, without knowing the value of x, we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get $5(2 + x) = 5 \cdot 2 + 5 \cdot x = 10 + 5x$. The Distributive Property is also responsible for combining 'like terms'. Why is $3x + 2x = 5x$? Because $3x + 2x = (3 + 2)x = 5x$.

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

Properties of Zero

Suppose a and b are real numbers.

- **Zero Product Property:** $ab = 0$ if and only if $a = 0$ or $b = 0$ (or both)

  Note: This not only says that $0 \cdot a = 0$ for any real number a, it also says that the *only* way to get an answer of '0' when multiplying two real numbers is to have one (or both) of the numbers be '0' in the first place.

- **Zeros in Fractions:** If $a \neq 0$, $\dfrac{0}{a} = 0 \cdot \left(\dfrac{1}{a}\right) = 0$.

  Note: The quantity $\dfrac{a}{0}$ is undefined.[a]

[a]The expression $\frac{0}{0}$ is technically an 'indeterminant form' as opposed to being strictly 'undefined' meaning that with Calculus we can make some sense of it in certain situations. We'll talk more about this in Chapter 7.

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve $x^2 + x - 6 = 0$ is by factoring[2] the left hand side of this equation to get $(x-2)(x+3) = 0$. From here, we apply the Zero Product Property and set each factor equal to zero. This yields $x - 2 = 0$ or $x + 3 = 0$ so $x = 2$ or $x = -3$. This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter 6.

Next up is a review of the arithmetic of 'negatives'. On page 17 we first introduced the dash which we all recognize as the 'negative' symbol in terms of the additive inverse. For example, the number -3 (read 'negative 3') is defined so that $3 + (-3) = 0$. We then defined subtraction using the concept of the additive inverse again so that, for example, $5-3 = 5+(-3)$. In this text we do not distinguish typographically between the dashes in the expressions '$5 - 3$' and '-3' even though they are mathematically quite different.[3] In the expression '$5 - 3$,' the dash is a *binary* operation (that is, an operation requiring *two* numbers) whereas in '-3', the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, 'Who cares?' Your calculator does - that's who! In the text we can write $-3 - 3 = -6$ but that will not work in your calculator. Instead you'd need to type $^{-}3 - 3$ to get -6 where the first dash comes from the '$+/-$' key and the second dash comes from the subtraction key.

Properties of Negatives

Given real numbers a and b we have the following.

- **Additive Inverse Properties:** $-a = (-1)a$ and $-(-a) = a$
- **Products of Negatives:** $(-a)(-b) = ab$.
- **Negatives and Products:** $-ab = -(ab) = (-a)b = a(-b)$.
- **Negatives and Fractions:** If b is nonzero, $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ and $\dfrac{-a}{-b} = \dfrac{a}{b}$.
- **'Distributing' Negatives:** $-(a + b) = -a - b$ and $-(a - b) = -a + b = b - a$.
- **'Factoring' Negatives:**[a] $-a - b = -(a + b)$ and $b - a = -(a - b)$.

[a] Or, as Carl calls it, reading 'Distributing' Negatives from right to left.

An important point here is that when we 'distribute' negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of -1 across each of these terms: $-(a + b) = (-1)(a+b) = (-1)(a) + (-1)(b) = (-a) + (-b) = -a - b$. Negatives do not 'distribute' across multiplication: $-(2 \cdot 3) \neq (-2) \cdot (-3)$. Instead, $-(2 \cdot 3) = (-2) \cdot (3) = (2) \cdot (-3) = -6$.

The same sort of thing goes for fractions: $-\frac{3}{5}$ can be written as $\frac{-3}{5}$ or $\frac{3}{-5}$, but not $\frac{-3}{-5}$.

[2] Don't worry. We'll review this in due course. And, yes, this is our old friend the Distributive Property!

[3] We're not just being lazy here. We looked at many of the big publishers' Precalculus books and none of them use different dashes, either.

Speaking of fractions, we now review their arithmetic.

Properties of Fractions

Suppose a, b, c and d are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

- **Identity Properties:** $a = \frac{a}{1}$ and $\frac{a}{a} = 1$.
- **Fraction Equality:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
- **Multiplication of Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. In particular: $\frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}$

  Note: A common denominator is **not** required to **multiply** fractions!
- **Division**[a] **of Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

  In particular: $1 \div \frac{a}{b} = \frac{b}{a}$ and $\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$

  Note: A common denominator is **not** required to **divide** fractions!
- **Addition and Subtraction of Fractions:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$.

  Note: A common denominator **is** required to **add or subtract** fractions!
- **Equivalent Fractions:** $\frac{a}{b} = \frac{ad}{bd}$, since $\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{d}{d} = \frac{ad}{bd}$

  Note: The *only* way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.
- **'Reducing'**[b] **Fractions:** $\frac{a\cancel{d}}{b\cancel{d}} = \frac{a}{b}$, since $\frac{ad}{bd} = \frac{a}{b} \cdot \frac{d}{d} = \frac{a}{b} \cdot 1 = \frac{a}{b}$.

  In particular, $\frac{ab}{b} = a$ since $\frac{ab}{b} = \frac{ab}{1 \cdot b} = \frac{a\cancel{b}}{1 \cdot \cancel{b}} = \frac{a}{1} = a$ and $\frac{b-a}{a-b} = \frac{(-1)\cancel{(a-b)}}{\cancel{(a-b)}} = -1$.

  Note: We may only cancel common **factors** from both numerator and denominator.

[a] The old 'invert and multiply' or 'fraction gymnastics' play.

[b] Or 'Canceling' Common Factors - this is really just reading the previous property 'from right to left'.

Students make so many mistakes with fractions that we feel it is necessary to pause the narrative for a moment and offer you the following examples. Please take the time to read these carefully. In the main body of the text we will skip many of the steps shown here and it is your responsibility to understand the arithmetic behind the computations we use throughout the text. We deliberately limited these examples to "nice" numbers (meaning that the numerators and denominators of the fractions are small integers) and will discuss more complicated matters later. In the upcoming example, we will make use of the Fundamental Theorem of Arithmetic which essentially says that every natural number has a unique prime factorization. Thus 'lowest terms' is clearly defined when reducing the fractions you're about to see.

Example 1.2.1. Perform the indicated operations and simplify. By 'simplify' here, we mean to have the final answer written in the form $\frac{a}{b}$ where a and b are integers which have no common factors. Said another way, we want $\frac{a}{b}$ in 'lowest terms'.

1. $\dfrac{1}{4}+\dfrac{6}{7}$
2. $\dfrac{5}{12}-\left(\dfrac{47}{30}-\dfrac{7}{3}\right)$
3. $\dfrac{\dfrac{7}{3-5}-\dfrac{7}{3-5.21}}{5-5.21}$
4. $\dfrac{\dfrac{12}{5}-\dfrac{7}{24}}{1+\left(\dfrac{12}{5}\right)\left(\dfrac{7}{24}\right)}$
5. $\dfrac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)}$
6. $\left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right)-\left(\dfrac{4}{5}\right)\left(-\dfrac{12}{13}\right)$

Solution.

1. It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no factors in common, the lowest common denominator is $4 \cdot 7 = 28$.

$$\begin{aligned}
\frac{1}{4}+\frac{6}{7} &= \frac{1}{4}\cdot\frac{7}{7}+\frac{6}{7}\cdot\frac{4}{4} && \text{Equivalent Fractions}\\
&= \frac{7}{28}+\frac{24}{28} && \text{Multiplication of Fractions}\\
&= \frac{31}{28} && \text{Addition of Fractions}
\end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

2. We could begin with the subtraction in parentheses, namely $\frac{47}{30}-\frac{7}{3}$, and then subtract that result from $\frac{5}{12}$. It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step.[4] The lowest common denominator[5] for all three fractions is 60.

$$\begin{aligned}
\frac{5}{12}-\left(\frac{47}{30}-\frac{7}{3}\right) &= \frac{5}{12}-\frac{47}{30}+\frac{7}{3} && \text{Distribute the Negative}\\
&= \frac{5}{12}\cdot\frac{5}{5}-\frac{47}{30}\cdot\frac{2}{2}+\frac{7}{3}\cdot\frac{20}{20} && \text{Equivalent Fractions}\\
&= \frac{25}{60}-\frac{94}{60}+\frac{140}{60} && \text{Multiplication of Fractions}\\
&= \frac{71}{60} && \text{Addition and Subtraction of Fractions}
\end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

[4] See the remark on page 17 about how we add $1+2+3$.

[5] We could have used $12 \cdot 30 \cdot 3 = 1080$ as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

3. What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction.[6] The longest division line[7] acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions). The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. To that end, we clean up the fractions in the numerator as follows.

$$\begin{aligned}
\frac{\frac{7}{3-5}-\frac{7}{3-5.21}}{5-5.21} &= \frac{\frac{7}{-2}-\frac{7}{-2.21}}{-0.21} \\
&= \frac{-\left(-\frac{7}{2}+\frac{7}{2.21}\right)}{0.21} && \text{Properties of Negatives} \\
&= \frac{\frac{7}{2}-\frac{7}{2.21}}{0.21} && \text{Distribute the Negative}
\end{aligned}$$

We are left with a compound fraction with decimals. We could replace 2.21 with $\frac{221}{100}$ but that would make a mess.[8] It's better in this case to eliminate the decimal by multiplying the numerator and denominator of the fraction with the decimal in it by 100 (since $2.21 \cdot 100 = 221$ is an integer) as shown below.

$$\frac{\frac{7}{2}-\frac{7}{2.21}}{0.21} = \frac{\frac{7}{2}-\frac{7\cdot 100}{2.21\cdot 100}}{0.21} = \frac{\frac{7}{2}-\frac{700}{221}}{0.21}$$

We now perform the subtraction in the numerator and replace 0.21 with $\frac{21}{100}$ in the denominator. This will leave us with one fraction divided by another fraction. We finish by performing the 'division by a fraction is multiplication by the reciprocal' trick and then cancel any factors that we can.

$$\begin{aligned}
\frac{\frac{7}{2}-\frac{700}{221}}{0.21} &= \frac{\frac{7}{2}\cdot\frac{221}{221}-\frac{700}{221}\cdot\frac{2}{2}}{\frac{21}{100}} = \frac{\frac{1547}{442}-\frac{1400}{442}}{\frac{21}{100}} \\
&= \frac{\frac{147}{442}}{\frac{21}{100}} = \frac{147}{442}\cdot\frac{100}{21} = \frac{14700}{9282} = \frac{350}{221}
\end{aligned}$$

The last step comes from the factorizations $14700 = 42 \cdot 350$ and $9282 = 42 \cdot 221$.

4. We are given another compound fraction to simplify and this time both the numerator and denominator contain fractions. As before, the longest division line acts as a grouping symbol to separate the

[6] Fractionception, perhaps?
[7] Also called a 'vinculum'.
[8] Try it if you don't believe us.

numerator from the denominator.

$$\frac{\frac{12}{5}-\frac{7}{24}}{1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}$$

Hence, one way to proceed is as before: simplify the numerator and the denominator then perform the 'division by a fraction is the multiplication by the reciprocal' trick. While there is nothing wrong with this approach, we'll use our Equivalent Fractions property to rid ourselves of the 'compound' nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the 'smaller' fractions - in this case, $24 \cdot 5 = 120$.

$$\begin{aligned}
\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5}-\frac{7}{24}\right)\cdot 120}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)\cdot 120} && \text{Equivalent Fractions} \\
&= \frac{\left(\frac{12}{5}\right)(120)-\left(\frac{7}{24}\right)(120)}{(1)(120)+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} && \text{Distributive Property} \\
&= \frac{\frac{12\cdot 120}{5}-\frac{7\cdot 120}{24}}{120+\frac{12\cdot 7\cdot 120}{5\cdot 24}} && \text{Multiply fractions} \\
&= \frac{\frac{12\cdot 24\cdot \cancel{5}}{\cancel{5}}-\frac{7\cdot 5\cdot \cancel{24}}{\cancel{24}}}{120+\frac{12\cdot 7\cdot \cancel{5}\cdot \cancel{24}}{\cancel{5}\cdot \cancel{24}}} && \text{Factor and cancel} \\
&= \frac{(12\cdot 24)-(7\cdot 5)}{120+(12\cdot 7)} \\
&= \frac{288-35}{120+84} \\
&= \frac{253}{204}
\end{aligned}$$

Since $253 = 11 \cdot 23$ and $204 = 2 \cdot 2 \cdot 3 \cdot 17$ have no common factors our result is in lowest terms which means we are done.

5. This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} = \frac{((2(2)+1)(-3-(-3))-5(4-7))}{(4-2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{aligned}
\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} &= \frac{(4+1)(-3+3)-5(-3)}{4-6} \\
&= \frac{(5)(0)+15}{-2} \\
&= \frac{15}{-2} \\
&= -\frac{15}{2} && \text{Properties of Negatives}
\end{aligned}$$

Since $15 = 3 \cdot 5$ and 2 have no common factors, we are done.

6. In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{aligned}
\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3\cdot 5}{5\cdot 13}-\frac{4\cdot(-12)}{5\cdot 13} && \text{Multiply fractions} \\
&= \frac{15}{65}-\frac{-48}{65} \\
&= \frac{15}{65}+\frac{48}{65} && \text{Properties of Negatives} \\
&= \frac{15+48}{65} && \text{Add numerators} \\
&= \frac{63}{65}
\end{aligned}$$

Since $64 = 3 \cdot 3 \cdot 7$ and $65 = 5 \cdot 13$ have no common factors, our answer $\frac{63}{65}$ is in lowest terms and we are done. □

Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favorite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that $2 \cdot 2 \cdot 2$ can be written as 2^3 because exponential notation expresses repeated multiplication. In the expression 2^3, 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1 \cdot (\text{three factors of two}) = 1 \cdot (2 \cdot 2 \cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1 \cdot (\text{zero factors of two}) = 1.$$

What about 2^{-3}? The '$-$' in the exponent indicates that we are 'taking away' three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1 \div (\text{three factors of two}) = 1 \div (2 \cdot 2 \cdot 2) = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

Properties of Integer Exponents

Suppose a and b are nonzero real numbers and n and m are integers.

- **Product Rules:** $(ab)^n = a^n b^n$ and $a^n a^m = a^{n+m}$.
- **Quotient Rules:** $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ and $\dfrac{a^n}{a^m} = a^{n-m}$.
- **Power Rule:** $(a^n)^m = a^{nm}$.
- **Negatives in Exponents:** $a^{-n} = \dfrac{1}{a^n}$.

  In particular, $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n = \dfrac{b^n}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$.
- **Zero Powers:** $a^0 = 1$.

  Note: The expression 0^0 is an indeterminate form.[a]
- **Powers of Zero:** For any *natural* number n, $0^n = 0$.

  Note: The expression 0^n for integers $n \leq 0$ is not defined.

[a]See the comment regarding '$\frac{0}{0}$' on page 18.

While it is important the state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that $(ab)^2 = a^2b^2$ (which some students refer to as 'distributing' the exponent to each factor) but you cannot do this sort of thing with addition. That is, in general, $(a + b)^2 \neq a^2 + b^2$. (For example, take $a = 3$ and $b = 4$.) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

Order of Operations Agreement

When evaluating an expression involving real numbers:

1. Evaluate any expressions in **p**arentheses (or other grouping symbols).
2. Evaluate **e**xponents.
3. Evaluate **m**ultiplication and **d**ivision as you read from left to right.
4. Evaluate **a**ddition and **s**ubtraction as you read from left to right.

We note that there are many useful mnemonic devices for remembering the order of operations.[a]

[a] Our favorite is '**P**lease **e**ntertain **m**y **d**ear **a**uld **S**asquatch.'

For example, $2 + 3 \cdot 4^2 = 2 + 3 \cdot 16 = 2 + 48 = 50$. Where students get into trouble is with things like -3^2. If we think of this as $0 - 3^2$, then it is clear that we evaluate the exponent first: $-3^2 = 0 - 3^2 = 0 - 9 = -9$. In general, we interpret $-a^n = -\left(a^n\right)$. If we want the 'negative' to also be raised to a power, we must write $(-a)^n$ instead. To summarize, $-3^2 = -9$ but $(-3)^2 = 9$.

Of course, many of the 'properties' we've stated in this section can be viewed as ways to circumvent the order of operations. We've already seen how the distributive property allows us to simplify $5(2 + x)$ by performing the indicated multiplication **before** the addition that's in parentheses. Similarly, consider trying to evaluate $2^{30172} \cdot 2^{-30169}$. The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute 2^{30172} is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get: $2^{30172-30169} = 2^3 = 8$.

Let's take a break and enjoy another example.

Example 1.2.2. Perform the indicated operations and simplify.

1. $\dfrac{(4-2)(2 \cdot 4) - (4)^2}{(4-2)^2}$
2. $12(-5)(-5+3)^{-4} + 6(-5)^2(-4)(-5+3)^{-5}$
3. $\dfrac{\left(\dfrac{5 \cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5 \cdot 3^{49}}{4^{34}}\right)}$
4. $\dfrac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}}$

Solution.

1. We begin working inside the parentheses then deal with the exponents before working through the other operations. As we saw in Example 1.2.1, the division here acts as a grouping symbol, so we

save the division to the end.

$$\begin{aligned}
\frac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2} &= \frac{(2)(8)-(4)^2}{(2)^2} = \frac{(2)(8)-16}{4} \\
&= \frac{16-16}{4} = \frac{0}{4} = 0
\end{aligned}$$

2. As before, we simplify what's in the parentheses first, then work our way through the exponents, multiplication, and finally, the addition.

$$\begin{aligned}
12(-5)(-5+3)^{-4}+6(-5)^2(-4)(-5+3)^{-5} &= 12(-5)(-2)^{-4}+6(-5)^2(-4)(-2)^{-5} \\
&= 12(-5)\left(\frac{1}{(-2)^4}\right)+6(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\
&= 12(-5)\left(\frac{1}{16}\right)+6(25)(-4)\left(\frac{1}{-32}\right) \\
&= (-60)\left(\frac{1}{16}\right)+(-600)\left(\frac{1}{-32}\right) \\
&= \frac{-60}{16}+\left(\frac{-600}{-32}\right) \\
&= \frac{-15\cdot\cancel{4}}{4\cdot\cancel{4}}+\frac{-75\cdot\cancel{8}}{-4\cdot\cancel{8}} \\
&= \frac{-15}{4}+\frac{-75}{-4} \\
&= \frac{-15}{4}+\frac{75}{4} \\
&= \frac{-15+75}{4} \\
&= \frac{60}{4} \\
&= 15
\end{aligned}$$

3. The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction. This gives us hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular) to help us out. Our first step here is to invert and multiply. We see immediately that the 5's cancel after which we group the powers of 3

together and the powers of 4 together and apply the properties of exponents.

$$\begin{aligned}
\frac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)} &= \frac{5\cdot 3^{51}}{4^{36}}\cdot\frac{4^{34}}{5\cdot 3^{49}} &&= \frac{\cancel{5}\cdot 3^{51}\cdot 4^{34}}{\cancel{5}\cdot 3^{49}\cdot 4^{36}} &&= \frac{3^{51}}{3^{49}}\cdot\frac{4^{34}}{4^{36}} \\
&= 3^{51-49}\cdot 4^{34-36} &&= 3^{2}\cdot 4^{-2} &&= 3^{2}\cdot\left(\frac{1}{4^{2}}\right) \\
&= 9\cdot\left(\frac{1}{16}\right) &&= \frac{9}{16}
\end{aligned}$$

4. We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we did in Example 1.2.1. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to simplify the fraction.

$$\begin{aligned}
\frac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}} &= \frac{2\left(\dfrac{12}{5}\right)}{1-\left(\dfrac{12}{5}\right)^{2}} &&= \frac{\left(\dfrac{24}{5}\right)}{1-\left(\dfrac{12^{2}}{5^{2}}\right)} &&= \frac{\left(\dfrac{24}{5}\right)}{1-\left(\dfrac{144}{25}\right)} \\
&= \frac{\left(\dfrac{24}{5}\right)\cdot 25}{\left(1-\dfrac{144}{25}\right)\cdot 25} &&= \frac{\left(\dfrac{24\cdot 5\cdot\cancel{5}}{\cancel{5}}\right)}{\left(1\cdot 25-\dfrac{144\cdot\cancel{25}}{\cancel{25}}\right)} &&= \frac{120}{25-144} \\
&= \frac{120}{-119} = -\frac{120}{119}
\end{aligned}$$

Since 120 and 119 have no common factors, we are done. □

One of the places where the properties of exponents play an important role is in the use of **Scientific Notation**. The basis for scientific notation is that since we use <u>dec</u>imals (base ten numerals) to represent real numbers, we can adjust where the decimal point lies by multiplying by an appropriate power of 10. This allows scientists and engineers to focus in on the 'significant' digits[9] of a number - the nonzero values - and adjust for the decimal places later. For instance, $-621 = -6.21 \times 10^{2}$ and $0.023 = 2.3 \times 10^{-2}$. Notice here that we revert to using the familiar '$\times$' to indicate multiplication.[10] In general, we arrange the real number so exactly one non-zero digit appears to the left of the decimal point. We make this idea precise in the following:

Definition 1.7. A real number is written in **Scientific Notation** if it has the form $\pm n.d_1 d_2 \ldots \times 10^{k}$ where n is a natural number, d_1, d_2, etc., are whole numbers, and k is an integer.

[9]Awesome pun!

[10]This is the 'notable exception' we alluded to earlier.

On calculators, scientific notation may appear using an 'E' or 'EE' as opposed to the $\times$ symbol. For instance, while we will write 6.02×10^{23} in the text, the calculator may display 6.02 E 23 or 6.02 EE 23.

Example 1.2.3. Perform the indicated operations and simplify. Write your final answer in scientific notation, rounded to two decimal places.

1. $\dfrac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}}$
2. $\left(2.13 \times 10^{53}\right)^{100}$

Solution.

1. As mentioned earlier, the point of scientific notation is to separate out the 'significant' parts of a calculation and deal with the powers of 10 later. In that spirit, we separate out the powers of 10 in both the numerator and the denominator and proceed as follows

$$\begin{aligned}
\frac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}} &= \frac{(6.626)(3.14)}{1.78} \cdot \frac{10^{-34} \cdot 10^{9}}{10^{23}} \\
&= \frac{20.80564}{1.78} \cdot \frac{10^{-34+9}}{10^{23}} \\
&= 11.685 \ldots \cdot \frac{10^{-25}}{10^{23}} \\
&= 11.685 \ldots \times 10^{-25-23} \\
&= 11.685 \ldots \times 10^{-48}
\end{aligned}$$

We are asked to write our final answer in scientific notation, rounded to two decimal places. To do this, we note that $11.685 \ldots = 1.1685 \ldots \times 10^{1}$, so

$$11.685 \ldots \times 10^{-48} = 1.1685 \ldots \times 10^{1} \times 10^{-48} = 1.1685 \ldots \times 10^{1-48} = 1.1685 \ldots \times 10^{-47}$$

Our final answer, rounded to two decimal places, is 1.17×10^{-47}.

We could have done that whole computation on a calculator so why did we bother doing any of this by hand in the first place? The answer lies in the next example.

2. If you try to compute $\left(2.13 \times 10^{53}\right)^{100}$ using most hand-held calculators, you'll most likely get an 'overflow' error. It is possible, however, to use the calculator in combination with the properties of exponents to compute this number. Using properties of exponents, we get:

$$\begin{aligned}
\left(2.13 \times 10^{53}\right)^{100} &= (2.13)^{100}\left(10^{53}\right)^{100} \\
&= \left(6.885 \ldots \times 10^{32}\right)\left(10^{53 \times 100}\right) \\
&= \left(6.885 \ldots \times 10^{32}\right)\left(10^{5300}\right) \\
&= 6.885 \ldots \times 10^{32} \cdot 10^{5300} \\
&= 6.885 \ldots \times 10^{5332}
\end{aligned}$$

To two decimal places our answer is 6.88×10^{5332}. $\square$

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

Definition 1.8. Let a be a real number and let n be a natural number. If n is odd, then the **principal $\mathbf{n^{th}}$ root** of a (denoted $\sqrt[n]{a}$) is the unique real number satisfying $\left(\sqrt[n]{a}\right)^n = a$. If n is even, $\sqrt[n]{a}$ is defined similarly provided $a \geq 0$ and $\sqrt[n]{a} \geq 0$. The number n is called the **index** of the root and the number a is called the **radicand**. For $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.

The reasons for the added stipulations for even-indexed roots in Definition 1.8 can be found in the Properties of Negatives. First, for all real numbers, $x^{\text{even power}} \geq 0$, which means it is never negative. Thus if a is a *negative* real number, there are no real numbers x with $x^{\text{even power}} = a$. This is why if n is even, $\sqrt[n]{a}$ only exists if $a \geq 0$. The second restriction for even-indexed roots is that $\sqrt[n]{a} \geq 0$. This comes from the fact that $x^{\text{even power}} = (-x)^{\text{even power}}$, and we require $\sqrt[n]{a}$ to have just one value. So even though $2^4 = 16$ and $(-2)^4 = 16$, we require $\sqrt[4]{16} = 2$ and ignore -2.

Dealing with odd powers is much easier. For example, $x^3 = -8$ has one and only one real solution, namely $x = -2$, which means not only does $\sqrt[3]{-8}$ exist, there is only one choice, namely $\sqrt[3]{-8} = -2$. Of course, when it comes to solving $x^{5213} = -117$, it's not so clear that there is one and only one real solution, let alone that the solution is $\sqrt[5213]{-117}$. Such pills are easier to swallow once we've thought a bit about such equations graphically,[11] and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' as opposed to a definition since they can be justified using the properties of exponents.

Theorem 1.1. Properties of Radicals: Let a and b be real numbers and let m and n be natural numbers. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
- **Quotient Rule:** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, provided $b \neq 0$.
- **Power Rule:** $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

The proof of Theorem 1.1 is based on the definition of the principal n$^{\text{th}}$ root and the Properties of Exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{ab}$ is the <u>unique</u> real number such that $(\sqrt[n]{ab})^n = ab$. Given that $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$ as well, it must be the case that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. If n is even, then $\sqrt[n]{ab}$ is the unique non-negative real number such that $(\sqrt[n]{ab})^n = ab$. Note that since n is even, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also non-negative thus $\sqrt[n]{a}\sqrt[n]{b} \geq 0$ as well. Proceeding as above, we find that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{a}$ is a real number to start with.[12] We leave that as an exercise as well.

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[11] See Chapter 6.

[12] Otherwise we'd run into an interesting paradox. See Section **??**.

We pause here to point out one of the most common errors students make when working with radicals. Obviously $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{9+16} = \sqrt{25} = 5$. Thus we can clearly see that $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3+4 = 7$ because we all know that $5 \neq 7$. The authors urge you to never consider 'distributing' roots or exponents. It's wrong and no good will come of it because in general $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Since radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

Definition 1.9. Let a be a real number, let m be an integer and let n be a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ whenever $\sqrt[n]{a}$ is a real number.[a]
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ whenever $\sqrt[n]{a}$ is a real number.

[a]If n is even we need $a \geq 0$.

It would make life really nice if the rational exponents defined in Definition 1.9 had all of the same properties that integer exponents have as listed on page 25 - but they don't. Why not? Let's look at an example to see what goes wrong. Consider the Product Rule which says that $(ab)^n = a^n b^n$ and let $a = -16$, $b = -81$ and $n = \frac{1}{4}$. Plugging the values into the Product Rule yields the equation $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$. The left side of this equation is $1296^{1/4}$ which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they apply to is non-negative? That works for some of the rules - we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression $\left(a^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(a^{2/3}\right)^{3/2} = a^{\frac{2}{3}\cdot\frac{3}{2}} = a^1 = a$. However, if we substitute $a = -1$ and apply Definition 1.9, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. Thus in this case we have $\left(a^{2/3}\right)^{3/2} \neq a$ even though all of the roots were defined. It is true, however, that $\left(a^{3/2}\right)^{2/3} = a$ and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.[13]

Example 1.2.4. Perform the indicated operations and simplify.

1. $\dfrac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$
2. $\dfrac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1 - \left(\dfrac{\sqrt{3}}{3}\right)^2}$
3. $\left(\sqrt[3]{-2} - \sqrt[3]{-54}\right)^2$
4. $2\left(\dfrac{9}{4} - 3\right)^{1/3} + 2\left(\dfrac{9}{4}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{9}{4} - 3\right)^{-2/3}$

[13]Much to Jeff's chagrin. He's fairly traditional and therefore doesn't care much for radicals.

Solution.

1. We begin in the numerator and note that the radical here acts a grouping symbol,[14] so our first order of business is to simplify the radicand.

$$\begin{aligned}
\frac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)} &= \frac{-(-4) - \sqrt{16 - 4(2)(-3)}}{2(2)} \\
&= \frac{-(-4) - \sqrt{16 - 4(-6)}}{2(2)} \\
&= \frac{-(-4) - \sqrt{16 - (-24)}}{2(2)} \\
&= \frac{-(-4) - \sqrt{16 + 24}}{2(2)} \\
&= \frac{-(-4) - \sqrt{40}}{2(2)}
\end{aligned}$$

   As you may recall, 40 can be factored using a perfect square as $40 = 4 \cdot 10$ so we use the product rule of radicals to write $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$. This lets us factor a '2' out of both terms in the numerator, eventually allowing us to cancel it with a factor of 2 in the denominator.

$$\begin{aligned}
\frac{-(-4) - \sqrt{40}}{2(2)} &= \frac{-(-4) - 2\sqrt{10}}{2(2)} &&= \frac{4 - 2\sqrt{10}}{2(2)} \\
&= \frac{2 \cdot 2 - 2\sqrt{10}}{2(2)} &&= \frac{2(2 - \sqrt{10})}{2(2)} \\
&= \frac{\cancel{2}(2 - \sqrt{10})}{\cancel{2}(2)} &&= \frac{2 - \sqrt{10}}{2}
\end{aligned}$$

   Since the numerator and denominator have no more common factors,[15] we are done.

2. Once again we have a compound fraction, so we first simplify the exponent in the denominator to see which factor we'll need to multiply by in order to clean up the fraction.

[14] The line extending horizontally from the square root symbol $\sqrt{\ }$ is, you guessed it, another vinculum.
[15] Do you see why we aren't 'canceling' the remaining 2's?

$$\begin{aligned}
\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{(\sqrt{3})^2}{3^2}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{3}{9}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1\cdot\cancel{3}}{3\cdot\cancel{3}}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1}{3}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)\cdot 3}{\left(1-\left(\frac{1}{3}\right)\right)\cdot 3} &&= \frac{\frac{2\cdot\sqrt{3}\cdot\cancel{3}}{\cancel{3}}}{1\cdot 3-\frac{1\cdot\cancel{3}}{\cancel{3}}} \\
&= \frac{2\sqrt{3}}{3-1} &&= \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}
\end{aligned}$$

3. Working inside the parentheses, we first encounter $\sqrt[3]{-2}$. While the -2 isn't a perfect cube,[16] we may think of $-2 = (-1)(2)$. Since $(-1)^3 = -1$, which *is* a perfect cube, we may write $\sqrt[3]{-2} = \sqrt[3]{(-1)(2)} = \sqrt[3]{-1}\sqrt[3]{2} = -\sqrt[3]{2}$. When it comes to $\sqrt[3]{54}$, we may write it as $\sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$. So,

$$\sqrt[3]{-2} - \sqrt[3]{-54} = -\sqrt[3]{2} - (-3\sqrt[3]{2}) = -\sqrt[3]{2} + 3\sqrt[3]{2}.$$

At this stage, we can simplify $-\sqrt[3]{2}+3\sqrt[3]{2} = 2\sqrt[3]{2}$. You may remember this as being called 'combining like radicals,' but it is in fact just another application of the distributive property:

$$-\sqrt[3]{2} + 3\sqrt[3]{2} = (-1)\sqrt[3]{2} + 3\sqrt[3]{2} = (-1+3)\sqrt[3]{2} = 2\sqrt[3]{2}.$$

Putting all this together, we get:

$$\begin{aligned}
(\sqrt[3]{-2} - \sqrt[3]{-54})^2 &= (-\sqrt[3]{2} + 3\sqrt[3]{2})^2 &&= (2\sqrt[3]{2})^2 \\
&= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} &&= 4\sqrt[3]{4}
\end{aligned}$$

There are no perfect integer cubes which are factors of 4 (apart from 1, of course), so we are done.

[16] Of an integer, that is!

4. We start working in the parentheses and get a common denominator to subtract the fractions:

$$\frac{9}{4} - 3 = \frac{9}{4} - \frac{3 \cdot 4}{1 \cdot 4} = \frac{9}{4} - \frac{12}{4} = \frac{-3}{4}$$

The denominators in the fractional exponents are odd, so we can proceed by using the properties of exponents:

$$\begin{aligned}
2\left(\frac{9}{4} - 3\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4} - 3\right)^{-2/3} &= 2\left(\frac{-3}{4}\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)^{-2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{-3}\right)^{2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{(4)^{2/3}}{(-3)^{2/3}}\right) \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 9 \cdot 1 \cdot 4^{2/3}}{4 \cdot 3 \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{\cancel{2} \cdot 3 \cdot \cancel{3} \cdot 4^{2/3}}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}}
\end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Since $4 = 2^2$, $4^{1/3} = (2^2)^{1/3} = 2^{2/3}$. Similarly, $4^{2/3} = (2^2)^{2/3} = 2^{4/3}$. The expressions $(-3)^{1/3}$ and $(-3)^{2/3}$ contain negative bases so we proceed with caution and convert them back to radical notation to get: $(-3)^{1/3} = \sqrt[3]{-3} = -\sqrt[3]{3} = -3^{1/3}$ and $(-3)^{2/3} = (\sqrt[3]{-3})^2 = (-\sqrt[3]{3})^2 = (\sqrt[3]{3})^2 = 3^{2/3}$. Hence:

$$\begin{aligned}
\frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} &= \frac{2 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3 \cdot 2^{4/3}}{2 \cdot 3^{2/3}} \\
&= \frac{2^1 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3^1 \cdot 2^{4/3}}{2^1 \cdot 3^{2/3}} \\
&= 2^{1-2/3} \cdot (-3^{1/3}) + 3^{1-2/3} \cdot 2^{4/3-1} \\
&= 2^{1/3} \cdot (-3^{1/3}) + 3^{1/3} \cdot 2^{1/3} \\
&= -2^{1/3} \cdot 3^{1/3} + 3^{1/3} \cdot 2^{1/3} \\
&= 0
\end{aligned}$$

□

We close this section with a note about simplifying. In the preceding examples we used "nice" numbers because we wanted to show as many properties as we could per example. This then begs the question "What happens when the numbers are *not* nice?" Unfortunately, the answer is "Not much simplifying can be done." Take, for example,

$$\frac{\sqrt{7}}{\pi} - \frac{3}{\pi^2} + \frac{4}{\sqrt{11}} = \frac{\pi\sqrt{77} - 3\sqrt{11} + 4\pi^2}{\pi^2\sqrt{11}}$$

Sadly, that's as good as it gets.

1.2.1 Exercises

In Exercises 1 - 33, perform the indicated operations and simplify.

1. $5 - 2 + 3$
2. $5 - (2 + 3)$
3. $\dfrac{2}{3} - \dfrac{4}{7}$
4. $\dfrac{3}{8} + \dfrac{5}{12}$
5. $\dfrac{5-3}{-2-4}$
6. $\dfrac{2(-3)}{3-(-3)}$
7. $\dfrac{2(3)-(4-1)}{2^2+1}$
8. $\dfrac{4-5.8}{2-2.1}$
9. $\dfrac{1-2(-3)}{5(-3)+7}$
10. $\dfrac{5(3)-7}{2(3)^2-3(3)-9}$
11. $\dfrac{2((-1)^2-1)}{((-1)^2+1)^2}$
12. $\dfrac{(-2)^2-(-2)-6}{(-2)^2-4}$
13. $\dfrac{3-\frac{4}{9}}{-2-(-3)}$
14. $\dfrac{\frac{2}{3}-\frac{4}{5}}{4-\frac{7}{10}}$
15. $\dfrac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}$
16. $\dfrac{1-\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1+\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$
17. $\left(\dfrac{2}{3}\right)^{-5}$
18. $3^{-1} - 4^{-2}$
19. $\dfrac{1+2^{-3}}{3-4^{-1}}$
20. $\dfrac{3 \cdot 5^{100}}{12 \cdot 5^{98}}$
21. $\sqrt{3^2+4^2}$
22. $\sqrt{12} - \sqrt{75}$
23. $(-8)^{2/3} - 9^{-3/2}$
24. $\left(-\frac{32}{9}\right)^{-3/5}$
25. $\sqrt{(3-4)^2+(5-2)^2}$
26. $\sqrt{(2-(-1))^2+\left(\frac{1}{2}-3\right)^2}$
27. $\sqrt{(\sqrt{5}-2\sqrt{5})^2+(\sqrt{18}-\sqrt{8})^2}$
28. $\dfrac{-12+\sqrt{18}}{21}$
29. $\dfrac{-2-\sqrt{(2)^2-4(3)(-1)}}{2(3)}$
30. $\dfrac{-(-4)+\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$
31. $2(-5)(-5+1)^{-1} + (-5)^2(-1)(-5+1)^{-2}$
32. $3\sqrt{2(4)+1} + 3(4)\left(\frac{1}{2}\right)(2(4)+1)^{-1/2}(2)$
33. $2(-7)\sqrt[3]{1-(-7)} + (-7)^2\left(\frac{1}{3}\right)(1-(-7))^{-2/3}(-1)$
34. With the help of your calculator, find $(3.14 \times 10^{87})^{117}$. Write your final answer, using scientific notation, rounded to two decimal places. (See Example 1.2.3.)
35. Prove the Quotient Rule and Power Rule stated in Theorem 1.1.
36. Discuss with your classmates how you might attempt to simplify the following.

    (a) $\sqrt{\dfrac{1-\sqrt{2}}{1+\sqrt{2}}}$

    (b) $\sqrt[5]{3} - \sqrt[3]{5}$

    (c) $\dfrac{\pi+7}{\pi}$

1.2.2 Answers

1. 6
2. 0
3. $\dfrac{2}{21}$
4. $\dfrac{19}{24}$
5. $-\dfrac{1}{3}$
6. -1
7. $\dfrac{3}{5}$
8. 18
9. $-\dfrac{7}{8}$
10. Undefined.
11. 0
12. Undefined.
13. $\dfrac{23}{9}$
14. $-\dfrac{4}{99}$
15. $-\dfrac{24}{7}$
16. 0
17. $\dfrac{243}{32}$
18. $\dfrac{13}{48}$
19. $\dfrac{9}{22}$
20. $\dfrac{25}{4}$
21. 5
22. $-3\sqrt{3}$
23. $\dfrac{107}{27}$
24. $-\dfrac{3\sqrt[5]{3}}{8} = -\dfrac{3^{6/5}}{8}$
25. $\sqrt{10}$
26. $\dfrac{\sqrt{61}}{2}$
27. $\sqrt{7}$
28. $\dfrac{-4+\sqrt{2}}{7}$
29. -1
30. $2+\sqrt{5}$
31. $\dfrac{15}{16}$
32. 13
33. $-\dfrac{385}{12}$
34. 1.38×10^{10237}

Chapter 2

Introduction to Functions

2.1 Functions and their Representations

2.1.1 Functions as Mappings

Mathematics can be thought of as the study of patterns. In most disciplines, Mathematics is used as a language to express, or codify, relationships between quantities - both algebraically and geometrically - with the ultimate goal of solving real-world problems. The fact that the same algebraic equation which models the growth of bacteria in a petri dish is also used to compute the account balance of a savings account or the potency of radioactive material used in medical treatments speaks to the universal nature of Mathematics. Indeed, Mathematics is more than just about solving a specific problem in a specific situation, it's about abstracting problems and creating universal tools which can be used by a variety of scientists and engineers to solve a variety of problems.

This power of abstraction has a tendency to create a language that is initially intimidating to students. Mathematical definitions are precise and adherence to that precision is often a source of confusion and frustration. It doesn't help matters that more often than not very common words are used in Mathematics with slightly different definitions than is commonly expected. The first 'universal tool' we wish to highlight - the concept of a 'function' - is a perfect example of this phenomenon in that we redefine a word that already has multiple meanings in English.

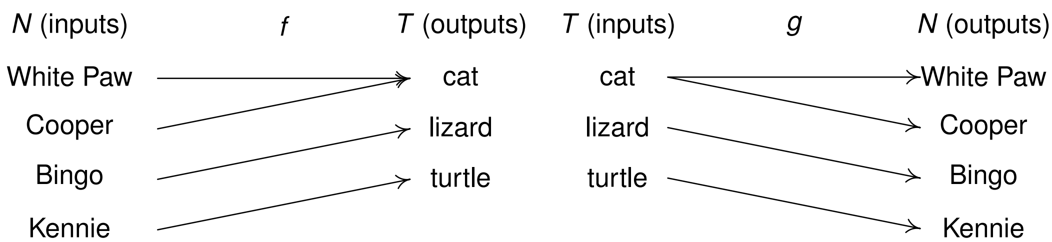
Definition 2.1. Given two sets[a] A and B, a **function** from A to B is a process by which each element of A is matched with (or 'mapped to') one and only one element of B.

[a] Please refer to Section 1.1 for a review of this terminology.

The grammar here '*from A to B*' is important. Thinking of a function as a process, we can view the elements of the set A as our starting materials, or *inputs* to the process. The function processes these inputs according to some specified rule and the result is a set of *outputs* - elements of the set B. In terms of inputs and outputs, Definition 2.1 says that a function is a process in which each *input* is matched to one and only one *output*.

For example, let's take a look at some of the pets in the Stitz household. Taylor's pets include White Paw and Cooper (both cats), Bingo (a lizard) and Kennie (a turtle). Let N be the set of pet names: $N = \{$White Paw, Cooper, Bingo, Kennie$\}$, and let T be the set of pet types: $T = \{$cat, lizard, turtle$\}$. Let f be the process that takes each pet's name as the input and returns that pet's type as the output. Let g be the reverse of f: that is, g takes each pet type as the input and returns the names of the pets of that type as the output. Note that both f and g are codifying the *same* given information about Taylor's pets, but one of them is a function and the other is not.

To help identify which process f or g is a function and why the other is not, we create **mapping diagrams** for f and g below. In each case, we organize the inputs in a column on the left and the outputs on a column on the right. We draw an arrow connecting each input to its corresponding output(s). Note that the arrows communicate the grammatical bias: the arrow originates at the input and points to the output.



The process f is a function since f matches each of its inputs (each pet name) to just one output (the pet's type). The fact that different inputs (White Paw and Cooper) are matched to the same output (cat) is fine. On the other hand, g matches the input 'cat' to the two different outputs 'White Paw' and 'Cooper', so g is not a function. Functions are favored in mathematical circles because they are processes which produce only one answer (output) for any given query (input). In this scenario, for instance, there is only one answer to the question: 'What type of pet is White Paw?' but there is more than one answer to the question 'Which of Taylor's pets are cats?'

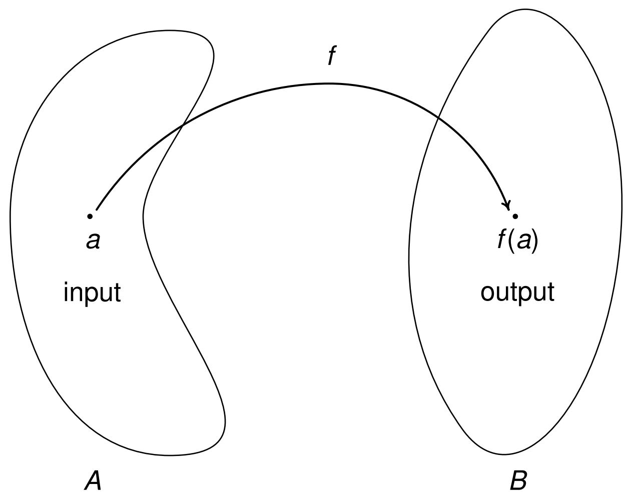
As you might expect, with functions being such an important concept in Mathematics, we need to build a vocabulary to assist us when discussing them. To that end, we have the following definitions.[1]

Definition 2.2. Suppose f is a function from A to B.

- If $a \in A$, we write $f(a)$ (read 'f of a') to denote the unique element of B to which f matches a.
  That is, if we view 'a' as the input to f, then '$f(a)$' is the output from f.
- The set A is called the **domain**.
  Said differently, the domain of a function is the set of inputs to the function.
- The set $\{f(a) \mid a \in A\}$ is called the **range** of f.
  Said differently, the range of a function is the set of outputs from the function.

[1] Please refer to Section 1.1 for a review of the terminology used in these definitions.

Some remarks about Definition 2.2 are in order. First, and most importantly, the notation '$f(a)$' in Definition 2.2 introduces yet another mathematical use for parentheses. Parentheses are used in some cases as grouping symbols, to represent ordered pairs, and to delineate intervals of real numbers. More often than not, the use of parentheses in expressions like '$f(a)$' is confused with multiplication. As always, paying attention to the context is key. If f is a function and 'a' is in the domain of f, then '$f(a)$' is the output from f when you input a. The diagram below provides a nice generic picture to keep in mind when thinking of a function as a mapping process with input 'a' and output '$f(a)$'.



In the preceding pet example, the symbol f(Bingo), read 'f of Bingo', is asking what type of pet Bingo is, so f(Bingo) = lizard. The fact that f is a function means f(Bingo) is unambiguous because f matches the name 'Bingo' to only one pet type, namely 'lizard'. In contrast, if we tried to use the notation 'g(cat)' to indicate what pet name g matched to 'cat', we have *two* possibilities, White Paw and Cooper, with no way to determine which one (or both) is indicated.

Continuing to apply Definition 2.2 to our pet example, we find that the domain of the function f is N, the set of pet names. Finding the range takes a little more work, mostly because it's easy to be caught off guard by the notation used in the definition of 'range'. The description of the range as '$\{f(a) \mid a \in A\}$' is an example of 'set-builder' notation. In English, '$\{f(a) \mid a \in A\}$' reads as 'the set of $f(a)$ such that a is in A'. In other words, the range consists of all of the outputs from f - all of the $f(a)$ values - as a varies through each of the elements in the domain A. Note that while every element of the set A is, by definition, an element of the domain of f, not every element of the set B is necessarily part of the range of f.[2]

In our pet example, we can obtain the range of f by looking at the mapping diagram or by constructing the set $\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\}$ which lists all of the outputs from f as we run through all of the inputs to f. Keep in mind that we list each element of a set only once so the range of f is:[3]

$$\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\} = \{\text{cat}, \text{lizard}, \text{turtle}\} = T.$$

[2] For purposes of completeness, the set B is called the **codomain** of f. For us, the concepts of domain and range suffice since our codomain will most always be the set of real numbers, $\mathbb{R}$.

[3] If instead of mapping N into T, we could have mapped N into $U = \{\text{cat}, \text{lizard}, \text{turtle}, \text{dog}\}$ in which case the range of f would not have been the entire codomain U.

If we let n denote a generic element of N then $f(n)$ is some element t in T, so we write $t = f(n)$. In this equation, n is called the **independent variable** and t is called the **dependent variable**.[4] Moreover, we say 't is a function of n', or, more specifically, 'the type of pet is a function of the pet name' meaning that every pet name n corresponds to one, and only one, pet type t. Even though f and t are different things,[5] it is very common for the function and its outputs to become more-or-less synonymous, even in what are otherwise precise mathematical definitions.[6] We will endeavor to point out such ambiguities as we move through the text.

While the concept of a function is very general in scope, we will be focusing primarily on functions of real numbers because most disciplines use real numbers to quantify data. Our next example explores a function defined using a table of numerical values.

Example 2.1.1. Suppose Skippy records the outdoor temperature every two hours starting at 6 a.m. and ending at 6 p.m. and summarizes the data in the table below:

time (hours after 6 a.m.)	outdoor temperature in degrees Fahrenheit
0	64
2	67
4	75
6	80
8	83
10	83
12	82

1. Explain why the recorded outdoor temperature is a function of the corresponding time.
2. Is time a function of the outdoor temperature? Explain.
3. Let f be the function which matches time to the corresponding recorded outdoor temperature.
   (a) Find and interpret the following:
      - $f(2)$
      - $f(4)$
      - $f(2+4)$
      - $f(2)+f(4)$
      - $f(2)+4$

   (b) Solve and interpret $f(t) = 83$.
   (c) State the range of f. What is lowest recorded temperature of the day? The highest?

[4]These adjectives stem from the fact that the value of t *depends* entirely on our (independent) choice of n.
[5]Specifically, f is a function so it requires and domain, a range and a rule of assignment whereas t is simply the output from f.
[6]In fact, it is not uncommon to see the name of the function as the same as the dependent variable. For example, writing '$y = y(x)$' would be a way to communicate the idea that 'y is a function of x'.

Solution.

1. The outdoor temperature is a function of time because each time value is associated with only one recorded temperature.

2. Time is not a function of the outdoor temperature because there are instances when different times are associated with a given temperature. For example, the temperature 83 corresponds to both of the times 8 and 10.

3. (a)
   - To find $f(2)$, we look in the table to find the recorded outdoor temperature that corresponds to when the time is 2. We get $f(2) = 67$ which means that 2 hours after 6 a.m. (i.e., at 8 a.m.), the temperature is 67°F.
   - Per the table, $f(4) = 75$, so the recorded outdoor temperature at 10 a.m. (4 hours after 6 a.m.) is 75°F.
   - From the table, we find $f(2 + 4) = f(6) = 80$, which means that at noon (6 hours after 6 a.m.), the recorded outdoor temperature is 80°F.
   - Using results from above we see that $f(2) + f(4) = 67 + 75 = 142$. When adding $f(2) + f(4)$, we are adding the recorded outdoor temperatures at 8 a.m. (2 hours after 6 a.m.) and 10 a.m. (4 hours after 6 AM), respectively, to get 142°F.
   - We compute $f(2) + 4 = 67 + 4 = 71$. Here, we are adding 4°F to the outdoor temperature recorded at 8 a.m..

   (b) Solving $f(t) = 83$ means finding all of the input (time) values t which produce an output value of 83. From the data, we see that the temperature is 83 when the time is 8 or 10, so the solution to $f(t) = 83$ is $t = 8$ or $t = 10$. This means the outdoor temperature is 83°F at 2 p.m. (8 hours after 6 a.m.) and at 4 p.m. (10 hours after 6 a.m.).

   (c) The range of f is the set of all of the outputs from f, or in this case, the outside recorded temperatures. Based on the data, we get $\{64, 67, 75, 80, 82, 83\}$. (Here again, we list elements of a set only once.) The lowest recorded temperature of the day is 64°F and the highest recorded temperature of the day is 83°F. □

A few remarks about Example 2.1.1 are in order. First, note that $f(2 + 4)$, $f(2) + f(4)$ and $f(2) + 4$ all work out to be numerically different, and more importantly, all represent different things.[7] One of the common mistakes students make is to misinterpret expressions like these, so it's important to pay close attention to the syntax here.

Next, when solving $f(t) = 83$, the variable 't' is being used as a convenient 'dummy' variable or placeholder in the sense that solving $f(t) = 83$ produces the same solutions as solving $f(x) = 83$, $f(w) = 83$, or even $f(?) = 83$. All of these equations are asking for the same thing: what inputs to f produce an output of 83. The choice of the letter 't' here makes sense since the inputs are **t**ime values. Throughout the text, we will endeavor to use meaningful labels when working in applied situations, but the fact remains that the choice of letters (or symbols) is completely arbitrary.

[7] You may be wondering why one would ever compute these quantities. Rest assured that we will use expressions like these in examples throughout the text. For now, it suffices just to know that they are different.

Finally, given that the range in this example was a finite set of real numbers, we could find the smallest and largest elements of it. Here, they correspond to the coolest and warmest temperatures of the day, respectively, but the meaning would change if the function related different quantities. In many applications involving functions, the end goal is to find the minimum or maximum values of the outputs of those functions (called **optimizing** the function) so for that reason, we have the following definition.

Definition 2.3. Suppose f is a function whose range is a set of real numbers containing m and M.

- The value m is called the **minimum**[a] of f if $m \leq f(x)$ for all x in the domain of f.

  That is, the minimum of f is the smallest output from f, if it exists.

- The value M is called the **maximum**[b] of f if $f(x) \leq M$ for all x in the domain of f.

  That is, the maximum of f is the largest output from f, if it exists.

- Taken together, the values m and M (if they exist) are called the **extrema**[c] of f.

[a] also called ‘absolute’ or ‘global’ minimum
[b] also called ‘absolute’ or ‘global’ maximum
[c] also called the ‘absolute’ or ‘global’ extrema or the ‘extreme values’

Definition 2.3 is an example where the name of the function, f, is being used almost synonymously with its outputs in that when we speak of ‘the minimum and maximum of the *function* f’ we are really talking about the minimum and maximum values of the *outputs* $f(x)$ as x varies through the domain of f. Thus we say that the maximum of f is 83 and the minimum of f is 64 when referring to the highest and lowest recorded temperatures in the previous example.

2.1.2 Algebraic Representations of Functions

By focusing our attention to functions that involve real numbers, we gain access to all of the structures and tools from prior courses in Algebra. In this subsection, we discuss how to represent functions algebraically using formulas and begin with the following example.

Example 2.1.2.

1. Let f be the function which takes a real number and performs the following sequence of operations:

   - Step 1: add 2
   - Step 2: multiply the result of Step 1 by 3
   - Step 3: subtract 1 from the result of Step 2.

   (a) Find and simplify $f(-5)$.

   (b) Find and simplify a formula for $f(x)$.

2. Let $h(t) = -t^2 + 3t + 4$.

   (a) Find and simplify the following:

      i. $h(-1)$, $h(0)$ and $h(2)$.
      ii. $h(2x)$ and $2h(x)$.
      iii. $h(t+2)$, $h(t)+2$ and $h(t)+h(2)$.

   (b) Solve $h(t) = 0$.

Solution.

1. (a) We take -5 and follow it through each step:

      - Step 1: adding 2 gives us $-5 + 2 = -3$.
      - Step 2: multiplying the result of Step 1 by 3 yields $(-3)(3) = -9$.
      - Step 3: subtracting 1 from the result of Step 2 produces $-9 - 1 = -10$.

      Hence, $f(-5) = -10$.

   (b) To find a formula for $f(x)$, we repeat the above process but use the variable 'x' in place of the number -5:

      - Step 1: adding 2 gives us the quantity $x + 2$.
      - Step 2: multiplying the result of Step 1 by 3 yields $(x + 2)(3) = 3x + 6$.
      - Step 3: subtracting 1 from the result of Step 2 produces $(3x + 6) - 1 = 3x + 5$.

      Hence, we have codified f using the formula $f(x) = 3x + 5$. In other words, the function f matches each real number 'x' with the value of the expression '$3x + 5$'. As a partial check of our answer, we use this formula to find $f(-5)$. We compute $f(-5)$ by substituting $x = -5$ into the formula $f(x)$ and find $f(-5) = 3(-5) + 5 = -10$ as before.

2. As before, representing the function h as $h(t) = -t^2 + 3t + 4$ means that h matches the real number t with the value of the expression $-t^2 + 3t + 4$.

   (a) To find $h(-1)$, we substitute -1 for t in the expression $-t^2 + 3t + 4$. It is highly recommended that you be generous with parentheses here in order to avoid common mistakes:

   $$\begin{aligned} h(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0. \end{aligned}$$

   Similarly, $h(0) = -(0)^2 + 3(0) + 4 = 4$, and $h(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$.

   (b) To find $h(2x)$, we substitute $2x$ for t:

   $$\begin{aligned} h(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4. \end{aligned}$$

The expression $2h(x)$ means that we multiply the expression $h(x)$ by 2. We first get $h(x)$ by substituting x for t: $h(x) = -x^2 + 3x + 4$. Hence,

$$\begin{aligned} 2h(x) &= 2\left(-x^2 + 3x + 4\right) \\ &= -2x^2 + 6x + 8. \end{aligned}$$

(c) To find $h(t+2)$, we substitute the quantity $t+2$ in place of t:

$$\begin{aligned} h(t+2) &= -(t+2)^2 + 3(t+2) + 4 \\ &= -\left(t^2 + 4t + 4\right) + (3t + 6) + 4 \\ &= -t^2 - 4t - 4 + 3t + 6 + 4 \\ &= -t^2 - t + 6. \end{aligned}$$

To find $h(t) + 2$, we add 2 to the expression for $h(t)$

$$\begin{aligned} h(t) + 2 &= \left(-t^2 + 3t + 4\right) + 2 \\ &= -t^2 + 3t + 6. \end{aligned}$$

From our work above, we see that $h(2) = 6$ so

$$\begin{aligned} h(t) + h(2) &= \left(-t^2 + 3t + 4\right) + 6 \\ &= -t^2 + 3t + 10. \end{aligned}$$

3. We know $h(-1) = 0$ from above, so $t = -1$ should be one of the answers to $h(t) = 0$. In order to see if there are any more, we set $h(t) = -t^2 + 3t + 4 = 0$. Factoring[8] gives $-(t+1)(t-4) = 0$, so we get $t = -1$ (as expected) along with $t = 4$. □

A few remarks about Example 2.1.2 are in order. First, note that $h(2x)$ and $2h(x)$ are different expressions. In the former, we are multiplying the *input* by 2; in the latter, we are multiplying the *output* by 2. The same goes for $h(t+2)$, $h(t)+2$ and $h(t)+h(2)$. The expression $h(t+2)$ calls for adding 2 to the input t and then performing the function h. The expression $h(t)+2$ has us performing the process h first, then adding 2 to the output $h(t)$. Finally, $h(t)+h(2)$ directs us to first find the outputs $h(t)$ and $h(2)$ and then add the results. As we saw in Example 2.1.1, we see here again the importance paying close attention to syntax.[9]

Let us return for a moment to the function f in Example 2.1.2 which we ultimately represented using the formula $f(x) = 3x + 5$. If we introduce the dependent variable y, we get the equation $y = f(x) = 3x + 5$, or, more simply $y = 3x + 5$. To say that the equation $y = 3x + 5$ describes y as a function of x means that for each choice of x, the formula $3x + 5$ determines only one associated y-value.

We could turn the tables and ask if the equation $y = 3x + 5$ describes x as a function of y. That is, for each value we pick for y, does the equation $y = 3x + 5$ produce only one associated x value? One way to proceed is to solve $y = 3x + 5$ for x and get $x = \frac{1}{3}(y-5)$. We see that for each choice of y, the expression $\frac{1}{3}(y-5)$ evaluates to just one number, hence, x is a function of y. If we give this function a name, say g, we have $x = g(y) = \frac{1}{3}(y-5)$, where in this equation, y is the independent variable and x is the dependent variable. We explore this idea in the next example.

[8] You may need to review Section 5.2.

[9] As was mentioned before, we will give meanings to the these quantities in other examples throughout the text.

Example 2.1.3.

1. Consider the equation $x^3 + y^2 = 25$.
   (a) Does this equation represent y as a function of x? Explain.
   (b) Does this equation represent x as a function of y? Explain.

2. Consider the equation $u^4 + t^3 u = 16$.
   (a) Does this equation represent t as a function of u? Explain.
   (b) Does this equation represent u as a function of t? Explain.

Solution.

1. (a) To say that $x^3 + y^2 = 25$ represents y as a function of x, we need to show that for each x we choose, the equation produces only one associated y-value. To help with this analysis, we solve the equation for y in terms of x.

$$\begin{aligned} x^3 + y^2 &= 25 \\ y^2 &= 25 - x^3 \\ y &= \pm\sqrt{25 - x^3} \quad \text{extract square roots. (See Section 8.1 for a review, if needed.)} \end{aligned}$$

The presence of the '$\pm$' indicates that there is a good chance that for some x-value, the equation will produce *two* corresponding y-values. Indeed, $x = 0$ produces $y = \pm\sqrt{25 - 0^3} = \pm 5$. Hence, $x^3 + y^2 = 25$ equation does *not* represent y as a function of x because $x = 0$ is matched with more than one y-value.

(b) To see if $x^3 + y^2 = 25$ represents x as a function of y, we solve the equation for x in terms of y:

$$\begin{aligned} x^3 + y^2 &= 25 \\ x^3 &= 25 - y^2 \\ x &= \sqrt[3]{25 - y^2} \quad \text{extract cube roots. (See Section 8.1 for a review, if needed.)} \end{aligned}$$

In this case, each choice of y produces only *one* corresponding value for x, so $x^3 + y^2 = 25$ represents x as a function of y.

2. (a) To see if $u^4 + t^3 u = 16$ represents t as a function of u, we proceed as above and solve for t in terms of u:

$$\begin{aligned} u^4 + t^3 u &= 16 \\ t^3 u &= 16 - u^4 \\ t^3 &= \frac{16 - u^4}{u} \qquad \text{assumes } u \neq 0 \\ t &= \sqrt[3]{\frac{16 - u^4}{u}} \quad \text{extract cube roots.} \end{aligned}$$

Although it's a bit cumbersome, as long as $u \neq 0$ the expression $\sqrt[3]{\frac{16-u^4}{u}}$ will produce just one value of t for each value of u. What if $u = 0$? In that case, the equation $u^4 + t^3 u = 16$ reduces to $0 = 16$ - which is never true - so we don't need to worry about that case.[10] Hence, $u^4 + t^3 u = 16$ represents t as a function of u.

(b) In order to determine if $u^4 + t^3 u = 16$ represents u as a function of t, we could attempt to solve $u^4 + t^3 u = 16$ for u in terms of t, but we won't get very far.[11] Instead, we take a different approach and experiment with looking for solutions for u for specific values of t. If we let $t = 0$, we get $u^4 = 16$ which gives $u = \pm\sqrt[4]{16} = \pm 2$. Hence, $t = 0$ corresponds to more than one u-value which means $u^4 + t^3 u = 16$ does not represent u as a function of t. □

We'll have more to say about using equations to describe functions in Section 9.3. For now, we turn our attention to a geometric way to represent functions.

2.1.3 Geometric Representations of Functions

In this section, we introduce how to graph functions. As we'll see in this and later sections, visualizing functions geometrically can assist us in both analyzing them and using them to solve associated application problems. Our playground, if you will, for the Geometry in this course is the Cartesian Coordinate Plane. The reader would do well to review Section **??** as needed.

Our path to the Cartesian Plane requires ordered pairs. In general, we can represent every function as a set of ordered pairs. Indeed, given a function f with domain A, we can represent $f = \{(a, f(a)) \mid a \in A\}$. That is, we represent f as a set of ordered pairs $(a, f(a))$, or, more generally, (input, output). For example, the function f which matches Taylor's pet's names to their associated pet type can be represented as:

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\}$$

Moving on, we next consider the function f from Example 2.1.1 which relates time to temperature. In this case, $f = \{(0, 64), (2, 67), (4, 75), (6, 80), (8, 83), (10, 83), (12, 82)\}$. This function has numerical values for both the domain and range so we can identify these ordered pairs with points in the Cartesian Plane. The first coordinates of these points (the abscissae) represent time values so we'll use t to label the horizontal axis. Likewise, we'll use T to label the vertical axis since the second coordinates of these points (the ordinates) represent temperature values. Note that labeling these axes in this way determines our independent and dependent variable names, t and T, respectively.

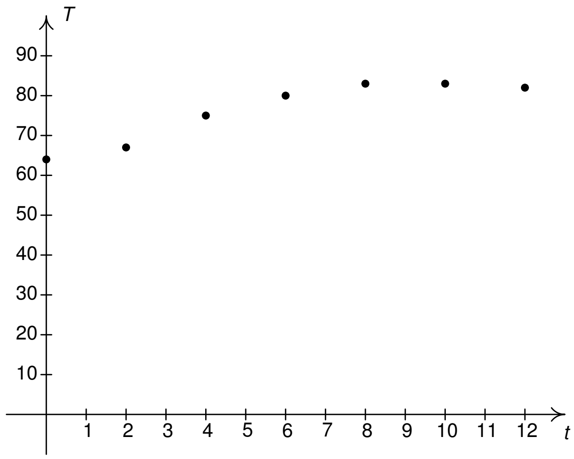
The plot of these points is called 'the **graph** of f'. More specifically, we could describe this plot as 'the graph of $f(t)$', because we have decided to name the independent variable t. Most specifically, we could describe the plot as 'the graph of $T = f(t)$', given that we have named the independent variable t and the dependent variable T.

On the next page we present two plots, both of which are graphs of the function f. In both cases, the vertical axis has been scaled in order to save space. In the graph on the left, the same increment on

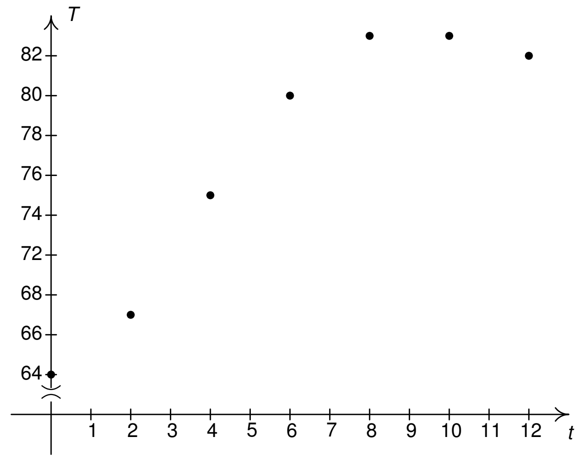
[10] Said differently, $u = 0$ is not in the domain of the function represented by the equation $u^4 + t^3 u = 16$.

[11] Try it for yourself!

the horizontal axis to measure 1 unit measures 10 units on the vertical axis whereas in the graph on the right, this ratio is 1 : 2. The '≍' symbol on the vertical axis in the graph on the right is used to indicate a jump in the vertical labeling. Both are perfectly accurate data plots, but they have different visual impacts. Note here that the extrema of f, 64 and 83, correspond to the lowest and highest points on the graph, respectively: $(0, 64)$, $(8, 83)$ and $(10, 83)$. More often than not, we will use the graph of a function to help us optimize that function.[12]



The graph of $T = f(t)$.



The graph of $T = f(t)$.

If you found yourself wanting to connect the dots in the graphs above, you're not alone. As it stands, however, the function f matches only seven inputs to seven outputs, so those seven points - and just those seven points - comprise the graph of f. That being said, common everyday experience tells us that while the data Skippy collected in his table gives some good information about the relationship between time and temperature on a given day, it is by no means a complete description of the relationship.

For example, Skippy's data cannot tell us what the temperature was at 7 a.m. or 12:13 p.m, although we are pretty sure there were outdoor temperatures at those times. Also, given that at some point it was 64°F and later on it was 83°F, it seems reasonable to assume that at some point it was 70°F or even 79.923°F.

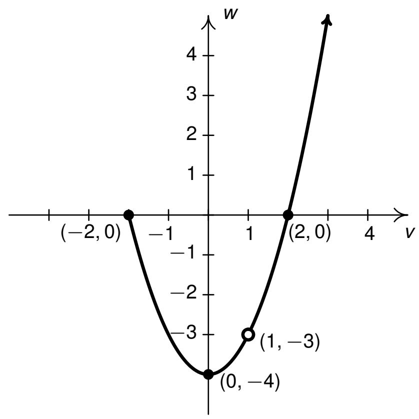
Skippy's temperature function f is an example of a **discrete** function in the sense that each of the data points are 'isolated' with measurable gaps in between. The idea of 'filling in' those gaps is a quest to find a **continuous** function to model this same phenomenon.[13] We'll return to this example in Sections 3.2 and 5.4 in an attempt to do just that.

In the meantime, our next example involves a function whose domain is (almost) an *interval* of real numbers and whose graph consists of a (mostly) *connected* arc.

[12]One major use of Calculus is to optimize functions analytically - that is, without a graph.

[13]Roughly speaking, a *continuous variable* is a variable which takes on values over an *interval* of real numbers as opposed to values in a discrete list. In this case we would think of time as a 'continuum' - an interval of real numbers as opposed to 7 or so isolated times. A *continuous function* is a function which takes an interval of real numbers and maps it in such a way that its graph is a connected curve with no holes or gaps. This is technically a Calculus idea, but we'll need to discuss the notion of continuity a few times in the text.

Example 2.1.4. Consider the graph below.



1. (a) Explain why this graph suggests that w is a function of v, $w = F(v)$.

   (b) Find $F(0)$ and solve $F(v) = 0$.

   (c) Find the domain and range of F using interval notation.[14] Find the extrema of F, if any exist.

2. Does this graph suggest v is a function of w? Explain.

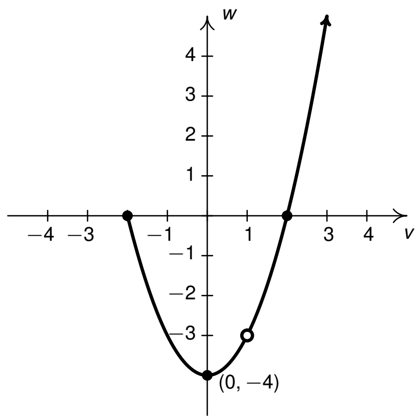
Solution. The challenge in working with only a graph is that unless points are specifically labeled (as some are in this case), we are forced to approximate values. In addition to the labeled points, there are other interesting features of the graph; a gap or 'hole' labeled $(1, -3)$ and an arrow on the upper right hand part of the curve. We'll have more to say about these two features shortly.

1. (a) In order for w to be a function of v, each v-value on the graph must be paired with only one w-value. What if this weren't the case? We'd have at least two points with the *same* v-coordinate with *different* w-coordinates. Graphically, we'd have two points on graph on the same vertical line, one above the other. This never happens so we may conclude that w is a function of v.

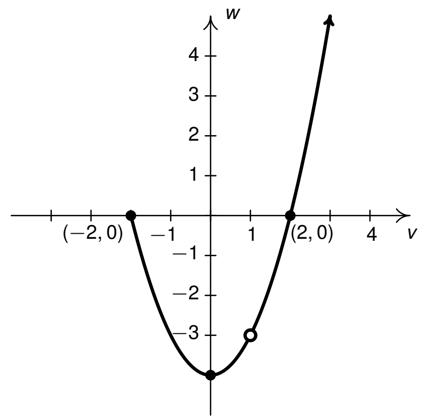
   (b) The value $F(0)$ is the output from F when $v = 0$. The points on the graph of F are of the form $(v, F(v))$ thus we are looking for the w-coordinate of the point on the graph where $v = 0$. Given that the point $(0, -4)$ is labeled on the graph, we can be sure $F(0) = -4$.

   To solve $F(v) = 0$, we are looking for the v-values where the output, or associated w value, is 0. Hence, we are looking for points on the graph with a w-coordinate of 0. We find two such points, $(-2, 0)$ and $(2, 0)$, so our solutions to $F(v) = 0$ are $v = \pm 2$. Pictures highlighting the relevant graphical features are given at the top of the next page.

[14]Please consult Section 1.1 for a review of interval notation if need be.

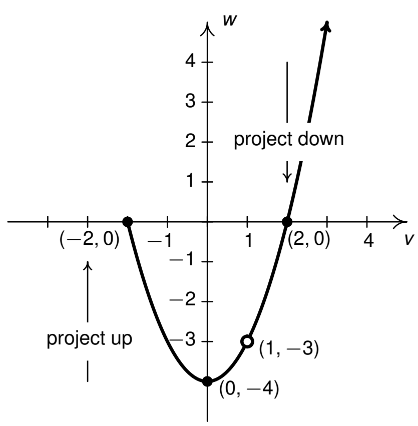


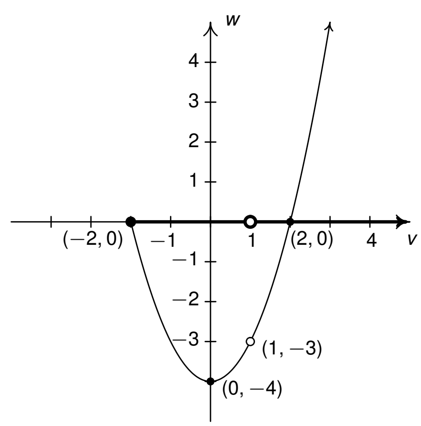
Finding $F(0) = -4$.



Solving $F(v) = 0$.

(c) The domain of F is the set of inputs to F. With v as the input here, we need to describe the set of v-values on the graph. We can accomplish this by **projecting** the graph to the v-axis and seeing what part of the v-axis is covered. The leftmost point on the graph is $(-2, 0)$, so we know that the domain starts at $v = -2$. The graph continues to the right until we encounter the 'hole' labeled at $(1, -3)$. This indicates one and only one point, namely $(1, -3)$ is missing from the curve which for us means $v = 1$ is not in the domain of F. The graph continues to the right and the arrow on the graph indicates that the graph goes upwards to the right indefinitely. Hence, our domain is $\{v \,|\, v \geq -2, \, v \neq 1\}$ which, in interval notation, is $[-2, 1) \cup (1, \infty)$. Pictures demonstrating the process of projecting the graph to the v-axis are shown below.

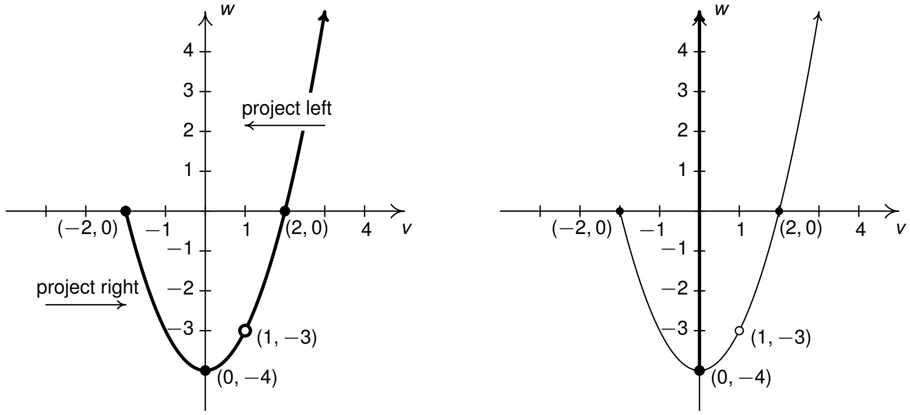




To find the range of F, we need to describe the set of outputs - in this case, the w-values on the graph. Here, we project the graph to the w-axis. Vertically, the graph starts at $(0, -4)$ so our range starts at $w = -4$. Note that even though there is a hole at $(1, -3)$, the w-value -3 is covered by what *appears* to be the point $(-1, -3)$ on the graph.[15] The arrow indicates that the graph extends upwards indefinitely so the range of F is $\{w \,|\, w \geq -4\}$ or, in interval notation, $[-4, \infty)$. Regarding extrema, F has a minimum of -4 when $v = 0$, but given that the graph extends upwards indefinitely, F has no maximum.

[15] For all we know, it could be $(-0.992, -3)$.

Pictures showing the projection of the graph onto the w-axis are given below.



2. Finally, to determine if v is a function of w, we look to see if each w-value is paired with only one v-value on the graph. We have points on the graph, namely $(-2, 0)$ and $(2, 0)$, that clearly show us that $w = 0$ is matched with the *two* v-values $v = 2$ and $v = -2$. Hence, v is not a function of w. $\square$

It cannot be stressed enough that when given a graphical representation of a function, certain assumptions must be made. In the previous example, for all we know, the minimum of the graph is at $(0.001, -4.0001)$ instead of $(0, -4)$. If we aren't given an equation or table of data, or if specific points aren't labeled, we really have no way to tell. We also are assuming that the graph depicted in the example, while ultimately made of infinitely many points, has no gaps or holes other than those noted. This allows us to make such bold claims as the existence of a point on the graph with a w-coordinate of -3.

Before moving on to our next example, it is worth noting that the geometric argument made in Example 2.1.4 to establish that w is a function of v can be generalized to any graph. This result is the celebrated Vertical Line Test and it enables us to detect functions geometrically. Note that the statement of the theorem resorts to the 'default' x and y labels on the horizontal and vertical axes, respectively.

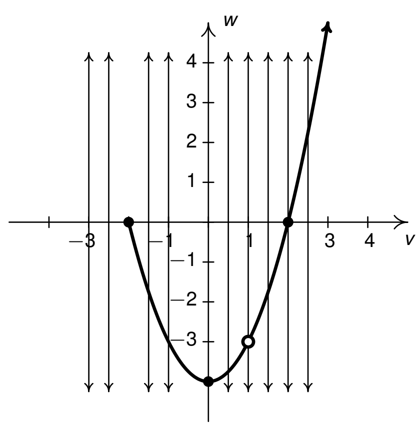
Theorem 2.1. The Vertical Line Test: A graph in the xy-plane[a] represents y as a function of x if and only if no vertical line intersects the graph more than once.

[a]That is, the horizontal axis is labeled with 'x' and the vertical axis is labeled with 'y'.

Let's take a minute to discuss the phrase 'if and only if' used in Theorem 2.1. The statement 'the graph represents y as a function of x *if and only if* no vertical line intersects the graph more than once' is actually saying two things. First, it's saying 'the graph represents y as a function of x *if* no vertical line intersects the graph more than once' and, second, 'the graph represents y as a function of x *only if* no vertical line intersects the graph more than once'.

Logically, these statements are saying two different things. The first says that if no vertical line crosses the graph more than once, then the graph represents y as a function of x. But the question remains: could a graph represent y as a function of x and yet there be a vertical line that intersects the graph more than once? The answer to this is 'no' because the second statement says that the *only* way the graph represents y as a function of x is the case when no vertical line intersects the graph more than once.

Applying the Vertical Line Test to the graph given in Example 2.1.4, we see below that all of the vertical lines meet the graph at most once (several are shown for illustration) showing w is a function of v. Notice that some of the lines ($x = -3$ and $x = 1$, for example) don't hit the graph at all. This is fine because the Vertical Line Test is looking for lines that hit the graph more than once. It does not say *exactly* once so missing the graph altogether is permitted.



There is also a geometric test to determine if the graph above represents v as a function of w. We introduce this aptly-named **Horizontal Line Test** in Exercise 57 and revisit it in Sections 9.3 and 9.4.

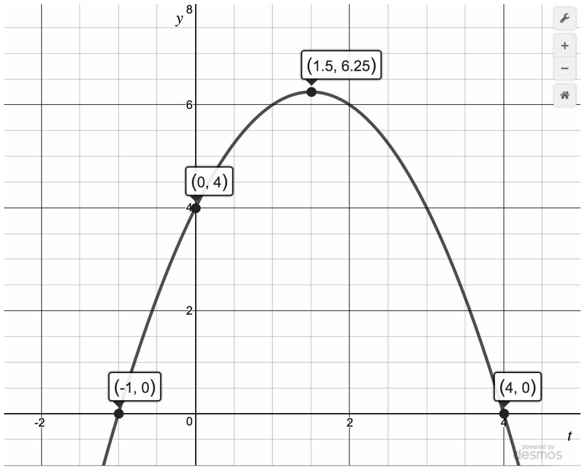
Our next example revisits the function h from Example 2.1.2 from a graphical perspective.

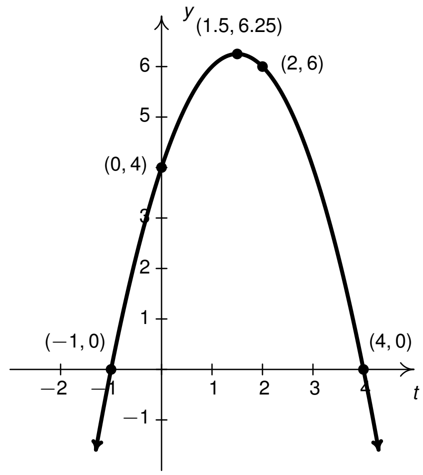
Example 2.1.5. With the help of a graphing utility graph $h(t) = -t^2 + 3t + 4$. From your graph, state the domain, range and extrema, if any exist.

Solution. The dependent variable wasn't specified so we use the default 'y' label for the vertical axis and set about graphing $y = h(t)$. From our work in Example 2.1.2, we already know $h(-1) = 0$, $h(0) = 4$, $h(2) = 6$ and $h(4) = 0$. These give us the points $(-1, 0)$, $(0, 4)$, $(2, 6)$ and $(4, 0)$, respectively. Using these as a guide, we can use desmos to produce the graph at the top of the next page on the left.[16]

As nice as the graph is, it is still technically incomplete. There is no restriction stated on the independent variable t so the domain of h is all real numbers. However, the graph as presented shows only the behavior of h between roughly $t = -2.5$ and $t = 4.25$. By zooming out, we see that the graph extends downwards indefinitely which we indicate by adding the arrows you see in the graph on the right. We find that the domain is $(-\infty, \infty)$ and the range is $(-\infty, 6.25]$. There is no minimum, but the maximum of h is 6.25 and it occurs at $t = 1.5$. The point $(1.5, 6.25)$ is shown on both graphs.

[16]The curve in this example is called a 'parabola'. In Section 5.4, we'll learn how to graph these accurately *by hand*.

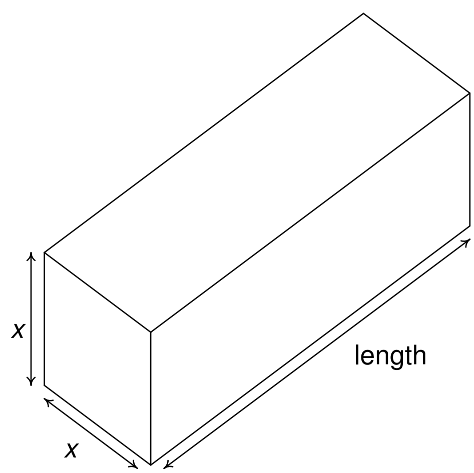




□

Our last example of the section uses the interplay between algebraic and graphical representations of a function to solve a real-world problem.

Example 2.1.6. The United States Postal Service mandates that when shipping parcels using 'Parcel Select' service, the sum of the length (the longest dimension) and the girth (the distance around the thickest part of the parcel perpendicular to the length) must not exceed 130 inches.[17] Suppose we wish to ship a rectangular box whose girth forms a square measuring x inches per side as shown below.



It turns out[18] that the volume of a box, $V(x)$, measured in cubic inches, whose length plus girth is exactly 130 inches is given by the formula: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$.

[17]See here.

[18]We'll skip the explanation for now because we want to focus on just the different representations of the function. Rest assured, you'll be asked to construct this very model in Exercise 56a in Section 6.1.

1. Find and interpret $V(5)$.

2. Make a table of values and use these along with a graphing utility to graph $y = V(x)$.

3. What is the largest volume box that can be shipped? What value of x maximizes the volume? Round your answers to two decimal places.

Solution.

1. To find $V(5)$, we substitute $x = 5$ into the expression $V(x)$: $V(5) = (5)^2(130 - 4(5)) = 25(110) = 2750$. Our result means that when the length and width of the square measure 5 inches, the volume of the resulting box is 2750 cubic inches.[19]

2. The domain of V is specified by the inequality $0 < x \leq 26$, so we can begin graphing V by sampling V at finitely many x-values in this interval to help us get a sense of the range of V. This, in turn, will help us determine an adequate viewing window on our graphing utility when the time comes.

   It seems natural to start with what's happening near $x = 0$. Even though the expression $x^2(130 - 4x)$ is defined when we substitute $x = 0$ (it reduces very quickly to 0), it would be incorrect to state $V(0) = 0$ because $x = 0$ is not in the domain of V. However, there is nothing stopping us from evaluating $V(x)$ at values x 'very close' to $x = 0$. A table of such values is given below.

x	$V(x)$
0.1	1.296
0.01	0.012996
0.001	0.000129996
10^{-23}	$\approx 1.3 \times 10^{-44}$

   There is no such thing as a 'smallest' positive number,[20] so we will have points on the graph of V to the right of $x = 0$ leading to the point $(0, 0)$. We indicate this behavior by putting a hole at $(0, 0)$.[21]

   Moving forward, we start with $x = 5$ and sample V at steps of 5 in its domain. Our goal is to graph $y = V(x)$, so we plot our points $(x, V(x))$ using the domain as a guide to help us set the horizontal bounds (i.e., the bounds on x) and the sample values from the range to help us set the vertical bounds (i.e., the bounds on y). The right endpoint, $x = 26$, is included in the domain $0 < x \leq 26$ so we finish the graph by plotting the point $(26, V(26)) = (26, 17576)$. At the top of the next page on the left is the table of data and on the right is a graph produced with some help from desmos.

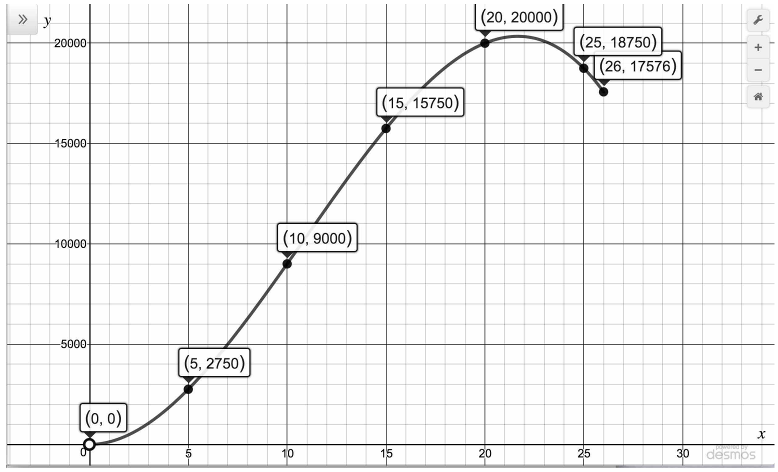
[19] Note that we have $V(5)$ and $25(110)$ in the same string of equality. The first set of parentheses is function notation and directs us to substitute 5 for x in the expression $V(x)$ while the second indicates multiplying 25 by 110. Context is key!

[20] If p is any positive real number, $0 < 0.5p < p$, so we can always find a smaller positive real number.

[21] What's really needed here is the precise definition of 'closeness' discussed in Calculus. This hand-waving will do for now.

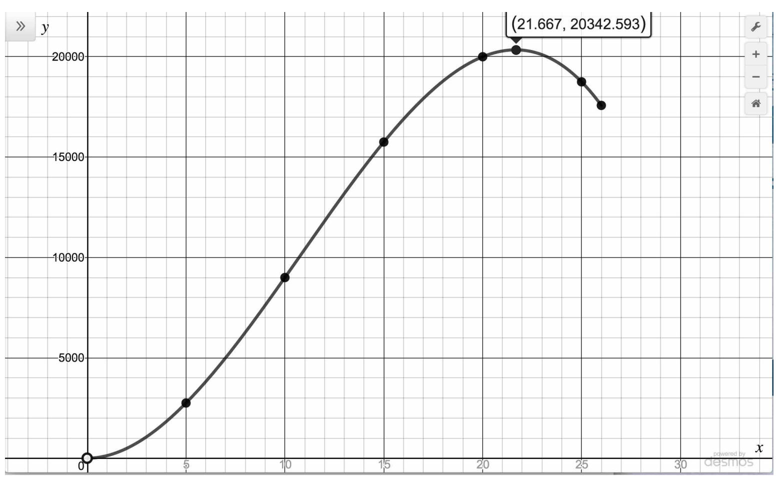
x	$V(x)$	$(x, V(x))$
≈ 0	≈ 0	hole at $(0, 0)$
5	2750	$(5, 2750)$
10	9000	$(10, 9000)$
15	15,750	$(15, 15{,}750)$
20	20,000	$(20, 20{,}000)$
25	18,750	$(25, 18{,}750)$
26	17,576	$(26, 17{,}576)$

Sampling V



The graph of $y = V(x)$

3. The largest volume in this case refers to the maximum of V. The biggest y-value in our table of data is 20,000 cubic inches which occurs at $x = 20$ inches, but the graph produced by the graphing utility indicates that there are points on the graph of V with y-values (hence $V(x)$ values) greater than 20,000. Indeed, the graph continues to rise to the right of $x = 20$ and the graphing utility reports the maximum y-value to be $y \approx 20{,}342.593$ when $x \approx 21.667$. Rounding to two decimal places, we find the maximum volume obtainable under these conditions is about 20,342.59 cubic inches which occurs when the length and width of the square side of the box are approximately 21.67 inches.[22]



Finding the maximum volume using the graph of $y = V(x)$.

[22]We could also find the length of the box in this case as well. The sum of length and girth is 130 inches so the length is 130 minus the girth, or $130 - 4x \approx 130 - 4(21.67) = 43.32$ inches.

It is worth noting that while the function V has a maximum, it did not have a minimum. Even though $V(x) > 0$ for all x in its domain,[23] the presence of the hole at $(0, 0)$ means that 0 is not in the range of V. Hence, based on our model, we can never make a box with a 'smallest' volume.[24] □

Example 2.1.6 typifies the interplay between Algebra and Geometry which lies ahead. Both the algebraic description of V: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$, and the graph of $y = V(x)$ were useful in describing aspects of the physical situation at hand. Wherever possible, we'll use the algebraic representations of functions to *analytically* produce *exact* answers to certain problems and use the graphical descriptions to check the reasonableness of our answers.

That being said, we'll also encounter problems which we simply *cannot* answer analytically (such as determining the maximum volume in the previous example), so we will be forced to resort to using technology (specifically graphing technology) in order to find *approximate* solutions. The most important thing to keep in mind is that while technology may *suggest* a result, it is ultimately Mathematics that *proves* it.

We close this section with a summary of the different ways to represent functions.

Ways to Represent a Function

Suppose f is a function with domain A. Then f can be represented:

- verbally; that is, by describing how the inputs are matched with their outputs.
- using a mapping diagram.
- as a set of ordered pairs of the form (input, output): $\{(a, f(a)) \mid a \in A\}$.

If f is a function whose domain and range are subsets of real numbers, then f can be represented:

- algebraically as a formula for $f(a)$.
- graphically by plotting the points $\{(a, f(a)) \mid a \in A\}$ in the plane.

Note: An important consequence of the last bulleted item is that the point (a, b) is on the graph of $y = f(x)$ if and only if $f(a) = b$.

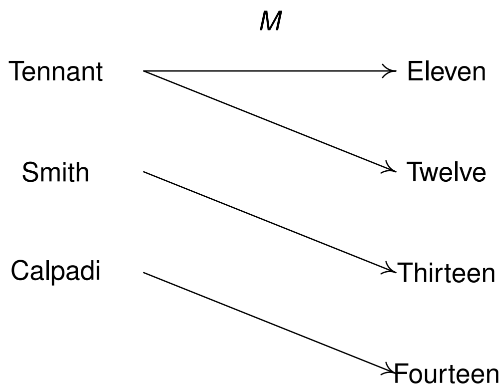
[23] said differently, the values of $V(x)$ are **bounded below** by 0.

[24] How realistic is this?

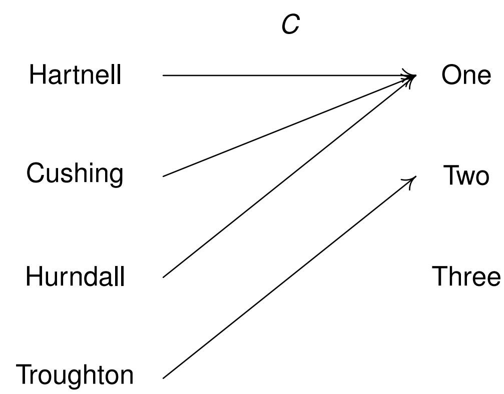
2.1.4 Exercises

In Exercises 1 - 2, determine whether or not the mapping diagram represents a function. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

1.



2.



In Exercises 3 - 4, determine whether or not the data in the given table represents y as a function of x. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

3.

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

4.

x	y
0	0
1	1
1	-1
2	2
2	-2
3	3
3	-3

5. Suppose W is the set of words in the English language and we set up a mapping from W into the set of natural numbers $\mathbb{N}$ as follows: word $\rightarrow$ number of letters in the word. Explain why this mapping is a function. What would you need to know to determine the range of the function?

6. Suppose L is the set of last names of all the people who have served or are currently serving as the President of the United States. Consider the mapping from L into $\mathbb{N}$ as follows: last name $\rightarrow$ number of their presidency. For example, Washington $\rightarrow$ 1 and Obama $\rightarrow$ 44. Is this mapping a function? What if we use full names instead of just last names? (**HINT:** Research Grover Cleveland.)

7. Under what conditions would the time of day be a function of the outdoor temperature?

For the functions f described in Exercises 8 - 13, find $f(2)$ and find and simplify an expression for $f(x)$ that takes a real number x and performs the following three steps in the order given:

8. (1) multiply by 2; (2) add 3; (3) divide by 4.
9. (1) add 3; (2) multiply by 2; (3) divide by 4.
10. (1) divide by 4; (2) add 3; (3) multiply by 2.
11. (1) multiply by 2; (2) add 3; (3) take the square root.
12. (1) add 3; (2) multiply by 2; (3) take the square root.
13. (1) add 3; (2) take the square root; (3) multiply by 2.

In Exercises 14 - 19, use the given function f to find and simplify the following:

- $f(3)$
- $f(-1)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

14. $f(x) = 2x + 1$
15. $f(x) = 3 - 4x$
16. $f(x) = 2 - x^2$
17. $f(x) = x^2 - 3x + 2$
18. $f(x) = 6$
19. $f(x) = 0$

In Exercises 20 - 25, use the given function f to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a) + f(2)$
- $f\left(\dfrac{2}{a}\right)$
- $\dfrac{f(a)}{2}$
- $f(a+h)$

20. $f(x) = 2x - 5$
21. $f(t) = 5 - 2t$
22. $f(w) = 2w^2 - 1$
23. $f(q) = 3q^2 + 3q - 2$
24. $f(r) = 117$
25. $f(z) = \dfrac{z}{2}$

In Exercises 26 - 29, use the given function f to find $f(0)$ and solve $f(x) = 0$

26. $f(x) = 2x - 1$
27. $f(x) = 3 - \frac{2}{5}x$
28. $f(x) = 2x^2 - 6$
29. $f(x) = x^2 - x - 12$

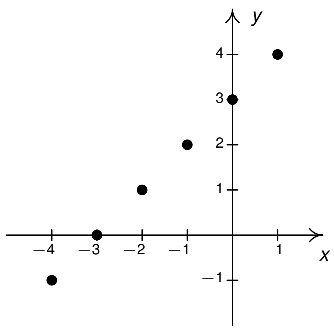
In Exercises 30 - 44, determine whether or not the equation represents y as a function of x.

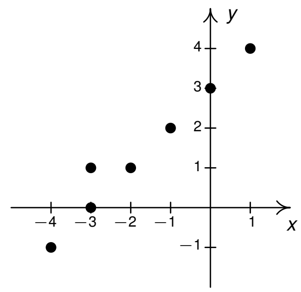
30. $y = x^3 - x$
31. $y = \sqrt{x - 2}$
32. $x^3 y = -4$
33. $x^2 - y^2 = 1$
34. $y = \dfrac{x}{x^2 - 9}$
35. $x = -6$
36. $x = y^2 + 4$
37. $y = x^2 + 4$
38. $x^2 + y^2 = 4$
39. $y = \sqrt{4 - x^2}$
40. $x^2 - y^2 = 4$
41. $x^3 + y^3 = 4$
42. $2x + 3y = 4$
43. $2xy = 4$
44. $x^2 = y^2$

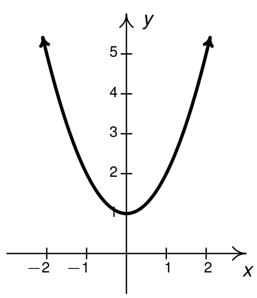
Exercises 45 - 56 give a set of points in the xy-plane. Determine if y is a function of x. If so, state the domain and range.

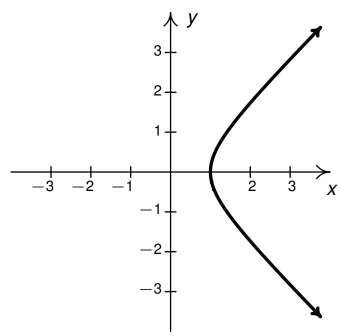
45. $\{(-3, 9),\ (-2, 4),\ (-1, 1),\ (0, 0),\ (1, 1),\ (2, 4),\ (3, 9)\}$
46. $\{(-3, 0), (1, 6), (2, -3), (4, 2), (-5, 6), (4, -9), (6, 2)\}$
47. $\{(-3, 0), (-7, 6), (5, 5), (6, 4), (4, 9), (3, 0)\}$
48. $\{(1, 2), (4, 4), (9, 6), (16, 8), (25, 10), (36, 12), ...\}$
49. $\{(x, y) \mid x$ is an odd integer, and y is an even integer$\}$
50. $\{(x, 1) \mid x$ is an irrational number$\}$
51. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), ...\}$
52. $\{... (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), ...\}$
53. $\{(-2, y) \mid -3 < y < 4\}$
54. $\{(x, 3) \mid -2 \leq x < 4\}$
55. $\{\left(x, x^2\right) \mid x$ is a real number$\}$
56. $\{\left(x^2, x\right) \mid x$ is a real number$\}$
57. The Vertical Line Test is a quick way to determine from a graph if the vertical axis variable is a function of the horizontal axis variable. If we are given a graph and asked to determine if the horizontal axis variable is a function of the vertical axis variable, we can use horizontal lines instead of vertical lines to check. Using Theorem 2.1 as a guide, formulate a 'Horizontal Line Test.' (We'll refer back to this exercise in Section 9.4.)

In Exercises 58 - 61, determine whether or not the graph suggests y is a function of x. For the ones which do, state the domain and range.

58. 

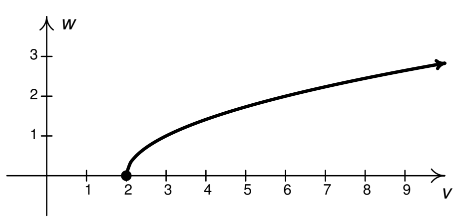
59. 

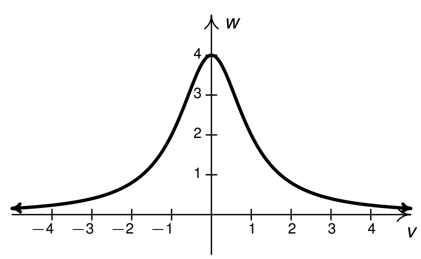
60. 

61. 

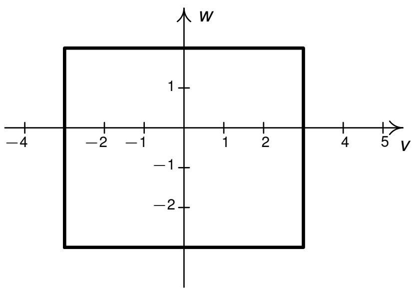
62. Determine which, if any, of the graphs in numbers 58 - 61 represent x as a function of y. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 63 - 66, determine whether or not the graph suggests w is a function of v . For the ones which do, state the domain and range.

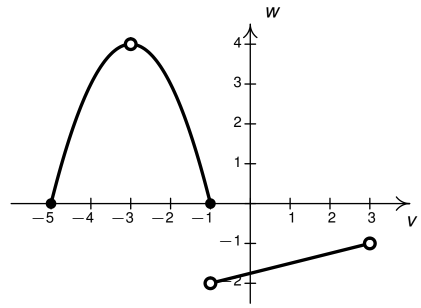
63. 

64. 

65.



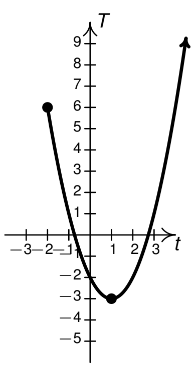
66.



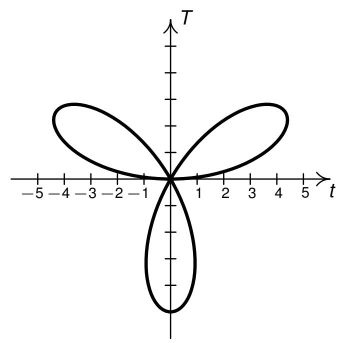
67. Determine which, if any, of the graphs in numbers 63 - 66 represent v as a function of w. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 68 - 71, determine whether or not the graph suggests T is a function of t. For the ones which do, state the domain and range.

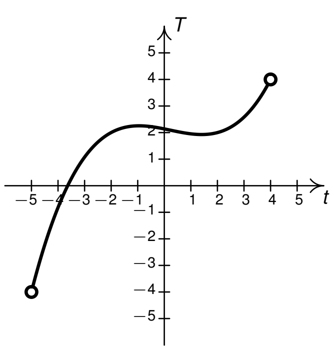
68.



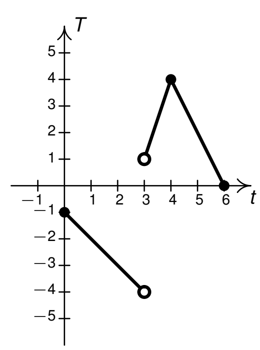
69.



70.



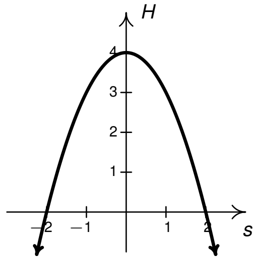
71.



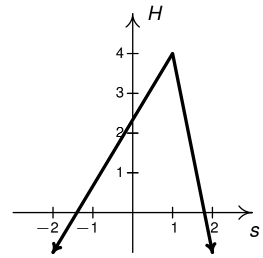
72. Determine which, if any, of the graphs in numbers 68 - 71 represent t as a function of T. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 73 - 76, determine whether or not the graph suggests H is a function of s. For the ones which do, state the domain and range.

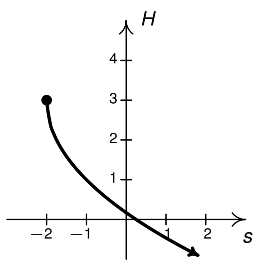
73.



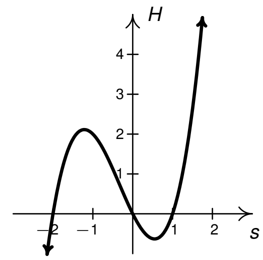
74.



75.



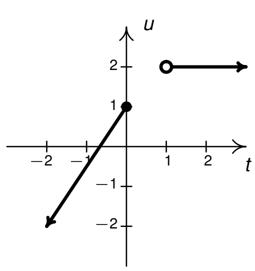
76.



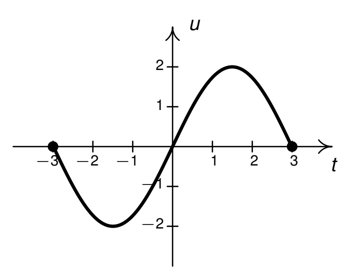
77. Determine which, if any, of the graphs in numbers 73 - 76 represent s as a function of H. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 78 - 81, determine whether or not the graph suggests u is a function of t. For the ones which do, state the domain and range.

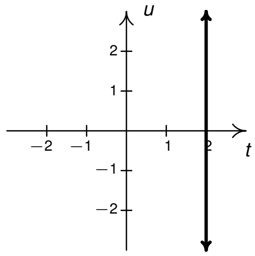
78.



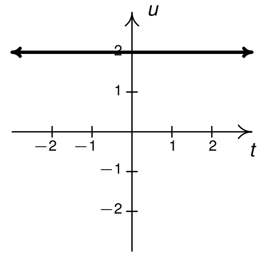
79.



80.

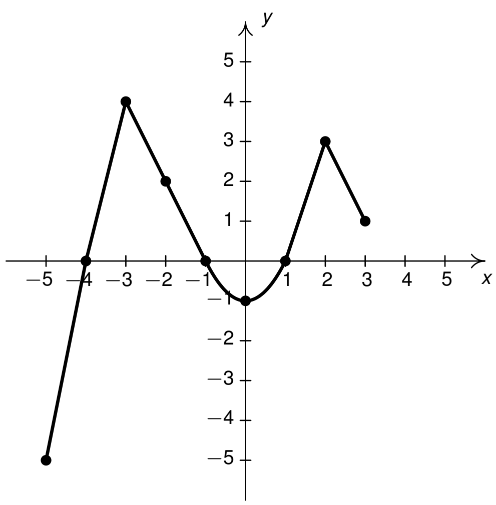


81.

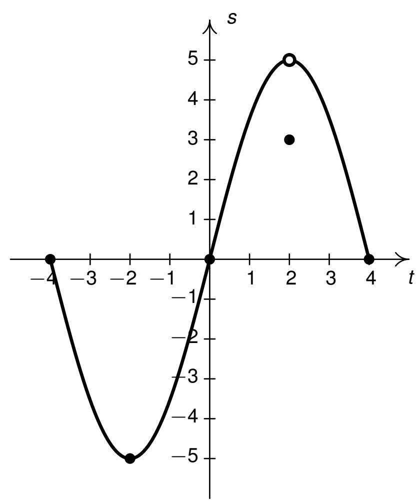


82. Determine which, if any, of the graphs in numbers 78 - 81 represent t as a function of u. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 83 - 92, use the graphs of f and g below to find the indicated values.



The graph of $y = f(x)$.



The graph of $s = g(t)$.

83. $f(-2)$

84. $g(-2)$

85. $f(2)$

86. $g(2)$

87. $f(0)$

88. $g(0)$

89. Solve $f(x) = 0$.

90. Solve $g(t) = 0$.

91. State the domain and range of f.

92. State the domain and range of g .

In Exercises 93 - 104, graph each function by making a table, plotting points, and using a graphing utility (if needed.) Use the independent variable as the horizontal axis label and the default 'y' label for the vertical axis label. State the domain and range of each function.

93. $f(x) = 2 - x$

94. $g(t) = \dfrac{t-2}{3}$

95. $h(s) = s^2 + 1$

96. $f(x) = 4 - x^2$

97. $g(t) = 2$

98. $h(s) = s^3$

99. $f(x) = x(x-1)(x+2)$

100. $g(t) = \sqrt{t-2}$

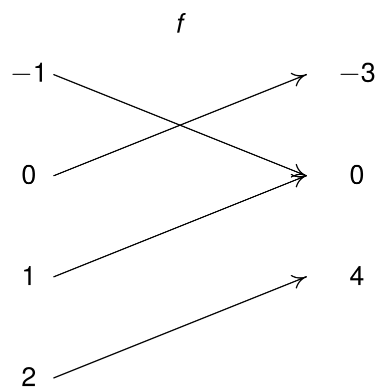
101. $h(s) = \sqrt{5-s}$

102. $f(x) = 3 - 2\sqrt{x+2}$

103. $g(t) = \sqrt[3]{t}$

104. $h(s) = \dfrac{1}{s^2+1}$

105. Consider the function f described below:



(a) State the domain and range of f.

(b) Find $f(0)$ and solve $f(x) = 0$.

(c) Write f as a set of ordered pairs.

(d) Graph f.

106. Let $g = \{(-1,4), (0,2), (2,3), (3,4)\}$

(a) State the domain and range of g.

(b) Create a mapping diagram for g.

(c) Find $g(0)$ and solve $g(x) = 0$.

(d) Graph g.

107. Let $F = \{(t, t^2) \mid t \text{ is a real number}\}$. Find $F(4)$ and solve $F(x) = 4$.

HINT: Elements of F are of the form $(x, F(x))$.

108. Let $G = \{(2t, t + 5) \mid t \text{ is a real number}\}$. Find $G(4)$ and solve $G(x) = 4$.

HINT: Elements of G are of the form $(x, G(x))$.

109. The area enclosed by a square, in square inches, is a function of the length of one of its sides ℓ, when measured in inches. This function is represented by the formula $A(\ell) = \ell^2$ for $\ell > 0$. Find $A(3)$ and solve $A(\ell) = 36$. Interpret your answers to each. Why is ℓ restricted to $\ell > 0$?

110. The area enclosed by a circle, in square meters, is a function of its radius r, when measured in meters. This function is represented by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is r restricted to $r > 0$?

111. The volume enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides s, when measured in centimeters. This function is represented by the formula $V(s) = s^3$ for $s > 0$. Find $V(5)$ and solve $V(s) = 27$. Interpret your answers to each. Why is s restricted to $s > 0$?

112. The volume enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet. This function is represented by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

113. The height of an object dropped from the roof of an eight story building is modeled by the function: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is t restricted to $0 \leq t \leq 2$?

114. The temperature in degrees Fahrenheit t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \leq t \leq 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.

115. The function $C(x) = x^2 - 10x + 27$ models the cost, in *hundreds* of dollars, to produce x *thousand* pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.

116. Using data from the Bureau of Transportation Statistics, the average fuel economy in miles per gallon for passenger cars in the US can be modeled by $E(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Use a calculator to find $E(0)$, $E(14)$ and $E(28)$. Round your answers to two decimal places and interpret your answers to each.

117. The perimeter of a square, in centimeters, is four times the length of one if its sides, also measured in centimeters. Represent the function P which takes as its input the length of the side of a square in centimeters, s and returns the perimeter of the square in inches, $P(s)$ using a formula.

118. The circumference of a circle, in feet, is π times the diameter of the circle, also measured in feet. Represent the function C which takes as its input the length of the diameter of a circle in feet, D and returns the circumference of a circle in inches, $C(D)$ using a formula.

119. Suppose $A(P)$ gives the amount of money in a retirement account (in dollars) after 30 years as a function of the amount of the monthly payment (in dollars), P.

   (a) What does $A(50)$ mean?

   (b) What is the significance of the solution to the equation $A(P) = 250000$? .

   (c) Explain what each of the following expressions mean: $A(P + 50)$, $A(P) + 50$, and $A(P) + A(50)$.

120. Suppose $P(t)$ gives the chance of precipitation (in percent) t hours after 8 AM.

   (a) Write an expression which gives the chance of precipitation at noon.

   (b) Write an inequality which determines when the chance of precipitation is more than 50%.

121. Explain why the graph in Exercise 63 suggests that not only is v as a function of w but also w is a function of v. Suppose $w = f(v)$ and $v = g(w)$. That is, f is the name of the function which takes v values as inputs and returns w values as outputs and g is the name of the function which does vice-versa. Find the domain and range of g and compare these to the domain and range of f.

122. Sketch the graph of a function with domain $(-\infty, 3) \cup [4, 5)$ with range $\{2\} \cup (5, \infty)$.

2.1.5 Answers

1. The mapping M is not a function since 'Tennant' is matched with both 'Eleven' and 'Twelve.'

2. The mapping C is a function since each input is matched with only one output. The domain of C is $\{$Hartnell, Cushing, Hurndall, Troughton$\}$ and the range is $\{$One, Two$\}$. We can represent C as the following set of ordered pairs: $\{$(Hartnell, One), (Cushing, One), (Hurndall, One), (Troughton, Two)$\}$

3. In this case, y is a function of x since each x is matched with only one y.

   The domain is $\{-3, -2, -1, 0, 1, 2, 3\}$ and the range is $\{0, 1, 2, 3\}$.

   As ordered pairs, this function is $\{(-3, 3), (-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2), (3, 3)\}$

4. In this case, y is not a function of x since there are x values matched with more than one y value. For instance, 1 is matched both to 1 and -1.

5. The mapping is a function since given any word, there is only one answer to 'how many letters are in the word?' For the range, we would need to know what the length of the longest word is and whether or not we could find words of all the lengths between 1 (the length of the word 'a') and it. See here.

6. Since Grover Cleveland was both the 22nd and 24th POTUS, neither mapping described in this exercise is a function.

7. The outdoor temperature could never be the same for more than two different times - so, for example, it could always be getting warmer or it could always be getting colder.

8. $f(2) = \frac{7}{4}$, $f(x) = \frac{2x+3}{4}$

9. $f(2) = \frac{5}{2}$, $f(x) = \frac{2(x+3)}{4} = \frac{x+3}{2}$

10. $f(2) = 7$, $f(x) = 2\left(\frac{x}{4} + 3\right) = \frac{1}{2}x + 6$

11. $f(2) = \sqrt{7}$, $f(x) = \sqrt{2x + 3}$

12. $f(2) = \sqrt{10}$, $f(x) = \sqrt{2(x + 3)} = \sqrt{2x + 6}$

13. $f(2) = 2\sqrt{5}$, $f(x) = 2\sqrt{x + 3}$

14. For $f(x) = 2x + 1$

    - $f(3) = 7$
    - $f(-1) = -1$
    - $f\left(\frac{3}{2}\right) = 4$
    - $f(4x) = 8x + 1$
    - $4f(x) = 8x + 4$
    - $f(-x) = -2x + 1$
    - $f(x - 4) = 2x - 7$
    - $f(x) - 4 = 2x - 3$
    - $f\left(x^2\right) = 2x^2 + 1$

15. For $f(x) = 3 - 4x$

    - $f(3) = -9$
    - $f(-1) = 7$
    - $f\left(\frac{3}{2}\right) = -3$

- $f(4x) = 3 - 16x$
- $4f(x) = 12 - 16x$
- $f(-x) = 4x + 3$
- $f(x-4) = 19 - 4x$
- $f(x) - 4 = -4x - 1$
- $f\left(x^2\right) = 3 - 4x^2$

16. For $f(x) = 2 - x^2$
    - $f(3) = -7$
    - $f(-1) = 1$
    - $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
    - $f(4x) = 2 - 16x^2$
    - $4f(x) = 8 - 4x^2$
    - $f(-x) = 2 - x^2$
    - $f(x-4) = -x^2 + 8x - 14$
    - $f(x) - 4 = -x^2 - 2$
    - $f\left(x^2\right) = 2 - x^4$

17. For $f(x) = x^2 - 3x + 2$
    - $f(3) = 2$
    - $f(-1) = 6$
    - $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
    - $f(4x) = 16x^2 - 12x + 2$
    - $4f(x) = 4x^2 - 12x + 8$
    - $f(-x) = x^2 + 3x + 2$
    - $f(x-4) = x^2 - 11x + 30$
    - $f(x) - 4 = x^2 - 3x - 2$
    - $f\left(x^2\right) = x^4 - 3x^2 + 2$

18. For $f(x) = 6$
    - $f(3) = 6$
    - $f(-1) = 6$
    - $f\left(\frac{3}{2}\right) = 6$
    - $f(4x) = 6$
    - $4f(x) = 24$
    - $f(-x) = 6$
    - $f(x-4) = 6$
    - $f(x) - 4 = 2$
    - $f\left(x^2\right) = 6$

19. For $f(x) = 0$
    - $f(3) = 0$
    - $f(-1) = 0$
    - $f\left(\frac{3}{2}\right) = 0$
    - $f(4x) = 0$
    - $4f(x) = 0$
    - $f(-x) = 0$
    - $f(x-4) = 0$
    - $f(x) - 4 = -4$
    - $f\left(x^2\right) = 0$

20. For $f(x) = 2x - 5$
    - $f(2) = -1$
    - $f(-2) = -9$
    - $f(2a) = 4a - 5$
    - $2f(a) = 4a - 10$
    - $f(a+2) = 2a - 1$
    - $f(a) + f(2) = 2a - 6$

- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a+h) = 2a + 2h - 5$

21. For $f(x) = 5 - 2x$
    - $f(2) = 1$
    - $f(-2) = 9$
    - $f(2a) = 5 - 4a$
    - $2f(a) = 10 - 4a$
    - $f(a+2) = 1 - 2a$
    - $f(a) + f(2) = 6 - 2a$
    - $f\left(\frac{2}{a}\right) = 5 - \frac{4}{a} = \frac{5a-4}{a}$
    - $\frac{f(a)}{2} = \frac{5-2a}{2}$
    - $f(a+h) = 5 - 2a - 2h$
22. For $f(x) = 2x^2 - 1$
    - $f(2) = 7$
    - $f(-2) = 7$
    - $f(2a) = 8a^2 - 1$
    - $2f(a) = 4a^2 - 2$
    - $f(a+2) = 2a^2 + 8a + 7$
    - $f(a) + f(2) = 2a^2 + 6$
    - $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
    - $\frac{f(a)}{2} = \frac{2a^2-1}{2}$
    - $f(a+h) = 2a^2+4ah+2h^2-1$
23. For $f(x) = 3x^2 + 3x - 2$
    - $f(2) = 16$
    - $f(-2) = 4$
    - $f(2a) = 12a^2 + 6a - 2$
    - $2f(a) = 6a^2 + 6a - 4$
    - $f(a+2) = 3a^2 + 15a + 16$
    - $f(a) + f(2) = 3a^2 + 3a + 14$
    - $f\left(\frac{2}{a}\right) = \frac{12}{a^2} + \frac{6}{a} - 2 = \frac{12+6a-2a^2}{a^2}$
    - $\frac{f(a)}{2} = \frac{3a^2+3a-2}{2}$
    - $f(a+h) = 3a^2 + 6ah + 3h^2 + 3a + 3h - 2$
24. For $f(x) = 117$
    - $f(2) = 117$
    - $f(-2) = 117$
    - $f(2a) = 117$
    - $2f(a) = 234$
    - $f(a+2) = 117$
    - $f(a) + f(2) = 234$
    - $f\left(\frac{2}{a}\right) = 117$
    - $\frac{f(a)}{2} = \frac{117}{2}$
    - $f(a+h) = 117$
25. For $f(x) = \frac{x}{2}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = a$
- $2f(a) = a$
- $f(a+2) = \frac{a+2}{2}$
- $f(a) + f(2) = \frac{a}{2} + 1 = \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = \frac{1}{a}$
- $\frac{f(a)}{2} = \frac{a}{4}$
- $f(a+h) = \frac{a+h}{2}$

26. For $f(x) = 2x - 1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$
27. For $f(x) = 3 - \frac{2}{5}x$, $f(0) = 3$ and $f(x) = 0$ when $x = \frac{15}{2}$
28. For $f(x) = 2x^2 - 6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$
29. For $f(x) = x^2 - x - 12$, $f(0) = -12$ and $f(x) = 0$ when $x = -3$ or $x = 4$
30. Function
31. Function
32. Function
33. Not a function
34. Function
35. Not a function
36. Not a function
37. Function
38. Not a function
39. Function
40. Not a function
41. Function
42. Function
43. Function
44. Not a function
45. Function
domain = $\{-3, -2, -1, 0, 1, 2, 3\}$
range = $\{0, 1, 4, 9\}$
46. Not a function
47. Function
domain = $\{-7, -3, 3, 4, 5, 6\}$
range = $\{0, 4, 5, 6, 9\}$
48. Function
domain = $\{1, 4, 9, 16, 25, 36, ...\}$
= $\{x \mid x$ is a perfect square$\}$
range = $\{2, 4, 6, 8, 10, 12, ...\}$
= $\{y \mid y$ is a positive even integer$\}$
49. Not a function
50. Function
domain = $\{x \mid x$ is irrational$\}$
range = $\{1\}$
51. Function
domain = $\{x \mid 1, 2, 4, 8, ...\}$
= $\{x \mid x = 2^n$ for some whole number $n\}$
range = $\{0, 1, 2, 3, ...\}$
= $\{y \mid y$ is any whole number$\}$
52. Function
domain = $\{x \mid x$ is any integer$\}$
range = $\{y \mid y$ is the square of an integer$\}$

53. Not a function

54. Function
domain = $\{x \mid -2 \leq x < 4\} = [-2, 4)$,
range = $\{3\}$

55. Function
domain = $\{x \mid x$ is a real number$\} = (-\infty, \infty)$
range = $\{y \mid y \geq 0\} = [0, \infty)$

56. Not a function

57. **Horizontal Line Test:** A graph on the xy-plane represents x as a function of y if and only if no **horizontal** line intersects the graph more than once.

58. Function
domain = $\{-4, -3, -2, -1, 0, 1\}$
range = $\{-1, 0, 1, 2, 3, 4\}$

59. Not a function

60. Function
domain = $(-\infty, \infty)$
range = $[1, \infty)$

61. Not a function

62.
- Number 58 represents x as a function of y.
  domain = $\{-1, 0, 1, 2, 3, 4\}$ and range = $\{-4, -3, -2, -1, 0, 1\}$
- Number 61 represents x as a function of y.
  domain = $(-\infty, \infty)$ and range = $[1, \infty)$

63. Function
domain = $[2, \infty)$
range = $[0, \infty)$

64. Function
domain = $(-\infty, \infty)$
range = $(0, 4]$

65. Not a function

66. Function
domain = $[-5, -3) \cup (-3, 3)$
range = $(-2, -1) \cup [0, 4)$

67. Only number 63 represents v as a function of w; domain = $[0, \infty)$ and range = $[2, \infty)$

68. Function
domain = $[-2, \infty)$
range = $[-3, \infty)$

69. Not a function

70. Function
domain = $(-5, 4)$
range = $(-4, 4)$

71. Function
domain = $[0, 3) \cup (3, 6]$
range = $(-4, -1] \cup [0, 4]$

72. None of numbers 68 - 71 represent t as a function of T.

73. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

74. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

75. Function
domain = $[-2, \infty)$
range = $(-\infty, 3]$

76. Function
domain = $(-\infty, \infty)$
range = $(-\infty, \infty)$

77. Only number 75 represents s as a function of H; domain = $(-\infty, 3]$ and range = $[-2, \infty)$

78. Function
domain = $(-\infty, 0] \cup (1, \infty)$
range = $(-\infty, 1] \cup \{2\}$

79. Function
domain = $[-3, 3]$
range = $[-2, 2]$

80. Not a function

81. Function
domain = $(-\infty, \infty)$
range = $\{2\}$

82. Only number 80 represents t as a function of u; domain = $(-\infty, \infty)$ and range=$\{2\}$.

83. $f(-2) = 2$

84. $g(-2) = -5$

85. $f(2) = 3$

86. $g(2) = 3$

87. $f(0) = -1$

88. $g(0) = 0$

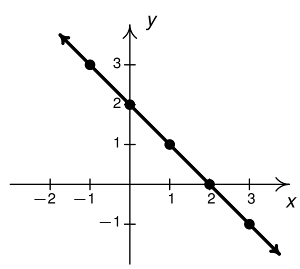
89. $x = -4, -1, 1$

90. $t = -4, 0, 4$

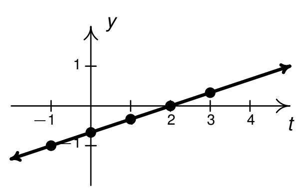
91. Domain: $[-5, 3]$, Range: $[-5, 4]$.

92. Domain: $[-4, 4]$, Range: $[-5, 5)$.

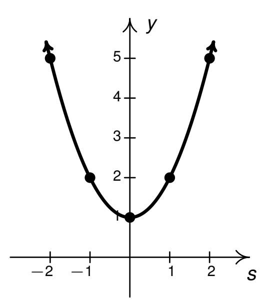
93. $f(x) = 2 - x$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



94. $g(t) = \dfrac{t-2}{3}$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



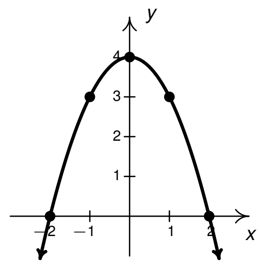
95. $h(s) = s^2 + 1$
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



96. $f(x) = 4 - x^2$

Domain: $(-\infty, \infty)$

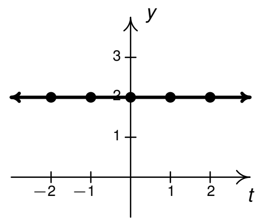
Range: $(-\infty, 4]$



97. $g(t) = 2$

Domain: $(-\infty, \infty)$

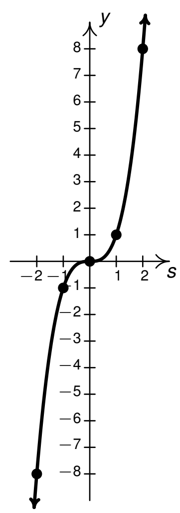
Range: $\{2\}$



98. $h(s) = s^3$

Domain: $(-\infty, \infty)$

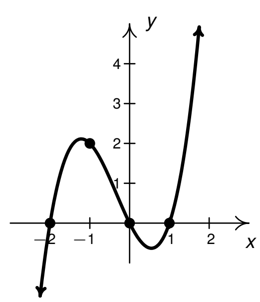
Range: $(-\infty, \infty)$



99. $f(x) = x(x-1)(x+2)$

Domain: $(-\infty, \infty)$

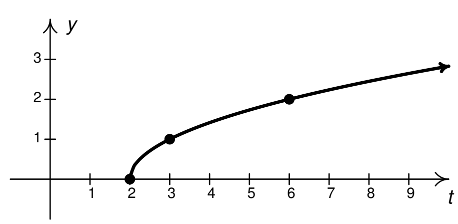
Range: $(-\infty, \infty)$



100. $g(t) = \sqrt{t-2}$

Domain: $[2, \infty)$

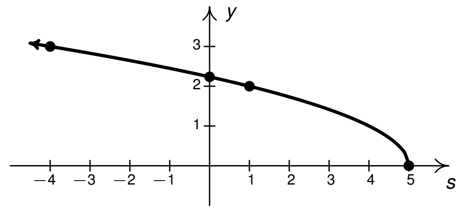
Domain: $[0, \infty)$



101. $h(s) = \sqrt{5-s}$

Domain: $(-\infty, 5]$

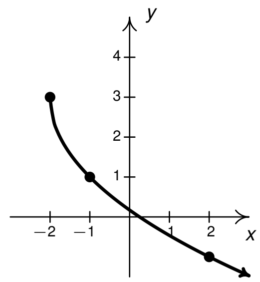
Range: $[0, \infty)$



102. $f(x) = 3 - 2\sqrt{x+2}$

Domain: $[-2, \infty)$

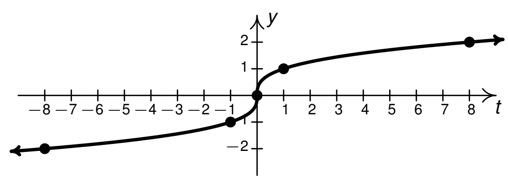
Range: $(-\infty, 3]$



103. $g(t) = \sqrt[3]{t}$

Domain: $(-\infty, \infty)$

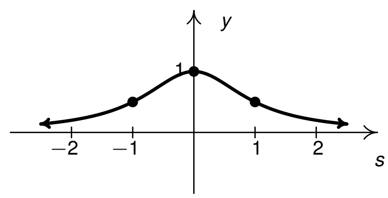
Range: $(-\infty, \infty)$



104. $h(s) = \dfrac{1}{s^2+1}$

Domain: $(-\infty, \infty)$

Range: $(0, 1]$

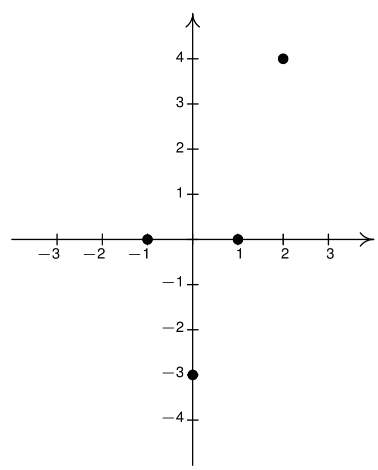


105. (a) domain = $\{-1, 0, 1, 2\}$, range = $\{-3, 0, 4\}$

(b) $f(0) = -3$, $f(x) = 0$ for $x = -1, 1$.

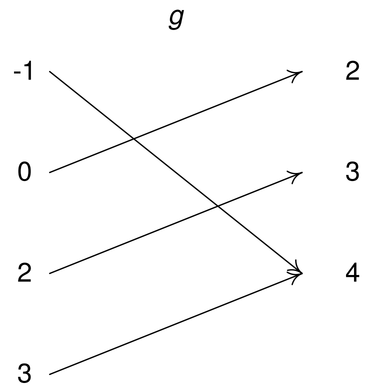
(c) $f = \{(-1, 0), (0, -3), (1, 0), (2, 4)\}$

(d)



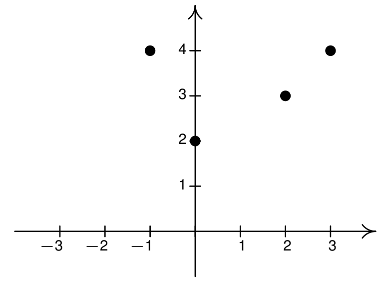
106. (a) domain = $\{-1, 0, 2, 3\}$, range = $\{2, 3, 4\}$

(b)



(c) Find $g(0) = 2$ and $g(x) = 0$ has no solutions.

(d)



107. $F(4) = 4^2 = 16$ (when $t = 4$), the solutions to $F(x) = 4$ are $x = \pm 2$ (when $t = \pm 2$).

108. $G(4) = 7$ (when $t = 2$), the solution to $G(t) = 4$ is $x = -2$ (when $t = -1$)

109. $A(3) = 9$, so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to $A(\ell) = 36$ are $\ell = \pm 6$. Since ℓ is restricted to $\ell > 0$, we only keep $\ell = 6$. This means for the area enclosed by the square to be 36 square inches, the length of the side needs to be 6 inches. Since ℓ represents a length, $\ell > 0$.

110. $A(2) = 4\pi$, so the area enclosed by a circle with radius 2 meters is 4π square meters. The solutions to $A(r) = 16\pi$ are $r = \pm 4$. Since r is restricted to $r > 0$, we only keep $r = 4$. This means for the area enclosed by the circle to be 16π square meters, the radius needs to be 4 meters. Since r represents a radius (length), $r > 0$.

111. $V(5) = 125$, so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to $V(s) = 27$ is $s = 3$. This means for the volume enclosed by the cube to be 27 cubic centimeters, the length of the side needs to 3 centimeters. Since x represents a length, $x > 0$.

112. $V(3) = 36\pi$, so the volume enclosed by a sphere with radius 3 feet is 36π cubic feet. The solution to $V(r) = \frac{32\pi}{3}$ is $r = 2$. This means for the volume enclosed by the sphere to be $\frac{32\pi}{3}$ cubic feet, the radius needs to 2 feet. Since r represents a radius (length), $r > 0$.

113. $h(0) = 64$, so at the moment the object is dropped off the building, the object is 64 feet off of the ground. The solutions to $h(t) = 0$ are $t = \pm 2$. Since we restrict $0 \leq t \leq 2$, we only keep $t = 2$. This means 2 seconds after the object is dropped off the building, it is 0 feet off the ground. Said differently, the object hits the ground after 2 seconds. The restriction $0 \leq t \leq 2$ restricts the time to be between the moment the object is released and the moment it hits the ground.

114. $T(0) = 3$, so at 6 AM (0 hours after 6 AM), it is 3° Fahrenheit. $T(6) = 33$, so at noon (6 hours after 6 AM), the temperature is 33° Fahrenheit. $T(12) = 27$, so at 6 PM (12 hours after 6 AM), it is 27° Fahrenheit.

115. $C(0) = 27$, so to make 0 pens, it costs[25] \$2700. $C(2) = 11$, so to make 2000 pens, it costs \$1100. $C(5) = 2$, so to make 5000 pens, it costs \$2000.

116. $E(0) = 16.00$, so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon. $E(14) = 20.81$, so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon. $E(28) = 22.64$, so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.

117. $P(s) = 4s$, $s > 0$.

118. $C(D) = \pi D$, $D > 0$.

119. (a) The amount in the retirement account after 30 years if the monthly payment is \$50.
     (b) The solution to $A(P) = 250000$ is what the monthly payment needs to be in order to have \$250,000 in the retirement account after 30 years.
     (c) $A(P + 50)$ is how much is in the retirement account in 30 years if \$50 is added to the monthly payment P. $A(P) + 50$ represents the amount of money in the retirement account after 30 years if \$$P$ is invested each month plus an additional \$50. $A(P) + A(50)$ is the sum of money from two retirement accounts after 30 years: one with monthly payment \$$P$ and one with monthly payment \$50.

120. (a) Since noon is 4 hours after 8 AM, $P(4)$ gives the chance of precipitation at noon.
     (b) We would need to solve $P(t) \geq 50\%$ or $P(t) \geq 0.5$.

121. The graph in question passes the horizontal line test meaning for each w there is only one v. The domain of g is $[0, \infty)$ (which is the range of f) and the range of g is $[2, \infty)$ which is the domain of f.

122. Answers vary.

[25] This is called the 'fixed' or 'start-up' cost. We'll revisit this concept in Example 3.2.3 in Section 3.2.

2.2 Graphs of Functions

Up until this point in the text, we have primarily focused on studying particular *families* of functions. These families and their relationships to one another provide useful *examples* of more abstract function structures and relationships. The notions introduced in this chapter will not only provide us a more formal vocabulary with which to describe the connections between the function families we have already studied, but, more importantly, give us additional lenses through which to view new families of functions that we'll encounter.

In this section, we review of the concepts associated with the graphs of functions. We introduced the notion of the graph of a function in Section 2.1, and the vast majority of the graphs we have encountered in this text were generated from an algebraic representation of a function. In this section, we define the functions geometrically from the outset and review the important concepts associated with the graphs of functions.

Recall the **domain** of a function is the set of inputs to the function and the **range** of a function is the set of outputs from the function. When graphing a function whose domain and range are subsets of real numbers, we plot the ordered pairs (input, output) on the Cartesian plane. Hence, the domain values are found on the horizontal axis while the range values are found on the vertical axis.

Recall from Definition 2.3 that the largest output from the function (if there is one) is called the **maximum** or, when there may be some confusion, the **absolute maximum** of the function. Likewise, the smallest output from the function (again, if there is one) is called the **minimum** or **absolute minimum**.

A concept related to 'absolute' maximum and minimum is the concept of 'local' maximum and minimum as described in Definition 6.7. Here, a point (a, b) on the graph of a function f is a **local maximum** if b is the maximum function value for some open interval in the domain containing a. The notion of 'local' here meaning instead of surveying the entire domain, we instead restrict our attention to inputs 'local' or 'near' the input a. The concept of **local minimum** is defined similarly.

Next, we review the notions of **increasing**, **decreasing**, and **constant** as described in Definition 3.6. Recall a function is increasing over an interval if, as the inputs increase, do the outputs. This means that, geometrically, the graph of the function rises as we move left to right. Similarly, a function is decreasing over an interval if the outputs decrease as the inputs increase. Geometrically, a decreasing function falls as we move left to right. Finally, a function is constant over an interval if the output is the same regardless of the input. If a function is constant over an interval, its graph remains 'flat' - a horizontal line.

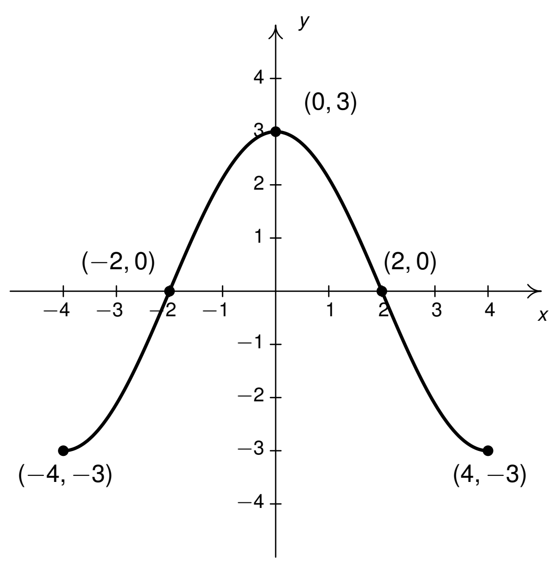
Last, and according to some[1] least, we briefly review the notion of symmetry in the graphs of functions. Recall from Definition 6.2 that a function f is called **even** if $f(-x) = f(x)$ for all x in the domain of f. The graphs of even functions are symmetric about the vertical (usually y-) axis. In a similar manner, Definition 6.3 tells us a function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f. Geometrically, the graphs of odd functions are symmetric about the origin.

The next example reviews all of the aforementioned concepts as well as many more.

Example 2.2.1. Given the graph of $y = f(x)$ below, answer all of the following questions.

[1] Jeff

1. Find the domain of f.
2. Find the range of f.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the x-intercepts, if any exist.
6. List the y-intercepts, if any exist.
7. Find the zeros of f.
8. Solve $f(x) < 0$.
9. Determine $f(2)$.
10. Solve $f(x) = -3$.
11. Find the number of solutions to $f(x) = 1$.
12. Does f appear to be even, odd, or neither?
13. List the local maximums, if any exist.
14. List the local minimums, if any exist.
15. List the intervals on which f is increasing.
16. List the intervals on which f is decreasing.



Solution.

1. To find the domain of f, we proceed as in Section 2.1. By projecting the graph to the x-axis, we see that the portion of the x-axis which corresponds to a point on the graph is everything from -4 to 4, inclusive. Hence, the domain is $[-4, 4]$.

2. To find the range, we project the graph to the y-axis. We see that the y values from -3 to 3, inclusive, constitute the range of f. Hence, our answer is $[-3, 3]$.

3. The maximum value of f is the largest y-coordinate which is 3.

4. The minimum value of f is the smallest y-coordinate which is -3.

5. The x-intercepts are the points on the graph with y-coordinate 0, namely $(-2, 0)$ and $(2, 0)$.

6. The y-intercept is the point on the graph with x-coordinate 0, namely $(0, 3)$.

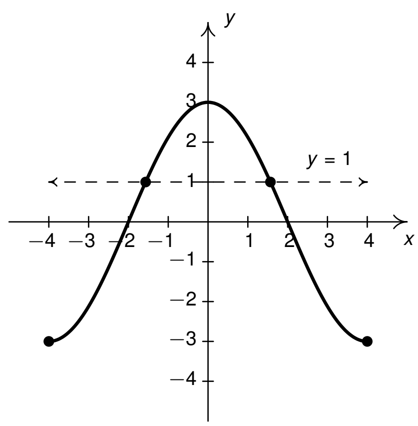
7. The zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$ which are $x = -2, 2$.

8. To solve $f(x) < 0$, we look for the x values of the points on the graph where the $y = f(x)$ is negative. Graphically, we are looking for where the graph is *below* the x-axis. This happens for the x values from -4 to -2 and again from 2 to 4. So our answer is $[-4, -2) \cup (2, 4]$.

9. Since the graph of f is the graph of the equation $y = f(x)$, $f(2)$ is the y-coordinate of the point which corresponds to $x = 2$. Since the point $(2, 0)$ is on the graph, we have $f(2) = 0$.

10. To solve $f(x) = -3$, we look where $y = f(x) = -3$. We find two points with a y-coordinate of -3, namely $(-4, -3)$ and $(4, -3)$. Hence, the solutions to $f(x) = -3$ are $x = \pm 4$.

11. As in the previous problem, to solve $f(x) = 1$, we look for points on the graph where the y-coordinate is 1. If we imagine the horizontal like $y = 1$ superimposed over the graph of f as sketched below, we get two intersections. Hence, even though these points aren't specified, we know there are *two* points on the graph of f whose y-coordinate is 1. Hence, there are two solutions to $f(x) = 1$.



12. The graph appears to be symmetric about the y-axis. This suggests[2] that f is even.

13. The function has its only local maximum at $(0, 3)$.

14. There are no local minimums. Why don't $(-4, -3)$ and $(4, -3)$ count? Let's consider the point $(-4, -3)$ for a moment. Recall that, in the definition of local minimum, there needs to be an open interval containing $x = -4$ which is in the domain of f. In this case, there is no open interval containing $x = -4$ which lies entirely in the domain of f, $[-4, 4]$. Because we are unable to fulfill the requirements of the definition for a local minimum, we cannot claim that f has one at $(-4, -3)$. The point $(4, -3)$ fails for the same reason $-$ no open interval around $x = 4$ stays within the domain of f.

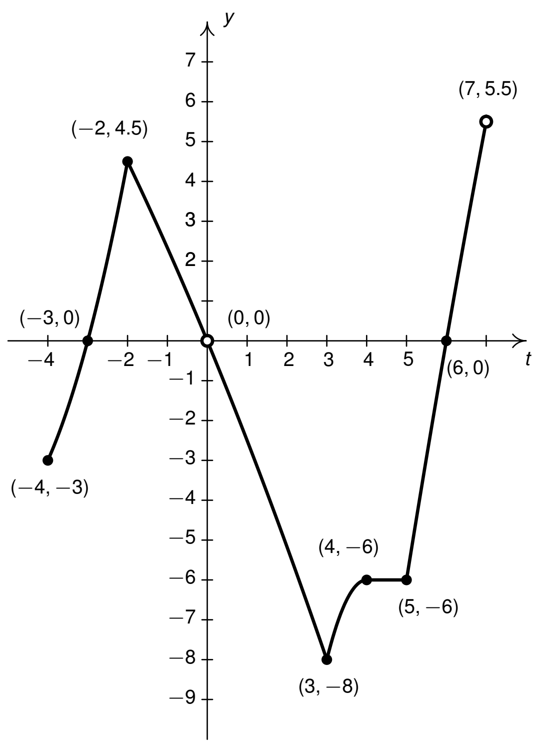
[2]but does not prove

15. As we move from left to right, the graph rises from $(-4,-3)$ to $(0,3)$. This means f is increasing on the interval $[-4,0]$. (Remember, the answer here is an interval on the x-axis.)

16. As we move from left to right, the graph falls from $(0,3)$ to $(4,-3)$. This means f is decreasing on the interval $[0,4]$. (Again, the answer here is an interval on the x-axis.) $\square$

Our next example involves a more complicated function and asks more complicated questions.

Example 2.2.2. Consider the graph of the function g below.



The graph of $y = g(t)$

1. Find the domain of g.
2. Find the range of g.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the local maximums, if any exist.
6. List the local minimums, if any exist.
7. Solve $(t^2-25)g(t) = 0$.
8. Solve $\dfrac{g(t)}{t^2+t-30} \geq 0$.

Solution.

1. Projecting the graph of g to the t-axis, we see the domain contains values of t from -4 up to, but not including $t = 0$ and values greater than $t = 0$ up to, but not including $t = 7$. Using interval notation, we write the domain as $[-4,0) \cup (0,7)$.

2. Projecting the graph of g to the y-axis, we see the range of g contains all real numbers from $y = -8$ up to, but not including, $y = 5.5$. Note that even though there is a hole in the graph at $(0, 0)$, the points $(-3, 0)$ and $(6, 0)$ put $y = 0$ in the range of g. Hence, the range of g is $[-8, 5.5)$.

3. Owing to the hole in the graph at $(7, 5.5)$, g has no maximum.[3]

4. The minimum of g is -8 which occurs at the point $(3, -8)$.

5. The point $(-2, 4.5)$ is clearly a local maximum, but there are actually infinitely many more. Per Definition 6.7, all points of the form $(t, -6)$ for $4 \leq t < 5$ are also local maximums. For each of these points, we can find an open interval on the t axis within which we produce no points on the graph higher than $(t, -6)$. (You may think about 'zooming in' on the point $(4.5, -6)$ to see how this works.)

6. The local minimums of the graph are $(3, -8)$ along with points of the form $(t, -6)$ for $4 < t \leq 5$. Note the point $(-4, -3)$ is not a local minimum since there is no open interval containing $t = -4$ which lies entirely within the domain of g.

7. To solve $(t^2 - 25)g(t) = 0$, we use the zero product property of real numbers[4] to conclude either $t^2 - 25 = 0$ or $g(t) = 0$.

   From $t^2 - 25 = 0$, we get $t = \pm 5$. However, since $t = -5$ isn't in the domain of g, it cannot be regarded as a solution to the equation $(t^2 - 25)g(t) = 0$. (If we substitute $t = -5$ into the equation, we'd get $((-5)^2 - 25)g(-5) = 0 \cdot g(-5)$. Since $g(-5)$ is undefined, so is $0 \cdot g(-5)$.)

   To solve $g(t) = 0$, we look for the zeros of g which are $t = -3$ and $t = 6$. (Again, there is a hole at $(0, 0)$, so $t = 0$ doesn't count as a zero.) Our final answer to $(t^2 - 25)g(t) = 0$ is $t = -3$, 5, or 6.

8. To solve $\frac{g(t)}{t^2+t-30} \geq 0$, we employ a sign diagram as we (most recently) have done in Section **??**.[5] To that end, we define $F(t) = \frac{g(t)}{t^2+t-30}$ and we set about finding the domain of f.

   First, we note that since F is defined in terms of g, the domain of F is restricted to some subset of the domain of g, namely $[-4, 0) \cup (0, 7)$. Since $t^2 + t - 30$ is in the denominator of $F(t)$, we must also exclude the values where $t^2 + t - 30 = (t+6)(t-5) = 0$. Hence, we must exclude $t = -6$ (which isn't in the domain of g in the first place) along with $t = 5$. Hence, the domain of F is $[-4, 0) \cup (0, 5) \cup (5, 7)$.

   Next, we find the zeros of F. Setting $F(t) = \frac{g(t)}{t^2+t-30} = 0$ amounts to solving $g(t) = 0$. Graphically, we see this occurs when $t = -3$ and $t = 6$. Hence, we need to select test values in each of the following intervals: $[-4, -3)$, $(-3, 0)$, $(0, 5)$, $(5, 6)$ and $(6, 7)$.

   For the interval $[-4, -3)$, we may choose $t = -4$. $F(-4) = \frac{g(-4)}{(-4)^2+(-4)-30} = \frac{-3}{-18} > 0$ so is $(+)$. For the interval $(-3, 0)$ we choose $t = -2$ and get $F(-2) = \frac{g(-2)}{(-2)^2+(-2)-30} = \frac{4.5}{-28} < 0$ so is $(-)$. For the interval $(0, 5)$, we choose $t = 3$ and find $F(3) = \frac{g(3)}{(3)^2+(3)-30} = \frac{-8}{-18} > 0$ which is $(+)$ again.

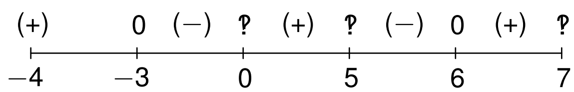
[3] There is no real number 'right before' 5.5 ...

[4] see Section 1.2, 18

[5] Note that g is continuous on its domain, and hence, it follows that $\frac{g(t)}{t^2+t-30}$ is, too. (Thank Calculus!) This means the Intermediate Value Theorem applies so a Sign Diagram approach is valid.

For the last two intervals, $(5, 6)$ and $(6, 7)$, we do not have specific function values for g. However, all we are interested in is the *sign* of the function over these intervals, and we can get that information about g graphically.

For the interval $(5, 6)$, we choose $t = 5.5$ as our test value. Since the graph of $y = g(t)$ is *below* the t-axis when $t = 5.5$. we know $g(5.5)$ is $(-)$. Hence, $F(5.5) = \frac{g(5.5)}{(5.5)^2+(5.5)-30} = \frac{(-)}{5.75} < 0$ so is $(-)$. Similarly, when $t = 6.5$, the graph of $y = g(t)$ is *above* the t-axis so $F(6.5) = \frac{g(6.5)}{(6.5)^2+(6.5)-30} = \frac{(+)}{18.75} > 0$ so is $(+)$. Putting all of this together, we get the sign diagram for $F(t) = \frac{g(t)}{t^2+t-30}$ below:

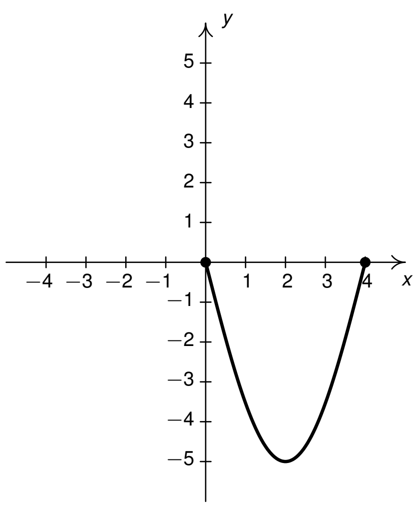


Hence, $F(t) \geq 0$ on $[-4, -3] \cup (0, 5) \cup [6, 7)$. □

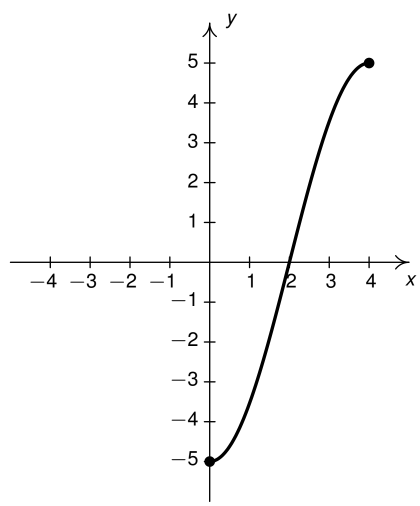
Our last example focuses on symmetry. The reader is encouraged to review the notes about symmetry as summarized on page **??** in Section **??**.

Example 2.2.3. Below are the partial graphs of functions f and g.

1. If possible, complete the graphs of f and g assuming both functions are even.
2. If possible, complete the graphs of f and g assuming both functions are odd.



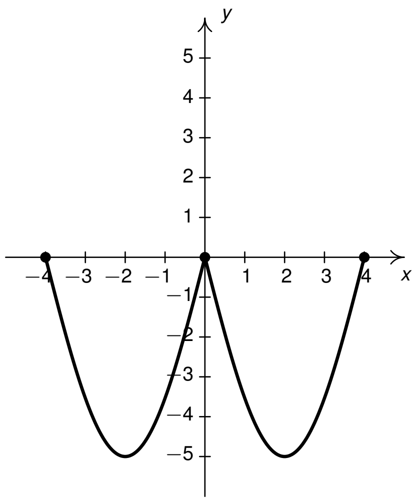
Partial graph of $y = f(x)$



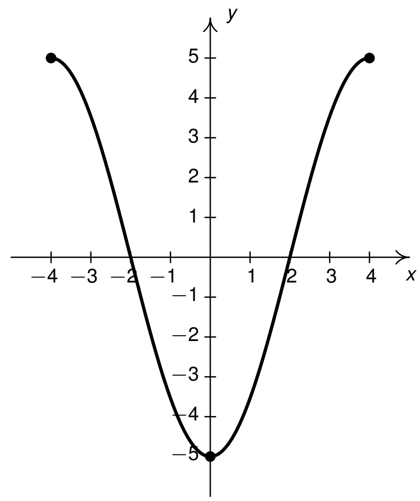
Partial graph of $y = g(x)$

Solution.

1. If f and g are even then their graphs are symmetric about the y-axis. Hence, to complete each graph, we reflect each point on the graphs of f and g about the y-axis.



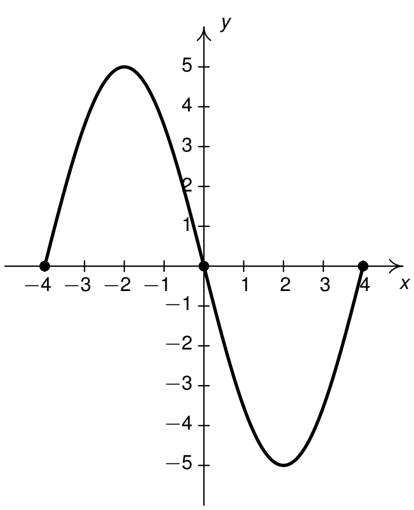
The graph of f assuming f is even.



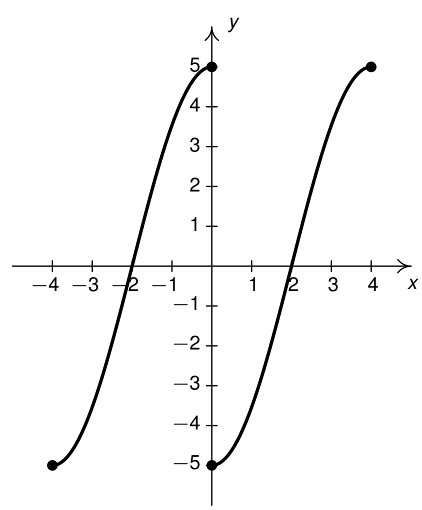
The graph of g assuming g is even.

2. If f and g are odd then their graphs are symmetric about the origin. Hence, to complete each graph, we imagine reflecting each of the points on their graphs through the origin. We complete the process on the graph of f with no issues.

   However, when attempting to do the same with the graph of the function g, we find the point $(0, -5)$ is reflected to the point $(0, 5)$. Hence, this new graph doesn't pass the vertical line test and hence is not a function. Therefore, g cannot be odd.[6]



The graph of f assuming f is odd.



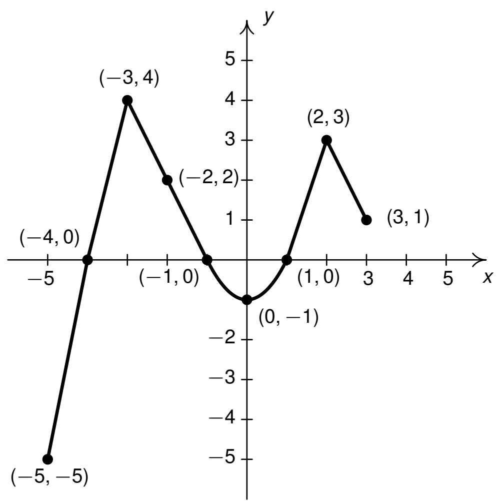
This graph fails the vertical line test.

□

[6]We leave it as an exercise to show that if a function f is odd and 0 is in the domain of f, then, necessarily, $f(0) = 0$.

2.2.1 Exercises

In Exercises 1 - 4, use the graph of $y = f(x)$ given below to answer the question.



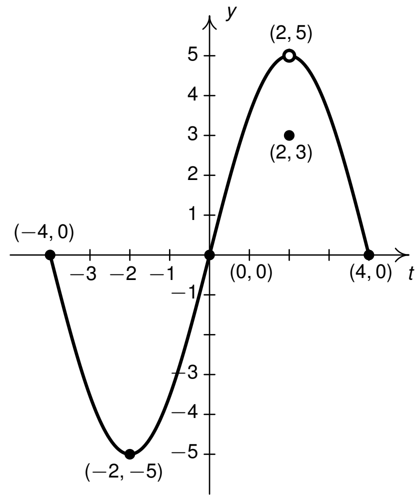
$y = f(x)$

1. Find the domain of f.
2. Find the range of f.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the local maximums, if any exist.
6. List the local minimums, if any exist.
7. List the intervals where f is increasing.
8. List the intervals where f is decreasing.
9. Determine $f(-2)$.
10. Solve $f(x) = 4$.
11. List the x-intercepts, if any exist.
12. List the y-intercepts, if any exist.
13. Find the zeros of f.
14. Solve $f(x) \geq 0$.
15. Find the number of solutions to $f(x) = 1$.
16. Find the number of solutions to $|f(x)| = 1$.
17. Solve $(x^2 - x - 2)f(x) = 0$
18. Solve $(x^2 - x - 2)f(x) > 0$

With help from your classmates:

19. Find the domain of $R(x) = \dfrac{1}{f(x)}$
20. Find the range of $R(x) = \dfrac{1}{f(x)}$

In Exercises 21 - 24, use the graph of $y = g(t)$ given below to answer the question.

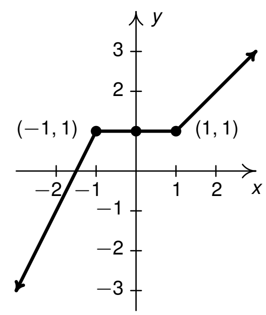


$y = g(t)$

21. Find the domain of g.
22. Find the range of g.
23. Find the maximum, if it exists.
24. Find the minimum, if it exists.
25. List the local maximums, if any exist.
26. List the local minimums, if any exist.
27. List the intervals where g is increasing.
28. List the intervals where g is decreasing.
29. Determine $g(2)$.
30. Solve $g(t) = -5$.
31. List the t-intercepts, if any exist.
32. List the y-intercepts, if any exist.
33. Find the zeros of g.
34. Solve $g(t) \leq 0$.
35. Find the domain of $G(t) = \dfrac{g(t)}{x+2}$.
36. Solve $\dfrac{g(t)}{x+2} \leq 0$.
37. How many solutions are there to $[g(t)]^2 = 9$?
38. Does g appear to be even, odd, or neither?
39. Prove that if f is an odd function and 0 is in the domain of f, then $f(0) = 0$.
40. Let $R(x)$ be the function defined as: $R(x) = 1$ if x is a rational number, $R(x) = 0$ if x is an irrational number. With help from your classmates, try to graph R. What difficulties do you encounter?

    NOTE: Between every pair of real numbers, there is both a rational and an irrational number ...

41. Consider the graph of the function f given below.



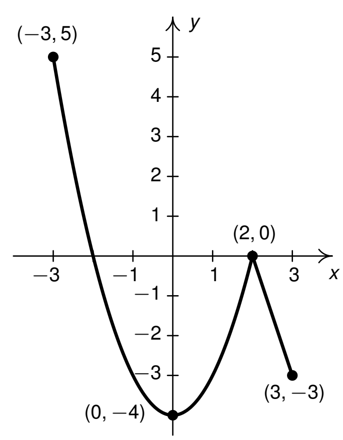
(a) Explain why f has a local maximum but not a local minimum at the point $(-1, 1)$.

(b) Explain why f has a local minimum but not a local maximum at the point $(1, 1)$.

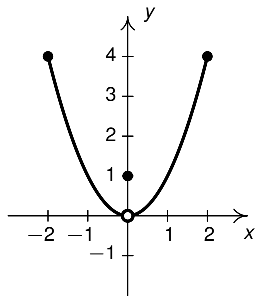
(c) Explain why f has a local maximum AND a local minimum at the point $(0, 1)$.

(d) Explain why f is constant on the interval $[-1, 1]$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1 < x < 1$.

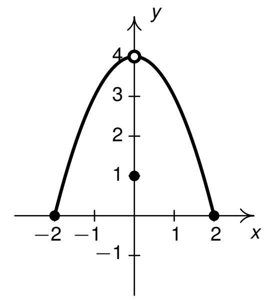
42. Explain why the function g whose graph is given below does <u>not</u> have a local maximum at $(-3, 5)$ nor does it have a local minimum at $(3, -3)$. Find its extrema, both local and absolute and find the intervals on which g is increasing and those on which g is decreasing.



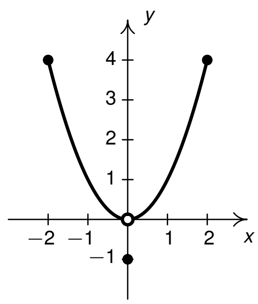
43. For each function below, find the local maximum or local minimum and list the interval over which the function is increasing and the interval over which the function is decreasing.



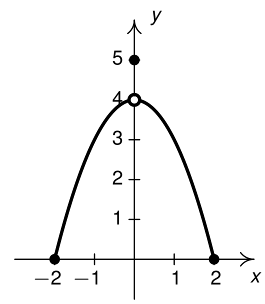
(a) Function I



(b) Function II



(c) Function III



(d) Function IV

2.2.2 Answers

1. $[-5, 3]$
2. $[-5, 4]$
3. $f(-3) = 4$
4. $f(-5) = -5$
5. $(-3, 4)$, $(2, 3)$
6. $(0, -1)$
7. $[-5, -3]$, $[0, 2]$
8. $[-3, 0]$, $[2, 3]$
9. $f(-2) = 2$
10. $x = -3$
11. $(-4, 0)$, $(-1, 0)$, $(1, 0)$
12. $(0, -1)$
13. -4, -1, 1
14. $[-4, -1]$, $[1, 3]$
15. 4
16. 6
17. $x = -4, -1, 1, 2$
18. $(-4, -1) \cup (-1, 1) \cup (2, 3)$
19. To find the domain of $R(x) = \frac{1}{f(x)}$, we start with the domain of f and exclude values where $f(x) = 0$. Hence, the domain of R is $[-5, -4) \cup (-4, -1) \cup (-1, 1) \cup (1, 3]$.
20. To find the range of $R(x) = \frac{1}{f(x)}$, we start with the range of f (excluding 0) and take reciprocals. If $-5 \leq y < 0$, then $\frac{1}{y} \leq -\frac{1}{5}$. If $0 < y \leq 4$, then $\frac{1}{y} \geq \frac{1}{4}$. Hence the range of R is $\left(-\infty, -\frac{1}{5}\right] \cup \left[\frac{1}{4}, \infty\right)$.
21. $[-4, 4]$
22. $[-5, 5)$
23. none
24. $g(-2) = -5$
25. none
26. $(-2, -5)$, $(2, 3)$
27. $[-2, 2)$
28. $[-4, -2]$, $(2, 4]$
29. $g(2) = 3$
30. $t = -2$
31. $(-4, 0)$, $(0, 0)$, $(4, 0)$
32. $(0, 0)$
33. -4, 0, 4
34. $[-4, 0] \cup \{4\}$
35. $[-4, -2) \cup (-2.4]$
36. $\{-4\} \cup (-2, 0] \cup \{4\}$
37. 5
38. Neither.

43. (a) Local maximum: $(0, 1)$, no local minimum. Increasing: $(0, 2]$, decreasing: $[-2, 0)$.
    (b) No local maximum, local minimum: $(0, 1)$. Increasing: $[-2, 0)$, decreasing: $(0, 2]$.
    (c) No local maximum, local minimum: $(0, -1)$. Increasing: $[0, 2]$, decreasing: $[-2, 0]$.
    (d) Local maximum: $(0, 5)$, no local minimum. Increasing: $[-2, 0]$, decreasing: $[0, 2]$.

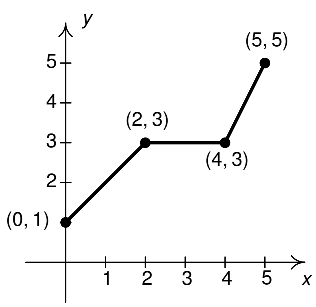
2.3 Transformations of Graphs

Theorems 4.4, 5.7, 6.1, 7.1, 8.2 and **??** all describe ways in which the graph of a function can changed, or 'transformed' to obtain the graph of a related function. The results and proofs of each of these theorems are virtually identical, and with the language of function composition, we can see better why.

Consider, for instance, Theorem **??**, in which we describe how to transform the graph of $f(x) = x^r$ to $F(x) = a(bx - h)^r + k$. We may think of F as being build up from f by composing f with linear functions. Specifically, if we let $i(x) = bx - h$, then $(f \circ i)(x) = f(i(x)) = f(bx - h) = (bx - h)^r$. If, additionally, we let and $j(x) = ax + k$, then $(j \circ (f \circ i))(x) = j((f \circ i)(x)) = j((bx - h)^r) = a(bx - h)^r + k = F(x)$. Hence, we can view $F = j \circ f \circ i$.

In this section, our goal is to generalize the aforementioned theorems to the graphs of *all* functions. Along the way, you'll see some very familiar arguments, but, additionally, we hope this section affords the reader an opportunity to not only see *how* these transformations work they way they do, but *why*.

Our motivational example for the results in this section is the graph of $y = f(x)$ below. While we could formulate an expression for $f(x)$ as a piecewise-defined function consisting of linear and constant parts, we wish to focus more on the geometry here. That being said, we do record some of the function values - the 'key points' if you will - to track through each transformation.



$y = f(x)$

x	$(x, f(x))$	$f(x)$
0	$(0, 1)$	1
2	$(2, 3)$	3
4	$(4, 3)$	3
5	$(5, 5)$	5

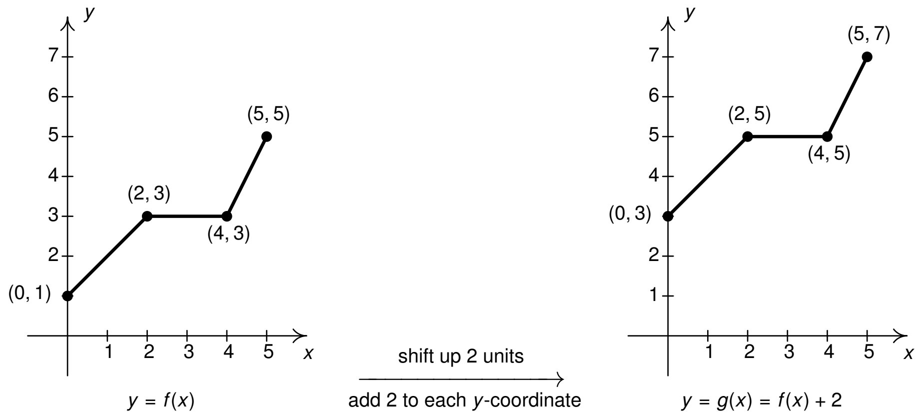
2.3.1 Vertical and Horizontal Shifts

Suppose we wished to graph $g(x) = f(x) + 2$. From a procedural point of view, we start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. Using the sample values for f from the table above we can create a table of values for g below, hence generating points on the graph of g.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	$1 + 2 = 3$	$(0, 3)$
2	$(2, 3)$	3	$3 + 2 = 5$	$(2, 5)$
4	$(4, 3)$	3	$3 + 2 = 5$	$(4, 5)$
5	$(5, 5)$	5	$5 + 2 = 7$	$(5, 7)$

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Hence, $g(a) = f(a) + 2 = b + 2$, so the point $(a, b+2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f.

Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* is moves or 'shifts' the entire graph of f up 2 units. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four 'key points' we moved in the same manner in which they were connected before.



You'll note that the domain of f and the domain of g are the same, namely $[0, 5]$, but that the range of f is $[1, 5]$ while the range of g is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range.

You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1, 3]$. In general, we have:

Theorem 2.2. Vertical Shifts. Suppose f is a function and k is a real number.
To graph $F(x) = f(x) + k$, add k to each of the y-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a vertical shift up k units if $k > 0$ or down k units if $k < 0$.

To prove Theorem 2.2, we first note that f and F have the same domain (why?) Let c be an element in the domain of F and, hence, the domain of f. The fact that f and F are *functions* guarantees there is *exactly one* point on each of their graphs corresponding to $x = c$. On $y = f(x)$, this point is $(c, f(c))$; on $y = F(x)$, this point is $(c, F(c)) = (c, f(c)+k)$. This sets up a nice correspondence between the two graphs and shows that each of the points on the graph of F can be obtained to by adding k to each of the y-coordinates of the corresponding point on the graph of f. This proves Theorem 2.2. In the language of 'inputs' and 'outputs', Theorem 2.2 says adding to the *output* of a function causes the graph to shift *vertically*.

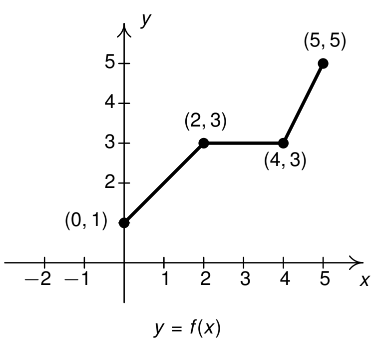
Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function. Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties. For instance, when we substitute $x = 4$ into the formula $g(x) = f(x + 2)$, we are asked to find $f(4 + 2) = f(6)$ which doesn't exist because the domain of f is only $[0, 5]$. The same thing happens when we attempt to find $g(5)$.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	$(0,1)$	1	$g(0) = f(0+2) = f(2) = 3$	$(0,3)$
2	$(2,3)$	3	$g(2) = f(2+2) = f(4) = 3$	$(2,3)$
4	$(4,3)$	3	$g(4) = f(4+2) = f(6) = ?$	
5	$(5,5)$	5	$g(5) = f(5+2) = f(7) = ?$	

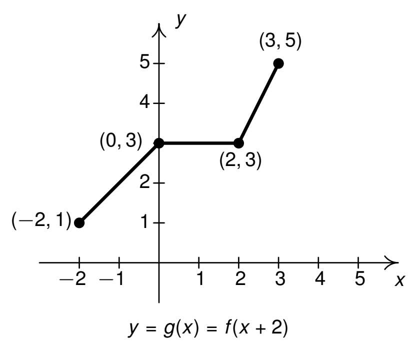
What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x+2$, works out to be 0. Solving $x + 2 = 0$ gives $x = -2$, and $g(-2) = f((-2) + 2) = f(0) = 1$ so $(-2, 1)$ on the graph of g. To use the fact $f(2) = 3$, we set $x + 2 = 2$ to get $x = 0$. Substituting gives $g(0) = f(0 + 2) = f(2) = 3$. Continuing in this fashion, we produce the table below.

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(-2+2) = f(0) = 1$	$(-2,1)$
0	2	$g(0) = f(0+2) = f(2) = 3$	$(0,3)$
2	4	$g(2) = f(2+2) = f(4) = 3$	$(2,3)$
3	5	$g(3) = f(3+2) = f(5) = 5$	$(3,5)$

In summary, the points $(0, 1)$, $(2, 3)$, $(4, 3)$ and $(5, 5)$ on the graph of $y = f(x)$ give rise to the points $(-2, 1)$, $(0, 3)$, $(2, 3)$ and $(3, 5)$ on the graph of $y = g(x)$, respectively. In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x + 2 = a$ gives $x = a - 2$ so that $g(a - 2) = f((a - 2) + 2) = f(a) = b$. As such, $(a - 2, b)$ is on the graph of $y = g(x)$. The point $(a - 2, b)$ is exactly 2 units to the *left* of the point (a, b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.



shift left 2 units $\longrightarrow$ subtract 2 from each x-coordinate



Note that while the ranges of f and g are the same, the domain of g is $[-2, 3]$ whereas the domain of f is $[0, 5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x - 2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

Theorem 2.3. Horizontal Shifts. Suppose f is a function and h is a real number.
To graph $F(x) = f(x - h)$, add h to each of the x-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a horizontal shift right h units if $h > 0$ or left h units if $h < 0$.

To prove Theorem 2.3, we first note the domains of f and F may be different. If c is in the domain of f, then the only number we know for sure is in the domain of F is $c + h$, since $F(c + h) = f((c + h) - h) = f(c)$. This sets up a nice correspondence between the domain of f and the domain of F which spills over to a correspondence between their graphs, The point $(c, f(c))$ is the one and only point on the graph of $y = f(x)$ corresponding to $x = c$ just as the point $(c + h, F(c + h)) = (c + h, f(c))$ is the one and only point on the graph of $y = F(x)$ corresponding to $x = c + h$. This correspondence shows we may obtain the graph of F by adding h to each x-coordinate of each point on the graph of f, which establishes the theorem. In words, Theorem 2.3 says that subtracting from the *input* to a function amounts to shifting the graph *horizontally*.

Theorems 2.2 and 2.3 present a theme which will run common throughout the section: changes to the *outputs* from a function result in some kind of *vertical change*; changes to the *inputs* to a function result in some kind of *horizontal* change. We demonstrate Theorems 2.2 and 2.3 in the example below.

Example 2.3.1. Use Theorems 2.2 and 2.3 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 3)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = f(x) + 5$ (b) $y = f(x + 5)$ (c) $f(x - 7) + 4$

2. Find a formula for a function $g(t)$ whose graph is the same as $f(t) = |t| - 2t$ but is shifted:

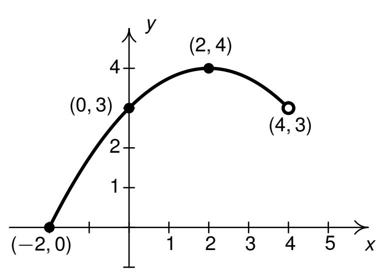
   (a) to the right 4 units. (b) down 2 units.

3. Predict how the graph of $F(x) = \dfrac{(x-2)^{\frac{2}{3}}}{x}$ relates to the graph of $f(x) = \dfrac{x^{\frac{2}{3}}}{x+2}$.

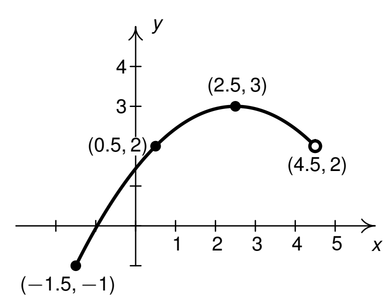
4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(x - 2)$ (b) $F(x) = f(x) + 1$ (c) $F(x) = f(x + 1) - 2$

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.



$y = f(x)$



$y = g(x)$

Solution.

1. (a) To apply Theorem 2.2, we identify $f(x) + 5 = f(x) + k$ so $k = 5$. Hence, we add 5 to the y-coordinate of $(-1, 3)$ and get $(-1, 3 + 5) = (-1, 8)$. To check our answer note since $(-1, 3)$ is on the graph of f this means $f(-1) = 3$. Substituting $x = -1$ into the formula $y = f(x) + 5$, we get $y = f(-1) + 5 = 3 + 5 = 8$. Hence, $(-1, 8)$ is on the graph of $f(x) + 5$.

   (b) We note that $f(x + 5)$ can be written as $f(x - (-5)) = f(x - h)$ so we apply Theorem 2.3 with $h = -5$. Adding -5 to (subtracting 5 from) the x-coordinate of $(-1, 3)$ gives $(-1 + (-5), 3) = (-6, 3)$. To check our answer, since $(-1, 3)$ is on the graph of f, $f(-1) = 3$. Substituting $x = -6$ into $y = f(x + 5)$ gives $y = f(-6 + 5) = f(-1) = 3$, proving $(-6, 3)$ is on the graph of $y = f(x + 5)$.

   (c) Note that the expression $f(x - 7) + 4$ differs from $f(x)$ in *two* ways indicating two different transformations. In situations like this, its best if we handle each transformation in turn, starting with the graph of $y = f(x)$ and 'building up' to the graph of $y = f(x - 7) + 4$.

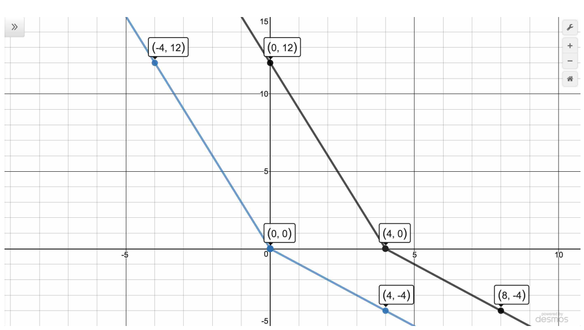
   We choose to work from the 'inside' (argument) out and use Theorem 2.3 to first get a point on the graph of $y = f(x - 7) = f(x - h)$. Identifying $h = 7$, we add 7 to the x-coordinate of $(-1, 3)$ to get $(-1 + 7, 3) = (6, 3)$. Hence, $(6, 3)$ is a point on the graph of $y = f(x - 7)$.

   Next, we apply Theorem 2.2 to graph $y = f(x - 7) + 4$ starting with $y = f(x - 7)$. Viewing $f(x - 7) + 4 = f(x - 7) + k$, we identify $k = 4$ and add 4 to the y-coordinate of $(6, 3)$ to get $(6, 3 + 4) = (6, 7)$. To check, we note that if we substitute $x = 6$ into $y = f(x - 7) + 4$, we get $y = f(6 - 7) + 4 = f(-1) + 4 = 3 + 4 = 7$.

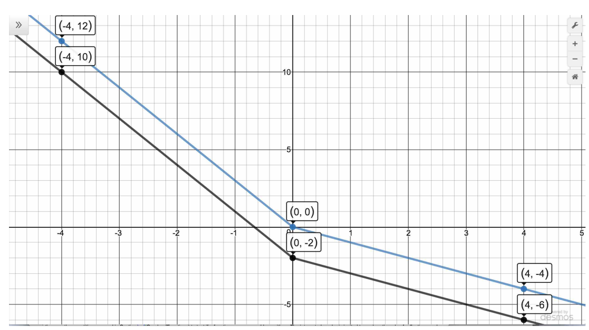
2. Here the independent variable is t instead of x which doesn't affect the geometry in any way since our convention is the independent variable is used to label the horizontal axis and the dependent variable is used to label the vertical axis.

   (a) Per Theorem 2.3, the graph of $g(t) = f(t - 4) = |t - 4| - 2(t - 4) = |t - 4| - 2t + 8$ should be the graph of $f(t) = |t| - 2t$ shifted to the right 4 units. Our check is below on the left.

   (b) Per Theorem 2.2, the graph of $g(t) = f(t) + (-2) = |t| - 2t + (-2) = |t| - 2t - 2$ should be the graph of $f(t) = |t| - 2t$ shifted down 2 units. Our check is below on the right.



$y = |t| - 2t$ (lighter color) and $y = |t - 4| - 2t + 8$

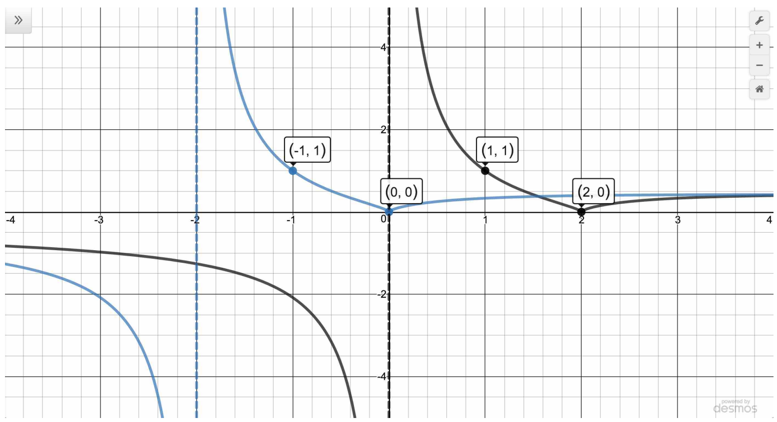


$y = |t| - 2t$ (lighter color) and $y = |t| - 2t - 2$

3. Comparing *formulas*, it appears as if $F(x) = f(x-2)$. We check:

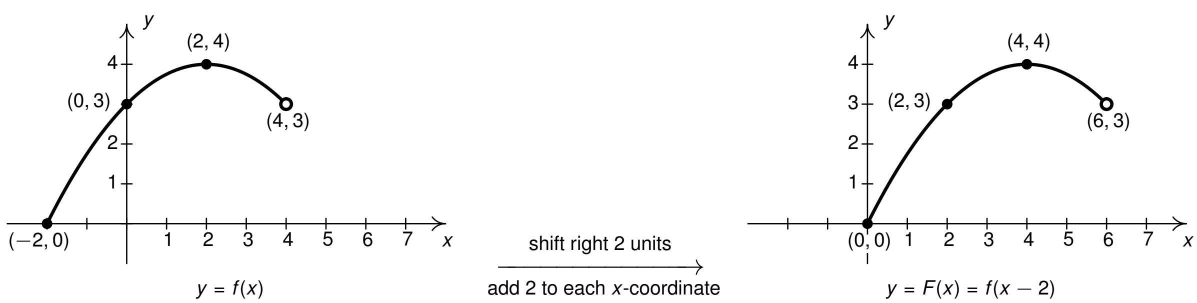
$$f(x-2) = \frac{(x-2)^{\frac{2}{3}}}{(x-2)+2} = \frac{(x-2)^{\frac{2}{3}}}{x} = F(x),$$

so, per Theorem 2.3, the graph of $y = F(x)$ should be the graph of $y = f(x)$ but shifted to the right 2 units. We graph both functions below to confirm our answer.



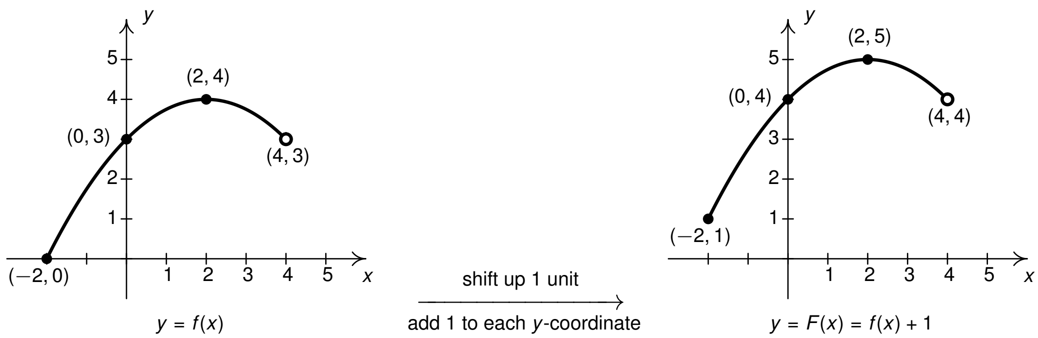
$y = \dfrac{x^{\frac{2}{3}}}{x+2}$ (lighter color) and $y = \dfrac{(x-2)^{\frac{2}{3}}}{x}$

4. (a) We recognize $F(x) = f(x-2) = f(x-h)$. With $h = 2$, Theorem 2.3 tells us to add 2 to each of the x-coordinates of the points on the graph of f, moving the graph of f to the *right* two units.



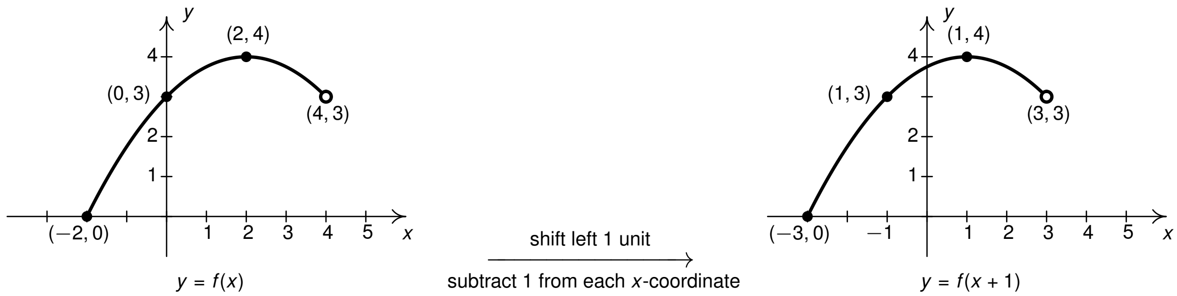
We can check our answer by showing each ordered pair (x, y) listed on our final graph satisfies the equation $y = f(x-2)$. Starting with $(0, 0)$, we substitute $x = 0$ into $y = f(x-2)$ and get $y = f(0-2) = f(-2)$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(0-2) = f(-2) = 0$, showing the point $(0, 0)$ is on the graph of $y = f(x-2)$. We invite the reader to check the remaining points.

(b) We have $F(x) = f(x) + 1 = f(x) + k$ where $k = 1$, so Theorem 2.2 tells us to move the graph of f *up* 1 unit by adding 1 to each of the y-coordinates of the points on the graph of f.

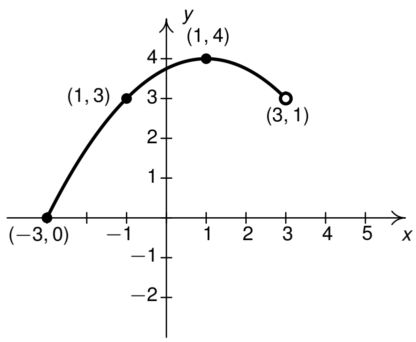


To check our answer, we proceed as above. Starting with the point $(-2, 1)$, we substitute $x = -2$ into $y = f(-2) + 1$ to get $y = f(-2) + 1$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(-2) + 1 = 0 + 1 = 1$. This proves $(-2, 1)$ is on the graph of $y = f(x) + 1$. We encourage the reader to check the remaining points in kind.

(c) We are asked to graph $F(x) = f(x+1) - 2$. As above, when we have more than one modification to do, we work from the inside out and build up to $F(x) = f(x + 1) - 2$ from $f(x)$ in stages. First, we apply Theorem 2.3 to graph $y = f(x + 1)$ from $y = f(x)$. Rewriting $f(x + 1) = f(x - (-1))$, we identify $h = -1$, so we add -1 to (subtract 1 from) each of the x-coordinates on the graph of f, shifting it to the *left* 1 unit.

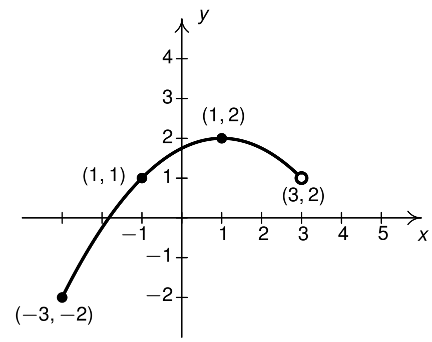


Next, we apply Theorem 2.2 to graph $y = f(x + 1) - 2$ starting with the graph of $y = f(x + 1)$. Writing $f(x + 1) - 2 = f(x + 1) + (-2) = f(x + 1) + k$, we identify $k = -2$ so Theorem 2.2 instructs us to add -2 to (subtract 2 from) each of the y-coordinates on the graph of $y = f(x+1)$, thereby shifting the graph *down* two units.



$y = f(x+1)$

shift down 2 units
$\longrightarrow$
subtract 2 from each y-coordinate



$y = f(x+1) - 2 = F(x)$

To check, we start with the point $(-3,-2)$. We find when we substitute $x = -3$ into the equation $y = f(x+1) - 2$ we get $y = f(-3+1) - 2 = f(-2) - 2$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$, so $y = f(-3+1) - 2 = f(-2) - 2 = 0 - 2 = -2$. This proves $(-3,-2)$ is on the graph of $y = f(x+1) - 2$. We leave the checks of the remaining points to the reader.

5. To write $g(x)$ in terms of $f(x)$, we note that based on points which are labeled, it appears as if the graph of g can be obtained from the graph of f by shifting the graph of f to the right 0.5 units and down 1 unit.

   Per Theorems 2.3 and 2.2, $g(x)$ must take the form $g(x) = f(x-h) + k$. Since the horizontal shift is to the *right* 0.5 units, $h = 0.5$ and since the vertical shift is *down* 1 unit, $k = -1$. Hence, we get $g(x) = f(x - 0.5) - 1$.

   We can check our answer by working through both transformations, in sequence, as in the previous example. To write $f(x)$ in terms of $g(x)$, we need to reverse the process - that is, we need to shift the graph of g *left* one half of a unit and *up* one unit. Theorems 2.3 and 2.2 suggest the formula $f(x) = g(x+0.5) + 1$. We leave it to the reader to check. □

2.3.2 Reflections about the Coordinate Axes

We now turn our attention to reflections. We know from Section **??** that to reflect a point (x, y) across the x-axis, we replace y with $-y$. If (x, y) is on the graph of f, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis.[1]

Theorem 2.4. Reflections. Suppose f is a function.
To graph $F(x) = -f(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by -1.
NOTE: This results in a reflection across the x-axis.
To graph $F(x) = f(-x)$, multiply each of the x-coordinates of the points on the graph of $y = f(x)$ by -1.
NOTE: This results in a reflection across the y-axis.

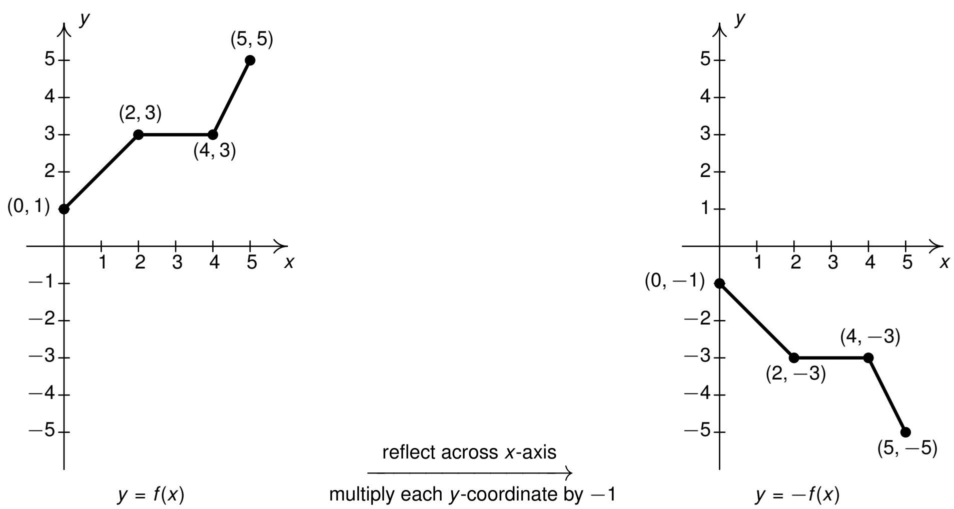
[1] The expressions $-f(x)$ and $f(-x)$ should look familiar - they are the quantities we used in Section 6.1 to determine if a function was even, odd or neither. We explore impact of symmetry on reflections in Exercise 74.

The proof of Theorem 2.4 follows in much the same way as the proofs of Theorems 2.2 and 2.3. If c is an element of the domain of f and $F(x) = -f(x)$, then the point $(c, f(c))$ corresponds to the point $(c, F(c)) = (c, -f(c))$. Comparing the corresponding points $(c, f(c))$ and $(c, -f(c))$, we see they only difference is the y-coordinates are the exact opposite - indicating they are mirror-images across the x-axis. Similarly, if c is an element in the domain of f, then c corresponds to the element $-c$ in the domain of $F(x) = f(-x)$ since $F(-c) = f(-(-c)) = f(c)$. Hence, the corresponding points here are $(c, f(c))$ and $(-c, F(-c)) = (-c, f(c))$. Comparing $(c, f(c))$ with $(-c, f(c))$, we see they are reflections about the y-axis.

Using the language of inputs and outputs, Theorem 2.4 says that multiplying the *outputs* from a function by -1 reflects its graph across the *horizontal* axis, while multiplying the *inputs* to a function by -1 reflects the graph across the *vertical* axis.

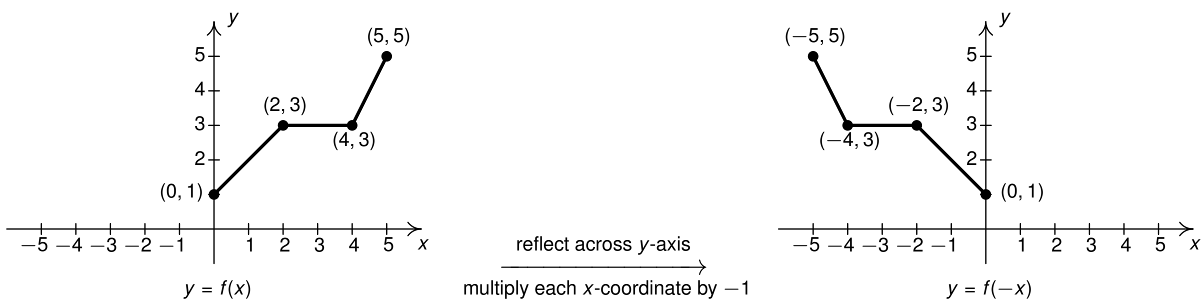
Applying Theorem 2.4 to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis.

x	$(x, f(x))$	$f(x)$	$g(x) = -f(x)$	$(x, g(x))$
0	$(0, 1)$	1	-1	$(0, -1)$
2	$(2, 3)$	3	-3	$(2, -3)$
4	$(4, 3)$	3	-3	$(4, -3)$
5	$(5, 5)$	5	-5	$(5, -5)$



By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.

x	$-x$	$g(x) = f(-x)$	$(x, g(x))$
0	0	$g(0) = f(-(-0)) = f(0) = 1$	$(0, 1)$
-2	2	$g(-2) = f(-(-2)) = f(2) = 3$	$(-2, 3)$
-4	4	$g(-4) = f(-(-4)) = f(4) = 3$	$(-4, 3)$
-5	5	$g(-5) = f(-(-5)) = f(5) = 5$	$(-5, 5)$



Example 2.3.2. Use Theorems 2.2, 2.3 and 2.4 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(2, -5)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = f(-x)$ (b) $y = -f(x + 2)$ (c) $f(8 - x)$

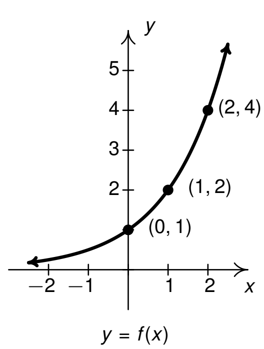
2. Find a formula for a function $H(s)$ whose graph is the same as $t = h(s) = s^3 - s^2$ but is reflected across the t-axis.

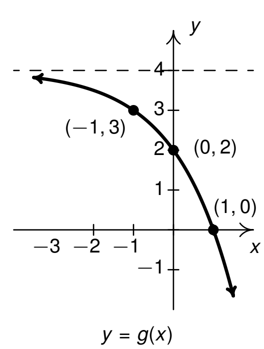
3. Predict how the graph of $G(t) = \dfrac{t+4}{t-3}$ relates to the graph of $g(t) = \dfrac{t+4}{3-t}$.

4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(-x) + 1$ (b) $F(x) = 1 - f(2 - x)$

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.





NOTE: The x-axis, $y = 0$, is a horizontal asymptote to the graph of $y = f(x)$ and the line $y = 4$ is a horizontal asymptote to the graph of $y = g(x)$.

Solution.

1. (a) To find a point on the graph of $y = f(-x)$, Theorem 2.4 tells us to multiply the x-coordinate of the point on the graph of $y = f(x)$ by -1: $((-1)2, -5) = (-2, -5)$.

   To check, since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, when we substitute $x = -2$ into $y = f(-x)$, we get $y = f(-(-2)) = f(2) = -5$, proving $(-2, -5)$ is on the graph of $y = f(-x)$.

   (b) To find a point on the graph of $y = -f(x+2)$, we first note we have two transformations at work here, so we work our way from the inside out and build $f(x)$ to $-f(x+2)$.

   First, we find a point on the graph of $y = f(x+2)$. Writing $f(x+2) = f(x-(-2))$, we apply Theorem 2.3 with $h = -2$ and add -2 to (or subtract 2 from) the x-coordinate of the point we know is on $y = f(x)$: $(2-2, -5) = (0, -5)$.

   Next we apply Theorem 2.4 to the graph of $y = f(x+2)$ to get a point on the graph of $y = -f(x+2)$ by multiplying the y-coordinate of $(0, -5)$ by -1: $(0, (-1)(-5)) = (0, 5)$.

   To check, recall $f(2) = -5$ so that when we substitute $x = 0$ into the equation $y = -f(x+2)$, we get $y = -f(0+2) = -f(2) = -(-5) = 5$, as required.

   (c) Rewriting $f(8-x) = f(-x+8)$ we see we have two transformations at play here: a reflection across the y-axis and a horizontal shift. Since both of these transformations affect the x-coordinates of the graph, the question becomes which transformation to address first. To help us with this decision, we attack the problem algebraically.

   Recall that since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, to get a point on the graph of $y = f(-x+8)$, we need to match up the arguments of $f(-x+8)$ and $f(2)$: $-x+8 = 2$.

   To solve this equation, we first subtract 8 from both sides to get $-x = -6$. Geometrically, subtracting 8 from the x-coordinate of $(2, -5)$, shifts the point $(2, -5)$ left 8 units to get the point $(-6, -5)$.

   Next, we multiply both sides of the equation $-x = -6$ by -1 to get $x = 6$. Geometrically, multiplying the x-coordinate of $(-6, -5)$ by -1 reflects the point $(-6, -5)$ across the y-axis to $(6, -5)$.

   To check we substitute $x = 6$ into $y = f(-x+8)$, and obtain $y = f(-6+8) = f(2) = -5$.

   Even though we have found our answer, we re-examine this process from a 'build' perspective. We began with a point on the graph of $y = f(x)$ and first shifted the graph to the left 8 units. Per Theorem 2.3, this point is on the graph of $y = f(x+8)$.

   Next we took a point on the graph of $y = f(x+8)$ and reflected it about the y-axis. Per Theorem 2.4, this put the point on the graph of $y = f(-x+8)$.

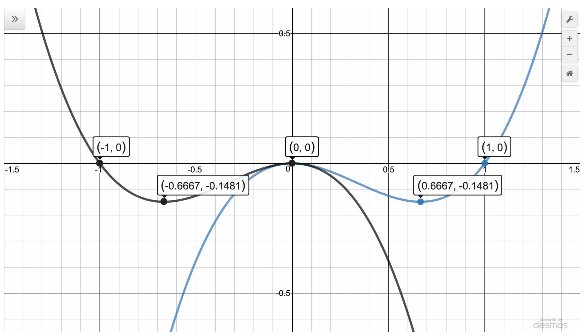
   In general, when faced with graphing functions in which there is both a horizontal shift and a reflection about the y-axis, we'll deal with the shift first.

2. In this example, the independent variable is s and the dependent variable is t. We are asked to reflect the graph of h about the t-axis, which in this case is the *vertical* axis. Hence, $H(s) = h(-s) = (-s)^3 - (-s)^2 = -s^3 - s^2$. Our confirmation is below on the left.

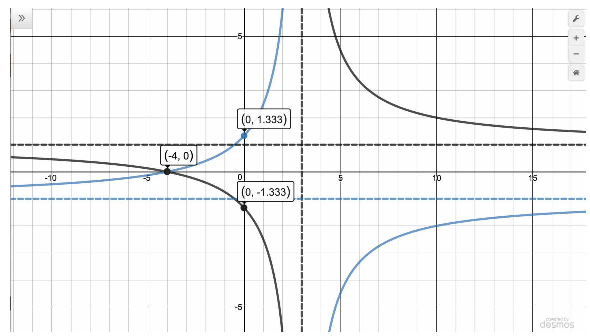
3. Comparing the formulas for $G(t) = \frac{t+4}{t-3}$ and $g(t) = \frac{t+4}{3-t}$, we have the same numerators, but in the denominator, we have $(t-3) = -(3-t)$:

$$G(t) = \frac{t+4}{t-3} = \frac{t+4}{-(3-t)} = -\frac{t+4}{3-t} = -g(t).$$

Hence, the graph of $y = G(t)$ should be the graph of $y = g(t)$ reflected across the t-axis. We check our answer below on the right.

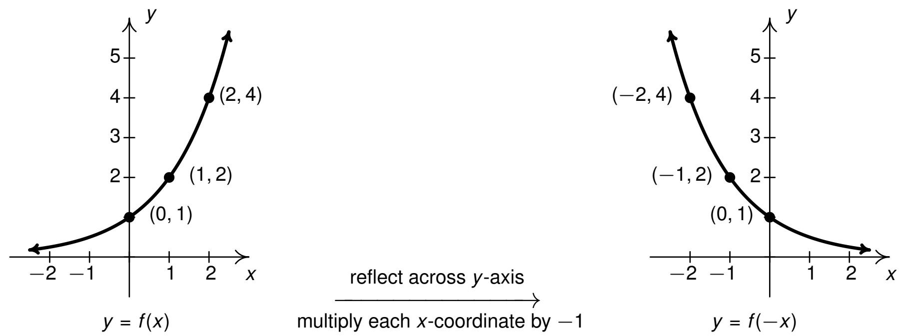


$t = s^3 - s^2$ (lighter color) and $t = -s^3 - s^2$

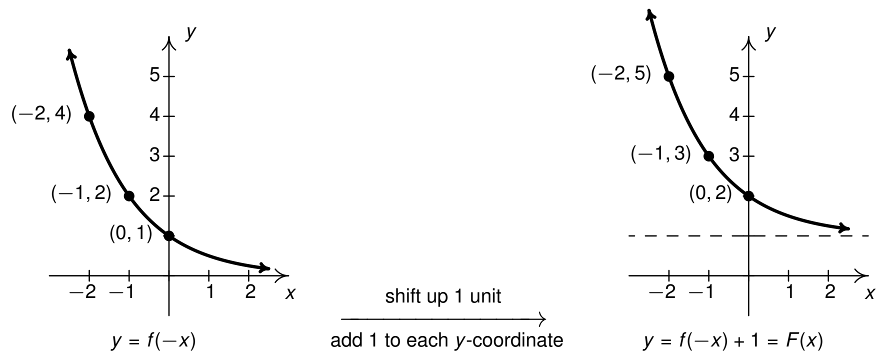


$y = \dfrac{t+4}{3-t}$ (lighter color) and $y = \dfrac{t+4}{t-3}$

4. (a) We have two transformations indicated with the formula $F(x) = f(-x) + 1$: a reflection across the y-axis and a vertical shift. Working from the inside out, we first tackle the reflection. Per Theorem 2.4, to obtain the graph of $y = f(-x)$ from $y = f(x)$, we multiply each of the x-coordinates of each of the points on the graph of $y = f(x)$ by (-1).



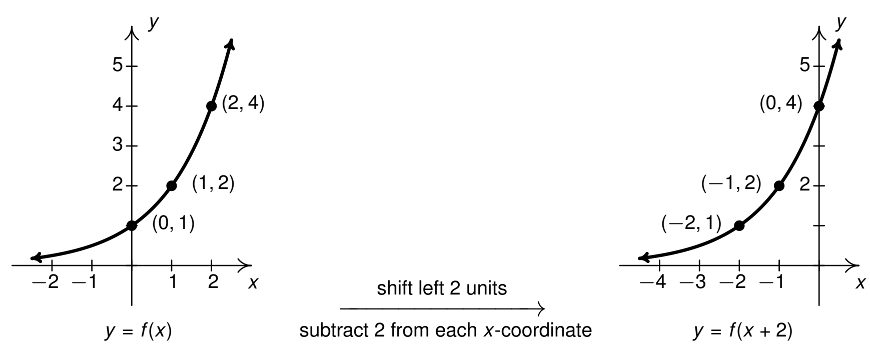
Next, we use Theorem 2.2 to obtain the graph of $y = f(-x) + 1$ from the graph of $y = f(-x)$ by adding 1 to each of the y-coordinates of each of the points on the graph of $y = f(-x)$. This shifts the graph of $y = f(-x)$ up one unit. Note, the horizontal asymptote $y = 0$ is also shifted up 1 unit to $y = 1$.



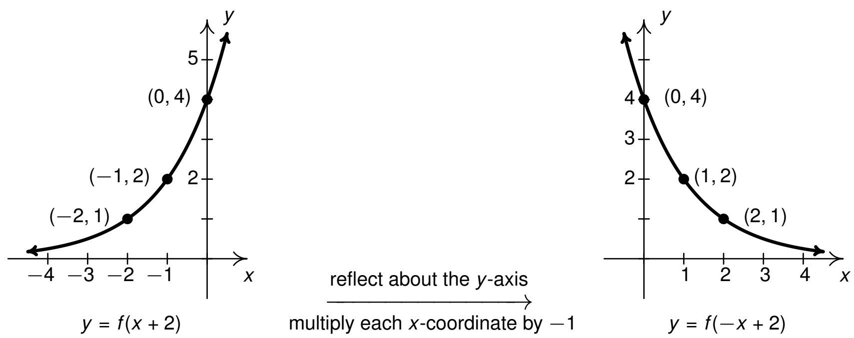
To check our answer, we begin with the point $(0, 2)$. Substituting $x = 0$ into $y = f(-x) + 1$, we get $y = f(-0)+1 = f(0)+1$. Since the point $(0, 1)$ is on the graph of f, we know $f(0) = 1$. Hence, $y = f(0) + 1 = 1 + 1 = 2$, so $(0, 2)$ is, indeed, on the graph of $y = f(-x) + 1$. We leave it to the reader to check the remaining points.

(b) In order to graph $F(x) = 1 - f(2 - x)$, we first rewrite as $F(x) = -f(-x + 2) + 1$ and note there are *four* modifications to the formula $f(x)$ indicated here.

Working from the inside out, we see we have a reflection about the y-axis indicated as well as a horizontal shift. From our work above, we know we first handle the shift: that is, we apply Theorem 2.3 to graph $y = f(x + 2) = f(x - (-2))$ by adding -2 to (subtracting 2 from) the x-coordinates of the points on the graph of $y = f(x)$.



Next, we use Theorem 2.4 to graph $y = f(-x + 2)$ starting with the graph of $y = f(x + 2)$ by multiplying each of the x-coordinates of the points of the graph of $y = f(x + 2)$ by -1. This reflects the graph of $f(x + 2)$ about the y-axis.

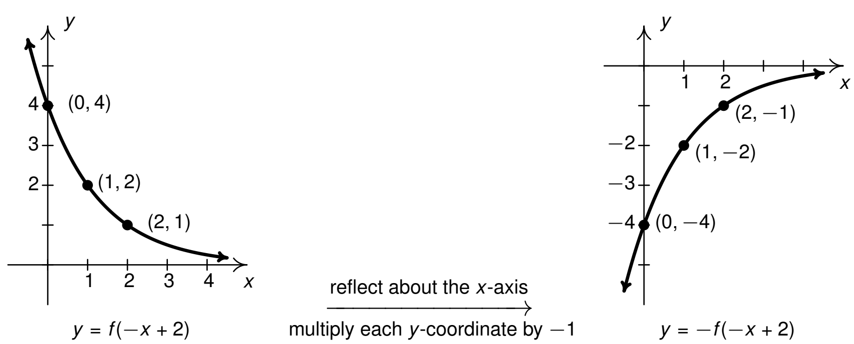


We have the graph of $y = f(-x+2)$ and need to build towards the graph of $y = -f(-x+2)+1$. The transformations that remain are a reflection about the x-axis and a vertical shift. The question is which to do first.

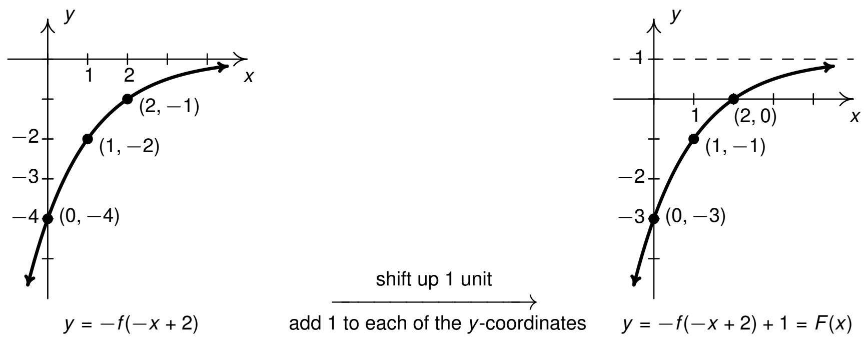
Once again, we can think algebraically about the problem. We know the point $(0,1)$ is on the graph of f which means $f(0) = 1$. This point corresponds to the point $(2,1)$ on the graph of $f(-x+2)$. Indeed, when we substitute $x = 2$ into $y = f(-x+2)$, we get $y = f(-2+2) = f(0) = 1$.

If we substitute $x = 2$ into the formula $y = -f(-x+2)+1$, we get $y = -f(-2+2)+1 = -f(0)+1 = -1(1)+1 = 0$. That is, we first multiply the y-coordinate of $(2,1)$ by -1 then add 1. This suggests we take care of the reflection about the x-axis first, then the vertical shift.

We proceed below to obtain the graph of $y = -f(-x+2)$ from $y = f(-x+2)$ by multiplying each of the y-coordinates on the graph of $y = f(-x+2)$ by -1. Note the horizontal asymptote remains unchanged: $y = (-1)(0) = 0$.



Finally, we take care of the vertical shift. Per Theorem 2.2, we graph $y = -f(-x+2)+1$ by adding 1 to the y-coordinates of each of the points on the graph of $y = -f(-x+2)$. This moves the graph up one unit, including the horizontal asymptote: $y = 0+1 = 1$.



To check, we begin with the point $(2, 0)$. Substituting $x = 2$ into $y = 1 - f(2 - x)$, we obtain $y = 1 - f(2 - 2) = 1 - f(0)$. Since $(0, 1)$ is on the graph of f, we know $f(0) = 1$. This means $y = 1 - f(2 - 2) = 1 - f(0) = 1 - 1 = 0$. This proves $(2, 0)$ is on the graph of $y = 1 - f(2 - x)$, and we recommend the reader check the remaining points.

5. With the transformations at our disposal, our task amounts to finding values of h and k and choosing between signs $\pm$ so that $g(x) = \pm f(\pm x - h) + k$.

   Based on the horizontal asymptote, $y = 4$, we choose $k = 4$. Note, however, in the graph of $y = f(x) + 4$, the entire graph is *above* the line $y = 4$. Since the graph of g approaches the asymptote from below, we know $y = -f(\pm x - h) + 4$.

   Hence, two of transformations applied to the graph of f are a reflection across the x-axis followed by a shift up 4 units. This means the point $(0, 1)$ on the graph of f must correspond to the point $(-1, 3)$ on the graph of g, since these are the points closest to the asymptote on each graph.

   Likewise, the points $(1, 2)$ and $(2, 4)$ on the graph of f must correspond to $(0, 2)$ and $(1, 0)$, respectively, on the graph of g. Looking at the x-coordinates only, we have $x = 0$ moves to $x = -1$, $x = 1$ moves to $x = 0$, and $x = 2$ moves to $x = 1$. Hence, the net effect on the x-values is a shift left 1 unit. Hence, we guess the formula for $g(x)$ to be $g(x) = -f(x + 1) + 4$.

   We can readily check by going through the transformations: first, shift left 1 unit; next, reflect across the x-axis; finally, shift up 4. We leave it to the reader to verify that tracking each of the points on the graph of f along with the horizontal asymptote through this sequence of transformations results in the graph of g.

   One way to recover the graph of f from the graph of g is to reverse the process by which we obtained g from f. The challenge here comes from the fact that two different operations were done which affected the y-values: reflection and shifting - and the order in which these are done matters.

   To motivate our methodology, let's consider a more down-to-earth example like putting on socks and then putting on shoes. Unless we're very talented, to reverse this process, we take off the shoes first, then the socks - that is, we undo each step in the reverse order.[2] In the same way, when we

[2]We'll have more to say about this sort of thing in Section 9.4.

think about reversing the steps transforming the graph of f to the graph of g, we need to undo each transformation in the opposite order.

To review, we obtained the graph of g from the graph of f by first shifting the graph to the left 1 unit, then reflecting the graph about the x-axis, then, finally, shifting the graph up 4 units. Hence, we first undo the vertical shift. Instead of shifting the graph *up* four units, we shift the graph *down* four units. This takes the graph of $y = g(x)$ to $y = g(x) - 4$.

Next, we have to undo the refection across the x-axis. Thinking at the level of points, to recover the point (a, b) from its reflection across the x-axis, $(a, -b)$, we simply reflect across the x-axis again: $(a, -(-b)) = (a, b)$. Per Theorem 2.4, this takes the graph the graph of $y = g(x) - 4$ to the graph of $y = -[g(x) - 4] = -g(x) + 4$.[3]

Last, to undo moving the graph to the *left* 1 unit, we move the graph of $y = -g(x) + 4$ to the *right* 1 unit. Per Theorem 2.3, we accomplish this by graphing $y = -g(x - 1) + 4$. We leave it to the reader to start with the graph of $y = g(x)$ and graph $y = -g(x - 1) + 4$ and show it matches the graph of $y = f(x)$. $\square$

Some remarks about Example 2.3.2 are in order. In number 1c above, to find a point on the graph of $y = f(-x+8)$, we took the given x-coordinate on our starting graph, 2, and subtracted 8 first then multiplied by -1. If this seems somehow 'backwards' it should.

When *evaluating* the expression $-x + 8$, the order of operations mandates we multiply by -1 first then add 8. Here, however, we weren't *evaluating* an expression - we were *solving* an equation: $-x + 8 = 2$, which meant we did the exact opposite steps in the opposite order.[4] This exemplifies a larger theme with transformations: when adjusting inputs, the resulting points on the graph are obtained by applying the opposite operations indicated by the formula in the opposite order of operations.

On the other hand, when it came to multiple transformations involving the y-coordinates, we followed the order of operations. As in 4b above, when it came to applying a reflection about the x-axis and a vertical shift, we applied the reflection first, then the shift. This is because instead of *solving* an *equation* to find the new y-coordinates, we were *simplifying* an *expression*. Again, this is an example of a much larger theme: when adjusting outputs, the resulting points on the graph are obtained by applying the stated operations in the usual order.

Last but not least, in number 5, to find f in terms of g, we reversed the steps used to transform f into g. Another tact is to approach the problem in the same way we approached transforming f into g: namely, starting with the graph of g, determine values h and k and signs $\pm$ so that $f(x) = \pm g(\pm x - h) + k$. We leave this to the reader.

[3]To see this better, let us temporarily write $F(x) = g(x) - 4$. Theorem 2.4 tells us to reflect the graph of F about the x-axis, graph $y = -F(x) = -[g(x) - 4] = -g(x) + 4$.

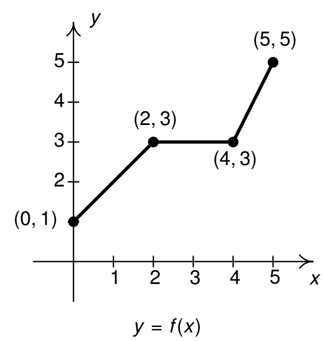
[4]Note that dividing by -1 is the same as multiplying by -1, so to keep with the 'opposite steps in opposite order' theme, we could more precisely say we subtracted 8 and *divided* by -1.

2.3.3 Scalings

We now turn our attention to our last class of transformations: **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**.

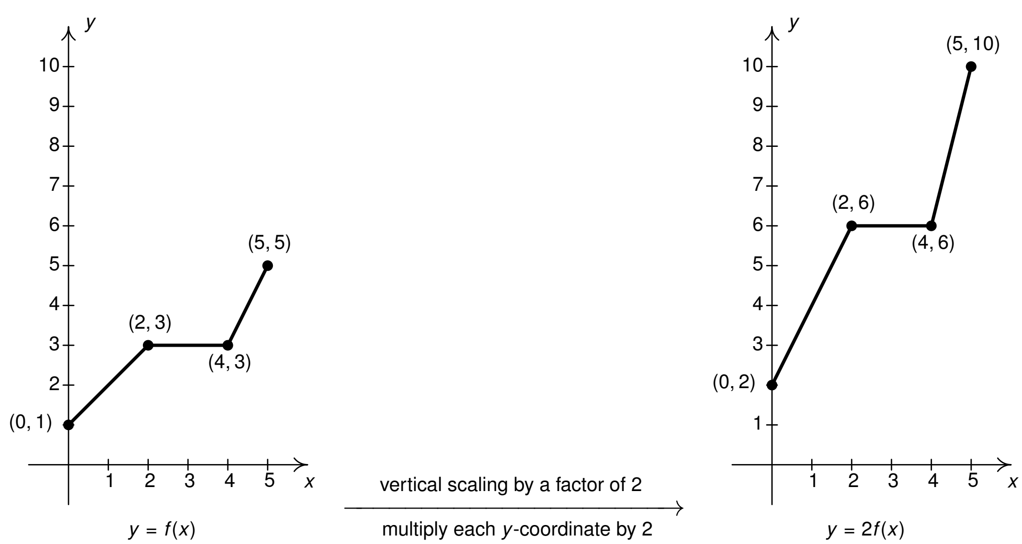
Simply put, rigid transformations preserve the distances between points on the graph - only their position and orientation in the plane change.[5] If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be affecting the distance between points. These sorts of transformations are hence called **non-rigid**. As always, we motivate the general theory with an example.

Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph, we can build a table of values for g as before.



x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0, 1)$	1	2	$(0, 2)$
2	$(2, 3)$	3	6	$(2, 6)$
4	$(4, 3)$	3	6	$(4, 6)$
5	$(5, 5)$	5	10	$(5, 10)$

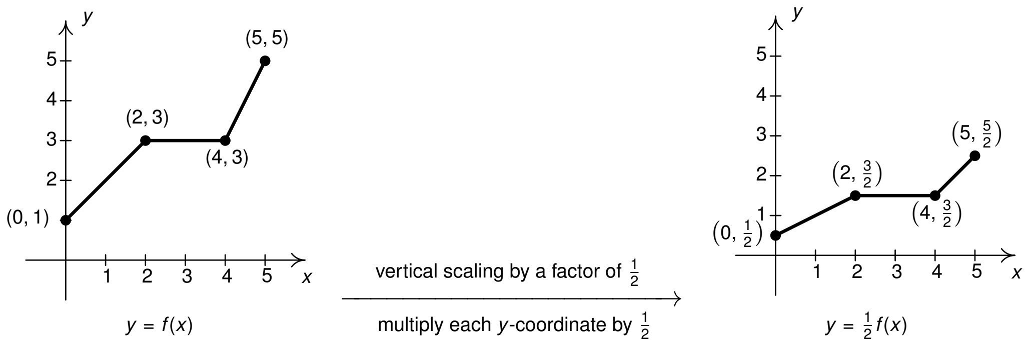
Graphing, we get:



[5]Another word that can be used here instead of 'rigid transformation' is 'isometry' - meaning 'same distance.'

In general, if (a, b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling[6] by a factor of 2.'

If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling[7] by a factor of $\frac{1}{2}$' as seen below.



These results are generalized in the following theorem.

Theorem 2.5. Vertical Scalings. Suppose f is a function and $a > 0$ is a real number.
To graph $F(x) = af(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by a.

- If $a > 1$, we say the graph of f has undergone a vertical stretch[a] by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrink[b] by a factor of $\frac{1}{a}$.

[a] expansion, dilation
[b] compression, contraction

The proof of Theorem 2.5 mimics the proofs of Theorems 2.2 and 2.4. If c is in the domain of f, then $(c, f(c))$ is on the graph of f and the corresponding point on the graph of $F(x) = af(x)$ is $(c, F(c)) = (c, af(c))$. Comparing the points $(c, f(c))$ and $(c, af(c))$ proves the theorem.

A few remarks about Theorem 2.5 are in order. First, a note about the verbiage. To the authors, the words 'stretch', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrink', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$'. This is why we have written the descriptions 'stretch by a factor of a' and 'shrink by a factor of $\frac{1}{a}$' in the statement of the theorem.

Second, in terms of inputs and outputs, Theorem 2.5 says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

[6] Also called a 'vertical stretch,' 'vertical expansion' or 'vertical dilation' by a factor of 2.
[7] Also called 'vertical shrink,' 'vertical compression' or 'vertical contraction' by a factor of 2.

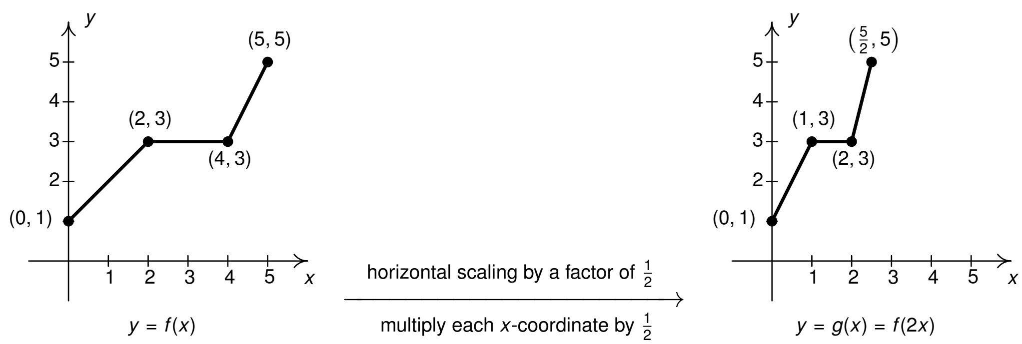
Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 2.3, as seen in the table on the left below.

We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2, 3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1, 3)$ is on the graph of g. Continuing in this fashion, we obtain the table on the lower right.

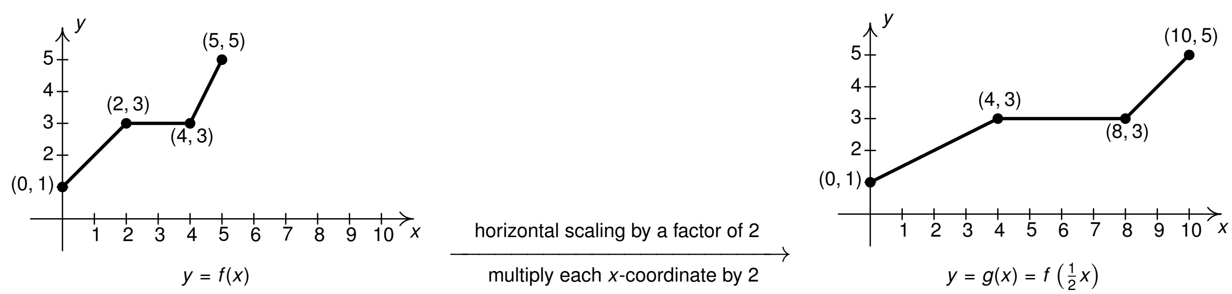
x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0, 1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0, 1)$
2	$(2, 3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5, 5)$	5	$f(2 \cdot 5) = f(10) = ?$	

x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(2 \cdot 0) = f(0) = 1$	$(0, 0)$
1	2	$g(1) = f(2 \cdot 1) = f(2) = 3$	$(1, 3)$
2	4	$g(2) = f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f\left(2 \cdot \frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if (a, b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling[8] by a factor of $\frac{1}{2}$.



If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[9] by a factor of 2, as demonstrated below.



We have the following theorem.

[8] Also called 'horizontal shrink,' 'horizontal compression' or 'horizontal contraction' by a factor of 2.
[9] Also called 'horizontal stretch,' 'horizontal expansion' or 'horizontal dilation' by a factor of 2.

Theorem 2.6. Horizontal Scalings. Suppose f is a function and $b > 0$ is a real number. To graph $F(x) = f(bx)$, divide each of the x-coordinates of the points on the graph of $y = f(x)$ by b.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretch[a] by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrink[b] by a factor of b.

[a]expansion, dilation
[b]compression, contraction

The proof of Theorem 2.6 follows closely the spirit of the proof of Theorems 2.3 and 2.4. If c is an element of the domain of f, them the number $\frac{c}{b}$ corresponds to a domain element of $F(x) = f(bx)$ since $F\left(\frac{c}{b}\right) = f\left(b \cdot \frac{c}{b}\right) = f(c)$. Hence, there is a correspondence between the point $(c, f(c))$ on the graph of f and the point $\left(\frac{c}{b}, F\left(\frac{c}{b}\right)\right) = \left(\frac{c}{b}, f(c)\right)$ on the graph of F. We can obtain $\left(\frac{c}{b}, f(c)\right)$ by dividing the x-coordinate of $(c, f(c))$ by b and the result follows.

Theorem 2.6 tells us that if we multiply the input to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

Example 2.3.3. Use Theorems 2.2, 2.3, 2.4, 2.5 and 2.6 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 4)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = 3f(x - 2)$ (b) $y = f\left(-\frac{1}{2}x\right)$ (c) $f(2x - 3) + 1$

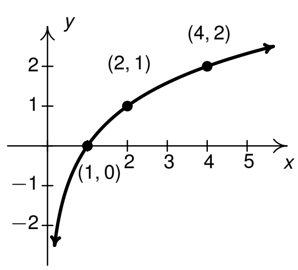
2. Find a formula for a function $G(t)$ whose graph is the same as $y = g(t) = \frac{2t+1}{t-1}$ but is vertically stretched by a factor of 4.

3. Predict how the graph of $H(s) = 8s^3 - 12s^2$ relates to the graph of $h(s) = s^3 - 3s^2$.

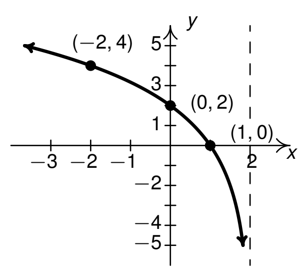
4. Below on the left is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = \dfrac{1 - f(x)}{2}$ (b) $F(x) = f\left(\dfrac{1 - x}{2}\right)$

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.



$y = f(x)$



$y = g(x)$

NOTE: The y-axis, $x = 0$, is a vertical asymptote to the graph of $y = f(x)$ and the line $x = 2$ is a vertical asymptote to the graph of $y = g(x)$.

Solution.

1. (a) As we examine the formula $y = 3f(x-2)$, we note two modifications from $y = f(x)$. Building from the inside out, we start with obtaining a point on the graph of $y = f(x-2)$.

   Per Theorem 2.3, this shifts all of the points on the graph of $y = f(x)$ 2 units to the right. Hence, the point $(-1, 4)$ on the graph of $y = f(x)$ moves to the point $(-1+2, 4) = (1, 4)$ on the graph of $y = f(x-2)$.

   To get a point on the graph of $y = 3f(x-2) = af(x-3)$, we apply Theorem 2.5 with $a = 3$ to the point $(1, 4)$ on the graph of $y = f(x-2)$ to get the point $(1, 3(4)) = (1, 12)$ on the graph of $y = 3f(x-2)$.

   To check, we note that since $(-1, 4)$ is on the graph of $y = f(x)$, we know $f(-1) = 4$. Hence, when we substitute $x = 1$ into the $y = 3f(x-2)$, we get $y = 3f(1-2) = 3f(-1) = 3(4) = 12$.

   (b) The formula $y = f\left(-\frac{1}{2}x\right)$ also indicates two transformations: a horizontal scaling, indicated by $\frac{1}{2}$ factor, as well as a reflection across the y-axis. The question before us is which to do first.

   If we return to algebra for inspiration, we know $f(-1) = 4$, so we match up the arguments of $f\left(-\frac{1}{2}x\right)$ and $f(-1)$ and get the equation $-\frac{1}{2}x = -1$. We solve this equation by multiplying both sides by -2: $x = (-2)(-1) = 2$. That is, we take the original x-value on the graph of $y = f(x)$ and multiply it by -2.

   If we think of $-2 = (-1)(2)$ then multiplying by the '2' in '$(-1)(2)$' produces a horizontal stretch by a factor of 2 while multiplying by the '-1' reflects the point across the y-axis.

   Applying the horizontal stretch first, we use Theorem 2.6 and start with the point $(-1, 4)$ on the graph of $y = f(x)$ and multiply the x-coordinate by 2 to obtain a point on the graph of $y = f\left(\frac{1}{2}x\right)$: $(-1(2), 4) = (-2, 4)$.

   Next, we take care of the reflection about the y-axis using Theorem 2.4 . Starting with $(-2, 4)$ on the graph of $y = f\left(\frac{1}{2}x\right)$, we multiply the x-coordinate by -1 to obtain a point on the graph of $y = f\left(\frac{1}{2}(-x)\right) = f\left(-\frac{1}{2}x\right)$: $((-1)(-2), 4) = (2, 4)$.

   To check, note when $x = 2$ is substituted into $y = f\left(-\frac{1}{2}x\right)$, we get $y = f\left(-\frac{1}{2}(2)\right) = f(-1) = 4$.

   Of course, we could have equally written the multiple $-2 = (2)(-1)$ and reversed these steps: doing the reflection first, then the horizontal scaling.

   Proceeding this way, we start with the point $(-1, 4)$ on the graph of $y = f(x)$ and reflect across the y-axis to obtain the point $((-1)(-1), 4) = (1, 4)$ on the graph of $y = f(-x)$.

   Next, we stretch the graph of $y = f(-x)$ by a factor of 2 by multiplying the x-coordinates of the points on the graph by 2 and obtain $(2(1), 4) = (2, 4)$ on the graph of $y = f\left(-\frac{1}{2}x\right)$.

   In general when it comes to reflections and scalings, whether horizontal or, as we'll see soon, vertical, either order will produce the same results.

   (c) The formula $f(2x-3)+1$ indicates *three* transformations: a horizontal shift, a horizontal scaling, and a vertical shift. As usual, we appeal to algebra to give us guidance on which horizontal transformation to apply first.

Since we know $f(-1) = 4$, we set $2x - 3 = -1$ and solve. Our first step is to add 3 to both sides: $2x = (-1) + 3 = 2$. Since we are adding 3 to the given x-value -1, this corresponds to a shift to the right 3 units, so the point $(-1, 4)$ is moved to the point $(2, 4)$.

Next, to solve $2x = 2$, we divide this new x-coordinate 2 by 2 and get $x = \frac{2}{2} = 1$ which corresponds to a horizontal compression by a factor of 2. This moves the point $(2, 4)$ to $(1, 4)$.

Hence, the algebra suggests we use Theorem 2.3 first and follow it up with Theorem 2.6. Starting with $(-1, 4)$ on the graph of $y = f(x)$, we shift to the right 3 units to obtain the point $(-1 + 3, 4) = (2, 4)$ on the graph of $y = f(x - 3)$.

Next, we start with the point $(2, 4)$ on the graph of $y = f(x-3)$ and horizontally shrink the x-axis by a factor of 2 to get the point $\left(\frac{2}{2}, 4\right) = (1, 4)$ on the graph of $y = f(2x - 3)$.

Last, but not least, we take care of the vertical shift using Theorem 2.2. Starting with the point $(1, 4)$ on the graph of $y = f(2x-3)$, we add 1 to the y-coordinate to get the point $(1, 4+1) = (1, 5)$ on the graph of $y = f(2x - 3) + 1$.

To check, we substitute $x = 1$ into the formula $y = f(2x - 3) + 1$ and get $y = f(2(1) - 3) + 1 = f(-1) + 1 = 4 + 1 = 5$, as required.

2. To vertically stretch the graph of $y = g(t)$ by 4, we use Theorem 2.5 with $a = 4$ to get

$$G(t) = 4g(t) = 4\frac{2t + 1}{t - 1} = \frac{4(2t + 1)}{t - 1} = \frac{8t + 4}{t - 1}.$$

We check our answer below on the left.

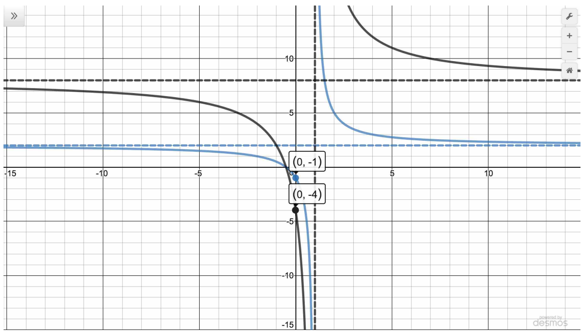
3. When comparing the formulas for $H(s) = 8s^3 - 12s^2$ and $h(s) = s^3 - 3s^2$, it doesn't appear as if any shifting or reflecting is going on (why not?)

We also note that since the coefficient of s^3 in the expression of $H(s)$ is 8 times that of the coefficient of s^3 in $h(s)$, but the coefficient of s^2 in $H(s)$ is only 4 times the coefficient of s^2 in $h(s)$, the change is not the result of a vertical scaling (again, why not?)

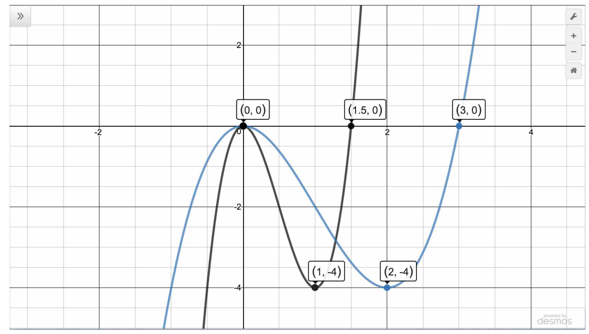
Hence, if anything, we are looking for a horizontal scaling. In other words, we are looking for a real number $b > 0$ so $h(bs) = H(s)$, that is, $(bs)^3 - 3(bs)^2 = b^3s^3 - 3b^2s^2 = 8s^3 - 12s^2$.

Matching up coefficients of s^3 gives $b^3 = 8$ so $b = 2$ which checks with the coefficients of s^2 : $3b^2 = 3(2)^2 = 12$.

Hence, we predict the graph of $y = H(s)$ to be the same as $y = h(s)$ except horizontally compressed by a factor of 2. Our check is below on the right.



$y = g(t) = \frac{2t+1}{t-1}$ (lighter color) and $y = 4g(t) = \dfrac{8t+4}{t-1}$



$y = h(s) = s^3 - 3s^2$ (lighter color) and $y = H(s) = 8s^3 - 12s^2$

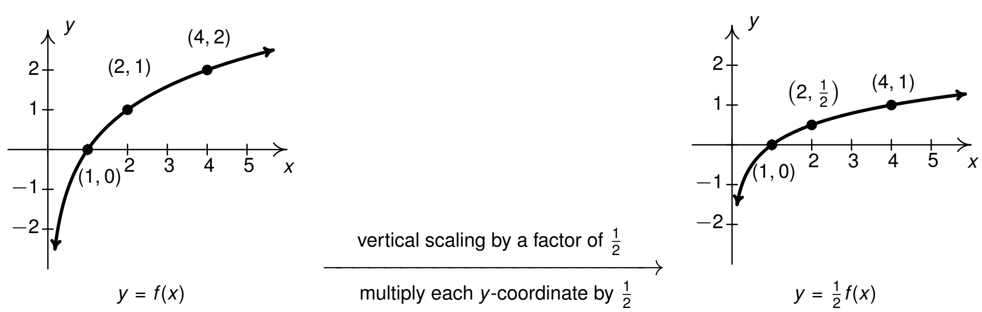
4. (a) We first rewrite the expression for $F(x) = \frac{1-f(x)}{2} = -\frac{1}{2}f(x) + \frac{1}{2}$ in order to use the theorems available to us. Note we have two modifications to the formula of $f(x)$ which correspond to three transformations.

Multiplying $f(x)$ by $-\frac{1}{2}$ indicates a vertical compression by a factor of 2 along with a reflection about the x-axis. Adding $\frac{1}{2}$ indicates a vertical shift up $\frac{1}{2}$ units.

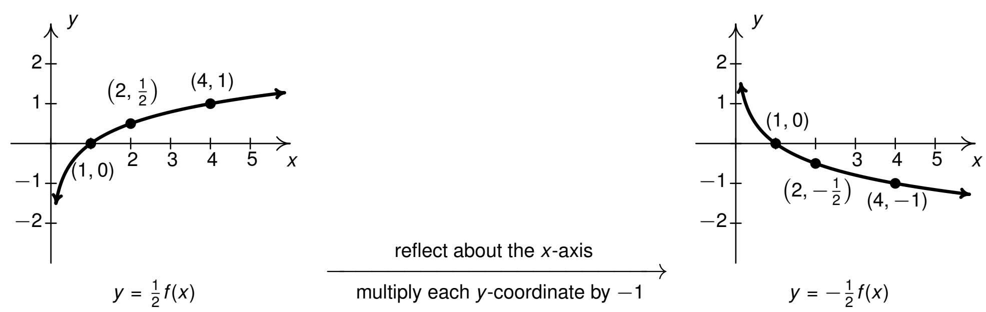
As always the question is which to do first. Once again, we look to algebra for the answer. Picking the point $(1, 0)$ on the graph of $f(x)$, we know $f(1) = 0$. To see which point this corresponds to on the graph of $y = F(x)$, we find $F(1) = -\frac{1}{2}f(1) + \frac{1}{2} = -\frac{1}{2}(0) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$.

Hence, we first multiplied the y-value 0 by $-\frac{1}{2}$. As above, we can think of $-\frac{1}{2} = (-1)\frac{1}{2}$ so that multiplying by $-\frac{1}{2}$ amounts to a vertical compression by a factor of 2 first, then the refection about the x-axis second. Lastly, adding the $\frac{1}{2}$ is the vertical shift up $\frac{1}{2}$ unit.

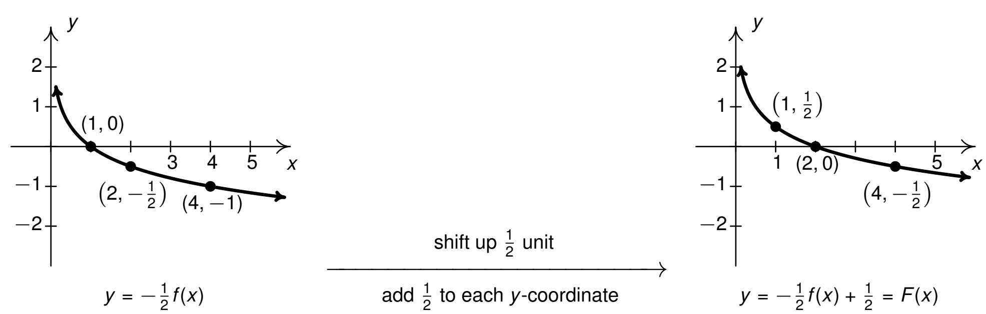
Beginning with the vertical scaling by a factor of $\frac{1}{2}$, we use Theorem 2.5 to graph $y = \frac{1}{2}f(x)$ starting from $y = f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = f(x)$ by $\frac{1}{2}$.



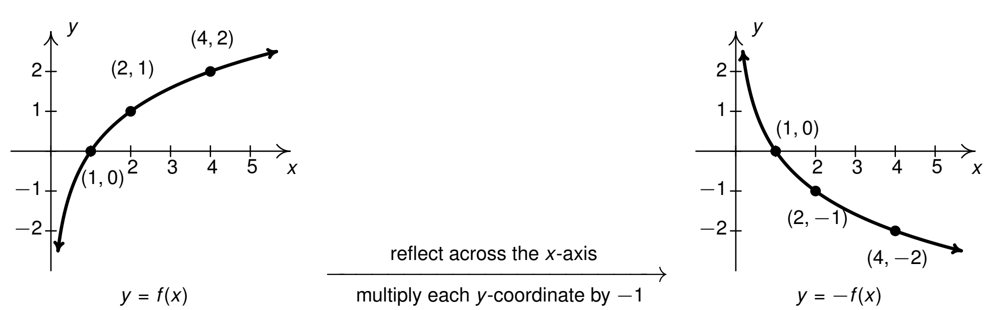
Next, we reflect the graph of $y = \frac{1}{2}f(x)$ across the x-axis to produce the graph of $y = -\frac{1}{2}f(x)$ by multiplying each of the y-coordinates of the points on the graph of $y = \frac{1}{2}f(x)$ by -1:



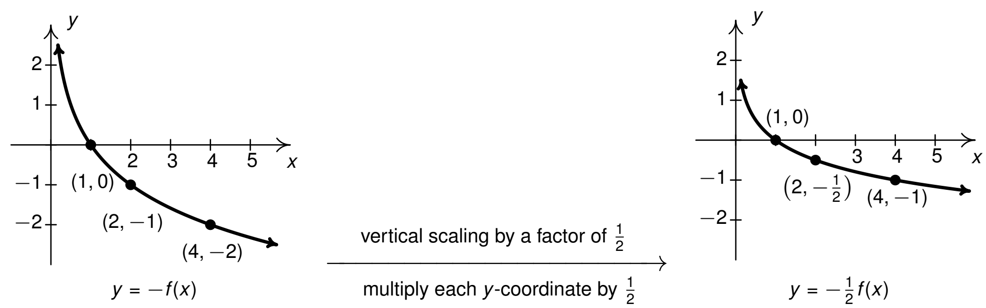
Finally, we shift the graph of $y = -\frac{1}{2}f(x)$ vertically up $\frac{1}{2}$ unit by adding $\frac{1}{2}$ to each of the y-coordinates of each of the points to obtain the graph of $y = -\frac{1}{2}f(x) + \frac{1}{2} = F(x)$.



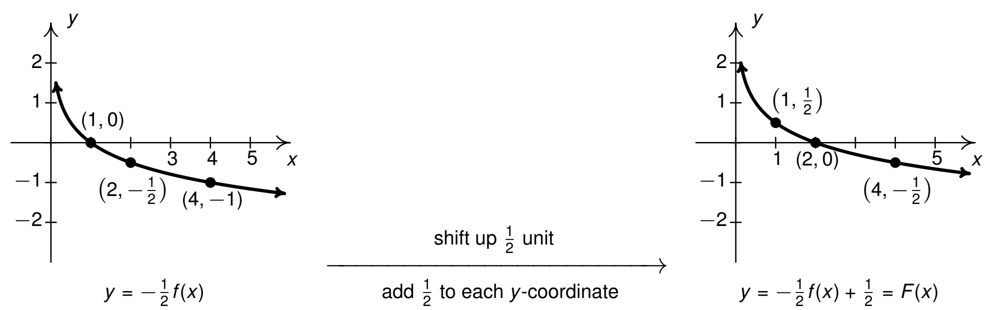
Note that as with horizontal scalings and reflections about the y-axis, the order of vertical scalings and reflections across the x-axis is interchangeable. Had we decided to think of the factor $-\frac{1}{2} = \frac{1}{2} \cdot (-1)$, we could have just as well started with the graph of $y = f(x)$ and produced the graph of $y = -f(x)$ first:



Next, we vertically scale the graph of $y = -f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = -f(x)$ by $\frac{1}{2}$ to obtain the graph of $y = -\frac{1}{2}f(x)$:



Notice we've reached the same graph of $y = -\frac{1}{2}f(x)$ that we had before, and, hence we arrive at the same final answer as before:



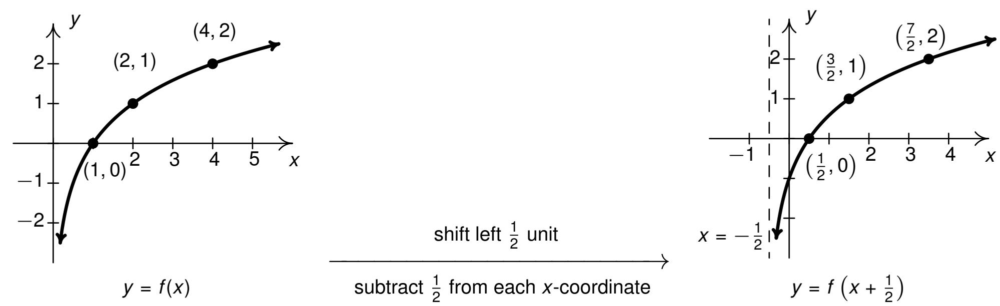
We check our answer as we have so many times before. We start with the point $\left(1, \frac{1}{2}\right)$ and substitute $x = 1$ into $y = \frac{1-f(x)}{2}$ to get $y = \frac{1-f(1)}{2}$. From the graph of f, we know $f(1) = 0$, so we get $y = \frac{1-f(1)}{2} = \frac{1-0}{2} = \frac{1}{2}$. This proves $\left(1, \frac{1}{2}\right)$ is on the graph of $y = \frac{1-f(x)}{2}$. We invite the reader to check the remaining points.

Note that in the preceding example, since none of the transformations included adjusting the x-coordinates of points, the vertical asymptote, $x = 0$ remained in place.

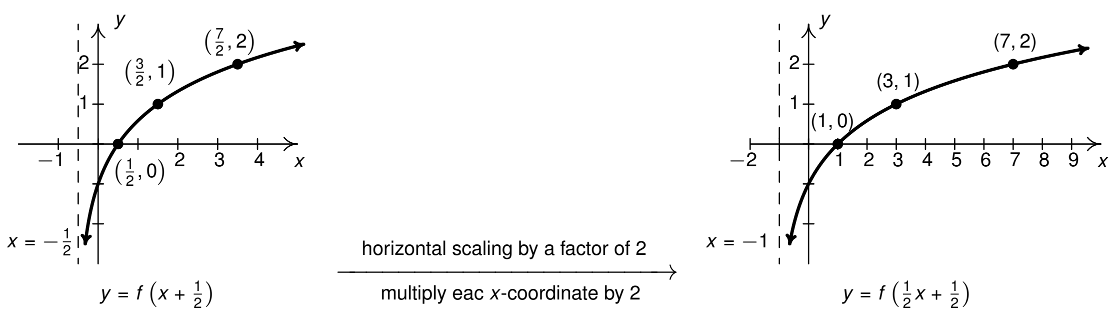
(b) As with the previous example, we first rewrite $F(x) = f\left(\frac{1-x}{2}\right) = F\left(-\frac{1}{2}x + \frac{1}{2}\right)$. Here again, we have two modifications to the formula $f(x)$, the $-\frac{1}{2}$ multiple indicating a horizontal scaling and a reflection across the y-axis and a horizontal shift.

Based on our experience from previous examples, we do the horizontal shift first, with the order of the scaling and reflection more or less irrelevant.

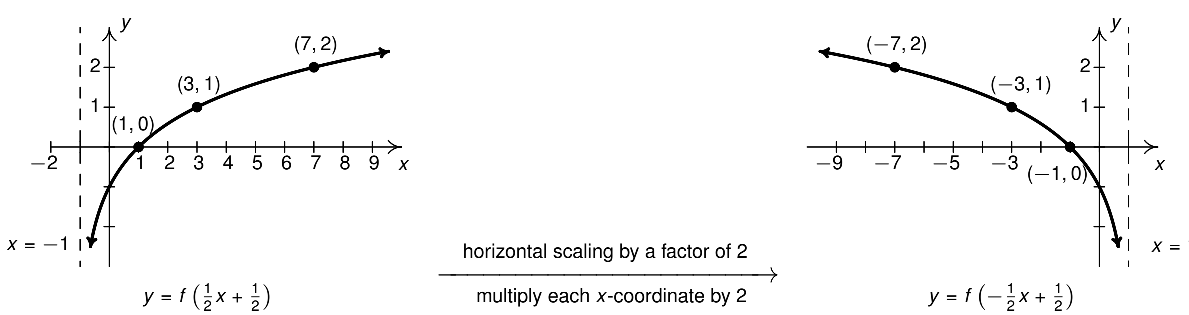
To produce the graph of $y = f\left(x + \frac{1}{2}\right)$ we subtract $\frac{1}{2}$ from each of the x-coordinates of each of the points on the graph of $y = f(x)$. This moves the graph to the left $\frac{1}{2}$ unit, including the vertical asymptote $x = 0$ which moves to $x = -\frac{1}{2}$.



Next, we graph $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ starting with $y = f\left(x + \frac{1}{2}\right)$ by horizontally expanding the graph by a factor of 2. That is, we multiply each x-coordinates on the graph of $y = f\left(x + \frac{1}{2}\right)$ by 2, including the vertical asymptote, $x = -\frac{1}{2}$ which moves to $x = 2\left(-\frac{1}{2}\right) = -1$.



Finally, we reflect the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ about the y-axis to graph $y = f\left(-\frac{1}{2}x + \frac{1}{2}\right)$. We accomplish this by multiplying each of the x-coordinates of each of the points on the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ by -1. This includes the vertical asymptote which is moved to $x = (-1)(-1) = 1$.



To check our answer, we begin with the point $(-1, 0)$ and substitute $x = -1$ into $y = f\left(\frac{1-x}{2}\right)$. We get $y = f\left(\frac{1-(-1)}{2}\right) = f\left(\frac{2}{2}\right) = f(1)$. From the graph of f, we know $f(1) = 0$, hence we have $y = f(1) = 0$, proving $(-1, 0)$ is on the graph of $y = f\left(\frac{1-x}{2}\right)$. The reader is encouraged to check the remaining points.

As mentioned previously, instead of doing the horizontal scaling first, then the reflection, we could have done the reflection first, then the scaling. We leave this to the reader to check.

5. To write $g(x)$ in terms of $f(x)$, we assume we can find real numbers a, b, h, and k and choose signs $\pm$ so that $g(x) = \pm af(\pm bx - h) + k$.

   The most notable change we see is the vertical asymptote $x = 0$ has moved to $x = 2$. Moreover, instead of the graph increasing off to the right, it is decreasing coming in from the left. This suggests a horizontal shift of 2 units as well as a reflection across the y-axis.

   Since we always shift first then reflect, we have a shift *left* of 2 units followed by a reflection about the y-axis. In other words, $g(x) = \pm af(-x + 2) + k$.

   Comparing y-values, the y-values on the graph of g appear to be exactly twice the corresponding values on the graph of f, indicating a vertical stretch by a factor of 2. Hence, we get $g(x) = 2f(-x+2)$. We leave it to the reader to check the graph of $y = 2f(-x + 2)$ matches the graph of $y = g(x)$.

   To write $f(x)$ in terms of $g(x)$, we reverse the steps done in obtaining the graph of $g(x)$ from $f(x)$ in the reverse order.

   Since to get from the graph of f to the graph of g, we: first, shifted left 2 units; second reflected across the y-axis; third, vertically stretched by a factor of 2, our first step in taking g back to f is to implement a vertical compression by a factor of 2. Hence, starting with the graph of $y = g(x)$, our first step results in the formula $y = \frac{1}{2}g(x)$.

   Next, we need to undo the reflection abut the y-axis. If the point (a, b) is reflected about the y-axis, we obtain the point $(-a, b)$. To return to the point (a, b), we reflect $(-a, b)$ across the y-axis again: $(-(-a), b) = (a, b)$. Hence, we take the graph of $y = \frac{1}{2}g(x)$ and reflect it across the y-axis to obtain $y = \frac{1}{2}g(-x)$.

   Our last step is to undo a horizontal shift to the left 2 units. The reverse of this process is shifting the graph to the *right* two units, so we get $y = \frac{1}{2}g(-(x - 2)) = \frac{1}{2}g(-x + 2)$.[10]

   We leave it to the reader to start with the graph of $y = g(x)$ and check the graph of $y = \frac{1}{2}g(-x + 2)$ matches the graph of $y = f(x)$. □

2.3.4 Transformations in Sequence

Now that we have studied three basic classes of transformations: shifts, reflections, and scalings, we present a result below which provides one algorithm to follow to transform the graph of $y = f(x)$ into the graph of $y = af(bx - h) + k$ without the need of using Theorems 2.2, 2.3, 2.4, 2.5 and 2.6 individually.

Theorem 2.7 is the ultimate generalization of Theorems 4.4, 5.7, 6.1, 7.1, 8.2 and **??**. We note the underlying assumption here is that regardless of the order or number of shifts, reflections and scalings applied to the graph of a function f, we can always represent the final result in the form $g(x) = af(bx-h)+k$.

[10] To see this better, let $F(x) = \frac{1}{2}g(-x)$. Per Theorem 2.3, the graph of $F(x - 2) = \frac{1}{2}g(-(x - 2)) = \frac{1}{2}g(-x + 2)$ is the same as the graph of F but shifted 2 units to the right.

Since each of these transformations can ultimately be traced back to composing f with linear functions,[11] this fact is verified by showing compositions of linear functions results in a linear function.[12]

Theorem 2.7. Transformations in Sequence. Suppose f is a function. If $a, b \neq 0$, then to graph $g(x) = af(bx - h) + k$ start with the graph of $y = f(x)$ and follow the steps below.

1. Add h to each of the x-coordinates of the points on the graph of f.

   NOTE: This results in a horizontal shift to the left if $h < 0$ or right if $h > 0$.

2. Divide the x-coordinates of the points on the graph obtained in Step 1 by b.

   NOTE: This results in a horizontal scaling, but includes a reflection about the y-axis if $b < 0$.

3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by a.

   NOTE: This results in a vertical scaling, but includes a reflection about the x-axis if $a < 0$.

4. Add k to each of the y-coordinates of the points on the graph obtained in Step 3.

   NOTE: This results in a vertical shift up if $k > 0$ or down if $k < 0$.

Theorem 2.7 can be established by generalizing the techniques developed in this section. Suppose $(c, f(c))$ is on the graph of f. To match up the inputs of $f(bx - h)$ and $f(c)$, we solve $bx - h = c$ and solve.

We first add the h (causing the horizontal shift) and then divide by b. If b is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{b}$. If $b < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{c+h}{b}$ as the input to g which corresponds to the input $x = c$ to f.

We now evaluate $g\left(\frac{c+h}{b}\right) = af\left(b \cdot \frac{c+h}{b} - h\right) + K = af(c + h - h) = af(c) + k$. We notice that the output from f is first multiplied by a. As with the constant b, if $a > 0$, this induces only a vertical scaling. If $a < 0$, then the -1 induces a reflection across the x-axis. Finally, we add k to the result, which is our vertical shift.

A less precise, but more intuitive way to paraphrase Theorem 2.7 is to think of the quantity $bx - h$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation.

What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations to simplify the new y-value: we first multiply by a then add k to find the corresponding y-coordinates on the graph of g.

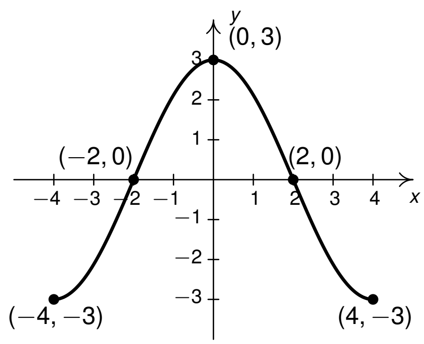
It needs to be stressed that our approach to handling multiple transformations, as summarized in Theorem 2.7 is only one approach. Your instructor may have a different algorithm. As always, the more you understand, the less you'll ultimately need to memorize, so whatever algorithm you choose to follow, it is worth thinking through each step both algebraically and geometrically.

[11] See the remarks at the beginning of the section.

[12] See Exercise 72.

We make good use of Theorem 2.7 in the following example.

Example 2.3.4. Below is the complete graph of $y = f(x)$. Use Theorem 2.7 to graph $g(x) = \dfrac{4 - 3f(1 - 2x)}{2}$.



Solution. We use Theorem 2.7 to track the five 'key points' $(-4,-3)$, $(-2,0)$, $(0,3)$, $(2,0)$ and $(4,-3)$ indicated on the graph of f to their new locations.

We first rewrite $g(x)$ in the form presented in Theorem 2.7, $g(x) = -\frac{3}{2}f(-2x+1)+2$. We set $-2x+1$ equal to the x-coordinates of the key points and solve.

For example, solving $-2x + 1 = -4$, we first subtract 1 to get $-2x = -5$ then divide by -2 to get $x = \frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process: dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in a table below on the left.

Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x = \frac{5}{2}$, and using the fact that $f(-4) = -3$, we get

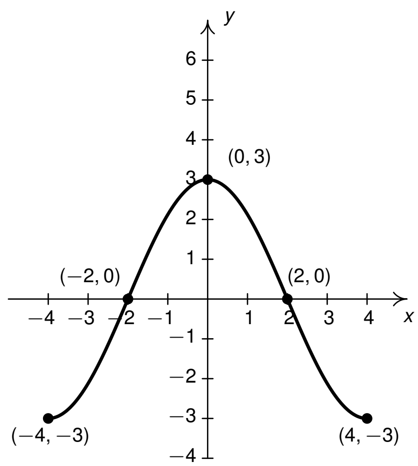
$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2 = -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

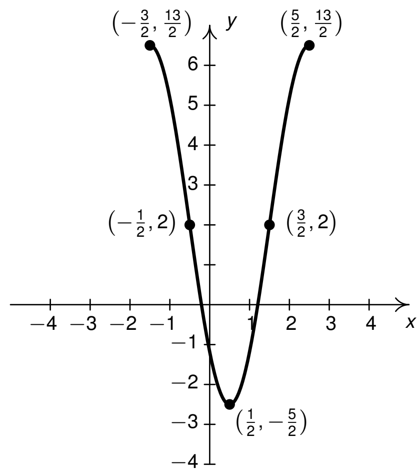
We see that the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we have a vertical stretching by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table below on the right.

$(c, f(c))$	c	$-2x+1=c$	x
$(-4,-3)$	-4	$-2x+1=-4$	$x=\frac{5}{2}$
$(-2,0)$	-2	$-2x+1=-2$	$x=\frac{3}{2}$
$(0,3)$	0	$-2x+1=0$	$x=\frac{1}{2}$
$(2,0)$	2	$-2x+1=2$	$x=-\frac{1}{2}$
$(4,-3)$	4	$-2x+1=4$	$x=-\frac{3}{2}$

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2},\frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2},2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2},-\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2},2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2},\frac{13}{2}\right)$

To graph g, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting f and g side-by-side gives





□

The reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g in Example 2.3.4. We have outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages. Our next example turns the tables and asks for the formula of a function given a desired sequence of transformations.

Example 2.3.5. Let $f(x) = x^2 - |x|$. Find and simplify the formula of the function $g(x)$ whose graph is the result of the graph of $y = f(x)$ undergoing the following sequence of transformations. Check your answer to each step using a graphing utility.

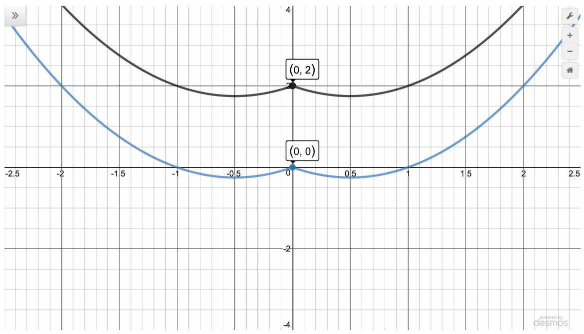
1. Vertical shift up 2 units.
2. Reflection across the x-axis.
3. Horizontal shift right 1 unit.
4. Horizontal compression by a factor of 2.
5. Vertical shift up 3 units.
6. Reflection across the y-axis.

Solution. To help keep us organized we will label each intermediary function. The function g_1 will be the result of applying the first transformation to f. The function g_2 will be the result of applying the first two transformations to f - which is also the result of applying the second transformation to g_1, and so on.[13]

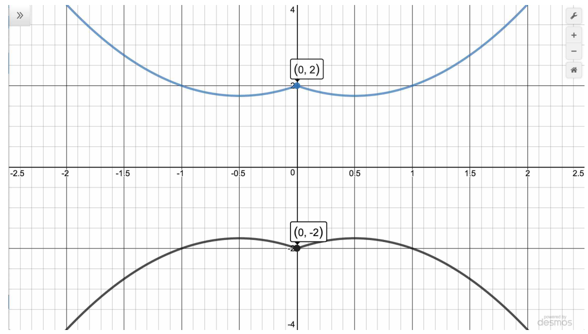
1. Per Theorem 2.2, $g_1(x) = f(x) + 2 = x^2 - |x| + 2$.

2. Per Theorem 2.4, $g_2(x) = -g_1(x) = -[x^2 - |x| + 2] = -x^2 + |x| - 2$.

[13]So, we can think of $g_0 = f$ and $g_6 = g$.



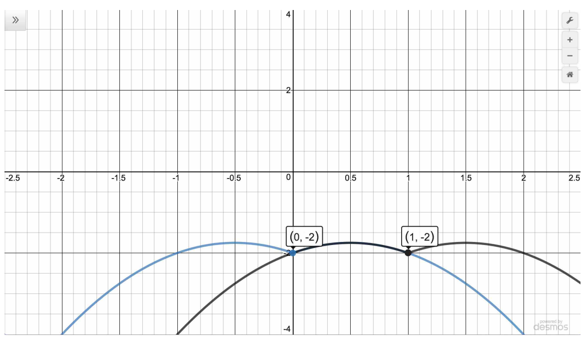
$y = f(x)$ (lighter color) and $y = g_1(x) = f(x) + 2$



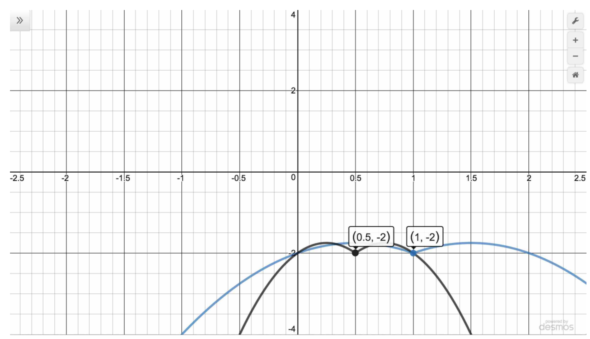
$y = g_1(x)$ (lighter color) and $y = g_2(x) = -g_1(x)$

3. Per Theorem 2.3, $g_3(x) = g_2(x-1) = -(x-1)^2 + |x-1| - 2$.

4. Per Theorem 2.6, $g_4(x) = g_3(2x) = -(2x-1)^2 + |2x-1| - 2$.



$y = g_2(x)$ (lighter color) and $y = g_3(x) = g_2(x-1)$

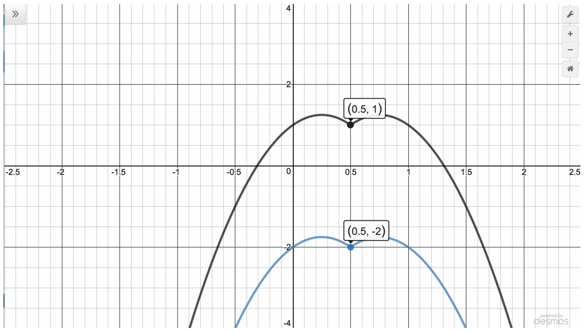


$y = g_3(x)$ (lighter color) and $y = g_4(x) = g_3(2x)$

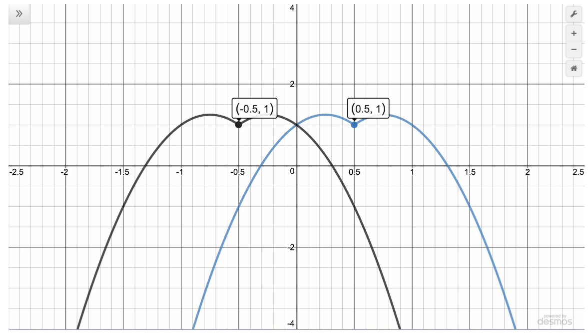
5. Per Theorem 2.2, $g_5(x) = g_4(x) + 3 = -(2x-1)^2 + |2x-1| - 2 + 3 = -(2x-1)^2 + |2x-1| + 1$.

6. Per Theorem 2.4, $g_6(x) = g_5(-x)$:

$$\begin{aligned} g_6(x) &= g_5(-x) \\ &= -(2(-x)-1)^2 + |2(-x)-1| + 1 \\ &= -(-2x-1)^2 + |-2x-1| + 1 \\ &= -[(-1)(2x+1)]^2 + |[(-1)(2x+1)]| + 1 \\ &= -(-1)^2(2x+1)^2 + |-1||2x+1| + 1 \\ &= -(2x+1)^2 + |2x+1| + 1 \end{aligned}$$



$y = g_4(x)$ (lighter color) and $y = g_5(x) = g_4(x) + 3$



$y = g_5(x)$ (lighter color) and $y = g_6(x) = g_5(-x)$

Hence, $g(x) = g_6(x) = -(2x+1)^2 + |2x+1| + 1$. $\square$

It is instructive to show that the expression $g(x)$ in Example 2.3.4 can be written as $g(x) = af(bx - h) + k$.

One way is to compare the graphs of f and g and work backwards. A more methodical way is to repeat the work of Example 2.3.4, but never substitute the formula for $f(x)$ as follows:

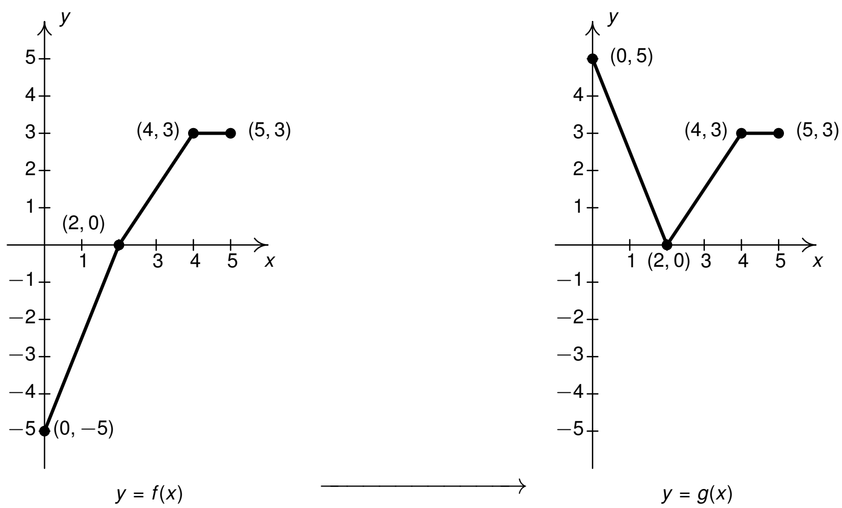
1. Per Theorem 2.2, $g_1(x) = f(x) + 2$.
2. Per Theorem 2.4, $g_2(x) = -g_1(x) = -[f(x) + 2] = -f(x) - 2$.
3. Per Theorem 2.3, $g_3(x) = g_2(x-1) = -f(x-1) - 2$.
4. Per Theorem 2.6, $g_4(x) = g_3(2x) = -f(2x-1) - 2$.
5. Per Theorem 2.2, $g_5(x) = g_4(x) + 3 = -f(2x-1) - 2 + 3 = -f(2x-1) + 1$.
6. Per Theorem 2.4, $g_6(x) = g_5(-x) = -f(2(-x) - 1) + 1 = -f(-2x-1) + 1$.

Hence $g(x) = -f(-2x-1) + 1$. Note we can show f is even,[14] so $f(-2x-1) = f(-(2x+1)) = f(2x+1)$ and obtain $g(x) = -f(2x+1) + 1$.

At the beginning of this section, we discussed how all of the transformations we'd be discussing are the result of composing given functions with linear functions. Not all transformations, not even all rigid transformations,[15] fall into these categories.

For example, consider the graphs of $y = f(x)$ and $y = g(x)$ below.

[14]Recall this means $f(-x) = f(x)$.
[15]See Section **??**.



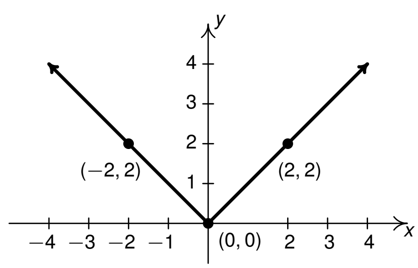
In Exercise 76, we explore a non-linear transformation and revisit the pair of functions f and g then.

2.3.5 Exercises

Suppose $(2, -3)$ is on the graph of $y = f(x)$. In Exercises 1 - 18, use Theorem 2.7 to find a point on the graph of the given transformed function.

1. $y = f(x) + 3$
2. $y = f(x + 3)$
3. $y = f(x) - 1$
4. $y = f(x - 1)$
5. $y = 3f(x)$
6. $y = f(3x)$
7. $y = -f(x)$
8. $y = f(-x)$
9. $y = f(x - 3) + 1$
10. $y = 2f(x + 1)$
11. $y = 10 - f(x)$
12. $y = 3f(2x) - 1$
13. $y = \frac{1}{2}f(4 - x)$
14. $y = 5f(2x + 1) + 3$
15. $y = 2f(1 - x) - 1$
16. $y = f\left(\dfrac{7 - 2x}{4}\right)$
17. $y = \dfrac{f(3x) - 1}{2}$
18. $y = \dfrac{4 - f(3x - 1)}{7}$

The complete graph of $y = f(x)$ is given below. In Exercises 19 - 27, use it and Theorem 2.7 to graph the given transformed function.



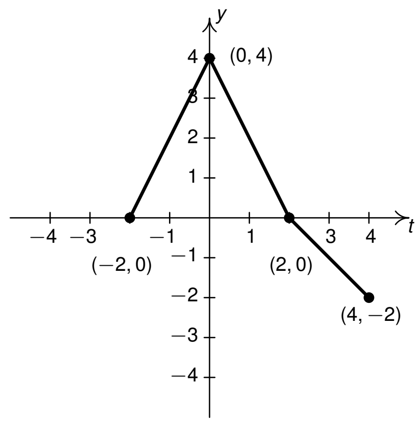
The graph of $y = f(x)$ for Ex. 19 - 27

19. $y = f(x) + 1$
20. $y = f(x) - 2$
21. $y = f(x + 1)$
22. $y = f(x - 2)$
23. $y = 2f(x)$
24. $y = f(2x)$
25. $y = 2 - f(x)$
26. $y = f(2 - x)$
27. $y = 2 - f(2 - x)$

28. Some of the answers to Exercises 19 - 27 above should be the same. Which ones match up? What properties of the graph of $y = f(x)$ contribute to the duplication?

29. The function f used in Exercises 19 - 27 should look familiar. What is $f(x)$? How does this this explain some of the duplication in the answers to Exercises 19 - 27 mentioned in Exercise 28?

The complete graph of $y = g(t)$ is given below. In Exercises 30 - 38, use it and Theorem 2.7 to graph the given transformed function.



The graph of $y = g(t)$ for Ex. 30 - 38

30. $y = g(t) - 1$

31. $y = g(t + 1)$

32. $y = \frac{1}{2}g(t)$

33. $y = g(2t)$

34. $y = -g(t)$

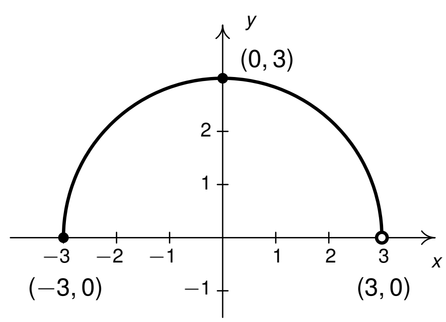
35. $y = g(-t)$

36. $y = g(t + 1) - 1$

37. $y = 1 - g(t)$

38. $y = \frac{1}{2}g(t + 1) - 1$

The complete graph of $y = f(x)$ is given below. In Exercises 39 - 50, use it and Theorem 2.7 to graph the given transformed function.



The graph of $y = f(x)$ for Ex. 39 - 50

39. $g(x) = f(x) + 3$

40. $h(x) = f(x) - \frac{1}{2}$

41. $j(x) = f\left(x - \frac{2}{3}\right)$

42. $a(x) = f(x + 4)$

43. $b(x) = f(x + 1) - 1$

44. $c(x) = \frac{3}{5}f(x)$

45. $d(x) = -2f(x)$

46. $k(x) = f\left(\frac{2}{3}x\right)$

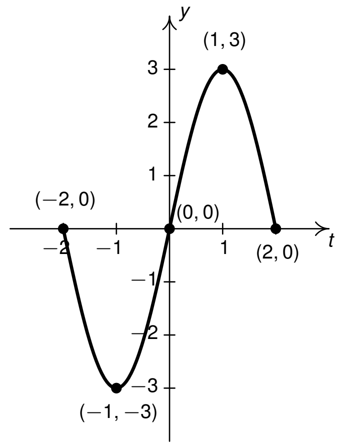
47. $m(x) = -\frac{1}{4}f(3x)$

48. $n(x) = 4f(x - 3) - 6$

49. $p(x) = 4 + f(1 - 2x)$

50. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

The complete graph of $y = S(t)$ is given below.



The graph of $y = S(t)$

The purpose of Exercises 51 - 54 is to build up to the graph of $y = \frac{1}{2}S(-t + 1) + 1$ one step at a time.

51. $y = S_1(t) = S(t + 1)$

52. $y = S_2(t) = S_1(-t) = S(-t + 1)$

53. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t + 1)$

54. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t + 1) + 1$

Let $f(x) = \sqrt{x}$. Find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

55. (1) shift right 2 units; (2) shift down 3 units

56. (1) shift down 3 units; (2) shift right 2 units

57. (1) reflect across the x-axis; (2) shift up 1 unit

58. (1) shift up 1 unit; (2) reflect across the x-axis

59. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units

60. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units

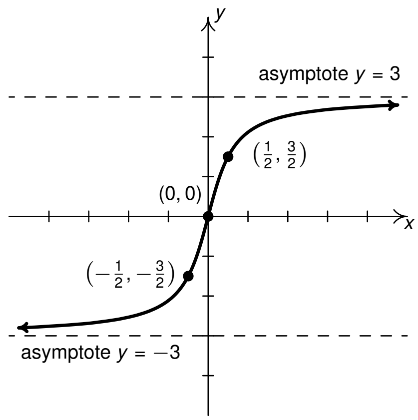
61. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units

62. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2

63. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit

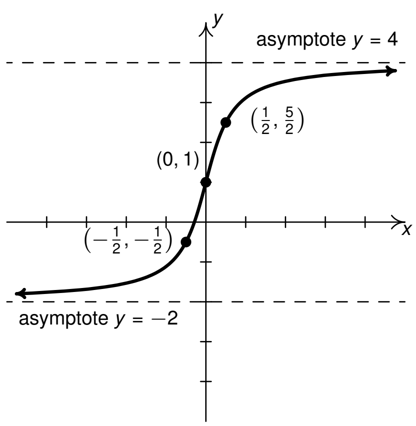
64. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

For Exercises 65 - 70, use the given of $y = f(x)$ to write each function in terms of $f(x)$.

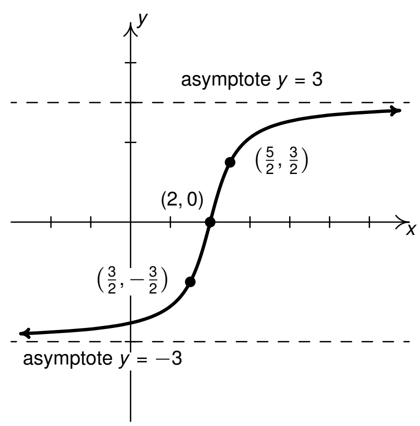


The graph of $y = f(x)$ for Ex. 65 - 70.

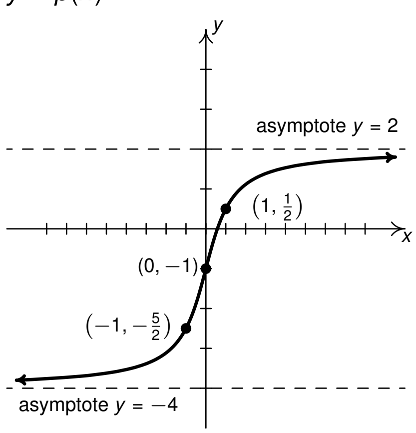
65. $y = g(x)$



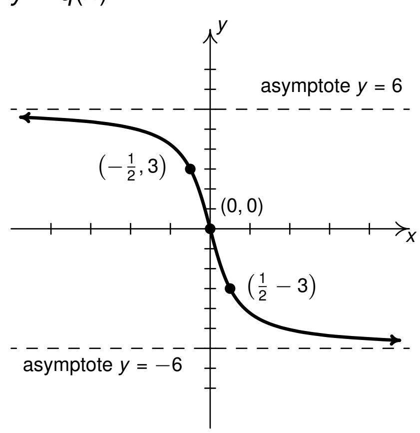
66. $y = h(x)$



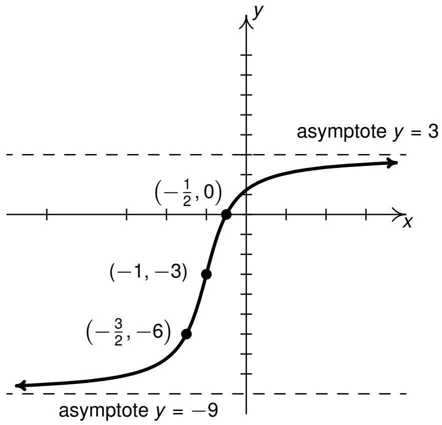
67. $y = p(x)$



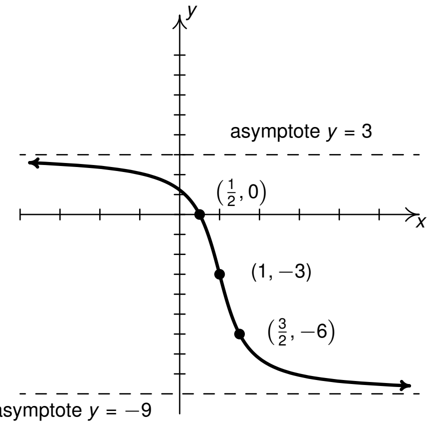
68. $y = q(x)$



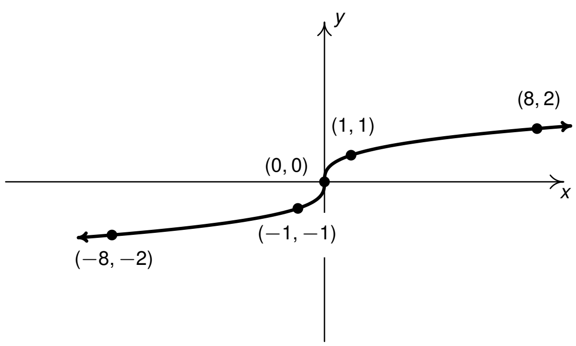
69. $y = r(x)$



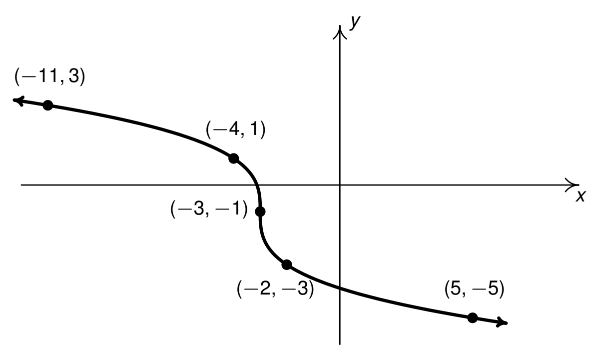
70. $y = s(x)$



71. The graph of $y = f(x) = \sqrt[3]{x}$ is given below on the left and the graph of $y = g(x)$ is given on the right. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.



$y = \sqrt[3]{x}$



$y = g(x)$

72. Show that the composition of two linear functions is a linear function. Hence any (finite) sequence of transformations discussed in this section can be combined into the form given in Theorem 2.7.

(HINT: Let $f(x) = ax + b$ and $g(x) = cx + d$. Find $(f \circ g)(x)$.)

73. For many common functions, the properties of Algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, $\sqrt{9x} = 3\sqrt{x}$, so a horizontal compression of $y = \sqrt{x}$ by a factor of 9 results in the same graph as a vertical stretch of $y = \sqrt{x}$ by a factor of 3.

With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$.

What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

74. Discuss the following questions with your classmates.

    - If f is even, what happens when you reflect the graph of $y = f(x)$ across the y-axis?
    - If f is odd, what happens when you reflect the graph of $y = f(x)$ across the y-axis?
    - If f is even, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
    - If f is odd, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
    - How would you describe symmetry about the origin in terms of reflections?

75. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates, determine the situations in which order does matter and those in which it does not.

76. This Exercise is a follow-up to Exercise 11 in Section 4.2.

    (a) For each of the following functions, use a graphing utility to compare the graph of $y = f(x)$ with the graphs of $y = |f(x)|$ and $y = f(|x|)$.

    - $f(x) = 3 - x$
    - $f(x) = x^2 - x - 6$
    - $f(x) = \sqrt{x+3} - 1$

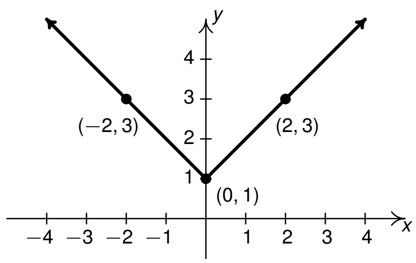
    (b) In general, how does the graph of $y = |f(x)|$ compare with that of $y = f(x)$? What about the graph of $y = f(|x|)$ and $y = f(x)$?

    (c) Referring to the functions f and g graphed on page 120, write g in terms of f.

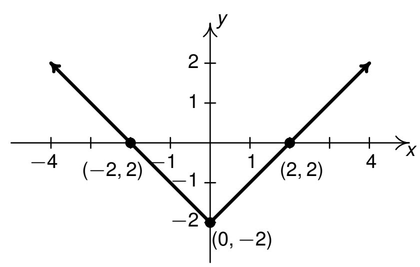
2.3.6 Answers

1. $(2, 0)$
2. $(-1, -3)$
3. $(2, -4)$
4. $(3, -3)$
5. $(2, -9)$
6. $\left(\frac{2}{3}, -3\right)$
7. $(2, 3)$
8. $(-2, -3)$
9. $(5, -2)$
10. $(1, -6)$
11. $(2, 13)$
12. $y = (1, -10)$
13. $\left(2, -\frac{3}{2}\right)$
14. $\left(\frac{1}{2}, -12\right)$
15. $(-1, -7)$
16. $\left(-\frac{1}{2}, -3\right)$
17. $\left(\frac{2}{3}, -2\right)$
18. $(1, 1)$

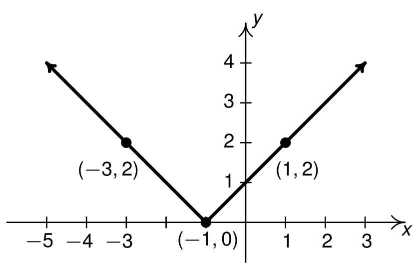
19. $y = f(x) + 1$



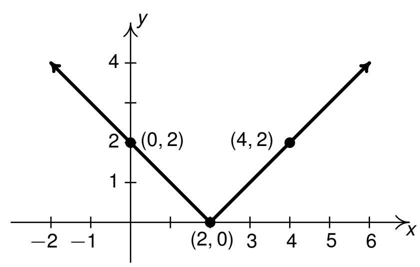
20. $y = f(x) - 2$



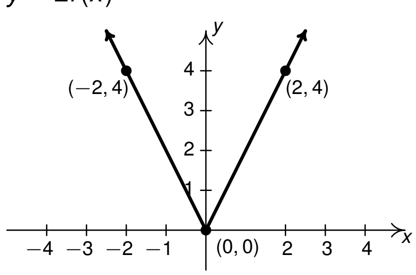
21. $y = f(x + 1)$



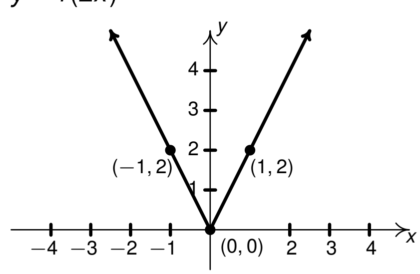
22. $y = f(x - 2)$



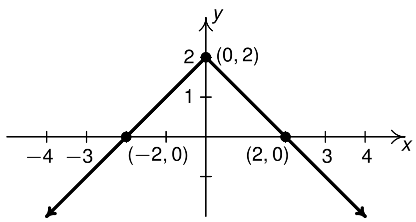
23. $y = 2f(x)$



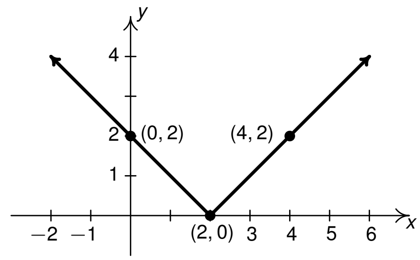
24. $y = f(2x)$



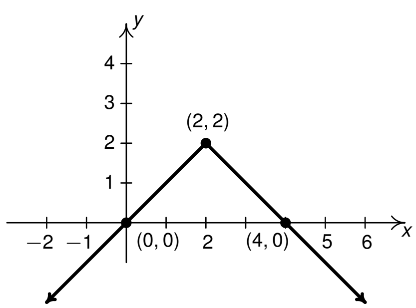
25. $y = 2 - f(x)$



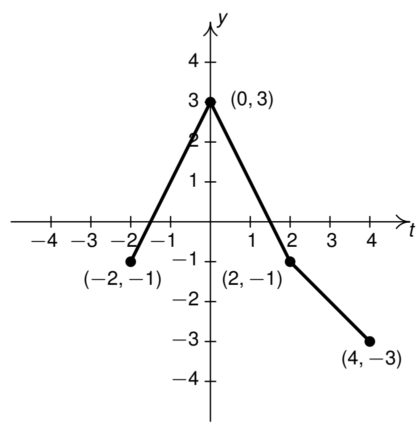
26. $y = f(2 - x)$



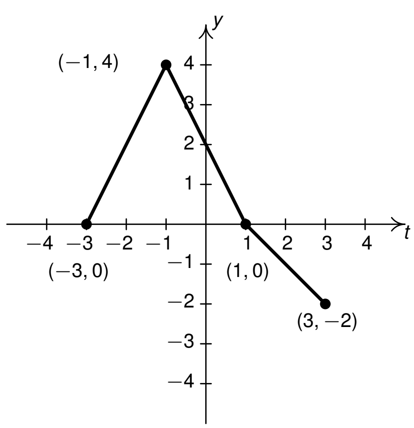
27. $y = 2 - f(2 - x)$



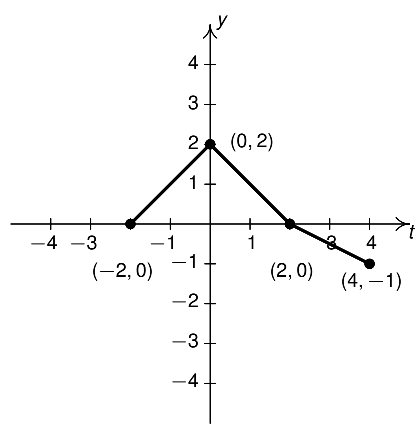
30. $y = g(t) - 1$



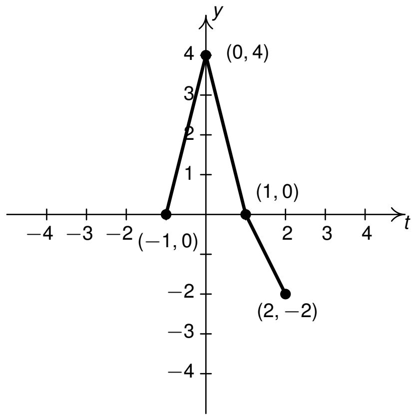
31. $y = g(t + 1)$



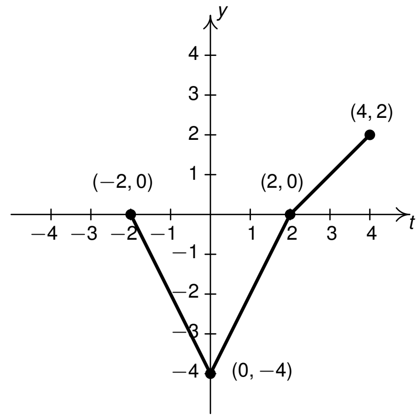
32. $y = \frac{1}{2}g(t)$



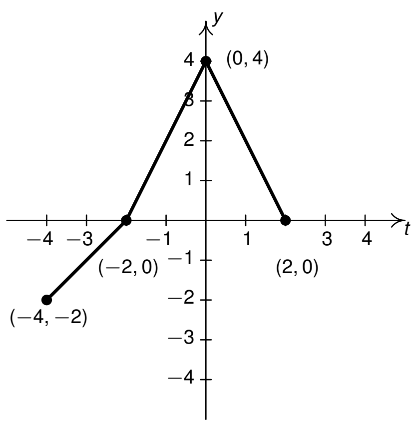
33. $y = g(2t)$



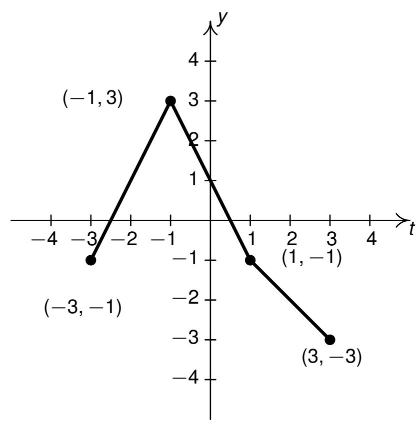
34. $y = -g(t)$



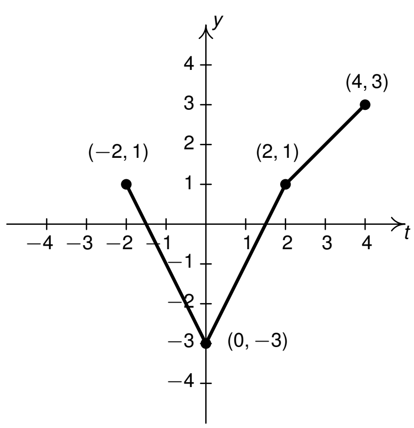
35. $y = g(-t)$



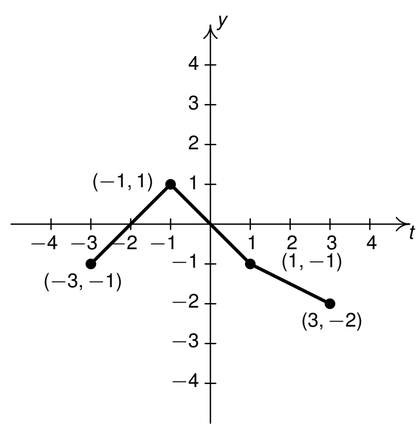
36. $y = g(t+1) - 1$



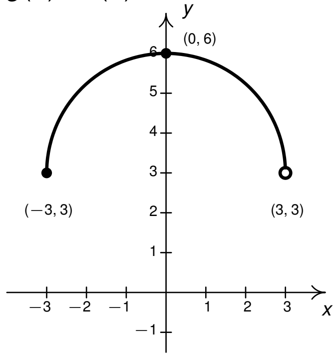
37. $y = 1 - g(t)$



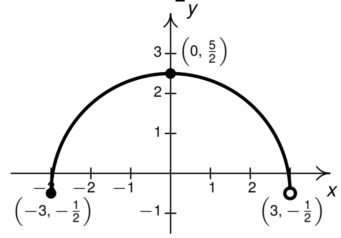
38. $y = \frac{1}{2}g(t+1) - 1$



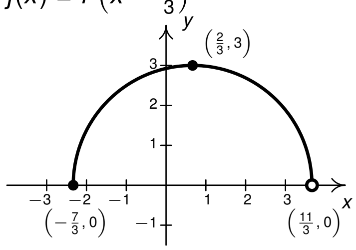
39. $g(x) = f(x) + 3$



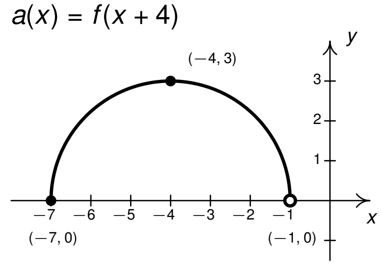
40. $h(x) = f(x) - \frac{1}{2}$



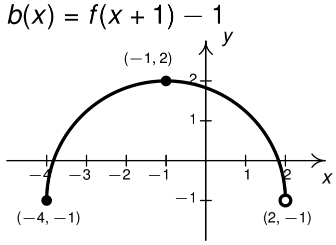
41. $j(x) = f\left(x - \frac{2}{3}\right)$



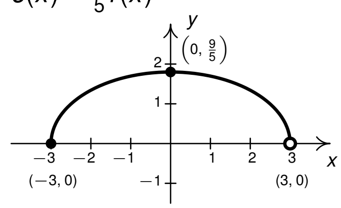
42. $a(x) = f(x + 4)$



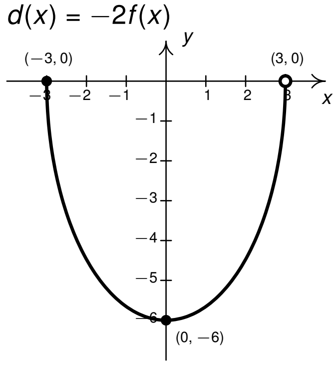
43. $b(x) = f(x + 1) - 1$



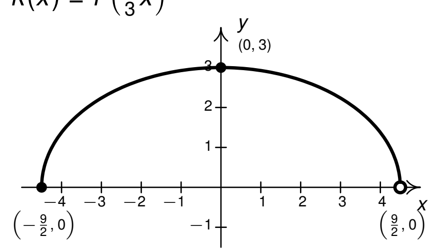
44. $c(x) = \frac{3}{5}f(x)$



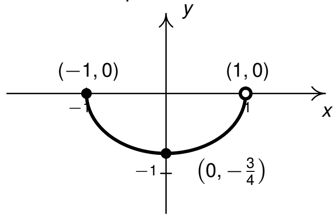
45. $d(x) = -2f(x)$



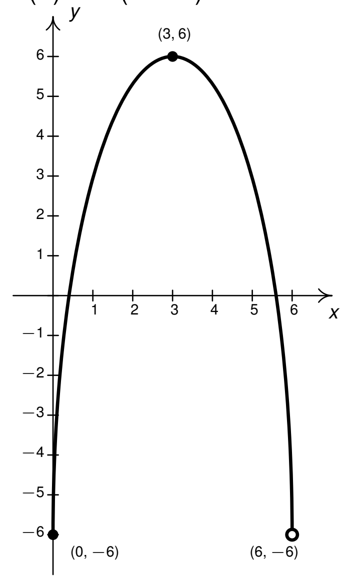
46. $k(x) = f\left(\frac{2}{3}x\right)$



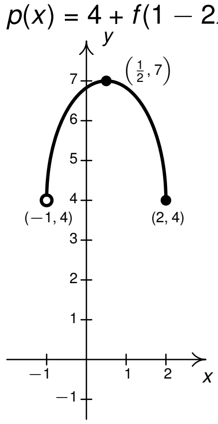
47. $m(x) = -\frac{1}{4}f(3x)$



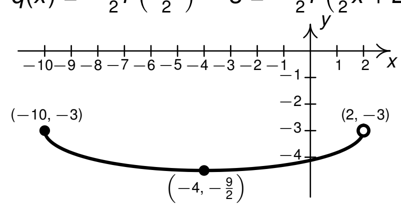
48. $n(x) = 4f(x-3) - 6$



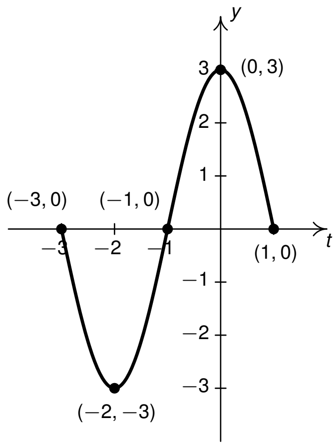
49. $p(x) = 4 + f(1-2x) = f(-2x+1) + 4$



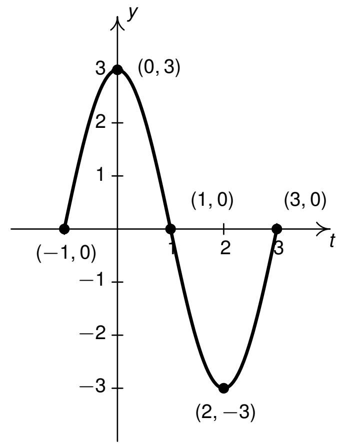
50. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3 = -\frac{1}{2}f\left(\frac{1}{2}x+2\right) - 3$



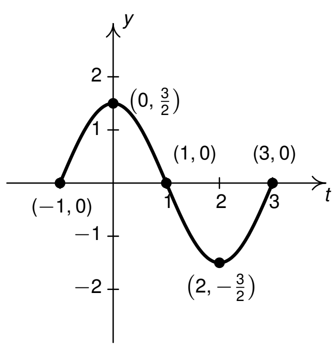
51. $y = S_1(t) = S(t+1)$



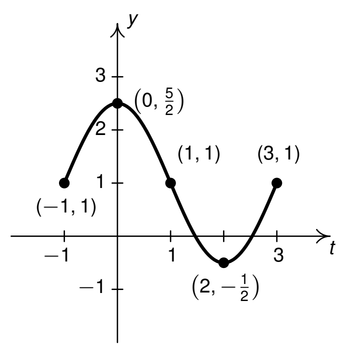
52. $y = S_2(t) = S_1(-t) = S(-t+1)$



53. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t+1)$



54. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t+1) + 1$



55. $g(x) = \sqrt{x-2} - 3$

56. $g(x) = \sqrt{x-2} - 3$

57. $g(x) = -\sqrt{x} + 1$

58. $g(x) = -(\sqrt{x} + 1) = -\sqrt{x} - 1$

59. $g(x) = \sqrt{-x+1} + 2$

60. $g(x) = \sqrt{-(x+1)} + 2 = \sqrt{-x-1} + 2$

61. $g(x) = 2\sqrt{x+3} - 4$

62. $g(x) = 2\left(\sqrt{x+3} - 4\right) = 2\sqrt{x+3} - 8$

63. $g(x) = \sqrt{2x-3} + 1$

64. $g(x) = \sqrt{2(x-3)} + 1 = \sqrt{2x-6} + 1$

65. $g(x) = f(x) + 1$

66. $h(x) = f(x-2)$

67. $p(x) = f\left(\frac{x}{2}\right) - 1$

68. $q(x) = -2f(x) = 2f(-x)$

69. $r(x) = 2f(x+1) - 3$

70. $s(x) = 2f(-x+1) - 3 = -2f(x-1) + 3$

71. $g(x) = -2\sqrt[3]{x+3} - 1$ or $g(x) = 2\sqrt[3]{-x-3} - 1$

Chapter 3

Linear Functions

3.1 Lineae Equations

3.1.1 Linear Equations

The first equations we wish to review are **linear** equations as defined below.

Definition 3.1. An equation is said to be **linear** in a variable x if it can be written in the form $ax = b$ where a and b are expressions which do not involve x and $a \neq 0$.

One key point about Definition 3.1 is that the exponent on the unknown 'x' in the equation is 1, that is $x = x^1$. Our main strategy for solving linear equations is summarized below.

Strategy for Solving Linear Equations

In order to solve an equation which is linear in a given variable, say x:

1. Isolate all of the terms containing x on one side of the equation, putting all of the terms not containing x on the other side of the equation.

2. Factor out the x and divide both sides of the equation by its coefficient.

We illustrate this process with a collection of examples below.

Example 3.1.1. Solve the following equations for the indicated variable. Check your answer.

1. Solve for x: $3x - 6 = 7x + 4$
2. Solve for t: $3 - 1.7t = \dfrac{t}{4}$
3. Solve for a: $\dfrac{1}{18}(7 - 4a) + 2 = \dfrac{a}{3} - \dfrac{4-a}{12}$
4. Solve for y: $8y\sqrt{3} + 1 = 7 - \sqrt{12}(5 - y)$
5. Solve for x: $\dfrac{3x-1}{2} = x\sqrt{50} + 4$
6. Solve for y: $x(4 - y) = 8y$

Solution.

1. The variable we are asked to solve for is x so our first move is to gather all of the terms involving x on one side and put the remaining terms on the other.[1]

$$\begin{array}{rcll}
3x - 6 & = & 7x + 4 & \\
(3x - 6) - 7x + 6 & = & (7x + 4) - 7x + 6 & \text{Subtract } 7x\text{, add } 6 \\
3x - 7x - 6 + 6 & = & 7x - 7x + 4 + 6 & \text{Rearrange terms} \\
-4x & = & 10 & 3x - 7x = (3-7)x = -4x \\
\dfrac{-4x}{-4} & = & \dfrac{10}{-4} & \text{Divide by the coefficient of } x \\
x & = & -\dfrac{5}{2} & \text{Reduce to lowest terms}
\end{array}$$

To check our answer, we substitute $x = -\frac{5}{2}$ into each side of the **orginial** equation to see the equation is satisfied. Sure enough, $3\left(-\frac{5}{2}\right) - 6 = -\frac{27}{2}$ and $7\left(-\frac{5}{2}\right) + 4 = -\frac{27}{2}$.

[1] In the margin notes, when we speak of operations, e.g.,'Subtract $7x$,' we mean to subtract $7x$ from **both** sides of the equation. The 'from both sides of the equation' is omitted in the interest of spacing.

2. In our next example, the unknown is t and we not only have a fraction but also a decimal to wrangle. Fortunately, with equations we can multiply both sides to rid us of these computational obstacles:

$$\begin{array}{rcll}
3 - 1.7t & = & \dfrac{t}{4} & \\
40(3 - 1.7t) & = & 40\left(\dfrac{t}{4}\right) & \text{Multiply by 40} \\
40(3) - 40(1.7t) & = & \dfrac{40t}{4} & \text{Distribute} \\
120 - 68t & = & 10t & \\
(120 - 68t) + 68t & = & 10t + 68t & \text{Add } 68t \text{ to both sides} \\
120 & = & 78t & 68t + 10t = (68 + 10)t = 78t \\
\dfrac{120}{78} & = & \dfrac{78t}{78} & \text{Divide by the coefficient of } t \\
\dfrac{120}{78} & = & t & \\
\dfrac{20}{13} & = & t & \text{Reduce to lowest terms}
\end{array}$$

To check, we again substitute $t = \frac{20}{13}$ into each side of the original equation. We find that $3 - 1.7\left(\frac{20}{13}\right) = 3 - \left(\frac{17}{10}\right)\left(\frac{20}{13}\right) = \frac{5}{13}$ and $\frac{(20/13)}{4} = \frac{20}{13} \cdot \frac{1}{4} = \frac{5}{13}$ as well.

3. To solve this next equation, we begin once again by clearing fractions. The least common denominator here is 36:

$$\begin{array}{rcll}
\dfrac{1}{18}(7 - 4a) + 2 & = & \dfrac{a}{3} - \dfrac{4 - a}{12} & \\
36\left(\dfrac{1}{18}(7 - 4a) + 2\right) & = & 36\left(\dfrac{a}{3} - \dfrac{4 - a}{12}\right) & \text{Multiply by 36} \\
\dfrac{36}{18}(7 - 4a) + (36)(2) & = & \dfrac{36a}{3} - \dfrac{36(4 - a)}{12} & \text{Distribute} \\
2(7 - 4a) + 72 & = & 12a - 3(4 - a) & \text{Distribute} \\
14 - 8a + 72 & = & 12a - 12 + 3a & \\
86 - 8a & = & 15a - 12 & 12a + 3a = (12 + 3)a = 15a \\
(86 - 8a) + 8a + 12 & = & (15a - 12) + 8a + 12 & \text{Add } 8a \text{ and } 12 \\
86 + 12 - 8a + 8a & = & 15a + 8a - 12 + 12 & \text{Rearrange terms} \\
98 & = & 23a & 15a + 8a = (15 + 8)a = 23a \\
\dfrac{98}{23} & = & \dfrac{23a}{23} & \text{Divide by the coefficient of } a \\
\dfrac{98}{23} & = & a &
\end{array}$$

The check, as usual, involves substituting $a = \frac{98}{23}$ into both sides of the original equation. The reader is encouraged to work through the (admittedly messy) arithmetic. Both sides work out to $\frac{199}{138}$.

4. The square roots may dishearten you but we treat them just like the real numbers they are. Our strategy is the same: get everything with the variable (in this case y) on one side, put everything else on the other and divide by the coefficient of the variable. We've added a few steps to the narrative that we would ordinarily omit just to help you see that this equation is indeed linear.

$$\begin{array}{rcll}
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5-y) & \\
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5)+\sqrt{12}y & \text{Distribute} \\
8y\sqrt{3}+1 & = & 7-(2\sqrt{3})5+(2\sqrt{3})y & \sqrt{12}=\sqrt{4\cdot 3}=2\sqrt{3} \\
8y\sqrt{3}+1 & = & 7-10\sqrt{3}+2y\sqrt{3} & \\
(8y\sqrt{3}+1)-1-2y\sqrt{3} & = & (7-10\sqrt{3}+2y\sqrt{3})-1-2y\sqrt{3} & \text{Subtract 1 and } 2y\sqrt{3} \\
8y\sqrt{3}-2y\sqrt{3}+1-1 & = & 7-1-10\sqrt{3}+2y\sqrt{3}-2y\sqrt{3} & \text{Rearrange terms} \\
(8\sqrt{3}-2\sqrt{3})y & = & 6-10\sqrt{3} & \\
6y\sqrt{3} & = & 6-10\sqrt{3} & \text{See note below} \\
\dfrac{6y\sqrt{3}}{6\sqrt{3}} & = & \dfrac{6-10\sqrt{3}}{6\sqrt{3}} & \text{Divide } 6\sqrt{3} \\
y & = & \dfrac{2\cdot\sqrt{3}\cdot\sqrt{3}-2\cdot 5\cdot\sqrt{3}}{2\cdot 3\cdot\sqrt{3}} & \\
y & = & \dfrac{\cancel{2}\cancel{\sqrt{3}}(\sqrt{3}-5)}{\cancel{2}\cdot 3\cdot\cancel{\sqrt{3}}} & \text{Factor and cancel} \\
y & = & \dfrac{\sqrt{3}-5}{3} &
\end{array}$$

In the list of computations above we marked the row $6y\sqrt{3} = 6 - 10\sqrt{3}$ with a note. That's because we wanted to draw your attention to this line without breaking the flow of the manipulations. The equation $6y\sqrt{3} = 6 - 10\sqrt{3}$ is in fact linear according to Definition 3.1: the variable is y, the value of A is $6\sqrt{3}$ and $B = 6 - 10\sqrt{3}$. Checking the solution, while not trivial, is good mental exercise. Each side works out to be $\frac{27-40\sqrt{3}}{3}$.

5. Proceeding as before, we simplify radicals and clear denominators. Once we gather all of the terms containing x on one side and move the other terms to the other, we factor out x to identify its

coefficient then divide to get our answer.

$$\begin{array}{rcll}
\dfrac{3x-1}{2} & = & x\sqrt{50}+4 & \\
\dfrac{3x-1}{2} & = & 5x\sqrt{2}+4 & \sqrt{50}=\sqrt{25\cdot 2} \\
2\left(\dfrac{3x-1}{2}\right) & = & 2\left(5x\sqrt{2}+4\right) & \text{Multiply by 2} \\
\dfrac{\cancel{2}\cdot(3x-1)}{\cancel{2}} & = & 2(5x\sqrt{2})+2\cdot 4 & \text{Distribute} \\
3x-1 & = & 10x\sqrt{2}+8 & \\
(3x-1)-10x\sqrt{2}+1 & = & (10x\sqrt{2}+8)-10x\sqrt{2}+1 & \text{Subtract } 10x\sqrt{2}\text{, add 1} \\
3x-10x\sqrt{2}-1+1 & = & 10x\sqrt{2}-10x\sqrt{2}+8+1 & \text{Rearrange terms} \\
3x-10x\sqrt{2} & = & 9 & \\
(3-10\sqrt{2})x & = & 9 & \text{Factor} \\
\dfrac{(3-10\sqrt{2})x}{3-10\sqrt{2}} & = & \dfrac{9}{3-10\sqrt{2}} & \text{Divide by the coefficient of } x \\
x & = & \dfrac{9}{3-10\sqrt{2}} &
\end{array}$$

The reader is encouraged to check this solution - it isn't as bad as it looks if you're careful! Each side works out to be $\dfrac{12+5\sqrt{2}}{3-10\sqrt{2}}$.

6. If we were instructed to solve our last equation for x, we'd be done in one step: divide both sides by $(4-y)$ - assuming $4-y\neq 0$, that is. Alas, we are instructed to solve for y, which means we have some more work to do.

$$\begin{array}{rcll}
x(4-y) & = & 8y & \\
4x-xy & = & 8y & \text{Distribute} \\
(4x-xy)+xy & = & 8y+xy & \text{Add } xy \\
4x & = & (8+x)y & \text{Factor}
\end{array}$$

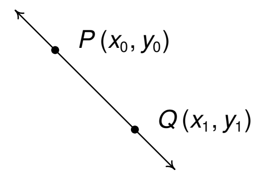
In order to finish the problem, we need to divide both sides of the equation by the coefficient of y which in this case is $8+x$. This expression contains a variable so we need to stipulate that we may perform this division only if $8+x\neq 0$, or, in other words, $x\neq -8$. Hence, we write our solution as:

$$y=\frac{4x}{8+x}, \quad \text{provided } x\neq -8$$

What happens if $x=-8$? Substituting $x=-8$ into the original equation gives $(-8)(4-y)=8y$ or $-32+8y=8y$. This reduces to $-32=0$, which is a contradiction. This means there is no solution when $x=-8$, so we've covered all the bases. Checking our answer requires some Algebra we haven't reviewed yet in this text, but the necessary skills *should* be lurking somewhere in the mathematical mists of your mind. The adventurous reader is invited to plug $y=\frac{4x}{8+x}$ into the original equation and show that both sides work out to $\frac{32x}{x+8}$. □

3.1.2 Graphing Lines

In Section **??**, we concerned ourselves with the finite line segment between two points P and Q. Specifically, we found its length (the distance between P and Q) and its midpoint. In this section, our focus will be on the *entire* line, and ways to describe it algebraically. Consider the generic situation below.



To give a sense of the 'steepness' of the line, we recall that we can compute the **slope** of the line as follows. (Read the character Δ as 'change in'.)

Equation 3.1. The **slope** m of the line containing the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x},$$

provided $x_1 \neq x_0$, that is, $\Delta x \neq 0$.

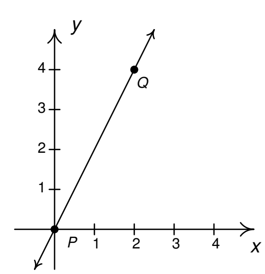
A couple of notes about Equation 3.1 are in order. First, don't ask why we use the letter 'm' to represent slope. There are many explanations out there, but apparently no one really knows for sure.[1] Secondly, the stipulation $x_1 \neq x_0$ (or $\Delta x \neq 0$) ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically when the 'change in x' is 0; the anxious reader can skip along to the next example.

Example 3.1.2. Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

1. $P(0, 0)$, $Q(2, 4)$
2. $P(-1, 2)$, $Q(3, 4)$
3. $P(-2, 3)$, $Q(2, -3)$
4. $P(-3, 2)$, $Q(4, 2)$
5. $P(2, 3)$, $Q(2, -1)$
6. $P(2, 3)$, $Q(2.1, -1)$

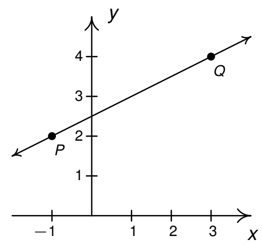
Solution. In each of these examples, we apply the slope formula, Equation 3.1.

1. $m = \dfrac{4 - 0}{2 - 0} = \dfrac{4}{2} = 2$

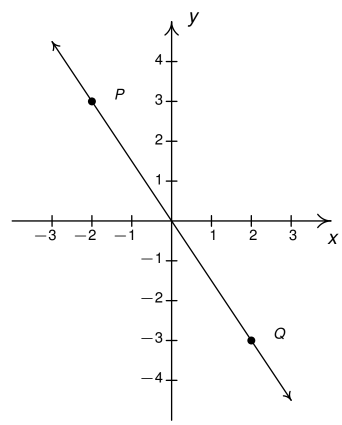


[1] See www.mathforum.org or www.mathworld.wolfram.com for discussions on this topic.

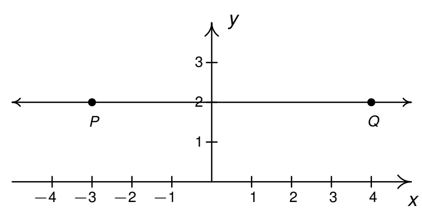
2. $m = \dfrac{4-2}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$



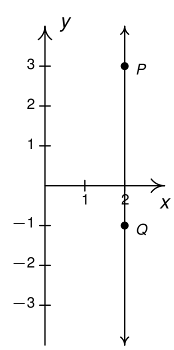
3. $m = \dfrac{-3-3}{2-(-2)} = \dfrac{-6}{4} = -\dfrac{3}{2}$



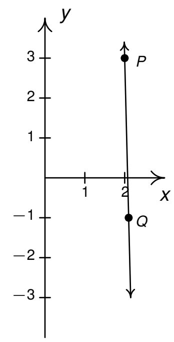
4. $m = \dfrac{2-2}{4-(-3)} = \dfrac{0}{7} = 0$



5. $m = \dfrac{-1-3}{2-2} = \dfrac{-4}{0}$, which is undefined



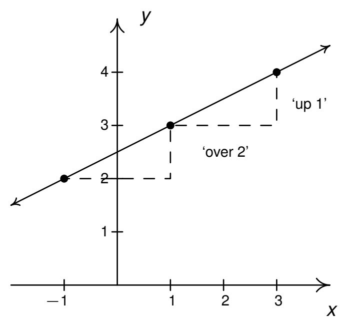
6. $m = \dfrac{-1-3}{2.1-2} = \dfrac{-4}{0.1} = -40$



□

A few comments about Example 3.1.2 are in order. First, if the slope is positive then the resulting line is said to be 'increasing', meaning as we move from left to right,[2] the y-values are getting larger.[3] Similarly, if the slope is negative, we say the line is 'decreasing', since as we move from left to right, the y-values are getting smaller. A slope of 0 results in a horizontal line which we say is 'constant', since the y-values here remain unchanged as we move from left to right, and an undefined slope results in a vertical line.[4]

Second, the larger the slope is in absolute value, the steeper the line. You may recall from Intermediate Algebra that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, if the slope works out to be $\frac{1}{2}$, we can interpret this as a 'rise' of 1 unit upward for every 'run' of 2 units to the right:



In this way, we may view the slope as 'the **rate of change** of y with respect to x'. From the expression

$$m = \frac{\Delta y}{\Delta x}$$

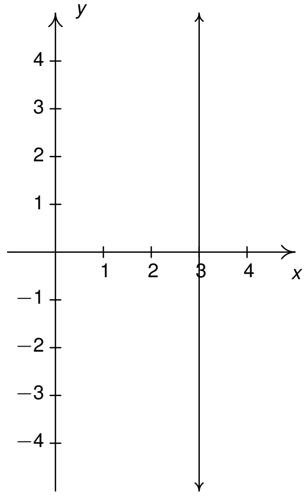
we get $\Delta y = m\Delta x$ so that the y-values change 'm' times as fast as the x-values. We'll have more to say about this concept in Section 3.2 when we explore applications of linear functions; presently, we will keep our attention focused on the analytic geometry of lines. To that end, our next task is to find algebraic equations that describe lines and we start with a discussion of vertical and horizontal lines.

[2]That is, as we increase the x-values . . .

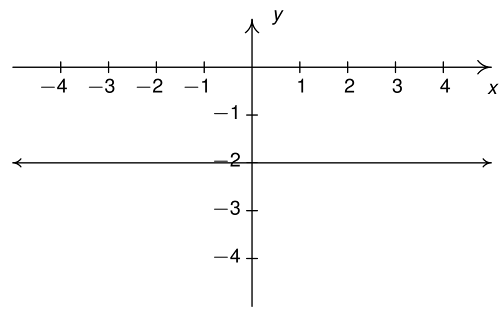
[3]We'll have more to say about this idea in Section 3.2.

[4]Some authors use the unfortunate moniker 'no slope' when a slope is undefined. It's easy to confuse the notions of 'no slope' with 'slope of 0'. For this reason, we will describe slopes of vertical lines as 'undefined'.

Consider the two lines shown below: V (for 'V'ertical Line) and H (for 'H'orizontal Line).



The line V



The line H

All of the points on the line V have an x-coordinate of 3. Conversely, any point with an x-coordinate of 3 lies on the line V. Said differently, the point (x, y) lies on V if and only if $x = 3$. Because of this, we say the equation $x = 3$ *describes* the line V, or, said differently, the *graph* of the equation $x = 3$ is the line V.

In Section 9.3, we'll spend a great deal of time talking about graphing equations. For now, it suffices to know that a graph of an equation is a plot of all of the points which make the equation true. So to graph $x = 3$, we plot all of the points (x, y) which satisfy $x = 3$ and this gives us our vertical line V.

Turning our attention to H, we note that every point on H has a y-coordinate of -2, and vice-versa. Hence the equation $y = -2$ describes the line H, or the graph of the equation $y = -2$ is H. In general:

Equation 3.2. Equations of Vertical and Horizontal Lines

- The graph of the equation $x = a$ in the xy-plane is a **vertical line** through $(a, 0)$.
- The graph of the equation $y = b$ in the xy-plane is a **horizontal line** through $(0, b)$.

Of course, we may be working on axes which aren't labeled with the 'usual' x's and y's. In this case, we understand Equation 3.2 to say 'horizontal axis label = a' describes a *vertical* line through $(a, 0)$ and 'vertical axis label = b' describes a *horizontal* line through $(0, b)$.

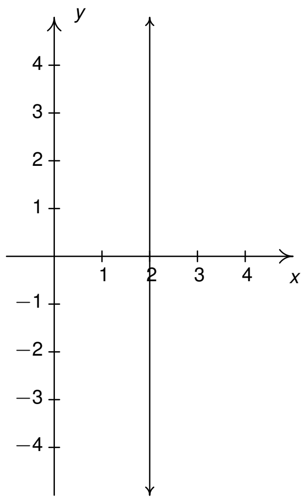
Example 3.1.3.

1. Graph the following equations in the xy-plane:

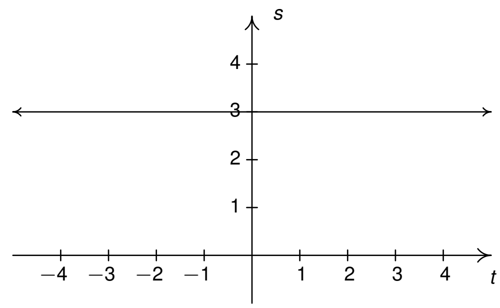
   (a) $y = 3$

   (b) $x = -117$

2. Find the equation of each of the given lines.



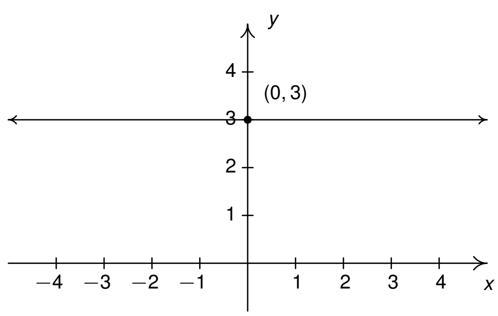
Line L_1



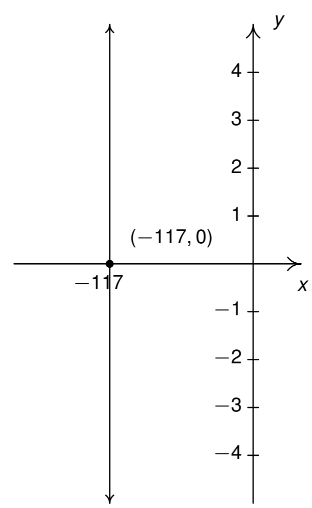
Line L_2

Solution.

1. Since we're in the familiar xy-plane, the graph of $y = 3$ is a horizontal line through $(0, 3)$, shown below on the left and the graph of $x = -117$ is a vertical line through $(-117, 0)$. We scale the x-axis differently than the y-axis to produce the graph below on the right.



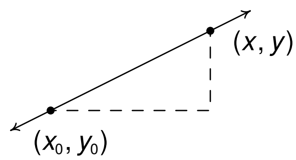
The line $y = 3$



The line $x = -117$

2. Since L_1 is a vertical line through $(2, 0)$, and the horizontal axis is labeled with 'x', the equation of L_1 is $x = 2$. Since L_2 is a horizontal line through $(0, 3)$ and the vertical axis is labeled as 's', the equation of this line is $s = 3$. □

Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of m and contains the point (x_0, y_0). Suppose (x, y) is another point on the line, as indicated below.



Equation 3.1 yields

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} \\ m(x - x_0) &= y - y_0 \\ y - y_0 &= m(x - x_0) \end{aligned}$$

which is known as the **point-slope form** of a line.

Equation 3.3. The **point-slope form** of the line with slope m containing the point (x_0, y_0) is the equation

$$y - y_0 = m(x - x_0)$$

A few remarks about Equation 3.3 are in order. First, note that if the slope $m = 0$, then the line is horizontal and Equation 3.3 reduces to $y - y_0 = 0$ or $y = y_0$, as prescribed by Equation 3.2.[5] Second, we may need to change the letters in Equation 3.3 from 'x' and 'y' depending on the context, so while Equation 3.3 should be committed to memory, it should be understood that 'x' refers to whichever variable is used to label the horizontal axis, and y refers to whichever variable is used to label the vertical axis. Lastly, while Equation 3.3 is, by far, the easiest way to *construct* the equation of a line given a point and a slope, more often than not, the equation is solved for y and simplified into the form below.

Equation 3.4. The **slope-intercept form** of the line with slope m and y-intercept $(0, b)$ is the equation

$$y = mx + b$$

Equation 3.4 is probably[6] a familiar sight from Intermediate Algebra. You may recall from that class that the 'intercept' in 'slope-intercept' comes from the fact that this line 'intercepts' or crosses the y-axis at the point $(0, b)$.[7] If we set the slope, $m = 0$, we obtain $y = b$, the formula for Horizontal Lines first introduced in Equation 3.2. Hence, any line which has a defined slope m can be represented in both point-slope and slope-intercept forms. The only exceptions are vertical lines.[8] There is one equation - the aptly named 'general form' - which describes every type of line and it is presented on the next page.

[5] Here we have y_0 as the constant whereas in the Equation we used the letter b. The form y = constant is what matters.
[6] Hopefully?
[7] We can verify this algebraically by setting $x = 0$ in the equation $y = mx + b$ and obtaining $y = b$.
[8] We'll have more to say about this in Section 3.2.

Equation 3.5. Every line may be represented by an equation of the form $Ax + By = C$, where A, B and C are real numbers for which A and B aren't both zero. This is called *a* **general form** of the line.

Note the indefinite article 'a' in Equation 3.5. The line $y = 5$ is a general form for the horizontal line through $(0, 5)$, but so are $3y = 15$ and $0.5y = 2.5$. The reader is left to ponder the use of the definite article 'the' in Equations 3.3 and 3.4. Regardless of *which* form the equation of a line takes, note that the variables involved are all raised to the first power.[9] For instance, there are no $\sqrt{x}$ terms, no y^2 terms or any variables appearing in denominators. Let's look at a few examples.

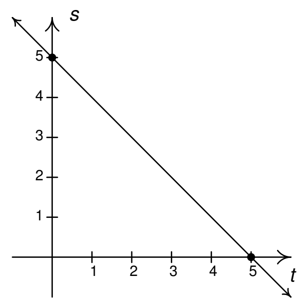
Example 3.1.4.

1. Graph the following equations in the xy-plane:

   (a) $y = 3x - 1$ (b) $2x + 4y = 3$

2. Find the slope-intercept form of the line containing the points $(-1, 3)$ and $(2, 1)$.

3. Find the slope-intercept form of the equation of the line below:



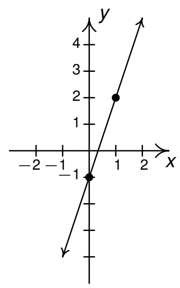
Solution.

1. To graph a line, we need just two points on that line. There are several ways to do this, and we showcase two of them here. For the first equation, we recognize that $y = 3x - 1$ is in slope-intercept form, $y = mx + b$, with $m = 3$ and $b = -1$. This immediately gives us one point on the graph – the y-intercept $(0, -1)$. From here, we use the slope $m = 3 = \frac{3}{1}$ and move one unit to the right and three units up, to obtain a second point on the line, $(1, 2)$. Connecting these points gives us the graph on the left at the top of the next page.

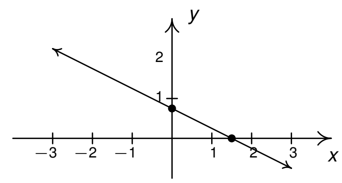
   The second equation, $2x + 4y = 3$, is a general form of a line. To get two points here, we choose 'convenient' values for one of the variables, and solve for the other variable. Choosing $x = 0$, for example, reduces $2x + 4y = 3$ to $4y = 3$, or $y = \frac{3}{4}$. This means the point $\left(0, \frac{3}{4}\right)$ is on the graph. Choosing $y = 0$ gives $2x = 3$, or $x = \frac{3}{2}$. This gives is a second point on the line, $\left(\frac{3}{2}, 0\right)$.[10] Our graph of $2x + 4y = 3$ is on the right at the top of the next page.

[9] Recall, $x = x^1$, $y = y^1$, etc.

[10] You may recall, that this is the x-intercept of the line.



$y = 3x - 1$



$2x + 4y = 3$

2. We'll assume we're using the familiar (x, y) axis labels and begin by finding the slope of the line using Equation 3.1: $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$. Next, we substitute this result, along with one of the given points, into the point-slope equation of the line, Equation 3.3. We have two options for the point (x_0, y_0). We'll use $(-1, 3)$ and leave it to the reader to check that using $(2, 1)$ results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned}
y - y_0 &= m\,(x - x_0) \\
y - 3 &= -\frac{2}{3}\,(x - (-1)) \\
y - 3 &= -\frac{2}{3}\,(x + 1) \\
y - 3 &= -\frac{2}{3}x - \frac{2}{3} \\
y &= -\frac{2}{3}x - \frac{2}{3} + 3 \\
y &= -\frac{2}{3}x + \frac{7}{3}.
\end{aligned}$$

We can check our answer by showing that both $(-1, 3)$ and $(2, 1)$ are on the graph of $y = -\frac{2}{3}x + \frac{7}{3}$ algebraically by showing that the equation holds true when we substitute $x = -1$ and $y = 3$ and when $x = 2$ and $y = 1$.

3. From the graph, we see that the points $(0, 5)$ and $(5, 0)$ are on the line, so we may proceed as we did in the previous problem. Here, however, we use 't' in place of 'x' and 's' in place of 'y' in accordance to the axis labels given. We find the slope $m = \frac{\Delta s}{\Delta t} = \frac{0-5}{5-0} = -1$. As before, we have two points to choose from to substitute into the point-slope formula, and, as before, we'll select one of them, $(0, 5)$ and leave the computations with $(5, 0)$ to the reader.

$$\begin{aligned}
s - s_0 &= m\,(t - t_0) \\
s - 5 &= (-1)\,(t - 0) \\
s - 5 &= -t \\
s &= -t + 5.
\end{aligned}$$

As before we can check this line contains both points $(t, s) = (0, 5)$ and $(t, s) = (5, 0)$ algebraically. □

While every point on a line holds value and meaning,[11] we've reminded you of certain points, called 'intercepts,' which hold special enough significance to be singled out. Formally, we define these as follows.

Definition 3.2. Given a graph of an equation in the xy-plane:

- A point on a graph which is also on the x-axis is called an ***x*-intercept** of the graph. To determine the x-intercept(s) of a graph, set $y = 0$ in the equation and solve for x.

  NOTE: x-intercepts always have the form: $(x_0, 0)$.

- A point on a graph which is also on the y-axis is called an ***y*-intercept** of the graph. To determine the y-intercept(s) of a graph, set $x = 0$ in the equation and solve for y.

  NOTE: y-intercepts always have the form: $(0, y_0)$.

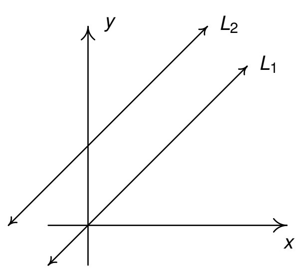
As usual, the labels of the axes in the problem will dictate the labels on the intercepts. If we're working in the vw-plane, for instance, there would be v- and w-intercepts.

The last little bit of analytic geometry we need to review about lines are the concepts of 'parallel' and 'perpendicular' lines. Parallel lines do not intersect,[12] and hence, parallel lines necessarily have the same slope. Perpendicular lines intersect at a right (90°) angle. The relationship between these slopes is somewhat more complicated, and is summarized below.

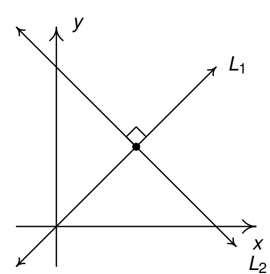
Theorem 3.1. Suppose line L_1 has slope m_1 and line L_2 has slope m_2:

- L_1 and L_2 are parallel (written $L_1 \parallel L_2$) if and only if $m_1 = m_2$.
- If $m_1 \neq 0$ and $m_2 \neq 0$ then L_1 and L_2 are perpendicular (written $L_1 \perp L_2$) if and only if $m_1 m_2 = -1$.

  NOTE: $m_1 m_2 = -1$ is equivalent to $m_2 = -\dfrac{1}{m_1}$, so that perpendicular lines have slopes which are 'opposite reciprocals' of one another.



$L_1 \parallel L_2$, $m_1 = m_2$



$L_1 \perp L_2$, $m_1 m_2 = -1$

A few remarks about Theorem 3.1 are in order. First off, the theorem assumes that the slopes of the lines exist. The reader is encouraged to think about the case when one (or both) of the slopes don't exist. Along those same lines, the reader is encouraged to think about why the stipulations $m_1 \neq 0$ and $m_2 \neq 0$ appear

[11] Lines missing points - even one - usually belie some algebraic pathology which we'll discuss in more detail in Chapter 7.

[12] Well, at least in Euclidean Geometry . . .

in the statement regarding slopes of perpendicular lines, and what happens in this case as well. (Think geometrically!) In Exercise 41, you'll prove the assertion about the slopes of perpendicular lines. For now, we accept it as true and use it in the following example.

Example 3.1.5. For line $y = 2x - 1$ and the point $(3, 4)$, find:

1. the equation of the line parallel to the given line which contains the given point.
2. the equation of the line perpendicular to the given line which contains the given point. Check your answers by graphing them, along with the original line, using a graphing utility.

Solution.

1. Since $y = 2x - 1$ is already in slope-intercept form, we have the slope $m = 2$. To find the line parallel to this line containing $(3, 4)$, we use the point-slope form with $m = 2$ to get:

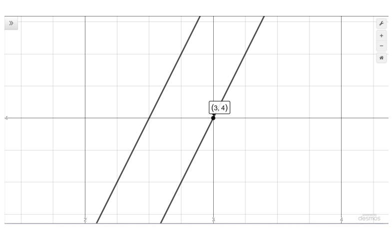
$$\begin{aligned} y - y_0 &= m\,(x - x_0) \\ y - 4 &= 2\,(x - 3) \\ y - 4 &= 2x - 6 \\ y &= 2x - 2 \end{aligned}$$

Algebraically, we can verify that the slope is indeed 2 and that when $x = 3$ we get $y = 4$. Using a graphing utility with a window centered at the point $(3, 4)$, we graph both $y = 2x - 1$ and $y = 2x - 2$ below on the left and observe that they appear to be parallel.

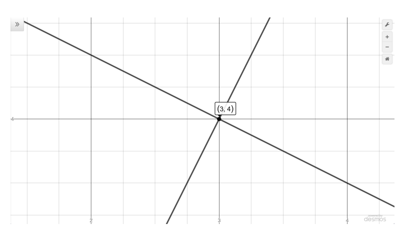
2. To find the line perpendicular to $y = 2x - 1$ containing $(3, 4)$, we use the slope $m = -\frac{1}{2}$ in the point-slope formula:

$$\begin{aligned} y - y_0 &= m\,(x - x_0) \\ y - 4 &= -\frac{1}{2}\,(x - 3) \\ y - 4 &= -\frac{1}{2}x + \frac{3}{2} \\ y &= -\frac{1}{2}x + \frac{11}{2} \end{aligned}$$

Algebraically, we check that the slope is $m = -\frac{1}{2}$ and when $x = 3$ we get $y = 4$ as required. When checking using our graphing utility, we centered the viewing window at $(3, 4)$ and had to 'square' it, removing its default aspect ratio, to truly observe the perpendicular nature of the lines.



$y = 2x - 1$ and
$\boldsymbol{y = 2x - 2}$

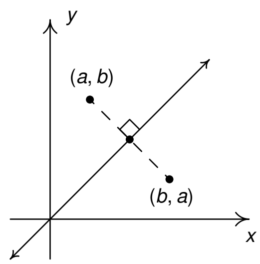


$y = 2x - 1$ and
$\boldsymbol{y = -\frac{1}{2}x + \frac{11}{2}}$

Our last example with lines sets up a fourth kind of symmetry which will be revisited in Section 9.4.

Example 3.1.6. Show that the points (a, b) and (b, a) in the xy-plane are symmetric about the line $y = x$.

Solution. If $a = b$ then $(a, b) = (a, a) = (b, a)$ and this point lies on the line $y = x$.[13] To prove the claim for the case when $a \neq b$, we will show that the line $y = x$ is a perpendicular bisector of the line segment with endpoints (a, b) and (b, a), as illustrated below.



To show the 'perpendicular' part, we first note the slope of the line containing (a, b) and (b, a) is

$$m = \frac{a-b}{b-a} = \frac{\cancel{(a-b)}}{-\cancel{(b-a)}} = -1$$

Since the slope of $y = x = 1x + 0$ is $m = 1$, we see that the slopes of these two lines are negative reciprocals. Hence, $y = x$ and the line segment with endpoints (a, b) and (b, a) are perpendicular. For the 'bisector' part, we use Equation **??** to find the midpoint of the line segment with endpoints (a, b) and (b, a):

$$\begin{aligned} M &= \left(\frac{a+b}{2}, \frac{b+a}{2}\right) \\ &= \left(\frac{a+b}{2}, \frac{a+b}{2}\right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y = x$. $\square$

[13] Please ask your instructor if lying on the line counts as being 'symmetric about the line' or not.

3.1.3 Exercises

In Exercises 1 - 10, find both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

1. $m = 3, \quad P(3, -1)$
2. $m = -2, \quad P(-5, 8)$
3. $m = -1, \quad P(-7, -1)$
4. $m = \frac{2}{3}, \quad P(-2, 1)$
5. $m = -\frac{1}{5}, \quad P(10, 4)$
6. $m = \frac{1}{7}, \quad P(-1, 4)$
7. $m = 0, \quad P(3, 117)$
8. $m = -\sqrt{2}, \quad P(0, -3)$
9. $m = -5, \quad P(\sqrt{3}, 2\sqrt{3})$
10. $m = 678, \quad P(-1, -12)$

In Exercises 11 - 20, find the slope-intercept form of the line which passes through the given points.

11. $P(0, 0), \ Q(-3, 5)$
12. $P(-1, -2), \ Q(3, -2)$
13. $P(5, 0), \ Q(0, -8)$
14. $P(3, -5), \ Q(7, 4)$
15. $P(-1, 5), \ Q(7, 5)$
16. $P(4, -8), \ Q(5, -8)$
17. $P\left(\frac{1}{2}, \frac{3}{4}\right), \ Q\left(\frac{5}{2}, -\frac{7}{4}\right)$
18. $P\left(\frac{2}{3}, \frac{7}{2}\right), \ Q\left(-\frac{1}{3}, \frac{3}{2}\right)$
19. $P\left(\sqrt{2}, -\sqrt{2}\right), \ Q\left(-\sqrt{2}, \sqrt{2}\right)$
20. $P\left(-\sqrt{3}, -1\right), \ Q\left(\sqrt{3}, 1\right)$

In Exercises 21 - 10, graph the line. Find the slope, y-intercept and x-intercept, if any exist.

21. $y = 2x - 1$
22. $y = 3 - x$
23. $y = 3$
24. $y = 0$
25. $y = \frac{2}{3}x + \frac{1}{3}$
26. $y = \dfrac{1 - x}{2}$
27. Graph $3v + 2w = 6$ on both the vw- and wv-axes. What characteristics to both graphs share? What's different?
28. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1, 3)$.

In Exercises 29 - 34, you are given a line and a point which is not on that line. Find the line parallel to the given line which passes through the given point.

29. $y = 3x + 2,\ P(0, 0)$

30. $y = -6x + 5,\ P(3, 2)$

31. $y = \frac{2}{3}x - 7,\ P(6, 0)$

32. $y = \dfrac{4 - x}{3},\ P(1, -1)$

33. $y = 6,\ P(3, -2)$

34. $x = 1,\ P(-5, 0)$

In Exercises 35 - 40, you are given a line and a point which is not on that line. Find the line perpendicular to the given line which passes through the given point.

35. $y = \frac{1}{3}x + 2,\ P(0, 0)$

36. $y = -6x + 5,\ P(3, 2)$

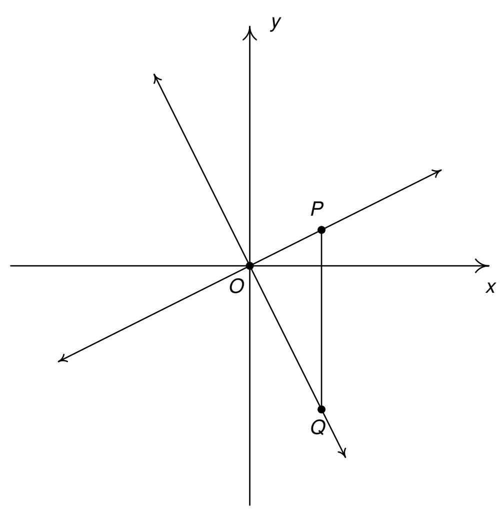
37. $y = \frac{2}{3}x - 7,\ P(6, 0)$

38. $y = \dfrac{4 - x}{3},\ P(1, -1)$

39. $y = 6,\ P(3, -2)$

40. $x = 1,\ P(-5, 0)$

41. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0, 0)$, $P(1, m_1)$ and $Q(1, m_2)$ gives us the following set up.



The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

3.1.4 Answers

1. $y+1=3(x-3)$
   $y=3x-10$

2. $y-8=-2(x+5)$
   $y=-2x-2$

3. $y+1=-(x+7)$
   $y=-x-8$

4. $y-1=\frac{2}{3}(x+2)$
   $y=\frac{2}{3}x+\frac{7}{3}$

5. $y-4=-\frac{1}{5}(x-10)$
   $y=-\frac{1}{5}x+6$

6. $y-4=\frac{1}{7}(x+1)$
   $y=\frac{1}{7}x+\frac{29}{7}$

7. $y-117=0$
   $y=117$

8. $y+3=-\sqrt{2}(x-0)$
   $y=-\sqrt{2}x-3$

9. $y-2\sqrt{3}=-5(x-\sqrt{3})$
   $y=-5x+7\sqrt{3}$

10. $y+12=678(x+1)$
    $y=678x+666$

11. $y=-\frac{5}{3}x$

12. $y=-2$

13. $y=\frac{8}{5}x-8$

14. $y=\frac{9}{4}x-\frac{47}{4}$

15. $y=5$

16. $y=-8$

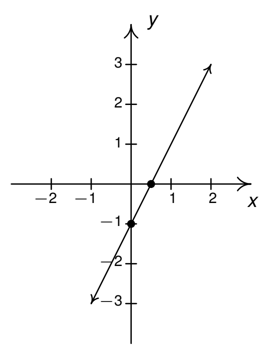
17. $y=-\frac{5}{4}x+\frac{11}{8}$

18. $y=2x+\frac{13}{6}$

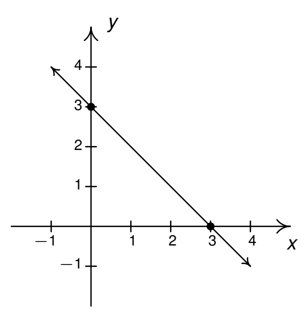
19. $y=-x$

20. $y=\frac{\sqrt{3}}{3}x$

21. $y=2x-1$
    slope: $m=2$
    y-intercept: $(0,-1)$
    x-intercept: $\left(\frac{1}{2},0\right)$



22. $y=3-x$
    slope: $m=-1$
    y-intercept: $(0,3)$
    x-intercept: $(3,0)$

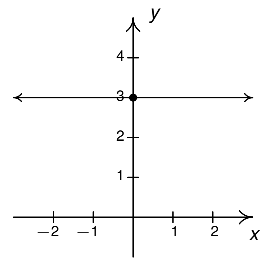


23. $y = 3$

slope: $m = 0$

y-intercept: $(0, 3)$

x-intercept: none

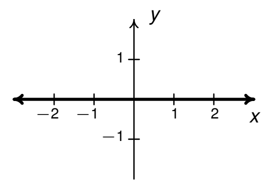


24. $y = 0$

slope: $m = 0$

y-intercept: $(0, 0)$

x-intercept: $\{(x, 0) \mid x \text{ is a real number}\}$

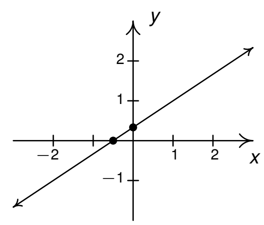


25. $y = \frac{2}{3}x + \frac{1}{3}$

slope: $m = \frac{2}{3}$

y-intercept: $\left(0, \frac{1}{3}\right)$

x-intercept: $\left(-\frac{1}{2}, 0\right)$

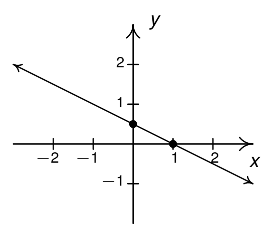


26. $y = \dfrac{1 - x}{2}$

slope: $m = -\frac{1}{2}$

y-intercept: $\left(0, \frac{1}{2}\right)$

x-intercept: $(1, 0)$

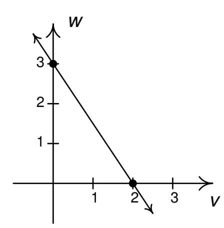


27. $w = -\frac{3}{2}v + 3$

slope: $m = -\frac{3}{2}$

w-intercept: $(0, 3)$

v-intercept: $(2, 0)$

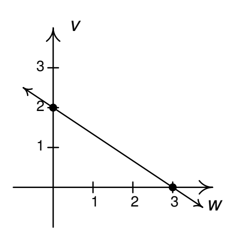


$v = -\frac{2}{3}w + 2$

slope: $m = -\frac{2}{3}$

v-intercept: $(0, 2)$

w-intercept: $(3, 0)$



28. $(-1, -1)$ and $\left(\frac{11}{5}, \frac{27}{5}\right)$

29. $y = 3x$

30. $y = -6x + 20$

31. $y = \frac{2}{3}x - 4$

32. $y = -\frac{1}{3}x - \frac{2}{3}$

33. $y = -2$

34. $x = -5$

35. $y = -3x$

36. $y = \frac{1}{6}x + \frac{3}{2}$

37. $y = -\frac{3}{2}x + 9$

38. $y = 3x - 4$

39. $x = 3$

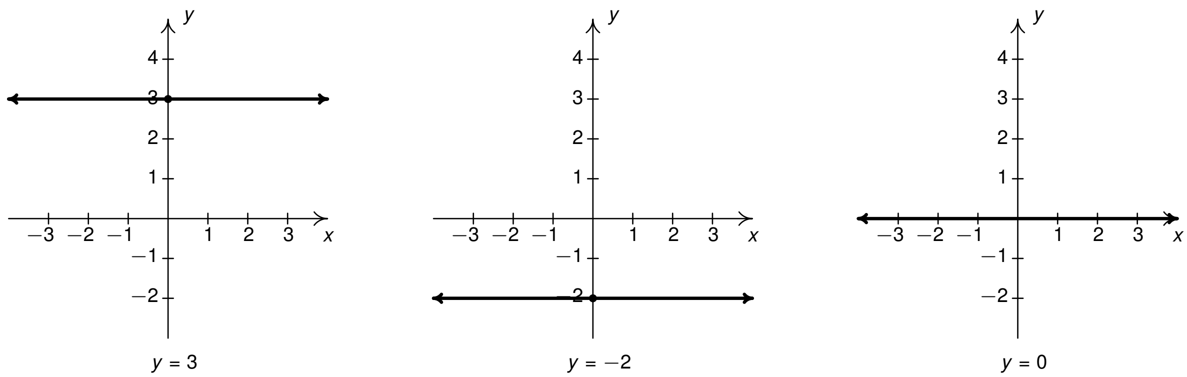
40. $y = 0$

3.2 Constant and Linear Functions

3.2.1 Constant Functions

Now that we have defined the concept of a function, we'll spend the rest of Chapter 2 revisiting families of curves from prior courses in Algebra by viewing them through a 'function lens'. We start with lines and refer the reader to Section 3.1.2 for a review of the basic properties of lines. The simplest lines are vertical and horizontal lines. We leave it to the reader (see Exercise 58) to think about why we eschew vertical lines in our discussion here, and begin with a functional description of horizontal lines.

Consider the horizontal lines graphed in the xy-plane as shown below. The Vertical Line Test, Theorem 2.1, tells us that each describes y as a function of x so the question becomes how to represent these functions algebraically. The key here is to remember that the equation relating the independent variable x, the dependent variable y, and the function f is given by $y = f(x)$.



In the graph on the left, y always equals 3 so we have $f(x) = 3$. Procedurally, '$f(x) = 3$' says that the rule f takes the input x, and, regardless of that input, gives the output 3. This is an example of what is called a **constant** function - a function which returns the *same* value regardless of the input. Likewise, the function represented by the graph in the middle is $f(x) = -2$, and the graph on the right (the x-axis) is the graph of $f(x) = 0$. In general, we have the following definition:

Definition 3.3. A **constant function** is a function of the form

$$f(x) = b$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Some remarks about Definition 3.3 are in order. First, note that we are using 'x' as the independent variable, 'f' as the function name, and the letter 'b' as a **parameter**. In this context, a parameter is a fixed, but arbitrary, constant used to describe a *family* of functions. Different values of b determine different constant functions. For example, $b = 3$ gives $f(x) = 3$, $b = -2$ gives $f(x) = -2$, and so on. Once b is chosen, however, it does not change as the independent variable, x, changes.

Also note that we are using the generic defaults for function names and independent variables, namely f and x, respectively. The functions $G(t) = \sqrt{\pi}$ and $Z(\rho) = 0$ are also fine examples of constant functions.

Recall that inherent in the definition of a function is the notion of domain, so we record (as part of the definition) that a constant function has domain $(-\infty, \infty)$. The range of a constant function is the set $\{b\}$. The value b in this case is both the maximum and minimum of f, attained at each value in its domain.[1]

The next example showcases an application of constant functions and introduces the notion of a **piecewise-defined** function.

Example 3.2.1. The price of admission to see a matinee showing at a local movie theater is a function of the age of the ticket holder. If a person is aged A years, the price per ticket is $p(A)$ dollars and is given by:

$$p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50 \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

1. Find and interpret $p(3)$, $p(6)$ and $p(62)$.

2. Explain the pricing structure verbally.

3. Graph p.

Solution. The function p described above is an example of a **piecewise-defined** function because the *rule* to determine outputs, not just the value of the output, changes depending on the inputs.

1. To find $p(3)$, we note that the value $A = 3$ satisfies the inequality $0 \leq A < 6$ so we use the rule $p(A) = 5.75$. Hence, $p(3) = 5.75$ which means a ticket for a 3 year old is \$5.75. The next age, $A = 6$, just barely satisfies the inequality $6 \leq A < 50$ so we use the rule $p(A) = 7.25$, This yields $p(6) = 7.25$ which means a ticket for a 6 year old is \$7.25. Lastly, $A = 62$ satisfies the inequality $A \geq 50$, so we are back to the rule $p(A) = 5.75$. Thus $p(62) = 5.75$ which means someone 62 years young gets in for \$5.75.

2. Now that we've had some practice interpreting function values, we can begin to verbalize what the function is really saying. In the first 'piece' of the function, the inequality $0 \leq A < 6$ describes ticket holders under the age of 6 years and the inequality $A \geq 50$ describes ticket holders fifty years old or or older. For folks in these two age demographics, $p(A) = 5.75$ so the price per ticket is \$5.75. For everyone else, that is for folks at least 6 but younger than 50, the price is \$7.25 per ticket.

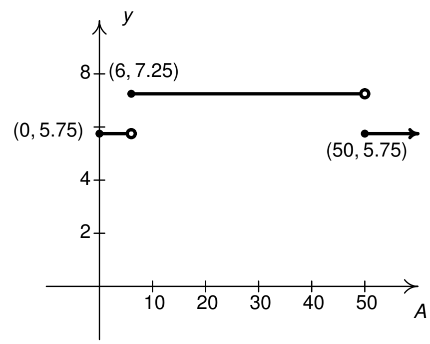
3. The independent variable here is specified as A, so we'll label our horizontal axis that way. The dependent variable remains unspecified so we can use the default y. The graph of $y = p(A)$ consists of three horizontal line pieces: the first is $y = 5.75$ for $0 \leq A < 6$, the second piece is $y = 7.25$ for $6 \leq A < 50$, and the last piece is $y = 5.75$ for $A \geq 50$.

   For the first piece, note that $A = 0$ is included in the inequality $0 \leq A < 6$ but $A = 6$ is not. For this reason, we have a point indicated at $(0, 5.75)$ but leave a hole[2] at $(6, 5.75)$. Similarly, to graph

[1] It gets much weirder than that as we explore other more complicated functions. The key is to pay attention to the precision in the definitions of the terms involved in the discussion. Stay tuned!

[2] See our discussion about holes in graphs in Example 2.1.6 in Section 2.1.

the second piece, we begin with a point at $(6, 7.25)$ and continue the horizontal line to a hole at $(50, 7.25)$. Lastly, we finish the graph with a point at $(50, 5.75)$ and continue to the right indefinitely.[3] Note the scaling on the horizontal axis compared to the vertical axis.



$$y = p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50 \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

□

One of the favorite piecewise-defined functions in mathematical circles is the **greatest integer of *x***, denoted by $\lfloor x \rfloor$. In Section 1.1.2 we defined the set of **integers** as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[4] The value $\lfloor x \rfloor$ is defined to be the largest integer k with $k \leq x$. That is, $\lfloor x \rfloor$ is the unique integer k such that $k \leq x < k + 1$. Said differently, given any real number x, if x is an integer, then $\lfloor x \rfloor = x$. If not, then x lies in an interval between two integers, k and $k + 1$ and we choose $\lfloor x \rfloor = k$, the left endpoint.

Example 3.2.2. Let $\lfloor x \rfloor$ denote the greatest integer function.

1. Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$ and $\lfloor \pi + 6 \rfloor$
2. Explain how we can view $\lfloor x \rfloor$ as a piecewise-defined function and use this to graph $y = \lfloor x \rfloor$.

Solution.

1. To find $\lfloor 0.785 \rfloor$, we note that $0 \leq 0.785 < 1$ so $\lfloor 0.785 \rfloor = 0$. Given that 117 is an integer, we have $\lfloor 117 \rfloor = 117$. To find $\lfloor -2.001 \rfloor$, we note that $-3 \leq -2.001 < -2$, so $\lfloor -2.001 \rfloor = -3$. Finally, with $\pi \approx 3.14$, we get $\pi + 6 \approx 9.14$ and $9 \leq \pi + 6 < 10$ so $\lfloor \pi + 6 \rfloor = 9$.

2. The first step in evaluating $\lfloor x \rfloor$ is to determine the interval $[k, k + 1)$ containing x so it seems reasonable that these are the intervals which produce the 'pieces'. In this case, there happen to be infinitely many pieces. The inequality '$k \leq x < k + 1$' includes the left endpoint but excludes the right endpoint, so we have points at the left endpoints of our horizontal line segments while we have holes at the right endpoints.

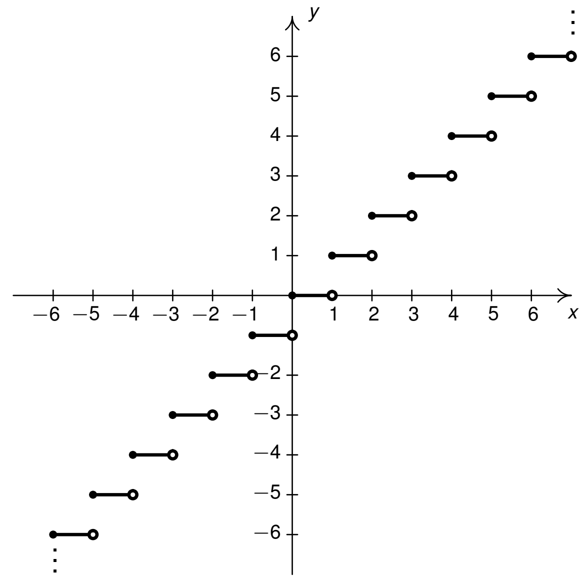
   A partial description of $\lfloor x \rfloor$ is given alongside a partial graph at the top of the next page. (A full description or a complete graph would require infinitely large paper!) We use the vertical dots $\vdots$ to indicate that both the rule and the graph continue indefinitely following the established pattern.[5]

[3] The domain of p is $[0, \infty)$ by definition, even though few 327 year olds are out and about these days.

[4] The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word *zahlen* means 'to count'.

[5] It is always dangerous to leave the rest of the pattern to the reader. See, for instance, this paper.

$$\lfloor x \rfloor = \begin{cases} \vdots & \\ -5 & \text{if } -5 \leq x < -4 \\ -4 & \text{if } -4 \leq x < -3 \\ -3 & \text{if } -3 \leq x < -2 \\ -2 & \text{if } -2 \leq x < -1 \\ -1 & \text{if } -1 \leq x < 0 \\ 0 & \text{if } \ \ 0 \leq x < 1 \\ 1 & \text{if } \ \ 1 \leq x < 2 \\ 2 & \text{if } \ \ 2 \leq x < 3 \\ 3 & \text{if } \ \ 3 \leq x < 4 \\ 4 & \text{if } \ \ 4 \leq x < 5 \\ 5 & \text{if } \ \ 5 \leq x < 6 \\ \vdots & \end{cases}$$



The graph of $y = \lfloor x \rfloor$.

□

3.2.2 Linear Functions

Now that we've discussed the functions which correspond to horizontal lines, $y = b$, we move to discussing the functions which can be represented by lines of the form $y = mx + b$ where $m \neq 0$. These functions are called **linear** functions and are described below.

Definition 3.4. A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

As with Definition 3.3, in Definition 3.4, x is the independent variable, f is the function name, and both m and b are parameters. Notice that m is restricted by $m \neq 0$ for if $m = 0$ then the function $f(x) = mx + b$ would reduce to the constant function $f(x) = b$. The domain of linear functions, like that of constant functions, is specified as $(-\infty, \infty)$

Recall[6] that the form of the line $y = mx + b$ is called the slope-intercept form of the line and the slope, m, and the y-intercept $(0, b)$, are easily determined when the line is written this way. Likewise, the form of the function in Definition 3.4, $f(x) = mx + b$, is often called the **slope-intercept form** of a linear function.

The graph of a linear function is the graph of the line $y = mx + b$. Lines are uniquely determined by two points, and two points of geometric interest are the axis intercepts. We've already reminded you of the y-intercept, $(0, b)$, which is obtained by setting $x = 0$. Similarly, to find the x-intercept, we set $y = 0$

[6] or see Section 3.1.2

and solve $mx + b = 0$ for x. We leave this to the reader in Exercise 38. In addition to having special graphical significance, axis intercepts quite often play important roles in applications involving both linear and non-linear functions. For that reason, we take the time to define them here using function notation.

Definition 3.5. Suppose f is a function represented by the graph of $y = f(x)$.

- If 0 is in the domain of f then the point $(0, f(0))$ is the y**-intercept** of the graph of $y = f(x)$.

  That is, $(0, f(0))$ is where the graph meets the y-axis.

- If 0 is in the range of f then the solutions to $f(x) = 0$ are called the **zeros** of f. If c is a zero of f then the point $(c, 0)$ is an x**-intercept** of the graph of $y = f(x)$.

  That is, $(c, 0)$ is where the graph meets the x-axis.

As is customary in this text, Definition 3.5 uses the default independent variable x, function name f, and dependent variable y, so these letters will change depending on the context. Also note that the 'zeros' of a function are the solutions to $f(x) = 0$ - so they are *real numbers*. The x-intercepts are, on the other hand, *points* on the graph. As a quick example, consider $f(x) = x - 3$. The zeros of f are found by solving $f(x) = 0$, or $x - 3 = 0$. We get one solution, $x = 3$. Therefore, $x = 3$ is the *zero* of f that corresponds graphically to the x-*intercept* $(3, 0)$.

We now turn our attention to slope. The role of slope, or more generally a 'rate of change', in Science and Mathematics cannot be overstated.[7] As you may recall, or quickly read about on page 138, the slope of a line that has been graphed in the xy-plane is defined geometrically as follows:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x},$$

where the capital Greek letter 'Δ' denotes 'change in.'[8] In this course, it is vital that we regard the slope of a linear function as a rate of change of *function outputs* to *function inputs*. That is, given the graph of a linear function $y = f(x) = mx + b$:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x} = \frac{\Delta\text{outputs}}{\Delta\text{inputs}}.$$

What is important to note here is that for linear functions, the rate of change m is constant for all values in the domain.[9] We'll see the importance of this statement in the upcoming examples.

Geometrically, the sign of the slope has a profound impact on the graph of the line. Recall that if the slope $m > 0$, the line rises as we read from left to right; if $m < 0$, the line falls as we read from left to right; if $m = 0$, we have a horizontal line and the graph plateaus. We define these notions more precisely for general functions in the following definition.

[7] The first half of any introductory Calculus course is about slope.

[8] More specifically, if (x_0, y_0) and (x_1, y_1) are two distinct points in the plane, then $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$.

[9] See Exercise 57 for more details.

Definition 3.6. Let f be a function defined on an interval I. Then f is said to be:

- **increasing** on I if, whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)

  NOTE: The graph of an increasing function **rises** as one moves from left to right.

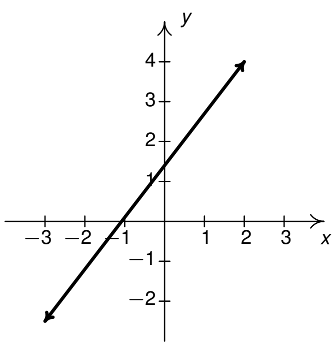
- **decreasing** on I if, whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)

  NOTE: The graph of a decreasing function **falls** as one moves from left to right.

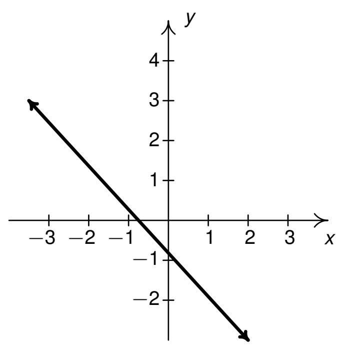
- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs.)

  NOTE: The graph of a function that is constant over an interval is a horizontal line.

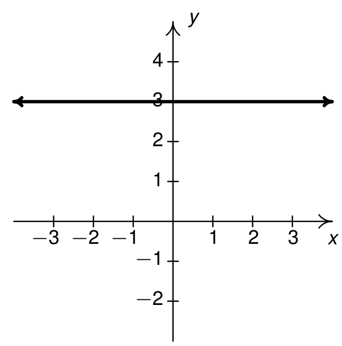
Again, as with Definition 3.5, Definition 3.6 applies to any function, not just linear and constant functions. Also, note that, like Definition 2.3, Definition 3.6 blurs the line between the function, f, and its outputs, $f(x)$, because the verbiage 'f is increasing' is really a statement about the outputs, $f(x)$. Finally, when we ask 'where' a function is increasing, decreasing or constant, we are looking for an interval of *inputs*. We'll have more to say about this in later sections, but for now, we summarize these ideas graphically below.



'increasing', $m > 0$



'decreasing', $m < 0$



'constant', $m = 0$

From the graphs above, we see that regardless if $m > 0$ or $m < 0$, the range of linear functions is $(-\infty, \infty)$. Therefore, linear functions have no maximum or minimum.[10]

Example 3.2.3. The cost, in dollars, to produce x PortaBoy[11] game systems for a local retailer is given by $C(x) = 80x + 150$ for $x \geq 0$.

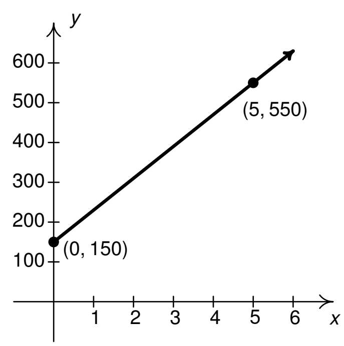
1. Find and interpret $C(0)$ and $C(5)$ and use these to graph $y = C(x)$.
2. Explain the significance of the restriction on the domain, $x \geq 0$.
3. Interpret the slope of $y = C(x)$ geometrically and as a rate of change.
4. How many PortaBoys can be produced for \$15,000?

[10]This is one of the more pedantic reasons why we distinguish between constant and linear functions. See the discussion concerning the range of a constant function on page 155.

[11]The similarity of this name to PortaJohn is deliberate.

Solution.

1. To find $C(0)$, we substitute 0 for x in the formula $C(x)$ and obtain: $C(0) = 80(0) + 150 = 150$. Given that x represents the number of PortaBoys produced and $C(x)$ represents the cost to produce said PortaBoys, $C(0) = 150$ means it costs \$150 even if we don't produce any PortaBoys at all. At first, this may not seem realistic, but that \$150 is often called the **fixed** or **start-up** cost of the venture. Things like re-tooling equipment, leasing space, or any other 'up front' costs get lumped into the fixed cost. To find $C(5)$, we substitute 5 for x in the formula $C(x)$: $C(5) = 80(5) + 150 = 550$. This means it costs \$550 to produce 5 PortaBoys for the local retailer. These two computations give us two points on the graph: $(0, C(0))$ and $(5, C(5))$. Along with the domain restriction $x \geq 0$, we get:



$y = C(x)$

2. In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems,[12] so $x \geq 0$.

3. The cost function $C(x) = 80x + 150$ is in slope-intercept form so we recognize the slope as the coefficient of x, $m = 80$. With $m > 0$, the function C is always increasing. This means that it costs more money to make more game systems. To interpret the slope as a rate of change, we note that the output, $C(x)$, is the cost in dollars, while the input, x, is the number of PortaBoys produced:

$$m = 80 = \frac{80}{1} = \frac{\Delta[C(x)]}{\Delta x} = \frac{\$80}{1 \text{ PortaBoy produced}}.$$

Hence, the cost to produce PortaBoys is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for the venture.

4. To find how many PortaBoys can be produced for \$15,000, we solve $C(x) = 15000$, which means $80x + 150 = 15000$. This yields $x = 185.625$. We can produce only a whole number amount of PortaBoys so we are left with two options: produce 185 or 186 PortaBoys. Given that $C(185) = 14950$ and $C(186) = 15030$, we would be over budget if we produced 186 PortaBoys. Hence, we can produce 185 PortaBoys for \$15,000 (with \$50 to spare). □

[12] Actually, it makes no sense to produce a fractional part of a game system, either, which we'll discuss later in this example.

A couple of remarks about Example 3.2.3 are in order. First, if x represents the number of PortaBoy game systems being produced, then x can really only take on whole number values. We will revisit this scenario in Section 5.4 where we will see how the approach presented here allows us to use more elegant techniques when analyzing the situation than a discrete data set would allow.[13]

Second, once we know that the variable cost is \$80 per PortaBoy, we can revisit a computation we did earlier in the example. We computed $C(185) = 14950$ and needed to compute $C(186)$. With 186 being just one more PortaBoy than 185, we can use the variable cost to get

$$C(186) = C(185) + 80(1) = 14950 + 80 = 15030,$$

which agrees with our earlier computation.[14] If we wanted to find $C(300)$, we could do something similar. Using $300 - 185 = 115$, we can find $C(300)$ as follows:

$$C(300) = C(185) + 80(115) = 14950 + 9200 = 24150.$$

In general, we could rewrite $C(x) = C(185) + 80(x - 115)$. This same reasoning shows that for any x_0 in the domain of C, we have $C(x) = C(x_0) + 80(x - x_0)$ - a fact we invite the reader to verify.[15]

Indeed, the computations above are at the heart of what it means to be a linear function: linear functions change at a constant rate known as the slope. To better see this algebraically, recall that given a point (x_0, y_0) on a line along with the slope, m, the **point-slope form of the line** is: $y - y_0 = m(x - x_0)$.[16] Rewriting, we get $y = y_0 + m(x - x_0)$ and setting $y = f(x)$ and $y_0 = f(x_0)$ yields:

Equation 3.6. The **point-slope form** of a linear function is

$$f(x) = f(x_0) + m(x - x_0)$$

A few remarks are in order. First note that if the point $(x_0, f(x_0))$ is the y-intercept $(0, b)$, Equation 3.6 immediately reduces to the slope-intercept form of the line: $f(x) = f(x_0) + m(x - x_0) = b + m(x - 0) = mx + b$, so you can use Equation 3.6 exclusively from this point forward.[17]

Second, if we write $\Delta x = x - x_0$, then $x = x_0 + \Delta x$ so we can rewrite Equation 3.6 as follows:

$$\begin{array}{ccccc} f(x_0 + \Delta x) & = & f(x_0) & + & m\Delta x \\ \text{(new output)} & = & \text{(known output)} & + & \text{(change in outputs)} \end{array}$$

In other words, changing the *input* by Δx results in changing the *output* by $m\Delta x$. This tracks since

$$m\Delta x = \frac{\Delta[f(x)]}{\Delta x}\Delta x = \Delta[f(x)] = \Delta\text{outputs}.$$

[13] This is an example of using a 'continuous' variable to model a 'discrete' scenario. Contrast this with the discussion following Example 2.1.1 in Section 2.1.

[14] The cost to produce 'just one more item' is called the **marginal cost**. The difference between variable and marginal costs in this case are the units used: the variable cost is \$80 per Portaboy whereas the marginal cost is simply \$80.

[15] In the case $x_0 = 0$, this formula reduces to $C(x) = C(0) + 80(x - 0) = 150 + 80x = 80x + 150$. To show the formula in general, consider $C(x_0) = 80x_0 + 150 \ldots$

[16] See Section 3.1.2 for a review of this form.

[17] In other words, the slope intercept form of a line is just a special case of the point-slope form.

The fact that we can write $\Delta\text{outputs} = m\Delta x$ for any choice of x_0 is another way to see that for linear functions, the rate of change is constant. That is, the rate of change, m, is the same for all values x_0 in the domain. We'll put Equation 3.6 to good use in the next example.

Example 3.2.4. The local retailer in Example 3.2.3 is trying to mathematically model the relationship between the number of PortaBoy systems sold and the price per system. Suppose 20 systems were sold when the price was \$220 per system but when the systems went on sale for \$190 each, sales doubled.

1. Find a formula for a linear function p which represents the price $p(x)$ as a function of the number of systems sold, x. Graph $y = p(x)$, find and interpret the intercepts, and determine a reasonable domain for p.

2. Interpret the slope of $p(x)$ in terms of price and game system sales.

3. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

4. How many systems would sell if the price per system were set at \$150?

Solution.

1. We are asked to find a linear function $p(x)$ ostensibly because the retailer has only two data points and two points are all that is needed to determine a unique line. We know that 20 PortaBoys were sold when the price was 220 dollars and double that, so 40 units, were sold when the price was 190 dollars. Using the language of function notation, these statements translate to $p(20) = 220$ and $p(40) = 190$, respectively. We first find the slope

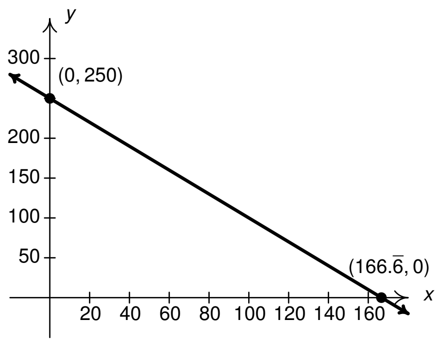
$$m = \frac{\Delta[p(x)]}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5$$

and then substitute it and a pair $(x_0, p(x_0))$ into the point-slope formula. We have two choices: $x_0 = 20$ and $p(x_0) = 220$ or $x_0 = 40$ and $p(x_0) = 190$. We'll choose the former and invite the reader to use the latter - both will result in the same simplified expression. The point-slope formula yields

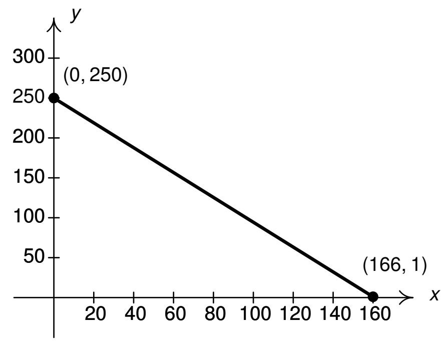
$$p(x) = p(x_0) + m(x - x_0) = 220 + (-1.5)(x - 40)$$

which simplifies to $p(x) = -1.5x + 250$. (To check this algebraically, we can verify that $p(20) = 220$ and $p(40) = 190$.) To find the y-intercept of the graph, we substitute $x = 0$ and find $p(0) = 250$. Hence our y-intercept is $(0, 250)$. To find the x-intercept, we set $p(x) = 0$. Solving $-1.5x + 250 = 0$ gives $x = 166.\overline{6}$, so our x-intercept is $(166.\overline{6}, 0)$.[18] The graph on the left is that of the line $y = -1.5x + 250$.

[18] The exact value is $x = \frac{500}{3}$. Recall that the bar over the 6 indicates that the decimal repeats. See page 6 for details.



$y = -1.5x + 250$



$y = p(x)$

To determine a reasonable domain for p, we certainly require $x \geq 0$, because we can't sell a negative number of game systems.[19] Next, we require $p(x) \geq 0$, otherwise we'd be *paying* customers to 'buy' PortaBoys. Solving $-1.5x + 250 \geq 0$ results in $x \leq 166.\overline{6}$. This shouldn't be too surprising since our graph passes through the x-axis at $(166.\overline{6}, 0)$, going from positive y-values (hence, positive $p(x)$ values) to negative y (hence negative $p(x)$ values).[20]

Given that x represents the number of PortaBoys sold, we need to choose to end the domain at either $x = 166$ or $x = 167$. We have that $p(166) = 1 > 0$ but $p(167) = -0.5 < 0$ so we settle on the domain $[0, 166]$. Our final answer is $p(x) = -1.5x + 250$ restricted to $0 \leq x \leq 166$ which is graphed above on the right.

2. The slope $m = -1.5$ represents the rate of change of the price of a system with respect to sales of PortaBoys. The slope is negative so we have that the price is *decreasing* at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)

3. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150) + 250 = 25$. That is, the price would have to be \$25 per system.

4. If the price of a PortaBoy were set at \$150, we'd have $p(x) = 150$, or $-1.5x + 250 = 150$. This yields $-1.5x = -100$ or $x = 66.\overline{6}$. Again our algebraic solution lies between two whole numbers, so we find $p(66) = 151$ and $p(67) = 149.5$. If the price were set at \$150, we'd sell 66 systems, since to sell 67 systems, we'd have to drop the price just under \$150. $\square$

The function p in Example 3.2.4 is called the **price-demand** function (or, sometimes called more simply a 'demand function') because it returns the price $p(x)$ associated with a certain demand x - that is, how many products will sell.[21] These functions, along with cost functions like the one in Example 3.2.3, will be revisited in Example 5.4.3.

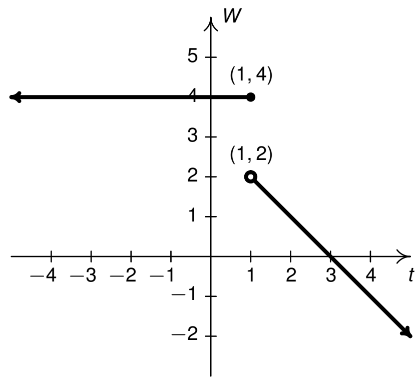
Our next two examples focus on writing formulas for piecewise-defined functions, the second of which models a real-world situation.

[19] ignoring returns, that is.

[20] We'll discuss these sorts of connections in greater depth in Section 4.2.

[21] It may seem counter-intuitive to express price as a function of demand. Shouldn't the price determine how many systems people will buy? We will address this issue later.

Example 3.2.5. Find a formula for the function L graphed below.



$W = L(t)$

Solution. From the graph of $W = L(t)$ we see that there are two distinct pieces. Taking note of the point at $(1, 4)$, we get $L(t) = 4$ for $t \leq 1$. To represent L for $t > 1$, we use the point-slope form of a linear function: $L(t) = L(t_0) + m(t - t_0)$. The only 'point' labeled with this part of the graph is the hole at $(1, 2)$ and it isn't technically part of the graph, so we will avoid using it.[22] Instead, we infer from the graph two other points: $(2, 1)$ and $(3, 0)$. We get the slope to be

$$m = \frac{\Delta W}{\Delta t} = \frac{\Delta[L(t)]}{\Delta t} = \frac{3-2}{0-1} = -1.$$

Next, we choose a point to plug into $L(t) = L(t_0) + m(t - t_0)$. We have two options: $t_0 = 2$ and $L(t_0) = 1$ or $t_0 = 3$ and $L(t_0) = 0$. Using the latter, we get $L(t) = 0 + (-1)(t - 3)$, or $L(t) = -t + 3$. Putting this together with the first part, we get:

$$L(t) = \begin{cases} 4 & \text{if } t \leq 1 \\ -t + 3 & \text{if } t > 1 \end{cases}$$

Note that when $t = 1$ is substituted into the expression $-t + 3$, we get 2, so the hole at $(1, 2)$ checks.[23] □

Example 3.2.6. A popular Fōn-i smartphone carrier offers the following smartphone data plan: use any amount of data up to and including 4 gigabytes for \$60 per month with an 'overage' charge of \$5 per gigabyte. Determine a formula that computes the cost in dollars as a function of using g gigabytes of data per month. Graph your answer.

Solution. It is clear from context that we are to use the variable g (for 'g'igabytes) as the independent variable. We are asked to compute the cost so it seems natural to name the function C. Hence, we are after a formula for $C(g)$. Knowing that g represents the amount of data used each month, we must have $g \geq 0$. In order to get a feel for the formula for $C(g)$, we can choose some specific values for g and determine the cost, $C(g)$. For example, if we use no data at all, 1 gigabyte of data, or 3.796 gigabytes

[22] We actually could use the point $(1, 2)$ to find the equation of the *line* containing $(1, 2)$ and, say $(3, 0)$, which is $y = -t + 3$. It's just that the graph of $L(t)$ and the line $y = -t + 3$ only agree for $t > 1$, so it would be incorrect to write $L(1) = 2$.

[23] Alternatively, for t values larger than 1 but getting closer and closer to 1, $L(t) \approx 2$.

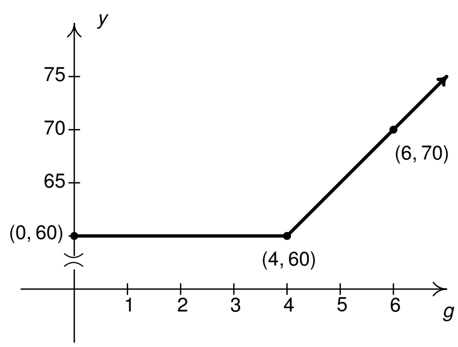
of data, the cost is the same: \$60. Indeed, per the plan, for any amount of data up to and including 4 gigabytes, the cost is \$60.

Translating this to function notation means $C(0) = 60$, $C(1) = 60$, $C(3.796) = 60$, and, in general, $C(g) = 60$ for $0 \leq g \leq 4$. What happens if we use more than 4 gigabytes? Let's say we use 6 gigabytes. Per the plan, we are charged \$60 for the first 4 and then \$5 for each gigabyte over 4. Using 6 gigabytes means that we are 2 gigabytes over and our overage charge is (\$5)(2) = \$10. The total cost is the base plus the overages or \$60 + \$10 = \$70. In general, if $g > 4$, the expression $(g - 4)$ computes the amount of data used over 4 gigabytes. Our base plus overage then comes to: $60 + 5(g - 4) = 5g + 40$. Putting this together with our previous work, we get

$$C(g) = \begin{cases} 60 & \text{if } 0 \leq g \leq 4 \\ 5g + 40 & \text{if } g > 4 \end{cases}$$

To graph C, we graph $y = C(g)$. For $0 \leq g \leq 4$, we have the horizontal line $y = 60$ from $(0, 60)$ to $(4, 60)$. For $g > 4$, we have the line $y = 5g + 40$. Even though the inequality $g > 4$ is strict, we nevertheless substitute $g = 4$ into the formula $y = 5g + 40$ and get $y = 60$. Normally, this would produce a hole at $(4, 60)$, but in this case, the point $(4, 60)$ is already on the graph from the first piece of the function. Essentially, the point $(4, 60)$ from $C(g) = 60$ for $0 \leq g \leq 4$ 'plugs' the hole from $C(g) = 5g + 40$ when $g > 4$.

We are graphing a line so we need to plot just one more point to determine the graph. From our work above, we know $C(6) = 70$, so we use $(6, 70)$ as our second point. Our graph is below. As with the graphs shown on page 47 from Example 2.1.1, we use '$\asymp$' to denote a break in the vertical axis in order to better display the graph.



$y = C(g)$

□

3.2.3 Exercises

In Exercises 1 - 6, graph the function. Find the slope and axis intercepts, if any.

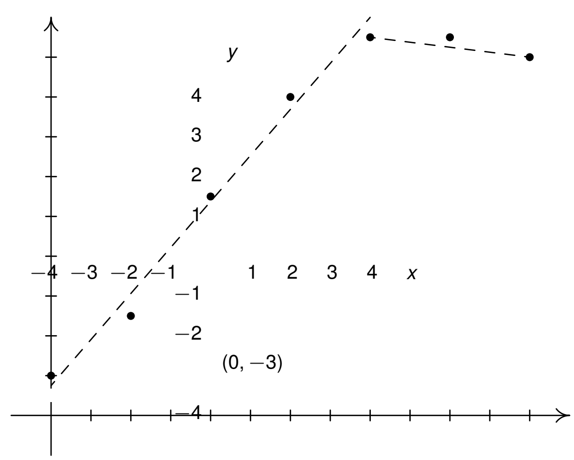
1. $f(x) = 2x - 1$
2. $g(t) = 3 - t$
3. $F(w) = 3$
4. $G(s) = 0$
5. $h(t) = \frac{2}{3}t + \frac{1}{3}$
6. $j(w) = \dfrac{1 - w}{2}$

In Exercises 7 - 10, graph the function. Find the domain, range, and axis intercepts, if any.

7. $f(x) = \begin{cases} 4 - x & \text{if} \quad x \leq 3 \\ 2 & \text{if} \quad x > 3 \end{cases}$
8. $g(x) = \begin{cases} 2 - x & \text{if} \quad x < 2 \\ x - 2 & \text{if} \quad x \geq 2 \end{cases}$
9. $F(t) = \begin{cases} -2t - 4 & \text{if} \quad t < 0 \\ 3t & \text{if} \quad t \geq 0 \end{cases}$
10. $G(t) = \begin{cases} -3 & \text{if} \quad t < 0 \\ 2t - 3 & \text{if} \quad 0 < t < 3 \\ 3 & \text{if} \quad t > 3 \end{cases}$
11. The **unit step function** is defined as $U(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 1. \end{cases}$
    (a) Graph $y = U(t)$.
    (b) State the domain and range of U.
    (c) List the interval(s) over which U is increasing, decreasing, and/or constant.
    (d) Write $U(t - 2)$ as a piecewise defined function and graph.

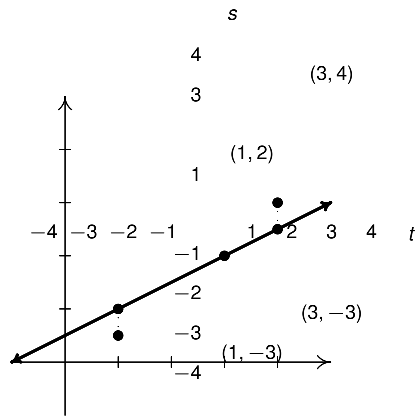
In Exercises 12 - 15, find a formula for the function.

12.



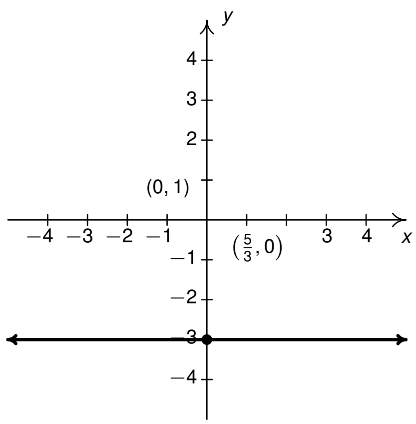
$y = f(x)$

13.



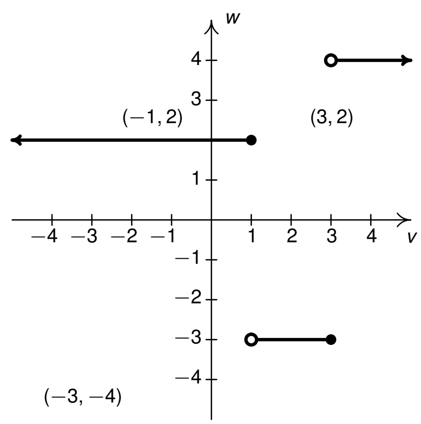
$s = F(t)$

14.



$y = L(x)$

15.



$w = g(v)$

16. For n copies of the book *Me and my Sasquatch*, a print on-demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n) = \begin{cases} 15n & \text{if} \quad 1 \leq n \leq 25 \\ 13.50n & \text{if} \quad 25 < n \leq 50 \\ 12n & \text{if} \quad n > 50 \end{cases}$$

(a) Find and interpret $C(20)$.

(b) How much does it cost to order 50 copies of the book? What about 51 copies?

(c) Your answer to 16b should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

17. An on-line comic book retailer charges shipping costs according to the following formula

$$S(n) = \begin{cases} 1.5n + 2.5 & \text{if} \quad 1 \leq n \leq 14 \\ 0 & \text{if} \quad n \geq 15 \end{cases}$$

where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

(a) What is the cost to ship 10 comic books?

(b) What is the significance of the formula $S(n) = 0$ for $n \geq 15$?

18. The cost in dollars $C(m)$ to talk m minutes a month on a mobile phone plan is modeled by

$$C(m) = \begin{cases} 25 & \text{if } \quad 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if } \quad m > 1000 \end{cases}$$

   (a) How much does it cost to talk 750 minutes per month with this plan?
   (b) How much does it cost to talk 20 hours a month with this plan?
   (c) Explain the terms of the plan verbally.

19. Jeff can walk comfortably at 3 miles per hour. Find an expression for a linear function $d(t)$ that represents the total distance Jeff can walk in t hours, assuming he doesn't take any breaks.

20. Carl can stuff 6 envelopes per *minute*. Find an expression for a linear function $E(t)$ that represents the total number of envelopes Carl can stuff after t *hours*, assuming he doesn't take any breaks.

21. A landscaping company charges \$45 per cubic yard of mulch plus a delivery charge of \$20. Find an expression for a linear function $C(x)$ which computes the total cost in dollars to deliver x cubic yards of mulch.

22. A plumber charges \$50 for a service call plus \$80 per hour. If she spends no longer than 8 hours a day at any one site, find an expression for a linear function $C(t)$ that computes her total daily charges in dollars as a function of the amount of time spent in hours, t at any one given location.

23. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of x dollars. Find an expression for a linear function $W(x)$ which computes her total weekly pay in dollars as a function of x. What must her weekly sales be in order for her to earn \$475.00 for the week?

24. An on-demand publisher charges \$22.50 to print a 600 page book and \$15.50 to print a 400 page book. Find an expression for a linear function which models the cost of a book in dollars $C(p)$ as a function of the number of pages p. Find and interpret both the slope of the linear function and $C(0)$.

25. The Topology Taxi Company charges \$2.50 for the first fifth of a mile and \$0.45 for each additional fifth of a mile. Find an expression for a linear function which models the taxi fare $F(m)$ as a function of the number of miles driven, m. Find and interpret both the slope of the linear function and $F(0)$.

26. Water freezes at 0° Celsius and 32° Fahrenheit and it boils at 100°C and 212°F.

   (a) Find an expression for a linear function $F(T)$ that computes temperature in the Fahrenheit scale as a function of the temperature T given in degrees Celsius. Use this function to convert 20°C into Fahrenheit.
   (b) Find an expression for a linear function $C(T)$ that computes temperature in the Celsius scale as a function of the temperature T given in degrees Fahrenheit. Use this function to convert 110°F into Celsius.
   (c) Is there a temperature T such that $F(T) = C(T)$?

27. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is $40^{\circ}F$ outside and only 5 times per hour if it's $70^{\circ}F$. Assuming that the number of howls per hour, N, can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only $20^{\circ}F$ outside. What troubles do you encounter when trying to determine a reasonable applied domain?

28. Economic forces have changed the cost function for PortaBoys to $C(x) = 105x + 175$. Rework Example 3.2.3 with this new cost function.

29. In response to the economic forces in Exercise 28 above, the local retailer sets the selling price of a PortaBoy at \$250. Remarkably, 30 units were sold each week. When the systems went on sale for \$220, 40 units per week were sold. Rework Example 3.2.4 with this new data.

30. A local pizza store offers medium two-topping pizzas delivered for \$6.00 per pizza plus a \$1.50 delivery charge per order. On weekends, the store runs a 'game day' special: if six or more medium two-topping pizzas are ordered, they are \$5.50 each with no delivery charge. Write a piecewise-defined linear function which calculates the cost in dollars $C(p)$ of p medium two-topping pizzas delivered during a weekend.

31. A restaurant offers a buffet which costs \$15 per person. For parties of 10 or more people, a group discount applies, and the cost is \$12.50 per person. Write a piecewise-defined linear function which calculates the total bill $T(n)$ of a party of n people who all choose the buffet.

32. A mobile plan charges a base monthly rate of \$10 for the first 500 minutes of air time plus a charge of 15¢ for each additional minute. Write a piecewise-defined linear function which calculates the monthly cost in dollars $C(m)$ for using m minutes of air time.

    HINT: You may wish to refer to number 18 for inspiration.

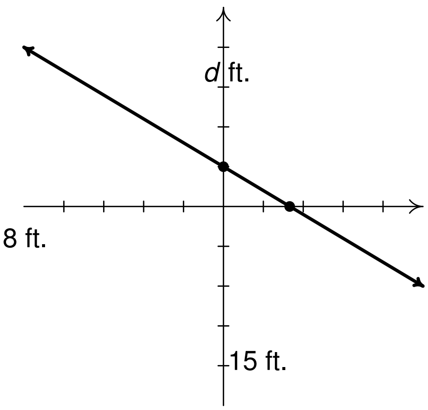
33. The local pet shop charges 12¢ per cricket up to 100 crickets, and 10¢ per cricket thereafter. Write a piecewise-defined linear function which calculates the price in dollars $P(c)$ of purchasing c crickets.

34. The cross-section of a swimming pool is below. Write a piecewise-defined linear function which describes the depth of the pool, D (in feet) as a function of:

    (a) the distance (in feet) from the edge of the shallow end of the pool, d.

    (b) the distance (in feet) from the edge of the deep end of the pool, s.

    (c) Graph each of the functions in (a) and (b). Discuss with your classmates how to transform one into the other and how they relate to the diagram of the pool.



37 ft.

s ft.

2 ft.

10 ft.

35. The function defined by $I(x) = x$ is called the Identity Function. Thinking from a procedural perspective, explain a possible origin of this name.

36. Why must the graph of a function $y = f(x)$ have at most one y-intercept?

    HINT: Consider what would happen graphically if there were more than one . . .

37. Why is a discussion of vertical lines omitted when discussing functions?

38. Find a formula for the x-intercept of the graph of $f(x) = mx + b$. Assume $m \neq 0$.

39. Suppose $(c, 0)$ is the x-intercept of a linear function f. Use the point-slope form of a liner function, Equation 3.6 to show $f(x) = m(x - c)$. This is the 'slope x-intercept' form of the linear function.

40. Prove that for all linear functions L with with slope 3, $L(120) = L(100) + 60$.

41. Find the slopes between the following points from the data set given in Example **??** and compare them with the slope of the corresponding regression line:

    (a) $(0, 64)$, $(4, 75)$ (b) $(4, 75)$, $(8, 83)$ (c) $(8, 83)$, $(10, 83)$ (d) $(10, 83)$, $(12, 82)$

42. According to this website[24], the census data for Lake County, Ohio is:

Year	1970	1980	1990	2000
Population	197200	212801	215499	227511

    (a) Find the least squares regression line for these data and comment on the goodness of fit.[25] Interpret the slope of the line of best fit.

    (b) Use the regression line to predict the population of Lake County in 2010. (The recorded figure from the 2010 census is 230,041)

[24] http://www.ohiobiz.com/census/Lake.pdf

[25] We'll develop more sophisticated models for the growth of populations in Chapter 10. For the moment, we use a theorem from Calculus to approximate those functions with lines.

(c) Use the regression line to predict when the population of Lake County will reach 250,000.

43. According to this website[26], the census data for Lorain County, Ohio is:

Year	1970	1980	1990	2000
Population	256843	274909	271126	284664

(a) Find the least squares regression line for these data and comment on the goodness of fit. Interpret the slope of the line of best fit.

(b) Use the regression line to predict the population of Lorain County in 2010. (The recorded figure from the 2010 census is 301,356)

(c) Use the regression line to predict when the population of Lake County will reach 325,000.

44. The chart below contains a portion of the fuel consumption information for a 2002 Toyota Echo that Jeffrey used to own. The first row is the cumulative number of gallons of gasoline that I had used and the second row is the odometer reading when I refilled the gas tank. So, for example, the fourth entry is the point (28.25, 1051) which says that I had used a total of 28.25 gallons of gasoline when the odometer read 1051 miles.

Gasoline Used (Gallons)	0	9.26	19.03	28.25	36.45	44.64	53.57	62.62	71.93	81.69	90.43
Odometer (Miles)	41	356	731	1051	1347	1631	1966	2310	2670	3030	3371

Find the least squares line for this data. Is it a good fit? What does the slope of the line represent? Do you and your classmates believe this model would have held for ten years had I not crashed the car on the Turnpike a few years ago?

45. Using the energy production data given below

Year	1950	1960	1970	1980	1990	2000
Production (in Quads)	35.6	42.8	63.5	67.2	70.7	71.2

(a) Plot the data using a graphing utility and explain why it does not appear to be linear.

(b) Discuss with your classmates why ignoring the first two data points may be justified from a historical perspective.

(c) Find the least squares regression line for the last four data points and comment on the goodness of fit. Interpret the slope of the line of best fit.

(d) Use the regression line to predict the annual US energy production in the year 2010.

(e) Use the regression line to predict when the annual US energy production will reach 100 Quads.

In Exercises 46 - 51, compute the average rate of change of the function over the specified interval.

[26] http://www.ohiobiz.com/census/Lorain.pdf

46. $f(x) = x^3$, $[-1, 2]$

47. $g(x) = \dfrac{1}{x}$, $[1, 5]$

48. $f(t) = \sqrt{t}$, $[0, 16]$

49. $g(t) = x^2$, $[-3, 3]$

50. $F(s) = \dfrac{s+4}{s-3}$, $[5, 7]$

51. $G(s) = 3s^2 + 2s - 7$, $[-4, 2]$

52. The height of an object dropped from the roof of a building is modeled by: $h(t) = -16t^2 + 64$, for $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground in feet t seconds after the object is dropped. Find and interpret the average rate of change of h over the interval $[0, 2]$.

53. Using data from Bureau of Transportation Statistics, the average fuel economy $F(t)$ in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Find and interpret the average rate of change of F over the interval $[0, 28]$.

54. The temperature $T(t)$ in degrees Fahrenheit t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

   (a) Find and interpret $T(4)$, $T(8)$ and $T(12)$.
   (b) Find and interpret the average rate of change of T over the interval $[4, 8]$.
   (c) Find and interpret the average rate of change of T from $t = 8$ to $t = 12$.
   (d) Find and interpret the average rate of temperature change between 10 AM and 6 PM.

55. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. Find and interpret the average rate of change as production is increased from making 3000 to 5000 pens.

56. Recall from Example **??** The formula $s(t) = -5t^2 + 100t$ for $0 \leq t \leq 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time after lift-off, t, in seconds.

   (a) Find and interpret the average rate of change of s over the following intervals:

      i. $[14.9, 15]$ ii. $[15, 15.1]$ iii. $[14.99, 15]$ iv. $[15, 15.01]$

   (b) What value does the average rate of change appear to be approaching as the interval shrinks closer to the value $t = 15$?
   (c) Find the equation of the line containing $(15, 375)$ with slope $m = -50$ and graph it along with s on the same set of axes using a graphing utility. What happens as you zoom in near $(15, 375)$?

57. Show the average rate of change of a function of the form $f(x) = mx + b$ over *any* interval is m.

58. Why doesn't the graph of the vertical line $x = b$ in the xy-plane represent y as a function of x?

59. With help from a graphing utility, graph the following pairs of functions on the same set of axes:[27]

- $f(x) = 2 - x$ and $g(x) = \lfloor 2 - x \rfloor$
- $f(x) = x^2 - 4$ and $g(x) = \lfloor x^2 - 4 \rfloor$
- $f(x) = x^3$ and $g(x) = \lfloor x^3 \rfloor$
- $f(x) = \sqrt{x} - 4$ and $g(x) = \lfloor \sqrt{x} - 4 \rfloor$

Choose more functions $f(x)$ and graph $y = f(x)$ alongside $y = \lfloor f(x) \rfloor$ until you can explain how, in general, one would obtain the graph of $y = \lfloor f(x) \rfloor$ given the graph of $y = f(x)$.

60. The Lagrange Interpolate function L for two points (x_0, y_0) and (x_1, y_1) where $x_0 \neq x_1$ is given by:

$$L(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

(a) For each of the following pairs of points, find $L(x)$ using the formula above and verify each of the points lies on the graph of $y = L(x)$.

i. $(-1, 3)$, $(2, 3)$ ii. $(-3, -2)$, $(5, -2)$ iii. $(-3, -2)$, $(0, 1)$ iv. $(-1, 5)$, $(2, -1)$

(b) Verify that, in general, $L(x_0) = y_0$ and $L(x_1) = y_1$.

(c) Show the point-slope form of a linear function, Equation 3.6 is equivalent to the formula given for $L(x)$ after making the identifications: $f(x_0) = y_0$ and $m = \dfrac{y_1 - y_0}{x_1 - x_0}$.

[27] See Example 3.2.2 for the definition of $\lfloor x \rfloor$.

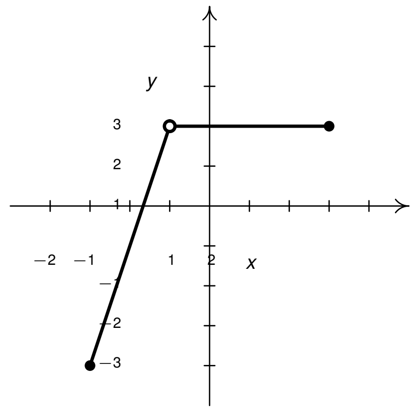
3.2.4 Answers

1. $f(x) = 2x - 1$

   slope: $m = 2$

   y-intercept: $(0, -1)$

   x-intercept: $\left(\frac{1}{2}, 0\right)$

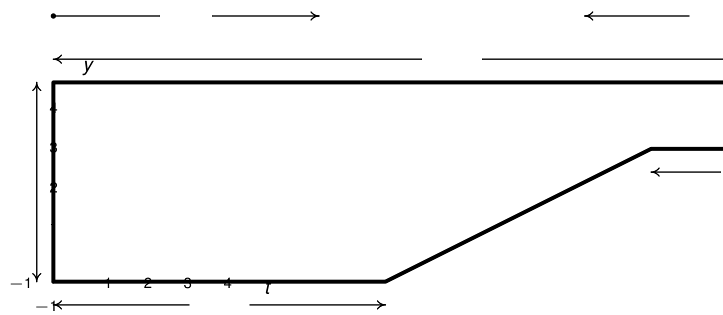


2. $g(t) = 3 - t$

   slope: $m = -1$

   y-intercept: $(0, 3)$

   t-intercept: $(3, 0)$

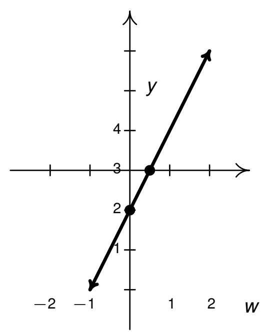


3. $F(w) = 3$

   slope: $m = 0$

   y-intercept: $(0, 3)$

   w-intercept: none

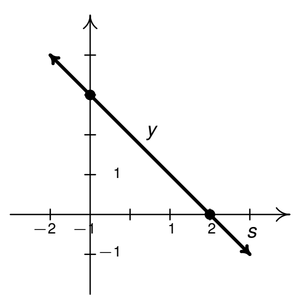


4. $G(s) = 0$

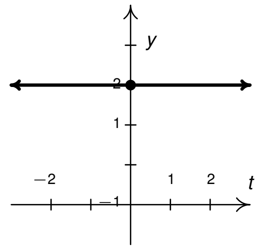
   slope: $m = 0$

   y-intercept: $(0, 0)$

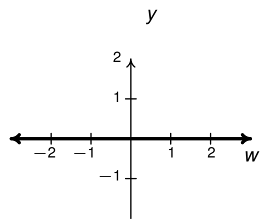
   s-intercept: $\{(s, 0) \mid s \text{ is a real number}\}$



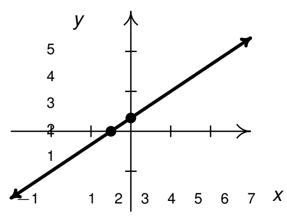
5. $h(t) = \frac{2}{3}x + \frac{1}{3}$
   slope: $m = \frac{2}{3}$
   y-intercept: $\left(0, \frac{1}{3}\right)$
   t-intercept: $\left(-\frac{1}{2}, 0\right)$



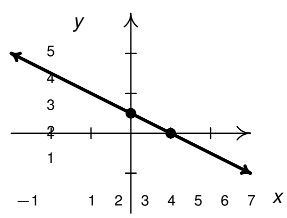
6. $j(w) = \dfrac{1 - w}{2}$
   slope: $m = -\frac{1}{2}$
   y-intercept: $\left(0, \frac{1}{2}\right)$
   w-intercept: $(1, 0)$



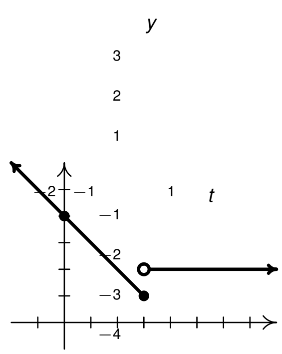
7.
   domain: $(-\infty, \infty)$
   range: $[1, \infty)$
   y-intercept: $(0, 4)$
   x-intercept: none



8.
   domain: $(-\infty, \infty)$
   range: $[0, \infty)$
   y-intercept: $(0, 2)$
   x-intercept: $(2, 0)$

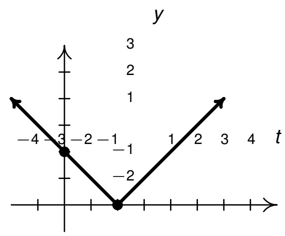


9.
   domain: $(-\infty, \infty)$
   range: $(-4, \infty)$
   y-intercept: $(0, 0)$
   t-intercepts: $(-2, 0)$, $(0, 0)$

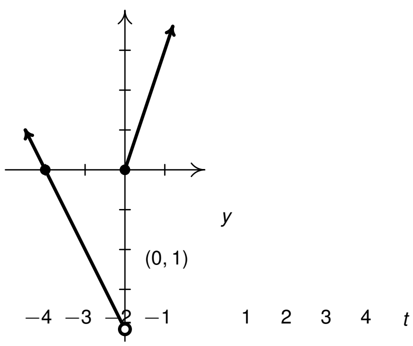


10.

domain: $(-\infty, \infty)$
range: $[-3, 3]$
y-intercept: $(0, -3)$
t-intercept: $\left(\frac{3}{2}, 0\right) = (1.5, 0)$



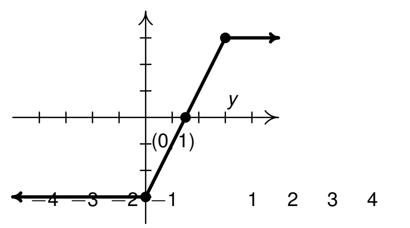
11. (a)



$y = U(t)$

(b) domain: $(-\infty, \infty)$, range: $\{0, 1\}$
(c) U is constant on $(-\infty, 0)$ and $[0, \infty)$.

(d) $U(t-2) = \begin{cases} 0 & \text{if } t < 2, \\ 1 & \text{if } t \geq 2. \end{cases}$



$y = U(t-2)$

12. $f(x) = -3$

13. $F(t) = \begin{cases} 2 & \text{if } t \leq 1, \\ -3 & \text{if } 1 < t \leq 3, \\ 4 & \text{if } t > 3. \end{cases}$

14. $L(x) = -\frac{3}{5}x + 1$

15. $g(v) = \begin{cases} 3v + 5 & \text{if } v \leq -1, \\ 2 & \text{if } -1 < v \leq 3, \end{cases}$

16. (a) $C(20) = 300$. It costs \$300 for 20 copies of the book.
(b) $C(50) = 675$, \$675. $C(51) = 612$, \$612.
(c) 56 books.

17. (a) $S(10) = 17.5$, \$17.50.
 (b) There is free shipping on orders of 15 or more comic books.

18. (a) $C(750) = 25$, \$25.
 (b) $C(1200) = 45$, \$45.
 (c) It costs \$25 for up to 1000 minutes and 10 cents per minute for each minute over 1000 minutes.

19. $d(t) = 3t$, $t \geq 0$.

20. $E(t) = 360t$, $t \geq 0$.

21. $C(x) = 45x + 20$, $x \geq 0$.

22. $C(t) = 80t + 50$, $0 \leq t \leq 8$.

23. $W(x) = 200 + .05x,\ x \geq 0$ She must make \$5500 in weekly sales.

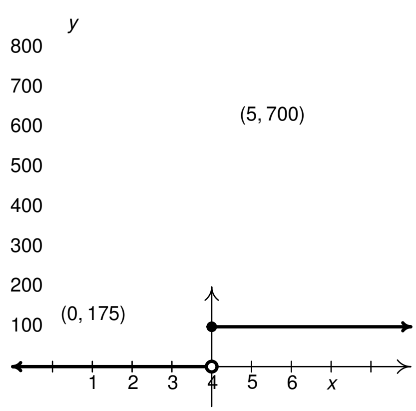
24. $C(p) = 0.035p + 1.5$ The slope 0.035 means it costs 3.5¢ per page. $C(0) = 1.5$ means there is a fixed, or start-up, cost of \$1.50 to make each book.

25. $F(m) = 2.25m + 2.05$ The slope 2.25 means it costs an additional \$2.25 for each mile beyond the first 0.2 miles. $F(0) = 2.05$, so according to the model, it would cost \$2.05 for a trip of 0 miles. Would this ever really happen? Depends on the driver and the passenger, we suppose.

26. (a) $F(T) = \frac{9}{5}T + 32$
 (b) $C(T) = \frac{5}{9}(T - 32) = \frac{5}{9}T - \frac{160}{9}$
 (c) $F(-40) = -40 = C(-40)$.

27. $N(T) = -\frac{2}{15}T + \frac{43}{3}$ and $N(20) = \frac{35}{3} \approx 12$ howls per hour.
 Having a negative number of howls makes no sense and since $N(107.5) = 0$ we can put an upper bound of $107.5^\circ F$ on the domain. The lower bound is trickier because there's nothing other than common sense to go on. As it gets colder, he howls more often. At some point it will either be so cold that he freezes to death or he's howling non-stop. So we're going to say that he can withstand temperatures no lower than $-42^\circ F$ so that the applied domain is $[-42, 107.5]$.

28. (a) $C(0) = 175$, so our start-up costs are \$175. $C(5) = 700$, so to produce 5 systems, it costs \$700.



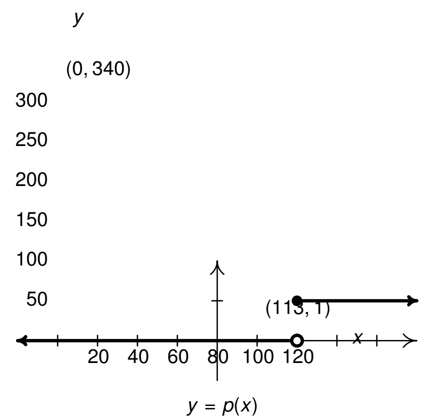
$y = C(x)$

(b) Since we can't make a negative number of game systems, $x \geq 0$.

(c) The slope is $m = 105$ so for each additional system produced, it costs an additional \$105.

(d) Solving $C(x) = 15000$ gives $x \approx 141.19$ so 141 can be produced for \$15,000.

29. (a) $p(x) = -3x + 340$, $0 \leq x \leq 113$.



(b) The slope is $m = -3$ so for each \$3 drop in price, we sell one additional game system.

(c) Since $x = 150$ is not in the domain of p, $p(150)$ is not defined. (In other words, under these conditions, it is impossible to sell 150 game systems.)

(d) Solving $p(x) = 150$ gives $x \approx 63.33$ so if the price \$150 per system, we would sell 63 systems.

30. $C(p) = \begin{cases} 6p + 1.5 & \text{if} \quad 1 \leq p \leq 5 \\ 5.5p & \text{if} \quad p \geq 6 \end{cases}$

31. $T(n) = \begin{cases} 15n & \text{if} \quad 1 \leq n \leq 9 \\ 12.5n & \text{if} \quad n \geq 10 \end{cases}$

32. $C(m) = \begin{cases} 10 & \text{if} \quad 0 \leq m \leq 500 \\ 10 + 0.15(m - 500) & \text{if} \quad m > 500 \end{cases}$

33. $P(c) = \begin{cases} 0.12c & \text{if} \quad 1 \leq c \leq 100 \\ 12 + 0.1(c - 100) & \text{if} \quad c > 100 \end{cases}$

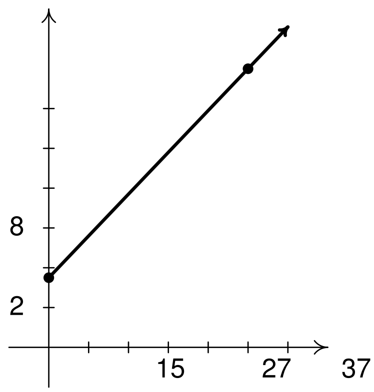
34. (a)

$$D(d) = \begin{cases} 8 & \text{if} \quad 0 \leq d \leq 15 \\ -\frac{1}{2}d + \frac{31}{2} & \text{if} \quad 15 \leq d \leq 27 \\ 2 & \text{if} \quad 27 \leq d \leq 37 \end{cases}$$

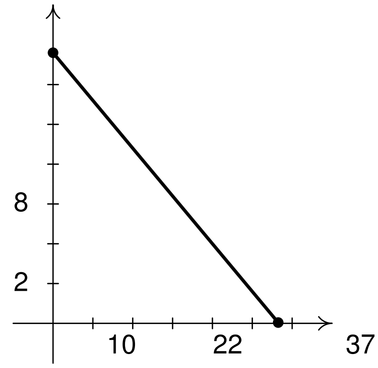
(b)

$$D(s) = \begin{cases} 2 & \text{if} \quad 0 \leq s \leq 10 \\ \frac{1}{2}s - 3 & \text{if} \quad 10 \leq s \leq 22 \\ 8 & \text{if} \quad 22 \leq s \leq 37 \end{cases}$$

(c)



$y = D(d)$



$y = D(s)$

35. Since $I(x) = x$ for all real numbers x, the function I doesn't change the 'identity' of the input at all.

36. If a graph contains more than one y-intercept, it would violate the Vertical Line Test since $x = 0$ would be matched with (at least) two different y-values.

37. Vertical Lines fail the Vertical Line Test.

38. $\left(-\frac{b}{m}, 0\right)$. (Note the importance here of $m \neq 0$.)

39. Plugging in $(c, 0)$ for $(x_0, f(x_0))$, we get $f(x) = f(x_0) + m(x - x_0) = 0 + m(x - c)$ or $f(x) = m(x - c)$.

40. Since L is linear with slope 3, $L(x) = L(x_0) + m\Delta x = L(100) + (3)(120 - 100) = L(100) + 60$.

41. (a) $m = \frac{75-64}{4-0} = 2.75$
    (b) $m = \frac{83-75}{8-4} = 2$
    (c) $m = \frac{83-83}{10-8} = 0$
    (d) $m = \frac{82-83}{12-10} = -0.5$

    The first two points contributed to a regression line slope of $m = 2.55$; the last two points contributed to a regression line slope of $m = -0.25$.

42. (a) $y = 936.31x - 1645322.6$ with $r = 0.9696$ which indicates a good fit. The slope 936.31 indicates Lake County's population is increasing at a rate of (approximately) 936 people per year.
    (b) According to the model, the population in 2010 will be 236,660.
    (c) According to the model, the population of Lake County will reach 250,000 sometime between 2024 and 2025.

43. (a) $y = 796.8x - 1309762.5$ with $r = 0.8916$ which indicates a reasonable fit. The slope 796.8 indicates Lorain County's population is increasing at a rate of (approximately) 797 people per year.
    (b) According to the model, the population in 2010 will be 291,805.

(c) According to the model, the population of Lake County will reach 325,000 sometime between 2051 and 2052.

44. The regression line is $y = 36.8x + 16.39$ with $r = .99987$, so this is an excellent fit. The slope 36.8 represents mileage in miles per gallon.

45. (c) $y = 0.266x - 459.86$ with $r = 0.9607$ which indicates a good fit. The slope 0.266 indicates the country's energy production is increasing at a rate of 0.266 Quad per year.

(d) According to the model, the production in 2010 will be 74.8 Quad.

(e) According to the model, the production will reach 100 Quad in the year 2105.

46. $\frac{2^3-(-1)^3}{2-(-1)} = 3$

47. $\frac{\frac{1}{5}-\frac{1}{1}}{5-1} = -\frac{1}{5}$

48. $\frac{\sqrt{16}-\sqrt{0}}{16-0} = \frac{1}{4}$

49. $\frac{3^2-(-3)^2}{3-(-3)} = 0$

50. $\dfrac{\frac{7+4}{7-3} - \frac{5+4}{5-3}}{7-5} = -\frac{7}{8}$

51. $\dfrac{(3(2)^2 + 2(2) - 7) - (3(-4)^2 + 2(-4) - 7)}{2-(-4)} = -4$

52. The average rate of change is $\frac{h(2)-h(0)}{2-0} = -32$. During the first two seconds after it is dropped, the object has fallen at an average rate of 32 feet per second.

53. The average rate of change is $\frac{F(28)-F(0)}{28-0} = 0.2372$. From 1980 to 2008, the average fuel economy of passenger cars in the US increased, on average, at a rate of 0.2372 miles per gallon per year.

54. (a) $T(4) = 56$, so at 10 AM (4 hours after 6 AM), it is 56°F. $T(8) = 64$, so at 2 PM (8 hours after 6 AM), it is 64°F. $T(12) = 56$, so at 6 PM (12 hours after 6 AM), it is 56°F.

(b) The average rate of change is $\frac{T(8)-T(4)}{8-4} = 2$. Between 10 AM and 2 PM, the temperature increases, on average, at a rate of 2°F per hour.

(c) The average rate of change is $\frac{T(12)-T(8)}{12-8} = -2$. Between 2 PM and 6 PM, the temperature decreases, on average, at a rate of 2°F per hour.

(d) The average rate of change is $\frac{T(12)-T(4)}{12-4} = 0$. Between 10 AM and 6 PM, the temperature, on average, remains constant.

55. The average rate of change is $\frac{C(5)-C(3)}{5-3} = -2$. As production is increased from 3000 to 5000 pens, the cost decreases at an average rate of \$200 per 1000 pens produced (20¢ per pen.)

56. (a) i. -49.5 so the average velocity of the rocket between 14.9 and 15 seconds after lift off is -49.5 feet per second (49.5 feet per second directed *downwards*.)

ii. -50.5 so the average velocity of the rocket between 14 and 15.1 seconds after lift off is -50.5 feet per second. (50.5 feet per second directed *downwards*.)

iii. -49.95 so the average velocity of the rocket between 14.99 and 15 seconds after lift off is -49.95 feet per second. (49.95 feet per second directed *downwards*.)

iv. -50.05 so the average velocity of the rocket between 15.01 and 15 seconds after lift off is -50.05 feet per second. (50.05 feet per second directed *downwards*.)

(b) The average rate of change seem to be approaching -50.

(c) Line: $y = -50(t - 15) + 375$ or $y = -50t + 1125$. Graphing this line along with the s on a graphing utility we find the two graphs become indistinguishable as we zoom in near $(15, 375)$.

60. (a)

i. $L(x) = 3$ ii. $L(x) = -2$ iii. $L(x) = x + 1$ iv. $L(x) = -2x + 3$

Chapter 4

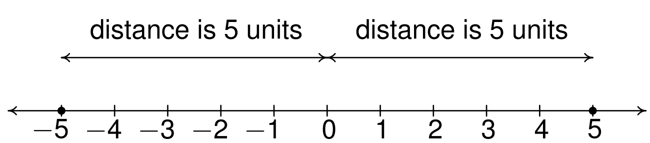
Absolute Value Functions

4.1 Absolute Value Equations and Inequalities

In this section, we review some of the basic concepts involving the absolute value of a real number x. There are a few different ways to define absolute value and in this section we choose the following definition. (Absolute value will be revisited in much greater depth in Section 4.2 where we present what one can think of as the "precise" definition.)

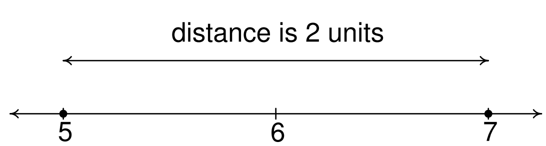
Definition 4.1. Absolute Value as Distance: For every real number x, the **absolute value** of x, denoted $|x|$, is the distance between x and 0 on the number line. More generally, if x and c are real numbers, $|x - c|$ is the distance between the numbers x and c on the number line.

For example, $|5| = 5$ and $|-5| = 5$, since each is 5 units from 0 on the number line:



Graphically why $|-5| = 5$ and $|5| = 5$

Computationally, the absolute value 'makes negative numbers positive', though we need to be a little cautious with this description. While $|-7| = 7$, $|5 - 7| \neq 5 + 7$. The absolute value acts as a grouping symbol, so $|5 - 7| = |-2| = 2$, which makes sense since 5 and 7 are two units away from each other on the number line:



Graphically why $|5 - 7| = 2$

We list some of the operational properties of absolute value below.

Theorem 4.1. Properties of Absolute Value: Let a and b be real numbers and let n be an integer.[a]

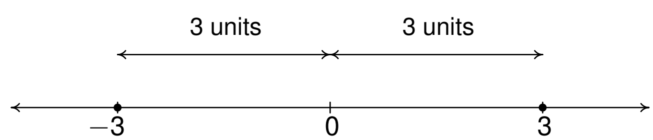
- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^n| = |a|^n$ whenever a^n is defined
- **Quotient Rule:** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, provided $b \neq 0$

[a]See page 4 if you don't remember what an integer is.

The proof of Theorem 4.1 is difficult, but not impossible, using the distance definition of absolute value or even the 'it makes negatives positive' notion. It is, however, much easier if one uses the "precise" definition given in Section 4.2 so we will revisit the proof then. For now, let's focus on how to solve basic equations and inequalities involving the absolute value.

4.1.1 Absolute Value Equations

Thinking of absolute value in terms of distance gives us a geometric way to interpret equations. For example, to solve $|x| = 3$, we are looking for all real numbers x whose distance from 0 is 3 units. If we move three units to the right of 0, we end up at $x = 3$. If we move three units to the left, we end up at $x = -3$. Thus the solutions to $|x| = 3$ are $x = \pm 3$.



The solutions to $|x| = 3$ are $x = \pm 3$.

Thinking this way gives us the following.

Theorem 4.2. Absolute Value Equations: Suppose x, y and c are real numbers.

- $|x| = 0$ if and only if $x = 0$.
- For $c > 0$, $|x| = c$ if and only if $x = c$ or $x = -c$.
- For $c < 0$, $|x| = c$ has no solution.
- $|x| = |y|$ if and only if $x = y$ or $x = -y$.

  (That is, if two numbers have the same absolute values, they are either the same number or exact opposites of each other.)

Theorem 4.2 is our main tool in solving equations involving the absolute value, since it allows us a way to rewrite such equations as compound linear equations.

Strategy for Solving Equations Involving Absolute Value

In order to solve an equation involving the absolute value of a quantity $|X|$:

1. Isolate the absolute value on one side of the equation so it has the form $|X| = c$.
2. Apply Theorem 4.2.

The techniques we use to 'isolate the absolute value' are precisely those we used in Section 3.1 to isolate the variable when solving linear equations. Time for some practice.

Example 4.1.1. Solve each of the following equations.

1. $|3x - 1| = 6$
2. $\dfrac{3 - |y + 5|}{2} = 1$
3. $3|2t + 1| - \sqrt{5} = 0$
4. $4 - |5w + 3| = 5$
5. $\left|3 - x\sqrt[3]{12}\right| = |4x + 1|$
6. $|t - 1| - 3|t + 1| = 0$

Solution.

1. The equation $|3x - 1| = 6$ is of already in the form $|X| = c$, so we know that either $3x - 1 = 6$ or $3x - 1 = -6$. Solving the former gives us at $x = \frac{7}{3}$ and solving the latter yields $x = -\frac{5}{3}$. We may check both of these solutions by substituting them into the original equation and showing that the arithmetic works out.

2. We begin solving $\frac{3-|y+5|}{2} = 1$ by isolating the absolute value to put it in the form $|X| = c$.

$$\begin{array}{rcll} \dfrac{3-|y+5|}{2} & = & 1 & \\ 3-|y+5| & = & 2 & \text{Multiply by 2} \\ -|y+5| & = & -1 & \text{Subtract 3} \\ |y+5| & = & 1 & \text{Divide by } -1 \end{array}$$

   At this point, we have $y + 5 = 1$ or $y + 5 = -1$, so our solutions are $y = -4$ or $y = -6$. We leave it to the reader to check both answers in the original equation.

3. As in the previous example, we first isolate the absolute value. Don't let the $\sqrt{5}$ throw you off - it's just another real number, so we treat it as such:

$$\begin{array}{rcll} 3|2t+1| - \sqrt{5} & = & 0 & \\ 3|2t+1| & = & \sqrt{5} & \text{Add } \sqrt{5} \\ |2t+1| & = & \dfrac{\sqrt{5}}{3} & \text{Divide by 3} \end{array}$$

   From here, we have that $2t + 1 = \frac{\sqrt{5}}{3}$ or $2t + 1 = -\frac{\sqrt{5}}{3}$. The first equation gives $t = \frac{\sqrt{5}-3}{6}$ while the second gives $t = \frac{-\sqrt{5}-3}{6}$ thus we list our answers as $t = \frac{-3\pm\sqrt{5}}{6}$. The reader should enjoy the challenge of substituting both answers into the original equation and following through the arithmetic to see that both answers work.

4. Upon isolating the absolute value in the equation $4 - |5w + 3| = 5$, we get $|5w + 3| = -1$. At this point, we know there cannot be any real solution. By definition, the absolute value is a *distance*, and as such is never negative. We write 'no solution' and carry on.

5. Our next equation already has the absolute value expressions (plural) isolated, so we work from the principle that if $|x| = |y|$, then $x = y$ or $x = -y$. Thus from $\left|3 - x\sqrt[3]{12}\right| = |4x + 1|$ we get two equations to solve:

$$3 - x\sqrt[3]{12} = 4x + 1, \qquad \text{and} \qquad 3 - x\sqrt[3]{12} = -(4x + 1)$$

   Notice that the right side of the second equation is $-(4x + 1)$ and not simply $-4x + 1$. Remember, the expression $4x + 1$ represents a single real number so in order to negate it we need to negate the *entire* expression $-(4x + 1)$. Moving along, when solving $3 - x\sqrt[3]{12} = 4x + 1$, we obtain $x = \frac{2}{4+\sqrt[3]{12}}$ and the solution to $3 - x\sqrt[3]{12} = -(4x + 1)$ is $x = \frac{4}{\sqrt[3]{12}-4}$. As usual, the reader is invited to check these answers by substituting them into the original equation.

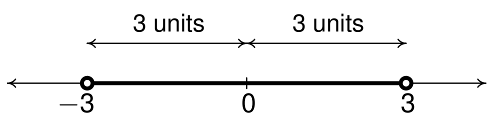
6. We start by isolating one of the absolute value expressions: $|t-1|-3|t+1| = 0$ gives $|t-1| = 3|t+1|$. While this *resembles* the form $|x| = |y|$, the coefficient 3 in $3|t+1|$ prevents it from being an exact match. Not to worry - since 3 is positive, $3 = |3|$ so

$$3|t+1| = |3||t+1| = |3(t+1)| = |3t+3|.$$

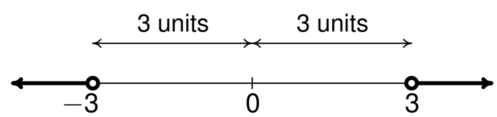
Hence, our equation becomes $|t-1| = |3t+3|$ which results in the two equations: $t-1 = 3t+3$ and $t-1 = -(3t+3)$. The first equation gives $t = -2$ and the second gives $t = -\frac{1}{2}$. The reader is encouraged to check both answers in the original equation. □

4.1.2 Absolute Value Inequalities

We now turn our attention to solving some basic inequalities involving the absolute value. Suppose we wished to solve $|x| < 3$. Geometrically, we are looking for all of the real numbers whose distance from 0 is *less* than 3 units. We get $-3 < x < 3$, or in interval notation, $(-3, 3)$. Suppose we are asked to solve $|x| > 3$ instead. Now we want the distance between x and 0 to be *greater* than 3 units. Moving in the positive direction, this means $x > 3$. In the negative direction, this puts $x < -3$. Our solutions would then satisfy $x < -3$ *or* $x > 3$. In interval notation, we express this as $(-\infty, -3) \cup (3, \infty)$.



The solution to $|x| < 3$ is $(-3, 3)$



The solution to $|x| > 3$ is $(-\infty, -3) \cup (3, \infty)$

Generalizing this notion, we get the following:

Theorem 4.3. Inequalities Involving Absolute Value: Let c be a real number.

- If $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- If $c \leq 0$, $|x| < c$ has no solution.
- If $c > 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
- If $c \leq 0$, $|x| > c$ is true for all real numbers.

If the inequality we're faced with involves '$\leq$' or '$\geq$,' we can combine the results of Theorem 4.3 with Theorem 4.2 as needed.

Strategy for Solving Inequalities Involving Absolute Value

In order to solve an inequality involving the absolute value of a quantity $|X|$:

1. Isolate the absolute value on one side of the inequality.
2. Apply Theorem 4.3.

Example 4.1.2. Solve the following inequalities.

1. $\left|x - \sqrt[4]{5}\right| > 1$
2. $\dfrac{4 - 2|2x+1|}{4} \geq -\sqrt{3}$
3. $|2x-1| \leq 3|4-8x| - 10$
4. $|2x-1| \leq 3|4-8x| + 10$
5. $2 < |x-1| \leq 5$
6. $|10x-5| + |10-5x| \leq 0$

Solution.

1. From Theorem 4.3, $\left|x - \sqrt[4]{5}\right| > 1$ is equivalent to $x - \sqrt[4]{5} < -1$ or $x - \sqrt[4]{5} > 1$. Solving this compound inequality, we get $x < -1 + \sqrt[4]{5}$ or $x > 1 + \sqrt[4]{5}$. Our answer, in interval notation, is: $\left(-\infty, -1+\sqrt[4]{5}\right) \cup \left(1+\sqrt[4]{5}, \infty\right)$. As with linear inequalities, we can only partially check our answer by selecting values of x both inside and outside of the solution intervals to see which values of x satisfy the original inequality and which do not.

2. Our first step in solving $\frac{4-2|2x+1|}{4} \geq -\sqrt{3}$ is to isolate the absolute value.

$$\begin{array}{rcll} \dfrac{4-2|2x+1|}{4} & \geq & -\sqrt{3} & \\ 4 - 2|2x+1| & \geq & -4\sqrt{3} & \text{Multiply by } 4 \\ -2|2x+1| & \geq & -4-4\sqrt{3} & \text{Subtract } 4 \\ |2x+1| & \leq & \dfrac{-4-4\sqrt{3}}{-2} & \text{Divide by } -2\text{, reverse the inequality} \\ |2x+1| & \leq & 2+2\sqrt{3} & \text{Reduce} \end{array}$$

Since we're dealing with '$\leq$' instead of just '$<$,' we can combine Theorems 4.3 and 4.2 to rewrite this last inequality as:[1] $-(2+2\sqrt{3}) \leq 2x+1 \leq 2+2\sqrt{3}$. Subtracting the '1' across both inequalities gives $-3-2\sqrt{3} \leq 2x \leq 1+2\sqrt{3}$, which reduces to $\frac{-3-2\sqrt{3}}{2} \leq x \leq \frac{1+2\sqrt{3}}{2}$. In interval notation this reads as $\left[\frac{-3-2\sqrt{3}}{2}, \frac{1+2\sqrt{3}}{2}\right]$.

3. There are two absolute values in $|2x-1| \leq 3|4-8x| - 10$, so we cannot directly apply Theorem 4.3 here. Notice, however, that $|4-8x| = |(-4)(2x-1)|$. Using this, we get:

$$\begin{array}{rcll} |2x-1| & \leq & 3|4-8x| - 10 & \\ |2x-1| & \leq & 3|(-4)(2x-1)| - 10 & \text{Factor} \\ |2x-1| & \leq & 3|-4||2x-1| - 10 & \text{Product Rule} \\ |2x-1| & \leq & 12|2x-1| - 10 & \\ -11|2x-1| & \leq & -10 & \text{Subtract } 12|2x-1| \\ |2x-1| & \geq & \dfrac{10}{11} & \text{Divide by } -11 \text{ and reduce} \end{array}$$

[1] Note the use of parentheses: $-(2+2\sqrt{3})$ as opposed to $-2+2\sqrt{3}$.

Now we are allowed to invoke Theorems 4.2 and 4.3 and write the equivalent compound inequality: $2x - 1 \leq -\frac{10}{11}$ *or* $2x - 1 \geq \frac{10}{11}$. We get $x \leq \frac{1}{22}$ *or* $x \geq \frac{21}{22}$, which when written with interval notation becomes $\left(-\infty, \frac{1}{22}\right] \cup \left[\frac{21}{22}, \infty\right)$.

4. The inequality $|2x-1| \leq 3|4-8x|+10$ differs from the previous example in exactly one respect: on the right side of the inequality, we have '+10' instead of '-10.' The steps to isolate the absolute value here are identical to those in the previous example, but instead of obtaining $|2x-1| \geq \frac{10}{11}$ as before, we obtain $|2x-1| \geq -\frac{10}{11}$. This latter inequality is *always* true. (Absolute value is, by definition, a distance and hence always 0 or greater.) Thus our solution to this inequality is all real numbers.

5. To solve $2 < |x-1| \leq 5$, we rewrite it as the compound inequality: $2 < |x-1|$ *and* $|x-1| \leq 5$. The first inequality, $2 < |x-1|$, can be re-written as $|x-1| > 2$ so it is equivalent to $x-1 < -2$ *or* $x - 1 > 2$. Thus the solution to $2 < |x-1|$ is $x < -1$ or $x > 3$, which in interval notation is $(-\infty, -1) \cup (3, \infty)$. For $|x-1| \leq 5$, we combine the results of Theorems 4.2 and 4.3 to get $-5 \leq x - 1 \leq 5$ so that $-4 \leq x \leq 6$, or $[-4, 6]$.

   Our solution to $2 < |x-1| \leq 5$ is comprised of values of x which satisfy both parts of the inequality, so we intersect $(-\infty, -1) \cup (3, \infty)$ with $[-4, 6]$ to get our final answer $[-4, -1) \cup (3, 6]$.

6. Our first hope when encountering $|10x-5| + |10-5x| \leq 0$ is that we can somehow combine the two absolute value quantities as we'd done in earlier examples. We leave it to the reader to show, however, that no matter what we try to factor out of the absolute value quantities, what remains inside the absolute values will always be different.

   At this point, we take a step back and look at the equation in a more general way: we are adding two absolute values together and wanting the result to be less than or equal to 0. The absolute value of anything is always 0 or greater, so there are no solutions to: $|10x-5| + |10-5x| < 0$.

   Is it possible that $|10x-5|+|10-5x| = 0$? Only if there is an x where $|10x-5| = 0$ and $|10-5x| = 0$ *at the same time*.[2] The first equation holds only when $x = \frac{1}{2}$, while the second holds only when $x = 2$. Alas, we have no solution.[3] □

The astute reader will have noticed by now that the authors have done nothing in the way of explaining *why* anyone would ever need to know this stuff. Go back and read the New Preface and the introduction to the Appendix. These sections are designed to review skills and concepts that you've already learned. Thus the deeper applications are in the main body of the text as opposed to here in the Appendix.

We close this section with an example of how the properties in Theorem 4.1 are used in Calculus. Here, 'ε' is the Greek letter 'epsilon' and it represents a positive real number. Those of you who will be taking Calculus in the future should become *very* familiar with this type of algebraic manipulation.

[2] Do you see why?

[3] Not for lack of trying, however!

$$\begin{aligned}
\left|\frac{8-4x}{3}\right| &< \varepsilon \\
\frac{|8-4x|}{|3|} &< \varepsilon && \text{Quotient Rule} \\
\frac{|-4(x-2)|}{3} &< \varepsilon && \text{Factor} \\
\frac{|-4||x-2|}{3} &< \varepsilon && \text{Product Rule} \\
\frac{4|x-2|}{3} &< \varepsilon \\
\frac{3}{4}\cdot\frac{4|x-2|}{3} &< \frac{3}{4}\cdot\varepsilon && \text{Multiply by } \frac{3}{4} \\
|x-2| &< \frac{3}{4}\varepsilon
\end{aligned}$$

4.1.3 Exercises

In Exercises 1 - 18, solve the equation.

1. $|x| = 6$
2. $|3t - 1| = 10$
3. $|4 - w| = 7$
4. $4 - |y| = 3$
5. $2|5m + 1| - 3 = 0$
6. $|7x - 1| + 2 = 0$
7. $\dfrac{5 - |x|}{2} = 1$
8. $\dfrac{2}{3}|5 - 2w| - \dfrac{1}{2} = 5$
9. $|3t - \sqrt{2}| + 4 = 6$
10. $\dfrac{|2v + 1| - 3}{4} = \dfrac{1}{2} - |2v + 1|$
11. $|2x + 1| = \dfrac{|2x + 1| - 3}{2}$
12. $\dfrac{|3 - 2y| + 4}{2} = 2 - |3 - 2y|$
13. $|3t - 2| = |2t + 7|$
14. $|3x + 1| = |4x|$
15. $|1 - \sqrt{2}y| = |y + 1|$
16. $|4 - x| - |x + 2| = 0$
17. $|2 - 5z| = 5|z + 1|$
18. $\sqrt{3}|w - 1| = 2|w + 1|$

In Exercises 19 - 30, solve the inequality. Write your answer using interval notation.

19. $|3x - 5| \leq 4$
20. $|7t + 2| > 10$
21. $|2w + 1| - 5 < 0$
22. $|2 - y| - 4 \geq -3$
23. $|3z + 5| + 2 < 1$
24. $2|7 - v| + 4 > 1$
25. $3 - |x + \sqrt{5}| < -3$
26. $|5t| \leq |t| + 3$
27. $|w - 3| < |3 - w|$
28. $2 \leq |4 - y| < 7$
29. $1 < |2w - 9| \leq 3$
30. $3 > 2|\sqrt{3} - x| > 1$
31. With help from your classmates, solve:
    (a) $|5 - |2x - 3|| = 4$
    (b) $|5 - |2x - 3|| < 4$

4.1.4 Answers

1. $x = -6$ or $x = 6$
2. $t = -3$ or $t = \frac{11}{3}$
3. $w = -3$ or $w = 11$
4. $y = -1$ or $y = 1$
5. $m = -\frac{1}{2}$ or $m = \frac{1}{10}$
6. No solution
7. $x = -3$ or $x = 3$
8. $w = -\frac{13}{8}$ or $w = \frac{53}{8}$
9. $t = \frac{\sqrt{2} \pm 2}{3}$
10. $v = -1$ or $v = 0$
11. No solution
12. $y = \frac{3}{2}$
13. $t = -1$ or $t = 9$
14. $x = -\frac{1}{7}$ or $x = 1$
15. $y = 0$ or $y = \frac{2}{\sqrt{2} - 1}$
16. $x = 1$
17. $z = -\frac{3}{10}$
18. $w = \frac{\sqrt{3} \pm 2}{\sqrt{3} \mp 2}$
    See footnote[4]
19. $\left[\frac{1}{3}, 3\right]$
20. $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
21. $(-3, 2)$
22. $(-\infty, 1] \cup [3, \infty)$
23. No solution
24. $(-\infty, \infty)$
25. $(-\infty, -6 - \sqrt{5}) \cup (6 - \sqrt{5}, \infty)$
26. $\left[-\frac{3}{4}, \frac{3}{4}\right]$
27. No solution
28. $(-3, 2] \cup [6, 11)$
29. $[3, 4) \cup (5, 6]$
30. $\left(\frac{2\sqrt{3} - 3}{2}, \frac{2\sqrt{3} - 1}{2}\right) \cup \left(\frac{2\sqrt{3} + 1}{2}, \frac{2\sqrt{3} + 3}{2}\right)$
31. (a) $x = -3$, or $x = 1$, or $x = 2$, or $x = 6$
    (b) $(-3, 1) \cup (2, 6)$

[4]That is, $w = \frac{\sqrt{3} + 2}{\sqrt{3} - 2}$ or $w = \frac{\sqrt{3} - 2}{\sqrt{3} + 2}$

4.2 Absolute Value Functions

4.2.1 Graphs of Absolute Value Functions

In Section 3.2, we revisited lines in a function context. In this section, we revisit the absolute value in a similar manner, so it may be useful to refresh yourself with the basics in Section 4.1. Recall that the absolute value of a real number x, denoted $|x|$, can be defined as the distance from x to 0 on the real number line.[1] This definition is very useful for several applications, and lends itself well to solving equations and inequalities such as $|x-2|+1=5$ or $2|t+1|>4$.

We now wish to explore solving more complicated equations and inequalities, such as $|x-2|+1=x$ and $2|t+1|\geq t+4$. We'll approach these types of problems from a function standpoint and use the interplay between the graphical and analytical representations of these functions to obtain solutions. The key to this section is understanding the absolute value from that function (or procedural) standpoint.

Consider a real number $x\geq 0$ such as $x=0$, $x=\pi$ or $x=117.42$. When computing absolute values, we find $|0|=0$, $|\pi|=\pi$ and $|117.42|=117.42$. In general, if $x\geq 0$, the absolute value function does nothing to change the input, so $|x|=x$. On the other hand, if $x<0$, say $x=-1$, $x=-\sqrt{42}$ or $x=-117.42$, we get $|-1|=1$, $|-\sqrt{42}|=\sqrt{42}$ and $|-117.42|=117.42$. That is, if $x<0$, $|x|$ returns the exact *opposite* of the input x, so $|x|=-x$.

Putting these two observations together, we have the following.

Definition 4.2. The **absolute value** of a real number x, denoted $|x|$, is given by

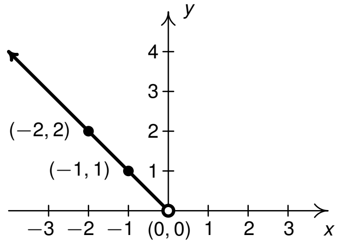
$$|x|=\begin{cases}-x & \text{if } x<0\\ x & \text{if } x\geq 0\end{cases}$$

In Definition 4.2, it is *absolutely* essential to read '$-x$' as 'the *opposite* of x' as *opposed* to 'negative x' in order to avoid serious errors later. To see that this description agrees with our previous experience, consider $|117.42|$. Given that $117.42\geq 0$, we use the rule $|x|=x$. Hence, $|117.42|=117.42$. Likewise, $|0|=0$. To compute $|-\sqrt{42}|$, we note that $-\sqrt{42}<0$ we use the rule $|x|=-x$ in this case. We get $|-\sqrt{42}|=-(-\sqrt{42})$ (the opposite of $-\sqrt{42}$), so $|-\sqrt{42}|=-(-\sqrt{42})=\sqrt{42}$.

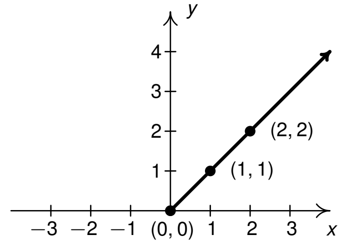
Another way to view Definition 4.2 is to think of $-x=(-1)x$ and $x=(1)x$. That is, $|x|$ multiplies negative inputs by -1 and non-negative inputs by 1. This viewpoint is especially useful in graphing $f(x)=|x|$. For $x<0$, $|x|=(-1)x$, so the graph of $y=|x|$ is the graph of $y=-x=(-1)x$: a line with slope -1 and y-intercept $(0,0)$. Likewise, for $x\geq 0$, $|x|=x$, so the graph of $y=|x|$ is the graph of $y=x=(1)x$: a line with slope 1 and y-intercept $(0,0)$.

At the top of the next page we graph each piece and then put them together. Note that when graphing $f(x)=|x|$ for $x<0$, we have a hole at $(0,0)$ because the inequality $x<0$ is strict. However, the point $(0,0)$ is included in the graph of $f(x)=|x|$ for $x\geq 0$, so there is no hole in our final graph.

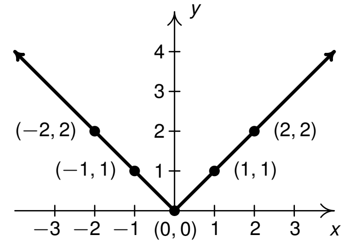
[1] More generally, $|x-c|$ is the distance from x to c on the number line.



$f(x) = |x|,\ x < 0$



$f(x) = |x|,\ x \geq 0$



$f(x) = |x|$

The graph of $f(x) = |x|$ is a very distinctive '$\vee$' shape and is worth remembering. The point $(0, 0)$ on the graph is called the **vertex**. This terminology makes sense from a geometric viewpoint because $(0, 0)$ is the point where two lines meet to form an angle. We will also see this term used in Section 5.4 where, more generally, it corresponds to the graphical location of the sole maximum or minimum of a quadratic function.

We put Definition 4.2 to good use in the next example and review the basics of graphing along the way.

Example 4.2.1. For each of the functions below, analytically find the zeros of the function and the axis intercepts of the graph, if any exist. Rewrite the function using Definition 4.2 as a piecewise-defined function and sketch its graph. From the graph, determine the vertex, find the range of the function and any extrema, and then list the intervals over which the function is increasing, decreasing or constant.

1. $f(x) = |x - 3|$
2. $g(t) = |t| - 3$
3. $h(u) = |2u - 1| - 3$
4. $i(w) = 4 - 2|3w - 1|$

Solution. In what follows below, we will be doing quite a bit of substitution. As we have mentioned before, when substituting one expression in for another, the use of parentheses or other grouping symbols is highly recommended. Also, the dependent variable wasn't specified so we use the default y in each case.

1. To find the zeros of f, we solve $f(x) = 0$ or $|x - 3| = 0$. We get $x = 3$ so the sole x-intercept of the graph of f is $(3, 0)$. To find the y-intercept, we compute $f(0) = |0 - 3| = 3$ and obtain $(0, 3)$. Using Definition 4.2 to rewrite the expression for $f(x)$ means that we substitute the expression $x - 3$ in for x and simplify. Note that when substituting the $x - 3$ in for x, we do so for *every* instance of x – both in the formula (output) as well as the inequality (input).

$$f(x) = |x-3| = \begin{cases} -(x-3) & \text{if } (x-3) < 0 \\ (x-3) & \text{if } (x-3) \geq 0 \end{cases} \longrightarrow f(x) = \begin{cases} -x+3 & \text{if } x < 3 \\ x-3 & \text{if } x \geq 3 \end{cases}$$

As both pieces of the graph of f are lines, we need just two points for each piece. We already have two points for the graph: $(0, 3)$ and $(3, 0)$. These two points both lie on the line $y = -x + 3$ but the strictness of the inequality means $f(x) = -x + 3$ only for $x < 3$, <u>not</u> $x = 3$, so we would have a hole at $(3, 0)$ instead of a point there. For $x \geq 3$, $f(x) = x - 3$, so the hole we thought we had at $(3, 0)$ gets plugged because $f(3) = 3 - 3 = 0$. We need just one more point for $f(x)$ where $x \geq 3$ and choose somewhat arbitrarily $x = 6$. We find $f(6) = |6 - 3| = 3$ so our final point on the graph is $(6, 3)$. Now that we have a complete graph,[2] we see that the vertex is $(3, 0)$ and the range is $[0, \infty)$.

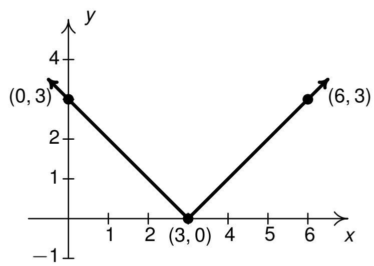
[2]We know it's complete because we did the Math - no trusting technology on this example!

The minimum of f is 0 when $x = 3$ and f has no maximum. Also, f is decreasing over $(-\infty, 3]$ and increasing on $[3, \infty)$. The graph is given below on the left.

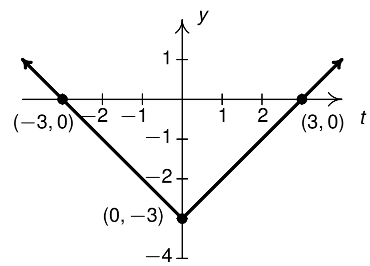
2. To find the zeros of g, we solve $g(t) = |t| - 3 = 0$ and get $|t| = 3$ or $t = \pm 3$. Hence, the t-intercepts of the graph of g are $(-3, 0)$ and $(3, 0)$. To find the y-intercept, we compute $g(0) = |0| - 3 = -3$ and get $(0, -3)$. To rewrite $g(t)$ has a piecewise defined function, we first substitute t in for x in Definition 4.2 to get a piecewise definition of $|t|$. This breaks the domain into two pieces: $t < 0$ and $t \geq 0$. For $t < 0$, $|t| = -t$, so $g(t) = |t| - 3 = (-t) - 3 = -t - 3$. Likewise, for $t \geq 0$, $|t| = t$ so $g(t) = |t| - 3 = t - 3$.

$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \quad \longrightarrow \quad g(t) = |t| - 3 = \begin{cases} -t - 3 & \text{if } t < 0 \\ t - 3 & \text{if } t \geq 0 \end{cases}$$

Once again, we have two lines to graph, but in this case we have three points: $(-3, 0)$, $(0, -3)$ and $(3, 0)$. Both $(-3, 0)$ and $(0, -3)$ lie on $y = -t - 3$, but $g(t) = -t - 3$ only for $t < 0$. This would yield a hole at $(0, -3)$, but, just like in the previous example, the hole is plugged thanks to the second piece of the function because $g(0) = 0 - 3 = -3$. We also pick up the second t-intercept, $(3, 0)$ and this helps us complete our graph. We see that the vertex is $(0, -3)$ and the range is $[-3, \infty)$. The minimum of g is -3 at $t = 0$ and there is no maximum. Also, g is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. The graph of g is shown below on the right.



$f(x) = |x - 3|$



$g(t) = |t| - 3$

3. Solving $h(u) = |2u - 1| - 3 = 0$ gives $|2u - 1| = 3$ or $2u - 1 = \pm 3$. We get two zeros: $u = -1$ and $u = 2$ which correspond to two u-intercepts: $(-1, 0)$ and $(2, 0)$. We find $h(0) = |2(0) - 1| - 3 = -2$ so our y-intercept is $(0, -2)$. To rewrite $h(u)$ as a piecewise defined function, we first rewrite $|2u - 1|$ as a piecewise function. Substituting the expression $2u - 1$ in for x in Definition 4.2 gives:

$$|2u - 1| = \begin{cases} -(2u - 1) & \text{if } 2u - 1 < 0 \\ 2u - 1 & \text{if } 2u - 1 \geq 0 \end{cases} \quad \longrightarrow \quad |2u - 1| = \begin{cases} -2u + 1 & \text{if } u < \frac{1}{2} \\ 2u - 1 & \text{if } u \geq \frac{1}{2} \end{cases}$$

Hence, for $u < \frac{1}{2}$, $|2u - 1| = -2u + 1$ so $h(u) = |2u - 1| - 3 = (-2u + 1) - 3 = -2u - 2$. Likewise, for $u \geq \frac{1}{2}$, $|2u - 1| = 2u - 1$ so $h(u) = |2u - 1| - 3 = (2u - 1) - 3 = 2u - 4$.

$$h(u) = |2u - 1| - 3 = \begin{cases} (-2u + 1) - 3 & \text{if } u < \frac{1}{2} \\ (2u - 1) - 3 & \text{if } u \geq \frac{1}{2} \end{cases} \quad \longrightarrow \quad h(u) = \begin{cases} -2u - 2 & \text{if } u < \frac{1}{2} \\ 2u - 4 & \text{if } u \geq \frac{1}{2} \end{cases}$$

We have three points to help us graph $y = h(u)$: $(-1, 0)$, $(0, -2)$ and $(2, 0)$. Unlike in the last two examples, these points do not give us information at the value $u = \frac{1}{2}$ where the rule for $h(u)$ changes. Substituting $u = \frac{1}{2}$ into the expression $-2u - 2$ gives -3, so from $h(u) = -2u - 2$, $u < \frac{1}{2}$, we get a hole at $\left(\frac{1}{2}, -3\right)$. However, this hole is filled because $h\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - 4 = -3$ and this produces the vertex at $\left(\frac{1}{2}, -3\right)$. The range of h is $[-3, \infty)$, with the minimum of h being -3 at $t = \frac{1}{2}$. Moreover, h is decreasing on $\left(-\infty, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, \infty\right)$. The graph of h is given below on the left.

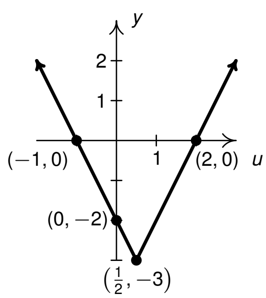
4. Solving $i(w) = 4 - 2|3w - 1| = 0$ yields $|3w - 1| = 2$ or $3w - 1 = \pm 2$. This gives two zeros, $w = -\frac{1}{3}$ and $w = 1$, which correspond to two w-intercepts, $\left(-\frac{1}{3}, 0\right)$ and $(1, 0)$. Also, $i(0) = 4 - 2|3(0) - 1| = 2$, so the y-intercept of the graph is $(0, 2)$. As in the previous example, the first step in rewriting $i(w)$ as a piecewise defined function is to rewrite $|3w - 1|$ as a piecewise function. Once again, we substitute the expression $3w - 1$ in for every occurrence of x in Definition 4.2:

$$|3w - 1| = \begin{cases} -(3w - 1) & \text{if } 3w - 1 < 0 \\ 3w - 1 & \text{if } 3w - 1 \geq 0 \end{cases} \longrightarrow |3w - 1| = \begin{cases} -3w + 1 & \text{if } w < \frac{1}{3} \\ 3w - 1 & \text{if } w \geq \frac{1}{3} \end{cases}$$

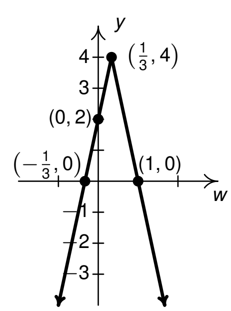
Thus for $w < \frac{1}{3}$, $|3w - 1| = -3w + 1$, so $i(w) = 4 - 2|3w - 1| = 4 - 2(-3w + 1) = 6w + 2$. Likewise, for $w \geq \frac{1}{3}$, $|3w - 1| = 3w - 1$ so $i(w) = 4 - 2|3w - 1| = 4 - 2(3w - 1) = -6w + 6$.

$$i(w) = 4 - 2|3w - 1| = \begin{cases} 4 - 2(-3w + 1) & \text{if } w < \frac{1}{3} \\ 4 - 2(3w - 1) & \text{if } w \geq \frac{1}{3} \end{cases} \longrightarrow i(w) = \begin{cases} 6w + 2 & \text{if } w < \frac{1}{3} \\ -6w + 6 & \text{if } w \geq \frac{1}{3} \end{cases}$$

As with the previous example, we have three points on the graph of i: $\left(-\frac{1}{3}, 0\right)$, $(0, 2)$ and $(1, 0)$, but no information about happens at $w = \frac{1}{3}$. Substituting this value of w into the formula $6w + 2$ would produce a hole at $\left(\frac{1}{3}, 4\right)$. As we've seen several times already, however, $i\left(\frac{1}{3}\right) = 4$ so we don't have a hole at $\left(\frac{1}{3}, 4\right)$ but, rather, the vertex. From the graph we see that the range of i is $(-\infty, 4]$ with the maximum of i being 4 when $w = \frac{1}{3}$. Also, i is increasing over $\left(-\infty, \frac{1}{3}\right]$ and decreasing on $\left[\frac{1}{3}, \infty\right)$. Its graph is given below on the right.



$h(u) = |2u - 1| - 3$



$i(w) = 4 - 2|3w - 1|$ □

As we take a step back and look at the graphs produced in Example 4.2.1, some patterns begin to emerge. Indeed, each of the graphs has the common '∨' shape (in the case of the function i it's a '∧') with the vertex located at the x-value where the rule for each function changes from one formula to the other. It turns out that, independent variable labels aside, each and every function in Example 4.2.1 can be rewritten in the form $F(x) = a|x - h| + k$ for real number parameters a, h and k.

Each of the functions from Example 4.2.1 is rewritten in this form below and we record the vertex along with the slopes of the lines in the graph.

- $f(x) = |x-3| = (1)|x-3| + 0$: $a = 1$, $h = 3$, $k = 0$;   vertex $(3, 0)$;   slopes ± 1
- $g(t) = |t| - 3 = (1)|t-0| + (-3)$: $a = 1$, $h = 0$, $k = -3$;   vertex $(0, -3)$;   slopes ± 1
- $h(u) = |2u-1| - 3 = 2\left|u - \frac{1}{2}\right| + (-3)$: $a = 2$, $h = \frac{1}{2}$, $k = -3$;   vertex $\left(\frac{1}{2}, -3\right)$;   slopes ± 2
- $i(w) = 4 - 2|3w-1| = -6\left|w - \frac{1}{3}\right| + 4$: $a = -6$, $h = \frac{1}{3}$, $k = 4$;   vertex $\left(\frac{1}{3}, 4\right)$;   slopes ± 6

These specific examples suggest the following theorem.

Theorem 4.4. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a|x-h| + k$ consists of parts of two lines with slopes $\pm a$ which meet at a vertex (h, k). If $a > 0$, the shape resembles '$\vee$'. If $a < 0$, the shape resembles '$\wedge$'. Moreover, the graph is symmetric about the line $x = h$.

Proof. What separates Mathematics from the other sciences is its ability to actually *prove* patterns like the one stated in the theorem above as opposed to just *verifying* it by working more examples. The proof of Theorem 4.4 uses the exact same concepts as were used in Example 4.2.1, just in a more general context by which we mean using letters as parameters instead of numbers.

The first step is to rewrite $|x-h|$ as a piecewise function.

$$|x-h| = \begin{cases} -(x-h) & \text{if } x-h < 0 \\ x-h & \text{if } x-h \geq 0 \end{cases} \quad \longrightarrow \quad |x-h| = \begin{cases} -x+h & \text{if } x < h \\ x-h & \text{if } x \geq h \end{cases}$$

We plug that work into $F(x)$ to rewrite it as a piecewise function. For $x < h$, we have $|x-h| = -x+h$, so

$$F(x) = a|x-h| + k = a(-x+h) + k = -ax + ah + k = -ax + (ah+k)$$

Similarly, for $x \geq h$, we have $|x-h| = x-h$, so

$$F(x) = a|x-h| + k = a(x-h) + k = ax - ah + k = ax + (-ah+k)$$

Hence,

$$F(x) = a|x-h| + k = \begin{cases} a(-x+h)+k & \text{if } x < h \\ a(x-h)+k & \text{if } x \geq h \end{cases} \quad \longrightarrow \quad F(x) = \begin{cases} -ax + (ah+k) & \text{if } x < h \\ ax + (-ah+k) & \text{if } x \geq h \end{cases}$$

All three parameters, a, h and k, are fixed (but arbitrary) real numbers. Thus, for any given choice of a, h and k the numbers $ah+k$ and $-ah+k$ are also just numbers as opposed to variables. This shows that the graph of F is comprised of pieces of two lines, $y = -ax + (ah+k)$ and $y = ax + (-ah+k)$, the former with slope $-a$ and the latter with slope a. Note that substituting $x = h$ into $y = -ax + (ah+k)$ produces $y = -ah + (ah+k) = k$ and substituting $x = h$ into $y = ax + (-ah+k)$ also produces $y = ah + (-ah+k) = k$. This tells us that the two linear pieces meet at the point (h, k).

If $a > 0$ then $-a < 0$ so the line $y = -ax + (ah + k)$, hence F, is decreasing on $(-\infty, h]$. Similarly, the line $y = ax + (-ah + k)$, hence F, is increasing on $[h, \infty)$. This produces a '$\vee$' shape. On the other hand, if $a < 0$ then $-a > 0$ which produces a '$\wedge$' shape because F is increasing on $(-\infty, h]$ followed by decreasing on $[h, \infty)$. (Said another way, $-a > 0$ means that the first linear piece has a positive slope and $a < 0$ means that the second piece has a negative slope.)

To show that the graph is symmetric about the line $x = h$, we need to show that if we move left or right the same distance away from $x = h$, then we get the same y-value on the graph. Suppose we move Δx to the right or left of h. The y-values are the function values so we need to show that $F(a + \Delta x) = F(a - \Delta x)$. Given that

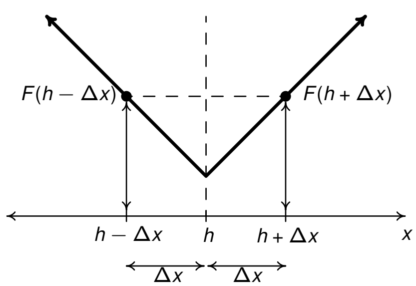
$$F(a + \Delta x) = a|a + \Delta x - a| + k = a|\Delta x| + k$$

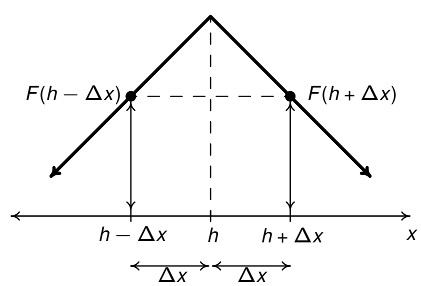
and

$$F(a - \Delta x) = a|a - \Delta x - a| + k = a|-\Delta x| + k = a|\Delta x| + k$$

we see that $F(a + \Delta x) = F(a - \Delta x)$. Thus we have shown that the y-values on the graph on either side of $x = h$ are equal provided we move the same distance away from $x = a$. This completes the proof. $\square$

The line $x = a$ in Theorem 4.4 is called the **axis of symmetry** of the graph of $y = F(x)$. This language is consistent with the basics of symmetry discussed in Section **??** and we will build upon our work here in several upcoming sections. For now, we simply present two graphs illustrating the concept of the axis of symmetry below.





While Theorem 4.4 and its proof are specific to the particular family of absolute value functions, there are ideas here that apply to all functions. Thus we wish to take a slight detour away from the main narrative to argue this result again from an even more generalized viewpoint. Our goal is to 'build' the formula $F(x) = a|x - h| + k$ from $f(x) = |x|$ in three stages, each corresponding to the role of one of the parameters a, h and k, and track the geometric changes that go along with each stage. We will revisit all of the ideas described below in complete generality in Section 2.3.

The graph of $f(x) = |x|$ consists of the points $\{(c, |c|) \mid c \in \mathbb{R}\}$.[3] Consider $F_1(x) = |x - h|$. The graph of F_1 is the set of points $\{(x, |x - h|) \mid x \in \mathbb{R}\}$. If we relabel $x - h = c$, then $x = c + h$, and as x varies through all of the real numbers, so does c and vice-versa.[4]

[3] See the box on page 55. Also, we use 'c' as our dummy variable to avoid the confusion that would arise by over-using 'x'.

[4] That is, every real number c can be written as $x - h$ for some x, and every real number x can be written as $c + h$ for some c.

Hence, we can write $\{(x, |x-h|) \mid x \in \mathbb{R}\} = \{(c+h, |c|) \mid c \in \mathbb{R}\}$. If we fix a y-coordinate, $|c|$, we see that the corresponding points on the graph of f and F_1, $(c, |c|)$ and $(c+h, |c|)$, respectively, differ only in that one is horizontally shifted by h. In other words, to get the graph of F_1, we simply take the graph of f and shift each point horizontally by adding h to the x-coordinate. Translating the graph in this manner preserves the '$\vee$' shape and symmetry, but moves the vertex from $(0, 0)$ to $(h, 0)$.

Next, we examine $F_2(x) = a|x-h|$ and compare its graph to that of $F_1(x) = |x-h|$. The graph of F_2 consists of the points $\{(x, a|x-h|) \mid x \in \mathbb{R}\}$ whereas the graph of F_1 consists of the points $\{(x, |x-h|) \mid x \in \mathbb{R}\}$. The only difference between the points $(x, |x-h|)$ and $(x, a|x-h|)$ is that the y-coordinate of one is a times the y-coordinate of the other. If $a > 0$, all we are doing is scaling the y-axis by a factor of a. As we've seen when plotting points and graphing functions, the scaling of the y-axis affects only the relative vertical displacement of points[5] and not the overall shape.

If $a < 0$, then in addition to scaling the vertical axis, we are reflecting the points across the x-axis.[6] Such a transformation doesn't change the '$\vee$' shape except for flipping it upside-down to make it a '$\wedge$'. In either case, the vertex $(h, 0)$ stays put at $(h, 0)$ because the y-value of the vertex is 0 and $a \cdot 0 = 0$ regardless if $a > 0$ or $a < 0$.

Last, we examine the graph of $F(x) = a|x-h|+k$ to see how it relates to the graph of $F_2(x) = a|x-h|$. The graph of F consists of the points $\{(x, a|x-h|+k) \mid x \in \mathbb{R}\}$ whereas the graph of F_2 consists of the points $\{(x, a|x-h|) \mid x \in \mathbb{R}\}$. The difference between the corresponding points $(x, a|x-h|)$ and $(x, a|x-h|+k)$ is the addition of k in the y-coordinate of the latter. Adding k to each of the y-values translates the graph of F_2 vertically by k units. The basic shape doesn't change but the vertex goes from $(h, 0)$ to (h, k).

In summary, the graph of $F(x) = a|x-h|+k$ can be obtained from the graph of $f(x) = |x|$ in three steps: first, add h to each of the x-coordinates; second, multiply each y-coordinate by a; and third, add k to each y-coordinate. Geometrically, these steps mean that we first move the graph left or right, then scale the y-axis by a factor of a (and reflect across the x-axis if $a < 0$), and then move the graph up or down. Throughout all of these *transformations*, the graph maintains its '$\vee$' or '$\wedge$' shape.

Of course, not every function involving absolute values can be written in the form given in Theorem 4.4. A good example of this is $G(x) = |x-2| - x$. However recognizing the ones that can be rewritten will greatly simplify the graphing process. In the next example, we graph four more absolute value functions, two using Theorem 4.4 and two using Definition 4.2.

Example 4.2.2.

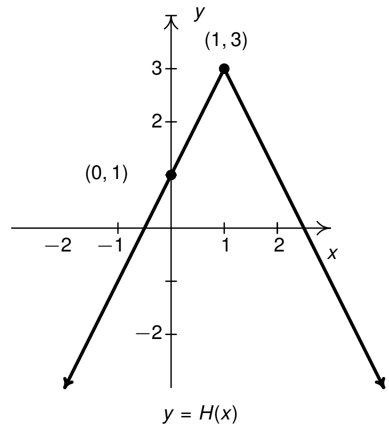
1. Graph each of the functions below using Theorem 4.4 or by rewriting it as a piecewise defined function using Definition 4.2. Find the zeros, axis-intercepts and the extrema (if any exist) and then list the intervals over which the function is increasing, decreasing or constant.

   (a) $F(x) = |x+3| + 2$ (b) $f(t) = \dfrac{4 - |5 - 3t|}{2}$ (c) $G(x) = |x-2| - x$ (d) $g(t) = |t-2| - |t|$

[5] See the discussion following Example 2.1.1 regarding the plot of Skippy's data.

[6] See the box on page **??** in Section **??**.

2. Use Theorem 4.4 to write a possible formula for $H(x)$ whose graph is given below:



Solution.

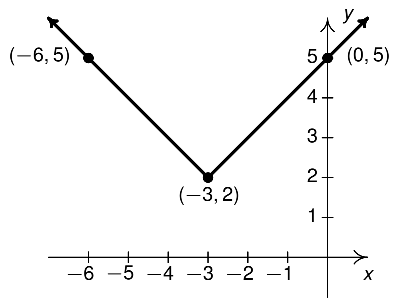
1. (a) Rewriting $F(x) = |x+3|+2 = (1)|x-(-3)|+2$, we have $F(x)$ in the form stated in Theorem 4.4 with $a = 1$, $h = -3$ and $k = 2$. The vertex is $(-3, 2)$ and the graph will be a '$\vee$' shape. Seeing as the vertex is already above the x-axis and the graph opens upwards, there are no x-intercepts on the graph of F, hence there are no zeros.[7] With $F(0) = 5$, the y-intercept is $(0, 5)$. To get a third point, we can pick an arbitrary x-value to the left of the vertex or we could use symmetry: three units to the *right* of the vertex the y-value is 5, so the same must be true three units to the *left* of the vertex, at $x = -6$. Sure enough, $F(-6) = |-6+3|+2 = |-3|+2 = 5$. The range of F is $[2, \infty)$ with its minimum of 2 when $x = -3$ and F decreasing on $(-\infty, -3]$ then increasing on $[-3, \infty)$. The graph is in the middle of the next page on the left.

   (b) We see in the formula for $f(t)$ that t appears only once to the first power inside the absolute values, so we proceed to rewrite it in the form $a|t-h|+k$:

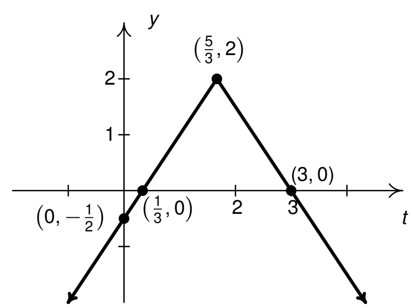
$$
\begin{aligned}
f(x) &= \frac{4-|5-3t|}{2} \\
&= -\frac{|5-3t|}{2} + \frac{4}{2} \\
&= \left(-\frac{1}{2}\right)\left|(-3)\left(t-\frac{5}{3}\right)\right| + 2 \\
&= \left(-\frac{1}{2}\right)|-3|\left|t-\frac{5}{3}\right| + 2 \\
&= -\frac{3}{2}\left|t-\frac{5}{3}\right| + 2.
\end{aligned}
$$

[7] Alternatively, setting $|x+3|+2=0$ gives $|x+3| = -2$. Absolute values are never negative thus we have no solution.

Matching up the constants in the formula $f(t)$ to the parameters of $F(x)$ in Theorem 4.4, we identify $a = -\frac{3}{2}$, $h = \frac{5}{3}$ and $k = 2$. Hence the vertex is $\left(\frac{5}{3}, 2\right)$, and the graph is shaped like '$\wedge$' comprised of pieces of lines with slopes $\pm\frac{3}{2}$. To find the zeros of f, we set $f(t) = 0$. (We can use either expression here.) Solving $-\frac{3}{2}\left|t - \frac{5}{3}\right| + 2 = 0$, we get $\left|t - \frac{5}{3}\right| = \frac{4}{3}$, so $t - \frac{5}{3} = \pm\frac{4}{3}$. Hence our zeros are $t = \frac{1}{3}$ and $t = 3$, producing the t-intercepts $\left(\frac{1}{3}, 0\right)$ and $(3, 0)$. Using either formula gives $f(0) = -\frac{1}{2}$, so our y-intercept is $\left(0, -\frac{1}{2}\right)$. Plotting the vertex, along with the intercepts, gives us enough information to produce the graph below on the right. The range is $(-\infty, 2]$ with a maximum of 2 at $t = \frac{5}{3}$ and f is increasing on $\left(-\infty, \frac{5}{3}\right]$ then decreasing on $\left[\frac{5}{3}, \infty\right)$.



$F(x) = |x + 3| + 2$



$f(t) = \dfrac{4 - |5 - 3t|}{2}$

(c) We are unable to apply Theorem 4.4 to $G(x) = |x - 2| - x$ because there is an x both inside and outside of the absolute value. We can, however, rewrite the function as a piecewise function using Definition 4.2. Our first step is to rewrite $|x - 2|$ as a piecewise function:

$$|x-2| = \begin{cases} -(x-2) & \text{if } x - 2 < 0 \\ x - 2 & \text{if } x - 2 \geq 0 \end{cases} \longrightarrow |x-2| = \begin{cases} -x+2 & \text{if } x < 2 \\ x - 2 & \text{if } x \geq 2 \end{cases}$$

Hence, for $x < 2$, $|x - 2| = -x + 2$ so $G(x) = |x - 2| - x = (-x + 2) - x = -2x + 2$. Likewise, for $x \geq 2$, $|x - 2| = x - 2$ so $G(x) = |x - 2| - x = x - 2 - x = -2$.

$$G(x) = |x-2| - x = \begin{cases} (-x+2) - x & \text{if } x < 2 \\ (x-2) - x & \text{if } x \geq 2 \end{cases} \longrightarrow G(x) = \begin{cases} -2x+2 & \text{if } x < 2 \\ -2 & \text{if } x \geq 2 \end{cases}$$

To find the zeros of G, we set $G(x) = 0$. Solving $|x - 2| - x = 0$ can be problematic, given that x is both inside and outside of the absolute values.[8] We can, however, use the piecewise description of $G(x)$. With $G(x) = -2x + 2$ for $x < 2$, we solve $-2x + 2 = 0$ to get $x = 1$. This works because $1 < 2$, so we have $x = 1$ as the zero of G corresponding to the x-intercept $(1, 0)$. The other piece of $G(x)$ is $G(x) = -2$ which is never 0. For the y-intercept, we find $G(0) = 2$, and get $(0, 2)$.

To graph $y = G(x)$, we have the line $y = -2x + 2$ which contains $(0, 2)$ and $(1, 0)$ and continues to a hole at $(2, -2)$. At this point, $G(x) = -2$ takes over and we have a horizontal line containing

[8]We'll return to this momentarily.

$(2, -2)$ extending indefinitely to the right. The range of G is $[-2, \infty)$ with a minimum value of -2 attained for all $x \geq 2$. Moreover, G is decreasing on $(-\infty, 2]$ and then constant on $[2, \infty)$. The graph is below on the left.

(d) Once again we are unable to use Theorem 4.4 because $g(t) = |t - 2| - |t|$ has two absolute values with no apparent way to combine them. Thus we proceed by re-writing the function g with two separate applications of Definition 4.2 to remove each instance of the absolute values. To start with we have:

$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \quad \text{and} \quad |t - 2| = \begin{cases} -t + 2 & \text{if } t < 2 \\ t - 2 & \text{if } t \geq 2 \end{cases}$$

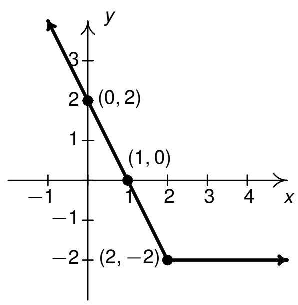
Taken together, these break the domain into *three* pieces: $t < 0$, $0 \leq t < 2$ and $t \geq 2$. For $t < 0$, $|t| = -t$ and $|t-2| = -t+2$. Therefore $g(t) = |t-2| - |t| = (-t+2) - (-t) = 2$ for $t < 0$. For $0 \leq t < 2$, $|t| = t$ and $|t - 2| = -t + 2$, so $g(t) = |t - 2| - |t| = (-t + 2) - t) = -2t + 2$.

Last, for $t \geq 2$, $|t| = t$ and $|t - 2| = t - 2$, so $g(t) = |t - 2| - |t| = (t - 2) - (t) = -2$. Putting all three parts together yields:

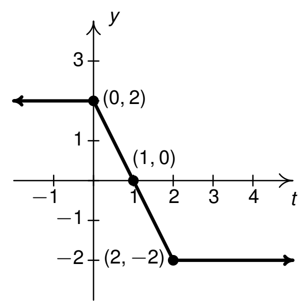
$$g(t) = |t - 2| - |t| = \begin{cases} (-t + 2) - (-t) & \text{if } t < 0 \\ (-t + 2) - (t) & \text{if } 0 \leq t < 2 \\ (t - 2) - (t) & \text{if } t \geq 2 \end{cases} = \begin{cases} 2 & \text{if } t < 0 \\ -2t + 2 & \text{if } 0 \leq t < 2 \\ -2 & \text{if } t \geq 2 \end{cases}$$

As with the previous example, we'll delay discussing the absolute value algebra needed to find the zeros of g and use the piecewise description instead. To graph g, we have the horizontal line $y = 2$ up to, but not including, the point $(0, 2)$. For $0 \leq t < 2$, we have the line $y = -2t + 2$ which has a y-intercept at $(0, 2)$ (thus picking up where the first part left off) and a t-intercept at $(1, 0)$. This piece ends with a hole at $(2, -2)$ which is promptly plugged by the horizontal line $y = -2$ for $t \geq 2$. Hence the only zero of t is $t = 1$.

The range of g is $[-2, 2]$ with a minimum of -2 achieved for all $t \geq 2$, and a maximum of 2 for $t \leq 0$. We note that g is constant on $(-\infty, 0]$ and $[2, \infty)$, but with different values, and g is decreasing on $[0, 2]$. The graph is given below on the right.



$G(x) = |x - 2| - x$



$g(t) = |t - 2| - |t|$

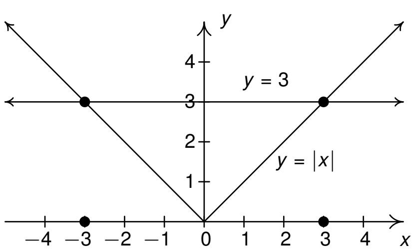
2. We are told to use Theorem 4.4 to find a formula for $H(x)$ so we start with $H(x) = a|x-h|+k$ and look for real numbers a, h and k that make sense. The vertex is labeled as $(1,3)$, meaning $h=1$ and $k=3$. Hence we know $H(x) = a|x-1|+3$, so all that is left for us to find is the value of a. The only other point labeled for us is $(0,1)$, meaning $H(0)=1$. Substituting $x=0$ into our formula for $H(x)$ gives: $H(0) = a|0-1|+3 = a+3$. Given that $H(0)=1$, we have $a+3=1$, so $a=-2$. Our final answer is $H(x) = -2|x-1|+3$. □

If nothing else, Example 4.2.2 demonstrates the value of *changing forms* of functions and the utility of the interplay between algebraic and graphical descriptions of functions. These themes resonate time and time again in this and later courses in Mathematics.

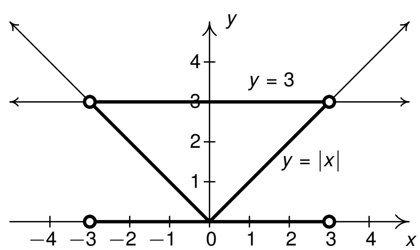
4.2.2 Graphical Solution Techniques for Equations and Inequalities

Consider the basic equation and related inequalities: $|x| = 3$, $|x| < 3$ and $|x| > 3$. At some point you learned how to solve these using properties of the absolute value inspired by the distance definition. (If not, see Section 4.1.) While there is nothing wrong with this understanding, we wish to use these problems to motivate powerful graphical techniques which we'll use to solve more complicated equations and inequalities in this section, and in many other sections of the textbook.

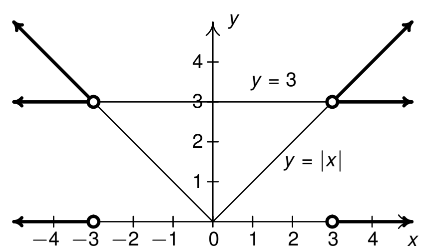
To that end, let's call $f(x) = |x|$ and $g(x) = 3$. If we graph $y = f(x)$ and $y = g(x)$ on the same set of axes then, by looking for x values where $f(x) = g(x)$, we are looking for x-values which have the same y-value on both graphs. That is, the solutions to $f(x) = g(x)$ are the x-coordinates of the *intersection points* of the two graphs. We graph $y = f(x) = |x|$ (the characteristic '$\vee$') along with $y = g(x) = 3$ (the horizontal line) below on the far left. Indeed, the two graphs intersect at $(-3,3)$ and $(3,3)$ so our solutions to $f(x) = g(x)$ are the x-values of these points, $x = \pm 3$.



Graphically solving $|x| = 3$.



Graphically solving $|x| < 3$.



Graphically solving $|x| > 3$.

Likewise, if we wish to solve $|x| < 3$, we can view this as a functional inequality $f(x) < g(x)$ which means we are looking for the x-values where the $f(x)$ values are less than the corresponding $g(x)$ values. On the graphs, this means we'd be looking for the x-values where the y-values of $y = f(x)$ are less than, hence *below*, those on the graph of $y = g(x)$.

In the middle picture above we see that the graph of f is below the graph of g between $x = -3$ and $x = 3$, so our solution is $-3 < x < 3$, or in interval notation, $(-3,3)$. Finally, the inequality $|x| > 3$ is equivalent to $f(x) > g(x)$ so we are looking for the x-values where the graph of f is *above* the graph of g.[9] The picture

[9]Solving $f(x) > g(x)$ is equivalent to solving $g(x) < f(x)$ - that is, finding where the graph of g is below the graph of f.

on the far right on the previous page shows that this is true for all $x < -3$ or for all $x > 3$. In interval notation, the solution set is $(-\infty, -3) \cup (3, \infty)$.

The methodology and reasoning behind solving the above equation and inequalities extend to any pair of functions f and g, since when graphed on the same set of axes, function outputs are always the dependent variable or the ordinate (second coordinate) of the ordered pairs which comprise the graph. In general:

Graphical Interpretation of Equations and Inequalities

Suppose f and g are functions whose domains and ranges are sets of real numbers.

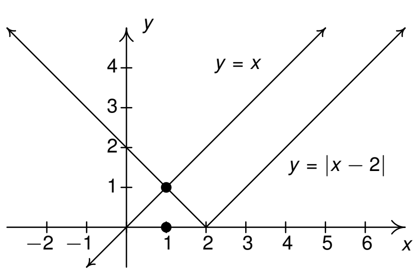
- The solutions to $f(x) = g(x)$ are the x-values where the graphs of f and g intersect.
- The solution to $f(x) < g(x)$ is the set of x-values where the graph of f is *below* the graph of g.
- The solution to $f(x) > g(x)$ is the set of x-values where the graph of f *above* the graph of g.

Let's return to Example 4.2.2 where were asked to find the zeros of the functions $G(x) = |x - 2| - x$ and $g(t) = |t - 2| - |t|$. In that Example, instead of tackling the algebra involving the absolute values head on we rewrote each function as a piecewise-defined function and obtained our solutions that way.

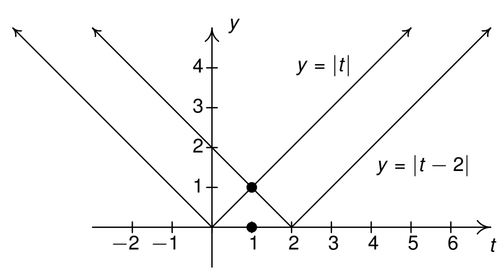
Let's see what this looks like graphically. Note that solving $|x-2|-x = 0$ is equivalent to solving $|x-2| = x$. We graphed $y = |x - 2|$ and $y = x$ on the same set of axes on the left of the top of the next page and it appears as if we have just one point of intersection, corresponding to just one solution.

Indeed, we can *show* that there is just one point of intersection. The graph of $y = |x - 2|$ is comprised of parts of two lines, $y = -(x - 2)$ and $y = x - 2$. The first line has a slope of -1 and the second has slope 1. The line $y = x$ also has a slope 1 meaning it and the 'right half' of $y = |x - 2|$ are parallel, so they never intersect. If our graphs are accurate enough, we may even be able to guess that the solution is $x = 1$, which we can verify by substituting $x = 1$ into $|x - 2| = x$ and seeing that it checks.

Likewise, solving $|t - 2| - |t| = 0$ is equivalent to solving $|t - 2| = |t|$. We graphed $y = |t - 2|$ and $y = |t|$ on the right at the top of the next page and used the same arguments to get the solution $t = 1$ here as well.



Graphically solving $|x - 2| = x$.



Graphically solving $|t - 2| = |t|$.

There is more to see here. Consider solving $|x - 2| - x = 0$ algebraically using the techniques from a previous Algebra course (or Section 4.1). Our first step would be to isolate the absolute value quantity: $|x - 2| = x$. We then 'drop' the absolute value by paying the price of a '$\pm$': $x - 2 = \pm x$. This gives us two equations: $x-2=x$ and $x-2=-x$. The first equation, $x-2=x$ reduces to $-2=0$ which has no solution. The second equation, $x-2=-x$, does have a solution, namely $x=1$.

How does the algebra tie into the graphs above? Instead of 'dropping' the absolute value and tagging the right hand side with a $\pm$, we can think about the piecewise definition of $|x-2|$ and write $|x-2|=\pm(x-2)$ depending on if $x<2$ or if $x\geq 2$. That is, $|x-2|=x$ is more precisely equivalent to the two equations: $-(x-2)=x$ which is valid for $x<2$ or $x-2=x$ which is valid for $x\geq 2$.

Graphically, the first equation is looking for intersection points between the 'left half' of the '$\vee$' of $y=|x-2|$ and the line $y=x$. Indeed, $-(x-2)=x$ is equivalent to $x-2=-x$ from which we obtain our solution $x=1$. Likewise, the second equation, $x-2=x$ is looking for intersection points of the 'right half' of the '$\vee$' and the line $y=x$, but there is none. The equation $-2=0$ is telling us that for us to have any solutions, the lines $y=x-2$ and $y=x$, which have the same slope, must also have the same y-intercepts: that is, -2 would have to equal 0 and that's just silly.

Similarly, when solving $|t-2|-|t|=0$ or $|t-2|=|t|$, we can use our graphs to prove that the only intersection point is when the 'left half' of $y=|t-2|$ intersects the 'right half' of $y=|t|$ - that is, when $-(t-2)=t$. The moral of the story is this: careful graphs can help us simplify the algebra, because we can narrow down the cases. This is especially useful in solving inequalities, as we'll see in our next example.

Example 4.2.3. Solve the following equations and inequalities.

1. $4-|x|=0.9x-3.6$
2. $|t-3|-|t|=3$
3. $|x+1|\geq \dfrac{x+4}{2}$
4. $2<|t-1|\leq 5$

Solution.

1. We begin by graphing $y=4-|x|$ and $y=0.9x-3.6$ to look for intersection points. Using Theorem 4.4, we know that the graph of $y=4-|x|=-|x|+4$ has a vertex at $(0,4)$ and is a '$\wedge$' shape, so there are x-intercepts to find. Solving $4-|x|=0$, we get $|x|=4$, or $x=\pm 4$. Hence, we have two x-intercepts: $(-4,0)$ and $(4,0)$.

   We know from Section 3.1.2 that the graph of $y=0.9x-3.6$ is a line with slope 0.9 and y-intercept $(0,-3.6)$. To find the x-intercept here we solve $0.9x-3.6=0$ and get $x=4$. Hence, $(4,0)$ is an x-intercept here as well, and we have stumbled upon one solution to $4-|x|=0.9x-3.6$, namely $x=4$. The question is if there are any other solutions. Our graph (below on the left) certainly looks as if there is just one intersection point, but we know from Theorem 4.4 that the slopes of the linear parts of $y=4-|x|$ are ± 1. The slope of $y=0.9x-3.6$ is 0.9 and $0.9\neq 1$ so we know that the left hand side of the'$\wedge$' must meet up with the graph of the line because they are not parallel.[10]

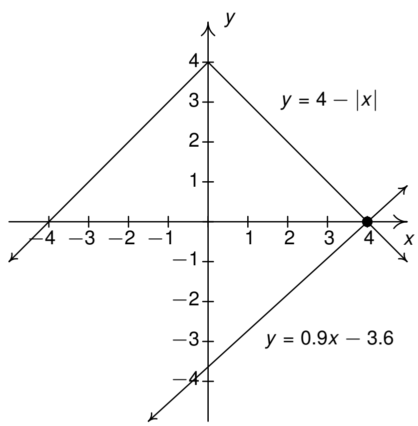
   Definition 4.2 tells us that when $x<0$, $|x|=-x$, so $4-|x|=4-(-x)=4+x$. Hence we set about solving $4+x=0.9x-3.6$ and get $x=-76$. Both $x=-76$ and $x=4$ check in our original equation, $4-|x|=0.9x-3.6$, so we have found our two solutions.[11]

[10] See Theorem 3.1.

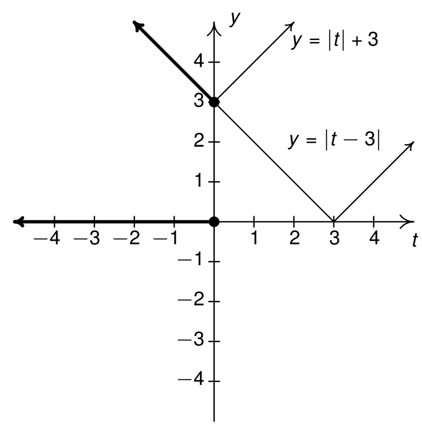
[11] Our picture shows only one of the solutions. We encourage you to take the time with a graphing utility to get the picture to show both points of intersection.

2. While we could graph $y = |t-3| - |t|$ and $y = 3$ to help us find solutions, we choose to rewrite the equation as $|t-3| = |t| + 3$. This way, we have somewhat easier graphs to deal with, namely $y = |t-3|$ and $y = |t| + 3$. The first graph, $y = |t-3|$, has a vertex at $(3, 0)$ and is shaped like a '$\vee$' with slopes ± 1 and a y-intercept of $(0, 3)$. The second graph, $y = |t| + 3$, has a vertex at $(0, 3)$ and is also shaped like a '$\vee$,' with slopes ± 1, and has no t-intercepts.

   To our surprise and delight, the graphs (below on the right) appear to overlap for $t \leq 0$. Indeed, for $t \leq 0$, $|t-3| = -(t-3) = -t+3$ and $|t| + 3 = -t+3$. Since the formulas are *identical* for these values of t, our solutions are all values of t with $t \leq 0$. Using interval notation, we state our solution as $(-\infty, 0]$. (The other parts of the graphs are non-intersecting parallel lines so we ignored them.)



Solving $4 - |x| = 0.9x - 3.6$.



Solving $|t-3| - |t| = 3$.

3. To solve $|x+1| \geq \frac{x+4}{2}$, we first graph $y = |x+1|$ and $y = \frac{x+4}{2} = \frac{1}{2}x + 2$. The former is '$\vee$' shaped with a vertex at $(-1, 0)$ and a y-intercept of $(0, 1)$. The latter is a line with y-intercept $(0, 2)$, slope $m = \frac{1}{2}$ and x-intercept $(-4, 0)$. The picture in the middle of the next page on the right shows two intersection points. To find these, we solve the equations: $-(x+1) = \frac{x+4}{2}$, obtaining $x = -2$, and $x + 1 = \frac{x+4}{2}$ obtaining $x = 2$.

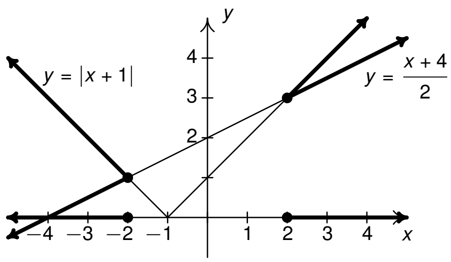
   Graphically, the inequality $|x+1| \geq \frac{x+4}{2}$ is looking for where the graph of $y = |x+1|$, the '$\vee$,' intersects $(=)$ or is above $(>)$ the line $y = \frac{x+4}{2}$. The graph shows this happening whenever $x \leq -2$ or $x \geq 2$. Using interval notation, our solution is $(-\infty, -2] \cup [2, \infty)$. While we cannot check every single x value individually, choosing test values $x < -2$, $x = 2$, $-2 < x < 2$, $x = 2$, and $x > 2$ to see if the original inequality $|x+1| \geq \frac{x+4}{2}$ holds would help us verify our solution.

4. Recall that the inequality $2 < |t-1| \leq 5$ is an example of a 'compound' inequality in that is two inequalities in one.[12] The values of t in the solution set need to satisfy $2 < |t-1|$ *and* $|t-1| \leq 5$. To help us sort through the cases, we graph the horizontal lines $y = 2$ and $y = 5$ along with the '$\vee$' shaped $y = |t-1|$ with vertex $(1, 0)$ and y-intercept $(0, 1)$.

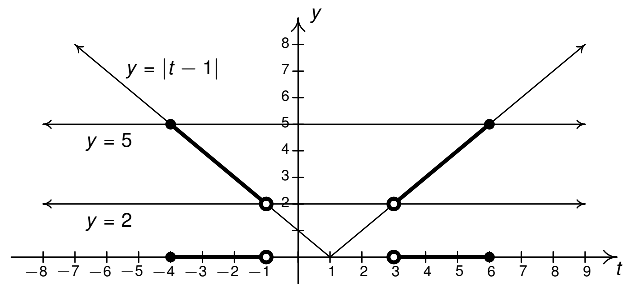
[12] See Example **??** for examples of linear compound inequalities.

Geometrically, we are looking for where $y = |t - 1|$ is strictly *above* the line $y = 2$ but *below* (or meets) the line $y = 5$. Solving $|t - 1| = 2$ gives $t = -1$ and $t = 3$ whereas solving $|t - 1| = 5$ gives $t = -4$ or $t = 6$. Per the graph (below on the right), we see that $y = |t - 1|$ lies between $y = 2$ and $y = 5$ when $-4 \leq t < -1$ and again when $3 < t \leq 6$.

In interval notation, our solution is $[-4, -1) \cup (3, 6]$. As with the previous example, it is impossible to check each and every one of these solutions, but choosing t values both in and around the solution intervals would give us some numerical confidence we have the correct and complete solution.



Solving $|x + 1| \geq \frac{x+4}{2}$



Solving $2 < |t - 1| \leq 5$

□

We will see the interplay of Algebra and Geometry throughout the rest of this course. In the Exercises, do not hesitate to use whatever mix of algebraic and graphical methods you deem necessary to solve the given equation or inequality. Indeed, there is great value in checking your algebraic answers graphically and vice-versa.

One of the classic applications of inequalities involving absolute values is the notion of tolerances.[13] Recall that for real numbers x and c, the quantity $|x - c|$ may be interpreted as the distance from x to c. Solving inequalities of the form $|x - c| \leq d$ for $d > 0$ can then be interpreted as finding all numbers x which lie within d units of c. We can think of the number d as a 'tolerance' and our solutions x as being within an accepted tolerance of c. We use this principle in the next example.

Example 4.2.4. Suppose a manufacturer needs to produce a 24 inch by 24 inch square piece of particle board as part of a home office desk kit. How close does the side of the piece of particle board need to be cut to 24 inches to guarantee that the area of the piece is within a tolerance of 0.25 square inches of the target area of 576 square inches?

Solution. Let x denote the length of the side of the square piece of particle board so that the area of the board is x^2 square inches. Our tolerance specifies that the area of the board, x^2, needs to be within 0.25 square inches of 576. Mathematically, this translates to $|x^2 - 576| \leq 0.25$. Rewriting, we get $-0.25 \leq x^2 - 576 \leq 0.25$, or $575.75 \leq x^2 \leq 576.25$. At this point, we take advantage of the fact that the square root is increasing.[14] Therefore, taking square roots preserves the inequality. When simplifying, we keep in mind that since x represents a length, $x > 0$.

[13] The underlying concept of Calculus can be phrased in terms of tolerances, so this is well worth your attention.
[14] This means that for $a, b \geq 0$, if $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$.

$$\begin{aligned}
575.75 \leq &\quad x^2 &\leq 576.25 & \\
\sqrt{575.75} \leq &\quad \sqrt{x^2} &\leq \sqrt{576.25} & \quad \text{(take square roots.)} \\
\sqrt{575.75} \leq &\quad |x| &\leq \sqrt{576.25} & \quad (\sqrt{x^2} = |x|) \\
\sqrt{575.75} \leq &\quad x &\leq \sqrt{576.25} & \quad (|x| = x \text{ since } x > 0)
\end{aligned}$$

The side of the piece of particle board must be between $\sqrt{575.75} \approx 23.995$ and $\sqrt{576.25} \approx 24.005$ inches, a tolerance of (approximately) 0.005 inches of the target length of 24 inches, to ensure that the area is within 0.25 square inches of 576. $\square$

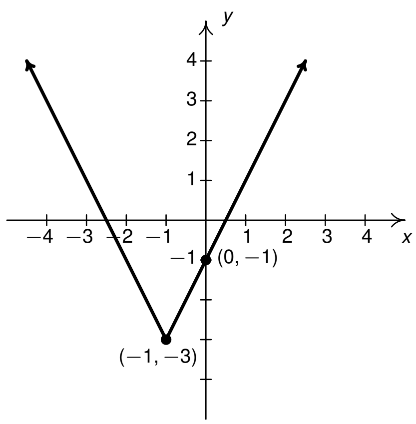
4.2.3 Exercises

In Exercises 1 - 6, graph the function using Theorem 4.4. Find the axis intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, the maximum and minimum of each function, if they exist, and list the intervals on which the function is increasing, decreasing or constant.

1. $f(x) = |x+4|$
2. $f(x) = |x| + 4$
3. $f(x) = |4x|$
4. $g(t) = -3|t|$
5. $g(t) = 3|t+4| - 4$
6. $g(t) = \dfrac{1}{3}|2t-1|$

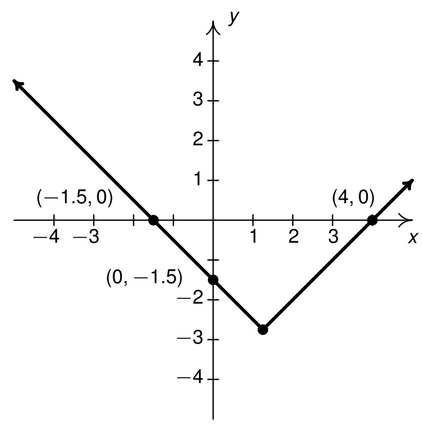
In Exercises 7 - 10, find a formula for each function below in the form $F(x) = a|x-h| + k$.

7.



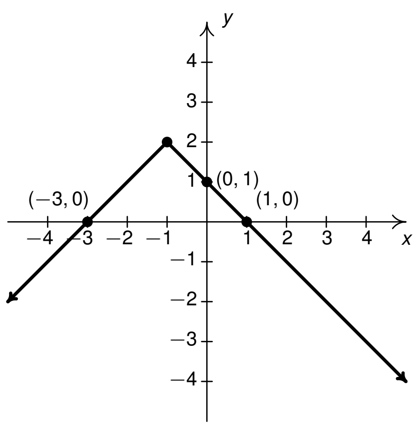
$y = F(x)$

8.



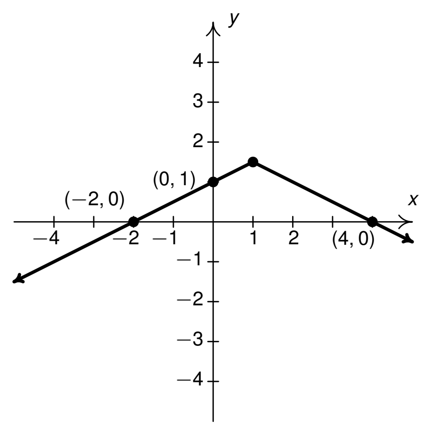
$y = F(x)$

9.



$y = F(x)$

10.



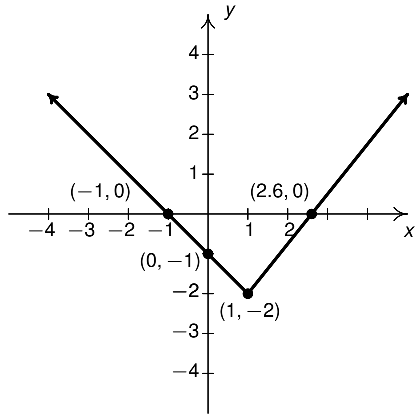
$y = F(x)$

11. With help from a graphing utility, graph the following pairs of functions on the same set of axes:

   - $f(x) = 2 - x$ and $g(x) = |2 - x|$
   - $f(x) = x^2 - 4$ and $g(x) = |x^2 - 4|$
   - $f(x) = x^3$ and $g(x) = |x^3|$
   - $f(x) = \sqrt{x} - 4$ and $g(x) = |\sqrt{x} - 4|$

   Choose more functions $f(x)$ and graph $y = f(x)$ alongside $y = |f(x)|$ until you can explain how, in general, one would obtain the graph of $y = |f(x)|$ given the graph of $y = f(x)$. How does your explanation tie in with with Definition 4.2?

12. Explain the function below cannot be written in the form $F(x) = a|x - h| + k$. Write $F(x)$ as a piecewise-defined linear function.



$y = F(x)$

In Exercises 13 - 18, graph the function by rewriting each function as a piecewise defined function using Definition 4.2. Find the axis intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, the maximum and minimum of each function, if they exist, and list the intervals on which the function is increasing, decreasing or constant.

13. $f(x) = x + |x| - 3$

14. $f(x) = |x + 2| - x$

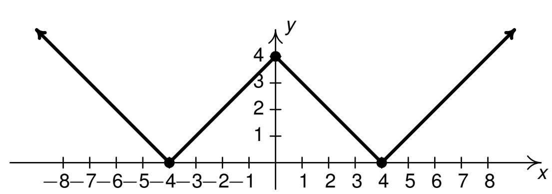
15. $f(x) = |x + 2| - |x|$

16. $g(t) = |t + 4| + |t - 2|$

17. $g(t) = \dfrac{|t + 4|}{t + 4}$

18. $g(t) = \dfrac{|2 - t|}{2 - t}$

19. With the help of your classmates, find an absolute value function whose graph is given below.



In Exercises 20 - 31, solve the equation.

20. $|x| = 6$
21. $|3x - 1| = 10$
22. $|4 - x| = 7$
23. $4 - |t| = 3$
24. $2|5t + 1| - 3 = 0$
25. $|7t - 1| + 2 = 0$
26. $\dfrac{5 - |w|}{2} = 1$
27. $\frac{2}{3}|5 - 2w| - \frac{1}{2} = 5$
28. $|w| = w + 3$
29. $|2x - 1| = x + 1$
30. $4 - |x| = 2x + 1$
31. $|x - 4| = x - 5$

Solve the equations in Exercises 32 - 37 using the property that if $|a| = |b|$ then $a = \pm b$.

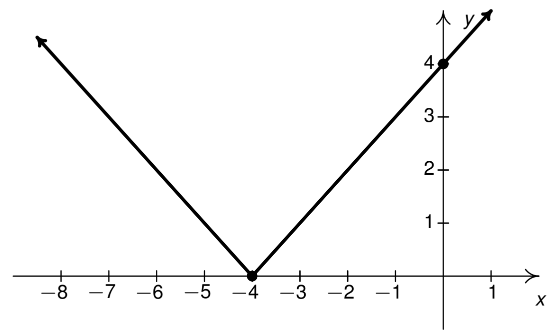
32. $|3x - 2| = |2x + 7|$
33. $|3x + 1| = |4x|$
34. $|1 - 2x| = |x + 1|$
35. $|4 - t| - |t + 2| = 0$
36. $|2 - 5t| = 5|t + 1|$
37. $3|t - 1| = 2|t + 1|$

In Exercises 38 - 53, solve the inequality. Write your answer using interval notation.

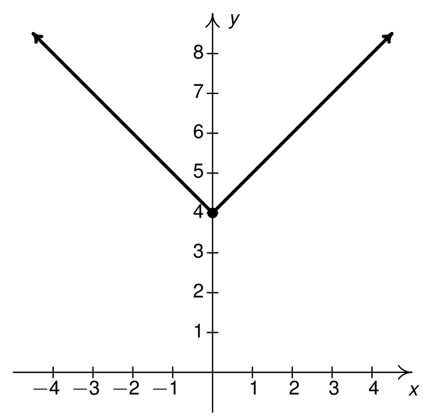
38. $|3x - 5| \leq 4$
39. $|7x + 2| > 10$
40. $|2t + 1| - 5 < 0$
41. $|2 - t| - 4 \geq -3$
42. $|3w + 5| + 2 < 1$
43. $2|7 - w| + 4 > 1$
44. $2 \leq |4 - x| < 7$
45. $1 < |2x - 9| \leq 3$
46. $|t + 3| \geq |6t + 9|$
47. $|t - 3| - |2t + 1| < 0$
48. $|1 - 2x| \geq x + 5$
49. $x + 5 < |x + 5|$
50. $x \geq |x + 1|$
51. $|2x + 1| \leq 6 - x$
52. $t + |2t - 3| < 2$
53. $|3 - t| \geq t - 5$
54. Show that if δ is a real number with $\delta > 0$, the solution to $|x - a| < \delta$ is the interval: $(a - \delta, a + \delta)$. That is, an interval centered at a with 'radius' δ.
55. The Triangle Inequality for real numbers states that for all real numbers x and a, $|x + a| \leq |x| + |a|$ and, moreover, $|x + a| = |x| + |a|$ if and only if x and a are both positive, both negative, or one or the other is 0. Graph each pair of functions below on the same pair of axes and use the graphs to verify the triangle inequality in each instance.
    - $f(x) = |x + 2|$ and $g(x) = |x| + 2$.
    - $f(x) = |x + 4|$ and $g(x) = |x| + 4$.

4.2.4 Answers

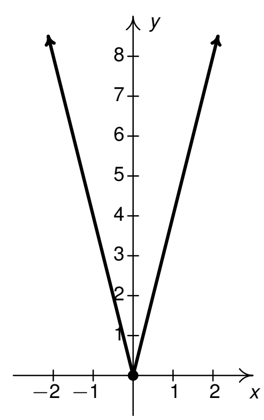
1. $f(x) = |x + 4|$
   x-intercept $(-4, 0)$
   y-intercept $(0, 4)$
   Domain $(-\infty, \infty)$
   Range $[0, \infty)$
   Decreasing on $(-\infty, -4]$
   Increasing on $[-4, \infty)$
   Minimum is 0 at $(-4, 0)$
   No maximum



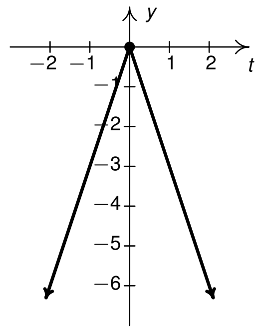
2. $f(x) = |x| + 4$
   No x-intercepts
   y-intercept $(0, 4)$
   Domain $(-\infty, \infty)$
   Range $[4, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Minimum is 4 at $(0, 4)$
   No maximum



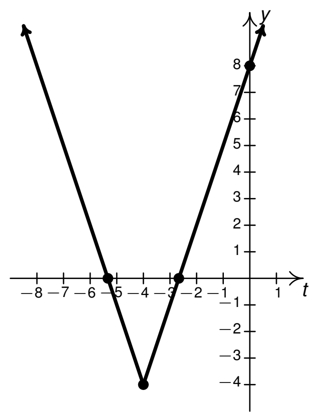
3. $f(x) = |4x|$
   x-intercept $(0, 0)$
   y-intercept $(0, 0)$
   Domain $(-\infty, \infty)$
   Range $[0, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Minimum is 0 at $(0, 0)$
   No maximum



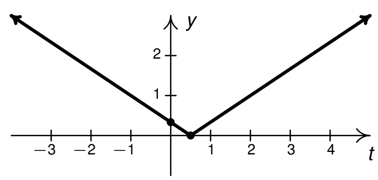
4. $g(t) = -3|t|$
   t-intercept $(0, 0)$
   y-intercept $(0, 0)$
   Domain $(-\infty, \infty)$
   Range $(-\infty, 0]$
   Increasing on $(-\infty, 0]$
   Decreasing on $[0, \infty)$
   Maximum is 0 at $(0, 0)$
   No minimum



5. $g(t) = 3|t+4| - 4$
   t-intercepts $\left(-\frac{16}{3}, 0\right)$, $\left(-\frac{8}{3}, 0\right)$
   y-intercept $(0, 8)$
   Domain $(-\infty, \infty)$
   Range $[-4, \infty)$
   Decreasing on $(-\infty, -4]$
   Increasing on $[-4, \infty)$
   Minimum is -4 at $(-4, -4)$
   No maximum



6. $g(t) = \frac{1}{3}|2t - 1|$
   t-intercepts $\left(\frac{1}{2}, 0\right)$
   y-intercept $\left(0, \frac{1}{3}\right)$
   Domain $(-\infty, \infty)$
   Range $[0, \infty)$
   Decreasing on $\left(-\infty, \frac{1}{2}\right]$
   Increasing on $\left[\frac{1}{2}, \infty\right)$
   Minimum is 0 at $\left(\frac{1}{2}, 0\right)$
   No maximum



7. $F(x) = 2|x+1| - 3$

8. $F(x) = |x - 1.25| - 2.75$

9. $F(x) = -|x+1| + 2$

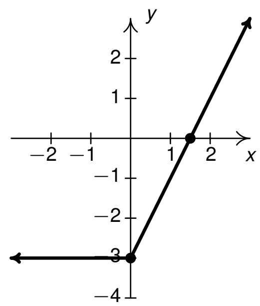
10. $F(x) = -\frac{1}{2}|x+1| + \frac{3}{2}$

11. In each case, the graph of g can be obtained from the graph of f by reflecting the portion of the graph of f which lies below the x-axis about the x-axis. This meshes with Definition 4.2 since what we are doing algebraically is making the negative y-values positive.

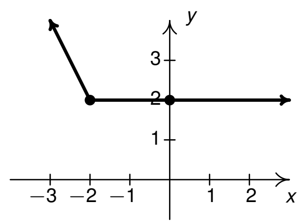
12. If $F(x) = a|x-h|+k$, then for the vertex to be at $(1,-2)$, $h=1$ and $k=-2$ so $F(x) = a|x-1|-2$. Since $(0,-1)$ is on the graph, $F(0) = -1$ so $-1 = a|0-1|-2$ which means $a = 1$. This means $F(x) = |x-1|-2$. However, $(2.6, 0)$ is also on the graph, so it should work out that $F(2.6) = 0$. However, we find $F(2.6) = |2.6-1|-2 = -0.4 \neq 0$.

$$F(x) = \begin{cases} -x-1 & \text{if } x \leq 1, \\ \frac{5}{4}x - \frac{13}{4} & \text{if } x \geq 1, \end{cases}$$

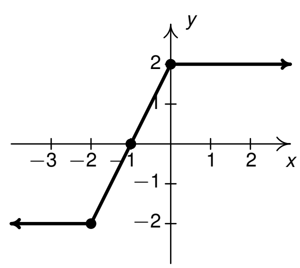
13. Re-write $f(x) = x + |x| - 3$ as
$f(x) = \begin{cases} -3 & \text{if} \quad x < 0 \\ 2x-3 & \text{if} \quad x \geq 0 \end{cases}$
x-intercept $\left(\frac{3}{2}, 0\right)$
y-intercept $(0,-3)$
Domain $(-\infty, \infty)$
Range $[-3, \infty)$
Increasing on $[0, \infty)$
Constant on $(-\infty, 0]$
Minimum is -3 at $(x,-3)$ where $x \leq 0$
No maximum



14. Re-write $f(x) = |x+2| - x$ as
$f(x) = \begin{cases} -2x-2 & \text{if} \quad x < -2 \\ 2 & \text{if} \quad x \geq -2 \end{cases}$
No x-intercepts
y-intercept $(0,2)$
Domain $(-\infty, \infty)$
Range $[2, \infty)$
Decreasing on $(-\infty, -2]$
Constant on $[-2, \infty)$
Minimum is 2 at every point $(x, 2)$ where $x \geq -2$
No maximum



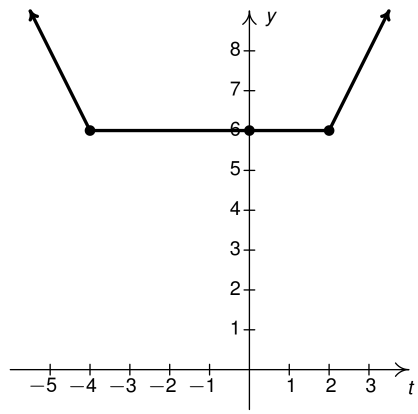
15. Re-write $f(x) = |x+2| - |x|$ as
$f(x) = \begin{cases} -2 & \text{if} \quad x < -2 \\ 2x+2 & \text{if} \quad -2 \leq x < 0 \\ 2 & \text{if} \quad x \geq 0 \end{cases}$
x-intercept $(-1, 0)$
y-intercept $(0,2)$
Domain $(-\infty, \infty)$
Range $[-2, 2]$
Increasing on $[-2, 0]$
Constant on $(-\infty, -2]$
Constant on $[0, \infty)$
Minimum is -2 at $(x,-2)$ where $x \leq -2$
Maximum is 2 at $(x, 2)$ where $x \geq 0$



16. Re-write $g(t) = |t+4| + |t-2|$ as

$$g(t) = \begin{cases} -2t-2 & \text{if} \quad t < -4 \\ 6 & \text{if} \quad -4 \leq t < 2 \\ 2t+2 & \text{if} \quad t \geq 2 \end{cases}$$

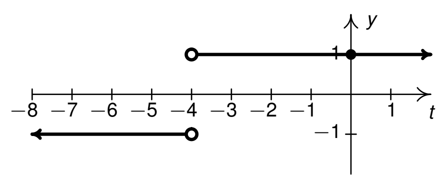
No t-intercept
y-intercept $(0, 6)$
Domain $(-\infty, \infty)$
Range $[6, \infty)$
Decreasing on $(-\infty, -4]$
Constant on $[-4, 2]$
Increasing on $[2, \infty)$
Minimum is 6 at $(t, 6)$ where $-4 \leq t \leq 2$
No maximum



17. Re-write $g(t) = \dfrac{|t+4|}{t+4}$ as

$$g(t) = \begin{cases} -1 & \text{if} \quad t < -4 \\ 1 & \text{if} \quad t > -4 \end{cases}$$

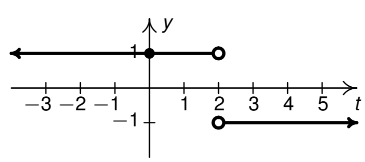
No t-intercept
y-intercept $(0, 1)$
Domain $(-\infty, -4) \cup (-4, \infty)$
Range $\{-1, 1\}$
Constant on $(-\infty, -4)$
Constant on $(-4, \infty)$
Minimum is -1 at every point $(t, -1)$ where $t < -4$
Maximum is 1 at $(t, 1)$ where $t > -4$



18. Re-write $g(t) = \dfrac{|2-t|}{2-t}$ as

$$g(t) = \begin{cases} 1 & \text{if} \quad t < 2 \\ -1 & \text{if} \quad t > 2 \end{cases}$$

No t-intercept
y-intercept $(0, 1)$
Domain $(-\infty, 2) \cup (2, \infty)$
Range $\{-1, 1\}$
Constant on $(-\infty, 2)$
Constant on $(2, \infty)$
Minimum is -1 at $(t, -1)$ where $t > 2$
Maximum is 1 at every point $(t, 1)$ where $t < 2$



19. $f(x) = ||x| - 4|$

20. $x = -6$ or $x = 6$

21. $x = -3$ or $x = \frac{11}{3}$

22. $x = -3$ or $x = 11$

23. $t = -1$ or $t = 1$

24. $t = -\frac{1}{2}$ or $t = \frac{1}{10}$

25. no solution

26. $w = -3$ or $w = 3$

27. $w = -\frac{13}{8}$ or $w = \frac{53}{8}$

28. $w = -\frac{3}{2}$

29. $x = 0$ or $x = 2$

30. $x = 1$

31. no solution

32. $x = -1$ or $x = 9$
33. $x = -\frac{1}{7}$ or $x = 1$
34. $x = 0$ or $x = 2$
35. $t = 1$
36. $t = -\frac{3}{10}$
37. $t = \frac{1}{5}$ or $t = 5$
38. $\left[\frac{1}{3}, 3\right]$
39. $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
40. $(-3, 2)$
41. $(-\infty, 1] \cup [3, \infty)$
42. No solution
43. $(-\infty, \infty)$
44. $(-3, 2] \cup [6, 11)$
45. $[3, 4) \cup (5, 6]$
46. $\left[-\frac{12}{7}, -\frac{6}{5}\right]$
47. $(-\infty, -4) \cup \left(\frac{2}{3}, \infty\right)$
48. $\left(-\infty, -\frac{4}{3}\right] \cup [6, \infty)$
49. $(-\infty, -5)$
50. No Solution.
51. $\left[-7, \frac{5}{3}\right]$
52. $\left(1, \frac{5}{3}\right)$
53. $(-\infty, \infty)$

Chapter 5

Quadratic Functions

5.1 Polynomial Arithmetic

In this section, we review the vocabulary and arithmetic of **polynomials**. We start by defining what is meant by the word 'polynomial' in general. A more narrow definition of a 'polynomial function' will be given in Chapter 6. The general definition suffices for the purposes of this review.

Definition 5.1. A **polynomial** is a sum of terms each of which is a real number or a real number multiplied by one or more variables to natural number powers.

Some examples of polynomials are $x^2+x\sqrt{3}+4$, $27x^2y+\frac{7x}{2}$ and 6. Things like $3\sqrt{x}$, $4x-\frac{2}{x+1}$ and $13x^{2/3}y^2$ are **not** polynomials. In the box below we review some of the terminology associated with polynomials.

Definition 5.2. Polynomial Vocabulary

- **Constant Terms:** Terms in polynomials without variables are called **constant** terms.
- **Coefficient:** In non-constant terms, the real number factor in the expression is called the **coefficient** of the term.
- **Degree:** The **degree** of a non-constant term is the sum of the exponents on the variables in the term; non-zero constant terms are defined to have degree 0. The degree of a polynomial is the highest degree of the nonzero terms.
- **Like Terms:** Terms in a polynomial are called **like** terms if they have the same variables each with the same corresponding exponents.
- **Simplified:** A polynomial is said to be **simplified** if all arithmetic operations have been completed and there are no longer any like terms.
- **Classification by Number of Terms:** A simplified polynomial is called a
  - **monomial** if it has exactly one nonzero term
  - **binomial** if it has exactly two nonzero terms
  - **trinomial** if it has exactly three nonzero terms

For example, $x^2 + x\sqrt{3} + 4$ is a trinomial of degree 2. The coefficient of x^2 is 1 and the constant term is 4. The polynomial $27x^2y + \frac{7x}{2}$ is a binomial of degree 3 ($x^2y = x^2y^1$) with constant term 0.

The concept of 'like' terms really amounts to finding terms which can be combined using the Distributive Property. For example, in the polynomial $17x^2y - 3xy^2 + 7xy^2$, $-3xy^2$ and $7xy^2$ are like terms, since they have the same variables with the same corresponding exponents. This allows us to combine these two terms as follows:

$$17x^2y - 3xy^2 + 7xy^2 = 17x^2y + (-3)xy^2 + 7xy^2 + 17x^2y + (-3 + 7)xy^2 = 17x^2y + 4xy^2$$

Note that even though $17x^2y$ and $4xy^2$ have the same variables, they are not like terms since in the first term we have x^2 and $y = y^1$ but in the second we have $x = x^1$ and $y = y^2$ so the corresponding exponents aren't the same. Hence, $17x^2y + 4xy^2$ is the simplified form of the polynomial.

There are four basic operations we can perform with polynomials: addition, subtraction, multiplication and division. The first three of these follow directly from properties of real number arithmetic and will be discussed together. Division, on the other hand, is a bit more complicated and will be discussed separately.

5.1.1 Polynomial Addition, Subtraction and Multiplication.

Adding and subtracting polynomials comes down to identifying like terms and then adding or subtracting the coefficients of those like terms. Multiplying polynomials comes to us courtesy of the Generalized Distributive Property.

Theorem 5.1. Generalized Distributive Property: To multiply a quantity of n terms by a quantity of m terms, multiply each of the n terms of the first quantity by each of the m terms in the second quantity and add the resulting $n \cdot m$ terms together.

In particular, Theorem 5.1 says that, before combining like terms, a product of an n-term polynomial and an m-term polynomial will generate $(n \cdot m)$-terms. For example, a binomial times a trinomial will produce six terms some of which may be like terms. Thus the simplified end result may have fewer than six terms but you will start with six terms.

A special case of Theorem 5.1 is the famous **F.O.I.L.**, listed here:[1]

Theorem 5.2. F.O.I.L: The terms generated from the product of two binomials: $(a + b)(c + d)$ can be verbalized as follows: "Take the sum of

- the product of the **F**irst terms a and c, ac
- the product of the **O**uter terms a and d, ad
- the product of the **I**nner terms b and c, bc
- the product of the **L**ast terms b and d, bd."

That is, $(a + b)(c + d) = ac + ad + bc + bd$.

Theorem 5.1 is best proved using the technique known as Mathematical Induction which is covered in Section **??**. The result is really nothing more than repeated applications of the Distributive Property so it seems reasonable and we'll use it without proof for now. The other major piece of polynomial multiplication is one of the Power Rules of Exponents from page 25 in Section 1.2, namely $a^n a^m = a^{n+m}$. The Commutative and Associative Properties of addition and multiplication are also used extensively. We put all of these properties to good use in the next example.

[1] We caved to peer pressure on this one. Apparently all of the cool Precalculus books have FOIL in them even though it's redundant once you know how to distribute multiplication across addition. In general, we don't like mechanical short-cuts that interfere with a student's understanding of the material and FOIL is one of the worst.

Example 5.1.1. Perform the indicated operations and simplify.

1. $\left(3x^2 - 2x + 1\right) - (7x - 3)$
2. $4xz^2 - 3z(xz - x + 4)$
3. $(2t + 1)(3t - 7)$
4. $\left(3y - \sqrt[3]{2}\right)\left(9y^2 + 3\sqrt[3]{2}y + \sqrt[3]{4}\right)$
5. $\left(4w - \dfrac{1}{2}\right)^2$
6. $\left[2(x + h) - (x + h)^2\right] - \left(2x - x^2\right)$

Solution.

1. We begin 'distributing the negative' as indicated on page 19 in Section 1.2, then we rearrange and combine like terms:

$$\begin{array}{rcll} \left(3x^2 - 2x + 1\right) - (7x - 3) & = & 3x^2 - 2x + 1 - 7x + 3 & \text{Distribute} \\ & = & 3x^2 - 2x - 7x + 1 + 3 & \text{Rearrange terms} \\ & = & 3x^2 - 9x + 4 & \text{Combine like terms} \end{array}$$

Our answer is $3x^2 - 9x + 4$.

2. Following in our footsteps from the previous example, we first distribute the $-3z$ through, then rearrange and combine like terms:

$$\begin{array}{rcll} 4xz^2 - 3z(xz - x + 4) & = & 4xz^2 - 3z(xz) + 3z(x) - 3z(4) & \text{Distribute} \\ & = & 4xz^2 - 3xz^2 + 3xz - 12z & \text{Multiply} \\ & = & xz^2 + 3xz - 12z & \text{Combine like terms} \end{array}$$

We get our final answer: $xz^2 + 3xz - 12z$.

3. At last, we have a chance to use our F.O.I.L. technique:

$$\begin{array}{rcll} (2t + 1)(3t - 7) & = & (2t)(3t) + (2t)(-7) + (1)(3t) + (1)(-7) & \text{F.O.I.L.} \\ & = & 6t^2 - 14t + 3t - 7 & \text{Multiply} \\ & = & 6t^2 - 11t - 7 & \text{Combine like terms} \end{array}$$

We get $6t^2 - 11t - 7$ as our final answer.

4. We use the Generalized Distributive Property here, multiplying each term in the second quantity first by $3y$, then by $-\sqrt[3]{2}$:

$$\begin{array}{rcl} \left(3y - \sqrt[3]{2}\right)\left(9y^2 + 3\sqrt[3]{2}y + \sqrt[3]{4}\right) & = & 3y\left(9y^2\right) + 3y\left(3\sqrt[3]{2}y\right) + 3y\left(\sqrt[3]{4}\right) \\ & & -\sqrt[3]{2}\left(9y^2\right) - \sqrt[3]{2}\left(3\sqrt[3]{2}y\right) - \sqrt[3]{2}\left(\sqrt[3]{4}\right) \\ & = & 27y^3 + 9y^2\sqrt[3]{2} + 3y\sqrt[3]{4} - 9y^2\sqrt[3]{2} - 3y\sqrt[3]{4} - \sqrt[3]{8} \\ & = & 27y^3 + 9y^2\sqrt[3]{2} - 9y^2\sqrt[3]{2} + 3y\sqrt[3]{4} - 3y\sqrt[3]{4} - 2 \\ & = & 27y^3 - 2 \end{array}$$

To our surprise and delight, this product reduces to $27y^3 - 2$.

5. Exponents do **not** distribute across powers[2] so we know that $\left(4w - \frac{1}{2}\right)^2 \neq (4w)^2 - \left(\frac{1}{2}\right)^2$. Instead, we proceed as follows:

$$\begin{array}{rcll} \left(4w - \dfrac{1}{2}\right)^2 & = & \left(4w - \dfrac{1}{2}\right)\left(4w - \dfrac{1}{2}\right) & \\ & = & (4w)(4w) + (4w)\left(-\dfrac{1}{2}\right) + \left(-\dfrac{1}{2}\right)(4w) + \left(-\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right) & \text{F.O.I.L.} \\ & = & 16w^2 - 2w - 2w + \dfrac{1}{4} & \text{Multiply} \\ & = & 16w^2 - 4w + \dfrac{1}{4} & \text{Combine like terms} \end{array}$$

Our (correct) final answer is $16w^2 - 4w + \frac{1}{4}$.

6. Our last example has two levels of grouping symbols. We begin simplifying the quantity inside the brackets, expanding $(x + h)^2$ in the same way we expanded $(4w - \frac{1}{2})^2$ in our previous example:

$$(x + h)^2 = (x + h)(x + h) = (x)(x) + (x)(h) + (h)(x) + (h)(h) = x^2 + 2xh + h^2$$

When we substitute this into our expression, we envelope it in parentheses, as usual, so that we don't forget to distribute the negative.

$$\begin{array}{rcll} \left[2(x + h) - (x + h)^2\right] - \left(2x - x^2\right) & = & \left[2(x + h) - \left(x^2 + 2xh + h^2\right)\right] - \left(2x - x^2\right) & \text{Substitute} \\ & = & \left[2x + 2h - x^2 - 2xh - h^2\right] - \left(2x - x^2\right) & \text{Distribute} \\ & = & 2x + 2h - x^2 - 2xh - h^2 - 2x + x^2 & \text{Distribute} \\ & = & 2x - 2x + 2h - x^2 + x^2 - 2xh - h^2 & \text{Rearrange terms} \\ & = & 2h - 2xh - h^2 & \text{Combine like terms} \end{array}$$

We find no like terms in $2h - 2xh - h^2$ so we are finished. □

We conclude our discussion of polynomial multiplication by showcasing two special products which happen often enough they should be committed to memory.

Theorem 5.3. Special Products: Let a and b be real numbers:

- **Perfect Square:** $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$
- **Difference of Two Squares:** $(a - b)(a + b) = a^2 - b^2$

The formulas in Theorem 5.3 can be verified by working through the multiplication.[3]

[2] See the remarks following the Properties of Exponents on 25.

[3] These are both special cases of F.O.I.L.

5.1.2 Polynomial Long Division.

We now turn our attention to polynomial long division. Dividing two polynomials follows the same algorithm, in principle, as dividing two natural numbers so we review that process first. Suppose we wished to divide 2585 by 79. The standard division tableau is given below.

$$\begin{array}{rr} & 32 \\ 79 & \overline{\big)\,2585} \\ & -\underline{237}\downarrow \\ & 215 \\ & -\underline{158} \\ & 57 \end{array}$$

In this case, 79 is called the **divisor**, 2585 is called the **dividend**, 32 is called the **quotient** and 57 is called the **remainder**. We can check our answer by showing:

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

or in this case, $2585 = (79)(32)+57\checkmark$. We hope that the long division tableau evokes warm, fuzzy memories of your formative years as opposed to feelings of hopelessness and frustration. If you experience the latter, keep in mind that the Division Algorithm essentially is a two-step process, iterated over and over again. First, we guess the number of times the divisor goes into the dividend and then we subtract off our guess. We repeat those steps with what's left over until what's left over (the remainder) is less than what we started with (the divisor). That's all there is to it!

The division algorithm for polynomials has the same basic two steps but when we subtract polynomials, we must take care to subtract *like terms* only. As a transition to polynomial division, let's write out our previous division tableau in expanded form.

$$\begin{array}{rr} & 3\cdot 10+2 \\ 7\cdot 10+9 & \overline{\big)\,2\cdot 10^3 + 5\cdot 10^2 + 8\cdot 10 + 5} \\ & -\underline{\left(2\cdot 10^3 + 3\cdot 10^2 + 7\cdot 10\right)}\ \downarrow \\ & 2\cdot 10^2 + 1\cdot 10 + 5 \\ & -\underline{\left(1\cdot 10^2 + 5\cdot 10 + 8\right)} \\ & 5\cdot 10 + 7 \end{array}$$

Written this way, we see that when we line up the digits we are really lining up the coefficients of the corresponding powers of 10 - much like how we'll have to keep the powers of x lined up in the same columns. The big difference between polynomial division and the division of natural numbers is that the value of x is an unknown quantity. So unlike using the known value of 10, when we subtract there can be no regrouping of coefficients as in our previous example. (The subtraction $215 - 158$ requires us to 'regroup' or 'borrow' from the tens digit, then the hundreds digit.) This actually makes polynomial division easier.[4]

[4] In our opinion - you can judge for yourself.

Before we dive into examples, we first state a theorem telling us when we can divide two polynomials, and what to expect when we do so.

Theorem 5.4. Polynomial Division: Let d and p be nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, q and r, such that $p = d \cdot q + r$, where either $r = 0$ or the degree of r is strictly less than the degree of d.

Essentially, Theorem 5.4 tells us that we can divide polynomials whenever the degree of the divisor is less than or equal to the degree of the dividend. We know we're done with the division when the polynomial left over (the remainder) has a degree strictly less than the divisor. It's time to walk through a few examples to refresh your memory.

Example 5.1.2. Perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

1. $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$
2. $(2t + 7) \div (3t - 4)$
3. $\left(6y^2 - 1\right) \div (2y + 5)$
4. $\left(w^3\right) \div \left(w^2 - \sqrt{2}\right)$.

Solution.

1. To begin $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$, we divide the first term in the dividend, namely x^3, by the first term in the divisor, namely x, and get $\frac{x^3}{x} = x^2$. This then becomes the first term in the quotient. We proceed as in regular long division at this point: we multiply the entire divisor, $x - 2$, by this first term in the quotient to get $x^2(x - 2) = x^3 - 2x^2$. We then subtract this result from the dividend.

$$\begin{array}{r|l} & \;\; x^2 \\ \cline{2-2} x-2 & \;\; x^3 + 4x^2 - 5x - 14 \\ & -\left(x^3 - 2x^2\right) \quad \downarrow \\ \cline{2-2} & \qquad\quad 6x^2 - 5x \end{array}$$

Now we 'bring down' the next term of the quotient, namely $-5x$, and repeat the process. We divide $\frac{6x^2}{x} = 6x$, and add this to the quotient polynomial, multiply it by the divisor (which yields $6x(x - 2) = 6x^2 - 12x$) and subtract.

$$\begin{array}{r|l} & \;\; x^2 + 6x \\ \cline{2-2} x-2 & \;\; x^3 + 4x^2 - 5x - 14 \\ & -\left(x^3 - 2x^2\right) \qquad\quad \downarrow \\ \cline{2-2} & \qquad\quad 6x^2 - 5x \quad\;\; \downarrow \\ & \qquad -\left(6x^2 - 12x\right) \;\; \downarrow \\ \cline{2-2} & \qquad\qquad\qquad 7x - 14 \end{array}$$

Finally, we 'bring down' the last term of the dividend, namely -14, and repeat the process. We divide $\frac{7x}{x} = 7$, add this to the quotient, multiply it by the divisor (which yields $7(x-2) = 7x - 14$) and subtract.

$$\begin{array}{rrrrrr}
 & & x^2 & + \; 6x & + \; 7 & \\ \cline{3-6}
x-2 & \big) & x^3 & + \; 4x^2 & - \; 5x & - \; 14 \\
 & - & \underline{\left(x^3 - 2x^2\right)} & & & \\
 & & & 6x^2 & - \; 5x & \\
 & & - & \underline{\left(6x^2 - 12x\right)} & & \\
 & & & & 7x & - \; 14 \\
 & & & - & \underline{(7x} & \underline{-14)} \\
 & & & & & 0
\end{array}$$

In this case, we get a quotient of $x^2 + 6x + 7$ with a remainder of 0. To check our answer, we compute

$$(x-2)\left(x^2 + 6x + 7\right) + 0 = x^3 + 6x^2 + 7x - 2x^2 - 12x - 14 = x^3 + 4x^2 - 5x - 14 \checkmark$$

2. To compute $(2t + 7) \div (3t - 4)$, we start as before. We find $\frac{2t}{3t} = \frac{2}{3}$, so that becomes the first (and only) term in the quotient. We multiply the divisor $(3t - 4)$ by $\frac{2}{3}$ and get $2t - \frac{8}{3}$. We subtract this from the divided and get $\frac{29}{3}$.

$$\begin{array}{rrrr}
 & & & \frac{2}{3} \\ \cline{3-4}
3t-4 & \big) & 2t & + \; 7 \\
 & - & \underline{\left(2t - \frac{8}{3}\right)} & \\
 & & & \frac{29}{3}
\end{array}$$

Our answer is $\frac{2}{3}$ with a remainder of $\frac{29}{3}$. To check our answer, we compute

$$(3t - 4)\left(\frac{2}{3}\right) + \frac{29}{3} = 2t - \frac{8}{3} + \frac{29}{3} = 2t + \frac{21}{3} = 2t + 7 \checkmark$$

3. When we set-up the tableau for $\left(6y^2 - 1\right) \div (2y + 5)$, we must first issue a 'placeholder' for the 'missing' y-term in the dividend, $6y^2 - 1 = 6y^2 + 0y - 1$. We then proceed as before. Since $\frac{6y^2}{2y} = 3y$, $3y$ is the first term in our quotient. We multiply $(2y + 5)$ times $3y$ and subtract it from the dividend.

We bring down the -1, and repeat.

$$\begin{array}{r|rrr}
 & 3y & - & \frac{15}{2} \\ \hline
2y+5 & 6y^2 + & 0y & - \; 1 \\
 & -\left(6y^2 + \right. & \left. 15y\right) & \downarrow \\ \cline{2-3}
 & & -15y & - \; 1 \\
 & & -\left(-15y - \right. & \left. \frac{75}{2}\right) \\ \cline{3-4}
 & & & \frac{73}{2}
\end{array}$$

Our answer is $3y - \frac{15}{2}$ with a remainder of $\frac{73}{2}$. To check our answer, we compute:

$$(2y+5)\left(3y - \frac{15}{2}\right) + \frac{73}{2} = 6y^2 - 15y + 15y - \frac{75}{2} + \frac{73}{2} = 6y^2 - 1 \checkmark$$

4. For our last example, we need 'placeholders' for both the divisor $w^2 - \sqrt{2} = w^2 + 0w - \sqrt{2}$ and the dividend $w^3 = w^3 + 0w^2 + 0w + 0$. The first term in the quotient is $\frac{w^3}{w^2} = w$, and when we multiply and subtract this from the dividend, we're left with just $0w^2 + w\sqrt{2} + 0 = w\sqrt{2}$.

$$\begin{array}{r|llll}
 & & & w & \\ \hline
w^2+0w-\sqrt{2} & w^3 & +0w^2 + & 0w & +0 \\
- & \left(w^3\right. & +0w^2 - & \left. w\sqrt{2}\right) & \downarrow \\ \cline{2-4}
 & & 0w^2 + & w\sqrt{2} & +0
\end{array}$$

Since the degree of $w\sqrt{2}$ (which is 1) is less than the degree of the divisor (which is 2), we are done.[5] Our answer is w with a remainder of $w\sqrt{2}$. To check, we compute:

$$\left(w^2 - \sqrt{2}\right) w + w\sqrt{2} = w^3 - w\sqrt{2} + w\sqrt{2} = w^3 \checkmark$$

□

[5]Since $\frac{0w^2}{w^2} = 0$, we could proceed, write our quotient as $w + 0$, and move on. . . but even pedants have limits.

5.1.3 Exercises

In Exercises 1 - 15, perform the indicated operations and simplify.

1. $(4 - 3x) + (3x^2 + 2x + 7)$
2. $t^2 + 4t - 2(3 - t)$
3. $q(200 - 3q) - (5q + 500)$
4. $(3y - 1)(2y + 1)$
5. $\left(3 - \dfrac{x}{2}\right)(2x + 5)$
6. $-(4t + 3)(t^2 - 2)$
7. $2w(w^3 - 5)(w^3 + 5)$
8. $(5a^2 - 3)(25a^4 + 15a^2 + 9)$
9. $(x^2 - 2x + 3)(x^2 + 2x + 3)$
10. $(\sqrt{7} - z)(\sqrt{7} + z)$
11. $(x - \sqrt[3]{5})^3$
12. $(x - \sqrt[3]{5})(x^2 + x\sqrt[3]{5} + \sqrt[3]{25})$
13. $(w - 3)^2 - (w^2 + 9)$
14. $(x+h)^2 - 2(x+h) - (x^2 - 2x)$
15. $(x - [2 + \sqrt{5}])(x - [2 - \sqrt{5}])$

In Exercises 16 - 27, perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

16. $(5x^2 - 3x + 1) \div (x + 1)$
17. $(3y^2 + 6y - 7) \div (y - 3)$
18. $(6w - 3) \div (2w + 5)$
19. $(2x + 1) \div (3x - 4)$
20. $(t^2 - 4) \div (2t + 1)$
21. $(w^3 - 8) \div (5w - 10)$
22. $(2x^2 - x + 1) \div (3x^2 + 1)$
23. $(4y^4 + 3y^2 + 1) \div (2y^2 - y + 1)$
24. $w^4 \div (w^3 - 2)$
25. $(5t^3 - t + 1) \div (t^2 + 4)$
26. $(t^3 - 4) \div (t - \sqrt[3]{4})$
27. $(x^2 - 2x - 1) \div (x - [1 - \sqrt{2}])$

In Exercises 28 - 33 verify the given formula by showing the left hand side of the equation simplifies to the right hand side of the equation.

28. **Perfect Cube:** $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
29. **Difference of Cubes:** $(a - b)(a^2 + ab + b^2) = a^3 - b^3$
30. **Sum of Cubes:** $(a + b)(a^2 - ab + b^2) = a^3 + b^3$
31. **Perfect Quartic:** $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
32. **Difference of Quartics:** $(a - b)(a + b)(a^2 + b^2) = a^4 - b^4$
33. **Sum of Quartics:** $(a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2) = a^4 + b^4$
34. With help from your classmates, determine under what conditions $(a + b)^2 = a^2 + b^2$. What about $(a + b)^3 = a^3 + b^3$? In general, when does $(a + b)^n = a^n + b^n$ for a natural number $n \geq 2$?

5.1.4 Answers

1. $3x^2 - x + 11$
2. $t^2 + 6t - 6$
3. $-3q^2 + 195q - 500$
4. $6y^2 + y - 1$
5. $-x^2 + \frac{7}{2}x + 15$
6. $-4t^3 - 3t^2 + 8t + 6$
7. $2w^7 - 50w$
8. $125a^6 - 27$
9. $x^4 + 2x^2 + 9$
10. $7 - z^2$
11. $x^3 - 3x^2\sqrt[3]{5} + 3x\sqrt[3]{25} - 5$
12. $x^3 - 5$
13. $-6w$
14. $h^2 + 2xh - 2h$
15. $x^2 - 4x - 1$
16. quotient: $5x - 8$, remainder: 9
17. quotient: $3y + 15$, remainder: 38
18. quotient: 3, remainder: 18
19. quotient: $\frac{2}{3}$, remainder: $\frac{11}{3}$
20. quotient: $\frac{t}{2} - \frac{1}{4}$, remainder: $-\frac{15}{4}$
21. quotient: $\frac{w^2}{5} + \frac{2w}{5} + \frac{4}{5}$, remainder: 0
22. quotient: $\frac{2}{3}$, remainder: $-x + \frac{1}{3}$
23. quotient: $2y^2 + y + 1$, remainder: 0
24. quotient: w, remainder: $2w$
25. quotient: $5t$, remainder: $-21t + 1$
26. quotient:[6] $t^2 + t\sqrt[3]{4} + 2\sqrt[3]{2}$, remainder: 0
27. quotient: $x - 1 - \sqrt{2}$, remainder: 0

[6]Note: $\sqrt[3]{16} = 2\sqrt[3]{2}$.

5.2 Basic Factoring Techniques

Now that we have reviewed the basics of polynomial arithmetic it's time to review the basic techniques of factoring polynomial expressions. Our goal is to apply these techniques to help us solve certain specialized classes of non-linear equations. Given that 'factoring' literally means to resolve a product into its factors, it is, in the purest sense, 'undoing' multiplication. If this sounds like division to you then you've been paying attention. Let's start with a numerical example.

Suppose we are asked to factor 16337. We could write $16337 = 16337 \cdot 1$, and while this is technically a factorization of 16337, it's probably not an answer the poser of the question would accept. Usually, when we're asked to factor a natural number, we are being asked to resolve it into to a product of so-called 'prime' numbers.[1] Recall that **prime numbers** are defined as natural numbers whose only (natural number) factors are themselves and 1. They are, in essence, the 'building blocks' of natural numbers as far as multiplication is concerned. Said differently, we can build - via multiplication - any natural number given enough primes.

So how do we find the prime factors of 16337? We start by dividing each of the primes: 2, 3, 5, 7, etc., into 16337 until we get a remainder of 0. Eventually, we find that $16337 \div 17 = 961$ with a remainder of 0, which means $16337 = 17 \cdot 961$. So factoring and division are indeed closely related - factors of a number are precisely the divisors of that number which produce a zero remainder.[2] We continue our efforts to see if 961 can be factored down further, and we find that $961 = 31 \cdot 31$. Hence, 16337 can be 'completely factored' as $17 \cdot 31^2$. (This factorization is called the **prime factorization** of 16337.)

In factoring natural numbers, our building blocks are prime numbers, so to be completely factored means that every number used in the factorization of a given number is prime. One of the challenges when it comes to factoring polynomial expressions is to explain what it means to be 'completely factored'. In this section, our 'building blocks' for factoring polynomials are 'irreducible' polynomials as defined below.

Definition 5.3. A polynomial is said to be **irreducible** if it cannot be written as the product of polynomials of lower degree.

While Definition 5.3 seems straightforward enough, sometimes a greater level of specificity is required. For example, $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. While $x - \sqrt{3}$ and $x + \sqrt{3}$ are perfectly fine polynomials, factoring which requires irrational numbers is usually saved for a more advanced treatment of factoring.[3] For now, we will restrict ourselves to factoring using rational coefficients. So, while the polynomial $x^2 - 3$ can be factored using irrational numbers, it is called irreducible **over the rationals**, since there are no polynomials with *rational* coefficients of smaller degree which can be used to factor it.[4]

Since polynomials involve terms, the first step in any factoring strategy involves pulling out factors which are common to all of the terms. For example, in the polynomial $18x^2y^3 - 54x^3y^2 - 12xy^2$, each coefficient is a multiple of 6 so we can begin the factorization as $6(3x^2y^3 - 9x^3y^2 - 2xy^2)$. The remaining coefficients: 3, 9 and 2, have no common factors so 6 was the greatest common factor. What about the variables? Each

[1] As mentioned in Section 1.2, this is possible, in only one way, thanks to the Fundamental Theorem of Arithmetic.

[2] We'll refer back to this when we get to Section 6.2.

[3] See Section 6.3.

[4] If this isn't immediately obvious, don't worry - in some sense, it shouldn't be. We'll talk more about this later.

term contains an x, so we can factor an x from each term. When we do this, we are effectively dividing each term by x which means the exponent on x in each term is reduced by 1: $6x(3xy^3 - 9x^2y^2 - 2y^2)$. Next, we see that each term has a factor of y in it. In fact, each term has at least *two* factors of y in it, since the lowest exponent on y in each term is 2. This means that we can factor y^2 from each term. Again, factoring out y^2 from each term is tantamount to dividing each term by y^2 so the exponent on y in each term is reduced by *two*: $6xy^2(3xy - 9x^2 - 2)$. Just like we checked our division by multiplication in the previous section, we can check our factoring here by multiplication, too. $6xy^2(3xy - 9x^2 - 2) = (6xy^2)(3xy) - (6xy^2)(9x^2) - (6xy^2)(2) = 18x^2y^3 - 54x^3y^2 - 12xy^2 \checkmark$. We summarize how to find the Greatest Common Factor (G.C.F.) of a polynomial expression below.

Finding the G.C.F. of a Polynomial Expression

- If the coefficients are integers, find the G.C.F. of the coefficients.

  NOTE 1: If all of the coefficients are negative, consider the negative as part of the G.C.F..

  NOTE 2: If the coefficients involve fractions, get a common denominator, combine numerators, reduce to lowest terms and apply this step to the polynomial in the numerator.

- If a variable is common to all of the terms, the G.C.F. contains that variable to the smallest exponent which appears among the terms.

For example, to factor $-\frac{3}{5}z^3 - 6z^2$, we would first get a common denominator and factor as:

$$-\frac{3}{5}z^3 - 6z^2 = \frac{-3z^3 - 30z^2}{5} = \frac{-3z^2(z+10)}{5} = -\frac{3z^2(z+10)}{5} = -\frac{3}{5}z^2(z+10)$$

We now list some common factoring formulas, each of which can be verified by multiplying out the right side of the equation. While they all should look familiar - this is a review section after all - some should look more familiar than others since they appeared as 'special product' formulas in the previous section.

Common Factoring Formulas

- **Perfect Square Trinomials:** $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$

- **Difference of Two Squares:** $a^2 - b^2 = (a-b)(a+b)$

  NOTE: In general, the sum of squares, $a^2 + b^2$ is irreducible over the rationals.

- **Sum of Two Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

  NOTE: In general, $a^2 - ab + b^2$ is irreducible over the rationals.

- **Difference of Two Cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

  NOTE: In general, $a^2 + ab + b^2$ is irreducible over the rationals.

The example on the next page gives us practice with these formulas.

Example 5.2.1. Factor the following polynomials completely over the rationals. That is, write each polynomial as a product polynomials of lowest degree which are irreducible over the rationals.

1. $18x^2 - 48x + 32$
2. $64y^2 - 1$
3. $75t^4 + 30t^3 + 3t^2$
4. $w^4z - wz^4$
5. $81 - 16t^4$
6. $x^6 - 64$

Solution.

1. Our first step is to factor out the G.C.F. which in this case is 2. To match what is left with one of the special forms, we rewrite $9x^2 = (3x)^2$ and $16 = 4^2$. Since the 'middle' term is $-24x = -2(4)(3x)$, we see that we have a perfect square trinomial.

$$\begin{array}{rcll} 18x^2 - 48x + 32 & = & 2(9x^2 - 24x + 16) & \text{Factor out G.C.F.} \\ & = & 2((3x)^2 - 2(4)(3x) + (4)^2) & \\ & = & 2(3x-4)^2 & \text{Perfect Square Trinomial: } a = 3x,\ b = 4 \end{array}$$

   Our final answer is $2(3x-4)^2$. To check our work, we multiply out $2(3x-4)^2$ to show that it equals $18x^2 - 48x + 32$.

2. For $64y^2 - 1$, we note that the G.C.F. of the terms is just 1, so there is nothing (of substance) to factor out of both terms. Since $64y^2 - 1$ is the difference of two terms, one of which is a square, we look to the Difference of Squares Formula for inspiration. Seeing $64y^2 = (8y)^2$ and $1 = 1^2$, we get

$$\begin{array}{rcll} 64y^2 - 1 & = & (8y)^2 - 1^2 & \\ & = & (8y-1)(8y+1) & \text{Difference of Squares, } a = 8y,\ b = 1 \end{array}$$

   As before, we can check our answer by multiplying out $(8y-1)(8y+1)$ to show that it equals $64y^2-1$.

3. The G.C.F. of the terms in $75t^4+30t^3+3t^2$ is $3t^2$, so we factor that out first. We identify what remains as a perfect square trinomial:

$$\begin{array}{rcll} 75t^4 + 30t^3 + 3t^2 & = & 3t^2(25t^2 + 10t + 1) & \text{Factor out G.C.F.} \\ & = & 3t^2((5t)^2 + 2(1)(5t) + 1^2) & \\ & = & 3t^2(5t+1)^2 & \text{Perfect Square Trinomial, } a = 5t,\ b = 1 \end{array}$$

   Our final answer is $3t^2(5t+1)^2$, which the reader is invited to check.

4. For $w^4z - wz^4$, we identify the G.C.F. as wz and once we factor it out a difference of cubes is revealed:

$$\begin{array}{rcll} w^4z - wz^4 & = & wz(w^3 - z^3) & \text{Factor out G.C.F.} \\ & = & wz(w-z)(w^2 + wz + z^2) & \text{Difference of Cubes, } a = w,\ b = z \end{array}$$

   Our final answer is $wz(w-z)(w^2+wz+z^2)$. The reader is strongly encouraged to multiply this out to see that it reduces to $w^4z - wz^4$.

5. The G.C.F. of the terms in $81 - 16t^4$ is just 1 so there is nothing of substance to factor out from both terms. With just a difference of two terms, we are limited to fitting this polynomial into either the Difference of Two Squares or Difference of Two Cubes formula. Since the variable here is t^4, and 4 is a multiple of 2, we can think of $t^4 = (t^2)^2$. This means that we can write $16t^4 = (4t^2)^2$ which is a perfect square. (Since 4 is not a multiple of 3, we cannot write t^4 as a perfect cube of a polynomial.) Identifying $81 = 9^2$ and $16t^4 = (4t^2)^2$, we apply the Difference of Squares Formula to get:

$$\begin{aligned} 81 - 16t^4 &= 9^2 - (4t^2)^2 \\ &= (9 - 4t^2)(9 + 4t^2) \quad \text{Difference of Squares, } a = 9,\ b = 4t^2 \end{aligned}$$

At this point, we have an opportunity to proceed further. Identifying $9 = 3^2$ and $4t^2 = (2t)^2$, we see that we have another difference of squares in the first quantity, which we can reduce. (The sum of two squares in the second quantity <u>cannot</u> be factored over the rationals.)

$$\begin{aligned} 81 - 16t^4 &= (9 - 4t^2)(9 + 4t^2) \\ &= (3^2 - (2t)^2)(9 + 4t^2) \\ &= (3 - 2t)(3 + 2t)(9 + 4t^2) \quad \text{Difference of Squares, } a = 3,\ b = 2t \end{aligned}$$

As always, the reader is encouraged to multiply out $(3 - 2t)(3 + 2t)(9 + 4t^2)$ to check the result.

6. With a G.C.F. of 1 and just two terms, $x^6 - 64$ is a candidate for both the Difference of Squares and the Difference of Cubes formulas. Notice that we can identify $x^6 = (x^3)^2$ and $64 = 8^2$ (both perfect squares), but also $x^6 = (x^2)^3$ and $64 = 4^3$ (both perfect cubes). If we follow the Difference of Squares approach, we get:

$$\begin{aligned} x^6 - 64 &= (x^3)^2 - 8^2 \\ &= (x^3 - 8)(x^3 + 8) \quad \text{Difference of Squares, } a = x^3 \text{ and } b = 8 \end{aligned}$$

At this point, we have an opportunity to use both the Difference and Sum of Cubes formulas:

$$\begin{aligned} x^6 - 64 &= (x^3 - 2^3)(x^3 + 2^3) \\ &= (x - 2)(x^2 + 2x + 2^2)(x + 2)(x^2 - 2x + 2^2) \quad \text{Sum / Difference of Cubes, } a = x,\ b = 2 \\ &= (x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4) \quad \text{Rearrange factors} \end{aligned}$$

From this approach, our final answer is $(x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4)$.

Following the Difference of Cubes Formula approach, we get

$$\begin{aligned} x^6 - 64 &= (x^2)^3 - 4^3 \\ &= (x^2 - 4)((x^2)^2 + 4x^2 + 4^2) \quad \text{Difference of Cubes, } a = x^2,\ b = 4 \\ &= (x^2 - 4)(x^4 + 4x^2 + 16) \end{aligned}$$

At this point, we recognize $x^2 - 4$ as a difference of two squares:

$$\begin{aligned} x^6 - 64 &= (x^2 - 2^2)(x^4 + 4x^2 + 16) \\ &= (x - 2)(x + 2)(x^4 + 4x^2 + 16) \quad \text{Difference of Squares, } a = x,\ b = 2 \end{aligned}$$

Unfortunately, the remaining factor $x^4 + 4x^2 + 16$ is not a perfect square trinomial - the middle term would have to be $8x^2$ for this to work - so our final answer using this approach is $(x-2)(x+2)(x^4 + 4x^2 + 16)$. This isn't as factored as our result from the Difference of Squares approach which was $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. While it is true that $x^4+4x^2+16 = (x^2-2x+4)(x^2+2x+4)$, there is no 'intuitive' way to motivate this factorization at this point.[5] The moral of the story? When given the option between using the Difference of Squares and Difference of Cubes, start with the Difference of Squares. Our final answer to this problem is $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. The reader is strongly encouraged to show that this reduces down to $x^6 - 64$ after performing all of the multiplication. □

The formulas on page 231, while useful, can only take us so far. Thus we need to review some additional factoring strategies which should be good friends from back in the day!

Additional Factoring Formulas

- **'un-F.O.I.L.ing':** Given a trinomial $Ax^2 + Bx + C$, try to reverse the F.O.I.L. process.

  That is, find a, b, c and d such that $Ax^2 + Bx + C = (ax + b)(cx + d)$.

  NOTE: This means $ac = A$, $bd = C$ and $B = ad + bc$.

- **Factor by Grouping:** If the expression contains four terms with no common factors among the four terms, try 'factor by grouping':

$$ac + bc + ad + bd = (a+b)c + (a+b)d = (a+b)(c+d)$$

The techniques of 'un-F.O.I.L.ing' and 'factoring by grouping' are difficult to describe in general but should make sense to you with enough practice. Be forewarned - like all 'Rules of Thumb', these strategies work just often enough to be useful, but you can be sure there are exceptions which will defy any advice given here and will require some 'inspiration' to solve.[6] Even though Chapter 6 will give us more powerful factoring methods, we'll find that, in the end, there is no single algorithm for factoring which works for every polynomial. In other words, there will be times when you just have to try something and see what happens.

Example 5.2.2. Factor the following polynomials completely over the integers.[7]

1. $x^2 - x - 6$
2. $2t^2 - 11t + 5$
3. $36 - 11y - 12y^2$
4. $18xy^2 - 54xy - 180x$
5. $2t^3 - 10t^2 + 3t - 15$
6. $x^4 + 4x^2 + 16$

[5] Of course, this begs the question, "How do we know $x^2 - 2x + 4$ and $x^2 + 2x + 4$ are irreducible?" (We were told so on page 231, but no reason was given.) Stay tuned! We'll get back to this in due course.

[6] Jeff will be sure to pepper the Exercises with these.

[7] This means that all of the coefficients in the factors will be integers. In a rare departure from form, Carl decided to avoid fractions in this set of examples. Don't get complacent, though, because fractions will return with a vengeance soon enough.

Solution.

1. The G.C.F. of the terms $x^2 - x - 6$ is 1 and $x^2 - x - 6$ isn't a perfect square trinomial (Think about why not.) so we try to reverse the F.O.I.L. process and look for integers a, b, c and d such that $(ax + b)(cx + d) = x^2 - x - 6$. To get started, we note that $ac = 1$. Since a and c are meant to be integers, that leaves us with either a and c both being 1, or a and c both being -1. We'll go with $a = c = 1$, since we can factor[8] the negatives into our choices for b and d. This yields $(x + b)(x + d) = x^2 - x - 6$. Next, we use the fact that $bd = -6$. The product is negative so we know that one of b or d is positive and the other is negative. Since b and d are integers, one of b or d is ± 1 and the other is ∓ 6 OR one of b or d is ± 2 and the other is ∓ 3. After some guessing and checking,[9] we find that $x^2 - x - 6 = (x + 2)(x - 3)$.

2. As with the previous example, we check the G.C.F. of the terms in $2t^2 - 11t + 5$, determine it to be 1 and see that the polynomial doesn't fit the pattern for a perfect square trinomial. We now try to find integers a, b, c and d such that $(at + b)(ct + d) = 2t^2 - 11t + 5$. Since $ac = 2$, we have that one of a or c is 2, and the other is 1. (Once again, we ignore the negative options.) At this stage, there is nothing really distinguishing a from c so we choose $a = 2$ and $c = 1$. Now we look for b and d so that $(2t + b)(t + d) = 2t^2 - 11t + 5$. We know $bd = 5$ so one of b or d is ± 1 and the other ± 5. Given that bd is positive, b and d must have the same sign. The negative middle term $-11t$ guides us to guess $b = -1$ and $d = -5$ so that we get $(2t - 1)(t - 5) = 2t^2 - 11t + 5$. We verify our answer by multiplying.[10]

3. Once again, we check for a nontrivial G.C.F. and see if $36 - 11y - 12y^2$ fits the pattern of a perfect square. Twice disappointed, we rewrite $36 - 11y - 12y^2 = -12y^2 - 11y + 36$ for notational convenience. We now look for integers a, b, c and d such that $-12y^2 - 11y + 36 = (ay + b)(cy + d)$. Since $ac = -12$, we know that one of a or c is ± 1 and the other ± 12 OR one of them is ± 2 and the other is ± 6 OR one of them is ± 3 while the other is ± 4. Since their product is -12, however, we know one of them is positive, while the other is negative. To make matters worse, the constant term 36 has its fair share of factors, too. Our answers for b and d lie among the pairs ± 1 and ± 36, ± 2 and ± 18, ± 4 and ± 9, or ± 6. Since we know one of a or c will be negative, we can simplify our choices for b and d and just look at the positive possibilities. After some guessing and checking,[11] we find $(-3y + 4)(4y + 9) = -12y^2 - 11y + 36$.

4. Since the G.C.F. of the terms in $18xy^2 - 54xy - 180x$ is $18x$, we begin the problem by factoring it out first: $18xy^2 - 54xy - 180x = 18x(y^2 - 3y - 10)$. We now focus our attention on $y^2 - 3y - 10$. We can take a and c to both be 1 which yields $(y + b)(y + d) = y^2 - 3y - 10$. Our choices for b and d are among the factor pairs of -10: ± 1 and ± 10 or ± 2 and ± 5, where one of b or d is positive and the other is negative. We find $(y - 5)(y + 2) = y^2 - 3y - 10$. Our final answer is $18xy^2 - 54xy - 180x = 18x(y - 5)(y + 2)$.

[8] Pun intended!

[9] The authors have seen some strange gimmicks that allegedly help students with this step. We don't like them so we're sticking with good old-fashioned guessing and checking.

[10] That's the 'checking' part of 'guessing and checking'.

[11] Some of these guesses can be more 'educated' than others. Since the middle term is relatively 'small,' we don't expect the 'extreme' factors of 36 and 12 to appear, for instance.

5. Since $2t^3 - 10t^2 - 3t + 15$ has four terms, we are pretty much resigned to factoring by grouping. The strategy here is to factor out the G.C.F. from two *pairs* of terms, and see if this reveals a common factor. If we group the first two terms, we can factor out a $2t^2$ to get $2t^3 - 10t^2 = 2t^2(t-5)$. We now try to factor something out of the last two terms that will leave us with a factor of $(t-5)$. Sure enough, we can factor out a -3 from both: $-3t + 15 = -3(t-5)$. Hence, we get

$$2t^3 - 10t^2 - 3t + 15 = 2t^2(t-5) - 3(t-5) = (2t^2-3)(t-5)$$

Now the question becomes can we factor $2t^2 - 3$ over the integers? This would require integers a, b, c and d such that $(at+b)(ct+d) = 2t^2 - 3$. Since $ab = 2$ and $cd = -3$, we aren't left with many options - in fact, we really have only four choices: $(2t-1)(t+3)$, $(2t+1)(t-3)$, $(2t-3)(t+1)$ and $(2t+3)(t-1)$. None of these produces $2t^2 - 3$ - which means it's irreducible over the integers - thus our final answer is $(2t^2-3)(t-5)$.

6. Our last example, $x^4 + 4x^2 + 16$, is our old friend from Example 5.2.1. As noted there, it is not a perfect square trinomial, so we could try to reverse the F.O.I.L. process. This is complicated by the fact that our highest degree term is x^4, so we would have to look at factorizations of the form $(x+b)(x^3+d)$ as well as $(x^2+b)(x^2+d)$. We leave it to the reader to show that neither of those work. This is an example of where 'trying something' pays off. Even though we've stated that it is not a perfect square trinomial, it's pretty close. Identifying $x^4 = (x^2)^2$ and $16 = 4^2$, we'd have $(x^2+4)^2 = x^4 + 8x^2 + 16$, but instead of $8x^2$ as our middle term, we only have $4x^2$. We could add in the extra $4x^2$ we need, but to keep the balance, we'd have to subtract it off. Doing so produces an unexpected opportunity:

$$\begin{array}{rcll} x^4 + 4x^2 + 16 & = & x^4 + 4x^2 + 16 + (4x^2 - 4x^2) & \text{Adding and subtracting the same term} \\ & = & x^4 + 8x^2 + 16 - 4x^2 & \text{Rearranging terms} \\ & = & (x^2+4)^2 - (2x)^2 & \text{Factoring perfect square trinomial} \\ & = & [(x^2+4) - 2x][(x^2+4) + 2x] & \text{Difference of Squares: } a = (x^2+4),\ b = 2x \\ & = & (x^2 - 2x + 4)(x^2 + 2x + 4) & \text{Rearraging terms} \end{array}$$

We leave it to the reader to check that neither $x^2 - 2x + 4$ nor $x^2 + 2x + 4$ factor over the integers, so we are done. □

5.2.1 Solving Equations by Factoring

Many students wonder why they are forced to learn how to factor. Simply put, factoring is our main tool for solving the non-linear equations which arise in many of the applications of Mathematics.[12] We use factoring in conjunction with the Zero Product Property of Real Numbers which was first stated on page 18 and is given here again for reference.

The Zero Product Property of Real Numbers: If a and b are real numbers with $ab = 0$ then either $a = 0$ or $b = 0$ or both.

[12] Also known as 'story problems' or 'real-world examples'.

Consider the equation $6x^2 + 11x = 10$. To see how the Zero Product Property is used to help us solve this equation, we first set the equation equal to zero and then apply the techniques from Example 5.2.2:

$$\begin{array}{rcll} 6x^2 + 11x & = & 10 & \\ 6x^2 + 11x - 10 & = & 0 & \text{Subtract 10 from both sides} \\ (2x+5)(3x-2) & = & 0 & \text{Factor} \\ 2x+5=0 & \text{or} & 3x-2=0 & \text{Zero Product Property} \\ & & & a = 2x+5,\, b = 3x-2 \\ x = -\frac{5}{2} & \text{or} & x = \frac{2}{3} & \end{array}$$

The reader should check that both of these solutions satisfy the original equation.

It is critical that you see the importance of setting the expression equal to 0 before factoring. Otherwise, we'd get something silly like:

$$\begin{array}{rcll} 6x^2 + 11x & = & 10 & \\ x(6x+11) & = & 10 & \text{Factor} \end{array}$$

What we **cannot** deduce from this equation is that $x = 10$ or $6x + 11 = 10$ or that $x = 2$ and $6x + 11 = 5$. (It's wrong and you should feel bad if you do it.) It is precisely because 0 plays such a special role in the arithmetic of real numbers (as the Additive Identity) that we can assume a factor is 0 when the product is 0. No other real number has that ability.

We summarize the **correct** equation solving strategy below.

Strategy for Solving Non-linear Equations

1. Put all of the nonzero terms on one side of the equation so that the other side is 0.
2. Factor.
3. Use the Zero Product Property of Real Numbers and set each factor equal to 0.
4. Solve each of the resulting equations.

Let's finish the section with a collection of examples in which we use this strategy.

Example 5.2.3. Solve the following equations.

1. $3x^2 = 35 - 16x$
2. $t = \dfrac{1+4t^2}{4}$
3. $(y-1)^2 = 2(y-1)$
4. $\dfrac{w^4}{3} = \dfrac{8w^3 - 12}{12} - \dfrac{w^2 - 4}{4}$
5. $z(z(18z+9) - 50) = 25$
6. $x^4 - 8x^2 - 9 = 0$

Solution.

1. We begin by gathering all of the nonzero terms to one side getting 0 on the other and then we proceed to factor and apply the Zero Product Property.

$$\begin{array}{rclr} 3x^2 & = & 35 - 16x & \\ 3x^2 + 16x - 35 & = & 0 & \text{Add } 16x\text{, subtract } 35 \\ (3x-5)(x+7) & = & 0 & \text{Factor} \\ 3x - 5 = 0 & \text{or} & x + 7 = 0 & \text{Zero Product Property} \\ x = \frac{5}{3} & \text{or} & x = -7 & \end{array}$$

We check our answers by substituting each of them into the original equation. Plugging in $x = \frac{5}{3}$ yields $\frac{25}{3}$ on both sides while $x = -7$ gives 147 on both sides.

2. To solve $t = \frac{1+4t^2}{4}$, we first clear fractions then move all of the nonzero terms to one side of the equation, factor and apply the Zero Product Property.

$$\begin{array}{rclr} t & = & \dfrac{1+4t^2}{4} & \\ 4t & = & 1 + 4t^2 & \text{Clear fractions (multiply by 4)} \\ 0 & = & 1 + 4t^2 - 4t & \text{Subtract 4} \\ 0 & = & 4t^2 - 4t + 1 & \text{Rearrange terms} \\ 0 & = & (2t-1)^2 & \text{Factor (Perfect Square Trinomial)} \end{array}$$

At this point, we get $(2t-1)^2 = (2t-1)(2t-1) = 0$, so, the Zero Product Property gives us $2t - 1 = 0$ in both cases.[13] Our final answer is $t = \frac{1}{2}$, which we invite the reader to check.

3. Following the strategy outlined above, the first step to solving $(y-1)^2 = 2(y-1)$ is to gather the nonzero terms on one side of the equation with 0 on the other side and factor.

$$\begin{array}{rclr} (y-1)^2 & = & 2(y-1) & \\ (y-1)^2 - 2(y-1) & = & 0 & \text{Subtract } 2(y-1) \\ (y-1)[(y-1)-2] & = & 0 & \text{Factor out G.C.F.} \\ (y-1)(y-3) & = & 0 & \text{Simplify} \\ y - 1 = 0 & \text{or} & y - 3 = 0 & \\ y = 1 & \text{or} & y = 3 & \end{array}$$

Both of these answers are easily checked by substituting them into the original equation.

An alternative method to solving this equation is to begin by dividing both sides by $(y-1)$ to simplify things outright. As we saw in Example 3.1.1, however, whenever we divide by a variable quantity, we make the explicit assumption that this quantity is nonzero. Thus we must stipulate that $y - 1 \neq 0$.

$$\begin{array}{rcll} \dfrac{(y-1)^2}{(y-1)} & = & \dfrac{2(y-1)}{(y-1)} & \text{Divide by } (y-1) \text{ - this assumes } (y-1) \neq 0 \\ y - 1 & = & 2 & \\ y & = & 3 & \end{array}$$

[13]More generally, given a positive power p, the only solution to $x^p = 0$ is $x = 0$.

Note that in this approach, we obtain the $y = 3$ solution, but we 'lose' the $y = 1$ solution. How did that happen? Assuming $y - 1 \neq 0$ is equivalent to assuming $y \neq 1$. This is an issue because $y = 1$ is a solution to the original equation and it was 'divided out' too early. The moral of the story? If you decide to divide by a variable expression, double check that you aren't excluding any solutions.[14]

4. Proceeding as before, we clear fractions, gather the nonzero terms on one side of the equation, have 0 on the other and factor.

$$\begin{array}{rcll}
\dfrac{w^4}{3} & = & \dfrac{8w^3-12}{12} - \dfrac{w^2-4}{4} & \\
12\left(\dfrac{w^4}{3}\right) & = & 12\left(\dfrac{8w^3-12}{12} - \dfrac{w^2-4}{4}\right) & \text{Multiply by 12} \\
4w^4 & = & (8w^3-12) - 3(w^2-4) & \text{Distribute} \\
4w^4 & = & 8w^3 - 12 - 3w^2 + 12 & \text{Distribute} \\
0 & = & 8w^3 - 12 - 3w^2 + 12 - 4w^4 & \text{Subtract } 4w^4 \\
0 & = & 8w^3 - 3w^2 - 4w^4 & \text{Gather like terms} \\
0 & = & w^2(8w - 3 - 4w^2) & \text{Factor out G.C.F.}
\end{array}$$

At this point, we apply the Zero Product Property to deduce that $w^2 = 0$ or $8w - 3 - 4w^2 = 0$. From $w^2 = 0$, we get $w = 0$. To solve $8w - 3 - 4w^2 = 0$, we rearrange terms and factor: $-4w^2 + 8w - 3 = (2w - 1)(-2w + 3) = 0$. Applying the Zero Product Property again, we get $2w - 1 = 0$ (which gives $w = \frac{1}{2}$), or $-2w + 3 = 0$ (which gives $w = \frac{3}{2}$). Our final answers are $w = 0$, $w = \frac{1}{2}$ and $w = \frac{3}{2}$. The reader is encouraged to check each of these answers in the original equation. (You need the practice with fractions!)

5. For our next example, we begin by subtracting the 25 from both sides then work out the indicated operations before factoring by grouping.

$$\begin{array}{rcll}
z(z(18z+9)-50) & = & 25 & \\
z(z(18z+9)-50) - 25 & = & 0 & \text{Subtract 25} \\
z(18z^2+9z-50) - 25 & = & 0 & \text{Distribute} \\
18z^3 + 9z^2 - 50z - 25 & = & 0 & \text{Distribute} \\
9z^2(2z+1) - 25(2z+1) & = & 0 & \text{Factor} \\
(9z^2-25)(2z+1) & = & 0 & \text{Factor}
\end{array}$$

At this point, we use the Zero Product Property and get $9z^2 - 25 = 0$ or $2z + 1 = 0$. The latter gives $z = -\frac{1}{2}$ whereas the former factors as $(3z - 5)(3z + 5) = 0$. Applying the Zero Product Property again gives $3z - 5 = 0$ (so $z = \frac{5}{3}$) or $3z + 5 = 0$ (so $z = -\frac{5}{3}$.) Our final answers are $z = -\frac{1}{2}$, $z = \frac{5}{3}$ and $z = -\frac{5}{3}$, each of which is good fun to check.

6. The nonzero terms of the equation $x^4 - 8x^2 - 9 = 0$ are already on one side of the equation so we proceed to factor. This trinomial doesn't fit the pattern of a perfect square so we attempt to reverse

[14] You will see other examples throughout this text where dividing by a variable quantity does more harm than good. Keep this basic one in mind as you move on in your studies - it's a good cautionary tale.

the F.O.I.L.ing process. With an x^4 term, we have two possible forms to try: $(ax^2+b)(cx^2+d)$ and $(ax^3+b)(cx+d)$. We leave it to you to show that $(ax^3+b)(cx+d)$ does not work and we show that $(ax^2+b)(cx^2+d)$ does.

Since the coefficient of x^4 is 1, we take $a=c=1$. The constant term is -9 so we know b and d have opposite signs and our choices are limited to two options: either b and d come from ± 1 and ± 9 OR one is 3 while the other is -3. After some trial and error, we get $x^4-8x^2-9=(x^2-9)(x^2+1)$. Hence $x^4-8x^2-9=0$ reduces to $(x^2-9)(x^2+1)=0$. The Zero Product Property tells us that either $x^2-9=0$ or $x^2+1=0$. To solve the former, we factor: $(x-3)(x+3)=0$, so $x-3=0$ (hence, $x=3$) or $x+3=0$ (hence, $x=-3$). The equation $x^2+1=0$ has no (real) solution, since for any real number x, x^2 is always 0 or greater. Thus x^2+1 is always positive. Our final answers are $x=3$ and $x=-3$. As always, the reader is invited to check both answers in the original equation. $\square$

5.2.2 Exercises

In Exercises 1 - 30, factor completely over the integers. Check your answer by multiplication.

1. $2x - 10x^2$
2. $12t^5 - 8t^3$
3. $16xy^2 - 12x^2y$
4. $5(m+3)^2 - 4(m+3)^3$
5. $(2x-1)(x+3) - 4(2x-1)$
6. $t^2(t-5) + t - 5$
7. $w^2 - 121$
8. $49 - 4t^2$
9. $81t^4 - 16$
10. $9z^2 - 64y^4$
11. $(y+3)^2 - 4y^2$
12. $(x+h)^3 - (x+h)$
13. $y^2 - 24y + 144$
14. $25t^2 + 10t + 1$
15. $12x^3 - 36x^2 + 27x$
16. $m^4 + 10m^2 + 25$
17. $27 - 8x^3$
18. $t^6 + t^3$
19. $x^2 - 5x - 14$
20. $y^2 - 12y + 27$
21. $3t^2 + 16t + 5$
22. $6x^2 - 23x + 20$
23. $35 + 2m - m^2$
24. $7w - 2w^2 - 3$
25. $3m^3 + 9m^2 - 12m$
26. $x^4 + x^2 - 20$
27. $4(t^2-1)^2 + 3(t^2-1) - 10$
28. $x^3 - 5x^2 - 9x + 45$
29. $3t^2 + t - 3 - t^3$
30. [15] $y^4 + 5y^2 + 9$

In Exercises 31 - 45, find all rational number solutions. Check your answers.

31. $(7x+3)(x-5) = 0$
32. $(2t-1)^2(t+4) = 0$
33. $(y^2+4)(3y^2+y-10) = 0$
34. $4t = t^2$
35. $y + 3 = 2y^2$
36. $26x = 8x^2 + 21$
37. $16x^4 = 9x^2$
38. $w(6w+11) = 10$
39. $2w^2 + 5w + 2 = -3(2w+1)$
40. $x^2(x-3) = 16(x-3)$
41. $(2t+1)^3 = (2t+1)$
42. $a^4 + 4 = 6 - a^2$
43. $\dfrac{8t^2}{3} = 2t + 3$
44. $\dfrac{x^3+x}{2} = \dfrac{x^2+1}{3}$
45. $\dfrac{y^4}{3} - y^2 = \dfrac{3}{2}(y^2+3)$
46. With help from your classmates, factor $4x^4 + 8x^2 + 9$.
47. With help from your classmates, find an equation which has 3, $-\frac{1}{2}$, and 117 as solutions.

[15] $y^4 + 5y^2 + 9 = (y^4 + 6y^2 + 9) - y^2$

5.2.3 Answers

1. $2x(1-5x)$
2. $4t^3(3t^2-2)$
3. $4xy(4y-3x)$
4. $-(m+3)^2(4m+7)$
5. $(2x-1)(x-1)$
6. $(t-5)(t^2+1)$
7. $(w-11)(w+11)$
8. $(7-2t)(7+2t)$
9. $(3t-2)(3t+2)(9t^2+4)$
10. $(3z-8y^2)(3z+8y^2)$
11. $-3(y-3)(y+1)$
12. $(x+h)(x+h-1)(x+h+1)$
13. $(y-12)^2$
14. $(5t+1)^2$
15. $3x(2x-3)^2$
16. $(m^2+5)^2$
17. $(3-2x)(9+6x+4x^2)$
18. $t^3(t+1)(t^2-t+1)$
19. $(x-7)(x+2)$
20. $(y-9)(y-3)$
21. $(3t+1)(t+5)$
22. $(2x-5)(3x-4)$
23. $(7-m)(5+m)$
24. $(-2w+1)(w-3)$
25. $3m(m-1)(m+4)$
26. $(x-2)(x+2)(x^2+5)$
27. $(2t-3)(2t+3)(t^2+1)$
28. $(x-3)(x+3)(x-5)$
29. $(t-3)(1-t)(1+t)$
30. $(y^2-y+3)(y^2+y+3)$
31. $x=-\frac{3}{7}$ or $x=5$
32. $t=\frac{1}{2}$ or $t=-4$
33. $y=\frac{5}{3}$ or $y=-2$
34. $t=0$ or $t=4$
35. $y=-1$ or $y=\frac{3}{2}$
36. $x=\frac{3}{2}$ or $x=\frac{7}{4}$
37. $x=0$ or $x=\pm\frac{3}{4}$
38. $w=-\frac{5}{2}$ or $w=\frac{2}{3}$
39. $w=-5$ or $w=-\frac{1}{2}$
40. $x=3$ or $x=\pm 4$
41. $t=-1$, $t=-\frac{1}{2}$, or $t=0$
42. $a=\pm 1$
43. $t=-\frac{3}{4}$ or $t=\frac{3}{2}$
44. $x=\frac{2}{3}$
45. $y=\pm 3$

5.3 Quadratic Equations

In Section 5.2.1, we reviewed how to solve basic non-linear equations by factoring. The astute reader should have noticed that all of the equations in that section were carefully constructed so that the polynomials could be factored using the integers. To demonstrate just how contrived the equations had to be, we can solve $2x^2 + 5x - 3 = 0$ by factoring, $(2x - 1)(x + 3) = 0$, from which we obtain $x = \frac{1}{2}$ and $x = -3$. If we change the 5 to a 6 and try to solve $2x^2 + 6x - 3 = 0$, however, we find that this polynomial doesn't factor over the integers and we are stuck. It turns out that there are two real number solutions to this equation, but they are *irrational* numbers, and the goal of this section is to review the techniques which allow us to find these solutions.[1] In this section, we focus our attention on **quadratic** equations.

Definition 5.4. An equation is said to be **quadratic** in a variable x if it can be written in the form $ax^2 + bx + c = 0$ where a, b and c are expressions which do not involve x and $a \neq 0$.

Think of quadratic equations as equations that are one degree up from linear equations - instead of the highest power of x being just $x = x^1$, it's x^2. The simplest class of quadratic equations to solve are the ones in which $b = 0$. In that case, we have the following.

Solving Quadratic Equations by Extracting Square Roots

If c is a real number with $c \geq 0$, the solutions to $x^2 = c$ are $x = \pm\sqrt{c}$.

Note: If $c < 0$, $x^2 = c$ has no real number solutions.

There are a couple different ways to see why Extracting Square Roots works, both of which are demonstrated by solving the equation $x^2 = 3$. If we follow the procedure outlined in the previous section, we subtract 3 from both sides to get $x^2 - 3 = 0$ and we now try to factor $x^2 - 3$. As mentioned in the remarks following Definition 5.3, we could think of $x^2 - 3 = x^2 - (\sqrt{3})^2$ and apply the Difference of Squares formula to factor $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. We solve $(x - \sqrt{3})(x + \sqrt{3}) = 0$ by using the Zero Product Property as before by setting each factor equal to zero: $x - \sqrt{3} = 0$ and $x + \sqrt{3} - 0$. We get the answers $x = \pm\sqrt{3}$. In general, if $c \geq 0$, then $\sqrt{c}$ is a real number, so $x^2 - c = x^2 - (\sqrt{c})^2 = (x - \sqrt{c})(x + \sqrt{c})$. Replacing the '3' with 'c' in the above discussion gives the general result.

Another way to view this result is to visualize 'taking the square root' of both sides: since $x^2 = c$, $\sqrt{x^2} = \sqrt{c}$. How do we simplify $\sqrt{x^2}$? We have to exercise a bit of caution here. Note that $\sqrt{(5)^2}$ and $\sqrt{(-5)^2}$ both simplify to $\sqrt{25} = 5$. In both cases, $\sqrt{x^2}$ returned a *positive* number, since the negative in -5 was 'squared away' *before* we took the square root. In other words, $\sqrt{x^2}$ is x if x is positive, or, if x is negative, we make x positive - that is, $\sqrt{x^2} = |x|$, the absolute value of x. So from $x^2 = 3$, we 'take the square root' of both sides of the equation to get $\sqrt{x^2} = \sqrt{3}$. This simplifies to $|x| = \sqrt{3}$, which by Theorem 4.2 is equivalent to $x = \sqrt{3}$ or $x = -\sqrt{3}$. Replacing the '3' in the previous argument with 'c,' gives the general result.

As you might expect, Extracting Square Roots can be applied to more complicated equations. Consider the equation below. We can solve it by Extracting Square Roots provided we first isolate the quantity that

[1]While our discussion in this section departs from factoring, we'll see in Chapter 6 that the same correspondence between factoring and solving equations holds whether or not the polynomial factors over the integers.

is being squared :

$$\begin{aligned}
2\left(x+\frac{3}{2}\right)^2-\frac{15}{2} &= 0 & \\
2\left(x+\frac{3}{2}\right)^2 &= \frac{15}{2} & \text{Add } \frac{15}{2} \\
\left(x+\frac{3}{2}\right)^2 &= \frac{15}{4} & \text{Divide by 2} \\
x+\frac{3}{2} &= \pm\sqrt{\frac{15}{4}} & \text{Extract Square Roots} \\
x+\frac{3}{2} &= \pm\frac{\sqrt{15}}{2} & \text{Property of Radicals} \\
x &= -\frac{3}{2}\pm\frac{\sqrt{15}}{2} & \text{Subtract } \frac{3}{2} \\
x &= -\frac{3\pm\sqrt{15}}{2} & \text{Add fractions}
\end{aligned}$$

Let's return to the equation $2x^2+6x-3=0$ from the beginning of the section. We leave it to the reader to expand the left side and show that

$$2\left(x+\frac{3}{2}\right)^2-\frac{15}{2}=2x^2+6x-3.$$

In other words, we can solve $2x^2+6x-3=0$ by *transforming* into an equivalent equation. This process, you may recall, is called 'Completing the Square.' We'll revisit Completing the Square in Section 5.4 in more generality and for a different purpose but for now we revisit the steps needed to complete the square to solve a quadratic equation.

Solving Quadratic Equations: Completing the Square

To solve a quadratic equation $ax^2+bx+c=0$ by Completing the Square:

1. Subtract the constant c from both sides.
2. Divide both sides by a, the coefficient of x^2. (Remember: $a\neq 0$.)
3. Add $\left(\frac{b}{2a}\right)^2$ to both sides of the equation. (That's half the coefficient of x, squared.)
4. Factor the left hand side of the equation as $\left(x+\frac{b}{2a}\right)^2$.
5. Extract Square Roots.
6. Subtract $\frac{b}{2a}$ from both sides.

To refresh our memories, we apply this method to solve $3x^2 - 24x + 5 = 0$:

$$\begin{array}{rcll}
3x^2 - 24x + 5 & = & 0 & \\
3x^2 - 24x & = & -5 & \text{Subtract } c = 55 \\
x^2 - 8x & = & -\dfrac{5}{3} & \text{Divide by } a = 3 \\
x^2 - 8x + 16 & = & -\dfrac{5}{3} + 16 & \text{Add } \left(\frac{b}{2a}\right)^2 = (-4)^2 = 16 \\
(x-4)^2 & = & \dfrac{43}{3} & \text{Factor: Perfect Square Trinomial} \\
x - 4 & = & \pm\sqrt{\dfrac{43}{3}} & \text{Extract Square Roots} \\
x & = & 4 \pm \sqrt{\dfrac{43}{3}} & \text{Add } 4
\end{array}$$

At this point, we use properties of fractions and radicals to 'rationalize' the denominator:[2]

$$\sqrt{\frac{43}{3}} = \sqrt{\frac{43 \cdot 3}{3 \cdot 3}} = \frac{\sqrt{129}}{\sqrt{9}} = \frac{\sqrt{129}}{3}$$

We can now get a common (integer) denominator which yields:

$$x = 4 \pm \sqrt{\frac{43}{3}} = 4 \pm \frac{\sqrt{129}}{3} = \frac{12 \pm \sqrt{129}}{3}$$

The key to Completing the Square is that the procedure always produces a perfect square trinomial. To see why this works *every single time*, we start with $ax^2 + bx + c = 0$ and follow the procedure:

$$\begin{array}{rcll}
ax^2 + bx + c & = & 0 & \\
ax^2 + bx & = & -c & \text{Subtract } c \\
x^2 + \dfrac{bx}{a} & = & -\dfrac{c}{a} & \text{Divide by } a \neq 0 \\
x^2 + \dfrac{bx}{a} + \left(\dfrac{b}{2a}\right)^2 & = & -\dfrac{c}{a} + \left(\dfrac{b}{2a}\right)^2 & \text{Add } \left(\dfrac{b}{2a}\right)^2
\end{array}$$

(Hold onto the line above for a moment.) Here's the heart of the method - we need to show that

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$$

To show this, we start with the right side of the equation and apply the Perfect Square Formula from Theorem 5.3

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 \checkmark$$

[2] Recall that this means we want to get a denominator with rational (more specifically, integer) numbers.

With just a few more steps we can solve the general equation $ax^2 + bx + c = 0$ so let's pick up the story where we left off. (The line on the previous page we told you to hold on to.)

$$\begin{aligned}
x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 & & \\
\left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} & & \text{Factor: Perfect Square Trinomial} \\
\left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} & & \text{Get a common denominator} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} & & \text{Add fractions} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} & & \text{Extract Square Roots} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} & & \text{Properties of Radicals} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} & & \text{Subtract } \frac{b}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} & & \text{Add fractions.}
\end{aligned}$$

Lo and behold, we have derived the legendary **Quadratic Formula**!

Theorem 5.5. Quadratic Formula: The solution(s) to $ax^2 + bx + c = 0$ with $a \neq 0$ is/are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can check our earlier solutions to $2x^2 + 6x - 3 = 0$ and $3x^2 - 24x + 5 = 0$ using the Quadratic Formula. For $2x^2 + 6x - 3 = 0$, we identify $a = 2$, $b = 6$ and $c = -3$. The quadratic formula gives:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(2)(-3)}}{2(2)} - \frac{-6 \pm \sqrt{36 + 24}}{4} = \frac{-6 \pm \sqrt{60}}{4}$$

Using properties of radicals ($\sqrt{60} = 2\sqrt{15}$), this reduces to $\frac{2(-3 \pm \sqrt{15})}{4} = \frac{-3 \pm \sqrt{15}}{2}$. We leave it to the reader to show these two answers are the same as $-\frac{3 \pm \sqrt{15}}{2}$, as required.[3]

For $3x^2 - 24x + 5 = 0$, we identify $a = 3$, $b = -24$ and $c = 5$. Here, we get:

$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(3)(5)}}{2(3)} = \frac{24 \pm \sqrt{516}}{6}$$

Since $\sqrt{516} = 2\sqrt{129}$, this reduces to $x = \frac{12 \pm \sqrt{129}}{3}$.

[3]Think about what $-(3 \pm \sqrt{15})$ is really telling you.

It is worth noting that the Quadratic Formula applies to all quadratic equations - even ones we could solve using other techniques. For example, to solve $2x^2 + 5x - 3 = 0$ we identify $a = 2$, $b = 5$ and $c = -3$. Plugging those into the Quadratic Formula yields:

$$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4}$$

At this point, we have $x = \frac{-5+7}{4} = \frac{1}{2}$ and $x = \frac{-5-7}{4} = \frac{-12}{4} = -3$ - the same two answers we obtained factoring. We can also use it to solve $x^2 = 3$, if we wanted to. From $x^2 - 3 = 0$, we have $a = 1$, $b = 0$ and $c = -3$. The Quadratic Formula produces

$$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(3)}}{2(1)} = \frac{\pm\sqrt{12}}{2} = \pm\frac{2\sqrt{3}}{2} = \pm\sqrt{3}$$

As this last example illustrates, while the Quadratic Formula *can* be used to solve every quadratic equation, that doesn't mean it *should* be used. Many times other methods are more efficient. We now provide a more comprehensive approach to solving Quadratic Equations.

Strategies for Solving Quadratic Equations

- If the variable appears in the squared term only, isolate it and Extract Square Roots.
- Otherwise, put the nonzero terms on one side of the equation so that the other side is 0.
  - Try factoring.
  - If the expression doesn't factor easily, use the Quadratic Formula.

The reader is encouraged to pause for a moment to think about why 'Completing the Square' doesn't appear in our list of strategies despite the fact that we've spent the majority of the section so far talking about it.[4] Let's get some practice solving quadratic equations, shall we?

Example 5.3.1. Find all real number solutions to the following equations.

1. $3 - (2w - 1)^2 = 0$
2. $5x - x(x - 3) = 7$
3. $(y - 1)^2 = 2 - \dfrac{y + 2}{3}$
4. $5(25 - 21x) = \dfrac{59}{4} - 25x^2$
5. $-4.9t^2 + 10t\sqrt{3} + 2 = 0$
6. $2x^2 = 3x^4 - 6$

Solution.

1. Since $3 - (2w - 1)^2 = 0$ contains a perfect square, we isolate it first then extract square roots:

$$\begin{array}{rclr}
3 - (2w-1)^2 & = & 0 & \\
3 & = & (2w-1)^2 & \text{Add } (2w-1)^2 \\
\pm\sqrt{3} & = & 2w - 1 & \text{Extract Square Roots} \\
1 \pm \sqrt{3} & = & 2w & \text{Add } 1 \\
\dfrac{1 \pm \sqrt{3}}{2} & = & w & \text{Divide by } 2
\end{array}$$

[4] Unacceptable answers include "Jeff and Carl are mean" and "It was one of Carl's Pedantic Rants".

We find our two answers $w = \frac{1 \pm \sqrt{3}}{2}$. The reader is encouraged to check both answers by substituting each into the original equation.[5]

2. To solve $5x - x(x-3) = 7$, we perform the indicated operations and set one side equal to 0.

$$\begin{array}{rclr} 5x - x(x-3) & = & 7 & \\ 5x - x^2 + 3x & = & 7 & \text{Distribute} \\ -x^2 + 8x & = & 7 & \text{Gather like terms} \\ -x^2 + 8x - 7 & = & 0 & \text{Subtract } 7 \end{array}$$

At this point, we attempt to factor and find $-x^2 + 8x - 7 = (x-1)(-x+7)$. Using the Zero Product Property, we get $x - 1 = 0$ or $-x + 7 = 0$. Our answers are $x = 1$ or $x = 7$, which are easily verified.

3. Even though we have a perfect square in $(y-1)^2 = 2 - \frac{y+2}{3}$, Extracting Square Roots won't help matters since we have a y on the other side of the equation. Our strategy here is to perform the indicated operations (and clear the fraction for good measure) and get 0 on one side of the equation.

$$\begin{array}{rclr} (y-1)^2 & = & 2 - \dfrac{y+2}{3} & \\ y^2 - 2y + 1 & = & 2 - \dfrac{y+2}{3} & \text{Perfect Square Trinomial} \\ 3(y^2 - 2y + 1) & = & 3\left(2 - \dfrac{y+2}{3}\right) & \text{Multiply by } 3 \\ 3y^2 - 6y + 3 & = & 6 - 3\left(\dfrac{y+2}{3}\right) & \text{Distribute} \\ 3y^2 - 6y + 3 & = & 6 - (y+2) & \\ 3y^2 - 6y + 3 - 6 + (y+2) & = & 0 & \text{Subtract } 6\text{, Add } (y+2) \\ 3y^2 - 5y - 1 & = & 0 & \end{array}$$

A cursory attempt at factoring bears no fruit, so we run this through the Quadratic Formula with $a = 3$, $b = -5$ and $c = -1$.

$$\begin{array}{rcl} y & = & \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} \\ y & = & \dfrac{5 \pm \sqrt{25 + 12}}{6} \\ y & = & \dfrac{5 \pm \sqrt{37}}{6} \end{array}$$

Since 37 is prime, we have no way to reduce $\sqrt{37}$. Thus, our final answers are $y = \frac{5 \pm \sqrt{37}}{6}$. The reader is encouraged to supply the details of the challenging verification of the answers.

[5]It's excellent practice working with radicals and fractions so we really, *really* want you to take the time to do it.

4. We proceed as before; our goal is to gather the nonzero terms on one side of the equation.

$$\begin{array}{rcll}
5(25-21x) & = & \frac{59}{4}-25x^2 & \\
125-105x & = & \frac{59}{4}-25x^2 & \text{Distribute} \\
4(125-105x) & = & 4\left(\frac{59}{4}-25x^2\right) & \text{Multiply by 4} \\
500-420x & = & 59-100x^2 & \text{Distribute} \\
500-420x-59+100x^2 & = & 0 & \text{Subtract 59, Add } 100x^2 \\
100x^2-420x+441 & = & 0 & \text{Gather like terms}
\end{array}$$

With highly composite numbers like 100 and 441, factoring seems inefficient at best,[6] so we apply the Quadratic Formula with $a = 100$, $b = -420$ and $c = 441$:

$$\begin{array}{rcl}
x & = & \frac{-(-420)\pm\sqrt{(-420)^2-4(100)(441)}}{2(100)} \\
 & = & \frac{420\pm\sqrt{176000-176400}}{200} \\
 & = & \frac{420\pm\sqrt{0}}{200} \\
 & = & \frac{420\pm 0}{200} \\
 & = & \frac{420}{200} \\
 & = & \frac{21}{10}
\end{array}$$

To our surprise and delight we obtain just one answer, $x = \frac{21}{10}$.

5. Our next equation $-4.9t^2 + 10t\sqrt{3} + 2 = 0$, already has 0 on one side of the equation, but with coefficients like -4.9 and $10\sqrt{3}$, factoring with integers is not an option. We could make things a *bit* easier by clearing the decimal (by multiplying through by 10) to get $-49t^2 + 100t\sqrt{3} + 20 = 0$ but we simply cannot rid ourselves of the irrational number $\sqrt{3}$. The Quadratic Formula is our only recourse. With $a = -49$, $b = 100\sqrt{3}$ and $c = 20$ we get:

[6] This is actually the Perfect Square Trinomial $(10x-21)^2$.

$$\begin{aligned}
t &= \frac{-100\sqrt{3} \pm \sqrt{(100\sqrt{3})^2 - 4(-49)(20)}}{2(-49)} \\
&= \frac{-100\sqrt{3} \pm \sqrt{30000 + 3920}}{-98} \\
&= \frac{-100\sqrt{3} \pm \sqrt{33920}}{-98} \\
&= \frac{-100\sqrt{3} \pm 8\sqrt{530}}{-98} \\
&= \frac{2(-50\sqrt{3} \pm 4\sqrt{530})}{2(-49)} \\
&= \frac{-50\sqrt{3} \pm 4\sqrt{530}}{-49} && \text{Reduce} \\
&= \frac{-(-50\sqrt{3} \pm 4\sqrt{530})}{49} && \text{Properties of Negatives} \\
&= \frac{50\sqrt{3} \mp 4\sqrt{530}}{49} && \text{Distribute}
\end{aligned}$$

You'll note that when we 'distributed' the negative in the last step, we changed the '$\pm$' to a '$\mp$.' While this is technically correct, at the end of the day both symbols mean 'plus or minus',[7] so we can write our answers as $t = \frac{50\sqrt{3} \pm 4\sqrt{530}}{49}$. Checking these answers are a true test of arithmetic mettle.

6. At first glance, the equation $2x^2 = 3x^4 - 6$ seems misplaced. The highest power of the variable x here is 4, not 2, so this equation isn't a quadratic equation - at least not in terms of the variable x. It is, however, an example of an equation that is 'Quadratic in Disguise'.[8] We introduce a new variable u to help us see the pattern - specifically we let $u = x^2$. Thus $u^2 = (x^2)^2 = x^4$. So in terms of the variable u, the equation $2x^2 = 3x^4 - 6$ is $2u = 3u^2 - 6$. The latter is a quadratic equation, which we can solve using the usual techniques:

$$\begin{aligned}
2u &= 3u^2 - 6 \\
0 &= 3u^2 - 2u - 6 && \text{Subtract } 2u
\end{aligned}$$

After a few attempts at factoring, we resort to the Quadratic Formula with $a = 3$, $b = -2$ and $c = -6$

[7] There are instances where we need both symbols, however. For example, the Sum and Difference of Cubes Formulas (page 231) can be written as a single formula: $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$. In this case, all of the 'top' symbols are read to give the sum formula; the 'bottom' symbols give the difference formula.

[8] More formally, **quadratic in form.** Carl likes 'Quadratics in Disguise' since it reminds him of the tagline of one of his beloved childhood cartoons and toy lines.

to get the following:

$$\begin{aligned}
u &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} & \\
&= \frac{2 \pm \sqrt{4+72}}{6} & \\
&= \frac{2 \pm \sqrt{76}}{6} & \\
&= \frac{2 \pm \sqrt{4 \cdot 19}}{6} & \\
&= \frac{2 \pm 2\sqrt{19}}{6} & \text{Properties of Radicals} \\
&= \frac{2(1 \pm \sqrt{19})}{2(3)} & \text{Factor} \\
&= \frac{1 \pm \sqrt{19}}{3} & \text{Reduce}
\end{aligned}$$

We've solved the equation for u, but what we still need to solve the original equation[9] - which means we need to find the corresponding values of x. Since $u = x^2$, we have two equations:

$$x^2 = \frac{1+\sqrt{19}}{3} \quad \text{or} \quad x^2 = \frac{1-\sqrt{19}}{3}$$

We can solve the first equation by extracting square roots to get $x = \pm\sqrt{\frac{1+\sqrt{19}}{3}}$. The second equation, however, has no real number solutions because $\frac{1-\sqrt{19}}{3}$ is a negative number. For our final answers we can rationalize the denominator[10] to get:

$$x = \pm\sqrt{\frac{1+\sqrt{19}}{3}} = \pm\sqrt{\frac{1+\sqrt{19}}{3} \cdot \frac{3}{3}} = \pm\frac{\sqrt{3+3\sqrt{19}}}{3}$$

As with the previous exercise, the very challenging check is left to the reader. □

Our last example above, the 'Quadratic in Disguise', hints that the Quadratic Formula is applicable to a wider class of equations than those which are strictly quadratic. We give some general guidelines to recognizing these beasts in the wild on the next page.

[9]Or, you've solved the equation for 'you' (u), now you have to solve it for your instructor (x).

[10]We'll say more about this technique in Section 8.1.

Identifying Quadratics in Disguise

An equation is a 'Quadratic in Disguise' if it can be written in the form: $ax^{2m} + bx^m + c = 0$.
In other words:

- There are exactly three terms, two with variables and one constant term.
- The exponent on the variable in one term is *exactly twice* the variable on the other term.

To transform a Quadratic in Disguise to a quadratic equation, let $u = x^m$ so $u^2 = (x^m)^2 = x^{2m}$. This transforms the equation into $au^2 + bu + c = 0$.

For example, $3x^6 - 2x^3 + 1 = 0$ is a Quadratic in Disguise, since $6 = 2 \cdot 3$. If we let $u = x^3$, we get $u^2 = (x^3)^2 = x^6$, so the equation becomes $3u^2 - 2u + 1 = 0$. However, $3x^6 - 2x^2 + 1 = 0$ is *not* a Quadratic in Disguise, since $6 \neq 2 \cdot 2$. The substitution $u = x^2$ yields $u^2 = (x^2)^2 = x^4$, not x^6 as required. We'll see more instances of 'Quadratics in Disguise' in later sections.

We close this section with a review of the **discriminant** of a quadratic equation as defined below.

Definition 5.5. The Discriminant: Given a quadratic equation $ax^2 + bx + c = 0$, the quantity $b^2 - 4ac$ is called the **discriminant** of the equation.

The discriminant is the radicand of the square root in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It *discriminates* between the nature and number of solutions we get from a quadratic equation. The results are summarized below.

Theorem 5.6. Discriminant Theorem: Given a Quadratic Equation $ax^2 + bx + c = 0$, let $D = b^2 - 4ac$ be the discriminant.

- If $D > 0$, there are two distinct real number solutions to the equation.
- If $D = 0$, there is one repeated real number solution.

  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-b \pm 0}{2a}$ reduce to $-\frac{b}{2a}$.
- If $D < 0$, there are no real solutions.

For example, the equation $x^2 + x - 1 = 0$ has two real number solutions since the discriminant works out to be $(1)^2 - 4(1)(-1) = 5 > 0$. This results in a $\pm\sqrt{5}$ in the Quadratic Formula which then generates two different answers. On the other hand, $x^2 + x + 1 = 0$ has no real solutions since here, the discriminant is $(1)^2 - 4(1)(1) = -3 < 0$ which generates a $\pm\sqrt{-3}$ in the Quadratic Formula. The equation $x^2 + 2x + 1 = 0$ has discriminant $(2)^2 - 4(1)(1) = 0$ so in the Quadratic Formula we get a $\pm\sqrt{0} = 0$ thereby generating just one solution. More can be said as well. For example, the discriminant of $6x^2 - x - 40 = 0$ is 961. This is a perfect square, $\sqrt{961} = 31$, which means our solutions are rational numbers. When our solutions are rational numbers, the quadratic actually factors nicely. In our example $6x^2 - x - 40 = (2x + 5)(3x - 8)$. Admittedly, if you've already computed the discriminant, you're most of the way done with the problem and probably wouldn't take the time to experiment with factoring the quadratic at this point – but we'll see another use for this analysis of the discriminant in Example 7.1.1.

5.3.1 Exercises

In Exercises 1 - 21, find all real solutions. Check your answers, as directed by your instructor.

1. $3\left(x-\frac{1}{2}\right)^2 = \frac{5}{12}$
2. $4-(5t+3)^2 = 3$
3. $3(y^2-3)^2-2 = 10$
4. $x^2+x-1 = 0$
5. $3w^2 = 2-w$
6. $y(y+4) = 1$
7. $\frac{z}{2} = 4z^2-1$
8. $0.1v^2+0.2v = 0.3$
9. $x^2 = x-1$
10. $3-t = 2(t+1)^2$
11. $(x-3)^2 = x^2+9$
12. $(3y-1)(2y+1) = 5y$
13. $w^4+3w^2-1 = 0$
14. $2x^4+x^2 = 3$
15. $(2-y)^4 = 3(2-y)^2+1$
16. $3x^4+6x^2 = 15x^3$
17. $6p+2 = p^2+3p^3$
18. $10v = 7v^3-v^5$
19. $y^2-\sqrt{8}y = \sqrt{18}y-1$
20. $x^2\sqrt{3} = x\sqrt{6}+\sqrt{12}$
21. $\frac{v^2}{3} = \frac{v\sqrt{3}}{2}+1$

In Exercises 22 - 27, find all real solutions and use a calculator to approximate your answers, rounded to two decimal places.

22. $5.54^2+b^2 = 36$
23. $\pi r^2 = 37$
24. $54 = 8r\sqrt{2}+\pi r^2$
25. $-4.9t^2+100t = 410$
26. $x^2 = 1.65(3-x)^2$
27. $(0.5+2A)^2 = 0.7(0.1-A)^2$

In Exercises 28 - 30, use Theorem 4.2 along with the techniques in this section to find all real solutions to the following.

28. $|x^2-3x| = 2$
29. $|2x-x^2| = |2x-1|$
30. $|x^2-x+3| = |4-x^2|$
31. Prove that for every nonzero number p, $x^2+xp+p^2 = 0$ has no real solutions.
32. Solve for t: $-\frac{1}{2}gt^2+vt+h = 0$. Assume $g > 0$, $v \geq 0$ and $h \geq 0$.

5.3.2 Answers

1. $x = \frac{3 \pm \sqrt{5}}{6}$
2. $t = -\frac{4}{5}, -\frac{2}{5}$
3. $y = \pm 1, \pm\sqrt{5}$
4. $x = \frac{-1 \pm \sqrt{5}}{2}$
5. $w = -1, \frac{2}{3}$
6. $y = -2 \pm \sqrt{5}$
7. $z = \frac{1 \pm \sqrt{65}}{16}$
8. $v = -3, 1$
9. No real solution.
10. $t = \frac{-5 \pm \sqrt{33}}{4}$
11. $x = 0$
12. $y = \frac{2 \pm \sqrt{10}}{6}$
13. $w = \pm\sqrt{\frac{\sqrt{13} - 3}{2}}$
14. $x = \pm 1$
15. $y = \frac{4 \pm \sqrt{6 + 2\sqrt{13}}}{2}$
16. $x = 0, \frac{5 \pm \sqrt{17}}{2}$
17. $p = -\frac{1}{3}, \pm\sqrt{2}$
18. $v = 0, \pm\sqrt{2}, \pm\sqrt{5}$
19. $y = \frac{5\sqrt{2} \pm \sqrt{46}}{2}$
20. $x = \frac{\sqrt{2} \pm \sqrt{10}}{2}$
21. $v = -\frac{\sqrt{3}}{2}, 2\sqrt{3}$
22. $b = \pm\frac{\sqrt{13271}}{50} \approx \pm 2.30$
23. $r = \pm\sqrt{\frac{37}{\pi}} \approx \pm 3.43$
24. $r = \frac{-4\sqrt{2} \pm \sqrt{54\pi + 32}}{\pi}$, $r \approx -6.32, 2.72$
25. $t = \frac{500 \pm 10\sqrt{491}}{49}$, $t \approx 5.68, 14.73$
26. $x = \frac{99 \pm 6\sqrt{165}}{13}$, $x \approx 1.69, 13.54$
27. $A = \frac{-107 \pm 7\sqrt{70}}{330}$, $A \approx -0.50, -0.15$
28. $x = 1, 2, \frac{3 \pm \sqrt{17}}{2}$
29. $x = \pm 1, 2 \pm \sqrt{3}$
30. $x = -\frac{1}{2}, 1, 7$
31. The discriminant is: $D = p^2 - 4p^2 = -3p^2 < 0$. Since $D < 0$, there are no real solutions.
32. $t = \frac{v \pm \sqrt{v^2 + 2gh}}{g}$

5.4 Quadratic Functions

5.4.1 Graphs of Quadratic Functions

You may recall studying quadratic equations in a previous Algebra course. If not, you may wish to refer to Section 5.3 to revisit this topic. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 5.6. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

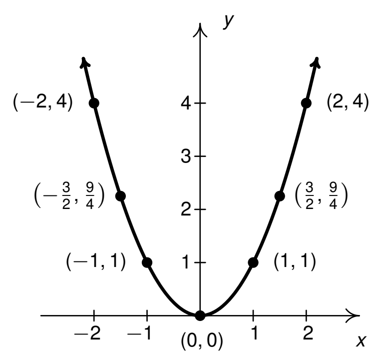
where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

As in Definitions 3.3 and 3.4, the independent variable in Definition 5.6 is x while the values a, b and c are parameters. Note that $a \neq 0$ - otherwise we would have a linear function (see Definition 3.4).

The most basic quadratic function is $f(x) = x^2$, the squaring function, whose graph appears below along with a corresponding table of values. Its shape may look familiar from your previous studies in Algebra – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola because it is the sole point where the function obtains its extreme value, in this case, a minimum of 0 when $x = 0$.

Indeed, the range of $f(x) = x^2$ appears to be $[0, \infty)$ from the graph. We can substantiate this algebraically since for all x, $f(x) = x^2 \geq 0$. This tells us that the range of f is a subset of $[0, \infty)$. To show that the range of f actually equals $[0, \infty)$, we need to show that every real number c in $[0, \infty)$ is in the range of f. That is, for every $c \geq 0$, we have to show c is an output from f. In other words, we have to show there is a real number x so that $f(x) = x^2 = c$. Choosing $x = \sqrt{c}$, we find $f(x) = f(\sqrt{c}) = (\sqrt{c})^2 = c$, as required.[1]

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



$f(x) = x^2$

The techniques we used to graph many of the absolute value functions in Section 4.2 can be applied to quadratic functions, too. In fact, knowing the graph of $f(x) = x^2$ enables us to graph *every* quadratic function, but there's some extra work involved. We start with the following theorem:

[1]This assumes, of course, $\sqrt{c}$ is a real number for all real numbers $c \geq 0$ . . .

Theorem 5.7. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x-h)^2 + k$ is a parabola with vertex (h, k). If $a > 0$, the graph resembles '$\smile$.' If $a < 0$, the graph resembles '$\frown$.' Moreover, the vertical line $x = h$ is the **axis of symmetry** of the graph of $y = F(x)$.

To prove Theorem 5.7 the reader is encouraged to revisit the discussion following the proof of Theorem 4.4, replacing every occurrence of absolute value notation with the squared exponent.[2] Alternatively, the reader can skip ahead and read the statement and proof of Theorem 6.1 in Section 6.1. In the meantime we put Theorem 5.7 to good use in the next example.

Example 5.4.1.

1. Graph the following functions using Theorem 5.7. Find the vertex, zeros and axis-intercepts (if any exist). Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

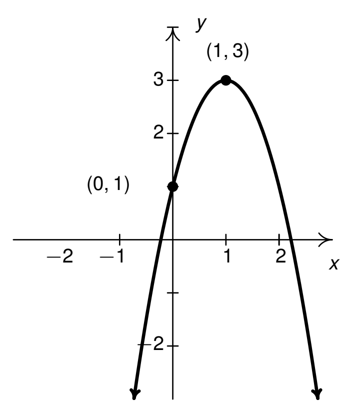
   (a) $f(x) = \dfrac{(x-3)^2}{2}$

   (b) $g(x) = (x+2)^2 - 3$

   (c) $h(t) = -2(t-3)^2 + 1$

   (d) $i(t) = \dfrac{(3-2t)^2 + 1}{2}$

2. Use Theorem 5.7 to write a possible formula for $H(x)$ whose graph is given below:



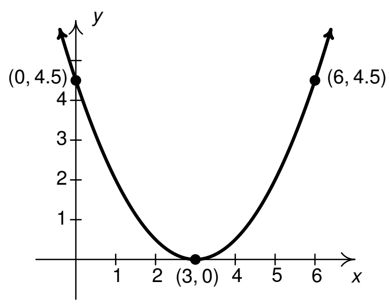
$y = H(x)$

Solution.

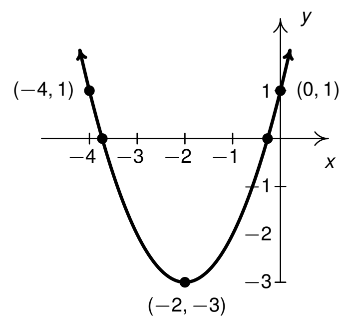
1. (a) For $f(x) = \frac{(x-3)^2}{2} = \frac{1}{2}(x-3)^2 + 0$, we identify $a = \frac{1}{2}$, $h = 3$ and $k = 0$. Thus the vertex is $(3, 0)$ and the parabola opens upwards. The only x-intercept is $(3, 0)$. Since $f(0) = \frac{1}{2}(0-3)^2 = \frac{9}{2}$, our y-intercept is $\left(0, \frac{9}{2}\right)$. To help us graph the function, it would be nice to have a third point and we'll use symmetry to find it. The y-value three units to the *left* of the vertex is 4.5, so the y-value must be 4.5 three units to the *right* of the vertex as well. Hence, we have our third point: $\left(6, \frac{9}{2}\right)$. From the graph, we get that the range is $[0, \infty)$ and see that f has the minimum value of 0 at $x = 3$ and no maximum. Also, f is decreasing on $(-\infty, 3]$ and increasing on $[3, \infty)$. The graph is the one on the left of the two on the next page.

[2]i.e., replace $|x|$ with x^2, $|c|$ with c^2, $|x-h|$ with $(x-h)^2$.

(b) For $g(x) = (x+2)^2 - 3 = (1)(x-(-2))^2 + (-3)$, we identify $a = 1$, $h = -2$ and $k = -3$. This means that the vertex is $(-2, -3)$ and the parabola opens upwards. Thus we have two x-intercepts. To find them, we set $y = g(x) = 0$ and solve. Doing so yields the equation $(x+2)^2 - 3 = 0$, or $(x+2)^2 = 3$. Extracting square roots gives us the two zeros of g: $x + 2 = \pm\sqrt{3}$, or $x = -2 \pm \sqrt{3}$. Our x-intercepts are $(-2 - \sqrt{3}, 0) \approx (-3.73, 0)$ and $(-2 + \sqrt{3}, 0) \approx (-0.27, 0)$. We find $g(0) = (0+2)^2 - 3 = 1$ so our y-intercept is $(0, 1)$. Using symmetry, we get $(-4, 1)$ as another point to help us graph. The range of g is $[-3, \infty)$. The minimum of g is -3 at $x = -2$, and g has no maximum. Moreover, g is decreasing on $(-\infty, -2]$ and g is increasing on $[-2, \infty)$. The graph is below on the right.



$f(x) = \frac{(x-3)^2}{2}$



$g(x) = (x+2)^2 - 3$

(c) Given $h(t) = -2(t-3)^2 + 1$, we identify $a = -2$, $h = 3$ and $k = 1$. Hence the vertex of the graph is $(3, 1)$ and the parabola opens downwards. Solving $h(t) = -2(t-3)^2 + 1 = 0$ gives $(t-3)^2 = \frac{1}{2}$. Extracting square roots[3] gives $t - 3 = \pm\frac{\sqrt{2}}{2}$, so that when we add 3 to each side,[4] we get $t = \frac{6 \pm \sqrt{2}}{2}$. Hence, our t-intercepts are $\left(\frac{6-\sqrt{2}}{2}, 0\right) \approx (2.29, 0)$ and $\left(\frac{6+\sqrt{2}}{2}, 0\right) \approx (3.71, 0)$. To find the y-intercept, we compute $h(0) = -2(0-3)^2 + 1 = -17$. Thus the y-intercept is $(0, -17)$. Using symmetry, we also have that $(6, -17)$ is on the graph which we show on the left side at the top of the next page.

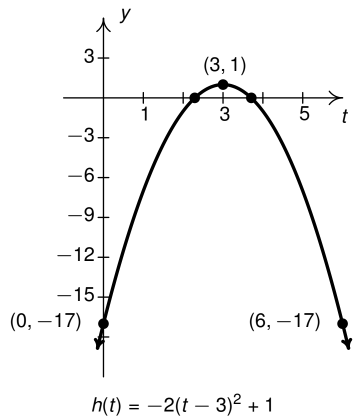
(d) We have some work ahead of us to put $i(t)$ into a form we can use to exploit Theorem 5.7:

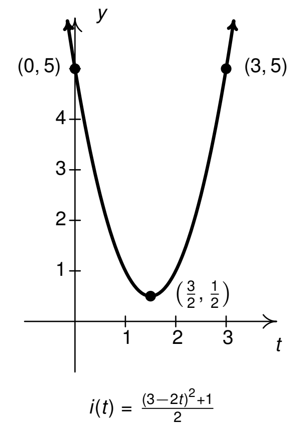
$$\begin{aligned} i(t) = \frac{(3-2t)^2 + 1}{2} &= \tfrac{1}{2}(-2t+3)^2 + \tfrac{1}{2} &&= \tfrac{1}{2}\left[-2\left(t - \tfrac{3}{2}\right)\right]^2 + \tfrac{1}{2} \\ &= \tfrac{1}{2}(-2)^2\left(t - \tfrac{3}{2}\right)^2 + \tfrac{1}{2} &&= 2\left(t - \tfrac{3}{2}\right)^2 + \tfrac{1}{2} \end{aligned}$$

We identify $a = 2$, $h = \frac{3}{2}$ and $k = \frac{1}{2}$. Hence our vertex is $\left(\frac{3}{2}, \frac{1}{2}\right)$ and the parabola opens upwards, meaning there are no t-intercepts. Since $i(0) = \frac{(3-2(0))^2+1}{2} = 5$, we get $(0, 5)$ as the y-intercept. Using symmetry, this means we also have $(3, 5)$ on the graph. The range is $\left[\frac{1}{2}, \infty\right)$ with the minimum of i, $\frac{1}{2}$, occurring when $t = \frac{3}{2}$. Also, i is decreasing on $\left(-\infty, \frac{3}{2}\right]$ and increasing on $\left[\frac{3}{2}, \infty\right)$. The graph is given on the right at the top of the next page.

[3] and rationalizing denominators!

[4] and get common denominators!





2. We are instructed to use Theorem 5.7, so we know $H(x) = a(x-h)^2+k$ for some choice of parameters a, h and k. The vertex is $(1, 3)$ so we know $h = 1$ and $k = 3$, and hence $H(x) = a(x-1)^2+3$. To find the value of a, we use the fact that the y-intercept, as labeled, is $(0, 1)$. This means $H(0) = 1$, or $a(0-1)^2+3 = 1$. This reduces to $a+3 = 1$ or $a = -2$. Our final answer[5] is $H(x) = -2(x-1)^2+3$. □

A few remarks about Example 5.4.1 are in order. First note that none of the functions are in the form of Definition 5.6. However, if we took the time to perform the indicated operations and simplify, we'd find:

- $f(x) = \frac{(x-3)^2}{2} = \frac{1}{2}x^2 - 3x + \frac{9}{2}$
- $g(x) = (x+2)^2 - 3 = x^2 + 4x + 1$
- $h(t) = -2(t-3)^2 + 1 = -2t^2 + 12t - 17$
- $i(t) = \frac{(3-2t)^2+1}{2} = 2t^2 - 6t + 5$

While the y-intercepts of the graphs of the each of the functions are easier to see when the formulas for the functions are written in the form of Definition 5.6, the vertex is not. For this reason, the form of the functions presented in Theorem 5.7 are given a special name.

Definition 5.7. Standard and General Form of Quadratic Functions:

- The **general form** of the quadratic function f is $f(x) = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.
- The **standard form** of the quadratic function f is $f(x) = a(x-h)^2 + k$, where a, h and k are real numbers with $a \neq 0$.

If we proceed as in the remarks following Example 5.4.1, we can convert any quadratic function given to us in standard form and convert to general form by performing the indicated operation and simplifying:

$$\begin{array}{rcl} f(x) & = & a(x-h)^2 + k \\ & = & a\left(x^2 - 2hx + h^2\right) + k \\ & = & ax^2 - 2ahx + ah^2 + k \\ & = & ax^2 + (-2ah)x + (ah^2 + k). \end{array}$$

[5]The reader is encouraged to compare this example with number 2 of Example 4.2.2.

With the identifications $b = -2ah$ and $c = ah^2 + k$, we have written $f(x)$ in the form $f(x) = ax^2 + bx + c$. Likewise, through a process known as 'completing the square', we can take any quadratic function written in general form and rewrite it in standard form. We briefly review this technique in the following example – for a more thorough review the reader should see Section 5.3.

Example 5.4.2. Graph the following functions. Find the vertex, zeros and axis-intercepts, if any exist. Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

1. $f(x) = x^2 - 4x + 3$.
2. $g(t) = 6 - 4t - 2t^2$

Solution.

1. We follow the procedure for completing the square in Section 5.3. The only difference here is instead of the quadratic equation being set to 0, it is equal to $f(x)$. This means when we are finished completing the square, we need to solve for $f(x)$.

$$\begin{array}{rcll} f(x) & = & x^2 - 4x + 3 & \\ f(x) - 3 & = & x^2 - 4x & \text{Subtract 3 from both sides.} \\ f(x) - 3 + (-2)^2 & = & x^2 - 4x + (-2)^2 & \text{Add } \left(\frac{1}{2}(-4)\right)^2 \text{ to both sides.} \\ f(x) + 1 & = & (x-2)^2 & \text{Factor the perfect square trinomial.} \\ f(x) & = & (x-2)^2 - 1 & \text{Solve for } f(x). \end{array}$$

   The reader is encouraged to start with $f(x) = (x-2)^2 - 1$, perform the indicated operations and simplify the result to $f(x) = x^2 - 4x + 3$. From the standard form, $f(x) = (x-2)^2 - 1$, we see that the vertex is $(2, 1)$ and that the parabola opens upwards. To find the zeros of f, we set $f(x) = 0$.

   We have two equivalent expressions for $f(x)$ so we could use either the general form or standard form. We solve the former and leave it to the reader to solve the latter to see that we get the same results either way. To solve $x^2 - 4x + 3 = 0$, we factor: $(x-3)(x-1) = 0$ and obtain $x = 1$ and $x = 3$. We get two x-intercepts, $(1, 0)$ and $(3, 0)$.

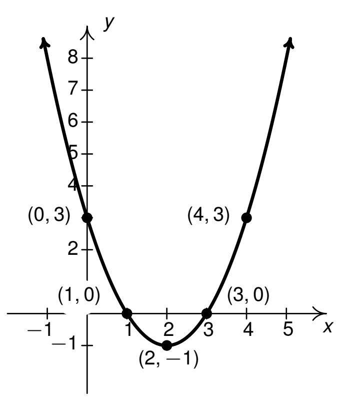
   To find the y-intercept, we need $f(0)$. Again, we could use either form of $f(x)$ for this and we choose the general form and find that the y-intercept is $(0, 3)$. From symmetry, we know the point $(4, 3)$ is also on the graph. We see that the range of f is $[-1, \infty)$ with the minimum -1 at $x = 2$. Finally, f is decreasing on $(-\infty, 2]$ and increasing from $[2, \infty)$. The graph is given on the left at the bottom the next page.

2. We first rewrite $g(t) = 6 - 4t - 2t^2$ as $g(t) = -2t^2 - 4t + 6$. As with the previous example, once we complete the square, we solve for $g(t)$:

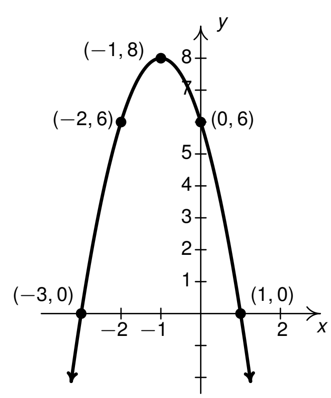
$$\begin{aligned}
g(t) &= -2t^2 - 4t + 6 && \\
g(t) - 6 &= -2t^2 - 4t && \text{Subtract 6 from both sides.} \\
\frac{g(t) - 6}{-2} &= \frac{-2t^2 - 4t}{-2} && \text{Divide both sides by } -2. \\
\frac{g(t) - 6}{-2} + (1)^2 &= t^2 + 2t + (1)^2 && \text{Add } \left(\tfrac{1}{2}(2)\right)^2 \text{ to both sides.} \\
\frac{g(t) - 6}{-2} + 1 &= (t + 1)^2 && \text{Factor the prefect square trinomial.} \\
\frac{g(t) - 6}{-2} &= (t + 1)^2 - 1 && \\
g(t) - 6 &= -2\left[(t + 1)^2 - 1\right] && \\
g(t) &= -2(t + 1)^2 + 2 + 6 && \\
g(t) &= -2(t + 1)^2 + 8 &&
\end{aligned}$$

We can check our answer by expanding $-2(t + 1)^2 + 8$ and show that it simplifies to $-2t^2 - 4t + 6$. From the standard form, we find that the vertex is $(-1, 8)$ and that the parabola opens downwards. Setting $g(t) = -2t^2 - 4t + 6 = 0$, we factor to get $-2(t - 1)(t + 3) = 0$ so $t = -3$ and $t = 1$. Hence, our two t-intercepts are $(-3, 0)$ and $(1, 0)$.

Since $g(0) = 6$, we get the y-intercept to be $(0, 6)$. Using symmetry, we also have the point $(-2, 6)$ on the graph. The range is $(-\infty, 8]$ with a maximum of 8 when $t = -1$. Finally we note that g is increasing on $(-\infty, -1]$ and decreasing on $[-1, \infty)$. The graph is below on the right.



$f(x) = x^2 - 4x + 3$



$g(x) = 6 - 4t - 2t^2$

$\square$

We now generalize the procedure demonstrated in Example 5.4.2. Let $f(x) = ax^2 + bx + c$ for $a \neq 0$:

$$
\begin{array}{rcll}
f(x) & = & ax^2 + bx + c & \\
f(x) - c & = & ax^2 + bx & \text{Subtract } c \text{ from both sides.} \\
\dfrac{f(x) - c}{a} & = & \dfrac{ax^2 + bx}{a} & \text{Divide both sides by } a \neq 0. \\
\dfrac{f(x) - c}{a} & = & x^2 + \dfrac{b}{a}x & \\
\dfrac{f(x) - c}{a} + \left(\dfrac{b}{2a}\right)^2 & = & x^2 + \dfrac{b}{a}x + \left(\dfrac{b}{2a}\right)^2 & \text{Add } \left(\dfrac{b}{2a}\right)^2 \text{ to both sides.} \\
\dfrac{f(x) - c}{a} + \dfrac{b^2}{4a^2} & = & \left(x + \dfrac{b}{2a}\right)^2 & \text{Factor the perfect square trinomial.} \\
\dfrac{f(x) - c}{a} & = & \left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2} & \text{Solve for } f(x). \\
f(x) - c & = & a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2}\right] & \\
f(x) - c & = & a\left(x + \dfrac{b}{2a}\right)^2 - a\dfrac{b^2}{4a^2} & \\
f(x) & = & a\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a} + c & \\
f(x) & = & a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a} & \text{Get a common denominator.}
\end{array}
$$

By setting $h = -\frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$, we have written the function in the form $f(x) = a(x-h)^2 + k$. This establishes the fact that every quadratic function can be written in standard form.[6] Moreover, writing a quadratic function in standard form allows us to identify the vertex rather quickly, and so our work also shows us that the vertex of $f(x) = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$. It is not worth memorizing the expression $\frac{4ac-b^2}{4a}$ especially since we can write this as $f\left(-\frac{b}{2a}\right)$. (This about this last statement for a moment.)

We summarize the information detailed above in the following box:

Equation 5.1. Vertex Formulas for Quadratic Functions: Suppose a, b, c, h and k are real numbers where $a \neq 0$.

- If $f(x) = a(x-h)^2 + k$ then the vertex of the graph of $y = f(x)$ is the point (h, k).
- If $f(x) = ax^2 + bx + c$ then the vertex of the graph of $y = f(x)$ is the point $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

[6] To avoid completing the square, we could solve the equations $b = -2ah$ and $c = ah^2 + k$ for h and k. See Exercise 54.

Completing the square is also the means by which we may derive the celebrated Quadratic Formula, a formula which returns the solutions to $ax^2 + bx + c = 0$ for $a \neq 0$. Before we state it here for reference, we wish to encourage the reader to pause a moment and read the derivation if the Quadratic Formula found in Section 5.3. The work presented in this section transforms the general form of a quadratic *function* into the standard form whereas the work in Section 5.3 finds a formula to solve an *equation*. There is great value in understanding the similarities and differences between the two approaches.

Equation 5.2. The Quadratic Formula: The zeros of the quadratic function $f(x) = ax^2 + bx + c$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is worth pointing out the symmetry inherent in Equation 5.2. We may rewrite the zeros as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

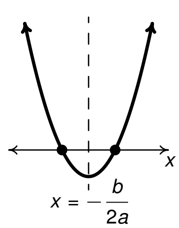
so that, if there are real zeros, they (like the rest of the parabola) are symmetric about the line $x = -\frac{b}{2a}$. Another way to view this symmetry is that the x-coordinate of the vertex is the average of the zeros. We encourage the reader to verify this fact in all of the preceding examples, where applicable.

Next, recall that if the quantity $b^2 - 4ac$ is strictly negative then we do not have any real zeros. This quantity is called the *discriminant* and is useful in determining the number and nature of solutions to a quadratic equation. We remind the reader of this below.

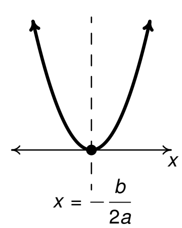
Equation 5.3. The Discriminant of a Quadratic Function: Given a quadratic function in general form $f(x) = ax^2 + bx + c$, the **discriminant** is the quantity $b^2 - 4ac$.

- If $b^2 - 4ac > 0$ then f has two unequal (distinct) real zeros.
- If $b^2 - 4ac = 0$ then f has one (repeated) real zero.
- If If $b^2 - 4ac < 0$ then f has two unequal (distinct) non-real zeros.

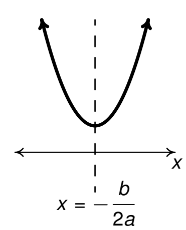
We'll talk more about what we mean by a 'repeated' zero and how to compute 'non-real' zeros in Chapter 6. For us, the discriminant has the graphical implication that if $b^2 - 4ac > 0$ then we have two x-intercepts; if $b^2 - 4ac = 0$ then we have just one x-intercept, namely, the vertex; and if $b^2 - 4ac < 0$ then we have no x-intercepts because the parabola lies entirely above or below the x-axis. We sketch each of these scenarios below assuming $a > 0$. (The sketches for $a < 0$ are similar - see Exercise 49.)



$b^2 - 4ac > 0$



$b^2 - 4ac = 0$



$b^2 - 4ac < 0$

We now revisit the economic scenario first described in Examples 3.2.3 and 3.2.4 where we were producing and selling PortaBoy game systems. Recall that the cost to produce x PortaBoys is denoted by $C(x)$ and the price-demand function, that is, the price to charge in order to sell x systems is denoted by $p(x)$. We introduce two more related functions below: the **revenue** and **profit** functions.

Definition 5.8. Revenue and Profit: Suppose $C(x)$ represents the cost to produce x units and $p(x)$ is the associated price-demand function. Under the assumption that we are producing the same number of units as are being sold:

- The **revenue** obtained by selling x units is $R(x) = x\, p(x)$.

  That is, revenue = (number of items sold) · (price per item).

- The **profit** made by selling x units is $P(x) = R(x) - C(x)$.

  That is, profit = (revenue) − (cost).

Said differently, the *revenue* is the amount of money *collected* by selling x items whereas the *profit* is how much money is *left over* after the costs are paid.

Example 5.4.3. In Example 3.2.3 the cost to produce x PortaBoy game systems for a local retailer was given by $C(x) = 80x + 150$ for $x \geq 0$ and in Example 3.2.4 the price-demand function was found to be $p(x) = -1.5x + 250$, for $0 \leq x \leq 166$.

1. Find formulas for the associated revenue and profit functions; include the domain of each.

2. Find and interpret $P(0)$.

3. Find and interpret the zeros of P.

4. Graph $y = P(x)$. Find the vertex and axis intercepts.

5. Interpret the vertex of the graph of $y = P(x)$.

6. What should the price per system be in order to maximize profit?

7. Find and interpret the average rate of change of P over the interval $[0, 57]$.

Solution.

1. The formula for the revenue function is $R(x) = x\,p(x) = x(-1.5x + 250) = -1.5x^2 + 250x$. Since the domain of p is restricted to $0 \leq x \leq 166$, so is the domain of R. To find the profit function $P(x)$, we subtract $P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150$. The cost function formula is valid for $x \geq 0$, but the revenue function is valid when $0 \leq x \leq 166$. Hence, the domain of P is likewise restricted to $[0, 166]$.

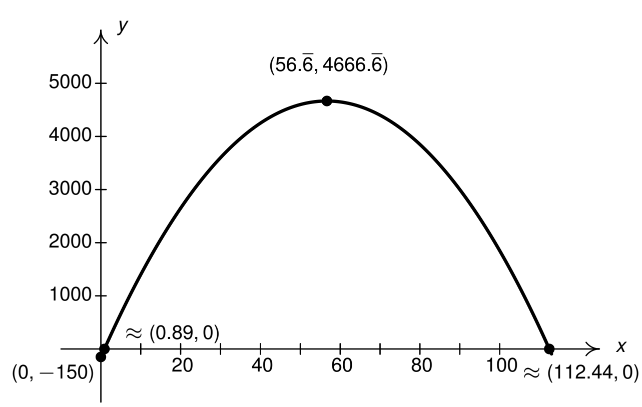
2. We find $P(0) = -1.5(0)^2 + 170(0) - 150 = -150$. This means that if we produce and sell 0 PortaBoy game systems, we have a profit of $-\$150$. Since profit = (revenue) − (cost), this means our costs exceed our revenue by \$150. This makes perfect sense, since if we don't sell any systems, our revenue is \$0 but our fixed costs (see Example 3.2.3) are \$150.

3. To find the zeros of P, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. Factoring here would be challenging to say the least, so we use the Quadratic Formula, Equation 5.2. Identifying $a = -1.5$, $b = 170$ and $c = -150$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} \\ &= \frac{170 \pm 20\sqrt{70}}{3} \\ &\approx 0.89,\ 112.44. \end{aligned}$$

Given that profit = (revenue) $-$ (cost), if profit = 0, then revenue = cost. Hence, the zeros of P are called the 'break-even' points - where just enough product is sold to recover the cost spent to make the product. Also, x represents a number of game systems, which is a whole number, so instead of using the exact values of the zeros, or even their approximations, we consider $x = 0$ and $x = 1$ along with $x = 112$ and $x = 113$. We find $P(0) = -150$, $P(1) = 18.5$, $P(112) = 74$ and $P(113) = -93.5$. These data suggest that, in order to be profitable, at least 1 but not more than 112 systems should be produced and sold, as borne out in the graph below.

4. Knowing the zeros of P, we have two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right) \approx (0.89, 0)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right) \approx (112.44, 0)$. Since $P(0) = -150$, we get the y-intercept is $(0, -150)$. To find the vertex, we appeal to Equation 5.1. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3} = 56.\overline{6}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3} = 4666.\overline{6}$. Hence, the vertex is $(56.\overline{6}, 4666.\overline{6})$. The domain is restricted $0 \leq x \leq 166$ and we find $P(166) = -13264$. Attempting to plot all of these points on the same graph to any sort of scale is challenging. Instead, we present a portion of the graph for $0 \leq x \leq 113$. Even with this, the intercepts near the origin are crowded.



$y = P(x)$

5. From the vertex, we see that the maximum of P is $4666.\overline{6}$ when $x = 56.\overline{6}$. As before, x represents the number of PortaBoy systems produced and sold, so we cannot produce and sell $56.\overline{6}$ systems. Hence, by comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

6. We've determined that we need to sell 57 PortaBoys to maximize profit, so we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. In other words, to sell 57 systems, and thereby maximize the profit, we should set the price at \$164.50 per system.

7. To find the average rate of change of P over $[0, 57]$, we compute

$$\frac{\Delta[P(x)]}{\Delta x} = \frac{P(57) - P(0)}{57 - 0} = \frac{4666.5 - (-150)}{57} = 84.5.$$

This means that as the number of systems produced and sold ranges from 0 to 57, the average profit per system is increasing at a rate of \$84.50. In other words, for each additional system produced and sold, the profit increased by \$84.50 on average. □

We hope Example 5.4.3 shows the value of using a continuous model to describe a discrete situation. True, we could have 'run the numbers' and computed $P(1)$, $P(2)$, ..., $P(166)$ to eventually determine the maximum profit, but the vertex formula made much quicker work of the problem.

Along these same lines, in our next example we revisit Skippy's temperature data from Example 2.1.1 in Section 2.1. We found a piecewise-linear model in Section 3.2 to model the temperature over the course the day and now we seek a quadratic function to do the job. The methodology used here is similar to that of the least squares regression line discussed in Section **??** but instead of finding the line closest to the data points, we want the *parabola* closest to them that comes from a function of the form $f(x) = ax^2 + bx + c$. The Mathematics required to find the desired quadratic function is beyond the scope of this text, but most graphing utilities can do these quickly. In the quadratic case, the machine will return a value of R^2 such that $0 \leq R^2 \leq 1$. The closer R^2 is to 1, the better the fit. (Again, how R^2 is computed is beyond this text.)

Example 5.4.4.

1. Use a graphing utility to fit a quadratic model to the time and temperature data in Example 2.1.1. Comment on the goodness of fit.

2. Use your model to predict the temperature at 7 AM and 3 PM. Round your answers to one decimal place. How do your results compare with those from Example **??**?

3. According to the model, what was the warmest temperature of the day? When did that occur? Round your answers to one decimal place.

Solution.

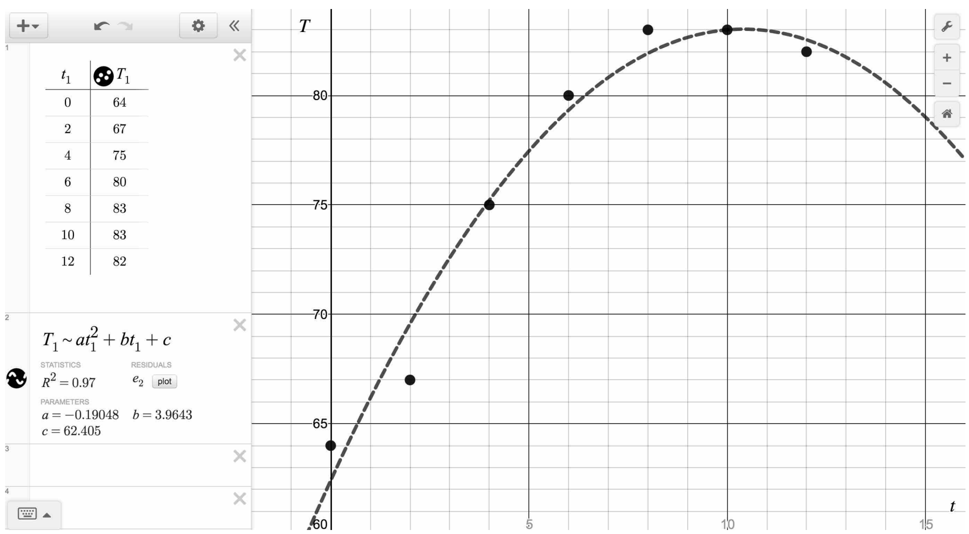
1. Entering the data in Desmos we find $T = F(t) = -0.1905t^2 + 3.9643t + 62.405$ with an R^2 value of 0.97, indicating a pretty strong fit.

2. Since 7 AM corresponds to $t = 1$, we find $T = F(1) \approx 66.18$. Hence our quadratic model predicts a temperature of 66.2° F at 7 AM - identical (when rounded) to the 66.2° F predicted in Example **??**. Similarly, 3 PM corresponds to $t = 9$, so we find $T = F(9) \approx 82.65$. Thus the model predicts an outdoor temperature of 82.6° F which is very close to the 82.9°F prediction from Example **??**.

3. The model is quadratic with $a < 0$ so the maximum (warmest) temperature can be determined by finding the vertex. We get

$$t = -\frac{b}{2a} = -\frac{3.9643}{2(-0.1905)} \approx -10.40, \quad T = F(-10.40) \approx 83.03,$$

or, in other words, the warmest temperature is 83.0° F at 4:24 PM (10.40 hours after 6 AM.)



□

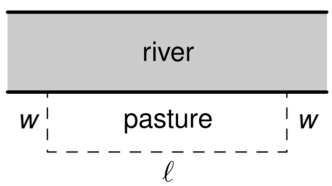
It is interesting how close the predictions from Examples **??** and 5.4.4 despite one using linear models and one using a quadratic model. Which model is the 'better' model? We leave that discussion to the reader and their classmates.

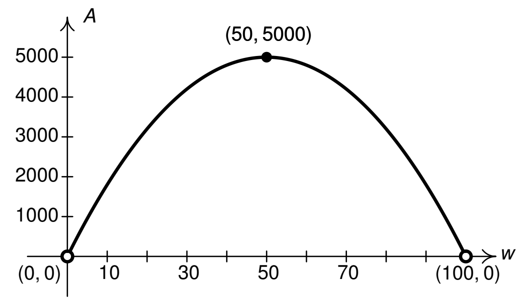
Our next example is classic application of optimizing a quadratic function.

Example 5.4.5. Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives so the time is right for him to pursue his dream of raising alpaca. He wishes to build a rectangular pasture and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a river (so that no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

Solution. We are asked to find the dimensions of the pasture which would give a maximum area, so we begin by sketching the diagram seen below on the left. We let w denote the width of the pasture and we let ℓ denote the length of the pasture. The units given to us in the statement of the problem are feet, so we assume that w and ℓ are measured in feet. The area of the pasture, which we'll call A, is related to w and ℓ by the equation $A = w\ell$. Since w and ℓ are both measured in feet, A has units of feet^2, or square feet.

We are also told that the total amount of fencing available is 200 feet, which means $w + \ell + w = 200$, or, $\ell + 2w = 200$. We now have two equations, $A = w\ell$ and $\ell + 2w = 200$. In order to use the tools given to us in this section to *maximize* A, we need to use the information given to write A as a function of just *one* variable, either w or ℓ. This is where we use the equation $\ell + 2w = 200$. Solving for ℓ, we find $\ell = 200 - 2w$, and we substitute this into our equation for A. We get $A = w\ell = w(200 - 2w) = 200w - 2w^2$. We now have A as a function of w, $A = f(w) = 200w - 2w^2 = -2w^2 + 200w$.





$A = f(w)$

Before we go any further, we need to find the applied domain of f so that we know what values of w make sense in this situation.[7] Given that w represents the width of the pasture we need $w > 0$. Likewise, ℓ represents the length of the pasture, so $\ell = 200 - 2w > 0$. Solving this latter inequality yields $w < 100$. Hence, the function we wish to maximize is $f(w) = -2w^2 + 200w$ for $0 < w < 100$. We know two things about the quadratic function f: the graph of $A = f(w)$ is a parabola and (since the coefficient of w^2 is -2) the parabola opens downwards.

This means that there is a maximum value to be found, and we know it occurs at the vertex. Using the vertex formula, we find $w = -\frac{200}{2(-2)} = 50$, and $A = f(50) = -2(50)^2 + 200(50) = 5000$. Since $w = 50$ lies in the applied domain, $0 < w < 100$, we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use $\ell = 200 - 2w$ and find $\ell = 200 - 2(50) = 100$, so the length of the pasture is 100 feet. The maximum area is $A = f(50) = 5000$, or 5000 square feet. If an average alpaca requires 25 square feet of pasture, Donnie can raise $\frac{5000}{25} = 200$ average alpaca. $\square$

The function f in Example 5.4.5 is called the **objective function** for this problem - it's the function we're trying to optimize. In the case above, we were trying to maximize f. The equation $\ell + 2w = 200$ along with the inequalities $w > 0$ and $\ell > 0$ are called the **constraints**. As we saw in this example, and as we'll see again and again, the constraint equation is used to rewrite the objective function in terms of just one of the variables where constraint inequalities, if any, help determine the applied domain.

[7] Donnie would be very upset if, for example, we told him the width of the pasture needs to be -50 feet.

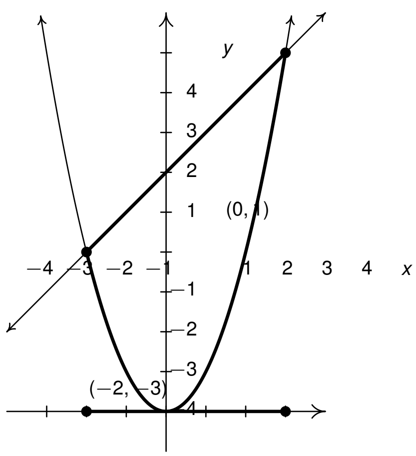
5.4.2 Exercises

In Exercises 1 - 9, graph the quadratic function. Find the vertex and axis intercepts of each graph, if they exist. State the domain and range, identify the maximum or minimum, and list the intervals over which the function is increasing or decreasing. If the function is given in general form, convert it into standard form; if it is given in standard form, convert it into general form.

1. $f(x) = x^2 + 2$
2. $f(x) = -(x+2)^2$
3. $f(x) = x^2 - 2x - 8$
4. $g(t) = -2(t+1)^2 + 4$
5. $g(t) = 2t^2 - 4t - 1$
6. $g(t) = -3t^2 + 4t - 7$
7. $h(s) = s^2 + s + 1$
8. $h(s) = -3s^2 + 5s + 4$
9. $h(s) = s^2 - \dfrac{1}{100}s - 1$

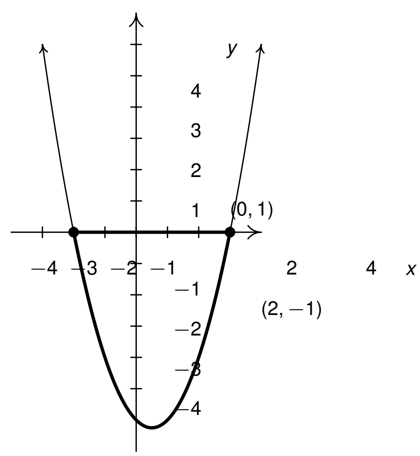
In Exercises 10 - 13, find a formula for each function below in the form $F(x) = a(x-h)^2 + k$.

10.



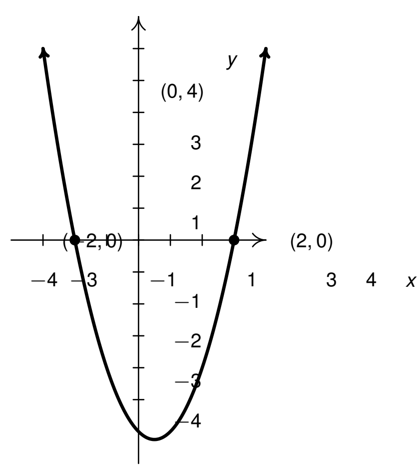
$y = F(x)$

11.

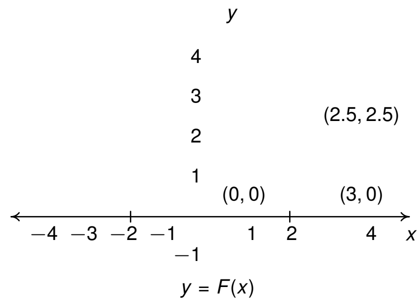


$y = F(x)$

12.



$y = F(x)$

13. 

In Exercises 14 - 29, solve the inequality. Write your answer using interval notation.

14. $x^2 + 2x - 3 \geq 0$
15. $16x^2 + 8x + 1 > 0$
16. $t^2 + 9 < 6t$
17. $9t^2 + 16 \geq 24t$
18. $u^2 + 4 \leq 4u$
19. $u^2 + 1 < 0$
20. $3x^2 \leq 11x + 4$
21. $x > x^2$
22. $2t^2 - 4t - 1 > 0$
23. $5t + 4 \leq 3t^2$
24. $2 \leq |x^2 - 9| < 9$
25. $x^2 \leq |4x - 3|$
26. $t^2 + t + 1 \geq 0$
27. $t^2 \geq |t|$
28. $x|x + 5| \geq -6$
29. $x|x - 3| < 2$

In Exercises 30 - 34, cost and price-demand functions are given. For each scenario,

- Find the profit function $P(x)$.
- Find the number of items which need to be sold in order to maximize profit.
- Find the maximum profit.
- Find the price to charge per item in order to maximize profit.
- Find and interpret break-even points.

30. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, for $0 \leq x \leq 15$.

31. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, for $0 \leq x \leq 35$.

32. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, for $0 \leq x \leq 30$.

33. The daily cost, in dollars, to produce x Sasquatch Berry Pies is $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, for $0 \leq x \leq 24$.

34. The monthly cost, in *hundreds* of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in *hundreds* of dollars per scooter, is $p(x) = 140 - 2x$, for $0 \leq x \leq 70$.

35. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking x cookies is $C(x) = 0.1x + 25$ and that the demand function for their cookies is $p = 10 - .01x$ for $0 \leq x \leq 1000$. How many cookies should they bake in order to maximize their profit?

36. Using data from Bureau of Transportation Statistics, the average fuel economy $F(t)$ in miles per gallon for passenger cars in the US t years after 1980 can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.

37. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

What is the warmest temperature of the day? When does this happen?

38. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. How many pens should be produced to minimize the cost? What is this minimum cost?

39. Skippy wishes to plant a vegetable garden along one side of his house. In his garage, he found 32 linear feet of fencing. Since one side of the garden will border the house, Skippy doesn't need fencing along that side. What are the dimensions of the garden which will maximize the area of the garden? What is the maximum area of the garden?

40. In the situation of Example 5.4.5, Donnie has a nightmare that one of his alpaca fell into the river. To avoid this, he wants to move his rectangular pasture *away* from the river so that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions maximize the area of the pasture now? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpaca can he raise now?

41. What is the largest rectangular area one can enclose with 14 inches of string?

42. The height of an object dropped from the roof of an eight story building is modeled by by the function $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground, in feet, t seconds after the object is dropped. How long before the object hits the ground?

43. The height $h(t)$ in feet of a model rocket above the ground t seconds after lift-off is given by the function $h(t) = -5t^2 + 100t$, for $0 \leq t \leq 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

44. Carl's friend Jason participates in the Highland Games. In one event, the hammer throw, the height $h(t)$ in feet of the hammer above the ground t seconds after Jason lets it go is modeled by the function $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

45. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t of a falling object is given by $s(t) = -4.9t^2 + v_0 t + s_0$ where s is in meters, t is in seconds, v_0 is the object's initial velocity in meters per second and s_0 is its initial position in meters.

    (a) What is the applied domain of this function?

    (b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.

    (c) Come up with a scenario in which $s_0 < 0$.

    (d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?

    (e) If the marble is shot from the top of a 25 meter tall tower, when does it hit the ground?

    (f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

46. The two towers of a suspension bridge are 400 feet apart. The parabolic cable[8] attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

47. On New Year's Day, Jeff started weighing himself every morning in order to have an interesting data set for this section of the book. (Discuss with your classmates if that makes him a nerd or a geek. Also, the professionals in the field of weight management strongly discourage weighing yourself every day. When you focus on the number and not your overall health, you tend to lose sight of your objectives. Jeff was making a noble sacrifice for science, but you should <u>not</u> try this at home.) The whole chart would be too big to put into the book neatly, so we've decided to give only a small portion of the data to you. This then becomes a Civics lesson in honesty, as you shall soon see. There are two charts given below. One has Jeff's weight for the first eight Thursdays of the year (January 1, 2009 was a Thursday and we'll count it as Day 1.) and the other has Jeff's weight for the first 10 Saturdays of the year.

Day # (Thursday)	1	8	15	22	29	36	43	50
My weight in pounds	238.2	237.0	235.6	234.4	233.0	233.8	232.8	232.0

Day # (Saturday)	3	10	17	24	31	38	45	52	59	66
My weight in pounds	238.4	235.8	235.0	234.2	236.2	236.2	235.2	233.2	236.8	238.2

    (a) Find the least squares line for the Thursday data and comment on its goodness of fit.

[8] The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line does not form a parabola. We shall see in Exercise **??** in Section **??** what shape a free hanging cable makes.

(b) Find the least squares line for the Saturday data and comment on its goodness of fit.

(c) Use Quadratic Regression to find a parabola which models the Saturday data and comment on its goodness of fit.

(d) Compare and contrast the predictions the three models make for Jeff's weight on January 1, 2010 (Day #366). Can any of these models be used to make a prediction of Jeff's weight 20 years from now? Explain your answer.

(e) Why is this a Civics lesson in honesty? Well, compare the two linear models you obtained above. One was a good fit and the other was not, yet both came from careful selections of real data. In presenting the tables to you, we've not lied about Jeff's weight, nor have you used any bad math to falsify the predictions. The word we're looking for here is 'disingenuous'. Look it up and then discuss the implications this type of data manipulation could have in a larger, more complex, politically motivated setting.

48. (Data that is neither linear nor quadratic.) We'll close this exercise set with two data sets that, for reasons presented later in the book, cannot be modeled correctly by lines or parabolas. It is a good exercise, though, to see what happens when you attempt to use a linear or quadratic model when it's not appropriate.

(a) This first data set came from a Summer 2003 publication of the Portage County Animal Protective League called "Tattle Tails". They make the following statement and then have a chart of data that supports it. "It doesn't take long for two cats to turn into 80 million. If two cats and their surviving offspring reproduced for ten years, you'd end up with 80,399,780 cats." We assume $N(0) = 2$.

Year x	1	2	3	4	5	6	7	8	9	10
Number of Cats $N(x)$	12	66	382	2201	12680	73041	420715	2423316	13968290	80399780

Use Quadratic Regression to find a parabola which models this data and comment on its goodness of fit. (Spoiler Alert: Does anyone know what type of function we need here?)

(b) This next data set comes from the U.S. Naval Observatory. That site has loads of awesome stuff on it, but for this exercise I used the sunrise/sunset times in Fairbanks, Alaska for 2009 to give you a chart of the number of hours of daylight they get on the 21^{st} of each month. We'll let $x = 1$ represent January 21, 2009, $x = 2$ represent February 21, 2009, and so on.

Month Number	1	2	3	4	5	6	7	8	9	10	11	12
Hours of Daylight	5.8	9.3	12.4	15.9	19.4	21.8	19.4	15.6	12.4	9.1	5.6	3.3

Use Quadratic Regression to find a parabola which models this data and comment on its goodness of fit. (Spoiler Alert: Does anyone know what type of function we need here?)

49. Redraw the three scenarios discussed in the discriminant box for $a < 0$.

50. Graph $f(x) = |1 - x^2|$

51. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$.

52. Let L be the line $y = 2x + 1$. Find a function $D(x)$ which measures the distance *squared* from a point on L to $(0, 0)$. Use this to find the point on L closest to $(0, 0)$.

53. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros then the x-coordinate of the vertex is the midpoint of the zeros.

54. On page 259, we argued that any quadratic function in standard form $f(x) = a(x - h)^2 + k$ can be converted to a quadratic function in general form $f(x) = ax^2 + bx + c$ by making the identifications $b = -2ah$ and $c = ah^2 + k$. In this exercise, we use same identifications to show every parabola given in general form can be converted to standard form without completing the square.

    Solve $b = -2ah$ for h and substitute the result into the equation $c = ah^2 + k$ and then solve for k. Show $h = -\frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$ so that

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

In Exercises 55 - 60, solve the quadratic equation for the indicated variable.

55. $x^2 - 10y^2 = 0$ for x
56. $y^2 - 4y = x^2 - 4$ for x
57. $x^2 - mx = 1$ for x
58. $y^2 - 3y = 4x$ for y
59. $y^2 - 4y = x^2 - 4$ for y
60. $-gt^2 + v_0t + s_0 = 0$ for t (Assume $g \neq 0$.)

61. (This is a follow-up to Exercise 60 in Section 3.2.) The Lagrange Interpolate function L for three points (x_0, y_0), (x_1, y_1), and (x_2, y_2) where x_0, x_1, and x_2 are three distinct real numbers is given by:

$$L(x) = y_0\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1\frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

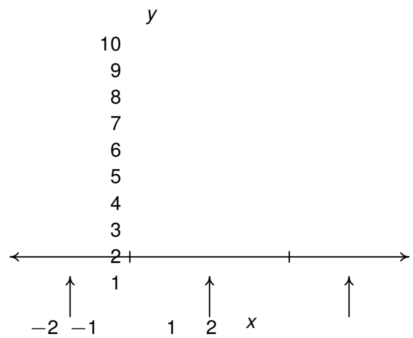
    (a) For each of the following sets of points, find $L(x)$ using the formula above and verify each of the points lies on the graph of $y = L(x)$.

        i. $(-1, 1)$, $(1, 1)$, $(2, 4)$
        ii. $(1, 3)$, $(2, 10)$, $(3, 21)$
        iii. $(0, 1)$, $(1, 5)$, $(2, 7)$

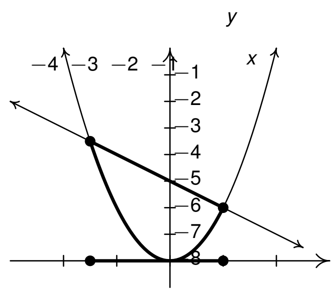
    (b) Verify that, in general, $L(x_0) = y_0$, $L(x_1) = y_1$, and $L(x_2) = y_2$.
    (c) Find $L(x)$ for the points $(-1, 6)$, $(1, 4)$ and $(3, 2)$. What happens?
    (d) Under what conditions will $L(x)$ produce a quadratic function? Make a conjecture, test some cases, and prove your answer.

5.4.3 Answers

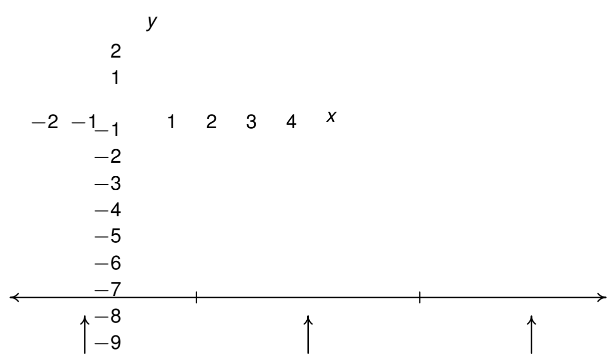
1. $f(x) = x^2 + 2$ (this is both forms!)
   No x-intercepts
   y-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $[2, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Vertex $(0, 2)$ is a minimum
   Axis of symmetry $x = 0$



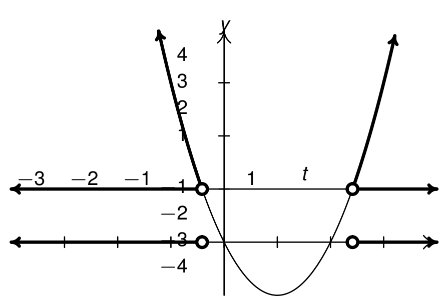
2. $f(x) = -(x + 2)^2 = -x^2 - 4x - 4$
   x-intercept $(-2, 0)$
   y-intercept $(0, -4)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 0]$
   Increasing on $(-\infty, -2]$
   Decreasing on $[-2, \infty)$
   Vertex $(-2, 0)$ is a maximum
   Axis of symmetry $x = -2$



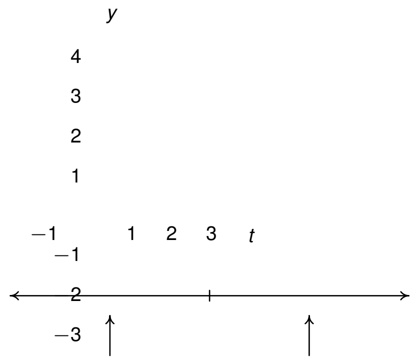
3. $f(x) = x^2 - 2x - 8 = (x - 1)^2 - 9$
   x-intercepts $(-2, 0)$ and $(4, 0)$
   y-intercept $(0, -8)$
   Domain: $(-\infty, \infty)$
   Range: $[-9, \infty)$
   Decreasing on $(-\infty, 1]$
   Increasing on $[1, \infty)$
   Vertex $(1, -9)$ is a minimum
   Axis of symmetry $x = 1$



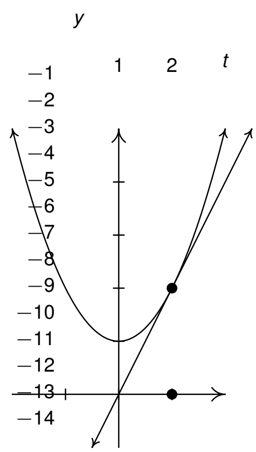
4. $g(t) = -2(t + 1)^2 + 4 = -2t^2 - 4t + 2$
   t-intercepts $(-1 - \sqrt{2}, 0)$ and $(-1 + \sqrt{2}, 0)$
   y-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 4]$
   Increasing on $(-\infty, -1]$
   Decreasing on $[-1, \infty)$
   Vertex $(-1, 4)$ is a maximum
   Axis of symmetry $t = -1$



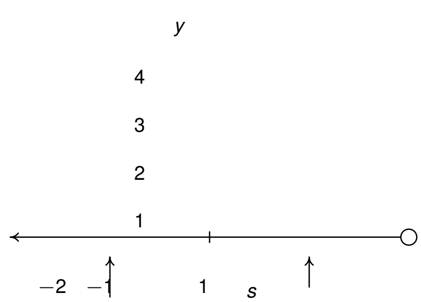
5. $g(t) = 2t^2 - tx - 1 = 2(t-1)^2 - 3$
   t-intercepts $\left(\frac{2-\sqrt{6}}{2}, 0\right)$ and $\left(\frac{2+\sqrt{6}}{2}, 0\right)$
   y-intercept $(0, -1)$
   Domain: $(-\infty, \infty)$
   Range: $[-3, \infty)$
   Increasing on $[1, \infty)$
   Decreasing on $(-\infty, 1]$
   Vertex $(1, -3)$ is a minimum
   Axis of symmetry $t = 1$



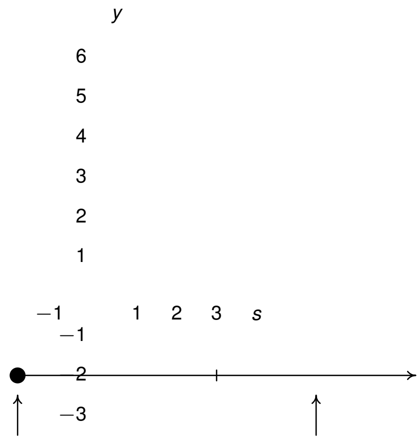
6. $g(t) = -3t^2 + 4t - 7 = -3\left(t - \frac{2}{3}\right)^2 - \frac{17}{3}$
   No t-intercepts
   y-intercept $(0, -7)$
   Domain: $(-\infty, \infty)$
   Range: $\left(-\infty, -\frac{17}{3}\right]$
   Increasing on $\left(-\infty, \frac{2}{3}\right]$
   Decreasing on $\left[\frac{2}{3}, \infty\right)$
   Vertex $\left(\frac{2}{3}, -\frac{17}{3}\right)$ is a maximum
   Axis of symmetry $t = \frac{2}{3}$



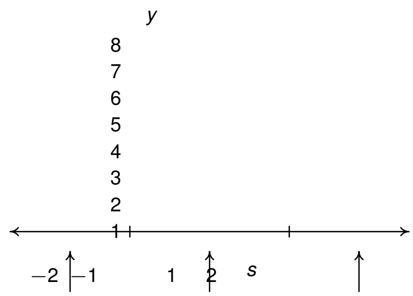
7. $h(s) = s^2 + s + 1 = \left(s + \frac{1}{2}\right)^2 + \frac{3}{4}$
   No s-intercepts
   y-intercept $(0, 1)$
   Domain: $(-\infty, \infty)$
   Range: $\left[\frac{3}{4}, \infty\right)$
   Increasing on $\left[-\frac{1}{2}, \infty\right)$
   Decreasing on $\left(-\infty, -\frac{1}{2}\right]$
   Vertex $\left(-\frac{1}{2}, \frac{3}{4}\right)$ is a minimum
   Axis of symmetry $s = -\frac{1}{2}$



8. $h(s) = -3s^2 + 5s + 4 = -3\left(s - \frac{5}{6}\right)^2 + \frac{73}{12}$
s-intercepts $\left(\frac{5-\sqrt{73}}{6}, 0\right)$ and $\left(\frac{5+\sqrt{73}}{6}, 0\right)$
y-intercept $(0, 4)$
Domain: $(-\infty, \infty)$
Range: $\left(-\infty, \frac{73}{12}\right]$
Increasing on $\left(-\infty, \frac{5}{6}\right]$
Decreasing on $\left[\frac{5}{6}, \infty\right)$
Vertex $\left(\frac{5}{6}, \frac{73}{12}\right)$ is a maximum
Axis of symmetry $s = \frac{5}{6}$



9. $h(s) = s^2 - \frac{1}{100}s - 1 = \left(s - \frac{1}{200}\right)^2 - \frac{40001}{40000}$
s-intercepts $\left(\frac{1+\sqrt{40001}}{200}\right)$ and $\left(\frac{1-\sqrt{40001}}{200}\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $\left[-\frac{40001}{40000}, \infty\right)$
Decreasing on $\left(-\infty, \frac{1}{200}\right]$
Increasing on $\left[\frac{1}{200}, \infty\right)$
Vertex $\left(\frac{1}{200}, -\frac{40001}{40000}\right)$ is a minimum[9]
Axis of symmetry $s = \frac{1}{200}$



10. $F(x) = (x+2)^2 - 3$

11. $F(x) = \frac{1}{2}(x-2)^2 - 1$

12. $F(x) = -x^2 + 4$

13. $F(x) = -2(x - 1.5)^2 + 4.5$

14. $(-\infty, -3] \cup [1, \infty)$

15. $\left(-\infty, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$

16. No solution

17. $(-\infty, \infty)$

18. $\{2\}$

19. No solution

20. $\left[-\frac{1}{3}, 4\right]$

21. $(0, 1)$

22. $\left(-\infty, 1 - \frac{\sqrt{6}}{2}\right) \cup \left(1 + \frac{\sqrt{6}}{2}, \infty\right)$

23. $\left(-\infty, \frac{5-\sqrt{73}}{6}\right] \cup \left[\frac{5+\sqrt{73}}{6}, \infty\right)$

24. $\left(-3\sqrt{2}, -\sqrt{11}\right] \cup [-\sqrt{7}, 0) \cup (0, \sqrt{7}] \cup \left[\sqrt{11}, 3\sqrt{2}\right)$

25. $[-2 - \sqrt{7}, -2 + \sqrt{7}] \cup [1, 3]$

[9] You'll need to use your calculator to zoom in far enough to see that the vertex is not the y-intercept.

26. $(-\infty, \infty)$

27. $(-\infty, -1] \cup \{0\} \cup [1, \infty)$

28. $[-6, -3] \cup [-2, \infty)$

29. $(-\infty, 1) \cup \left(2, \frac{3+\sqrt{17}}{2}\right)$

30.
- $P(x) = -2x^2 + 28x - 26$, for $0 \leq x \leq 15$.
- 7 T-shirts should be made and sold to maximize profit.
- The maximum profit is \$72.
- The price per T-shirt should be set at \$16 to maximize profit.
- The break even points are $x = 1$ and $x = 13$, so to make a profit, between 1 and 13 T-shirts need to be made and sold.

31.
- $P(x) = -x^2 + 25x - 100$, for $0 \leq x \leq 35$
- Since the vertex occurs at $x = 12.5$, and it is impossible to make or sell 12.5 bottles of tonic, maximum profit occurs when either 12 or 13 bottles of tonic are made and sold.
- The maximum profit is \$56.
- The price per bottle can be either \$23 (to sell 12 bottles) or \$22 (to sell 13 bottles.) Both will result in the maximum profit.
- The break even points are $x = 5$ and $x = 20$, so to make a profit, between 5 and 20 bottles of tonic need to be made and sold.

32.
- $P(x) = -3x^2 + 72x - 240$, for $0 \leq x \leq 30$
- 12 cups of lemonade need to be made and sold to maximize profit.
- The maximum profit is 192¢ or \$1.92.
- The price per cup should be set at 54¢ per cup to maximize profit.
- The break even points are $x = 4$ and $x = 20$, so to make a profit, between 4 and 20 cups of lemonade need to be made and sold.

33.
- $P(x) = -0.5x^2 + 9x - 36$, for $0 \leq x \leq 24$
- 9 pies should be made and sold to maximize the daily profit.
- The maximum daily profit is \$4.50.
- The price per pie should be set at \$7.50 to maximize profit.
- The break even points are $x = 6$ and $x = 12$, so to make a profit, between 6 and 12 pies need to be made and sold daily.

34.
- $P(x) = -2x^2 + 120x - 1000$, for $0 \leq x \leq 70$
- 30 scooters need to be made and sold to maximize profit.
- The maximum monthly profit is 800 hundred dollars, or \$80,000.
- The price per scooter should be set at 80 hundred dollars, or \$8000 per scooter.

- The break even points are $x = 10$ and $x = 50$, so to make a profit, between 10 and 50 scooters need to be made and sold monthly.

35. 495 cookies

36. The vertex is (approximately) $(29.60, 22.66)$, which corresponds to a maximum fuel economy of 22.66 miles per gallon, reached sometime between 2009 and 2010 (29 – 30 years after 1980.) Unfortunately, the model is only valid up until 2008 (28 years after 1908.) So, at this point, we are using the model to *predict* the maximum fuel economy.

37. 64° at 2 PM (8 hours after 6 AM.)

38. 5000 pens should be produced for a cost of \$200.

39. 8 feet by 16 feet; maximum area is 128 square feet.

40. 50 feet by 50 feet; maximum area is 2500 feet; he can raise 100 average alpacas.

41. The largest rectangle has area 12.25 square inches.

42. 2 seconds.

43. The rocket reaches its maximum height of 500 feet 10 seconds after lift-off.

44. The hammer reaches a maximum height of approximately 13.62 feet. The hammer is in the air approximately 1.61 seconds.

45. (a) The applied domain is $[0, \infty)$.
    (d) The height function is this case is $s(t) = -4.9t^2 + 15t$. The vertex of this parabola is approximately $(1.53, 11.48)$ so the maximum height reached by the marble is 11.48 meters. It hits the ground again when $t \approx 3.06$ seconds.
    (e) The revised height function is $s(t) = -4.9t^2 + 15t + 25$ which has zeros at $t \approx -1.20$ and $t \approx 4.26$. We ignore the negative value and claim that the marble will hit the ground after 4.26 seconds.
    (f) Shooting down means the initial velocity is negative so the height functions becomes $s(t) = -4.9t^2 - 15t + 25$.

46. Make the vertex of the parabola $(0, 10)$ so that the point on the top of the left-hand tower where the cable connects is $(-200, 100)$ and the point on the top of the right-hand tower is $(200, 100)$. Then the parabola is given by $p(x) = \frac{9}{4000}x^2 + 10$. Standing 50 feet to the right of the left-hand tower means you're standing at $x = -150$ and $p(-150) = 60.625$. So the cable is 60.625 feet above the bridge deck there.

47. (a) The line for the Thursday data is $y = -.12x + 237.69$. We have $r = -.9568$ and $r^2 = .9155$ so this is a really good fit.

(b) The line for the Saturday data is $y = -0.000693x + 235.94$. We have $r = -0.008986$ and $r^2 = 0.0000807$ which is horrible. This data is not even close to linear.

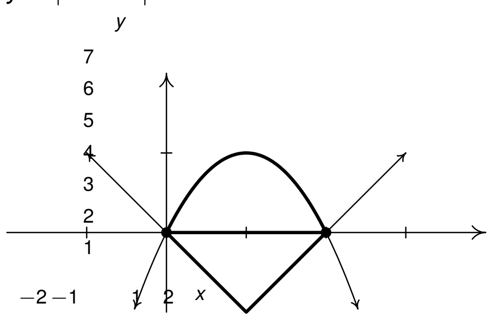
(c) The parabola for the Saturday data is $y = 0.003x^2 - 0.21x + 238.30$. We have $R^2 = .47497$ which isn't good. Thus the data isn't modeled well by a quadratic function, either.

(d) The Thursday linear model had my weight on January 1, 2010 at 193.77 pounds. The Saturday models give 235.69 and 563.31 pounds, respectively. The Thursday line has my weight going below 0 pounds in about five and a half years, so that's no good. The quadratic has a positive leading coefficient which would mean unbounded weight gain for the rest of my life. The Saturday line, which mathematically does not fit the data at all, yields a plausible weight prediction in the end. I think this is why grown-ups talk about "Lies, Damned Lies and Statistics."

48. (a) The quadratic model for the cats in Portage county is $y = 1917803.54x^2 - 16036408.29x + 24094857.7$. Although $R^2 = .70888$ this is not a good model because it's so far off for small values of x. The model gives us 24,094,858 cats when $x = 0$ but we know $N(0) = 2$.

(b) The quadratic model for the hours of daylight in Fairbanks, Alaska is $y = .51x^2 + 6.23x - .36$. Even with $R^2 = .92295$ we should be wary of making predictions beyond the data. Case in point, the model gives -4.84 hours of daylight when $x = 13$. So January 21, 2010 will be "extra dark"? Obviously a parabola pointing down isn't telling us the whole story.

50. $y = |1 - x^2|$



51. $\left(\dfrac{3-\sqrt{7}}{2}, \dfrac{-1+\sqrt{7}}{2}\right), \left(\dfrac{3+\sqrt{7}}{2}, \dfrac{-1-\sqrt{7}}{2}\right)$

52. $D(x) = x^2 + (2x+1)^2 = 5x^2 + 4x + 1$ is minimized when $x = -\frac{2}{5}$. Hence to find the point on $y = 2x + 1$ closest to $(0, 0)$ we substitute $x = -\frac{2}{5}$ into $y = 2x + 1$ to get $\left(-\frac{2}{5}, \frac{1}{5}\right)$.

55. $x = \pm y\sqrt{10}$

56. $x = \pm(y - 2)$

57. $x = \dfrac{m \pm \sqrt{m^2+4}}{2}$

58. $y = \dfrac{3 \pm \sqrt{16x+9}}{2}$

59. $y = 2 \pm x$

60. $t = \dfrac{v_0 \pm \sqrt{v_0^2 + 4gs_0}}{2g}$

61. (a) i. $L(x) = x^2$ ii. $L(x) = 2x^2 + x$ iii. $L(x) = -x^2 + 5x + 1$

(c) The three points lie on the same line and we get $L(x) = -x + 5$.

(d) To obtain a quadratic function, we require that the points are not collinear (i.e., they do not all lie on the same line.)

Chapter 6

Polynomial Functions

6.1 Graphs of Polynomial Functions

In Chapter 2, we studied functions of the form $f(x) = b$ (constant functions), $f(x) = mx + b$, $m \neq 0$ (linear functions), and $f(x) = ax^2 + bx + c$, $a \neq 0$ (quadratic functions). In each case, we learned how to construct graphs, find zeros, describe behavior, and use the functions in each family to model real-world phenomena. One might wonder about functions of the form $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, or functions containing even higher powers of x. These are the **polynomial functions** and are the subject of study in this chapter.[1] As you may recall, *poly*nomials are the result of adding *mon*omials, so we begin our study of polynomial functions with monomial functions.

6.1.1 Monomial Functions

Definition 6.1. A **monomial function** is a function of the form

$$f(x) = b \qquad \text{or} \qquad f(x) = ax^n,$$

where a and b are real numbers, $a \neq 0$ and $n \in \mathbb{N}$. The domain of a monomial function is $(-\infty, \infty)$.

Monomial functions, by definition, contain the constant functions along with a two parameter family of functions, $f(x) = ax^n$. We use x as the default independent variable here with a and n as parameters. From Section 1.1.2, we recall that the set $\mathbb{N} = \{1, 2, 3, ...\}$ is the set of natural numbers, so examples of monomial functions include $f(x) = 2x = 2x^1$, $g(t) = -0.1t^2$, and $H(s) = \sqrt{2}\,s^{117}$. Note that the function $f(x) = x^0$ is *not* a monomial function. Even though $x^0 = 1$ for all *nonzero* values of x, 0^0 is undefined,[2] and hence $f(x) = x^0$ does *not* have a domain of $(-\infty, \infty)$.[3]

We begin our study of the graphs of polynomial functions by studying graphs of monomial functions. Starting with $f(x) = x^n$ where n is even, we investigate the cases $n = 2$, 4 and 6 at the top of the next page.

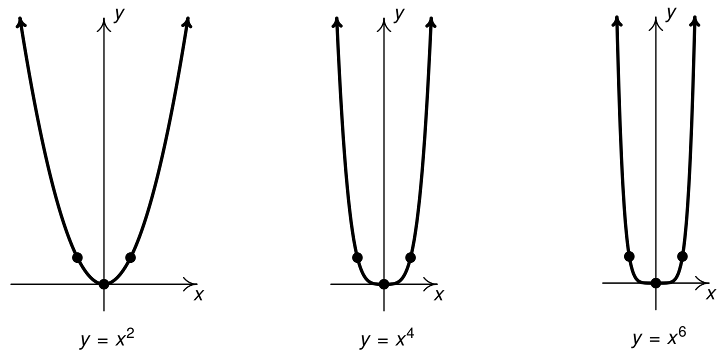
[1] You've seen polynomials before - see Section 5.1, for instance. Here, we restrict our attention to polynomial *functions* which for us means *one* independent variable instead of expressions with more than one variable.

[2] More specifically, 0^0 is an *indeterminate form.* These are studied extensively in Calculus.

[3] This is why we do not describe monomial functions as having the form $f(x) = ax^n$ for any *whole* number n. See Section 1.1.2

Numerically, we see that if $-1 < x < 1$, x^n becomes much smaller as n increases whereas if $x < -1$ or $x > 1$, x^n becomes much larger as n increases. These trends manifest themselves geometrically as the graph 'flattening' for $|x| < 1$ and 'narrowing' for $|x| > 1$ as n increases.[4]

x	x^2	x^4	x^6
-2	4	16	64
-1	1	1	1
-0.5	0.25	0.0625	0.015625
0	0	0	0
0.5	0.25	0.0625	0.015625
1	1	1	1
2	4	16	64



From the graphs, it appears as if the range of each of these functions is $[0, \infty)$. When n is even, $x^n \geq 0$ for all x so the range of $f(x) = x^n$ is contained in $[0, \infty)$. To show that the range of f is all of $[0, \infty)$, we note that the equation $x^n = c$ for $c \geq 0$ has (at least) one solution for every even integer n, namely $x = \sqrt[n]{c}$. (See Section 8.1 for a review of this notation.) Hence, $f(\sqrt[n]{c}) = (\sqrt[n]{c})^n = c$ which shows that every non-negative real number is in the range of f.[5]

Another item worthy of note is the symmetry about the line $x = 0$ a.k.a the y-axis. (See Definition **??** for a review of this concept.) With n being even, $f(-x) = (-x)^n = x^n = f(x)$. At the level of points, we have that for all x, $(-x, f(-x)) = (-x, f(x))$. Hence for every point $(x, f(x))$ on the graph of f, the point symmetric about the y-axis, $(-x, f(x))$ is on the graph, too. We give this sort of symmetry a name honoring its roots here with even-powered monomial functions:

Definition 6.2. A function f is said to be **even** if $f(-x) = f(x)$ for all x in the domain of f.
NOTE: A function f is even if and only if the graph of $y = f(x)$ is symmetric about the y-axis.

An investigation of the odd powered monomial functions ($n \geq 3$) yields similar results with the major difference being that when a negative number is raised to an odd natural number power the result is still negative. Numerically we see that for $|x| > 1$ the values of $|x^n|$ increase as n increases and the values of $|x^n|$ get closer to 0 as n increases. This translates graphically into a flattening behavior on the interval $(-1, 1)$ and a narrowing elsewhere. The graphs are shown on the top of the next page.

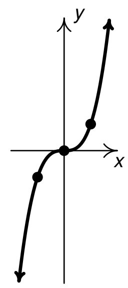
The range of these functions appear to be all real numbers, $(-\infty, \infty)$ which is algebraically sound as the equation $x^n = c$ has a solution for every real number,[6] namely $x = \sqrt[n]{c}$. Hence, for every real number c, choose $x = \sqrt[n]{c}$ so that $f(x) = f(\sqrt[n]{c}) = (\sqrt[n]{c})^n = c$. This shows that every real number is in the range of f.

[4] Recall that $|x| < 1$ is equivalent to $-1 < x < 1$ and $|x| > 1$ is equivalent to $x < -1$ or $x > 1$. Using absolute values allow us to describe these sets of real numbers more succinctly.

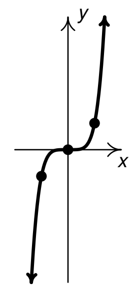
[5] This should sound familiar - see the comments regarding the range of $f(x) = x^2$ in Section 5.4.

[6] Do you see the importance of n being odd here?

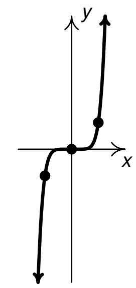
x	x^3	x^5	x^7
-2	-8	-32	-128
-1	-1	-1	-1
-0.5	0.125	-0.03125	-0.0078125
0	0	0	0
0.5	0.125	0.03125	0.0078125
1	1	1	1
2	8	32	128



$y = x^3$



$y = x^5$



$y = x^7$

Here, since n is odd, $f(-x) = (-x)^n = -x^n = -f(x)$. This means that whenever $(x, f(x))$ is on the graph, so is the point symmetric about the origin, $(-x, -f(x))$. (Again, see Definition **??**.) We generalize this property below. Not surprisingly, we name it in honor of its odd powered heritage:

Definition 6.3. A function f is said to be **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
NOTE: A function f is odd if and only if the graph of $y = f(x)$ is symmetric about the origin.

The most important thing to take from the discussion above is the basic shape and common points on the graphs of $y = x^n$ for each of the families when n even and n is odd. While symmetry is nice and should be noted when present, even and odd symmetry are comparatively rare. The point of Definitions 6.2 and 6.3is to give us the vocabulary to point out the symmetry when appropriate.

Moving on, we take a cue from Theorem 4.4 and prove the following.

Theorem 6.1. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x-h)^n + k$ can be obtained from the graph of $f(x) = x^n$ by performing the following operations, in sequence:

1. add h to the x-coordinates of each of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^n$ to $y = (x-h)^n$.

2. multiply the y-coordinates of each of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (x-h)^n$ to $y = a(x-h)^n$.

3. add k to the y-coordinates of each of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a(x-h)^n$ to $y = a(x-h)^n + k$

Proof. Our goal is to start with the graph of $f(x) = x^n$ and build it up to the graph of $F(x) = a(x-h)^n + k$. We begin by examining $F_1(x) = (x-h)^n$. The graph of $f(x) = x^n$ can be described as the set of points $\{(c, c^n) \mid c \in \mathbb{R}\}$.[7] Likewise, the graph of F_1 can be described as the set of points $\{(x, (x-h)^n) \mid x \in \mathbb{R}\}$.

[7] We are using the dummy variable c here instead of x for reasons that will become apparent shortly.

If we relabel $c = x - h$ so that $x = c + h$, then as x varies through all real numbers so does c.[8] Hence, we can describe the graph of F_1 as $\{(c + h, c^n) \mid c \in \mathbb{R}\}$. This means that we can obtain the graph of F_1 from the graph of f by adding h to each of the x-coordinates of the points on the graph of f and that establishes the first step of the theorem.

Next, we consider the graph of $F_2(x) = a(x-h)^n$ as compared to the graph of $F_1(x) = (x-h)^n$. The graph of F_1 is the set of points $\{(x, (x-h)^n \mid x \in \mathbb{R}\}$ while the graph of F_2 is the set of points $\{(x, a(x-h)^n) \mid x \in \mathbb{R}\}$. The only difference between the points $(x, (x - h)^n)$ and $(x, a(x - h)^n)$ is that the y-coordinate in the latter is a times the y-coordinate of the former.

In other words, to produce the graph of F_2 from the graph of F_1, we take the y-coordinate of each point on the graph of F_1 and multiply it by a to get the corresponding point on the graph of F_2. If $a > 0$, all we are doing is scaling the y-axis by a. If $a < 0$, then, in addition to scaling the y-axis, we are also reflecting each point across the x-axis. In either case, we have established the second step of the theorem.

Last, we compare the graph of $F(x) = a(x - h)^n + k$ to that of $F_2(x) = a(x - h)^n$. Once again, we view the graphs as sets of points in the plane. The graph of F_2 is $\{(x, a(x - h)^n) \mid x \in \mathbb{R}\}$ and the graph of F is$\{(x, a(x - h)^n + k) \mid x \in \mathbb{R}\}$. Looking at the corresponding points, $(x, a(x - h)^n)$ and $(x, a(x - h)^n + k)$, we see that we can obtain all of the points on the graph of F by adding k to each of the y-coordinates to points on the graph of F_2. This is equivalent to shifting every point vertically by k units which establishes the third and final step in the theorem. $\square$

This argument should sound familiar. The proof we presented above is more-or-less the same argument we presented after the proof of Theorem 4.4 in Section 4.2 but with '$|\cdot|$' replaced by '$(\cdot)^n$.' Also note that using $n = 2$ in Theorem 6.1 establishes Theorem 5.7 in Section 5.4.

We now use Theorem 6.1 to graph two different "transformed" monomial functions. To provide the reader an opportunity to compare and contrast the graphical behaviors exhibited in the case when n is even versus when n is odd, we graph one of each case.

Example 6.1.1. Use Theorem 6.1 to graph the following functions. Label at least three points on each graph. State the domain and range using interval notation.

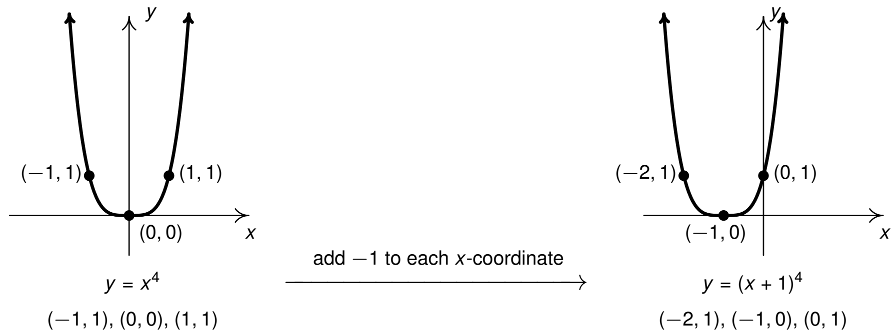
1. $f(x) = -2(x + 1)^4 + 3$
2. $g(t) = \dfrac{(2t - 1)^3}{5}$

Solution.

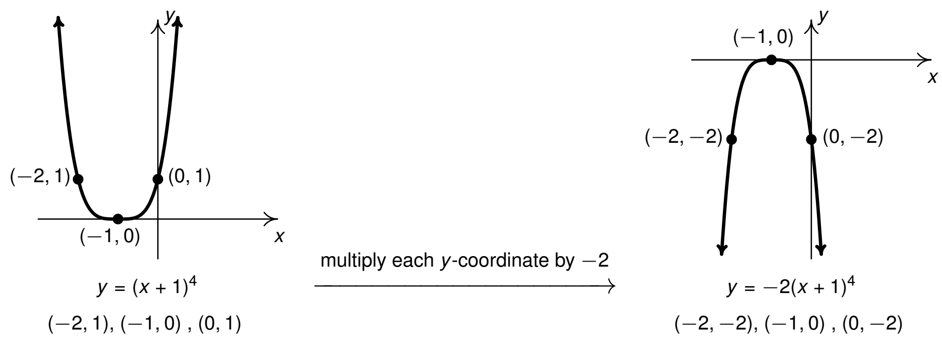
1. For $f(x) = -2(x + 1)^4 + 3 = -2(x - (-1))^4 + 3$, we identify $n = 4$, $a = -2$, $h = -1$, and $k = 3$. Thus to graph f, we start with $y = x^4$ and perform the following steps, in sequence, tracking the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ through each step:

   Step 1: add -1 to the x-coordinates of each of the points on the graph of $y = x^4$:

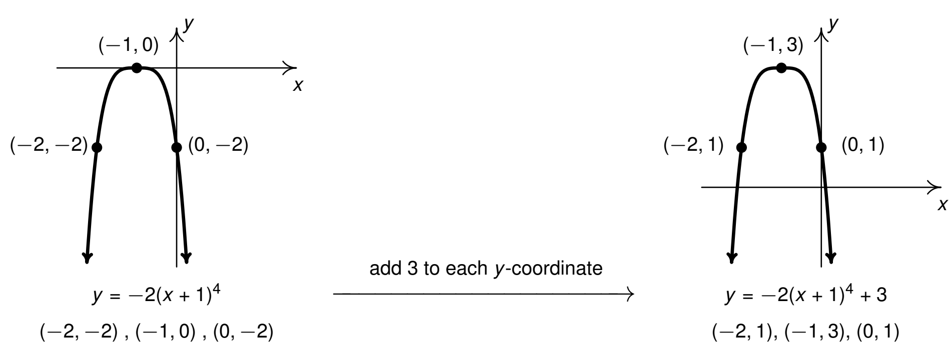
[8] That is, for a fixed number h every real number c can be written as $x - h$ for some real number x, and every real number x can be written as $c + h$ for some real number c.



Step 2: multiply the y-coordinates of each of the points on the graph of $y = (x+1)^4$ by -2:



Step 3: add 3 to the y-coordinates of each of the points on the graph of $y = -2(x+1)^4$:



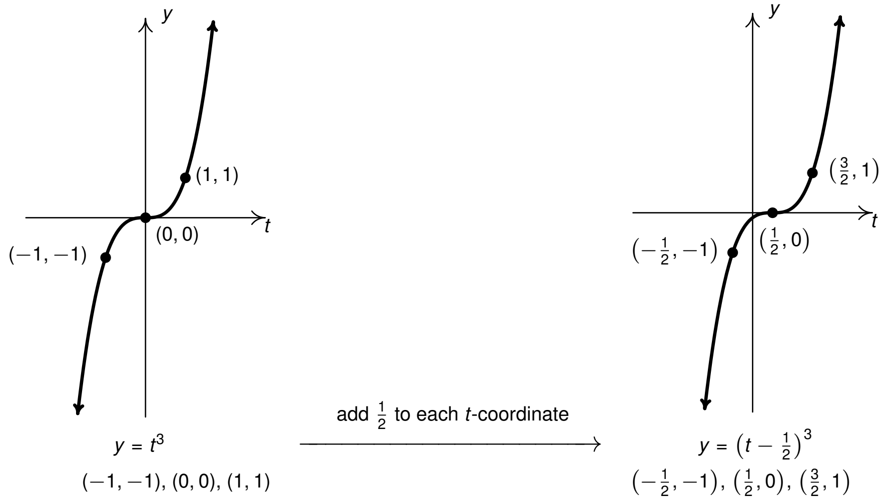
The domain here is $(-\infty, \infty)$ while the range is $(-\infty, 3]$.

2. To use Theorem 6.1 to graph $g(t) = \dfrac{(2t-1)^3}{5}$, we must rewrite the expression for $g(t)$:

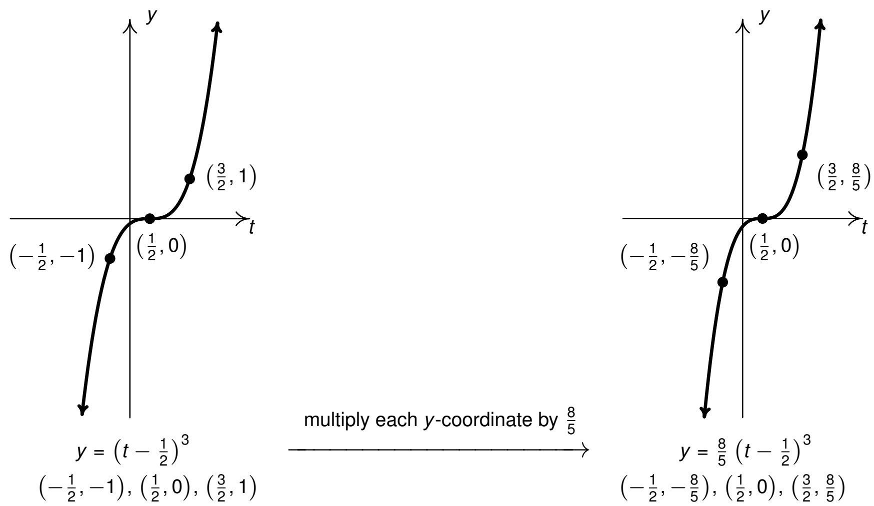
$$g(t) = \frac{(2t-1)^3}{5} = \frac{1}{5}\left(2\left(t - \frac{1}{2}\right)\right)^3 = \frac{1}{5}(2)^3\left(t - \frac{1}{2}\right)^3 = \frac{8}{5}\left(t - \frac{1}{2}\right)^3$$

We identify $n = 3$, $h = \frac{1}{2}$ and $a = \frac{8}{5}$. Hence, we start with the graph of $y = t^3$ and perform the following steps, in sequence, tracking the points $(-1,-1)$, $(0,0)$ and $(1,1)$ through each step:

Step 1: add $\frac{1}{2}$ to each of the t-coordinates of each of the points on the graph of $y = t^3$:



Step 2: multiply each of the y-coordinates of the graph of $y = \left(t - \frac{1}{2}\right)^3$ by $\frac{8}{5}$.



Both the domain and range of g is $(-\infty, \infty)$. $\square$

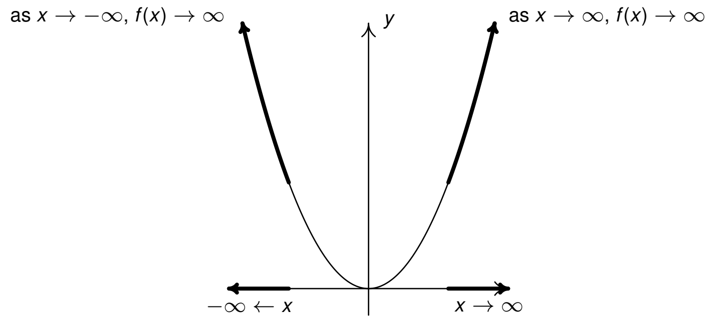
Example 6.1.1 demonstrates two big ideas in mathematics: first, resolving a complex problem into smaller, simpler steps, and, second, the value of changing form.[9]

Next we wish to focus on the so-called **end behavior** presented in each case.[10] The end behavior of a function is a way to describe what is happening to the outputs from a function as the inputs approach the 'ends' of the domain. Since domain of monomial functions is $(-\infty, \infty)$, we are looking to see what these

[9]We've seen the importance of changing form several times already, but it never hurts to point it out.
[10]Sometimes called the 'long run' behavior.

functions do as their inputs ‘approach’ $\pm\infty$. The best we can do is sample inputs and outputs and infer general behavior from these observations.[11] The good news is we’ve wrestled with this concept before. Indeed, every time we add ‘arrows’ to the graph of a function, we’ve indicated its end behavior. Let’s revisit the graph of $f(x) = x^2$ using the table below.

x	$f(x) = x^2$
-1000	1000000
-100	10000
-10	100
0	0
10	100
100	10000
1000	1000000

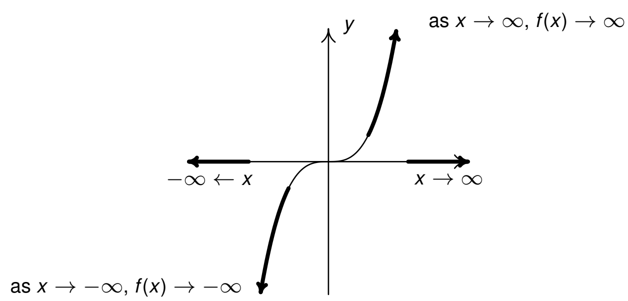


$f(x) = x^2$

As x takes on smaller and smaller values,[12], we see $f(x)$ takes on larger and larger positive values. The smaller x we use, the larger the $f(x)$ becomes, seemingly without bound.[13] We codify this behavior by writing as $x \to -\infty$, $f(x) \to \infty$. Graphically, the farther to the left we travel on the x-axis, the farther up the y-axis the function values travel. This is why we use an ‘arrow’ on the graph in Quadrant II heading upwards to the left. Similarly, we write as $x \to \infty$, $f(x) \to \infty$ since as the x values increase, so do the $f(x)$ values - seemingly without bound. Graphically we indicate this by an arrow on the graph in Quadrant I heading upwards to the right. This behavior holds for all functions $f(x) = x^n$ where $n \geq 2$ is even.

Repeating this investigation for for $f(x) = x^3$, we find as $x \to -\infty$, $f(x)$ becomes unbounded in the negative direction, so we write $f(x) \to -\infty$. As $x \to \infty$, $f(x)$ becomes unbounded in the positive direction, so we write $f(x) \to \infty$. This trend holds for all functions $f(x) = x^n$ where n is odd.

x	$f(x) = x^3$
-1000	-1000000000
-100	-1000000
-10	-1000
0	0
10	1000
100	1000000
1000	1000000000



$f(x) = x^3$

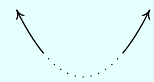
Theorem 6.2 summarizes the end behavior of monomial functions. The results are a consequence of Theorem 6.1 in that the end behavior of a function of the form $y = ax^n$ only differs from that of $y = x^n$ if there is a reflection, that is, if $a < 0$.

[11]and let Calculus students prove our claims.
[12]said differently, negative values that are larger in absolute value
[13]That is, the $f(x)$ values grow larger than any positive number. They are ‘*unbounded*.’

Theorem 6.2. End Behavior of Monomial Functions:
Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \in \mathbb{N}$.

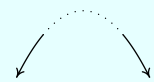
- If n is even:

  if $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$:

  $a > 0$

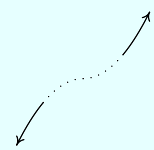
  for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$:

  $a < 0$

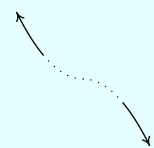
- If n is odd:

  for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$:

  $a > 0$

  for $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$:

  $a < 0$

6.1.2 Polynomial Functions

We are now in the position to discuss **polynomial** functions. Simply stated, *poly*nomial functions are sums of *mon*omial functions. The challenge becomes how to describe one of these beasts in general. Up until now, we have used distinct letters to indicate different parameters in our definitions of function families. In other words, we define constant functions as $f(x) = b$, linear functions as $f(x) = mx + b$, and quadratic functions as $f(x) = ax^2 + bx + c$. We even hinted at a function of the form $f(x) = ax^3 + bx^2 + cx + d$. What happens if we wanted to describe a generic polynomial that required, say, 117 different parameters? Our work around is to use subscripted parameters, a_k, that denote the coefficient of x^k. For example, instead of writing a quadratic as $f(x) = ax^2 + bx + c$, we describe it as $f(x) = a_2x^2 + a_1x + a_0$, where a_2, a_1, and a_0 are real numbers and $a_2 \neq 0$. As an added example, consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. We can re-write the formula for f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. and identify $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. This is the notation we use in the following definition.

Definition 6.4. A **polynomial function** is a function of the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers and $n \in \mathbb{N}$. The domain of a polynomial function is $(-\infty, \infty)$.

As usual, x is used in Definition 6.4 as the independent variable with the a_k each being a parameter. Even though we specify $n \in \mathbb{N}$ so $n \geq 1$, the value of the a_k are unrestricted. Hence, any constant function $f(x) = b$ can be written as $f(x) = 0x + a_0$, and so they are polynomials. Polynomials have an associated vocabulary,[14] and hence, so do polynomial functions.

Definition 6.5.

- Given $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with $n \in \mathbb{N}$ and $a_n \neq 0$, we say
  - The natural number n is called the **degree** of the polynomial f.
  - The term $a_n x^n$ is called the **leading term** of the polynomial f.
  - The real number a_n is called the **leading coefficient** of the polynomial f.
  - The real number a_0 is called the **constant term** of the polynomial f.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.
- If $f(x) = 0$, we say f has no degree.[a]

[a] Some authors say $f(x) = 0$ has degree $-\infty$ for reasons not even we will go into.

Again, constant functions are split off in their own separate case Definition 6.5 because of the ambiguity of 0^0. (See the remarks following Definition 6.1.) A consequence of Definition 6.5 is that we can now think of nonzero constant functions as 'zeroth' degree polynomial functions, linear functions as 'first' degree polynomial functions, and quadratic functions as 'second' degree polynomial functions.

Example 6.1.2. Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(t) = 12t - t^3$
3. $H(w) = \dfrac{4 - w}{5}$
4. $p(z) = (2z - 1)^3(z - 2)(3z + 2)$

Solution.

1. There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of Definition 6.5, and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.

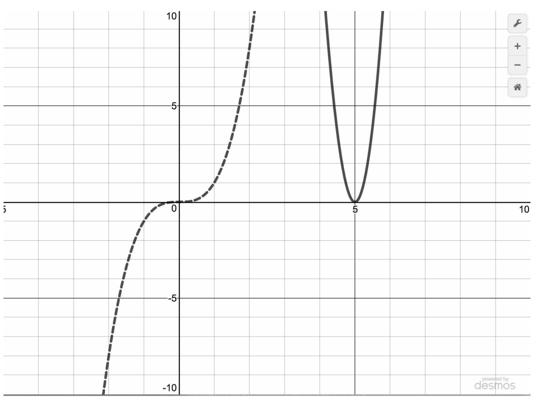
2. Two changes here: first, the independent variable is t, not x. Second, the form given in Definition 6.5 specifies the function be written in descending order of the powers of x, or in this case, t. To that end, we re-write $g(t) = 12t - t^3 = -t^3 + 12t$, and see that the degree of g is 3, the leading term is $-t^3$, the leading coefficient is -1 and the constant term is 0.

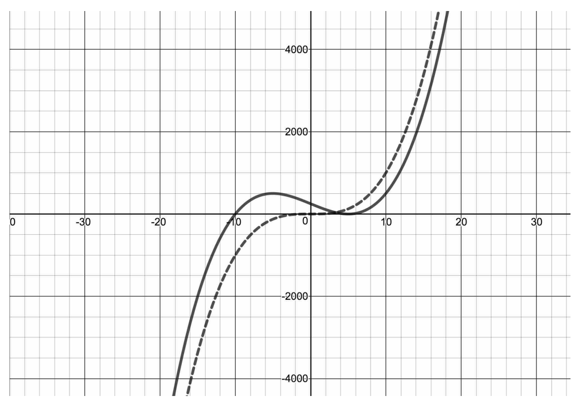
[14] See Section 5.1.

3. We need to rewrite the formula for $H(w)$ so that it resembles the form given in Definition 6.5: $H(w) = \frac{4-w}{5} = \frac{4}{5} - \frac{w}{5} = -\frac{1}{5}w + \frac{4}{5}$. We see the degree of H is 1, the leading term is $-\frac{1}{5}w$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.

4. It may seem that we have some work ahead of us to get p in the form of Definition 6.5. However, it is possible to glean the information requested about p without multiplying out the entire expression $(2z-1)^3(z-2)(3z+2)$. The leading term of p will be the term which has the highest power of z. The way to get this term is to multiply the terms with the highest power of z from each factor together - in other words, the leading term of $p(z)$ is the product of the leading terms of the *factors* of $p(z)$. Hence, the leading term of p is $(2z)^3(z)(3z) = 24z^5$. This means that the degree of p is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar operation. The constant term of p is obtained by multiplying the constant terms from each of the *factors*: $(-1)^3(-2)(2) = 4$. $\square$

We now turn our attention to graphs of polynomial functions. Since polynomial functions are sums of monomial functions, it stands to reason that some of of the properties of those graphs carry over to more general polynomials. We first discuss end behavior. Consider $f(x) = x^3 - 75x + 250$. Below is the graph of $f(x)$ (solid line) along with the graph of its leading term, $y = x^3$ (dashed line.) Below on the left is a view 'near' the origin while below on the right is a 'zoomed out' view. Near the origin, the graphs have little in common, but as we look farther out, it becomes that the functions begin to look quite similar.





This observation is borne out numerically as well. Based on the table below, as $x \to \pm\infty$, it certainly appears as if $f(x) \approx g(x)$. One way to think about what is happening numerically is that the leading term x^3 *dominates* the lower order terms $-75x$ and 250 as $x \to \pm\infty$. In other words, x^3 grows so much faster than $-75x$ and 250 that these 'lower order terms' don't contribute anything of significance to the x^3 so $f(x) \approx x^3$. Another way to see this is to rewrite $f(x)$ as[15]

$$f(x) = x^3 - 75x + 250 = x^3\left(1 - \frac{75}{x^2} + \frac{250}{x^3}\right).$$

As $x \to \pm\infty$, both $\frac{75}{x^2}$ and $\frac{250}{x^3}$ have constant numerators but denominators that are becoming unbounded.

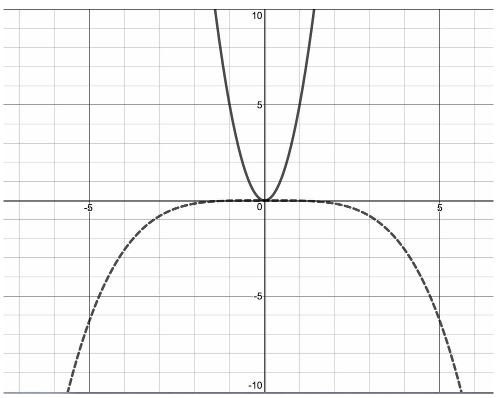
[15]Since we are considering $x \to \pm\infty$, we are not concerned with x even being close to 0, so these fractions will all be defined.

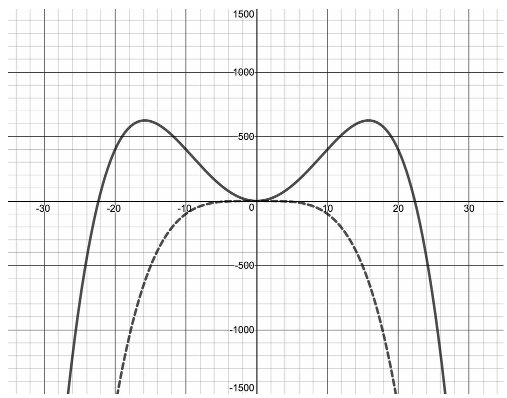
As such, both $\frac{75}{x^2}$ and $\frac{250}{x^3} \to 0$. Therefore, as $x \to \pm\infty$,

$$f(x) = x^3 - 75x + 250 = x^3\left(1 - \frac{75}{x^2} + \frac{250}{x^3}\right) \approx x^3(1 + 0 + 0) = x^3.$$

x	$f(x) = x^3 - 75x + 250$	x^3	$-75x$	250	$\frac{75}{x^2}$	$\frac{250}{x^3}$
-1000	$\approx -1 \times 10^9$	-1×10^9	75000	250	7.5×10^{-5}	-2.5×10^{-7}
-100	$\approx -9.9 \times 10^5$	-1×10^6	7500	250	0.0075	-2.5×10^{-4}
-10	0	-1000	750	250	0.75	-0.25
10	500	1000	-750	250	0.75	0.25
100	$\approx 9.9 \times 10^5$	1×10^6	-7500	250	0.0075	2.5×10^{-4}
1000	$\approx 1 \times 10^9$	1×10^9	-75000	250	7.5×10^{-5}	2.5×10^{-7}

Next, consider $g(x) = -0.01x^4 + 5x^2$. Following the logic of the above example, we would expect the end behavior of $y = g(x)$ to mimic that of $y = -0.01x^4$. When we graph $y = g(x)$ (solid line) on the same set of axes as $y = -0.01x^4$ (dashed line), a view near the origin seems to suggest the exact opposite. However, zooming out reveals that the two graphs do share the same end behavior.[16]





Algebraically, for $x \to \pm\infty$, even with the small coefficient of -0.01, $-0.01x^4$ dominates the $5x^2$ term so $g(x) \approx -0.01x^4$. More precisely,

$$g(x) = -0.01x^4 + 5x^2 = x^4\left(-0.01 + \frac{5}{x^2}\right) \approx x^4(-0.01 + 0) = -0.01x^4.$$

The results of these last two examples generalize below in Theorem 6.3.

Theorem 6.3. End Behavior for Polynomial Functions:
The end behavior of polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with $a_n \neq 0$ matches the end behavior of $y = a_n x^n$.
That is, the end behavior of a polynomial function is determined by its leading term.

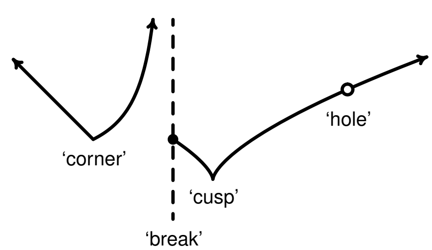
We argue Theorem 6.3 using an argument similar to ones used above. As $x \to \pm\infty$,

[16]Or at least they appear to within the limits of the technology.

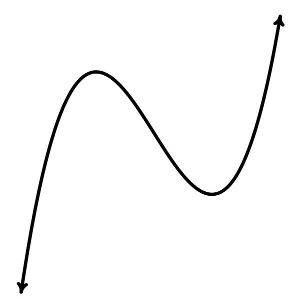
$$f(x) = x^n\left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \approx x^n(a_n + 0 + \ldots 0) = a_n x^n$$

If this argument looks a little fuzzy, it should. In Calculus, we have the tools necessary to more explicitly state what we mean by ≈ 0. For now, we'll rely on number sense and algebraic intuition.[17]

Now that we know how to determine the end behavior of polynomial functions, it's time to investigate what happens 'in between' the ends. First and foremost, polynomial functions are **continuous**. Recall from Section 5.4 that, informally, graphs of continuous functions have no 'breaks' or 'holes' in them.[18] Since monomial functions are continuous (as far as we can tell) and polynomials are sums of monomial functions, it turns out that polynomial functions are continuous as well. Moreover, the graphs of monomial functions, hence polynomial functions, are **smooth**. Once again, 'smoothness' is a concept defined precisely in Calculus, but for us, functions have no 'corners' or 'sharp turns'. Below we find the graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison. The function whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in the graph on the right. The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteristic '$\vee$' shape, cannot be a polynomial function, even though it is a piecewise-defined function comprised of polynomial functions. Knowing polynomial functions are continuous and smooth gives us an idea of how to 'connect the dots' when sketching the graph from points that we're able to find analytically such as intercepts.



Pathologies not found on graphs of polynomials functions.



The graph of a polynomial function.

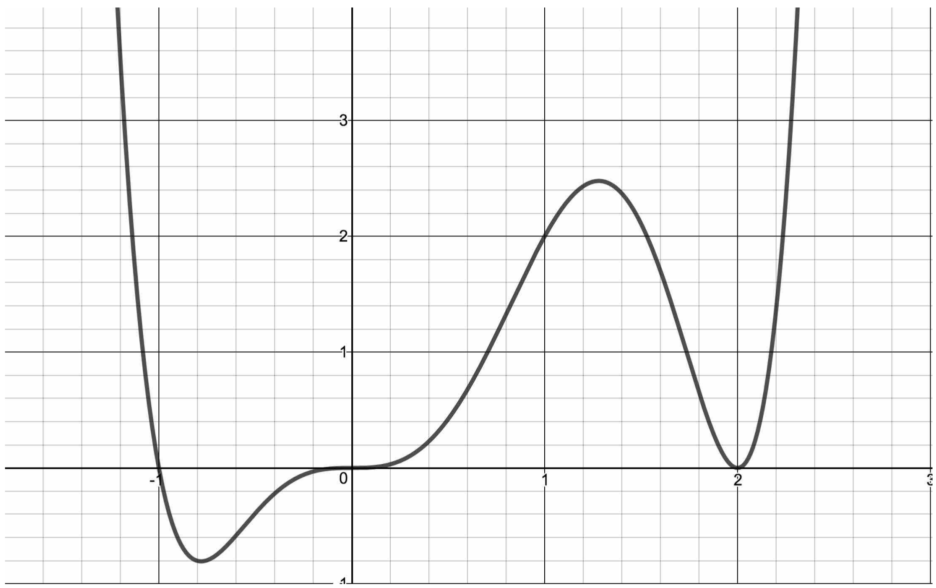
Speaking of intercepts, we next focus our attention on the behavior of the graphs of polynomial functions near their zeros. Recall a zero c of a function f is a solution to $f(x) = 0$. Geometrically, the zeros of a function are the x-coordinates of the x-intercepts of the graph of $y = f(x)$. Consider the polynomial function $f(x) = x^3(x-2)^2(x+1)$. To find the zeros of f, we set $f(x) = x^3(x-2)^2(x+1) = 0$. Since the expression $f(x)$ is already factored, we set each factor equal to zero.[19] Solving $x^3 = 0$ gives $x = 0$, $(x-2)^2 = 0$ gives $x = 2$, and $x + 1 = 0$ gives $x = -1$. Hence, our zeros are $x = -1$, $x = 0$, and $x = 2$. Below, we graph $y = f(x)$ and observe the x-intercepts $(-1, 0)$, $(0, 0)$ and $(2, 0)$. We first note that the graph

[17]Both of which, by the way, can lead one astray, so we must proceed cautiously.

[18]Again, the formal definition of 'continuity' and properties of continuous functions are discussed in Calculus.

[19]in accordance with the Zero Product Property of the Real Numbers - see Section 1.2.

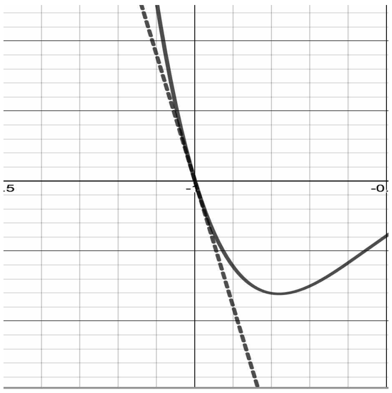
crosses through the x-axis at $(-1, 0)$ and $(0, 0)$, but the graph *touches* and *rebounds* at $(2, 0)$. Moreover, at $(-1, 0)$, the graph crosses through the axis is a fairly 'linear' fashion whereas there is a substantial amount of 'flattening' going on near $(0, 0)$. Or aim is to explain these observations and generalize them.



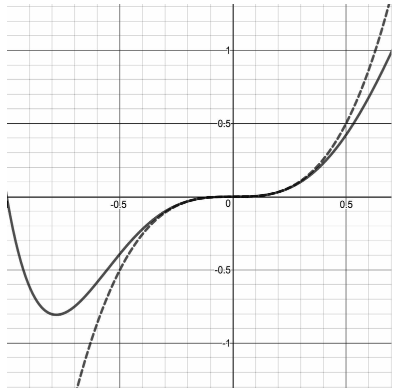
First, let's look at what's happening with the formula $f(x) = x^3(x-2)^2(x+1)$ when $x \approx -1$. We know the x-intercept at $(-1, 0)$ is due to the presence of the $(x+1)$ factor in the expression for $f(x)$. So, in this sense, the factor $(x+1)$ is determining a major piece of the behavior of the graph near $x = -1$. For that reason, we focus instead on the other two factors to see what contribution they make. We find when $x \approx -1$, $x^3 \approx (-1)^3 = -1$ and $(x-2)^2 \approx (-1-2)^2 = 9$. Hence, $f(x) = x^3(x-3)^2(x+1) \approx (-1)^3(-1-2)^2(x+1) = -9(x+1)$. Below on the left is a graph of $y = f(x)$ (the solid line) and the graph of $y = -9(x+1)$ (the dashed line.) Sure enough, these graphs approximate one another near $x = -1$.

Likewise, let's look near $x = 0$. The x-intercept $(0, 0)$ is due to the x^3 term. For $x \approx 0$, $(x-2)^2 \approx (0-2)^2 = 4$ and $(x+1) \approx (0+1) = 1$, so $f(x) = x^3(x-3)^2(x+1) \approx x^3(-2)^2(1) = 4x^3$. Below in the center picture, we have the graph of $y = f(x)$ (again, the solid line) and $y = 4x^3$ (the dashed line) near $x = 0$. Once again, the graphs verify our analysis.

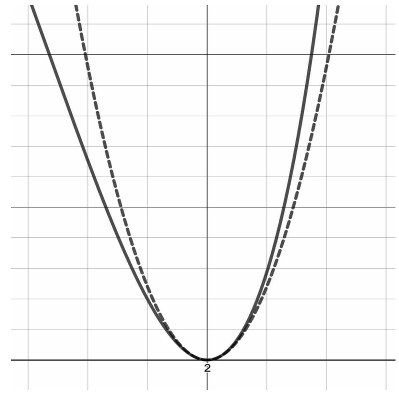
Last, but not least, we analyze f near $x = 2$. Here, the intercept $(2, 0)$ is due to the $(x-2)^2$ factor, so we look at the x^3 and $(x+1)$ factors. If $x \approx 2$, $x^3 \approx (2)^3 = 8$ and $(x+1) \approx (2+1) = 3$. Hence, $f(x) = x^3(x-3)^2(x+1) \approx (2)^3(x-2)^2(2+1) = 24(x-2)^2$. Sure enough, as evidenced below on the right, the graphs of $y = f(x)$ and $y = 24(x-2)^2$.



$y = f(x)$ and $y = -9(x + 1)$



$y = f(x)$ and $y = 4x^3$



$y = f(x)$ and $y = 24(x - 2)^2$

We generalize our observations in Theorem 6.4 below. Like many things we've seen in this text, a more precise statement and proof can be found in a course on Calculus.

Theorem 6.4. Suppose f is a polynomial function and $f(x) = (x - c)^m q(x)$ where $m \in \mathbb{N}$ and $q(c) \neq 0$. Then the the graph of $y = f(x)$ near $(c, 0)$ resembles that of $y = q(c)(x - c)^m$.

Let's see how Theorem 6.4 applies to our findings regarding $f(x) = x^3(x - 2)^2(x + 1)$. For $c = -1$, $(x - c) = (x - (-1)) = (x + 1)$. We rewrite $f(x) = x^3(x - 2)^2(x + 1) = (x - (-1))^1 \left[x^3(x - 2)^2\right]$ and identify $m = 1$ and $q(x) = x^3(x - 2)^2$. We find $q(c) = q(-1) = (-1)^3(-1 - 2)^2 = -9$ so Theorem 6.4 says that near $(-1, 0)$, the graph of $y = f(x)$ resembles $y = q(-1)(x - (-1))^1 = -9(x + 1)$. For $c = 0$, $(x - c) = (x - 0) = x$ and we can rewrite $f(x) = x^3(x - 2)^2(x + 1) = (x - 0)^3 \left[(x - 2)^2(x + 1)\right]$. We identify $m = 3$ and $q(x) = (x - 2)^2(x + 1)$. In this case $q(c) = q(0) = (0 - 2)^2(0 + 1) = 4$, so Theorem 6.4 guarantees the graph of $y = f(x)$ near $x = 0$ resembles $y = q(0)(x - 0)^3 = 4x^3$. Lastly, for $c = 2$, we see $f(x) = (x - 2)^2 \left[x^3(x + 1)\right]$ and we identify $m = 2$ and $q(x) = x^3(x + 1)$. We find $q(2) = 2^3(2 + 1) = 24$, so Theorem 6.4 guarantees the graph of $y = f(x)$ resembles $y = 24(x - 2)^2$ near $x = 2$.

As we already mentioned, the formal statement and proof of Theorem 6.4 require Calculus. For now, we can understand the theorem as follows. If we factor a polynomial function as $f(x) = (x - c)^m q(x)$ where $m \geq 1$, then $x = c$ is a zero of f, since $f(c) = (c - c)^m q(c) = 0 \cdot q(c) = 0$. The stipulation that $q(c) \neq 0$ means that we have essentially factored the expression $f(x) = (x - c)^m q(x) = (\text{going to } 0) \cdot (\text{not going to } 0)$. Thinking back to Theorem 6.1, the graph $y = q(c)(x - c)^m$ has an x-intercept at $(c, 0)$, a basic overall shape determined by the exponent m, and end behavior determined by the sign of $q(c)$. The fact that *if* $x = c$ is a zero *then* we are guaranteed we can factor $f(x) = (x - c)^m q(x)$ were $q(c) \neq 0$ and, moreover, such a factorization is unique (so that there's only one value of m possible for each zero) is a consequence of two theorems, Theorem 6.6 and The Factor Theorem, Theorem 6.8 which we'll review in Section 6.2. For now, we assume such a factorization is unique in order to define the following.

Definition 6.6. Suppose f is a polynomial function and $m \in \mathbb{N}$. If $f(x) = (x - c)^m q(x)$ where $q(c) \neq 0$, we say $x = c$ is a zero of **multiplicity** m.

So, for $f(x) = x^3(x - 2)^2(x + 1) = (x - 0)^3(x - 2)^2(x - (-1))^1$, $x = 0$ is a zero of multiplicity 3, $x = 2$ is a zero of multiplicity 2, and $x = -1$ is a zero of multiplicity 1. Theorems 6.3 and 6.4 give us the following:

Theorem 6.5. The Role of Multiplicity: Suppose f is a polynomial function and $x = c$ is a zero of multiplicity m.

- If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$.
- If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$.

Our next example showcases how all of the above theory can assist in sketching relatively good graphs of polynomial functions without the assistance of technology.

Example 6.1.3. Let $p(x) = (2x - 1)(x + 1)(1 - x^4)$.

1. Find all real zeros of p and state their multiplicities.
2. Describe the behavior of the graph of $y = p(x)$ near each of the x-intercepts.
3. Determine the end behavior and y-intercept of the graph of $y = p(x)$.
4. Sketch $y = p(x)$ and check your answer using a graphing utility.

Solution.

1. To find the zeros of p, we set $p(x) = (2x - 1)(x + 1)(1 - x^4) = 0$. Since the expression $p(x)$ is already (partially) factored, we set each factor equal to 0 and solve. From $(2x - 1) = 0$, we get $x = \frac{1}{2}$; from $(x + 1) = 0$ we get $x = -1$; and from solving $1 - x^4 = 0$ we get $x = \pm 1$. Hence, the zeros are $x = -1$, $x = \frac{1}{2}$, and $x = 1$. In order to determine the multiplicities, we need to factor $p(x)$ as so we can identify the m and $q(x)$ as described in Definition 6.6. The zero $x = -1$ corresponds to the factor $(x + 1)$. Notice, however, that writing $p(x) = (x+1)^1 \left[(2x - 1)(1 - x^4)\right]$ with $m = 1$ and $q(x) = (2x-1)(1-x^4)$ does *not* satisfy Definition 6.6 since here, $q(-1) = (2(-1) - 1)(1 - (-1)^4) = 0$. Indeed, we can factor $(1 - x^4) = (1 - x^2)(1 + x^2) = (1 - x)(1 + x)(x^2 + 1)$ so that

   $$p(x) = (2x-1)(x+1)(1-x^4) = (2x-1)(x+1)(1-x)(1+x)(x^2+1) = (x+1)^2 \left[(2x - 1)(1 - x)(x^2 + 1)\right].$$

   Identifying $q(x) = (2x-1)(1-x)(x^2+1)$, we find $q(-1) = (2(-1)-1)(1-(-1))((-1)^2+1) = -12 \neq 0$, which means the multiplicity of $x = -1$ is $m = 2$.

   The zero $x = \frac{1}{2}$ came from the factor $(2x - 1) = 2(x - \frac{1}{2})$, so we have

   $$p(x) = (2x - 1)(x + 1)^2(1 - x)(x^2 + 1) = (x - \frac{1}{2})^1 \left[2(x + 1)^2(1 - x)(x^2 + 1)\right].$$

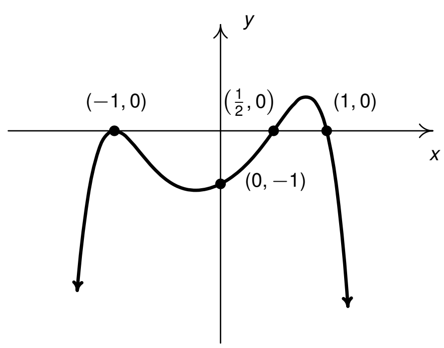
   If we identify $q(x) = 2(x + 1)^2(1 - x)(x^2 + 1)$, we find $q(\frac{1}{2}) = \frac{45}{16} \neq 0$ so multiplicity here is $m = 1$.

   Last but not least, we turn our attention to our last zero, $x = 1$, which we obtained from solving $1 - x^4 = 0$. However, from $p(x) = (2x - 1)(x + 1)^2(1 - x)(x^2 + 1)$, we see the zero $x = 1$ corresponds to the factor $(1 - x) = -(x - 1)$. We have $p(x) = (x - 1)^1 \left[-(2x - 1)(x + 1)^2(x^2 + 1)\right]$. Identifying $q(x) = -(2x - 1)(x + 1)^2(x^2 + 1)$, we see $q(1) = -8$, so the multiplicity $m = 1$ here as well.

2. From Theorem 6.5, since the multiplicities of $x = \frac{1}{2}$ and $x = 1$ are both *odd*, we know the graph of $y = p(x)$ *crosses* through the x-axis at $(\frac{1}{2}, 0)$ and $(1, 0)$. More specifically, since the multiplicity for both of these zeros is 1, the graph will look locally linear at these points. More specifically, based on our calculations above, near $x = \frac{1}{2}$, the graph will resemble the increasing line $y = \frac{45}{16}(x - \frac{1}{2})$, and near $x = 1$, the graph will resemble the decreasing line $y = -8(x - 1)$. Since the multiplicity of $x = -1$ is *even*, we know the graph of $y = p(x)$ *touches* and *rebounds* at $(-1, 0)$. Since the multiplicity of $x = -1$ is 2, it will look locally like a parabola. More specifically, the graph near $x = -1$ will resemble $y = -12(x + 1)^2$.

3. Per Theorem 6.3, the end behavior of $y = p(x)$, matches the end behavior of its leading term. As in Example 6.1.2, we multiply the leading terms from each factor together to obtain the leading term for $p(x) : p(x) = (2x-1)(x+1)(1-x^4) = (2x)(x)(-x^4) + \ldots = -2x^6 + \ldots$. Since the degree here, 6, is even and the leading coefficient $-2 < 0$, we know as $x \to \pm\infty$, $p(x) \to -\infty$. To find the y-intercept, we find $p(0) = (2(0) - 1)(0 + 1)(1 - 0^4) = -1$, hence, the y-intercept is $(0, -1)$.

4. From the end behavior, $x \to -\infty$, $p(x) \to -\infty$, we start the graph in Quadrant III and head towards $(-1, 0)$. At $(-1, 0)$, we 'bounce' off of the x-axis and head towards the y-intercept, $(0, -1)$. We then head towards $\left(\frac{1}{2}, 0\right)$ and cross through the x-axis there. Finally, we head back to the x-axis and cross through at $(1, 0)$. Owing to the end behavior $x \to \infty$, $p(x) \to -\infty$, we exit the picture in Quadrant IV. Since polynomial functions are continuous and smooth, we have no holes or gaps in the graph, and all the 'turns' are rounded (no abrupt turns or corners.) We produce something resembling the graph below.



$y = p(x)$

□

A couple of remarks about Example 6.1.3 are in order. First, notice that the factor $(x^2 + 1)$ was more of a spectator in our discussion of the zeros of p. Indeed, if we set $x^2 + 1 = 0$, we have $x^2 = -1$ which provides no *real* solutions.[20] That being said, the factor $x^2 + 1$ *does* affect the shape of the graph (See Exercise 60.) Next, when connecting up the graph from $(-1, 0)$ to $(0, -1)$ to $\left(\frac{1}{2}, 0\right)$, there really is no way for us to know how low the graph goes, or where the lowest point is between $x = -1$ and $x = \frac{1}{2}$ unless we plot more

[20]The solutions are $x = \pm i$ - see Section **??**.

points. Likewise, we have no idea how high the graph gets between $x = \frac{1}{2}$ and $x = 1$. While there are ways to determine these points analytically, more often than not, finding them requires Calculus. Since these points do play an important role in many applications, we'll need to discuss them in this course and, when required, we'll use technology to find them. For that reason, we have the following definition:

Definition 6.7. Suppose f is a function with $f(a) = b$.

- We say f has a **local minimum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \leq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local minimum value of f.'

  That is, b is the minimum $f(x)$ value over an *open interval* containing a.

  Graphically, no points 'near' a local minimum are lower than (a, b).

- We say f has a **local maximum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \geq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local maximum value of f.'

  That is, b is the maximum $f(x)$ value over an *open interval* containing a.

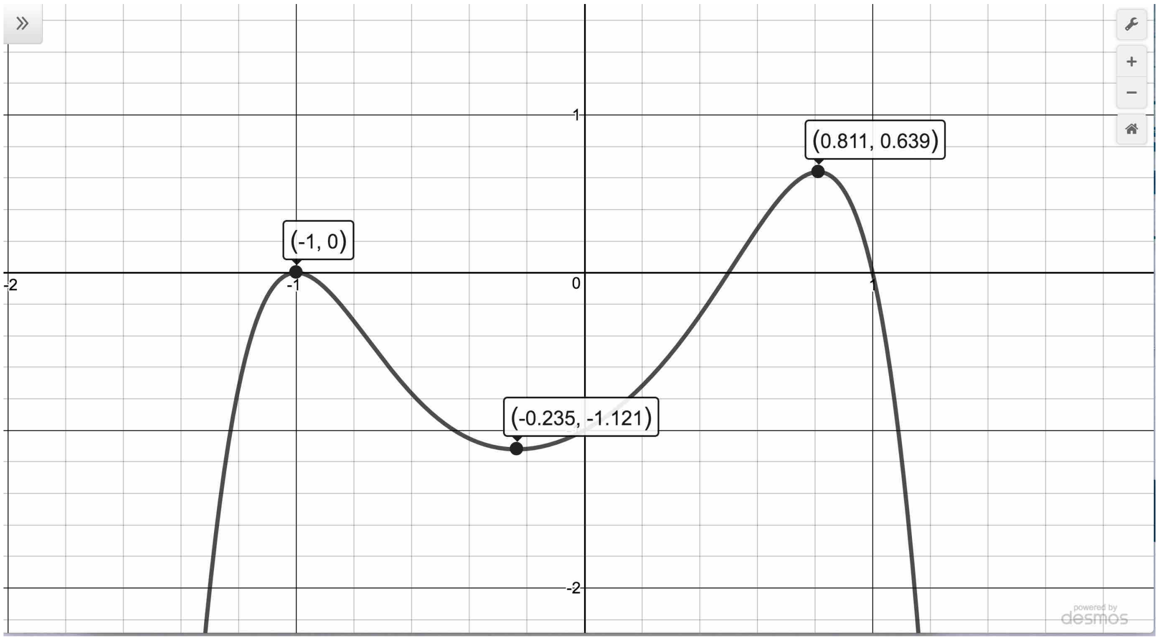
  Graphically, no points 'near' a local maximum are higher than (a, b).

Taken together, the local maximums and local minimums of a function, if they exist, are called the **local extrema** of the function.

Once again, the terminology used in Definition 6.7 blurs the line between the function f and its outputs, $f(x)$. Also, some textbooks use the terms 'relative' minimum and 'relative' maximum instead of the adjective 'local.' Lastly, note the definition of local extrema requires an *open* interval exist in the domain containing a in order for $(a, f(a))$ to be a candidate for a local maximum or local minimum. We'll have more to say about this in later chapters. If our open interval happens to be $(-\infty, \infty)$, then our local extrema are the extrema of f - we'll see an example of this momentarily.

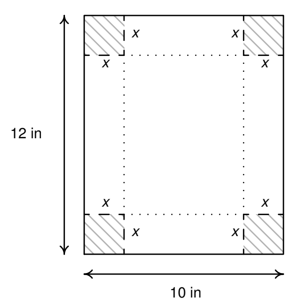
Below we use a graphing utility to graph $y = p(x) = (2x - 1)(x + 1)(1 - x^4)$. We first consider the point $(-1, 0)$. Even though there are points on the graph of $y = p(x)$ that are higher than $(-1, 0)$, locally, $(-1, 0)$ is the top of a hill. To satisfy Definition 6.7, we need to provide an open interval on which $p(-1) = 0$ is the largest, or maximum function value. Note the definition requires us to provide *just one* open interval. One that works is the interval $(-1.5, -0.5)$. We could use any smaller interval or go as large as $\left(-\infty, \frac{1}{2}\right)$ (can you see why?) Next we encounter a 'low' point at approximately $(-0.2353, -1.1211)$. More specifically, for all x in the interval, say, $(-0.5, 0)$, $p(x) \geq 1.1211$, Hence, we have a local minimum at $(-0.2353, -1.1211)$. Lastly, at $(0.811, 0.639)$, we are back to a high point. In fact, 0.639 isn't just a local maximum value, based on the graph, it is *the* maximum of p. Here, we may choose the open interval $(-\infty, \infty)$ as the open interval required by Definition 6.7, since for all x, $p(x) \leq 0.639$. It is important to note that there is no minimum value of p despite there being a local minimum value.[21]

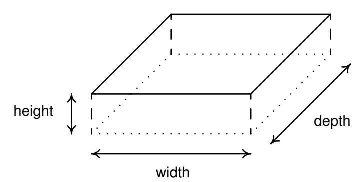
[21] Some books use the adjectives 'global' or 'absolute' when describing the extreme values of a function to distinguish them from their local counterparts.



We close this section with a classic application of a third degree polynomial function.

Example 6.1.4. A box with no top is to be fashioned from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let x denote the length of the side of the square which is removed from each corner.





1. Find an expression for $V(x)$, the volume of the box produced by removing squares of edge length x. Include an appropriate domain.

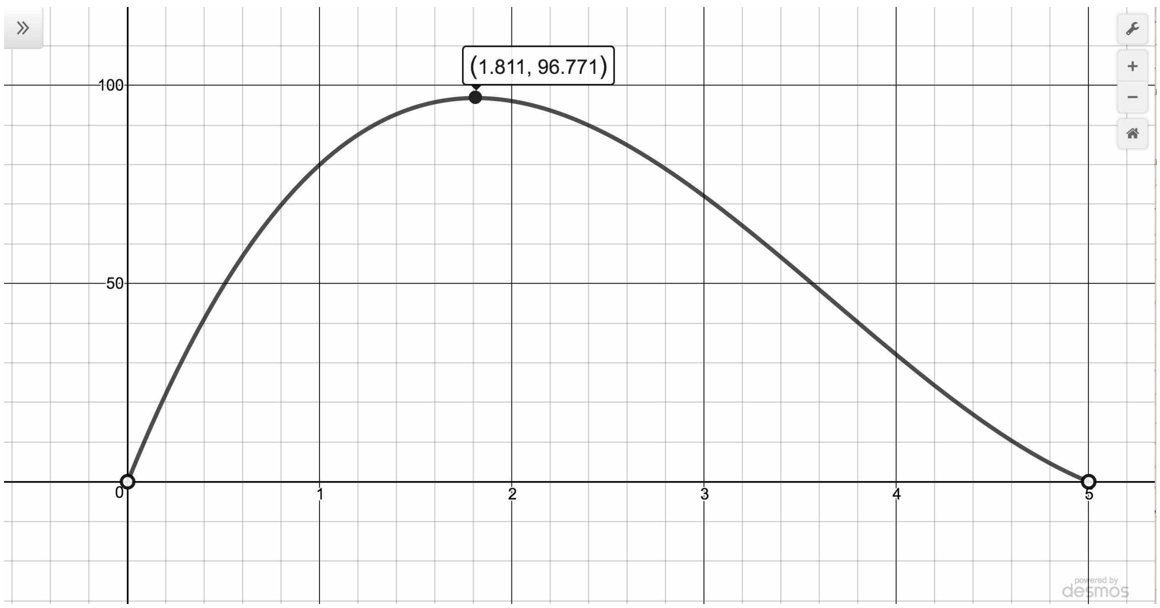
2. Use a graphing utility to help you determine the value of x which produces the box with the largest volume. What is the largest volume? Round your answers to two decimal places.

Solution.

1. From Geometry, we know that Volume = width $\times$ height $\times$ depth. The key is to find each of these quantities in terms of x. From the figure, we see that the height of the box is x itself. The cardboard

piece is initially 10 inches wide. Removing squares with a side length of x inches from each corner leaves $10 - 2x$ inches for the width.[22] As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12 - 2x$ inches remaining. Hence, we get $V(x) = x(10 - 2x)(12 - 2x)$. To find a suitable applied domain, we note that to make a box at all we need $x > 0$. Also the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10 - 2x > 0$ or $x < 5$. Hence, our applied domain is $0 < x < 5$.

2. Using a graphing utility, we find a local maximum at approximately $(1.811, 96.771)$. Because the domain of V is restricted to the interval $(0, 5)$, the maximum of V is here as well.



This means the maximum volume attainable is approximately 96.77 cubic inches when we remove squares approximately 1.81 inches per side. $\square$

Notice that there is a very slight, but important, difference between the function $V(x) = x(10-2x)(12-2x)$, $0 < x < 5$ from Example 6.1.4 and the function $p(x) = x(10 - 2x)(12 - 2x)$: their domains. The domain of V is restricted to the interval $(0, 5)$ while the domain of p is $(-\infty, \infty)$. Indeed, the function V has a maximum of (approximately) 96.771 at (approximately) $x = 1.811$ whereas for the function p, 96.771 is a local maximum value only. We leave it to the reader to verify that V has neither a minimum nor a local minimum.

[22]There's no harm in taking an extra step here and making sure this makes sense. If we chopped out a 1 inch square from each side, then the width would be 8 inches, so chopping out x inches would leave $10 - 2x$ inches.

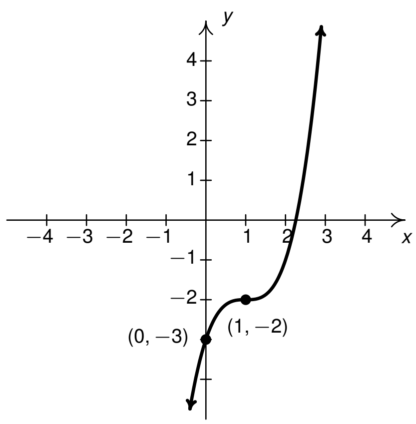
6.1.3 Exercises

In Exercises 1 - 6, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 6.1. Track at least three points of your choice through the transformations. State the domain and range of g.

1. $f(x) = x^3$, $F(x) = (x+2)^3 + 1$
2. $f(x) = x^4$, $F(x) = (x+2)^4 + 1$
3. $f(x) = x^4$, $F(x) = 2 - 3(x-1)^4$
4. $f(x) = x^5$, $F(x) = -x^5 - 3$
5. $f(x) = x^5$, $F(x) = (x+1)^5 + 10$
6. $f(x) = x^6$, $F(x) = 8 - x^6$

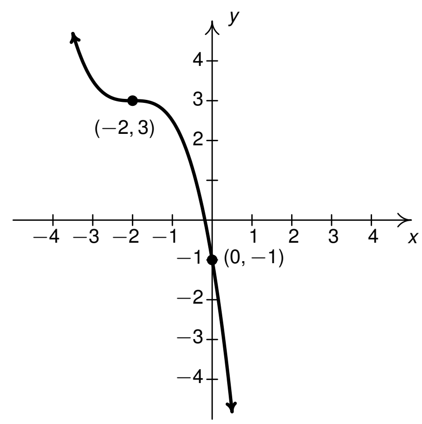
In Exercises 7 - 8, find a formula for each function below in the form $F(x) = a(x-h)^3 + k$.

7.



$y = F(x)$

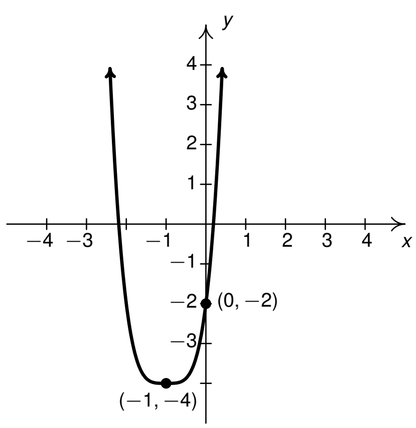
8.



$y = F(x)$

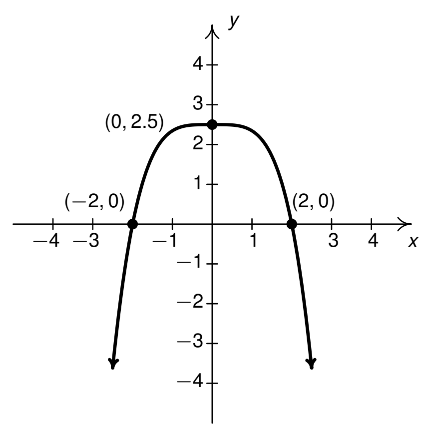
In Exercises 9 - 10, find a formula for each function below in the form $F(x) = a(x-h)^4 + k$.

9.



$y = F(x)$

10.



$y = F(x)$

In Exercises 11 - 20, find the degree, the leading term, the leading coefficient, the constant term and the end behavior of the given polynomial function.

11. $f(x) = 4 - x - 3x^2$
12. $g(x) = 3x^5 - 2x^2 + x + 1$
13. $q(r) = 1 - 16r^4$
14. $Z(b) = 42b - b^3$
15. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
16. $s(t) = -4.9t^2 + v_0 t + s_0$
17. $P(x) = (x-1)(x-2)(x-3)(x-4)$
18. $p(t) = -t^2(3-5t)(t^2+t+4)$
19. $f(x) = -2x^3(x+1)(x+2)^2$
20. $G(t) = 4(t-2)^2\left(t + \frac{1}{2}\right)$

In Exercises 21 - 30, find the real zeros of the given polynomial and their corresponding multiplicities. Use this information along with end behavior to provide a rough sketch of the graph of the polynomial function. Compare your answer with the result from a graphing utility.

21. $a(x) = x(x+2)^2$
22. $g(t) = t(t+2)^3$
23. $f(z) = -2(z-2)^2(z+1)$
24. $g(x) = (2x+1)^2(x-3)$
25. $F(t) = t^3(t+2)^2$
26. $P(z) = (z-1)(z-2)(z-3)(z-4)$
27. $Q(x) = (x+5)^2(x-3)^4$
28. $h(t) = t^2(t-2)^2(t+2)^2$
29. $H(z) = (3-z)(z^2+1)$
30. $Z(x) = x(42 - x^2)$

In Exercises 31 - 45, determine analytically if the following functions are even, odd or neither. Confirm your answer using a graphing utility.

31. $f(x) = 7x$
32. $g(t) = 7t + 2$
33. $p(z) = 7$
34. $F(s) = 3s^2 - 4$
35. $h(t) = 4 - t^2$
36. $g(x) = x^2 - x - 6$
37. $f(x) = 2x^3 - x$
38. $p(z) = -z^5 + 2z^3 - z$
39. $G(t) = t^6 - t^4 + t^2 + 9$
40. $G(s) = s(s^2 - 1)$
41. $f(x) = (x^2+1)(x-1)$
42. $H(t) = (t^2-1)(t^4+t^2+3)$
43. $g(t) = t(t-2)(t+2)$
44. $P(z) = (2z^5 - 3z)(5z^3 + z)$
45. $f(x) = 0$

46. Suppose $p(x)$ is a polynomial function written in the form of Definition 6.4.

    (a) If the nonzero terms of $p(x)$ consist of even powers of x (or a constant), explain why p is even.
    (b) If the nonzero terms of $p(x)$ consist of odd powers of x, explain why p is odd.
    (c) If $p(x)$ the nonzero terms of $p(x)$ contain at least one odd power of x and one even power of x (or a constant term), then p is neither even nor odd.

47. Use the results of Exercise 46 to determine whether the following functions are even, odd, or neither.

    (a) $p(x) = 3x^4 + x^2 - 1$ (b) $F(s) = s^3 - 14s$ (c) $f(t) = 2t^5 - t^2 + 1$ (d) $g(x) = x^3(x^2 + 1)$

48. Show $f(x) = |x|$ is an even function.

49. Rework Example 6.1.4 assuming the box is to be made from an 8.5 inch by 11 inch sheet of paper. Using scissors and tape, construct the box. Are you surprised?[23]

50. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[24] What trends do you observe? How do your answers manifest themselves graphically?

$f(x)$	$[-0.1, 0]$	$[0, 0.1]$	$[0.9, 1]$	$[1, 1.1]$	$[1.9, 2]$	$[2, 2.1]$
1						
x						
x^2						
x^3						
x^4						
x^5						

51. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[25] What trends do you observe? How do your answers manifest themselves graphically?

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
1				
x				
x^2				
x^3				
x^4				
x^5				

In Exercises 52 - 54, suppose the revenue R, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $R(x) = -5x^3 + 35x^2 + 155x$ for $0 \leq x \leq 10.07$.

52. Use a graphing utility to graph $y = R(x)$ and determine the number of TVs which should be sold to maximize revenue. What is the maximum revenue?

53. Assume the cost, in *thousands* of dollars, to produce x *hundred* LCD TVs is given by the function $C(x) = 200x + 25$ for $x \geq 0$. Find and simplify an expression for the profit function $P(x)$.

    (Remember: Profit = Revenue - Cost.)

[23] Consider decorating the box and presenting it to your instructor. If done well enough, maybe your instructor will issue you some bonus points. Or maybe not.

[24] See Definition **??** in Section **??** for a review of this concept, as needed.

[25] See Definition **??** in Section **??** for a review of this concept, as needed.

54. Use a graphing utility to graph $y = P(x)$ and determine the number of TVs which should be sold to maximize profit. What is the maximum profit?

55. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy (from Example 3.2.3) revised their cost function and now use $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$. As before, $C(x)$ is the cost to make x PortaBoy Game Systems. Market research indicates that the demand function $p(x) = -1.5x + 250$ remains unchanged. Use a graphing utility to find the production level x that maximizes the *profit* made by producing and selling x PortaBoy game systems.

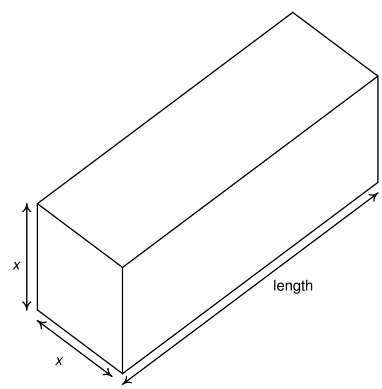
56. According to US Postal regulations, a rectangular shipping box must satisfy the following inequality: "Length + Girth $\leq$ 130 inches" for Parcel Post and "Length + Girth $\leq$ 108 inches" for other services.

    Let's assume we have a closed rectangular box with a square face of side length x as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square, $4x$.

    (a) Assuming that we'll be mailing a box via Parcel Post where Length + Girth = 130 inches, express the length of the box in terms of x and then express the volume V of the box in terms of x.

    (b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

    (c) Repeat parts 56a and 56b if the box is shipped using "other services".



57. This exercise revisits the data set from Exercise 48b in Section 5.4. In that exercise, you were given a chart of the number of hours of daylight they get on the 21$^{\text{st}}$ of each month in Fairbanks, Alaska based on the 2009 sunrise and sunset data found on the U.S. Naval Observatory website. Here $x = 1$ represents January 21, 2009, $x = 2$ represents February 21, 2009, and so on.

Month Number	1	2	3	4	5	6	7	8	9	10	11	12
Hours of Daylight	5.8	9.3	12.4	15.9	19.4	21.8	19.4	15.6	12.4	9.1	5.6	3.3

- Find cubic (third degree) and quartic (fourth degree) polynomials which model this data and comment on the goodness of fit for each. What can we say about using either model to make predictions about the year 2020? (Hint: Think about the end behavior of polynomials.)
- Use the models to see how many hours of daylight they got on your birthday and then check the website to see how accurate the models are.
- Sasquatch are largely nocturnal, so what days of the year according to your models allow for at least 14 hours of darkness for field research on the elusive creatures?

58. An electric circuit is built with a variable resistor installed. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given in the table below. [26]

Resistance: ($k\Omega$)	1.012	2.199	3.275	4.676	6.805	9.975
Power: (mW)	1.063	1.496	1.610	1.613	1.505	1.314

(a) Make a scatter diagram of the data using the Resistance as the independent variable and Power as the dependent variable.

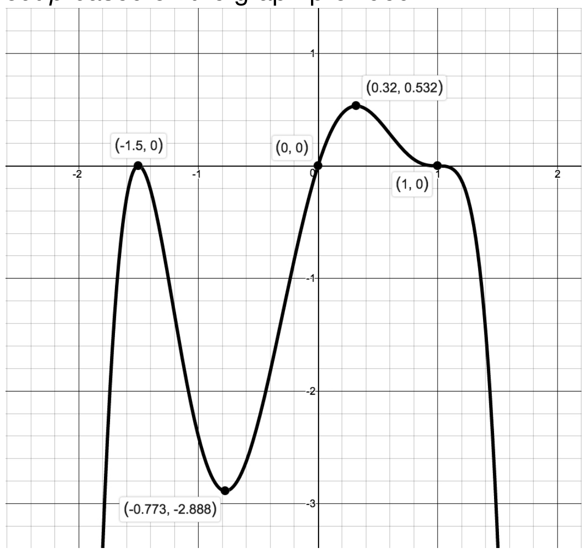
(b) Use your calculator to find quadratic (2nd degree), cubic (3rd degree) and quartic (4th degree) regression models for the data and judge the reasonableness of each.

(c) For each of the models found above, find the predicted maximum power that can be delivered to the load. What is the corresponding resistance value?

(d) Discuss with your classmates the limitations of these models - in particular, discuss the end behavior of each.

[26]The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

59. Below is a graph of a polynomial function $y = p(x)$ as generated by a graphing utility. Answer the following questions about p based on the graph provided.



(a) Describe the end behavior of $y = p(x)$.

(b) List the real zeros of p along with their respective multiplicities.

(c) List the local minimums and local maximums of the graph of $y = p(x)$.

(d) What can be said about the degree of and leading coefficient $p(x)$?

(e) It turns out that $p(x)$ is a seventh degree polynomial.[27] How can this be?

60. (This Exercise is a follow up to Example 6.1.3.) Use a graphing utility to compare and contrast the graphs of $f(x) = (2x - 1)(x + 1)^2(1 - x)(x^2 + 1)$ and $g(x) = (2x - 1)(x + 1)^2(1 - x)$.

61. Use the graph of $y = p(x) = (2x - 1)(x + 1)(1 - x^4)$ on page 299 to estimate the largest open interval containing $x = -0.235$ which satisfies the the criteria for 'local minimum' in Definition 6.7.

62. In light of Definition 6.7, explain why *every* point on the graph of a constant function is both a local maximum and a local minimum.

63. This exercise involves the greatest integer function, $f(x) = \lfloor x \rfloor$, introduced in Example 3.2.2. Explain why the points (k, k) for integers k are local maximums but not local minimums.

64. Use Theorems 6.3 and 6.4 prove Theorem 6.5.

[27] to be exact, $p(x) = -0.1\,(x + 1.5)^2\,(3x)\,(x - 1)^3\,(x + 5)$.

65. Here are a few other questions for you to discuss with your classmates.

    (a) How many and how few local extrema could a polynomial of degree n have?
    (b) Could a polynomial have two local maxima but no local minima?
    (c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?
    (d) Can a polynomial have local extrema without having any real zeros?
    (e) Why must every polynomial of odd degree have at least one real zero?
    (f) Can a polynomial have two distinct real zeros and no local extrema?
    (g) Can an x-intercept yield a local extrema? Can it yield an absolute extrema?
    (h) If the y-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

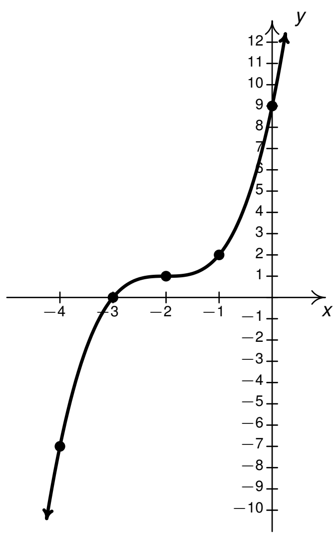
66. (This is a follow-up to Exercises 60 in Section 3.2 and 61 in Section 5.4.) The Lagrange Interpolate function L for four points: (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) where x_0, x_1, x_2, and x_3 are four distinct real numbers is given by the formula:

$$\begin{aligned} L(x) &= y_0\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &\quad + y_2\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \end{aligned}$$

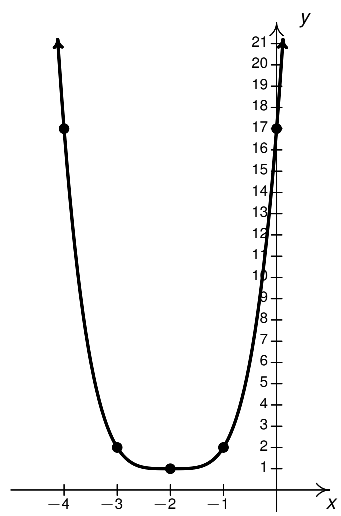
    (a) Choose four points with different x-values and construct the Lagrange Interpolate for those points. Verify each of the points lies on the polynomial.
    (b) Verify that, in general, $L(x_0) = y_0$, $L(x_1) = y_1$, $L(x_2) = y_2$, and $L(x_3) = y_3$.
    (c) Find $L(x)$ for the points $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 4)$. What happens?
    (d) Find $L(x)$ for the points $(-1, 0)$, $(0, 1)$, $(1, 2)$ and $(2, 3)$. What happens?
    (e) Generalize the formula for $L(x)$ to five points. What's the pattern?

6.1.4 Answers

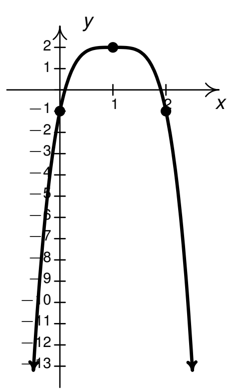
1. $F(x) = (x+2)^3 + 1$
   domain: $(-\infty, \infty)$
   range: $(-\infty, \infty)$



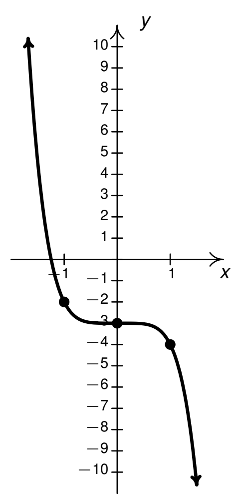
2. $F(x) = (x+2)^4 + 1$
   domain: $(-\infty, \infty)$
   range: $[1, \infty)$



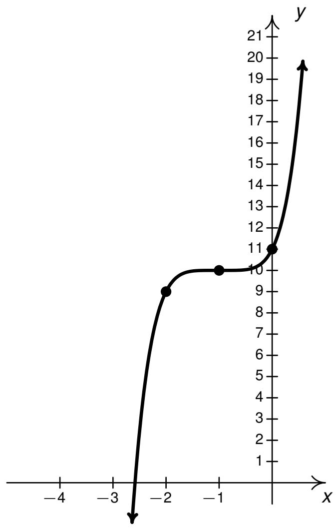
3. $F(x) = 2 - 3(x-1)^4$
   domain: $(-\infty, \infty)$
   range: $(-\infty, 2]$



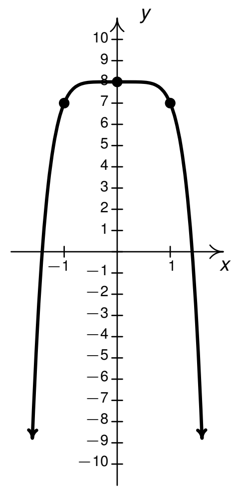
4. $F(x) = -x^5 - 3$
   domain: $(-\infty, \infty)$
   range: $(-\infty, \infty)$



5. $F(x) = (x+1)^5 + 10$
domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$



6. $F(x) = 8 - x^6$
domain: $(-\infty, \infty)$
range: $(-\infty, 8]$



7. $F(x) = (x-1)^3 - 2$

8. $F(x) = -\frac{1}{2}(x+2)^3 + 3$

9. $F(x) = 2(x+1)^4 - 4$

10. $F(x) = -0.15625x^4 + 2.5$

11. $f(x) = 4 - x - 3x^2$
Degree 2
Leading term $-3x^2$
Leading coefficient -3
Constant term 4
As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to -\infty$

12. $g(x) = 3x^5 - 2x^2 + x + 1$
Degree 5
Leading term $3x^5$
Leading coefficient 3
Constant term 1
As $x \to -\infty$, $g(x) \to -\infty$
As $x \to \infty$, $g(x) \to \infty$

13. $q(r) = 1 - 16r^4$
Degree 4
Leading term $-16r^4$
Leading coefficient -16
Constant term 1
As $r \to -\infty$, $q(r) \to -\infty$
As $r \to \infty$, $q(r) \to -\infty$

14. $Z(b) = 42b - b^3$
Degree 3
Leading term $-b^3$
Leading coefficient -1
Constant term 0
As $b \to -\infty$, $Z(b) \to \infty$
As $b \to \infty$, $Z(b) \to -\infty$

15. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
Degree 17
Leading term $\sqrt{3}x^{17}$
Leading coefficient $\sqrt{3}$
Constant term $\frac{1}{3}$
As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to \infty$

16. $s(t) = -4.9t^2 + v_0 t + s_0$
Degree 2
Leading term $-4.9t^2$
Leading coefficient -4.9
Constant term s_0
As $t \to -\infty$, $s(t) \to -\infty$
As $t \to \infty$, $s(t) \to -\infty$

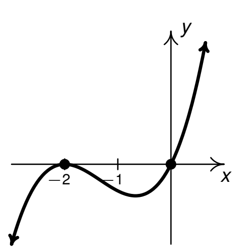
17. $P(x) = (x-1)(x-2)(x-3)(x-4)$
Degree 4
Leading term x^4
Leading coefficient 1
Constant term 24
As $x \to -\infty$, $P(x) \to \infty$
As $x \to \infty$, $P(x) \to \infty$

18. $p(t) = -t^2(3-5t)(t^2+t+4)$
Degree 5
Leading term $5t^5$
Leading coefficient 5
Constant term 0
As $t \to -\infty$, $p(t) \to -\infty$
As $t \to \infty$, $p(t) \to \infty$

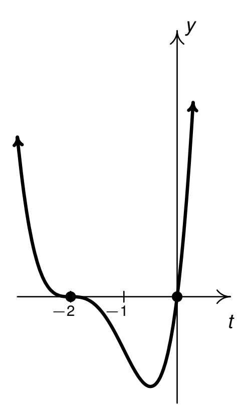
19. $f(x) = -2x^3(x+1)(x+2)^2$
Degree 6
Leading term $-2x^6$
Leading coefficient -2
Constant term 0
As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to -\infty$

20. $G(t) = 4(t-2)^2\left(t+\frac{1}{2}\right)$
Degree 3
Leading term $4t^3$
Leading coefficient 4
Constant term 8
As $t \to -\infty$, $G(t) \to -\infty$
As $t \to \infty$, $G(t) \to \infty$

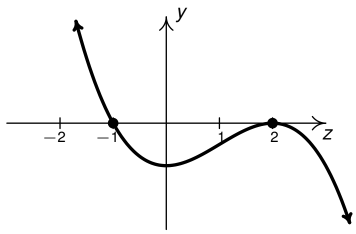
21. $a(x) = x(x+2)^2$
$x = 0$ multiplicity 1
$x = -2$ multiplicity 2



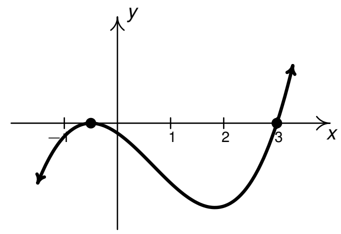
22. $g(t) = t(t+2)^3$
$t = 0$ multiplicity 1
$t = -2$ multiplicity 3



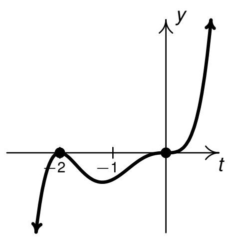
23. $f(z) = -2(z-2)^2(z+1)$
$z = 2$ multiplicity 2
$z = -1$ multiplicity 1



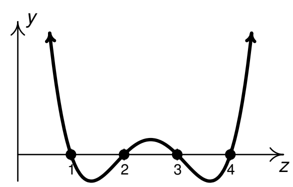
24. $g(x) = (2x+1)^2(x-3)$
$x = -\frac{1}{2}$ multiplicity 2
$x = 3$ multiplicity 1



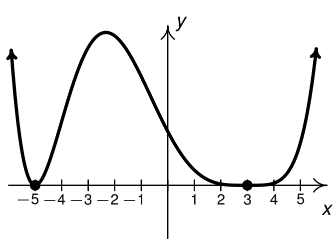
25. $F(t) = t^3(t+2)^2$
$t = 0$ multiplicity 3
$t = -2$ multiplicity 2



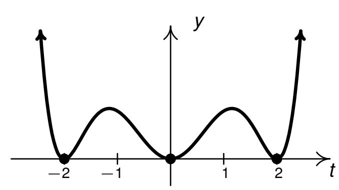
26. $P(z) = (z-1)(z-2)(z-3)(z-4)$
$z = 1$ multiplicity 1
$z = 2$ multiplicity 1
$z = 3$ multiplicity 1
$z = 4$ multiplicity 1



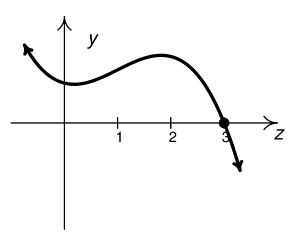
27. $Q(x) = (x+5)^2(x-3)^4$
$x = -5$ multiplicity 2
$x = 3$ multiplicity 4



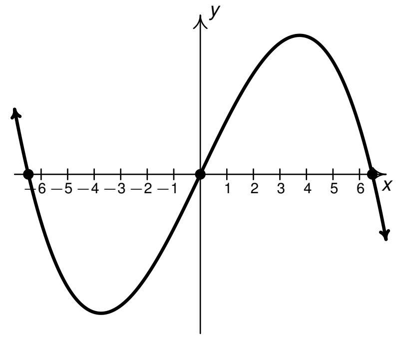
28. $f(t) = t^2(t-2)^2(t+2)^2$
$t = -2$ multiplicity 2
$t = 0$ multiplicity 2
$t = 2$ multiplicity 2



29. $H(z) = (3-z)\left(z^2+1\right)$
$z = 3$ multiplicity 1



30. $Z(x) = x(42 - x^2)$
    $x = -\sqrt{42}$ multiplicity 1
    $x = 0$ multiplicity 1
    $x = \sqrt{42}$ multiplicity 1



31. odd
32. neither
33. even
34. even
35. even
36. neither
37. odd
38. odd
39. even
40. odd
41. neither
42. even
43. odd
44. even
45. even **and** odd

47. (a) even (b) odd (c) neither (d) odd[28]

48. For $f(x) = |x|$, $f(-x) = |-x| = |(-1)x| = |-1||x| = (1)|x| = |x|$. Hence, $f(-x) = f(x)$.

49. $V(x) = x(8.5 - 2x)(11 - 2x) = 4x^3 - 39x^2 + 93.5x$, $0 < x < 4.25$. Volume is maximized when $x \approx 1.58$, so we get the dimensions of the box with maximum volume are: height ≈ 1.58 inches, width ≈ 5.34 inches, and depth ≈ 7.84 inches. The maximum volume is ≈ 66.15 cubic inches.

50. Each of these average rates of change indicate slope of the curve over the given interval. Smaller slopes correspond to 'flatter' curves and higher slopes correspond to 'steeper' curves.

$f(x)$	$[-0.1, 0]$	$[0, 0.1]$	$[0.9, 1]$	$[1, 1.1]$	$[1.9, 2]$	$[2, 2.1]$
1	0	0	0	0	0	0
x	1	1	1	1	1	1
x^2	-0.1	0.1	1.9	2.1	3.9	4.1
x^3	0.01	0.01	2.71	3.31	11.41	12.61
x^4	-0.001	0.001	3.439	4.641	29.679	34.481
x^5	0.0001	0.0001	4.0951	6.1051	72.3901	88.4101

51. As we sample points closer to $x = 1$, the slope of the curve approaches the exponent on x.

[28] You need to first multiply out the expression for $g(x)$ so it is in the form prescribed by Definition 6.4.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
1	0	0	0	0
x	1	1	1	1
x^2	2	2	2	2
x^3	3.01	3.0001	≈ 3	≈ 3
x^4	4.04	4.0004	≈ 4	≈ 4
x^5	5.1001	≈ 5.001	≈ 5	≈ 5

52. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 6.305$ and $y \approx 1115.417$. Since x represents the number of TVs sold in hundreds, $x = 6.305$ corresponds to 630.5 TVs. Since we can't sell half of a TV, we compare $R(6.30) \approx 1115.415$ and $R(6.31) \approx 1115.416$, so selling 631 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is \$1,115,416.

53. $P(x) = R(x) - C(x) = -5x^3 + 35x^2 - 45x - 25$, $0 \leq x \leq 10.07$.

54. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 3.897$ and $y \approx 35.255$. Since x represents the number of TVs sold in hundreds, $x = 3.897$ corresponds to 389.7 TVs. Since we can't sell 0.7 of a TV, we compare $P(3.89) \approx 35.254$ and $P(3.90) \approx 35.255$, so selling 390 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is \$35,255.

55. Making and selling 71 PortaBoys yields a maximized profit of \$5910.67.

56. (a) To maximize the volume, we assume we start with the maximum Length + Girth of 130, so the length is $130 - 4x$. The volume of a rectangular box is 'length $\times$ width $\times$ height' so we get $V(x) = x^2(130 - 4x) = -4x^3 + 130x^2$.

    (b) Using a graphing utility, we get a (local) maximum of $y = V(x)$ at $(21.67, 20342.59)$. Hence, the maximum volume is 20342.59in.^{3} using a box with dimensions 21.67in. $\times$ 21.67in. $\times$ 43.32in..

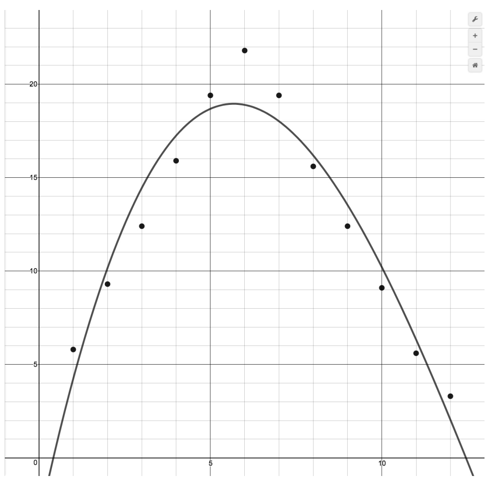
    (c) If we start with Length + Girth = 108 then the length is $108 - 4x$ so $V(x) = -4x^3 + 108x^2$. Graphing $y = V(x)$ shows a (local) maximum at $(18.00, 11664.00)$ so the dimensions of the box with maximum volume are 18.00in.$\times$18.00in.$\times$36in. for a volume of 11664.00in.^{3}. (Calculus will confirm that the measurements which maximize the volume are <u>exactly</u> 18in. by 18in. by 36in., however, as I'm sure you are aware by now, we treat all numerical results as approximations and list them as such.)

57. • The cubic regression model is $p_3(x) = 0.0226x^3 - 0.9508x^2 + 8.615x - 3.446$. It has $R^2 = 0.9377$ which isn't bad. The graph of $y = p_3(x)$ along with the data is shown below on the left. Note p_3 hits the x-axis at about $x = 12.45$ making this a bad model for future predictions.

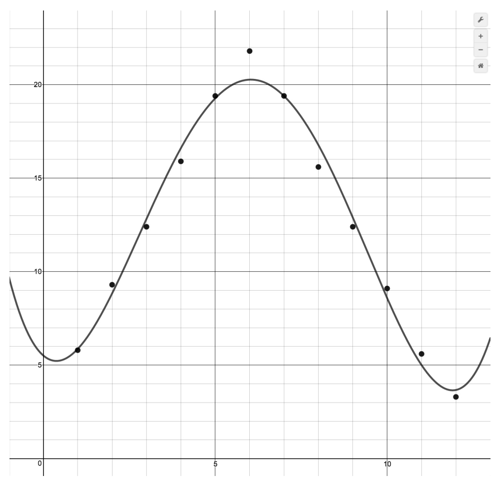
    • To use the model to approximate the number of hours of sunlight on your birthday, you'll have to figure out what decimal value of x is close enough to your birthday and then plug it into the model. Jeff's birthday is July 31 which is 10 days after July 21 ($x = 7$). Assuming 30 days in a month, I think $x = 7.33$ should work for my birthday and $p_3(7.33) \approx 17.5$. The website says there will be about 18.25 hours of daylight that day.

- To have 14 hours of darkness we need 10 hours of daylight. We see that $p_3(1.96) \approx 10$ and $p_3(10.05) \approx 10$ so it seems reasonable to say that we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to February 21, 2009 ($x = 2$) and then again from October 21,2009 ($x = 10$) to December 21, 2009 ($x = 12$).

- The quartic regression model is $p_4(x) = 0.0144x^4 - 0.3507x^3 + 2.259x^2 - 1.571x + 5.513$. It has $R^2 = 0.9859$ which is good. The graph of $y = p_4(x)$ along with data is shown below on the right. Note $p_4(15)$ is above 24 making this a bad model as well for future predictions.
- Here, $p_4(7.33) \approx 18.71$ so this model more accurately predicts the number of hours of daylight on Jeff's birthday.
- This model says we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to about March 1, 2009 ($x = 2.30$) and then again from October 10, 2009 ($x = 9.667$) to December 21, 2009 ($x = 12$).



$y = p_3(x)$

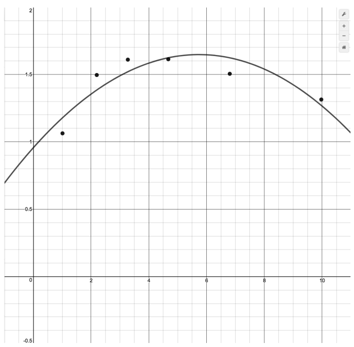


$y = p_4(x)$

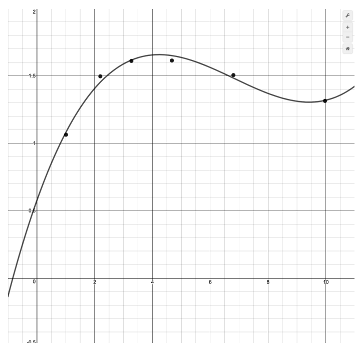
58. (a) The scatter plot is shown below with each of the three regression models.

(b) The quadratic model is $P_2(x) = -0.021x^2 + 0.241x + 0.956$, $R^2 = 0.7771$.
The cubic model is $P_3(x) = 0.005x^3 - 0.103x^2 + 0.602x + 0.573$, $R^2 = 0.9815$.
The quartic model is $P_4(x) = -0.000969x^4 + 0.0253x^3 - 0.240x^2 + 0.944x + 0.330$, $R^2 = 0.9993$.

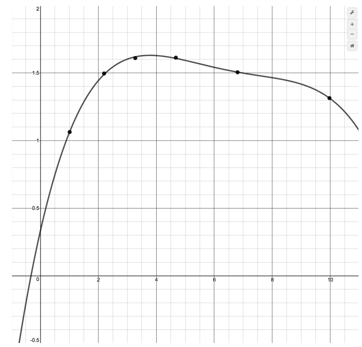
(c) The models give maximums: $P_2(5.737) \approx 1.648$, $P_3(4.232) \approx 1.657$ and $P_4(3.784) \approx 1.630$.



$y = P_2(x)$



$y = P_3(x)$



$y = P_4(x)$

59. (a) as $x \to -\infty$, $p(x) \to -\infty$ and as $x \to \infty$, $p(x) \to -\infty$

(b) The zeros appear to be: $x = -1.5$, even multiplicity - probably 2 since it doesn't 'look like' the graph is very flat near $x = 2$; $x = 0$, odd multiplicity - probably 1 since the graph seems fairly linear as it passes through the origin; $x = 1$ odd multiplicity - probably 3 or higher since the graph seems fairly 'flat' near $x = 1$.

(c) local minimum: approximately $(-0.773, -2.888)$; local maximums: approximately $(-1.5, 0)$, and $(0.32, 0.532)$

(d) Based on the graph, even degree (at least 6 based on multiplicities) with a negative leading coefficient based on the end behavior.

(e) We only have a *portion* of the graph represented here.

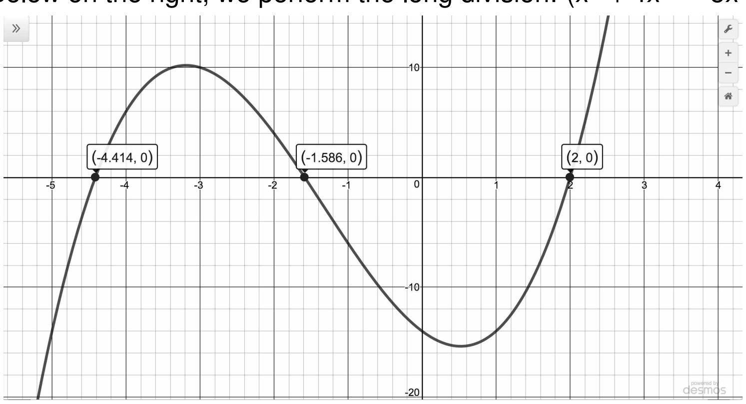
61. We are looking for the largest open interval containing $x = -0.235$ for which the graph of $y = p(x)$ is at or above $y = -1.121$. Since each of the gridlines on the x-axis correspond to 0.2 units, we approximate this interval as $(-1.25\,\text{ish}, 1.1\,\text{ish})$.

66. (c) $L(x) = x^2$ (d) $L(x) = x + 1$

6.2 The Remainder and Factor Theorems

In Section 6.1 we saw how much of the 'local' behavior of the graph of a polynomial function is determined by the zeros of the polynomial function. In that section, the polynomial functions we were given were mostly, if not completely, factored which greatly simplified the process for determining zeros. In this section, we revisit the relationship between zeros and factors with the ultimate aim of taking a polynomial function given to us in the form stated in Definition 6.4 and determining its zeros.

We start by way of example: suppose we wish to determine the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all of the factoring techniques we learned (and forgot!) in Intermediate Algebra, this equation foils[1] us at every turn. Knowing that the zeros of f correspond to x-intercepts on the graph of $y = f(x)$, we use a graphing utility to produce the graph below on the left. The graph suggests that the function has three zeros, one of which appears to be $x = 2$ and two others for whom we are provided what we assume to be decimal approximations: $x \approx -4.414$ and $x \approx -1.586$. We can verify if these are zeros easily enough. We find $f(2) = (2)^2 + 4(2)^2 - 5(2) - 14 = 0$, but $f(-4.414) \approx 0.0039$ and $f(-1.586) \approx 0.0022$, While these last two values are probably by some measures, 'close' to 0, they are not *exactly* equal to 0. The question becomes: is there a way to use the fact that $x = 2$ is a zero to obtain the other two zeros? Based on our experience, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. In other words, we should expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)\, q(x)$, where $q(x)$ is some other polynomial. How could we find such a $q(x)$, if it even exists? The answer comes from our old friend, polynomial division. (See Section 5.1.2.) Below on the right, we perform the long division: $(x^3 + 4x^2 - 5x - 14) \div (x - 2)$ and obtain $x^2 + 6x + 7$.



$$
\begin{array}{r|rrrr}
\multicolumn{2}{r}{x^2} & +\,6x & +\,7 & \\
\cline{2-5}
x-2 & x^3 & +\,4x^2 & -\,5x & -\,14 \\
 & -\left(x^3\right. & \left.-\,2x^2\right) & & \\
\cline{2-3}
 & & 6x^2 & -\,5x & \\
 & & -\left(6x^2\right. & \left.-\,12x\right) & \\
\cline{3-4}
 & & & 7x & -\,14 \\
 & & & -\,(7x & -\,14) \\
\cline{4-5}
 & & & & 0
\end{array}
$$

Said differently, $f(x) = x^3 + 4x^2 - 5x - 14 = (x - 2)\left(x^2 + 6x + 7\right)$. Using this form of $f(x)$, we find the zeros by solving $(x - 2)\left(x^2 + 6x + 7\right) = 0$. Setting each factor equal to 0, we get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2 + 6x + 7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. Sure enough, $-3 - \sqrt{2} \approx -4.414$ and $-3 + \sqrt{2} \approx -1.586$. We leave it to the reader to show $f(-3 - \sqrt{2}) = 0$ and $f(-3 + \sqrt{2}) = 0$. (See Exercise 36.)

The point of this section is to generalize the technique applied here. First up is a friendly reminder of what we can expect when we divide polynomials.

[1] pun intended

Theorem 6.6. Polynomial Division:
Suppose $d(x)$ and $p(x)$ are nonzero polynomial functions where the degree of p is greater than or equal to the degree of d. There exist two unique polynomial functions, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

As you may recall, all of the polynomials in Theorem 6.6 have special names. The polynomial p is called the **dividend**; d is the **divisor**; q is the **quotient**; r is the **remainder**. If $r(x) = 0$ then d is called a **factor** of p. The word 'unique' here is critical in that it guarantees there is only *one* quotient and remainder for each division problem.[2] The proof of Theorem 6.6 is usually relegated to a course in Abstract Algebra, but we can still use the result to establish two important facts which are the basis of the rest of the chapter.

Theorem 6.7. The Remainder Theorem: Suppose p is a polynomial function of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$. Said differently, there is a polynomial function $q(x)$ such that:

$$p(x) = (x - c)q(x) + p(c)$$

The proof of Theorem 6.7 is a direct consequence of Theorem 6.6. Since $x - c$ has degree 1, when a polynomial function is divided by $x - c$, the remainder is either 0 or degree 0 (i.e., a nonzero constant.) In either case, $p(x) = (x - c)\,q(x) + r$, where r, the remainder, is a real number, possibly 0. It follows that $p(c) = (c - c)\,q(c) + r = 0 \cdot q(c) + r = r$, so we get $r = p(c)$ as required. There is one last 'low hanging fruit'[3] to collect which we present below.

Theorem 6.8. The Factor Theorem:
Suppose p is a nonzero polynomial function. The real number c is a zero of p if and only if $(x - c)$ is a factor of $p(x)$.

Once again, we see the phrase 'if and only if' which means there are really two things being said in The Factor Theorem: if $(x - c)$ is a factor of $p(x)$, then c is a zero of p and the *only* way c is a zero of p is if $(x - c)$ is a factor of $p(x)$. We argue the Factor Theorem as follows: if $(x - c)$ is a factor of $p(x)$, then $p(x) = (x - c)\,q(x)$ for some polynomial q. Hence, $p(c) = (c - c)\,q(c) = 0$, so c is a zero of p. Conversely, suppose c is a zero of p, so $p(c) = 0$. The Remainder Theorem tells us $p(x) = (x - c)q(x) + p(c) = (x - c)q(x) + 0 = (x - c)q(x)$. Hence, $(x - c)$ is a factor of $p(x)$.

We have enough theory to explain why the concept of multiplicity (Definition 6.6) is well-defined. If c is a zero of p, then The Factor Theorem tells us there is a polynomial function q_1 so that $p(x) = (x - c)q_1(x)$. If $q_1(c) = 0$, then we apply the Factor Theorem to q_1 and find a polynomial q_2 so that $q_1(x) = (x - c)q_2(x)$. Hence, we have

$$p(x) = (x - c)q_1(x) = (x - c)(x - c)q_2(x) = (x - c)^2 q_2(x).$$

We now 'rinse and repeat' this process. Since the degree of p is a finite number, this process has to end at some point. That is we arrive at a factorization $p(x) = (x - c)^m q(x)$ where $q(c) \neq 0$. Suppose we arrive at a different factorization of p using other methods. That is, we find $p(x) = (x - c)^k Q(x)$, where Q is a polynomial function with $Q(c) \neq 0$. Then we have $(x - c)^m q(x) = (x - c)^k Q(x)$. If $m \neq k$, then either $m < k$

[2] Hence the use of the definite article 'the' when speaking of *the* quotient and *the* remainder.
[3] Jeff hates this expression and Carl included it just to annoy him.

or $m > k$. Assuming the former, then we may divide both sides by $(x-c)^m$ to get: $q(x) = (x-c)^{k-m}Q(x)$. Since $k > m$, $k - m > 0$ and we would have $q(c) = (c-c)^{k-m}Q(c) = 0$, a contradiction since we are assuming $q(c) \neq 0$. The assumption that $m > k$ likewise ends in a contradiction. Therefore, we have $m = k$, so $p(x) = (x-c)^m q(x) = (x-c)^m Q(x)$. By the uniqueness guaranteed in Theorem 6.6, we must have that $q(x) = Q(x)$. Hence, we have shown the number m, as well as the quotient polynomial $q(x)$ are unique. The process outlined above, in which we coax out factors of $p(x)$ one at a time until we have all of them serves as a template for our work to come.

Of the things The Factor Theorem tells us, the most pragmatic is that we had better find a more efficient way to divide polynomial functions by quantities of the form $x - c$. Fortunately, people like Ruffini and Horner have already blazed this trail. Let's take a closer look at the long division we performed at the beginning of the section and try to streamline it. First off, let's change all of the subtractions into additions by distributing through the -1s.

$$
\begin{array}{r|rrrr}
 & x^2 & +\,6x & +\,7 & \\
\cline{2-5}
x-2 & x^3 & +\,4x^2 & -\,5x & -14 \\
 & -x^3 & +\,2x^2 & & \\
\cline{2-3}
 & & 6x^2 & -\,5x & \\
 & & -6x^2 & +\,12x & \\
\cline{3-4}
 & & & 7x & -14 \\
 & & & -7x & +14 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and -14) aren't really necessary to recopy, so we omit them, too.

$$
\begin{array}{r|rrrr}
 & x^2 & +\,6x & +\,7 & \\
\cline{2-5}
x-2 & x^3 & +4x^2 & -\,5x & -14 \\
 & & 2x^2 & & \\
\cline{2-3}
 & & 6x^2 & & \\
 & & & 12x & \\
\cline{3-4}
 & & & 7x & \\
 & & & & 14 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

Let's move terms up a bit and copy the x^3 into the last row.

$$\begin{array}{r|rrrr} \multicolumn{1}{r}{} & & x^2 & +\ 6x & +\ 7 \\ \cline{2-5} x-2 & x^3 & +4x^2 & -\ 5x & -14 \\ & & 2x^2 & 12x & 14 \\ \cline{2-5} & x^3 & 6x^2 & 7x & 0 \end{array}$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$\begin{array}{r|rrrr} -2 & x^3 & +4x^2 & -\ 5x & -14 \\ & & 2x^2 & 12x & 14 \\ \cline{2-5} & x^3 & 6x^2 & 7x & 0 \end{array}$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \cline{2-5} & 1 & 6 & 7 & 0 \end{array}$$

We have constructed a **synthetic division tableau** for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then 'bring down' the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \cline{2-5} & & & & \end{array} \qquad\qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \cline{2-5} & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was 'brought down' to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \cline{2-5} & 1 & & & \end{array} \qquad\qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \cline{2-5} & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x - c$. It is important to note that it works *only* for these kinds of divisors.[4] Also take note that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division. While the authors have done their best to indicate where the algorithm comes from, there is no substitute for working through it yourself.

Example 6.2.1. Use synthetic division to perform the following polynomial divisions. Identify the quotient and remainder. Write the dividend, quotient and remainder in the form given in Theorem 6.6.

1. $\left(5x^3 - 2x^2 + 1\right) \div (x - 3)$
2. $\left(t^3 + 8\right) \div (t + 2)$
3. $\dfrac{4 - 8z - 12z^2}{2z - 3}$

Solution.

1. When setting up the synthetic division tableau, the coefficients of even 'missing' terms need to be accounted for, so we enter 0 for the coefficient of x in the dividend.

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \boxed{118} \end{array}$$

Since the dividend was a third degree polynomial function, the quotient is a second degree (quadratic) polynomial function with coefficients 5, 13 and 39: $q(x) = 5x^2+13x+39$. The remainder is $r(x) = 118$. According to Theorem 6.6, we have $5x^3 - 2x^2 + 1 = (x - 3)\left(5x^2 + 13x + 39\right) + 118$, which we leave to the reader to check.

[4] You'll need to use good old-fashioned polynomial long division for divisors of degree larger than 1.

2. To use synthetic division here, we rewrite $t+2$ as $t-(-2)$ and proceed as before

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

We get the quotient $q(t) = t^2 - 2t + 4$ and the remainder $r(t) = 0$. Relating the dividend, quotient and remainder gives: $t^3 + 8 = (t+2)\left(t^2 - 2t + 4\right)$, which is a specific instance of the 'sum of cubes' formula some of you may recall from Intermediate Algebra.

3. To divide $4 - 8z - 12z^2$ by $2z - 3$, two things must be done. First, we write the dividend in descending powers of z as $-12z^2 - 8z + 4$. Second, since synthetic division works only for factors of the form $z - c$, we factor $2z - 3$ as $2\left(z - \frac{3}{2}\right)$. Hence, we are dividing $-12z^2 - 8z + 4$ by two factors: 2 and $\left(z - \frac{3}{2}\right)$. Dividing first by 2, we obtain $-6z^2 - 4z + 2$. Next, we divide $-6z^2 - 4z + 2$ by $\left(z - \frac{3}{2}\right)$:

$$\begin{array}{r|rrr} \frac{3}{2} & -6 & -4 & 2 \\ & \downarrow & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & \boxed{-\frac{35}{2}} \end{array}$$

Hence, $-6z^2 - 4z + 2 = \left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2}$. However when it comes to writing the dividend, quotient and remainder in the form given in Theorem 6.6, we need to find $q(z)$ and $r(z)$ so that $-12z^2 - 8z + 4 = (2z - 3)q(z) + r(z)$. Hence, starting with $-6z^2 - 4z + 2 = \left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2}$, we multiply 2 back on both sides:

$$\begin{array}{rcl} -6z^2 - 4z + 2 & = & \left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2} \\ 2\left(-6z^2 - 4z + 2\right) & = & 2\left[\left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2}\right] \\ -12z^2 - 8z + 4 & = & 2\left(z - \frac{3}{2}\right)(-6z - 13) - 2\left(\frac{35}{2}\right) \\ -12z^2 - 8z + 4 & = & (2z - 3)(-6z - 13) - 35 \end{array}$$

At this stage, we have written $-12z^2 - 8z + 4$ in the **form** $(2z - 3)q(z) + r(z)$, so we identify the quotient as $q(z) = -6z - 13$ and the remainder is $r(z) = -35$. But how can we be sure these are the same quotient and remainder polynomial functions we would have obtained if we had taken the time to do the long division in the first place? Because of the word 'unique' in Theorem 6.6. The theorem states that there is only *one* way to decompose $-12z^2 - 8z + 4$ as $(2z - 3)q(z) + r(z)$. Since we have found such a way, we can be sure it is the only way.[5] □

The next example pulls together all of the concepts discussed in this section.

Example 6.2.2. Let $p(x) = 2x^3 - 5x + 3$.

1. Find $p(-2)$ using The Remainder Theorem. Check your answer by substitution.

[5] But it wouldn't hurt to check, just this once.

2. Verify $x = 1$ is a zero of p and use this information to all the real zeros of p.

Solution.

1. The Remainder Theorem states $p(-2)$ is the remainder when $p(x)$ is divided by $x - (-2)$. We set up our synthetic division tableau below. We are careful to record the coefficient of x^2 as 0:

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -5 & 3 \\ & \downarrow & -4 & 8 & -6 \\ \hline & 2 & -4 & 3 & \boxed{-3} \end{array}$$

According to the Remainder Theorem, $p(-2) = -3$. We can check this by direct substitution into the formula for $p(x)$: $p(-2) = 2(-2)^3 - 5(-2) + 3 = -16 + 10 + 3 = -3$.

2. We verify $x = 1$ is a zero of p by evaluating $p(1) = 2(1)^3 - 5(1) + 3 = 0$. To see if there are any more real zeros, we need to solve $p(x) = 2x^3 - 5x + 3 = 0$. From the Factor Theorem, we know since $p(1) = 0$, we can factor $p(x)$ as $(x - 1)q(x)$. To find $q(x)$, we use synthetic division:

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -5 & 3 \\ & \downarrow & 2 & 2 & -3 \\ \hline & 2 & 2 & -3 & \boxed{0} \end{array}$$

As promised, our remainder is 0, and we get $p(x) = (x - 1)\left(2x^2 + 2x - 3\right)$. Setting this form of $p(x)$ equal to 0 we get $(x - 1)\left(2x^2 + 2x - 3\right) = 0$. We recover $x = 1$ from setting $x - 1 = 0$ but we also obtain $x = \frac{-1\pm\sqrt{7}}{2}$ from $2x^2 + 2x - 3 = 0$, courtesy of the Quadratic Formula. □

Our next example demonstrates how we can extend the synthetic division tableau to accommodate zeros of multiplicity greater than 1.

Example 6.2.3. Let $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$. Show $x = \frac{1}{2}$ is a zero of multiplicity 2 and find all of the remaining real zeros of p.

Solution. While computing $p\left(\frac{1}{2}\right) = 0$ shows $x = \frac{1}{2}$ is a zero of p, to prove it has multiplicity 2, we need to factor $p(x) = \left(x - \frac{1}{2}\right)^2 q(x)$ with $q\left(\frac{1}{2}\right) \neq 0$,. We set up for synthetic division, but instead of stopping after the first division, we continue the tableau downwards and divide $\left(x - \frac{1}{2}\right)$ directly into the quotient we obtained from the first division as follows:

$$\begin{array}{r|rrrrr} \frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\ & \downarrow & 2 & -1 & -6 & 3 \\ \hline \frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\ & \downarrow & 2 & 0 & -6 & \\ \hline & 4 & 0 & -12 & \boxed{0} & \end{array}$$

We get:[6] $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2 \left(4x^2 - 12\right)$. Note if we let $q(x) = 4x^2 - 12$, then $q\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 - 12 = -11 \neq 0$ which proves $x = \frac{1}{2}$ is a zero of p of multiplicity 2. To find the remaining zeros of p, we set the quotient $4x^2 - 12 = 0$, so $x^2 = 3$ and extract square roots to get $x = \pm\sqrt{3}$. $\square$

A couple of things about the last example are worth mentioning. First, the extension of the synthetic division tableau for repeated divisions will be a common site in the sections to come. Typically, we will start with a higher order polynomial and peel off one zero at a time until we are left with a quadratic, whose roots can always be found using the Quadratic Formula. Secondly, we found $x = \pm\sqrt{3}$ are zeros of p. The Factor Theorem guarantees $\left(x - \sqrt{3}\right)$ and $\left(x - \left(-\sqrt{3}\right)\right)$ are both factors of p. We can certainly put the Factor Theorem to the test and continue the synthetic division tableau from above to see what happens.

$$
\begin{array}{r|rrrrr}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
\sqrt{3} & 4 & 0 & -12 & \boxed{0} & \\
 & \downarrow & 4\sqrt{3} & 12 & & \\
\hline
-\sqrt{3} & 4 & 4\sqrt{3} & \boxed{0} & & \\
 & \downarrow & -4\sqrt{3} & & & \\
\hline
 & 4 & \boxed{0} & & &
\end{array}
$$

This gives us

$$
\begin{aligned}
p(x) &= 4x^4 - 4x^3 - 11x^2 + 12x - 3 \\
&= \left(x - \tfrac{1}{2}\right)^2 \left(x - \sqrt{3}\right)\left(x - \left(-\sqrt{3}\right)\right)(4) \\
&= 4\left(x - \tfrac{1}{2}\right)^2 \left(x - \sqrt{3}\right)\left(x - \left(-\sqrt{3}\right)\right)
\end{aligned}
$$

We have shown that p is a product of its leading coefficient times linear factors of the form $(x - c)$ where c are zeros of p. It may surprise and delight the reader that, in theory, all polynomials can be reduced to this kind of factorization. We leave that discussion to Section **??**, because the zeros may not be real numbers. Our final theorem in the section gives us an upper bound on the number of real zeros.

Theorem 6.9. Suppose f is a polynomial of degree $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

Theorem 6.9 is a consequence of the Factor Theorem and polynomial multiplication. Every zero c of f gives us a factor of the form $(x - c)$ for $f(x)$. Since f has degree n, there can be at most n of these factors. The next section provides us some tools which not only help us determine where the real zeros are to be found, but which real numbers they may be.

We close this section with a summary of several concepts previously presented. You should take the time to look back through the text to see where each concept was first introduced and where each connection to the other concepts was made.

[6] For those wanting more detail: the first division gives: $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - 12x + 6\right)$. The second division gives: $4x^3 - 2x^2 - 12x + 6 = \left(x - \frac{1}{2}\right)\left(4x^2 - 12\right)$.

Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x - c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

6.2.1 Exercises

In Exercises 1 - 14, use synthetic division to perform the following polynomial divisions. Identify the quotient and remainder. Write the dividend, quotient and remainder in the form given in Theorem 6.6.

1. $\left(3x^2 - 2x + 1\right) \div (x - 1)$
2. $\left(x^2 - 5\right) \div (x - 5)$
3. $\left(3 - 4t - 2t^2\right) \div (t + 1)$
4. $\left(4t^2 - 5t + 3\right) \div (t + 3)$
5. $\left(z^3 + 8\right) \div (z + 2)$
6. $\left(4z^3 + 2z - 3\right) \div (z - 3)$
7. $\left(18x^2 - 15x - 25\right) \div \left(x - \frac{5}{3}\right)$
8. $\left(4x^2 - 1\right) \div \left(x - \frac{1}{2}\right)$
9. $\left(2t^3 + t^2 + 2t + 1\right) \div \left(t + \frac{1}{2}\right)$
10. $\left(3t^3 - t + 4\right) \div \left(t - \frac{2}{3}\right)$
11. $\left(2z^3 - 3z + 1\right) \div \left(z - \frac{1}{2}\right)$
12. $\left(4z^4 - 12z^3 + 13z^2 - 12z + 9\right) \div \left(z - \frac{3}{2}\right)$
13. $\left(x^4 - 6x^2 + 9\right) \div \left(x - \sqrt{3}\right)$
14. $\left(x^6 - 6x^4 + 12x^2 - 8\right) \div \left(x + \sqrt{2}\right)$

In Exercises 15 - 24, find $p(c)$ using the Remainder Theorem. If $p(c) = 0$, use the Factor Theorem to partially factor the polynomial function.

15. $p(x) = 2x^2 - x + 1$, $c = 4$
16. $p(x) = 4x^2 - 33x - 180$, $c = 12$
17. $p(t) = 2t^3 - t + 6$, $c = -3$
18. $p(t) = t^3 + 2t^2 + 3t + 4$, $c = -1$
19. $p(z) = 3z^3 - 6z^2 + 4z - 8$, $c = 2$
20. $p(z) = 8z^3 + 12z^2 + 6z + 1$, $c = -\frac{1}{2}$
21. $p(x) = x^4 - 2x^2 + 4$, $c = \frac{3}{2}$
22. $p(x) = 6x^4 - x^2 + 2$, $c = -\frac{2}{3}$
23. $p(t) = t^4 + t^3 - 6t^2 - 7t - 7$, $c = -\sqrt{7}$
24. $p(t) = t^2 - 4t + 1$, $c = 2 - \sqrt{3}$

In Exercises 25 - 34, you are given a polynomial function and one of its zeros. Find the remaining real zeros and factor the polynomial.

25. $x^3 - 6x^2 + 11x - 6, \quad c = 1$
26. $x^3 - 24x^2 + 192x - 512, \quad c = 8$
27. $3t^3 + 4t^2 - t - 2, \quad c = \frac{2}{3}$
28. $2t^3 - 3t^2 - 11t + 6, \quad c = \frac{1}{2}$
29. $z^3 + 2z^2 - 3z - 6, \quad c = -2$
30. $2z^3 - z^2 - 10z + 5, \quad c = \frac{1}{2}$
31. $4x^4 - 28x^3 + 61x^2 - 42x + 9$, $c = \frac{1}{2}$ is a zero of multiplicity 2
32. $t^5 + 2t^4 - 12t^3 - 38t^2 - 37t - 12$, $c = -1$ is a zero of multiplicity 3
33. $125z^5 - 275z^4 - 2265z^3 - 3213z^2 - 1728z - 324$, $c = -\frac{3}{5}$ is a zero of multiplicity 3
34. $x^2 - 2x - 2, \quad c = 1 - \sqrt{3}$

35. Find a quadratic polynomial with integer coefficients which has $x = \frac{3}{5} \pm \frac{\sqrt{29}}{5}$ as its real zeros.

36. For $f(x) = x^3 + 4x^2 - 5x - 14$, show $f(-3 - \sqrt{2}) = 0$ and $f(-3 + \sqrt{2}) = 0$ two ways:

    (a) By direct substitution.

    (b) Using synthetic division and the Factor Theorem

37. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ be a polynomial function with the property that $a_n + a_{n-1} + ... + a_1 + a_0 = 0$. (That is, the sum of the coefficients and the constant term is 0.)

    Prove that $(x - 1)$ is a factor of $f(x)$.

    HINT: Show $f(1) = 0$ and invoke the Factor Theorem

38. Verify the result in number 37 with the functions: $f(x) = x^3 - 2x + 1$ and $f(x) = 3x^4 - x - 2$.

39. Suppose a is a nonzero real number. Find the quotients below, using synthetic division as required.

    - $\dfrac{x - a}{x - a}$
    - $\dfrac{x^2 - a^2}{x - a}$
    - $\dfrac{x^3 - a^3}{x - a}$
    - $\dfrac{x^4 - a^4}{x - a}$
    - $\dfrac{x^5 - a^5}{x - a}$

    Based on the pattern that evolves, find the quotient: $\dfrac{x^{10} - a^{10}}{x - a}$. What about $\dfrac{x^n - a^n}{x - a}$?

40. Use your result from number 39 to rewrite the sum: $1 + r + r^2 + \cdots + r^{n-2} + r^{n-1}$ as a quotient. What assumptions need to be made about r?

6.2.2 Answers

1. $\left(3x^2 - 2x + 1\right) = (x - 1)\,(3x + 1) + 2$
2. $\left(x^2 - 5\right) = (x - 5)\,(x + 5) + 20$
3. $\left(3 - 4t - 2t^2\right) = (t + 1)\,(-2t - 2) + 5$
4. $\left(4t^2 - 5t + 3\right) = (t + 3)\,(4t - 17) + 54$
5. $\left(z^3 + 8\right) = (z + 2)\left(z^2 - 2z + 4\right) + 0$
6. $\left(4z^3 + 2z - 3\right) = (z - 3)\left(4z^2 + 12z + 38\right) + 111$
7. $\left(18x^2 - 15x - 25\right) = \left(x - \frac{5}{3}\right)(18x + 15) + 0$
8. $\left(4x^2 - 1\right) = \left(x - \frac{1}{2}\right)(4x + 2) + 0$
9. $\left(2t^3 + t^2 + 2t + 1\right) = \left(t + \frac{1}{2}\right)\left(2t^2 + 2\right) + 0$
10. $\left(3t^3 - t + 4\right) = \left(t - \frac{2}{3}\right)\left(3t^2 + 2t + \frac{1}{3}\right) + \frac{38}{9}$
11. $\left(2z^3 - 3z + 1\right) = \left(z - \frac{1}{2}\right)\left(2z^2 + z - \frac{5}{2}\right) - \frac{1}{4}$
12. $\left(4z^4 - 12z^3 + 13z^2 - 12z + 9\right) = \left(z - \frac{3}{2}\right)\left(4z^3 - 6z^2 + 4z - 6\right) + 0$
13. $\left(x^4 - 6x^2 + 9\right) = \left(x - \sqrt{3}\right)\left(x^3 + \sqrt{3}\,x^2 - 3x - 3\sqrt{3}\right) + 0$
14. $\left(x^6 - 6x^4 + 12x^2 - 8\right) = \left(x + \sqrt{2}\right)\left(x^5 - \sqrt{2}\,x^4 - 4x^3 + 4\sqrt{2}\,x^2 + 4x - 4\sqrt{2}\right) + 0$
15. $p(4) = 29$
16. $p(12) = 0$, $p(x) = (x - 12)(4x + 15)$
17. $p(-3) = -45$
18. $p(-1) = 2$
19. $p(2) = 0$, $p(z) = (z - 2)\left(3z^2 + 4\right)$
20. $p\left(-\frac{1}{2}\right) = 0$, $p(z) = \left(z + \frac{1}{2}\right)\left(8z^2 + 8z + 2\right)$
21. $p\left(\frac{3}{2}\right) = \frac{73}{16}$
22. $p\left(-\frac{2}{3}\right) = \frac{74}{27}$
23. $p(-\sqrt{7}) = 0$, $p(t) = (t + \sqrt{7})\left(t^3 + (1 - \sqrt{7})t^2 + (1 - \sqrt{7})t - \sqrt{7}\right)$
24. $p(2 - \sqrt{3}) = 0$, $p(t) = (t - (2 - \sqrt{3}))(t - (2 + \sqrt{3}))$
25. $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$
26. $x^3 - 24x^2 + 192x - 512 = (x - 8)^3$
27. $3t^3 + 4t^2 - t - 2 = 3\left(t - \frac{2}{3}\right)(t + 1)^2$
28. $2t^3 - 3t^2 - 11t + 6 = 2\left(t - \frac{1}{2}\right)(t + 2)(t - 3)$

29. $z^3 + 2z^2 - 3z - 6 = (z+2)(z+\sqrt{3})(z-\sqrt{3})$

30. $2z^3 - z^2 - 10z + 5 = 2\left(z - \frac{1}{2}\right)(z+\sqrt{5})(z-\sqrt{5})$

31. $4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4\left(x - \frac{1}{2}\right)^2 (x-3)^2$

32. $t^5 + 2t^4 - 12t^3 - 38t^2 - 37t - 12 = (t+1)^3(t+3)(t-4)$

33. $125z^5 - 275z^4 - 2265z^3 - 3213z^2 - 1728z - 324 = 125\left(z + \frac{3}{5}\right)^3 (z+2)(z-6)$

34. $x^2 - 2x - 2 = (x - (1-\sqrt{3}))(x - (1+\sqrt{3}))$

35. $p(x) = 5x^2 - 6x - 4$

38.
- For $f(x) = x^3 - 2x + 1$, the coefficients $1 + (-2) + 1 = 0$ and $f(x) = (x-1)(x^2+x-1)$.
- For $f(x) = 3x^4 - x - 2$ the coefficients $3 + (-1) + (-2) = 0$ and $f(x) = (x-1)(3x^3+3x^2+3x+2)$.

39.
- $\dfrac{x-a}{x-a} = 1$
- $\dfrac{x^2-a^2}{x-a} = x + a$
- $\dfrac{x^3-a^3}{x-a} = x^2 + ax + a^2$
- $\dfrac{x^4-a^4}{x-a} = x^3 + ax^2 + a^2x + a^3$
- $\dfrac{x^5-a^5}{x-a} = x^4 + ax^3 + a^2x^2 + a^3x + a^4$

Following the pattern:

- $\dfrac{x^{10}-a^{10}}{x-a} = x^9 + ax^8 + a^2x^7 + a^3x^6 + a^4x^5 + a^5x^4 + a^6x^3 + a^7x^2 + a^8x + a^9$
- $\dfrac{x^n-a^n}{x-a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + ... + a^{n-2}x + a^{n-1}$

40. Put $x = 1$ and $a = r$ so that $1 + r + r^2 + \cdots + r^{n-2} + r^{n-1} = \dfrac{1-r^n}{1-r}$. Here, $r \neq 1$ as otherwise we'd be dividing by 0.

6.3 Real Zeros of Polynomials

In Section 6.2, we found that we can use synthetic division to determine if a given real number is a zero of a polynomial function. This section presents results which will help us determine good candidates to test using synthetic division. There are two approaches to the topic of finding the real zeros of a polynomial. The first approach is to use a little bit of Mathematics followed by a good use of technology like graphing utilities. The second approach makes good use of mathematical machinery (theorems) only. For completeness, we include the two approaches but in separate subsections. Both approaches benefit from the following two theorems, the first of which is due to the famous mathematician Augustin Cauchy. It gives us an interval on which *all* of the real zeros of a polynomial can be found.

Theorem 6.10. Cauchy's Bound: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}$. Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.

There's a lot going on in the statement of Cauchy's Bound, so we'll get right to an example and show how it is used. For those wanting a proof of Cauchy's Bound, see Exercise **??** in Section **??**.

Example 6.3.1. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Determine an interval which contains all of the real zeros of f.

Solution. To find the M stated in Cauchy's Bound, we take the absolute value of the leading coefficient, in this case $|2| = 2$ and divide it into the largest (in absolute value) of the remaining coefficients, in this case $|-6| = 6$. This yields $M = 3$ so it is guaranteed that all of the real zeros of f lie in the interval $[-4, 4]$. □

Whereas the previous result tells us *where* we can find the real zeros of a polynomial, the next theorem gives us a list of *possible* real zeros.

Theorem 6.11. Rational Zeros Theorem: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and a_0, a_1, $\ldots a_n$ are integers. If r is a rational zero of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

The Rational Zeros Theorem gives us a list of numbers to try in our synthetic division and that is a lot nicer than simply guessing. If none of the numbers in the list are zeros, then either the polynomial has no real zeros at all, or all of the real zeros are irrational numbers. To see why the Rational Zeros Theorem works, suppose c is a zero of f and $c = \frac{p}{q}$ in lowest terms. This means p and q have no common factors. Since $f(c) = 0$, we have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_1 \left(\frac{p}{q}\right) + a_0 = 0.$$

Multiplying both sides of this equation by q^n, we clear the denominators to get

$$a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 pq^{n-1} + a_0 q^n = 0$$

Rearranging this equation, we get

$$a_n p^n = -a_{n-1}p^{n-1}q - \ldots - a_1 pq^{n-1} - a_0 q^n$$

Now, the left hand side is an integer multiple of p, and the right hand side is an integer multiple of q. (Can you see why?) This means $a_n p^n$ is both a multiple of p and a multiple of q. Since p and q have no common factors, a_n must be a multiple of q. If we rearrange the equation

$$a_n p^n + a_{n-1} p^{n-1} q + \ldots + a_1 p q^{n-1} + a_0 q^n = 0$$

as

$$a_0 q^n = -a_n p^n - a_{n-1} p^{n-1} q - \ldots - a_1 p q^{n-1}$$

we can play the same game and conclude a_0 is a multiple of p, and we have the result.

Example 6.3.2. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Zeros Theorem to list all of the possible rational zeros of f.

Solution. To generate a complete list of rational zeros, we need to take each of the factors of constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$. The factors of -3 are ± 1 and ± 3. Since the Rational Zeros Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Zeros Theorem gives the list $\left\{\pm \frac{1}{1}, \pm \frac{1}{2}, \pm \frac{3}{1}, \pm \frac{3}{2}\right\}$ or $\left\{\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 3\right\}$. □

Our discussion now diverges between those who wish to use technology and those who do not.

6.3.1 For Those Wishing to use a Graphing Utility

At this stage, we know not only the interval in which all of the zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ are located, but we also know some potential candidates. We can now use our calculator to help us determine all of the real zeros of f, as illustrated in the next example.

Example 6.3.3. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Graph $y = f(x)$ using a graphing utility over the interval obtained in Example 6.3.1.
2. Use the graph to shorten the list of possible rational zeros obtained in Example 6.3.2.
3. Use synthetic division to find the real zeros of f, and state their multiplicities.

Solution.

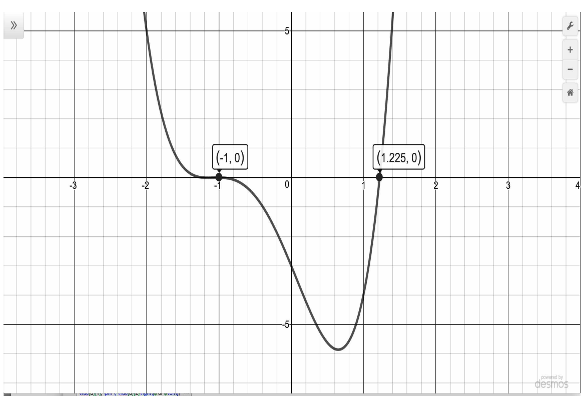
1. In Example 6.3.1, we determined all of the real zeros of f lie in the interval $[-4, 4]$, so we restrict our attention to that portion of the x-axis.
2. In Example 6.3.2, we learned that any rational zero of f must be in the list $\left\{\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 3\right\}$. From the graph, it looks as if we can rule out any of the positive rational zeros, since the graph seems to cross the x-axis at $x \approx 1.225$. On the negative side, $x = -1$ looks good. Indeed, the shape of the graph near $(-1, 0)$ suggests that if $x = -1$ is a zero, it is of multiplicity at least three. We set about synthetically dividing:

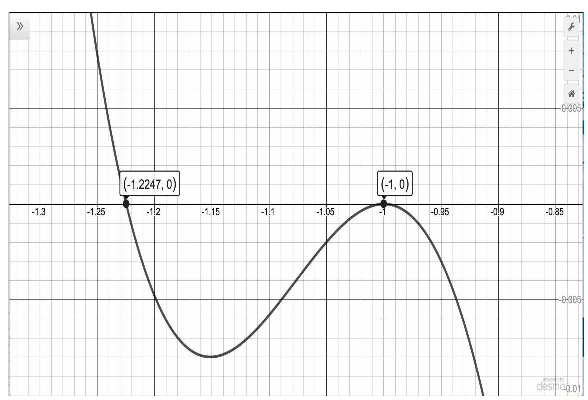
$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \boxed{0} \end{array}$$

Since f is a fourth degree polynomial, we know that our quotient is a third degree polynomial. If we can do one more successful division, we will have reduced the quotient to a quadratic, and we can use the quadratic formula, if needed, to find the two remaining zeros. Continuing with $x = -1$:

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

Our quotient polynomial is now $2x^2 - 3$. Setting this to zero gives $2x^2 - 3 = 0$, or $x^2 = \frac{3}{2}$, which gives us $x = \pm\frac{\sqrt{6}}{2}$. Based on our division work, we know that -1 has a multiplicity of *at least* 2. The Factor Theorem tells us our remaining zeros, $\pm\frac{\sqrt{6}}{2}$, each have multiplicity at least 1. However, Theorem 6.9 tells us f can have at most 4 real zeros, counting multiplicity, and so we conclude that -1 is of multiplicity *exactly* 2 and $\pm\frac{\sqrt{6}}{2} \approx \pm 1.225$ each has multiplicity 1. Thus, we were incorrect in thinking -1 was a zero of multiplicity 3. Sure enough, if we adjust zoom in near $(-1, 0)$ using graphing utility, we find the graph of $y = f(x)$ touches and rebounds from the x-axis at $(-1, 0)$, typical behavior near a zero of even multiplicity. The lesson here is, once again, technology may *suggest* a result, but it is only the mathematics which can *prove* (or in this case, *disprove*) it.





$\square$

Our next example shows how even a mild-mannered polynomial can cause problems.

Example 6.3.4. Let $f(x) = x^4 + x^2 - 12$.

1. Use Cauchy's Bound to determine an interval in which all of the real zeros of f lie.

2. Use the Rational Zeros Theorem to determine a list of possible rational zeros of f.

3. Graph $y = f(x)$ using a graphing utility.

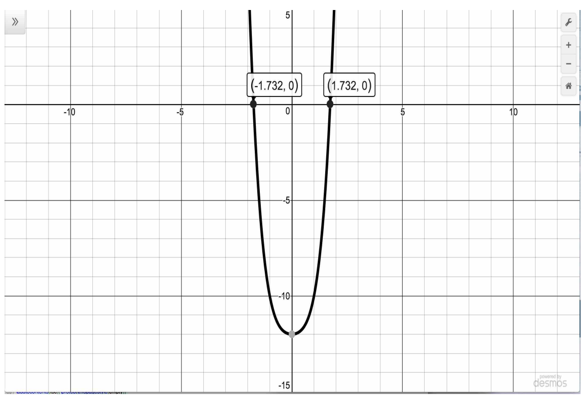
4. Find all of the real zeros of f and their multiplicities.

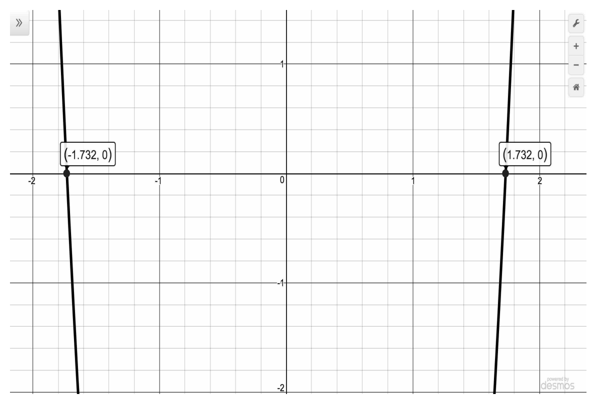
Solution.

1. Applying Cauchy's Bound, we find $M = 12$, so all of the real zeros lie in the interval $[-13, 13]$.

2. Applying the Rational Zeros Theorem with constant term $a_0 = -12$ and leading coefficient $a_4 = 1$, we get the list $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$.

3. Graphing $y = f(x)$ on the interval $[-13, 13]$ produces the graph below on the left. Zooming in a bit gives the graph below on the right. Based on the graph, none of our rational zeros will work. (Do you see why not?)





4. From the graph, we know f has two real zeros, one positive, and one negative. Our only hope at this point is to try and find the zeros of f by setting $f(x) = x^4 + x^2 - 12 = 0$ and solving. If we stare at this equation long enough, we may recognize it as a 'quadratic in disguise' or 'quadratic in form'. (See Section 5.3.) In other words, we have three terms: x^4, x^2 and 12, and the exponent on the first term, x^4, is exactly twice that of the second term, x^2. We may rewrite this as $\left(x^2\right)^2 + \left(x^2\right) - 12 = 0$. To better see the forest for the trees, we momentarily replace x^2 with the variable u. In terms of u, our equation becomes $u^2 + u - 12 = 0$, which we can readily factor as $(u + 4)(u - 3) = 0$. In terms of x, this means $x^4 + x^2 - 12 = \left(x^2 - 3\right)\left(x^2 + 4\right) = 0$. We get $x^2 = 3$, which gives us $x = \pm\sqrt{3}$, or $x^2 = -4$, which admits no real solutions. Since $\sqrt{3} \approx 1.73$, the two zeros match what we expected from the graph. Turning our attention now to multiplicities, the Factor Theorem guarantees that since $x = \pm\sqrt{3}$ are zeros, $(x - \sqrt{3})$ and $(x + \sqrt{3})$ are factors of $f(x)$. We've already partially factored $f(x)$ as $f(x) = \left(x^2 - 3\right)\left(x^2 + 4\right)$. Since $x^2 + 4$ as no real zeros, we know both $(x - \sqrt{3})$ and $(x + \sqrt{3})$ must divide $x^2 - 3$. By Theorem 6.9, $x^2 - 3$ can only have a total of two zeros, including multiplicities, so we are forced to conclude $x = \pm\sqrt{3}$ are each zeros of multiplicity 1 of $x^2 - 3$, and hence, $f(x)$.[1] $\square$

[1]Alternatively, we could recognize $x^2 - 3 = x^2 - (\sqrt{3})^2 = (x - \sqrt{3})(x + \sqrt{3})$, but the above argument works for all quadratic functions, even those which aren't as easy to factor.

A couple of remarks are in order. First, the graph of $f(x) = x^4 + x^2 - 12$ appears to be symmetric about the y-axis. Sure enough, we find $f(-x) = (-x)^4 + (-x)^2 - 12 = x^4 + x^2 = 12 = f(x)$ proving f is, indeed, an even function, thus *proving* the symmetry *suggested* by the graph. Second, the technique used to factor $f(x)$ in Example 6.3.4 is called ***u*-substitution**. We shall this technique now and then in the sections to come, so it is worth taking the time to let this idea sink in. In general, substitution can help us identify a 'quadratic in disguise' - in essence, it helps us 'see the forest for the trees.' Last, but not least, it is entirely possible that a polynomial has no real roots at all, or worse, it has real roots but none of the techniques discussed in this section can help us find them exactly. In the latter case, we are forced to approximate using technology.

6.3.2 For Those Wishing NOT to use a Graphing Calculator

Suppose we wish to find the zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ *without* using the calculator. In this subsection, we present some more advanced mathematical tools (theorems) to help us. Our first result is due to René Descartes.

Theorem 6.12. Descartes' Rule of Signs: Suppose $f(x)$ is the formula for a polynomial function written with descending powers of x.

- If P denotes the number of variations of sign in the formula for $f(x)$, then the number of positive real zeros (counting multiplicity) is one of the numbers $\{P, P-2, P-4, \ldots\}$.
- If N denotes the number of variations of sign in the formula for $f(-x)$, then the number of negative real zeros (counting multiplicity) is one of the numbers $\{N, N-2, N-4, \ldots\}$.

A few remarks are in order. First, to use Descartes' Rule of Signs, we need to understand what is meant by a '**variation in sign**' of a polynomial function. Consider $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. If we focus on only the *signs* of the coefficients, we start with a (+), followed by another (+), then switch to $(-)$, and stay $(-)$ for the remaining two coefficients. Since the signs of the coefficients switched *once* as we read from left to right, we say that $f(x)$ has *one* variation in sign. When we speak of the variations in sign of a polynomial function f we assume the formula for $f(x)$ is written with descending powers of x, as in Definition 6.4, and concern ourselves only with the nonzero coefficients. Second, unlike the Rational Zeros Theorem, Descartes' Rule of Signs gives us an estimate to the *number* of positive and negative real zeros, not the actual *value* of the zeros. Lastly, Descartes' Rule of Signs counts multiplicities. This means that, for example, if one of the zeros has multiplicity 2, Descsartes' Rule of Signs would count this as *two* zeros. Lastly, note that the number of positive or negative real zeros always starts with the number of sign changes and decreases by an even number. For example, if $f(x)$ has 7 sign changes, then, counting multplicities, f has either 7, 5, 3 or 1 positive real zero. This implies that the graph of $y = f(x)$ crosses the positive x-axis at least once. If $f(-x)$ results in 4 sign changes, then, counting multiplicities, f has 4, 2 or 0 negative real zeros; hence, the graph of $y = f(x)$ may not cross the negative x-axis at all. The proof of Descartes' Rule of Signs is a bit technical, and can be found here.

Example 6.3.5. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use Descartes' Rule of Signs to determine the possible number and location of the real zeros of f.

Solution. As noted above, the variations of sign of $f(x)$ is 1. This means, counting multiplicities, f has exactly 1 positive real zero. Since $f(-x) = 2(-x)^4 + 4(-x)^3 - (-x)^2 - 6(-x) - 3 = 2x^4 - 4x^3 - x^2 + 6x - 3$ has 3 variations in sign, f has either 3 negative real zeros or 1 negative real zero, counting multiplicities. $\square$

Cauchy's Bound gives us a general bound on the zeros of a polynomial function. Our next result helps us determine bounds on the real zeros of a polynomial as we synthetically divide which are often sharper[2] bounds than Cauchy's Bound.

Theorem 6.13. Upper and Lower Bounds: Suppose f is a polynomial of degree $n \geq 1$.

- If $c > 0$ is synthetically divided into f and all of the numbers in the final line of the division tableau have the same signs, then c is an upper bound for the real zeros of f. That is, there are no real zeros greater than c.
- If $c < 0$ is synthetically divided into f and the numbers in the final line of the division tableau alternate signs, then c is a lower bound for the real zeros of f. That is, there are no real zeros less than c.

  NOTE: If the number 0 occurs in the final line of the division tableau in either of the above cases, it can be treated as $(+)$ or $(-)$ as needed.

The Upper and Lower Bounds Theorem works because of Theorem 6.6. For the upper bound part of the theorem, suppose $c > 0$ is divided into f and the resulting line in the division tableau contains, for example, all nonnegative numbers. This means $f(x) = (x - c)q(x) + r$, where the coefficients of the quotient polynomial and the remainder are nonnegative. (Note that the leading coefficient of q is the same as f so $q(x)$ is not the zero polynomial.) If $b > c$, then $f(b) = (b - c)q(b) + r$, where $(b - c)$ and $q(b)$ are both positive and $r \geq 0$. Hence $f(b) > 0$ which shows b cannot be a zero of f. Thus no real number $b > c$ can be a zero of f, as required. A similar argument proves $f(b) < 0$ if all of the numbers in the final line of the synthetic division tableau are non-positive. To prove the lower bound part of the theorem, we note that a lower bound for the negative real zeros of $f(x)$ is an upper bound for the positive real zeros of $f(-x)$, since all we are doing is reflecting the numbers across the $x = 0$. Applying the upper bound portion to $f(-x)$ gives the result. (Do you see where the alternating signs come in?) With the additional mathematical machinery of Descartes' Rule of Signs and the Upper and Lower Bounds Theorem, we can find the real zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ without the use of a graphing utility.

Example 6.3.6. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Find all of the real zeros of f and their multiplicities.
2. Sketch the graph of $y = f(x)$.

Solution.

1. We know from Cauchy's Bound that all of the real zeros lie in the interval $[-4, 4]$ and that our possible rational zeros are $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$ and ± 3. Descartes' Rule of Signs guarantees us at least one negative real zero and exactly one positive real zero, counting multiplicity. We try our positive rational

[2] That is, better, or more accurate.

zeros, starting with the smallest, $\frac{1}{2}$. Since the remainder isn't zero, we know $\frac{1}{2}$ isn't a zero. Sadly, the final line in the division tableau has both positive and negative numbers, so $\frac{1}{2}$ is not an upper bound. The only information we get from this division is courtesy of the Remainder Theorem which tells us $f\left(\frac{1}{2}\right) = -\frac{45}{8}$ so the point $\left(\frac{1}{2}, -\frac{45}{8}\right)$ is on the graph of f. We continue to our next possible zero, 1. As before, the only information we can glean from this is that $(1, -4)$ is on the graph of f. When we try our next possible zero, $\frac{3}{2}$, we get that it is not a zero, and we also see that it is an upper bound on the zeros of f, since all of the numbers in the final line of the division tableau are positive. This means there is no point trying our last possible rational zero, 3. Descartes' Rule of Signs guaranteed us a positive real zero, and at this point we have shown this zero is irrational.[3]

$$\begin{array}{c|ccccc} \frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 1 & \frac{5}{2} & \frac{3}{4} & -\frac{21}{8} \\ \hline & 2 & 5 & \frac{3}{2} & -\frac{21}{4} & \boxed{-\frac{45}{8}} \end{array} \qquad \begin{array}{c|ccccc} 1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 2 & 6 & 5 & -1 \\ \hline & 2 & 6 & 5 & -1 & \boxed{-4} \end{array} \qquad \begin{array}{c|ccccc} \frac{3}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 3 & \frac{21}{2} & \frac{57}{4} & \frac{99}{8} \\ \hline & 2 & 7 & \frac{19}{2} & \frac{33}{4} & \boxed{\frac{75}{8}} \end{array}$$

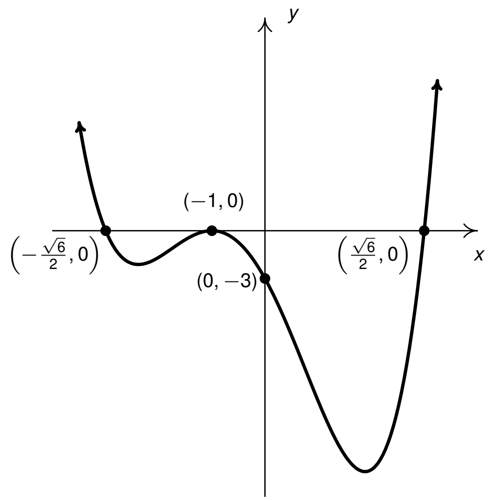
We now turn our attention to negative real zeros. We try the largest possible zero, $-\frac{1}{2}$. Synthetic division shows us it is not a zero, nor is it a lower bound (since the numbers in the final line of the division tableau do not alternate), so we proceed to -1. This division shows -1 is a zero. Descartes' Rule of Signs told us that we may have up to three negative real zeros, counting multiplicity, so we try -1 again, and it works once more. At this point, we have taken f, a fourth degree polynomial, and performed two successful divisions. Our quotient polynomial is quadratic, so we look at it to find the remaining zeros.

$$\begin{array}{c|ccccc} -\frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -1 & -\frac{3}{2} & \frac{5}{4} & \frac{19}{8} \\ \hline & 2 & 3 & -\frac{5}{2} & -\frac{19}{4} & \boxed{-\frac{5}{8}} \end{array} \qquad \begin{array}{c|ccccc} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

Setting the quotient polynomial equal to zero yields $2x^2 - 3 = 0$, so that $x^2 = \frac{3}{2}$, or $x = \pm\frac{\sqrt{6}}{2}$. Descartes' Rule of Signs tells us that the positive real zero we found, $\frac{\sqrt{6}}{2}$, has multiplicity 1. Descartes also tells us the total multiplicity of negative real zeros is 3, which forces -1 to be a zero of multiplicity 2 and $-\frac{\sqrt{6}}{2}$ to have multiplicity 1.

2. We know the end behavior of $y = f(x)$ resembles that of its leading term $y = 2x^4$. This means that the graph enters the scene in Quadrant II and exits in Quadrant I. Since $\pm\frac{\sqrt{6}}{2}$ are zeros of multiplicity 1, we have that the graph crosses through the x-axis at the points $\left(-\frac{\sqrt{6}}{2}, 0\right)$ and $\left(\frac{\sqrt{6}}{2}, 0\right)$ in a fairly linear fashion. Since -1 is a zero of multiplicity 2, the graph of $y = f(x)$ touches and rebounds off the x-axis at $(-1, 0)$ in a parabolic manner. Last, but not least, since $f(0) = -3$, we get the y-intercept is $(0, -3)$. Putting all of this together results in the graph below.

[3]Since polynomials are continuous, we know the zero lies between 1 and $\frac{3}{2}$, since $f(1) < 0$ and $f\left(\frac{3}{2}\right) > 0$.



$y = f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$

□

6.3.3 The Intermediate Value Theorem and Inequalities

As we mentioned in Section 6.1, polynomial functions are continuous. An important property of continuous functions is that they cannot change sign between two values unless there is a zero in between. We used this property of quadratic functions when constructing sign diagrams to help us solve inequalities (see Section **??**.) This property is a version of the celebrated **Intermediate Value Theorem**.

Theorem 6.14. The Intermediate Value Theorem (Zero Version): If f is continuous over an interval containing a and b and $f(a)$ and $f(b)$ have different signs, then f has a zero between a and b. That is, for at least one value c between a and b, $f(c) = 0$.

The Intermediate Value Theorem is discussed in greater detail in Calculus, and its proof is usually delayed until a formal analysis course. It is an example of an 'existence' theorem - it tells us that, under suitable conditions, a zero exists - but offers us no algorithm to find it.[4] Its use to us in this section is that it provides the justification needed to create sign diagrams for general polynomial functions in the same manner in which we constructed them for quadratic functions.

Steps for Constructing a Sign Diagram for a Polynomial Function

Suppose f is a polynomial function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine and record the sign of $f(x)$ for each test value in step 2.

[4]See the notes on the 'Bisection Method' at the end of this section.

The Intermediate Value Theorem justifies the use of just one 'test' value in the algorithm above, since a continuous function cannot change signs on an interval without there being a zero on that interval. Since we have found the zeros in Step 1 of the algorithm and used these to create the intervals for Step 2, there cannot be any sign changes on any of the intervals in Step 2.

Not surprisingly, we use sign diagrams to solve inequalities involving higher order polynomial functions in the same way we used them to solve inequalities involving quadratic functions. We reproduce our algorithm from section **??** for reference.

Solving Inequalities using Sign Diagrams

To solve an inequality using a sign diagram:

1. Rewrite the inequality so a function $f(x)$ is being compared to '0.'
2. Make a sign diagram for f.
3. Record the solution.

Example 6.3.7.

1. Find all of the real solutions to the equation $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$.
2. Solve the inequality $2x^5 + 6x^3 + 3 \leq 3x^4 + 8x^2$.
3. Interpret your answer to part 2 graphically, and verify using a graphing utility.

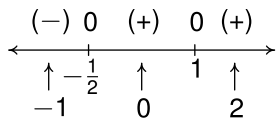
Solution.

1. Finding the real solutions to $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$ is the same as finding the real solutions to $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 = 0$. In other words, we are looking for the real zeros of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$. Using the techniques developed in this section, we get

$$\begin{array}{r|rrrrrr}
1 & 2 & -3 & 6 & -8 & 0 & 3 \\
 & \downarrow & 2 & -1 & 5 & -3 & -3 \\
\hline
1 & 2 & -1 & 5 & -3 & -3 & \boxed{0} \\
 & \downarrow & 2 & 1 & 6 & 3 & \\
\hline
-\frac{1}{2} & 2 & 1 & 6 & 3 & \boxed{0} & \\
 & \downarrow & -1 & 0 & -3 & & \\
\hline
 & 2 & 0 & 6 & \boxed{0} & &
\end{array}$$

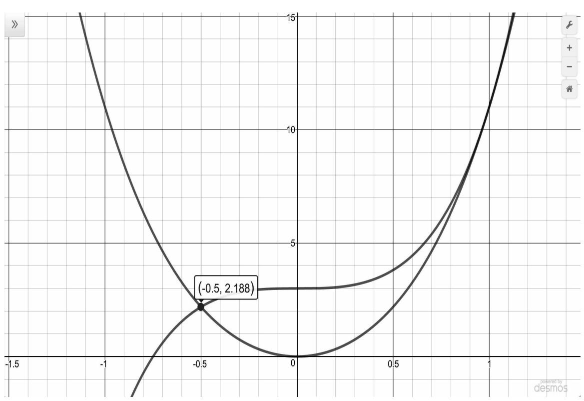
The quotient polynomial is $2x^2 + 6$ which has no real zeros so we get $x = -\frac{1}{2}$ and $x = 1$.

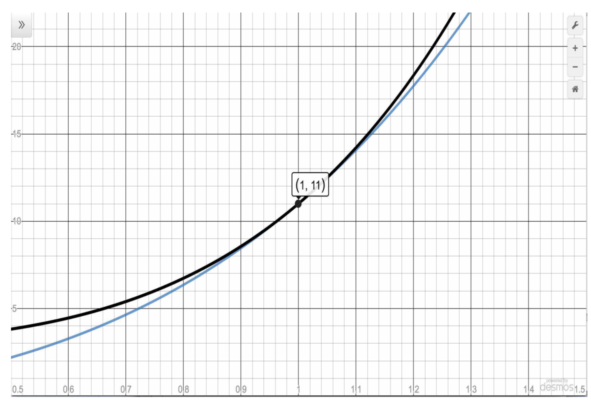
2. Our first step is to rewrite this inequality so as to compare a function $f(x)$ to 0. We have two options, but choose $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 \leq 0$, since we found the zeros of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$ to be $x = -\frac{1}{2}$ and $x = 1$. We construct our sign diagram below using the test values -1, 0, and 2.



The solution to $p(x) < 0$ is $\left(-\infty, -\frac{1}{2}\right)$, and we know $p(x) = 0$ at $x = -\frac{1}{2}$ and $x = 1$. Hence, the solution to $p(x) \leq 0$ is $\left(-\infty, -\frac{1}{2}\right] \cup \{1\}$.

3. To interpret this solution graphically, we set $f(x) = 2x^5 + 6x^3 + 3$ and $g(x) = 3x^4 + 8x^2$. Recall from Section 4.2 the solution to $f(x) \leq g(x)$ is the set of x values for which the graph of f is below the graph of g (where $f(x) < g(x)$) along with the x values where the two graphs intersect ($f(x) = g(x)$). Graphing f and g using a graphing utility produces the graph below on the left. (The end behavior should tell you which is which.) We see that the graph of f is below the graph of g on $\left(-\infty, -\frac{1}{2}\right)$. However, it is difficult to see what is happening near $x = 1$. Zooming in (and making the graph of g lighter), we see that the graphs of f and g do intersect at $x = 1$, but the graph of g remains below the graph of f on either side of $x = 1$.





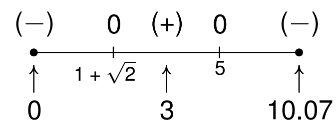
□

Note that we could have used end behavior and the concept of multiplicity to create the sign diagram used in Example 6.3.7 as follows. We know the end behavior of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$ matches that of $y = 2x^5$ which means as $x \to -\infty$, $p(x) \to -\infty$. This means for the interval $\left(-\infty, -\frac{1}{2}\right)$, $p(x) < 0$ or $(-)$. From our work finding the zeros of p, we can deduce the multiplicity of the zero $x = -\frac{1}{2}$ is 1 which means the graph of $y = p(x)$ crosses through the x-axis at $\left(-\frac{1}{2}, 0\right)$, hence, changing sign from $(-)$ to $(+)$. Finally, we can deduce the multiplicity of the zero $x = 1$ is 2 which means the graph of $y = p(x)$ rebounds here, meaning the sign of $p(x)$ for $x > 1$ is $(+)$. This matches the end behavior, since as $x \to \infty$, $p(x) \to \infty$. The reader is encouraged to tackle any given problem using whatever tools are comfortable and convenient, but it also never hurts to think outside the box and revisit a problem from a variety of perspectives.

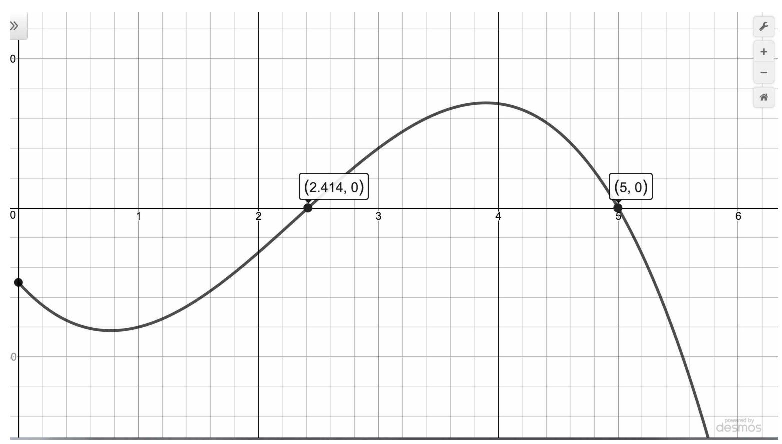
Next up is an application problem torn from page 304 in the Exercises of Section 6.1.

Example 6.3.8. Suppose the profit P, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $P(x) = -5x^3 + 35x^2 - 45x - 25$, $0 \leq x \leq 10.07$. How many TVs should be produced to make a profit? Check your answer using a graphing utility.

Solution. To 'make a profit' means to solve $P(x) = -5x^3 + 35x^2 - 45x - 25 > 0$, which we do analytically using a sign diagram. To simplify things, we first factor out the -5 common to all the coefficients to get $-5\left(x^3 - 7x^2 + 9x - 5\right) > 0$, so we can just focus on finding the zeros of $f(x) = x^3 - 7x^2 + 9x + 5$. The possible rational zeros of f are ± 1 and ± 5, and going through the usual computations, we find $x = 5$ is the only rational zero. Using this, we factor $f(x) = x^3 - 7x^2 + 9x + 5 = (x - 5)\left(x^2 - 2x - 1\right)$, and we find the remaining zeros by applying the Quadratic Formula to $x^2 - 2x - 1 = 0$. We find three real zeros, $x = 1 - \sqrt{2} = -0.414\ldots$, $x = 1 + \sqrt{2} = 2.414\ldots$, and $x = 5$, of which only the last two fall in the applied domain of $[0, 10.07]$. We choose $x = 0$, $x = 3$ and $x = 10.07$ as our test values and plug them into the function $P(x) = -5x^3 + 35x^2 - 45x - 25$ (not $f(x) = x^3 - 7x^2 + 9x - 5$) to get the sign diagram below.



We see immediately that $P(x) > 0$ on $(1 + \sqrt{2}, 5)$. Since x measures the number of TVs in *hundreds*, $x = 1 + \sqrt{2}$ corresponds to $241.4\ldots$ TVs. Since we can't produce a fractional part of a TV, we need to choose between producing 241 and 242 TVs. From the sign diagram, we see that $P(2.41) < 0$ but $P(2.42) > 0$ so, in this case we take the next *larger* integer value and set the minimum production to 242 TVs. At the other end of the interval, we have $x = 5$ which corresponds to 500 TVs. Here, we take the next *smaller* integer value, 499 TVs to ensure that we make a profit. Hence, in order to make a profit, at least 242, but no more than 499 TVs need to be produced. We graph $y = P(x)$ below using a graphing utility and see $P(x) > 0$ between $x \approx 2.414$ and $x = 5$, as predicted.



□

It would be a sin of omission if the authors left the reader with the impression that the theory in this section is compete in that given *any* polynomial function, provided here are the tools to find all of its real zeros exactly.

The reality is this couldn't be further from the truth. In general, no matter how many theorems you throw at a polynomial, it may well be impossible to express its zeros exactly. The polynomial $f(x) = x^5 - x - 1$ is one such beast.[5] According to Descartes' Rule of Signs, f has exactly one positive real zero, and it could have two negative real zeros, or none at all. The Rational Zeros Test gives us ± 1 as rational zeros to try but neither of these work since $f(1) = f(-1) = -1$. If we try the substitution technique we used in Example 6.3.4, we find $f(x)$ has three terms, but the exponent on the x^5 isn't exactly twice the exponent on x. How could we go about approximating the positive zero? We use the **Bisection Method**.

The first step in the Bisection Method is to find an interval on which f changes sign. We know $f(1) = -1$ and we find $f(2) = 29$. By the Intermediate Value Theorem, we know that the zero of f lies in the interval $[1, 2]$. Next, we 'bisect' this interval by finding the midpoint, 1.5. We compute $f(1.5) \approx 5.09$. Once again, the Intermediate Value Theorem guarantees our zero is between 1 and 1.5, since f changes sign on this interval. Now, we 'bisect' the interval $[1, 1.5]$ and find $f(1.25) \approx 0.80$, so now we have the zero between 1 and 1.25. Bisecting $[1, 1.25]$, we find $f(1.125) \approx -0.32$, which means the zero of f is between 1.125 and 1.25. We continue in this fashion until we have 'sandwiched' the zero between two numbers whose digits agree to a desired amount.[6] You can think of the Bisection Method as reversing the sign diagram process: instead of finding the zeros and checking the sign of f using test values, we are using test values to determine where the signs switch to find the zeros. It is a slow and tedious, yet fool-proof, method for *approximating* a real zero when the other analytical methods fail us.

[5] See this page.

[6] We ask you to approximate this zero to three decimal places using the Bisection Method in Exercise 64.

6.3.4 Exercises

In Exercises 1 - 10, for the given polynomial:

- Use Cauchy's Bound to find an interval containing all of the real zeros.
- Use the Rational Zeros Theorem to make a list of possible rational zeros.
- Use Descartes' Rule of Signs to list the possible number of positive and negative real zeros, counting multiplicities.

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
3. $p(z) = z^4 - 9z^2 - 4z + 12$
4. $p(z) = z^3 + 4z^2 - 11z + 6$
5. $g(t) = t^3 - 7t^2 + t - 7$
6. $g(t) = -2t^3 + 19t^2 - 49t + 20$
7. $f(x) = -17x^3 + 5x^2 + 34x - 10$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
9. $p(z) = 3z^3 + 3z^2 - 11z - 10$
10. $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$

In Exercises 11 - 30, find the real zeros of the polynomial using the techniques specified by your instructor. State the multiplicity of each real zero.

11. $f(x) = x^3 - 2x^2 - 5x + 6$
12. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
13. $p(z) = z^4 - 9z^2 - 4z + 12$
14. $p(z) = z^3 + 4z^2 - 11z + 6$
15. $g(t) = t^3 - 7t^2 + t - 7$
16. $g(t) = -2t^3 + 19t^2 - 49t + 20$
17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
18. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
19. $p(z) = 3z^3 + 3z^2 - 11z - 10$
20. $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$
21. $g(t) = 9t^3 - 5t^2 - t$
22. $g(t) = 6t^4 - 5t^3 - 9t^2$
23. $f(x) = x^4 + 2x^2 - 15$
24. $f(x) = x^4 - 9x^2 + 14$
25. $p(z) = 3z^4 - 14z^2 - 5$
26. $p(z) = 2z^4 - 7z^2 + 6$
27. $g(t) = t^6 - 3t^3 - 10$
28. $g(t) = 2t^6 - 9t^3 + 10$
29. $f(x) = x^5 - 2x^4 - 4x + 8$
30. $f(x) = 2x^5 + 3x^4 - 18x - 27$

In Exercises 31 - 33, use your calculator,[7] to help you find the real zeros of the polynomial. State the multiplicity of each real zero.

31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$

32. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$

33. $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$

34. Find the real zeros of $f(x) = x^3 - \frac{1}{12}x^2 - \frac{7}{72}x + \frac{1}{72}$ by first finding a polynomial $q(x)$ with integer coefficients such that $q(x) = N \cdot f(x)$ for some integer N. (Recall that the Rational Zeros Theorem required the polynomial in question to have integer coefficients.) Show that f and q have the same real zeros.

In Exercises 35 - 44, find the real solutions of the polynomial equation. (See Example 6.3.7.)

35. $9x^3 = 5x^2 + x$

36. $9x^2 + 5x^3 = 6x^4$

37. $z^3 + 6 = 2z^2 + 5z$

38. $z^4 + 2z^3 = 12z^2 + 40z + 32$

39. $t^3 - 7t^2 = 7 - t$

40. $2t^3 = 19t^2 - 49t + 20$

41. $x^3 + x^2 = \dfrac{11x + 10}{3}$

42. $x^4 + 2x^2 = 15$

43. $14z^2 + 5 = 3z^4$

44. $2z^5 + 3z^4 = 18z + 27$

In Exercises 45 - 54, solve the polynomial inequality and state your answer using interval notation.

45. $-2x^3 + 19x^2 - 49x + 20 > 0$

46. $x^4 - 9x^2 \leq 4x - 12$

47. $(z - 1)^2 \geq 4$

48. $4z^3 \geq 3z + 1$

49. $t^4 \leq 16 + 4t - t^3$

50. $3t^2 + 2t < t^4$

51. $\dfrac{x^3 + 2x^2}{2} < x + 2$

52. $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$

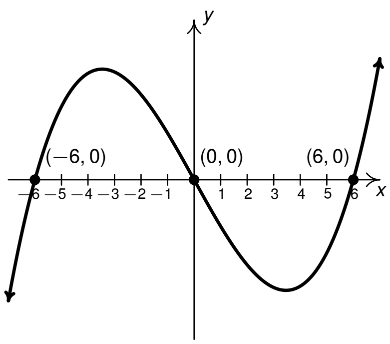
53. $2z^4 > 5z^2 + 3$

54. $z^6 + z^3 \geq 6$

[7] You *can* do these without your calculator, but it may test your mettle!

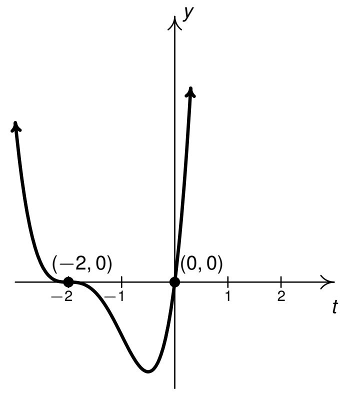
In Exercises 55 - 60, use the the graph of the given polynomial function to solve the stated inequality.

55. Solve $f(x) < 0$.



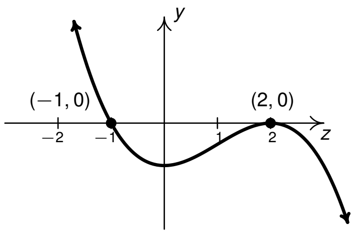
$y = f(x)$

56. Solve $g(t) > 0$.



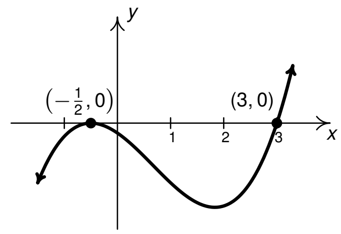
$y = g(t)$

57. Solve $p(z) \geq 0$



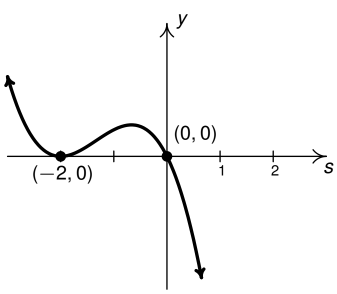
$y = p(z)$

58. Solve $f(x) < 0$.



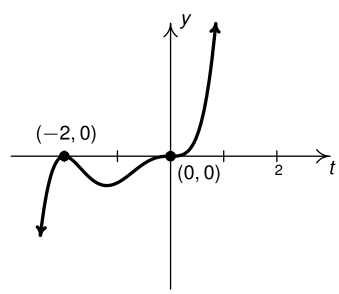
$y = f(x)$

59. Solve $F(s) \leq 0$.



$y = F(s)$

60. Solve $G(t) \geq 0$.



$y = G(t)$

61. Use the Intermediate Value Theorem, Theorem 6.14 to prove that $f(x) = x^3 - 9x + 5$ has a real zero in each of the following intervals: $[-4, -3]$, $[0, 1]$ and $[2, 3]$.

62. Use the concepts of End Behavior and the Intermediate Value Theorem to prove any odd-degree polynomial function with real number coefficients has at least one real zero.

63. Find an even-degree polynomial function with real number coefficients which has no real zeros.

64. Continue the Bisection Method as introduced on 341 to approximate the real zero of $f(x) = x^5 - x - 1$ to three decimal places.

65. In this exercise, we prove $\sqrt{2}$ is an irrational number and approximate its value. Let $f(x) = x^2 - 2$.

    (a) Use Decartes' Rule of Signs to prove f has exactly one positive real zero.

    (b) Use the Intermediate Value Theorem to prove f has a zero in $[1, 2]$.

    (c) Use the Rational Zeros Theorem to prove f has no rational zeros.

    (d) Use the Bisection Method (see 341) to approximate the zero of f on $[1, 2]$ to three decimal places.

66. Generalize the argument given in Exercise 65c to prove:

    (a) If N is not the perfect square of an integer, then $\sqrt{N}$ is irrational. (HINT: Consider $f(x) = x^2 - N$.)

    (b) For natural numbers $n \geq 2$, if N is not the perfect n^{th} power of an integer, then $\sqrt[n]{N}$ is irrational. (HINT: Consider $f(x) = x^n - N$.)

67. In Example 6.1.4 in Section 6.1, a box with no top is constructed from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. We determined the volume of that box (in cubic inches) is given by the function $V(x) = 4x^3 - 44x^2 + 120x$, where x denotes the length of the side of the square which is removed from each corner (in inches), $0 < x < 5$. Solve the inequality $V(x) \geq 80$ analytically and interpret your answer in the context of that example.

68. From Exercise 55 in Section 6.1, $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$ models the cost, in dollars, to produce x PortaBoy game systems. If the production budget is \$5000, find the number of game systems which can be produced and still remain under budget.

69. Let $f(x) = 5x^7 - 33x^6 + 3x^5 - 71x^4 - 597x^3 + 2097x^2 - 1971x + 567$. With the help of your classmates, find the x- and y- intercepts of the graph of f. Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Sketch the graph of f, using more than one picture if necessary to show all of the important features of the graph.

70. With the help of your classmates, create a list of five polynomials with different degrees whose real zeros cannot be found using any of the techniques in this section.

6.3.5 Answers

1. For $f(x) = x^3 - 2x^2 - 5x + 6$
   - All of the real zeros lie in the interval $[-7, 7]$
   - Possible rational zeros are ± 1, ± 2, ± 3, ± 6
   - There are 2 or 0 positive real zeros; there is 1 negative real zero
2. For $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
   - All of the real zeros lie in the interval $[-41, 41]$
   - Possible rational zeros are ± 1, ± 2, ± 4, ± 8, ± 16, ± 32
   - There is 1 positive real zero; there are 3 or 1 negative real zeros
3. For $p(z) = z^4 - 9z^2 - 4z + 12$
   - All of the real zeros lie in the interval $[-13, 13]$
   - Possible rational zeros are ± 1, ± 2, ± 3, ± 4, ± 6, ± 12
   - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros
4. For $p(z) = z^3 + 4z^2 - 11z + 6$
   - All of the real zeros lie in the interval $[-12, 12]$
   - Possible rational zeros are ± 1, ± 2, ± 3, ± 6
   - There are 2 or 0 positive real zeros; there is 1 negative real zero
5. For $g(t) = t^3 - 7t^2 + t - 7$
   - All of the real zeros lie in the interval $[-8, 8]$
   - Possible rational zeros are ± 1, ± 7
   - There are 3 or 1 positive real zeros; there are no negative real zeros
6. For $g(t) = -2t^3 + 19t^2 - 49t + 20$
   - All of the real zeros lie in the interval $\left[-\frac{51}{2}, \frac{51}{2}\right]$
   - Possible rational zeros are $\pm\frac{1}{2}$, ± 1, ± 2, $\pm\frac{5}{2}$, ± 4, ± 5, ± 10, ± 20
   - There are 3 or 1 positive real zeros; there are no negative real zeros
7. For $f(x) = -17x^3 + 5x^2 + 34x - 10$
   - All of the real zeros lie in the interval $[-3, 3]$
   - Possible rational zeros are $\pm\frac{1}{17}$, $\pm\frac{2}{17}$, $\pm\frac{5}{17}$, $\pm\frac{10}{17}$, ± 1, ± 2, ± 5, ± 10
   - There are 2 or 0 positive real zeros; there is 1 negative real zero

8. For $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
   - All of the real zeros lie in the interval $\left[-\frac{4}{3}, \frac{4}{3}\right]$
   - Possible rational zeros are $\pm\frac{1}{36}, \pm\frac{1}{18}, \pm\frac{1}{12}, \pm\frac{1}{9}, \pm\frac{1}{6}, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1$
   - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros

9. For $p(z) = 3z^3 + 3z^2 - 11z - 10$
   - All of the real zeros lie in the interval $\left[-\frac{14}{3}, \frac{14}{3}\right]$
   - Possible rational zeros are $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm 1, \pm 2, \pm 5, \pm 10$
   - There is 1 positive real zero; there are 2 or 0 negative real zeros

10. For $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$
    - All of the real zeros lie in the interval $\left[-\frac{9}{2}, \frac{9}{2}\right]$
    - Possible rational zeros are $\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3$
    - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros

11. $f(x) = x^3 - 2x^2 - 5x + 6$
    $x = -2$, $x = 1$, $x = 3$ (each has mult. 1)

12. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
    $x = -2$ (mult. 3), $x = 4$ (mult. 1)

13. $p(z) = z^4 - 9z^2 - 4z + 12$
    $z = -2$ (mult. 2), $z = 1$ (mult. 1), $z = 3$ (mult. 1)

14. $p(z) = z^3 + 4z^2 - 11z + 6$
    $z = -6$ (mult. 1), $z = 1$ (mult. 2)

15. $g(t) = t^3 - 7t^2 + t - 7$
    $t = 7$ (mult. 1)

16. $g(t) = -2t^3 + 19t^2 - 49t + 20$
    $t = \frac{1}{2}$, $t = 4$, $t = 5$ (each has mult. 1)

17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
    $x = \frac{5}{17}$, $x = \pm\sqrt{2}$ (each has mult. 1)

18. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
    $x = \frac{1}{2}$ (mult. 2), $x = -\frac{1}{3}$ (mult. 2)

19. $p(z) = 3z^3 + 3z^2 - 11z - 10$
    $z = -2$, $z = \frac{3\pm\sqrt{69}}{6}$ (each has mult. 1)

20. $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$
$z = -1$, $z = \frac{1}{2}$, $z = \pm\sqrt{3}$ (each mult. 1)

21. $g(t) = 9t^3 - 5t^2 - t$
$t = 0$, $t = \frac{5 \pm \sqrt{61}}{18}$ (each has mult. 1)

22. $g(t) = 6t^4 - 5t^3 - 9t^2$
$t = 0$ (mult. 2), $t = \frac{5 \pm \sqrt{241}}{12}$ (each has mult. 1)

23. $f(x) = x^4 + 2x^2 - 15$
$x = \pm\sqrt{3}$ (each has mult. 1)

24. $f(x) = x^4 - 9x^2 + 14$
$x = \pm\sqrt{2}$, $x = \pm\sqrt{7}$ (each has mult. 1)

25. $p(z) = 3z^4 - 14z^2 - 5$
$z = \pm\sqrt{5}$ (each has mult. 1)

26. $p(z) = 2z^4 - 7z^2 + 6$
$z = \pm\frac{\sqrt{6}}{2}$, $z = \pm\sqrt{2}$ (each has mult. 1)

27. $g(t) = t^6 - 3t^3 - 10$
$t = \sqrt[3]{-2} = -\sqrt[3]{2}$, $t = \sqrt[3]{5}$ (each has mult. 1)

28. $g(t) = 2t^6 - 9t^3 + 10$
$t = \frac{\sqrt[3]{20}}{2}$, $t = \sqrt[3]{2}$ (each has mult. 1)

29. $f(x) = x^5 - 2x^4 - 4x + 8$
$x = 2$, $x = \pm\sqrt{2}$ (each has mult. 1)

30. $f(x) = 2x^5 + 3x^4 - 18x - 27$
$x = -\frac{3}{2}$, $x = \pm\sqrt{3}$ (each has mult. 1)

31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$
$x = -4$ (mult. 3), $x = 6$ (mult. 2)

32. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$
$x = \frac{3}{5}$ (mult. 2), $x = 1$ (mult. 3)

33. $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$
$x = \frac{2}{3}$, $x = \frac{3}{2}$, $x = \frac{5}{3}$, $x = \frac{3}{5}$ (each has mult. 1)

34. We choose $q(x) = 72x^3 - 6x^2 - 7x + 1 = 72 \cdot f(x)$. Clearly $f(x) = 0$ if and only if $q(x) = 0$ so they have the same real zeros. In this case, $x = -\frac{1}{3}$, $x = \frac{1}{6}$ and $x = \frac{1}{4}$ are the real zeros of both f and q.

35. $x = 0, \frac{5 \pm \sqrt{61}}{18}$

36. $x = 0, \frac{5 \pm \sqrt{241}}{12}$

37. $z = -2, 1, 3$

38. $z = -2, 4$

39. $t = 7$

40. $t = \frac{1}{2}, 4, 5$

41. $x = -2, \frac{3 \pm \sqrt{69}}{6}$

42. $x = \pm\sqrt{3}$

43. $z = \pm\sqrt{5}$

44. $z = -\frac{3}{2}, \pm\sqrt{3}$

45. $(-\infty, \frac{1}{2}) \cup (4, 5)$

46. $\{-2\} \cup [1, 3]$

47. $(-\infty, -1] \cup [3, \infty)$

48. $\left\{-\dfrac{1}{2}\right\} \cup [1, \infty)$

49. $[-2, 2]$

50. $(-\infty, -1) \cup (-1, 0) \cup (2, \infty)$

51. $(-\infty, -2) \cup \left(-\sqrt{2}, \sqrt{2}\right)$

52. $\{2\} \cup [4, \infty)$

53. $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

54. $(-\infty, -\sqrt[3]{3}\,) \cup (\sqrt[3]{2}, \infty)$

55. $f(x) < 0$ on $(-\infty, -6) \cup (0, 6)$

56. $g(t) > 0$ on $(-\infty, -2) \cup (0, \infty)$

57. $p(z) \geq 0$ on $(-\infty, -1] \cup \{2\}$

58. $f(x) < 0$ on $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 3\right)$

59. $F(s) \leq 0$ on $\{-2\} \cup [0, \infty)$

60. $G(t) \geq 0$ on $\{-2\} \cup [0, \infty)$

61. Since $f(-4) = -23$, $f(-3) = 5$, $f(0) = 5$, $f(1) = -3$, $f(2) = -5$ and $f(3) = 5$ the Intermediate Value Theorem gives that $f(x) = x^3 - 9x + 5$ has real zeros in the intervals $[-4, -3]$, $[0, 1]$ and $[2, 3]$.

62. An odd degree polynomial function f has 'mismatched' end behavior. That is, the end behavior of $f(x)$ is either: $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$ or as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$. This means at some point, $f(x) > 0$ and at some other point $f(x) < 0$. The Intermediate Value Theorem guarantees at least one place where $f(x) = 0$.

63. The function $f(x) = x^2 + 1$ has no real zeros.

64. $x \approx 1.167$.

65. (a) $f(x)$ has only one variation in sign, so the result follows from Descartes' Rule of Signs.
    (b) $f(1) = -1 < 0$ and $f(2) = 2 > 0$ so the Intermediate Value Theorem promises a zero in $[1, 2]$.
    (c) The Rational Zeros Theorem gives the only possible rational zeros of f are ± 1 and ± 2. Since $f(\pm 1) = -1$ and $f(\pm 2) = 2$, f has no rational zeros.
    (d) The zero of f is $\sqrt{2} \approx 1.414$.

66. $V(x) \geq 80$ on $[1, 5-\sqrt{5}] \cup [5+\sqrt{5}, \infty)$. Only the portion $[1, 5-\sqrt{5}]$ lies in the applied domain, however. In the context of the problem, this says for the volume of the box to be at least 80 cubic inches, the square removed from each corner needs to have a side length of at least 1 inch, but no more than $5-\sqrt{5} \approx 2.76$ inches.

67. $C(x) \leq 5000$ on (approximately) $(-\infty, 82.18]$. The portion of this which lies in the applied domain is $(0, 82.18]$. Since x represents the number of game systems, we check $C(82) = 4983.04$ and $C(83) = 5078.11$, so to remain within the production budget, anywhere between 1 and 82 game systems can be produced.

Chapter 7

Rational Functions

7.1 Rational Expressions and Equations

We now turn our attention to rational expressions - that is, algebraic fractions - and equations which contain them. The reader is encouraged to keep in mind the properties of fractions listed on page 20 because we will need them along the way. Before we launch into reviewing the basic arithmetic operations of rational expressions, we take a moment to review how to simplify them properly. As with numeric fractions, we 'cancel common *factors*,' not common *terms*. That is, in order to simplify rational expressions, we first *factor* the numerator and denominator. For example:

$$\frac{x^4 + 5x^3}{x^3 - 25x} \neq \frac{x^4 + 5\cancel{x^3}}{\cancel{x^3} - 25x}$$

but, rather

$$\begin{aligned}
\frac{x^4 + 5x^3}{x^3 - 25x} &= \frac{x^3(x+5)}{x(x^2-25)} && \text{Factor G.C.F.} \\
&= \frac{x^3(x+5)}{x(x-5)(x+5)} && \text{Difference of Squares} \\
&= \frac{\overset{x^2}{\cancel{x^3}}\cancel{(x+5)}}{\cancel{x}(x-5)\cancel{(x+5)}} && \text{Cancel common factors} \\
&= \frac{x^2}{x-5}
\end{aligned}$$

This equivalence holds provided the factors being canceled aren't 0. Since a factor of x and a factor of $x+5$ were canceled, $x \neq 0$ and $x + 5 \neq 0$, so $x \neq -5$. We usually stipulate this as:

$$\frac{x^4 + 5x^3}{x^3 - 25x} = \frac{x^2}{x-5}, \qquad \text{provided } x \neq 0,\ x \neq -5$$

While we're talking about common mistakes, please notice that

$$\frac{5}{x^2+9} \neq \frac{5}{x^2} + \frac{5}{9}$$

Just like their numeric counterparts, you don't add algebraic fractions by *adding denominators* of fractions with *common numerators* - it's the other way around:[1]

$$\frac{x^2+9}{5} = \frac{x^2}{5} + \frac{9}{5}$$

It's time to review the basic arithmetic operations with rational expressions.

[1]One of the most common errors students make on college Mathematics placement tests is that they forget how to add algebraic fractions correctly. This places many students into remedial classes even though they are probably ready for college-level Math. We urge you to really study this section with great care so that you don't fall into that trap.

Example 7.1.1. Perform the indicated operations and simplify.

1. $\dfrac{2x^2-5x-3}{x^4-4} \div \dfrac{x^2-2x-3}{x^5+2x^3}$
2. $\dfrac{5}{w^2-9} - \dfrac{w+2}{w^2-9}$
3. $\dfrac{3}{y^2-8y+16} + \dfrac{y+1}{16y-y^3}$
4. $\dfrac{\dfrac{2}{4-(x+h)} - \dfrac{2}{4-x}}{h}$
5. $2t^{-3} - (3t)^{-2}$
6. $10x(x-3)^{-1} + 5x^2(-1)(x-3)^{-2}$

Solution.

1. As with numeric fractions, we divide rational expressions by 'inverting and multiplying'. Before we get too carried away however, we factor to see what, if any, factors cancel.

$$\begin{array}{rcll} \dfrac{2x^2-5x-3}{x^4-4} \div \dfrac{x^2-2x-3}{x^5+2x^3} & = & \dfrac{2x^2-5x-3}{x^4-4} \cdot \dfrac{x^5+2x^3}{x^2-2x-3} & \text{Invert and multiply} \\ & = & \dfrac{(2x^2-5x-3)(x^5+2x^3)}{(x^4-4)(x^2-2x-3)} & \text{Multiply fractions} \\ & = & \dfrac{(2x+1)(x-3)x^3(x^2+2)}{(x^2-2)(x^2+2)(x-3)(x+1)} & \text{Factor} \\ & = & \dfrac{(2x+1)\cancel{(x-3)}x^3\cancel{(x^2+2)}}{(x^2-2)\cancel{(x^2+2)}\cancel{(x-3)}(x+1)} & \text{Cancel common factors} \\ & = & \dfrac{x^3(2x+1)}{(x+1)(x^2-2)} & \text{Provided } x \neq 3 \end{array}$$

The '$x \neq 3$' is mentioned since a factor of $(x-3)$ was canceled as we reduced the expression. We also canceled a factor of (x^2+2). Why is there no stipulation as a result of canceling this factor? Because $x^2+2 \neq 0$ for all real x. (See Section **??** for details.) At this point, we *could* go ahead and multiply out the numerator and denominator to get

$$\frac{x^3(2x+1)}{(x+1)(x^2-2)} = \frac{2x^4+x^3}{x^3+x^2-2x-2}$$

but for most of the applications where this kind of algebra is needed (solving equations, for instance), it is best to leave things factored. Your instructor will let you know whether to leave your answer in factored form or not.[2]

[2]Speaking of factoring, do you remember why x^2-2 can't be factored over the integers?

2. As with numeric fractions we need common denominators in order to subtract. This is already the case here so we proceed by subtracting the numerators.

$$\begin{aligned} \frac{5}{w^2-9} - \frac{w+2}{w^2-9} &= \frac{5-(w+2)}{w^2-9} && \text{Subtract fractions} \\ &= \frac{5-w-2}{w^2-9} && \text{Distribute} \\ &= \frac{3-w}{w^2-9} && \text{Combine like terms} \end{aligned}$$

At this point, we need to see if we can reduce this expression so we proceed to factor. It first appears as if we have no common factors among the numerator and denominator until we recall the property of 'factoring negatives' from Page 19: $3 - w = -(w - 3)$. This yields:

$$\begin{aligned} \frac{3-w}{w^2-9} &= \frac{-(w-3)}{(w-3)(w+3)} && \text{Factor} \\ &= \frac{-\cancel{(w-3)}}{\cancel{(w-3)}(w+3)} && \text{Cancel common factors} \\ &= \frac{-1}{w+3} && \text{Provided } w \neq 3 \end{aligned}$$

The stipulation $w \neq 3$ comes from the cancellation of the $(w - 3)$ factor.

3. In this next example, we are asked to add two rational expressions with *different* denominators. As with numeric fractions, we must first find a *common denominator*. To do so, we start by factoring each of the denominators.

$$\begin{aligned} \frac{3}{y^2-8y+16} + \frac{y+1}{16y-y^3} &= \frac{3}{(y-4)^2} + \frac{y+1}{y(16-y^2)} && \text{Factor} \\ &= \frac{3}{(y-4)^2} + \frac{y+1}{y(4-y)(4+y)} && \text{Factor some more} \end{aligned}$$

To find the common denominator, we examine the factors in the first denominator and note that we need a factor of $(y - 4)^2$. We now look at the second denominator to see what other factors we need. We need a factor of y and $(4 + y) = (y + 4)$. What about $(4 - y)$? As mentioned in the last example, we can factor this as: $(4 - y) = -(y - 4)$. Using properties of negatives, we 'migrate' this negative out to the front of the fraction, turning the addition into subtraction. We find the (least) common denominator to be $(y - 4)^2y(y + 4)$. We can now proceed to multiply the numerator and denominator of each fraction by whatever factors are missing from their respective denominators to produce equivalent expressions with common denominators.

$$\begin{aligned}
\frac{3}{(y-4)^2} + \frac{y+1}{y(4-y)(4+y)} &= \frac{3}{(y-4)^2} + \frac{y+1}{y(-(y-4))(y+4)} \\
&= \frac{3}{(y-4)^2} - \frac{y+1}{y(y-4)(y+4)} \\
&= \frac{3}{(y-4)^2} \cdot \frac{y(y+4)}{y(y+4)} - \frac{y+1}{y(y-4)(y+4)} \cdot \frac{(y-4)}{(y-4)} && \text{Equivalent Fractions} \\
&= \frac{3y(y+4)}{(y-4)^2 y(y+4)} - \frac{(y+1)(y-4)}{y(y-4)^2(y+4)} && \text{Multiply Fractions}
\end{aligned}$$

At this stage, we can subtract numerators and simplify. We'll keep the denominator factored (in case we can reduce down later), but in the numerator, since there are no common factors, we proceed to perform the indicated multiplication and combine like terms.

$$\begin{aligned}
\frac{3y(y+4)}{(y-4)^2 y(y+4)} - \frac{(y+1)(y-4)}{y(y-4)^2(y+4)} &= \frac{3y(y+4) - (y+1)(y-4)}{(y-4)^2 y(y+4)} && \text{Subtract numerators} \\
&= \frac{3y^2 + 12y - (y^2 - 3y - 4)}{(y-4)^2 y(y+4)} && \text{Distribute} \\
&= \frac{3y^2 + 12y - y^2 + 3y + 4}{(y-4)^2 y(y+4)} && \text{Distribute} \\
&= \frac{2y^2 + 15y + 4}{y(y+4)(y-4)^2} && \text{Gather like terms}
\end{aligned}$$

We would like to factor the numerator and cancel factors it has in common with the denominator. After a few attempts, it appears as if the numerator doesn't factor, at least over the integers. As a check, we compute the discriminant of $2y^2 + 15y + 4$ and get $15^2 - 4(2)(4) = 193$. This isn't a perfect square so we know that the quadratic equation $2y^2 + 15y + 4 = 0$ has irrational solutions. This means $2y^2 + 15y + 4$ can't factor over the integers[3] so we are done.

4. In this example, we have a compound fraction, and we proceed to simplify it as we did its numeric counterparts in Example 1.2.1. Specifically, we start by multiplying the numerator and denominator of the 'big' fraction by the least common denominator of the 'little' fractions inside of it - in this case we need to use $(4 - (x + h))(4 - x)$ - to remove the compound nature of the 'big' fraction. Once we

[3]See the remarks following Theorem 5.6.

have a more normal looking fraction, we can proceed as we have in the previous examples.

$$
\begin{aligned}
\frac{\dfrac{2}{4-(x+h)}-\dfrac{2}{4-x}}{h} &= \frac{\left(\dfrac{2}{4-(x+h)}-\dfrac{2}{4-x}\right)}{h}\cdot\frac{(4-(x+h))(4-x)}{(4-(x+h))(4-x)} && \text{Equivalent fractions}\\
&= \frac{\left(\dfrac{2}{4-(x+h)}-\dfrac{2}{4-x}\right)\cdot(4-(x+h))(4-x)}{h(4-(x+h))(4-x)} && \text{Multiply}\\
&= \frac{\dfrac{2(4-(x+h))(4-x)}{4-(x+h)}-\dfrac{2(4-(x+h))(4-x)}{4-x}}{h(4-(x+h))(4-x)} && \text{Distribute}\\
&= \frac{\dfrac{2\cancel{(4-(x+h))}(4-x)}{\cancel{(4-(x+h))}}-\dfrac{2(4-(x+h))\cancel{(4-x)}}{\cancel{(4-x)}}}{h(4-(x+h))(4-x)} && \text{Reduce}\\
&= \frac{2(4-x)-2(4-(x+h))}{h(4-(x+h))(4-x)}
\end{aligned}
$$

Now we can clean up and factor the numerator to see if anything cancels. (This why we kept the denominator factored.)

$$
\begin{aligned}
\frac{2(4-x)-2(4-(x+h))}{h(4-(x+h))(4-x)} &= \frac{2[(4-x)-(4-(x+h))]}{h(4-(x+h))(4-x)} && \text{Factor out G.C.F.}\\
&= \frac{2[4-x-4+(x+h)]}{h(4-(x+h))(4-x)} && \text{Distribute}\\
&= \frac{2[4-4-x+x+h]}{h(4-(x+h))(4-x)} && \text{Rearrange terms}\\
&= \frac{2h}{h(4-(x+h))(4-x)} && \text{Gather like terms}\\
&= \frac{2\cancel{h}}{\cancel{h}(4-(x+h))(4-x)} && \text{Reduce}\\
&= \frac{2}{(4-(x+h))(4-x)} && \text{Provided } h\neq 0
\end{aligned}
$$

Your instructor will let you know if you are to expand the denominator or not.[4]

5. At first glance, it doesn't seem as if there is anything that can be done with $2t^{-3}-(3t)^{-2}$ because the exponents on the variables are different. However, since the exponents are negative, these are actually rational expressions. In the first term, the -3 exponent applies to the t *only* but in the second

[4]We'll keep it factored because in Calculus it needs to be factored.

term, the exponent -2 applies to *both* the 3 and the t, as indicated by the parentheses. One way to proceed is as follows:

$$\begin{aligned} 2t^{-3} - (3t)^{-2} &= \frac{2}{t^3} - \frac{1}{(3t)^2} \\ &= \frac{2}{t^3} - \frac{1}{9t^2} \end{aligned}$$

We see that we are being asked to subtract two rational expressions with different denominators, so we need to find a common denominator. The first fraction contributes a t^3 to the denominator, while the second contributes a factor of 9. Thus our common denominator is $9t^3$, so we are missing a factor of '9' in the first denominator and a factor of 't' in the second.

$$\begin{aligned} \frac{2}{t^3} - \frac{1}{9t^2} &= \frac{2}{t^3} \cdot \frac{9}{9} - \frac{1}{9t^2} \cdot \frac{t}{t} && \text{Equivalent Fractions} \\ &= \frac{18}{9t^3} - \frac{t}{9t^3} && \text{Multiply} \\ &= \frac{18 - t}{9t^3} && \text{Subtract} \end{aligned}$$

We find no common factors among the numerator and denominator so we are done.

A second way to approach this problem is by factoring. We can extend the concept of the 'Polynomial G.C.F.' to these types of expressions and we can follow the same guidelines as set forth on page 231 to factor out the G.C.F. of these two terms. The key ideas to remember are that we take out each factor with the *smallest* exponent and that factoring is the same as dividing. We first note that $2t^{-3} - (3t)^{-2} = 2t^{-3} - 3^{-2}t^{-2}$ and we see that the smallest power on t is -3. Thus we want to factor out t^{-3} from both terms. It's clear that this will leave 2 in the first term, but what about the second term? Since factoring is the same as dividing, we would be dividing the second term by t^{-3} which thanks to the properties of exponents is the same as *multiplying* by $\frac{1}{t^{-3}} = t^3$. The same holds for 3^{-2}. Even though there are no factors of 3 in the first term, we can factor out 3^{-2} by multiplying it by $\frac{1}{3^{-2}} = 3^2 = 9$. We put these ideas together below.

$$\begin{aligned} 2t^{-3} - (3t)^{-2} &= 2t^{-3} - 3^{-2}t^{-2} && \text{Properties of Exponents} \\ &= 3^{-2}t^{-3}(2(3)^2 - t^1) && \text{Factor} \\ &= \frac{1}{3^2}\frac{1}{t^3}(18 - t) && \text{Rewrite} \\ &= \frac{18 - t}{9t^3} && \text{Multiply} \end{aligned}$$

While both ways are valid, one may be more of a natural fit than the other depending on the circumstances and temperament of the student.

6. As with the previous example, we show two different yet equivalent ways to approach simplifying $10x(x-3)^{-1} + 5x^2(-1)(x-3)^{-2}$. First up is what we'll call the 'common denominator approach' where we rewrite the negative exponents as fractions and proceed from there.

- *Common Denominator Approach*:

$$\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= \frac{10x}{x-3}+\frac{5x^2(-1)}{(x-3)^2} \\
&= \frac{10x}{x-3}\cdot\frac{x-3}{x-3}-\frac{5x^2}{(x-3)^2} && \text{Equivalent Fractions} \\
&= \frac{10x(x-3)}{(x-3)^2}-\frac{5x^2}{(x-3)^2} && \text{Multiply} \\
&= \frac{10x(x-3)-5x^2}{(x-3)^2} && \text{Subtract} \\
&= \frac{5x(2(x-3)-x)}{(x-3)^2} && \text{Factor out G.C.F.} \\
&= \frac{5x(2x-6-x)}{(x-3)^2} && \text{Distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Combine like terms}
\end{aligned}$$

  Both the numerator and the denominator are completely factored with no common factors so we are done.

- *'Factoring Approach'*: In this case, the G.C.F. is $5x(x-3)^{-2}$. Factoring this out of both terms gives:

$$\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= 5x(x-3)^{-2}(2(x-3)^1-x) && \text{Factor} \\
&= \frac{5x}{(x-3)^2}(2x-6-x) && \text{Rewrite, distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Multiply}
\end{aligned}$$

  As expected, we got the same reduced fraction as before. □

Next, we review the solving of equations which involve rational expressions. As with equations involving numeric fractions, our first step in solving equations with algebraic fractions is to clear denominators. In doing so, we run the risk of introducing what are known as **extraneous** solutions - 'answers' which don't satisfy the original equation. As we illustrate the techniques used to solve these basic equations, see if you can find the step which creates the problem for us.

Example 7.1.2. Solve the following equations.

1. $1+\dfrac{1}{x}=x$
2. $\dfrac{t^3-2t+1}{t-1}=\dfrac{1}{2}t-1$
3. $\dfrac{3}{1-w\sqrt{2}}-\dfrac{1}{2w+5}=0$
4. $3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x)=0$
5. Solve $x=\dfrac{2y+1}{y-3}$ for y.
6. Solve $\dfrac{1}{f}=\dfrac{1}{S_1}+\dfrac{1}{S_2}$ for S_1.

Solution.

1. Our first step is to clear the fractions by multiplying both sides of the equation by x. In doing so, we are implicitly assuming $x\neq 0$; otherwise, we would have no guarantee that the resulting equation is equivalent to our original equation.[5]

$$\begin{array}{rcll}
1+\dfrac{1}{x} & = & x & \\
\left(1+\dfrac{1}{x}\right)x & = & (x)x & \text{Provided } x\neq 0 \\
1(x)+\dfrac{1}{x}(x) & = & x^2 & \text{Distribute} \\
x+\dfrac{x}{x} & = & x^2 & \text{Multiply} \\
x+1 & = & x^2 & \\
0 & = & x^2-x-1 & \text{Subtract } x\text{, subtract } 1 \\
x & = & \dfrac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)} & \text{Quadratic Formula} \\
x & = & \dfrac{1\pm\sqrt{5}}{2} & \text{Simplify}
\end{array}$$

We obtain two answers, $x=\frac{1\pm\sqrt{5}}{2}$. Neither of these are 0 thus neither contradicts our assumption that $x\neq 0$. The reader is invited to check both of these solutions.[6]

[5] See page ??.

[6] The check relies on being able to 'rationalize' the denominator - a skill we haven't reviewed yet. (Come back after you've read Section 8.1.1 if you want to!) Additionally, the positive solution to this equation is the famous Golden Ratio.

2. To solve the equation, we clear denominators. Here, we need to assume $t-1\neq 0$, or $t\neq 1$.

$$
\begin{array}{rcll}
\dfrac{t^3-2t+1}{t-1} & = & \dfrac{1}{2}t-1 & \\
\left(\dfrac{t^3-2t+1}{t-1}\right)\cdot 2(t-1) & = & \left(\dfrac{1}{2}t-1\right)\cdot 2(t-1) & \text{Provided } t\neq 1\\
\dfrac{(t^3-2t+1)(2\cancel{(t-1)})}{\cancel{(t-1)}} & = & \dfrac{1}{\cancel{2}}t(\cancel{2}(t-1))-1(2(t-1)) & \text{Multiply, distribute}\\
2(t^3-2t+1) & = & t^2-t-2t+2 & \text{Distribute}\\
2t^3-4t+2 & = & t^2-3t+2 & \text{Distribute, combine like terms}\\
2t^3-t^2-t & = & 0 & \text{Subtract } t^2\text{, add } 3t\text{, subtract } 2\\
t(2t^2-t-1) & = & 0 & \text{Factor}\\
t=0 & \text{or} & 2t^2-t-1=0 & \text{Zero Product Property}\\
t=0 & \text{or} & (2t+1)(t-1)=0 & \text{Factor}\\
t=0 & \text{or} & 2t+1=0 \quad \text{or} \quad t-1=0 & \\
t & = & 0,\ -\dfrac{1}{2} \text{ or } 1 &
\end{array}
$$

We assumed that $t\neq 1$ in order to clear denominators. Sure enough, the candidate $t=1$ doesn't check in the original equation since it causes division by 0. In this case, we call $t=1$ an *extraneous* solution. Note that $t=1$ *does* work in every equation *after* we clear denominators. In general, multiplying by variable expressions can produce these 'extra' solutions, which is why checking our answers is always encouraged.[7] The other two candidates, $t=0$ and $t=-\frac{1}{2}$, are solutions.

3. As before, we begin by clearing denominators. Here, we assume $1-w\sqrt{2}\neq 0$ (so $w\neq\frac{1}{\sqrt{2}}$) and $2w+5\neq 0$ (so $w\neq-\frac{5}{2}$).

$$
\begin{array}{rcll}
\dfrac{3}{1-w\sqrt{2}}-\dfrac{1}{2w+5} & = & 0 & \\
\left(\dfrac{3}{1-w\sqrt{2}}-\dfrac{1}{2w+5}\right)(1-w\sqrt{2})(2w+5) & = & 0(1-w\sqrt{2})(2w+5) & w\neq\dfrac{1}{\sqrt{2}},-\dfrac{5}{2}\\
\dfrac{3\cancel{(1-w\sqrt{2})}(2w+5)}{\cancel{(1-w\sqrt{2})}}-\dfrac{1(1-w\sqrt{2})\cancel{(2w+5)}}{\cancel{(2w+5)}} & = & 0 & \text{Distribute}\\
3(2w+5)-(1-w\sqrt{2}) & = & 0 &
\end{array}
$$

The result is a *linear* equation in w so we gather the terms with w on one side of the equation and

[7]Contrast this with what happened in Example 5.2.3 when we divided by a variable and 'lost' a solution.

put everything else on the other. We factor out w and divide by its coefficient.

$$\begin{array}{rcll}
3(2w+5) - (1 - w\sqrt{2}) & = & 0 & \\
6w + 15 - 1 + w\sqrt{2} & = & 0 & \text{Distribute} \\
6w + w\sqrt{2} & = & -14 & \text{Subtract 14} \\
(6+\sqrt{2})w & = & -14 & \text{Factor} \\
w & = & -\dfrac{14}{6+\sqrt{2}} & \text{Divide by } 6+\sqrt{2}
\end{array}$$

This solution is different than our excluded values, $\frac{1}{\sqrt{2}}$ and $-\frac{5}{2}$, so we keep $w = -\frac{14}{6+\sqrt{2}}$ as our final answer. The reader is invited to check this in the original equation.

4. To solve our next equation, we have two approaches to choose from: we could rewrite the quantities with negative exponents as fractions and clear denominators, or we can factor. We showcase each technique below.

- *Clearing Denominators Approach*: We rewrite the negative exponents as fractions and clear denominators. In this case, we multiply both sides of the equation by $(x^2+4)^2$, which is never 0. (Think about that for a moment.) As a result, we need not exclude any x values from our solution set.

$$\begin{array}{rcll}
3(x^2+4)^{-1} + 3x(-1)(x^2+4)^{-2}(2x) & = & 0 & \\
\dfrac{3}{x^2+4} + \dfrac{3x(-1)(2x)}{(x^2+4)^2} & = & 0 & \text{Rewrite} \\
\left(\dfrac{3}{x^2+4} - \dfrac{6x^2}{(x^2+4)^2}\right)(x^2+4)^2 & = & 0(x^2+4)^2 & \text{Multiply} \\
\dfrac{3\cancel{(x^2+4)^2}^{(x^2+4)}}{\cancel{(x^2+4)}} - \dfrac{6x^2\cancel{(x^2+4)^2}}{\cancel{(x^2+4)^2}} & = & 0 & \text{Distribute} \\
3(x^2+4) - 6x^2 & = & 0 & \\
3x^2 + 12 - 6x^2 & = & 0 & \text{Distribute} \\
-3x^2 & = & -12 & \text{Combine like terms, subtract 12} \\
x^2 & = & 4 & \text{Divide by } -3 \\
x & = & \pm\sqrt{4} = \pm 2 & \text{Extract square roots}
\end{array}$$

We leave it to the reader to show that both $x = -2$ and $x = 2$ satisfy the original equation.

- *Factoring Approach*: Since the equation is already set equal to 0, we're ready to factor. Following the guidelines presented in Example 7.1.1, we factor out $3(x^2+4)^{-2}$ from both terms and look to see if more factoring can be done:

$$\begin{array}{rcll}
3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x) & = & 0 & \\
3(x^2+4)^{-2}((x^2+4)^1+x(-1)(2x)) & = & 0 & \text{Factor} \\
3(x^2+4)^{-2}(x^2+4-2x^2) & = & 0 & \\
3(x^2+4)^{-2}(4-x^2) & = & 0 & \text{Gather like terms} \\
3(x^2+4)^{-2}=0 & \text{or} & 4-x^2=0 & \text{Zero Product Property} \\
\dfrac{3}{x^2+4}=0 & \text{or} & 4=x^2 &
\end{array}$$

The first equation yields no solutions (Think about this for a moment.) while the second gives us $x=\pm\sqrt{4}=\pm 2$ as before.

5. We are asked to solve this equation for y so we begin by clearing fractions with the stipulation that $y-3\neq 0$ or $y\neq 3$. We are left with a linear equation in the variable y. To solve this, we gather the terms containing y on one side of the equation and everything else on the other. Next, we factor out the y and divide by its coefficient, which in this case turns out to be $x-2$. In order to divide by $x-2$, we stipulate $x-2\neq 0$ or, said differently, $x\neq 2$.

$$\begin{array}{rcll}
x & = & \dfrac{2y+1}{y-3} & \\
x(y-3) & = & \left(\dfrac{2y+1}{y-3}\right)(y-3) & \text{Provided } y\neq 3 \\
xy-3x & = & \dfrac{(2y+1)\cancel{(y-3)}}{\cancel{(y-3)}} & \text{Distribute, multiply} \\
xy-3x & = & 2y+1 & \\
xy-2y & = & 3x+1 & \text{Add } 3x\text{, subtract } 2y \\
y(x-2) & = & 3x+1 & \text{Factor} \\
y & = & \dfrac{3x+1}{x-2} & \text{Divide provided } x\neq 2
\end{array}$$

We highly encourage the reader to check the answer algebraically to see where the restrictions on x and y come into play.[8]

6. Our last example comes from physics and the world of photography.[9] We take a moment here to note that while superscripts in Mathematics indicate exponents (powers), subscripts are used primarily to distinguish one or more variables. In this case, S_1 and S_2 are two *different* variables (much like x and y) and we treat them as such. Our first step is to clear denominators by multiplying both sides by fS_1S_2 - provided each is nonzero. We end up with an equation which is linear in S_1 so we proceed

[8] It involves simplifying a compound fraction!

[9] See this article on focal length.

as in the previous example.

$$\begin{array}{rcll}
\dfrac{1}{f} & = & \dfrac{1}{S_1} + \dfrac{1}{S_2} & \\
\left(\dfrac{1}{f}\right)(fS_1S_2) & = & \left(\dfrac{1}{S_1} + \dfrac{1}{S_2}\right)(fS_1S_2) & \text{Provided } f \neq 0,\, S_1 \neq 0,\, S_2 \neq 0 \\
\dfrac{fS_1S_2}{f} & = & \dfrac{fS_1S_2}{S_1} + \dfrac{fS_1S_2}{S_2} & \text{Multiply, distribute} \\
\dfrac{\cancel{f}S_1S_2}{\cancel{f}} & = & \dfrac{f\cancel{S_1}S_2}{\cancel{S_1}} + \dfrac{fS_1\cancel{S_2}}{\cancel{S_2}} & \text{Cancel} \\
S_1S_2 & = & fS_2 + fS_1 & \\
S_1S_2 - fS_1 & = & fS_2 & \text{Subtract } fS_1 \\
S_1(S_2 - f) & = & fS_2 & \text{Factor} \\
S_1 & = & \dfrac{fS_2}{S_2 - f} & \text{Divide provided } S_2 \neq f
\end{array}$$

As always, the reader is highly encouraged to check the answer.[10] □

[10] . . . and see what the restriction $S_2 \neq f$ means in terms of focusing a camera!

7.1.1 Exercises

In Exercises 1 - 18, perform the indicated operations and simplify.

1. $\dfrac{x^2-9}{x^2} \cdot \dfrac{3x}{x^2-x-6}$

2. $\dfrac{t^2-2t}{t^2+1} \div (3t^2-2t-8)$

3. $\dfrac{4y-y^2}{2y+1} \div \dfrac{y^2-16}{2y^2-5y-3}$

4. $\dfrac{x}{3x-1} - \dfrac{1-x}{3x-1}$

5. $\dfrac{2}{w-1} - \dfrac{w^2+1}{w-1}$

6. $\dfrac{2-y}{3y} - \dfrac{1-y}{3y} + \dfrac{y^2-1}{3y}$

7. $b + \dfrac{1}{b-3} - 2$

8. $\dfrac{2x}{x-4} - \dfrac{1}{2x+1}$

9. $\dfrac{m^2}{m^2-4} + \dfrac{1}{2-m}$

10. $\dfrac{\frac{2}{x}-2}{x-1}$

11. $\dfrac{\frac{3}{2-h}-\frac{3}{2}}{h}$

12. $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$

13. $3w^{-1}-(3w)^{-1}$

14. $-2y^{-1}+2(3-y)^{-2}$

15. $3(x-2)^{-1}-3x(x-2)^{-2}$

16. $\dfrac{t^{-1}+t^{-2}}{t^{-3}}$

17. $\dfrac{2(3+h)^{-2}-2(3)^{-2}}{h}$

18. $\dfrac{(7-x-h)^{-1}-(7-x)^{-1}}{h}$

In Exercises 19 - 27, find all real solutions. Be sure to check for extraneous solutions.

19. $\dfrac{x}{5x+4} = 3$

20. $\dfrac{3y-1}{y^2+1} = 1$

21. $\dfrac{1}{w+3} + \dfrac{1}{w-3} = \dfrac{w^2-3}{w^2-9}$

22. $\dfrac{2x+17}{x+1} = x+5$

23. $\dfrac{t^2-2t+1}{t^3+t^2-2t} = 1$

24. $\dfrac{-y^3+4y}{y^2-9} = 4y$

25. $w+\sqrt{3} = \dfrac{3w-w^3}{w-\sqrt{3}}$

26. $\dfrac{2}{x\sqrt{2}-1} - 1 = \dfrac{3}{x\sqrt{2}+1}$

27. $\dfrac{x^2}{(1+x\sqrt{3})^2} = 3$

In Exercises 28 - 30, use Theorem 4.2 along with the techniques in this section to find all real solutions.

28. $\left|\dfrac{3n}{n-1}\right| = 3$

29. $\left|\dfrac{2x}{x^2-1}\right| = 2$

30. $\left|\dfrac{2t}{4-t^2}\right| = \left|\dfrac{2}{t-2}\right|$

In Exercises 31 - 33, find all real solutions and use a calculator to approximate your answers, rounded to two decimal places.

31. $2.41 = \dfrac{0.08}{4\pi R^2}$

32. $\dfrac{x^2}{(2.31-x)^2} = 0.04$

33. $1 - \dfrac{6.75\times 10^{16}}{c^2} = \dfrac{1}{4}$

In Exercises 34 - 39, solve the given equation for the indicated variable.

34. Solve for y: $\dfrac{1-2y}{y+3} = x$

35. Solve for y: $x = 3 - \dfrac{2}{1-y}$

36.[11] Solve for T_2: $\dfrac{V_1}{T_1} = \dfrac{V_2}{T_2}$

37. Solve for t_0: $\dfrac{t_0}{1 - t_0 t_1} = 2$

38. Solve for x: $\dfrac{1}{x - v_r} + \dfrac{1}{x + v_r} = 5$

39. Solve for R: $P = \dfrac{25R}{(R+4)^2}$

[11] Recall: subscripts on variables have no intrinsic mathematical meaning; they're just used to distinguish one variable from another. In other words, treat quantities like 'V_1' and 'V_2' as two different variables as you would 'x' and 'y.'

7.1.2 Answers

1. $\dfrac{3(x+3)}{x(x+2)}$, $x \neq 3$
2. $\dfrac{t}{(3t+4)(t^2+1)}$, $t \neq 2$
3. $-\dfrac{y(y-3)}{y+4}$, $y \neq -\dfrac{1}{2}, 3, 4$
4. $\dfrac{2x-1}{3x-1}$
5. $-w-1$, $w \neq 1$
6. $\dfrac{y}{3}$, $y \neq 0$
7. $\dfrac{b^2-5b+7}{b-3}$
8. $\dfrac{4x^2+x+4}{(x-4)(2x+1)}$
9. $\dfrac{m+1}{m+2}$, $m \neq 2$
10. $-\dfrac{2}{x}$, $x \neq 1$
11. $\dfrac{3}{4-2h}$, $h \neq 0$
12. $-\dfrac{1}{x(x+h)}$, $h \neq 0$
13. $\dfrac{8}{3w}$
14. $-\dfrac{2(y^2-7y+9)}{y(y-3)^2}$
15. $-\dfrac{6}{(x-2)^2}$
16. t^2+t, $t \neq 0$
17. $-\dfrac{2(h+6)}{9(h+3)^2}$, $h \neq 0$
18. $\dfrac{1}{(7-x)(7-x-h)}$, $h \neq 0$
19. $x = -\dfrac{6}{7}$
20. $y = 1, 2$
21. $w = -1$
22. $x = -6, 2$
23. No solution.
24. $y = 0, \pm 2\sqrt{2}$
25. $w = -\sqrt{3}, -1$
26. $x = -\dfrac{3\sqrt{2}}{2}, \sqrt{2}$
27. $x = -\dfrac{\sqrt{3}}{2}, -\dfrac{\sqrt{3}}{4}$
28. $n = \dfrac{1}{2}$
29. $x = \dfrac{1 \pm \sqrt{5}}{2}, \dfrac{-1 \pm \sqrt{5}}{2}$
30. $t = -1$
31. $R = \pm\sqrt{\dfrac{0.08}{9.64\pi}} \approx \pm 0.05$
32. $x = -\dfrac{231}{400} \approx -0.58$, $x = \dfrac{77}{200} \approx 0.38$
33. $c = \pm\sqrt{\dfrac{4 \cdot 6.75 \times 10^{16}}{3}} = \pm 3.00 \times 10^8$ (You actually didn't *need* a calculator for this!)
34. $y = \dfrac{1-3x}{x+2}$, $y \neq -3$, $x \neq -2$
35. $y = \dfrac{x-1}{x-3}$, $y \neq 1$, $x \neq 3$
36. $T_2 = \dfrac{V_2 T_1}{V_1}$, $T_1 \neq 0$, $T_2 \neq 0$, $V_1 \neq 0$
37. $t_0 = \dfrac{2}{2t_1+1}$, $t_1 \neq -\dfrac{1}{2}$
38. $x = \dfrac{1 \pm \sqrt{25v_r^2+1}}{5}$, $x \neq \pm v_r$.
39. $R = \dfrac{-(8P-25) \pm \sqrt{(8P-25)^2 - 64P^2}}{2P} = \dfrac{(25-8P) \pm 5\sqrt{25-16P}}{2P}$, $P \neq 0$, $R \neq -4$

7.2 Introduction to Rational Functions

If we add, subtract, or multiply polynomial functions, the result is another polynomial function. When we divide polynomial functions, however, we may not get a polynomial function. The result of dividing two polynomials is a **rational function**, so named because rational functions are *ratios* of polynomials.

Definition 7.1. A **rational function** is a function which is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions.[a]

[a]According to this definition, all polynomial functions are also rational functions. (Take $q(x) = 1$).

7.2.1 Laurent Monomial Functions

As with polynomial functions, we begin our study of rational functions with what are, in some sense, the building blocks of rational functions, **Laurent monomial functions**.

Definition 7.2. A **Laurent monomial function** is either a monomial function (see Definition 6.1) or a function of the form $f(x) = \dfrac{a}{x^n} = ax^{-n}$ for $n \in \mathbb{N}$.

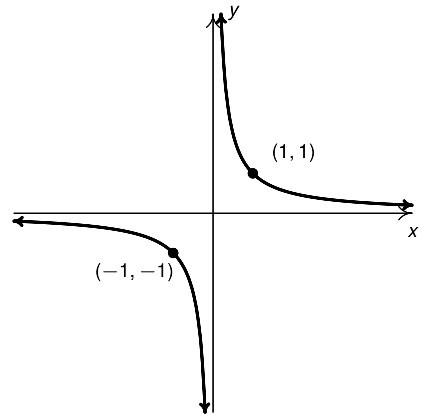
Laurent monomial functions are named in honor of Pierre Alphonse Laurent and generalize the notion of 'monomial function' from Chapter 6 to terms with negative exponents. Our study of these functions begins with an analysis of $r(x) = \frac{1}{x} = x^{-1}$, the reciprocal function. The first item worth noting is that $r(0)$ is not defined owing to the presence of x in the denominator. That is, the domain of r is $\{x \in \mathbb{R} \mid x \neq 0\}$ or, using interval notation, $(-\infty, 0) \cup (0, \infty)$. Of course excluding 0 from the domain of r serves only to pique our curiosity about the behavior of $r(x)$ when $x \approx 0$. Thinking from a number sense perspective, the closer the denominator of $\frac{1}{x}$ is to 0, the larger the value of the fraction (in absolute value.)[1] So it stands to reason that as x gets closer and closer to 0, the values for $r(x) = \frac{1}{x}$ should grow larger and larger (in absolute value.) This is borne out in the table below on the left where it is apparent that for $x \approx 0$, $r(x)$ is becoming unbounded.

As we investigate the end behavior of r, we find that as $x \to \pm\infty$, $r(x) \approx 0$. Again, number sense agrees here with the data, since as the denominator of $\frac{1}{x}$ becomes unbounded, the value of the fraction should diminish. That being said, we could ask if the graph ever reaches the x-axis. If we attempt to solve $y = r(x) = \frac{1}{x} = 0$. we arrive at the contradiction $1 = 0$ hence, 0 is not in the range of r. Every other real number besides 0 is in the range of r, however. To see this, let $c \neq 0$ be a real number. Then $\frac{1}{c}$ is defined and, moreover, $r\left(\frac{1}{c}\right) = \frac{1}{(1/c)} = c$. This shows c is in the range of r. Hence, the range of r is $\{y \in \mathbb{R} \mid y \neq 0\}$ or, using interval notation, $(-\infty, 0) \cup (0, \infty)$.

[1]Technically speaking, -1×10^{117} is a 'small' number (since it is very far to the left on the number line.) However, it's absolute value, 1×10^{117} is very large.

x	$r(x) = \frac{1}{x}$
-0.01	-100
-0.001	-1000
-0.0001	-10000
-0.00001	-100000
0	undefined
0.00001	100000
0.0001	10000
0.001	1000
0.01	100

x	$r(x) = \frac{1}{x}$
-100000	-0.00001
-10000	-0.0001
-1000	-0.001
-100	-0.01
0	undefined
100	0.01
1000	0.001
10000	0.0001
100000	0.00001



$y = r(x) = \frac{1}{x}$

In order to more precisely describe the behavior near 0, we say 'as x approaches 0 *from the left*,' written as $x \to 0^-$, the function values $r(x) \to -\infty$. By 'from the left' we mean we are considering x-values slightly to the *left* of 0 on the number line, such as $x = -0.001$ and $x = -0.0001$ in the table above. If we think of these numbers as all being x-values where $x =$ '$0-$ a little bit', the the '$-$' in the notation '$x \to 0^-$' makes better sense. The notation to describe the $r(x)$ values, $r(x) \to -\infty$, is used here in the same manner as it was in Section 6.1. That is, as $x \to 0^-$, the values $r(x)$ are becoming unbounded in the negative direction. Similarly, we say 'as x approaches 0 *from the right*,' that is as $x \to 0^+$, $r(x) \to \infty$. Here 'from the right' means we are using x values slightly to the *right* of 0 on the number line: numbers such as $x = 0.001$ which could be described as '0 + a little bit.' For these values of x, the values of $r(x)$ become unbounded (in the positive direction) so we write $r(x) \to \infty$ here.

We can also use this notation to describe the end behavior, but here the numerical roles are reversed. We see as $x \to -\infty$, $r(x) \to 0^-$ and as $x \to \infty$, $r(x) \to 0^+$.

The way we describe what is happening graphically is to say the line $x = 0$ is a **vertical asymptote** to the graph of $y = r(x)$ an the line $y = 0$ is a **horizontal asymptote** to the graph of $y = r(x)$. Roughly speaking, asymptotes are lines which approximate functions as either the inputs our outputs become unbounded. While defined more precisely using the language of Calculus, we do our best to formally define vertical and horizontal asymptotes below.

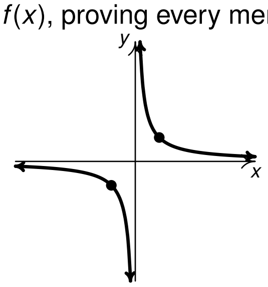
Definition 7.3. The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

Definition 7.4. The line $y = c$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to c$.

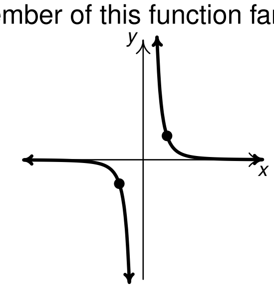
Note that in Definition 7.4, we write $f(x) \to c$ (not $f(x) \to c^+$ or $f(x) \to c^-$) because we are unconcerned from which direction the values $f(x)$ approach the value c, just as long as they do so. As we shall see, the graphs of rational functions may, in fact, *cross* their horizontal asymptotes. If this happens, however, it does so only a *finite* number of times (at least in this chapter), and so for each choice of $x \to -\infty$ and $x \to \infty$, $f(x)$ will approach c from either below (in the case $f(x) \to c^-$) or above (in the case $f(x) \to c^+$.)

We leave $f(x) \to c$ generic in our definition, however, to allow this concept to apply to less tame specimens in the Precalculus zoo, one that cross horizontal asymptotes an infinite number of times.[2]
The behaviors illustrated in the graph $r(x) = \frac{1}{x}$ are typical of functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ for natural numbers, n. As with the monomial functions discussed in Section 6.1, the patterns that develop primarily depend on whether n is odd or even. Having thoroughly discussed the graph of $y = \frac{1}{x} = x^{-1}$, we graph it along with $y = \frac{1}{x^3} = x^{-3}$ and $y = \frac{1}{x^5} = x^{-5}$ below. Note the points $(-1, -1)$ and $(1, 1)$ are common to all three graphs as are the asymptotes $x = 0$ and $y = 0$. As the n increases, the graphs become steeper for $|x| < 1$ and flatten out more quickly for $|x| > 1$. Both the domain and range in each case appears to be $(-\infty, 0) \cup (0, \infty)$. Indeed, owing to the x in the denominator of $f(x) = \frac{1}{x^n}$, $f(0)$, and only $f(0)$, is undefined. Hence the domain is $(-\infty, 0) \cup (0, \infty)$. When thinking about the range, note the equation $f(x) = \frac{1}{x^n} = c$ has the solution $x = \sqrt[n]{\frac{1}{c}}$ as long as $c \neq 0$. Thus means $f\left(\sqrt[n]{\frac{1}{c}}\right) = c$ for every nonzero real number c. If $c = 0$, we are in the same situation as before: $\frac{1}{x^n} = 0$ has no real solution. This establishes the range is $(-\infty, 0) \cup (0, \infty)$. Finally, each of the graphs appear to be symmetric about the origin. Indeed, since n is odd, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = -x^{-n} = -f(x)$, proving every member of this function family is odd.

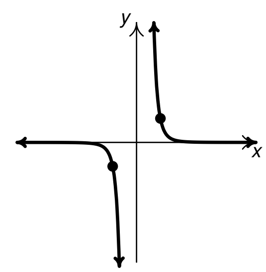
x	$\frac{1}{x} = x^{-1}$	$\frac{1}{x^3} = x^{-3}$	$\frac{1}{x^5} = x^{-5}$
-10	-0.1	-0.001	-0.00001
-1	-1	-1	-1
-0.1	-10	-1000	-100000
0	undefined	undefined	undefined
0.1	10	1000	100000
1	1	1	1
10	0.1	0.001	0.00001



$y = \frac{1}{x} = x^{-1}$



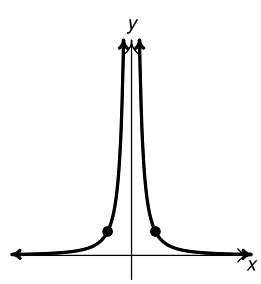
$y = \frac{1}{x^3} = x^{-3}$



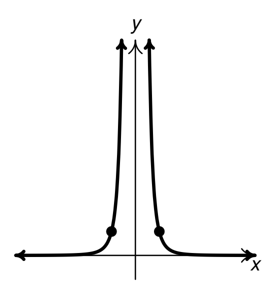
$y = \frac{1}{x^5} = x^{-5}$

We repeat the same experiment with functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ where n is even. $y = \frac{1}{x^2} = x^2$, $y = \frac{1}{x^4} = x^{-4}$ and $y = \frac{1}{x^6} = x^{-6}$. These graphs all share the points $(-1, 1)$ and $(1, 1)$, and asymptotes $x = 0$ and $y = 0$. The same remarks about the steepness for $|x| < 1$ and the flattening for $|x| > 1$ also apply. For the same reasons as given above, the domain of each of these functions is $(-\infty, 0) \cup (0, \infty)$. When it comes to the range, the fact n is even tells us tere are solutions to $\frac{1}{x^n} = c$ only if $c > 0$. It follows that the range is $(0, \infty)$ for each of these functions. Concerning symmetry, as n is even, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = x^{-n} = f(x)$, proving each member of this function family is even. Hence, all of the graphs of these functions is symmetric about the y-axis.

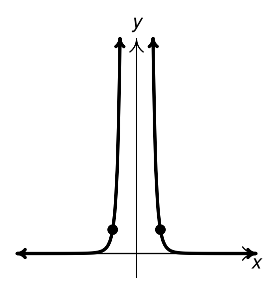
x	$\frac{1}{x^2} = x^{-2}$	$\frac{1}{x^4} = x^{-4}$	$\frac{1}{x^6} = x^{-6}$
-10	0.01	0.0001	1×10^{-6}
-1	1	1	1
-0.1	100	10000	1×10^{6}
0	undefined	undefined	undefined
0.1	100	10000	1×10^{6}
1	1	1	1
10	0.01	0.0001	1×10^{-6}



$y = \frac{1}{x^2} = x^{-2}$



$y = \frac{1}{x^3} = x^{-3}$



$y = \frac{1}{x^6} = x^{-6}$

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[2]See Exercise **??** in Section **??**.

Not surprisingly, we have an analog to Theorem 6.1 for this family of Laurent monomial functions.

Theorem 7.1. For real numbers a, h, and k with $a \neq 0$, the graph of $F(x) = \frac{a}{(x-h)^n} + k = a(x-h)^{-n} + k$ can be obtained from the graph of $f(x) = \frac{1}{x^n} = x^{-n}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^{-n}$ to $y = (x-h)^{-n}$.

   The vertical asymptote moves from $x = 0$ to $x = h$.

2. multiply the y-coordinates of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (x-h)^{-n}$ to $y = a(x-h)^{-n}$.

3. add k to each of the y-coordinates of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a(x-h)^{-n}$ to $y = a(x-h)^{-n} + k$.

   The horizontal asymptote moves from $y = 0$ to $y = k$.

The proof of Theorem 7.1 is *identical* to the proof of Theorem 6.1 - just replace x^n with x^{-n}. We nevertheless encourage the reader to work through the details[3] and compare the results of this theorem with Theorems 4.4, 5.7, and 6.1.
We put Theorem 7.1 to good use in the following example.

Example 7.2.1. Use Theorem 7.1 to graph the following. Label at least two points and the asymptotes. State the domain and range using interval notation.

1. $f(x) = (2x-3)^{-2}$
2. $g(t) = \dfrac{2t-1}{t+1}$

Solution.

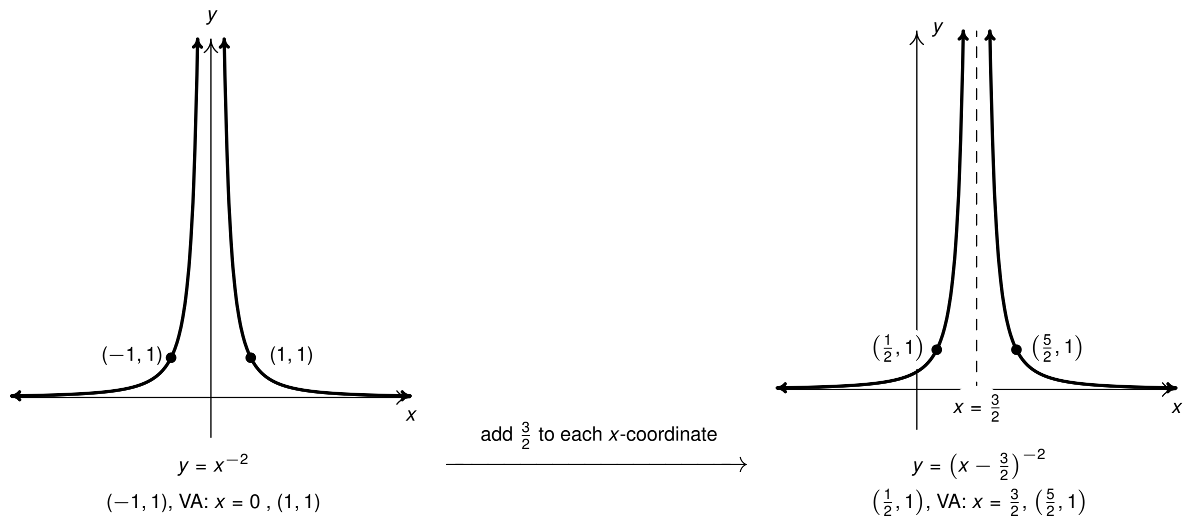
1. In order to use Theorem 7.1, we first must put $f(x) = (2x-3)^{-2}$ into the form prescribed by the theorem. To that end, we factor:

$$f(x) = \left(2\left[x - \frac{3}{2}\right]\right)^{-2} = 2^{-2}\left(x - \frac{3}{2}\right)^{-2} = \frac{1}{4}\left(x - \frac{3}{2}\right)^{-2}$$

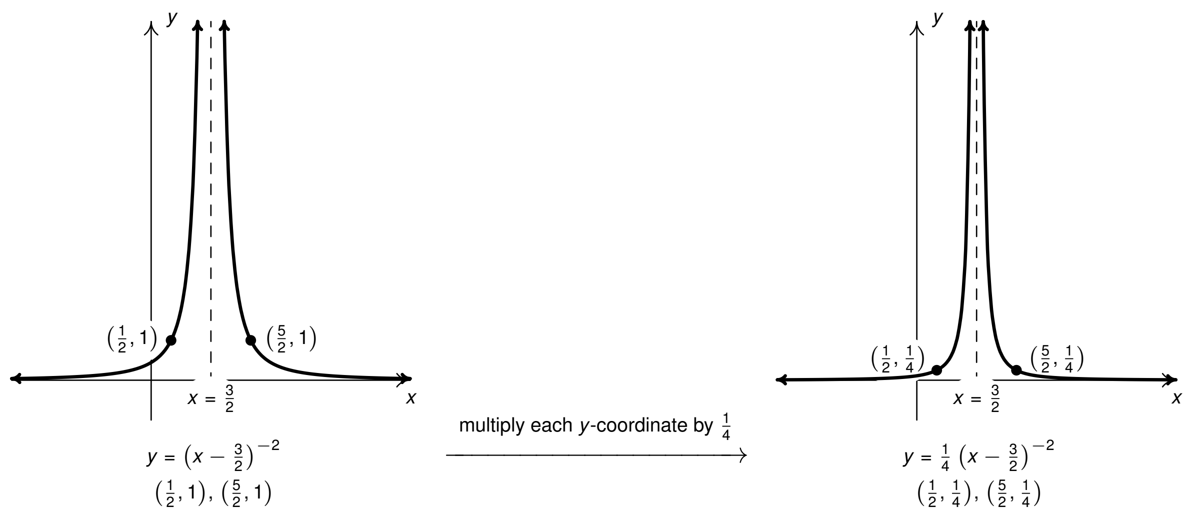
   We identify $n = 2$, $a = \frac{1}{4}$ and $h = \frac{3}{2}$ (and $k = 0$.) Per the theorem, we begin with the graph of $y = x^{-2}$ and track the two points $(-1, 1)$ and $(1, 1)$ along with the vertical and horizontal asymptotes $x = 0$ and $y = 0$, respectively through each step.

[3] We are, in fact, building to Theorem 2.7 in Section 2.3, so the more you see the forest for the trees, the better off you'll be when the time comes to generalize these moves to all functions.

Step 1: add $\frac{3}{2}$ to each of the x-coordinates of each of the points on the graph of $y = x^{-2}$. This moves the vertical asymptote from $x = 0$ to $x = \frac{3}{2}$ (which we represent by a dashed line.)



Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \left(x - \frac{3}{2}\right)^{-2}$ by $\frac{1}{4}$.



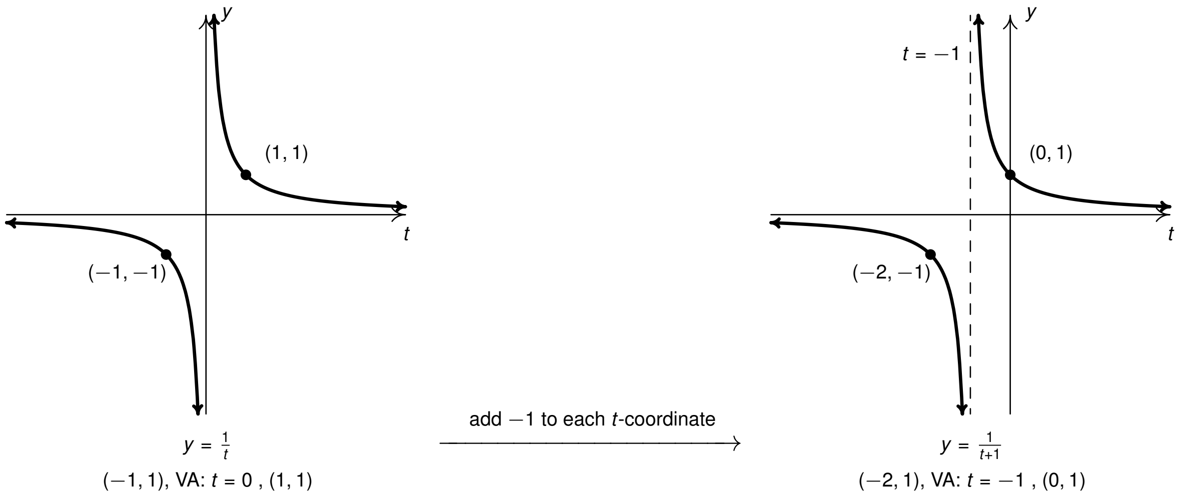
Since we did not shift the graph vertically, the horizontal asymptote remains $y = 0$. We can determine the domain and range of f by tracking the changes to the domain and range of our progenitor function, $y = x^{-2}$. We get the domain and range of f is $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$ and the range of f is $(-\infty, 0) \cup (0, \infty)$.

2. Using either long or synthetic division, we get

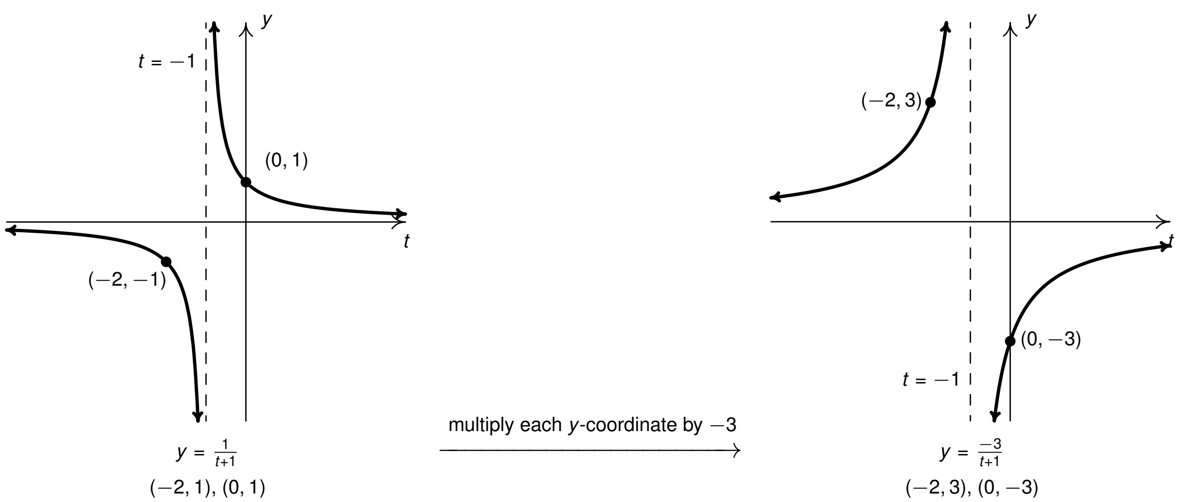
$$g(t) = \frac{2t - 1}{t + 1} = -\frac{3}{t + 1} + 2 = \frac{-3}{(t - (-1))^1} + 2$$

so we identify $n = 1$, $a = -3$, $h = -1$, and $k = 2$. We start with the graph of $y = \frac{1}{t}$ with points $(-1, -1)$, $(1, 1)$ and asymptotes $t = 0$ and $y = 0$ and track these through each of the steps.

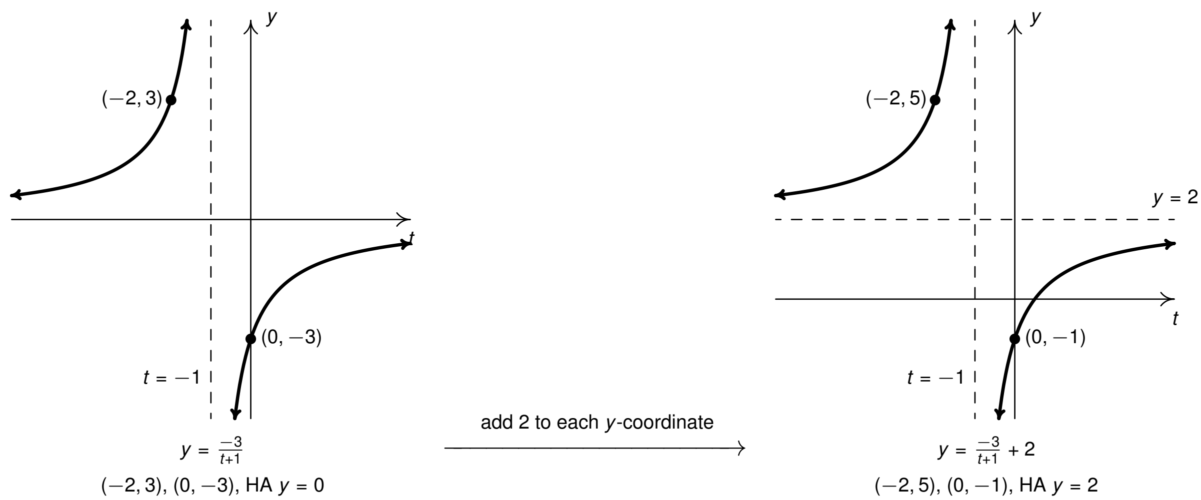
Step 1: Add -1 to each of the t-coordinates of each of the points on the graph of $y = \frac{1}{t}$. This moves the vertical asymptote from $t = 0$ to $t = -1$.



Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \frac{1}{t+1}$ by -3.



Step 3: add 2 to each of the y-coordinates of each of the points on the graph of $y = \frac{-3}{t+1}$. This moves the horizontal asymptote from $y = 0$ to $y = 2$.



As above, we determine the domain and range of g by tracking the changes in the domain and range of $y = \frac{1}{t}$. We find the domain of g is $(-\infty, -1) \cup (-1, \infty)$ and the range is $(-\infty, 2) \cup (2, \infty)$. $\square$

In Example 7.2.1, we once again see the benefit of changing the form of a function to make use of an important result. A natural question to ask is to what extent general rational functions can be rewritten to use Theorem 7.1. In the same way polynomial functions are sums of monomial functions, it turns out, allowing for non-real number coefficients, that every rational function can be written as a sum of (possibly shifted) Laurent monomial functions.[4]

7.2.2 Local Behavior near Excluded Values

We take time now to focus on behaviors of the graphs of rational functions near excluded values. We've already seen examples of one type of behavior: vertical asymptotes. Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises from a family of equations cheerily named 'doomsday' equations.[5]

Example 7.2.2. A mathematical model for the population $P(t)$, in thousands, of a certain species of bacteria, t days after it is introduced to an environment is given by $P(t) = \frac{100}{(5-t)^2}$, $0 \leq t < 5$.

1. Find and interpret $P(0)$.
2. When will the population reach 100,000?
3. Graph $y = P(t)$.
4. Find and interpret the behavior of P as $t \to 5^-$.

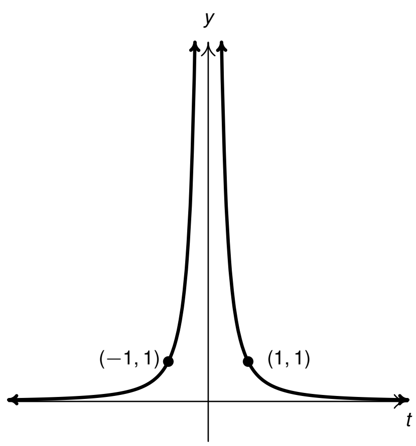
[4]i.e., Laurent 'Polynomials.' This result is a combination of Theorems **??** in Section **??** and Theorem **??** in Section **??**.
[5]These functions arise in Differential Equations. The unfortunate name will make sense shortly.

Solution.

1. Substituting $t = 0$ gives $P(0) = \frac{100}{(5-0)^2} = 4$. Since t represents the number of days *after* the bacteria are introduced into the environment, $t = 0$ corresponds to the day the bacteria are introduced. Since $P(t)$ is measured in *thousands*, $P(t) = 4$ means 4000 bacteria are initially introduced into the environment.

2. To find when the population reaches 100,000, we first need to remember that $P(t)$ is measured in *thousands*. In other words, 100,000 bacteria corresponds to $P(t) = 100$. Hence, we need to solve $P(t) = \frac{100}{(5-t)^2} = 100$. Clearing denominators and dividing by 100 gives $(5-t)^2 = 1$, which, after extracting square roots, produces $t = 4$ or $t = 6$. Of these two solutions, only $t = 4$ in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.

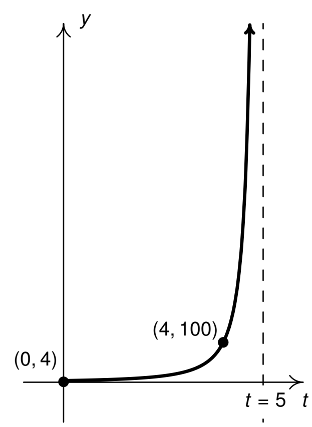
3. After a slight re-write, we have $P(t) = \frac{100}{(5-t)^2} = \frac{100}{[(-1)(t-5)]^2} = \frac{100}{(t-5)^2}$. Using Theorem 7.1, we start with the graph of $y = \frac{1}{t^2}$ below on the left. After shifting the graph to the right 5 units and stretching it vertically by a factor of 100 (note, the graphs are not to scale!), we restrict the domain to $0 \leq t < 5$ to arrive at the graph of $y = P(t)$ below on the right.



$y = \frac{1}{t^2}$

$(-1, 1)$, VA: $t = 0$, $(1, 1)$

Theorem 7.1 $\longrightarrow$



$y = P(t)$

$(4, 100)$, VA: $t = 5$

4. We see that as $t \to 5^-$, $P(t) \to \infty$. This means that the population of bacteria is increasing without bound as we near 5 days, which cannot actually happen. For this reason, $t = 5$ is called the 'doomsday' for this population. There is no way any environment can support infinitely many bacteria, so shortly before $t = 5$ the environment would collapse. $\square$

Will all values excluded from the domain of a rational function produce vertical asymptotes in the graph? The short answer is 'no.' There are milder interruptions that can occur - holes in the graph - which we explore in our next example. To this end, we formalize the notion of *average velocity* - a concept we first encountered in Example **??** in Section 3.2. In that example, the function $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$

gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. The function s an example of a **position function** since it provides information about *where* the rocket is at time t. In that example, we interpreted the average rate of change of s over an interval as the average velocity of the rocket over that interval. The average velocity provides two pieces of information: the average speed of the rocket along with the rocket's direction. Suppose we have a position function s defined over an interval containing some fixed time t_0. We can define the average velocity as a function of any time t other than t_0:

Definition 7.5. Suppose $s(t)$ gives the position of an object at time t and t_0 is a fixed time in the domain of s. The **average velocity** between time t and time t_0 is given by

$$\overline{v}(t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0},$$

provided $t \neq t_0$.

It is clear why we must exclude $t = t_0$ from the domain of $\overline{v}$ in Definition 7.5 since otherwise we would have a 0 in the denominator. What is interesting in this case however, is that substituting $t = t_0$ also produces 0 in the *numerator*. (Do you see why?) While '$\frac{0}{0}$' is undefined, it is more precisely called an 'indeterminate form' and is studied extensively in Calculus. We can nevertheless explore this function in the next example.

Example 7.2.3. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Find and simplify an expression for the average velocity of the rocket between times t and 15.
2. Find and interpret $\overline{v}(14)$.
3. Graph $y = \overline{v}(t)$. Interpret the intercepts.
4. Interpret the behavior of $\overline{v}$ as $t \to 15$.

Solution.

1. Using Definition 7.5 with $t_0 = 15$, we get:

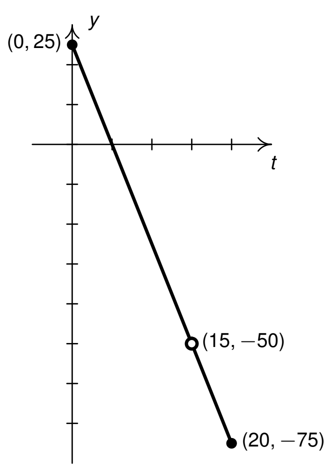
$$\begin{aligned}
\overline{v}(t) &= \frac{s(t) - s(15)}{t - 15} \\
&= \frac{(-5t^2 + 100t) - 375}{t - 15} \\
&= \frac{-5(t^2 - 20t + 75)}{t - 15} \\
&= \frac{-5(t - 15)(t - 5)}{t - 15} \\
&= \frac{-5\cancel{(t - 15)}(t - 5)}{\cancel{(t - 15)}} \\
&= -5(t - 5) = -5t + 25 \quad t \neq 15
\end{aligned}$$

Since domain of s is $0 \leq t \leq 20$, our final answer is $\overline{v}(t) = -5t + 25$, for $t \in [0, 15) \cup (15, 20]$.

2. We find $\overline{v}(14) = -5(14) + 25 = -45$. This means between 14 and 15 seconds after launch, the rocket was traveling, on average a speed 45 feet per second *downwards*, or falling back to the Moon's surface.

3. The graph of $\overline{v}(t)$ is a portion of the line $y = -5t + 25$. Since the domain of s is $[0, 20]$ and $\overline{v}(t)$ is not defined when $t = 15$, our graph is the line segment starting at $(0, 25)$ and ending at $(20, 75)$ with a hole at $(15, 50)$. The y-intercept is $(0, 25)$ which means on average, the rocket is traveling 25 feet per second *upwards*.[6] To get the t-intercept, we set $\overline{v}(t) = -5t + 25 = 0$ and obtain $t = 5$. Hence, $\overline{v}(5) = 0$ or the average velocity between times $t = 5$ and $t = 15$ is 0. As you may recall, this is due to the rocket being at the same altitude (375 feet) at both times, hence, $\Delta[s(t)]$ and, hence $\overline{v}(t) = 0$.

4. From the graph, we see as $t \to 15$, $\overline{v}(t) \to -50$. (This is also borne out in the numerically in the tables below.) This means as we sample the average velocity between time $t_0 = 15$ and and times closer and closer to 15, the average velocity approaches -50. This value is how we define the **instantaneous velocity** - that is, *at* $t = 15$ seconds, the rocket is falling at a rate of 50 feet per second towards the surface of the Moon.



$y = \overline{v}(t)$

t	$\overline{v}(t)$
14.9	-49.5
14.99	-49.95
14.999	-49.995
15	undefined
15.001	-50.005
15.01	-50.05
15.1	-50.5

□

If nothing else, Example 7.2.3 shows us that just because a value is excluded from the domain of a rational function doesn't mean there will be a vertical asymptote to the graph there. In this case, the factor $(t - 15)$ cancelled from the denominator, thereby effectively removing the threat of dividing by 0. It turns out, this situation generalizes to the theorem below.

[6]Note that the rocket has already started its descent at $t = 10$ seconds (see Example **??** in Section 3.2.) However, the rocket is still at a higher altitude at when $t = 15$ than $t = 0$ which produces a positive *average* velocity.

Theorem 7.2. Location of Vertical Asymptotes and Holes:[a] Suppose r is a rational function which can be written as $r(x) = \frac{p(x)}{q(x)}$ where p and q have no common zeros.[b] Let c be a real number which is not in the domain of r.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a hole at $\left(c, \frac{p(c)}{q(c)}\right)$
- If $q(c) = 0$, then the line $x = c$ is a vertical asymptote to the graph of $y = r(x)$.

[a] Or, 'How to tell your asymptote from a hole in the graph.'
[b] In other words, $r(x)$ is in lowest terms.

In English, Theorem 7.2 says that if $x = c$ is not in the domain of r but, when we simplify $r(x)$, it no longer makes the denominator 0, then we have a hole at $x = c$. Otherwise, the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$. Like many properties of rational functions, we owe Theorem 7.2 to Calculus, but that won't stop us from putting Theorem 7.2 to good use in the following example.

Example 7.2.4. For each function below:

- determine the values excluded from the domain.
- determine whether each excluded value corresponds to a vertical asymptote or hole in the graph.
- verify your answers using a graphing utility.
- describe the behavior of the graph near each excluded value using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

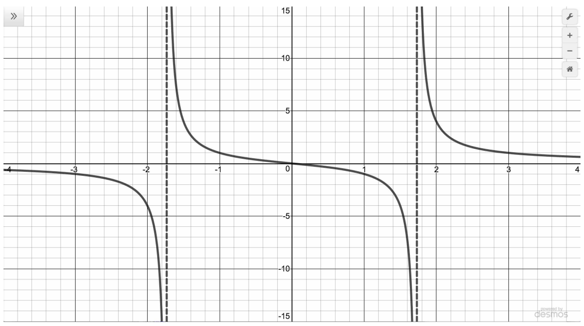
1. $f(x) = \dfrac{2x}{x^2 - 3}$
2. $g(t) = \dfrac{t^2 - t - 6}{t^2 - 9}$
3. $h(t) = \dfrac{t^2 - t - 6}{t^2 + 9}$
4. $r(t) = \dfrac{t^2 - t - 6}{t^2 + 6t + 9}$

Solution.

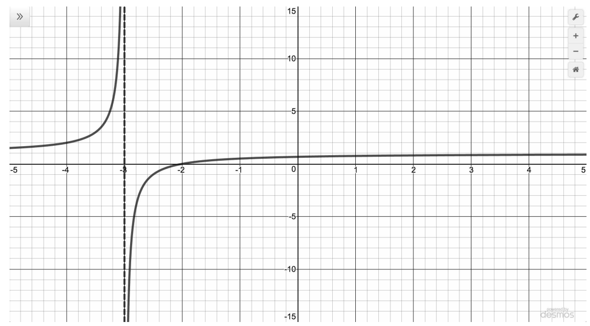
1. To use Theorem 7.2, we first find all of the real numbers which aren't in the domain of f. To do so, we solve $x^2 - 3 = 0$ and get $x = \pm\sqrt{3}$. Since the expression $f(x)$ is in lowest terms (can you see why?), there is no cancellation possible, and we conclude that the lines $x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes to the graph of $y = f(x)$. The graphing utility verifies this claim, and from the graph, we see that as $x \to -\sqrt{3}^-$, $f(x) \to -\infty$, as $x \to -\sqrt{3}^+$, $f(x) \to \infty$, as $x \to \sqrt{3}^-$, $f(x) \to -\infty$, and finally as $x \to \sqrt{3}^+$, $f(x) \to \infty$. As a side note, the graph of f appears to be symmetric about the origin. Sure enough, we find: $f(-x) = \frac{2(-x)}{(-x)^2-3} = -\frac{2x}{x^2-3} = -f(x)$, proving f is odd.

2. As above, we find the values excluded from the domain of g by setting the denominator equal to 0. Solving $t^2 - 9 = 0$ gives $t = \pm 3$. In lowest terms $g(t) = \frac{t^2-t-6}{t^2-9} = \frac{(t-3)(t+2)}{(t-3)(t+3)} = \frac{t+2}{t+3}$. Since $t = -3$ continues to be a zero of the denominator in the reduced formula, we know the line $t = -3$ is a

vertical asymptote to the graph of $y = g(t)$. Since $t = 3$ does not produce a '0' in the denominator of the reduced formula, we have a hole at $t = 3$. To find the y-coordinate of the hole, we substitute $t = 3$ into the reduced formula: $\frac{t+2}{t+3} = \frac{3+2}{3+3} = \frac{5}{6}$ so the hole is at $\left(3, \frac{5}{6}\right)$. Graphing g we can definitely see the vertical asymptote $t = -3$: as $t \to -3^{-}$, $g(t) \to \infty$ and as $t \to -3^{+}$, $g(t) \to -\infty$. Near $t = 3$, the graph seems to have no interruptions (but we know g is undefined at $t = 3$.) Since g appears to be increasing on $(-3, \infty)$, we write as $t \to 3^{-}$, $g(t) \to \frac{5}{6}^{-}$, and as $t \to 3^{+}$, $g(t) \to \frac{5}{6}^{+}$.



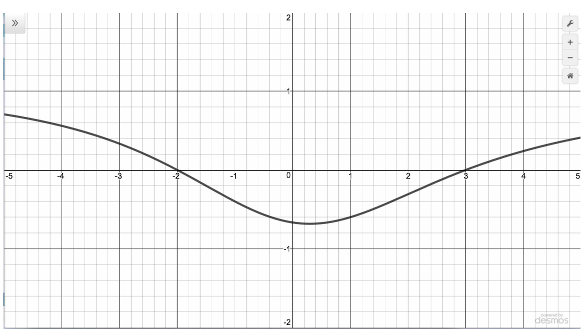
The graph of $y = f(x)$



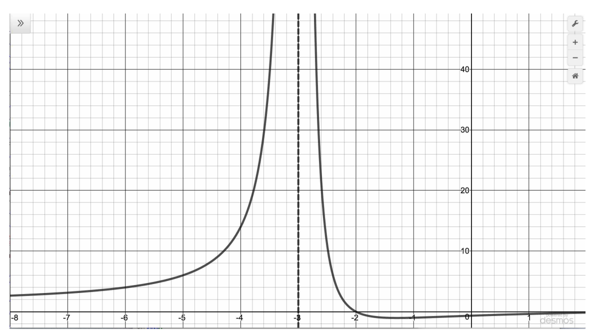
The graph of $y = g(t)$

3. Setting the denominator of the expression for $h(t)$ to 0 gives $t^2+9 = 0$, which ahs has no real solutions. Accordingly, the graph of $y = h(t)$ (at least as much as we can discern from the technology) is devoid of both vertical asymptotes and holes.

4. Setting the denominator of $r(t)$ to zero gives the equation $t^2 + 6t + 4 = 0$. We get the (repeated!) solution $t = -2$. Simplifying, we see $r(t) = \frac{t^2-t-6}{t^2+4t+4} = \frac{(t-3)(t+2)}{(t+2)^2} = \frac{t-3}{t+2}$. Since $t = -2$ continues to produce a 0 in the denominator of the reduced function, we know $t = -2$ is a vertical asymptote to the graph. The calculator bears this out, and, moreover, we see that as $t \to -2^{-}$, $r(t) \to \infty$ and as $t \to -2^{+}$, $r(t) \to -\infty$.



The graph of $y = h(t)$



The graph of $y = r(t)$

□

7.2.3 End Behavior

Now that we've thoroughly discussed behavior near values excluded from the domains of rational functions, focus our attention on end behavior. We have already seen one example of this in the form of horizontal asymptotes. Our next example of the section gives us a real-world application of a horizontal asymptote.[7]

Example 7.2.5. The number of students $N(t)$ at local college who have had the flu t months after the semester begins can be modeled by:

$$N(t) = \frac{1500t + 50}{3t + 1}, \quad t \geq 0.$$

1. Find and interpret $N(0)$.

2. How long will it take until 300 students will have had the flu?

3. Use Theorem 7.1 to graph $y = N(t)$.

4. Find and interpret the behavior of N as $t \to \infty$.

Solution.

1. Substituting $t = 0$ gives $N(0) = \frac{1500(0)+50}{1+3(0)} = 50$. Since t represents the number of months since the beginning of the semester, $t = 0$ describes the state of the flu outbreak at the beginning of the semester. Hence, at the beginning of the semester, 50 students have had the flu.

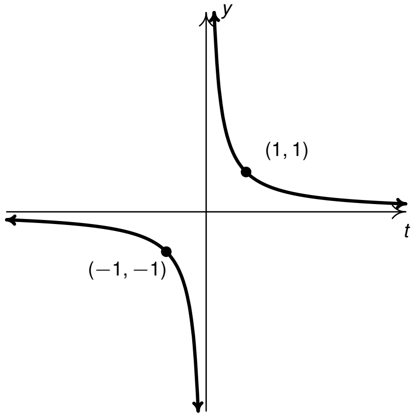
2. We set $N(t) = \frac{1500t+50}{3t+1} = 300$ and solve. Clearing denominators gives $1500t + 50 = 300(3t + 1)$ from which we get $t = \frac{5}{12}$. This means it will take $\frac{5}{12}$ months, or about 13 days, for 300 students to have had the flu.

3. To graph $y = N(t)$, we first use long division to rewrite $N(t) = \frac{-450}{3t+1} + 500$. From there, we get

$$N(t) = -\frac{450}{3t + 1} + 500 = \frac{-450}{3\left(t + \frac{1}{3}\right)} + 500 = \frac{-150}{t + \frac{1}{3}} + 500$$

Using Theorem 7.1, we start with the graph of $y = \frac{1}{t}$ below on the left and perform the following steps: shift the graph to the left by $\frac{1}{3}$ units, stretch the graph vertically by a factor of 150, reflect the graph across the t-axis, and finally, shift the graph up 500 units. As the domain of N is $t \geq 0$, we obtain the graph below on the right.

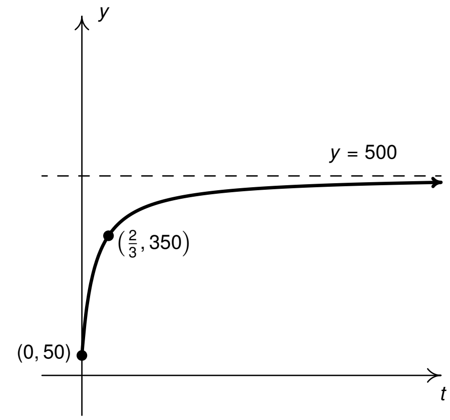
[7] Though the population below is more accurately modeled with the functions in Chapter 10, we approximate it (using Calculus, of course!) using a rational function.



$y = \frac{1}{t}$

$(-1, 1)$, $(1, 1)$, HA: $y = 0$

Theorem 7.1
$\longrightarrow$



$y = N(t)$

$\left(\frac{2}{3}, 350\right)$, HA: $y = 500$

4. From the graph, we see as $t \to \infty$, $N(t) \to 500$. (More specifically, 500^-.) This means as time goes by, only a total of 500 students will have ever had the flu. ☐

We determined the horizontal asymptote to the graph of $y = N(t)$ in Example 7.2.5 by rewriting $N(t)$ into a form compatible with Theorem 7.1, and while there is nothing wrong with this approach, it will simply not work for general rational functions which cannot be rewritten this way. To that end, we revisit this problem using Theorem 6.3 from Section 6.1. The end behavior of the numerator of $N(t) = \frac{1500t+50}{3t+1}$ is determined by its leading term, $1500t$, and the end behavior of the denominator is likewise determined by its leading term, $3t$. Hence, as $t \to \pm\infty$,

$$N(t) = \frac{1500t + 50}{3t + 1} \approx \frac{1500t}{3t} = 500.$$

Hence as $t \to \pm\infty$, $y = N(t) \to 500$, producing the horizontal asymptote $y = 500$. This same reasoning can be used in general to argue the following theorem.

Theorem 7.3. Location of Horizontal Asymptotes: Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = \frac{a}{b}$ is the[a] horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = r(x)$ has no horizontal asymptotes.

[a]The use of the definite article will be justified momentarily.

So see why Theorem 7.3 works, suppose $r(x) = \frac{p(x)}{q(x)}$ where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$. As $x \to \pm\infty$, Theorem 6.3 gives $r(x) \approx \frac{ax^n}{bx^m}$, where n and m are the degrees of $p(x)$ and $q(x)$, respectively.

If the degree of $p(x)$ and the degree of $q(x)$ are the same, then $n = m$ so that $r(x) \approx \frac{ax^n}{bx^n} = \frac{a}{b}$, which means $y = \frac{a}{b}$ is the horizontal asymptote in this case.

If the degree of $p(x)$ is less than the degree of $q(x)$, then $n < m$, so $m - n$ is a positive number, and hence, $r(x) \approx \frac{ax^n}{bx^m} = \frac{a}{bx^{m-n}} \to 0$. As $x \to \pm\infty$, $r(x)$ is more or less a fraction with a constant numerator, a, but a denominator which is unbounded. Hence, $r(x) \to 0$ producing the horizontal asymptote $y = 0$.

If the degree of $p(x)$ is greater than the degree of $q(x)$, then $n > m$, and hence $n - m$ is a positive number and $r(x) \approx \frac{ax^n}{bx^m} = \frac{ax^{n-m}}{b}$, which is a monomial function from Section 6.1. As such, r becomes unbounded as $x \to \pm\infty$.

Note that in the two cases which produce horizontal asymptotes, the behavior of r is identical as $x \to \infty$ and $x \to -\infty$. Hence, if the graph of a rational function has a horizontal asymptote, there is only one.[8]

We put Theorem 7.3 to good use in the following example.

Example 7.2.6. For each function below:

- use Theorem 6.3 to analytically determine the horizontal asymptotes to the graph, if any.
- check your answers Theorem 7.3 and a graphing utility.
- describe the end behavior of the graph using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $F(s) = \dfrac{5s}{s^2 + 1}$
2. $g(x) = \dfrac{x^2 - 4}{x + 1}$
3. $h(t) = \dfrac{6t^3 - 3t + 1}{5 - 2t^3}$
4. $r(x) = 2 - \dfrac{3x^2}{1 - x^2}$

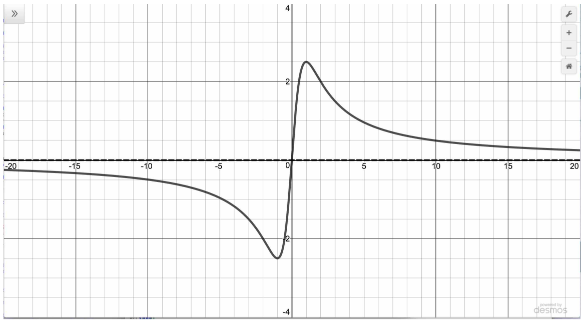
Solution.

1. Using Theorem 6.3, we get as $s \pm \infty$, $F(s) = \frac{5s}{s^2+1} \approx \frac{5s}{s^2} = \frac{5}{s}$. Hence, as $s \to \infty$, $F(s) \to 0$, so $y = 0$ is a horizontal asymptote to the graph of $y = F(s)$. To check, we note that since the degree of the numerator of $F(s)$, 1, is less than the degree of the denominator, 2 Theorem 7.3 gives $y = 0$ is the horizontal asymptote. Graphically, we see as $s \to \pm\infty$, $F(s) \to 0$. More specifically, as $s \to -\infty$, $F(s) \to 0^-$ and as $s \to \infty$, $F(s) \to 0^+$. As a side note, the graph of F appears to be symmetric about the origin. Indeed, $F(-s) = \frac{5(-s)}{(-s)^2+1} = -\frac{5s}{s^2+1}$ proving F is odd.

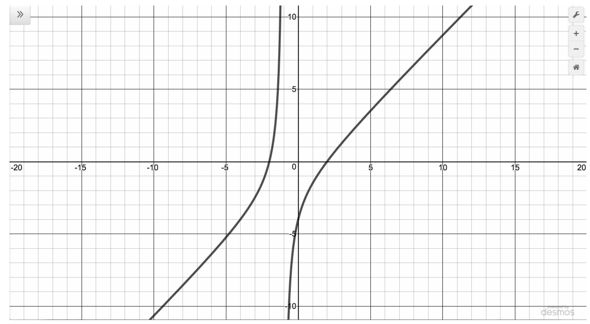
2. As $x \to \pm\infty$, $g(x) = \frac{x^2-4}{x+1} \approx \frac{x^2}{x} = x$, and while $y = x$ is a line, it is not a horizontal line. Hence, we conclude the graph of $y = g(x)$ has no horizontal asymptotes. Sure enough, Theorem 7.3 supports this since the degree of the numerator of $g(x)$ is 2 which is greater than the degree of the denominator, 1. By, there is no horizontal asymptote. From the graph, we see that the graph of $y = g(x)$ doesn't appear to level off to a constant value, so there is no horizontal asymptote.[9]

[8] We will (first) encounter functions with more than one horizontal asymptote in Chapter 8.2.

[9] Sit tight! We'll revisit this function and its end behavior shortly.



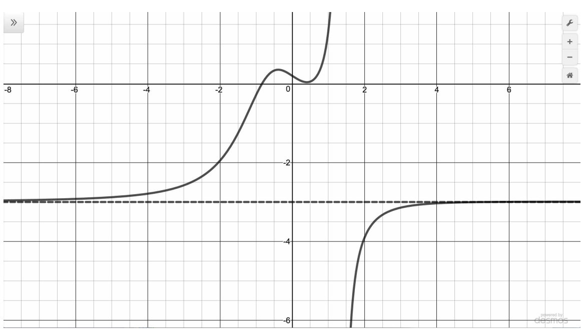
The graph of $y = F(s)$



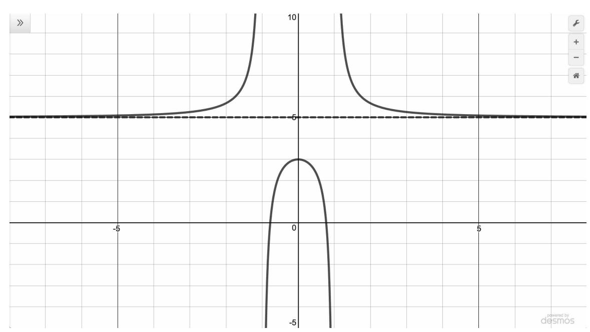
The graph of $y = g(x)$

3. We have $h(t) = \frac{6t^3-3t+1}{5-2t^3} \approx \frac{6t^3}{-2t^3} = -3$ as $t \to \pm\infty$, indicating a horizontal asymptote $y = -3$. Sure enough, since the degrees of the numerator and denominator of $h(t)$ are both three, Theorem 7.3 tells us $y = \frac{6}{-2} = -3$ is the horizontal asymptote. We see from the graph of $y = h(t)$ that as $t \to -\infty$, $h(t) \to -3^{+}$, and as $t \to \infty$, $h(t) \to -3^{-}$.

4. If we apply Theorem 6.3 to the term $\frac{3x^2}{1-x^2}$ in the expression for $r(x)$, we find $\frac{3x^2}{1-x^2} \approx \frac{3x^2}{-x^2} = -3$ as $x \to \pm\infty$. It seems reasonable to conclude, then, that $r(x) = 2 - \frac{3x^2}{1-x^2} \approx 2-(-3) = 5$ so $y = 5$ is our horizontal asymptote. In order to use Theorem 7.3 as stated, however, we need to rewrite the expression $r(x)$ with a single denominator: $r(x) = 2 - \frac{3x^2}{1-x^2} = \frac{2(1-x^2)-3x^2}{1-x^2} = \frac{2-5x^2}{1-x^2}$. Now we apply Theorem 7.3 and note since the numerator and denominator have the same degree, we are guaranteed the horizontal asymptote is $y = \frac{-5}{-1} = 5$. These calculations are borne out graphically below where it appears as if as $x \to \pm\infty$, $r(x) \to 5^{+}$. As a final note, the graph of r appears to be symmetric about the y axis. We find $r(-x) = 2 - \frac{3(-x)^2}{1-(-x)^2} = 2 - \frac{3x^2}{1-x^2} = r(x)$, proving r is even.



The graph of $y = h(t)$



The graph of $y = r(x)$

$\square$

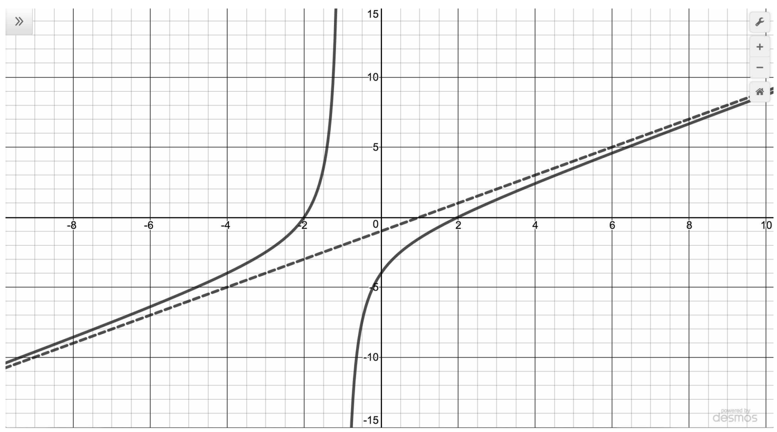
We close this section with a discussion of the *third* (and final!) kind of asymptote which can be associated with the graphs of rational functions. Let us return to the function $g(x) = \frac{x^2-4}{x+1}$ in Example 7.2.6. Performing long division,[10] we get $g(x) = \frac{x^2-4}{x+1} = x - 1 - \frac{3}{x+1}$. Since the term $\frac{3}{x+1} \to 0$ as $x \to \pm\infty$, it stands to reason

[10]See the remarks following Theorem 7.3.

that as x becomes unbounded, the function values $g(x) = x - 1 - \frac{3}{x+1} \approx x - 1$. Geometrically, this means that the graph of $y = g(x)$ should resemble the line $y = x - 1$ as $x \to \pm\infty$. We see this play out both numerically and graphically below. (As usual, we the asymptote $y = x - 1$ is denoted by a dashed line.)

x	$g(x)$	$x - 1$
-10	≈ -10.6667	-11
-100	≈ -100.9697	-101
-1000	≈ -1000.9970	-1001
-10000	≈ -10000.9997	-10001

x	$g(x)$	$x - 1$
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999



The way we symbolize the relationship between the end behavior of $y = g(x)$ with that of the line $y = x - 1$ is to write 'as $x \to \pm\infty$, $g(x) \to x - 1$' in order to have some notational consistency with what we have done earlier in this section when it comes to end behavior.[11] In this case, we say the line $y = x - 1$ is a **slant asymptote**[12] to the graph of $y = g(x)$. Informally, the graph of a rational function has a slant asymptote if, as $x \to \infty$ or as $x \to -\infty$, the graph resembles a non-horizontal, or 'slanted' line. Formally, we define a slant asymptote as follows.

Definition 7.6. The line $y = mx + b$ where $m \neq 0$ is called a **slant asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to mx + b$.

A few remarks are in order. First, note that the stipulation $m \neq 0$ in Definition 7.6 is what makes the 'slant' asymptote 'slanted' as opposed to the case when $m = 0$ in which case we'd have a horizontal asymptote. Secondly, while we have motivated what me mean intuitively by the notation '$f(x) \to mx + b$,' like so many ideas in this section, the formal definition requires Calculus. Another way to express this sentiment, however, is to rephrase '$f(x) \to mx + b$' as '$[f(x) - (mx + b)] \to 0$.' In other words, the graph of $y = f(x)$ has the *slant* asymptote $y = mx + b$ if and only if the graph of $y = f(x) - (mx + b)$ has a *horizontal* asymptote $y = 0$. If we wanted to, we could introduce the notations $f(x) \to (mx + b)^{+}$ to mean $[f(x) - (mx + b)] \to 0^{+}$ and $f(x) \to (mx + b)^{-}$ to mean $[f(x) - (mx + b)] \to 0^{-}$, but these non-standard notations.[13]

[11] Other notations include $g(x) \asymp x - 1$ or $g(x) \sim x - 1$.

[12] Also called an 'oblique' asymptote in some, ostensibly higher class (and more expensive), texts.

[13] With the introduction of the symbol '‽' in the next section, the authors feel we are in enough trouble already.

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of $g(x) = \frac{x^2-4}{x+1}$, the degree of the numerator $x^2 - 4$ is 2, which is *exactly one more* than the degree if its denominator $x + 1$ which is 1. This results in a *linear* quotient polynomial, and it is this quotient polynomial which is the slant asymptote. Generalizing this situation gives us the following theorem.[14]

Theorem 7.4. Determination of Slant Asymptotes: Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where the degree of p is *exactly* one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

In the same way that Theorem 7.3 gives us an easy way to see if the graph of a rational function $r(x) = \frac{p(x)}{q(x)}$ has a horizontal asymptote by comparing the degrees of the numerator and denominator, Theorem 7.4 gives us an easy way to check for slant asymptotes. Unlike Theorem 7.3, which gives us a quick way to *find* the horizontal asymptotes (if any exist), Theorem 7.4 gives us no such 'short-cut'. If a slant asymptote exists, we have no recourse but to use long division to find it.[15]

Example 7.2.7. For each of the following functions:

- find the slant asymptote, if it exists.
- verify your answer using a graphing utility.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$
2. $g(t) = \dfrac{t^2 - 4}{t - 2}$
3. $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$
4. $r(t) = 2t - 1 + \dfrac{4t^3}{1 - t^2}$

Solution.

1. The degree of the numerator is 2 and the degree of the denominator is 1, so Theorem 7.4 guarantees us a slant asymptote. To find it, we divide $1 - x = -x + 1$ into $x^2 - 4x + 2$ and get a quotient of $-x + 3$, so our slant asymptote is $y = -x + 3$. We confirm this graphically below.

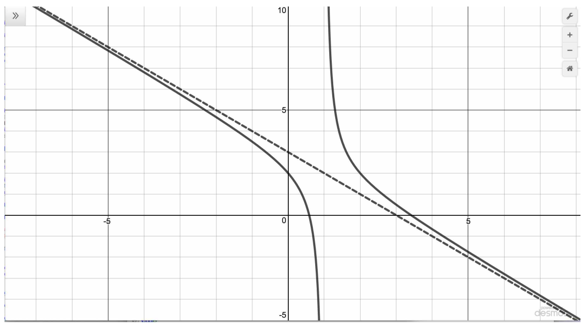
2. As with the previous example, the degree of the numerator $g(t) = \frac{t^2-4}{t-2}$ is 2 and the degree of the denominator is 1, so Theorem 7.4 applies. In this case,

$$g(t) = \frac{t^2 - 4}{t - 2} = \frac{(t + 2)(t - 2)}{(t - 2)} = \frac{(t + 2)\cancel{(t - 2)}}{\cancel{(t - 2)}^{\,1}} = t + 2, \quad t \neq 2$$

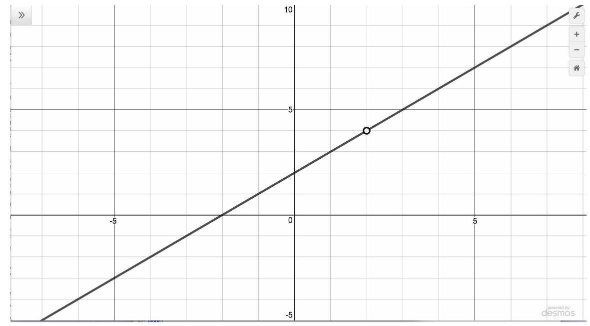
so we have that the slant asymptote $y = t+2$ is identical to the graph of $y = g(t)$ except at $t = 2$ (where the latter has a 'hole' at $(2, 4)$.) While the word 'asymptote' has the connotation of 'approaching but not equaling,' Definitions 7.6 and 7.4 allow for these extreme cases.

[14]Once again, this theorem is brought to you courtesy of Theorem 5.4 and Calculus.

[15]That's OK, though. In the next section, we'll use long division to analyze end behavior and it's worth the effort!



The graph of $y = f(x)$



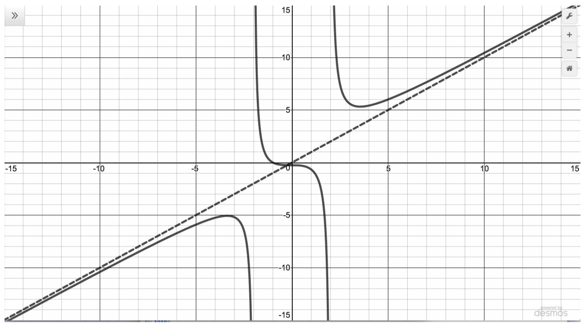
The graph of $y = g(t)$

3. For $h(x) = \frac{x^3+1}{x^2-4}$, the degree of the numerator is 3 and the degree of the denominator is 2 so again, we are guaranteed the existence of a slant asymptote. The long division $\left(x^3+1\right) \div \left(x^2-4\right)$ gives a quotient of just x, so our slant asymptote is the line $y = x$. The graphing utility confirms this. Note the graph of h appears to be symmetric about the origin. We check $h(-x) = \frac{(-x)^3+1}{(-x)^2-4} = \frac{-x^3+1}{x^2-4} = -\frac{x^3-1}{x^2-4}$. However, $-h(x) = -\frac{x^3+1}{x^2-4}$, so it appears as if $h(-x) \neq -h(x)$ for all x. Checking $x = 1$, we find $h(1) = -\frac{2}{3}$ but $h(-1) = 0$ which shows the graph of h, is in fact, *not* symmetric about the origin.

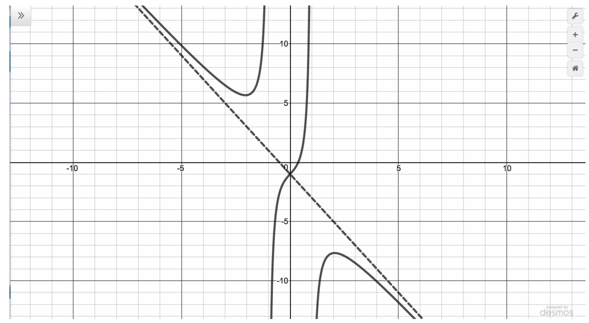
4. For our last example, $r(t) = 2t - 1 + \frac{4t^3}{1-t^2}$, the expression $r(t)$ is not in the form to apply Theorem 7.4 directly. We can, nevertheless, appeal to the spirit of the theorem and use long division to rewrite the term $\frac{4t^3}{1-t^2} = -4t + \frac{4t}{1-t^2}$. We then get:

$$
\begin{aligned}
r(t) &= 2t - 1 + \frac{4t^3}{1-t^2} \\
&= 2t - 1 - 4t + \frac{4t}{1-t^2} \\
&= -2t - 1 + \frac{4t}{1-t^2}
\end{aligned}
$$

As $t \to \pm\infty$, Theorem 6.3 gives $\frac{4t}{1-t^2} \approx \frac{4t}{-t^2} = -\frac{4}{t} \to 0$. Hence, as $t \to \pm\infty$, $r(t) \to -2t - 1$, so $y = -2t - 1$ is the slant asymptote to the graph as confirmed by the graphing utility below. from a distance, the graph of r appears to be symmetric about the origin. However, if we look carefully, we see the y-intercept is $(0, -1)$, as borne out by the computation $r(0) = -1$. Hence r cannot be odd. (Do you see why?)



The graph of $y = h(x)$



The graph of $y = r(t)$

□

Our last example gives a real-world application of a slant asymptote. The problem features the concept of **average profit**. The average profit, denoted $\overline{P}(x)$, is the total profit, $P(x)$, divided by the number of items sold, x. In English, the average profit tells us the profit made per item sold. It, along with average cost, is defined below.

Definition 7.7. Let $C(x)$ and $P(x)$ represent the cost and profit to make and sell x items, respectively.

- The **average cost**, $\overline{C}(x) = \dfrac{C(x)}{x}$, $x > 0$.

  NOTE: The average cost is the cost per item produced.

- The **average profit**, $\overline{P}(x) = \dfrac{P(x)}{x}$, $x > 0$.

  NOTE: The average profit is the profit per item sold.

You'll explore average cost (and its relation to variable cost) in Exercise 37. For now, we refer the reader to to Example 5.4.3 in Section 5.4.

Example 7.2.8. Recall the profit (in dollars) when x PortaBoy game systems are produced and sold is given by $P(x) = -1.5x^2 + 170x - 150$, $0 \leq x \leq 166$.

1. Find and simplify an expression for the average profit, $\overline{P}(x)$. What is the domain of $\overline{P}$?

2. Find and interpret $\overline{P}(50)$.

3. Determine the slant asymptote to the graph of $y = \overline{P}(x)$. Check your answer using a graphing utility.

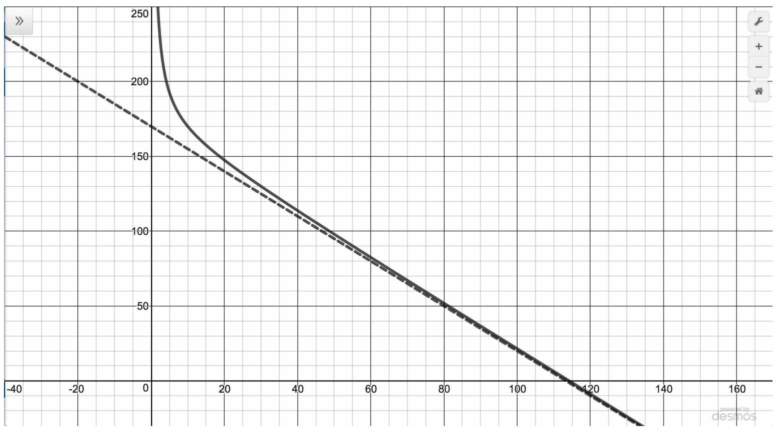
4. Interpret the slope of the slant asymptote.

Solution.

1. We find $\overline{P}(x) = \frac{P(x)}{x} = \frac{-1.5x^2+170x-150}{x} = -1.5x + 170 + \frac{150}{x}$. Since the domain of P is $[0, 166]$ but $x \neq 0$, the domain of $\overline{P}$ is $(0, 166]$.

2. We find $\overline{P}(50) = -1.5(50) + 170 + \frac{150}{50} = 98$. This means that when 50 PortaBoy systems are sold, the average profit is \$98 per system.

3. Technically, the graph of $y = \overline{P}(x)$ has no slant asymptote since the domain of the function is restricted to $(0, 166]$. That being said, if we were to let $x \to \infty$, the term $\frac{150}{x} \to 0$, so we'd have $\overline{P}(x) \to -1.5x + 170$. This means the slant asymptote would be $y = -1.5x + 170$. We graph $y = \overline{P}(x)$ and $y = -1.5x + 170$.



4. The slope of the slant asymptote $y = -1.5x + 170$ is -1.5. Since, ostensibly $\overline{P}(x) \approx -1.5x + 170$, this means that, as we sell more systems, the average profit is decreasing at about a rate of \$1.50 per system. If the number 1.5 sounds familiar to this problem situation, it should. In Example 3.2.4 in Section 3.2, we determined the slope of the demand function to be -1.5. In that situation, the -1.5 meant that in order to sell an additional system, the price had to drop by \$1.50. The fact the average profit is decreasing at more or less this same rate means the loss in profit per system can be attributed to the reduction in price needed to sell each additional system.[16] $\square$

[16] We generalize this result in Exercise 38.)

7.2.4 Exercises

(Review of Long Division):[17] In Exercises 1 - 6, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

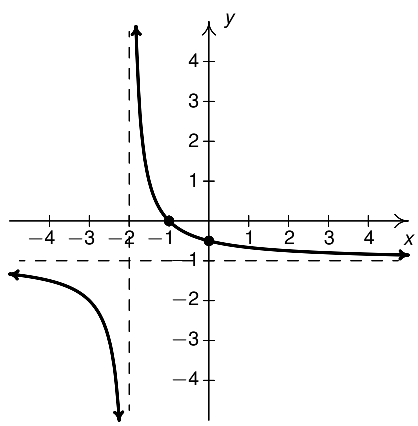
1. $\left(4x^2 + 3x - 1\right) \div (x - 3)$
2. $\left(2x^3 - x + 1\right) \div \left(x^2 + x + 1\right)$
3. $\left(5x^4 - 3x^3 + 2x^2 - 1\right) \div \left(x^2 + 4\right)$
4. $\left(-x^5 + 7x^3 - x\right) \div \left(x^3 - x^2 + 1\right)$
5. $\left(9x^3 + 5\right) \div (2x - 3)$
6. $\left(4x^2 - x - 23\right) \div \left(x^2 - 1\right)$

In Exercises 7 - 10, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 7.1. Track at least two points and the asymptotes. State the domain and range using interval notation.

7. $f(x) = \dfrac{1}{x}$, $F(x) = \dfrac{1}{x-2} + 1$
8. $f(x) = \dfrac{1}{x}$, $F(x) = \dfrac{2x}{x+1}$
9. $f(x) = x^{-1}$, $F(x) = 4x(2x+1)^{-1}$
10. $f(x) = x^{-2}$, $F(x) = -(x-1)^{-2} + 3$

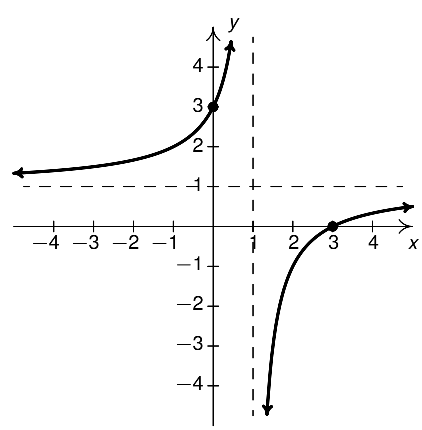
In Exercises 11 - 12, find a formula for each function below in the form $F(x) = \dfrac{a}{x-h} + k$.

11. $y = F(x)$



x-intercept $(-1, 0)$, y-intercept $\left(0, -\frac{1}{2}\right)$

12. $y = F(x)$



x-intercept $(3, 0)$, y-intercept $(0, 3)$

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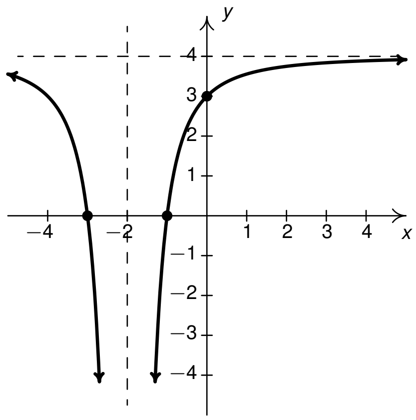
[17] For more review, see Section 5.1.2.

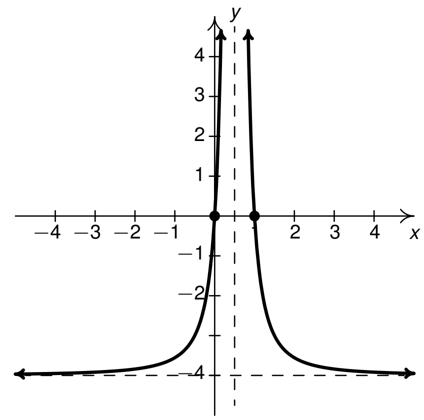
In Exercises 13 - 14, find a formula for each function below in the form $F(x) = \dfrac{a}{(x-h)^2} + k$.

13. $y = F(x)$



x-intercepts $(-3,0)$, $(-1,0)$, y-intercept $(0,3)$

14. $y = F(x)$



x-intercepts $(0,0)$, $(1,0)$, Vertical Asymptote: $x = \frac{1}{2}$

In Exercises 15 - 32, for the given rational function:

- State the domain.
- Identify any vertical asymptotes of the graph.
- Identify any holes in the graph.
- Find the horizontal asymptote, if it exists.
- Find the slant asymptote, if it exists.
- Graph the function using a graphing utility and describe the behavior near the asymptotes.

15. $f(x) = \dfrac{x}{3x-6}$

16. $f(x) = \dfrac{3+7x}{5-2x}$

17. $f(x) = \dfrac{x}{x^2+x-12}$

18. $g(t) = \dfrac{t}{t^2+1}$

19. $g(t) = \dfrac{t+7}{(t+3)^2}$

20. $g(t) = \dfrac{t^3+1}{t^2-1}$

21. $r(z) = \dfrac{4z}{z^2+4}$

22. $r(z) = \dfrac{4z}{z^2-4}$

23. $r(z) = \dfrac{z^2-z-12}{z^2+z-6}$

24. $f(x) = \dfrac{3x^2-5x-2}{x^2-9}$

25. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

26. $f(x) = \dfrac{x^3-3x+1}{x^2+1}$

27. $g(t) = \dfrac{2t^2+5t-3}{3t+2}$

28. $g(t) = \dfrac{-t^3+4t}{t^2-9}$

29. $g(t) = \dfrac{-5t^4-3t^3+t^2-10}{t^3-3t^2+3t-1}$

30. $r(z) = \dfrac{z^3}{1-z}$

31. $r(z) = \dfrac{18-2z^2}{z^2-9}$

32. $r(z) = \dfrac{z^3-4z^2-4z-5}{z^2+z+1}$

33. The cost $C(p)$ in dollars to remove p% of the invasive Ippizuti fish species from Sasquatch Pond is:

$$C(p) = \frac{1770p}{100 - p}, \quad 0 \leq p < 100$$

(a) Find and interpret $C(25)$ and $C(95)$.

(b) What does the vertical asymptote at $x = 100$ mean within the context of the problem?

(c) What percentage of the Ippizuti fish can you remove for \$40000?

34. In the scenario of Example 7.2.3, $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. For each of the times t_0 listed below, find and simplify a the formula for the average velocity $\overline{v}(t)$ between t and t_0 (see Definition 7.5) and use $\overline{v}(t)$ to find and interpret the instantaneous velocity of the rocket at $t = t_0$.

(a) $t_0 = 5$ (b) $t_0 = 9$ (c) $t_0 = 10$ (d) $t_0 = 11$

35. The population of Sasquatch in Portage County t years after the year 1803 is modeled by the function

$$P(t) = \frac{150t}{t + 15}.$$

Find and interpret the horizontal asymptote of the graph of $y = P(t)$ and explain what it means.

36. The cost in dollars, $C(x)$ to make x dOpi media players is $C(x) = 100x + 2000$, $x \geq 0$. You may wish to review the concepts of fixed and variable costs introduced in Example 3.2.3 in Section 3.2.2.

(a) Find a formula for the average cost $\overline{C}(x)$.

(b) Find and interpret $\overline{C}(1)$ and $\overline{C}(100)$.

(c) How many dOpis need to be produced so that the average cost per dOpi is \$200?

(d) Interpret the behavior of $\overline{C}(x)$ as $x \to 0^+$.

(e) Interpret the behavior of $\overline{C}(x)$ as $x \to \infty$.

37. This exercise explores the relationships between fixed cost, variable cost, and average cost. The reader is encouraged to revisit Example 3.2.3 in Section 3.2.2 as needed. Suppose the cost in dollars $C(x)$ to make x items is given by $C(x) = mx + b$ where m and b are positive real numbers.

(a) Show the fixed cost (the money spent even if no items are made) is b.

(b) Show the variable cost (the increase in cost per item made) is m.

(c) Find a formula for the average cost when making x items, $\overline{C}(x)$.

(d) Show $\overline{C}(x) > m$ for all $x > 0$ and, moreover, $\overline{C}(x) \to m^+$ as $x \to \infty$.

(e) Interpret $\overline{C}(x) \to m^+$ both geometrically and in terms of fixed, variable, and average costs.

38. Suppose the price-demand function for a particular product is given by $p(x) = mx + b$ where x is the number of items made and sold for $p(x)$ dollars. Here, $m < 0$ and $b > 0$. If the cost (in dollars) to make x of these products is also a linear function $C(x)$, show that the graph of the average profit function $\overline{P}(x)$ has a slant asymptote with slope m and interpret.

39. In Exercise 58 in Section 6.1, we fit a few polynomial models to the following electric circuit data. The circuit was built with a variable resistor. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given below.[18]

Resistance: ($k\Omega$)	1.012	2.199	3.275	4.676	6.805	9.975
Power: (mW)	1.063	1.496	1.610	1.613	1.505	1.314

Using some fundamental laws of circuit analysis mixed with a healthy dose of algebra, we can derive the actual formula relating power $P(x)$ to resistance x:

$$P(x) = \frac{25x}{(x+3.9)^2}, \quad x \geq 0.$$

(a) Graph the data along with the function $y = P(x)$ using a graphing utility.

(b) Use a graphing utility to approximate the maximum power that can be delivered to the load. What is the corresponding resistance value?

(c) Find and interpret the end behavior of $P(x)$ as $x \to \infty$.

40. Let $f(x) = \dfrac{ax^2 - c}{x+3}$. Find values for a and c so the graph of f has a hole at $(-3, 12)$.

41. Let $f(x) = \dfrac{ax^n - 4}{2x^2+1}$.

(a) Find values for a and n so the graph of $y = f(x)$ has the horizontal asymptote $y = 3$.

(b) Find values for a and n so the graph of $y = f(x)$ has the slant asymptote $y = 5x$.

42. Suppose p is a polynomial function and a is a real number. Define $r(x) = \dfrac{p(x) - p(a)}{x - a}$. Use the Factor Theorem, Theorem 6.8, to prove the graph of $y = r(x)$ has a hole at $x = a$.

43. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[19] What trends do you observe? How do your answers manifest themselves graphically? How do you results compare with those of Exercise 51 in Section 6.1?

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
x^{-1}				
x^{-2}				
x^{-3}				
x^{-4}				

[18]The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

[19]See Definition **??** in Section **??** for a review of this concept, as needed.

44. In his now famous 1919 dissertation The Learning Curve Equation, Louis Leon Thurstone presents a rational function which models the number of words a person can type in four minutes as a function of the number of pages of practice one has completed.[20] Using his original notation and original language, we have $Y = \frac{L(X+P)}{(X+P)+R}$ where L is the predicted practice limit in terms of speed units, X is pages written, Y is writing speed in terms of words in four minutes, P is equivalent previous practice in terms of pages and R is the rate of learning. In Figure 5 of the paper, he graphs a scatter plot and the curve $Y = \frac{216(X+19)}{X+148}$. Discuss this equation with your classmates. How would you update the notation? Explain what the horizontal asymptote of the graph means. You should take some time to look at the original paper. Skip over the computations you don't understand yet and try to get a sense of the time and place in which the study was conducted.

[20] This paper, which is now in the public domain and can be found here, is from a bygone era when students at business schools took typing classes on manual typewriters.

7.2.5 Answers

1. $4x^2 + 3x - 1 = (x - 3)(4x + 15) + 44$

2. $2x^3 - x + 1 = \left(x^2 + x + 1\right)(2x - 2) + (-x + 3)$

3. $5x^4 - 3x^3 + 2x^2 - 1 = \left(x^2 + 4\right)\left(5x^2 - 3x - 18\right) + (12x + 71)$

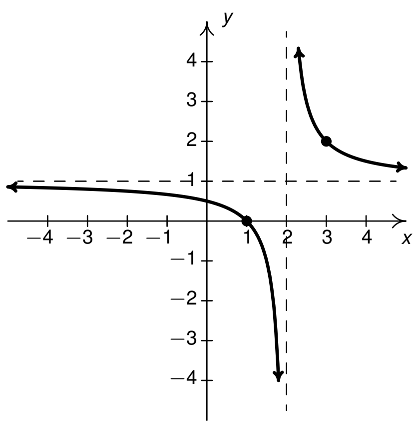
4. $-x^5 + 7x^3 - x = \left(x^3 - x^2 + 1\right)\left(-x^2 - x + 6\right) + \left(7x^2 - 6\right)$

5. $9x^3 + 5 = (2x - 3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$

6. $4x^2 - x - 23 = \left(x^2 - 1\right)(4) + (-x - 19)$

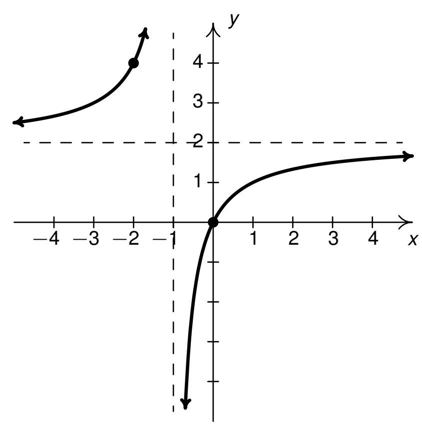
7. $F(x) = \dfrac{1}{x - 2} + 1$

   Domain: $(-\infty, 2) \cup (2, \infty)$
   Range: $(-\infty, 1) \cup (1, \infty)$
   Vertical asymptote: $x = 2$
   Horizontal asymptote: $y = 1$



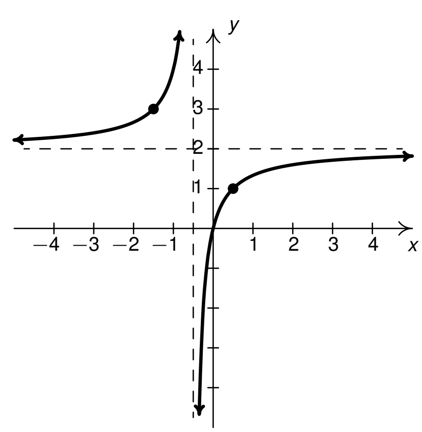
8. $F(x) = \dfrac{2x}{x + 1} = \dfrac{-2}{x + 1} + 2$

   Domain: $(-\infty, -1) \cup (-1, \infty)$
   Range: $(-\infty, 2) \cup (2, \infty)$
   Vertical asymptote: $x = -1$
   Horizontal asymptote: $y = 2$



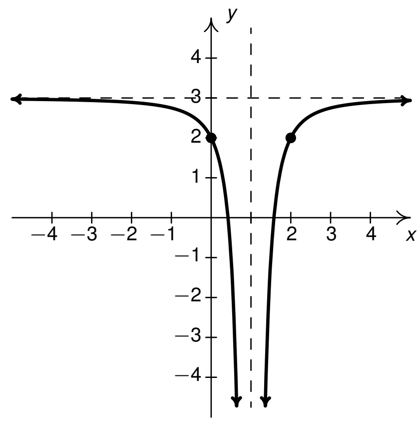
9. $F(x) = 4x(2x + 1)^{-1} = \dfrac{4x}{2x + 1} = \dfrac{-1}{x + \frac{1}{2}} + 2$

   Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$
   Range: $(-\infty, 2) \cup (2, \infty)$
   Vertical asymptote: $y = 2$
   Horizontal asymptote: $x = -\frac{1}{2}$



10. $F(x) = -(x-1)^{-2} + 3 = \dfrac{-1}{(x-1)^2} + 3$

Domain: $(-\infty, 1) \cup (1, \infty)$
Range: $(-\infty, 3) \cup (3, \infty)$
Vertical asymptote: $x = 1$
Horizontal asymptote: $y = 3$



11. $F(x) = \dfrac{1}{x+2} - 1$

12. $F(x) = \dfrac{-2}{x-1} + 1$

13. $F(x) = \dfrac{-4}{(x+2)^2} + 4$

14. $F(x) = \dfrac{1}{\left(x - \frac{1}{2}\right)^2} - 4$

15. $f(x) = \dfrac{x}{3x-6}$
Domain: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-$, $f(x) \to -\infty$
As $x \to 2^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = \frac{1}{3}$
As $x \to -\infty$, $f(x) \to \frac{1}{3}^-$
As $x \to \infty$, $f(x) \to \frac{1}{3}^+$

16. $f(x) = \dfrac{3+7x}{5-2x}$
Domain: $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$
Vertical asymptote: $x = \frac{5}{2}$
As $x \to \frac{5}{2}^-$, $f(x) \to \infty$
As $x \to \frac{5}{2}^+$, $f(x) \to -\infty$
No holes in the graph
Horizontal asymptote: $y = -\frac{7}{2}$
As $x \to -\infty$, $f(x) \to -\frac{7}{2}^+$
As $x \to \infty$, $f(x) \to -\frac{7}{2}^-$

17. $f(x) = \dfrac{x}{x^2+x-12} = \dfrac{x}{(x+4)(x-3)}$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -4$, $x = 3$
As $x \to -4^-$, $f(x) \to -\infty$
As $x \to -4^+$, $f(x) \to \infty$
As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

18. $g(t) = \dfrac{t}{t^2+1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $t \to -\infty$, $g(t) \to 0^-$
As $t \to \infty$, $g(t) \to 0^+$

19. $g(t) = \dfrac{t+7}{(t+3)^2}$
Domain: $(-\infty, -3) \cup (-3, \infty)$
Vertical asymptote: $t = -3$
As $t \to -3^-$, $g(t) \to \infty$
As $t \to -3^+$, $g(t) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
[21]As $t \to -\infty$, $g(t) \to 0^-$
As $t \to \infty$, $g(t) \to 0^+$

20. $g(t) = \dfrac{t^3+1}{t^2-1} = \dfrac{t^2-t+1}{t-1}$
Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
Vertical asymptote: $t = 1$
As $t \to 1^-$, $g(t) \to -\infty$
As $t \to 1^+$, $g(t) \to \infty$
Hole at $(-1, -\frac{3}{2})$
Slant asymptote: $y = t$
As $t \to -\infty$, the graph is below $y = t$
As $t \to \infty$, the graph is above $y = t$

21. $r(z) = \dfrac{4z}{z^2+4}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $z \to -\infty$, $r(z) \to 0^-$
As $z \to \infty$, $r(z) \to 0^+$

22. $r(z) = \dfrac{4z}{z^2-4} = \dfrac{4z}{(z+2)(z-2)}$
Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
Vertical asymptotes: $z = -2$, $z = 2$
As $z \to -2^-$, $r(z) \to -\infty$
As $z \to -2^+$, $r(z) \to \infty$
As $z \to 2^-$, $r(z) \to -\infty$
As $z \to 2^+$, $r(z) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $z \to -\infty$, $r(z) \to 0^-$
As $z \to \infty$, $r(z) \to 0^+$

23. $r(z) = \dfrac{z^2-z-12}{z^2+z-6} = \dfrac{z-4}{z-2}$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
Vertical asymptote: $z = 2$
As $z \to 2^-$, $r(z) \to \infty$
As $z \to 2^+$, $r(z) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
As $z \to -\infty$, $r(z) \to 1^+$
As $z \to \infty$, $r(z) \to 1^-$

24. $f(x) = \dfrac{3x^2-5x-2}{x^2-9} = \dfrac{(3x+1)(x-2)}{(x+3)(x-3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -3$, $x = 3$
As $x \to -3^-$, $f(x) \to \infty$
As $x \to -3^+$, $f(x) \to -\infty$
As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 3$
As $x \to -\infty$, $f(x) \to 3^+$
As $x \to \infty$, $f(x) \to 3^-$

[21]This is hard to see on the calculator, but trust me, the graph is below the t-axis to the left of $t = -7$.

25. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty,-1)\cup(-1,2)\cup(2,\infty)$
Vertical asymptote: $x=2$
As $x\to 2^-$, $f(x)\to-\infty$
As $x\to 2^+$, $f(x)\to\infty$
Hole at $(-1,0)$
Slant asymptote: $y=x+3$
As $x\to-\infty$, the graph is below $y=x+3$
As $x\to\infty$, the graph is above $y=x+3$

26. $f(x) = \dfrac{x^3-3x+1}{x^2+1}$
Domain: $(-\infty,\infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y=x$
As $x\to-\infty$, the graph is above $y=x$
As $x\to\infty$, the graph is below $y=x$

27. $g(t) = \dfrac{2t^2+5t-3}{3t+2}$
Domain: $\left(-\infty,-\frac{2}{3}\right)\cup\left(-\frac{2}{3},\infty\right)$
Vertical asymptote: $t=-\frac{2}{3}$
As $t\to-\frac{2}{3}^-$, $g(t)\to\infty$
As $t\to-\frac{2}{3}^+$, $g(t)\to-\infty$
No holes in the graph
Slant asymptote: $y=\frac{2}{3}t+\frac{11}{9}$
As $t\to-\infty$, the graph is above $y=\frac{2}{3}t+\frac{11}{9}$
As $t\to\infty$, the graph is below $y=\frac{2}{3}t+\frac{11}{9}$

28. $g(t) = \dfrac{-t^3+4t}{t^2-9} = \dfrac{-t^3+4t}{(t-3)(t+3)}$
Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
Vertical asymptotes: $t=-3$, $t=3$
As $t\to-3^-$, $g(t)\to\infty$
As $t\to-3^+$, $g(t)\to-\infty$
As $t\to 3^-$, $g(t)\to\infty$
As $t\to 3^+$, $g(t)\to-\infty$
No holes in the graph
Slant asymptote: $y=-t$
As $t\to-\infty$, the graph is above $y=-t$
As $t\to\infty$, the graph is below $y=-t$

29. $g(t) = \dfrac{-5t^4-3t^3+t^2-10}{t^3-3t^2+3t-1} = \dfrac{-5t^4-3t^3+t^2-10}{(t-1)^3}$
Domain: $(-\infty,1)\cup(1,\infty)$
Vertical asymptotes: $t=1$
As $t\to 1^-$, $g(t)\to\infty$
As $t\to 1^+$, $g(t)\to-\infty$
No holes in the graph
Slant asymptote: $y=-5t-18$
As $t\to-\infty$, the graph is above $y=-5t-18$
As $t\to\infty$, the graph is below $y=-5t-18$

30. $r(z) = \dfrac{z^3}{1-z}$
Domain: $(-\infty,1)\cup(1,\infty)$
Vertical asymptote: $z=1$
As $z\to 1^-$, $r(z)\to\infty$
As $z\to 1^+$, $r(z)\to-\infty$
No holes in the graph
No horizontal or slant asymptote
As $z\to-\infty$, $r(z)\to-\infty$
As $z\to\infty$, $r(z)\to-\infty$

31. $r(z) = \dfrac{18-2z^2}{z^2-9} = -2$
Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
No vertical asymptotes
Holes in the graph at $(-3,-2)$ and $(3,-2)$
Horizontal asymptote $y=-2$
As $z\to\pm\infty$, $r(z)=-2$

32. $r(z) = \dfrac{z^3 - 4z^2 - 4z - 5}{z^2 + z + 1} = z - 5$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y = z - 5$
$r(z) = z - 5$ everywhere.

33. (a) $C(25) = 590$ means it costs \$590 to remove 25% of the fish and and $C(95) = 33630$ means it would cost \$33630 to remove 95% of the fish from the pond.
(b) The vertical asymptote at $x = 100$ means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.
(c) For \$40000 you could remove about 95.76% of the fish.

34. (a) $\overline{v}(t) = \frac{s(t)-s(5)}{t-5} = \frac{-5t^2+100t-375}{t-5} = -5t + 75,\ t \neq 5$. The instantaneous velocity of the rocket when $t_0 = 5$ is $-5(5) + 75 = 50$ meaning it is traveling 50 feet per second upwards.
(b) $\overline{v}(t) = \frac{s(t)-s(9)}{t-9} = \frac{-5t^2+100t-495}{t-9} = -5t + 55,\ t \neq 9$. The instantaneous velocity of the rocket when $t_0 = 9$ is $-5(9) + 55 = 10$, so the rocket has slowed to 10 feet per second (but still heading up.)
(c) $\overline{v}(t) = \frac{s(t)-s(10)}{t-10} = \frac{-5t^2+100t-495}{t-10} = -5t + 50,\ t \neq 10$. The instantaneous velocity of the rocket when $t_0 = 10$ is $-5(10) + 50 = 0$, so the rocket has momentarily stopped! In Example **??**, we learned the rocket reaches its maximum height when $t = 10$ seconds, which means the rocket must change direction from heading up to coming back down, so it makes sense that for this instant, its velocity is 0.
(d) $\overline{v}(t) = \frac{s(t)-s(11)}{t-11} = \frac{-5t^2+100t-495}{t-11} = -5t + 45,\ t \neq 11$. The instantaneous velocity of the rocket when $t_0 = 11$ is $-5(11) + 45 = -10$ meaning the rocket has, indeed, changed direction and is heading downwards at a rate of 10 feet per second. (Note the symmetry here between this answer and our answer when $t = 9$.)

35. The horizontal asymptote of the graph of $P(t) = \frac{150t}{t+15}$ is $y = 150$ and it means that the model predicts the population of Sasquatch in Portage County will never exceed 150.

36. (a) $\overline{C}(x) = \frac{100x+2000}{x} = 100 + \frac{2000}{x},\ x > 0$.
(b) $\overline{C}(1) = 2100$ and $\overline{C}(100) = 120$. When just 1 dOpi is produced, the cost per dOpi is \$2100, but when 100 dOpis are produced, the cost per dOpi is \$120.
(c) $\overline{C}(x) = 200$ when $x = 20$. So to get the cost per dOpi to \$200, 20 dOpis need to be produced.
(d) As $x \to 0^+$, $\overline{C}(x) \to \infty$. This means that as fewer and fewer dOpis are produced, the cost per dOpi becomes unbounded. In this situation, there is a fixed cost of \$2000 ($C(0) = 2000$), we are trying to spread that \$2000 over fewer and fewer dOpis.
(e) As $x \to \infty$, $\overline{C}(x) \to 100^+$. This means that as more and more dOpis are produced, the cost per dOpi approaches \$100, but is always a little more than \$100. Since \$100 is the variable cost per dOpi ($C(x) = \underline{100}x + 2000$), it means that no matter how many dOpis are produced, the average cost per dOpi will always be a bit higher than the variable cost to produce a dOpi. As before, we can attribute this to the \$2000 fixed cost, which factors into the average cost per dOpi no matter how many dOpis are produced.

37. (a) The cost to make 0 items is $C(0) = m(0) + b = b$. Hence, so the fixed costs are b.

    (b) $C(x) = mx + b$ is a linear function with slope $m > 0$. Hence, the cost increases at a rate of m dollars per item made. Hence, the variable cost is m.

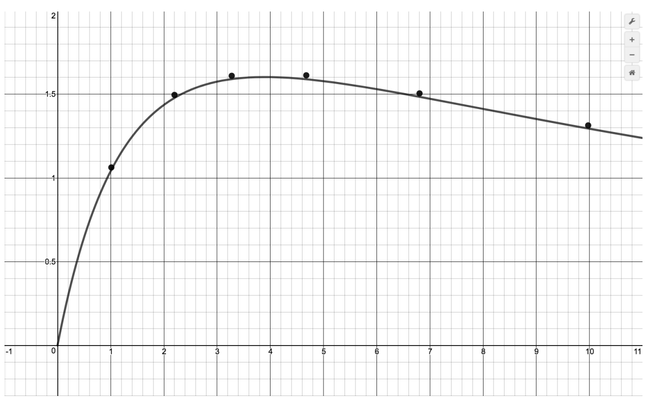
    (c) $\overline{C}(x) = \frac{C(x)}{x} = \frac{mx+b}{x} = m + \frac{b}{x}$ for $x > 0$.

    (d) Since $b > 0$, $\overline{C}(x) = m + \frac{b}{x} > m$ for $x > 0$. As $x \to \infty$, $\frac{b}{x} \to 0$ so $\overline{C}(x) = m + \frac{b}{x} \to m$.

    (e) Geometrically, the graph of $y = \overline{C}(x)$ has a horizontal asymptote $y = m$, the variable cost. In terms of costs, as more items are produced, the affect of the fixed cost on the average cost, $\frac{b}{x}$ falls away so that the average cost per item approaches the variable cost to make each item.

38. If $p(x) = mx + b$ and $C(x)$ is linear, say $C(x) = rx + s$, then we can compute the the profit function (in general) as: $P(x) = xp(x) - C(x) = x(mx + b) - (rx + s)$ which simplifies to $P(x) = mx^2 + (b - r)x - s$. Hence, the average profit $\overline{P}(x) = \frac{P(x)}{x} = \frac{mx^2+(b-r)x-s}{x} = mx + (b - r) - \frac{s}{x}$. We see that as $x \to \infty$, $\frac{s}{x} \to 0$ so $\overline{P}(x) \approx mx + (b - r)$. Hence, $y = mx + (b - r)$ is the slant asymptote to $y = \overline{P}(x)$. This means that as more items are sold, the average profit is decreasing at approximately the same rate as the price function is decreasing, m dollars per item. That is, to sell one additional item, we drop the price $p(x)$ by m dollars which results in a drop in the average profit by approximately m dollars.

39. (a)



    (b) The maximum power is approximately 1.603 mW which corresponds to 3.9 $k\Omega$.

    (c) As $x \to \infty$, $P(x) \to 0^+$ which means as the resistance increases without bound, the power diminishes to zero.

40. $a = -2$ and $c = -18$ so $f(x) = \dfrac{-2x^2 + 18}{x + 3}$.

41. (a) $a = 6$ and $n = 2$ so $f(x) = \dfrac{6x^2 - 4}{2x^2 + 1}$ (b) $a = 10$ and $n = 3$ so $f(x) = \dfrac{10x^3 - 4}{2x^2 + 1}$.

42. If we define $f(x) = p(x) - p(a)$ then f is a polynomial function with $f(a) = p(a) - p(a) = 0$. The Factor Theorem guarantees $(x - a)$ is a factor of $f(x)$, that is, $f(x) = p(x) - p(a) = (x - a)q(x)$ for some polynomial $q(x)$. Hence, $r(x) = \frac{p(x)-p(a)}{x-a} = \frac{(x-a)q(x)}{x-a} = q(x)$ so the graph of $y = r(x)$ is the same as the graph of the polynomial $y = q(x)$ except for a hole when $x = a$.

43. The slope of the curves near $x = 1$ matches the exponent on x. This exactly what we saw in Exercise 51 in Section 6.1.

$f(x)$	[0.9, 1.1]	[0.99, 1.01]	[0.999, 1.001]	[0.9999, 1.0001]
x^{-1}	-1.0101	-1.0001	≈ -1	≈ -1
x^{-2}	-2.0406	-2.0004	≈ -2	≈ -2
x^{-3}	-3.1021	-3.0010	≈ -3	≈ -3
x^{-4}	-4.2057	-4.0020	≈ -4	≈ -4

Chapter 8

Root, Radical and Power Functions

8.1 Radical Equations

In this section we review simplifying expressions and solving equations involving radicals. In addition to the product, quotient and power rules stated in Theorem 1.1 in Section 1.2, we present the following result which states that n$^{\text{th}}$ roots and n$^{\text{th}}$ powers more or less 'undo' each other.[1]

Theorem 8.1. Simplifying n$^{\text{th}}$ powers of n$^{\text{th}}$ roots and n$^{\text{th}}$ roots of n$^{\text{th}}$ powers: Suppose n is a natural number, a is a real number and $\sqrt[n]{a}$ is a real number. Then

- $(\sqrt[n]{a})^n = a$
- if n is odd, $\sqrt[n]{a^n} = a$; if n is even, $\sqrt[n]{a^n} = |a|$.

Since $\sqrt[n]{a}$ is *defined* so that $(\sqrt[n]{a})^n = a$, the first claim in the theorem is just a re-wording of Definition 1.8. The second part of the theorem breaks down along odd/even exponent lines due to how exponents affect negatives. To see this, consider the specific cases of $\sqrt[3]{(-2)^3}$ and $\sqrt[4]{(-2)^4}$.

In the first case, $\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$, so we have an instance of when $\sqrt[n]{a^n} = a$. The reason that the cube root 'undoes' the third power in $\sqrt[3]{(-2)^3} = -2$ is because the negative is preserved when raised to the third (odd) power. In $\sqrt[4]{(-2)^4}$, the negative 'goes away' when raised to the fourth (even) power: $\sqrt[4]{(-2)^4} = \sqrt[4]{16}$. According to Definition 1.8, the fourth root is defined to give only *non-negative* numbers, so $\sqrt[4]{16} = 2$. Here we have a case where $\sqrt[4]{(-2)^4} = 2 = |-2|$, not -2.

In general, we need the absolute values to simplify $\sqrt[n]{a^n}$ only when n is even because a negative to an even power is always positive. In particular, $\sqrt{x^2} = |x|$, not just 'x' (unless we *know* $x \geq 0$.)[2] We practice these formulas in the following example.

Example 8.1.1. Perform the indicated operations and simplify.

[1] See Sections 8.2.2 and 9.4 for a more precise understanding of what we mean here.

[2] This discussion should sound familiar - see the discussion following Definition 1.9 and the discussion following 'Extracting the Square Root' on page 243.

1. $\sqrt{x^2+1}$
2. $\sqrt{t^2-10t+25}$
3. $\sqrt[3]{48x^{14}}$
4. $\sqrt[4]{\dfrac{\pi r^4}{L^8}}$
5. $2x\sqrt[3]{x^2-4}+2\left(\dfrac{1}{2(\sqrt[3]{x^2-4})^2}\right)(2x)$
6. $\sqrt{(\sqrt{18y}-\sqrt{8y})^2+(\sqrt{20}-\sqrt{80})^2}$

Solution.

1. We told you back on page 31 that roots do not 'distribute' across addition and since x^2+1 cannot be factored over the real numbers, $\sqrt{x^2+1}$ cannot be simplified. It may seem silly to start with this example but it is extremely important that you understand what maneuvers are legal and which ones are not.[3]

2. Again we note that $\sqrt{t^2-10t+25} \neq \sqrt{t^2}-\sqrt{10t}+\sqrt{25}$, since radicals do *not* distribute across addition and subtraction.[4] In this case, however, we can factor the radicand and simplify as
$$\sqrt{t^2-10t+25} = \sqrt{(t-5)^2} = |t-5|$$
Without knowing more about the value of t, we have no idea if $t-5$ is positive or negative so $|t-5|$ is our final answer.[5]

3. To simplify $\sqrt[3]{48x^{14}}$, we need to look for perfect cubes in the radicand. For the cofficient, we have $48 = 8\cdot 6 = 2^3\cdot 6$. To find the largest perfect cube factor in x^{14}, we divide 14 (the exponent on x) by 3 (since we are looking for a perfect *cube*). We get 4 with a remainder of 2. This means $14 = 4\cdot 3+2$, so $x^{14} = x^{4\cdot 3+2} = x^{4\cdot 3}x^2 = (x^4)^3x^2$. Putting this altogether gives:
$$\begin{array}{rcll} \sqrt[3]{48x^{14}} & = & \sqrt[3]{2^3\cdot 6\cdot (x^4)^3x^2} & \text{Factor out perfect cubes} \\ & = & \sqrt[3]{2^3}\sqrt[3]{(x^4)^3}\sqrt[3]{6x^2} & \text{Rearrange factors, Product Rule of Radicals} \\ & = & 2x^4\sqrt[3]{6x^2} & \end{array}$$

4. In this example, we are looking for perfect fourth powers in the radicand. In the numerator r^4 is clearly a perfect fourth power. For the denominator, we take the power on the L, namely 12, and divide by 4 to get 3. This means $L^8 = L^{2\cdot 4} = (L^2)^4$. We get
$$\begin{array}{rcll} \sqrt[4]{\dfrac{\pi r^4}{L^{12}}} & = & \dfrac{\sqrt[4]{\pi r^4}}{\sqrt[4]{L^{12}}} & \text{Quotient Rule of Radicals} \\ & = & \dfrac{\sqrt[4]{\pi}\sqrt[4]{r^4}}{\sqrt[4]{(L^2)^4}} & \text{Product Rule of Radicals} \\ & = & \dfrac{\sqrt[4]{\pi}|r|}{|L^2|} & \text{Simplify} \end{array}$$

---

[3]You really do need to understand this otherwise horrible evil will plague your future studies in Math. If you say something totally wrong like $\sqrt{x^2+1} = x+1$ then you may never pass Calculus. PLEASE be careful!

[4]Let $t=1$ and see what happens to $\sqrt{t^2-10t+25}$ versus $\sqrt{t^2}-\sqrt{10t}+\sqrt{25}$.

[5]In general, $|t-5| \neq |t|-|5|$ and $|t-5| \neq t+5$ so watch what you're doing!

Without more information about r, we cannot simplify $|r|$ any further. However, we can simplify $|L^2|$. Regardless of the choice of L, $L^2 \geq 0$. Actually, $L^2 > 0$ because L is in the denominator which means $L \neq 0$. Hence, $|L^2| = L^2$. Our answer simplifies to:

$$\frac{\sqrt[4]{\pi}|r|}{|L^2|} = \frac{|r|\sqrt[4]{\pi}}{L^2}$$

5. After a quick cancellation (two of the 2's in the second term) we need to obtain a common denominator. Since we can view the first term as having a denominator of 1, the common denominator is precisely the denominator of the second term, namely $(\sqrt[3]{x^2-4})^2$. With common denominators, we proceed to add the two fractions. Our last step is to factor the numerator to see if there are any cancellation opportunities with the denominator.

$$\begin{aligned}
2x\sqrt[3]{x^2-4} + 2\left(\frac{1}{2(\sqrt[3]{x^2-4})^2}\right)(2x) &= 2x\sqrt[3]{x^2-4} + \cancel{2}\left(\frac{1}{\cancel{2}(\sqrt[3]{x^2-4})^2}\right)(2x) && \text{Reduce} \\
&= 2x\sqrt[3]{x^2-4} + \frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Mutiply} \\
&= (2x\sqrt[3]{x^2-4}) \cdot \frac{(\sqrt[3]{x^2-4})^2}{(\sqrt[3]{x^2-4})^2} + \frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Equivalent fractions} \\
&= \frac{2x(\sqrt[3]{x^2-4})^3}{(\sqrt[3]{x^2-4})^2} + \frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Multiply} \\
&= \frac{2x(x^2-4)}{(\sqrt[3]{x^2-4})^2} + \frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Simplify} \\
&= \frac{2x(x^2-4)+2x}{(\sqrt[3]{x^2-4})^2} && \text{Add} \\
&= \frac{2x(x^2-4+1)}{(\sqrt[3]{x^2-4})^2} && \text{Factor} \\
&= \frac{2x(x^2-3)}{(\sqrt[3]{x^2-4})^2}
\end{aligned}$$

We cannot reduce this any further because $x^2 - 3$ is irreducible over the rational numbers.

6. We begin by working inside each set of parentheses, using the product rule for radicals and combin-

ing like terms.

$$
\begin{aligned}
\sqrt{(\sqrt{18y}-\sqrt{8y})^2+(\sqrt{20}-\sqrt{80})^2} &= \sqrt{(\sqrt{9\cdot 2y}-\sqrt{4\cdot 2y})^2+(\sqrt{4\cdot 5}-\sqrt{16\cdot 5})^2} \\
&= \sqrt{(\sqrt{9}\sqrt{2y}-\sqrt{4}\sqrt{2y})^2+(\sqrt{4}\sqrt{5}-\sqrt{16}\sqrt{5})^2} \\
&= \sqrt{(3\sqrt{2y}-2\sqrt{2y})^2+(2\sqrt{5}-4\sqrt{5})^2} \\
&= \sqrt{(\sqrt{2y})^2+(-2\sqrt{5})^2} \\
&= \sqrt{2y+(-2)^2(\sqrt{5})^2} \\
&= \sqrt{2y+4\cdot 5} \\
&= \sqrt{2y+20}
\end{aligned}
$$

To see if this simplifies any further, we factor the radicand: $\sqrt{2y+20}=\sqrt{2(y+10)}$. Finding no perfect square factors, we are done. □

Theorem 8.1 allows us to generalize the process of 'Extracting Square Roots' to 'Extracting n[th] Roots' which in turn allows us to solve equations[6] of the form $X^n = c$.

Extracting n[th] roots:

- If c is a real number and n is odd then the real number solution to $X^n = c$ is $X = \sqrt[n]{c}$.
- If $c \geq 0$ and n is even then the real number solutions to $X^n = c$ are $X = \pm\sqrt[n]{c}$.

  Note: If $c < 0$ and n is even then $X^n = c$ has no real number solutions.

Essentially, we solve $X^n = c$ by 'taking the n[th] root' of both sides: $\sqrt[n]{X^n} = \sqrt[n]{c}$. Simplifying the left side gives us just X if n is odd or $|X|$ if n is even. In the first case, $X = \sqrt[n]{c}$, and in the second, $X = \pm\sqrt[n]{c}$. Putting this together with the other part of Theorem 8.1, namely $(\sqrt[n]{a})^n = a$, gives us a strategy for solving equations which involve n[th] powers and n^{th} roots.

Strategies for Solving Power and Radical Equations

- If the equation involves an n[th] power and the variable appears in only one term, isolate the term with the n[th] power and extract n[th] roots.
- If the equation involves an n[th] root and the variable appears in that n[th] root, isolate the n[th] root and raise both sides of the equation to the n[th] power.

  Note: When raising both sides of an equation to an *even* power, be sure to check for extraneous solutions.

[6] Well, not entirely. The equation $x^7 = 1$ has seven answers: $x = 1$ and six complex number solutions which we'll find using techniques in Section **??**.

The note about 'extraneous solutions' can be demonstrated by the basic equation: $\sqrt{x} = -2$. This equation has no solution since, by definition, $\sqrt{x} \geq 0$ for all real numbers x. However, if we square both sides of this equation, we get $(\sqrt{x})^2 = (-2)^2$ or $x = 4$. However, $x = 4$ doesn't check in the original equation, since $\sqrt{4} = 2$, not -2. Once again, the root[7] of all of our problems lies in the fact that a *negative* number to an *even* power results in a *positive* number. In other words, raising both sides of an equation to an even power does *not* produce an equivalent equation, but rather, an equation which may possess *more* solutions than the original. Hence the cautionary remark above about extraneous solutions.

Example 8.1.2. Solve the following equations.

1. $(5x+3)^4 = 16$
2. $1 - \dfrac{(5-2w)^3}{7} = 9$
3. $t + \sqrt{2t+3} = 6$
4. $\sqrt{2} - 3\sqrt[3]{2y+1} = 0$
5. $\sqrt{4x-1} + 2\sqrt{1-2x} = 1$
6. $\sqrt[4]{n^2+2} + n = 0$

For the remaining problems, assume that all of the variables represent positive real numbers.[8]

7. Solve for r: $V = \dfrac{4\pi}{3}(R^3 - r^3)$.
8. Solve for M_1: $\dfrac{r_1}{r_2} = \sqrt{\dfrac{M_2}{M_1}}$
9. Solve for v: $m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$. Again, assume that no arithmetic rules are violated.

Solution.

1. In our first equation, the quantity containing x is already isolated, so we extract fourth roots. The exponent is even, so when the roots are extracted we need both the positive and negative roots.

$$\begin{array}{rcll} (5x+3)^4 & = & 16 & \\ 5x+3 & = & \pm\sqrt[4]{16} & \text{Extract fourth roots} \\ 5x+3 & = & \pm 2 & \\ 5x+3=2 & \text{or} & 5x+3=-2 & \\ x = -\dfrac{1}{5} & \text{or} & x=-1 & \end{array}$$

We leave it to the reader to verify that both of these solutions satisfy the original equation.

2. In this example, we first need to isolate the quantity containing the variable w. Here, third (cube)

[7] Pun intended!

[8] That is, you needn't worry that you're multiplying or dividing by 0 or that you're forgetting absolute value symbols.

roots are required and since the exponent (index) is odd, we do not need the $\pm$:

$$\begin{array}{rcll}
1-\dfrac{(5-2w)^3}{7} & = & 9 & \\
-\dfrac{(5-2w)^3}{7} & = & 8 & \text{Subtract 1} \\
(5-2w)^3 & = & -56 & \text{Multiply by } -7 \\
5-2w & = & \sqrt[3]{-56} & \text{Extract cube root} \\
5-2w & = & \sqrt[3]{(-8)(7)} & \\
5-2w & = & \sqrt[3]{-8}\sqrt[3]{7} & \text{Product Rule} \\
5-2w & = & -2\sqrt[3]{7} & \\
-2w & = & -5-2\sqrt[3]{7} & \text{Subtract 5} \\
w & = & \dfrac{-5-2\sqrt[3]{7}}{-2} & \text{Divide by } -2 \\
w & = & \dfrac{5+2\sqrt[3]{7}}{2} & \text{Properties of Negatives}
\end{array}$$

The reader should check the answer because it provides a hearty review of arithmetic.

3. To solve $t+\sqrt{2t+3}=6$, we first isolate the square root, then proceed to square both sides of the equation. In doing so, we run the risk of introducing extraneous solutions so checking our answers here is a necessity.

$$\begin{array}{rcll}
t+\sqrt{2t+3} & = & 6 & \\
\sqrt{2t+3} & = & 6-t & \text{Subtract } t \\
(\sqrt{2t+3})^2 & = & (6-t)^2 & \text{Square both sides} \\
2t+3 & = & 36-12t+t^2 & \text{F.O.I.L. / Perfect Square Trinomial} \\
0 & = & t^2-14t+33 & \text{Subtract } 2t \text{ and } 3 \\
0 & = & (t-3)(t-11) & \text{Factor}
\end{array}$$

From the Zero Product Property, we know either $t-3=0$ (which gives $t=3$) or $t-11=0$ (which gives $t=11$). When checking our answers, we find $t=3$ satisfies the original equation, but $t=11$ does not.[9] So our final answer is $t=3$ only.

4. In our next example, we locate the variable (in this case y) beneath a cube root, so we first isolate

[9] It is worth noting that when $t=11$ is substituted into the original equation, we get $11+\sqrt{25}=6$. If the $+\sqrt{25}$ were $-\sqrt{25}$, the solution would check. Once again, when squaring both sides of an equation, we lose track of $\pm$, which is what lets extraneous solutions in the door.

that root and cube both sides.

$$\begin{array}{rcll}
\sqrt{2}-3\sqrt[3]{2y+1} & = & 0 & \\
-3\sqrt[3]{2y+1} & = & -\sqrt{2} & \text{Subtract } \sqrt{2} \\
\sqrt[3]{2y+1} & = & \dfrac{-\sqrt{2}}{-3} & \text{Divide by } -3 \\
\sqrt[3]{2y+1} & = & \dfrac{\sqrt{2}}{3} & \text{Properties of Negatives} \\
(\sqrt[3]{2y+1})^3 & = & \left(\dfrac{\sqrt{2}}{3}\right)^3 & \text{Cube both sides} \\
2y+1 & = & \dfrac{(\sqrt{2})^3}{3^3} & \\
2y+1 & = & \dfrac{2\sqrt{2}}{27} & \\
2y & = & \dfrac{2\sqrt{2}}{27}-1 & \text{Subtract 1} \\
2y & = & \dfrac{2\sqrt{2}}{27}-\dfrac{27}{27} & \text{Common denominators} \\
2y & = & \dfrac{2\sqrt{2}-27}{27} & \text{Subtract fractions} \\
y & = & \dfrac{2\sqrt{2}-27}{54} & \text{Divide by 2 } \left(\text{multiply by } \frac{1}{2}\right)
\end{array}$$

Since we raised both sides to an *odd* power, we don't need to worry about extraneous solutions but we encourage the reader to check the solution just for the fun of it.

5. In the equation $\sqrt{4x-1}+2\sqrt{1-2x}=1$, we have not one but two square roots. We begin by isolating one of the square roots and squaring both sides.

$$\begin{array}{rcll}
\sqrt{4x-1}+2\sqrt{1-2x} & = & 1 & \\
\sqrt{4x-1} & = & 1-2\sqrt{1-2x} & \text{Subtract } 2\sqrt{1-2x} \text{ from both sides} \\
(\sqrt{4x-1})^2 & = & (1-2\sqrt{1-2x})^2 & \text{Square both sides} \\
4x-1 & = & 1-4\sqrt{1-2x}+(2\sqrt{1-2x})^2 & \text{F.O.I.L. / Perfect Square Trinomial} \\
4x-1 & = & 1-4\sqrt{1-2x}+4(1-2x) & \\
4x-1 & = & 1-4\sqrt{1-2x}+4-8x & \text{Distribute} \\
4x-1 & = & 5-8x-4\sqrt{1-2x} & \text{Gather like terms}
\end{array}$$

At this point, we have just one square root so we proceed to isolate it and square both sides a second time.[10]

$$\begin{array}{rcll} 4x-1 & = & 5-8x-4\sqrt{1-2x} & \\ 12x-6 & = & -4\sqrt{1-2x} & \text{Subtract 5, add } 8x \\ (12x-6)^2 & = & (-4\sqrt{1-2x})^2 & \text{Square both sides} \\ 144x^2-144x+36 & = & 16(1-2x) & \\ 144x^2-144x+36 & = & 16-32x & \\ 144x^2-112x+20 & = & 0 & \text{Subtract 16, add } 32x \\ 4(36x^2-28x+5) & = & 0 & \text{Factor} \\ 4(2x-1)(18x-5) & = & 0 & \text{Factor some more} \end{array}$$

From the Zero Product Property, we know either $2x-1=0$ or $18x-5=0$. The former gives $x=\frac{1}{2}$ while the latter gives us $x=\frac{5}{18}$. Since we squared both sides of the equation (twice!), we need to check for extraneous solutions. We find $x=\frac{5}{18}$ to be extraneous, so our only solution is $x=\frac{1}{2}$.

6. As usual, our first step in solving $\sqrt[4]{n^2+2}+n=0$ is to isolate the radical. We then proceed to raise both sides to the fourth power to eliminate the fourth root:

$$\begin{array}{rcll} \sqrt[4]{n^2+2}+n & = & 0 & \\ \sqrt[4]{n^2+2} & = & -n & \text{Subtract } n \\ (\sqrt[4]{n^2+2})^4 & = & (-n)^4 & \text{Raise both sides to the } 4^{\text{th}} \text{ power} \\ n^2+2 & = & n^4 & \text{Properties of Negatives} \\ 0 & = & n^4-n^2-2 & \text{Subtract } n^2 \text{ and 2} \\ 0 & = & (n^2-2)(n^2+1) & \text{Factor - this is a ‘Quadratic in Disguise’} \end{array}$$

At this point, the Zero Product Property gives either $n^2-2=0$ or $n^2+1=0$. From $n^2-2=0$, we get $n^2=2$, so $n=\pm\sqrt{2}$. From $n^2+1=0$, we get $n^2=-1$, which gives no real solutions.[11] Since we raised both sides to an even (the fourth) power, we need to check for extraneous solutions. We find that $n=-\sqrt{2}$ works but $n=\sqrt{2}$ is extraneous.

7. In this problem, we are asked to solve for r. While there are a lot of letters in this equation[12], r

[10] To avoid complications with fractions, we'll forego dividing by the coefficient of $\sqrt{1-2x}$, namely -4. This is perfectly fine so long as we don't forget to square it when we square both sides of the equation.

[11] Why is that again?

[12] including a Greek letter, no less!

appears in only one term: r^3. Our strategy is to isolate r^3 then extract the cube root.

$$\begin{array}{rclr}
V & = & \frac{4\pi}{3}(R^3 - r^3) & \\
3V & = & 4\pi(R^3 - r^3) & \text{Multiply by 3 to clear fractions} \\
3V & = & 4\pi R^3 - 4\pi r^3 & \text{Distribute} \\
3V - 4\pi R^3 & = & -4\pi r^3 & \text{Subtract } 4\pi R^3 \\
\frac{3V - 4\pi R^3}{-4\pi} & = & r^3 & \text{Divide by } -4\pi \\
\frac{4\pi R^3 - 3V}{4\pi} & = & r^3 & \text{Properties of Negatives} \\
\sqrt[3]{\frac{4\pi R^3 - 3V}{4\pi}} & = & r & \text{Extract the cube root}
\end{array}$$

The check is, as always, left to the reader and highly encouraged.

8. The equation we are asked to solve in this example is from the world of Chemistry and is none other than Graham's Law of Effusion. As was mentioned in Example 7.1.2, subscripts in Mathematics are used to distinguish between variables and have no arithmetic significance. In this example, r_1, r_2, M_1 and M_2 are as different as x, y, z and 117. Since we are asked to solve for M_1, we locate M_1 and see it is in the denominator of a fraction which is inside of a square root. We eliminate the square root by squaring both sides and proceed from there.

$$\begin{array}{rclr}
\frac{r_1}{r_2} & = & \sqrt{\frac{M_2}{M_1}} & \\
\left(\frac{r_1}{r_2}\right)^2 & = & \left(\sqrt{\frac{M_2}{M_1}}\right)^2 & \text{Square both sides} \\
\frac{r_1^2}{r_2^2} & = & \frac{M_2}{M_1} & \\
r_1^2 M_1 & = & M_2 r_2^2 & \text{Multiply by } r_2^2 M_1 \text{ to clear fractions, assume } r_2, M_1 \neq 0 \\
M_1 & = & \frac{M_2 r_2^2}{r_1^2} & \text{Divide by } r_1^2\text{, assume } r_1 \neq 0
\end{array}$$

As the reader may expect, checking the answer amounts to a good exercise in simplifying rational and radical expressions. The fact that we are assuming all of the variables represent positive real numbers comes in to play, as well.

9. Our last equation to solve comes from Einstein's Special Theory of Relativity and relates the mass of an object to its velocity as it moves.[13] We are asked to solve for v which is located in just one term, namely v^2, which happens to lie in a fraction underneath a square root which is itself a denominator.

[13] See this article on the Lorentz Factor.

We have quite a lot of work ahead of us!

$$\begin{aligned}
m &= \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \\
m\sqrt{1-\frac{v^2}{c^2}} &= m_0 && \text{Multiply by } \sqrt{1-\frac{v^2}{c^2}} \text{ to clear fractions} \\
\left(m\sqrt{1-\frac{v^2}{c^2}}\right)^2 &= m_0^2 && \text{Square both sides} \\
m^2\left(1-\frac{v^2}{c^2}\right) &= m_0^2 && \text{Properties of Exponents} \\
m^2 - \frac{m^2v^2}{c^2} &= m_0^2 && \text{Distribute} \\
-\frac{m^2v^2}{c^2} &= m_0^2 - m^2 && \text{Subtract } m^2 \\
m^2v^2 &= -c^2(m_0^2 - m^2) && \text{Multiply by } -c^2 \ (c^2 \neq 0) \\
m^2v^2 &= -c^2m_0^2 + c^2m^2 && \text{Distribute} \\
v^2 &= \frac{c^2m^2 - c^2m_0^2}{m^2} && \text{Rearrange terms, divide by } m^2 \ (m^2 \neq 0) \\
v &= \sqrt{\frac{c^2m^2 - c^2m_0^2}{m^2}} && \text{Extract Square Roots, } v > 0 \text{ so no } \pm \\
v &= \frac{\sqrt{c^2(m^2 - m_0^2)}}{\sqrt{m^2}} && \text{Properties of Radicals, factor} \\
v &= \frac{|c|\sqrt{m^2 - m_0^2}}{|m|} \\
v &= \frac{c\sqrt{m^2 - m_0^2}}{m} && c > 0 \text{ and } m > 0 \text{ so } |c| = c \text{ and } |m| = m
\end{aligned}$$

Checking the answer algebraically would earn the reader great honor and respect on the Algebra battlefield so it is highly recommended.

8.1.1 Rationalizing Denominators and Numerators

In Section 5.3, there were a few instances where we needed to 'rationalize' a denominator - that is, take a fraction with radical in the denominator and re-write it as an equivalent fraction without a radical in the denominator. There are various reasons for wanting to do this,[14] but the most pressing reason is that rationalizing denominators - and numerators as well - gives us an opportunity for more practice with

[14]Before the advent of the handheld calculator, rationalizing denominators made it easier to get decimal approximations to fractions containing radicals. However, some (admittedly more abstract) applications remain today – one of which we'll explore in Section **??**; one you'll see in Calculus.

fractions and radicals. To refresh your memory, we rationalize a denominator and a numerator below:

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \frac{7\sqrt[3]{4}}{3} = \frac{7\sqrt[3]{4}\sqrt[3]{2}}{3\sqrt[3]{2}} = \frac{7\sqrt[3]{8}}{3\sqrt[3]{2}} = \frac{7 \cdot 2}{3\sqrt[3]{2}} = \frac{14}{3\sqrt[3]{2}}$$

In general, if the fraction contains either a single term numerator or denominator with an undesirable n$^{\text{th}}$ root, we multiply the numerator and denominator by whatever is required to obtain a perfect n$^{\text{th}}$ power in the radicand that we want to eliminate. If the fraction contains two terms the situation is somewhat more complicated. To see why, consider the fraction $\frac{3}{4-\sqrt{5}}$. Suppose we wanted to rid the denominator of the $\sqrt{5}$ term. We could try as above and multiply numerator and denominator by $\sqrt{5}$ but that just yields:

$$\frac{3}{4-\sqrt{5}} = \frac{3\sqrt{5}}{(4-\sqrt{5})\sqrt{5}} = \frac{3\sqrt{5}}{4\sqrt{5}-\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{4\sqrt{5}-5}$$

We haven't removed $\sqrt{5}$ from the denominator - we've just shuffled it over to the other term in the denominator. As you may recall, the strategy here is to multiply both the numerator and the denominator by what's called the **conjugate**.

Definition 8.1. Congugate of a Square Root Expression: If a, b and c are real numbers with $c > 0$ then the quantities $(a + b\sqrt{c})$ and $(a - b\sqrt{c})$ are **conjugates** of one another.[a] Conjugates multiply according to the Difference of Squares Formula:

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - (b\sqrt{c})^2 = a^2 - b^2c$$

[a]As are $(b\sqrt{c} - a)$ and $(b\sqrt{c} + a)$ because $(b\sqrt{c} - a)(b\sqrt{c} + a) = b^2c - a^2$.

That is, to get the conjugate of a two-term expression involving a square root, you change the '$-$' to a '+,' or vice-versa. For example, the conjugate of $4 - \sqrt{5}$ is $4 + \sqrt{5}$, and when we multiply these two factors together, we get $(4 - \sqrt{5})(4 + \sqrt{5}) = 4^2 - (\sqrt{5})^2 = 16 - 5 = 11$. Hence, to eliminate the $\sqrt{5}$ from the denominator of our original fraction, we multiply both the numerator and the denominator by the *conjugate* of $4 - \sqrt{5}$ to get:

$$\frac{3}{4-\sqrt{5}} = \frac{3(4+\sqrt{5})}{(4-\sqrt{5})(4+\sqrt{5})} = \frac{3(4+\sqrt{5})}{4^2-(\sqrt{5})^2} = \frac{3(4+\sqrt{5})}{16-5} = \frac{12+3\sqrt{5}}{11}$$

What if we had $\sqrt[3]{5}$ instead of $\sqrt{5}$? We could try multiplying $4 - \sqrt[3]{5}$ by $4 + \sqrt[3]{5}$ to get

$$(4 - \sqrt[3]{5})(4 + \sqrt[3]{5}) = 4^2 - (\sqrt[3]{5})^2 = 16 - \sqrt[3]{25},$$

which leaves us with a cube root. What we need to undo the cube root is a perfect cube, which means we look to the Difference of Cubes Formula for inspiration: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. If we take $a = 4$ and $b = \sqrt[3]{5}$, we multiply

$$(4 - \sqrt[3]{5})(4^2 + 4\sqrt[3]{5} + (\sqrt[3]{5})^2) = 4^3 + 4^2\sqrt[3]{5} + 4\sqrt[3]{5} - 4^2\sqrt[3]{5} - 4(\sqrt[3]{5})^2 - (\sqrt[3]{5})^3 = 64 - 5 = 59$$

So if we were charged with rationalizing the denominator of $\frac{3}{4-\sqrt[3]{5}}$, we'd have:

$$\frac{3}{4-\sqrt[3]{5}} = \frac{3(4^2+4\sqrt[3]{5}+(\sqrt[3]{5})^2)}{(4-\sqrt[3]{5})(4^2+4\sqrt[3]{5}+(\sqrt[3]{5})^2)} = \frac{48+12\sqrt[3]{5}+3\sqrt[3]{25}}{59}$$

This sort of thing extends to n$^{\text{th}}$ roots since $(a-b)$ is a factor of $a^n - b^n$ for all natural numbers n, but in practice, we'll stick with square roots with just a few cube roots thrown in for a challenge.[15]

Example 8.1.3. Rationalize the indicated numerator or denominator:

1. Rationalize the denominator: $\dfrac{3}{\sqrt[5]{24x^2}}$
2. Rationalize the numerator: $\dfrac{\sqrt{9+h}-3}{h}$

Solution.

1. We are asked to rationalize the denominator, which in this case contains a fifth root. That means we need to work to create fifth powers of each of the factors of the radicand. To do so, we first factor the radicand: $24x^2 = 8 \cdot 3 \cdot x^2 = 2^3 \cdot 3 \cdot x^2$. To obtain fifth powers, we need to multiply by $2^2 \cdot 3^4 \cdot x^3$ inside the radical.

$$\begin{aligned}
\frac{3}{\sqrt[5]{24x^2}} &= \frac{3}{\sqrt[5]{2^3 \cdot 3 \cdot x^2}} && \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^3 \cdot 3 \cdot x^2}\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}} && \text{Equivalent Fractions} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^3 \cdot 3 \cdot x^2 \cdot 2^2 \cdot 3^4 \cdot x^3}} && \text{Product Rule} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^5 \cdot 3^5 \cdot x^5}} && \text{Property of Exponents} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^5}\sqrt[5]{3^5}\sqrt[5]{x^5}} && \text{Product Rule} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{2 \cdot 3 \cdot x} && \text{Product Rule} \\
&= \frac{\cancel{3}\sqrt[5]{4 \cdot 81 \cdot x^3}}{2 \cdot \cancel{3} \cdot x} && \text{Reduce} \\
&= \frac{\sqrt[5]{324x^3}}{2x} && \text{Simplify}
\end{aligned}$$

2. Here, we are asked to rationalize the *numerator*. Since it is a two term numerator involving a square root, we multiply both numerator and denominator by the conjugate of $\sqrt{9+h}-3$, namely $\sqrt{9+h}+3$.

[15] To see what to do about fourth roots, use long division to find $(a^4 - b^4) \div (a-b)$, and apply this to $4 - \sqrt[4]{5}$.

After simplifying, we find an opportunity to reduce the fraction:

$$\begin{aligned}
\frac{\sqrt{9+h}-3}{h} &= \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)} && \text{Equivalent Fractions} \\
&= \frac{(\sqrt{9+h})^2-3^2}{h(\sqrt{9+h}+3)} && \text{Difference of Squares} \\
&= \frac{(9+h)-9}{h(\sqrt{9+h}+3)} && \text{Simplify} \\
&= \frac{h}{h(\sqrt{9+h}+3)} && \text{Simplify} \\
&= \frac{\cancelto{1}{h}}{\cancel{h}(\sqrt{9+h}+3)} && \text{Reduce} \\
&= \frac{1}{\sqrt{9+h}+3}
\end{aligned}$$

We close this section with an awesome example from Calculus.

Example 8.1.4. Simplify the compound fraction $\dfrac{\dfrac{1}{\sqrt{2(x+h)+1}}-\dfrac{1}{\sqrt{2x+1}}}{h}$ then rationalize the numerator of the result.

Solution. We start by multiplying the top and bottom of the 'big' fraction by $\sqrt{2x+2h+1}\sqrt{2x+1}$.

$$\begin{aligned}
\frac{\dfrac{1}{\sqrt{2(x+h)+1}}-\dfrac{1}{\sqrt{2x+1}}}{h} &= \frac{\dfrac{1}{\sqrt{2x+2h+1}}-\dfrac{1}{\sqrt{2x+1}}}{h} \\
&= \frac{\left(\dfrac{1}{\sqrt{2x+2h+1}}-\dfrac{1}{\sqrt{2x+1}}\right)\sqrt{2x+2h+1}\sqrt{2x+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} \\
&= \frac{\dfrac{\cancel{\sqrt{2x+2h+1}}\sqrt{2x+1}}{\cancel{\sqrt{2x+2h+1}}}-\dfrac{\sqrt{2x+2h+1}\cancel{\sqrt{2x+1}}}{\cancel{\sqrt{2x+1}}}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} \\
&= \frac{\sqrt{2x+1}-\sqrt{2x+2h+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}}
\end{aligned}$$

Next, we multiply the numerator and denominator by the conjugate of $\sqrt{2x+1}-\sqrt{2x+2h+1}$, namely

$\sqrt{2x+1} + \sqrt{2x+2h+1}$, simplify and reduce:

$$\begin{aligned}
\frac{\sqrt{2x+1} - \sqrt{2x+2h+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} &= \frac{(\sqrt{2x+1} - \sqrt{2x+2h+1})(\sqrt{2x+1} + \sqrt{2x+2h+1})}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{(\sqrt{2x+1})^2 - (\sqrt{2x+2h+1})^2}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{(2x+1) - (2x+2h+1)}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{2x+1-2x-2h-1}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{-2\cancel{h}}{\cancel{h}\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{-2}{\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})}
\end{aligned}$$

While the denominator is quite a bit more complicated than what we started with, we have done what was asked of us. In the interest of full disclosure, the reason we did all of this was to cancel the original '*h*' from the denominator. That's an awful lot of effort to get rid of just one little *h*, but you'll see the significance of this in Calculus. □

8.1.2 Exercises

In Exercises 1 - 13, perform the indicated operations and simplify.

1. $\sqrt{9x^2}$
2. $\sqrt[3]{8t^3}$
3. $\sqrt{50y^6}$
4. $\sqrt{4t^2+4t+1}$
5. $\sqrt{w^2-16w+64}$
6. $\sqrt{(\sqrt{12x}-\sqrt{3x})^2+1}$
7. $\sqrt{\dfrac{c^2-v^2}{c^2}}$
8. $\sqrt[3]{\dfrac{24\pi r^5}{L^3}}$
9. $\sqrt[4]{\dfrac{32\pi\varepsilon^8}{\rho^{12}}}$
10. $\sqrt{x}-\dfrac{x+1}{\sqrt{x}}$
11. $3\sqrt{1-t^2}+3t\left(\dfrac{1}{2\sqrt{1-t^2}}\right)(-2t)$
12. $2\sqrt[3]{1-z}+2z\left(\dfrac{1}{3\left(\sqrt[3]{1-z}\right)^2}\right)(-1)$
13. $\dfrac{3}{\sqrt[3]{2x-1}}+(3x)\left(-\dfrac{1}{3\left(\sqrt[3]{2x-1}\right)^4}\right)(2)$

In Exercises 14 - 25, find all real solutions.

14. $(2x+1)^3+8=0$
15. $\dfrac{(1-2y)^4}{3}=27$
16. $\dfrac{1}{1+2t^3}=4$
17. $\sqrt{3x+1}=4$
18. $5-\sqrt[3]{t^2+1}=1$
19. $x+1=\sqrt{3x+7}$
20. $y+\sqrt{3y+10}=-2$
21. $3t+\sqrt{6-9t}=2$
22. $2x-1=\sqrt{x+3}$
23. $w=\sqrt[4]{12-w^2}$
24. $\sqrt{x-2}+\sqrt{x-5}=3$
25. $\sqrt{2x+1}=3+\sqrt{4-x}$

In Exercises 26 - 29, solve each equation for the indicated variable. Assume all quantities represent positive real numbers.

26. Solve for h: $I=\dfrac{bh^3}{12}$.
27. Solve for a: $I_0=\dfrac{5\sqrt{3}a^4}{16}$
28. Solve for g: $T=2\pi\sqrt{\dfrac{L}{g}}$
29. Solve for v: $L=L_0\sqrt{1-\dfrac{v^2}{c^2}}$.

In Exercises 30 - 35, rationalize the numerator or denominator, and simplify.

30. $\dfrac{4}{3-\sqrt{2}}$
31. $\dfrac{7}{\sqrt[3]{12x^7}}$
32. $\dfrac{\sqrt{x}-\sqrt{c}}{x-c}$
33. $\dfrac{\sqrt{2x+2h+1}-\sqrt{2x+1}}{h}$
34. $\dfrac{\sqrt[3]{x+1}-2}{x-7}$
35. $\dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$

8.1.3 Answers

1. $3|x|$
2. $2t$
3. $5|y^3|\sqrt{2}$
4. $|2t+1|$
5. $|w-8|$
6. $\sqrt{3x+1}$
7. $\dfrac{\sqrt{c^2-v^2}}{|c|}$
8. $\dfrac{2r\sqrt[3]{3\pi r^2}}{L}$
9. $\dfrac{2\varepsilon^2\sqrt[4]{2\pi}}{|\rho^3|}$
10. $-\dfrac{1}{\sqrt{x}}$
11. $\dfrac{3-6t^2}{\sqrt{1-t^2}}$
12. $\dfrac{6-8z}{3(\sqrt[3]{1-z})^2}$
13. $\dfrac{4x-3}{(2x-1)\sqrt[3]{2x-1}}$
14. $x=-\dfrac{3}{2}$
15. $y=-1,2$
16. $t=-\dfrac{\sqrt[3]{3}}{2}$
17. $x=5$
18. $t=\pm 3\sqrt{7}$
19. $x=3$
20. $y=-3$
21. $t=-\dfrac{1}{3},\dfrac{2}{3}$
22. $x=\dfrac{5+\sqrt{57}}{8}$
23. $w=\sqrt{3}$
24. $x=6$
25. $x=4$
26. $h=\sqrt[3]{\dfrac{12I}{b}}$
27. $a=\dfrac{2\sqrt[4]{I_0}}{\sqrt[4]{5\sqrt{3}}}$
28. $g=\dfrac{4\pi^2L}{T^2}$
29. $v=\dfrac{c\sqrt{L_0^2-L^2}}{L_0}$
30. $\dfrac{12+4\sqrt{2}}{7}$
31. $\dfrac{7\sqrt[3]{18x^2}}{6x^3}$
32. $\dfrac{1}{\sqrt{x}+\sqrt{c}}$
33. $\dfrac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}}$
34. $\dfrac{1}{(\sqrt[3]{x+1})^2+2\sqrt[3]{x+1}+4}$
35. $\dfrac{1}{(\sqrt[3]{x+h})^2+\sqrt[3]{x+h}\sqrt[3]{x}+(\sqrt[3]{x})^2}$

8.2 Root and Radical Functions

In Sections 3.2, 4.2 and 5.4, we studied constant, linear, absolute value,[1] and quadratic functions. Constant, linear and quadratic functions were specific examples of polynomial functions, which we studied in generality in Chapter 6. Chapter 6 culminated with the Real Factorization Theorem, Theorem **??**, which says that all polynomial functions with real coefficients can be thought of as products of linear and quadratic functions. Our next step was to enlarge our field[2] of study to rational functions in Chapter 7. Being quotients of polynomials, we can ultimately view this family of functions as being built up of linear and quadratic functions as well. So in some sense, Sections 3.2, 4.2 and 5.4 along with Chapters 6 and 7 can be thought of as an exhaustive study of linear and quadratic[3] functions. We now turn our attention to functions involving radicals which cannot be written in terms of linear functions. For a more detailed review of the basics of roots and radicals, we refer the reader to Sections 1.2 and 8.1.

[1] These were introduced, as you may recall, as piecewise-defined linear functions.

[2] This is a really bad math pun.

[3] If we broaden our concept of functions to allow for complex valued coefficients, the Complex Factorization Theorem, Theorem **??**, tells us every function we have studied thus far is a combination of linear functions.

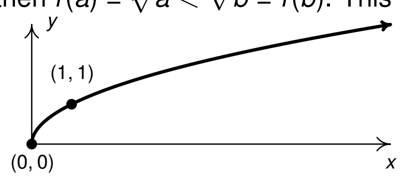
8.2.1 Root Functions

As with polynomial functions and rational functions, we begin our study of functions involving radical with a special family of functions: the (principal) root functions.

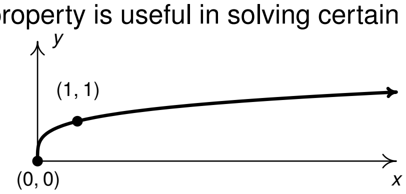
Definition 8.2. Let $n \in \mathbb{N}$ with $n \geq 2$. The n**th (principal) root function** is the function $f(x) = \sqrt[n]{x}$.
NOTE: If n is even, the domain of f is $[0, \infty)$; if n is odd, the domain of f is $(-\infty, \infty)$.

The domain restriction for even indexed roots means that, once again, we are restricting our attention to *real* numbers.[4] We graph a few members of the root function family below, and quickly notice that, as with the monomial, and, more generally, the Laurent monomial functions, the behavior of the root functions depends primarily on whether the root is even or odd.

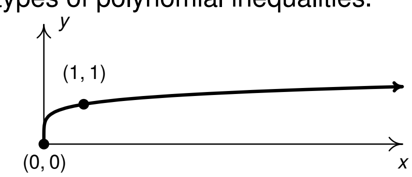
In addition to having the common domain of $[0, \infty)$, the graphs of $f(x) = \sqrt[n]{x}$ for even indices n all share the points $(0, 0)$ and $(1, 1)$. As n increases, the functions become 'steeper' near the y-axis and 'flatter' as $x \to \infty$. To show $f(x) \to \infty$ as $x \to \infty$, we show, more generally, the range of f is $[0, \infty)$. Indeed, if $c \geq 0$ is a real number, then $f(c^n) = \sqrt[n]{c^n} = c$ so c is in the range of f. Note that f is increasing: that is, if $a < b$, then $f(a) = \sqrt[n]{a} < \sqrt[n]{b} = f(b)$. This property is useful in solving certain types of polynomial inequalities.[5]



$y = \sqrt{x}$

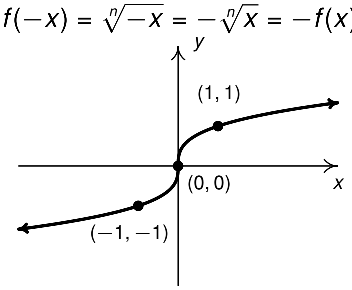


$y = \sqrt[4]{x}$

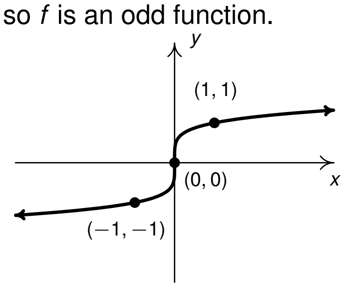


$y = \sqrt[6]{x}$

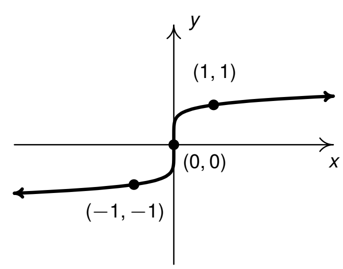
The functions $f(x) = \sqrt[n]{x}$ for odd natural numbers $n \geq 3$ also follow a predictable trend - steepening near $x = 0$ and flattening as $x \to \pm\infty$. The range for these functions is $(-\infty, \infty)$ since if c is any real number, $f(c^n) = \sqrt[n]{c^n} = c$, so c is in the range of f. Like the even indexed roots, the odd indexed roots are also increasing. Moreover, these graphs appear to be symmetric about the origin. Sure enough, when n is odd, $f(-x) = \sqrt[n]{-x} = -\sqrt[n]{x} = -f(x)$ so f is an odd function.



$y = \sqrt[3]{x}$



$y = \sqrt[5]{x}$



$y = \sqrt[7]{x}$

At this point, you're probably expecting a theorem like Theorems 4.4, 5.7, 6.1, 7.1 - that is, a theorem which tells us how to obtain the graph of $F(x) = a\sqrt[n]{x - h} + k$ from the graph of $f(x) = \sqrt[n]{x}$ - and you would not be wrong. Here, however, we need to add an extra parameter 'b' to the recipe and discuss functions of the form $F(x) = a\sqrt[n]{bx - h} + k$. The reason is that, with all of the previous function families, we were always able to factor out the coefficient of x. We list some examples of this below, and invite the reader to revisit

[4] Although we discussed imaginary numbers in Section **??**, we restrict our attention to real numbers in this section. See the epilogue on page **??** for more details.

[5] See Exercise 13.

other examples in the text:

- $F(x) = |6 - 2x| = |-2x + 6| = |-2(x+3)| = |-2||x+3| = 2|x+3|$.
- $F(x) = (2x-1)^2 + 1 = \left[2\left(x - \frac{1}{2}\right)\right]^2 + 1 = (2)^2\left(x - \frac{1}{2}\right)^2 + 1 = 4\left(x - \frac{1}{2}\right)^2 + 1$
- $F(x) = \frac{2}{(1-x)^3} - 5 = \frac{2}{[(-1)(x-1)]^3} - 5 = \frac{2}{(-1)^3(x-1)^3} - 5 = \frac{2}{-(x-1)^3} - 5 = \frac{-2}{(x-1)^3} - 5$.

For a function like $F(x) = \sqrt{4x-12} + 1 = \sqrt{4(x-6)} + 1 = \sqrt{4}\sqrt{x-3} + 1 = 2\sqrt{x-3} + 1$, this approach works fine. However, if the coefficient of x is *negative*, for example, $F(x) = \sqrt{1-x} = \sqrt{(-1)(x-1)}$ we get stuck the product rule for radicals doesn't extend to negative quantities when the index is even.[6] Hence we add an extra parameter which means we have an extra step. We state Theorem 8.2 below.

Theorem 8.2. For real numbers a, b, h, and k with $a, b \neq 0$, the graph of $F(x) = a\sqrt[n]{bx-h} + k$ can be obtained from the graph of $f(x) = \sqrt[n]{x}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{x}$ to $y = \sqrt[n]{x-h}$.

2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{x-h}$ to $y = \sqrt[n]{bx-h}$.

3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h}$.

4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a\sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h} + k$.

Proof. As usual, we 'build' the graph of $F(x) = a\sqrt[n]{bx-h} + k$ starting with the graph of $f(x) = \sqrt[n]{x}$ one step at a time. First, we consider the graph of $F_1(x) = \sqrt{x-h}$. A generic point on the graph of F_1 looks like $(x, \sqrt[n]{x-h})$. Note that if n is odd, x can be any real number whereas if n is even $x - h \geq 0$ so $x \geq h$. If we let $c = x - h$, then $x = c + h$ and we can change (dummy) variables[7] and obtain a new representation of the point: $(c + h, \sqrt[n]{c})$. Note that if n is odd, x and c vary through all real numbers; if n is even, $x \geq h$ and, hence, $c \geq 0$. Since a generic point on the graph of $f(x) = \sqrt[n]{x}$ can be represented as $(c, \sqrt[n]{c})$ for applicable values of c, we see that we can obtain every point on the graph of F_1 by adding h to each x-coordinate of the graph of f, establishing step 1 of the theorem.
Proceeding to (the new!) step 2, a point on the graph of $F_2(x) = \sqrt[n]{bx-h}$ has the form $(x, \sqrt[n]{bx-h})$. If n is odd, as usual, x can vary through all real numbers. If n is even, we require $bx - h \geq 0$ or $bx \geq h$.

[6]Since, otherwise, $-1 = i^2 = i \cdot i = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$, a contradition.
[7]again this is because every real number can be represented as both $x - h$ for some value x and as $c + h$ for some value c.

If $b > 0$, this gives $x \geq \frac{h}{b}$. If, on the other hand, $b < 0$, the we have $x \leq \frac{h}{b}$. Let $c = bx$ and since by assumption $b \neq 0$, we have $x = \frac{c}{b}$. Once again, we change dummy variables from x to c and describe a generic point on the graph of F_2 as $\left(\frac{c}{b}, \sqrt[n]{c-h}\right)$. If n is odd, x and c can vary through all real numbers. If n is even and $b > 0$, then $x \geq \frac{h}{b}$ and, hence, $c = bx \geq h$; if $b < 0$, then $x \leq \frac{h}{b}$ also gives $c = bx \geq h$. Since a generic point on the graph of F_1 can be represented as $(c, \sqrt{c-h})$ for applicable values of c, we see we can obtain every point on the graph of F_2 by dividing every x-coordinate on the graph of F_1 by b, as per step 2 of the theorem.

The proof of steps 3 and 4 of Theorem 8.2 are identical to the proof of Theorem 6.1 (just with $\sqrt[n]{\cdot}$ instead of $(\cdot)^n$) so we invite the reader to work through the details on their own. $\square$

We demonstrate Theorem 8.2 in the following example.

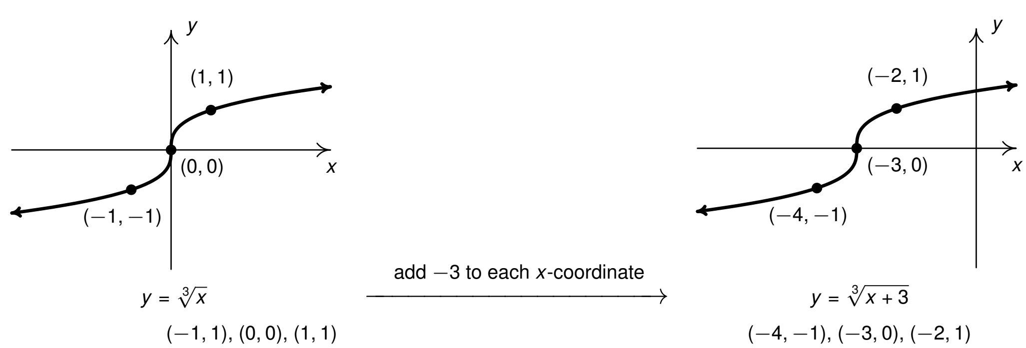
Example 8.2.1. Theorem 8.2 to graph the following functions. Label at least three points on the graph. State the domain and range using interval notation.

1. $f(x) = 1 - 2\sqrt[3]{x+3}$
2. $g(t) = \dfrac{\sqrt{1-2t}}{4}$

Solution.

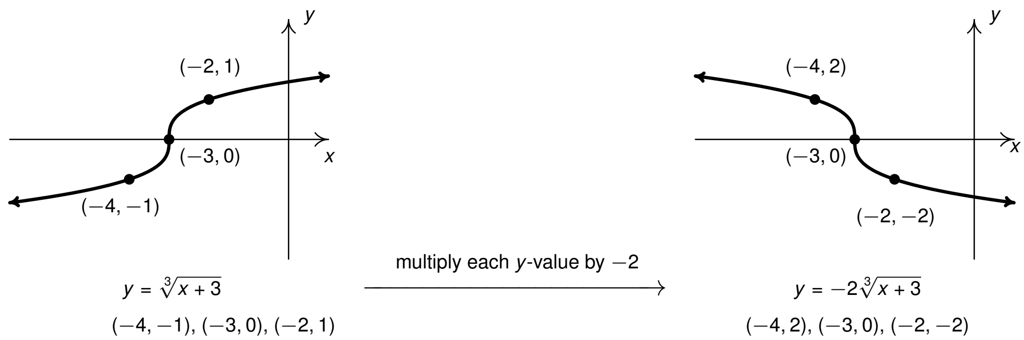
1. We begin by rewriting the expression for $f(x)$ in the form prescribed Theorem 8.2: $f(x) = -2\sqrt[3]{x+3} + 1$. We identify $n = 3$, $a = -2$, $b = 1$, $h = -3$ and $k = 1$.

   Step 1: add -3 to each of the x-coordinates of each of the points on the graph of $y = \sqrt[3]{x}$:

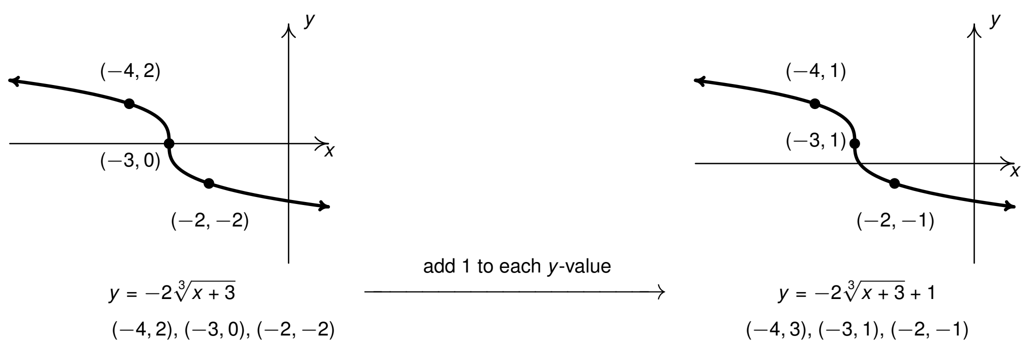


   Since $b = 1$, we can proceed to Step 3 (since dividing a real by 1 just results in the same real number.)

   Step 3: multiply each of the y-coordinates of each point on the graph of $y = \sqrt[3]{x+3}$ by -2:



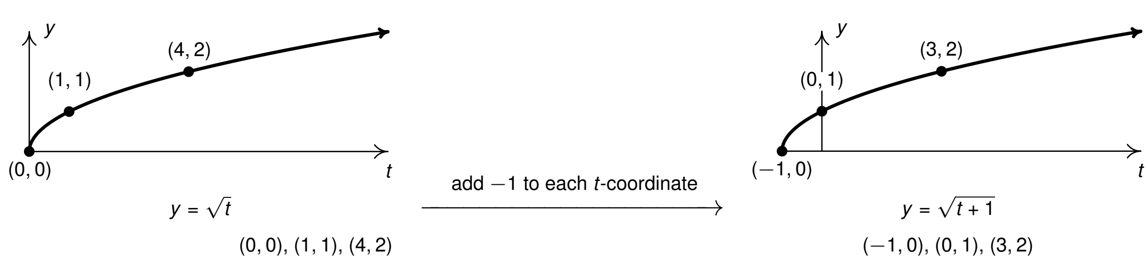
Step 4: add 1 to y-coordinates of each point on the graph of $y = -2\sqrt[3]{x+3}$:



We get the domain and range of f are $(-\infty, \infty)$.

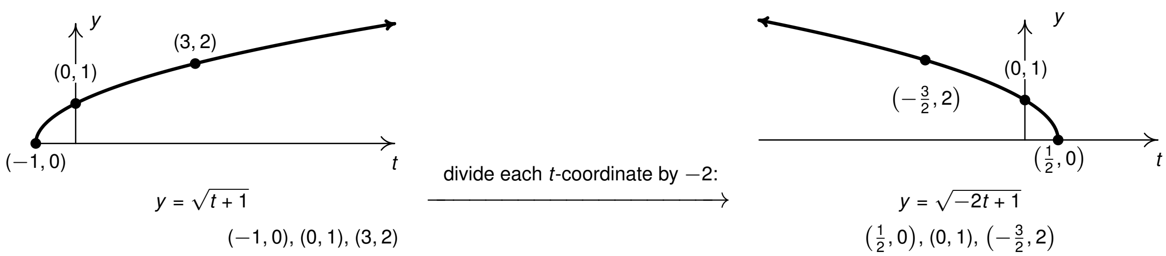
2. For $g(t) = \dfrac{\sqrt{1-2t}}{4} = \frac{1}{4}\sqrt{-2t+1}$, we identify $n = 2$, $a = \frac{1}{4}$, $b = -2$, $h = -1$ and $k = 0$. Since we are asked to label *three* points on the graph, we track $(4, 2)$ along with $(0, 0)$ and $(1, 1)$.[8]

   Step 1: add -1 to each of the t-coordinates of each of the points on the graph of $y = \sqrt{t}$:

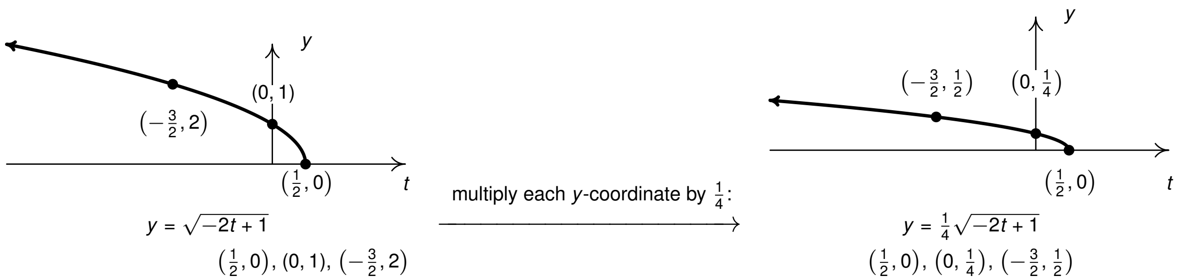


   Step 2: divide each of the t-coordinates of each of the points on the graph of $y = \sqrt{t+1}$ by -2:

[8] As $\sqrt{4} = 2$, we know $(4, 2)$ is on the graph of $y = \sqrt{t}$.



Step 3: multiply each of the y-coordinates of each of the points on the graph of $y = \sqrt{-2t+1}$ by $\frac{1}{4}$:



We get the domain is $\left(-\infty, \frac{1}{2}\right]$ and the range is $[0, \infty)$. □

8.2.2 Other Functions involving Radicals

Now that we have some practice with basic root functions, we turn our attention to more general functions involving radicals. In general, Calculus is the best tool with which to study these functions. Nevertheless, we will use what algebra we know in combination with a graphing utility to help us visualize these functions and preview concepts which are studied in greater depth in later courses. In the table below, we summarize some of the properties of radicals from elsewhere in this text (and Intermediate Algebra) we will be using in the coming examples.

Theorem 8.3. Some Useful Properties of Radicals: Suppose $\sqrt[n]{x}$, $\sqrt[n]{a}$, and $\sqrt[n]{b}$ are real numbers.[a]
Simplifying n th powers and n th roots:[b]

- $\left(\sqrt[n]{x}\right)^n = x$.
- if n is odd, then $\sqrt[n]{x^n} = x$
- if n is even, then $\sqrt[n]{x^n} = |x|$.

Root Functions Preserve Inequality:[c] if $a \leq b$, then $\sqrt[n]{a} \leq \sqrt[n]{b}$.

[a] i.e., if n is odd, x, a, and b can be any real numbers; if, on the other hand n is even, $x \geq 0$, $a \geq 0$, and $b \geq 0$.
[b] a.k.a., 'Inverse Properties.' See Section 9.4.
[c] i.e., root functions are increasing.

Example 8.2.2. For the following functions:

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.

- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase.
  - intervals of decrease.
- Construct a sign diagram for each function using the intercepts and graph.[9]

1. $f(x) = 3x\sqrt[3]{2-x}$
2. $g(t) = \sqrt[3]{\dfrac{8t}{t+1}}$
3. $h(x) = \dfrac{3x}{\sqrt{x^2+1}}$
4. $r(t) = t^{-1}\sqrt{16t^4-1}$

Solution.

1. When looking for the domain, we have two thing to watch out for: denominators (which we must make sure aren't 0) and even indexed radicals (whose radicands we must ensure are nonnegative.) Looking at the expression for $f(x)$, we have no denominators nor do we have an even indexed radical, so we are confident the domain is all real numbers, $(-\infty, \infty)$.

   To find the x-intercepts, we find the zeros of f by solving $f(x) = 3x\sqrt[3]{2-x} = 0$. Using the zero product property, we get $3x = 0$ or $\sqrt[3]{2-x} = 0$. The former gives $x = 0$ and to solve the latter, we cube both sides and get $2 - x = 0$ or $x = 2$. Hence, the x-intercepts are $(0, 0)$ and $(2, 0)$. Since $(0, 0)$ is also on the y-axis and functions can can have at most one y-intercept, we know $(0, 0)$ is the only y-intercept.[10] That being said, we can quickly verify $f(0) = 3(0)\sqrt[3]{2-0} = 0$.

   To determine the end behavior, we consider $f(x)$ as $x \to \pm\infty$. Using 'number sense,' [11] we have $f(x) = 3x\sqrt[3]{2-x} = 3x\sqrt[3]{-x+2} \approx (\text{big } (+))\sqrt[3]{\text{big } (-)} = (\text{big } (+))(\text{big } (-)) = \text{big } (-)$, so $f(x) \to -\infty$. As $x \to -\infty$ we get $f(x) = 3x\sqrt[3]{-x+2} \approx (\text{big } (-))\sqrt[3]{\text{big } (+)} = (\text{big } (-))(\text{big } (+)) = \text{big } (-)$, so $f(x) \to -\infty$ here, too.

   We graph f below on the left. From the graph, the range appears to be $(-\infty, 3.572]$ with a local maximum (which also happens to be *the* maximum) at $(1.5, 3.572)$. We also see f appears to be increasing on $(-\infty, 1.5)$ and decreasing on $(1.5, \infty)$. It is also worth noting that there appears to be 'unusual steepness' near the x-intercept $(2, 0)$. We invite the reader to zoom in on the graph near $(2, 0)$ to see that the function is 'locally vertical.'[12]

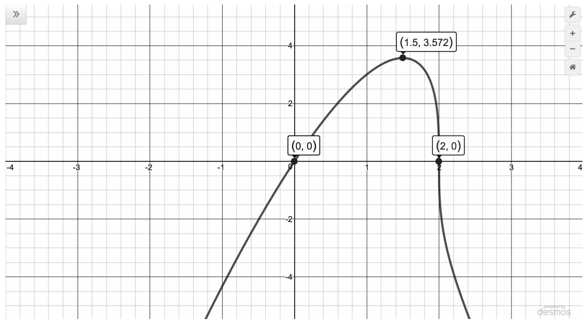
   To create a sign diagram for $f(x)$, we note that the function has zeros $x = 0$ and $x = 2$. For $x < 0$, $f(x) < 0$ or $(-)$, for $0 < x < 2$, $f(x) > 0$ or $(+)$, and for $x > 2$, $f(x) < 0$ or $(-)$. The sign diagram for $f(x)$ is below on the right.

[9] We'll revisit sign diagrams for these functions in Section **??** where we will use them to solve inequalities (surprised?)

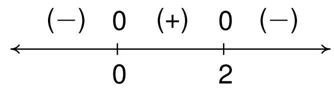
[10] Why is this, again?

[11] remember this means we use the adjective 'big' here to mean large in *absolute value*

[12] Of course, the Vertical Line Test prohibits the graph from actually *being* a vertical line. This behavior is more precisely defined and more closely studied in Calculus.



The graph of $y = f(x)$



Sign Diagram for $f(x)$

2. The index of the radical in the expression for $g(t)$ is odd, so our only concern is the denominator. Setting $t + 1 = 0$ gives $t = -1$, which we exclude, so our domain is $\{t \in \mathbb{R} \,|\, t \neq -1\}$ or using interval notation, $(-\infty, -1) \cup (-1, \infty)$. If we take the time to analyze the behavior of g near $t = -1$, we find that as $t \to -1^-$, $g(t) = \sqrt[3]{\frac{8t}{t+1}} \approx \sqrt[3]{\frac{-8}{\text{small }(-)}} \approx \sqrt[3]{\text{big}(+)} = \text{big }(+)$. That is, as $t \to -1^-$, $g(t) \to \infty$. Likewise, as $t \to -1^+$, $g(t) \approx \sqrt[3]{\frac{-8}{\text{small }(+)}} \approx \sqrt[3]{\text{big }(-)} = \text{big }(-)$. This suggests as $t \to -1^+$, $g(t) \to -\infty$. This behavior points to a vertical asymptote, $t = -1$.

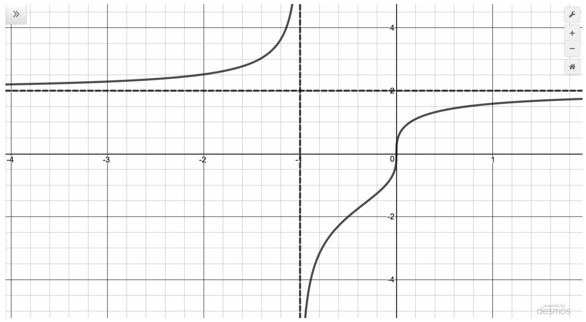
   To find the t-intercepts of the graph of g, we find the zeros of g by setting $g(t) = \sqrt[3]{\frac{8t}{t+1}} = 0$. Cubing both sides and clearing denominators gives $8t = 0$ or $t = 0$. Hence our t-, and in this case, y-intercept is $(0, 0)$.

   To determine the end behavior, we note that as $t \to \pm\infty$, $\frac{8t}{t+1} \to \frac{8}{1} = 8$. Hence, it stands to reason that as $t \to \pm\infty$, $g(t) = \sqrt[3]{\frac{8t}{t+1}} \to \sqrt[3]{8} = 2$. This suggests the graph of $y = g(t)$ has a horizontal asymptote at $y = 2$.

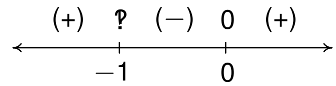
   We graph $y = g(t)$ below on the left. The graph confirms our suspicions about the asymptotes $t = -1$ and $y = 2$. Moreover, the range appears to be $(-\infty, 2) \cup (2, \infty)$. We could check if the graph ever crosses its horizontal asymptote by attempting to solve $g(t) = \sqrt[3]{\frac{8t}{t+1}} = 2$. Cubing both sides and clearing denominators gives $8t = 8(t + 1)$ which gives $0 = 8$, a contradiction. This proves 2 is not in the range, as we had suspected.

   Scanning the graph, there appears to be no local extrema, and, moreover, the graph suggests g is increasing on $(-\infty, -1)$ and again on $(-1, \infty)$. As with the previous example, the graph appears locally vertical near its intercept $(0, 0)$.

   To create a sign diagram for $g(t)$, we note that the function is undefined when $t = -1$ (so we place a '?' above it) and has a zero $t = 0$. When $t < -1$, $g(t) > 0$ or (+), for $-1 < t < 0$, $g(t) < 0$ or $(-)$, and for $t > 0$, $g(t) > 0$ or (+). Below on the right is a sign diagram for $g(t)$.



The graph of $y = g(t)$



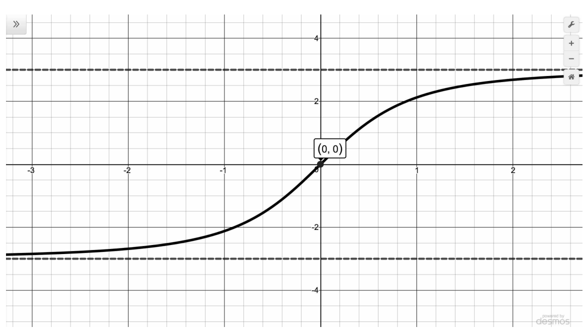
Sign Diagram for $g(t)$

3. The expression for $h(x) = \frac{3x}{\sqrt{x^2+1}}$ has both a denominator and an even-indexed radical, so we have to be extra cautious here. Fortunately for us, the quantity $x^2 + 1 > 0$ for al real numbers x. Not only does this mean $\sqrt{x^2+1}$ is always defined, it also tells us $\sqrt{x^2+1} > 0$ for all x, too. This means the domain of h is all real numbers, $(-\infty, \infty)$.

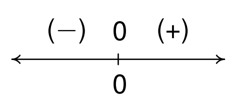
   Solving for the zeros of h gives only $x = 0$, and we find, once again, $(0, 0)$ is both our lone x- and y-intercept. Moving on to end behavior, as $x \to \pm\infty$, the term x^2 is the dominant term in the radicand in the denominator. As such, $h(x) = \frac{3x}{\sqrt{x^2+1}} \approx \frac{3x}{\sqrt{x^2}} = \frac{3x}{|x|}$. As $x \to \infty$, $|x| = x$ (since $x > 0$), so $h(x) \approx \frac{3x}{x} = 3$, so $h(x) \to 3$. Likewise, as $x \to -\infty$, $|x| = -x$ (since $x < 0$) and hence, $h(x) \approx \frac{3x}{-x} = -3$, so $h(x) \to -3$. This analysis suggests the graph of $y = h(x)$ has not one, but *two* horizontal asymptotes.[13] The graph of h below on the left bears this out.

   From the graph, we see the range of h appears to be $(-3, 3)$. Attempting to solve $h(x) = \frac{3x}{\sqrt{x^2+1}} = \pm 3$ gives, in either case, $9x^2 = 9(x^2 + 1)$ which reduces to $0 = 9$, a contradiction. Hence, the graph of $y = h(x)$ never reaches its horizontal asymptotes. Moreover, h appears to be always increasing, with no local extrema or 'unusual' steepness. One last remark: it appears as if the graph of h is symmetric about the origin. We check $h(-x) = \frac{3(-x)}{\sqrt{(-x)^2+1}} = -\frac{3x}{\sqrt{x^2+1}} = -h(x)$ which verifies h is odd.

   Since the domain of h is all real number and the only zero of h is $x = 0$, the sign diagram for $h(x)$ is fairly straight forward. For $x < 0$, $h(x) < 0$ or $(-)$ and for $x > 0$, $h(x) > 0$ or $(+)$. The sign diagram for $h(x)$ is below on the right.



The graph of $y = h(x)$



Sign Diagram for $h(x)$

[13]We warned you this was coming . . . see the discussion following Theorem 7.3 in Section 7.2.

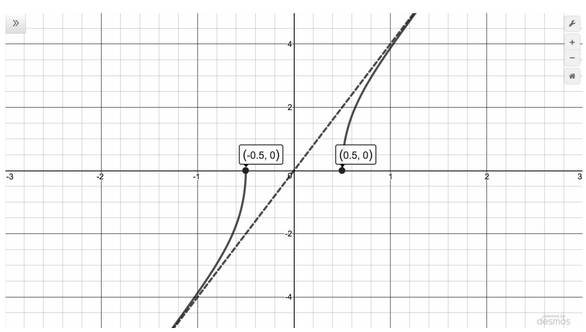
4. The first thing to note about the expression $r(t) = t^{-1}\sqrt{16t^4 - 1}$ is that $t^{-1} = \frac{1}{t}$. Hence, we must exclude $t = 0$ from the domain straight away. Next, we have an even-indexed radical expression: $\sqrt{16t^4 - 1}$. In order for this to return a real number, we require $16t^4 - 1 \geq 0$. Instead of using a sign diagram to solve this,[14] we opt instead to *carefully* use properties of radicals. Isolating t^4, we have $t^4 \geq \frac{1}{16}$. Since the root functions are increasing, we can apply the fourth root to both sides and preserve the inequality: $\sqrt[4]{t^4} \geq \sqrt[4]{\frac{1}{16}}$ which gives[15] $|t| \geq \frac{1}{2}$. Note that since $t = 0$ does *not* satisfy this inequality, restricting t in this manner takes care of *both* domain issues, so the domain is $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$.

   Next, we look for zeros. Setting $r(t) = t^{-1}\sqrt{16t^4 - 1} = \frac{\sqrt{16t^4-1}}{t} = 0$ gives $\sqrt{16t^4 - 1} = 0$. After squaring both sides, we get $16t^4 - 1 = 0$ or $t^4 = \frac{1}{16}$. Extracting fourth roots, we get $t = \pm\frac{1}{2}$. Both of these are (barely!) in the domain of r, so our t intercepts are $\left(-\frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, 0\right)$. Note, the graph of r has no y-intercept, since $r(0)$ is undefined ($t = 0$ is not in the domain of r).

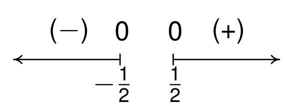
   Concerning end behavior, we note the term $16t^4$ dominates the radicand $\sqrt{16t^4 - 1}$ as $t \to \pm\infty$, hence, $r(t) = \frac{\sqrt{16t^4-1}}{t} \approx \frac{\sqrt{16t^4}}{t} = \frac{4t^2}{t} = 4t$. This suggests the graph of $y = r(t)$ has a slant asymptote with slope 4.[16]

   We graph $y = r(t)$ below on the left. We see the range appears to be all real numbers, $(-\infty, \infty)$. It appears as if r is increasing on $\left(-\infty, -\frac{1}{2}\right]$ and again on $\left[\frac{1}{2}, \infty\right)$. The graph does appear to be asymptotic to $y = 4t$, and it also appears to be symmetric about the origin. Sure enough, we find $r(-t) = \frac{\sqrt{16(-t)^4-1}}{-t} = -\frac{\sqrt{16t^4-1}}{t} = -r(t)$, proving r is an odd function.

   To construct the sign diagram for $r(t)$ we note r has two zeros, $t = \pm\frac{1}{2}$. For $t < \frac{1}{2}$, $r(t) < 0$ or $(-)$ and when $t > \frac{1}{2}$, $r(t) > 0$ or $(+)$. When $-\frac{1}{2} < t < \frac{1}{2}$, r is undefined so we have removed that segment from the diagram, as seen below on the right.



The graph of $y = r(t)$



Sign Diagram for $r(t)$.

□

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[14] See Section 6.3

[15] Recall: $\sqrt[n]{x^n} = |x|$, not x, if n is even.

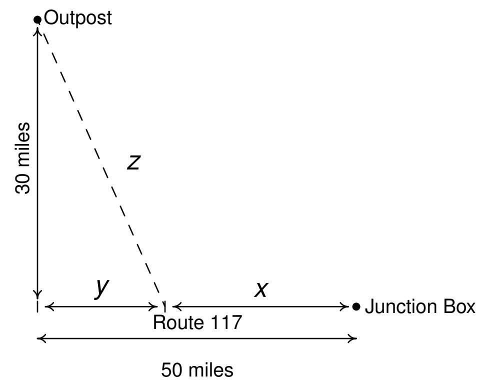
[16] Note: this analysis suggests the slant asymptote is $y = 4t + b$, but from this analysis, we cannot determine the value of b. As with slant asymptotes in Section 7.2, we'd need to perform a more detailed analysis which we omit in this case owing to the complexity of the function.

We end this section with a classic application of root functions.

Example 8.2.3. Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles down the road from the post, as indicated in the diagram below. Suppose it costs \$15 per mile to run cable along the road and \$20 per mile to run cable off of the road.

1. Find an expression $C(x)$ which computes the cost of connecting the Junction Box to the Outpost as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the Outpost. Determine a reasonable applied domain for the problem.

2. Use your calculator to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off of the road?



Solution.

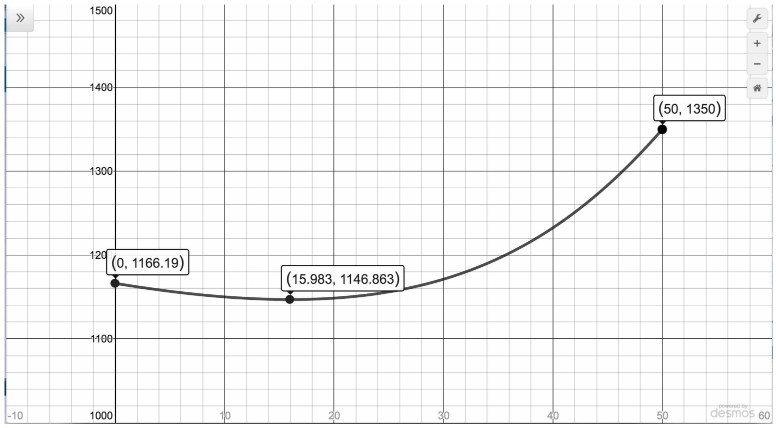
1. The cost is broken into two parts: the cost to run cable along Route 117 at \$15 per mile, and the cost to run it off road at \$20 per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $15x + 20z$.

   Our next goal is to determine z in terms of x. The diagram suggests we can use the Pythagorean Theorem to get $y^2 + 30^2 = z^2$. But we also see $x + y = 50$ so that $y = 50 - x$. Substituting $(50 - x)$ in for y we obtain $z^2 = (50 - x)^2 + 900$. Solving for z, we obtain $z = \pm\sqrt{(50 - x)^2 + 900}$. Since z represents a distance, we choose $z = \sqrt{(50 - x)^2 + 900}$.

   Hence, the cost as a function of x is given by $C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}$. From the context of the problem, we have $0 \leq x \leq 50$.

2. We graph $y = C(x)$ below and find our (local) minimum to be at the point $(15.98, 1146.86)$. Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route

117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is \$1146.86. The ability to stream live SasquatchCasts? Priceless.



$\square$

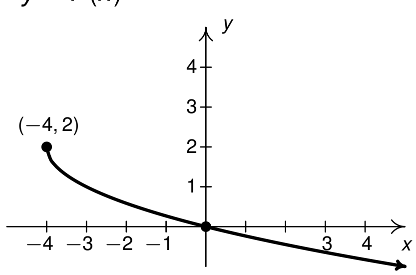
8.2.3 Exercises

In Exercises 1 - 8, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 8.2. Track at least two points and state the domain and range using interval notation.

1. $f(x) = \sqrt{x}$, $F(x) = \sqrt{x+3} - 2$
2. $f(x) = \sqrt{x}$, $F(x) = \sqrt{4-x} - 1$
3. $f(x) = \sqrt[3]{x}$, $F(x) = \sqrt[3]{x-1} - 2$
4. $f(x) = \sqrt[3]{x}$, $F(x) = -\sqrt[3]{8x+8} + 4$
5. $f(x) = \sqrt[4]{x}$, $F(x) = \sqrt[4]{x-1} - 2$
6. $f(x) = \sqrt[4]{x}$, $F(x) = -3\sqrt[4]{x-7} + 1$
7. $f(x) = \sqrt[5]{x}$, $F(x) = \sqrt[5]{x+2} + 3$
8. $f(x) = \sqrt[8]{x}$, $F(x) = \sqrt[8]{-x} - 2$

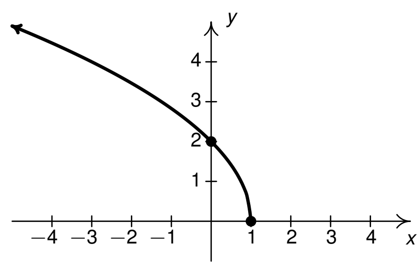
In Exercises 9 - 10, find a formula for each function below in the form $F(x) = a\sqrt{bx-h} + k$.
NOTE: There may be more than one solution!

9. $y = F(x)$



x,y-intercept $(0, 0)$

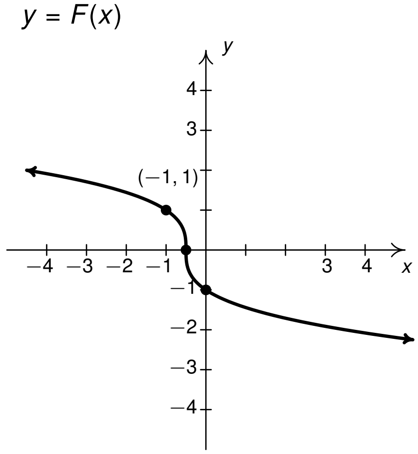
10. $y = F(x)$



x-intercept $(1, 0)$, y-intercept $(0, 2)$

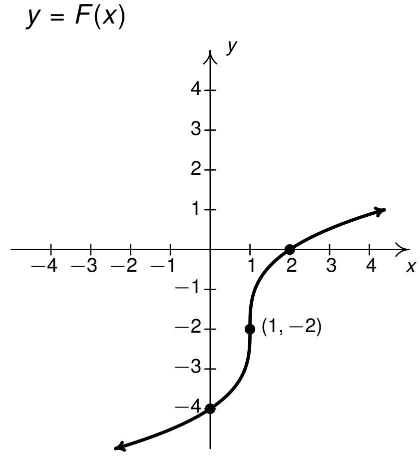
In Exercises 11 - 12, find a formula for each function below in the form $F(x) = a\sqrt[3]{bx-h} + k$.
NOTE: There may be more than one solution!

11. $y = F(x)$



x-intercept $\left(-\frac{1}{2}, 0\right)$,$y$-intercept $(0, -1)$

12. $y = F(x)$



x-intercept $(2, 0)$, y-intercept $(0, -4)$

13. Use the fact that the nth root functions are increasing to solve the following polynomial inequalities:

(a) $x^3 \leq 64$ (b) $2 - t^5 < 34$ (c) $\dfrac{(2z+1)^3}{4} \geq 2$

For the following inequalities, remember $\sqrt[n]{x^n} = |x|$ if n is even:

(d) $x^4 \leq 16$ (e) $6 - t^6 < -58$ (f) $\dfrac{(2z+1)^4}{3} \geq 27$

For each function in Exercises 14 - 21 below

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase/decrease.
  - any 'unusual steepness' or 'local' verticality.
  - vertical asymptotes.
  - horizontal / slant asymptotes.
- Construct a sign diagram for each function using the intercepts and graph.
- Comment on any observed symmetry.

14. $f(x) = \sqrt{1 - x^2}$
15. $f(x) = \sqrt{x^2 - 1}$
16. $g(t) = t\sqrt{1 - t^2}$
17. $g(t) = t\sqrt{t^2 - 1}$
18. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
19. $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$
20. $g(t) = \sqrt{t(t+5)(t-4)}$
21. $g(t) = \sqrt[3]{t^3 + 3t^2 - 6t - 8}$
22. Rework Example 8.2.3 so that the outpost is 10 miles from Route 117 and the nearest junction box is 30 miles down the road for the post.
23. The volume V of a right cylindrical cone depends on the radius of its base r and its height h and is given by the formula $V = \frac{1}{3}\pi r^2 h$. The surface area S of a right cylindrical cone also depends on r and h according to the formula $S = \pi r\sqrt{r^2 + h^2}$. In the following problems, suppose a cone is to have a volume of 100 cubic centimeters.

(a) Use the formula for volume to find the height as a function of r, $h(r)$.

(b) Use the formula for surface area along with your answer to 23a to find the surface area as a function of r, $S(r)$.

(c) Use your calculator to find the values of r and h which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

24. The period of a pendulum in seconds is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(for small displacements) where L is the length of the pendulum in meters and $g = 9.8$ meters per second per second is the acceleration due to gravity. My Seth-Thomas antique schoolhouse clock needs $T = \frac{1}{2}$ second and I can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should I set the pendulum?

25. According to Einstein's Theory of Special Relativity, the observed mass of an object is a function of how fast the object is traveling. Specifically, if m_r is the mass of the object at rest, v is the speed of the object and c is the speed of light, then the observed mass of the object $m(v)$ is given by:

$$m(v) = \frac{m_r}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

(a) Find the applied domain of the function.

(b) Compute $m(.1c)$, $m(.5c)$, $m(.9c)$ and $m(.999c)$.

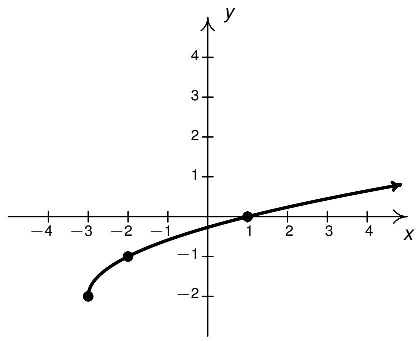
(c) As $v \to c^-$, what happens to $m(x)$?

(d) How slowly must the object be traveling so that the observed mass is no greater than 100 times its mass at rest?

26. Find the inverse of $k(x) = \dfrac{2x}{\sqrt{x^2 - 1}}$.

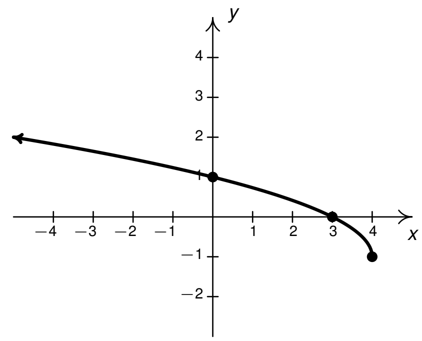
8.2.4 Answers

1. $F(x) = \sqrt{x+3} - 2$



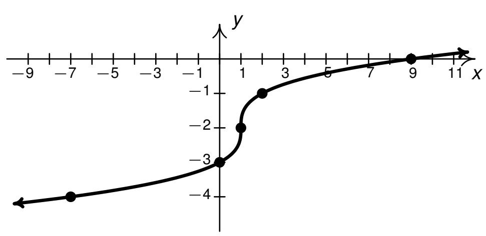
Domain: $[-3, \infty)$, Range: $[-2, \infty)$

2. $F(x) = \sqrt{4-x} - 1 = \sqrt{-x+4} - 1$



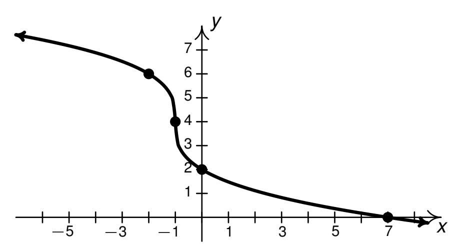
Domain: $(-\infty, 4]$, Range: $[-1, \infty)$

3. $F(x) = \sqrt[3]{x-1} - 2$



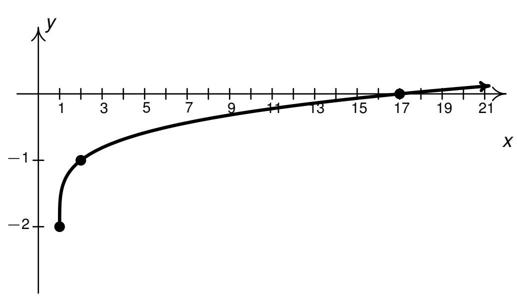
Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$

4. $F(x) = -\sqrt[3]{8x+8} + 4$



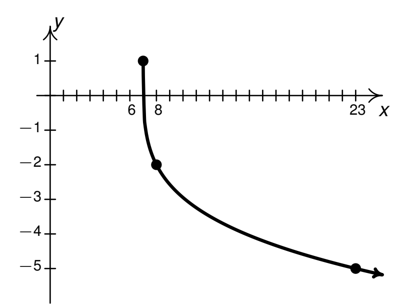
Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$

5. $F(x) = \sqrt[4]{x-1} - 2$



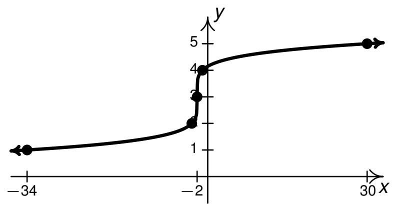
Domain: $[1, \infty)$, Range: $[-2, \infty)$

6. $F(x) = -3\sqrt[4]{x-7} + 1$



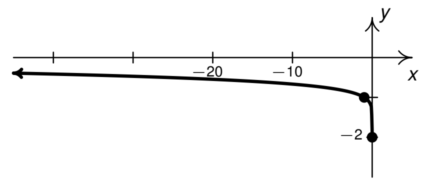
Domain: $[7, \infty)$, Range: $(-\infty, 1]$

7. $F(x) = \sqrt[5]{x+2} + 3$



Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$

8. $F(x) = \sqrt[8]{-x} - 2$



Domain: $(-\infty, 0]$, Range: $[-2, \infty)$

9. One solution is: $F(x) = -\sqrt{x+4} + 2$

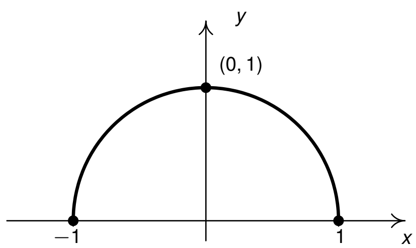
10. One solution is: $F(x) = 2\sqrt{-x+1}$

11. One solution is: $F(x) = -\sqrt[3]{2x+1}$

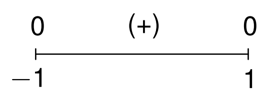
12. One solution is: $F(x) = 2\sqrt[3]{x-1} - 2$

13. (a) $(-\infty, 4]$ (b) $(-2, \infty)$ (c) $\left[\frac{1}{2}, \infty\right)$
(d) $[-2, 2]$ (e) $(-\infty, -2) \cup (2, \infty)$ (f) $(-\infty, -2] \cup [1, \infty)$

14. $f(x) = \sqrt{1 - x^2}$
Domain: $[-1, 1]$
Intercepts: $(-1, 0)$, $(1, 0)$
Graph:

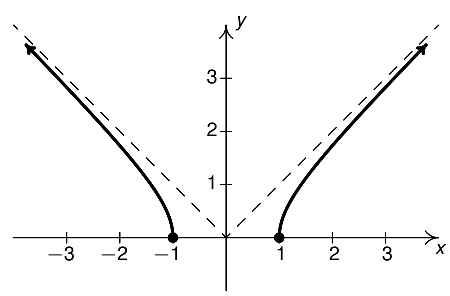


Range: $[0, 1]$
Local maximum: $(0, 1)$
Increasing: $[-1, 0]$, Decreasing: $[0, 1]$
Unusual steepness[17] at $x = -1$ and $x = 1$
Sign Diagram:



Note: f is even.

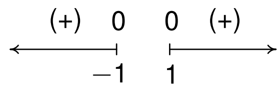
15. $f(x) = \sqrt{x^2 - 1}$
Domain: $(-\infty, -1] \cup [1, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$
Graph:



[17] You may need to zoom in to see this.
[18] You may need to zoom in to see this.

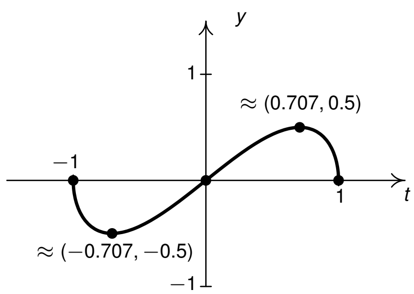
As $x \to \pm\infty$, $f(x) \to \infty$
Range: $[0, \infty)$
Increasing: $[1, \infty)$, Decreasing: $(-\infty, -1]$
Unusual steepness[18] at $x = -1$ and $x = 1$
Using Calculus, one can show $y = \pm x$ are slant asymptotes to the graph.

Sign Diagram:

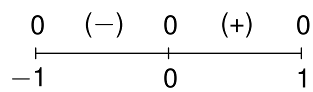


Note: f is even.

16. $g(t) = t\sqrt{1 - t^2}$
Domain: $[-1, 1]$
Intercepts: $(-1, 0)$, $(0, 0)$, $(1, 0)$
Graph:

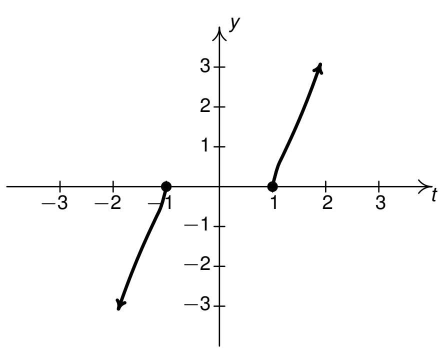


Range: $\approx [-0.5, 0.5]$
Local minimum $\approx (-0.707, -0.5)$
Local maximum: $\approx (0.707, 0.5)$
Increasing: $\approx [-0.707, 0.707]$
Decreasing: $\approx [-1, -0.707]$, $[0.707, 1]$
Unusual steepness at $t = -1$ and $t = 1$
Sign Diagram:

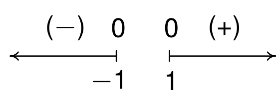


Note: g is odd.

17. $g(t) = t\sqrt{t^2 - 1}$
Domain: $(-\infty, -1] \cup [1, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$
Graph:

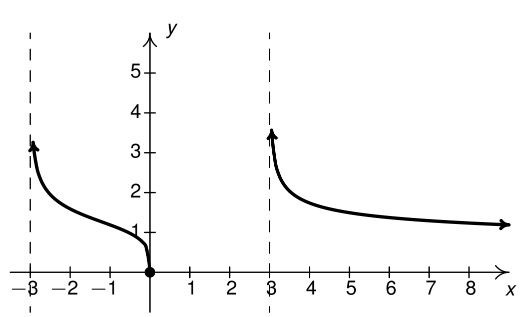


As $t \to -\infty$, $g(t) \to -\infty$
As $t \to \infty$, $g(t) \to \infty$
Range: $(-\infty, \infty)$
Increasing: $(-\infty, -1]$, $[1, \infty)$
Unusual steepness at $t = -1$ and $t = 1$
Sign Diagram:



Note: g is odd.

18. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
Domain: $(-3, 0] \cup (3, \infty)$
Graph:

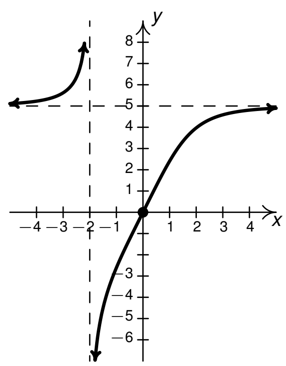


Intercept: $(0,0)$
As $x \to \infty$, $f(x) \to 0$
Range: $[0, \infty)$
Decreasing: $(-3, 0]$, $(3, \infty)$
Unusual steepness at $x = 0$
Vertical asymptotes: $x = -3$ and $x = 3$

Horizontal asymptote: $y = 0$
Sign Diagram:

‽ (+) 0 ‽ (+)
-3 0 3

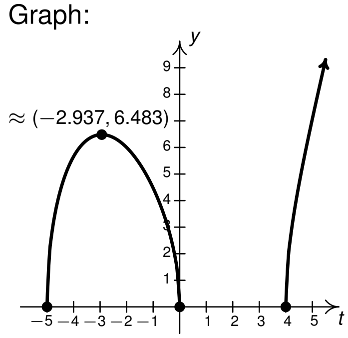
19. $f(x) = \dfrac{5x}{\sqrt[3]{x^3+8}}$
Graph:



Domain: $(-\infty, -2) \cup (-2, \infty)$
Intercept: $(0,0)$
As $x \to \pm\infty$, $f(x) \to 5$
Range: $(-\infty, 5) \cup (5, \infty)$
Increasing: $(-\infty, -2)$, $(-2, \infty)$
Vertical asymptote $x = -2$
Horizontal asymptote $y = 5$
Sign Diagram:

(+) ‽ (−) 0 (+)
-2 0

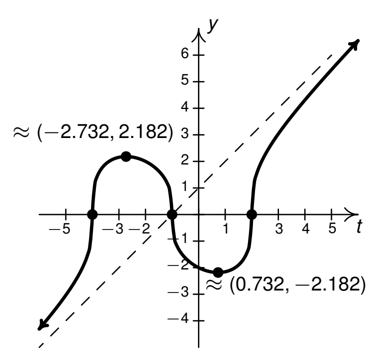
20. $g(t) = \sqrt{t(t+5)(t-4)}$
Domain: $[-5, 0] \cup [4, \infty)$
Intercepts $(-5, 0)$, $(0, 0)$, $(4, 0)$
As $t \to \infty$, $g(t) \to \infty$
Graph:



Range: $[0, \infty)$
Local maximum $\approx (-2.937, 6.483)$
Increasing: $\approx [-5, -2.937]$, $[4, \infty)$
Decreasing: $\approx [-2.937, 0]$
Unusual steepness at $t = -5$, $t = 0$ and $t = 4$
Sign Diagram:

0 (+) 0 0 (+)
-5 0 4

21. $g(t) = \sqrt[3]{t^3 + 3t^2 - 6t - 8}$
Domain: $(-\infty, \infty)$
Intercepts: $(-4, 0)$, $(-1, 0)$, $(0, -2)$, $(2, 0)$
Graph:



as $t \to -\infty$, $g(t) \to -\infty$
as $t \to \infty$, $g(t) \to \infty$
Range: $(-\infty, \infty)$
Local maximum: $\approx (-2.732, 2.182)$
Local minimum: $\approx (0.732, -2.182)$
Increasing: $\approx (-\infty, -2.732]$, $[0.732, \infty)$
Decreasing: $\approx [-2.732, 0.732]$

Unusual steepness at $t = -4$, $t = -1$ and $t = 2$
Using Calculus it can be shown that $y = t + 1$ is a slant asymptote of this graph.
Sign Diagram:

$$\begin{array}{ccccccc} (-) & 0 & (+) & 0 & (-) & 0 & (+) \\ & -4 & & -1 & & 2 & \end{array}$$

22. $C(x) = 15x + 20\sqrt{100 + (30 - x)^2}$, $0 \leq x \leq 30$. The calculator gives the absolute minimum at approximately $(18.66, 582.29)$. This means to minimize the cost, approximately 18.66 miles of cable should be run along Route 117 before turning off the road and heading towards the outpost. The minimum cost to run the cable is approximately \$582.29.

23. (a) $h(r) = \frac{300}{\pi r^2}$, $r > 0$.
    (b) $S(r) = \pi r \sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \frac{\sqrt{\pi^2 r^6 + 90000}}{r}$, $r > 0$
    (c) The calculator gives the absolute minimum at the point $\approx (4.07, 90.23)$. This means the radius should be (approximately) 4.07 centimeters and the height should be 5.76 centimeters to give a minimum surface area of 90.23 square centimeters.

24. $9.8\left(\frac{1}{4\pi}\right)^2 \approx 0.062$ meters or 6.2 centimeters

25. (a) $[0, c)$
    (b)
$$m(.1c) = \frac{m_r}{\sqrt{.99}} \approx 1.005 m_r \qquad m(.5c) = \frac{m_r}{\sqrt{.75}} \approx 1.155 m_r$$
$$m(.9c) = \frac{m_r}{\sqrt{.19}} \approx 2.294 m_r \qquad m(.999c) = \frac{m_r}{\sqrt{.0.001999}} \approx 22.366 m_r$$
    (c) As $v \to c^-$, $m(x) \to \infty$
    (d) If the object is traveling no faster than approximately 0.99995 times the speed of light, then its observed mass will be no greater than $100 m_r$.

26. $k^{-1}(x) = \dfrac{x}{\sqrt{x^2 - 4}}$

Chapter 9

Further Topics on Functions

9.1 Function Arithmetic

As we mentioned in Section 2.2, in this chapter, we are studying functions in a more abstract and general setting. In this section, we begin our study of what can be considered as the *algebra of functions* by defining *function arithmetic.*

Given two real numbers, we have four primary arithmetic operations available to us: addition, subtraction, multiplication, and division (provided we don't divide by 0.) Since the functions we study in this text have ranges which are sets of real numbers, it makes sense we can extend these arithmetic notions to functions.

For example, to add two functions means we add their outputs; to subtract two functions, we subtract their outputs, and so on and so forth. More formally, given two functions f and g, we *define* a new function $f+g$ whose rule is determined by adding the outputs of f and g. That is $(f+g)(x) = f(x)+g(x)$. While this looks suspiciously like some kind of distributive property, it is nothing of the sort. The '+' sign in the expression '$f+g$' is part of the *name* of the function we are defining,[1] whereas the plus sign '+' sign in the expression $f(x)+g(x)$ represents real number addition: we are adding the output from f, $f(x)$ with the output from g, $g(x)$ to determine the output from the sum function, $(f+g)(x)$.

Of course, in order to define $(f+g)(x)$ by the formula $(f+g)(x) = f(x)+g(x)$, both $f(x)$ and $g(x)$ need to be defined in the first place; that is, x must be in the domain of f *and* the domain of g. You'll recall[2] this means x must be in the *intersection* of the domains of f and g. We define the following.

[1] We could have just as easily called this new function $S(x)$ for 'sum' of f and g and defined S by $S(x) = f(x)+g(x)$.

[2] see Section 1.1.

Definition 9.1. Suppose f and g are functions and x is in both the domain of f and the domain of g.

- The **sum** of f and g, denoted $f + g$, is the function defined by the formula

$$(f+g)(x) = f(x) + g(x)$$

- The **difference** of f and g, denoted $f - g$, is the function defined by the formula

$$(f-g)(x) = f(x) - g(x)$$

- The **product** of f and g, denoted fg, is the function defined by the formula

$$(fg)(x) = f(x)g(x)$$

- The **quotient** of f and g, denoted $\dfrac{f}{g}$, is the function defined by the formula

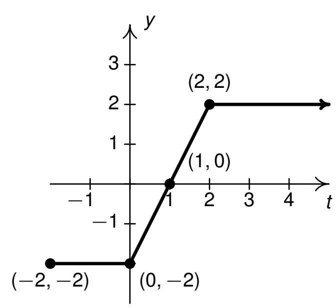
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

provided $g(x) \neq 0$.

We put these definitions to work for us in the next example.

Example 9.1.1. Consider the following functions:

- $f(x) = 6x^2 - 2x$
- $g(t) = 3 - \dfrac{1}{t}$, $t > 0$
- $h = \left\{(-3, 2), (-2, 0.4), (0, \sqrt{2}), (3, -6)\right\}$
- s whose graph is given below:



$y = s(t)$

1. Find and simplify the following function values:

(a) $(f+g)(1)$ (b) $(s-f)(-1)$ (c) $(fg)(2)$ (d) $\left(\dfrac{s}{h}\right)(0)$

(e) $((s+g)+h)(3)$ (f) $(s+(g+h))(3)$ (g) $\left(\dfrac{f+h}{s}\right)(3)$ (h) $(f(g-h))(-2)$

2. Find the domain of each of the following functions:

   (a) hg (b) $\dfrac{f}{s}$

3. Find expressions for the functions below. State the domain for each.

   (a) $(fg)(x)$ (b) $\left(\dfrac{g}{f}\right)(t)$

Solution.

1. (a) By definition, $(f+g)(1) = f(1) + g(1)$. We find $f(1) = 6(1)^2 - 2(1) = 4$ and $g(1) = 3 - \frac{1}{1} = 2$. So we get $(f+g)(1) = 4 + 2 = 6$.

   (b) To find $(s-f)(-1) = s(-1) - f(-1)$, we need both $s(-1)$ and $f(-1)$. To get $s(-1)$, we look to the graph of $y = s(t)$ and look for the y-coordinate of the point on the graph with the t-coordinate of -1. While not labeled directly, we infer the point $(-1,-2)$ is on the graph which means $s(-1) = -2$. For $f(-1)$, we compute: $f(-1) = 6(-1)^2 - 2(-1) = 8$. Putting it all together, we get $(s-f)(-1) = (-2) - (8) = -10$.

   (c) Since $(fg)(2) = f(2)g(2)$, we first compute $f(2)$ and $g(2)$. We find $f(2) = 6(2)^2 - 2(2) = 20$ and $g(2) = 2 + \frac{1}{2} = \frac{5}{2}$, so $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.

   (d) By definition, $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)}$. Since $(0,-2)$ is on the graph of $y = s(t)$, so we know $s(0) = -2$. Likewise, the ordered pair $(0, \sqrt{2}) \in h$, so $h(0) = \sqrt{2}$. We get $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$.

   (e) The expression $((s+g)+h)(3)$ involves *three* functions. Fortunately, they are grouped so that we can apply Definition 9.1 by first considering the sum of the two functions $(s+g)$ and h, then to the sum of the two functions s and g: $((s+g)+h)(3) = (s+g)(3) + h(3) = (s(3)+g(3)) + h(3)$. To get $s(3)$, we look to the graph of $y = s(t)$. We infer the point $(3,2)$ is on the graph of s, so $s(3) = 2$. We compute $g(3) = 3 - \frac{1}{3} = \frac{8}{3}$. To find $h(3)$, we note $(3,-6) \in h$, so $h(3) = -6$. Hence, $((s+g)+h)(3) = (s+g)(3) + h(3) = (s(3)+g(3)) + h(3) = \left(2 + \frac{8}{3}\right) + (-6) = -\frac{4}{3}$.

   (f) The expression $(s+(g+h))(3)$ is very similar to the previous problem, $((s+g)+h)(3)$ except that the g and h are grouped together here instead of the s and g. We proceed as above applying Definition 9.1 twice and find $(s+(g+h))(3) = s(3) + (g+h)(3) = s(3) + (g(3)+h(3))$. Substituting the values for $s(3)$, $g(3)$ and $h(3)$, we get $(s+(g+h))(3) = 2 + \left(\frac{8}{3} + (-6)\right) = -\frac{4}{3}$, which, not surprisingly, matches our answer to the previous problem.

   (g) Once again, we find the expression $\left(\frac{f+h}{s}\right)(3)$ has more than two functions involved. As with all fractions, we treat '$-$' as a grouping symbol and interpret $\left(\frac{f+h}{s}\right)(3) = \frac{(f+h)(3)}{s(3)} = \frac{f(3)+h(3)}{s(3)}$. We compute $f(3) = 6(3)^2 - 2(3) = 48$ and have $h(3) = -6$ and $s(3) = 2$ from above. Hence, $\left(\frac{f+h}{s}\right)(3) = \frac{f(3)+h(3)}{s(3)} = \frac{48+(-6)}{2} = 21$.

(h) We need to need to exercise caution in parsing $(f(g-h))(-2)$. In this context, f, g, and h are all functions, so we interpret $(f(g-h))$ as the function and -2 as the argument. We view the function $f(g-h)$ as the product of f and the function $g-h$. Hence, $(f(g-h))(-2) = f(-2)[(g-h)(-2)] = f(-2)[g(-2)-h(-2)]$. We compute $f(-2) = 6(-2)^2 - 2(-2) = 28$, and $g(-2) = 3 - \frac{1}{-2} = 3 + \frac{1}{2} = \frac{7}{2} = 3.5$. Since $(-2, 0.4) \in h$, $h(-2) = 0.4$. Putting this altogether, we get $(f(g-h))(-2) = f(-2)[(g-h)(-2)] = f(-2)[g(-2)-h(-2)] = 28(3.5-0.4) = 28(3.1) = 86.8$.

2. (a) To find the domain of hg, we need to find the real numbers in both the domain of h and the domain of g. The domain of h is $\{-3, -2, 0, 3\}$ and the domain of g is $\{t \in \mathbb{R} \mid t > 0\}$ so the only real number in common here is 3. Hence, the domain of hg is $\{3\}$, which may be small, but it's better than nothing.[3]

(b) To find the domain of $\frac{f}{s}$, we first note the domain of f is all real numbers, but that the domain of s, based on the graph, is just $[-2, \infty)$. Moreover, $s(t) = 0$ when $t = 1$, so we must exclude this value from the domain of $\frac{f}{s}$. Hence, we are left with $[-2, 1) \cup (1, \infty)$.

3. (a) By definition, $(fg)(x) = f(x)g(x)$. We are given $f(x) = 6x^2 - 2x$ and $g(t) = 3 - \frac{1}{t}$ so $g(x) = 3 - \frac{1}{x}$. Hence,

$$\begin{array}{rcll} (fg)(x) & = & f(x)g(x) & \\ & = & \left(6x^2-2x\right)\left(3-\dfrac{1}{x}\right) & \\ & = & 18x^2 - 6x^2\left(\dfrac{1}{x}\right) - 2x(3) + 2x\left(\dfrac{1}{x}\right) & \text{distribute} \\ & = & 18x^2 - 6x - 6x + 2 & \\ & = & 18x^2 - 12x + 2 & \end{array}$$

To find the domain of fg, we note the domain of f is all real numbers, $(-\infty, \infty)$ whereas the domain of g is restricted to $\{t \in \mathbb{R} \mid t > 0\} = (0, \infty)$. Hence, the domain of fg is likewise restricted to $(0, \infty)$. Note if we relied solely on the **simplified formula** for $(fg)(x) = 18x^2 - 12x + 2$, we would have obtained the *incorrect* answer for the domains of fg.

(b) To find an expression for $\left(\frac{g}{f}\right)(t) = \frac{f(t)}{g(t)}$ we first note $f(t) = 6t^2 - 2t$ and $g(t) = 3 - \frac{1}{t}$. Hence:

$$\begin{array}{rcll} \left(\dfrac{g}{f}\right)(t) & = & \dfrac{g(t)}{f(t)} & \\ & = & \dfrac{3-\frac{1}{t}}{6t^2-2t} = \dfrac{3-\frac{1}{t}}{6t^2-2t}\cdot\dfrac{t}{t} & \text{simplify compound fractions} \\ & = & \dfrac{\left(3-\frac{1}{t}\right)t}{\left(6t^2-2t\right)t} = \dfrac{3t-1}{\left(6t^2-2t\right)t} & \\ & = & \dfrac{3t-1}{2t^2(3t-1)} = \dfrac{\cancel{(3t-1)}^{\,1}}{2t^2\cancel{(3t-1)}} & \text{factor and cancel} \end{array}$$

[3] Since $(hg)(3) = h(3)g(3) = (-6)\left(\frac{8}{3}\right) = -16$, we can write $hg = \{(3, -16)\}$.

$$= \frac{1}{2t^2}$$

Hence, $\left(\frac{g}{f}\right)(t) = \frac{1}{2t^2} = \frac{1}{2}t^{-2}$. To find the domain of $\frac{g}{f}$, a real number must be both in the domain of g, $(0, \infty)$, and the domain of f, $(-\infty, \infty)$ so we start with the set $(0, \infty)$. Additionally, we require $f(t) \neq 0$. Solving $f(t) = 0$ amounts to solving $6t^2 - 2t = 0$ or $2t(3t - 1) = 0$. We find $t = 0$ or $t = \frac{1}{3}$ which means we need to exclude these values from the domain. Hence, our final answer for the domain of $\frac{g}{f}$ is $\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$. Note that, once again, using the *simplified formula* for $\left(\frac{g}{f}\right)(t)$ to determine the domain of $\frac{g}{f}$, would have produced erroneous results. $\square$

A few remarks are in order. First, in number 1 parts 1e through 1h, we first encountered combinations of *three* functions despite Definition 9.1 only addressing combinations of *two* functions at a time. It turns out that function arithmetic inherits many of the same properties of real number arithmetic. For example, we showed above that $((s + g) + h)(3) = (s + (g + h))(3)$. In general, given any three functions f, g, and h, $(f + g) + h = f + (g + h)$ that is, function addition is *assocative*. To see this, choose an element x common to the domains of f, g, and h. Then

$$\begin{array}{rcll}
((f+g)+h)(x) & = & (f+g)(x)+h(x) & \text{definiton of } ((f+g)+h)(x) \\
 & = & (f(x)+g(x))+h(x) & \text{definition of } (f+g)(x) \\
 & = & f(x)+(g(x)+h(x)) & \text{associative property of real number addition} \\
 & = & f(x)+(g+h)(x) & \text{definition of } (g+h)(x) \\
 & = & (f+(g+h))(x) & \text{definition of } (f+(g+h))(x)
\end{array}$$

The key step to the argument is that $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$ which is true courtesy of the associative property of real number addition. And just like with real number addition, because function addition is associative, we may write $f + g + h$ instead of $(f + g) + h$ or $f + (g + h)$ even though, when it comes down to computations, we can only add two things together at a time.[4]

For completeness, we summarize the properties of function arithmetic in the theorem below. The proofs of the properties all follow along the same lines as the proof of the associative property and are left to the reader. We investigate some additional properties in the exercises.

[4] Addition is a 'binary' operation - meaning it is defined only on two objects at once. Even though we write $1+2+3 = 6$, mentally, we add just two of numbers together at any given time to get our answer: for example, $1 + 2 + 3 = (1 + 2) + 3 = 3 + 3 = 6$.

Theorem 9.1. Suppose f, g and h are functions.

- **Commutative Law of Addition:** $f + g = g + f$
- **Associative Law of Addition:** $(f + g) + h = f + (g + h)$
- **Additive Identity:** The function $Z(x) = 0$ satisfies: $f + Z = Z + f = f$ for all functions f.
- **Additive Inverse:** The function $F(x) = -f(x)$ for all x in the domain of f satisfies:

$$f + F = F + f = Z.$$

- **Commutative Law of Multiplication:** $fg = gf$
- **Associative Law of Multiplication:** $(fg)h = f(gh)$
- **Multiplicative Identity:** The function $I(x) = 1$ satisfies: $fI = If = f$ for all functions f.
- **Multiplicative Inverse:** If $f(x) \neq 0$ for all x in the domain of f, then $F(x) = \dfrac{1}{f(x)}$ satisfies:

$$fF = Ff = I$$

- **Distributive Law of Multiplication over Addition:** $f(g + h) = fg + fh$

In the next example, we decompose given functions into sums, differences, products and/or quotients of other functions. Note that there are infinitely many different ways to do this, including some trivial ones. For example, suppose we were instructed to decompose $f(x) = x + 2$ into a sum or difference of functions. We could write $f = g + h$ where $g(x) = x$ and $h(x) = 2$ or we could choose $g(x) = 2x + 3$ and $h(x) = -x - 1$. More simply, we could write $f = g + h$ where $g(x) = x + 2$ and $h(x) = 0$. We'll call this last decomposition a 'trivial' decomposition. Likewise, if we ask for a decomposition of $f(x) = 2x$ as a product, a nontrivial solution would be $f = gh$ where $g(x) = 2$ and $h(x) = x$ whereas a trivial solution would be $g(x) = 2x$ and $h(x) = 1$. In general, non-trivial solutions to decomposition problems avoid using the additive identity, 0, for sums and differences and the multiplicative identity, 1, for products and quotients.

Example 9.1.2. 1. For $f(x) = x^2 - 2x$, find functions g, h and k to decompose f nontrivially as:

(a) $f = g - h$ (b) $f = g + h$ (c) $f = gh$ (d) $f = g(h - k)$

2. For $F(t) = \dfrac{2t + 1}{\sqrt{t^2 - 1}}$, find functions G, H and K to decompost F nontrivially as:

(a) $F = \dfrac{G}{H}$ (b) $F = GH$ (c) $F = G + H$ (d) $F = \dfrac{G + H}{K}$

Solution.

1. (a) To decompose $f = g - h$, we need functions g and h so $f(x) = (g - h)(x) = g(x) - h(x)$. Given $f(x) = x^2 - 2x$, one option is to let $g(x) = x^2$ and $h(x) = 2x$. To check, we find $(g - h)(x) = g(x) - h(x) = x^2 - 2x = f(x)$ as required. In addition to checking the formulas match up, we also need to check domains. There isn't much work here since the domains of g and h are all real numbers which combine to give the domain of f which is all real numbers.

   (b) In order to write $f = g + h$, we need $f(x) = (g + h)(x) = g(x) + h(x)$. One way to accomplish this is to write $f(x) = x^2 - 2x = x^2 + (-2x)$ and identify $g(x) = x^2$ and $h(x) = -2x$. To check, $(g + h)(x) = g(x) + h(x) = x^2 - 2x = f(x)$. Again, the domains for both g and h are all real numbers which combine to give f its domain of all real numbers.

   (c) To write $f = gh$, we require $f(x) = (gh)(x) = g(x)h(x)$. In other words, we need to factor $f(x)$. We find $f(x) = x^2 - 2x = x(x - 2)$, so one choice is to select $g(x) = x$ and $h(x) = x - 2$. Then $(gh)(x) = g(x)h(x) = x(x - 2) = x^2 - 2x = f(x)$, as required. As above, the domains of g and h are all real numbers which combine to give f the correct domain of $(-\infty, \infty)$.

   (d) We need to be careful here interpreting the equation $f = g(h-k)$. What we have is an equality of *functions* so the parentheses here *do not* represent function notation here, but, rather function *multiplication*. The way to parse $g(h-k)$, then, is the function g *times* the function $h-k$. Hence, we seek functions g, h, and k so that $f(x) = [g(h-k)](x) = g(x)[(h-k)(x)] = g(x)(h(x) - k(x))$. From the previous example, we know we can rewrite $f(x) = x(x - 2)$, so one option is to set $g(x) = h(x) = x$ and $k(x) = 2$ so that $[g(h - k)](x) = g(x)[(h - k)(x)] = g(x)(h(x) - k(x)) = x(x - 2) = x^2 - 2x = f(x)$, as required. As above, the domain of all constituent functions is $(-\infty, \infty)$ which matches the domain of f.

2. (a) To write $F = \frac{G}{H}$, we need $G(t)$ and $H(t)$ so $F(t) = \left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)}$. We choose $G(t) = 2t + 1$ and $H(t) = \sqrt{t^2 - 1}$. Sure enough, $\left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)} = \frac{2t+1}{\sqrt{t^2-1}} = F(t)$ as required. When it comes to the domain of F, owing to the square root, we require $t^2 - 1 \geq 0$. Since we have a denominator as well, we require $\sqrt{t^2 - 1} \neq 0$. The former requirement is the same restriction on H, and the latter requirement comes from Definition 9.1. Starting with the domain of G, all real numbers, and working through the details, we arrive at the correct domain of F, $(-\infty, -1) \cup (1, \infty)$.

   (b) Next, we are asked to find functions G and H so $F(t) = (GH)(t) = G(t)H(t)$. This means we need to rewrite the expression for $F(t)$ as a product. One way to do this is to convert radical notation to exponent notation:

$$F(t) = \frac{2t + 1}{\sqrt{t^2 - 1}} = \frac{2t + 1}{\left(t^2 - 1\right)^{\frac{1}{2}}} = (2t + 1)\left(t^2 - 1\right)^{-\frac{1}{2}}.$$

   Choosing $G(t) = 2t + 1$ and $H(t) = \left(t^2 - 1\right)^{-\frac{1}{2}}$, we see $(GH)(t) = G(t)H(t) = (2t + 1)\left(t^2 - 1\right)^{-\frac{1}{2}}$ as required. The domain restrictions on F stem from the presence of the square root in the denominator - both are addressed when finding the domain of H. Hence, we obtain the correct domain of F.

   (c) To express F as a sum of functions G and H, we could rewrite

$$F(t) = \frac{2t + 1}{\sqrt{t^2 - 1}} = \frac{2t}{\sqrt{t^2 - 1}} + \frac{1}{\sqrt{t^2 - 1}},$$

so that $G(t) = \frac{2t}{\sqrt{t^2-1}}$ and $H(t) = \frac{1}{\sqrt{t^2-1}}$. Indeed, $(G+H)(t) = G(t) + H(t) = \frac{2t}{\sqrt{t^2-1}} + \frac{1}{\sqrt{t^2-1}} = \frac{2t+1}{\sqrt{t^2-1}} = F(t)$, as required. Moreover, the domain restrictions for F are the same for both G and H, so we get agreement on the domain, as required.

(d) Last, but not least, to write $F = \frac{G+H}{K}$, we require $F(t) = \left(\frac{G+H}{K}\right)(t) = \frac{(G+H)(t)}{K(t)} = \frac{G(t)+H(t)}{K(t)}$. Identifying $G(t) = 2t$, $H(t) = 1$, and $K(t) = \sqrt{t^2-1}$, we get

$$\left(\frac{G+H}{K}\right)(t) = \frac{(G+H)(t)}{K(t)} = \frac{G(t)+H(t)}{K(t)} = \frac{2t+1}{\sqrt{t^2-1}} = F(t).$$

Concerning domains, the domain of both G and H are all real numbers, but the domain of K is restricted to $t^2 - 1 \geq 0$. Coupled with the restriction stated in Definition 9.1 that $K(t) \neq 0$, we recover the domain of F, $(-\infty, -1) \cup (1, \infty)$. □

9.1.1 Difference Quotients

Recall in Section **??** the concept of the average rate of change of a function over the interval $[a, b]$ is the slope between the two points $(a, f(a))$ and $(b, f(b))$ and is given by

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

Consider a function f defined over an interval containing x and $x + h$ where $h \neq 0$. The average rate of change of f over the interval $[x, x + h]$ is thus given by the formula:[5]

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+h) - f(x)}{h}, \quad h \neq 0.$$

The above is an example of what is traditionally called the **difference quotient** or **Newton quotient** of f, since it is the *quotient* of two *differences*, namely $\Delta[f(x)]$ and Δx. Another formula for the difference quotient sticks keeps with the notation Δx instead of h:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0.$$

It is important to understand that in this formulation of the difference quotient, the variables 'x' and 'Δx' are distinct - that is they do not combine as like terms.

In Section 7.2, the average rate of change of position function s can be interpreted as the average velocity (see Definition 7.5.) We can likewise re-cast this definition. After relabeling $t = t_0 + \Delta t$, we get

$$\overline{v}(\Delta t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t_0+\Delta t) - s(t_0)}{\Delta t}, \quad \Delta t \neq 0,$$

which measures the average velocity between time t_0 and time $t_0 + \Delta t$ as a function of Δt.

Note that, regardless of which form the difference quotient takes, when h, Δx, or Δt is 0, the difference quotient returns the indeterminant form '$\frac{0}{0}$.' As we've seen with rational functions in Section 7.2, when this

[5] assuming $h > 0$; otherwise, we the interval is $[x + h, x]$. We get the same formula for the difference quotient either way.

happens, we can reduce the fraction to lowest terms to see if we have a vertical asymptote or hole in the graph. With this in mind, when we speak of 'simplifying the difference quotient,' we mean to manipulate the expression until the factor of 'h' or 'Δx' cancels out from the denominator.

Our next example invites us to simplify three difference quotients, each cast slightly differently. In each case, the bulk of the work involves Intermediate Algebra. We refer the reader to Sections 7.1 and 8.1 for additional review, if needed.

Example 9.1.3. Find and simplify the indicated difference quotients for the following functions:

1. For $f(x) = x^2 - x - 2$, find and simplify:

   (a) $\dfrac{f(3+h) - f(3)}{h}$ (b) $\dfrac{f(x+h) - f(x)}{h}$.

2. For $g(x) = \dfrac{3}{2x+1}$, find and simplify:

   (a) $\dfrac{g(\Delta x) - g(0)}{\Delta x}$ (b) $\dfrac{g(x+\Delta x) - g(x)}{\Delta x}$.

3. $r(t) = \sqrt{t}$, find and simplify:

   (a) $\dfrac{r(9+\Delta t) - r(9)}{\Delta t}$ (b) $\dfrac{r(t+\Delta t) - r(t)}{\Delta t}$.

Solution.

1. (a) For our first difference quotient, we find $f(3+h)$ by substituting the quantity $(3+h)$ in for x:

$$\begin{aligned} f(3+h) &= (3+h)^2 - (3+h) - 2 \\ &= 9 + 6h + h^2 - 3 - h - 2 \\ &= 4 + 5h + h^2 \end{aligned}$$

Since $f(3) = (3)^2 - 3 - 2 = 4$, we get for the difference quotient:

$$\begin{aligned} \frac{f(3+h) - f(3)}{h} &= \frac{(4 + 5h + h^2) - 4}{h} \\ &= \frac{5h + h^2}{h} \\ &= \frac{h(5+h)}{h} && \text{factor} \\ &= \frac{\cancel{h}(5+h)}{\cancel{h}} && \text{cancel} \\ &= 5 + h \end{aligned}$$

(b) For the second difference quotient, we first find $f(x+h)$, we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\left(x^2 + 2xh + h^2 - x - h - 2\right) - \left(x^2 - x - 2\right)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h - 2 - x^2 + x + 2}{h} \\ &= \frac{2xh + h^2 - h}{h} \\ &= \frac{h\,(2x + h - 1)}{h} && \text{factor} \\ &= \frac{\cancel{h}\,(2x + h - 1)}{\cancel{h}} && \text{cancel} \\ &= 2x + h - 1. \end{aligned}$$

Note if we substitute $x = 3$ into this expression, we obtain $5 + h$ which agrees with our answer from the first difference quotient.

2. (a) Rewriting $\Delta x = 0 + \Delta x$, we see the first expression really is a difference quotient:

$$\frac{g(\Delta x) - g(0)}{\Delta x} = \frac{g(0 + \Delta x) - g(0)}{\Delta x}.$$

Since $g(\Delta x) = \frac{3}{2\Delta x + 1}$ and $g(0) = \frac{3}{2(0)+1} = 3$, our difference quotient is:

$$\begin{aligned}
\frac{g(0 + \Delta x) - g(0)}{\Delta x} &= \frac{\frac{3}{2\Delta x + 1} - 3}{\Delta x} \\
&= \frac{\frac{3}{2\Delta x + 1} - 3}{\Delta x} \cdot \frac{(2\Delta x + 1)}{(2\Delta x + 1)} \\
&= \frac{3 - 3(2\Delta x + 1)}{\Delta x(2\Delta x + 1)} \\
&= \frac{3 - 6\Delta x - 3}{\Delta x(2\Delta x + 1)} \\
&= \frac{-6\Delta x}{\Delta x(2\Delta x + 1)} \\
&= \frac{-6\cancel{\Delta x}}{\cancel{\Delta x}(2\Delta x + 1)} \\
&= \frac{-6}{2\Delta x + 1}.
\end{aligned}$$

(b) For our next difference quotient, we first find $g(x + \Delta x)$ by replacing every occurrence of x in the formula for $g(x)$ with the quantity $(x + \Delta x)$:

$$\begin{aligned}
g(x + \Delta x) &= \frac{3}{2(x + \Delta x) + 1} \\
&= \frac{3}{2x + 2\Delta x + 1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{g(x + \Delta x) - g(x)}{\Delta x} &= \frac{\frac{3}{2x + 2\Delta x + 1} - \frac{3}{2x + 1}}{\Delta x} \\
&= \frac{\frac{3}{2x + 2\Delta x + 1} - \frac{3}{2x + 1}}{\Delta x} \cdot \frac{(2x + 2\Delta x + 1)(2x + 1)}{(2x + 2\Delta x + 1)(2x + 1)} \\
&= \frac{3(2x + 1) - 3(2x + 2\Delta x + 1)}{\Delta x(2x + 2\Delta x + 1)(2x + 1)} \\
&= \frac{6x + 3 - 6x - 6\Delta x - 3}{\Delta x(2x + 2\Delta x + 1)(2x + 1)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{-6\Delta x}{\Delta x(2x+2\Delta x+1)(2x+1)} \\
&= \frac{-6\cancel{\Delta x}}{\cancel{\Delta x}(2x+2\Delta x+1)(2x+1)} \\
&= \frac{-6}{(2x+2\Delta x+1)(2x+1)}.
\end{aligned}$$

Since we have managed to cancel the factor 'Δx' from the denominator, we are done. Substituting $x = 0$ into our final expression gives $\frac{-6}{2\Delta x+1}$ thus checking our previous answer.

3. (a) We start with $r(9+\Delta t) = \sqrt{9+\Delta t}$ and $r(9) = \sqrt{9} = 3$ and get:

$$\frac{r(9+\Delta t) - r(9)}{\Delta t} = \frac{\sqrt{9+\Delta t} - 3}{\Delta t}.$$

In order to cancel the factor 'Δt' from the *denominator*, we set about rationalizing the *numerator* by multiplying both numerator and denominator by the conjugate of the numerator, $\sqrt{9+\Delta t} - 3$:

$$\begin{aligned}
\frac{r(9+\Delta t) - r(9)}{\Delta t} &= \frac{\sqrt{9+\Delta t} - 3}{\Delta t} \\
&= \frac{\left(\sqrt{9+\Delta t} - 3\right)}{\Delta t} \cdot \frac{\left(\sqrt{9+\Delta t} + 3\right)}{\left(\sqrt{9+\Delta t} + 3\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{9+\Delta t}\right)^2 - (3)^2}{\Delta t\left(\sqrt{9+\Delta t} + 3\right)} && \text{Difference of Squares.} \\
&= \frac{(9+\Delta t) - 9}{\Delta t\left(\sqrt{9+\Delta t} + 3\right)} \\
&= \frac{\Delta t}{\Delta t\left(\sqrt{9+\Delta t} + 3\right)} \\
&= \frac{\cancel{\Delta t}^{\,1}}{\cancel{\Delta t}\left(\sqrt{9+\Delta t} + 3\right)} \\
&= \frac{1}{\sqrt{9+\Delta t} + 3}
\end{aligned}$$

(b) As one might expect, we use this same strategy to simplify our final different quotient. We have:

$$
\begin{aligned}
\frac{r(t+\Delta t)-r(t)}{\Delta t} &= \frac{\sqrt{t+\Delta t}-\sqrt{t}}{\Delta t} \\
&= \frac{\left(\sqrt{t+\Delta t}-\sqrt{t}\right)}{\Delta t}\cdot\frac{\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}{\left(\sqrt{t+\Delta t}+\sqrt{t}\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{t+\Delta t}\right)^2-\left(\sqrt{t}\right)^2}{\Delta t\left(\sqrt{t+\Delta t}+\sqrt{t}\right)} && \text{Difference of Squares.} \\
&= \frac{(t+\Delta t)-t}{\Delta t\left(\sqrt{t+\Delta t}+\sqrt{t}\right)} \\
&= \frac{\Delta t}{\Delta t\left(\sqrt{t+\Delta t}+\sqrt{t}\right)} \\
&= \frac{\cancel{\Delta t}^{\,1}}{\cancel{\Delta t}\left(\sqrt{t+\Delta t}+\sqrt{t}\right)} \\
&= \frac{1}{\sqrt{t+\Delta t}+\sqrt{t}}
\end{aligned}
$$

Since we have canceled the original 'Δt' factor from the denominator, we are done. Setting $t = 9$ in this expression, we get $\frac{1}{\sqrt{9+\Delta t}+3}$ which agrees with our previous answer. □

We close this section revisiting the situation in Example 7.2.3.

Example 9.1.4. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Find, and simplify: $\overline{v}(\Delta t) = \dfrac{s(15+\Delta t)-s(15)}{\Delta t}$, for $\Delta t \neq 0$.
2. Find and interpret $\overline{v}(-1)$.
3. Graph $y = \overline{v}(t)$.
4. Describe the behavior of $\overline{v}$ as $\Delta t \to 0$ and interpret.

Solution.

1. To find $\overline{v}(\Delta t)$, we first find $s(15 + \Delta t)$:

$$\begin{aligned} s(15+\Delta t) &= -5(15+\Delta t)^2 + 100(15+\Delta t) \\ &= -5(225 + 30\Delta t + (\Delta t)^2) + 1500 + 100\Delta t \\ &= -5(\Delta t)^2 - 50\Delta t + 375 \end{aligned}$$

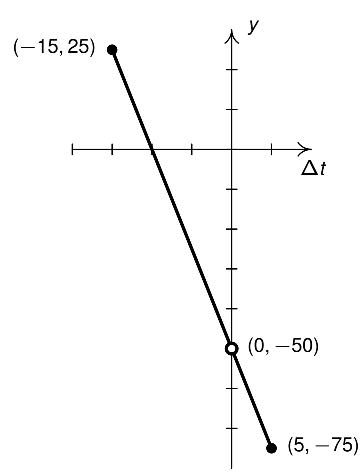
   Since $s(15) = -5(15)^2 + 100(15) = 375$, we get:

$$\begin{aligned} \overline{v}(\Delta t) &= \frac{s(15+\Delta t) - s(15)}{\Delta t} \\ &= \frac{(-5(\Delta t)^2 - 50\Delta t + 375) - 375}{\Delta t} \\ &= \frac{\Delta t(-5\Delta t - 50)}{\Delta t} \\ &= \frac{\cancel{\Delta t}(-5\Delta t - 50)}{\cancel{\Delta t}} \\ &= -5\Delta t - 50 \qquad\qquad \Delta t \neq 0 \end{aligned}$$

   In addition to the restriction $\Delta t \neq 0$, we also know the domain of s is $0 \leq t \leq 20$. Hence, we also require $0 \leq 15 + \Delta t \leq 20$ or $-15 \leq \Delta \leq 5$. Our final answer is $\overline{v}(\Delta t) = -5\Delta t - 50$, for $\Delta t \in [-15, 0) \cup (0, 5]$

2. We find $\overline{v}(-1) = -5(-1) - 50 = -45$. This means the average velocity over between time $t = 15 + (-1) = 14$ seconds and $t = 15$ seconds is -45 feet per second. This indicates the rocket is, on average, heading *downwards* at a rate of 45 feet per second.

3. The graph of $y = -5\Delta t - 50$ is a line with slope -5 and y-intercept $(0, -50)$. However, since the domain of $\overline{v}$ is $[-15, 0) \cup (0, 5]$, we the graph of $\overline{v}$ is a line *segment* from $(-15, 25)$ to $(5, -75)$ with a hole at $(0, -50)$.



$y = \overline{v}(\Delta t)$

4. As $\Delta t \to 0$, $\overline{v}(\Delta t) \to -50$ meaning as we approach $t = 15$, the velocity of the rocket approaches -50 feet per second. Recall from Example 7.2.3 that this is the so-called *instantaneous velocity* of the rocket *at* $t = 15$ seconds. That is, 15 seconds after lift-off, the rocket is heading back towards the surface of the moon at a rate of 15 feet per second. □

The reader is invited to compare Example 7.2.3 in Section 7.2 with Exercise 9.1.4 above. We obtain the exact same information because we are asking the *exact same* questions - they are just framed differently.

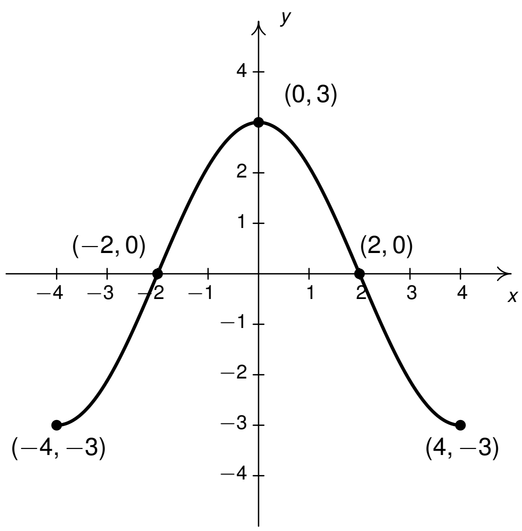
9.1.2 Exercises

In Exercises 1 - 10, use the pair of functions f and g to find the following values if they exist.

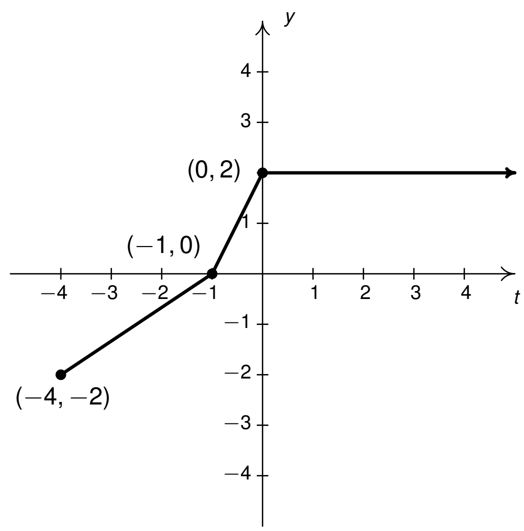
- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x+1$ and $g(t) = 4-t$
2. $f(x) = x^2$ and $g(t) = -2t+1$
3. $f(x) = x^2 - x$ and $g(t) = 12 - t^2$
4. $f(x) = 2x^3$ and $g(t) = -t^2 - 2t - 3$
5. $f(x) = \sqrt{x+3}$ and $g(t) = 2t-1$
6. $f(x) = \sqrt{4-x}$ and $g(t) = \sqrt{t+2}$
7. $f(x) = 2x$ and $g(t) = \dfrac{1}{2t+1}$
8. $f(x) = x^2$ and $g(t) = \dfrac{3}{2t-3}$
9. $f(x) = x^2$ and $g(t) = \dfrac{1}{t^2}$
10. $f(x) = x^2+1$ and $g(t) = \dfrac{1}{t^2+1}$

Exercises 11 - 20 refer to the functions f and g whose graphs are below.



$y = f(x)$



$y = g(t)$

11. $(f+g)(-4)$
12. $(f+g)(0)$
13. $(f-g)(4)$
14. $(fg)(-4)$
15. $(fg)(-2)$
16. $(fg)(4)$
17. $\left(\dfrac{f}{g}\right)(0)$
18. $\left(\dfrac{f}{g}\right)(2)$
19. $\left(\dfrac{g}{f}\right)(-1)$
20. Find the domains of $f+g$, $f-g$, fg, $\dfrac{f}{g}$ and $\dfrac{g}{f}$.

In Exercises 21 - 32, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined by

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

Compute the indicated value if it exists.

21. $(f+g)(-3)$
22. $(f-g)(2)$
23. $(fg)(-1)$
24. $(g+f)(1)$
25. $(g-f)(3)$
26. $(gf)(-3)$
27. $\left(\frac{f}{g}\right)(-2)$
28. $\left(\frac{f}{g}\right)(-1)$
29. $\left(\frac{f}{g}\right)(2)$
30. $\left(\frac{g}{f}\right)(-1)$
31. $\left(\frac{g}{f}\right)(3)$
32. $\left(\frac{g}{f}\right)(-3)$

In Exercises 33 - 42, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

33. $f(x) = 2x+1$ and $g(x) = x-2$
34. $f(x) = 1-4x$ and $g(x) = 2x-1$
35. $f(x) = x^2$ and $g(x) = 3x-1$
36. $f(x) = x^2 - x$ and $g(x) = 7x$
37. $f(x) = x^2 - 4$ and $g(x) = 3x+6$
38. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
39. $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{2}{x}$
40. $f(x) = x-1$ and $g(x) = \dfrac{1}{x-1}$
41. $f(x) = x$ and $g(x) = \sqrt{x+1}$
42. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

In Exercises 43 - 47, write the given function as a nontrivial decomposition of functions as directed.

43. For $p(z) = 4z - z^3$, find functions f and g so that $p = f - g$.
44. For $p(z) = 4z - z^3$, find functions f and g so that $p = f + g$.
45. For $g(t) = 3t|2t-1|$, find functions f and h so that $g = fh$.
46. For $r(x) = \dfrac{3-x}{x+1}$, find functions f and g so $r = \dfrac{f}{g}$.
47. For $r(x) = \dfrac{3-x}{x+1}$, find functions f and g so $r = fg$.

48. Can $f(x) = x$ be decomposed as $f = g - h$ where $g(x) = x + \frac{1}{x}$ and $h(x) = \frac{1}{x}$?

49. Discuss with your classmates how to phrase the quantities revenue and profit in Definition 5.8 terms of function arithmetic as defined in Definition 9.1.

In Exercises 50 - 59, find and simplify the difference quotients:

- $\dfrac{f(2+h) - f(2)}{h}$
- $\dfrac{f(x+h) - f(x)}{h}$

50. $f(x) = 2x - 5$
51. $f(x) = -3x + 5$
52. $f(x) = 6$
53. $f(x) = 3x^2 - x$
54. $f(x) = -x^2 + 2x - 1$
55. $f(x) = 4x^2$
56. $f(x) = x - x^2$
57. $f(x) = x^3 + 1$
58. $f(x) = mx + b$ where $m \neq 0$
59. $f(x) = ax^2 + bx + c$ where $a \neq 0$

In Exercises 60 - 67, find and simplify the difference quotients:

- $\dfrac{f(-1+\Delta x) - f(-1)}{\Delta x}$
- $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

60. $f(x) = \dfrac{2}{x}$
61. $f(x) = \dfrac{3}{1-x}$
62. $f(x) = \dfrac{1}{x^2}$
63. $f(x) = \dfrac{2}{x+5}$
64. $f(x) = \dfrac{1}{4x-3}$
65. $f(x) = \dfrac{3x}{x+2}$
66. $f(x) = \dfrac{x}{x-9}$
67. $f(x) = \dfrac{x^2}{2x+1}$

In Exercises 68 - 74, find and simplify the difference quotients:

- $\dfrac{g(\Delta t) - g(0)}{\Delta t}$
- $\dfrac{g(t+\Delta t) - g(t)}{\Delta t}$

68. $g(t) = \sqrt{9-t}$
69. $g(t) = \sqrt{2t+1}$
70. $g(t) = \sqrt{-4t+5}$
71. $g(t) = \sqrt{4-t}$

72. $g(t) = \sqrt{at+b}$, where $a \neq 0$.

73. $g(t) = t\sqrt{t}$

74. $g(t) = \sqrt[3]{t}$. **HINT:** $(a-b)\left(a^2+ab+b^2\right) = a^3 - b^3$

75. In this exercise, we explore decomposing a function into its positive and negative parts. Given a function f, we define the **positive part** of f, denoted f_+ and **negative part** of f, denoted f_- by:

$$f_+(x) = \frac{f(x) + |f(x)|}{2}, \qquad \text{and} \qquad f_-(x) = \frac{f(x) - |f(x)|}{2}.$$

(a) Using a graphing utility, graph each of the functions f below along with f_+ and f_-.

- $f(x) = x - 3$
- $f(x) = x^2 - x - 6$
- $f(x) = 4x - x^3$

Why is f_+ called the 'positive part' of f and f_- called the 'negative part' of f?

(b) Show that $f = f_+ + f_-$.

(c) Use Definition 4.2 to rewrite the expressions for $f_+(x)$ and $f_-(x)$ as piecewise defined functions.

9.1.3 Answers

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$
   - $(f + g)(2) = 9$
   - $(f - g)(-1) = -7$
   - $(g - f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = \frac{35}{4}$
   - $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$

2. For $f(x) = x^2$ and $g(x) = -2x + 1$
   - $(f + g)(2) = 1$
   - $(f - g)(-1) = -2$
   - $(g - f)(1) = -2$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{5}{4}$

3. For $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
   - $(f + g)(2) = 10$
   - $(f - g)(-1) = -9$
   - $(g - f)(1) = 11$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{47}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$

4. For $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$
   - $(f + g)(2) = 5$
   - $(f - g)(-1) = 0$
   - $(g - f)(1) = -8$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{17}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{3}{16}$

5. For $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 1$
   - $(f + g)(2) = 3 + \sqrt{5}$
   - $(f - g)(-1) = 3 + \sqrt{2}$
   - $(g - f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
   - $\left(\frac{g}{f}\right)(-2) = -5$

6. For $f(x) = \sqrt{4 - x}$ and $g(x) = \sqrt{x + 2}$
   - $(f + g)(2) = 2 + \sqrt{2}$
   - $(f - g)(-1) = -1 + \sqrt{5}$
   - $(g - f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = \frac{\sqrt{35}}{2}$
   - $\left(\frac{f}{g}\right)(0) = \sqrt{2}$
   - $\left(\frac{g}{f}\right)(-2) = 0$

7. For $f(x) = 2x$ and $g(x) = \frac{1}{2x+1}$
   - $(f+g)(2) = \frac{21}{5}$
   - $(f-g)(-1) = -1$
   - $(g-f)(1) = -\frac{5}{3}$
   - $(fg)\left(\frac{1}{2}\right) = \frac{1}{2}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

8. For $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$
   - $(f+g)(2) = 7$
   - $(f-g)(-1) = \frac{8}{5}$
   - $(g-f)(1) = -4$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{3}{8}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{3}{28}$

9. For $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$
   - $(f+g)(2) = \frac{17}{4}$
   - $(f-g)(-1) = 0$
   - $(g-f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = 1$
   - $\left(\frac{f}{g}\right)(0)$ is undefined.
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

10. For $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x^2+1}$
    - $(f+g)(2) = \frac{26}{5}$
    - $(f-g)(-1) = \frac{3}{2}$
    - $(g-f)(1) = -\frac{3}{2}$
    - $(fg)\left(\frac{1}{2}\right) = 1$
    - $\left(\frac{f}{g}\right)(0) = 1$
    - $\left(\frac{g}{f}\right)(-2) = \frac{1}{25}$

11. $(f+g)(-4) = -5$
12. $(f+g)(0) = 5$
13. $(f-g)(4) = -5$
14. $(fg)(-4) = 6$
15. $(fg)(-2) = 0$
16. $(fg)(4) = -6$
17. $\left(\dfrac{f}{g}\right)(0) = \dfrac{3}{2}$
18. $\left(\dfrac{f}{g}\right)(2) = 0$
19. $\left(\dfrac{g}{f}\right)(-1) = 0$
20. The domains of $f+g$, $f-g$ and fg are all $[-4, 4]$. The domain of $\frac{f}{g}$ is $[-4, -1) \cup (-1, 4]$ and the domain of $\frac{g}{f}$ is $[-4, -2) \cup (-2, 2) \cup (2, 4]$.
21. $(f+g)(-3) = 2$
22. $(f-g)(2) = 3$
23. $(fg)(-1) = 0$
24. $(g+f)(1) = 0$
25. $(g-f)(3) = 3$
26. $(gf)(-3) = -8$
27. $\left(\frac{f}{g}\right)(-2)$ does not exist
28. $\left(\frac{f}{g}\right)(-1) = 0$
29. $\left(\frac{f}{g}\right)(2) = 4$
30. $\left(\frac{g}{f}\right)(-1)$ does not exist
31. $\left(\frac{g}{f}\right)(3) = -2$
32. $\left(\frac{g}{f}\right)(-3) = -\frac{1}{2}$

33. For $f(x) = 2x + 1$ and $g(x) = x - 2$

- $(f + g)(x) = 3x - 1$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x + 3$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 2x^2 - 3x - 2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$
  Domain: $(-\infty, 2) \cup (2, \infty)$

34. For $f(x) = 1 - 4x$ and $g(x) = 2x - 1$

- $(f + g)(x) = -2x$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = 2 - 6x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -8x^2 + 6x - 1$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{1-4x}{2x-1}$
  Domain: $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

35. For $f(x) = x^2$ and $g(x) = 3x - 1$

- $(f + g)(x) = x^2 + 3x - 1$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 3x + 1$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 - x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$
  Domain: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

36. For $f(x) = x^2 - x$ and $g(x) = 7x$

- $(f + g)(x) = x^2 + 6x$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 8x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 7x^3 - 7x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-1}{7}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

37. For $f(x) = x^2 - 4$ and $g(x) = 3x + 6$

- $(f + g)(x) = x^2 + 3x + 2$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 3x - 10$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$
  Domain: $(-\infty, -2) \cup (-2, \infty)$

38. For $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$

- $(f + g)(x) = x - 3$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = -2x^2 + x + 15$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -x^4 + x^3 + 15x^2 - 9x - 54$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = -\frac{x+2}{x+3}$
  Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

39. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$

- $(f + g)(x) = \frac{x^2+4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(f - g)(x) = \frac{x^2-4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(fg)(x) = 1$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

40. For $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$

- $(f + g)(x) = \frac{x^2-2x+2}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $(f - g)(x) = \frac{x^2-2x}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $(fg)(x) = 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$
- $\left(\frac{f}{g}\right)(x) = x^2 - 2x + 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$

41. For $f(x) = x$ and $g(x) = \sqrt{x+1}$

- $(f + g)(x) = x + \sqrt{x+1}$
  Domain: $[-1, \infty)$
- $(f - g)(x) = x - \sqrt{x+1}$
  Domain: $[-1, \infty)$
- $(fg)(x) = x\sqrt{x+1}$
  Domain: $[-1, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$
  Domain: $(-1, \infty)$

42. For $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

- $(f + g)(x) = 2\sqrt{x-5}$
  Domain: $[5, \infty)$
- $(f - g)(x) = 0$
  Domain: $[5, \infty)$
- $(fg)(x) = x - 5$
  Domain: $[5, \infty)$
- $\left(\frac{f}{g}\right)(x) = 1$
  Domain: $(5, \infty)$

43. One solution is $f(z) = 4z$ and $g(z) = z^3$.

44. One solution is $f(z) = 4z$ and $g(z) = -z^3$.

45. One solution is $f(t) = 3t$ and $h(t) = |2t - 1|$

46. One solution is $f(x) = 3 - x$ and $g(x) = x + 1$.

47. One solution is $f(x) = 3 - x$ and $g(x) = (x + 1)^{-1}$.

48. No. The equivalence does not hold when $x = 0$.

50. 2, 2.

51. -3, -3.

52. 0, 0

53. $3h + 11$, $6x + 3h - 1$

54. $-h - 2$, $-2x - h + 2$

55. $4h + 16$, $8x + 4h$

56. $-h - 3$, $-2x - h + 1$

57. $h^2 + 6h + 12$, $3x^2 + 3xh + h^2$

58. m, m

59. $ah + 4a + b$, $2ax + ah + b$

60. $\dfrac{2}{\Delta x - 1}, \dfrac{-2}{x(x + \Delta x)}$

61. $\dfrac{-3}{2(\Delta x - 2)}, \dfrac{3}{(x + \Delta x - 1)(x - 1)}$

62. $\dfrac{2 - \Delta x}{(\Delta x - 1)^2}, \dfrac{-(2x + \Delta x)}{x^2(x + \Delta x)^2}$

63. $\dfrac{-1}{2(\Delta x + 4)}, \dfrac{-2}{(x + 5)(x + \Delta x + 5)}$

64. $\dfrac{4}{7(4\Delta x - 7)}, \dfrac{-4}{(4x - 3)(4x + 4\Delta x - 3)}$

65. $\dfrac{6}{\Delta x + 1}, \dfrac{6}{(x + 2)(x + \Delta x + 2)}$

66. $\dfrac{9}{10(\Delta x - 10)}, \dfrac{-9}{(x - 9)(x + \Delta x - 9)}$

67. $\dfrac{\Delta x}{2\Delta x - 1}, \dfrac{2x^2 + 2x\Delta x + 2x + \Delta x}{(2x + 1)(2x + 2\Delta x + 1)}$

68. $\dfrac{-1}{\sqrt{9 - \Delta t} + 3}, \dfrac{-1}{\sqrt{9 - t - \Delta t} + \sqrt{9 - t}}$

69. $\dfrac{2}{\sqrt{2\Delta t + 1} + 1}, \dfrac{2}{\sqrt{2t + 2\Delta t + 1} + \sqrt{2t + 1}}$

70. $\dfrac{-4}{\sqrt{5 - 4\Delta t} + \sqrt{5}}, \dfrac{-4}{\sqrt{-4t - 4\Delta t + 5} + \sqrt{-4t + 5}}$

71. $\dfrac{-1}{\sqrt{4 - \Delta t} + 2}, \dfrac{-1}{\sqrt{4 - t - \Delta t} + \sqrt{4 - t}}$

72. $\dfrac{a}{\sqrt{a\Delta t + b} + \sqrt{b}}, \dfrac{a}{\sqrt{at + a\Delta t + b} + \sqrt{at + b}}$

73. $(\Delta t)^{\frac{1}{2}}, \dfrac{3t^2 + 3t\Delta t + (\Delta t)^2}{(t + \Delta t)^{3/2} + t^{3/2}}$

74. $\dfrac{1}{(\Delta t)^{2/3}}, \dfrac{1}{(t + \Delta t)^{2/3} + (t + \Delta t)^{1/3}t^{1/3} + t^{2/3}}$

75. (b) $(f_+ + f_-)(x) = f_+(x) + f_-(x) = \dfrac{f(x) + |f(x)|}{2} + \dfrac{f(x) - |f(x)|}{2} = \dfrac{2f(x)}{2} = f(x)$.

(c)

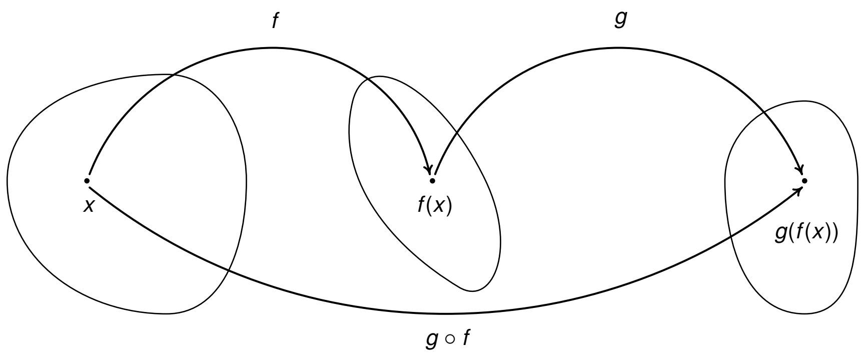
$$f_+(x) = \begin{cases} 0 & \text{if } f(x) < 0 \\ f(x) & \text{if } f(x) \geq 0 \end{cases}, \qquad f_-(x) = \begin{cases} f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}$$

9.2 Function Composition

In Section 9.1, we saw how the arithmetic of real numbers carried over into an arithmetic of functions. In this section, we discuss another way to combine functions which is unique to functions and isn't shared with real numbers - function **composition**.

Definition 9.2. Let f and g be functions where the real number x is in the domain of f and the real number $f(x)$ is in the domain of g. The **composite** of g with f, denoted $g \circ f$, and read 'g composed with f' is defined by the formula: $(g \circ f)(x) = g(f(x))$.

To compute $(g \circ f)(x)$, we use the formula given in Defintion 9.2: $(g \circ f)(x) = g(f(x))$. However, from a procedural viewpoint, Defintion 9.2 tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f then applying the procedure g. Abstractly, we have



In the expression $g(f(x))$, the function f is often called the 'inside' function while g is often called the 'outside' function. When evaluating composite function values we present two methods in the example below: the 'inside out' and 'outside in' methods.

Example 9.2.1. Let $f(x) = x^2 - 4x$, $g(t) = 2 - \sqrt{t+3}$, and $h(s) = \dfrac{2s}{s+1}$.
In numbers 1 - 3, find the indicated function value.

1. $(g \circ f)(1)$
2. $(f \circ g)(1)$
3. $(g \circ g)(6)$

In numbers 4 - 10, find and simplify the indicated composite functions. State the domain of each.

4. $(g \circ f)(x)$
5. $(f \circ g)(t)$
6. $(g \circ h)(s)$
7. $(h \circ g)(t)$
8. $(h \circ h)(x)$
9. $(h \circ (g \circ f))(x)$
10. $((h \circ g) \circ f)(x)$

Solution.

1. Using Definition 9.2, $(g \circ f)(1) = g(f(1))$. Since $f(1) = (1)^2 - 4(1) = -3$ and $g(-3) = 2 - \sqrt{(-3)+3} = 2$, we have $(g \circ f)(1) = g(f(1)) = g(-3) = 2$.

2. By definition, $(f \circ g)(1) = f(g(1))$. We find $g(1) = 2 - \sqrt{1+3} = 0$, and $f(0) = (0)^2 - 4(0) = 0$, so $(f \circ g)(1) = f(g(1)) = f(0) = 0$. Comparing this with our answer to the last problem, we see that $(g \circ f)(1) \neq (f \circ g)(1)$ which tells us function composition is not commutative,[1]

3. Since $(g \circ g)(6) = g(g(6))$, we 'iterate' the process g: that is, we apply the process g to 6, then apply the process g again. We find $g(6) = 2 - \sqrt{6+3} = -1$, and $g(-1) = 2 - \sqrt{(-1)+3} = 2 - \sqrt{2}$, so $(g \circ g)(6) = g(g(6)) = g(-1) = 2 - \sqrt{2}$.

4. By definition, $(g \circ f)(x) = g(f(x))$. We now illustrate *two* ways to approach this problem.

   - *inside out*: We substitute $f(x) = x^2 - 4x$ in for t in the expression $g(t)$ and get

   $$(g \circ f)(x) = g(f(x)) = g\left(x^2 - 4x\right) = 2 - \sqrt{\left(x^2 - 4x\right) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

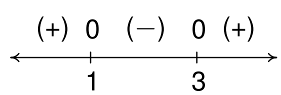
     Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   - *outside in*: We use the formula for g first to get

   $$(g \circ f)(x) = g(f(x)) = 2 - \sqrt{f(x) + 3} = 2 - \sqrt{\left(x^2 - 4x\right) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

     We get the same answer as before, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   To find the domain of $g \circ f$, we need to find the elements in the domain of f whose outputs $f(x)$ are in the domain of g. Since the domain of f is all real numbers, we focus on finding the range elements compatible with g. Owing to the presence of the square root in the formula $g(t) = 2 - \sqrt{t+3}$ we require $t \geq -3$. Hence, we need $f(x) \geq -3$ or $x^2 - 4x \geq -3$. To solve this inequality we rewrite as $x^2 - 4x + 3 \geq 0$ and use a sign diagram. Letting $r(x) = x^2 - 4x + 3$, we find the zeros of r to be $x = 1$ and $x = 3$ and obtain

   Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

5. To find $(f \circ g)(t)$, we find $f(g(t))$.

   - *inside out*: We substitute the expression $g(t) = 2 - \sqrt{t+3}$ in for x in the formula $f(x)$ and get

   $$\begin{aligned}
   (f \circ g)(t) &= f(g(t)) = f\left(2 - \sqrt{t+3}\right) \\
   &= \left(2 - \sqrt{t+3}\right)^2 - 4\left(2 - \sqrt{t+3}\right) \\
   &= 4 - 4\sqrt{t+3} + \left(\sqrt{t+3}\right)^2 - 8 + 4\sqrt{t+3} \\
   &= 4 + t + 3 - 8 \\
   &= t - 1
   \end{aligned}$$

[1] That is, in general, $g \circ f \neq f \circ g$. This shouldn't be too surprising, since, in general, the order of processes matters: adding eggs to a cake batter then baking the cake batter has a much different outcome than baking the cake batter then adding eggs.

- *outside in*: We use the formula for $f(x)$ first to get

$$\begin{array}{rcll}(f \circ g)(t) & = & f(g(t)) = (g(t))^2 - 4\,(g(t)) & \\ & = & \left(2 - \sqrt{t+3}\right)^2 - 4\left(2 - \sqrt{t+3}\right) & \\ & = & t - 1 & \text{same algebra as before}\end{array}$$

Thus we get $(f \circ g)(t) = t - 1$. To find the domain of $f \circ g$, we look for the elements t in the domain of g whose outputs, $g(t)$ are in the domain of f. As mentioned previously, the domain of g is limited by the presence of the square root to $\{t \in \mathbb{R} \,|\, t \geq -3\}$ while the domain of f is all real numbers. Hence, the domain of $f \circ g$ is restricted only by the domain of g and is $\{t \in \mathbb{R} \,|\, t \geq -3\}$ or, using interval notation, $[-3, \infty)$. Note that as with Example 9.1.1 in Section 9.1, had we used the simplified formula for $(f \circ g)(t) = t - 1$ to determine domain, we would have arrived at the incorrect answer.

6. To find $(g \circ h)(s)$, we compute $g(h(s))$.

- *inside out*: We substitute $h(s)$ in for t in the expression $g(t)$ to get

$$\begin{array}{rcll}(g \circ h)(s) & = & g(h(s)) = g\left(\dfrac{2s}{s+1}\right) & \\ & = & 2 - \sqrt{\left(\dfrac{2s}{s+1}\right) + 3} & \\ & = & 2 - \sqrt{\dfrac{2s}{s+1} + \dfrac{3(s+1)}{s+1}} & \text{get common denominators} \\ & = & 2 - \sqrt{\dfrac{5s+3}{s+1}} & \end{array}$$

- *outside in*: We use the formula for $g(t)$ first to get

$$\begin{array}{rcll}(g \circ h)(s) & = & g(h(s)) = 2 - \sqrt{h(s) + 3} & \\ & = & 2 - \sqrt{\left(\dfrac{2s}{s+1}\right) + 3} & \\ & = & 2 - \sqrt{\dfrac{5s+3}{s+1}} & \text{get common denominators as before}\end{array}$$

To find the domain of $g \circ h$, we need the elements in the domain of h so that $h(s)$ is in the domain of g. Owing to the $s + 1$ in the denominator of the expression $h(s)$, we require $s \neq -1$. Once again, because of the square root in $g(t) = 2 - \sqrt{t+3}$, we need $t \geq -3$ or, in this case $h(s) \geq -3$. To use a sign diagram to solve, we rearrange this inequality:

$$\begin{aligned}
\frac{2s}{s+1} &\geq -3 \\
\frac{2s}{s+1} + 3 &\geq 0 \\
\frac{5s+3}{s+1} &\geq 0 \quad \text{get common denominators as before}
\end{aligned}$$

Defining $r(s) = \frac{5s+3}{s+1}$, we see r is undefined at $s = -1$ (a carry over from the domain restriction of h) and $r(s) = 0$ at $s = -\frac{3}{5}$. Our sign diagram is

$$\begin{array}{ccccc} (+) & \text{?} & (-) & 0 & (+) \\ \hline & -1 & & -\frac{3}{5} & \end{array}$$

hence our domain is $(-\infty, -1) \cup \left[-\frac{3}{5}, \infty\right)$.

7. We find $(h \circ g)(t)$ by finding $h(g(t))$.

   - *inside out*: We substitute the expression $g(t)$ for s in the formula $h(s)$

$$\begin{aligned}
(h \circ g)(t) &= h(g(t)) = h\left(2 - \sqrt{t+3}\right) \\
&= \frac{2\left(2 - \sqrt{t+3}\right)}{\left(2 - \sqrt{t+3}\right) + 1} \\
&= \frac{4 - 2\sqrt{t+3}}{3 - \sqrt{t+3}}
\end{aligned}$$

   - *outside in*: We use the formula for $h(s)$ first to get

$$\begin{aligned}
(h \circ g)(t) &= h(g(t)) = \frac{2\,(g(t))}{(g(t)) + 1} \\
&= \frac{2\left(2 - \sqrt{t+3}\right)}{\left(2 - \sqrt{t+3}\right) + 1} \\
&= \frac{4 - 2\sqrt{t+3}}{3 - \sqrt{t+3}}
\end{aligned}$$

To find the domain of $h \circ g$, we need the elements of the domain of g so that $g(t)$ is in the domain of h. As we've seen already, for t to be in the domain of g, $t \geq -3$. For s to be in the domain of h, $s \neq -1$, so we require $g(t) \neq -1$. Hence, we solve $g(t) = 2 - \sqrt{t+3} = -1$ with the intent of excluding the solutions. Isolating the radical expression gives $\sqrt{t+3} = 3$ or $t = 6$. Sure enough, we check $g(6) = -1$ so we exclude $t = 6$ from the domain of $h \circ g$. Our final answer is $[-3, 6) \cup (6, \infty)$.

8. To find $(h \circ h)(s)$ we find $h(h(s))$:

   - *inside out*: We substitute the expression $h(s)$ for s in the expression $h(s)$ into h to get

$$
\begin{aligned}
(h \circ h)(s) &= h(h(s)) = h\left(\frac{2s}{s+1}\right) \\
&= \frac{2\left(\frac{2s}{s+1}\right)}{\left(\frac{2s}{s+1}\right)+1} \\
&= \frac{\frac{4s}{s+1}}{\frac{2s}{s+1}+1} \cdot \frac{(s+1)}{(s+1)} \\
&= \frac{\frac{4s}{s+1} \cdot (s+1)}{\left(\frac{2s}{s+1}\right) \cdot (s+1) + 1 \cdot (s+1)} \\
&= \frac{\frac{4s}{\cancel{(s+1)}} \cdot \cancelto{1}{(s+1)}}{\frac{2s}{\cancel{(s+1)}} \cdot \cancelto{1}{(s+1)} + s + 1} \\
&= \frac{4s}{3s+1}
\end{aligned}
$$

   - *outside in*: This approach yields

$$
\begin{aligned}
(h \circ h)(s) &= h(h(s)) = \frac{2(h(s))}{h(s)+1} \\
&= \frac{2\left(\frac{2s}{s+1}\right)}{\left(\frac{2s}{s+1}\right)+1} \\
&= \frac{4s}{3s+1} \qquad \text{same algebra as before}
\end{aligned}
$$

To find the domain of $h \circ h$, we need to find the elements in the domain of h so that the outputs, $h(s)$ are also in the domain of h. The only domain restriction for h comes from the denominator: $s \neq -1$, so in addition to this, we also need $h(s) \neq -1$. To this end, we solve $h(s) = -1$ and exclude the answers. Solving $\frac{2s}{s+1} = -1$ gives $s = -\frac{1}{3}$. The domain of $h \circ h$ is $(-\infty, -1) \cup \left(-1, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, \infty\right)$.

9. The expression $(h \circ (g \circ f))(x)$ indicates that we first find the composite, $g \circ f$ and compose the function h with the result. We know from number 4 that $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$ with domain $(-\infty, 1] \cup [3, \infty)$. We now proceed as usual.

   - *inside out*: We substitute the expression $(g \circ f)(x)$ for s in the expression $h(s)$ first to get

$$\begin{aligned} (h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h\left(2 - \sqrt{x^2 - 4x + 3}\right) \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

   - *outside in*: We use the formula for $h(s)$ first to get

$$\begin{aligned} (h \circ (g \circ f))(x) &= h((g \circ f)(x)) = \frac{2\left((g \circ f)(x)\right)}{((g \circ f)(x)) + 1} \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

   To find the domain of $h \circ (g \circ f)$, we need the domain elements of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, so that $(g \circ f)(x)$ is in the domain of h. As we've seen several times already, the only domain restriction for h is $s \neq -1$, so we set $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3} = -1$ and exclude the solutions. We get $\sqrt{x^2 - 4x + 3} = 3$, and, after squaring both sides, we have $x^2 - 4x + 3 = 9$. We solve $x^2 - 4x - 6 = 0$ using the quadratic formula and obtain $x = 2 \pm \sqrt{10}$. The reader is encouraged to check that both of these numbers satisfy the original equation, $2 - \sqrt{x^2 - 4x + 3} = -1$ and also belong to the domain of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, and so must be excluded from our final answer.[2] Our final domain for $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

10. The expression $((h \circ g) \circ f)(x)$ indicates that we first find the composite $h \circ g$ and then compose that with f. From number 7, we have

$$(h \circ g)(t) = \frac{4 - 2\sqrt{t + 3}}{3 - \sqrt{t + 3}}$$

   with domain $[-3, 6) \cup (6, \infty)$.

   - *inside out*: We substitute the expression $f(x)$ for t in the expression $(h \circ g)(t)$ to get

[2] We can approximate $\sqrt{10} \approx 3$ so $2 - \sqrt{10} \approx -1$ and $2 + \sqrt{10} \approx 5$.

$$\begin{aligned}((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = (h \circ g)\left(x^2 - 4x\right) \\ &= \frac{4 - 2\sqrt{\left(x^2 - 4x\right) + 3}}{3 - \sqrt{\left(x^2 - 4x\right) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

- *outside in*: We use the formula for $(h \circ g)(t)$ first to get

$$\begin{aligned}((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = \frac{4 - 2\sqrt{(f(x)) + 3}}{3 - \sqrt{f(x)) + 3}} \\ &= \frac{4 - 2\sqrt{\left(x^2 - 4x\right) + 3}}{3 - \sqrt{\left(x^2 - 4x\right) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

Since the domain of f is all real numbers, the challenge here to find the domain of $(h \circ g) \circ f$ is to determine the values $f(x)$ which are in the domain of $h \circ g$, $[-3, 6) \cup (6, \infty)$. At first glance, it appears as if we have two (or three!) inequalities to solve: $-3 \leq f(x) < 6$ and $f(x) > 6$. Alternatively, we could solve $f(x) = x^2 - 4x \geq -3$ and exclude the solutions to $f(x) = x^2 - 4x = 6$ which is not only easier from a procedural point of view, but also easier since we've already done both calculations. In number 4, we solved $x^2 - 4x \geq -3$ and obtained the solution $(-\infty, 1] \cup [3, \infty)$ and in number 9, we solved $x^2 - 4x - 6 = 0$ and obtained $x = 2 \pm \sqrt{10}$. Hence, the domain of $(h \circ g) \circ f$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$. □

As previously mentioned, it should be clear from Example 9.2.1 that, in general, $g \circ f \neq f \circ g$, in other words, function composition is not *commutative*. However, numbers 9 and 10 demonstrate the **associative** property of function composition. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. We summarize the important properties of function composition in the theorem below.

Theorem 9.2. Properties of Function Composition: Suppose f, g, and h are functions.

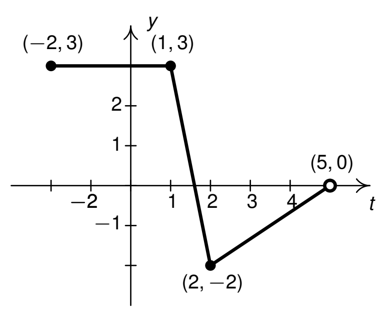
- **Associative Law of Composition:** $h \circ (g \circ f) = (h \circ g) \circ f$, provided the composite functions are defined.
- **Composition Identity:** The function $I(x) = x$ satisfies: $I \circ f = f \circ I = f$ for all functions, f.

By repeated applications of Definition 9.2, we find $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$. Similarly, $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$. This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality $h \circ (g \circ f) = (h \circ g) \circ f$. A consequence of the associativity of function composition is that there is no need for parentheses when we write $h \circ g \circ f$. The second property can also be verified using Definition 9.2. Recall that the function $I(x) = x$ is called the *identity function* and was introduced in Exercise 35 in Section 3.2. If we compose the function I with a function f, then we have $(I \circ f)(x) = I(f(x)) = f(x)$, and a similar computation shows $(f \circ I)(x) = f(I(x)) = f(x)$. This establishes that we have an identity for function composition much in the same way the function $I(x) = 1$ is an identity for function multiplication.

As we know, not all functions are described by formulas, and, moreover, not all functions are described by just *one* formula. The next example applies the concept of function composition to functions represented in various and sundry ways.

Example 9.2.2. Consider the following functions:

- $f(x) = 6x - x^2$
- $g(t) \begin{cases} 2t - 1 & \text{if } -1 \leq t < 3, \\ t^2 & \text{if } t \geq 3. \end{cases}$
- $h = \{(-3, 1), (-2, 6), (0, -2), (1, 5), (3, -1)\}$
- s whose graph is given below:



$y = s(t)$

1. Find and simplify the following function values:

   (a) $(g \circ f)(2)$ (b) $(h \circ g)(-1)$ (c) $(h \circ s)(-2)$ (d) $(f \circ s)(0)$

2. Find and simplify a formula for $(g \circ f)(x)$.

3. Write $s \circ h$ as a set of ordered pairs.

Solution.

1. (a) To find $(g \circ f)(2) = g(f(2))$ we first find $f(2) = 6(2) - (2)^2 = 8$. Since $8 \geq 3$, we use the rule $g(t) = t^2$ so $g(8) = (8)^2 = 64$. Hence, $(g \circ f)(3) = g(f(3)) = g(8) = 64$.

   (b) Since $(h \circ g)(-1) = h(g(-1))$ we first need $g(-1)$. Since $-1 \leq -1 < 3$, we use the rule $g(t) = 2t - 1$ and find $g(-1) = 2(-1) - 1 = -3$. Next, we need $h(-3)$. Since $(-3, 1) \in h$, we have that $h(-3) = 1$. Putting it all together, we find $(h \circ g)(-1) = h(g(-1)) = h(-3) = 1$.

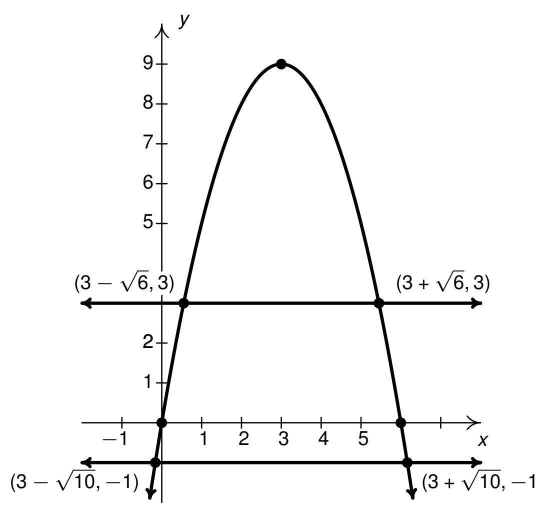
(c) To find $(h \circ s)(-2) = h(s(-2))$, we first need $s(-2)$. We see the point $(-2,3)$ is on the graph of s, so $s(-2) = 3$. Next, we see $(3,-1) \in h$, so $h(3) = -1$. Hence, $(h \circ s)(-2) = h(s(-2)) = h(3) = -1$.

(d) To find $(f \circ s)(0) = f(s(0))$ we infer from the graph of s that it contains the point $(0,3)$, so $s(0) = 3$. Sine $f(3) = 6(3) - (3)^2 = 9$, we have $(f \circ s)(0) = f(s(0)) = f(3) = 9$.

2. To find a formula for $(g \circ f)(x) = g(f(x))$, we substitute $f(x) = 6x - x^2$ in for t in the formula for $g(t)$:

$$(g \circ f)(x) = g(f(x)) = g(6x - x^2) = \begin{cases} 2(6x - x^2) - 1 & \text{if } -1 \leq 6x - x^2 < 3, \\ (6x - x^2)^2 & \text{if } 6x - x^2 \geq 3. \end{cases}$$

Simplifying each expression, we get $2(6x - x^2) - 1 = -2x^2 + 12x - 1$ for the first piece and $(6x - x^2)^2 = x^4 - 12x^3 + 36x^2$ for the second piece. The real challenge comes in solving the inequalities $-1 \leq 6x - x^2 < 3$ and $6x - x^2 \geq 3$. While we could solve each individually using a sign diagram, a graphical approach works best here. We graph the parabola $y = 6x - x^2$, finding the vertex is $(3,9)$ with intercepts $(0,0)$ and $(6,0)$ along with the horizontal lines $y = -1$ and $y = 3$ below. We determine the intersection points by solving $6x - x^2 = -1$ and $6x - x^2 = 3$. Using the quadratic formula, we find the solutions to each equation are $x = 3 \pm \sqrt{10}$ and $x = 3 \pm \sqrt{6}$, respectively.

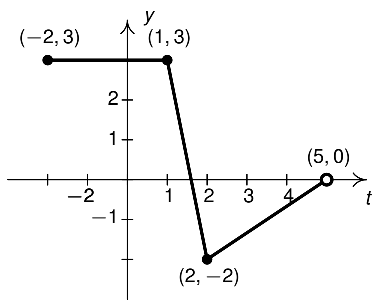


$y = 6x - x^2$, $y = -1$, $y = 3$

From the graph, we see the parabola $y = 6x - x^2$ is between the lines $y = -1$ and $y = 3$ from $x = 3 - \sqrt{10}$ to $x = 3 - \sqrt{6}$ and again from $x = 3 + \sqrt{6}$ to $x = 3 + \sqrt{10}$. Hence the solution to $-1 \leq 6x - x^2 < 3$ is $[3 - \sqrt{10}, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, 3 + \sqrt{10}]$. We also note $y = 6x - x^2$ is above the line $y = 3$ for all x between $x = 3 - \sqrt{6}$ and $3 + \sqrt{6}$. Hence, the solution to $6x - x^2 \geq 3$ is $[3 - \sqrt{6}, 3 + \sqrt{6}]$. Hence,

$$(g \circ f)(x) = \begin{cases} -2x^2 + 12x - 1 & \text{if } x \in [3 - \sqrt{10}, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, 3 + \sqrt{10}], \\ x^4 - 12x^3 + +36x^2 & \text{if } x \in [3 - \sqrt{6}, 3 + \sqrt{6}]. \end{cases}$$

3. Last but not least, we are tasked with representing $s \circ h$ as a set of ordered pairs. Since h is described by the discrete set of points $h = \{(-3, 1), (-2, 6), (0, -2), (1, 5), (3, -1)\}$, we find $s \circ h$ point by point. We keep the graph of s handy and construct the table below to help us organize our work.



$y = s(t)$

x	$h(x)$	$s(h(x))$
-3	1	3
-2	6	undefined
0	-2	3
1	5	undefined
3	-1	3

Since neither 6 nor 5 are in the domain of s, -2 and 1 are not in the domain of $s \circ h$. Hence, we get $s \circ h = \{(-3, 3), (0, 3), (3, 3)\}$. □

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates. As with Example 9.1.2, we want to avoid trivial decompositions, which, when it comes to function composition, are those involving the identity function $I(x) = x$ as described in Theorem 9.2.

Example 9.2.3.

1. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

   (a) $F(x) = |3x - 1|$ (b) $G(t) = \dfrac{2}{t^2 + 1}$ (c) $H(s) = \dfrac{\sqrt{s} + 1}{\sqrt{s} - 1}$

2. For $F(x) = \sqrt{\dfrac{2x - 1}{x^2 + 4}}$, find functions f, g, and h to decompose F nontrivially as $F = f \circ \left(\dfrac{g}{h}\right)$.

Solution. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. (a) Our goal is to express the function F as $F = g \circ f$ for functions g and f. From Definition 9.2, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the 'inside' function and g as being the 'outside' function. Looking at $F(x) = |3x - 1|$ from an 'inside versus outside' perspective, we can think of $3x - 1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x - 1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, this checks: $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x - 1| = F(x)$.

(b) We attack deconstructing G from an operational approach. Given an input t, the first step is to square t, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of *three* functions: f, g and h. Our first function, f, is the function that squares its input, $f(t) = t^2$. The next function is the function that adds 1 to its input, $g(t) = t + 1$. Our last function takes its input and divides it into 2, $h(t) = \frac{2}{t}$. The claim is that $G = h \circ g \circ f$ which checks:

$$(h \circ g \circ f)(t) = h(g(f(t))) = h(g\left(t^2\right)) = h\left(t^2 + 1\right) = \frac{2}{t^2 + 1} = G(x).$$

(c) If we look $H(s) = \frac{\sqrt{s}+1}{\sqrt{s}-1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{s}$ is a simple piece of the larger function. If we define $f(s) = \sqrt{s}$, we have $H(s) = \frac{f(s)+1}{f(s)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula for $g(s)$ by looking at what is being done to $f(s)$. We take $g(s) = \frac{s+1}{s-1}$, and check below:

$$(g \circ f)(s) = g(f(s)) = \frac{f(s) + 1}{f(s) - 1} = \frac{\sqrt{s} + 1}{\sqrt{s} - 1} = H(s).$$

$\square$

2. To write $F = f \circ \left(\frac{g}{h}\right)$ means

$$F(x) = \sqrt{\frac{2x - 1}{x^2 + 4}} = \left(f \circ \left(\frac{g}{h}\right)\right)(x) = f\left(\left(\frac{g}{h}\right)(x)\right) = f\left(\frac{g(x)}{h(x)}\right).$$

Working from the inside out, we have a rational expression with numerator $g(x)$ and denominator $h(x)$. Looking at the formula for $F(x)$, one choice is $g(x) = 2x - 1$ and $h(x) = x^2 + 4$. Making these identifications, we have

$$F(x) = \sqrt{\frac{2x - 1}{x^2 + 4}} = \sqrt{\frac{g(x)}{h(x)}}.$$

Since F takes the square root of $\frac{g(x)}{h(x)}$, the our last function f is the function that takes the square root of its input, i.e., $f(x) = \sqrt{x}$. We leave it to the reader to check that, indeed, $F = f \circ \left(\frac{g}{h}\right)$. $\square$

We close this section of a real-world application of function composition.

Example 9.2.4. The surface area of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula $r(t) = 3t^2$, where t is measured in seconds, $t \geq 0$, and r is measured in inches. Find and interpret $(S \circ r)(t)$.

Solution. If we look at the functions $S(r)$ and $r(t)$ individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time, t, we could find the radius at that time, $r(t)$ and feed that into $S(r)$ to find the surface area at that time. From this we see that the surface area S is ultimately a function of time t and we find $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi\left(3t^2\right)^2 = 36\pi t^4$. This formula allows us to compute the surface area directly given the time without going through the 'intermediary variable' r. $\square$

9.2.1 Exercises

In Exercises 1 - 12, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

1. $f(x) = x^2$, $g(t) = 2t + 1$
2. $f(x) = 4 - x$, $g(t) = 1 - t^2$
3. $f(x) = 4 - 3x$, $g(t) = |t|$
4. $f(x) = |x - 1|$, $g(t) = t^2 - 5$
5. $f(x) = 4x + 5$, $g(t) = \sqrt{t}$
6. $f(x) = \sqrt{3 - x}$, $g(t) = t^2 + 1$
7. $f(x) = 6 - x - x^2$, $g(t) = t\sqrt{t + 10}$
8. $f(x) = \sqrt[3]{x + 1}$, $g(t) = 4t^2 - t$
9. $f(x) = \dfrac{3}{1 - x}$, $g(t) = \dfrac{4t}{t^2 + 1}$
10. $f(x) = \dfrac{x}{x + 5}$, $g(t) = \dfrac{2}{7 - t^2}$
11. $f(x) = \dfrac{2x}{5 - x^2}$, $g(t) = \sqrt{4t + 1}$
12. $f(x) = \sqrt{2x + 5}$, $g(t) = \dfrac{10t}{t^2 + 1}$

In Exercises 13 - 24, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(t)$
- $(f \circ f)(x)$

13. $f(x) = 2x + 3$, $g(t) = t^2 - 9$
14. $f(x) = x^2 - x + 1$, $g(t) = 3t - 5$
15. $f(x) = x^2 - 4$, $g(t) = |t|$
16. $f(x) = 3x - 5$, $g(t) = \sqrt{t}$
17. $f(x) = |x + 1|$, $g(t) = \sqrt{t}$
18. $f(x) = 3 - x^2$, $g(t) = \sqrt{t + 1}$
19. $f(x) = |x|$, $g(t) = \sqrt{4 - t}$
20. $f(x) = x^2 - x - 1$, $g(t) = \sqrt{t - 5}$
21. $f(x) = 3x - 1$, $g(t) = \dfrac{1}{t + 3}$
22. $f(x) = \dfrac{3x}{x - 1}$, $g(t) = \dfrac{t}{t - 3}$
23. $f(x) = \dfrac{x}{2x + 1}$, $g(t) = \dfrac{2t + 1}{t}$
24. $f(x) = \dfrac{2x}{x^2 - 4}$, $g(t) = \sqrt{1 - t}$

In Exercises 25 - 30, use $f(x) = -2x$, $g(t) = \sqrt{t}$ and $h(s) = |s|$ to find and simplify expressions for the following functions and state the domain of each using interval notation.

25. $(h \circ g \circ f)(x)$
26. $(h \circ f \circ g)(t)$
27. $(g \circ f \circ h)(s)$
28. $(g \circ h \circ f)(x)$
29. $(f \circ h \circ g)(t)$
30. $(f \circ g \circ h)(s)$

In Exercises 31 - 43, let f be the function defined by

$$f = \{(-3, 4), (-2, 2), (-1, 0), (0, 1), (1, 3), (2, 4), (3, -1)\}$$

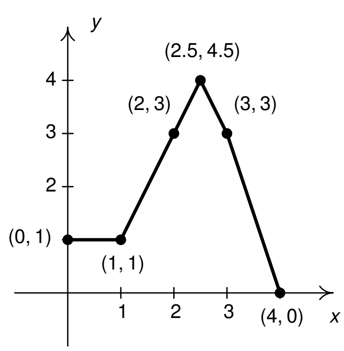
and let g be the function defined by

$$g = \{(-3, -2), (-2, 0), (-1, -4), (0, 0), (1, -3), (2, 1), (3, 2)\}.$$

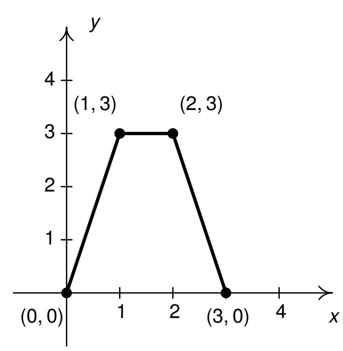
Find the following, if it exists.

31. $(f \circ g)(3)$
32. $f(g(-1))$
33. $(f \circ f)(0)$
34. $(f \circ g)(-3)$
35. $(g \circ f)(3)$
36. $g(f(-3))$
37. $(g \circ g)(-2)$
38. $(g \circ f)(-2)$
39. $g(f(g(0)))$
40. $f(f(f(-1)))$
41. $f(f(f(f(f(1)))))$
42. $(\underbrace{g \circ g \circ \cdots \circ g}_{n \text{ times}})(0)$
43. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 44 - 50, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the following if it exists.



$y = f(x)$



$y = g(x)$

44. $(g \circ f)(1)$
45. $(f \circ g)(3)$
46. $(g \circ f)(2)$
47. $(f \circ g)(0)$
48. $(f \circ f)(4)$
49. $(g \circ g)(1)$
50. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 51 - 60, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

51. $p(x) = (2x+3)^3$
52. $P(x) = \left(x^2 - x + 1\right)^5$
53. $h(t) = \sqrt{2t-1}$
54. $H(t) = |7-3t|$
55. $r(s) = \dfrac{2}{5s+1}$
56. $R(s) = \dfrac{7}{s^2-1}$
57. $q(z) = \dfrac{|z|+1}{|z|-1}$
58. $Q(z) = \dfrac{2z^3+1}{z^3-1}$
59. $v(x) = \dfrac{2x+1}{3-4x}$
60. $w(x) = \dfrac{x^2}{x^4+1}$
61. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.
62. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?
63. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?
64. Let $f(x) = 3x+1$ and let $g(x) = \begin{cases} 2x-1 & \text{if } x \leq 3 \\ 4-x & \text{if } x > 3 \end{cases}$. Find expressions for $(f \circ g)(x)$ and $(g \circ f)(x)$.
65. The volume V of a cube is a function of its side length x. Let's assume that $x = t+1$ is also a function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.
66. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour x is given by $x(t) = -4t^2 + 20t + 92$, where t is the number of hours since 10 AM, $0 \leq t \leq 4$.
    (a) Find an expression for the revenue per hour R as a function of x.
    (b) Find and simplify $(R \circ x)(t)$. What does this represent?
    (c) What is the revenue per hour at noon?
67. Discuss with your classmates how 'real-world' processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

9.2.2 Answers

1. For $f(x) = x^2$ and $g(t) = 2t + 1$,
   - $(g \circ f)(0) = 1$
   - $(g \circ f)(-3) = 19$
   - $(f \circ g)(-1) = 1$
   - $(f \circ g)\left(\frac{1}{2}\right) = 4$
   - $(f \circ f)(2) = 16$
   - $(f \circ f)(-2) = 16$

2. For $f(x) = 4 - x$ and $g(t) = 1 - t^2$,
   - $(g \circ f)(0) = -15$
   - $(g \circ f)(-3) = -48$
   - $(f \circ g)(-1) = 4$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{13}{4}$
   - $(f \circ f)(2) = 2$
   - $(f \circ f)(-2) = -2$

3. For $f(x) = 4 - 3x$ and $g(t) = |t|$,
   - $(g \circ f)(0) = 4$
   - $(g \circ f)(-3) = 13$
   - $(f \circ g)(-1) = 1$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{5}{2}$
   - $(f \circ f)(2) = 10$
   - $(f \circ f)(-2) = -26$

4. For $f(x) = |x - 1|$ and $g(t) = t^2 - 5$,
   - $(g \circ f)(0) = -4$
   - $(g \circ f)(-3) = 11$
   - $(f \circ g)(-1) = 5$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{23}{4}$
   - $(f \circ f)(2) = 0$
   - $(f \circ f)(-2) = 2$

5. For $f(x) = 4x + 5$ and $g(t) = \sqrt{t}$,
   - $(g \circ f)(0) = \sqrt{5}$
   - $(g \circ f)(-3)$ is not real
   - $(f \circ g)(-1)$ is not real
   - $(f \circ g)\left(\frac{1}{2}\right) = 5 + 2\sqrt{2}$
   - $(f \circ f)(2) = 57$
   - $(f \circ f)(-2) = -7$

6. For $f(x) = \sqrt{3 - x}$ and $g(t) = t^2 + 1$,
   - $(g \circ f)(0) = 4$
   - $(g \circ f)(-3) = 7$
   - $(f \circ g)(-1) = 1$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt{7}}{2}$
   - $(f \circ f)(2) = \sqrt{2}$
   - $(f \circ f)(-2) = \sqrt{3 - \sqrt{5}}$

7. For $f(x) = 6 - x - x^2$ and $g(t) = t\sqrt{t + 10}$,
   - $(g \circ f)(0) = 24$
   - $(g \circ f)(-3) = 0$
   - $(f \circ g)(-1) = 0$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{27 - 2\sqrt{42}}{8}$
   - $(f \circ f)(2) = 6$
   - $(f \circ f)(-2) = -14$

8. For $f(x) = \sqrt[3]{x+1}$ and $g(t) = 4t^2 - t$,

   - $(g \circ f)(0) = 3$
   - $(f \circ g)(-1) = \sqrt[3]{6}$
   - $(f \circ f)(2) = \sqrt[3]{\sqrt[3]{3}+1}$
   - $(g \circ f)(-3) = 4\sqrt[3]{4} + \sqrt[3]{2}$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt[3]{12}}{2}$
   - $(f \circ f)(-2) = 0$

9. For $f(x) = \frac{3}{1-x}$ and $g(t) = \frac{4t}{t^2+1}$,

   - $(g \circ f)(0) = \frac{6}{5}$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = \frac{3}{4}$
   - $(g \circ f)(-3) = \frac{48}{25}$
   - $(f \circ g)\left(\frac{1}{2}\right) = -5$
   - $(f \circ f)(-2)$ is undefined

10. For $f(x) = \frac{x}{x+5}$ and $g(t) = \frac{2}{7-t^2}$,

    - $(g \circ f)(0) = \frac{2}{7}$
    - $(f \circ g)(-1) = \frac{1}{16}$
    - $(f \circ f)(2) = \frac{2}{37}$
    - $(g \circ f)(-3) = \frac{8}{19}$
    - $(f \circ g)\left(\frac{1}{2}\right) = \frac{8}{143}$
    - $(f \circ f)(-2) = -\frac{2}{13}$

11. For $f(x) = \frac{2x}{5-x^2}$ and $g(t) = \sqrt{4t+1}$,

    - $(g \circ f)(0) = 1$
    - $(f \circ g)(-1)$ is not real
    - $(f \circ f)(2) = -\frac{8}{11}$
    - $(g \circ f)(-3) = \sqrt{7}$
    - $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{3}$
    - $(f \circ f)(-2) = \frac{8}{11}$

12. For $f(x) = \sqrt{2x+5}$ and $g(t) = \frac{10t}{t^2+1}$,

    - $(g \circ f)(0) = \frac{5\sqrt{5}}{3}$
    - $(f \circ g)(-1)$ is not real
    - $(f \circ f)(2) = \sqrt{11}$
    - $(g \circ f)(-3)$ is not real
    - $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{13}$
    - $(f \circ f)(-2) = \sqrt{7}$

13. For $f(x) = 2x + 3$ and $g(t) = t^2 - 9$

    - $(g \circ f)(x) = 4x^2 + 12x$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = 2t^2 - 15$, domain: $(-\infty, \infty)$
    - $(f \circ f)(x) = 4x + 9$, domain: $(-\infty, \infty)$

14. For $f(x) = x^2 - x + 1$ and $g(t) = 3t - 5$

    - $(g \circ f)(x) = 3x^2 - 3x - 2$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = 9t^2 - 33t + 31$, domain: $(-\infty, \infty)$
    - $(f \circ f)(x) = x^4 - 2x^3 + 2x^2 - x + 1$, domain: $(-\infty, \infty)$

15. For $f(x) = x^2 - 4$ and $g(t) = |t|$

    - $(g \circ f)(x) = |x^2 - 4|$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = |t|^2 - 4 = t^2 - 4$, domain: $(-\infty, \infty)$
    - $(f \circ f)(x) = x^4 - 8x^2 + 12$, domain: $(-\infty, \infty)$

16. For $f(x) = 3x - 5$ and $g(t) = \sqrt{t}$

    - $(g \circ f)(x) = \sqrt{3x - 5}$, domain: $\left[\frac{5}{3}, \infty\right)$
    - $(f \circ g)(t) = 3\sqrt{t} - 5$, domain: $[0, \infty)$
    - $(f \circ f)(x) = 9x - 20$, domain: $(-\infty, \infty)$

17. For $f(x) = |x + 1|$ and $g(t) = \sqrt{t}$

    - $(g \circ f)(x) = \sqrt{|x + 1|}$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = |\sqrt{t} + 1| = \sqrt{t} + 1$, domain: $[0, \infty)$
    - $(f \circ f)(x) = ||x + 1| + 1| = |x + 1| + 1$, domain: $(-\infty, \infty)$

18. For $f(x) = 3 - x^2$ and $g(t) = \sqrt{t + 1}$

    - $(g \circ f)(x) = \sqrt{4 - x^2}$, domain: $[-2, 2]$
    - $(f \circ g)(t) = 2 - t$, domain: $[-1, \infty)$
    - $(f \circ f)(x) = -x^4 + 6x^2 - 6$, domain: $(-\infty, \infty)$

19. For $f(x) = |x|$ and $g(t) = \sqrt{4 - t}$

    - $(g \circ f)(x) = \sqrt{4 - |x|}$, domain: $[-4, 4]$
    - $(f \circ g)(t) = |\sqrt{4 - t}| = \sqrt{4 - t}$, domain: $(-\infty, 4]$
    - $(f \circ f)(x) = ||x|| = |x|$, domain: $(-\infty, \infty)$

20. For $f(x) = x^2 - x - 1$ and $g(t) = \sqrt{t - 5}$

    - $(g \circ f)(x) = \sqrt{x^2 - x - 6}$, domain: $(-\infty, -2] \cup [3, \infty)$
    - $(f \circ g)(t) = t - 6 - \sqrt{t - 5}$, domain: $[5, \infty)$
    - $(f \circ f)(x) = x^4 - 2x^3 - 2x^2 + 3x + 1$, domain: $(-\infty, \infty)$

21. For $f(x) = 3x - 1$ and $g(t) = \frac{1}{t+3}$

    - $(g \circ f)(x) = \frac{1}{3x+2}$, domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
    - $(f \circ g)(t) = -\frac{t}{t+3}$, domain: $(-\infty, -3) \cup (-3, \infty)$
    - $(f \circ f)(x) = 9x - 4$, domain: $(-\infty, \infty)$

22. For $f(x) = \frac{3x}{x-1}$ and $g(t) = \frac{t}{t-3}$

- $(g \circ f)(x) = x$, domain: $(-\infty, 1) \cup (1, \infty)$
- $(f \circ g)(t) = t$, domain: $(-\infty, 3) \cup (3, \infty)$
- $(f \circ f)(x) = \frac{9x}{2x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$

23. For $f(x) = \frac{x}{2x+1}$ and $g(t) = \frac{2t+1}{t}$

- $(g \circ f)(x) = \frac{4x+1}{x}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right), \cup (0, \infty)$
- $(f \circ g)(t) = \frac{2t+1}{5t+2}$, domain: $\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, 0\right) \cup (0, \infty)$
- $(f \circ f)(x) = \frac{x}{4x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$

24. For $f(x) = \frac{2x}{x^2-4}$ and $g(t) = \sqrt{1-t}$

- $(g \circ f)(x) = \sqrt{\frac{x^2-2x-4}{x^2-4}}$, domain: $(-\infty, -2) \cup \left[1-\sqrt{5}, 2\right) \cup \left[1+\sqrt{5}, \infty\right)$
- $(f \circ g)(t) = -\frac{2\sqrt{1-t}}{t+3}$, domain: $(-\infty, -3) \cup (-3, 1]$
- $(f \circ f)(x) = \frac{4x-x^3}{x^4-9x^2+16}$, domain: $\left(-\infty, -\frac{1+\sqrt{17}}{2}\right) \cup \left(-\frac{1+\sqrt{17}}{2}, -2\right) \cup \left(-2, \frac{1-\sqrt{17}}{2}\right) \cup \left(\frac{1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2}\right) \cup \left(\frac{-1+\sqrt{17}}{2}, 2\right) \cup \left(2, \frac{1+\sqrt{17}}{2}\right) \cup \left(\frac{1+\sqrt{17}}{2}, \infty\right)$

25. $(h \circ g \circ f)(x) = |\sqrt{-2x}| = \sqrt{-2x}$, domain: $(-\infty, 0]$

26. $(h \circ f \circ g)(t) = |-2\sqrt{t}| = 2\sqrt{t}$, domain: $[0, \infty)$

27. $(g \circ f \circ h)(s) = \sqrt{-2|s|}$, domain: $\{0\}$

28. $(g \circ h \circ f)(x) = \sqrt{|-2x|} = \sqrt{2|x|}$, domain: $(-\infty, \infty)$

29. $(f \circ h \circ g)(t) = -2|\sqrt{t}| = -2\sqrt{t}$, domain: $[0, \infty)$

30. $(f \circ g \circ h)(s) = -2\sqrt{|s|}$, , domain: $(-\infty, \infty)$

31. $(f \circ g)(3) = f(g(3)) = f(2) = 4$

32. $f(g(-1)) = f(-4)$ which is undefined

33. $(f \circ f)(0) = f(f(0)) = f(1) = 3$

34. $(f \circ g)(-3) = f(g(-3)) = f(-2) = 2$

35. $(g \circ f)(3) = g(f(3)) = g(-1) = -4$

36. $g(f(-3)) = g(4)$ which is undefined

37. $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$

38. $(g \circ f)(-2) = g(f(-2)) = g(2) = 1$

39. $g(f(g(0))) = g(f(0)) = g(1) = -3$

40. $f(f(f(-1))) = f(f(0)) = f(1) = 3$

41. $f(f(f(f(f(1))))) = f(f(f(f(3)))) = f(f(f(-1))) = f(f(0)) = f(1) = 3$

42. $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0) = 0$

43.
    - The domain of $f \circ g$ is $\{-3, -2, 0, 1, 2, 3\}$ and the range of $f \circ g$ is $\{1, 2, 3, 4\}$.
    - The domain of $g \circ f$ is $\{-2, -1, 0, 1, 3\}$ and the range of $g \circ f$ is $\{-4, -3, 0, 1, 2\}$.

44. $(g \circ f)(1) = 3$

45. $(f \circ g)(3) = 1$

46. $(g \circ f)(2) = 0$

47. $(f \circ g)(0) = 1$

48. $(f \circ f)(4) = 1$

49. $(g \circ g)(1) = 0$

50.
    - The domain of $f \circ g$ is $[0, 3]$ and the range of $f \circ g$ is $[1, 4.5]$.
    - The domain of $g \circ f$ is $[0, 2] \cup [3, 4]$ and the range is $[0, 3]$.

51. Let $f(x) = 2x + 3$ and $g(x) = x^3$, then $p(x) = (g \circ f)(x)$.

52. Let $f(x) = x^2 - x + 1$ and $g(x) = x^5$, $P(x) = (g \circ f)(x)$.

53. Let $f(t) = 2t - 1$ and $g(t) = \sqrt{t}$, then $h(t) = (g \circ f)(t)$.

54. Let $f(t) = 7 - 3t$ and $g(t) = |t|$, then $H(t) = (g \circ f)(t)$.

55. Let $f(s) = 5s + 1$ and $g(s) = \frac{2}{s}$, then $r(s) = (g \circ f)(s)$.

56. Let $f(s) = s^2 - 1$ and $g(s) = \frac{7}{s}$, then $R(s) = (g \circ f)(s)$.

57. Let $f(z) = |z|$ and $g(z) = \frac{z+1}{z-1}$, then $q(z) = (g \circ f)(z)$.

58. Let $f(z) = z^3$ and $g(z) = \frac{2z+1}{z-1}$, then $Q(z) = (g \circ f)(z)$.

59. Let $f(x) = 2x$ and $g(x) = \frac{x+1}{3-2x}$, then $v(x) = (g \circ f)(x)$.

60. Let $f(x) = x^2$ and $g(x) = \frac{x}{x^2+1}$, then $w(x) = (g \circ f)(x)$.

61. $F(x) = \sqrt{\frac{x^3+6}{x^3-9}} = (h(g(f(x)))$ where $f(x) = x^3$, $g(x) = \frac{x+6}{x-9}$ and $h(x) = \sqrt{x}$.

62. $F(x) = 3\sqrt{-x+2} - 4 = k(j(f(h(g(x)))))$

63. One solution is $F(x) = -\frac{1}{2}(2x - 7)^3 + 1 = k(j(f(h(g(x)))))$ where $g(x) = 2x$, $h(x) = x - 7$, $j(x) = -\frac{1}{2}x$ and $k(x) = x + 1$. You could also have $F(x) = H(f(G(x)))$ where $G(x) = 2x - 7$ and $H(x) = -\frac{1}{2}x + 1$.

64. $(f \circ g)(x) = \begin{cases} 6x - 2 & \text{if } x \leq 3 \\ 13 - 3x & \text{if } x > 3 \end{cases}$ and $(g \circ f)(x) = \begin{cases} 6x + 1 & \text{if } x \leq \dfrac{2}{3} \\ 3 - 3x & \text{if } x > \dfrac{2}{3} \end{cases}$

65. $V(x) = x^3$ so $V(x(t)) = (t + 1)^3$

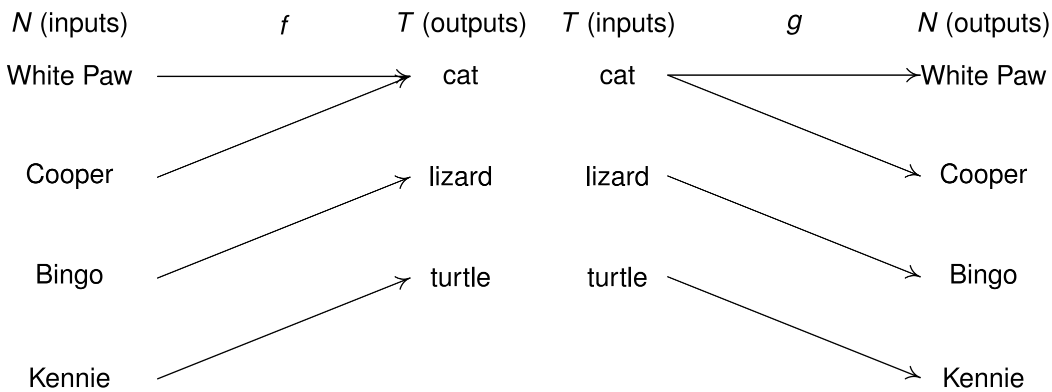
66.
    (a) $R(x) = 2x$
    (b) $(R \circ x)(t) = -8t^2 + 40t + 184$, $0 \leq t \leq 4$. This gives the revenue per hour as a function of time.
    (c) Noon corresponds to $t = 2$, so $(R \circ x)(2) = 232$. The hourly revenue at noon is \$232 per hour.

9.3 Relations and Implicit Functions

Up until now in this text, we have been exclusively special kinds of mappings called *functions*. In this section, we broaden our horizons to study more general mappings called *relations*. The reader is encouraged to revisit Definition 2.1 in Section 2.1 before proceeding with the definition of *relation* below.

Definition 9.3. Given two sets A and B, a **relation** from A to B is a process by which elements of A are matched with (or 'mapped to') elements of B.

Unlike Definition 2.1, Definition 9.3 puts no conditions on the process which maps elements of A to elements of B. This means that while all functions are relations, not all relations need be functions. For example, consider the mappings f and g below from Section 2.1.



Both f and g are relations. More specifically, f is a *function* from N to T while g is merely *relation* from T to N. As with functions, we may describe general relations in a variety of different ways: verbally, as mapping diagrams, or a set of ordered pairs. For example, just as we may describe the function f above as

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\},$$

we may represent g as

$$g = \{(\text{cat}, \text{White Paw}), (\text{cat}, \text{Cooper}), (\text{lizard}, \text{Bingo}), (\text{turtle}, \text{Kennie})\}.$$

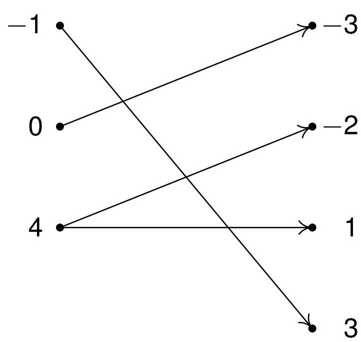
Note here the grammar 'g is a relation *from T to N*' is evidenced by the elements of T being listed first in the ordered pairs (i.e., the abscissae) and the elements of N being listed second (i.e., the ordinates.)

Unlike functions, we do not use function notation when describing the input/output relationship for general relations. For example, we may write '$f(\text{White Paw}) = \text{cat}$' since f maps the input 'White Paw' to only one output, 'cat.' However, $g(\text{cat})$ is ambiguous since it could mean 'White Paw' or 'Cooper.'[1]

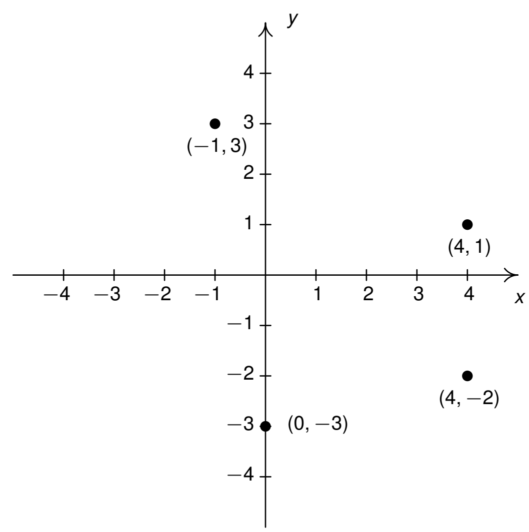
As with functions, our focus in this course will rest with relations of real numbers. Consider the relation R described as follows: $R = \{(-1, 3), (0, -3), (4, -2), (4, 1)\}$. Below on the left is a mapping diagram of R.

[1] In more advanced texts, we would write 'cat g White Paw' and 'cat g Cooper' to indicate g maps 'cat' to both 'White Paw' and 'Cooper.' Our study of relations, however, isn't deep enough to necessitate introducing and using this notation. Similarly, we won't introduce the notions of 'domain,' 'codomain,' and 'range' for relations, either.

However, since R relates real numbers, we can also create the graph of R in the same way we graphed functions - by interpreting the ordered pairs which comprise R as points in the plane. Since we have no context, we use the default labels 'x' for the horizontal axis and 'y' for the vertical axis.



A Mapping Diagram of R.



The graph of R.

Our next example focuses on using relations to describe sets of points in the plane and vice-versa.

Example 9.3.1.

1. Graph the following relations.

   (a) $S = \{(k, 2^k) \mid k = 0, \pm 1, \pm 2\}$

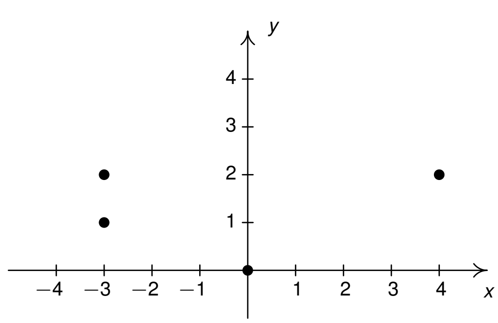
   (b) $P = \{(j, j^2) \mid j \text{ is an integer}\}$

   (c) $V = \{(3, y) \mid y \text{ is a real number}\}$

   (d) $R = \{(x, y) \mid x \text{ is a real number}, 1 < y \leq 3\}$

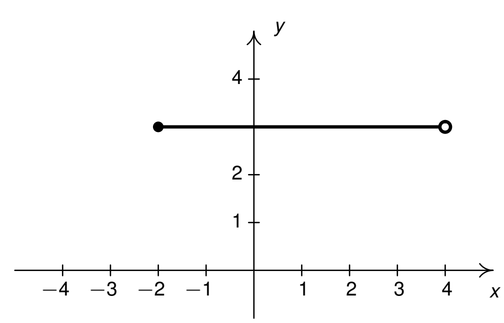
2. Find a roster or set-builder description for each of the relations below.

   (a)



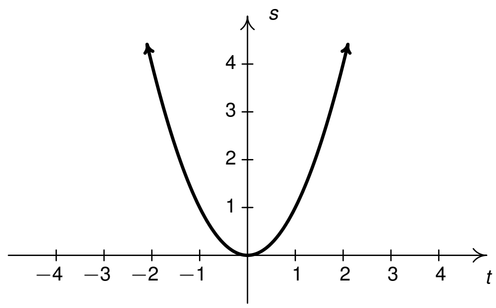
The graph of A

   (b)



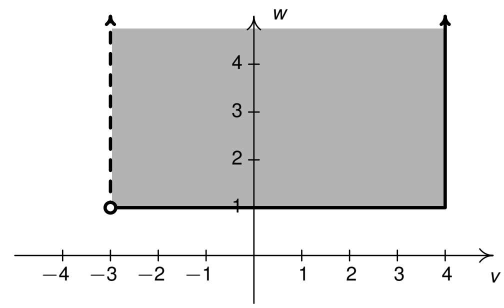
The graph of H

(c)



The graph of Q

(d)



The graph of T

Solution.

1. (a) The relation S is described using *set-builder notation*.[2] To generate the ordered pairs which belong to S, we substitute the given values of k, $k = 0, \pm 1, \pm 2$, into the formula $\left(k, 2^k\right)$.

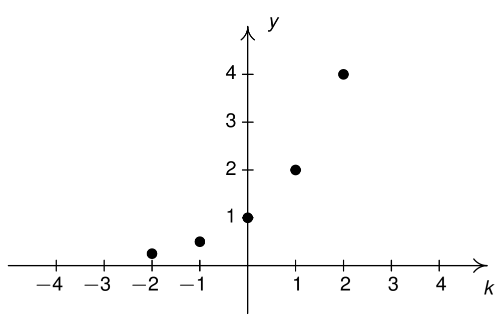
   Starting with $k = 0$, we get $\left(0, 2^0\right) = (0, 1)$. For $k = 1$, we get $\left(1, 2^1\right) = (1, 2)$, and for $k = -1$, we get $\left(-1, 2^{-1}\right) = \left(-1, \frac{1}{2}\right)$. Continuing, we get $\left(2, 2^2\right) = (2, 4)$ for $k = 2$ and, finally $\left(-2, 2^{-2}\right) = \left(-2, \frac{1}{4}\right)$ for $k = -2$. Hence, a roster description of S is $S = \left\{\left(-2, \frac{1}{4}\right), \left(-1, \frac{1}{2}\right), (0, 1), (1, 2), (2, 4)\right\}$.

   When we graph S, we label the horizontal axis as the k-axis, since 'k' was the variable chosen used to generate the ordered pairs and keep the default label 'y' for the vertical axis. The graph of S is below on the left.

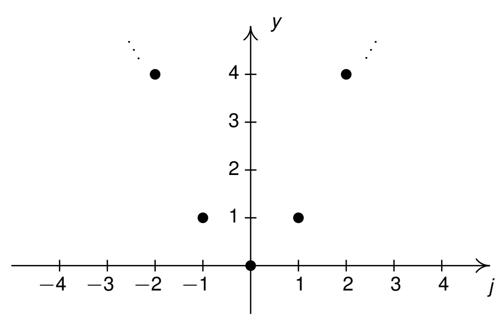
   (b) To graph the relation $P = \left\{\left(j, j^2\right) \mid j \text{ is an integer}\right\}$, we proceed as above when we graphed the relation S. Here, j is restricted to being an integer, which means $j = 0, \pm 1, \pm 2$, etc.

   Plugging in these sample values for j, we obtain the ordered pairs $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, $(-2, 4)$, etc. Since the variable j takes on only integer values, we could write P using the roster notation: $P = \{(0, 0), (\pm 1, 1), (\pm 2, 4), \ldots\}$.

   We plot a few of these points and use some periods of ellipsis to indicate the complete graph contains additional points not in the current field of view. The graph of P is below on the right.



The graph of S



The graph of P

[2]See Section 1.1 to review this, if needed.

(c) Next, we come to the relation V, described, once again, using set-builder notation. In this case, V consists of all ordered pairs of the form $(3, y)$ where y is free to be whatever real number we like, without any restriction.[3] For example, $(3, 0)$, $(3, -1)$, and $(3, 117)$ all belong to V as do $\left(3, \frac{1}{2}\right)$, $(3, -1.0342)$, $(3, \sqrt{2})$, etc.

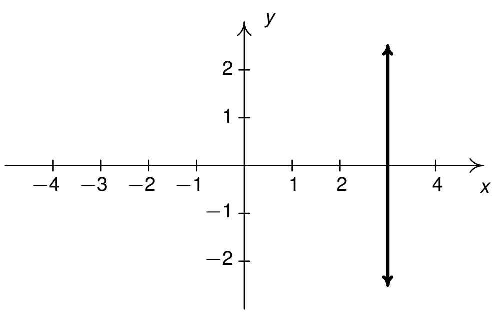
After plotting some sample points, becomes apparent that the ordered pairs which belong to V correspond to points which lie on the vertical line $x = 3$, and vice-versa. That is, every point on the line $x = 3$ has coordinates which correspond to an ordered pair belonging to V. The graph of V is below on the left.

(d) In the relation $R = \{(x, y) \mid 1 < y \leq 3\}$, we see y is restricted by the inequality $1 < y \leq 3$, but x is free to be whatever it likes.

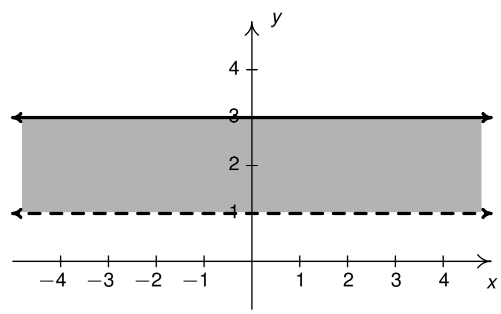
Since x is unrestricted, this means whatever the graph of R is, it will extend indefinitely off to the right and left. The restriction $y > 1$ means all points on the graph of R have a y-coordinate larger than one, so they are *above* the horizontal line $y = 1$. The restriction $y \leq 3$, on the other hand, means all the points on the graph of R have a y-coordinate less than or equal to 3, meaning they are either *on* or *below* the horizontal line $y = 3$.

In other words, the graph of R is the region in the plane between $y = 1$ and $y = 3$, including $y = 3$ but not $y = 1$. We signify this by *shading* the region between these two horizontal lines.

How do we communicate $y = 1$ is not part of the graph? One way is to visualize putting 'holes' all along the line $y = 1$ to indicate this is not part of the graph. In practice, however, this looks cluttered and could be confusing. Instead, we 'dash' the line $y = 1$ as seen below on the right.



The graph of V



The graph of R

2. (a) Since A consists of finitely many points, we can describe A using the roster method:

$$A = \{(-3, 2), (-3, 1), (0, 0), (4, 2)\}.$$

(b) The graph of H appears to be a portion of the horizontal line $y = 3$ from $x = -2$ (including $x = -2$) up to, but not including $x = 4$. Since it is impossible[4] to *list* each and every one of these points, we'll opt to describe H using set-builder as opposed to the roster method. Taking a cue from the description of the relations V and R above, we write $H = \{(x, 3) \mid -2 \leq x < 4\}$.

[3] We'll revisit the concept of a 'free variable' in Section **??**.

[4] Really impossible. The interested reader is encouraged to research **countable** versus **uncountable** sets.

(c) The graph of Q appears to be the graph of the function $s = f(t) = t^2$. Again, as the graph consists of infinitely many points, we will use set-builder notation to describe Q out of necessity.

There are a couple of different ways to do this. Taking a cue from the relation P above, we could write $Q = \{(t, t^2) \mid t \text{ is a real number}\}$. Alternatively, we could introduce the dependent variable, s into the description by writing $Q = \{(t, s) \mid s = t^2\}$ where here the assumption is x takes in all real number values.

(d) As with the relation R above, the relation T describes a region in the plane, The v-values appear to range between -3 (not including -3) and up to, and including, $v = 4$. The only restriction on the w-values is that $w \geq 1$, so we have $T = \{(v, w) \mid -3 < v \leq 4, w \geq 1\}$. $\Box$

As with functions, we can describe relations algebraically using equations. For example, the equation $v^2 + w^3 = 1$ relates two variables v and w each of which represent real numbers. More formally, we can express this sentiment by defining the relation $R = \{(v, w) \mid v^2 + w^3 = 1\}$. An ordered pair $(v, w) \in R$ means v and w are *related* by the equation $v^2 + w^3 = 1$; that is, the pair (v, w) *satisfy* the equation.

For example, to show $(3, -2) \in R$, we check that when we substitute $v = 3$ and $w = -2$, the equation $v^2 + w^3 = 1$ is true. Sure enough, $(3)^2 + (-2)^3 = 9 - 8 = 1$. Hence, R maps 3 to -2. Note, however, that $(-2, 3) \notin R$ since $(-2)^2 + (3)^3 = -8 + 27 \neq 1$ which means R does not map -2 to 3.

When asked to 'graph the equation' $v^2 + w^3 = 1$, we really have two options. We could graph the relation R above. In this case, we would be graphing $v^2 + w^3 = 1$ on the vw-plane.[5] Alternatively, we could define $S = \{(w, v) \mid v^2 + w^3 = 1\}$ and graph S. This is equivalent to graphing $v^2 + w^3 = 1$ on the wv-plane. We do both in our next example.

Example 9.3.2. Graph the equation $v^2 + w^3 = 1$ in the vw- and wv-planes. Include the axis-intercepts.
Solution.

- *graphing in the vw-plane:* We begin by finding the axis intercepts of the graph. To obtain a point on the v-axis, we require $w = 0$. To see if we have any v-intercepts on the graph of the equation $v^2 + w^3 = 1$, we substitute $w = 0$ into the equation and solve for v: $v^2 + (0)^3 = 1$. We get $v^2 = 1$ or $v = \pm 1$ so our two v-intercepts, as described in the vw-plane, are $(1, 0)$ and $(-1, 0)$.

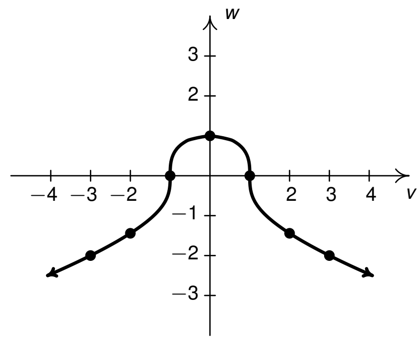
  Likewise, to find w-intercepts of the graph, we substitute $v = 0$ into the equation $v^2 + w^3 = 1$ and get $w^3 = 1$ or $w = 1$. Hence, he have only one w-intercept, $(0, 1)$.

  One way to efficiently produce additional points is to solve the equation $v^2 + w^3 = 1$ for one of the variables, say w, in terms of the other, v. In this way, we are treating w as the dependent variable and v as the independent variable. From $v^2 + w^3 = 1$, we get $w^3 = 1 - v^2$ or $w = \sqrt[3]{1 - v^2}$.

  We now substitute a value in for v, determine the corresponding value w, and plot the resulting point (v, w). We summarize our results below on the left. By plotting additional points (or getting help from a graphing utility), we produce the graph below on the right.

[5] Recall this means the horizontal axis is labeled 'v' and the vertical axis is labeled 'w.'

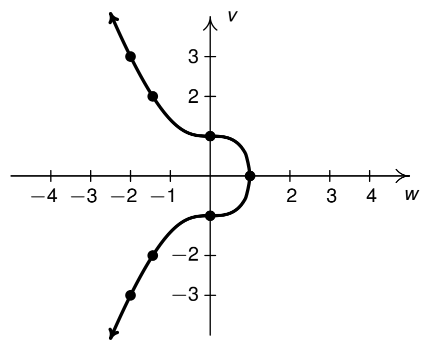
v	w	(v, w)
-3	-2	$(-3, -2)$
-2	$-\sqrt[3]{3}$	$(-2, -\sqrt[3]{3})$
-1	0	$(-1, 0)$
0	1	$(0, 1)$
1	0	$(1, 0)$
2	$-\sqrt[3]{3}$	$(2, -\sqrt[3]{3})$
3	-2	$(3, -2)$



$v^2 + w^3 = 1$

- *graphing in the wv-plane:* To graph $v^2 + w^3 = 1$ in the wv-plane, all we need to do is reverse the coordinates of the ordered pairs we obtained for our graph in the vw-plane. In particular, the v-intercepts are written $(0, 1)$ and $(0, -1)$ and the w-intercept is written $(1, 0)$. Using the table below on the left we produce the graph below on the right.

v	w	(w, v)
-3	-2	$(-2, -3)$
-2	$-\sqrt[3]{3}$	$(-\sqrt[3]{3}, -2)$
-1	0	$(0, -1)$
0	1	$(1, 0)$
1	0	$(0, 1)$
2	$-\sqrt[3]{3}$	$(-\sqrt[3]{3}, 2)$
3	-2	$(-2, 3)$



$v^2 + w^3 = 1$

□

Note that regardless of which geometric depiction we choose for $v^2 + w^3 = 1$, the graph appears to be symmetric about the w-axis. To prove this is the case, consider a generic point (v, w) on the graph of $v^2 + w^3 = 1$ in the vw-plane.

To show the point symmetric about the w-axis, $(-v, w)$ is also on the graph of $v^2 + w^3 = 1$, we need to show that the coordinates of the point $(-v, w)$ satisfy the equation $v^2 + w^3 = 1$. That is, we need to show $(-v)^2 + w^3 = 1$. Since $(-v)^2 + w^3 = v^2 + w^3$, and we know by assumption $v^2 + w^3 = 1$, we get $(-v)^2 + w^3 = v^2 + w^3 = 1$, proving $(-v, w)$ is also on the graph of the equation.

The key reason our proof above is successful is that algebraically, the equation $v^2 + w^3 = 1$ is unchanged if v is replaced with $-v$. Geometrically, this means the graph is the same if it undergoes a reflection across the w-axis. We generalize this reasoning in the following result. Note that, as usual, we default to the more common x and y-axis labels.

Theorem 9.3. Testing the Graph of an Equation for Symmetry:
To test the graph of an equation in the xy-plane for symmetry:

- about the x-axis: substitute $(x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the x-axis.
- about the y-axis: substitute $(-x, y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the y-axis.
- about the origin: substitute $(-x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Parts of Theorem 9.3 should look familiar from our work with even and odd functions. Indeed if a function f is even, $f(-x) = f(x)$. Hence, the equation $y = f(-x)$ reduces to the equation $y = f(x)$, so the graph of f is symmetric about the y-axis.

Likewise if f is odd, then $f(-x) = -f(x)$. In this case, the equation $-y = f(-x)$ reduces to $-y = -f(x)$, or $y = f(x)$, proving the graph is symmetric about the origin.

When it comes to symmetry about the x-axis, most of the time this indicates a violation of the Vertical Line Test, which is why we haven't discussed that particular kind of symmetry until now.

We put Theorem 9.3 to good use in the following example.

Example 9.3.3. Graph each of the equations below in the xy-plane. Find the axis intercepts, if any, and prove any symmetry suggested by the graphs.

1. $x^2 - y^2 = 4$
2. $(x - 1)^2 + 4y^2 = 16$

Solution.

1. We begin graphing $x^2 - y^2 = 4$ by checking for axis intercepts. To check for x-intercepts, we set $y = 0$ and solve $x^2 - (0)^2 = 4$. We get $x = \pm 2$ and obtain two x-intercepts $(-2, 0)$ and $(2, 0)$.

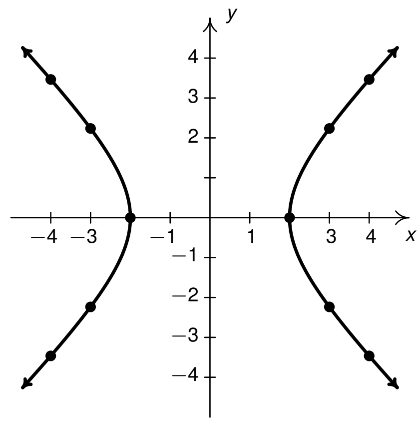
   When looking for y-intercepts, we set $x = 0$ and get $(0)^2 - y^2 = 4$ or $y^2 = -4$. Since this equation has no real number solutions, we have no y-intercepts.

   In order to produce more points on the graph, we solve $x^2 - y^2 = 4$ for y and obtain $y = \pm\sqrt{x^2 - 4}$. Since we know $x^2 - 4 \geq 0$ in order to produce real number results for y, we restrict our attention to $x \leq -2$ and $x \geq 2$. Doing so produces the table below on the left. Using these, we construct the graph below the right.

   The graph certainly appears to be symmetric about both axes and the origin. To prove this, we note that the equation $x^2 - (-y)^2 = 4$ quickly reduces to $x^2 - y^2 = 4$, proving the graph is symmetric about the x-axis.

   Likewise, the equations $(-x)^2 - y^2 = 4$ and $(-x)^2 - (-y)^2 = 4$ also reduce to $x^2 - y^2 = 4$, proving the graph is, indeed, symmetric about the y-axis and origin, respectively.

x	y	(x,y)
-4	$\pm 2\sqrt{3}$	$(-4, \pm 2\sqrt{3})$
-3	$\pm\sqrt{5}$	$(-3, \pm\sqrt{5})$
-2	0	$(-2, 0)$
2	0	$(2, 0)$
3	$\pm\sqrt{5}$	$(3, \pm\sqrt{5})$
4	$\pm 2\sqrt{3}$	$(4, \pm 2\sqrt{3})$



$x^2 - y^2 = 4$

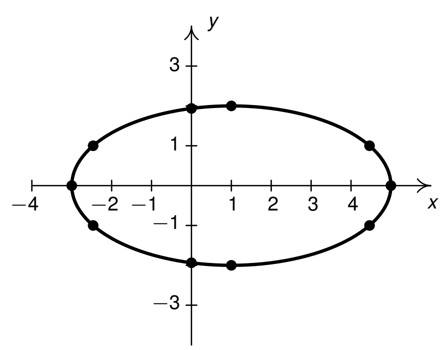
2. To determine if there are any x-intercepts on the graph of $(x-1)^2 + 4y^2 = 16$, we set $y = 0$ and solve $(x-1)^2 + 4(0)^2 = 16$. This reduces to $(x-1)^2 = 16$ which gives $x = -3$ and $x = 5$. Hence, we have two x-intercepts, $(-3, 0)$ and $(5, 0)$.

   Looking for y-intercepts, we set $x = 0$ and solve $(0-1)^2 + 4y^2 = 16$ or $1 + 4y^2 = 16$. This gives $y^2 = \frac{15}{4}$ so $y = \pm\frac{\sqrt{15}}{2}$. Hence, we have two y-intercepts: $\left(0, \pm\frac{\sqrt{15}}{2}\right)$.

   In this case, it is slightly easier[6] to solve for x in terms of y. From $(x-1)^2 + 4y^2 = 16$ we get $(x-1)^2 = 16 - 4y^2$ which gives $x = 1 \pm \sqrt{16 - 4y^2}$.

   Since we know $16 - 4y^2 \geq 0$ to produce real number results for x, we require $-2 \leq y \leq 2$. Selecting values in that range produces the table below on the left. Plotting these points, along with the y-intercepts produces the graph on the right.

y	x	(x,y)
-2	1	$(1, -2)$
-1	$1 \pm 2\sqrt{3}$	$(1 \pm 2\sqrt{3}, -1)$
0	$1 \pm 4 = -3, 5$	$(-3, 0), (5, 0)$
1	$1 \pm 2\sqrt{3}$	$(1 \pm 2\sqrt{3}, 1)$
2	1	$(1, 2)$



$(x-1)^2 + 4y^2 = 16$

   The graph certainly appears to be symmetric about the x-axis. To check, we substitute $(-y)$ in for y and get $(x-1)^2 + 4(-y)^2 = 16$ which reduces to $(x-1)^2 + 4y^2 = 16$.

   Owing to the placement of the x-intercepts, $(-3, 0)$ and $(5, 0)$, the graph is most certainly not symmetric about the y-axis nor about the origin. □

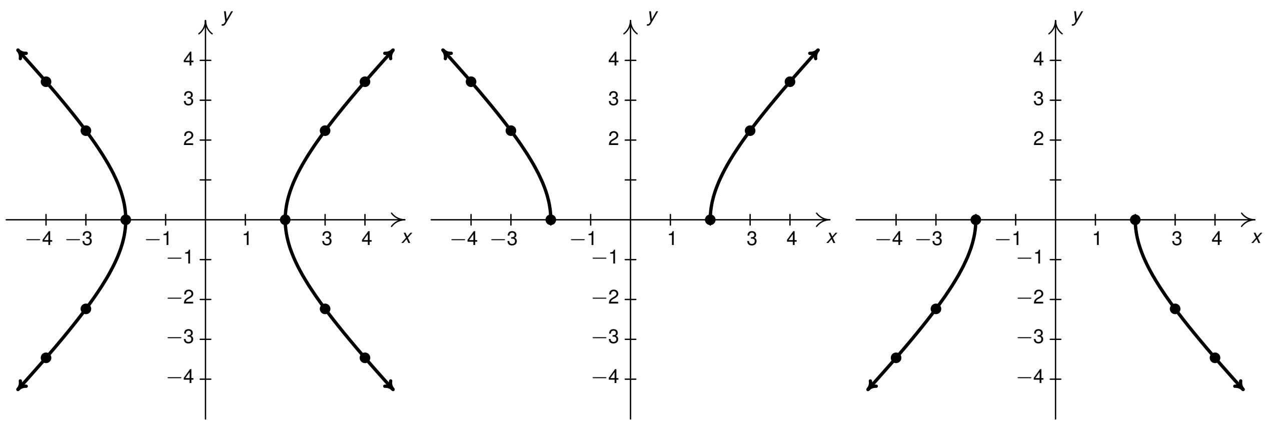
[6] Read this as we're avoiding fractions.

Looking at the graphs of the equations $x^2 - y^2 = 4$ and $(x-1)^2 + 4y^2 = 16$ in Example 9.3.3, it is evident neither of these equations represents y as a function of x nor x as a function of y. (Do you see why?)

With the concept of 'function' being touted in the opening remarks of Section 2.1 as being one of the 'universal tools' with which scientists and engineers solve a wide variety of problems, you may well wonder if we can't somehow apply what we know about functions to these sorts of relations. It turns out that while, taken all at once, these equations do not describe functions, taken in parts, they do.

For example, consider the equation $x^2 - y^2 = 4$. Solving for y, we obtained $y = \pm\sqrt{x^2-4}$. Defining $f_1(x) = \sqrt{x^2-4}$ and $f_2(x) = -\sqrt{x^2-4}$, we get a functional description for the upper and lower halves, or *branches* of the curve, respectively.[7]

If, for instance, we wanted to analyze this curve near $(3, -\sqrt{5})$, we could use the *function* f_2 and all the associated function tools[8] to do just that.



$x^2 - y^2 = 4$ $\qquad$ $f_1(x) = \sqrt{x^2-4}$ $\qquad$ $f_2(x) = -\sqrt{x^2-4}$

In this way we say the equation $x^2 - y^2 = 4$ *implicitly* describes y as a function of x meaning that given any point (x_0, y_0) on $x^2 - y^2 = 4$, we can find a function f defined (on an interval) containing x_0 so that $f(x_0) = y_0$ and whose graph lies on the curve $x^2 - y^2 = 4$.

Note that in this case, we are fortunate to have two *explicit* formulas for functions that cover the entire curve, namely $f_1(x) = \sqrt{x^2-4}$ and $f_2(x) = -\sqrt{x^2-4}$. We explore this concept further in the next example.

Example 9.3.4. Consider the graph of the relation R below.

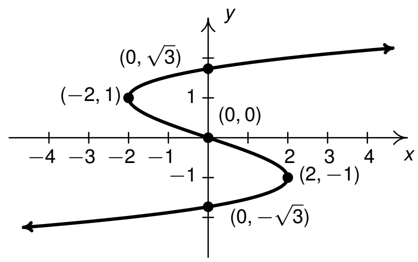
1. Explain why this curve does not represent y as a function of x.

2. Resolve the graph of R into two or more graphs of implicitly defined functions.

3. Explain why this curve represents x as a function of y and find a formula for $x = g(y)$.

[7] There are many more ways to break this relation into functional parts. We could, for instance, go piecewise and take portions of the graph which lie in Quadrants I and III as one function and leave the parts in Quadrants II and IV as the other; we could look at this as being comprised of *four* functions, and so on.

[8] including, when the time comes, Calculus



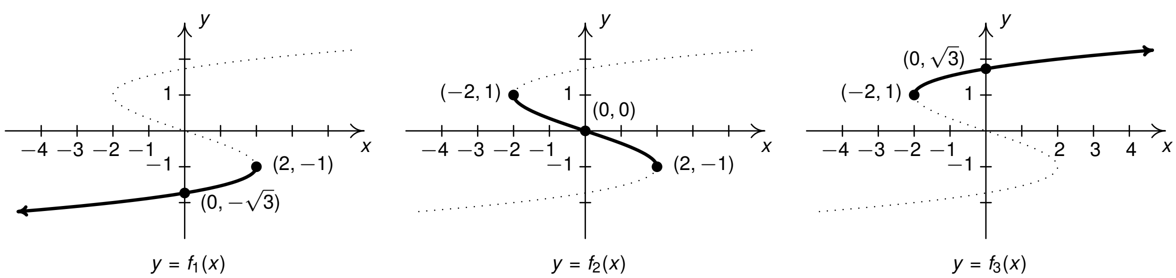
The graph of R

Solution.

1. Using the Vertical Line Test, Theorem 2.1, we find several instances where vertical lines intersect the graph of R more than once. The y-axis, $x = 0$ is one such line. We have $x = 0$ matched with *three* different y-values: $-\sqrt{3}$, 0, and $\sqrt{3}$.

2. Since the maximum number of times a vertical line intersects the graph of R is three, it stands to reason we need to resolve the graph of R into at least three pieces.

   One strategy is to begin at the far left and begin tracing the graph until it begins to 'double back' and repeat y-coordinates. Doing so we get three functions (represented by the bold solid lines) below.



3. To verify that R represents x as a function of y, we check to see if any y-value has more than one x associated with it. One way to do this is to employ the the Horizontal Line Test (Exercise 57 in Section 2.1.) Since every horizontal line intersects the graph at most once, x is a function of y.

   Using Theorem **??** from Chapter 6, we get $x = (1)y(y - \sqrt{3})(y + \sqrt{3}) = y^3 - 3y$, a fact we can readily check using a graphing utility. □

Not all equations implicitly define y as a function of x. For a quick example, take $x = 117$ or any other vertical line. Even if an equation implicitly describes y as a function of x near one point, there's no guarantee we can find an explicit algebraic representation for that function.[9]

While the theory of implicit functions is well beyond the scope of this text, we will nevertheless see this concept come into play in Section 9.4. For our purposes, it suffices to know that just because a relation is not a function doesn't mean we cannot find a way to apply what we know about functions to analyze the relation locally through a functional lens.

[9] An example of this is $y^5 - y - x = 1$ near $(-1, 0)$.

9.3.1 Exercises

In Exercises 1 - 20, graph the given relation in the xy-plane.

1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$

2. $\{(-2,0),\ (-1,1),\ (-1,-1),\ (0,2),\ (0,-2),\ (1,3),\ (1,-3)\}$

3. $\{(m, 2m) \mid m = 0, \pm 1, \pm 2\}$

4. $\left\{\left(\frac{6}{k}, k\right) \mid k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}$

5. $\left\{\left(n, 4 - n^2\right) \mid n = 0, \pm 1, \pm 2\right\}$

6. $\left\{\left(\sqrt{j}, j\right) \mid j = 0, 1, 4, 9\right\}$

7. $\{(x, -2) \mid x > -4\}$

8. $\{(x, 3) \mid x \leq 4\}$

9. $\{(-1, y) \mid y > 1\}$

10. $\{(2, y) \mid y \leq 5\}$

11. $\{(-2, y) \mid -3 < y \leq 4\}$

12. $\{(3, y) \mid -4 \leq y < 3\}$

13. $\{(x, 2) \mid -2 \leq x < 3\}$

14. $\{(x, -3) \mid -4 < x \leq 4\}$

15. $\{(x, y) \mid x > -2\}$

16. $\{(x, y) \mid x \leq 3\}$

17. $\{(x, y) \mid y < 4\}$

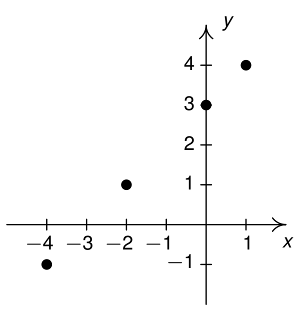
18. $\{(x, y) \mid x \leq 3,\ y < 2\}$

19. $\{(x, y) \mid x > 0,\ y < 4\}$

20. $\{(x, y) \mid -\sqrt{2} \leq x \leq \frac{2}{3},\ \pi < y \leq \frac{9}{2}\}$

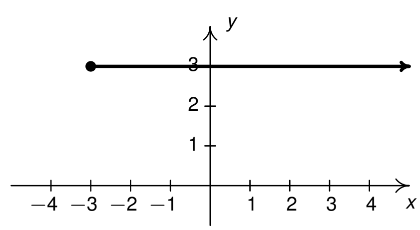
In Exercises 21 - 30, describe the given relation using either the roster or set-builder method.

21.



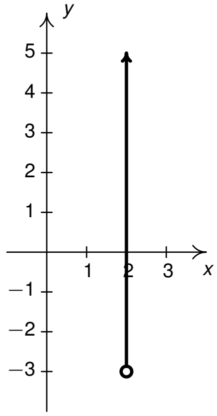
Relation A

22.



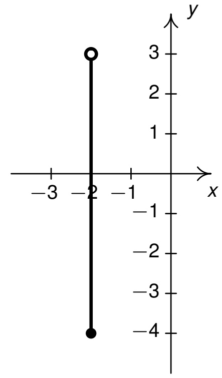
Relation B

23.



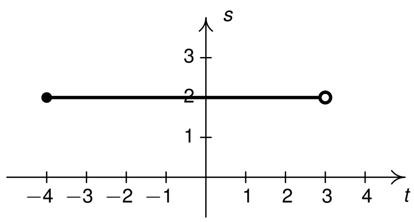
Relation *C*

24.



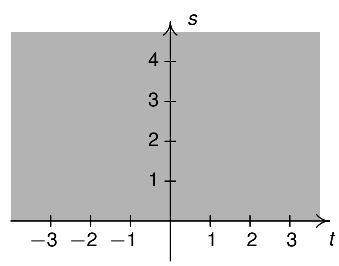
Relation *D*

25.



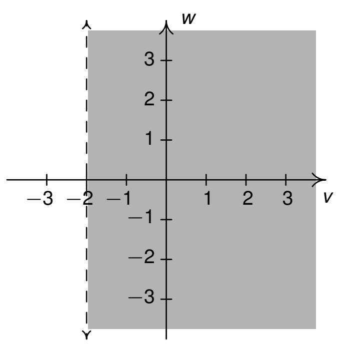
Relation *E*

26.



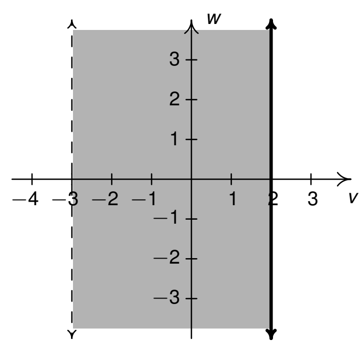
Relation *F*

27.



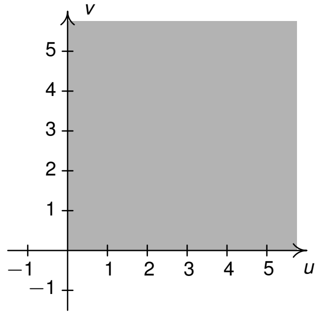
Relation *G*

28.



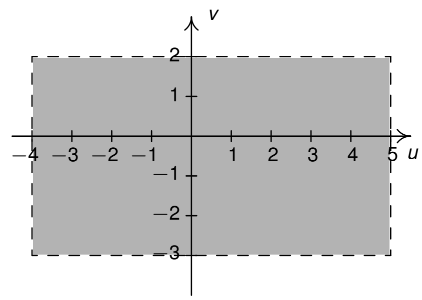
Relation *H*

29.



Relation I

30.



Relation J

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. Discuss with your classmates how you might graph the relations given in Exercises 31 - 34. Note that in the notation below we are using the ellipsis, '. . . ,' to denote that the list does not end, but rather, continues to follow the established pattern indefinitely.

For the relations in Exercises 31 and 32, give two examples of points which belong to the relation and two points which do not belong to the relation.

31. $\{(x, y) \mid x$ is an odd integer, and y is an even integer.$\}$

32. $\{(x, 1) \mid x$ is an irrational number $\}$

33. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), ...\}$

34. $\{... , (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), ...\}$

For each equation given in Exercises 35 - 38:

- Graph the equation in the xy-plane by creating a table of points.
- Find the axis intercepts, if they exist.
- Test the equation for symmetry. If the equation fails a symmetry test, find a point on the graph of the equation whose symmetric point is not on the graph of the equation.
- Determine if the equation describes y as a function of x. If not, describe the graph of the equation using two or more explicit functions of x. Check your answers using a graphing utility.

35. $(x + 2)^2 + y^2 = 16$

36. $x^2 - y^2 = 1$

37. $4y^2 - 9x^2 = 36$

38. $x^3y = -4$

For each equation given in Exercises 39 - 42:

- Graph the equation in the vw-plane by creating a table of points.
- Find the axis intercepts, if they exist.
- Test the equation for symmetry. If the equation fails a symmetry test, find a point on the graph of the equation whose symmetric point is not on the graph of the equation.
- Determine if the equation describes w as a function of v. If not, describe the graph of the equation using two or more explicit functions of v. Check your answers using a graphing utility.

39. $v + w^2 = 4$

40. $v^3 + w^3 = 8$

41. $v^2w^3 = 8$

42.[10] $v^4 - 2v^2w + w^2 = 16$

The procedures which we have outlined in the Examples of this section and used in Exercises 35 - 38 all rely on the fact that the equations were "well-behaved". Not everything in Mathematics is quite so tame, as the following equations will show you. Discuss with your classmates how you might approach graphing the equations given in Exercises 43 - 46. What difficulties arise when trying to apply the various tests and procedures given in this section? For more information, including pictures of the curves, each curve name is a link to its page at www.wikipedia.org. For a much longer list of fascinating curves, click here.

43. $x^3 + y^3 - 3xy = 0$ Folium of Descartes

44. $x^4 = x^2 + y^2$ Kampyle of Eudoxus

45. $y^2 = x^3 + 3x^2$ Tschirnhausen cubic

46. $(x^2 + y^2)^2 = x^3 + y^3$ Crooked egg

47. With the help of your classmates, find examples of equations whose graphs possess

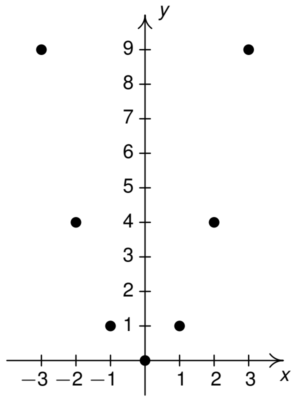
    - symmetry about the x-axis only
    - symmetry about the y-axis only
    - symmetry about the origin only
    - symmetry about the x-axis, y-axis, and origin

    Can you find an example of an equation whose graph possesses exactly *two* of the symmetries listed above? Why or why not?

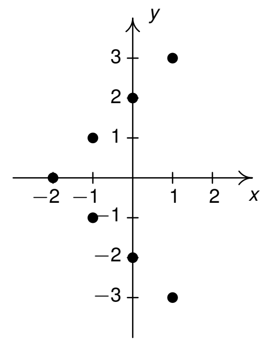
[10]HINT: $v^4 - 2v^2w + w^2 = \left(v^2 - w\right)^2 \ldots$

9.3.2 Answers

1.



2.



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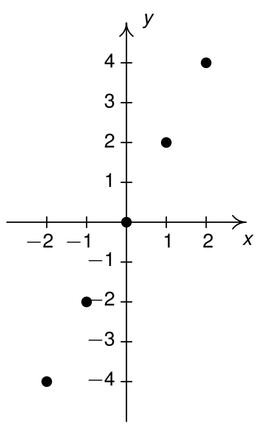

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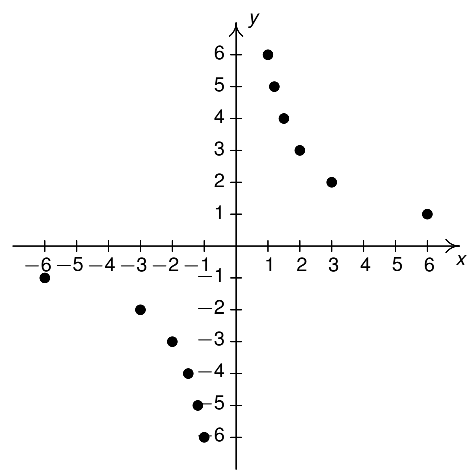

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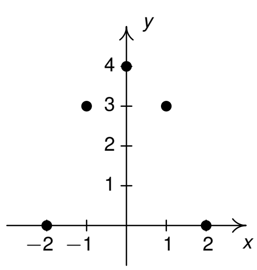

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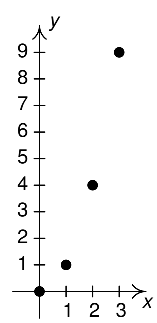
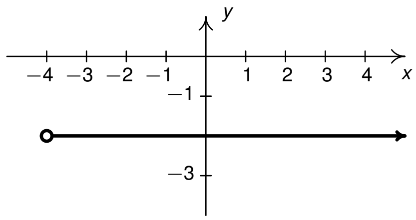
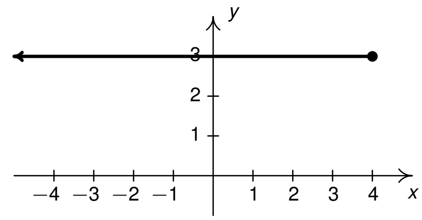


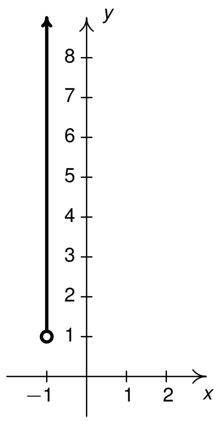
7.


y
−4 −3 −2 −1 1 2 3 4 x
−1
−3

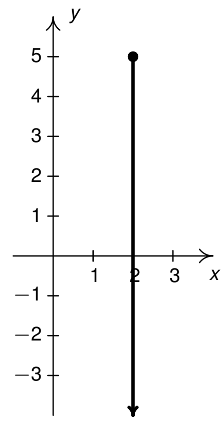
8.


y
3
2
1
−4 −3 −2 −1 1 2 3 4 x

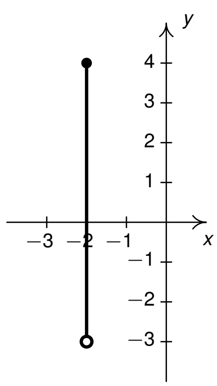
9.


y
8
7
6
5
4
3
2
1
−1 1 2 x

10.


y
5
4
3
2
1
1 2 3 x
−1
−2
−3

11.


y
4
3
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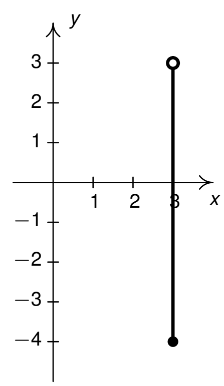
1
−3 −2 −1 x
−1
−2
−3

12.


y
3
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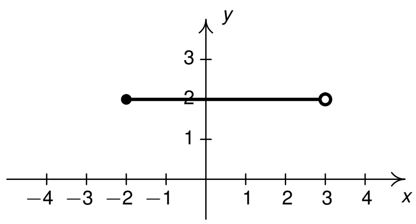
1
1 2 3 x
−1
−2
−3
−4

13.



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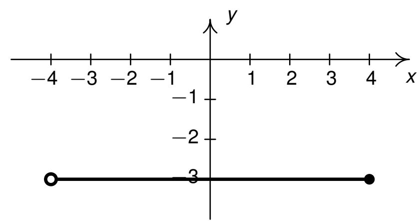

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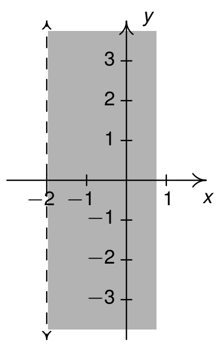

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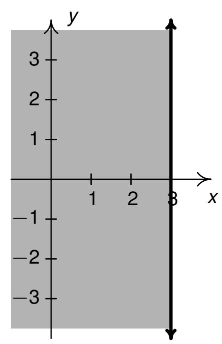

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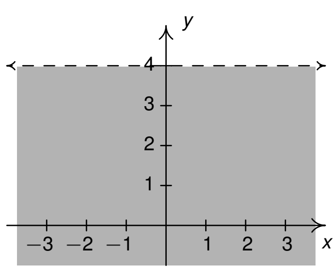

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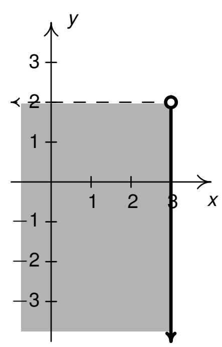

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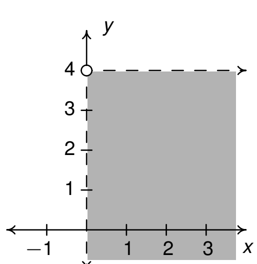

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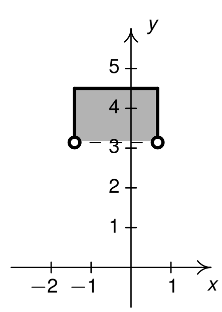


21. $A = \{(-4,-1), (-2,1), (0,3), (1,4)\}$

22. $B = \{(x,3) \mid x \geq -3\}$

23. $C = \{(2,y) \mid y > -3\}$

24. $D = \{(-2,y) \mid -4 \leq y < 3\}$

25. $E = \{(t,2) \mid -4 < t \leq 3\}$

26. $F = \{(t,s) \mid s \geq 0\}$

27. $G = \{(v,w) \mid v > -2\}$

28. $H = \{(v,w) \mid -3 < v \leq 2\}$

29. $I = \{(u,v) \mid u \geq 0, v \geq 0\}$

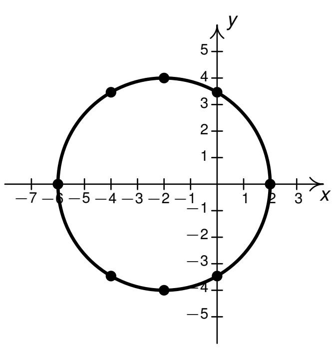
30. $J = \{(u,v) \mid -4 < u < 5,\ -3 < v < 2\}$

35. $(x+2)^2 + y^2 = 16$
Re-write as $y = \pm\sqrt{16-(x+2)^2}$.

x-intercepts: $(-6,0)$, $(2,0)$

y-intercepts: $\left(0, \pm 2\sqrt{3}\right)$

x	y	(x,y)
-6	0	$(-6,0)$
-4	$\pm 2\sqrt{3}$	$\left(-4, \pm 2\sqrt{3}\right)$
-2	± 4	$(-2, \pm 4)$
0	$\pm 2\sqrt{3}$	$\left(0, \pm 2\sqrt{3}\right)$
2	0	$(2,0)$



The graph is symmetric about the x-axis

The graph is not symmetric about the y-axis: $(-6,0)$ is on the graph but $(6,0)$ is not.

The graph is not symmetric about the origin: $(-6,0)$ is on the graph but $(6,0)$ is not.

The equation does not describe y as a function of x.

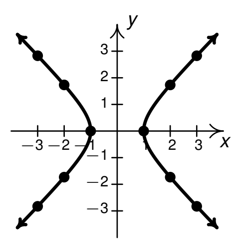
The graph of the equation is the graphs of $f_1(x) = \sqrt{16-(x+2)^2}$ together with $f_2(x) = -\sqrt{16-(x+2)^2}$.

36. $x^2 - y^2 = 1$
Re-write as: $y = \pm\sqrt{x^2-1}$.

x-intercepts: $(-1,0)$, $(1,0)$

The graph has no y-intercepts

x	y	(x,y)
-3	$\pm\sqrt{8}$	$(-3, \pm\sqrt{8})$
-2	$\pm\sqrt{3}$	$(-2, \pm\sqrt{3})$
-1	0	$(-1,0)$
1	0	$(1,0)$
2	$\pm\sqrt{3}$	$(2, \pm\sqrt{3})$
3	$\pm\sqrt{8}$	$(3, \pm\sqrt{8})$



The graph is symmetric about the x-axis.

The graph is symmetric about the y-axis.

The graph is symmetric about the origin.

The equation does not describe y as a function of x.

The graph of the equation is the graphs of $f_1(x) = \sqrt{x^2-1}$ together with $f_2(x) = -\sqrt{x^2-1}$.

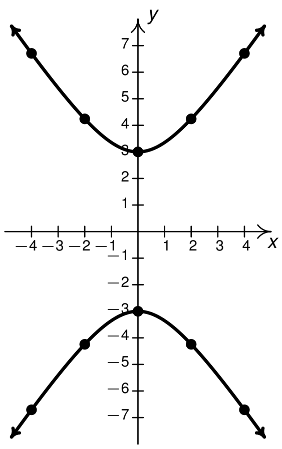
37. $4y^2 - 9x^2 = 36$

Re-write as: $y = \pm\dfrac{\sqrt{9x^2+36}}{2}$.

The graph has no x-intercepts

y-intercepts: $(0, \pm 3)$

x	y	(x, y)
-4	$\pm 3\sqrt{5}$	$\left(-4, \pm 3\sqrt{5}\right)$
-2	$\pm 3\sqrt{2}$	$\left(-2, \pm 3\sqrt{2}\right)$
0	± 3	$(0, \pm 3)$
2	$\pm 3\sqrt{2}$	$\left(2, \pm 3\sqrt{2}\right)$
4	$\pm 3\sqrt{5}$	$\left(4, \pm 3\sqrt{5}\right)$



The graph is symmetric about the x-axis.

The graph is symmetric about the y-axis.

The graph is symmetric about the origin.

The equation does not describe y as a function of x.

The graph of the equation is the graphs of $f_1(x) = \dfrac{\sqrt{9x^2+36}}{2}$ together with $f_2(x) = -\dfrac{\sqrt{9x^2+36}}{2}$.

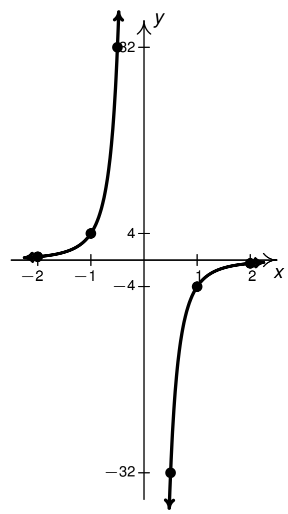
38. $x^3 y = -4$

Re-write as: $y = -\dfrac{4}{x^3} = -4x^{-3}$.

The graph has no x-intercepts

The graph has no y-intercepts

x	y	(x, y)
-2	$\frac{1}{2}$	$(-2, \frac{1}{2})$
-1	4	$(-1, 4)$
$-\frac{1}{2}$	32	$(-\frac{1}{2}, 32)$
$\frac{1}{2}$	-32	$(\frac{1}{2}, -32)$
1	-4	$(1, -4)$
2	$-\frac{1}{2}$	$(2, -\frac{1}{2})$



The graph is not symmetric about the x-axis: $(1, -4)$ is on the graph but $(1, 4)$ is not.

The graph is not symmetric about the y-axis: $(1, -4)$ is on the graph but $(-1, -4)$ is not.

The graph is symmetric about the origin.

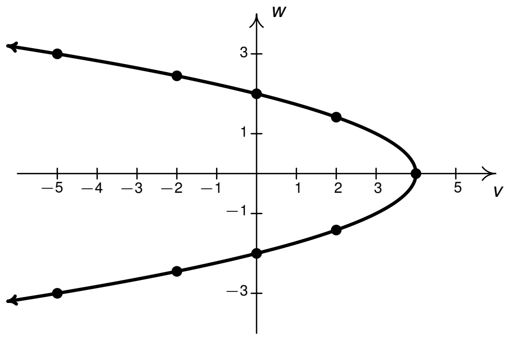
The equation does describe y as a function of x, namely $y = f(x) = -4x^{-3}$.

39. $v + w^2 = 4$
Re-write as $w = \pm\sqrt{4 - v}$.

v-intercept: $(4, 0)$

w-intercepts: $(0, \pm 2)$

v	w	(x, y)
-5	± 3	$(-5, \pm 3)$
-2	$\pm\sqrt{6}$	$\left(-2, \pm\sqrt{6}\right)$
0	± 2	$(0, \pm 2)$
2	$\pm\sqrt{2}$	$\left(1, \pm\sqrt{3}\right)$
4	0	$(4, 0)$



The graph is symmetric about the v-axis

The graph is not symmetric about the w-axis: $(4, 0)$ is on the graph but $(-4, 0)$ is not.

The graph is not symmetric about the origin: $(4, 0)$ is on the graph but $(-4, 0)$ is not.

The equation does not describe w as a function of v.

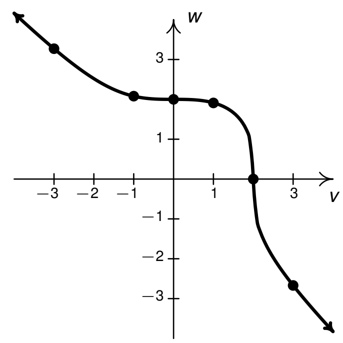
The graph of the equation is the graphs of $f_1(v) = \sqrt{4 - v}$ together with $f_2(v) = -\sqrt{4 - v}$.

40. $v^3 + w^3 = 8$
Re-write as: $w = \sqrt[3]{8 - v^3}$.

v-intercept: $(2, 0)$

w-intercept: $(0, 2)$

v	w	(v, w)
-3	$\sqrt[3]{35}$	$(-3, \sqrt[3]{35})$
-1	$\sqrt[3]{9}$	$(-1, \sqrt[3]{9})$
0	2	$(0, 2)$
1	$\sqrt[3]{7}$	$(1, \sqrt[3]{7})$
2	0	$(2, 0)$
3	$-\sqrt[3]{19}$	$(3, -\sqrt[3]{19})$



The graph is not symmetric about the v-axis: $(0, 2)$ is on the graph but $(0, -2)$ is not.

The graph is not symmetric about the w-axis: $(2, 0)$ is on the graph but $(-2, 0)$ is not.

The graph is not symmetric about the origin: $(0, 2)$ is on the graph but $(0, -2)$ is not.

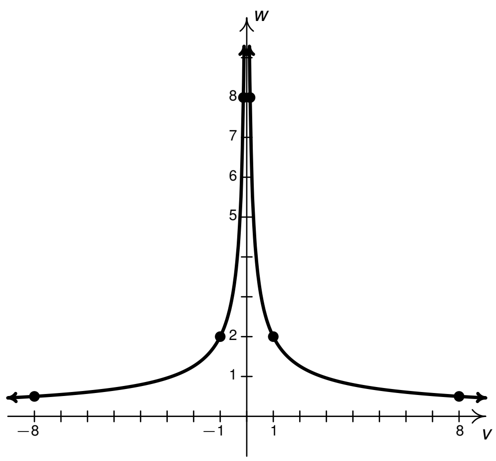
The equation does describe w as a function of v, namely $w = f(v) = \sqrt[3]{8 - v^3}$.

41. $v^2w^3 = 8$
Re-write as $w = \dfrac{2}{\sqrt[3]{v^2}} = 2v^{-\frac{2}{3}}$.

The graph has no v-intercepts.

The graph has no w-intercepts.

v	w	(x,y)
-8	$\frac{1}{2}$	$\left(-8,\frac{1}{2}\right)$
-1	2	$(-1,2)$
$-\frac{1}{8}$	8	$\left(-\frac{1}{8},8\right)$
$\frac{1}{8}$	8	$\left(\frac{1}{8},8\right)$
1	2	$(1,2)$
8	$\frac{1}{2}$	$\left(8,\frac{1}{2}\right)$



The graph is not symmetric about the v-axis: $(-1,2)$ is on the graph but $(-1,-2)$ is not.

The graph is symmetric about the w-axis.

The graph is not symmetric about the origin: $(-1,2)$ is on the graph but $(-1,-2)$ is not.

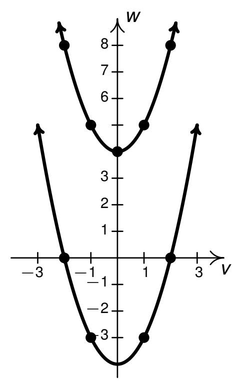
The equation does describe w as a function of v, namely $w = f(v) = 2v^{-\frac{2}{3}}$.

42. $v^4 - 2v^2w + w^2 = 16$
Re-write as: $\left(v^2 - w\right)^2 = 16$
Extracting square roots gives:
$w = v^2 + 4$ and $w = v^2 - 4$

v-intercepts: $(-2,0),(2,0)$.

w-intercepts: $(0,-4),(0,4)$

v	w	(v,w)
-2	8	$(-2,8)$
-2	0	$(-2,0)$
-1	5	$(-1,5)$
-1	-3	$(-1,-3)$
0	± 4	$(0,\pm 4)$
1	5	$(1,5)$
1	-3	$(1,-3)$
2	8	$(2,8)$
2	0	$(2,0)$



The graph is not symmetric about the v-axis: $(1,5)$ is on the graph but $(1,-5)$ is not.

The graph is symmetric about the w-axis.

The graph is not symmetric about the origin: $(1,5)$ is on the graph but $(-1,-5)$ is not.

The equation does not describe w as a function of v.

The graph of the equation is the graphs of $f_1(v) = v^2 + 4$ together with $f_2(v) = v^2 - 4$.

9.4 Inverse Functions

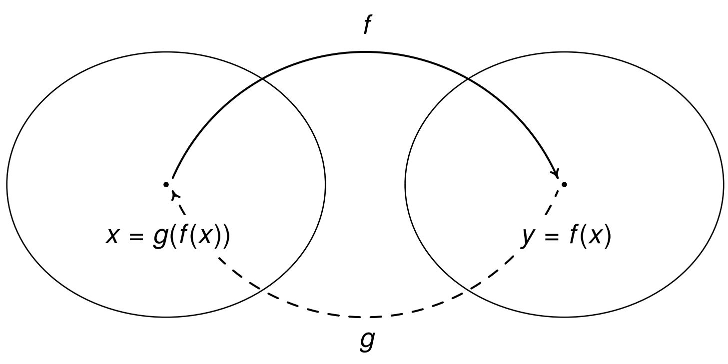
In Section 2.1, we defined functions as processes. In this section, we seek to reverse, or 'undo' those processes. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like baking a cake) are not.

Consider the function $f(x) = 3x + 4$. Starting with a real number input x, we apply two steps in the following sequence: first we multiply the input by 3 and, second, we add 4 to the result.

To reverse this process, we seek a function g which will undo each of these steps and take the output from f, $3x + 4$, and return the input x. If we think of the two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes and then we take off the socks. In much the same way, the function g should undo each step of f but in the opposite order. That is, the function g should first *subtract* 4 from the input x then *divide* the result by 3. This leads us to the formula $g(x) = \frac{x-4}{3}$.

Let's check to see if the function g does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f. To check that g does the job for all x in the domain of f, we take the generic output from f, $f(x) = 3x + 4$, and substitute that into g. That is, we simplify $g(f(x)) = g(3x + 4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3.

Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \frac{x-4}{3}$, and substitute that into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$. Using the language of function composition developed in Section 9.2, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively.[1] Abstractly, we can visualize the relationship between f and g in the diagram below.



The main idea to get from the diagram is that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of the section.

[1] At the level of functions, $g \circ f = f \circ g = I$, where I is the identity function as defined as $I(x) = x$ for all real numbers, x.

Definition 9.4. Suppose f and g are two functions such that

1. $(g \circ f)(x) = x$ for all x in the domain of f

   and

2. $(f \circ g)(x) = x$ for all x in the domain of g

then f and g are **inverses** of each other and the functions f and g are said to be **invertible**.

If we abstract one step further, we can express the sentiment in Definition 9.4 by saying that f and g are inverses if and only if $g \circ f = I_1$ and $f \circ g = I_2$ where I_1 is the identity function restricted[2] to the domain of f and I_2 is the identity function restricted to the domain of g.

In other words, $I_1(x) = x$ for all x in the domain of f and $I_2(x) = x$ for all x in the domain of g. Using this description of inverses along with the properties of function composition listed in Theorem 9.2, we can show that function inverses are unique.[3]

Suppose g and h are both inverses of a function f. By Theorem 9.4, the domain of g is equal to the domain of h, since both are the range of f. This means the identity function I_2 applies both to the domain of h and the domain of g. Thus $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$, as required.

We summarize the important properties of invertible functions in the following theorem.[4] Apart from introducing notation, each of the results below are immediate consequences of the idea that inverse functions map the outputs from a function f back to their corresponding inputs.

Theorem 9.4. Properties of Inverse Functions: Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read 'f-inverse')
- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}
- $f(a) = c$ if and only if $a = f^{-1}(c)$

  NOTE: In particular, for all y in the range of f, the solution to $f(x) = y$ is $x = f^{-1}(y)$.
- (a, c) is on the graph of f if and only if (c, a) is on the graph of f^{-1}

  NOTE: This means graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across $y = x$.[a]
- f^{-1} is an invertible function and $(f^{-1})^{-1} = f$.

[a] See Example **??** in Section **??** and Example 3.1.6 in Section 3.1.2.

[2] The identity function I, first introduced in Exercise 35 in Section 3.2 and mentioned in Theorem 9.2, has a domain of all real numbers. Since the domains of f and g may not be all real numbers, we need the restrictions listed here.

[3] In other words, invertible functions have exactly one inverse.

[4] In the interests of full disclosure, the authors would like to admit that much of the discussion in the previous paragraphs could have easily been avoided had we appealed to the description of a function as a set of ordered pairs. We make no apology for our discussion from a function composition standpoint, however, since it exposes the reader to more abstract ways of thinking of functions and inverses. We will revisit this concept again in Chapter **??**.

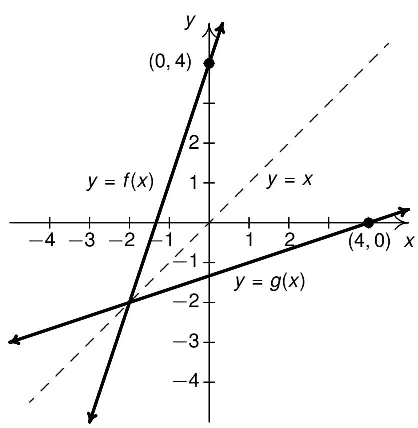
The notation f^{-1} is an unfortunate choice since you've been programmed since Elementary Algebra to think of this as $\frac{1}{f}$. This is most definitely *not* the case since, for instance, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$.

Why does this confusing notation persist? As we mentioned in Section 9.2, the identity function I is to function composition what the real number 1 is to real number multiplication. The choice of notation f^{-1} alludes to the property that $f^{-1} \circ f = I_1$ and $f \circ f^{-1} = I_2$, in much the same way as $3^{-1} \cdot 3 = 1$ and $3 \cdot 3^{-1} = 1$.

Before we embark on an example, we demonstrate the pertinent parts of Theorem 9.4 to the inverse pair $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$. Suppose we wanted to solve $3x + 4 = 7$. Going through the usual machinations, we obtain $x = 1$.

If we view this equation as $f(x) = 7$, however, then we are looking for the input x corresponding to the output $f(x) = 7$. This is exactly the question f^{-1} was built to answer. In other words, the solution to $f(x) = 7$ is $x = f^{-1}(7) = 1$. In other words, the formula $f^{-1}(x)$ encodes all of the algebra required to 'undo' what the formula $f(x)$ does to x. More generally, any time you have ever solved an equation, you have really been working through an inverse problem.

We also note the graphs of $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$ are easily seen to be reflections across the line $y = x$ as seen below. In particular, note that the y-intercept $(0, 4)$ on the graph of $y = f(x)$ corresponds to the x-intercept on the graph of $y = f^{-1}(x)$. Indeed, the point $(0, 4)$ on the graph of $y = f(x)$ can be interpreted as $(0, 4) = (0, f(0)) = (f^{-1}(4), 4)$ just as the point $(4, 0)$ on the graph of $y = f^{-1}(x)$ can be interpreted as $(4, 0) = (4, f^{-1}(4)) = (f(0), 0)$.



Graphs of inverse functions $y = f(x) = 3x + 4$ and $y = f^{-1}(x) = \dfrac{x-4}{3}$.

Example 9.4.1. For each pair of functions f and g below:

1. Verify each pair of functions f and g are inverses: (a) algebraically and (b) graphically.
2. Use the fact f and g are inverses to solve $f(x) = 5$ and $g(x) = -3$

- $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$
- $f(t) = \dfrac{2t}{t+1}$ and $g(t) = \dfrac{t}{2-t}$

Solution.

Solution for $f(x) = \sqrt[3]{x-1} + 2$ *and* $g(x) = (x-2)^3 + 1$.

1. (a) To verify $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$ are inverses, we appeal to Definition 9.4 and show $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$ for all real numbers, x.

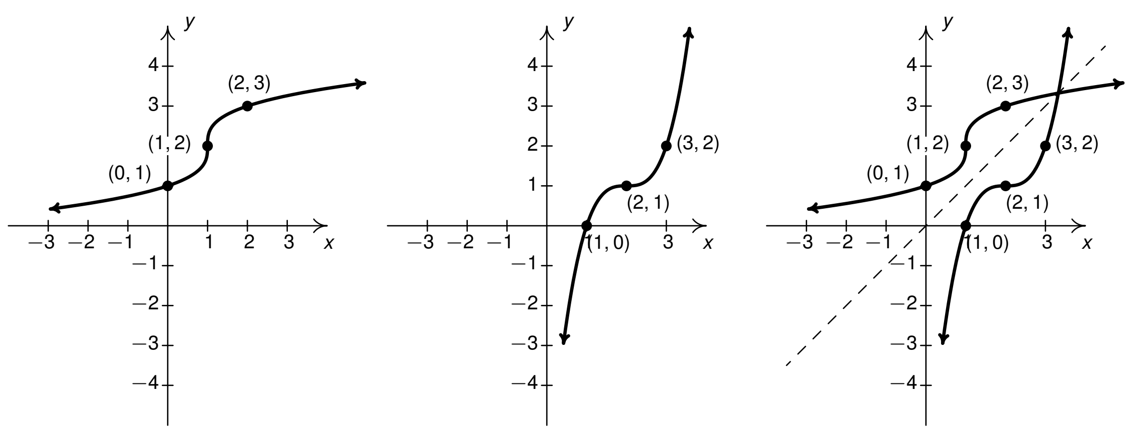
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt[3]{x-1} + 2) \\ &= [(\sqrt[3]{x-1} + 2) - 2]^3 + 1 \\ &= (\sqrt[3]{x-1})^3 + 1 \\ &= x - 1 + 1 \\ &= x \checkmark\end{aligned} \qquad \begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f((x-2)^3 + 1) \\ &= \sqrt[3]{[(x-2)^3 + 1] - 1} + 2 \\ &= \sqrt[3]{(x-2)^3} + 2 \\ &= x - 4 + 4 \\ &= x \checkmark\end{aligned}$$

Since the root here, 3, is odd, Theorem 8.3 gives $(\sqrt[3]{x-1})^3 = x - 1$ and $\sqrt[3]{(x-2)^3} = x - 2$.

(b) To show f and g are inverses graphically, we graph $y = f(x)$ and $y = g(x)$ on the same set of axes and check to see if they are reflections about the line $y = x$.

The graph of $y = f(x) = \sqrt[3]{x-1} + 2$ appears below on the left courtesy of Theorem 8.2 in Section 8.2. The graph of $y = g(x) = (x-2)^3 + 1$ appears below in the middle thanks to Theorem 6.1 in Section 6.1.

We can immediately see three pairs of corresponding points: $(0, 1)$ and $(1, 0)$, $(1, 2)$ and $(2, 1)$, $(2, 3)$ and $(3, 2)$. When graphed on the same pair of axes, the two graphs certainly appear to be symmetric about the line $y = x$, as required.



$y = f(x)$ $\qquad$ $y = g(x)$ $\qquad$ $y = f(x)$, $y = g(x)$, $y = x$

2. Since f and g are inverses, the solution to $f(x) = 5$ is $x = f^{-1}(5) = g(5) = (5-2)^3 + 1 = 28$. To check, we find $f(28) = \sqrt[3]{28-1} + 2 = \sqrt[3]{27} + 2 = 3 + 2 = 5$, as required.

   Likewise, the solution to $g(x) = -3$ is $x = g^{-1}(-3) = f(-3) = \sqrt[3]{(-3)-1} + 2 = 2 - \sqrt[3]{4}$. Once again, to check, we find $g(2 - \sqrt[3]{4}) = (2 - \sqrt[3]{4} - 2)^3 + 1 = (-\sqrt[3]{4})^3 + 1 = -4 + 1 = -3$.

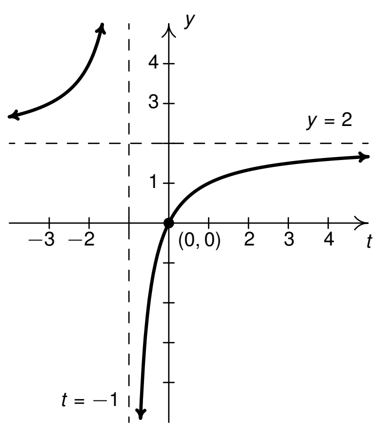
Solution for $f(t) = \dfrac{2t}{t+1}$ *and* $g(t) = \dfrac{t}{2-t}$.

1. (a) Note the domain of f excludes $t = -1$ and the domain of g excludes $t = 2$. Hence, when simplifying $(g \circ f)(t)$ and $(f \circ g)(t)$, we tacitly assume $t \neq -1$ and $t \neq 2$, respectively.

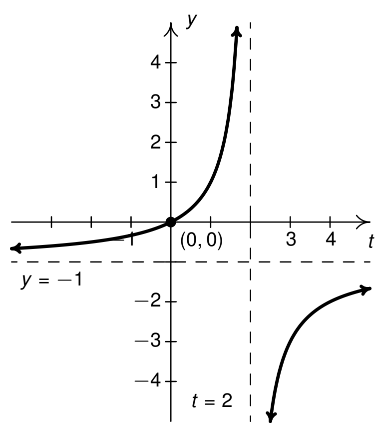
$$\begin{aligned}
(g \circ f)(t) &= g(f(t)) \\
&= g\left(\frac{2t}{t+1}\right) \\
&= \frac{\dfrac{2t}{t+1}}{2 - \dfrac{2t}{t+1}} \\
&= \frac{\dfrac{2t}{t+1}}{2 - \dfrac{2t}{t+1}} \cdot \frac{(t+1)}{(t+1)} \\
&= \frac{2t}{2(t+1) - 2t} \\
&= \frac{2t}{2t + 2 - 2t} \\
&= \frac{2t}{2} \\
&= t \checkmark
\end{aligned}$$

$$\begin{aligned}
(f \circ g)(t) &= f(g(t)) \\
&= f\left(\frac{t}{2-t}\right) \\
&= \frac{2\left(\dfrac{t}{2-t}\right)}{\left(\dfrac{t}{2-t}\right) + 1} \\
&= \frac{2\left(\dfrac{t}{2-t}\right)}{\left(\dfrac{t}{2-t}\right) + 1} \cdot \frac{(2-t)}{(2-t)} \\
&= \frac{2t}{t + (1)(2-t)} \\
&= \frac{2t}{t + 2 - t)} \\
&= \frac{2t}{2} \\
&= t \checkmark
\end{aligned}$$

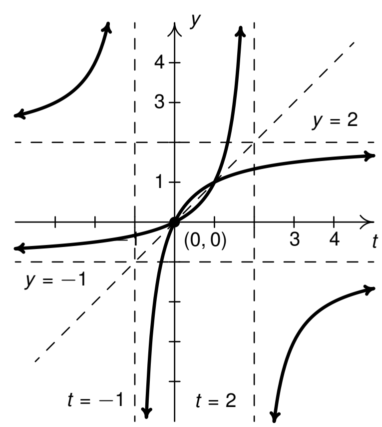
(b) We graph $y = f(t)$ and $y = g(t)$ using the techniques discussed in Sections 7.2 and **??**.



$y = f(t)$



$y = g(t)$



$y = f(t)$, $y = g(t)$, and $y = x$

We find the graph of f has a vertical asymptote $t = -1$ and a horizontal asymptote $y = 2$. Corresponding to the *vertical* asymptote $t = -1$ on the graph of f, we find the graph of g has a *horizontal* asymptote $y = -1$.

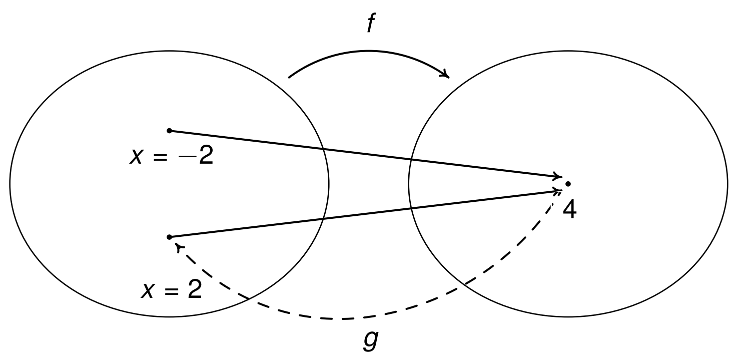
Likewise, the *horizontal* asymptote $y = 2$ on the graph of f corresponds to the *vertical* asymptote $t = 2$ on the graph of g. Both graphs share the intercept $(0, 0)$. When graphed together on the same set of axes, the graphs of f and g do appear to be symmetric about the line $y = t$.

2. Don't let the fact that f and g in this case were defined using the independent variable, 't' instead of 'x' deter you in your efforts to solve $f(x) = 5$. Remember that, ultimately, the function f here is the *process* represented by the formula $f(t)$, and is the same process (with the same inverse!) regardless of the letter used as the independent variable. Hence, the solution to $f(x) = 5$ is $x = f^{-1}(1) = g(5)$. We get $g(5) = \frac{5}{2-5} = -\frac{5}{3}$.

   To check, we find $f\left(-\frac{5}{3}\right) = \left(-\frac{10}{3}\right) / \left(-\frac{2}{3}\right) = 5$. Similarly, we solve $g(x) = -3$ by finding $x = g^{-1}(-3) = f(-3) = \frac{-6}{-2} = 3$. Sure enough, we find $g(3) = \frac{3}{2-3} = -3$. □

We now investigate under what circumstances a function is invertible. As a way to motivate the discussion, we consider $f(x) = x^2$. A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. However, $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to x unless $x \geq 0$.

For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$. That is, g failed to return the input -2 from its output 4. Instead, g matches the output 4 to a *different* input, namely 2, which satisfies $f(2) = 4$. Schematically:



We see from the diagram that since both $f(-2)$ and $f(2)$ are 4, it is impossible to construct a *function* which takes 4 back to *both* $x = 2$ and $x = -2$ since, by definition, a function can match 4 with only *one* number.

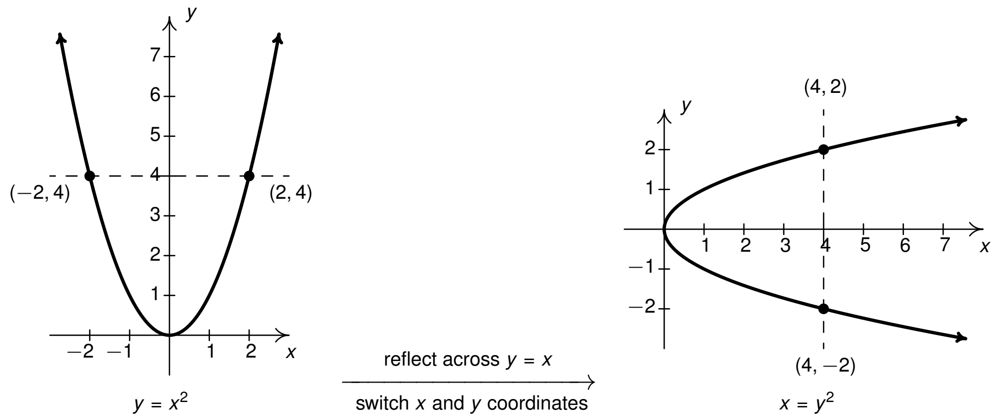
In general, in order for a function to be invertible, each output can come from only *one* input. Since, by definition, a function matches up each input to only *one* output, invertible functions have the property that they match one input to one output and vice-versa. We formalize this concept below.

Definition 9.5. A function f is said to be **one-to-one** if whenever $f(a) = f(b)$, then $a = b$.

Note that an equivalent way to state Definition 9.5 is that a function is one-to-one if *different* inputs go to *different* outputs. That is, if $a \neq b$, then $f(a) \neq f(b)$.

Before we solidify the connection between invertible functions and one-to-one functions, we take a moment to see what goes wrong graphically when trying to find the inverse of $f(x) = x^2$.

Per Theorem 9.4, the graph of $y = f^{-1}(x)$, if it exists, is obtained from the graph of $y = x^2$ by reflecting $y = x^2$ about the line $y = x$. Procedurally, this is accomplished by interchanging the x and y coordinates of each point on the graph of $y = x^2$. Algebraically, we are swapping the variables 'x' and 'y' which results in the equation $x = y^2$ whose graph is below on the right.



We see immediately the graph of $x = y^2$ fails the Vertical Line Test, Theorem 2.1. In particular, the vertical line $x = 4$ intersects the graph at two points, $(4, -2)$ and $(4, 2)$ meaning the relation described by $x = y^2$ matches the x-value 4 with two different y-values, -2 and 2.

Note that the *vertical* line $x = 4$ and the points $(4, \pm 2)$ on the graph of $x = y^2$ correspond to the *horizontal* line $y = 4$ and the points $(\pm 2, 4)$ on the graph of $y = x^2$ which brings us right back to the concept of one-to-one. The fact that both $(-2, 4)$ and $(2, 4)$ are on the graph of f means $f(-2) = f(2) = 4$. Hence, f takes different inputs, -2 and 2, to the same output, 4, so f is not one-to-one.

Recall the Horizontal Line Test from Exercise 57 in Section 2.1. Applying that result to the graph of f we say the graph of f 'fails' the Horizontal Line Test since the horizontal line $y = 4$ intersects the graph of $y = x^2$ more than once. This means that the equation $y = x^2$ does not represent x is not a function of y.

Said differently, the Horizontal Line Test detects when there is at least one y-value (4) which is matched to more than one x-value (± 2). In other words, the Horizontal Line Test can be used to detect whether or not a function is one-to-one.

So, to review, $f(x) = x^2$ is not invertible, not one-to-one, and its graph fails the Horizontal Line Test. It turns out that these three attributes: being invertible, one-to-one, and having a graph that passes the Horizontal Line Test are mathematically equivalent. That is to say if one if these things is true about a function, then they all are; it also means that, as in this case, if one of these things *isn't* true about a function, then *none* of them are. We summarize this result in the following theorem.

Theorem 9.5. Equivalent Conditions for Invertibility:
For a function f, either all of the following statements are true or none of them are:

- f is invertible.
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.[a]

[a] i.e., no horizontal line intersects the graph more than once.

To prove Theorem 9.5, we first suppose f is invertible. Then there is a function g so that $g(f(x)) = x$ for all x in the domain of f. If $f(a) = f(b)$, then $g(f(a)) = g(f(b))$. Since $g(f(x)) = x$, the equation $g(f(a)) = g(f(b))$ reduces to $a = b$. We've shown that if $f(a) = f(b)$, then $a = b$, proving f is one-to-one.

Next, assume f is one-to-one. Suppose a horizontal line $y = c$ intersects the graph of $y = f(x)$ at the points (a, c) and (b, c). This means $f(a) = c$ and $f(b) = c$ so $f(a) = f(b)$. Since f is one-to-one, this means $a = b$ so the points (a, c) and (b, c) are actually one in the same. This establishes that each horizontal line can intersect the graph of f at most once, so the graph of f passes the Horizontal Line Test.

Last, but not least, suppose the graph of f passes the Horizontal Line Test. Let c be a real number in the range of f. Then the horizontal line $y = c$ intersects the graph of $y = f(x)$ just *once*, say at the point $(a, c) = (a, f(a))$. Define the mapping g so that $g(c) = g(f(a)) = a$. The mapping g is a *function* since each horizontal line $y = c$ where c is in the range of f intersects the graph of f only *once*. By construction, we have the domain of g is the range of f and that for all x in the domain of $f, g(f(x)) = x$. We leave it to the reader to show that for all x in the domain of g, $f(g(x)) = x$, too.

Hence, we've shown: first, if f invertible, then f is one-to-one; second, if f is one-to-one, then the graph of f passes the Horizontal Line Test; and third, if f passes the Horizontal Line Test, then f is invertible. Hence if f is satisfies any one of these three conditions, we can show f must satisfy the other two.[5]

We put this result to work in the next example.

Example 9.4.2. Determine if the following functions are one-to-one: (a) analytically using Definition 9.5 and (b) graphically using the Horizontal Line Test. For the functions that are one-to-one, graph the inverse.

1. $f(x) = x^2 - 2x + 4$
2. $g(t) = \dfrac{2t}{1-t}$
3. $F = \{(-1, 1), (0, 2), (1, -3), (2, 1)\}$
4. $G = \{(t^3 + 1, 2t) \mid t \text{ is a real number.}\}$

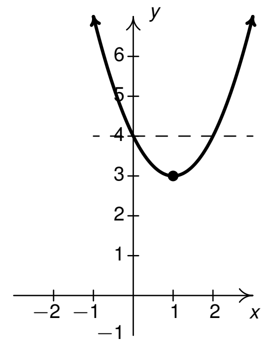
Solution.

1. (a) To determine whether or not f is one-to-one analytically, we assume $f(a) = f(b)$ and work to see if we can deduce $a = b$. As we work our way through the problem below on the left, we encounter a quadratic equation. We rewrite the equation so it equals 0 and factor by grouping. We get $a = b$ as one possibility, but we also get the possibility that $a = 2 - b$. This suggests that f may not be one-to-one. Taking $b = 0$, we get $a = 0$ or $a = 2$. Since $f(0) = 4$ and $f(2) = 4$, we have two different inputs with the same output, proving f is neither one-to-one nor invertible.

   (b) We note that f is a quadratic function and we graph $y = f(x)$ using the techniques presented in Section 5.4 below on the right. We see the graph fails the Horizontal Line Test quite often - in particular, crossing the line $y = 4$ at the points $(0, 4)$ and $(2, 4)$.

[5] For example, if we know f is one-to-one, we showed the graph of f passes the HLT which, in turn, guarantees f is invertible.

$$
\begin{array}{rcl}
f(a) & = & f(b) \\
a^2 - 2a + 4 & = & b^2 - 2b + 4 \\
a^2 - 2a & = & b^2 - 2b \\
a^2 - b^2 - 2a + 2b & = & 0 \\
(a+b)(a-b) - 2(a-b) & = & 0 \\
(a-b)((a+b)-2) & = & 0 \\
a - b = 0 & \text{or} & a + b - 2 = 0 \\
a = b & \text{or} & a = 2 - b
\end{array}
$$

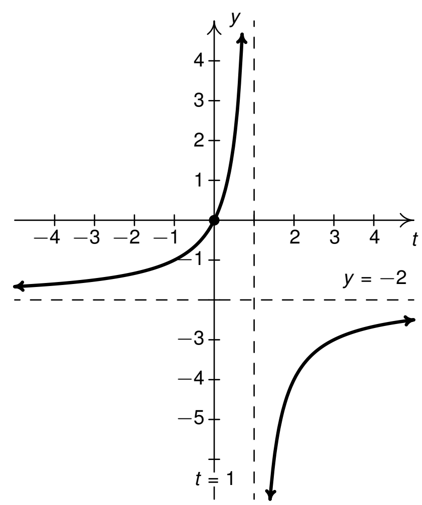


$y = f(x)$

2. (a) We begin with the assumption that $g(a) = g(b)$ for a, b in the domain of g (That is, we assume $a \neq 1$ and $b \neq 1$.) Through our work below on the left, we deduce $a = b$, proving g is one-to-one.

   (b) We graph $y = g(t)$ below on the right using the procedure outlined in Section **??**. We find the sole intercept is $(0, 0)$ with asymptotes $t = 1$ and $y = -2$. Based on our graph, the graph of g appears to pass the Horizontal Line Test, verifying g is one-to-one.

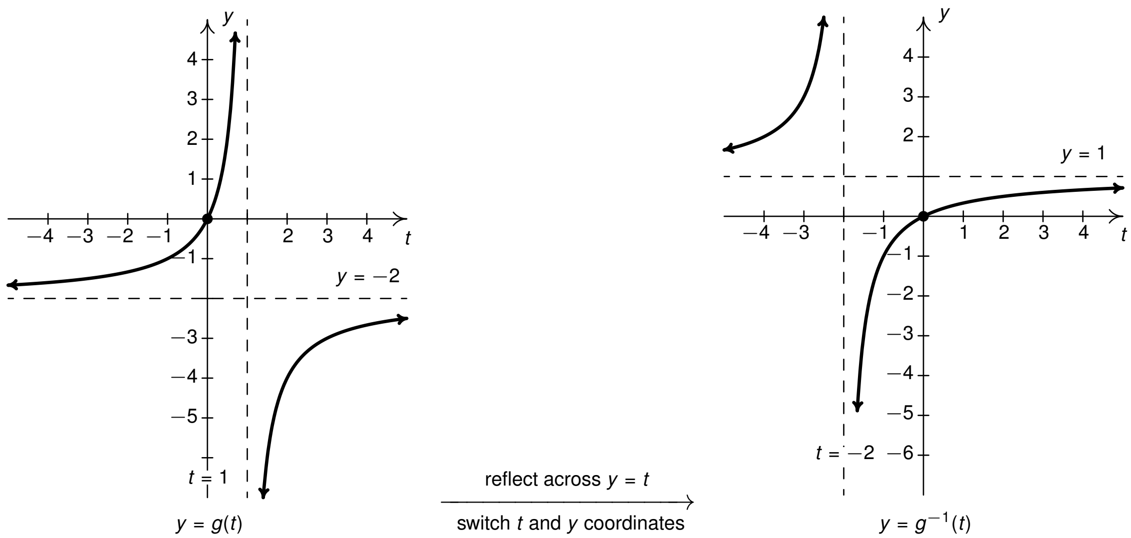
$$
\begin{array}{rcl}
g(a) & = & g(b) \\
\dfrac{2a}{1-a} & = & \dfrac{2b}{1-b} \\
2a(1-b) & = & 2b(1-a) \\
2a - 2ab & = & 2b - 2ba \\
2a & = & 2b \\
a & = & b \checkmark
\end{array}
$$



$y = g(t)$

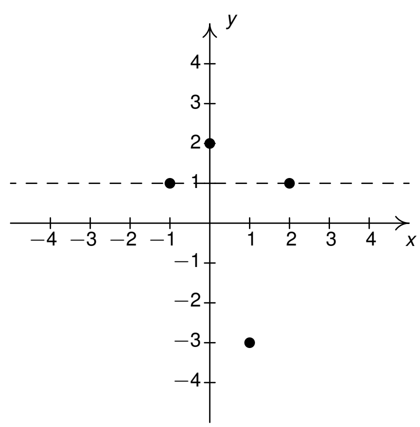
Since g is one-to-one, g is invertible. Even though we do not have a *formula* for $g^{-1}(t)$, we can nevertheless sketch the graph of $y = g^{-1}(t)$ by reflecting the graph of $y = g(t)$ across $y = t$.

Corresponding to the *vertical* asymptote $t = 1$ on the graph of g, the graph of $y = g^{-1}(t)$ will have a *horizontal* asymptote $y = 1$. Similarly, the *horizontal* asymptote $y = -2$ on the graph of g corresponds to a *vertical* asymptote $t = -2$ on the graph of g^{-1}. The point $(0, 0)$ remains unchanged when we switch the t and y coordinates, so it is on both the graph of g and g^{-1}.



3. (a) The function F is given to us as a set of ordered pairs. Recall each ordered pair is of the form $(a, F(a))$. Since $(-1, 1)$ and $(2, 1)$ are both elements of F, this means $F(-1) = 1$ and $F(2) = 1$. Hence, we have two distinct inputs, -1 an 2 with the same output, 1, so F is not one-to-one and, hence, not invertible.

   (b) To graph F, we plot the points in F below on the left. We see the horizontal line $y = 1$ crosses the graph more than once. Hence, the graph of F fails the Horizontal Line Test.



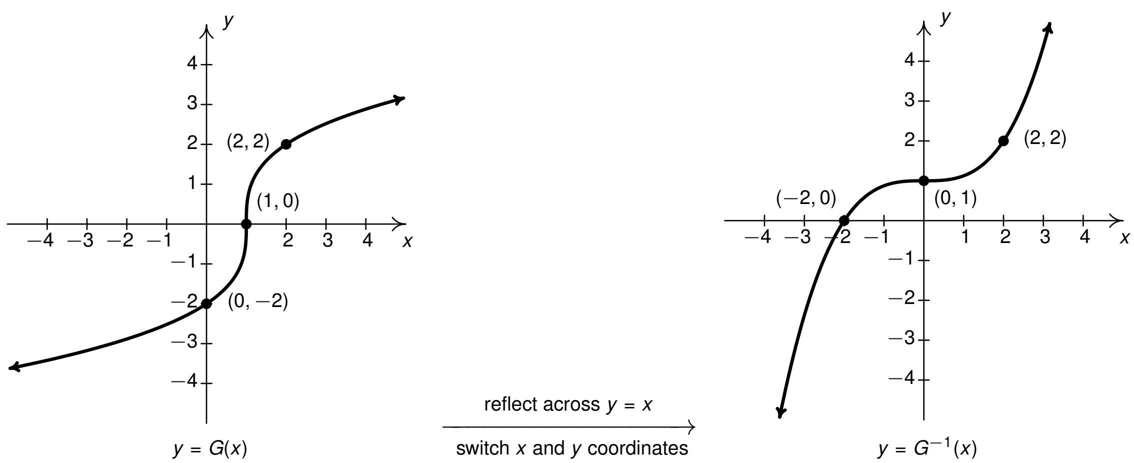
$y = F(x)$

4. Like the function F above, the function G is described as a set of ordered pairs. Before we set about determining whether or not G is one-to-one, we take a moment to show G is, in fact, a function. That is, we must show that each real number input to G is matched to only one output.

   We are given $G = \left\{(t^3 + 1, 2t) \mid t \text{ is a real number.}\right\}$. and we know that when represented in this way, each ordered pair is of the form (input, output). Hence, the inputs to G are of the form $t^3 + 1$ and the outputs from G are of the form $2t$. To establish G is a function, we must show that each input produces only one output. If it should happen that $a^3 + 1 = b^3 + 1$, then we must show $2a = 2b$. The equation $a^3 + 1 = b^3 + 1$ gives $a^3 = b^3$, or $a = b$. From this it follows that $2a = 2b$ so G is a function.

(a) To show G is one-to-one, we must show that if two outputs from G are the same, the corresponding inputs must also be the same. That is, we must show that if $2a = 2b$, then $a^3 + 1 = b^3 + 1$. We see almost immediately that if $2a = 2b$ then $a = b$ so $a^3 + 1 = b^3 + 1$ as required. This shows G is one-to-one and, hence, invertible.

(b) We graph G below on the left by plotting points in the default xy-plane by choosing different values for t. For instance, $t = 0$ corresponds to the point $(0^3+1, 2(0)) = (1, 0)$, $t = 1$ corresponds to the point $(1^3 + 1, 2(1)) = (2, 2)$, $t = -1$ corresponds to the point $((-1)^3 + 1, 2(-1)) = (0, -2)$, etc.[6] Our graph appears to pass the Horizontal Line Test, confirming G is one-to-one. We obtain the graph of G^{-1} below on the right by reflecting the graph of G about the line $y = x$.



□

In Example 9.4.2, we showed the functions G and g are invertible and graphed their inverses. While graphs are perfectly fine representations of functions, we have seen where they aren't the most accurate. Ideally, we would like to represent G^{-1} and g^{-1} in the same manner in which G and g are presented to us. The key to doing this is to recall that inverse functions take outputs back to their associated inputs.

Consider $G = \left\{(t^3 + 1, 2t) \,|\, t \text{ is a real number.}\right\}$. As mentioned in Example 9.4.2, the ordered pairs which comprise G are in the form (input, output). Hence to find a compatible description for G^{-1}, we simply interchange the expressions in each of the coordinates to obtain $G^{-1} = \left\{(2t, t^3 + 1) \,|\, t \text{ is a real number.}\right\}$.

Since the function g was defined in terms of a formula we would like to find a formula representation for g^{-1}. We apply the same logic as above. Here, the input, represented by the independent variable t, and the output, represented by the dependent variable y, are related by the equation $y = g(t)$. Hence, to

[6] Foreshadowing Section **??**, we could let $x = t^3 + 1$ so that $t = \sqrt[3]{x-1}$. Hence, $y = 2t = 2\sqrt[3]{x-1}$.

exchange inputs and outputs, we interchange the 't' and 'y' variables. Doing so, we obtain the equation $t = g(y)$ which is an *implicit* description for g^{-1}. Solving for y gives an explicit formula for g^{-1}, namely $y = g^{-1}(t)$. We demonstrate this technique below.

$$\begin{array}{rcll} y & = & g(t) & \\ y & = & \dfrac{2t}{1-t} & \\ t & = & \dfrac{2y}{1-y} & \text{interchange variables: } t \text{ and } y \\ t(1-y) & = & 2y & \\ t - ty & = & 2y & \\ t & = & ty + 2y & \\ t & = & y(t+2) & \text{factor} \\ y & = & \dfrac{t}{t+2} & \end{array}$$

We claim $g^{-1}(t) = \frac{t}{t+2}$, and leave the algebraic verification of this to the reader.

We generalize this approach below. As always, we resort to the default 'x' and 'y' labels for the independent and dependent variables, respectively.

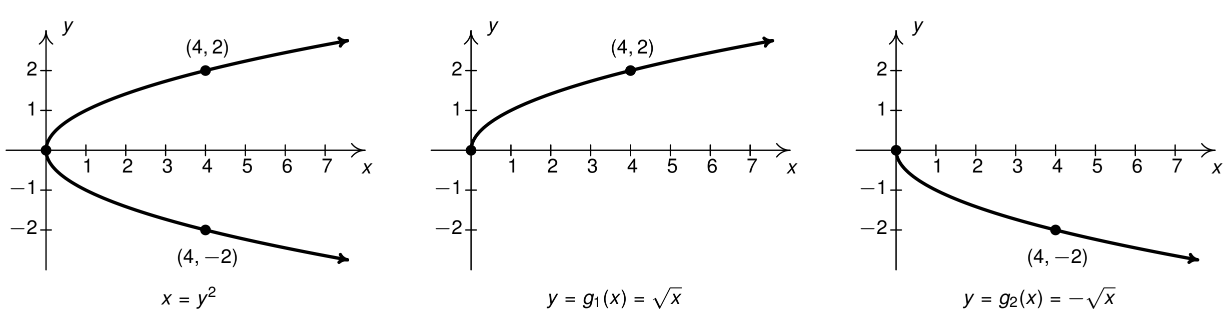
Steps for finding a formula for the Inverse of a one-to-one function

1. Write $y = f(x)$
2. Interchange x and y
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$

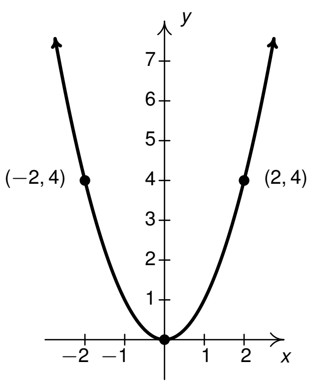
We now return to $f(x) = x^2$. We know that f is not one-to-one, and thus, is not invertible, but our goal here is to see what way to see what goes wrong algebraically.

If we attempt to follow the algorithm above to find a formula for $f^{-1}(x)$, we start with the equation $y = x^2$ and interchange the variables 'x' and 'y' to produce the equation $x = y^2$. Solving for y gives $y = \pm\sqrt{x}$. It's this '$\pm$' which is causing the problem for us since this produces *two* y-values for any $x > 0$.

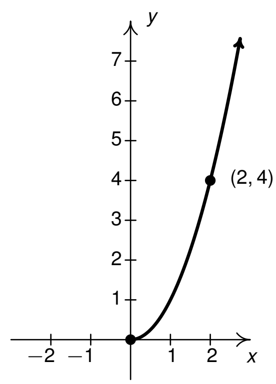
Using the language of Section 9.3, the equation $x = y^2$ implicitly defines *two* functions, $g_1(x) = \sqrt{x}$ and $g_2(x) = -\sqrt{x}$, each of which represents the top and bottom halves, respectively, of the graph of $x = y^2$.



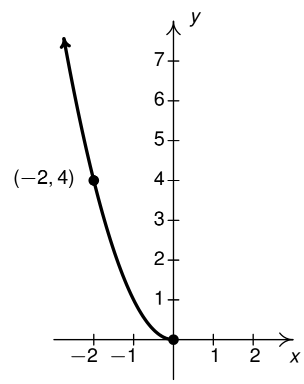
Hence, in some sense, we have two *partial* inverses for $f(x) = x^2$: $g_1(x) = \sqrt{x}$ returns the *positive* inputs from f and $g_2(x) = -\sqrt{x}$ returns the *negative* inputs to f. In order to view each of these functions as strict inverses, however, we need to split f into two parts: $f_1(x) = x^2$ for $x \geq 0$ and $f_2(x) = x^2$ for $x \leq 0$.



$y = f(x) = x^2$



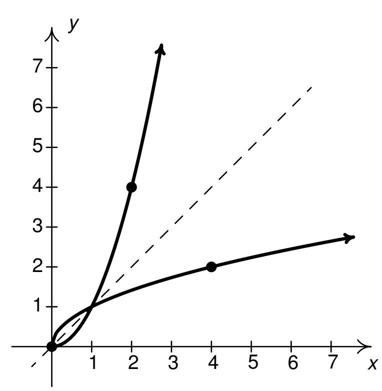
$y = f_1(x) = x^2,\ x \geq 0$



$y = f_2(x) = x^2,\ x \leq 0$

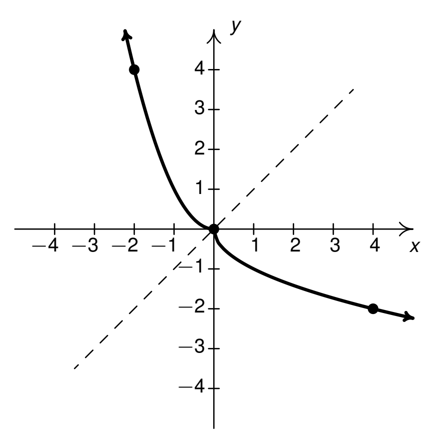
We claim that f_1 and g_1 are an inverse function pair as are f_2 and g_2. Indeed, we find:

$$\begin{aligned}(g_1 \circ f_1)(x) &= g_1(f_1(x)) \\ &= g_1(x^2) \\ &= \sqrt{x^2} \\ &= |x| = x, \text{ as } x \geq 0.\end{aligned} \qquad \begin{aligned}(f_1 \circ g_1)(x) &= f_1(g_1(x)) \\ &= f_1(\sqrt{x}) \\ &= (\sqrt{x})^2 \\ &= x\end{aligned}$$



$y = f_1(x) = x^2,\ x \geq 0$ and $y = g_1(x) = \sqrt{x}$

$$\begin{aligned}(g_2 \circ f_2)(x) &= g_2(f_2(x)) \\ &= g_2(x^2) \\ &= -\sqrt{x^2} \\ &= -|x| \\ &= -(-x) = x, \text{ as } x \leq 0.\end{aligned} \qquad \begin{aligned}(f_2 \circ g_2)(x) &= f_2(g_2(x)) \\ &= f_2(-\sqrt{x}) \\ &= (-\sqrt{x})^2 \\ &= (\sqrt{x})^2 \\ &= x\end{aligned}$$



$y = f_2(x) = x^2,\ x \leq 0$ and $y = g_2(x) = -\sqrt{x}$

Hence, by restricting the domain of f we are able to produce invertible functions. Said differently, in much the same way the equation $x = y^2$ implicitly describes a pair of *functions*, the equation $y = x^2$ implicitly describes a pair of *invertible* functions.

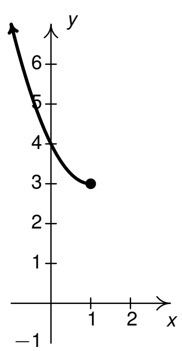
Our next example continues the theme of restricting the domain of a function to find inverse functions.

Example 9.4.3. Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4$, $x \leq 1$.
2. $k(t) = \sqrt{t+2} - 1$

Solution.

1. The function j is a restriction of the function f from Example 9.4.2. Since the domain of j is restricted to $x \leq 1$, we are selecting only the 'left half' of the parabola. Hence, the graph of j, seen below on the left, passes the Horizontal Line Test and thus j is invertible. Below on the right, we find an explicit formula for $j^{-1}(x)$ using our standard algorithm.[7]



$y = j(x)$

$$\begin{array}{rcll}
y & = & j(x) & \\
y & = & x^2 - 2x + 4, \quad x \leq 1 & \\
x & = & y^2 - 2y + 4, \quad y \leq 1 & \text{switch } x \text{ and } y \\
0 & = & y^2 - 2y + 4 - x & \\
y & = & \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} & \text{quadratic formula, } c = 4 - x \\
y & = & \dfrac{2 \pm \sqrt{4x - 12}}{2} & \\
y & = & \dfrac{2 \pm \sqrt{4(x-3)}}{2} & \\
y & = & \dfrac{2 \pm 2\sqrt{x-3}}{2} & \\
y & = & \dfrac{2\left(1 \pm \sqrt{x-3}\right)}{2} & \\
y & = & 1 \pm \sqrt{x-3} & \\
y & = & 1 - \sqrt{x-3} & \text{since } y \leq 1.
\end{array}$$

Hence, $j^{-1}(x) = 1 - \sqrt{x-3}$.

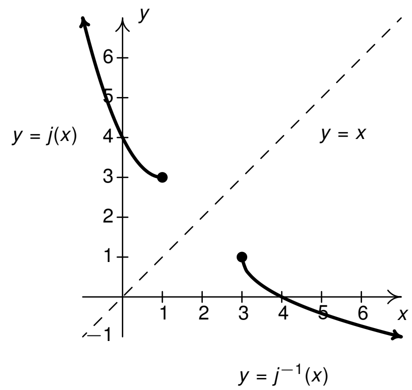
To check our answer algebraically, we simplify $(j^{-1} \circ j)(x)$ and $(j \circ j^{-1})(x)$ Note the importance of the domain restriction $x \leq 1$ when simplifying $(j^{-1} \circ j)(x)$.

[7] Here, we use the Quadratic Formula to solve for y. For 'completeness,' we note you can (and should!) also consider solving for y by 'completing' the square.

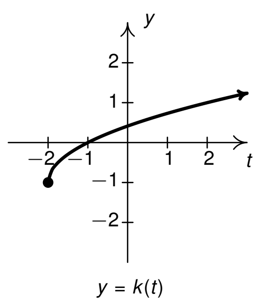
$$\begin{aligned}
\left(j^{-1} \circ j\right)(x) &= j^{-1}(j(x)) \\
&= j^{-1}\left(x^2 - 2x + 4\right), \quad x \leq 1 \\
&= 1 - \sqrt{\left(x^2 - 2x + 4\right) - 3} \\
&= 1 - \sqrt{x^2 - 2x + 1} \\
&= 1 - \sqrt{(x-1)^2} \\
&= 1 - |x - 1| \\
&= 1 - (-(x-1)) \text{ since } x \leq 1 \\
&= x \checkmark
\end{aligned}$$

$$\begin{aligned}
\left(j \circ j^{-1}\right)(x) &= j\left(j^{-1}(x)\right) \\
&= j\left(1 - \sqrt{x-3}\right) \\
&= \left(1 - \sqrt{x-3}\right)^2 - 2\left(1 - \sqrt{x-3}\right) + 4 \\
&= 1 - 2\sqrt{x-3} + \left(\sqrt{x-3}\right)^2 - 2 \\
&\quad + 2\sqrt{x-3} + 4 \\
&= 1 + x - 3 - 2 + 4 \\
&= x \checkmark
\end{aligned}$$

We graph both j and j^{-1} on the axes below. They appear to be symmetric about the line $y = x$.



2. Graphing $y = k(t) = \sqrt{t+2} - 1$, we see k is one-to-one, so we proceed to find an formula for k^{-1}.



$$\begin{aligned}
y &= k(t) \\
y &= \sqrt{t+2} - 1 \\
t &= \sqrt{y+2} - 1 \quad \text{switch } t \text{ and } y \\
t + 1 &= \sqrt{y+2} \\
(t+1)^2 &= \left(\sqrt{y+2}\right)^2 \\
t^2 + 2t + 1 &= y + 2 \\
y &= t^2 + 2t - 1
\end{aligned}$$

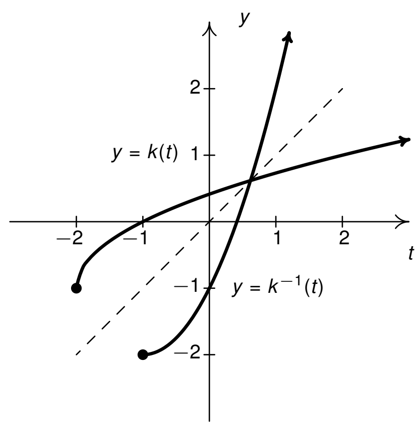
We have $k^{-1}(t) = t^2 + 2t - 1$. Based on our experience, we know something isn't quite right. We determined k^{-1} is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted.

Theorem 9.4 tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1, \infty)$, which means we restrict the domain of k^{-1} to $t \geq -1$.

We now check that this works in our compositions. Note the importance of the domain restriction, $t \geq -1$ when simplifying $(k \circ k^{-1})(t)$.

$$\begin{aligned}
\left(k^{-1} \circ k\right)(t) &= k^{-1}(k(t)) \\
&= k^{-1}\left(\sqrt{t+2}-1\right) \\
&= \left(\sqrt{t+2}-1\right)^2 + 2\left(\sqrt{t+2}-1\right) - 1 \\
&= \left(\sqrt{t+2}\right)^2 - 2\sqrt{t+2} + 1 \\
&\quad + 2\sqrt{t+2} - 2 - 1 \\
&= t + 2 - 2 \\
&= t \checkmark
\end{aligned}
\qquad
\begin{aligned}
\left(k \circ k^{-1}\right)(t) &= k\left(t^2 + 2t - 1\right), \quad t \geq -1 \\
&= \sqrt{\left(t^2 + 2t - 1\right) + 2} - 1 \\
&= \sqrt{t^2 + 2t + 1} - 1 \\
&= \sqrt{(t+1)^2} - 1 \\
&= |t+1| - 1 \\
&= t + 1 - 1, \text{ since } t \geq -1 \\
&= t \checkmark
\end{aligned}$$

Graphically, everything checks out, provided that we remember the domain restriction on k^{-1} means we take the right half of the parabola.



□

Our last example of the section gives an application of inverse functions. Recall in Example 3.2.4 in Section 3.2, we modeled the demand for PortaBoy game systems as the price per system, $p(x)$ as a function of the number of systems sold, x. In the following example, we find $p^{-1}(x)$ and interpret what it means.

Example 9.4.4. Recall the price-demand function for PortaBoy game systems is modeled by the formula $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$ where x represents the number of systems sold (the demand) and $p(x)$ is the price per system, in dollars.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.

2. Find and interpret $p^{-1}(220)$.

3. Recall from Section 5.4 that the profit P, in dollars, as a result of selling x systems is given by $P(x) = -1.5x^2 + 170x - 150$. Find and interpret $\left(P \circ p^{-1}\right)(x)$.

4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example 5.4.3.

Solution.

1. Recall the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0, 250)$ to $(166, 1)$, and as such passes the Horizontal Line Test. Hence, p is one-to-one. We find the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of p^{-1} should match the range of p, which is $[1, 250]$, and as such, we restrict the domain of p^{-1} to $1 \leq x \leq 250$.

2. We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. Since the function p took as inputs the number of systems sold and returned the price per system as the output, p^{-1} takes the price per system as its input and returns the number of systems sold as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold in if the price is set at \$220 per system.

3. We compute $\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$. After a hefty amount of Elementary Algebra,[8] we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$.

   To understand what this means, recall that the original profit function P gave us the profit as a function of the number of systems sold. The function p^{-1} gives us the number of systems sold as a function of the price. Hence, when we compute $(P \circ p^{-1})(x) = P(p^{-1}(x))$, we input a price per system, x into the function p^{-1}.

   The number $p^{-1}(x)$ is the number of systems sold at that price. This number is then fed into P to return the profit obtained by selling $p^{-1}(x)$ systems. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the profit (in dollars) as a function of the price per system, x.

4. We know from Section 5.4 that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the x-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation 5.1, $x = -\frac{b}{2a} = 165$.

   Hence, weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example 5.4.3, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately. □

[8] It is good review to actually do this!

9.4.1 Exercises

In Exercises 1 - 8, verify the given pairs of functions are inverses algebraically and graphically.

1. $f(x) = 2x + 7$ and $g(x) = \dfrac{x-7}{2}$
2. $f(x) = \dfrac{5-3x}{4}$ and $g(x) = -\dfrac{4}{3}x + \dfrac{5}{3}$.
3. $f(t) = \dfrac{5}{t-1}$ and $g(t) = \dfrac{t+5}{t}$
4. $f(t) = \dfrac{t}{t-1}$ and $g(t) = f(t) = \dfrac{t}{t-1}$
5. $f(x) = \sqrt{4-x}$ and $g(x) = -x^2 + 4$, $x \geq 0$
6. $f(x) = 1 - \sqrt{x+1}$ and $g(x) = x^2 - 2x$, $x \leq 1$.
7. $f(t) = (t-1)^3 + 5$ and $g(t) = \sqrt[3]{t-5} + 1$
8. $f(t) = -\sqrt[4]{t-2}$ and $g(t) = t^4 + 2$, $t \leq 0$.

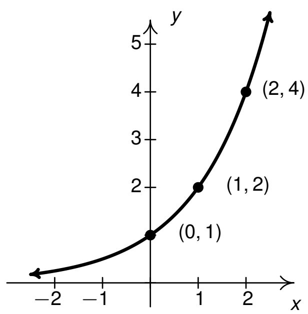
In Exercises 9 - 28, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify the range of the function is the domain of its inverse and vice-versa.

9. $f(x) = 6x - 2$
10. $f(x) = 42 - x$
11. $g(t) = \dfrac{t-2}{3} + 4$
12. $g(t) = 1 - \dfrac{4+3t}{5}$
13. $f(x) = \sqrt{3x-1} + 5$
14. $f(x) = 2 - \sqrt{x-5}$
15. $g(t) = 3\sqrt{t-1} - 4$
16. $g(t) = 1 - 2\sqrt{2t+5}$
17. $f(x) = \sqrt[5]{3x-1}$
18. $f(x) = 3 - \sqrt[3]{x-2}$
19. $g(t) = t^2 - 10t$, $t \geq 5$
20. $g(t) = 3(t+4)^2 - 5$, $t \leq -4$
21. $f(x) = x^2 - 6x + 5$, $x \leq 3$
22. $f(x) = 4x^2 + 4x + 1$, $x < -1$
23. $g(t) = \dfrac{3}{4-t}$
24. $g(t) = \dfrac{t}{1-3t}$
25. $f(x) = \dfrac{2x-1}{3x+4}$
26. $f(x) = \dfrac{4x+2}{3x-6}$
27. $g(t) = \dfrac{-3t-2}{t+3}$
28. $g(t) = \dfrac{t-2}{2t-1}$
29. Explain why each set of ordered pairs below represents a one-to-one function and find the inverse.
    (a) $F = \{(0,0), (1,1), (2,-1), (3,2), (4,-2), (5,3), (6,-3)\}$
    (b) $G = \{(0,0), (1,1), (2,-1), (3,2), (4,-2), (5,3), (6,-3), \ldots\}$
    NOTE: The difference between F and G is the '....'
    (c) $P = \{(2t^5, 3t-1) \mid t \text{ is a real number.}\}$
    (d) $Q = \{(n, n^2) \mid n \text{ is a } \textit{natural} \text{ number.}\}$[9]

[9]Recall this means $n = 0, 1, 2, \ldots$.

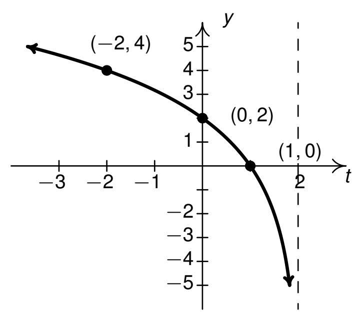
In Exercises 30 - 33, explain why each graph represents[10] a one-to-one function and graph its inverse.

30. $y = f(x)$



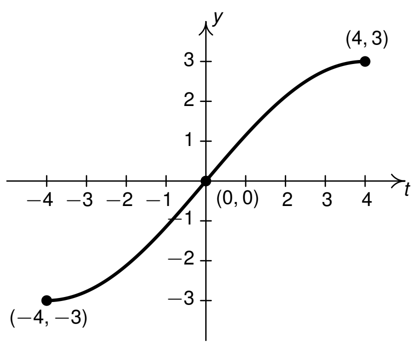
Asymptote: $y = 0$.

31. $y = g(t)$



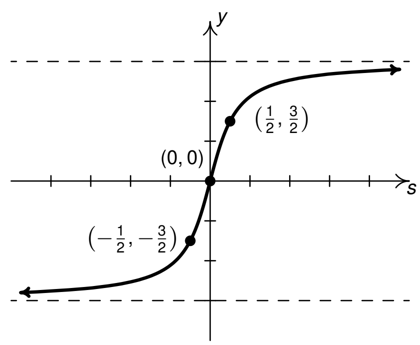
Asymptote: $t = 2$.

32. $y = S(t)$



Domain: $[-4, 4]$.

33. $y = R(s)$



Asymptotes: $y = \pm 3$.

34. The price of a dOpi media player, in dollars per dOpi, is given as a function of the weekly sales x according to the formula $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

    (a) Find $p^{-1}(x)$ and state its domain.

    (b) Find and interpret $p^{-1}(105)$.

    (c) The profit (in dollars) made from producing and selling x dOpis per week is given by the formula $P(x) = -15x^2 + 350x - 2000$, for $0 \leq x \leq 30$. Find $\left(P \circ p^{-1}\right)(x)$ and determine what price per dOpi would yield the maximum profit. What is the maximum profit? How many dOpis need to be produced and sold to achieve the maximum profit?

35. Show that the Fahrenheit to Celsius conversion function found in Exercise 26 in Section 3.2.2 is invertible and that its inverse is the Celsius to Fahrenheit conversion function.

36. Analytically show that the function $f(x) = x^3 + 3x + 1$ is one-to-one. Use Theorem 9.4 to help you compute $f^{-1}(1)$, $f^{-1}(5)$, and $f^{-1}(-3)$. What happens when you attempt to find a formula for $f^{-1}(x)$?

[10] or, more precisely, *appears* to represent . . .

37. Let $f(x) = \dfrac{2x}{x^2 - 1}$.

    (a) Graph $y = f(x)$ using the techniques in Section **??**. Check your answer using a graphing utility.

    (b) Verify that f is one-to-one on the interval $(-1, 1)$.

    (c) Use the procedure outlined on Page 512 to find the formula for $f^{-1}(x)$ for $-1 < x < 1$.

    (d) Since $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.

38. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

39. If f is odd and invertible, prove that f^{-1} is also odd.

40. Let f and g be invertible functions. With the help of your classmates show that $(f \circ g)$ is one-to-one, hence invertible, and that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

With help from your classmates, find the inverses of the functions in Exercises 41 - 44.

41. $f(x) = ax + b, \;\; a \neq 0$

42. $f(x) = a\sqrt{x - h} + k, \;\; a \neq 0, x \geq h$

43. $f(x) = ax^2 + bx + c$ where $a \neq 0, \; x \geq -\dfrac{b}{2a}$.

44. $f(x) = \dfrac{ax + b}{cx + d}$, (See Exercise 45 below.)

45. What conditions must you place on the values of a, b, c and d in Exercise 44 in order to guarantee that the function is invertible?

46. The function given in number 4 is an example of a function which is its own inverse.

    (a) Algebraically verify every function of the form: $f(x) = \dfrac{ax + b}{cx - a}$ is its own inverse. What assumptions do you need to make about the values of a, b, and c?

    (b) Under what conditions is $f(x) = mx + b$, $m \neq 0$ its own inverse? Prove your answer.

9.4.2 Answers

9. $f^{-1}(x) = \dfrac{x+2}{6}$

10. $f^{-1}(x) = 42 - x$

11. $g^{-1}(t) = 3t - 10$

12. $g^{-1}(t) = -\frac{5}{3}t + \frac{1}{3}$

13. $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}$, $x \geq 5$

14. $f^{-1}(x) = (x-2)^2 + 5$, $x \leq 2$

15. $g^{-1}(t) = \frac{1}{9}(t+4)^2 + 1$, $t \geq -4$

16. $g^{-1}(t) = \frac{1}{8}(t-1)^2 - \frac{5}{2}$, $t \leq 1$

17. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$

18. $f^{-1}(x) = -(x-3)^3 + 2$

19. $g^{-1}(t) = 5 + \sqrt{t+25}$

20. $g^{-1}(t) = -\sqrt{\frac{t+5}{3}} - 4$

21. $f^{-1}(x) = 3 - \sqrt{x+4}$

22. $f^{-1}(x) = -\frac{\sqrt{x}+1}{2}$, $x > 1$

23. $g^{-1}(t) = \dfrac{4t-3}{t}$

24. $g^{-1}(t) = \dfrac{t}{3t+1}$

25. $f^{-1}(x) = \dfrac{4x+1}{2-3x}$

26. $f^{-1}(x) = \dfrac{6x+2}{3x-4}$

27. $g^{-1}(t) = \dfrac{-3t-2}{t+3}$

28. $g^{-1}(t) = \dfrac{t-2}{2t-1}$

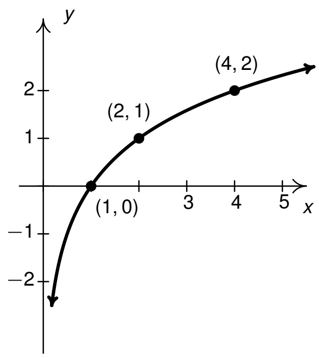
29. (a) None of the first coordinates of the ordered pairs in F are repeated, so F is a function and none of the second coordinates of the ordered pairs of F are repeated, so F is one-to-one. $F^{-1} = \{(0,0), (1,1), (-1,2), (2,3), (-2,4), (3,5), (-3,6)\}$

    (b) Because of the '...' it is helpful to determine a formula for the matching. For the even numbers n, $n = 0, 2, 4, ...$, the ordered pair $\left(n, -\frac{n}{2}\right)$ is in G. For the odd numbers $n = 1, 3, 5, ...$, the ordered pair $\left(n, \frac{n+1}{2}\right)$ is in G. Hence, given any input to G, n, whether it be even or odd, there is only one output from G, either $-\frac{n}{2}$ or $\frac{n+1}{2}$, both of which are functions of n. To show G is one to one, we note that if the output from G is 0 or less, then it must be of the form $-\frac{n}{2}$ for an even number n. Moreover, if $-\frac{n}{2} = -\frac{m}{2}$, then $n = m$. In the case we are looking at outputs from G which are greater than 0, then it must be of the form $\frac{n+1}{2}$ for an odd number n. In this, too, if $\frac{n+1}{2} = \frac{m+1}{2}$, then $n = m$. Hence, in any case, if the outputs from G are the same, then the inputs to G had to be the same so G is one-to-one and $G^{-1} = \{(0,0), (1,1), (-1,2), (2,3), (-2,4), (3,5), (-3,6), ...\}$

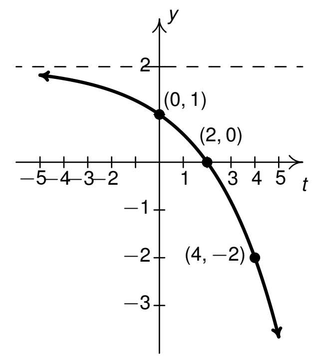
    (c) To show P is a function we note that if we have the same inputs to P, say $2t^5 = 2u^5$, then $t = u$. Hence the corresponding outputs, $2t-1$ and $3u-1$, are equal, too. To show P is one-to-one, we note that if we have the same outputs from P, $3t-1 = 3u-1$, then $t = u$. Hence, the corresponding inputs $2t^5$ and $2u^5$ are equal, too. Hence P is one-to-one and $P^{-1} = \{(3t-1, 2t^5) \mid t \text{ is a real number.}\}$

(d) To show Q is a function, we note that if we have the same inputs to Q, say $n = m$, then the outputs from Q, namely n^2 and m^2 are equal. To show Q is one-to-one, we note that if we get the same output from Q, namely $n^2 = m^2$, then $n = \pm m$. However since n and m are *natural* numbers, both n and m are positive so $n = m$. Hence Q is one-to-one and $Q^{-1} = \{(n^2, n) \mid n \text{ is a } \textit{natural} \text{ number.}\}$.

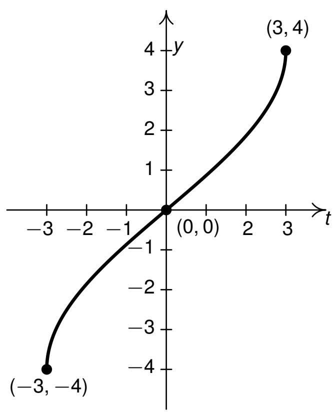
30. $y = f^{-1}(x)$. Asymptote: $x = 0$.



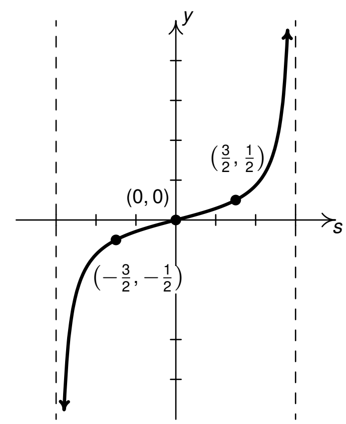
31. $y = g^{-1}(t)$. Asymptote: $y = 2$.



32. $y = S^{-1}(t)$. Domain $[-3, 3]$.



33. $y = R^{-1}(s)$. Asymptotes: $s = \pm 3$.



34. (a) $p^{-1}(x) = \frac{450-x}{15}$. The domain of p^{-1} is the range of p which is $[0, 450]$

(b) $p^{-1}(105) = 23$. This means that if the price is set to \$105 then 23 dOpis will be sold.

(c) $\left(P \circ p^{-1}\right)(x) = -\frac{1}{15}x^2 + \frac{110}{3}x - 5000$, $0 \leq x \leq 450$.

The graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards with vertex $\left(275, \frac{125}{3}\right) \approx (275, 41.67)$. This means that the maximum profit is a whopping \$41.67 when the price per dOpi is set to \$275. At this price, we can produce and sell $p^{-1}(275) = 11.\overline{6}$ dOpis. Since we cannot sell part of a system, we need to adjust the price to sell either 11 dOpis or 12 dOpis. We find $p(11) = 285$ and $p(12) = 270$, which means we set the price per dOpi at either \$285 or \$270, respectively. The profits at these prices are $\left(P \circ p^{-1}\right)(285) = 35$ and $\left(P \circ p^{-1}\right)(270) = 40$, so it looks as if the maximum profit is \$40 and it is made by producing and selling 12 dOpis a week at a price of \$270 per dOpi.

36. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$

46. (b) If $b = 0$, then $m = \pm 1$. If $b \neq 0$, then $m = -1$ and b can be any real number.

Chapter 10

Exponential and Logarithmic Functions

10.1 Exponential Functions

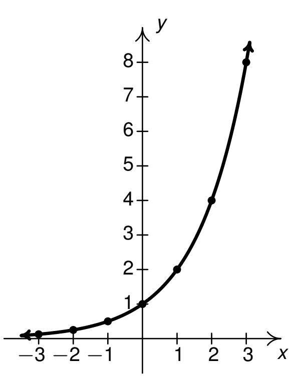
Of all of the functions we study in this text, exponential functions are possibly the ones which impact everyday life the most. This section introduces us to these functions while the rest of the chapter will more thoroughly explore their properties.

Up to this point, we have dealt with functions which involve terms like x^3, $x^{\frac{3}{2}}$, or x^π - in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant.

In this chapter, we study functions of the form $f(x) = b^x$ where the base b is a constant and the exponent x is the variable. We start our exploration of these functions with the time-honored classic, $f(x) = 2^x$.

We make a table of function values, plot enough points until we are more or less confident with the shape of the curve, and connect the dots in a pleasing fashion.

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(3, 8)$



$y = f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ are in order. As $x \to -\infty$ and takes on values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$. In other words, as $x \to -\infty$, $2^x \approx \frac{1}{\text{very big }(+)} \approx$ very small (+) That is, as $x \to -\infty$, $2^x \to 0^+$. This produces the x-axis, $y = 0$ as a horizontal asymptote to the graph as $x \to -\infty$.

On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $2^x \to \infty$.

We note that by 'connecting the dots in a pleasing fashion,' we are implicitly using the fact that $f(x) = 2^x$ is not only defined for all real numbers,[1] but is also *continuous*. Moreover, we are assuming $f(x) = 2^x$ is increasing: that is, if $a < b$, then $2^a < 2^b$. While these facts are true, the proofs of these properties are best left to Calculus. For us, we assume these properties in order to state the domain of f is $(-\infty, \infty)$, the range of f is $(0, \infty)$ and, since f is increasing, f is one-to-one, hence invertible.

Suppose we wish to study the family of functions $f(x) = b^x$. Which bases b make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ since $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if x is any rational number with an even denominator,[2] then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$.

What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and 0^0 is an indeterminant form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. Since we know everything about this function, we ignore this case.

The only other base we exclude is $b = 1$, since the function $f(x) = 1^x = 1$ for all real numbers x, since, once again, a function we have already studied. We are now ready for our definition of exponential functions.

Definition 10.1. An **exponential function** is the function of the form

$$f(x) = b^x$$

where b is a real number, $b > 0$, $b \neq 1$. The domain of an exponential function $(-\infty, \infty)$.
NOTE: More specifically, $f(x) = b^x$ is called the '*base b exponential function*.'

We leave it to the reader to verify[3] that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$.

What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$, where $f(x) = 2^x$. Per Section 2.3, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis.

$y = f(x) = 2^x$ $\qquad$ $y = g(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$

We see that the domain and range of g match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like f, g is also one-to-one. Whereas f is always increasing, g is always decreasing. As a result, as $x \to -\infty$,

[1] See the discussion of real number exponents in Section **??**.
[2] or, as we defined real number exponents in Section **??**, if x is an irrational number ...
[3] Meaning, graph some more examples on your own.

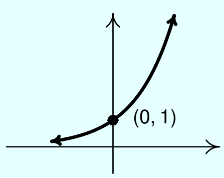
$g(x) \to \infty$, and on the flip side, as $x \to \infty$, $g(x) \to 0^+$. It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph of $y = b^x$ behaves similarly to the graph of g.

We summarize the basic properties of exponential functions in the following theorem.

Theorem 10.1. Properties of Exponential Functions: Suppose $f(x) = b^x$.

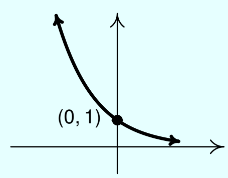
- The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- $(0, 1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- f is one-to-one, continuous and smooth[a]

- If $b > 1$:
  - f is always increasing
  - As $x \to -\infty$, $f(x) \to 0^+$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = b^x$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0^+$
  - The graph of f resembles:



$y = b^x$, $0 < b < 1$

[a]Recall that this means the graph of f has no sharp turns or corners.

Exponential functions also inherit the basic properties of exponents from Theorem **??**. We formalize these below and use them as needed in the coming examples.

Theorem 10.2. (Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u + w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u - w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

In addition to base 2 which is important to computer scientists,[4] two other bases are used more often than not in scientific and economic circles. The first is base 10. Base 10 is called the '**common base**' and is important in the study of intensity (sound intensity, earthquake intensity, acidity, etc.)

[4]The digital world is comprised of bytes which take on one of *two* values: 0 or 'off' and 1 or 'on.'

The second base is an irrational number, e. Like $\sqrt{2}$ or π, the decimal expansion of e neither terminates nor repeats, so we represent this number by the letter 'e.' A decimal approximation of e is $e \approx 2.718$, so the function $f(x) = e^x$ is an increasing exponential function.

The number e is called the '**natural base**' for lots of reasons, one of which is that it 'naturally' arises in the study of growth functions in Calculus. We will more formally discuss the origins of e in Section **??**.

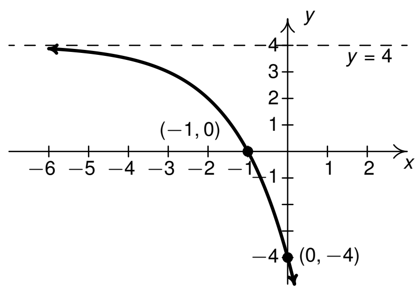
It is time for an example.

Example 10.1.1.

1. Graph the following functions by starting with a basic exponential function and using transformations, Theorem 2.7. Track at least three points and the horizontal asymptote through the transformations.

   (a) $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ (b) $G(t) = 2 - e^{-t}$

2. Find a formula for the graph of the function below. Assume the base of the exponential is 2.



$y = F(x)$

Solution.

1. (a) Since the base of the exponent in $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ is $\frac{1}{3}$, we start with the graph of $f(x) = \left(\frac{1}{3}\right)^x$.

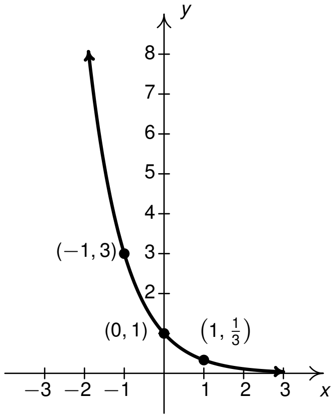
   To use Theorem 2.7, we first need to choose some 'control points' on the graph of $f(x) = \left(\frac{1}{3}\right)^x$. Since we are instructed to track three points (and the horizontal asymptote, $y = 0$) through the transformations, we choose the points corresponding to $x = -1$, $x = 0$, and $x = 1$: $(-1, 3)$, $(0, 1)$, and $\left(1, \frac{1}{3}\right)$, respectively.

   Next, we need determine how to modify $f(x) = \left(\frac{1}{3}\right)^x$ to obtain $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$. The key is to recognize the argument, or 'inside' of the function is the exponent and the 'outside' is anything outside the base of $\frac{1}{3}$. Using these principles as a guide, we find $F(x) = 2f(x - 1)$.

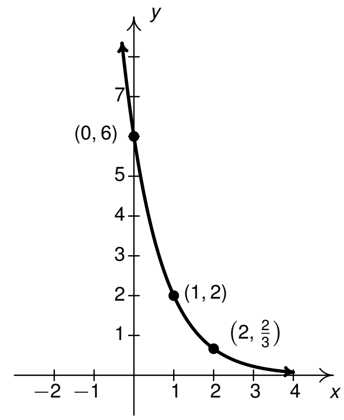
   Per Theorem 2.7, we first add 1 to the x-coordinates of the points on the graph of $y = f(x)$, shifting the graph to the right 1 unit. Next, multiply the y-coordinates of each point on this new graph by 2, vertically stretching the graph by a factor of 2.

   Looking point by point, we have $(-1, 3) \to (0, 3) \to (0, 6)$, $(0, 1) \to (1, 1) \to (1, 2)$, and $\left(1, \frac{1}{3}\right) \to \left(2, \frac{1}{3}\right) \to \left(2, \frac{2}{3}\right)$. The horizontal asymptote, $y = 0$ remains unchanged under the horizontal shift and the vertical stretch since $2 \cdot 0 = 0$.

Below we graph $y = f(x) = \left(\frac{1}{3}\right)^x$ on the left $y = F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ on the right.



$y = f(x) = \left(\frac{1}{3}\right)^x$ $\xrightarrow{\text{Theorem 2.7}}$ $y = F(x) = 2\left(\frac{1}{3}\right)^{x-1}$



As always we can check our answer by verifying each of the points $(0, 6)$, $(1, 2)$, $\left(2, \frac{2}{3}\right)$ is on the graph of $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ by checking $F(0) = 6$, $F(1) = 2$, and $F(2) = \frac{2}{3}$.

We can check the end behavior as well, that is, as $x \to -\infty$, $F(x) \to \infty$ and as $x \to \infty$, $F(x) \to 0$. We leave these calculations to the reader.

(b) Since the base of the exponential in $G(t) = 2 - e^{-t}$ is e, we start with the graph of $g(t) = e^t$.

Note that since e is an irrational number, we will use the approximation $e \approx 2.718$ when *plotting* points. However, when it comes to tracking and labeling said points, we do so with *exact* coordinates, that is, in terms of e.

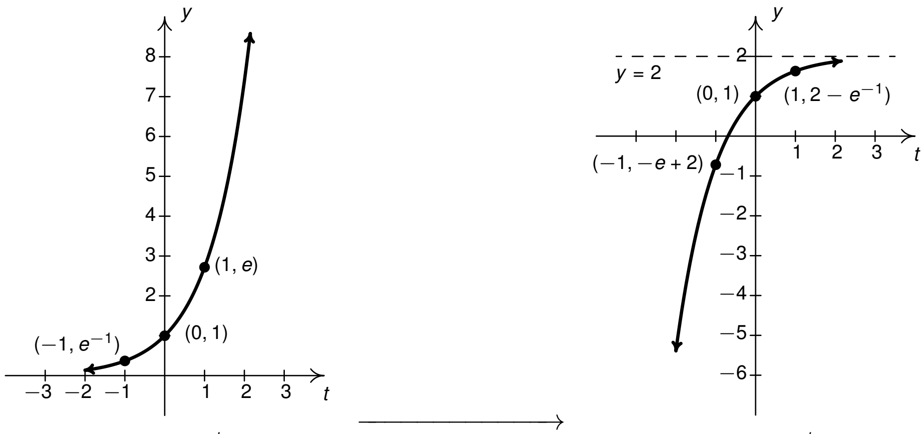
We choose points corresponding to $t = -1$, $t = 0$, and $t = 1$: $(-1, e^{-1}) \approx (-1, 0.368)$, $(0, 1)$, and $(1, e) \approx (1, 2.718)$, respectively.

Next, we need to determine how the formula for $G(t) = 2 - e^{-t}$ can be obtained from the formula $g(t) = e^t$. Rewriting $G(t) = -e^{-t} + 2$, we find $G(t) = -g(-t) + 2$.

Following Theorem 2.7, we first multiply the t-coordinates of the graph of $y = g(t)$ by -1, effecting a reflection across the y-axis. Next, we multiply each of the y-coordinates by -1 which reflects the graph about the t-axis. Finally, we add 2 to each of the y-coordinates of the graph from the second step which shifts the graph up 2 units.

Tracking points, we have $(-1, e^{-1}) \to (1, e^{-1}) \to (1, -e^{-1}) \to (1, -e^{-1} + 2) \approx (1, 1.632)$, $(0, 1) \to (0, 1) \to (0, -1) \to (0, 1)$, and $(1, e) \to (-1, e) \to (-1, -e) \to (-1, -e + 2) \approx (-1, -0.718)$. The horizontal asymptote is unchanged by the reflections, but is shifted up 2 units $y = 0 \to y = 2$.

We graph $g(t) = e^t$ below on the left and the transformed function $G(t) = -e^{-t} + 2$ below on the left. As usual, we can check our answer by verifying the indicated points do, in fact, lie on the graph of $y = G(t)$ along with checking end behavior. We leave these details to the reader.



$y = g(t) = e^t$ Theorem 2.7 $y = G(t) = -e^{-t} + 2$

2. Since we are told to assume the base of the exponential function is 2, we assume the function $F(x)$ is the result of the transforming the graph of $f(x) = 2^x$ using Theorem 2.7. This means we are tasked with finding values for a, b, h, and k so that $F(x) = af(bx - h) + k = a \cdot 2^{bx-h} + k$.

   Since the horizontal asymptote to the graph of $y = f(x) = 2^x$ is $y = 0$ and the horizontal asymptote to the graph $y = F(x)$ is $y = 4$, we know the vertical shift is 4 units up, so $k = 4$.

   Next, looking at how the graph of F approaches the vertical asymptote, it stands to reason the graph of $f(x) = 2^x$ undergoes a reflection across x-axis, meaning $a < 0$. For simplicity, we assume $a = -1$ and set see if we can find values for b and h that go along with this choice.

   Since $(-1, 0)$ and $(0, -4)$ on the graph of $F(x) = -2^{bx-h} + 4$, we know $F(-1) = 0$ and $F(0) = -4$. From $F(-1) = 0$, we have $-2^{-b-h} + 4 = 0$ or $2^{-b-h} = 4 = 2^2$. Hence, $-b - h = 2$ is one solution.[5]

   Next, using $F(0) = -4$, we get $-2^{-h} + 4 = -4$ or $2^{-h} = 8 = 2^3$. From this, we have $-h = 3$ so $h = -3$. Putting this together with $-b - h = 2$, we get $-b + 3 = 2$ so $b = 1$.

   Hence, one solution to the problem is $F(x) = -2^{x+3} + 4$. To check our answer, we leave it to the reader verify $F(-1) = 0$, $F(0) = -4$, as $x \to -\infty$, $F(x) \to 4$ and as $x \to \infty$, $F(x) \to -\infty$.

   Since we made a simplifying assumption ($a = -1$), we may well wonder if our solution is the *only* solution. Indeed, we started with what amounts to three pieces of information and set out to determine the value of four constants. We leave this for a thoughtful discussion in Exercise 14.

Our next example showcases an important application of exponential functions: economic depreciation.

[5]This is the *only* solution. Since $f(x) = 2^x$, the equation $2^{-b-h} = 2^2$ is equivalent to the functional equation $f(-b - h) = f(2)$. Since f is one-to-one, we know this is true *only* when $-b - h = 2$.

Example 10.1.2. The value of a car can be modeled by $V(t) = 25(0.8)^t$, where $t \geq 0$ is number of years the car is owned and $V(t)$ is the value in thousands of dollars.

1. Find and interpret $V(0)$, $V(1)$, and $V(2)$.
2. Find and interpret the average rate of change of V over the intervals $[0, 1]$ and $[0, 2]$ and $[1, 2]$.
3. Find and interpret $\dfrac{V(1)}{V(0)}$, $\dfrac{V(2)}{V(1)}$ and $\dfrac{V(2)}{V(0)}$.
4. For $t \geq 0$, find and interpret $\dfrac{V(t+1)}{V(t)}$ and $\dfrac{V(t+k)}{V(t)}$.
5. Find and interpret $\dfrac{V(1)-V(0)}{V(0)}$, $\dfrac{V(2)-V(1)}{V(1)}$, and $\dfrac{V(2)-V(0)}{V(0)}$.
6. For $t \geq 0$, find and interpret $\dfrac{V(t+1)-V(t)}{V(t)}$ and $\dfrac{V(t+k)-V(t)}{V(t)}$.
7. Graph $y = V(t)$ starting with the graph of $y = V(t)$ and using transformations.
8. Interpret the horizontal asymptote of the graph of $y = V(t)$.
9. Using a graphing utility, determine how long it takes for the car to depreciate to (a) one half its original value and (b) one quarter of its original value. Round your answers to the nearest hundredth.

Solution.

1. We find $V(0) = 25(0.8)^0 = 25 \cdot 1 = 25$, $V(1) = 25(0.8)^1 = 25 \cdot 0.8 = 20$ and $V(2) = 25(0.8)^2 = 25 \cdot 0.64 = 16$. Since t represents the number of years the car has been owned, $t = 0$ corresponds to the purchase price of the car. Since $V(t)$ returns the value of the car in *thousands* of dollars, $V(0) = 25$ means the car is worth \$25,000 when first purchased. Likewise, $V(1) = 20$ and $V(2) = 16$ means the car is worth \$20,000 after one year of ownership and \$16,000 after two years, respectively.

2. Recall to find the average rate of change of V over an interval $[a, b]$, we compute: $\frac{V(b)-V(a)}{b-a}$. For the interval $[0, 1]$, we find $\frac{V(1)-V(0)}{1-0} = \frac{20-25}{1} = -5$, which means over the course of the first year of ownership, the value of the car depreciated, on average, at a rate of \$5000 per year.

   For the interval $[0, 1]$, we compute $\frac{V(2)-V(0)}{2-0} = \frac{16-25}{2} = -4.5$, which means over the course of the first two years of ownership, the car lost, on average, \$4500 per year in value.

   Finally, we find for the interval $[1, 2]$, $\frac{V(2)-V(1)}{2-1} = \frac{16-20}{1} = -4$, meaning the car lost, on average, \$4000 in value per year between the first and second years.

   Notice that the car lost more value over the first year (\$5000) than it did the second year (\$4000), and these losses average out to the average yearly loss over the first two years (\$4500 per year.)[6]

[6] It turns out for any function f, the average rate of change over the interval $[x, x + 2]$ is the average of the average rates of change of f over $[x, x + 1]$ and $[x + 1, x + 2]$. See Exercise 23.

3. We compute: $\frac{V(1)}{V(0)} = \frac{20}{25} = 0.8$, $\frac{V(2)}{V(1)} = \frac{16}{20} = 0.8$, and $\frac{V(2)}{V(0)} = \frac{16}{25} = 0.64$.

   The ratio $\frac{V(1)}{V(0)} = 0.8$ can be rewritten as $V(1) = 0.8V(0)$ which means that the value of the car after 1 year, $V(1)$ is 0.8 times, or 80% the initial value of the car, $V(0)$.

   Similarly, the ratio $\frac{V(2)}{V(1)} = 0.8$ rewritten as $V(2) = 0.8V(1)$ means the value of the car after 2 years, $V(2)$ is 0.8 times, or 80% the value of the car after one year, $V(1)$.

   Finally, the ratio $\frac{V(2)}{V(0)} = 0.64$, or $V(2) = 0.64V(0)$ means the value of the car after 2 years, $V(2)$ is 0.64 times, or 64% of the initial value of the car, $V(0)$.

   Note that this last result tracks with the previous answers. Since $V(1) = 0.8V(0)$ and $V(2) = 0.8V(1)$, we get $V(2) = 0.8V(1) = 0.8(0.8V(0)) = 0.64V(0)$. Also note it is no coincidence that the base of the exponential, 0.8 has shown up in these calculations, as we'll see in the next problem.

4. Using properties of exponents, we find

$$\frac{V(t+1)}{V(t)} = \frac{25(0.8)^{t+1}}{25(0.8)^{t}} = (0.8)^{t+1-t} = 0.8$$

   Rewriting, we have $V(t+1) = 0.8V(t)$. This means after one year, the value of the car $V(t+1)$ is only 80% of the value it was a year ago, $V(t)$.

   Similarly, we find

$$\frac{V(t+k)}{V(t)} = \frac{25(0.8)^{t+k}}{25(0.8)^{t}} = (0.8)^{t+k-t} = (0.8)^{k}$$

   which, rewritten, says $V(t+k) = V(t)(0.8)^k$. This means in k years' time, the value of the car $V(t+k)$ is only $(0.8)^k$ times what it was worth k years ago, $V(t)$.

   These results shouldn't be too surprising. Verbally, the function $V(t) = 25(0.8)^t$ says to multiply 25 by 0.8 multiplied by itself t times. Therefore, for each additional year, we are multiplying the value of the car by an additional factor of 0.8.

5. We compute $\frac{V(1)-V(0)}{V(0)} = \frac{20-25}{25} = -0.2$, $\frac{V(2)-V(1)}{V(1)} = \frac{16-20}{20} = -0.2$, and $\frac{V(2)-V(0)}{V(0)} = \frac{16-25}{25} = -0.36$.

   The ratio $\frac{V(1)-V(0)}{V(0)}$ computes the ratio of *difference* in the value of the car after the first year of ownership, $V(1) - V(0)$, to the initial value, $V(0)$. We find this to be -0.2 or a 20% decrease in value. This makes sense since we know from our answer to number 3, the value of the car after 1 year, $V(1)$ is 80% of the initial value, $V(0)$. Indeed:

$$\frac{V(1) - V(0)}{V(0)} = \frac{V(1)}{V(0)} - \frac{V(0)}{V(0)} = \frac{V(1)}{V(0)} - 1,$$

   and since $\frac{V(1)}{V(0)} = 0.8$, we get $\frac{V(1)-V(0)}{V(0)} = 1 - 0.8 = -0.2$.

   Likewise, the ratio $\frac{V(2)-V(1)}{V(1)} = -0.2$ means the value of the car has lost 20% of its value over the course of the second year of ownership.

Finally, the ratio $\frac{V(2)-V(0)}{V(0)} = -0.36$ means that over the first two years of ownership, the car value has depreciated 36% of its initial purchase price. Again, this tracks with the result of number 3 which tells us that after two years, the car is only worth 64% of its initial purchase price.

6. Using properties of fractions and exponents, we get:

$$\frac{V(t+1)-V(t)}{V(t)} = \frac{25(0.8)^{t+1}-25(0.8)^t}{25(0.8)^t} = \frac{25(0.8)^{t+1}}{25(0.8)^t} - \frac{25(0.8)^t}{25(0.8)^t} = 0.8 - 1 = -0.2,$$

so after one year, the value of the car $V(t+1)$ has lost 20% of the value it was a year ago, $V(t)$.

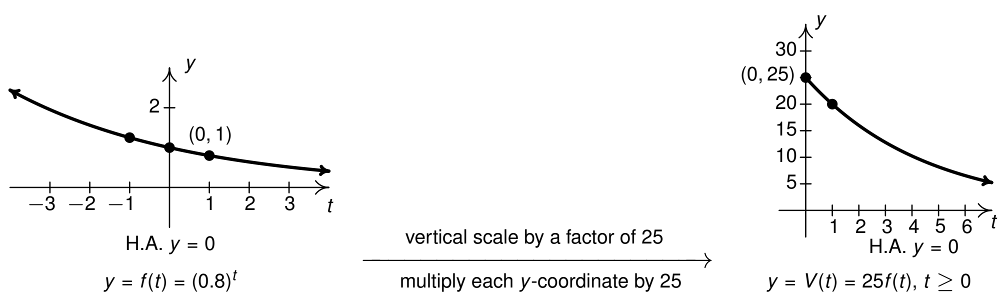
Similarly, we find:

$$\frac{V(t+k)-V(t)}{V(t)} = \frac{25(0.8)^{t+k}-25(0.8)^t}{25(0.8)^t} = \frac{25(0.8)^{t+1}}{25(0.8)^t} - \frac{25(0.8)^t}{25(0.8)^t} = (0.8)^k - 1,$$

so after k years' time, the value of the car $V(t)$ has decreased by $((0.8)^k - 1) \cdot 100\%$ of the value k years ago, $V(t)$.

7. To graph $y = 25(0.8)^t$, we start with the basic exponential function $f(t) = (0.8)^t$. Since the base $b = 0.8$ satisfies $0 < b < 1$, the graph of $y = f(t)$ is decreasing. We plot the y-intercept $(0,1)$ and two other points, $(-1, 1.25)$ and $(1, 0.8)$, and label the horizontal asymptote $y = 0$.

To obtain the graph of $y = 25(0.8)^t = 25f(t)$, we multiply all of the y values in the graph by 25 (including the y value of the horizontal asymptote) in accordance with Theorem 2.5 to obtain the points $(-1, 31.25)$, $(0, 25)$ and $(1, 20)$. The horizontal asymptote remains the same, since $25 \cdot 0 = 0$. Finally, we restrict the domain to $[0, \infty)$ to fit with the applied domain given to us.



8. We see from the graph of V that its horizontal asymptote is $y = 0$. This means as the car gets older, its value diminishes to 0.

9. We know the value of the car, brand new, is \$25,000, so when we are asked to find when the car depreciates to one half and one quarter of this value, we are trying to find when the value of the car dips to \$12,500 and \$6,125, respectively. Since $V(t)$ is measured in *thousands* of dollars, we this translates to solving the equations $V(t) = 12.5$ and $V(t) = 6.125$.

Since we have yet to develop any analytic means to solve equations like $25(0.8)^t = 12.5$ (since t is in the exponent here), we are forced to approximate solutions to this equation numerically[7] or use a graphing utility. Choosing the latter, we graph $y = V(t)$ along with the lines $y = 12.5$ and $y = 6.125$ and look for intersection points.

We find $y = V(t)$ and $y = 12.5$ intersect at (approximately) (3.10612.5) which means the car depreciates to half its initial value in (approximately) 3.11 years. Similarly, we find the car depreciates to one-quarter its initial value after (approximately) 6.23 years.[8]

□

Some remarks about Example 10.1.2 are in order. First the function in the previous example is called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' and we shall see several different examples of those in Section **??**.

Second, as seen in numbers 3 and 4, $V(t+1) = 0.8V(t)$. That is to say, the function V has a *constant unit multiplier*, in this case, 0.8 because to obtain the function value $V(t+1)$, we *multiply* the function value $V(t)$ by b. It is not coincidence that the multiplier here is the base of the exponential, 0.8.

Indeed, exponential functions of the form $f(x) = a \cdot b^x$ have a constant unit multiplier, b. To see this, note

$$\frac{f(x+1)}{f(x)} = \frac{a \cdot b^{x+1}}{a \cdot b^x} = b^1 = b.$$

[7]Since exponential functions are continuous we could use the Bisection Method to solve $f(t) = 25(0.8)^t - 12.5 = 0$. See the discussion on page 341 in Section 6.3 for more details.

[8]It turns out that it takes exactly twice as long for the car to depreciate to one-quarter of its initial value as it takes to depreciate to half its initial value. Can you see why?

Hence $f(x+1) = f(x) \cdot b$. This will prove useful to us in Section **??** when making decisions about whether or not a data set represents exponential growth or decay.

More generally, one can show (see Exercise 24) for any real number x_0 that $f(x_0 + \Delta x) = f(x_0) b^{\Delta x}$. That is, to obtain $f(x_0 + \Delta x)$ from $f(x_0)$, we *multiply* by Δx factors of the constant unit multiplier, b. This is at the heart of what it means to be an exponential function.

If this discussion seems familiar, it should. For linear functions, $f(x) = mx + b$, we can obtain the slope m by computing $f(x+1) - f(x)$. To see this, note $f(x+1) - f(x) = (m(x+1) + b) - (mx + b) = m$ so that $f(x+1) = f(x) + m$. In this way, we see that the slope m is the constant unit *addend* in that in order to obtain $f(x+1)$, we *add* m to the function value $f(x)$.

This notion is solidified in the point-slope form of a linear function, Equation 3.6. For for any real numbers x and x_0, we have $f(x) = f(x_0) + m(x - x_0)$. If we let $x = x_0 + \Delta x$, we get $f(x_0 + \Delta x) = f(x_0) + m\Delta x$. In other words, to obtain $f(x_0 + \Delta x)$ from $f(x_0)$, we *add* m times Δx.

Taking inspiration from linear functions, we define the 'point-base' form of an exponential function below.

Definition 10.2. The **point-base form** of the exponential function $f(x) = a \cdot b^x$ is

$$f(x) = f(x_0) b^{x - x_0}$$

Just as the point-slope form of a linear function is helpful in building linear models, the point-base form of an exponential function will prove useful in building exponential models.

Next, while we saw in Example 10.1.2 number 2, exponential functions, unlike linear functions, do not have a constant rate of change. However, in numbers 5 and 6, we see that in some cases, they do have a constant *relative* rate of change. We define this notion below.

Definition 10.3. Let f be a function defined on the interval $[a, b]$ where $f(a) \neq 0$.
The **relative rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta[f(x)]}{f(a)} = \frac{f(b) - f(a)}{f(a)}.$$

For exponential functions of the form $f(x) = a \cdot b^x$, we compute the relative rate of change over the interval $[x, x+1]$ and find it is constant:

$$\frac{f(x+1) - f(x)}{f(x)} = \frac{f(x+1)}{f(x)} - \frac{f(x)}{f(x)} = b - 1,$$

where we are using the fact that $\frac{f(x+1)}{f(x)} = b$.

One way to interpret this result is when comparing $f(x)$ to $f(x+1)$, the exponential function grows (if $b > 1$) or decays (if $b < 1$) by $(b - 1) \cdot 100\%$. In our example, $V(t) = 25(0.8)^t$ so $b = 0.8$ and, as we saw, the relative rate of change from $V(t)$ to $V(t+1)$ was $0.8 - 1 = -0.2$, meaning the value of the car over the course of one year depreciates by 20%.

We close this section with another important application of exponential functions, Newton's Law of Cooling.

Example 10.1.3. According to Newton's Law of Cooling[9] the temperature of coffee $T(t)$ (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$.

1. Find and interpret $T(0)$.
2. Sketch the graph of $y = T(t)$ using transformations.
3. Find and interpret the behavior of $T(t)$ as $t \to \infty$.

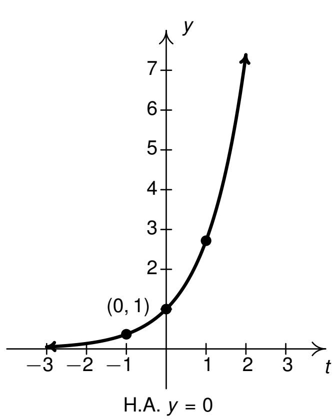
Solution.

1. Since $T(0) = 70 + 90e^{-0.1(0)} = 160$, the temperature of the coffee when it is served is 160°F.
2. To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As in Example 10.1.1, we track the points $(-1, e^{-1}) \approx (-1, 0.368)$, $(0, 1)$, and $(1, e) \approx (1, 2.718)$, along wth the horizontal asymptote $y = 0$ through each of transformations.

   To use Theorem 2.7, we rewrite $T(t) = 70 + 90e^{-0.1t} = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$. Following Theorem 2.7, we first divide the t-coordinates of each point on the graph of $y = f(t)$ by -0.1 which results in a horizontal expansion by a factor of 10 as well as a reflection about the y-axis.

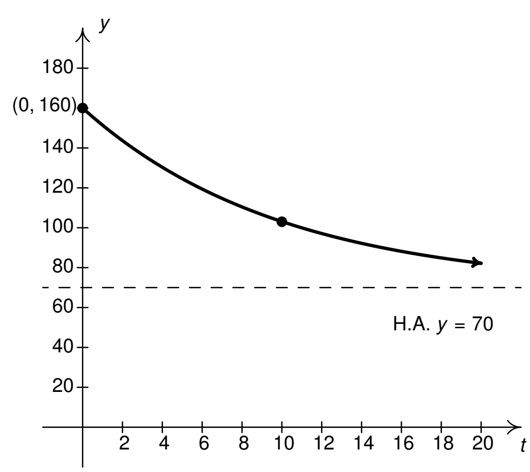
   Next, we multiply the y-values of the points on this new graph by 90 which effects a vertical stretch by a factor of 90. Last but not least, we add 70 to all of the y-coordinates of the points on this second graph, which shifts the graph upwards 70 units.

   Tracking points, we have $(-1, e^{-1}) \to (10, e^{-1}) \to (10, 90e^{-1}) \to (10, 90e^{-1} + 70) \approx (10, 103.112)$, $(0, 1) \to (0, 1) \to (0, 90) \to (0, 160)$, and $(1, e) \to (-10, e) \to (-10, 90e) \to (-10, 90e + 70) \approx (-10, 314.62)$. The horizontal asymptote $y = 0$ is unaffected by the horizontal expansion, reflection about the y-axis, and the vertical stretch. The vertical shift moves the horizontal asymptote up 70 units, $y = 0 \to y = 70$. After restricting the domain to $t \geq 0$, we get the graph below on the right.



$y = f(t) = e^t$

$\xrightarrow{\text{Theorem 2.7}}$



$y = T(t)$

[9]We will discuss this in greater detail in Section **??**.

3. We can determine the behavior of $T(t)$ as $t \to \infty$ two ways. First, we can employ the 'number sense' developed in Chapter 7.

   That is, as $t \to \infty$, We get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big }(-)}$. Since $e > 1$, $e^{\text{very big }(-)} \approx$ very small (+) The larger t becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx$ very small (+). Hence, $T(t) = 70 + 90e^{-0.1t} \approx 70 +$ very small (+) ≈ 70.

   Alternatively, we can look to the graph of $y = T(t)$. We know the horizontal asymptote is $y = 70$ which means as $t \to \infty$, $T(t) \approx 70$.

   In either case, we find that as time goes by, the temperature of the coffee is cooling to 70° Fahrenheit, ostensibly room temperature. $\square$

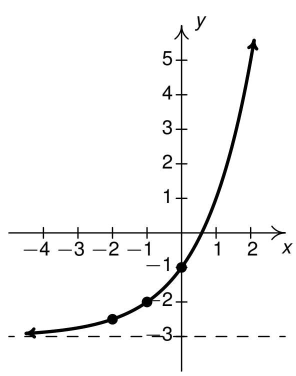
10.1.1 Exercises

In Exercises 1 - 8, sketch the graph of g by starting with the graph of f and using transformations. Track at least three points of your choice and the horizontal asymptote through the transformations. State the domain and range of g.

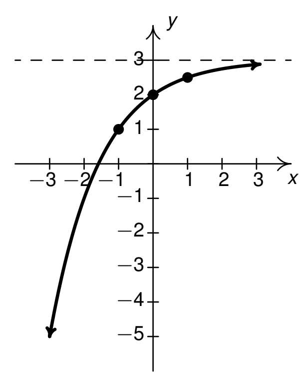
1. $f(x) = 2^x$, $g(x) = 2^x - 1$
2. $f(x) = \left(\frac{1}{3}\right)^x$, $g(x) = \left(\frac{1}{3}\right)^{x-1}$
3. $f(x) = 3^x$, $g(x) = 3^{-x} + 2$
4. $f(x) = 10^x$, $g(x) = 10^{\frac{x+1}{2}} - 20$
5. $f(t) = (0.5)^t$, $g(t) = 100(0.5)^{0.1t}$
6. $f(t) = (1.25)^t$, $g(t) = 1 - (1.25)^{t-2}$
7. $f(x) = e^t$, $g(x) = 8 - e^{-t}$
8. $f(x) = e^t$, $g(x) = 10e^{-0.1t}$

In Exercises, 9 - 12, the graph of an exponential function is given. Find a formula for the function in the form $F(x) = a \cdot 2^{bx-h} + k$.

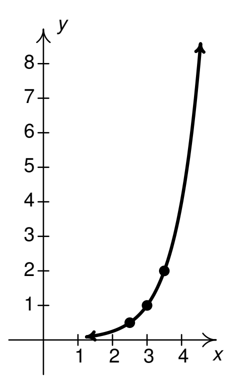
9. Points: $\left(-2, -\frac{5}{2}\right)$, $(-1, -2)$, $(0, -1)$,
   Asymptote: $y = -3$.



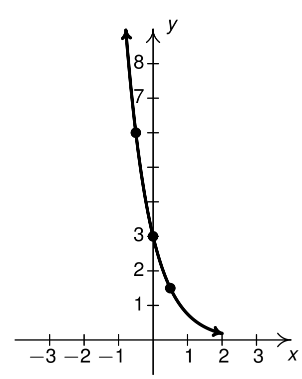
10. Points: $(-1, 1)$, $(0, 2)$, $\left(1, \frac{5}{2}\right)$,
    Asymptote: $y = 3$.



11. Points: $\left(\frac{5}{2}, \frac{1}{2}\right)$, $(3, 1)$, $\left(\frac{7}{2}, 2\right)$,
    Asymptote: $y = 0$.



12. Points: $\left(-\frac{1}{2}, 6\right)$, $(0, 3)$, $\left(\frac{1}{2}, \frac{3}{2}\right)$,
    Asymptote: $y = 0$.



13. Find a formula for each graph in Exercises 9 - 12 of the form $G(x) = a \cdot 4^{bx-h} + k$. Did you change your solution methodology? What is the relationship between your answers for $F(x)$ and $G(x)$ for each graph?

14. In Example 10.1.1 number 2, we obtained the solution $F(x) = -2^{x+3} + 4$ as one formula for the given graph by making a simplifying assumption that $a = -1$. This exercises explores if there are any other solutions for different choices of a.

    (a) Show $G(x) = -4 \cdot 2^{x+1} + 4$ also fits the data for the given graph, and use properties of exponents to show $G(x) = F(x)$. (Use the fact that $4 = 2^2 \ldots$)

    (b) With help from your classmates, find solutions to Example 10.1.1 number 2 using $a = -8$, $a = -16$ and $a = -\frac{1}{2}$. Show all your solutions can be rewritten as: $F(x) = -2^{x+3} + 4$.

    (c) Using properties of exponents and the fact that the range of 2^x is $(0, \infty)$, show that any function of the form $f(x) = -a \cdot 2^{bx-h} + k$ for $a > 0$ can be rewritten as $f(x) = -2^c \, 2^{bx-h} + k = -2^{bx-h+c} + k$. Relabeling, this means every function of the form $f(x) = -a \cdot 2^{bx-h} + k$ with four parameters (a, b, h, and k) can be rewritten as $f(x) = -2^{bx-H} + k$, a formula with just three parameters: b, H, and k. Conclude that *every* solution to Example 10.1.1 number 2 reduces to $F(x) = -2^{x+3} + 4$.

In Exercises 15 - 20, write the given function as a nontrivial decomposition of functions as directed.

15. For $f(x) = e^{-x} + 1$, find functions g and h so that $f = g + h$.

16. For $f(x) = e^{2x} - x$, find functions g and h so that $f = g - h$.

17. For $f(t) = t^2 e^{-t}$, find functions g and h so that $f = gh$.

18. For $r(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, find functions f and g so $r = \dfrac{f}{g}$.

19. For $k(x) = e^{-x^2}$, find functions f and g so that $k = g \circ f$.

20. For $s(x) = \sqrt{e^{2x} - 1}$, find functions f and g so $s = g \circ f$.

21. The amount of money in a savings account, $A(t)$, in dollars, t years after an initial investment is made is given by: $A(t) = 500(1.05)^t$, for $t \geq 0$.

    (a) Find and interpret $A(0)$, $A(1)$, and $A(2)$.

    (b) Find and interpret the relative rate of change of A over the intervals $[0, 1]$, $[1, 2]$, $[0, 2]$.

    (c) Find, simplify, and interpret the relative rate of change of A over the $[t, t + 1]$. Assume $t \geq 0$.

    (d) Use a graphing utility to estimate how long until the savings account is worth \$1500. Round your answer to the nearest year.

22. Based on census data,[10] the population of Lake County, Ohio, in 2010 was 230,041 and in 2015, the population was 229,437.

    (a) Show the percentage change in the population from 2010 to 2015 is approximately -0.263%.

    (b) If this percentage change remains constant, predict the population of Lake County in 2020.

    (c) Assuming this percentage change per five years remains constant, find an expression for the population $P(t)$ of Lake County where t is the number of five year intervals after 2010. (So $t = 0$ corresponds to 2010, $t = 1$ corresponds to 2015, $t = 2$ corresponds to 2020, etc.)

       HINT: Definitions 10.2 and 10.3 and ensuing discussion on that page is useful here.

    (d) Use your answer to 22c to predict the population of Lake County in the year 2017.

    (e) Let $A(t)$ represent the population of Lake County t years after 2010 where the we approximate the percentage change in population per year as $-\frac{0.263\%}{5} = -0.0526\%$. Find a formula for $A(t)$ and compare your predictions with $A(t)$ to those given by $P(t)$. In particular, what population does each model give for the year 2050? Discuss any discrepancies with your classmates.

23. Show that the average rate of change of a function over the interval $[x, x+2]$ is average of the average rates of change of the function over the intervals $[x, x+1]$ and $[x+1, x+2]$. Can the same be said for the average rate of change of the function over $[x, x+3]$ and the average of the average rates of change over $[x, x+1]$, $[x+1, x+2]$, and $[x+2, x+3]$? Generalize.

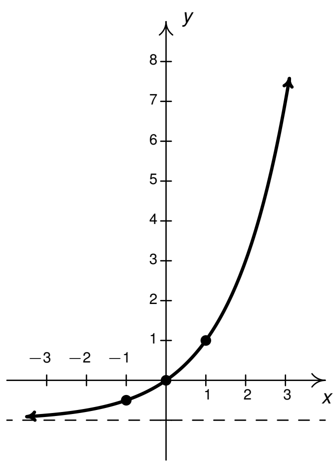
24. If $f(x) = b^x$ where $b > 0$, $b \neq 1$, show $f(x_0 + \Delta x) = f(x_0)b^{\Delta x}$.

25. Which is larger: e^{π} or π^{e}? How do you know? Can you find a proof that doesn't use technology?

[10] See here.

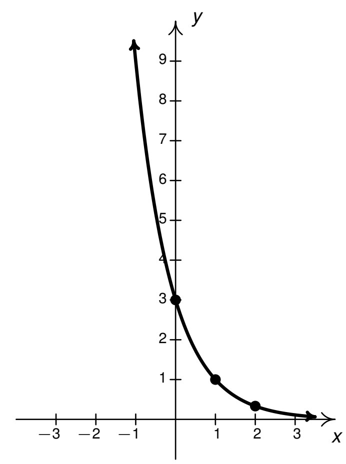
10.1.2 Answers

1. Domain of g: $(-\infty, \infty)$
   Range of g: $(-1, \infty)$
   Points: $\left(-1, -\frac{1}{2}\right)$, $(0, 0)$, $(1, 1)$
   Asymptote: $y = -1$



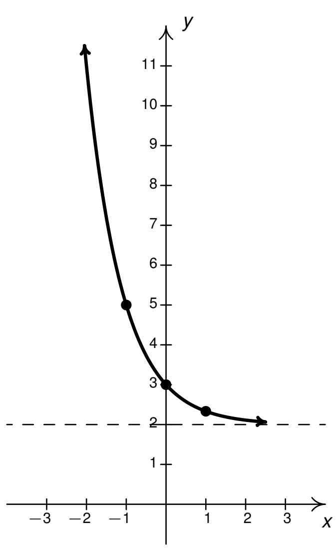
$y = g(x) = 2^x - 1$

2. Domain of g: $(-\infty, \infty)$
   Range of g: $(0, \infty)$
   Points: $(0, 3)$, $(1, 1)$, $\left(2, \frac{1}{3}\right)$
   Asymptote: $y = 0$



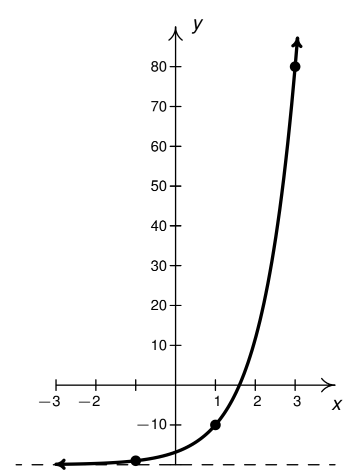
$y = g(x) = \left(\frac{1}{3}\right)^{x-1}$

3. Domain of g: $(-\infty, \infty)$
   Range of g: $(2, \infty)$
   Points: $\left(1, \frac{7}{3}\right)$, $(0, 3)$, $(-1, 5)$
   Asymptote: $y = 2$



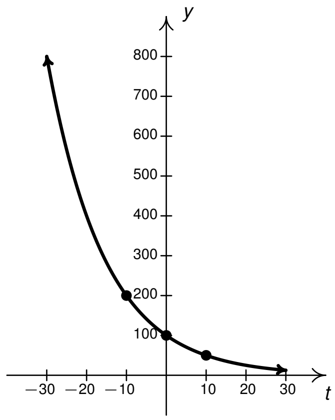
$y = g(x) = 3^{-x} + 2$

4. Domain of g: $(-\infty, \infty)$
   Range of g: $(-20, \infty)$
   Points: $(-1, -19)$, $(1, -10)$, $(3, 80)$
   Asymptote: $y = -20$



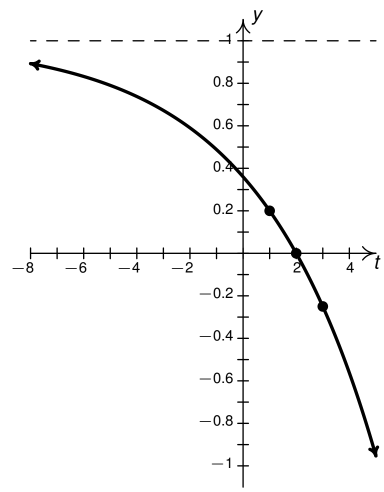
$y = g(x) = 10^{\frac{x+1}{2}} - 20$

5. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$
Points: $(-10, 200)$, $(0, 100)$, $(10, 50)$
Asymptote: $y = 0$



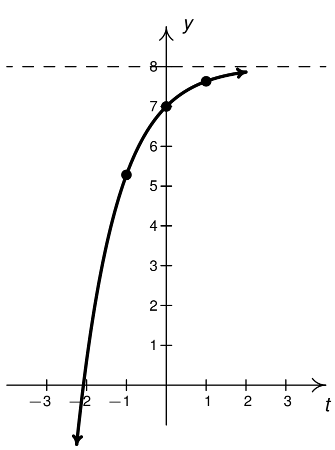
$y = g(t) = 100(0.5)^{0.1t}$

6. Domain of g: $(-\infty, \infty)$
Range of g: $(-\infty, 1)$
Points: $(1, 0.2)$, $(2, 0)$, $(3, -0.25)$
Asymptote: $y = 1$



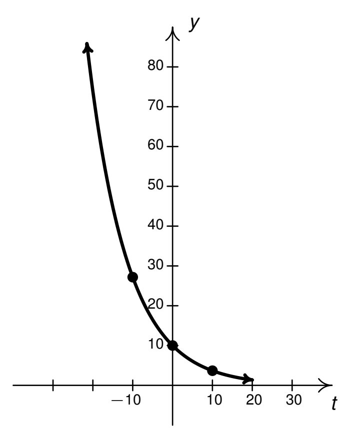
$y = g(t) = 1 - (1.25)^{t-2}$

7. Domain of g: $(-\infty, \infty)$
Range of g: $(-\infty, 8)$
Points: $\left(1, 8 - e^{-1}\right) \approx (1, 7.63)$,
$(0, 7)$, $(-1, 8 - e) \approx (1, 5.28)$
Asymptote: $y = 8$



$y = g(t) = 8 - e^{-t}$

8. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$
Points: $\left(10, 10e^{-1}\right) \approx (10, 3.68)$
$(0, 10)$, $(-10, 10e) \approx (-10, 27.18)$
Asymptote: $y = 0$



$y = g(t) = 10e^{-0.1t}$

9. $F(x) = 2^{x+1} - 3$ 10. $F(x) = -2^{-x} + 3$ 11. $F(x) = 2^{2x-6}$ 12. $F(x) = 3 \cdot 2^{-2x}$

13. Since $2 = 4^{\frac{1}{2}}$, one way to obtain the formulas for $G(x)$ is to use properties of exponents. For example, $F(x) = 2^{x+1} - 3 = \left(4^{\frac{1}{2}}\right)^{x+1} - 3 = 4^{\frac{1}{2}(x+1)} - 3 = 4^{\frac{1}{2}x+\frac{1}{2}} - 3$. In order, the formulas for $G(x)$ are:
   - $G(x) = 4^{\frac{1}{2}x+\frac{1}{2}} - 3$
   - $G(x) = -4^{-\frac{1}{2}x} + 3$
   - $G(x) = 4^{x-3}$
   - $G(x) = 3 \cdot 4^{-x}$

15. One solution is $g(x) = e^{-x}$ and $h(x) = 1$.

16. One solution is $g(x) = e^{2x}$ and $h(x) = x$.

17. One solution is $g(t) = t^2$ and $h(t) = e^{-t}$.

18. One solution is $f(x) = e^x - e^{-x}$ and $g(x) = e^x + e^{-x}$.

19. One solution is $f(x) = -x^2$ and $g(x) = e^x$.

20. One solution is $f(x) = e^{2x} - 1$ and $g(x) = \sqrt{x}$.

21. (a) $A(0) = 500$, so the initial balance in the savings account is \$500. $A(1) = 525$, so after 1 year, there is \$525 in the savings account. $A(2) = 551.25$, so after 2 years, there is \$551.25 in the savings account.
    (b) The relative rate of change of A over the intervals $[0, 1]$ and $[1, 2]$ is 0.05 which means the savings account is growing by 5% each year for those two years. Over the interval $[0, 2]$, the relative rate of change is 0.1025 meaning the account has grown by 10.25% over the course of the first two years. Note this is greater than the sum of the two rates 5% + 5% = 10%. This is due to the 'compounding effect' and will be discussed in greater detail in Section **??**.
    (c) The relative rate of change of A over the $[t, t+1]$ is 0.05. This means over the course of one year, the savings account grows by 5%.
    (d) Graphing $y = A(t)$ and $y = 1500$, we find they intersect when $t \approx 22.5$ so it takes approximately $22 - 23$ years for the savings account to grow to \$1500 in value.

22. (a) $\dfrac{229437 - 230041}{230041} \approx 0.263\%$.
    (b) Since 2020 is five years after 2015, we expect the population to decrease by 0.263% of 229437, or approximately 603 people. Hence, we approximate the population in 2020 as 228834.
    (c) $P(t) = 230041(1 - 0.00263)^t = 230041(0.99737)^t$, $t \geq 0$.
    (d) Since 2017 is 7 years after 2010, we set $t = \frac{7}{5} = 1.4$ and find $P(1.4) \approx 229194$. So the population is approximately 229, 194 in 2017.
    (e) $A(t) = 230041(1 - 0.0005626)^t = 230041(0.999474)^t$, $t \geq 0$. Since 2050 is 40 years after 2010, using the model $P(t)$, we divide $\frac{40}{5} = 8$ and find $P(8) \approx 225,245$. On the other hand, $A(40) \approx 225,250$. This is more than roundoff error. There is a compounding effect which makes the functions $A(t)$ and $P(t)$ different. [11]

[11] See number 21 above or, for more, see Section **??**.

10.2 Logarithmic Functions

In Section 10.1, we saw exponential functions $f(x) = b^x$ are are one-to-one which means they are invertible. In this section, we explore their inverses, the *logarithmic functions* which are called 'logs' for short.

Definition 10.4. For the exponential function $f(x) = b^x$, $f^{-1}(x) = \log_b(x)$ is called the **base *b* logarithm function**. We read '$\log_b(x)$' as 'log base b of x.'

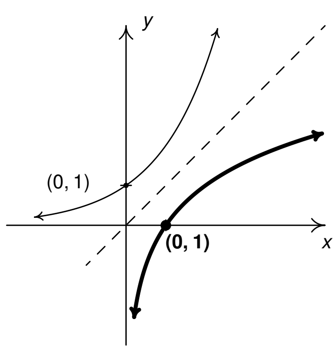
We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

Definition 10.5.

- The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written $\log(x)$.
- The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written $\ln(x)$.

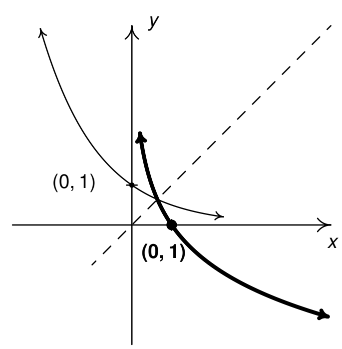
Since logs are defined as the inverses of exponential functions, we can use Theorems 9.4 and 10.1 to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function, namely $(0, \infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty, \infty)$.

Moreover, since we know the basic shapes of $y = f(x) = b^x$ for the different cases of b, we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of f across the line $y = x$. The y-intercept $(0, 1)$ on the graph of f corresponds to an x-intercept of $(1, 0)$ on the graph of f^{-1}. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs.



$y = b^x$, $b > 1$

$\boldsymbol{y = \log_b(x)}$, $b > 1$



$y = b^x$, $0 < b < 1$

$\boldsymbol{y = \log_b(x)}$, $0 < b < 1$

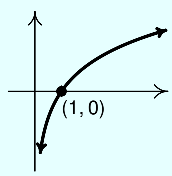
Procedurally, logarithmic functions 'undo' the exponential functions. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$.

More generally, $\log_2(x)$ is the exponent you put on 2 to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

Theorem 10.3. Properties of Logarithmic Functions: Suppose $f(x) = \log_b(x)$.

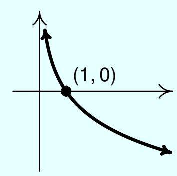
- The domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.
- $(1, 0)$ is on the graph of f and $x = 0$ is a vertical asymptote of the graph of f.
- f is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b\left(b^x\right) = x$ for all real numbers x and $b^{\log_b(x)} = x$ for all $x > 0$

- If $b > 1$:
  - f is always increasing
  - As $x \to 0^+$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = \log_b(x)$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to 0^+$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to -\infty$
  - The graph of f resembles:



$y = \log_b(x)$, $0 < b < 1$

As we have mentioned, Theorem 10.3 is a consequence of Theorems 9.4 and 10.1. However, it is worth the reader's time to understand Theorem 10.3 from an exponent perspective.

As an example, we know that the domain of $g(x) = \log_2(x)$ is $(0, \infty)$. Why? Because the range of $f(x) = 2^x$ is $(0, \infty)$. In a way, this says everything, but at the same time, it doesn't.

To really *understand* why the domain of $g(x) = \log_2(x)$ is $(0, \infty)$, consider trying to compute $\log_2(-1)$. We are searching for the exponent we put on 2 to give us -1. In other words, we are looking for x that satisfies $2^x = -1$. There is no such real number, since all powers of 2 are positive.

While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem 10.3 at this level instead of just merely memorizing the facts.

Our first example gives us practice computing logarithms as well as constructing basic graphs.

Example 10.2.1.

1. Simplify the following.

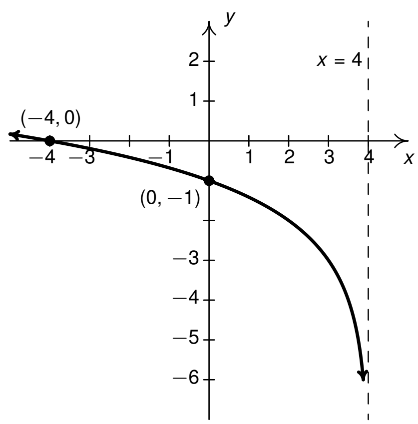
   (a) $\log_3(81)$ (b) $\log_2\left(\dfrac{1}{8}\right)$ (c) $\log_{\sqrt{5}}(25)$ (d) $\ln\left(\sqrt[3]{e^2}\right)$

   (a) $\log(0.001)$ (b) $2^{\log_2(8)}$ (c) $117^{-\log_{117}(6)}$

2. Graph the following functions by starting with a basic logarithmic function and using transformations, Theorem 2.7. Track at least three points and the vertical asymptote through the transformations.

   (a) $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ (b) $G(t) = -\ln(2-t)$

3. Find a formula for the graph of the function below. Assume the base of the logarithm is 2.



$y = F(x)$

Solution.

1. (a) The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.

   (b) To find $\log_2\left(\frac{1}{8}\right)$, we need rewrite $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.

   (c) To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = \left(\sqrt{5}\right)^2$, so we have $25 = \left(\left(\sqrt{5}\right)^2\right)^2 = \left(\sqrt{5}\right)^4$. We get $\log_{\sqrt{5}}(25) = 4$.

   (d) First, recall that the notation $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on e to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.

   (e) Rewriting $\log(0.001)$ as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.

(f) We can use Theorem 10.3 directly to simplify $2^{\log_2(8)} = 8$.

We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Since $8 = 2^3$, we have $\log_2(8) = 3$.

We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.

(g) From Theorem 10.3, we know $117^{\log_{117}(6)} = 6$,[1] but we cannot directly apply this formula to the expression $117^{-\log_{117}(6)}$ without first using a property of exponents. (Can you see why?)

Rather, we find: $117^{-\log_{117}(6)} = \frac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$.

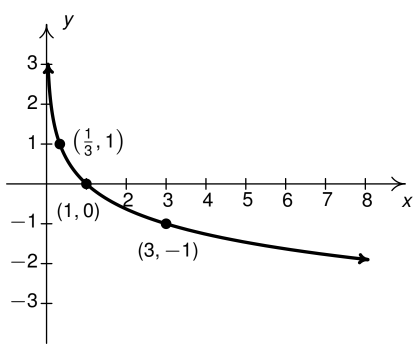
2. (a) To graph $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ we start with the graph of $f(x) = \log_{\frac{1}{3}}(x)$. and use Theorem 2.7.

First we choose some 'control points' on the graph of $f(x) = \log_{\frac{1}{3}}(x)$. Since we are instructed to track three points (and the vertical asymptote, $x = 0$) through the transformations, we choose the points corresponding to powers of $\frac{1}{3}$: $\left(\frac{1}{3}, 1\right)$, $(1, 0)$, and $(3, -1)$, respectively.

Next, we note $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1 = f\left(\frac{x}{2}\right) + 1$. Per Theorem 2.7, we first multiply the x-coordinates of the points on the graph of $y = f(x)$ by 2, horizontally expanding the graph by a factor of 2. Next, we add 1 to the y-coordinates of each point on this new graph, vertically shifting the graph up 1.

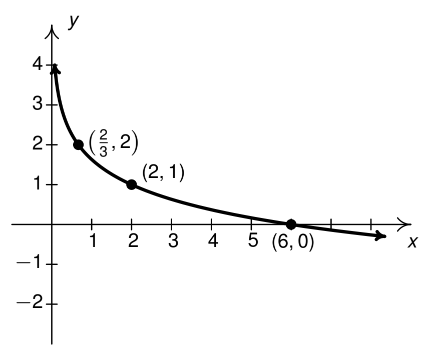
Looking at each point, we get $\left(\frac{1}{3}, 1\right) \to \left(\frac{2}{3}, 1\right) \to \left(\frac{2}{3}, 2\right)$, $(1, 0) \to (2, 0) \to (2, 1)$, and $(3, -1) \to (6, -1) \to (6, 0)$. The horizontal asymptote, $x = 0$ remains unchanged under the horizontal stretch and the vertical shift.

Below we graph $y = f(x) = \log_{\frac{1}{3}}(x)$ on the left and $y = F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ on the right.



$y = f(x) = \log_{\frac{1}{3}}(x)$

Theorem 2.7



$y = F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$

As always we can check our answer by verifying each of the points $\left(\frac{2}{3}, 2\right)$, $(2, 1)$, , and $(6, 0)$, is on the graph of $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ by checking $F\left(\frac{2}{3}\right) = 2$, $F(2) = 1$, and $F(6) = 0$. We can check the end behavior as well, that is, as $x \to 0^+$, $F(x) \to \infty$ and as $x \to \infty$, $F(x) \to -\infty$. We leave these calculations to the reader.

[1] It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117, so we get 6.

(b) Since the base of $G(t) = -\ln(2-t)$ is e, we start with the graph of $g(t) = \ln(t)$. As usual, since e is an irrational number, we use the approximation $e \approx 2.718$ when plotting points, but label points using exact coordinates in terms of e.

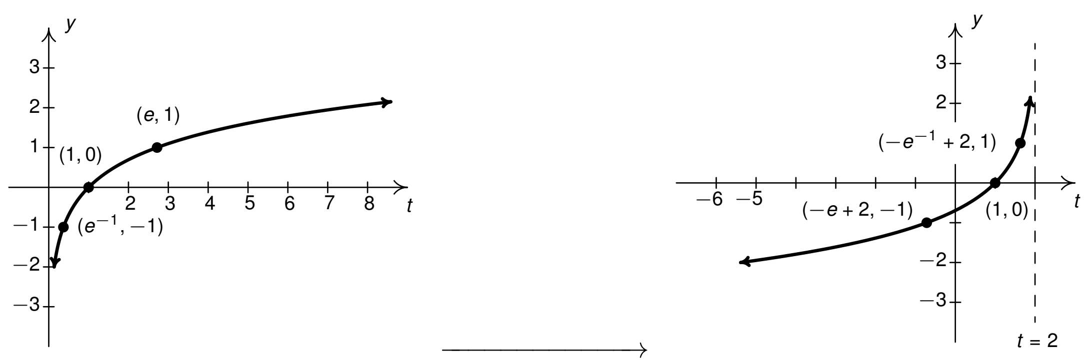
We choose points corresponding to powers of e on the graph of $g(t) = \ln(t)$: $(e^{-1}, -1) \approx (0.368, -1)$, $(1, 0)$, and $(e, 1) \approx (2.718, 1)$, respectively.

Since $G(t) = -\ln(2-t) = -\ln(-t+2) = -g(-t+2)$, Theorem 2.7 instructs us to first subtract 2 from each of the t-coordinates of the points on the graph of $g(t) = \ln(t)$, shifting the graph to the left two units.

Next, we multiply (divide) the t-coordinates of points on this new graph by -1 which reflects the graph across the y-axis. Lastly, we multiply each of the y-coordinates of this second graph by -1, reflecting it across the t-axis.

Tracking points, we have $(e^{-1}, -1) \to (e^{-1}-2, -1) \to (-e^{-1}+2, -1) \to (-e^{-1}+2, 1) \approx (1.632, 1)$, $(1, 0) \to (-1, 0) \to (1, 0) \to (1, 0)$, and $(e, 1) \to (e-2, 1) \to (-e+2, 1) \to (-e+2, -1) \approx (-0.718, -1)$. The vertical asymptote is affected by the horizontal shift and the reflection about the y-axis only: $t = 0 \to t = -2 \to t = 2$.

We graph $g(t) = \ln(t)$ below on the left and the transformed function $G(t) = -\ln(-t+2)$ below on the right. As usual, we can check our answer by verifying the indicated points do, in fact, lie on the graph of $y = G(t)$ along with checking the behavior as $t \to -\infty$ and $t \to 2^-$.



$y = g(t) = \ln(t)$ $\xrightarrow{\text{Theorem 2.7}}$ $y = G(t) = -\ln(-t+2)$

3. Since we are told to assume the base of the exponential function is 2, we assume the function $F(x)$ is the result of the transforming the graph of $f(x) = \log_2(x)$ using Theorem 2.7. This means we are tasked with finding values for a, b, h, and k so that $F(x) = af(bx-h)+k = a\log_2(bx-h)+k$.

Since the vertical asymptote to the graph of $y = f(x) = \log_2(x)$ is $x = 0$ and the vertical asymptote to the graph $y = F(x)$ is $x = 4$, we know we have a vertical shift of 4 units. Moreover, since the curve approaches the vertical asymptote from the *left*, we also know we have a reflection about the y-axis, so $b < 0$. Since the recipe in Theorem 2.7 instructs us to perform the vertical shift *before* the reflection across the y-axis, we take $h = -4$ and assume for simplicity $b = -1$ so $F(x) = a\log_2(-x+4)+k$.

To determine a and k, we make use of the two points on the graph. Since $(-4, 0)$ is on the graph of F, $F(-4) = a\log_2(-(-4)+4)+k = 0$. This reduces to $a\log_2(8)+k = 0$ or $3a+k = 0$. Next, we use the point $(0, -1)$ to get $F(0) = a\log_2(-(0)+4)+k = -1$. This reduces to $a\log_2(4)+k = -1$ or $2a+k = -1$. From $3a+k = 0$, we get $k = -3a$ which when substituted into $2a+k = -1$ gives $2a+(-3a) = -1$ or $a = 1$. Hence, $k = -3a = -3(1) = -3$.

Putting all of this work together we find $F(x) = \log_2(-x+4)-3$. As always, we can check our answer by verifying $F(-4) = 0$, $F(0) = -1$, $F(x) \to \infty$ as $x \to -\infty$ and $F(x) \to -\infty$ as $x \to 4^-$. We leave these details to the reader.[2]

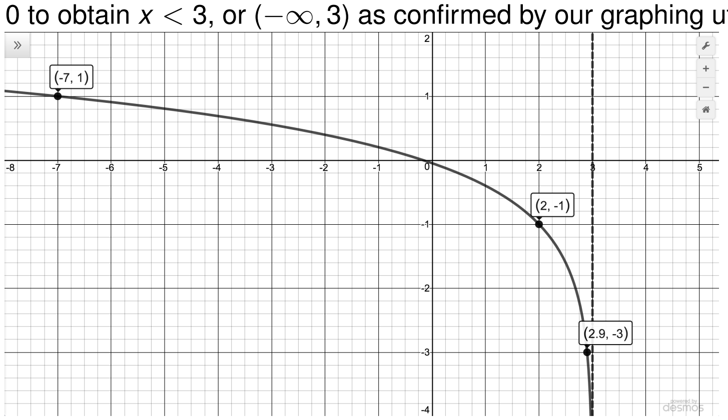
Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even indexed radicals. With the introduction of logs, we now have another restriction. Since the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the argument of the log[3] must be strictly positive.

Example 10.2.2. Find the domain each function analytically and check your answer using a graphing utility.

1. $f(x) = 2\log(3-x)-1$
2. $g(x) = \ln\left(\dfrac{x}{x-1}\right)$

Solution.

1. We set $3-x > 0$ to obtain $x < 3$, or $(-\infty, 3)$ as confirmed by our graphing utility below.



Note that in this case, we can graph f using transformations, which we do so here for extra practice.

Taking a cue from Theorem 2.7, we rewrite $f(x) = 2\log_{10}(-x+3)-1$ and view this function as a transformed version of $h(x) = \log_{10}(x)$.

To graph $y = \log(x) = \log_{10}(x)$, We select three points to track corresponding to powers of 10: $(0.1, -1)$, $(1, 0)$ and $(10, 1)$, along with the vertical asymptote $x = 0$.

[2]As with Exercise 10.1.1 in Section 10.1, we may well wonder if our solution to this problem is the *only* solution since we made a simplifying assumption that $b = -1$. We leave this for a thoughtful discussion in Exercise 40 in Section 10.3.

[3] that is, what's 'inside' the log

Since $f(x) = 2h(-x+3) - 1$, Theorem 2.7 tells us that to obtain the destinations of these points, we first subtract 3 from the x-coordinates (shifting the graph left 3 units), then divide (multiply) by the x-coordinates by -1 (causing a reflection across the y-axis).

Next, we multiply the y-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the y-coordinates which shifts the graph down 1 unit.

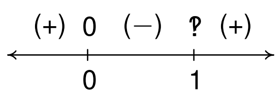
Tracking points, we find: $(0.1, -1) \to (-2.9, -1) \to (2.9, -1) \to (2.9, -2) \to (2.9, -3)$, $(1, 0) \to (-2, 0) \to (2, 0) \to (2, 0) \to (2, -1)$, and $(10, 1) \to (7, 1) \to (-7, 1) \to (-7, 2) \to (-7, 1)$. The vertical shift and reflection about the y-axis affects the vertical asymptote: $x = 0 \to x = -3 \to x = 3$.

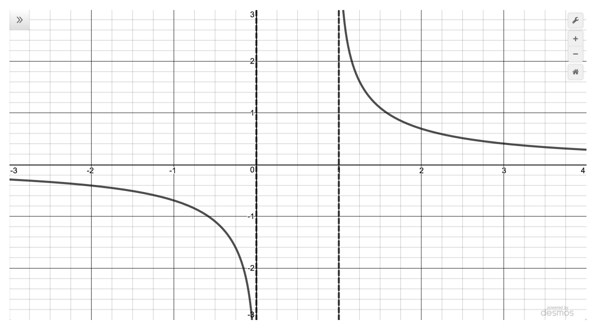
Plotting these three points along with the vertical asymptote produces the graph of f as seen above.

2. To find the domain of g, we need to solve the inequality $\frac{x}{x-1} > 0$ using a sign diagram.[4]

   If we define $r(x) = \frac{x}{x-1}$, we find r is undefined at $x = 1$ and $r(x) = 0$ when $x = 0$. Choosing some test values, we generate the sign diagram below on the left.

   We find $\frac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ which is the domain of g. The graph below confirms this.





   We can tell from the graph of g that it is not the result of Section 2.3 transformations being applied to the graph $y = \ln(x)$, (do you see why?) so barring a more detailed analysis using Calculus, producing a graph using a graphing utility is the best we can do.

   One thing worthy of note, however, is the end behavior of g. The graph suggests that as $x \to \pm\infty$, $g(x) \to 0$. We can verify this analytically. Using results from Chapter 7 and continuity, we know that as $x \to \pm\infty$, $\frac{x}{x-1} \approx 1$. Hence, it makes sense that $g(x) = \ln\left(\frac{x}{x-1}\right) \approx \ln(1) = 0$. $\square$

While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example reviews not only the major topics of this section, but reviews the salient points from Section 9.4.

[4] See Section **??** for a review of this process, if needed.

Example 10.2.3. Let $f(x) = 2^{x-1} - 3$.

1. Graph f using transformations and state the domain and range of f.
2. Explain why f is invertible and find a formula for $f^{-1}(x)$.
3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
5. Graph f and f^{-1} on the same set of axes and check for symmetry about the line $y = x$.
6. Use f or f^{-1} to solve the following equations. Check your answers algebraically.

   (a) $2^{x-1} - 3 = 4$ (b) $\log_2(t+3) + 1 = 0$

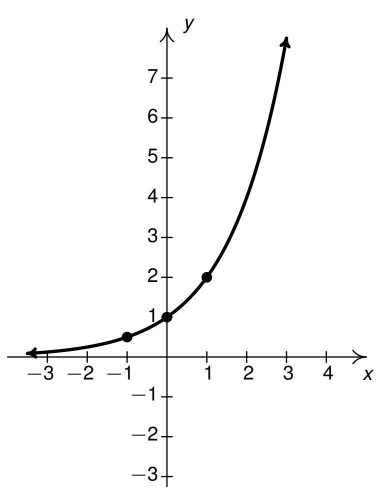
Solution.

1. To graph $f(x) = 2^{x-1} - 3$ using Theorem 2.7, we first identify $g(x) = 2^x$ and note $f(x) = g(x-1) - 3$. Choosing the 'control points' of $\left(-1, \frac{1}{2}\right)$, $(0,1)$ and $(1,2)$ on the graph of g along with the horizontal asymptote $y = 0$, we implement the algorithm set forth in Theorem 2.7.

   First, we first add 1 to the x-coordinates of the points on the graph of g which shifts the the graph of g to the right one unit. Next, we subtract 3 from each of the y-coordinates on this new graph, shifting the graph down 3 units to get the graph of f.

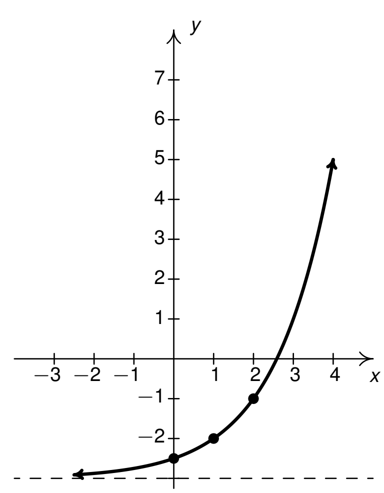
   Looking point-by-point, we have $\left(-1, \frac{1}{2}\right) \to \left(0, \frac{1}{2}\right) \to \left(0, -\frac{5}{2}\right)$, $(0,1) \to (1,1) \to (1,-2)$, and, finally, $(1,2) \to (2,2) \to (2,-1)$. The horizontal asymptote is affected only by the vertical shift, $y = 0 \to y = -3$.

   From the graph of f, we get the domain is $(-\infty, \infty)$ and the range is $(-3, \infty)$.



$y = h(x) = 2^x$

$\longrightarrow$



$y = f(x) = 2^{x-1} - 3$

2. The graph of f passes the Horizontal Line Test so f is one-to-one, hence invertible.

   To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the x and y, then proceed to solve for y. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for f^{-1} procedurally.

   Thinking of f as a process, the formula $f(x) = 2^{x-1} - 3$ takes an input x and applies the steps: first subtract 1. Second put the result of the first step as the exponent on 2. Last, subtract 3 from the result of the second step.

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm.

   By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should: first, add 3. Second, take the logarithm base 2 of the result of the first step. Lastly, add 1 to the result of the second ste. In symbols, $f^{-1}(x) = \log_2(x+3) + 1$.

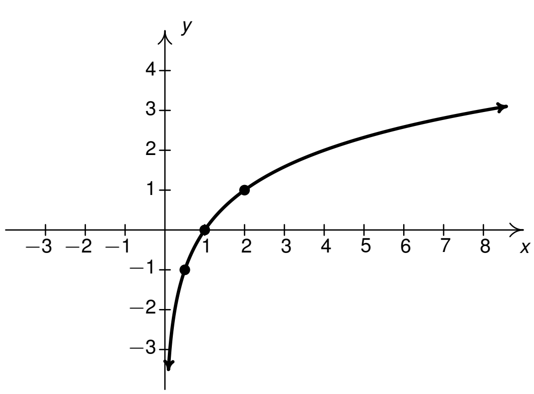
3. To graph $f^{-1}(x) = \log_2(x+3) + 1$ using Theorem 2.7, we start with $j(x) = \log_2(x)$ and track the points $\left(\frac{1}{2}, -1\right)$, $(1, 0)$ and $(2, 1)$ on the graph of j along with the vertical asymptote $x = 0$ through the transformations.

   Since $f^{-1}(x) = j(x+3) + 1$, we first subtract 3 from each of the x-coordinates of each of the points on the graph of $y = j(x)$ shifting the graph of j to the left three units. We then add 1 to each of the y-coordinates of the points on this new graph, shifting the graph up one unit.

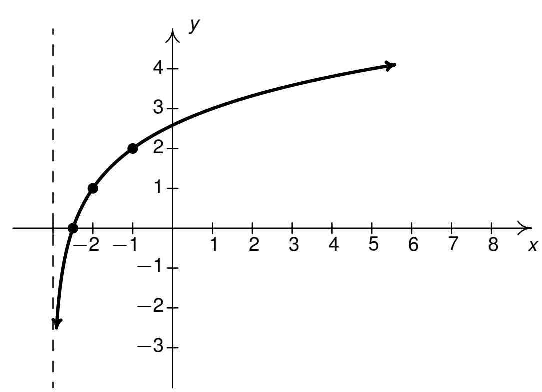
   Tracking points, we get $\left(\frac{1}{2}, -1\right) \to \left(-\frac{5}{2}, -1\right) \to \left(-\frac{5}{2}, 0\right)$, $(1, 0) \to (-2, 1) \to (-2, 2)$, and $(2, 1) \to (-1, 1) \to (-1, 2)$.

   The vertical asymptote is only affected by the horizontal shift, so we have $x = 0 \to x = -3$.

   From the graph below, we get the domain of f^{-1} is $(-3, \infty)$, which matches the range of f, and the range of f^{-1} is $(-\infty, \infty)$, which matches the domain of f, in accordance with Theorem 9.4.



$y = j(x) = \log_2(x)$



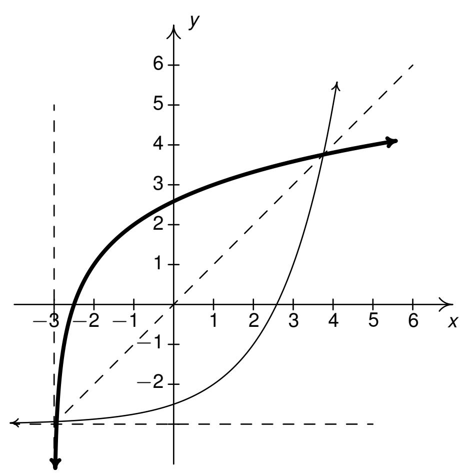
$y = f^{-1}(x) = \log_2(x+3) + 1$

4. We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x+3)+1$ satisfy the composition requirement for inverses. When simplifying $(f^{-1} \circ f)(x)$ we assume x can be any real number while when simplifying $(f \circ f^{-1})(x)$, we restrict our attention to $x > -3$. (Do you see why?)

   Note the use of the inverse properties of exponential and logarithmic functions from Theorem 10.3 when it comes to simplifying expressions of the form $\log_2(2^u)$ and $2^{\log_2(u)}$.

$$\begin{aligned} \left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\ &= f^{-1}\left(2^{x-1} - 3\right) \\ &= \log_2\left(\left[2^{x-1} - 3\right] + 3\right) + 1 \\ &= \log_2\left(2^{x-1}\right) + 1 \\ &= (x-1) + 1 \\ &= x \ \checkmark \end{aligned} \qquad \begin{aligned} \left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\ &= f\left(\log_2(x+3) + 1\right) \\ &= 2^{\left(\log_2(x+3)+1\right)-1} - 3 \\ &= 2^{\log_2(x+3)} - 3 \\ &= (x+3) - 3 \\ &= x \ \checkmark \end{aligned}$$

5. Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and observe the symmetry about the line $y = x$.



$y = f(x) = 2^{x-1} - 3$

$\boldsymbol{y = f^{-1}(x) = \log_2(x+3) + 1}$

1. Viewing $2^{x-1} - 3 = 4$ as $f(x) = 4$, we apply f^{-1} to 'undo' f to get $f^{-1}(f(x)) = f^{-1}(4)$, which reduces to $x = f^{-1}(4)$. Since we have shown (algebraically and graphically!) that $f^{-1}(x) = \log_2(x+3)+1$, we get $x = f^{-1}(4) = \log_2(4+3) + 1 = \log_2(7) + 1$.

   Alternatively, we know from Theorem 9.4 that $f(x) = 4$ is equivalent to $x = f^{-1}(4)$ directly.

   Note that since, by definition, $2^{\log_2(7)} = 7$, $2^{(\log_2(7)+1)-1} - 3 = 2^{\log_2(7)} - 3 = 7 - 3 = 4$, as required.

2. Since we may think of the equation $\log_2(t+3) + 1 = 0$ as $f^{-1}(t) = 0$, we can solve this equation by applying f to both sides to get $f(f^{-1}(t)) = f(0)$ or $t = 2^{0-1} - 3 = \frac{1}{2} - 3 = -\frac{5}{2}$.

   Since $\log_2(2^{-1}) = -1$, we get $\log_2\left(-\frac{5}{2} + 3\right) + 1 = \log_2\left(\frac{1}{2}\right) + 1 = \log_2(2^{-1}) - 1 + 1 = 0$, as required. $\square$

10.2.1 Exercises

In Exercises 1 - 15, use the property: $b^a = c$ if and only if $\log_b(c) = a$ from Theorem 10.3 to rewrite the given equation in the other form. That is, rewrite the exponential equations as logarithmic equations and rewrite the logarithmic equations as exponential equations.

1. $2^3 = 8$
2. $5^{-3} = \frac{1}{125}$
3. $4^{5/2} = 32$
4. $\left(\frac{1}{3}\right)^{-2} = 9$
5. $\left(\frac{4}{25}\right)^{-1/2} = \frac{5}{2}$
6. $10^{-3} = 0.001$
7. $e^0 = 1$
8. $\log_5(25) = 2$
9. $\log_{25}(5) = \frac{1}{2}$
10. $\log_3\left(\frac{1}{81}\right) = -4$
11. $\log_{\frac{4}{3}}\left(\frac{3}{4}\right) = -1$
12. $\log(100) = 2$
13. $\log(0.1) = -1$
14. $\ln(e) = 1$
15. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$

In Exercises 16 - 42, evaluate the expression without using a calculator.

16. $\log_3(27)$
17. $\log_6(216)$
18. $\log_2(32)$
19. $\log_6\left(\frac{1}{36}\right)$
20. $\log_8(4)$
21. $\log_{36}(216)$
22. $\log_{\frac{1}{5}}(625)$
23. $\log_{\frac{1}{6}}(216)$
24. $\log_{36}(36)$
25. $\log\left(\frac{1}{1000000}\right)$
26. $\log(0.01)$
27. $\ln\left(e^3\right)$
28. $\log_4(8)$
29. $\log_6(1)$
30. $\log_{13}\left(\sqrt{13}\right)$
31. $\log_{36}\left(\sqrt[4]{36}\right)$
32. $7^{\log_7(3)}$
33. $36^{\log_{36}(216)}$
34. $\log_{36}\left(36^{216}\right)$
35. $\ln\left(e^5\right)$
36. $\log\left(\sqrt[9]{10^{11}}\right)$
37. $\log\left(\sqrt[3]{10^5}\right)$
38. $\ln\left(\frac{1}{\sqrt{e}}\right)$
39. $\log_5\left(3^{\log_3(5)}\right)$
40. $\log\left(e^{\ln(100)}\right)$
41. $\log_2\left(3^{-\log_3(2)}\right)$
42. $\ln\left(42^{6\log(1)}\right)$

In Exercises 43 - 57, find the domain of the function.

43. $f(x) = \ln(x^2 + 1)$
44. $f(x) = \log_7(4x + 8)$
45. $g(t) = \ln(4t - 20)$
46. $g(t) = \log\left(t^2 + 9t + 18\right)$

47. $f(x) = \log\left(\dfrac{x+2}{x^2-1}\right)$

48. $f(x) = \log\left(\dfrac{x^2+9x+18}{4x-20}\right)$

49. $g(t) = \ln(7-t) + \ln(t-4)$

50. $g(t) = \ln(4t-20) + \ln\left(t^2+9t+18\right)$

51. $f(x) = \log\left(x^2+x+1\right)$

52. $f(x) = \sqrt[4]{\log_4(x)}$

53. $g(t) = \log_9(|t+3|-4)$

54. $g(t) = \ln(\sqrt{t-4}-3)$

55. $f(x) = \dfrac{1}{3-\log_5(x)}$

56. $f(x) = \dfrac{\sqrt{-1-x}}{\log_{\frac{1}{2}}(x)}$

57. $f(x) = \ln(-2x^3-x^2+13x-6)$

In Exercises 58 - 65, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the vertical asymptote through the transformations. State the domain and range of g.

58. $f(x) = \log_2(x)$, $g(x) = \log_2(x+1)$

59. $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{3}}(x)+1$

60. $f(x) = \log_3(x)$, $g(x) = -\log_3(x-2)$

61. $f(x) = \log(x)$, $g(x) = 2\log(x+20)-1$

62. $g(t) = \log_{0.5}(t)$, $g(t) = 10\log_{0.5}\left(\frac{t}{100}\right)$

63. $g(t) = \log_{1.25}(t)$, $g(t) = \log_{1.25}(-t+1)+2$

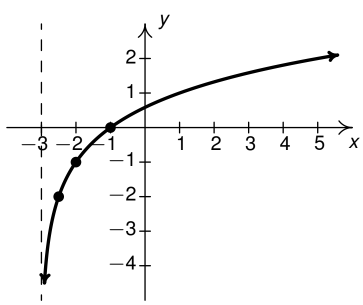
64. $g(t) = \ln(t)$, $g(t) = -\ln(8-t)$

65. $g(t) = \ln(t)$, $g(t) = -10\ln\left(\frac{t}{10}\right)$

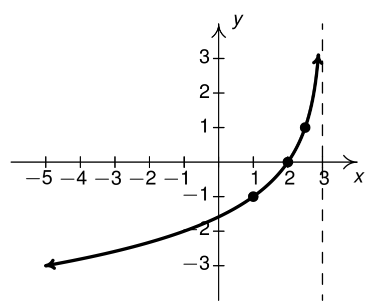
66. Verify that each function in Exercises 58 - 65 is the inverse of the corresponding function in Exercises 1 - 8 in Section 10.1. (Match up #1 and #58, and so on.)

In Exercises, 67 - 70, the graph of a logarithmic function is given. Find a formula for the function in the form $F(x) = a \cdot \log_2(bx-h)+k$.

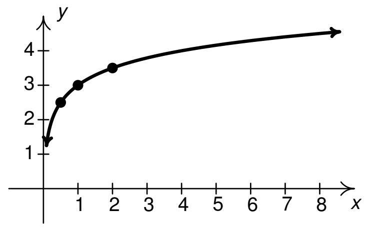
67. Points: $\left(-\frac{5}{2},-2\right)$, $(-2,-1)$, $(-1,0)$,
Asymptote: $x=-3$.



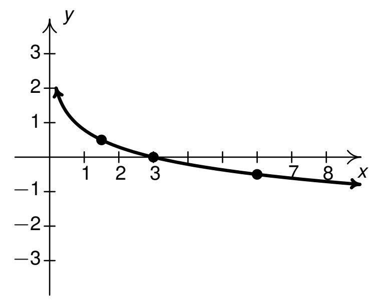
68. Points: $(1,-1)$, $(2,0)$, $\left(\frac{5}{2},1\right)$
Asymptote: $x=3$.



69. Points: $\left(\frac{1}{2}, \frac{5}{2}\right)$, $(1, 3)$, $\left(2, \frac{7}{2}\right)$,
Asymptote: $x = 0$.



70. Points: $\left(6, -\frac{1}{2}\right)$, $(3, 0)$, $\left(\frac{3}{2}, \frac{1}{2}\right)$,
Asymptote: $x = 0$.



71. Find a formula for each graph in Exercises 67 - 70 of the form $G(x) = a \cdot \log_4(bx - h) + k$.

In Exercises 72 - 75, find the inverse of the function from the 'procedural perspective' discussed in Example 10.2.3 and graph the function and its inverse on the same set of axes.

72. $f(x) = 3^{x+2} - 4$

73. $f(x) = \log_4(x - 1)$

74. $g(t) = -2^{-t} + 1$

75. $g(t) = 5\log(t) - 2$

In Exercises 76 - 81, write the given function as a nontrivial decomposition of functions as directed.

76. For $f(x) = \log_2(x + 3) + 4$, find functions g and h so that $f = g + h$.

77. For $f(x) = \log(2x) - e^{-x}$, find functions g and h so that $f = g - h$.

78. For $f(t) = 3t\log(t)$, find functions g and h so that $f = gh$.

79. For $r(x) = \dfrac{\ln(x)}{x}$, find functions f and g so $r = \dfrac{f}{g}$.

80. For $k(t) = \ln(t^2 + 1)$, find functions f and g so that $k = g \circ f$.

81. For $p(z) = (\ln(z))^2$, find functions f and g so $p = g \circ f$.

(Logarithmic Scales) In Exercises 82 - 84, we introduce three widely used measurement scales which involve common logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay attention to the subtle differences.

82. Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology[5] or the U.S. Geological Survey's Earthquake Hazards Program found here and present only a simplified version of the Richter scale. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a "magnitude 0 event",

[5]Rock-solid, perhaps?

which was chosen to be a seismograph reading of 0.001 millimeters recorded on a seismometer 100 kilometers from the earthquake's epicenter. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log\left(\frac{x}{0.001}\right)$$

where x is the seismograph reading in millimeters of the earthquake recorded 100 kilometers from the epicenter.

(a) Show that $M(0.001) = 0$.

(b) Compute $M(80,000)$.

(c) Show that an earthquake which registered 6.7 on the Richter scale had a seismograph reading ten times larger than one which measured 5.7.

(d) Find two news stories about recent earthquakes which give their magnitudes on the Richter scale. How many times larger was the seismograph reading of the earthquake with larger magnitude?

83. While the decibel scale can be used in many disciplines,[6] we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound. The Sound Intensity Level L (measured in decibels) of a sound intensity I (measured in watts per square meter) is given by

$$L(I) = 10\log\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares I to baseline: $10^{-12}\,\frac{W}{m^2}$ is the threshold of human hearing.

(a) Compute $L(10^{-6})$.

(b) Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity I is needed to produce this level?

(c) Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

84. The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\text{pH} = -\log[H^+]$ where $[H^+]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

(a) The hydrogen ion concentration of pure water is $[H^+] = 10^{-7}$. Find its pH.

(b) Find the pH of a solution with $[H^+] = 6.3 \times 10^{-13}$.

(c) The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

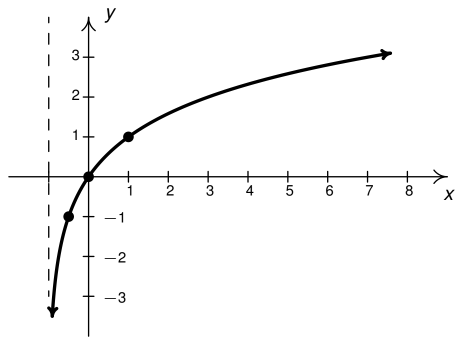
85. Use the definition of logarithm to explain why $\log_b 1 = 0$ and $\log_b b = 1$ for every $b > 0$, $b \neq 1$.

[6] See this webpage for more information.

10.2.2 Answers

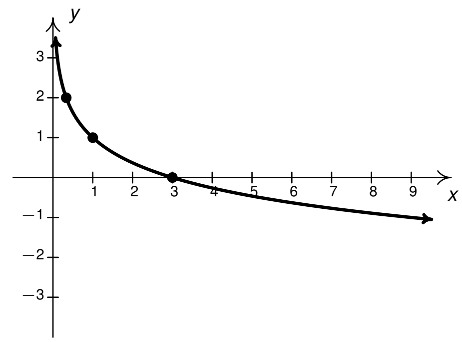
1. $\log_2(8) = 3$
2. $\log_5\left(\frac{1}{125}\right) = -3$
3. $\log_4(32) = \frac{5}{2}$
4. $\log_{\frac{1}{3}}(9) = -2$
5. $\log_{\frac{4}{25}}\left(\frac{5}{2}\right) = -\frac{1}{2}$
6. $\log(0.001) = -3$
7. $\ln(1) = 0$
8. $5^2 = 25$
9. $(25)^{\frac{1}{2}} = 5$
10. $3^{-4} = \frac{1}{81}$
11. $\left(\frac{4}{3}\right)^{-1} = \frac{3}{4}$
12. $10^2 = 100$
13. $10^{-1} = 0.1$
14. $e^1 = e$
15. $e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$
16. $\log_3(27) = 3$
17. $\log_6(216) = 3$
18. $\log_2(32) = 5$
19. $\log_6\left(\frac{1}{36}\right) = -2$
20. $\log_8(4) = \frac{2}{3}$
21. $\log_{36}(216) = \frac{3}{2}$
22. $\log_{\frac{1}{5}}(625) = -4$
23. $\log_{\frac{1}{6}}(216) = -3$
24. $\log_{36}(36) = 1$
25. $\log \frac{1}{1000000} = -6$
26. $\log(0.01) = -2$
27. $\ln\left(e^3\right) = 3$
28. $\log_4(8) = \frac{3}{2}$
29. $\log_6(1) = 0$
30. $\log_{13}\left(\sqrt{13}\right) = \frac{1}{2}$
31. $\log_{36}\left(\sqrt[4]{36}\right) = \frac{1}{4}$
32. $7^{\log_7(3)} = 3$
33. $36^{\log_{36}(216)} = 216$
34. $\log_{36}\left(36^{216}\right) = 216$
35. $\ln(e^5) = 5$
36. $\log\left(\sqrt[9]{10^{11}}\right) = \frac{11}{9}$
37. $\log\left(\sqrt[3]{10^5}\right) = \frac{5}{3}$
38. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$
39. $\log_5\left(3^{\log_3 5}\right) = 1$
40. $\log\left(e^{\ln(100)}\right) = 2$
41. $\log_2\left(3^{-\log_3(2)}\right) = -1$
42. $\ln\left(42^{6\log(1)}\right) = 0$
43. $(-\infty, \infty)$
44. $(-2, \infty)$
45. $(5, \infty)$
46. $(-\infty, -6) \cup (-3, \infty)$
47. $(-2, -1) \cup (1, \infty)$
48. $(-6, -3) \cup (5, \infty)$
49. $(4, 7)$
50. $(5, \infty)$
51. $(-\infty, \infty)$
52. $[1, \infty)$
53. $(-\infty, -7) \cup (1, \infty)$
54. $(13, \infty)$
55. $(0, 125) \cup (125, \infty)$
56. No domain
57. $(-\infty, -3) \cup \left(\frac{1}{2}, 2\right)$

58. Domain of g: $(-1, \infty)$
Range of g: $(-\infty, \infty)$
Points: $\left(-\frac{1}{2}, -1\right)$, $(0, 0)$, $(1, 1)$
Asymptote: $x = -1$



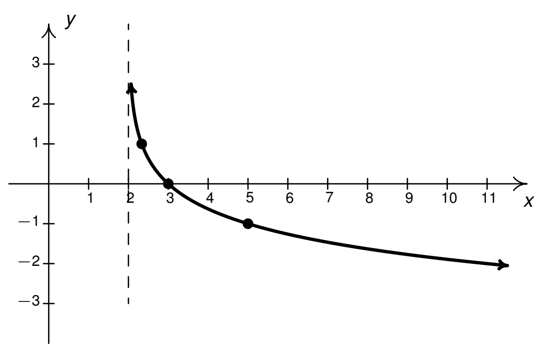
$y = g(x) = \log_2(x + 1)$

59. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$
Points: $\left(\frac{1}{3}, 2\right)$, $(1, 1)$, $(3, 0)$
Asymptote: $x = 0$



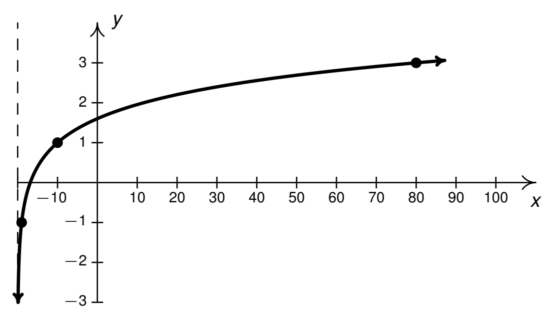
$y = g(x) = \log_{\frac{1}{3}}(x) + 1$

60. Domain of g: $(2, \infty)$
Range of g: $(-\infty, \infty)$
Points: $\left(\frac{7}{3}, 1\right)$, $(3, 0)$, $(5, -1)$
Asymptote: $x = 2$



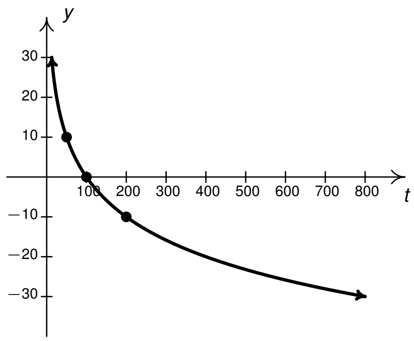
$y = g(x) = -\log_3(x - 2)$

61. Domain of g: $(-20, \infty)$
Range of g: $(-\infty, \infty)$
Points: $(-19, -1)$, $(-10, 1)$, $(80, 3)$
Asymptote: $x = -20$



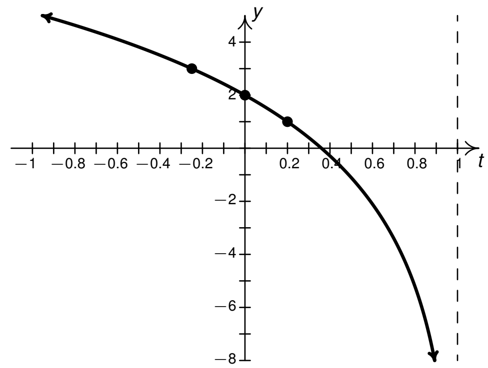
$y = g(x) = 2\log(x + 20) - 1$

62. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$
Points: $(50, 10)$, $(100, 0)$, $(200, -10)$
Asymptote: $t = 0$



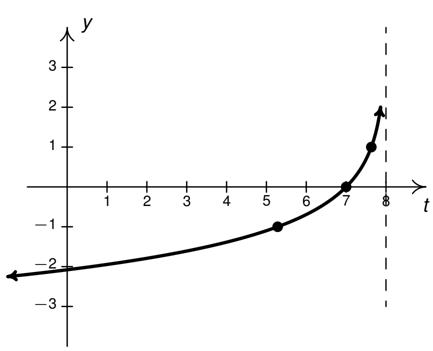
$y = g(t) = 10 \log_{0.5} \left(\frac{t}{100}\right)$

63. Domain of g: $(-\infty, 1)$
Range of g: $(-\infty, \infty)$
Points: $(-0.25, 3)$, $(0, 2)$, $(0.2, 1)$
Asymptote: $t = 1$



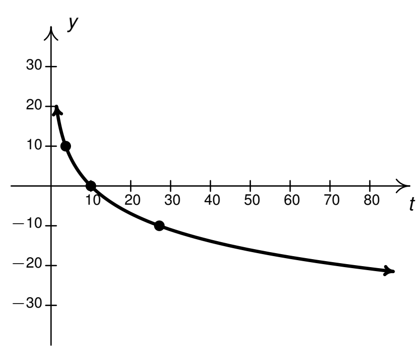
$y = g(t) = \log_{1.25}(-t + 1) + 2$

64. Domain of g: $(-\infty, 8)$
Range of g:$(-\infty, \infty)$
Points: $(8 - e, -1) \approx (5.28, -1)$,
$(7, 0)$, $(8 - e^{-1}, 1) \approx (7.63, 1)$
Asymptote: $t = 8$



$y = g(t) = -\ln(8 - t)$

65. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$
Points: $(10e^{-1}, 10) \approx (3.68.10)$
$(10, 0)$, $(10e, -10) \approx (27.18, -10)$
Asymptote: $t = 0$



$y = g(t) = -10 \ln \left(\frac{t}{10}\right)$

67. $F(x) = \log_2(x+3) - 1$

68. $F(x) = -\log_2(-x+3)$

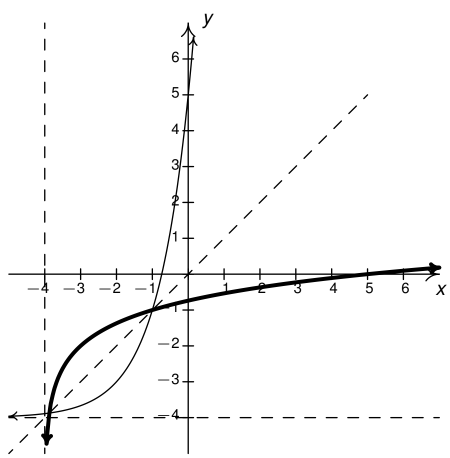
69. $F(x) = \frac{1}{2}\log_2(x) + 3$

70. $F(x) = -\frac{1}{2}\log_2\left(\frac{x}{3}\right)$

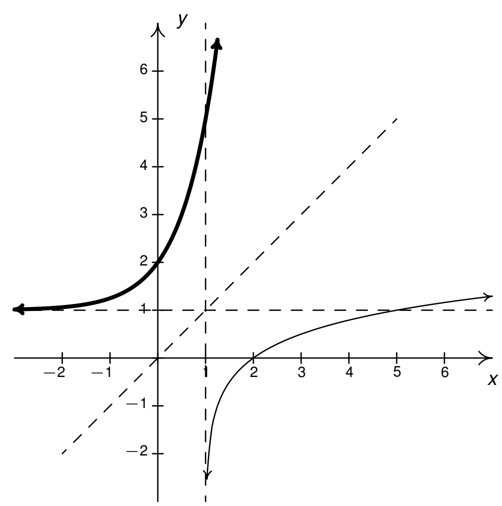
71. In order, the formulas for $G(x)$ are:

- $G(x) = 2\log_4(x+3) - 1$
- $G(x) = -2\log_4(-x+3)$
- $G(x) = \log_4(x) + 3$
- $G(x) = -\log_4\left(\frac{x}{3}\right)$

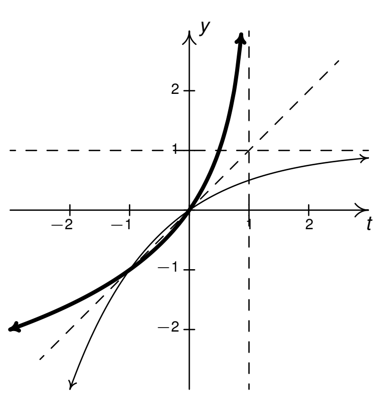
72. $y = f(x) = 3^{x+2} - 4$
$y = f^{-1}(x) = \log_3(x+4) - 2$



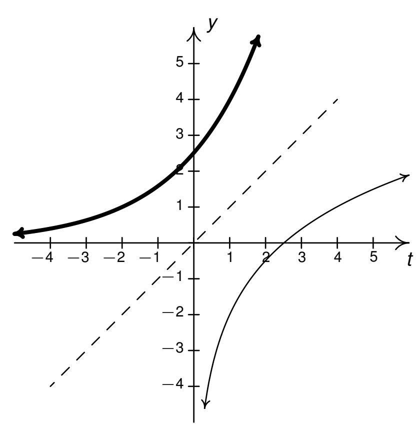
73. $y = f(x) = \log_4(x-1)$
$y = f^{-1}(x) = 4^x + 1$



74. $y = g(t) = -2^{-t} + 1$
$y = g^{-1}(t) = -\log_2(-t+1)$



75. $y = g(t) = 5\log(t) - 2$
$y = g^{-1}(t) = 10^{\frac{t+2}{5}}$



76. One solution is $g(x) = \log_2(x + 3)$ and $h(x) = 4$.

77. One solution is $g(x) = \log(2x)$ and $h(x) = e^{-x}$.

78. One solution is $g(t) = 3t$ and $h(t) = \log(t)$.

79. One solution is $f(x) = \ln(x)$ and $g(x) = x$.

80. One solution is $f(t) = t^2 + 1$ and $g(t) = \ln(t)$.

81. One solution is $f(z) = \ln(z)$ and $g(z) = z^2$.

82. (a) $M(0.001) = \log\left(\frac{0.001}{0.001}\right) = \log(1) = 0$.
    (b) $M(80,000) = \log\left(\frac{80,000}{0.001}\right) = \log(80,000,000) \approx 7.9$.

83. (a) $L(10^{-6}) = 60$ decibels.
    (b) $I = 10^{-.5} \approx 0.316$ watts per square meter.
    (c) Since $L(1) = 120$ decibels and $L(100) = 140$ decibels, a sound with intensity level 140 decibels has an intensity 100 times greater than a sound with intensity level 120 decibels.

84. (a) The pH of pure water is 7.
    (b) If $[H^+] = 6.3 \times 10^{-13}$ then the solution has a pH of 12.2.
    (c) $[H^+] = 10^{-0.7} \approx .1995$ moles per liter.

10.3 Properties of Logarithms

In Section 10.2, we introduced the logarithmic functions as inverses of exponential functions and discussed a few of their functional properties from that perspective. In this section, we explore the algebraic properties of logarithms. Historically, these have played a huge role in the scientific development of our society since, among other things, they were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing.

As we shall see shortly, logs inherit analogs of all of the properties of exponents you learned in Elementary and Intermediate Algebra. We first extract two properties from Theorem 10.3 to remind us of the definition of a logarithm as the inverse of an exponential function.

Theorem 10.4. (Inverse Properties of Exponential and Logarithmic Functions)
Let $b > 0$, $b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b\left(b^x\right) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

Next, we spell out what it means for exponential and logarithmic functions to be one-to-one.

Theorem 10.5. (One-to-one Properties of Exponential and Logarithmic Functions)
Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0$, $b \neq 1$. Then f and g are one-to-one and

- $b^u = b^w$ if and only if $u = w$ for all real numbers u and w.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all real numbers $u > 0$, $w > 0$.

Next, we re-state Theorem 10.2 for reference below.

Theorem 10.6. (Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u + w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u - w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $\left(b^u\right)^w = b^{uw}$

To each of these properties of listed in Theorem 10.2, there corresponds an analogous property of logarithmic functions. We list these below in our next theorem.

Theorem 10.7. (Algebraic Properties of Logarithmic Functions) Let $g(x) = \log_b(x)$ be a logarithmic function ($b > 0$, $b \neq 1$) and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $g(uw) = g(u) + g(w)$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $g\left(\frac{u}{w}\right) = g(u) - g(w)$. In other words, $\log_b\left(\frac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $g\left(u^w\right) = wg(u)$. In other words, $\log_b\left(u^w\right) = w\log_b(u)$

There are a couple of different ways to understand why Theorem 10.7 is true. For instance, consider the product rule: $\log_b(uw) = \log_b(u) + \log_b(w)$.

Let $a = \log_b(uw)$, $c = \log_b(u)$, and $d = \log_b(w)$. Then, by definition, $b^a = uw$, $b^c = u$ and $b^d = w$. Hence, $b^a = uw = b^c b^d = b^{c+d}$, so that $b^a = b^{c+d}$.

By the one-to-one property of b^x, $b^a = b^{c+d}$ gives $a = c + d$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$. The remaining properties are proved similarly.

From a purely functional approach, we can see the properties in Theorem 10.7 as an example of how inverse functions interchange the roles of inputs in outputs.

For instance, the Product Rule for exponential functions given in Theorem 10.2, $f(u + w) = f(u)f(w)$, says that adding inputs results in multiplying outputs.

Hence, whatever f^{-1} is, it must take the products of outputs from f and return them to the sum of their respective inputs. Since the outputs from f are the inputs to f^{-1} and vice-versa, we have that that f^{-1} must take products of its inputs to the sum of their respective outputs. This is precisely one way to interpret the Product Rule for Logarithmic functions: $g(uw) = g(u) + g(w)$.

The reader is encouraged to view the remaining properties listed in Theorem 10.7 similarly.

The following examples help build familiarity with these properties. In our first example, we are asked to 'expand' the logarithms. This means that we read the properties in Theorem 10.7 from left to right and rewrite products inside the log as sums outside the log, quotients inside the log as differences outside the log, and powers inside the log as factors outside the log.[1]

Example 10.3.1. Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\log_2\left(\frac{8}{x}\right)$
2. $\log_{0.1}\left(10x^2\right)$
3. $\ln\left(\frac{3}{et}\right)^2$
4. $\log\sqrt[3]{\frac{100x^2}{yz^5}}$
5. $\log_{117}\left(x^2 - 4\right)$

[1] Interestingly enough, it is the exact *opposite* process (which we will practice later) that is most useful in Algebra, the utility of expanding logarithms becomes apparent in Calculus.

Solution.

1. To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule identifying $u = 8$ and $w = x$ and simplify.

$$\begin{array}{rcll} \log_2\left(\dfrac{8}{x}\right) & = & \log_2(8) - \log_2(x) & \text{Quotient Rule} \\ & = & 3 - \log_2(x) & \text{Since } 2^3 = 8 \\ & = & -\log_2(x) + 3 & \end{array}$$

2. In the expression $\log_{0.1}\left(10x^2\right)$, we have a power (the x^2) and a product, and the question becomes which property, Power Rule or Product Rule to use first.

   In order to use the Power Rule, the *entire* quantity inside the log must be raised to the same exponent. Since the exponent 2 applies only to the x, we first apply the Product Rule with $u = 10$ and $w = x^2$. Once the x^2 is by itself inside the log, we apply the Power Rule with $u = x$ and $w = 2$.

$$\begin{array}{rcll} \log_{0.1}\left(10x^2\right) & = & \log_{0.1}(10) + \log_{0.1}\left(x^2\right) & \text{Product Rule} \\ & = & \log_{0.1}(10) + 2\log_{0.1}(x) & \text{Power Rule} \\ & = & -1 + 2\log_{0.1}(x) & \text{Since } (0.1)^{-1} = 10 \\ & = & 2\log_{0.1}(x) - 1 & \end{array}$$

3. We have a power, quotient and product occurring in $\ln\left(\frac{3}{et}\right)^2$. Since the exponent 2 applies to the entire quantity inside the logarithm, we begin with the Power Rule with $u = \frac{3}{et}$ and $w = 2$.

   Next, we see the Quotient Rule is applicable, with $u = 3$ and $w = et$, so we replace $\ln\left(\frac{3}{et}\right)$ with the quantity $\ln(3) - \ln(et)$.

   Since $\ln\left(\frac{3}{et}\right)$ is being multiplied by 2, the entire quantity $\ln(3) - \ln(et)$ is multiplied by 2.

   Finally, we apply the Product Rule with $u = e$ and $w = x$, and replace $\ln(et)$ with the quantity $\ln(e) + \ln(t)$, and simplify, keeping in mind that the natural log is log base e.

$$\begin{array}{rcll} \ln\left(\dfrac{3}{et}\right)^2 & = & 2\ln\left(\dfrac{3}{et}\right) & \text{Power Rule} \\ & = & 2\left[\ln(3) - \ln(et)\right] & \text{Quotient Rule} \\ & = & 2\ln(3) - 2\ln(et) & \\ & = & 2\ln(3) - 2\left[\ln(e) + \ln(t)\right] & \text{Product Rule} \\ & = & 2\ln(3) - 2\ln(e) - 2\ln(t) & \\ & = & 2\ln(3) - 2 - 2\ln(t) & \text{Since } e^1 = e \\ & = & -2\ln(t) + 2\ln(3) - 2 & \end{array}$$

4. In Theorem 10.7, there is no mention of how to deal with radicals. However, thinking back to Definition 1.9, we can rewrite the cube root as a $\frac{1}{3}$ exponent. We begin by using the Power Rule[2], and we keep in mind that the common log is log base 10.

$$
\begin{array}{rcll}
\log \sqrt[3]{\dfrac{100x^2}{yz^5}} & = & \log\left(\dfrac{100x^2}{yz^5}\right)^{1/3} & \\
& = & \frac{1}{3}\log\left(\dfrac{100x^2}{yz^5}\right) & \text{Power Rule} \\
& = & \frac{1}{3}\left[\log\left(100x^2\right) - \log\left(yz^5\right)\right] & \text{Quotient Rule} \\
& = & \frac{1}{3}\log\left(100x^2\right) - \frac{1}{3}\log\left(yz^5\right) & \\
& = & \frac{1}{3}\left[\log(100) + \log\left(x^2\right)\right] - \frac{1}{3}\left[\log(y) + \log\left(z^5\right)\right] & \text{Product Rule} \\
& = & \frac{1}{3}\log(100) + \frac{1}{3}\log\left(x^2\right) - \frac{1}{3}\log(y) - \frac{1}{3}\log\left(z^5\right) & \\
& = & \frac{1}{3}\log(100) + \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) & \text{Power Rule} \\
& = & \frac{2}{3} + \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) & \text{Since } 10^2 = 100 \\
& = & \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) + \frac{2}{3} &
\end{array}
$$

5. At first it seems as if we have no means of simplifying $\log_{117}\left(x^2-4\right)$, since none of the properties of logs addresses the issue of expanding a difference *inside* the logarithm. However, we may factor $x^2 - 4 = (x+2)(x-2)$ thereby introducing a product which gives us license to use the Product Rule. Assuming both $x + 2 > 0$ and $x - 2 > 0$, that is, $x > 2$ we expand as follows.

$$
\begin{array}{rcll}
\log_{117}\left(x^2-4\right) & = & \log_{117}[(x+2)(x-2)] & \text{Factor} \\
& = & \log_{117}(x+2) + \log_{117}(x-2) & \text{Product Rule}
\end{array}
$$

□

A couple of remarks about Example 10.3.1 are in order. First, if we take a step back and look at each problem in the foregoing example, a general rule of thumb to determine which log property to apply first when faced with a multi-step problem is to apply the logarithm properties in the 'reverse order of operations.' For example, if we were to substitute a number for x into the expression $\log_{0.1}\left(10x^2\right)$, we would first square the x, then multiply by 10. The last step is the multiplication, which tells us the first log property to apply is the Product Rule. The last property of logarithm to apply would the be the power rule applied to $\log_{0.1}(x^2)$. Second, the equivalence $\log_{117}\left(x^2-4\right) = \log_{117}(x+2) + \log_{117}(x-2)$ is valid only if $x > 2$. Indeed, the functions $f(x) = \log_{117}\left(x^2-4\right)$ and $g(x) = \log_{117}(x+2) + \log_{117}(x-2)$ have different domains, and, hence,

[2] At this point in the text, the reader is encouraged to carefully read through each step and think of which quantity is playing the role of u and which is playing the role of w as we apply each property.

are different functions.[3] In general, when using log properties to expand a logarithm, we may very well be restricting the domain as we do so.

One last comment before we move to reassembling logs from their various bits and pieces. The authors are well aware of the propensity for some students to become overexcited and invent their own properties of logs like $\log_{117}\left(x^2-4\right) = \log_{117}\left(x^2\right) - \log_{117}(4)$, which simply isn't true, in general. The unwritten[4] property of logarithms is that if it isn't written in a textbook, it probably isn't true.

Example 10.3.2. Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x-1) - \log_3(x+1)$
2. $\log(x) + 2\log(y) - \log(z)$
3. $4\log_2(x) + 3$
4. $-\ln(t) - \frac{1}{2}$

Solution. Whereas in Example 10.3.1 we read the properties in Theorem 10.7 from left to right to expand logarithms, in this example we read them from right to left.

1. The difference of logarithms requires the Quotient Rule: $\log_3(x-1) - \log_3(x+1) = \log_3\left(\frac{x-1}{x+1}\right)$.
2. In the expression, $\log(x) + 2\log(y) - \log(z)$, we have both a sum and difference of logarithms.

   Before we use the product rule to combine $\log(x) + 2\log(y)$, we note that we need to apply the Power Rule to rewrite the coefficient 2 as the power on y. We then apply the Product and Quotient Rules as we move from left to right.

$$\begin{aligned} \log(x) + 2\log(y) - \log(z) &= \log(x) + \log\left(y^2\right) - \log(z) && \text{Power Rule} \\ &= \log\left(xy^2\right) - \log(z) && \text{Product Rule} \\ &= \log\left(\frac{xy^2}{z}\right) && \text{Quotient Rule} \end{aligned}$$

3. We begin rewriting $4\log_2(x) + 3$ by applying the Power Rule: $4\log_2(x) = \log_2\left(x^4\right)$.

   In order to continue, we need to rewrite 3 as a logarithm base 2. From Theorem 10.4, we know $3 = \log_2\left(2^3\right)$. Rewriting 3 this way paves the way to use the Product Rule.

$$\begin{aligned} 4\log_2(x) + 3 &= \log_2\left(x^4\right) + 3 && \text{Power Rule} \\ &= \log_2\left(x^4\right) + \log_2\left(2^3\right) && \text{Since } 3 = \log_2\left(2^3\right) \\ &= \log_2\left(x^4\right) + \log_2(8) \\ &= \log_2\left(8x^4\right) && \text{Product Rule} \end{aligned}$$

[3] We leave it to the reader to verify the domain of f is $(-\infty, -2) \cup (2, \infty)$ whereas the domain of g is $(2, \infty)$.

[4] The authors relish the irony involved in writing what follows.

4. To get started with $-\ln(t) - \frac{1}{2}$, we rewrite $-\ln(t)$ as $(-1)\ln(t)$. We can then use the Power Rule to obtain $(-1)\ln(t) = \ln\left(t^{-1}\right)$.

   As in the previous problem, in order to continue, we need to rewrite $\frac{1}{2}$ as a natural logarithm. Theorem 10.4 gives us $\frac{1}{2} = \ln\left(e^{1/2}\right) = \ln\left(\sqrt{e}\right)$. Hence,

$$\begin{array}{rcll} -\ln(t) - \frac{1}{2} & = & (-1)\ln(t) - \frac{1}{2} & \\ & = & \ln\left(t^{-1}\right) - \frac{1}{2} & \text{Power Rule} \\ & = & \ln\left(t^{-1}\right) - \ln\left(e^{1/2}\right) & \text{Since } \frac{1}{2} = \ln\left(e^{1/2}\right) \\ & = & \ln\left(t^{-1}\right) - \ln\left(\sqrt{e}\right) & \\ & = & \ln\left(\dfrac{t^{-1}}{\sqrt{e}}\right) & \text{Quotient Rule} \\ & = & \ln\left(\dfrac{1}{t\sqrt{e}}\right) & \end{array}$$

□

As we would expect, the rule of thumb for re-assembling logarithms is the opposite of what it was for dismantling them. That is, to rewrite an expression as a single logarithm, we apply log properties following the usual order of operations: first, rewrite coefficients of logs as powers using the Power Rule, then rewrite addition and subtraction using the Product and Quotient Rules, respectively, as written from left to right.

Additionally, we find that using log properties in this fashion can increase the domain of the expression. For example, we leave it to the reader to verify the domain of $f(x) = \log_3(x-1) - \log_3(x+1)$ is $(1, \infty)$ but the domain of $g(x) = \log_3\left(\frac{x-1}{x+1}\right)$ is $(-\infty, -1) \cup (1, \infty)$. We'll need to keep this in mind in Section 10.5 since such manipulations can result in extraneous solutions.

The two logarithm buttons commonly found on calculators are the 'LOG' and 'LN' buttons which correspond to the common and natural logs, respectively. Suppose we wanted an approximation to $\log_2(7)$. The answer should be a little less than 3, (Can you explain why?) but how do we coerce the calculator into telling us a more accurate answer? We need the following theorem.

Theorem 10.8. (Change of Base Formulas) Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x \log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$.

To prove these formulas, consider $b^{x \log_b(a)}$. Using the Power Rule, we can rewrite $x \log_b(a)$ as $\log_b\left(a^x\right)$. Following this with the Inverse Properties in Theorem 10.4, we get

$$b^{x \log_b(a)} = b^{\log_b\left(a^x\right)} = a^x.$$

To verify the logarithmic form of the property, we use the Power Rule and an Inverse Property to get:

$$\log_a(x) \cdot \log_b(a) = \log_b\left(a^{\log_a(x)}\right) = \log_b(x).$$

We get the result by dividing both sides of the equation $\log_a(x) \cdot \log_b(a) = \log_b(x)$ by $\log_b(a)$.

Of course, the authors can't help but point out the inverse relationship between these two change of base formulas. To change the base of an exponential expression, we *multiply* the *input* by the factor $\log_b(a)$. To change the base of a logarithmic expression, we *divide* the *output* by the factor $\log_b(a)$.

While, in the grand scheme of things, both change of base formulas are really saying the same thing, the logarithmic form is the one usually encountered in Algebra while the exponential form isn't usually introduced until Calculus.

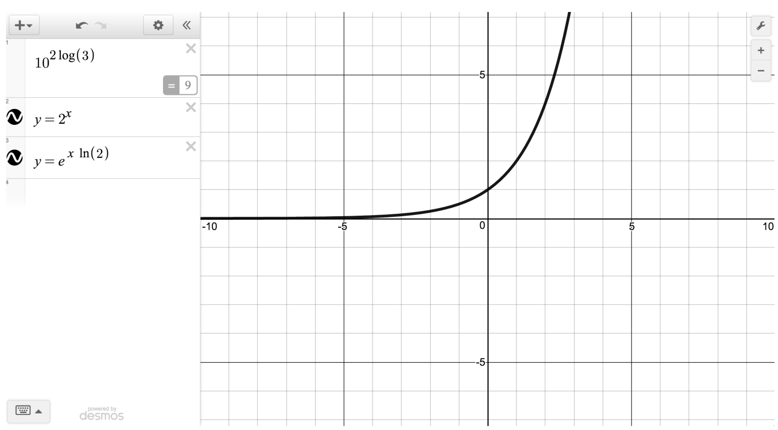
Example 10.3.3. Use an appropriate change of base formula to convert the following expressions to ones with the indicated base. Verify your answers using a graphing utility, as appropriate.

1. 3^2 to base 10
2. 2^x to base e
3. $\log_4(5)$ to base e
4. $\ln(x)$ to base 10

Solution.

1. We apply the Change of Base formula with $a = 3$ and $b = 10$ to obtain $3^2 = 10^{2\log(3)}$. Typing the latter into a graphing utility produces an answer of 9 as seen below.

2. Here, $a = 2$ and $b = e$ so we have $2^x = e^{x\ln(2)}$. Using a graphing utility, we find the graphs of $f(x) = 2^x$ and $g(x) = e^{x\ln(2)}$ appear to overlap perfectly.

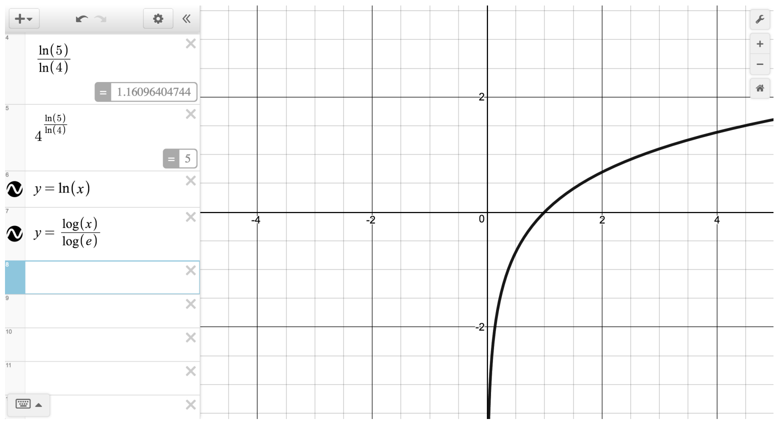


3. Applying the change of base with $a = 4$ and $b = e$ leads us to write $\log_4(5) = \frac{\ln(5)}{\ln(4)}$. Evaluating this gives the numerical approximation $\frac{\ln(5)}{\ln(4)} \approx 1.16$.

   To check our answer we know that, by definition, $\log_4(5)$ is the exponent we put on 4 to get 5, so a number a little larger than 1 seems reasonable.

Taking this one step further, we use a graphing utility and find $4^{\frac{\ln(5)}{\ln(4)}} = 5$, which means if the machine is lying to us about the first answer it gave us, at least it is being consistent.

4. We write $\ln(x) = \log_e(x) = \frac{\log(x)}{\log(e)}$. We graph both $f(x) = \ln(x)$ and $g(x) = \frac{\log(x)}{\log(e)}$ and find both graphs appear to be identical.



$\square$

What Theorem 10.8 really tells us is that all exponential and logarithmic functions are just scalings of one another. Not only does this explain why their graphs have similar shapes, but it also tells us that we could do all of mathematics with a single base, be it 10, 0.42, π, or 117.

As mentioned in Section 10.1, the 'natural' base, base e, features prominently in mathematical applications as we'll see in Section **??**. Hence, we conclude this section by specifying Theorem 10.8 to this case.

Theorem 10.9. Conversion to the Natural Base: Suppose $b > 0$, $b \neq 1$. Then

- $b^x = e^{x \ln(b)}$ for all real numbers x.
- $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ for all real numbers $x > 0$.

10.3.1 Exercises

In Exercises 1 - 15, expand the given logarithm and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\ln(x^3y^2)$
2. $\log_2\left(\dfrac{128}{x^2+4}\right)$
3. $\log_5\left(\dfrac{z}{25}\right)^3$
4. $\log(1.23 \times 10^{37})$
5. $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$
6. $\log_5\left(x^2-25\right)$
7. $\log_{\sqrt{2}}\left(4x^3\right)$
8. $\log_{\frac{1}{3}}(9x(y^3-8))$
9. $\log\left(1000x^3y^5\right)$
10. $\log_3\left(\dfrac{x^2}{81y^4}\right)$
11. $\ln\left(\sqrt[4]{\dfrac{xy}{ez}}\right)$
12. $\log_6\left(\dfrac{216}{x^3y}\right)^4$
13. $\log\left(\dfrac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$
14. $\log_{\frac{1}{2}}\left(\dfrac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$
15. $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

In Exercises 16 - 29, use the properties of logarithms to write the expression as a single logarithm.

16. $4\ln(x)+2\ln(y)$
17. $\log_2(x)+\log_2(y)-\log_2(z)$
18. $\log_3(x)-2\log_3(y)$
19. $\frac{1}{2}\log_3(x)-2\log_3(y)-\log_3(z)$
20. $2\ln(x)-3\ln(y)-4\ln(z)$
21. $\log(x)-\frac{1}{3}\log(z)+\frac{1}{2}\log(y)$
22. $-\frac{1}{3}\ln(x)-\frac{1}{3}\ln(y)+\frac{1}{3}\ln(z)$
23. $\log_5(x)-3$
24. $3-\log(x)$
25. $\log_7(x)+\log_7(x-3)-2$
26. $\ln(x)+\frac{1}{2}$
27. $\log_2(x)+\log_4(x)$
28. $\log_2(x)+\log_4(x-1)$
29. $\log_2(x)+\log_{\frac{1}{2}}(x-1)$

In Exercises 30 - 33, use the appropriate change of base formula to convert the given expression to an expression with the indicated base.

30. 7^{x-1} to base e
31. $\log_3(x+2)$ to base 10
32. $\left(\dfrac{2}{3}\right)^x$ to base e
33. $\log(x^2+1)$ to base e

In Exercises 34 - 39, use the appropriate change of base formula to approximate the logarithm.

34. $\log_3(12)$
35. $\log_5(80)$
36. $\log_6(72)$
37. $\log_4\left(\dfrac{1}{10}\right)$
38. $\log_{\frac{3}{5}}(1000)$
39. $\log_{\frac{2}{3}}(50)$

40. In Example 10.2.1 number 3 in Section 10.2, we obtained the solution $F(x) = \log_2(-x+4) - 3$ as one formula for the given graph by making a simplifying assumption that $b = -1$. This exercises explores if there are any other solutions for different choices of b.

    (a) Show $G(x) = \log_2(-2x+8) - 4$ also fits the data for the given graph.

    (b) Use properties of logarithms to show $G(x) = \log_2(-2x+8) - 4 = \log_2(-x+4) - 3 = F(x)$.

    (c) With help from your classmates, find solutions to Example 10.2.1 number 3 in Section 10.2 by assuming $b = -4$ and $b = -8$. In each case, use properties of logarithms to show the solutions reduce to $F(x) = \log_2(-x+4) - 3$.

    (d) Using properties of logarithms and the fact that the range of $\log_2(x)$ is all real numbers, show that any function of the form $f(x) = a\log_2(bx-h) + k$ where $a \neq 0$ can be rewritten as:

    $$f(x) = a\left(\log_2(bx-h) + \frac{k}{a}\right) = a(\log_2(bx-h) + \log_2(p)) = a\log_2(p(bx-h)) = a\log_2(pbx-ph),$$

    where $\frac{k}{a} = \log_2(p)$ for some positive real number p. Relabeling, we get every function of the form $f(x) = a\log_2(bx-h) + k$ with four parameters (a, b, h, and k) can be rewritten as $f(x) = a\log_2(Bx-H)$, a formula with just three parameters: a, B, and H.

    Show *every* solution to Example 10.2.1 number 3 in Section 10.2 can be written in the form $f(x) = \log_2\left(-\frac{1}{8}x + \frac{1}{2}\right)$ and that, in particular, $F(x) = \log_2(-x+4) - 3 = \log_2\left(-\frac{1}{8}x + \frac{1}{2}\right) = f(x)$. Hence, there is really just one solution to Example 10.2.1 number 3 in Section 10.2.

41. The Henderson-Hasselbalch Equation: Suppose HA represents a weak acid. Then we have a reversible chemical reaction

    $$HA \rightleftharpoons H^+ + A^-.$$

    The acid disassociation constant, K_a, is given by

    $$K_a = \frac{[H^+][A^-]}{[HA]} = [H^+]\frac{[A^-]}{[HA]},$$

    where the square brackets denote the concentrations just as they did in Exercise 84 in Section 10.2. The symbol $\mathrm{p}K_a$ is defined similarly to pH in that $\mathrm{p}K_a = -\log(K_a)$. Using the definition of pH from Exercise 84 and the properties of logarithms, derive the Henderson-Hasselbalch Equation:

    $$\mathrm{pH} = \mathrm{p}K_a + \log\frac{[A^-]}{[HA]}$$

42. Compare and contrast the graphs of $y = \ln(x^2)$ and $y = 2\ln(x)$.

43. Prove the Quotient Rule and Power Rule for Logarithms.

44. Give numerical examples to show that, in general,

    (a) $\log_b(x + y) \neq \log_b(x) + \log_b(y)$

    (b) $\log_b(x - y) \neq \log_b(x) - \log_b(y)$

    (c) $\log_b\left(\dfrac{x}{y}\right) \neq \dfrac{\log_b(x)}{\log_b(y)}$

45. Research the history of logarithms including the origin of the word 'logarithm' itself. Why is the abbreviation of natural log 'ln' and not 'nl'?

46. There is a scene in the movie 'Apollo 13' in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

10.3.2 Answers

1. $3\ln(x) + 2\ln(y)$
2. $7 - \log_2(x^2 + 4)$
3. $3\log_5(z) - 6$
4. $\log(1.23) + 37$
5. $\frac{1}{2}\ln(z) - \ln(x) - \ln(y)$
6. $\log_5(x-5) + \log_5(x+5)$
7. $3\log_{\sqrt{2}}(x) + 4$
8. $-2 + \log_{\frac{1}{3}}(x) + \log_{\frac{1}{3}}(y-2) + \log_{\frac{1}{3}}(y^2+2y+4)$
9. $3 + 3\log(x) + 5\log(y)$
10. $2\log_3(x) - 4 - 4\log_3(y)$
11. $\frac{1}{4}\ln(x) + \frac{1}{4}\ln(y) - \frac{1}{4} - \frac{1}{4}\ln(z)$
12. $12 - 12\log_6(x) - 4\log_6(y)$
13. $\frac{5}{3} + \log(x) + \frac{1}{2}\log(y)$
14. $-2 + \frac{2}{3}\log_{\frac{1}{2}}(x) - \log_{\frac{1}{2}}(y) - \frac{1}{2}\log_{\frac{1}{2}}(z)$
15. $\frac{1}{3}\ln(x) - \ln(10) - \frac{1}{2}\ln(y) - \frac{1}{2}\ln(z)$
16. $\ln(x^4y^2)$
17. $\log_2\left(\frac{xy}{z}\right)$
18. $\log_3\left(\frac{x}{y^2}\right)$
19. $\log_3\left(\frac{\sqrt{x}}{y^2z}\right)$
20. $\ln\left(\frac{x^2}{y^3z^4}\right)$
21. $\log\left(\frac{x\sqrt{y}}{\sqrt[3]{z}}\right)$
22. $\ln\left(\sqrt[3]{\frac{z}{xy}}\right)$
23. $\log_5\left(\frac{x}{125}\right)$
24. $\log\left(\frac{1000}{x}\right)$
25. $\log_7\left(\frac{x(x-3)}{49}\right)$
26. $\ln\left(x\sqrt{e}\right)$
27. $\log_2\left(x^{3/2}\right)$
28. $\log_2\left(x\sqrt{x-1}\right)$
29. $\log_2\left(\frac{x}{x-1}\right)$
30. $7^{x-1} = e^{(x-1)\ln(7)}$
31. $\log_3(x+2) = \frac{\log(x+2)}{\log(3)}$
32. $\left(\frac{2}{3}\right)^x = e^{x\ln\left(\frac{2}{3}\right)}$
33. $\log(x^2+1) = \frac{\ln(x^2+1)}{\ln(10)}$
34. $\log_3(12) \approx 2.26186$
35. $\log_5(80) \approx 2.72271$
36. $\log_6(72) \approx 2.38685$
37. $\log_4\left(\frac{1}{10}\right) \approx -1.66096$
38. $\log_{\frac{3}{5}}(1000) \approx -13.52273$
39. $\log_{\frac{2}{3}}(50) \approx -9.64824$

10.4 Equations and Inequalities involving Exponential Functions

In this section we will develop techniques for solving equations involving exponential functions. Consider the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we have $2^x = 2^7$. The one-to-one property of exponential functions, detailed in Theorem 10.5, tells us that $2^x = 2^7$ if and only if $x = 7$. This means that not only is $x = 7$ a solution to $2^x = 2^7$, it is the *only* solution.

Now suppose we change the problem ever so slightly to $2^x = 129$. We could use one of the inverse properties of exponentials and logarithms listed in Theorem 10.4 to write $129 = 2^{\log_2(129)}$. We'd then have $2^x = 2^{\log_2(129)}$, which means our solution is $x = \log_2(129)$.

After all, the definition of $\log_2(129)$ is 'the exponent we put on 2 to get 129.' Indeed we could have obtained this solution directly by rewriting the equation $2^x = 129$ in its logarithmic form $\log_2(129) = x$. Either way, in order to get a reasonable decimal approximation to this number, we'd use the change of base formula, Theorem 10.8, to give us something more calculator friendly. Typically this means we convert our answer to base 10 or base e, and we choose the latter: $\log_2(129) = \frac{\ln(129)}{\ln(2)} \approx 7.011$.

Still another way to obtain this answer is to 'take the natural log' of both sides of the equation. Since $f(x) = \ln(x)$ is a *function*, as long as two quantities are equal, their natural logs are equal.[1]

We then use the Power Rule to write the exponent x as a factor then divide both sides by the constant $\ln(2)$ to obtain our answer.[2]

$$\begin{array}{rcll} 2^x & = & 129 & \\ \ln\left(2^x\right) & = & \ln(129) & \text{Take the natural log of both sides.} \\ x\ln(2) & = & \ln(129) & \text{Power Rule} \\ x & = & \dfrac{\ln(129)}{\ln(2)} & \end{array}$$

We summarize our two strategies for solving equations featuring exponential functions below.

Steps for Solving an Equation involving Exponential Functions

1. Isolate the exponential function.
2. (a) If convenient, express both sides with a common base and equate the exponents.
   (b) Otherwise, take the natural log of both sides of the equation and use the Power Rule.

Example 10.4.1. Solve the following equations. Check your answer using a graphing utility.

1. $2^{3x} = 16^{1-x}$
2. $2000 = 1000 \cdot 3^{-0.1t}$
3. $9 \cdot 3^x = 7^{2x}$
4. $75 = \frac{100}{1+3e^{-2t}}$
5. $25^x = 5^x + 6$
6. $\frac{e^x - e^{-x}}{2} = 5$

[1] This is also the 'if' part of the statement $\log_b(u) = \log_b(w)$ if and only if $u = w$ in Theorem 10.5.

[2] Please resist the temptation to divide both sides by 'ln' instead of ln(2). Just like it wouldn't make sense to divide both sides by the square root symbol '$\sqrt{\ }$' when solving $x\sqrt{2} = 5$, it makes no sense to divide by 'ln'.

Solution.

1. Since 16 is a power of 2, we can rewrite $2^{3x} = 16^{1-x}$ as $2^{3x} = \left(2^4\right)^{1-x}$. Using properties of exponents, we get $2^{3x} = 2^{4(1-x)}$.

   Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives $x = \frac{4}{7}$.

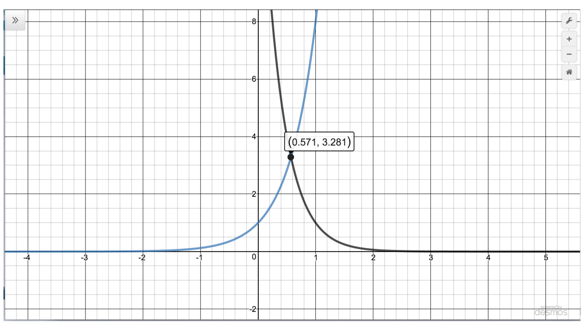
   Graphing $f(x) = 2^{3x}$ and $g(x) = 16^{1-x}$ and see that they intersect at $x \approx 0.571 \approx \frac{4}{7}$.

2. We begin solving $2000 = 1000 \cdot 3^{-0.1t}$ by dividing both sides by 1000 to isolate the exponential which yields $3^{-0.1t} = 2$.

   Since it is inconvenient to write 2 as a power of 3, we use the natural log to get $\ln\left(3^{-0.1t}\right) = \ln(2)$.

   Using the Power Rule, we get $-0.1t\ln(3) = \ln(2)$, so we divide both sides by $-0.1\ln(3)$ and obtain $t = -\frac{\ln(2)}{0.1\ln(3)} = -\frac{10\ln(2)}{\ln(3)}$.

   We see the graphs of $f(x) = 2000$ and $g(x) = 1000 \cdot 3^{-0.1x}$ intersect at $x \approx -6.309 \approx -\frac{10\ln(2)}{\ln(3)}$.



Checking $2^{3x} = 16^{1-x}$

Checking $2000 = 1000 \cdot 3^{-0.1t}$

3. We first note that we can rewrite the equation $9 \cdot 3^x = 7^{2x}$ as $3^2 \cdot 3^x = 7^{2x}$ to obtain $3^{x+2} = 7^{2x}$.

   Since it is not convenient to express both sides as a power of 3 (or 7 for that matter) we use the natural log: $\ln\left(3^{x+2}\right) = \ln\left(7^{2x}\right)$.

   The power rule gives $(x+2)\ln(3) = 2x\ln(7)$. Even though this equation appears very complicated, keep in mind that $\ln(3)$ and $\ln(7)$ are just constants.

   The equation $(x+2)\ln(3) = 2x\ln(7)$ is actually a linear equation (do you see why?) and as such we gather all of the terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

$$\begin{array}{rcll}
(x+2)\ln(3) & = & 2x\ln(7) & \\
x\ln(3) + 2\ln(3) & = & 2x\ln(7) & \\
2\ln(3) & = & 2x\ln(7) - x\ln(3) & \\
2\ln(3) & = & x(2\ln(7) - \ln(3)) & \text{Factor.} \\
x & = & \frac{2\ln(3)}{2\ln(7)-\ln(3)} &
\end{array}$$

We see the graphs of $f(x) = 9 \cdot 3^x$ and $g(x) = 7^{2x}$ intersect at $x \approx 0.787 \approx \frac{2\ln(3)}{2\ln(7) - \ln(3)}$.

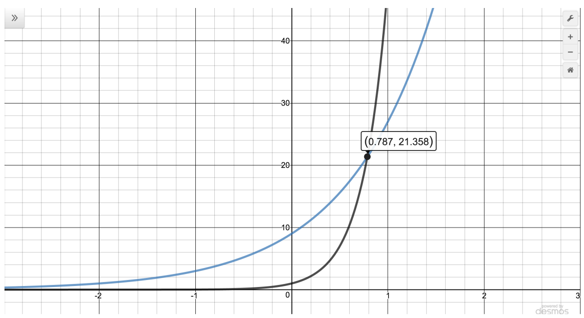
4. Our objective in solving $75 = \frac{100}{1+3e^{-2t}}$ is to first isolate the exponential.

   To that end, we clear denominators and get $75\left(1 + 3e^{-2t}\right) = 100$, or $75 + 225e^{-2t} = 100$. We get $225e^{-2t} = 25$, so finally, $e^{-2t} = \frac{1}{9}$.

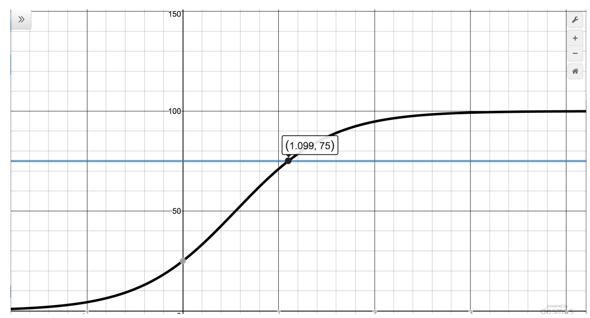
   Taking the natural log of both sides gives $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$. Since natural log is log base e, $\ln\left(e^{-2t}\right) = -2t$. Likewise, we use the Power Rule to rewrite $\ln\left(\frac{1}{9}\right) = -\ln(9)$.

   Putting these two steps together, we simplify $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$ to $-2t = -\ln(9)$. We arrive at our solution, $t = \frac{\ln(9)}{2}$ which simplifies to $t = \ln(3)$. (Can you explain why?)

   To check, we see the graphs of $f(x) = 75$ and $g(x) = \frac{100}{1+3e^{-2x}}$, intersect at $x \approx 1.099 \approx \ln(3)$.



Checking $9 \cdot 3^x = 7^{2x}$



Checking $75 = \frac{100}{1+3e^{-2t}}$

5. We start solving $25^x = 5^x + 6$ by rewriting $25 = 5^2$ so that we have $\left(5^2\right)^x = 5^x + 6$, or $5^{2x} = 5^x + 6$.

   Even though we have a common base, having two terms on the right hand side of the equation foils our plan of equating exponents or taking logs.

   If we stare at this long enough, we notice that we have three terms with the exponent on one term exactly twice that of another. To our surprise and delight, we have a 'quadratic in disguise'.

   Letting $u = 5^x$, we have $u^2 = (5^x)^2 = 5^{2x}$ so the equation $5^{2x} = 5^x + 6$ becomes $u^2 = u + 6$. Solving this as $u^2 - u - 6 = 0$ gives $u = -2$ or $u = 3$. Since $u = 5^x$, we have $5^x = -2$ or $5^x = 3$.

   Since $5^x = -2$ has no real solution,[3] we focus on $5^x = 3$. Since it isn't convenient to express 3 as a power of 5, we take natural logs and get $\ln(5^x) = \ln(3)$ so that $x\ln(5) = \ln(3)$ or $x = \frac{\ln(3)}{\ln(5)}$.

   We see the graphs of $f(x) = 25^x$ and $g(x) = 5^x + 6$ intersect at $x \approx 0.683 \approx \frac{\ln(3)}{\ln(5)}$.

6. Clearing the denominator in $\frac{e^x - e^{-x}}{2} = 5$ gives $e^x - e^{-x} = 10$, at which point we pause to consider how to proceed. Rewriting $e^{-x} = \frac{1}{e^x}$, we see we have another denominator to clear: $e^x - \frac{1}{e^x} = 10$.

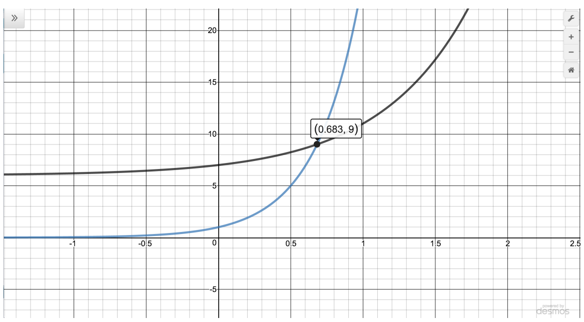
[3] Why not?

Doing so gives $e^{2x} - 1 = 10e^{x}$, which, once again fits the criteria of being a 'quadratic in disguise.'

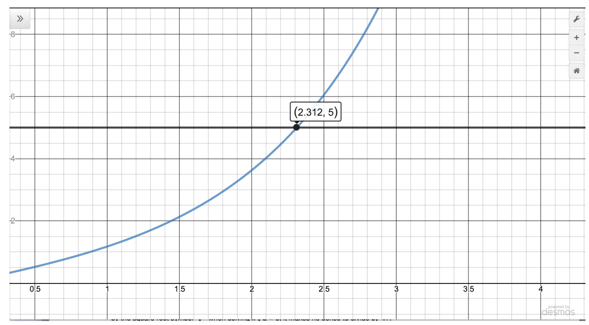
If we let $u = e^{x}$, then $u^{2} = e^{2x}$ so the equation $e^{2x} - 1 = 10e^{x}$ can be viewed as $u^{2} - 1 = 10u$. Solving $u^{2} - 10u - 1 = 0$ using the quadratic formula gives $u = 5 \pm \sqrt{26}$.

From this, we have $e^{x} = 5 \pm \sqrt{26}$. Since $5 - \sqrt{26} < 0$, we get no real solution to $e^{x} = 5 - \sqrt{26}$ (why not?) but for $e^{x} = 5 + \sqrt{26}$, we take natural logs to obtain $x = \ln\left(5 + \sqrt{26}\right)$.

We see the graphs of $f(x) = \frac{e^{x} - e^{-x}}{2}$ and $g(x) = 5$ intersect at $x \approx 2.312 \approx \ln\left(5 + \sqrt{26}\right)$.



Checking $25^{x} = 5^{x} + 6$



Checking $\frac{e^{x} - e^{-x}}{2} = 5$

□

Note that verifying our solutions to the equations in Example 10.4.1 *analytically* holds great educational value, since it reviews many of the properties of logarithms and exponents in tandem.

For example, to verify our solution to $2000 = 1000 \cdot 3^{-0.1t}$, we substitute $t = -\frac{10\ln(2)}{\ln(3)}$ and check:

$$\begin{array}{rcll}
2000 & \stackrel{?}{=} & 1000 \cdot 3^{-0.1\left(-\frac{10\ln(2)}{\ln(3)}\right)} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\frac{\ln(2)}{\ln(3)}} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\log_{3}(2)} & \text{Change of Base} \\
2000 & \stackrel{?}{=} & 1000 \cdot 2 & \text{Inverse Property} \\
2000 & \stackrel{\checkmark}{=} & 2000 &
\end{array}$$

We strongly encourage the reader to check the remaining equations analytically as well.

Since exponential functions are continuous on their domains, the Intermediate Value Theorem 6.14 applies. This allows us to solve inequalities using sign diagrams as demonstrated below.

Example 10.4.2. Solve the following inequalities. Check your answer graphically.

1. $2^{x^{2} - 3x} - 16 \geq 0$
2. $\dfrac{e^{x}}{e^{x} - 4} \leq 3$
3. $te^{2t} < 4t$

Solution.

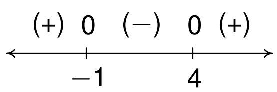
1. Since we already have 0 on one side of the inequality, we set $r(x) = 2^{x^2-3x} - 16$.

   The domain of r is all real numbers, so to construct our sign diagram, we need to find the zeros of r.

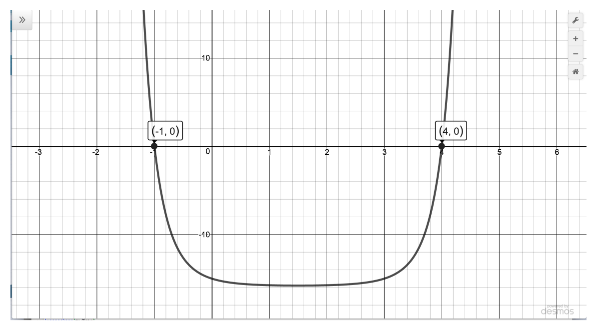
   Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$. By the one-to-one property of exponential functions, $x^2 - 3x = 4$ which gives $x = 4$ and $x = -1$.

   From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which is our solution.

   Graphing $r(x) = 2^{x^2-3x} - 16$, we find it is on or above the line $y = 0$ (the x-axis) precisely on the intervals $(-\infty, -1]$ and $[4, \infty)$ which checks our answer.



A Sign Diagram for $r(x) = 2^{x^2-3x} - 16$



Checking $2^{x^2-3x} - 16 \geq 0$

2. The first step we need to take to solve $\frac{e^x}{e^x-4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

$$\begin{aligned}
\frac{e^x}{e^x-4} &\leq 3 \\
\frac{e^x}{e^x-4} - 3 &\leq 0 \\
\frac{e^x}{e^x-4} - \frac{3\left(e^x-4\right)}{e^x-4} &\leq 0 \quad \text{Common denomintors.} \\
\frac{12-2e^x}{e^x-4} &\leq 0
\end{aligned}$$

   We set $r(x) = \frac{12-2e^x}{e^x-4}$ and we note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$.

   To find the zeros of r, we solve $r(x) = 0$ and obtain $12 - 2e^x = 0$. We find $e^x = 6$, or $x = \ln(6)$.

   When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$.

Recall that the function $\ln(x)$ is increasing[4] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$.

To determine the sign of $r(\ln(3))$, we remember that $e^{\ln(3)} = 3$ and get

$$r(\ln(3)) = \frac{12 - 2e^{\ln(3)}}{e^{\ln(3)} - 4} = \frac{12 - 2(3)}{3 - 4} = -6.$$

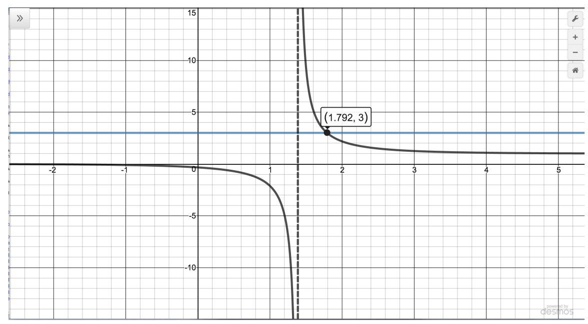
We determine the signs of $r(\ln(5))$ and $r(\ln(7))$ similarly.[5] From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$.

Using a graphing utility, we find the graph of $f(x) = \frac{e^x}{e^x - 4}$ is below the graph of $g(x) = 3$ on $(-\infty, \ln(4)) \cup (\ln(6), \infty)$, and they intersect at $x \approx 1.792 \approx \ln(6)$.

$(-)$ ‽ $(+)$ 0 $(-)$

$\ln(4)$ $\ln(6)$

A Sign Diagram for $r(x) = \dfrac{12 - 2e^x}{e^x - 4}$



Checking $\dfrac{e^x}{e^x - 4} \leq 3$

3. As before, we start solving $te^{2t} < 4t$ by getting 0 on one side of the inequality, $te^{2t} - 4t < 0$.

   We set $r(t) = te^{2t} - 4t$ and since there are no denominators, even-indexed radicals, or logs, the domain of r is all real numbers.

   Setting $r(t) = 0$ produces $te^{2t} - 4t = 0$. We factor to get $t\left(e^{2t} - 4\right) = 0$ which gives $t = 0$ or $e^{2t} - 4 = 0$.

   To solve the latter, we isolate the exponential and take logs to get $2t = \ln(4)$, or $t = \frac{\ln(4)}{2}$ which simplifies to $t = \ln(2)$. (Can you see why?)

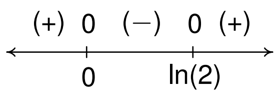
   As in the previous example, we need to be careful about choosing test values. Since $\ln(1) = 0$, we choose $\ln\left(\frac{1}{2}\right)$, $\ln\left(\frac{3}{2}\right)$ and $\ln(3)$. Evaluating,[6] we get

$$\begin{array}{rcll} r\left(\ln\left(\frac{1}{2}\right)\right) & = & \ln\left(\frac{1}{2}\right) e^{2\ln\left(\frac{1}{2}\right)} - 4\ln\left(\frac{1}{2}\right) & \\ & = & \ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{2}\right)^2} - 4\ln\left(\frac{1}{2}\right) & \text{Power Rule} \\ & = & \ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{4}\right)} - 4\ln\left(\frac{1}{2}\right) & \\ & = & \frac{1}{4}\ln\left(\frac{1}{2}\right) - 4\ln\left(\frac{1}{2}\right) = -\frac{15}{4}\ln\left(\frac{1}{2}\right) & \end{array}$$

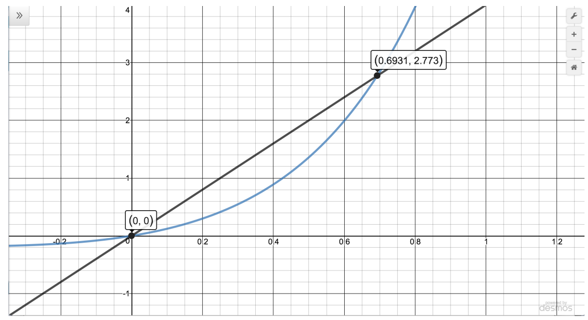
[4]This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.
[5]We could, of course, use the calculator, but what fun would that be?
[6]A calculator can be used at this point. As usual, we proceed without apologies, with the analytical method.

Since $\frac{1}{2} < 1$, $\ln\left(\frac{1}{2}\right) < 0$ and we get $r(\ln\left(\frac{1}{2}\right))$ is $(+)$. Proceeding similarly, we find $r\left(\ln\left(\frac{3}{2}\right)\right) < 0$ and $r(\ln(3)) > 0$. Our solution corresponds to $r(t) < 0$ which occurs on $(0, \ln(2))$.

The graphing utility confirms that the graph of $f(t) = te^{2t}$ is below the graph of $g(t) = 4t$ on $(0, \ln(2))$.[7]



A Sign Diagram for $r(t) = te^{2t} - 4t$



Checking $te^{2t} < 4t$

□

We note here that while sign diagrams will *always* work for solving inequalities involving exponential functions, as we've seen previously, there are circumstances in which we can short-cut this method.

For example, consider number 1 from Example 10.4.2 above: $2^{x^2-3x} - 16 \geq 0$. Since the base $2 > 1$, $\log_2(x)$ is an *increasing* function meaning it preserves inequalities.

We can use this to our advantage in this case and eliminate the exponential from the inequality altogether:

$$\begin{array}{rcll} 2^{x^2-3x} - 16 & \geq & 0 & \\ 2^{x^2-3x} & \geq & 16 & \\ \log_2\left(2^{x^2-3x}\right) & \geq & \log_2(16) & f(x) = \log_2(x) \text{ is increasing so if } b \geq a, \log_2(b) \geq \log_2(a). \\ x^2 - 3x & \geq & 4 & \end{array}$$

Hence, we've reduced our given inequality to $x^2 - 3x \geq 4$. As seen in Section 5.4, we can solve this inequality by completing the square, graphing, or a sign diagram, whichever strikes the reader's fancy.

Our next example is a follow-up to Example 10.1.3 in Section 10.1.

Example 10.4.3. Recall from Example 10.1.3 the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$. When will the coffee be warmer than 100°F?

Solution. We need to find when $T(t) > 100$, that is, we need to solve $70 + 90e^{-0.1t} > 100$.

To use a sign diagram, we need to get 0 on one side of the inequality. Subtracting 100 from both sides of $70 + 90e^{-0.1t} > 100$ produces $90e^{-0.1t} - 30 > 0$.

Identifying $r(t) = 90e^{-0.1t} - 30$, we note from the context of the problem the domain of r is $[0, \infty)$, so to build the sign diagram, we proceed to find the zeros of r.

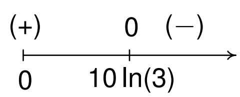
[7]Note: $\ln(2) \approx 0.693$.

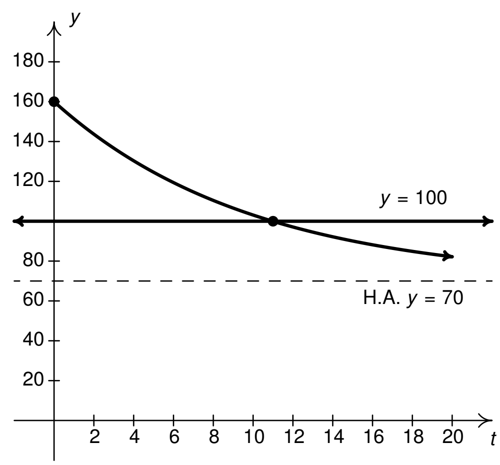
Solving $90e^{-0.1t} - 30 = 0$ results in $e^{-0.1t} = \frac{1}{3}$ so that $t = -10\ln\left(\frac{1}{3}\right)$ which reduces to $t = 10\ln(3)$.

If we wish to avoid using the calculator to choose test values, we note that $f(x) = \ln(x)$ is increasing. As a result, since $1 < 3$, $0 = \ln(1) < \ln(3)$ which proves $10\ln(3) > 0$. Hence, we may choose $t = 0$ as a test value in $[0, 10\ln(3))$. Since $3 < 4$, $\ln(3) < \ln(4)$, so $10\ln(3) < 10\ln(4)$. Hence, we may choose $10\ln(4)$ as test value for the interval $(10\ln(3), \infty)$.

We find $r(0) > 0$ and $r(10\ln(4)) < 0$ which gives the sign diagram below. We see $r(t) > 0$ on $[0, 10\ln(3))$.

We graph $y = T(t)$ from Example 10.1.3 below on the right along with with the horizontal line $y = 100$. We see the graph of T is above the horizontal line to the left of the intersection point, which we leave to the reader to show is $(10\ln(3), 100)$.





$y = T(t)$

Hence, the coffee is warmer than 100°F up to $10\ln(3) \approx 11$ minutes after it is served, or, said differently, it takes approximately 11 minutes for the coffee to cool to under 100°F. □

We note that, once again, we can short-cut the sign diagram in Example 10.4.3 to solve $70 + 90e^{-0.1t} > 100$. Since $\ln(x)$ is increasing, it preserves inequality. This means we can solve this inequality as follows.

$$\begin{array}{rcll}
70 + 90e^{-0.1t} & > & 100 & \\
90e^{-0.1t} & > & 30 & \\
e^{-0.1t} & > & \frac{1}{3} & \\
\ln\left(e^{-0.1t}\right) & > & \ln\left(\frac{1}{3}\right) & f(x) = \ln(x) \text{ is increasing so if } b \geq a,\ \ln(b) \geq \ln(a). \\
-0.1t & > & -\ln(3) & \ln\left(\frac{1}{3}\right) = \ln\left(3^{-1}\right) = -\ln(3). \\
t & < & \frac{-\ln(3)}{-0.1} = 10\ln(3) &
\end{array}$$

Since we are given $t \geq 0$, we arrive at the same answer $0 \leq t < 10\ln(3)$ or $[0, 10\ln(3))$.

Note the importance, once again, of having a base larger than 1 so that the corresponding logarithmic function is *increasing*. We can still adapt this strategy to exponential functions whose base is less than 1, but we need to remember the corresponding logarithmic function is *decreasing* so it *reverses* inequalities.

We close this section by finding a function inverse.

Example 10.4.4. The function $f(x) = \dfrac{5e^x}{e^x+1}$ is one-to-one.

1. Find a formula for $f^{-1}(x)$.
2. Solve $\dfrac{5e^x}{e^x+1} = 4$.

Solution.

1. We start by writing $y = f(x)$, and interchange the roles of x and y. To solve for y, we first clear denominators and then isolate the exponential function.

$$\begin{array}{rcll} y & = & \dfrac{5e^x}{e^x+1} & \\ x & = & \dfrac{5e^y}{e^y+1} & \text{Switch } x \text{ and } y \\ x\left(e^y+1\right) & = & 5e^y & \\ xe^y + x & = & 5e^y & \\ x & = & 5e^y - xe^y & \\ x & = & e^y(5-x) & \\ e^y & = & \dfrac{x}{5-x} & \\ \ln\left(e^y\right) & = & \ln\left(\dfrac{x}{5-x}\right) & \\ y & = & \ln\left(\dfrac{x}{5-x}\right) & \end{array}$$

   We claim $f^{-1}(x) = \ln\left(\frac{x}{5-x}\right)$. To verify this analytically, we would need to verify the compositions $\left(f^{-1}\circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. We leave this, as well as a graphical check, to the reader in Exercise 56.

2. We recognize the equation $\frac{5e^x}{e^x+1} = 4$ as $f(x) = 4$. Hence, our solution is $x = f^{-1}(4) = \ln\left(\frac{4}{5-4}\right) = \ln(4)$.

   We can check this fairly quickly algebraically. Using $e^{\ln(4)} = 4$, we find $\frac{5e^{\ln(4)}}{e^{\ln(4)}+1} = \frac{5(4)}{4+1} = \frac{20}{5} = 4$. $\square$

10.4.1 Exercises

In Exercises 1 - 33, solve the equation analytically.

1. $2^{4x} = 8$
2. $3^{(x-1)} = 27$
3. $5^{2x-1} = 125$
4. $4^{2t} = \frac{1}{2}$
5. $8^{t} = \frac{1}{128}$
6. $2^{(t^3-t)} = 1$
7. $3^{7x} = 81^{4-2x}$
8. $9 \cdot 3^{7x} = \left(\frac{1}{9}\right)^{2x}$
9. $3^{2x} = 5$
10. $5^{-t} = 2$
11. $5^{t} = -2$
12. $3^{(t-1)} = 29$
13. $(1.005)^{12x} = 3$
14. $e^{-5730k} = \frac{1}{2}$
15. $2000e^{0.1t} = 4000$
16. $500\left(1 - e^{2t}\right) = 250$
17. $70 + 90e^{-0.1t} = 75$
18. $30 - 6e^{-0.1t} = 20$
19. $\dfrac{100e^{x}}{e^{x}+2} = 50$
20. $\dfrac{5000}{1+2e^{-3t}} = 2500$
21. $\dfrac{150}{1+29e^{-0.8t}} = 75$
22. $25\left(\frac{4}{5}\right)^{x} = 10$
23. $e^{2x} = 2e^{x}$
24. $7e^{2t} = 28e^{-6t}$
25. $3^{(x-1)} = 2^{x}$
26. $3^{(x-1)} = \left(\frac{1}{2}\right)^{(x+5)}$
27. $7^{3+7x} = 3^{4-2x}$
28. $e^{2t} - 3e^{t} - 10 = 0$
29. $e^{2t} = e^{t} + 6$
30. $4^{t} + 2^{t} = 12$
31. $e^{x} - 3e^{-x} = 2$
32. $e^{x} + 15e^{-x} = 8$
33. $3^{x} + 25 \cdot 3^{-x} = 10$

In Exercises 34 - 41, solve the inequality analytically.

34. $e^{x} > 53$
35. $1000\,(1.005)^{12t} \geq 3000$
36. $2^{(x^3-x)} < 1$
37. $25\left(\dfrac{4}{5}\right)^{x} \geq 10$
38. $\dfrac{150}{1+29e^{-0.8t}} \leq 130$
39. $70 + 90e^{-0.1t} \leq 75$
40. $e^{-x} - xe^{-x} \geq 0$
41. $(1 - e^{t})t^{-1} \leq 0$

In Exercises 42 - 47, use a graphing utility to help you solve the equation or inequality.

42. $2^{x} = x^{2}$
43. $e^{t} = \ln(t) + 5$
44. $e^{\sqrt{x}} = x + 1$
45. $e^{-2t} - te^{-t} \geq 0$
46. $3^{(x-1)} < 2^{x}$
47. $e^{t} < t^{3} - t$

In Exercises 48 - 53, find the domain of the function.

48. $T(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

49. $C(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

50. $s(t) = \sqrt{e^{2t} - 3}$

51. $c(t) = \sqrt[3]{e^{2t} - 3}$

52. $L(x) = \log\left(3 - e^x\right)$

53. $\ell(x) = \ln\left(\dfrac{e^{2x}}{e^x - 2}\right)$

54. Since $f(x) = \ln(x)$ is a strictly increasing function, if $0 < a < b$ then $\ln(a) < \ln(b)$. Use this fact to solve the inequality $e^{(3x-1)} > 6$ without a sign diagram. Use this technique to solve the inequalities in Exercises 34 - 41. (NOTE: Isolate the exponential function first!)

55. Compute the inverse of $f(x) = \dfrac{e^x - e^{-x}}{2}$. State the domain and range of both f and f^{-1}.

56. In Example 10.4.4, we found that the inverse of $f(x) = \dfrac{5e^x}{e^x + 1}$ was $f^{-1}(x) = \ln\left(\dfrac{x}{5 - x}\right)$ but we left a few loose ends for you to tie up.

    (a) Algebraically check our answer by verifying: $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.

    (b) Find the range of f by finding the domain of f^{-1}.

    (c) With help of a graphing utility, graph $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same set of axes. How does this help to verify our answer?

    (d) Let $g(x) = \dfrac{5x}{x + 1}$ and $h(x) = e^x$. Show that $f = g \circ h$ and that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$.
    NOTE: We know this is true in general by Exercise 40 in Section 9.4, but it's nice to see a specific example of the property.

57. With the help of your classmates, solve the inequality $e^x > x^n$ for a variety of natural numbers n. What might you conjecture about the "speed" at which $f(x) = e^x$ grows versus any polynomial?

10.4.2 Answers

1. $x = \frac{3}{4}$
2. $x = 4$
3. $x = 2$
4. $t = -\frac{1}{4}$
5. $t = -\frac{7}{3}$
6. $t = -1,\ 0,\ 1$
7. $x = \frac{16}{15}$
8. $x = -\frac{2}{11}$
9. $x = \dfrac{\ln(5)}{2\ln(3)}$
10. $t = -\dfrac{\ln(2)}{\ln(5)}$
11. No solution.
12. $t = \dfrac{\ln(29) + \ln(3)}{\ln(3)}$
13. $x = \dfrac{\ln(3)}{12\ln(1.005)}$
14. $k = \dfrac{\ln\left(\frac{1}{2}\right)}{-5730} = \dfrac{\ln(2)}{5730}$
15. $t = \dfrac{\ln(2)}{0.1} = 10\ln(2)$
16. $t = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln(2)$
17. $t = \dfrac{\ln\left(\frac{1}{18}\right)}{-0.1} = 10\ln(18)$
18. $t = -10\ln\left(\frac{5}{3}\right) = 10\ln\left(\frac{3}{5}\right)$
19. $x = \ln(2)$
20. $t = \frac{1}{3}\ln(2)$
21. $t = \dfrac{\ln\left(\frac{1}{29}\right)}{-0.8} = \dfrac{5}{4}\ln(29)$
22. $x = \dfrac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)} = \dfrac{\ln(2) - \ln(5)}{\ln(4) - \ln(5)}$
23. $x = \ln(2)$
24. $t = -\frac{1}{8}\ln\left(\frac{1}{4}\right) = \frac{1}{4}\ln(2)$
25. $x = \dfrac{\ln(3)}{\ln(3) - \ln(2)}$
26. $x = \dfrac{\ln(3) + 5\ln\left(\frac{1}{2}\right)}{\ln(3) - \ln\left(\frac{1}{2}\right)} = \dfrac{\ln(3) - 5\ln(2)}{\ln(3) + \ln(2)}$
27. $x = \dfrac{4\ln(3) - 3\ln(7)}{7\ln(7) + 2\ln(3)}$
28. $t = \ln(5)$
29. $t = \ln(3)$
30. $t = \dfrac{\ln(3)}{\ln(2)}$
31. $x = \ln(3)$
32. $x = \ln(3), \ln(5)$
33. $x = \dfrac{\ln(5)}{\ln(3)}$
34. $(\ln(53), \infty)$
35. $\left[\dfrac{\ln(3)}{12\ln(1.005)}, \infty\right)$
36. $(-\infty, -1) \cup (0, 1)$
37. $\left(-\infty, \dfrac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)}\right] = \left(-\infty, \dfrac{\ln(2) - \ln(5)}{\ln(4) - \ln(5)}\right]$
38. $\left(-\infty, \dfrac{\ln\left(\frac{2}{377}\right)}{-0.8}\right] = \left(-\infty, \frac{5}{4}\ln\left(\dfrac{377}{2}\right)\right]$
39. $\left[\dfrac{\ln\left(\frac{1}{18}\right)}{-0.1}, \infty\right) = [10\ln(18), \infty)$

40. $(-\infty, 1]$

41. $(-\infty, 0) \cup (0, \infty)$

42. $x \approx -0.76666$, $x = 2$, $x = 4$

43. $x \approx 0.01866$, $x \approx 1.7115$

44. $x = 0$

45. $\approx [0.567, \infty)$

46. $\approx (-\infty, 2.7095)$

47. $\approx (2.3217, 4.3717)$

48. $(-\infty, \infty)$

49. $(-\infty, 0) \cup (0, \infty)$

50. $\left(\frac{1}{2}\ln(3), \infty\right)$

51. $(-\infty, \infty)$

52. $(-\infty, \ln(3))$

53. $(\ln(2), \infty)$

54. $x > \frac{1}{3}(\ln(6) + 1)$, so $\left(\frac{1}{3}(\ln(6) + 1), \infty\right)$

55. $f^{-1} = \ln\left(x + \sqrt{x^2 + 1}\right)$. Both f and f^{-1} have domain $(-\infty, \infty)$ and range $(-\infty, \infty)$.

10.5 Equations and Inequalities involving Logarithmic Functions

In Section 10.4 we solved equations and inequalities involving exponential functions using one of two basic strategies. We now turn our attention to equations and inequalities involving logarithmic functions, and not surprisingly, there are two basic strategies to choose from.

For example, per Theorem 10.5, the *only* solution to $\log_2(x) = \log_2(5)$ is $x = 5$. Now consider $\log_2(x) = 3$. To use Theorem 10.5, we need to rewrite 3 as a logarithm base 2. Theorem 10.4 gives us $3 = \log_2\left(2^3\right) = \log_2(8)$. Hence, $\log_2(x) = 3$ is equivalent to $\log_2(x) = \log_2(8)$ so that $x = 8$.

A second approach to solving $\log_2(x) = 3$ us to apply the corresponding exponential function, $f(x) = 2^x$ to both sides: $2^{\log_2(x)} = 2^3$ so $x = 2^3 = 8$.

A third approach to solving $\log_2(x) = 3$ is to use Theorem 10.4 to rewrite $\log_2(x) = 3$ as $2^3 = x$, so $x = 8$.

In the grand scheme of things, all three approaches we have presented to solve $\log_2(x) = 3$ are mathematically equivalent, so we opt to choose the last approach in our summary below.

Steps for Solving an Equation involving Logarithmic Functions

1. Isolate the logarithmic function.
2. (a) If convenient, express both sides as logs with the same base and equate arguments.
   (b) Otherwise, rewrite the log equation as an exponential equation.

Example 10.5.1. Solve the following equations. Check your solutions graphically using a calculator.

1. $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$
2. $2 - \ln(t-3) = 1$
3. $\log_6(x+4) + \log_6(3-x) = 1$
4. $\log_7(1-2t) = 1 - \log_7(3-t)$
5. $\log_2(x+3) = \log_2(6-x) + 3$
6. $1 + 2\log_4(t+1) = 2\log_2(t)$

Solution.

1. Since we have the same base on both sides of the equation $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$, we equate the arguments (what's inside) of the logs to get $1 - 3x = x^2 - 3$. Solving $x^2 + 3x - 4 = 0$ gives $x = -4$ and $x = 1$.

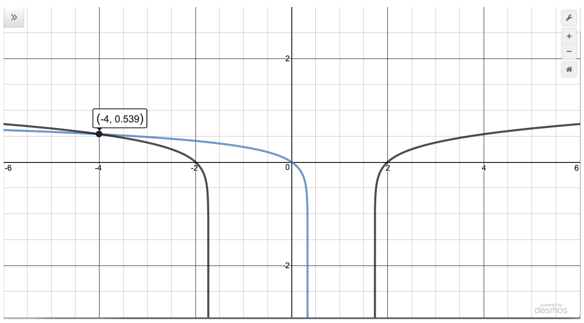
   To check these answers using a graphing utility, we make use of the change of base formula and graph $f(x) = \frac{\ln(1-3x)}{\ln(117)}$ and $g(x) = \frac{\ln\left(x^2-3\right)}{\ln(117)}$. We see these graphs intersect only at $x = -4$. however.

   To see what happened to the solution $x = 1$, we substitute it into our original equation to obtain $\log_{117}(-2) = \log_{117}(-2)$. While these expressions look identical, neither is a real number,[1] which means $x = 1$ is not in the domain of the original equation, and is not a solution.

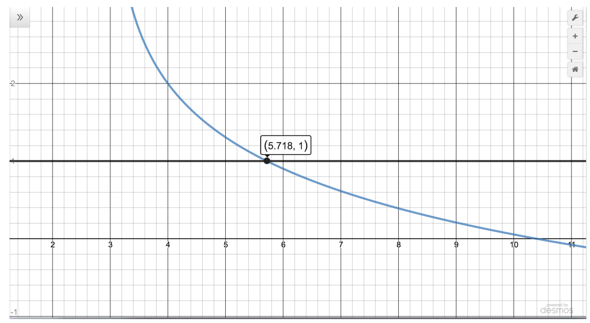
[1] They do, however, represent the same **family** of complex numbers. We refer the reader to a course in Complex Variables.

2. To solve $2 - \ln(t-3) = 1$, we first isolate the logarithm and get $\ln(t-3) = 1$. Rewriting $\ln(t-3) = 1$ as an exponential equation, we get is $e^1 = t - 3$, so $t = e + 3$.

   A graphing utility shows the graphs of $f(t) = 2 - \ln(t-3)$ and $g(t) = 1$ intersect at $t \approx 5.718 \approx e + 3$.



Checking $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$



Checking $2 - \ln(t-3) = 1$

3. We start solving $\log_6(x+4) + \log_6(3-x) = 1$ by using the Product Rule for logarithms to rewrite the equation as $\log_6\left[(x+4)(3-x)\right] = 1$.

   Rewriting as an exponential equation gives $6^1 = (x+4)(3-x)$ which reduces to $x^2 + x - 6 = 0$. We get two solutions: $x = -3$ and $x = 2$.

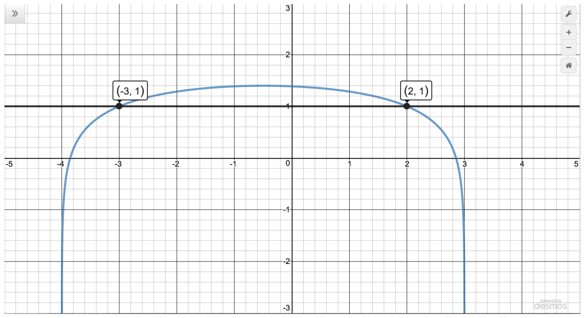
   Using the change of base formula, we graph $y = f(x) = \frac{\ln(x+4)}{\ln(6)} + \frac{\ln(3-x)}{\ln(6)}$ and $y = g(x) = 1$ and we see the graphs intersect twice, at $x = -3$ and $x = 2$, as required.

4. Taking a cue from the previous problem, we begin solving $\log_7(1-2t) = 1 - \log_7(3-t)$ by first collecting the logarithms on the same side, $\log_7(1-2t) + \log_7(3-t) = 1$, and then using the Product Rule to get $\log_7[(1-2t)(3-t)] = 1$.

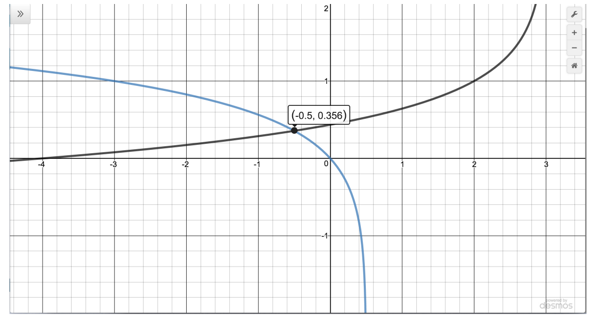
   Rewriting as an exponential equation gives $7^1 = (1-2t)(3-t)$ which gives the quadratic equation $2t^2 - 7t - 4 = 0$. Solving, we find $t = -\frac{1}{2}$ and $t = 4$.

   Once again, we use the change of base formula and find the graphs of $y = f(t) = \frac{\ln(1-2t)}{\ln(7)}$ and $y = g(t) = 1 - \frac{\ln(3-t)}{\ln(7)}$ intersect only at $t = -\frac{1}{2}$.

   Checking $t = 4$ in the original equation produces $\log_7(-7) = 1 - \log_7(-1)$, showing $t = 4$ is not in the domain of f nor g.



Checking $\log_6(x+4) + \log_6(3-x) = 1$



Checking $\log_7(1-2t) = 1 - \log_7(3-t)$

5. Our first step in solving $\log_2(x+3) = \log_2(6-x) + 3$ is to gather the logarithms to one side of of the equation: $\log_2(x+3) - \log_2(6-x) = 3$.

   The Quotient Rule gives $\log_2\left(\frac{x+3}{6-x}\right) = 3$ which, as an exponential equation is $2^3 = \frac{x+3}{6-x}$.

   Clearing denominators, we get $8(6-x) = x+3$, which reduces to $x = 5$.

   Using the change of base once again, we graph $f(x) = \frac{\ln(x+3)}{\ln(2)}$ and $g(x) = \frac{\ln(6-x)}{\ln(2)} + 3$ and find they intersect at $x = 5$.

6. Our first step in solving $1 + 2\log_4(t+1) = 2\log_2(t)$ is to gather the logs on one side of the equation. We obtain $1 = 2\log_2(t) - 2\log_4(t+1)$ but find we need a common base to combine the logs.

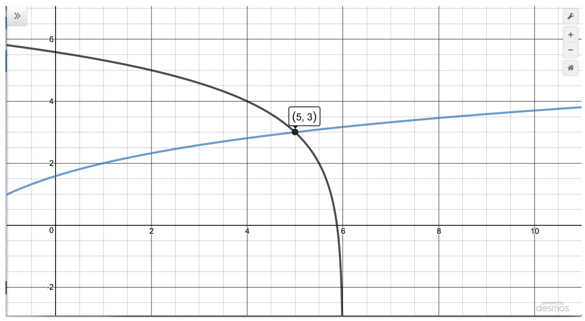
   Since 4 is a power of 2, we use change of base to convert $\log_4(t+1) = \frac{\log_2(t+1)}{\log_2(4)} = \frac{1}{2}\log_2(t+1)$. Hence, our original equation becomes

$$
\begin{array}{rcll}
1 & = & 2\log_2(t) - 2\left(\frac{1}{2}\log_2(t+1)\right) & \\
1 & = & 2\log_2(t) - \log_2(t+1) & \\
1 & = & \log_2\left(t^2\right) - \log_2(t+1) & \text{Power Rule} \\
1 & = & \log_2\left(\dfrac{t^2}{t+1}\right) & \text{Quotient Rule}
\end{array}
$$

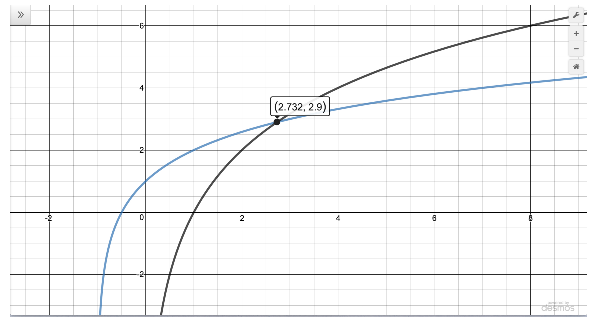
   Rewriting $1 = \log_2\left(\frac{t^2}{t+1}\right)$ in exponential form gives $\frac{t^2}{t+1} = 2$ or $t^2 - 2t - 2 = 0$. Using the quadratic formula, we obtain $t = 1 \pm \sqrt{3}$.

   One last time, we use the change of base formula and graph $f(t) = 1 + \frac{2\ln(t+1)}{\ln(4)}$ and $g(t) = \frac{2\ln(t)}{\ln(2)}$. We see the graphs intersect only at $t \approx 2.732 \approx 1 + \sqrt{3}$.

   Note the solution $t = 1 - \sqrt{3} < 0$ Hence if substituted into the original equation, the term $2\log_2\left(1-\sqrt{3}\right)$ is undefined, which explains why the graphs below intersect only once.



Checking $\log_2(x+3) = \log_2(6-x) + 3$



Checking $1 + 2\log_4(t+1) = 2\log_2(t)$

$\square$

If nothing else, Example 10.5.1 demonstrates the importance of checking for extraneous solutions[2] when solving equations involving logarithms. Even though we checked our answers graphically, extraneous solutions are easy to spot: any supposed solution which causes the argument of a logarithm to be negative must be discarded.

While identifying extraneous solutions is important, it is equally important to understand which machinations create the opportunity for extraneous solutions to appear. In the case of Example 10.5.1, extraneous solutions, by and large, result from using the Power, Product, or Quotient Rules. We encourage the reader to take the time to track each extraneous solution found in Example 10.5.1 backwards through the solution process to see at precisely which step it fails to be a solution.

As with the equations in Example 10.4.1, much can be learned from checking all of the answers in Example 10.5.1 analytically. We leave this to the reader and turn our attention to inequalities involving logarithmic functions. Since logarithmic functions are continuous on their domains, we can use sign diagrams.

Example 10.5.2. Solve the following inequalities. Check your answer graphically using a calculator.

1. $\dfrac{1}{\ln(x)+1} \leq 1$
2. $\left(\log_2(x)\right)^2 < 2\log_2(x) + 3$
3. $t\log(t+1) \geq t$

Solution.

1. We start solving $\frac{1}{\ln(x)+1} \leq 1$ by getting 0 on one side of the inequality: $\frac{1}{\ln(x)+1} - 1 \leq 0$.

   Getting a common denominator yields $\frac{1}{\ln(x)+1} - \frac{\ln(x)+1}{\ln(x)+1} \leq 0$ which reduces to $\frac{-\ln(x)}{\ln(x)+1} \leq 0$, or $\frac{\ln(x)}{\ln(x)+1} \geq 0$.

   We define $r(x) = \frac{\ln(x)}{\ln(x)+1}$ and set about finding the domain and the zeros of r. Due to the appearance of the term $\ln(x)$, we require $x > 0$. In order to keep the denominator away from zero, we solve $\ln(x) + 1 = 0$ so $\ln(x) = -1$, so $x = e^{-1} = \frac{1}{e}$. Hence, the domain of r is $\left(0, \frac{1}{e}\right) \cup \left(\frac{1}{e}, \infty\right)$.

   To find the zeros of r, we set $r(x) = \frac{\ln(x)}{\ln(x)+1} = 0$ so that $\ln(x) = 0$, and we find $x = e^0 = 1$.

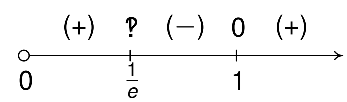
[2] Recall that an extraneous solution is an answer obtained analytically which does not satisfy the original equation.

In order to determine test values for r without resorting to the calculator, we need to find numbers between 0, $\frac{1}{e}$, and 1 which have a base of e. Since $e \approx 2.718 > 1$, $0 < \frac{1}{e^2} < \frac{1}{e} < \frac{1}{\sqrt{e}} < 1 < e$.

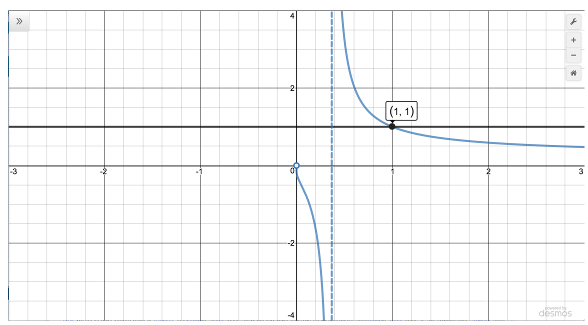
To determine the sign of $r\left(\frac{1}{e^2}\right)$, note $\ln\left(\frac{1}{e^2}\right) = \ln\left(e^{-2}\right) = -2$. Hence, $r\left(\frac{1}{e^2}\right) = \frac{-2}{-2+1} = 2 > 0$. The rest of the test values are determined similarly.

From our sign diagram, we find $r(x) \geq 0$ on $\left(0, \frac{1}{e}\right) \cup [1, \infty)$, which is our solution.

Graphing $f(x) = \frac{1}{\ln(x)+1}$ and $g(x) = 1$, we see the graph of f is below the graph of g on these intervals, and that the graphs intersect at $x = 1$.



A sign diagram for $r(x) = \dfrac{\ln(x)}{\ln(x)+1}$



Checking $\dfrac{1}{\ln(x)+1} \leq 1$

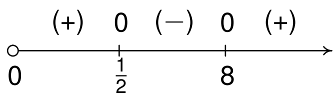
2. Moving all of the nonzero terms of $\left(\log_2(x)\right)^2 < 2\log_2(x) + 3$ to one side of the inequality in order to make use of a sign diagram, we have $\left(\log_2(x)\right)^2 - 2\log_2(x) - 3 < 0$.

   Defining $r(x) = \left(\log_2(x)\right)^2 - 2\log_2(x) - 3$, we get the domain of r is $(0, \infty)$, due to the presence of the logarithm. To find the zeros of r, we set $r(x) = \left(\log_2(x)\right)^2 - 2\log_2(x) - 3 = 0$ which we identify as a 'quadratic in disguise.'

   Setting $u = \log_2(x)$, our equation becomes $u^2 - 2u - 3 = 0$. Factoring gives us $u = -1$ and $u = 3$. Since $u = \log_2(x)$, we get $\log_2(x) = -1$, or $x = 2^{-1} = \frac{1}{2}$, and $\log_2(x) = 3$, which gives $x = 2^3 = 8$.

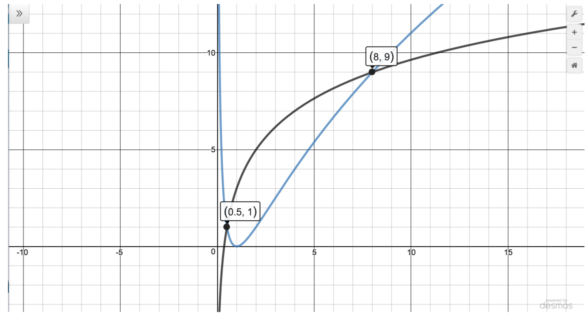
   We use test values which are powers of 2: $0 < \frac{1}{4} < \frac{1}{2} < 1 < 8 < 16$ to create the sign diagram below. From our sign diagram, we see $r(x) < 0$, which corresponds to our solution, on $\left(\frac{1}{2}, 8\right)$.

   Geometrically, the graph of $f(x) = \left(\frac{\ln(x)}{\ln(2)}\right)^2$ is below the graph of $y = g(x) = \frac{2\ln(x)}{\ln(2)} + 3$ on $\left(\frac{1}{2}, 8\right)$.



A sign diagram for

$r(x) = \left(\log_2(x)\right)^2 - 2\log_2(x) - 3$



Checking $\left(\log_2(x)\right)^2 < 2\log_2(x) + 3$

3. We begin to solve $t\log(t+1) \geq t$ by subtracting t from both sides to get $t\log(t+1) - t \geq 0$.

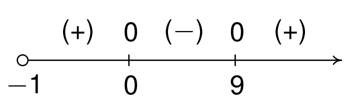
   We define $r(t) = t\log(t+1) - t$ and due to the presence of the logarithm, we require $t > -1$.

   To find the zeros of r, we set $r(t) = t\log(t+1) - t = 0$. Factoring, we get $t\,(\log(t+1) - 1) = 0$, which gives $t = 0$ or $\log(t+1) - 1 = 0$.

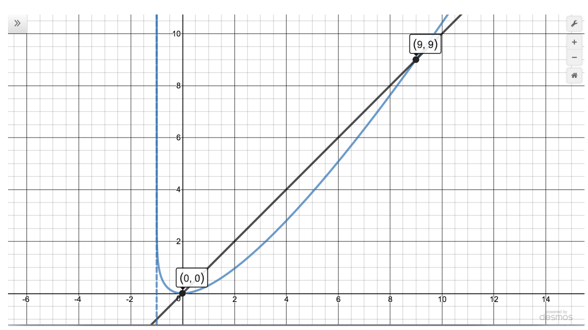
   From $\log(t+1) - 1 = 0$ we get $\log(t+1) = 1$, which we rewrite as $t + 1 = 10^1$. Hence, $t = 9$.

   We select test values t so that $t+1$ is a power of 10. Using $-1 < -0.9 < 0 < \sqrt{10} - 1 < 9 < 99$, our sign diagram gives the solution as $(-1, 0] \cup [9, \infty)$.

   We find the graphs of $y = f(t) = t\log(t+1)$ and $y = g(t) = t$ intersect at $t = 0$ and $t = 9$ with the graph of f above the graph of g on the given solution intervals.



A sign diagram for
$r(t) = t\log(t+1) - t$



Checking $t\log(t+1) \geq t$

□

Our next example revisits the concept of pH first seen in Exercise 84 in Section 10.2.

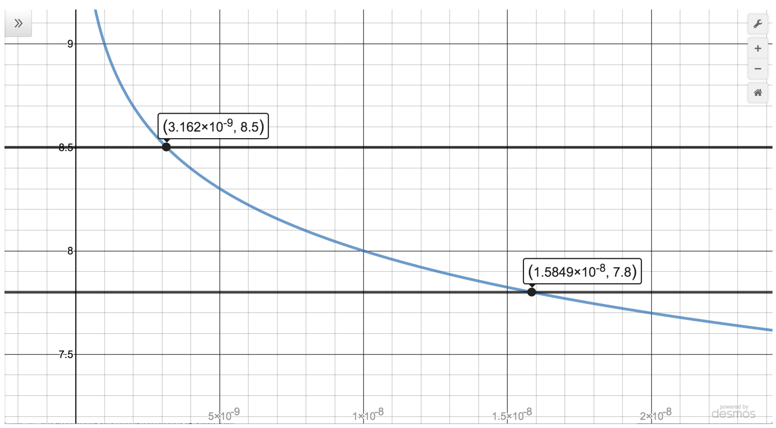
Example 10.5.3. In order to successfully breed Ippizuti fish the pH of a freshwater tank must be at least 7.8 but can be no more than 8.5. Determine the corresponding range of hydrogen ion concentration, and check your answer using a calculator.

Solution. Recall from Exercise 84 in Section 10.2 that $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter.

We require $7.8 \leq -\log[\text{H}^+] \leq 8.5$ or $-8.5 \leq \log[\text{H}^+] \leq -7.8$. One way to proceed is to break this compound inequality into two inequalities, solve each using a sign diagram, and take the intersection of the solution sets.[3]

On the other hand, we take advantage of the fact that $F(x) = 10^x$ is an increasing function, meaning that if $a \leq b \leq c$, then $10^a \leq 10^b \leq 10^c$. This property allows us to solve our inequality in one step: from $-8.5 \leq \log[\text{H}^+] \leq -7.8$, we get $10^{-8.5} \leq 10^{\log[\text{H}^+]} \leq 10^{-7.8}$, so our solution is $\leq [\text{H}^+] \leq 10^{-7.8}$. (Your Chemistry professor may want the answer written as $3.16 \times 10^{-9} \leq [\text{H}^+] \leq 1.58 \times 10^{-8}$.) Using interval notation, our answer is $\left[10^{-8.5}, 10^{-7.8}\right]$.

After *very* carefully adjusting the viewing window on the graphing utility, we see the graph of $f(x) = -\log(x)$ lies between the lines $y = 7.8$ and $y = 8.5$ on the interval $[3.162 \times 10^{-9}, 1.5849 \times 10^{-8}]$.



$\square$

We close this section by finding an inverse of a one-to-one function which involves logarithms.

Example 10.5.4. The function $f(x) = \dfrac{\log(x)}{1 - \log(x)}$ is one-to-one.

1. Find a formula for $f^{-1}(x)$ and check your answer graphically using a graphing utility.

2. Solve $\dfrac{\log(x)}{1 - \log(x)} = 1$

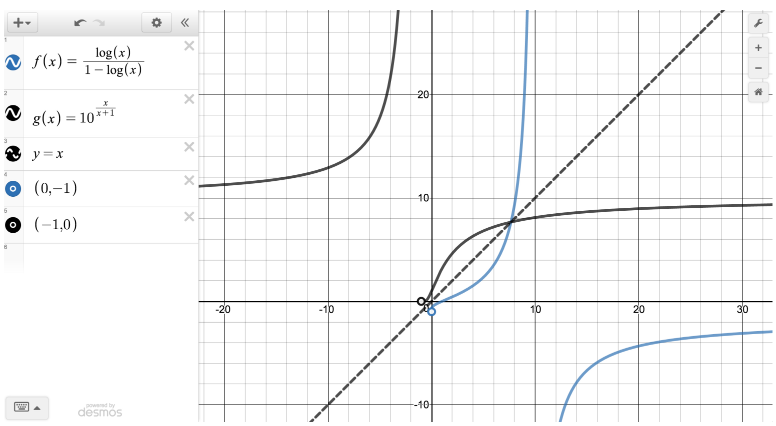
[3] Refer to page 2 for a discussion of what this means.

Solution.

1. We first write $y = f(x)$ then interchange the x and y and solve for y.

$$\begin{aligned}
y &= f(x) \\
y &= \frac{\log(x)}{1 - \log(x)} \\
x &= \frac{\log(y)}{1 - \log(y)} && \text{Interchange } x \text{ and } y. \\
x\,(1 - \log(y)) &= \log(y) \\
x - x\log(y) &= \log(y) \\
x &= x\log(y) + \log(y) \\
x &= (x+1)\log(y) \\
\frac{x}{x+1} &= \log(y) \\
y &= 10^{\frac{x}{x+1}} && \text{Rewrite as an exponential equation.}
\end{aligned}$$

We have $f^{-1}(x) = 10^{\frac{x}{x+1}}$. Graphing f and f^{-1} on the same viewing window produces the required symmetry about $y = x$.



2. Recognizing $\frac{\log(x)}{1-\log(x)} = 1$ as $f(x) = 1$, we have $x = f^{-1}(1) = 10^{\frac{1}{1+1}} = 10^{\frac{1}{2}} = \sqrt{10}$.

   To check our answer algebraically, first recall $\log(\sqrt{10}) = \log_{10}(\sqrt{10})$. Next, we know $\sqrt{10} = 10^{\frac{1}{2}}$. Hence, $\log_{10}\left(10^{\frac{1}{2}}\right) = \frac{1}{2} = 0.5$. It follows that $\frac{\log(\sqrt{10})}{1-\log(\sqrt{10})} = \frac{0.5}{1-0.5} = \frac{0.5}{0.5} = 1$, as required. $\square$

10.5.1 Exercises

In Exercises 1 - 24, solve the equation analytically.

1. $\log(3x-1) = \log(4-x)$
2. $\log_2\left(x^3\right) = \log_2(x)$
3. $\ln\left(8-t^2\right) = \ln(2-t)$
4. $\log_5\left(18-t^2\right) = \log_5(6-t)$
5. $\log_3(7-2x) = 2$
6. $\log_{\frac{1}{2}}(2x-1) = -3$
7. $\ln\left(t^2-99\right) = 0$
8. $\log(t^2-3t) = 1$
9. $\log_{125}\left(\dfrac{3x-2}{2x+3}\right) = \dfrac{1}{3}$
10. $\log\left(\dfrac{x}{10^{-3}}\right) = 4.7$
11. $-\log(x) = 5.4$
12. $10\log\left(\dfrac{x}{10^{-12}}\right) = 150$
13. $6-3\log_5(2t) = 0$
14. $3\ln(t)-2 = 1-\ln(t)$
15. $\log_3(t-4)+\log_3(t+4) = 2$
16. $\log_5(2t+1)+\log_5(t+2) = 1$
17. $\log_{169}(3x+7)-\log_{169}(5x-9) = \dfrac{1}{2}$
18. $\ln(x+1)-\ln(x) = 3$
19. $2\log_7(t) = \log_7(2)+\log_7(t+12)$
20. $\log(t)-\log(2) = \log(t+8)-\log(t+2)$
21. $\log_3(x) = \log_{\frac{1}{3}}(x)+8$
22. $\ln(\ln(x)) = 3$
23. $(\log(t))^2 = 2\log(t)+15$
24. $\ln(t^2) = (\ln(t))^2$

In Exercises 25 - 30, solve the inequality analytically.

25. $\dfrac{1-\ln(t)}{t^2} < 0$
26. $t\ln(t)-t > 0$
27. $10\log\left(\dfrac{x}{10^{-12}}\right) \geq 90$
28. $5.6 \leq \log\left(\dfrac{x}{10^{-3}}\right) \leq 7.1$
29. $2.3 < -\log(x) < 5.4$
30. $\ln(t^2) \leq (\ln(t))^2$

In Exercises 31 - 34, use a graphing utility to help you solve the equation or inequality.

31. $\ln(t) = e^{-t}$

32. $\ln(x) = \sqrt[4]{x}$

33. $\ln(t^2 + 1) \geq 5$

34. $\ln(-2x^3 - x^2 + 13x - 6) < 0$

In Exercises 35 - 40, find the domain of the function.

35. $r(x) = \dfrac{x}{1 - \ln(x)}$

36. $R(x) = \dfrac{x\ln(x)}{1 - \ln(x)}$

37. $s(t) = \sqrt{2 - \log(t)}$

38. $c(t) = (2\ln(t) - 1)^{\frac{2}{3}}$

39. $\ell(t) = \ln(\ln(t))$

40. $L(x) = \log\left(\dfrac{x\ln(x)}{1 - \ln(x)}\right)$

41. Since $f(x) = e^x$ is a strictly increasing function, if $a < b$ then $e^a < e^b$. Use this fact to solve the inequality $\ln(2x + 1) < 3$ without a sign diagram. Use this technique to solve the inequalities in Exercises 27 - 29. (Compare this to Exercise 54 in Section 10.4.)

42. Solve $\ln(3 - y) - \ln(y) = 2x + \ln(5)$ for y.

43. In Example 10.5.4 we found the inverse of $f(x) = \dfrac{\log(x)}{1 - \log(x)}$ to be $f^{-1}(x) = 10^{\frac{x}{x+1}}$.

    (a) Algebraically check our answer by verifying $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
    (b) Find the range of f by finding the domain of f^{-1}.
    (c) Let $g(x) = \dfrac{x}{1 - x}$ and $h(x) = \log(x)$. Show that $f = g \circ h$ and $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$.

        NOTE: We know this is true in general by Exercise 40 in Section 9.4, but it's nice to see a specific example of the property.

44. Let $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$. Compute $f^{-1}(x)$ and find its domain and range.

45. Explain the equation in Exercise 10 and the inequality in Exercise 28 above in terms of the Richter scale for earthquake magnitude. (See Exercise 82 in Section 10.1.)

46. Explain the equation in Exercise 12 and the inequality in Exercise 27 above in terms of sound intensity level as measured in decibels. (See Exercise 83 in Section 10.1.)

47. Explain the equation in Exercise 11 and the inequality in Exercise 29 above in terms of the pH of a solution. (See Exercise 84 in Section 10.1.)

48. With the help of your classmates, solve the inequality $\sqrt[n]{x} > \ln(x)$ for a variety of natural numbers n. What might you conjecture about the "speed" at which $f(x) = \ln(x)$ grows versus any principal n^{th} root function?

10.5.2 Answers

1. $x = \frac{5}{4}$
2. $x = 1$
3. $t = -2$
4. $t = -3,\ 4$
5. $x = -1$
6. $x = \frac{9}{2}$
7. $t = \pm 10$
8. $t = -2,\ 5$
9. $x = -\frac{17}{7}$
10. $x = 10^{1.7}$
11. $x = 10^{-5.4}$
12. $x = 10^{3}$
13. $t = \frac{25}{2}$
14. $t = e^{3/4}$
15. $t = 5$
16. $t = \frac{1}{2}$
17. $x = 2$
18. $x = \frac{1}{e^3-1}$
19. $t = 6$
20. $t = 4$
21. $x = 81$
22. $x = e^{e^3}$
23. $t = 10^{-3},\ 10^{5}$
24. $t = 1,\ x = e^2$
25. (e, ∞)
26. (e, ∞)
27. $\left[10^{-3}, \infty\right)$
28. $\left[10^{2.6}, 10^{4.1}\right]$
29. $\left(10^{-5.4}, 10^{-2.3}\right)$
30. $(0, 1] \cup [e^2, \infty)$
31. $t \approx 1.3098$
32. $x \approx 4.177,\ x \approx 5503.665$
33. $\approx (-\infty, -12.1414) \cup (12.1414, \infty)$
34. $\approx (-3.0281, -3) \cup (0.5, 0.5991) \cup (1.9299, 2)$
35. $(-\infty, e) \cup (e, \infty)$
36. $(0, e) \cup (e, \infty)$
37. $(0, 100]$
38. $(0, \infty)$
39. $(1, \infty)$
40. $(1, e)$
41. $-\dfrac{1}{2} < x < \dfrac{e^3-1}{2}$, so $\left(-\dfrac{1}{2}, \dfrac{e^3-1}{2}\right)$
42. $y = \dfrac{3}{5e^{2x}+1}$
44. $f^{-1}(x) = \dfrac{e^{2x}-1}{e^{2x}+1} = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$.

    To see why we rewrite this in this form, see Exercise **??** in Section **??**.

    The domain of f^{-1} is $(-\infty, \infty)$ and its range is the same as the domain of f, namely $(-1, 1)$.