# Functions

based on Precalculus
Version $4 - \epsilon$

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# Chapter 1

# Basic Concepts and Review

## 1.1 Basic Set Theory and Interval Notation

### 1.1.1 Some Basic Set Theory Notions

We begin this section with the definition of a concept that is central to all of Mathematics.

[redacted]

For example, the collection of letters that make up the word "smolko" is well-defined and is a set, but the collection of the worst Math teachers in the world is **not** well-defined and therefore is **not** a set.[1]

In general, there are three ways to describe sets and those methods are listed below.

[redacted]

Let $S$ be the set described *verbally* as the set of letters that make up the word "smolko". A *roster* description of $S$ is $\{s, m, o, l, k\}$. Note that we listed 'o' only once, even though it appears twice in the word "smolko". Also, the order of the elements doesn't matter, so $\{k, l, m, o, s\}$ is also a roster description of $S$. A *set-builder* description of $S$ is: $\{x \mid x \text{ is a letter in the word "smolko"}\}$. The way to read this is 'The set of elements $x$ such that $x$ is a letter in the word "smolko".' In each of the above cases, we may use the familiar equals sign '=' and write $S = \{s, m, o, l, k\}$ or $S = \{x \mid x \text{ is a letter in the word "smolko"}\}$.

[1] For a more thought-provoking example, consider the collection of all things that do not contain themselves - this leads to the famous paradox known as Russell's Paradox.

Notice that $m$ is in $S$ but many other letters, such as $q$, are not in $S$. We express these ideas of set inclusion and exclusion mathematically using the symbols $m \in S$ (read '$m$ is in $S$') and $q \notin S$ (read '$q$ is not in $S$'). More precisely, we have the following.

[redacted]

Now let's consider the set $C = \{x \mid x \text{ is a consonant in the word "smolko"}\}$. A roster description of $C$ is $C = \{s, m, l, k\}$. Note that by construction, every element of $C$ is also in $S$. We express this relationship by stating that the set $C$ is a **subset** of the set $S$, which is written in symbols as $C \subseteq S$. The more formal definition is given at the top of the next page.

[redacted]

In our previous example, $C \subseteq S$ yet not vice-versa since $o \in S$ but $o \notin C$. Additionally, the set of vowels $V = \{a, e, i, o, u\}$, while it does have an element in common with $S$, is not a subset of $S$. (As an added note, $S$ is not a subset of $V$, either.) We could, however, *build* a set which contains both $S$ and $V$ as subsets by gathering all of the elements in both $S$ and $V$ together into a single set, say $U = \{s, m, o, l, k, a, e, i, u\}$. Then $S \subseteq U$ and $V \subseteq U$. The set $U$ we have built is called the **union** of the sets $S$ and $V$ and is denoted $S \cup V$. Furthermore, $S$ and $V$ aren't completely *different* sets since they both contain the letter 'o.' The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of $S$ and $V$ is $\{o\}$, written $S \cap V = \{o\}$. We formalize these ideas below.

[redacted]

The key words in Definition 1.4 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). Please note that this mathematical use of the word 'or' differs than how we use 'or' in spoken English. In Math, we use the *inclusive or* which allows for the element to be in both sets. At a restaurant if you're asked "Do you want fries or a salad?" you must pick one and only one. This is known as the *exclusive or* and it plays a role in other Math classes. For our purposes it is good enough to say that for an element to belong to the union of two sets it must belong to *at least one* of them.

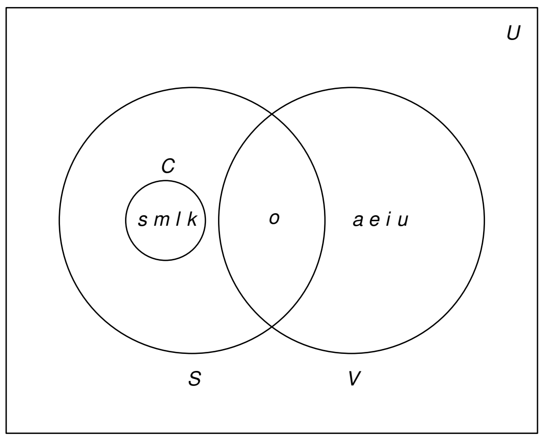
Returning to the sets $C$ and $V$ above, $C \cup V = \{s, m, l, k, a, e, i, o, u\}$.[2] Their intersection, however, creates a bit of notational awkwardness since $C$ and $V$ have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

[2]Which just so happens to be the same set as $S \cup V$.

[redacted]

As promised, the empty set is the set containing no elements since no matter what '$x$' is, '$x = x$.' Like the number '0,' the empty set plays a vital role in mathematics.[3] We introduce it here more as a symbol of convenience as opposed to a contrivance[4] because saying that $C \cap V = \emptyset$ is unambiguous whereas $\{\}$ looks like a typographical error.

A nice way to visualize the relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets $S$, $C$ and $V$ is drawn at the top of the next page.



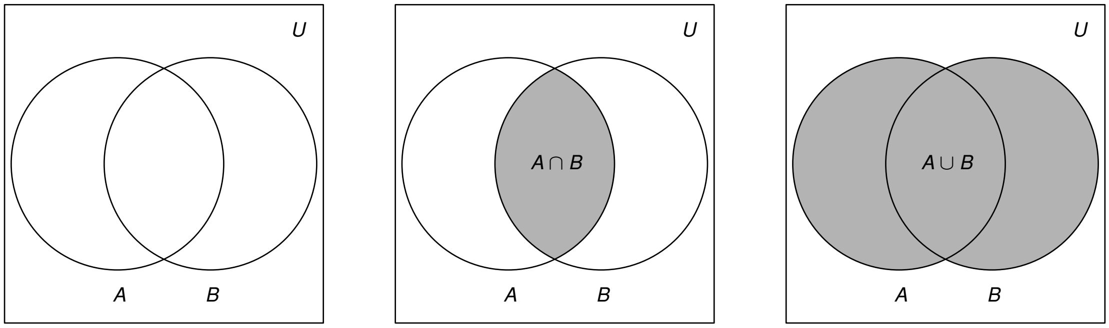

A Venn Diagram for $C$, $S$ and $V$.

In the Venn Diagram above we have three circles - one for each of the sets $C$, $S$ and $V$. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set $C$ is completely inside the circle representing $S$. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing $S$ and $V$ overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that since $C \cap V = \emptyset$, the circles which represent $C$ and $V$ have no overlap whatsoever.

All of these circles lie in a rectangle labeled $U$ for the 'universal' set. A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or $U$ as the set of letters in the entire alphabet. The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no' and we refer you once again to Russell's Paradox. The usual triptych of Venn Diagrams indicating generic sets $A$ and $B$ along with $A \cap B$ and $A \cup B$ is given below.

[3] Sadly, the full extent of the empty set's role will not be explored in this text.

[4] Actually, the empty set can be used to generate numbers - mathematicians can create something from nothing!




Sets $A$ and $B$. $A \cap B$ is shaded. $A \cup B$ is shaded.

The one major limitation of Venn Diagrams is that they become unwieldy if more than four sets need to be drawn simultaneously within the same universal set. This idea is explored in the Exercises.

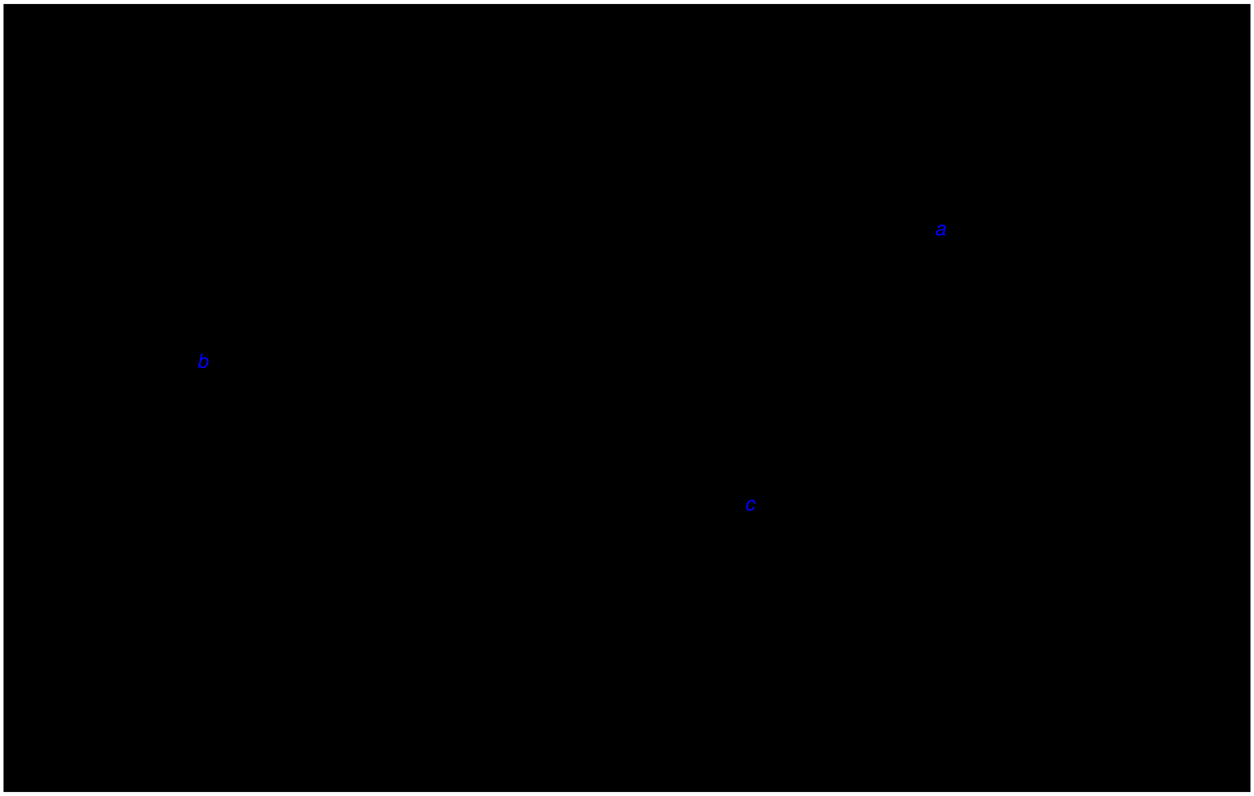
### 1.1.2 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Much of the "real world" can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete[5] definition of a real number is given below.

[redacted]

Certain subsets of the real numbers are worthy of note and are listed below. In fact, in more advanced texts,[6] the real numbers are *constructed* from some of these subsets.

[5] Math pun intended!
[6] See, for instance, Landau's Foundations of Analysis.




Note that every natural number is a whole number which, in turn, is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and since every rational number is a real number[7] the sets $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are nested like Matryoshka dolls. More formally, these sets form a subset chain: $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above.

It is time to put all of this together in an example.

**Example 1.1.1.**

1. Write a roster description for $P = \{2^n \mid n \in \mathbb{N}\}$ and $E = \{2n \mid n \in \mathbb{Z}\}$.

2. Write a verbal description for $S = \{x^2 \mid x \in \mathbb{R}\}$.

3. Let $A = \left\{-117, \frac{4}{5}, 0.20\overline{2002}, 0.202002000200002\ldots\right\}$.

   (a) Which elements of $A$ are natural numbers? Rational numbers? Real numbers?

   (b) Find $A \cap \mathbb{W}$, $A \cap \mathbb{Z}$ and $A \cap \mathbb{P}$.

4. What is another name for $\mathbb{N} \cup \mathbb{Q}$? What about $\mathbb{Q} \cup \mathbb{P}$?

[7] Thanks to long division!

**Solution.**

1. To find roster descriptions for each of these sets, we need to list their elements. Starting with the set $P = \{2^n \,|\, n \in \mathbb{N}\}$, we substitute natural number values $n$ into the formula $2^n$. For $n = 1$ we get $2^1 = 2$, for $n = 2$ we get $2^2 = 4$, for $n = 3$ we get $2^3 = 8$ and for $n = 4$ we get $2^4 = 16$. Hence $P$ describes the powers of 2, so a roster description for $P$ is $P = \{2, 4, 8, 16, ...\}$ where the '...' indicates the that pattern continues.[8]

   Proceeding in the same way, we generate elements in $E = \{2n \,|\, n \in \mathbb{Z}\}$ by plugging in integer values of $n$ into the formula $2n$. Starting with $n = 0$ we obtain $2(0) = 0$. For $n = 1$ we get $2(1) = 2$, for $n = -1$ we get $2(-1) = -2$ for $n = 2$, we get $2(2) = 4$ and for $n = -2$ we get $2(-2) = -4$. As $n$ moves through the integers, $2n$ produces all of the *even* integers.[9] A roster description for $E$ is $E = \{0, \pm 2, \pm 4, ...\}$.

2. One way to verbally describe $S$ is to say that $S$ is the 'set of all squares of real numbers'. While this isn't incorrect, we'd like to take this opportunity to delve a little deeper.[10] What makes the set $S = \{x^2 \,|\, x \in \mathbb{R}\}$ a little trickier to wrangle than the sets $P$ or $E$ above is that the dummy variable here, $x$, runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way.[11] Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in $S$. For $x = -2$, $x^2 = (-2)^2 = 4$ so 4 is in $S$, as are $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $(\sqrt{117})^2 = 117$. Even things like $(-\pi)^2$ and $(0.101001000100001\ldots)^2$ are in $S$.

   So suppose $s \in S$. What can be said about $s$? We know there is some real number $x$ so that $s = x^2$. Since $x^2 \geq 0$ for any real number $x$, we know $s \geq 0$. This tells us that everything in $S$ is a non-negative real number.[12] This begs the question: are <u>all</u> of the non-negative real numbers in $S$? Suppose $n$ is a non-negative real number, that is, $n \geq 0$. If $n$ were in $S$, there would be a real number $x$ so that $x^2 = n$. As you may recall, we can solve $x^2 = n$ by 'extracting square roots': $x = \pm\sqrt{n}$. Since $n \geq 0$, $\sqrt{n}$ is a real number.[13] Moreover, $(\sqrt{n})^2 = n$ so $n$ is the square of a real number which means $n \in S$. Hence, $S$ is the set of non-negative real numbers.

3. (a) The set $A$ contains no natural numbers.[14] Clearly $\frac{4}{5}$ is a rational number as is $-117$ (which can be written as $\frac{-117}{1}$). It's the last two numbers listed in $A$, $0.20\overline{2002}$ and $0.202002000200002\ldots$, that warrant some discussion. First, recall that the 'line' over the digits 2002 in $0.20\overline{2002}$ (called the vinculum) indicates that these digits repeat, so it is a rational number.[15] As for the number $0.202002000200002\ldots$, the '...' indicates the pattern of adding an extra '0' followed by a '2' is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal

[8] This isn't the most *precise* way to describe this set - it's always dangerous to use '...' since we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

[9] This shouldn't be too surprising, since an even integer is *defined* to be an integer multiple of 2.

[10] Think of this as an opportunity to stop and smell the mathematical roses.

[11] This is a nontrivial statement. Interested readers are directed to a discussion of Cantor's Diagonal Argument.

[12] This means $S$ is a subset of the non-negative real numbers.

[13] This is called the 'square root closed property' of the non-negative real numbers.

[14] Carl was tempted to include $0.\overline{9}$ in the set $A$, but thought better of it. See Section **??** for details.

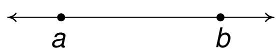
[15] So $0.20\overline{2002} = 0.20200220022002\ldots$.

is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of $A$ are real numbers, since all of them can be expressed as decimals (remember that $\frac{4}{5} = 0.8$).

(b) The set $A \cap \mathbb{W} = \{x \mid x \in A \text{ and } x \in \mathbb{W}\}$ is another way of saying we are looking for the set of numbers in $A$ which are whole numbers. Since $A$ contains no whole numbers, $A \cap \mathbb{W} = \emptyset$. Similarly, $A \cap \mathbb{Z}$ is looking for the set of numbers in $A$ which are integers. Since $-117$ is the only integer in $A$, $A \cap \mathbb{Z} = \{-117\}$. For the set $A \cap \mathbb{P}$, as discussed in part (a), the number $0.202002000200002\ldots$ is irrational, so $A \cap \mathbb{P} = \{0.202002000200002\ldots\}$.

4. The set $\mathbb{N} \cup \mathbb{Q} = \{x \mid x \in \mathbb{N} \text{ or } x \in \mathbb{Q}\}$ is the union of the set of natural numbers with the set of rational numbers. Since every natural number is a rational number, $\mathbb{N}$ doesn't contribute any new elements to $\mathbb{Q}$, so $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$.[16] For the set $\mathbb{Q} \cup \mathbb{P}$, we note that every real number is either rational or not, hence $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$, pretty much by the definition of the set $\mathbb{P}$. $\square$

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers $a$ and $b$, we write $a < b$ if $a$ is located to the left of $b$ on the number line, as shown below.




The real number line with two numbers $a$ and $b$ where $a < b$.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that $\mathbb{R}$ is complete. This means that there are no 'holes' or 'gaps' in the real number line.[17] Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you'll find a solid line segment (or interval) consisting of infinitely many real numbers. The next result tells us what types of numbers we can expect to find.

[redacted]

The root word 'dense' here communicates the idea that rationals and irrationals are 'thoroughly mixed' into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you've done that, try doing the same thing for the numbers $0.\overline{9}$ and 1. ('Try' is the operative word, here.[18])

The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers $a$ and $b$, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). This property is given below.

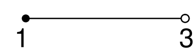
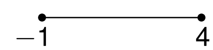
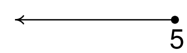
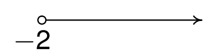
[16] In fact, anytime $A \subseteq B$, $A \cup B = B$ and vice-versa. See the exercises.

[17] Alas, this intuitive feel for what it means to be 'complete' is as good as it gets at this level. Completeness is given a much more precise meaning later in courses like Analysis and Topology.

[18] Again, see Section **??** for details.

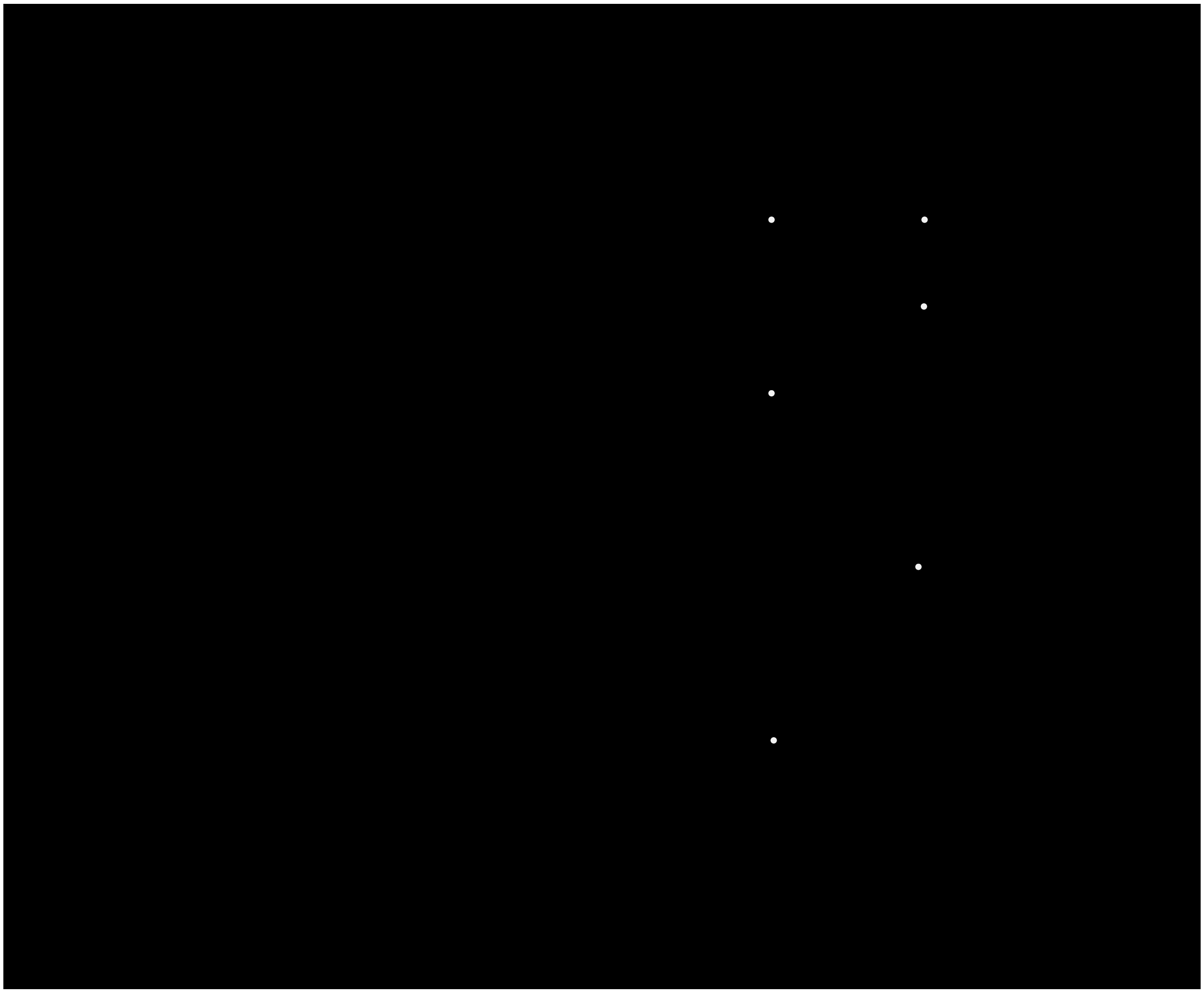
[redacted]

Segments of the real number line are called **intervals**. They play a huge role not only in this text but also in the Calculus curriculum so we need a concise way to describe them. We start by examining a few examples of the **interval notation** associated with some specific sets of numbers.

| Set of Real Numbers | Interval Notation | Region on the Real Number Line |
|---|---|---|
| $\{x \mid 1 \leq x < 3\}$ | $[1, 3)$ |  |
| $\{x \mid -1 \leq x \leq 4\}$ | $[-1, 4]$ |  |
| $\{x \mid x \leq 5\}$ | $(-\infty, 5]$ |  |
| $\{x \mid x > -2\}$ | $(-2, \infty)$ |  |

As you can glean from the table, for intervals with finite endpoints we start by writing 'left endpoint, right endpoint'. We use square brackets, '[' or ']', if the endpoint is included in the interval. This corresponds to a 'filled-in' or 'closed' dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, '(' or ')' that correspond to an 'open' circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol $\infty$ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions. We summarize all of the possible cases in one convenient table below.[19]

[19] The importance of understanding interval notation in this book and also in Calculus cannot be overstated so please do yourself a favor and memorize this chart.



Intervals of the forms $(a, b), (-\infty, b)$ and $(a, \infty)$ are said to be **open** intervals. Those of the forms $[a, b], (-\infty, b]$ and $[a, \infty)$ are said to be **closed** intervals.

Unfortunately, the words 'open' and 'closed' are not antonyms here because the empty set $\emptyset$ and the set $(-\infty, \infty)$ are simultaneously open and closed[20] while the intervals $(a, b]$ and $[a, b)$ are neither open nor closed. The inclusion or exclusion of an endpoint might seem like a terribly small thing to fuss about but these sorts of technicalities in the language become important in Calculus so we feel the need to put this material in the Precalculus book.

We close this section with an example that ties together some of the concepts presented earlier. Specifically, we demonstrate how to use interval notation along with the concepts of union and intersection to describe a variety of sets on the real number line. In many sections of the text to come you will need to be fluent with this notation so take the time to study it deeply now.

[20]You don't need to worry about that fact until you take an advanced course in Topology.

**Example 1.1.2.**

1. Express the following sets of numbers using interval notation.

   (a) $\{x \mid x \leq -2 \text{ or } x \geq 2\}$

   (b) $\{x \mid x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\}$

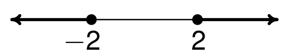
   (c) $\{x \mid x \neq \pm 3\}$

   (d) $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$

2. Let $A = [-5, 3)$ and $B = (1, \infty)$. Find $A \cap B$ and $A \cup B$.

**Solution.**

1. (a) The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. The 'or' in $\{x \mid x \leq -2 \text{ or } x \geq 2\}$ tells us that we are looking for the union of these two intervals, so our answer is $(-\infty, -2] \cup [2, \infty)$.

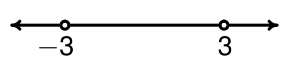



$(-\infty, -2] \cup [2, \infty)$

   (b) For the set $\{x \mid x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\}$, we need the real numbers less than (to the left of) $\sqrt{3}$ that are simultaneously greater than (to the right of) $-\frac{8}{5}$, including $-\frac{8}{5}$ but excluding $\sqrt{3}$. This yields $\{x \mid x < \sqrt{3} \text{ and } x \geq -\frac{8}{5}\} = \left[-\frac{8}{5}, \sqrt{3}\right)$.

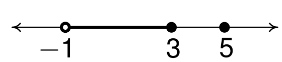

$\left[-\frac{8}{5}, \sqrt{3}\right)$

   (c) For the set $\{x \mid x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. (Refer back to page 4 for a discussion about $x = \pm 3$) This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.




$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

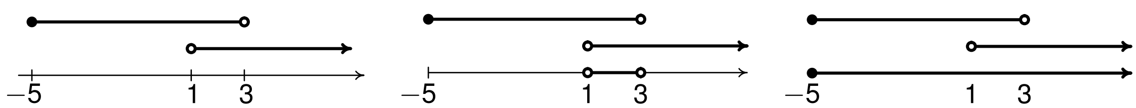
   (d) Graphing the set $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$ yields the interval $(-1, 3]$ along with the single number 5. While we *could* express this single point as $[5, 5]$, it is customary to write a single point as a 'singleton set', so in our case we have the set $\{5\}$. This means that our final answer is written $\{x \mid -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.




$(-1, 3] \cup \{5\}$

2. We start by graphing $A = [-5, 3)$ and $B = (1, \infty)$ on the number line. To find $A \cap B$, we need to find the numbers common to both $A$ and $B$; in other words, we need to find the overlap of the two intervals. Clearly, everything between 1 and 3 is in both $A$ and $B$. However, since 1 is in $A$ but not in $B$, 1 is not in the intersection. Similarly, since 3 is in $B$ but not in $A$, it isn't in the intersection either. Hence, $A \cap B = (1, 3)$.

   To find $A \cup B$, we need to find the numbers in at least one of $A$ or $B$. Graphically, we shade $A$ and $B$ along with it. Notice here that even though 1 isn't in $B$, it is in $A$, so it's in the union along with all of the other elements of $A$ between $-5$ and 1. A similar argument goes for the inclusion of 3 in the union. The result of shading both $A$ and $B$ together gives us $A \cup B = [-5, \infty)$.




$A = [-5, 3)$, $B = (1, \infty)$ $\qquad$ $A \cap B = (1, 3)$ $\qquad$ $A \cup B = [-5, \infty)$ $\qquad \square$

### 1.1.3 Exercises

1. Find a verbal description for $O = \{2n - 1 \mid n \in \mathbb{N}\}$

2. Find a roster description for $X = \{z^2 \mid z \in \mathbb{Z}\}$

3. Let $A = \left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \sqrt{3}, 5.2020020002\ldots, \frac{20}{10}, 117\right\}$

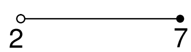
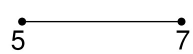
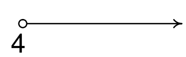
   (a) List the elements of $A$ which are natural numbers.

   (b) List the elements of $A$ which are irrational numbers.

   (c) Find $A \cap \mathbb{Z}$

   (d) Find $A \cap \mathbb{Q}$

4. Fill in the chart below.

| Set of Real Numbers | Interval Notation | Region on the Real Number Line |
|---|---|---|
| $\{x \mid -1 \leq x < 5\}$ | | |
| | $[0, 3)$ | |
| | |   |
| $\{x \mid -5 < x \leq 0\}$ | | |
| | $(-3, 3)$ | |
| | |   |
| $\{x \mid x \leq 3\}$ | | |
| | $(-\infty, 9)$ | |
| | |   |
| $\{x \mid x \geq -3\}$ | | |

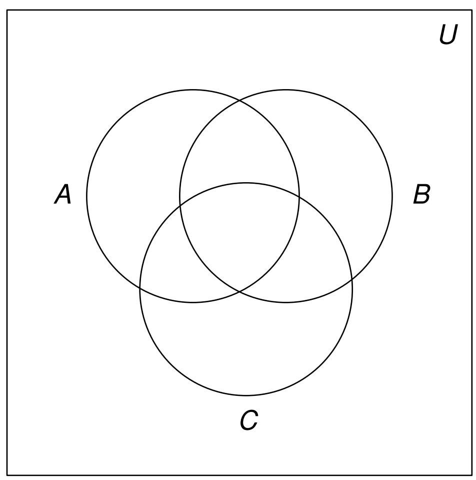
In Exercises **??** - **??**, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

5. $(-1,5]\cap[0,8)$
6. $(-1,1)\cup[0,6]$
7. $(-\infty,4]\cap(0,\infty)$
8. $(-\infty,0)\cap[1,5]$
9. $(-\infty,0)\cup[1,5]$
10. $(-\infty,5]\cap[5,8)$

In Exercises **??** - **??**, write the set using interval notation.

11. $\{x \mid x \neq 5\}$
12. $\{x \mid x \neq -1\}$
13. $\{x \mid x \neq -3,\ 4\}$
14. $\{x \mid x \neq 0,\ 2\}$
15. $\{x \mid x \neq 2,\ -2\}$
16. $\{x \mid x \neq 0,\ \pm 4\}$
17. $\{x \mid x \leq -1 \text{ or } x \geq 1\}$
18. $\{x \mid x < 3 \text{ and } x \geq 2\}$
19. $\{x \mid x \leq -3 \text{ or } x > 0\}$
20. $\{x \mid x \leq 2 \text{ and } x > 3\}$
21. $\{x \mid x > 2 \text{ or } x = \pm 1\}$
22. $\{x \mid 3 < x < 13 \text{ and } x \neq 4\}$

For Exercises **??** - **??**, use the blank Venn Diagram below with $A$, $B$, and $C$ in it as a guide to help you shade the following sets.




23. $A\cup C$
24. $B\cap C$
25. $(A\cup B)\cup C$
26. $(A\cap B)\cap C$
27. $A\cap(B\cup C)$
28. $(A\cap B)\cup(A\cap C)$
29. Explain how your answers to problems **??** and **??** show $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$. Phrased differently, this shows 'intersection *distributes* over union.' Discuss with your classmates if 'union' distributes over 'intersection.' Use a Venn Diagram to support your answer.
30. Show that $A\subseteq B$ if and only if $A\cup B = B$.
31. Let $A = \{1,3,5,7,9\}$, $B = \{2,4,6,8,10\}$, $C = \{1,6,9\}$ and $D = \{2,7,10\}$. Draw one Venn Diagram that shows all four of these sets. What sort of difficulties do you encounter?

### 1.1.4 Answers

1. $O$ is the odd natural numbers.

2. $X = \{0, 1, 4, 9, 16, ...\}$

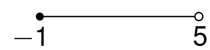
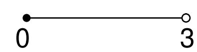
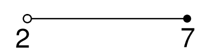
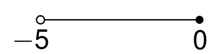
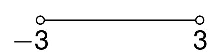
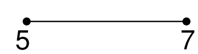
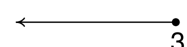
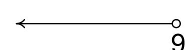
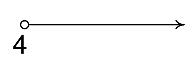
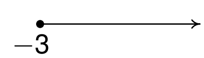
3. (a) $\frac{20}{10} = 2$ and 117

   (b) $\sqrt{3}$ and 5.2020020002

   (c) $\left\{-3, \frac{20}{10}, 117\right\}$

   (d) $\left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \frac{20}{10}, 117\right\}$

4.

| Set of Real Numbers | Interval Notation | Region on the Real Number Line |
|---|---|---|
| $\{x \mid -1 \leq x < 5\}$ | $[-1, 5)$ |  |
| $\{x \mid 0 \leq x < 3\}$ | $[0, 3)$ |  |
| $\{x \mid 2 < x \leq 7\}$ | $(2, 7]$ |  |
| $\{x \mid -5 < x \leq 0\}$ | $(-5, 0]$ |  |
| $\{x \mid -3 < x < 3\}$ | $(-3, 3)$ |  |
| $\{x \mid 5 \leq x \leq 7\}$ | $[5, 7]$ |  |
| $\{x \mid x \leq 3\}$ | $(-\infty, 3]$ |  |
| $\{x \mid x < 9\}$ | $(-\infty, 9)$ |  |
| $\{x \mid x > 4\}$ | $(4, \infty)$ |  |
| $\{x \mid x \geq -3\}$ | $[-3, \infty)$ |  |

5. $(-1, 5] \cap [0, 8) = [0, 5]$

6. $(-1, 1) \cup [0, 6] = (-1, 6]$

7. $(-\infty, 4] \cap (0, \infty) = (0, 4]$

8. $(-\infty, 0) \cap [1, 5] = \emptyset$

9. $(-\infty, 0) \cup [1, 5] = (-\infty, 0) \cup [1, 5]$

10. $(-\infty, 5] \cap [5, 8) = \{5\}$

11. $(-\infty, 5) \cup (5, \infty)$

12. $(-\infty, -1) \cup (-1, \infty)$

13. $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

14. $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

15. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

16. $(-\infty, -4) \cup (-4, 0) \cup (0, 4) \cup (4, \infty)$

17. $(-\infty, -1] \cup [1, \infty)$

18. $[2, 3)$

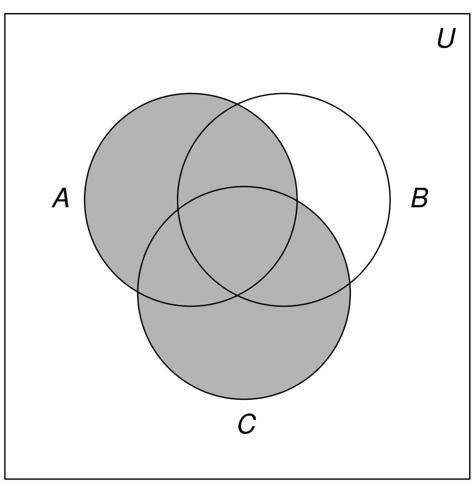
19. $(-\infty, -3] \cup (0, \infty)$

20. $\emptyset$

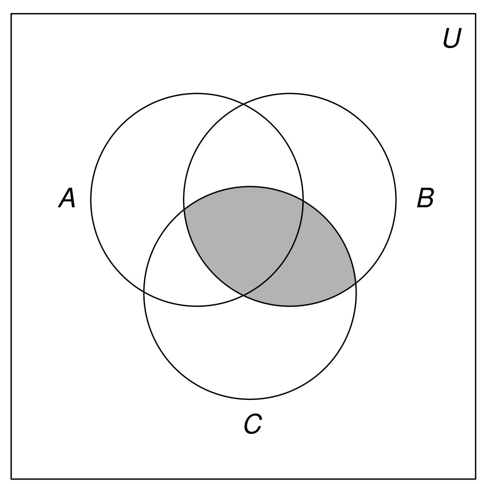
21. $\{-1\} \cup \{1\} \cup (2, \infty)$

22. $(3, 4) \cup (4, 13)$

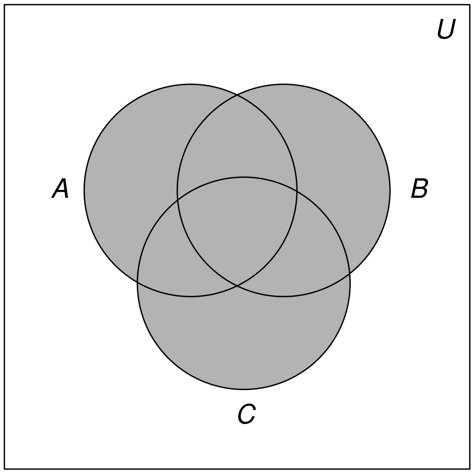
23. $A \cup C$



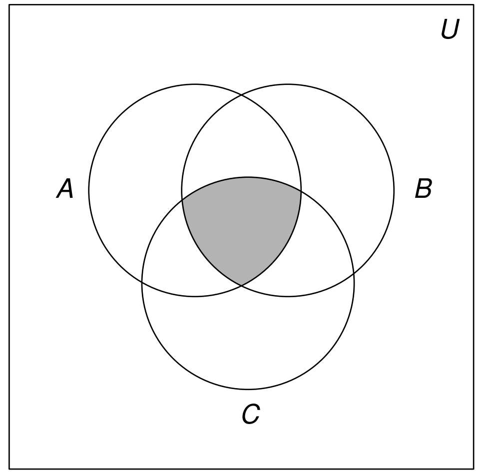

24. $B \cap C$



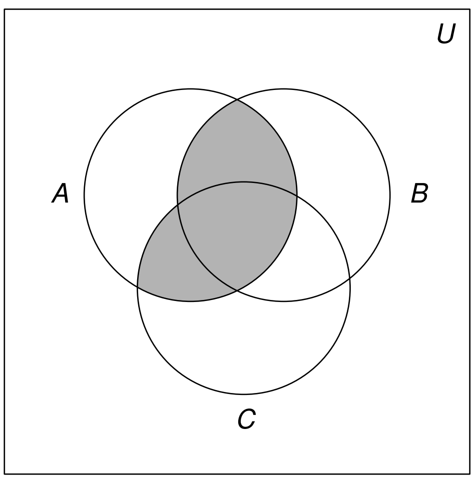
25. $(A \cup B) \cup C$



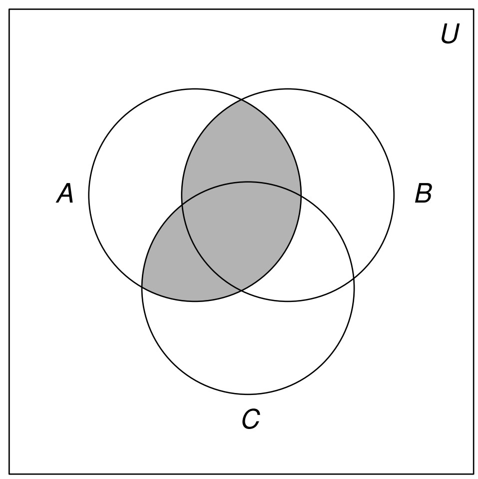

26. $(A \cap B) \cap C$



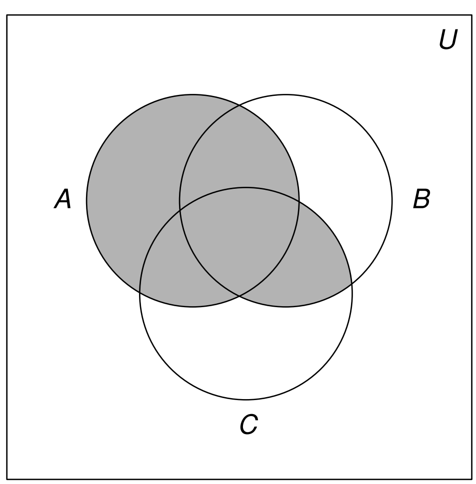

27. $A \cap (B \cup C)$




28. $(A \cap B) \cup (A \cap C)$




29. Yes, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



## 1.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra. There are two primary operations one can perform with real numbers: addition and multiplication. We'll start with the properties of addition.

[redacted]

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers $a$ and $b$ a variety of ways: $ab$, $a \cdot b$, $a(b)$, $(a)(b)$ and so on. We'll refrain from using $a \times b$ for real number multiplication in this text with one notable exception in Definition **??**.

[redacted]

While most students and some faculty tend to skip over these properties or give them a cursory glance at best,[1] it is important to realize that the properties stated above are what drive the symbolic manipulation in all of Algebra. When listing a tally of more than two numbers, $1+2+3$ for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as $(1+2)+3$ or $1+(2+3)$. This brings up a note about 'grouping symbols'. Recall that parentheses and brackets

[1] Not unlike how Carl approached all the Elven poetry in The Lord of the Rings.

are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example, $1 + 2 \cdot 3 = 1 + 6 = 7$, but $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$. As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first: $(1 + 2) \cdot 3 = 1 \cdot 3 + 2 \cdot 3 = 3 + 6 = 9$. More generally, we have the following.

[redacted]

It is worth pointing out that we didn't really need to list the Distributive Property both for $a(b+c)$ (distributing from the left) and $(a+b)c$ (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, 'factoring' is really the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students *see* these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression $5(2 + x)$, without knowing the value of $x$, we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get $5(2 + x) = 5 \cdot 2 + 5 \cdot x = 10 + 5x$. The Distributive Property is also responsible for combining 'like terms'. Why is $3x + 2x = 5x$? Because $3x + 2x = (3 + 2)x = 5x$.

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

[redacted]

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve $x^2 + x - 6 = 0$ is by factoring[2] the left hand side of this equation to get $(x-2)(x+3) = 0$. From here, we apply the Zero Product Property and set each factor equal to zero. This yields $x - 2 = 0$ or $x + 3 = 0$ so $x = 2$ or $x = -3$. This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter **??**.

Next up is a review of the arithmetic of 'negatives'. On page **??** we first introduced the dash which we all recognize as the 'negative' symbol in terms of the additive inverse. For example, the number $-3$ (read 'negative 3') is defined so that $3 + (-3) = 0$. We then defined subtraction using the concept of the additive inverse again so that, for example, $5-3 = 5+(-3)$. In this text we do not distinguish typographically between the dashes in the expressions '$5 - 3$' and '$-3$' even though they are mathematically quite different.[3] In the expression '$5 - 3$,' the dash is a *binary* operation (that is, an operation requiring *two* numbers) whereas in '$-3$', the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, 'Who cares?' Your calculator does - that's who! In the text we can write $-3 - 3 = -6$ but that will not work in your calculator. Instead you'd need to type $^{-}3 - 3$ to get $-6$ where the first dash comes from the '$+/-$' key and the second dash comes from the subtraction key.

[redacted]

An important point here is that when we 'distribute' negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of $-1$ across each of these terms: $-(a + b) = (-1)(a+b) = (-1)(a) + (-1)(b) = (-a) + (-b) = -a - b$. Negatives do not 'distribute' across multiplication: $-(2 \cdot 3) \neq (-2) \cdot (-3)$. Instead, $-(2 \cdot 3) = (-2) \cdot (3) = (2) \cdot (-3) = -6$.

The same sort of thing goes for fractions: $-\frac{3}{5}$ can be written as $\frac{-3}{5}$ or $\frac{3}{-5}$, but not $\frac{-3}{-5}$.

[2] Don't worry. We'll review this in due course. And, yes, this is our old friend the Distributive Property!

[3] We're not just being lazy here. We looked at many of the big publishers' Precalculus books and none of them use different dashes, either.

Speaking of fractions, we now review their arithmetic.

[redacted]

Students make so many mistakes with fractions that we feel it is necessary to pause the narrative for a moment and offer you the following examples. Please take the time to read these carefully. In the main body of the text we will skip many of the steps shown here and it is your responsibility to understand the arithmetic behind the computations we use throughout the text. We deliberately limited these examples to “nice” numbers (meaning that the numerators and denominators of the fractions are small integers) and will discuss more complicated matters later. In the upcoming example, we will make use of the Fundamental Theorem of Arithmetic which essentially says that every natural number has a unique prime factorization. Thus ‘lowest terms’ is clearly defined when reducing the fractions you’re about to see.

**Example 1.2.1.** Perform the indicated operations and simplify. By 'simplify' here, we mean to have the final answer written in the form $\frac{a}{b}$ where $a$ and $b$ are integers which have no common factors. Said another way, we want $\frac{a}{b}$ in 'lowest terms'.

1. $\dfrac{1}{4} + \dfrac{6}{7}$
2. $\dfrac{5}{12} - \left(\dfrac{47}{30} - \dfrac{7}{3}\right)$
3. $\dfrac{\dfrac{7}{3-5} - \dfrac{7}{3-5.21}}{5-5.21}$
4. $\dfrac{\dfrac{12}{5} - \dfrac{7}{24}}{1 + \left(\dfrac{12}{5}\right)\left(\dfrac{7}{24}\right)}$
5. $\dfrac{(2(2)+1)(-3-(-3)) - 5(4-7)}{4-2(3)}$
6. $\left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right) - \left(\dfrac{4}{5}\right)\left(-\dfrac{12}{13}\right)$

**Solution.**

1. It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no factors in common, the lowest common denominator is $4 \cdot 7 = 28$.

$$\begin{aligned} \frac{1}{4} + \frac{6}{7} &= \frac{1}{4} \cdot \frac{7}{7} + \frac{6}{7} \cdot \frac{4}{4} && \text{Equivalent Fractions} \\ &= \frac{7}{28} + \frac{24}{28} && \text{Multiplication of Fractions} \\ &= \frac{31}{28} && \text{Addition of Fractions} \end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

2. We could begin with the subtraction in parentheses, namely $\frac{47}{30} - \frac{7}{3}$, and then subtract that result from $\frac{5}{12}$. It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step.[4] The lowest common denominator[5] for all three fractions is 60.

$$\begin{aligned} \frac{5}{12} - \left(\frac{47}{30} - \frac{7}{3}\right) &= \frac{5}{12} - \frac{47}{30} + \frac{7}{3} && \text{Distribute the Negative} \\ &= \frac{5}{12} \cdot \frac{5}{5} - \frac{47}{30} \cdot \frac{2}{2} + \frac{7}{3} \cdot \frac{20}{20} && \text{Equivalent Fractions} \\ &= \frac{25}{60} - \frac{94}{60} + \frac{140}{60} && \text{Multiplication of Fractions} \\ &= \frac{71}{60} && \text{Addition and Subtraction of Fractions} \end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

[4]See the remark on page **??** about how we add $1 + 2 + 3$.

[5]We could have used $12 \cdot 30 \cdot 3 = 1080$ as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

3. What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction.[6] The longest division line[7] acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions). The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. To that end, we clean up the fractions in the numerator as follows.

$$\begin{aligned}
\frac{\dfrac{7}{3-5}-\dfrac{7}{3-5.21}}{5-5.21} &= \frac{\dfrac{7}{-2}-\dfrac{7}{-2.21}}{-0.21} \\
&= \frac{-\left(-\dfrac{7}{2}+\dfrac{7}{2.21}\right)}{0.21} && \text{Properties of Negatives} \\
&= \frac{\dfrac{7}{2}-\dfrac{7}{2.21}}{0.21} && \text{Distribute the Negative}
\end{aligned}$$

We are left with a compound fraction with decimals. We could replace 2.21 with $\frac{221}{100}$ but that would make a mess.[8] It's better in this case to eliminate the decimal by multiplying the numerator and denominator of the fraction with the decimal in it by 100 (since $2.21 \cdot 100 = 221$ is an integer) as shown below.

$$\frac{\dfrac{7}{2}-\dfrac{7}{2.21}}{0.21} = \frac{\dfrac{7}{2}-\dfrac{7\cdot 100}{2.21\cdot 100}}{0.21} = \frac{\dfrac{7}{2}-\dfrac{700}{221}}{0.21}$$

We now perform the subtraction in the numerator and replace 0.21 with $\frac{21}{100}$ in the denominator. This will leave us with one fraction divided by another fraction. We finish by performing the 'division by a fraction is multiplication by the reciprocal' trick and then cancel any factors that we can.

$$\begin{aligned}
\frac{\dfrac{7}{2}-\dfrac{700}{221}}{0.21} &= \frac{\dfrac{7}{2}\cdot\dfrac{221}{221}-\dfrac{700}{221}\cdot\dfrac{2}{2}}{\dfrac{21}{100}} = \frac{\dfrac{1547}{442}-\dfrac{1400}{442}}{\dfrac{21}{100}} \\
&= \frac{\dfrac{147}{442}}{\dfrac{21}{100}} = \frac{147}{442}\cdot\frac{100}{21} = \frac{14700}{9282} = \frac{350}{221}
\end{aligned}$$

The last step comes from the factorizations $14700 = 42 \cdot 350$ and $9282 = 42 \cdot 221$.

4. We are given another compound fraction to simplify and this time both the numerator and denominator contain fractions. As before, the longest division line acts as a grouping symbol to separate the

[6] Fractionception, perhaps?

[7] Also called a 'vinculum'.

[8] Try it if you don't believe us.

numerator from the denominator.

$$\frac{\frac{12}{5}-\frac{7}{24}}{1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}$$

Hence, one way to proceed is as before: simplify the numerator and the denominator then perform the 'division by a fraction is the multiplication by the reciprocal' trick. While there is nothing wrong with this approach, we'll use our Equivalent Fractions property to rid ourselves of the 'compound' nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the 'smaller' fractions - in this case, $24 \cdot 5 = 120$.

$$\begin{aligned}
\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5}-\frac{7}{24}\right)\cdot 120}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)\cdot 120} && \text{Equivalent Fractions} \\
&= \frac{\left(\frac{12}{5}\right)(120)-\left(\frac{7}{24}\right)(120)}{(1)(120)+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} && \text{Distributive Property} \\
&= \frac{\frac{12\cdot 120}{5}-\frac{7\cdot 120}{24}}{120+\frac{12\cdot 7\cdot 120}{5\cdot 24}} && \text{Multiply fractions} \\
&= \frac{\frac{12\cdot 24\cdot \cancel{5}}{\cancel{5}}-\frac{7\cdot 5\cdot \cancel{24}}{\cancel{24}}}{120+\frac{12\cdot 7\cdot \cancel{5}\cdot \cancel{24}}{\cancel{5}\cdot \cancel{24}}} && \text{Factor and cancel} \\
&= \frac{(12\cdot 24)-(7\cdot 5)}{120+(12\cdot 7)} \\
&= \frac{288-35}{120+84} \\
&= \frac{253}{204}
\end{aligned}$$

Since $253 = 11 \cdot 23$ and $204 = 2 \cdot 2 \cdot 3 \cdot 17$ have no common factors our result is in lowest terms which means we are done.

5. This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} = \frac{((2(2)+1)(-3-(-3))-5(4-7))}{(4-2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{aligned}
\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} &= \frac{(4+1)(-3+3)-5(-3)}{4-6} \\
&= \frac{(5)(0)+15}{-2} \\
&= \frac{15}{-2} \\
&= -\frac{15}{2} && \text{Properties of Negatives}
\end{aligned}$$

Since $15 = 3 \cdot 5$ and 2 have no common factors, we are done.

6. In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{aligned}
\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3\cdot 5}{5\cdot 13}-\frac{4\cdot(-12)}{5\cdot 13} && \text{Multiply fractions} \\
&= \frac{15}{65}-\frac{-48}{65} \\
&= \frac{15}{65}+\frac{48}{65} && \text{Properties of Negatives} \\
&= \frac{15+48}{65} && \text{Add numerators} \\
&= \frac{63}{65}
\end{aligned}$$

Since $64 = 3 \cdot 3 \cdot 7$ and $65 = 5 \cdot 13$ have no common factors, our answer $\frac{63}{65}$ is in lowest terms and we are done. □

Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favorite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that $2 \cdot 2 \cdot 2$ can be written as $2^3$ because exponential notation expresses repeated multiplication. In the expression $2^3$, 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1 \cdot (\text{three factors of two}) = 1 \cdot (2 \cdot 2 \cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1 \cdot (\text{zero factors of two}) = 1.$$

What about $2^{-3}$? The '$-$' in the exponent indicates that we are 'taking away' three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1 \div (\text{three factors of two}) = 1 \div (2 \cdot 2 \cdot 2) = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

[redacted]

While it is important the state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that $(ab)^2 = a^2b^2$ (which some students refer to as 'distributing' the exponent to each factor) but you cannot do this sort of thing with addition. That is, in general, $(a + b)^2 \neq a^2 + b^2$. (For example, take $a = 3$ and $b = 4$.) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

[redacted]

For example, $2+3\cdot 4^2 = 2+3\cdot 16 = 2+48 = 50$. Where students get into trouble is with things like $-3^2$. If we think of this as $0-3^2$, then it is clear that we evaluate the exponent first: $-3^2 = 0-3^2 = 0-9 = -9$. In general, we interpret $-a^n = -\left(a^n\right)$. If we want the 'negative' to also be raised to a power, we must write $(-a)^n$ instead. To summarize, $-3^2 = -9$ but $(-3)^2 = 9$.

Of course, many of the 'properties' we've stated in this section can be viewed as ways to circumvent the order of operations. We've already seen how the distributive property allows us to simplify $5(2+x)$ by performing the indicated multiplication **before** the addition that's in parentheses. Similarly, consider trying to evaluate $2^{30172}\cdot 2^{-30169}$. The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute $2^{30172}$ is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get: $2^{30172-30169} = 2^3 = 8$.

Let's take a break and enjoy another example.

**Example 1.2.2.** Perform the indicated operations and simplify.

1. $\dfrac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2}$
2. $12(-5)(-5+3)^{-4}+6(-5)^2(-4)(-5+3)^{-5}$
3. $\dfrac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)}$
4. $\dfrac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}}$

**Solution.**

1. We begin working inside the parentheses then deal with the exponents before working through the other operations. As we saw in Example **??**, the division here acts as a grouping symbol, so we

save the division to the end.

$$\begin{aligned} \frac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2} &= \frac{(2)(8)-(4)^2}{(2)^2} = \frac{(2)(8)-16}{4} \\ &= \frac{16-16}{4} = \frac{0}{4} = 0 \end{aligned}$$

2. As before, we simplify what's in the parentheses first, then work our way through the exponents, multiplication, and finally, the addition.

$$\begin{aligned} 12(-5)(-5+3)^{-4} + 6(-5)^2(-4)(-5+3)^{-5} &= 12(-5)(-2)^{-4} + 6(-5)^2(-4)(-2)^{-5} \\ &= 12(-5)\left(\frac{1}{(-2)^4}\right) + 6(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\ &= 12(-5)\left(\frac{1}{16}\right) + 6(25)(-4)\left(\frac{1}{-32}\right) \\ &= (-60)\left(\frac{1}{16}\right) + (-600)\left(\frac{1}{-32}\right) \\ &= \frac{-60}{16} + \left(\frac{-600}{-32}\right) \\ &= \frac{-15\cdot \cancel{4}}{4\cdot \cancel{4}} + \frac{-75\cdot \cancel{8}}{-4\cdot \cancel{8}} \\ &= \frac{-15}{4} + \frac{-75}{-4} \\ &= \frac{-15}{4} + \frac{75}{4} \\ &= \frac{-15+75}{4} \\ &= \frac{60}{4} \\ &= 15 \end{aligned}$$

3. The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction. This gives us hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular) to help us out. Our first step here is to invert and multiply. We see immediately that the 5's cancel after which we group the powers of 3

together and the powers of 4 together and apply the properties of exponents.

$$\begin{aligned}
\frac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)} &= \frac{5\cdot 3^{51}}{4^{36}}\cdot\frac{4^{34}}{5\cdot 3^{49}} &&= \frac{\cancel{5}\cdot 3^{51}\cdot 4^{34}}{\cancel{5}\cdot 3^{49}\cdot 4^{36}} &&= \frac{3^{51}}{3^{49}}\cdot\frac{4^{34}}{4^{36}} \\
&= 3^{51-49}\cdot 4^{34-36} &&= 3^2\cdot 4^{-2} &&= 3^2\cdot\left(\frac{1}{4^2}\right) \\
&= 9\cdot\left(\frac{1}{16}\right) &&= \frac{9}{16}
\end{aligned}$$

4. We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we did in Example **??**. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to simplify the fraction.

$$\begin{aligned}
\frac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}} &= \frac{2\left(\dfrac{12}{5}\right)}{1-\left(\dfrac{12}{5}\right)^{2}} &&= \frac{\left(\dfrac{24}{5}\right)}{1-\left(\dfrac{12^2}{5^2}\right)} &&= \frac{\left(\dfrac{24}{5}\right)}{1-\left(\dfrac{144}{25}\right)} \\
&= \frac{\left(\dfrac{24}{5}\right)\cdot 25}{\left(1-\dfrac{144}{25}\right)\cdot 25} &&= \frac{\left(\dfrac{24\cdot 5\cdot\cancel{5}}{\cancel{5}}\right)}{\left(1\cdot 25-\dfrac{144\cdot\cancel{25}}{\cancel{25}}\right)} &&= \frac{120}{25-144} \\
&= \frac{120}{-119} = -\frac{120}{119}
\end{aligned}$$

Since 120 and 119 have no common factors, we are done. □

One of the places where the properties of exponents play an important role is in the use of **Scientific Notation**. The basis for scientific notation is that since we use decimals (base ten numerals) to represent real numbers, we can adjust where the decimal point lies by multiplying by an appropriate power of 10. This allows scientists and engineers to focus in on the 'significant' digits[9] of a number - the nonzero values - and adjust for the decimal places later. For instance, $-621 = -6.21 \times 10^2$ and $0.023 = 2.3 \times 10^{-2}$. Notice here that we revert to using the familiar '$\times$' to indicate multiplication.[10] In general, we arrange the real number so exactly one non-zero digit appears to the left of the decimal point. We make this idea precise in the following:

[redacted]

[9] Awesome pun!

[10] This is the 'notable exception' we alluded to earlier.

On calculators, scientific notation may appear using an 'E' or 'EE' as opposed to the $\times$ symbol. For instance, while we will write $6.02 \times 10^{23}$ in the text, the calculator may display 6.02 E 23 or 6.02 EE 23.

**Example 1.2.3.** Perform the indicated operations and simplify. Write your final answer in scientific notation, rounded to two decimal places.

1. $\dfrac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}}$
2. $\left(2.13 \times 10^{53}\right)^{100}$

**Solution.**

1. As mentioned earlier, the point of scientific notation is to separate out the 'significant' parts of a calculation and deal with the powers of 10 later. In that spirit, we separate out the powers of 10 in both the numerator and the denominator and proceed as follows

$$\begin{aligned} \frac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}} &= \frac{(6.626)(3.14)}{1.78} \cdot \frac{10^{-34} \cdot 10^{9}}{10^{23}} \\ &= \frac{20.80564}{1.78} \cdot \frac{10^{-34+9}}{10^{23}} \\ &= 11.685 \ldots \cdot \frac{10^{-25}}{10^{23}} \\ &= 11.685 \ldots \times 10^{-25-23} \\ &= 11.685 \ldots \times 10^{-48} \end{aligned}$$

   We are asked to write our final answer in scientific notation, rounded to two decimal places. To do this, we note that $11.685 \ldots = 1.1685 \ldots \times 10^{1}$, so

$$11.685 \ldots \times 10^{-48} = 1.1685 \ldots \times 10^{1} \times 10^{-48} = 1.1685 \ldots \times 10^{1-48} = 1.1685 \ldots \times 10^{-47}$$

   Our final answer, rounded to two decimal places, is $1.17 \times 10^{-47}$.

   We could have done that whole computation on a calculator so why did we bother doing any of this by hand in the first place? The answer lies in the next example.

2. If you try to compute $\left(2.13 \times 10^{53}\right)^{100}$ using most hand-held calculators, you'll most likely get an 'overflow' error. It is possible, however, to use the calculator in combination with the properties of exponents to compute this number. Using properties of exponents, we get:

$$\begin{aligned} \left(2.13 \times 10^{53}\right)^{100} &= (2.13)^{100}\left(10^{53}\right)^{100} \\ &= \left(6.885 \ldots \times 10^{32}\right)\left(10^{53 \times 100}\right) \\ &= \left(6.885 \ldots \times 10^{32}\right)\left(10^{5300}\right) \\ &= 6.885 \ldots \times 10^{32} \cdot 10^{5300} \\ &= 6.885 \ldots \times 10^{5332} \end{aligned}$$

   To two decimal places our answer is $6.88 \times 10^{5332}$. □

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

[redacted]

The reasons for the added stipulations for even-indexed roots in Definition **??** can be found in the Properties of Negatives. First, for all real numbers, $x^{\text{even power}} \geq 0$, which means it is never negative. Thus if $a$ is a *negative* real number, there are no real numbers $x$ with $x^{\text{even power}} = a$. This is why if $n$ is even, $\sqrt[n]{a}$ only exists if $a \geq 0$. The second restriction for even-indexed roots is that $\sqrt[n]{a} \geq 0$. This comes from the fact that $x^{\text{even power}} = (-x)^{\text{even power}}$, and we require $\sqrt[n]{a}$ to have just one value. So even though $2^4 = 16$ and $(-2)^4 = 16$, we require $\sqrt[4]{16} = 2$ and ignore $-2$.

Dealing with odd powers is much easier. For example, $x^3 = -8$ has one and only one real solution, namely $x = -2$, which means not only does $\sqrt[3]{-8}$ exist, there is only one choice, namely $\sqrt[3]{-8} = -2$. Of course, when it comes to solving $x^{5213} = -117$, it's not so clear that there is one and only one real solution, let alone that the solution is $\sqrt[5213]{-117}$. Such pills are easier to swallow once we've thought a bit about such equations graphically,[11] and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' as opposed to a definition since they can be justified using the properties of exponents.

[redacted]

The proof of Theorem **??** is based on the definition of the principal n$^{\text{th}}$ root and the Properties of Exponents. To establish the product rule, consider the following. If $n$ is odd, then by definition $\sqrt[n]{ab}$ is the unique real number such that $(\sqrt[n]{ab})^n = ab$. Given that $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$ as well, it must be the case that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. If $n$ is even, then $\sqrt[n]{ab}$ is the unique non-negative real number such that $(\sqrt[n]{ab})^n = ab$. Note that since $n$ is even, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also non-negative thus $\sqrt[n]{a}\sqrt[n]{b} \geq 0$ as well. Proceeding as above, we find that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{a}$ is a real number to start with.[12] We leave that as an exercise as well.

[11] See Chapter **??**.

[12] Otherwise we'd run into an interesting paradox. See Section **??**.

We pause here to point out one of the most common errors students make when working with radicals. Obviously $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{9+16} = \sqrt{25} = 5$. Thus we can clearly see that $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ because we all know that $5 \neq 7$. The authors urge you to never consider 'distributing' roots or exponents. It's wrong and no good will come of it because in general $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Since radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

[redacted]

It would make life really nice if the rational exponents defined in Definition **??** had all of the same properties that integer exponents have as listed on page **??** - but they don't. Why not? Let's look at an example to see what goes wrong. Consider the Product Rule which says that $(ab)^n = a^n b^n$ and let $a = -16$, $b = -81$ and $n = \frac{1}{4}$. Plugging the values into the Product Rule yields the equation $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$. The left side of this equation is $1296^{1/4}$ which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they apply to is non-negative? That works for some of the rules - we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression $\left(a^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(a^{2/3}\right)^{3/2} = a^{\frac{2}{3}\cdot\frac{3}{2}} = a^1 = a$. However, if we substitute $a = -1$ and apply Definition **??**, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. Thus in this case we have $\left(a^{2/3}\right)^{3/2} \neq a$ even though all of the roots were defined. It is true, however, that $\left(a^{3/2}\right)^{2/3} = a$ and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.[13]

**Example 1.2.4.** Perform the indicated operations and simplify.

1. $\dfrac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$

2. $\dfrac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1 - \left(\dfrac{\sqrt{3}}{3}\right)^2}$

3. $\left(\sqrt[3]{-2} - \sqrt[3]{-54}\right)^2$

4. $2\left(\dfrac{9}{4} - 3\right)^{1/3} + 2\left(\dfrac{9}{4}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{9}{4} - 3\right)^{-2/3}$

[13] Much to Jeff's chagrin. He's fairly traditional and therefore doesn't care much for radicals.

**Solution.**

1. We begin in the numerator and note that the radical here acts a grouping symbol,[14] so our first order of business is to simplify the radicand.

$$\begin{aligned}
\frac{-(-4)-\sqrt{(-4)^2-4(2)(-3)}}{2(2)} &= \frac{-(-4)-\sqrt{16-4(2)(-3)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16-4(-6)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16-(-24)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16+24}}{2(2)} \\
&= \frac{-(-4)-\sqrt{40}}{2(2)}
\end{aligned}$$

As you may recall, 40 can be factored using a perfect square as $40 = 4 \cdot 10$ so we use the product rule of radicals to write $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$. This lets us factor a '2' out of both terms in the numerator, eventually allowing us to cancel it with a factor of 2 in the denominator.

$$\begin{aligned}
\frac{-(-4)-\sqrt{40}}{2(2)} &= \frac{-(-4)-2\sqrt{10}}{2(2)} &&= \frac{4-2\sqrt{10}}{2(2)} \\
&= \frac{2\cdot 2-2\sqrt{10}}{2(2)} &&= \frac{2(2-\sqrt{10})}{2(2)} \\
&= \frac{\cancel{2}(2-\sqrt{10})}{\cancel{2}(2)} &&= \frac{2-\sqrt{10}}{2}
\end{aligned}$$

Since the numerator and denominator have no more common factors,[15] we are done.

2. Once again we have a compound fraction, so we first simplify the exponent in the denominator to see which factor we'll need to multiply by in order to clean up the fraction.

[14] The line extending horizontally from the square root symbol $\sqrt{\ }$ is, you guessed it, another vinculum.

[15] Do you see why we aren't 'canceling' the remaining 2's?

$$\begin{aligned}
\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{(\sqrt{3})^2}{3^2}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{3}{9}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1\cdot\cancel{3}}{3\cdot\cancel{3}}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1}{3}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)\cdot 3}{\left(1-\left(\frac{1}{3}\right)\right)\cdot 3} &&= \frac{\frac{2\cdot\sqrt{3}\cdot\cancel{3}}{\cancel{3}}}{1\cdot 3-\frac{1\cdot\cancel{3}}{\cancel{3}}} \\
&= \frac{2\sqrt{3}}{3-1} &&= \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}
\end{aligned}$$

3. Working inside the parentheses, we first encounter $\sqrt[3]{-2}$. While the $-2$ isn't a perfect cube,[16] we may think of $-2 = (-1)(2)$. Since $(-1)^3 = -1$, which *is* a perfect cube, we may write $\sqrt[3]{-2} = \sqrt[3]{(-1)(2)} = \sqrt[3]{-1}\sqrt[3]{2} = -\sqrt[3]{2}$. When it comes to $\sqrt[3]{54}$, we may write it as $\sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$. So,

$$\sqrt[3]{-2} - \sqrt[3]{-54} = -\sqrt[3]{2} - (-3\sqrt[3]{2}) = -\sqrt[3]{2} + 3\sqrt[3]{2}.$$

At this stage, we can simplify $-\sqrt[3]{2}+3\sqrt[3]{2} = 2\sqrt[3]{2}$. You may remember this as being called 'combining like radicals,' but it is in fact just another application of the distributive property:

$$-\sqrt[3]{2} + 3\sqrt[3]{2} = (-1)\sqrt[3]{2} + 3\sqrt[3]{2} = (-1+3)\sqrt[3]{2} = 2\sqrt[3]{2}.$$

Putting all this together, we get:

$$\begin{aligned}
(\sqrt[3]{-2} - \sqrt[3]{-54})^2 &= (-\sqrt[3]{2} + 3\sqrt[3]{2})^2 &&= (2\sqrt[3]{2})^2 \\
&= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} &&= 4\sqrt[3]{4}
\end{aligned}$$

There are no perfect integer cubes which are factors of 4 (apart from 1, of course), so we are done.

[16] Of an integer, that is!

4. We start working in the parentheses and get a common denominator to subtract the fractions:

$$\frac{9}{4} - 3 = \frac{9}{4} - \frac{3 \cdot 4}{1 \cdot 4} = \frac{9}{4} - \frac{12}{4} = \frac{-3}{4}$$

The denominators in the fractional exponents are odd, so we can proceed by using the properties of exponents:

$$\begin{aligned}
2\left(\frac{9}{4}-3\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4}-3\right)^{-2/3} &= 2\left(\frac{-3}{4}\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)^{-2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{-3}\right)^{2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{(4)^{2/3}}{(-3)^{2/3}}\right) \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 9 \cdot 1 \cdot 4^{2/3}}{4 \cdot 3 \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{\cancel{2} \cdot 3 \cdot \cancel{3} \cdot 4^{2/3}}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}}
\end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Since $4 = 2^2$, $4^{1/3} = (2^2)^{1/3} = 2^{2/3}$. Similarly, $4^{2/3} = (2^2)^{2/3} = 2^{4/3}$. The expressions $(-3)^{1/3}$ and $(-3)^{2/3}$ contain negative bases so we proceed with caution and convert them back to radical notation to get: $(-3)^{1/3} = \sqrt[3]{-3} = -\sqrt[3]{3} = -3^{1/3}$ and $(-3)^{2/3} = (\sqrt[3]{-3})^2 = (-\sqrt[3]{3})^2 = (\sqrt[3]{3})^2 = 3^{2/3}$. Hence:

$$\begin{aligned}
\frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} &= \frac{2 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3 \cdot 2^{4/3}}{2 \cdot 3^{2/3}} \\
&= \frac{2^1 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3^1 \cdot 2^{4/3}}{2^1 \cdot 3^{2/3}} \\
&= 2^{1-2/3} \cdot (-3^{1/3}) + 3^{1-2/3} \cdot 2^{4/3-1} \\
&= 2^{1/3} \cdot (-3^{1/3}) + 3^{1/3} \cdot 2^{1/3} \\
&= -2^{1/3} \cdot 3^{1/3} + 3^{1/3} \cdot 2^{1/3} \\
&= 0
\end{aligned}$$

□

We close this section with a note about simplifying. In the preceding examples we used "nice" numbers because we wanted to show as many properties as we could per example. This then begs the question "What happens when the numbers are *not* nice?" Unfortunately, the answer is "Not much simplifying can be done." Take, for example,

$$\frac{\sqrt{7}}{\pi} - \frac{3}{\pi^2} + \frac{4}{\sqrt{11}} = \frac{\pi\sqrt{77} - 3\sqrt{11} + 4\pi^2}{\pi^2\sqrt{11}}$$

Sadly, that's as good as it gets.

### 1.2.1 Exercises

In Exercises **??** - **??**, perform the indicated operations and simplify.

1. $5 - 2 + 3$
2. $5 - (2 + 3)$
3. $\dfrac{2}{3} - \dfrac{4}{7}$
4. $\dfrac{3}{8} + \dfrac{5}{12}$
5. $\dfrac{5 - 3}{-2 - 4}$
6. $\dfrac{2(-3)}{3 - (-3)}$
7. $\dfrac{2(3) - (4 - 1)}{2^2 + 1}$
8. $\dfrac{4 - 5.8}{2 - 2.1}$
9. $\dfrac{1 - 2(-3)}{5(-3) + 7}$
10. $\dfrac{5(3) - 7}{2(3)^2 - 3(3) - 9}$
11. $\dfrac{2((-1)^2 - 1)}{((-1)^2 + 1)^2}$
12. $\dfrac{(-2)^2 - (-2) - 6}{(-2)^2 - 4}$
13. $\dfrac{3 - \frac{4}{9}}{-2 - (-3)}$
14. $\dfrac{\frac{2}{3} - \frac{4}{5}}{4 - \frac{7}{10}}$
15. $\dfrac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2}$
16. $\dfrac{1 - \left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1 + \left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$
17. $\left(\dfrac{2}{3}\right)^{-5}$
18. $3^{-1} - 4^{-2}$
19. $\dfrac{1 + 2^{-3}}{3 - 4^{-1}}$
20. $\dfrac{3 \cdot 5^{100}}{12 \cdot 5^{98}}$
21. $\sqrt{3^2 + 4^2}$
22. $\sqrt{12} - \sqrt{75}$
23. $(-8)^{2/3} - 9^{-3/2}$
24. $\left(-\frac{32}{9}\right)^{-3/5}$
25. $\sqrt{(3 - 4)^2 + (5 - 2)^2}$
26. $\sqrt{(2 - (-1))^2 + \left(\frac{1}{2} - 3\right)^2}$
27. $\sqrt{(\sqrt{5} - 2\sqrt{5})^2 + (\sqrt{18} - \sqrt{8})^2}$
28. $\dfrac{-12 + \sqrt{18}}{21}$
29. $\dfrac{-2 - \sqrt{(2)^2 - 4(3)(-1)}}{2(3)}$
30. $\dfrac{-(-4) + \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$
31. $2(-5)(-5 + 1)^{-1} + (-5)^2(-1)(-5 + 1)^{-2}$
32. $3\sqrt{2(4) + 1} + 3(4)\left(\frac{1}{2}\right)(2(4) + 1)^{-1/2}(2)$
33. $2(-7)\sqrt[3]{1 - (-7)} + (-7)^2\left(\frac{1}{3}\right)(1 - (-7))^{-2/3}(-1)$
34. With the help of your calculator, find $(3.14 \times 10^{87})^{117}$. Write your final answer, using scientific notation, rounded to two decimal places. (See Example **??**.)
35. Prove the Quotient Rule and Power Rule stated in Theorem **??**.
36. Discuss with your classmates how you might attempt to simplify the following.

   (a) $\sqrt{\dfrac{1 - \sqrt{2}}{1 + \sqrt{2}}}$

   (b) $\sqrt[5]{3} - \sqrt[3]{5}$

   (c) $\dfrac{\pi + 7}{\pi}$

### 1.2.2 Answers

1. 6
2. 0
3. $\frac{2}{21}$
4. $\frac{19}{24}$
5. $-\frac{1}{3}$
6. $-1$
7. $\frac{3}{5}$
8. 18
9. $-\frac{7}{8}$
10. Undefined.
11. 0
12. Undefined.
13. $\frac{23}{9}$
14. $-\frac{4}{99}$
15. $-\frac{24}{7}$
16. 0
17. $\frac{243}{32}$
18. $\frac{13}{48}$
19. $\frac{9}{22}$
20. $\frac{25}{4}$
21. 5
22. $-3\sqrt{3}$
23. $\frac{107}{27}$
24. $-\frac{3\sqrt[5]{3}}{8} = -\frac{3^{6/5}}{8}$
25. $\sqrt{10}$
26. $\frac{\sqrt{61}}{2}$
27. $\sqrt{7}$
28. $\frac{-4+\sqrt{2}}{7}$
29. $-1$
30. $2+\sqrt{5}$
31. $\frac{15}{16}$
32. 13
33. $-\frac{385}{12}$
34. $1.38 \times 10^{10237}$

# Chapter 2

# Introduction to Functions

## 2.1 Functions and their Representations

### 2.1.1 Functions as Mappings

Mathematics can be thought of as the study of patterns. In most disciplines, Mathematics is used as a language to express, or codify, relationships between quantities - both algebraically and geometrically - with the ultimate goal of solving real-world problems. The fact that the same algebraic equation which models the growth of bacteria in a petri dish is also used to compute the account balance of a savings account or the potency of radioactive material used in medical treatments speaks to the universal nature of Mathematics. Indeed, Mathematics is more than just about solving a specific problem in a specific situation, it's about abstracting problems and creating universal tools which can be used by a variety of scientists and engineers to solve a variety of problems.

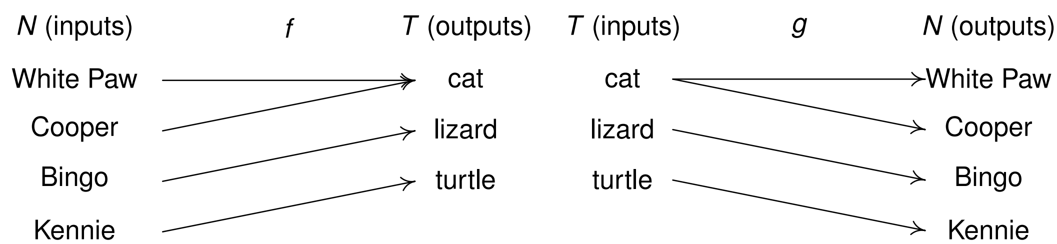
This power of abstraction has a tendency to create a language that is initially intimidating to students. Mathematical definitions are precise and adherence to that precision is often a source of confusion and frustration. It doesn't help matters that more often than not very common words are used in Mathematics with slightly different definitions than is commonly expected. The first 'universal tool' we wish to highlight - the concept of a 'function' - is a perfect example of this phenomenon in that we redefine a word that already has multiple meanings in English.

[redacted]

The grammar here '*from $A$ to $B$*' is important. Thinking of a function as a process, we can view the elements of the set $A$ as our starting materials, or *inputs* to the process. The function processes these inputs according to some specified rule and the result is a set of *outputs* - elements of the set $B$. In terms of inputs and outputs, Definition **??** says that a function is a process in which each *input* is matched to one and only one *output*.

For example, let's take a look at some of the pets in the Stitz household. Taylor's pets include White Paw and Cooper (both cats), Bingo (a lizard) and Kennie (a turtle). Let $N$ be the set of pet names: $N = \{\text{White Paw, Cooper, Bingo, Kennie}\}$, and let $T$ be the set of pet types: $T = \{\text{cat, lizard, turtle}\}$. Let $f$ be the process that takes each pet's name as the input and returns that pet's type as the output. Let $g$ be the reverse of $f$: that is, $g$ takes each pet type as the input and returns the names of the pets of that type as the output. Note that both $f$ and $g$ are codifying the *same* given information about Taylor's pets, but one of them is a function and the other is not.

To help identify which process $f$ or $g$ is a function and why the other is not, we create **mapping diagrams** for $f$ and $g$ below. In each case, we organize the inputs in a column on the left and the outputs on a column on the right. We draw an arrow connecting each input to its corresponding output(s). Note that the arrows communicate the grammatical bias: the arrow originates at the input and points to the output.



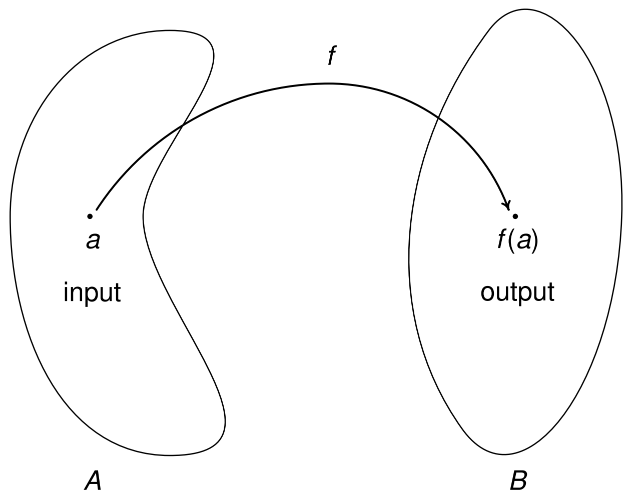

The process $f$ is a function since $f$ matches each of its inputs (each pet name) to just one output (the pet's type). The fact that different inputs (White Paw and Cooper) are matched to the same output (cat) is fine. On the other hand, $g$ matches the input 'cat' to the two different outputs 'White Paw' and 'Cooper', so $g$ is not a function. Functions are favored in mathematical circles because they are processes which produce only one answer (output) for any given query (input). In this scenario, for instance, there is only one answer to the question: 'What type of pet is White Paw?' but there is more than one answer to the question 'Which of Taylor's pets are cats?'

As you might expect, with functions being such an important concept in Mathematics, we need to build a vocabulary to assist us when discussing them. To that end, we have the following definitions.[1]

[redacted]

[1] Please refer to Section 1.1 for a review of the terminology used in these definitions.

Some remarks about Definition **??** are in order. First, and most importantly, the notation '$f(a)$' in Definition **??** introduces yet another mathematical use for parentheses. Parentheses are used in some cases as grouping symbols, to represent ordered pairs, and to delineate intervals of real numbers. More often than not, the use of parentheses in expressions like '$f(a)$' is confused with multiplication. As always, paying attention to the context is key. If $f$ is a function and '$a$' is in the domain of $f$, then '$f(a)$' is the output from $f$ when you input $a$. The diagram below provides a nice generic picture to keep in mind when thinking of a function as a mapping process with input '$a$' and output '$f(a)$'.




In the preceding pet example, the symbol $f$(Bingo), read '$f$ of Bingo', is asking what type of pet Bingo is, so $f$(Bingo) = lizard. The fact that $f$ is a function means $f$(Bingo) is unambiguous because $f$ matches the name 'Bingo' to only one pet type, namely 'lizard'. In contrast, if we tried to use the notation '$g$(cat)' to indicate what pet name $g$ matched to 'cat', we have *two* possibilities, White Paw and Cooper, with no way to determine which one (or both) is indicated.

Continuing to apply Definition **??** to our pet example, we find that the domain of the function $f$ is $N$, the set of pet names. Finding the range takes a little more work, mostly because it's easy to be caught off guard by the notation used in the definition of 'range'. The description of the range as '$\{f(a) \mid a \in A\}$' is an example of 'set-builder' notation. In English, '$\{f(a) \mid a \in A\}$' reads as 'the set of $f(a)$ such that $a$ is in $A$'. In other words, the range consists of all of the outputs from $f$ - all of the $f(a)$ values - as $a$ varies through each of the elements in the domain $A$. Note that while every element of the set $A$ is, by definition, an element of the domain of $f$, not every element of the set $B$ is necessarily part of the range of $f$.[2]

In our pet example, we can obtain the range of $f$ by looking at the mapping diagram or by constructing the set $\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\}$ which lists all of the outputs from $f$ as we run through all of the inputs to $f$. Keep in mind that we list each element of a set only once so the range of $f$ is:[3]

$$\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\} = \{\text{cat}, \text{lizard}, \text{turtle}\} = T.$$

[2]For purposes of completeness, the set $B$ is called the **codomain** of $f$. For us, the concepts of domain and range suffice since our codomain will most always be the set of real numbers, $\mathbb{R}$.

[3]If instead of mapping $N$ into $T$, we could have mapped $N$ into $U = \{\text{cat}, \text{lizard}, \text{turtle}, \text{dog}\}$ in which case the range of $f$ would not have been the entire codomain $U$.

If we let $n$ denote a generic element of $N$ then $f(n)$ is some element $t$ in $T$, so we write $t = f(n)$. In this equation, $n$ is called the **independent variable** and $t$ is called the **dependent variable**.[4] Moreover, we say '$t$ is a function of $n$', or, more specifically, 'the type of pet is a function of the pet name' meaning that every pet name $n$ corresponds to one, and only one, pet type $t$. Even though $f$ and $t$ are different things,[5] it is very common for the function and its outputs to become more-or-less synonymous, even in what are otherwise precise mathematical definitions.[6] We will endeavor to point out such ambiguities as we move through the text.

While the concept of a function is very general in scope, we will be focusing primarily on functions of real numbers because most disciplines use real numbers to quantify data. Our next example explores a function defined using a table of numerical values.

**Example 2.1.1.** Suppose Skippy records the outdoor temperature every two hours starting at 6 a.m. and ending at 6 p.m. and summarizes the data in the table below:

| time (hours after 6 a.m.) | outdoor temperature in degrees Fahrenheit |
|---|---|
| 0 | 64 |
| 2 | 67 |
| 4 | 75 |
| 6 | 80 |
| 8 | 83 |
| 10 | 83 |
| 12 | 82 |

1. Explain why the recorded outdoor temperature is a function of the corresponding time.

2. Is time a function of the outdoor temperature? Explain.

3. Let $f$ be the function which matches time to the corresponding recorded outdoor temperature.

   (a) Find and interpret the following:

   - $f(2)$
   - $f(4)$
   - $f(2+4)$
   - $f(2)+f(4)$
   - $f(2)+4$

   (b) Solve and interpret $f(t) = 83$.

   (c) State the range of $f$. What is lowest recorded temperature of the day? The highest?

[4]These adjectives stem from the fact that the value of $t$ *depends* entirely on our (independent) choice of $n$.

[5]Specifically, $f$ is a function so it requires and domain, a range and a rule of assignment whereas $t$ is simply the output from $f$.

[6]In fact, it is not uncommon to see the name of the function as the same as the dependent variable. For example, writing '$y = y(x)$' would be a way to communicate the idea that '$y$ is a function of $x$'.

**Solution.**

1. The outdoor temperature is a function of time because each time value is associated with only one recorded temperature.

2. Time is not a function of the outdoor temperature because there are instances when different times are associated with a given temperature. For example, the temperature 83 corresponds to both of the times 8 and 10.

3. (a) • To find $f(2)$, we look in the table to find the recorded outdoor temperature that corresponds to when the time is 2. We get $f(2) = 67$ which means that 2 hours after 6 a.m. (i.e., at 8 a.m.), the temperature is 67°F.
   - Per the table, $f(4) = 75$, so the recorded outdoor temperature at 10 a.m. (4 hours after 6 a.m.) is 75°F.
   - From the table, we find $f(2 + 4) = f(6) = 80$, which means that at noon (6 hours after 6 a.m.), the recorded outdoor temperature is 80°F.
   - Using results from above we see that $f(2) + f(4) = 67 + 75 = 142$. When adding $f(2) + f(4)$, we are adding the recorded outdoor temperatures at 8 a.m. (2 hours after 6 a.m.) and 10 a.m. (4 hours after 6 AM), respectively, to get 142°F.
   - We compute $f(2) + 4 = 67 + 4 = 71$. Here, we are adding 4°F to the outdoor temperature recorded at 8 a.m..

   (b) Solving $f(t) = 83$ means finding all of the input (time) values $t$ which produce an output value of 83. From the data, we see that the temperature is 83 when the time is 8 or 10, so the solution to $f(t) = 83$ is $t = 8$ or $t = 10$. This means the outdoor temperature is 83°F at 2 p.m. (8 hours after 6 a.m.) and at 4 p.m. (10 hours after 6 a.m.).

   (c) The range of $f$ is the set of all of the outputs from $f$, or in this case, the outside recorded temperatures. Based on the data, we get $\{64, 67, 75, 80, 82, 83\}$. (Here again, we list elements of a set only once.) The lowest recorded temperature of the day is 64°F and the highest recorded temperature of the day is 83°F. □

A few remarks about Example **??** are in order. First, note that $f(2 + 4)$, $f(2) + f(4)$ and $f(2) + 4$ all work out to be numerically different, and more importantly, all represent different things.[7] One of the common mistakes students make is to misinterpret expressions like these, so it's important to pay close attention to the syntax here.

Next, when solving $f(t) = 83$, the variable '$t$' is being used as a convenient 'dummy' variable or placeholder in the sense that solving $f(t) = 83$ produces the same solutions as solving $f(x) = 83$, $f(w) = 83$, or even $f(?) = 83$. All of these equations are asking for the same thing: what inputs to $f$ produce an output of 83. The choice of the letter '$t$' here makes sense since the inputs are **t**ime values. Throughout the text, we will endeavor to use meaningful labels when working in applied situations, but the fact remains that the choice of letters (or symbols) is completely arbitrary.

[7] You may be wondering why one would ever compute these quantities. Rest assured that we will use expressions like these in examples throughout the text. For now, it suffices just to know that they are different.

Finally, given that the range in this example was a finite set of real numbers, we could find the smallest and largest elements of it. Here, they correspond to the coolest and warmest temperatures of the day, respectively, but the meaning would change if the function related different quantities. In many applications involving functions, the end goal is to find the minimum or maximum values of the outputs of those functions (called **optimizing** the function) so for that reason, we have the following definition.

[redacted]

Definition **??** is an example where the name of the function, $f$, is being used almost synonymously with its outputs in that when we speak of 'the minimum and maximum of the *function* $f$' we are really talking about the minimum and maximum values of the *outputs* $f(x)$ as $x$ varies through the domain of $f$. Thus we say that the maximum of $f$ is 83 and the minimum of $f$ is 64 when referring to the highest and lowest recorded temperatures in the previous example.

### 2.1.2 Algebraic Representations of Functions

By focusing our attention to functions that involve real numbers, we gain access to all of the structures and tools from prior courses in Algebra. In this subsection, we discuss how to represent functions algebraically using formulas and begin with the following example.

**Example 2.1.2.**

1. Let $f$ be the function which takes a real number and performs the following sequence of operations:

    - Step 1: add 2
    - Step 2: multiply the result of Step 1 by 3
    - Step 3: subtract 1 from the result of Step 2.

    (a) Find and simplify $f(-5)$.

    (b) Find and simplify a formula for $f(x)$.

2. Let $h(t) = -t^2 + 3t + 4$.

   (a) Find and simplify the following:

      i. $h(-1)$, $h(0)$ and $h(2)$.

      ii. $h(2x)$ and $2h(x)$.

      iii. $h(t+2)$, $h(t) + 2$ and $h(t) + h(2)$.

   (b) Solve $h(t) = 0$.

**Solution.**

1. (a) We take $-5$ and follow it through each step:

      - Step 1: adding 2 gives us $-5 + 2 = -3$.
      - Step 2: multiplying the result of Step 1 by 3 yields $(-3)(3) = -9$.
      - Step 3: subtracting 1 from the result of Step 2 produces $-9 - 1 = -10$.

      Hence, $f(-5) = -10$.

   (b) To find a formula for $f(x)$, we repeat the above process but use the variable '$x$' in place of the number $-5$:

      - Step 1: adding 2 gives us the quantity $x + 2$.
      - Step 2: multiplying the result of Step 1 by 3 yields $(x+2)(3) = 3x + 6$.
      - Step 3: subtracting 1 from the result of Step 2 produces $(3x + 6) - 1 = 3x + 5$.

      Hence, we have codified $f$ using the formula $f(x) = 3x + 5$. In other words, the function $f$ matches each real number '$x$' with the value of the expression '$3x + 5$'. As a partial check of our answer, we use this formula to find $f(-5)$. We compute $f(-5)$ by substituting $x = -5$ into the formula $f(x)$ and find $f(-5) = 3(-5) + 5 = -10$ as before.

2. As before, representing the function $h$ as $h(t) = -t^2 + 3t + 4$ means that $h$ matches the real number $t$ with the value of the expression $-t^2 + 3t + 4$.

   (a) To find $h(-1)$, we substitute $-1$ for $t$ in the expression $-t^2 + 3t + 4$. It is highly recommended that you be generous with parentheses here in order to avoid common mistakes:

   $$\begin{aligned} h(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0. \end{aligned}$$

   Similarly, $h(0) = -(0)^2 + 3(0) + 4 = 4$, and $h(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$.

   (b) To find $h(2x)$, we substitute $2x$ for $t$:

   $$\begin{aligned} h(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4. \end{aligned}$$

The expression $2h(x)$ means that we multiply the expression $h(x)$ by 2. We first get $h(x)$ by substituting $x$ for $t$: $h(x) = -x^2 + 3x + 4$. Hence,

$$\begin{aligned} 2h(x) &= 2\left(-x^2 + 3x + 4\right) \\ &= -2x^2 + 6x + 8. \end{aligned}$$

(c) To find $h(t+2)$, we substitute the quantity $t + 2$ in place of $t$:

$$\begin{aligned} h(t+2) &= -(t+2)^2 + 3(t+2) + 4 \\ &= -\left(t^2 + 4t + 4\right) + (3t + 6) + 4 \\ &= -t^2 - 4t - 4 + 3t + 6 + 4 \\ &= -t^2 - t + 6. \end{aligned}$$

To find $h(t) + 2$, we add 2 to the expression for $h(t)$

$$\begin{aligned} h(t) + 2 &= \left(-t^2 + 3t + 4\right) + 2 \\ &= -t^2 + 3t + 6. \end{aligned}$$

From our work above, we see that $h(2) = 6$ so

$$\begin{aligned} h(t) + h(2) &= \left(-t^2 + 3t + 4\right) + 6 \\ &= -t^2 + 3t + 10. \end{aligned}$$

3. We know $h(-1) = 0$ from above, so $t = -1$ should be one of the answers to $h(t) = 0$. In order to see if there are any more, we set $h(t) = -t^2 + 3t + 4 = 0$. Factoring[8] gives $-(t+1)(t-4) = 0$, so we get $t = -1$ (as expected) along with $t = 4$. □

A few remarks about Example **??** are in order. First, note that $h(2x)$ and $2h(x)$ are different expressions. In the former, we are multiplying the *input* by 2; in the latter, we are multiplying the *output* by 2. The same goes for $h(t + 2)$, $h(t) + 2$ and $h(t) + h(2)$. The expression $h(t + 2)$ calls for adding 2 to the input $t$ and then performing the function $h$. The expression $h(t) + 2$ has us performing the process $h$ first, then adding 2 to the output $h(t)$. Finally, $h(t) + h(2)$ directs us to first find the outputs $h(t)$ and $h(2)$ and then add the results. As we saw in Example **??**, we see here again the importance paying close attention to syntax.[9]

Let us return for a moment to the function $f$ in Example **??** which we ultimately represented using the formula $f(x) = 3x + 5$. If we introduce the dependent variable $y$, we get the equation $y = f(x) = 3x + 5$, or, more simply $y = 3x + 5$. To say that the equation $y = 3x + 5$ describes $y$ as a function of $x$ means that for each choice of $x$, the formula $3x + 5$ determines only one associated $y$-value.

We could turn the tables and ask if the equation $y = 3x + 5$ describes $x$ as a function of $y$. That is, for each value we pick for $y$, does the equation $y = 3x + 5$ produce only one associated $x$ value? One way to proceed is to solve $y = 3x + 5$ for $x$ and get $x = \frac{1}{3}(y - 5)$. We see that for each choice of $y$, the expression $\frac{1}{3}(y - 5)$ evaluates to just one number, hence, $x$ is a function of $y$. If we give this function a name, say $g$, we have $x = g(y) = \frac{1}{3}(y - 5)$, where in this equation, $y$ is the independent variable and $x$ is the dependent variable. We explore this idea in the next example.

[8]You may need to review Section **??**.

[9]As was mentioned before, we will give meanings to the these quantities in other examples throughout the text.

**Example 2.1.3.**

1. Consider the equation $x^3 + y^2 = 25$.
   (a) Does this equation represent $y$ as a function of $x$? Explain.
   (b) Does this equation represent $x$ as a function of $y$? Explain.
2. Consider the equation $u^4 + t^3 u = 16$.
   (a) Does this equation represent $t$ as a function of $u$? Explain.
   (b) Does this equation represent $u$ as a function of $t$? Explain.

**Solution.**

1. (a) To say that $x^3 + y^2 = 25$ represents $y$ as a function of $x$, we need to show that for each $x$ we choose, the equation produces only one associated $y$-value. To help with this analysis, we solve the equation for $y$ in terms of $x$.

$$\begin{array}{rcll} x^3 + y^2 & = & 25 & \\ y^2 & = & 25 - x^3 & \\ y & = & \pm\sqrt{25 - x^3} & \text{extract square roots. (See Section \textbf{??} for a review, if needed.)} \end{array}$$

   The presence of the '$\pm$' indicates that there is a good chance that for some $x$-value, the equation will produce *two* corresponding $y$-values. Indeed, $x = 0$ produces $y = \pm\sqrt{25 - 0^3} = \pm 5$. Hence, $x^3 + y^2 = 25$ equation does *not* represent $y$ as a function of $x$ because $x = 0$ is matched with more than one $y$-value.

   (b) To see if $x^3 + y^2 = 25$ represents $x$ as a function of $y$, we solve the equation for $x$ in terms of $y$:

$$\begin{array}{rcll} x^3 + y^2 & = & 25 & \\ x^3 & = & 25 - y^2 & \\ x & = & \sqrt[3]{25 - y^2} & \text{extract cube roots. (See Section \textbf{??} for a review, if needed.)} \end{array}$$

   In this case, each choice of $y$ produces only *one* corresponding value for $x$, so $x^3 + y^2 = 25$ represents $x$ as a function of $y$.

2. (a) To see if $u^4 + t^3 u = 16$ represents $t$ as a function of $u$, we proceed as above and solve for $t$ in terms of $u$:

$$\begin{array}{rcll} u^4 + t^3 u & = & 16 & \\ t^3 u & = & 16 - u^4 & \\ t^3 & = & \dfrac{16 - u^4}{u} & \text{assumes } u \neq 0 \\ t & = & \sqrt[3]{\dfrac{16 - u^4}{u}} & \text{extract cube roots.} \end{array}$$

Although it's a bit cumbersome, as long as $u \neq 0$ the expression $\sqrt[3]{\frac{16-u^4}{u}}$ will produce just one value of $t$ for each value of $u$. What if $u = 0$? In that case, the equation $u^4 + t^3u = 16$ reduces to $0 = 16$ - which is never true - so we don't need to worry about that case.[10] Hence, $u^4 + t^3u = 16$ represents $t$ as a function of $u$.

(b) In order to determine if $u^4 + t^3u = 16$ represents $u$ as a function of $t$, we could attempt to solve $u^4 + t^3u = 16$ for $u$ in terms of $t$, but we won't get very far.[11] Instead, we take a different approach and experiment with looking for solutions for $u$ for specific values of $t$. If we let $t = 0$, we get $u^4 = 16$ which gives $u = \pm\sqrt[4]{16} = \pm 2$. Hence, $t = 0$ corresponds to more than one $u$-value which means $u^4 + t^3u = 16$ does not represent $u$ as a function of $t$. □

We'll have more to say about using equations to describe functions in Section **??**. For now, we turn our attention to a geometric way to represent functions.

### 2.1.3 Geometric Representations of Functions

In this section, we introduce how to graph functions. As we'll see in this and later sections, visualizing functions geometrically can assist us in both analyzing them and using them to solve associated application problems. Our playground, if you will, for the Geometry in this course is the Cartesian Coordinate Plane. The reader would do well to review Section **??** as needed.

Our path to the Cartesian Plane requires ordered pairs. In general, we can represent every function as a set of ordered pairs. Indeed, given a function $f$ with domain $A$, we can represent $f = \{(a, f(a)) \mid a \in A\}$. That is, we represent $f$ as a set of ordered pairs $(a, f(a))$, or, more generally, (input, output). For example, the function $f$ which matches Taylor's pet's names to their associated pet type can be represented as:

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\}$$

Moving on, we next consider the function $f$ from Example **??** which relates time to temperature. In this case, $f = \{(0, 64), (2, 67), (4, 75), (6, 80), (8, 83), (10, 83), (12, 82)\}$. This function has numerical values for both the domain and range so we can identify these ordered pairs with points in the Cartesian Plane. The first coordinates of these points (the abscissae) represent time values so we'll use $t$ to label the horizontal axis. Likewise, we'll use $T$ to label the vertical axis since the second coordinates of these points (the ordinates) represent temperature values. Note that labeling these axes in this way determines our independent and dependent variable names, $t$ and $T$, respectively.

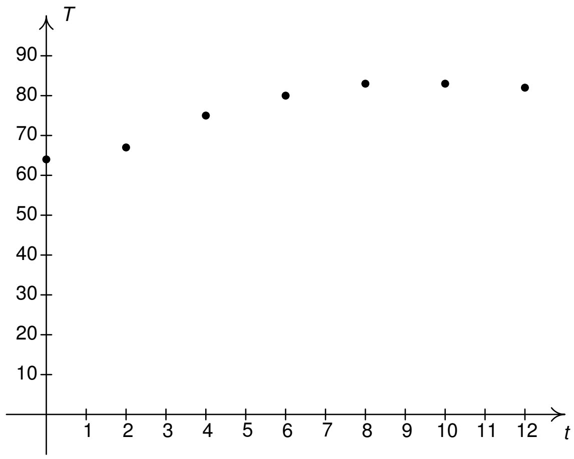
The plot of these points is called 'the **graph** of $f$'. More specifically, we could describe this plot as 'the graph of $f(t)$', because we have decided to name the independent variable $t$. Most specifically, we could describe the plot as 'the graph of $T = f(t)$', given that we have named the independent variable $t$ and the dependent variable $T$.

On the next page we present two plots, both of which are graphs of the function $f$. In both cases, the vertical axis has been scaled in order to save space. In the graph on the left, the same increment on

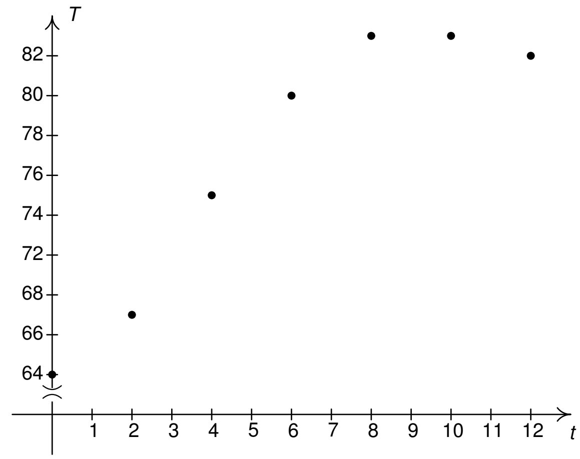
[10] Said differently, $u = 0$ is not in the domain of the function represented by the equation $u^4 + t^3u = 16$.

[11] Try it for yourself!

the horizontal axis to measure 1 unit measures 10 units on the vertical axis whereas in the graph on the right, this ratio is 1 : 2. The '$\asymp$' symbol on the vertical axis in the graph on the right is used to indicate a jump in the vertical labeling. Both are perfectly accurate data plots, but they have different visual impacts. Note here that the extrema of $f$, 64 and 83, correspond to the lowest and highest points on the graph, respectively: $(0, 64)$, $(8, 83)$ and $(10, 83)$. More often than not, we will use the graph of a function to help us optimize that function.[12]




The graph of $T = f(t)$.




The graph of $T = f(t)$.

If you found yourself wanting to connect the dots in the graphs above, you're not alone. As it stands, however, the function $f$ matches only seven inputs to seven outputs, so those seven points - and just those seven points - comprise the graph of $f$. That being said, common everyday experience tells us that while the data Skippy collected in his table gives some good information about the relationship between time and temperature on a given day, it is by no means a complete description of the relationship.

For example, Skippy's data cannot tell us what the temperature was at 7 a.m. or 12:13 p.m, although we are pretty sure there were outdoor temperatures at those times. Also, given that at some point it was 64°F and later on it was 83°F, it seems reasonable to assume that at some point it was 70°F or even 79.923°F.

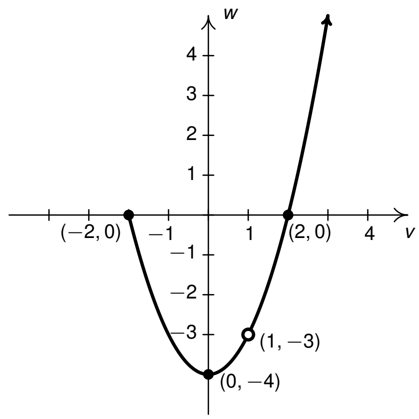
Skippy's temperature function $f$ is an example of a **discrete** function in the sense that each of the data points are 'isolated' with measurable gaps in between. The idea of 'filling in' those gaps is a quest to find a **continuous** function to model this same phenomenon.[13] We'll return to this example in Sections **??** and **??** in an attempt to do just that.

In the meantime, our next example involves a function whose domain is (almost) an *interval* of real numbers and whose graph consists of a (mostly) *connected* arc.

[12] One major use of Calculus is to optimize functions analytically - that is, without a graph.

[13] Roughly speaking, a *continuous variable* is a variable which takes on values over an *interval* of real numbers as opposed to values in a discrete list. In this case we would think of time as a 'continuum' - an interval of real numbers as opposed to 7 or so isolated times. A *continuous function* is a function which takes an interval of real numbers and maps it in such a way that its graph is a connected curve with no holes or gaps. This is technically a Calculus idea, but we'll need to discuss the notion of continuity a few times in the text.

**Example 2.1.4.** Consider the graph below.




1. (a) Explain why this graph suggests that $w$ is a function of $v$, $w = F(v)$.

   (b) Find $F(0)$ and solve $F(v) = 0$.

   (c) Find the domain and range of $F$ using interval notation.[14] Find the extrema of $F$, if any exist.

2. Does this graph suggest $v$ is a function of $w$? Explain.

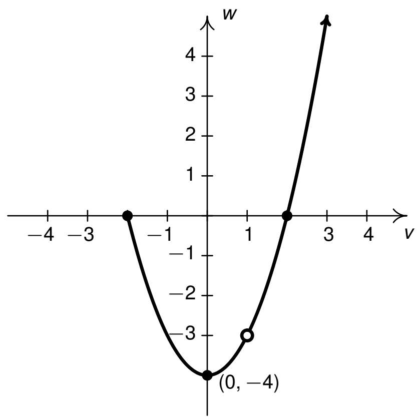
**Solution.** The challenge in working with only a graph is that unless points are specifically labeled (as some are in this case), we are forced to approximate values. In addition to the labeled points, there are other interesting features of the graph; a gap or 'hole' labeled $(1, -3)$ and an arrow on the upper right hand part of the curve. We'll have more to say about these two features shortly.

1. (a) In order for $w$ to be a function of $v$, each $v$-value on the graph must be paired with only one $w$-value. What if this weren't the case? We'd have at least two points with the *same* $v$-coordinate with *different* $w$-coordinates. Graphically, we'd have two points on graph on the same vertical line, one above the other. This never happens so we may conclude that $w$ is a function of $v$.

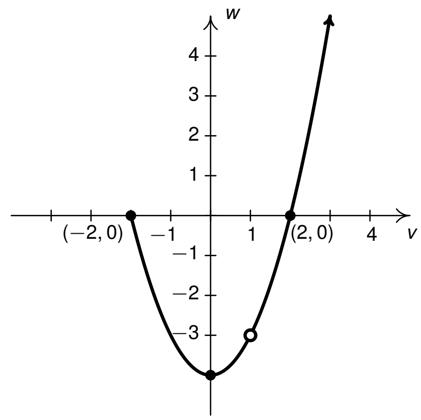
   (b) The value $F(0)$ is the output from $F$ when $v = 0$. The points on the graph of $F$ are of the form $(v, F(v))$ thus we are looking for the $w$-coordinate of the point on the graph where $v = 0$. Given that the point $(0, -4)$ is labeled on the graph, we can be sure $F(0) = -4$.

   To solve $F(v) = 0$, we are looking for the $v$-values where the output, or associated $w$ value, is 0. Hence, we are looking for points on the graph with a $w$-coordinate of 0. We find two such points, $(-2, 0)$ and $(2, 0)$, so our solutions to $F(v) = 0$ are $v = \pm 2$. Pictures highlighting the relevant graphical features are given at the top of the next page.

[14]Please consult Section 1.1 for a review of interval notation if need be.

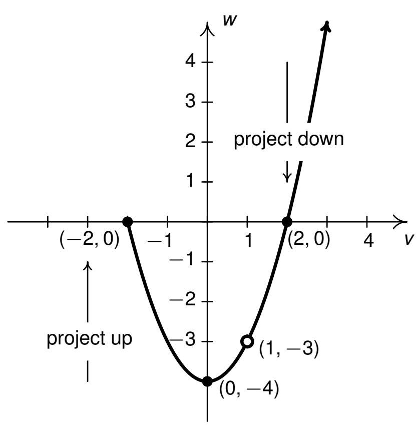


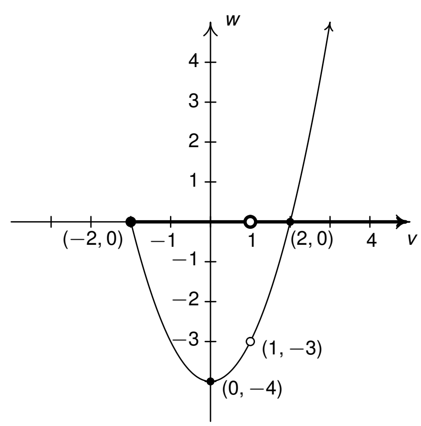

Finding $F(0) = -4$.




Solving $F(v) = 0$.

(c) The domain of $F$ is the set of inputs to $F$. With $v$ as the input here, we need to describe the set of $v$-values on the graph. We can accomplish this by **projecting** the graph to the $v$-axis and seeing what part of the $v$-axis is covered. The leftmost point on the graph is $(-2, 0)$, so we know that the domain starts at $v = -2$. The graph continues to the right until we encounter the 'hole' labeled at $(1, -3)$. This indicates one and only one point, namely $(1, -3)$ is missing from the curve which for us means $v = 1$ is not in the domain of $F$. The graph continues to the right and the arrow on the graph indicates that the graph goes upwards to the right indefinitely. Hence, our domain is $\{v \mid v \geq -2, v \neq 1\}$ which, in interval notation, is $[-2, 1) \cup (1, \infty)$. Pictures demonstrating the process of projecting the graph to the $v$-axis are shown below.

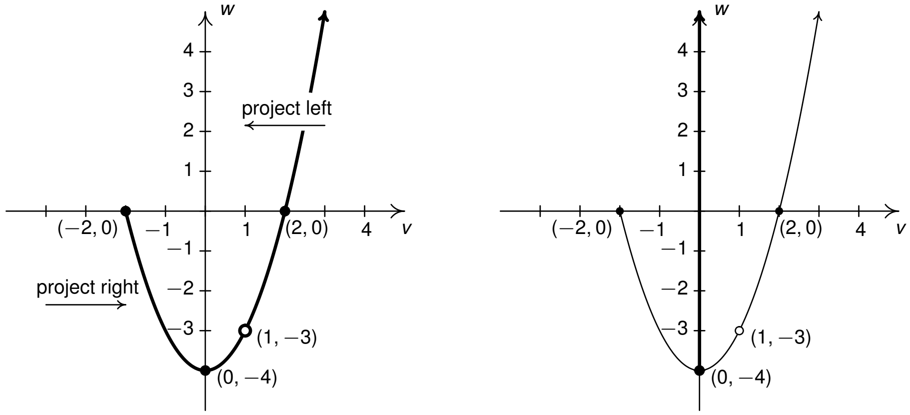



To find the range of $F$, we need to describe the set of outputs - in this case, the $w$-values on the graph. Here, we project the graph to the $w$-axis. Vertically, the graph starts at $(0, -4)$ so our range starts at $w = -4$. Note that even though there is a hole at $(1, -3)$, the $w$-value $-3$ is covered by what *appears* to be the point $(-1, -3)$ on the graph.[15] The arrow indicates that the graph extends upwards indefinitely so the range of $F$ is $\{w \mid w \geq -4\}$ or, in interval notation, $[-4, \infty)$. Regarding extrema, $F$ has a minimum of $-4$ when $v = 0$, but given that the graph extends upwards indefinitely, $F$ has no maximum.

[15] For all we know, it could be $(-0.992, -3)$.

Pictures showing the projection of the graph onto the $w$-axis are given below.




2. Finally, to determine if $v$ is a function of $w$, we look to see if each $w$-value is paired with only one $v$-value on the graph. We have points on the graph, namely $(-2, 0)$ and $(2, 0)$, that clearly show us that $w = 0$ is matched with the *two* $v$-values $v = 2$ and $v = -2$. Hence, $v$ is not a function of $w$. $\square$

It cannot be stressed enough that when given a graphical representation of a function, certain assumptions must be made. In the previous example, for all we know, the minimum of the graph is at $(0.001, -4.0001)$ instead of $(0, -4)$. If we aren't given an equation or table of data, or if specific points aren't labeled, we really have no way to tell. We also are assuming that the graph depicted in the example, while ultimately made of infinitely many points, has no gaps or holes other than those noted. This allows us to make such bold claims as the existence of a point on the graph with a $w$-coordinate of $-3$.

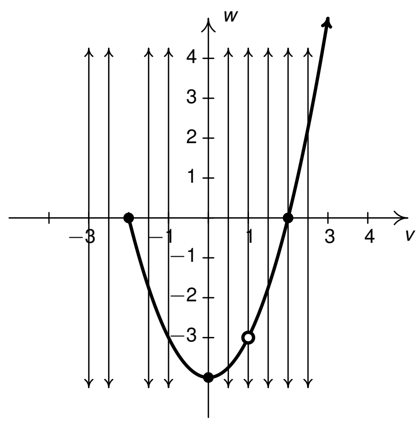
Before moving on to our next example, it is worth noting that the geometric argument made in Example **??** to establish that $w$ is a function of $v$ can be generalized to any graph. This result is the celebrated Vertical Line Test and it enables us to detect functions geometrically. Note that the statement of the theorem resorts to the 'default' $x$ and $y$ labels on the horizontal and vertical axes, respectively.

[redacted]

Let's take a minute to discuss the phrase 'if and only if' used in Theorem **??**. The statement 'the graph represents $y$ as a function of $x$ *if and only if* no vertical line intersects the graph more than once' is actually saying two things. First, it's saying 'the graph represents $y$ as a function of $x$ *if* no vertical line intersects the graph more than once' and, second, 'the graph represents $y$ as a function of $x$ *only if* no vertical line intersects the graph more than once'.

Logically, these statements are saying two different things. The first says that if no vertical line crosses the graph more than once, then the graph represents $y$ as a function of $x$. But the question remains: could a graph represent $y$ as a function of $x$ and yet there be a vertical line that intersects the graph more than once? The answer to this is 'no' because the second statement says that the *only* way the graph represents $y$ as a function of $x$ is the case when no vertical line intersects the graph more than once.

Applying the Vertical Line Test to the graph given in Example **??**, we see below that all of the vertical lines meet the graph at most once (several are shown for illustration) showing $w$ is a function of $v$. Notice that some of the lines ($x = -3$ and $x = 1$, for example) don't hit the graph at all. This is fine because the Vertical Line Test is looking for lines that hit the graph more than once. It does not say *exactly* once so missing the graph altogether is permitted.




There is also a geometric test to determine if the graph above represents $v$ as a function of $w$. We introduce this aptly-named **Horizontal Line Test** in Exercise **??** and revisit it in Sections **??** and **??**.

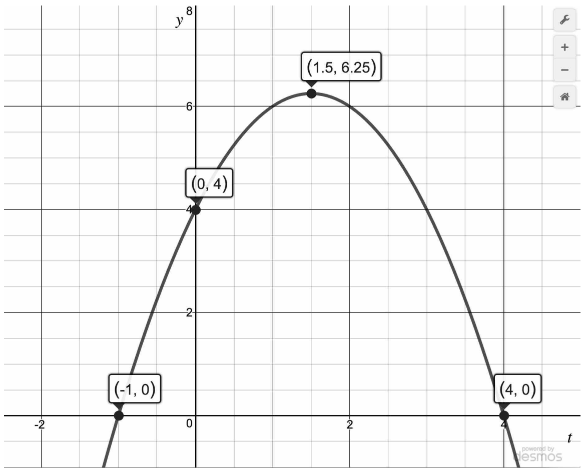
Our next example revisits the function $h$ from Example **??** from a graphical perspective.

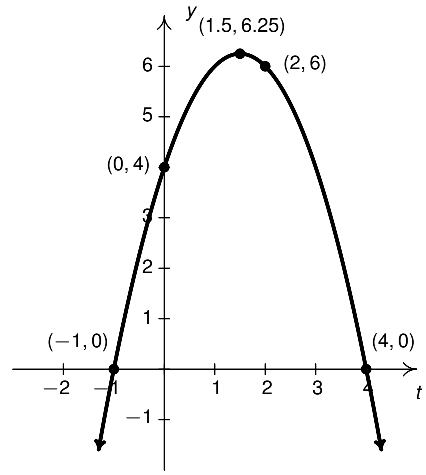
**Example 2.1.5.** With the help of a graphing utility graph $h(t) = -t^2 + 3t + 4$. From your graph, state the domain, range and extrema, if any exist.

**Solution.** The dependent variable wasn't specified so we use the default '$y$' label for the vertical axis and set about graphing $y = h(t)$. From our work in Example **??**, we already know $h(-1) = 0$, $h(0) = 4$, $h(2) = 6$ and $h(4) = 0$. These give us the points $(-1, 0)$, $(0, 4)$, $(2, 6)$ and $(4, 0)$, respectively. Using these as a guide, we can use desmos to produce the graph at the top of the next page on the left.[16]

As nice as the graph is, it is still technically incomplete. There is no restriction stated on the independent variable $t$ so the domain of $h$ is all real numbers. However, the graph as presented shows only the behavior of $h$ between roughly $t = -2.5$ and $t = 4.25$. By zooming out, we see that the graph extends downwards indefinitely which we indicate by adding the arrows you see in the graph on the right. We find that the domain is $(-\infty, \infty)$ and the range is $(-\infty, 6.25]$. There is no minimum, but the maximum of $h$ is 6.25 and it occurs at $t = 1.5$. The point $(1.5, 6.25)$ is shown on both graphs.

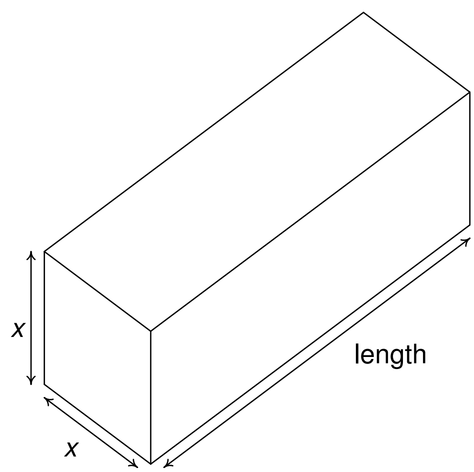
[16]The curve in this example is called a 'parabola'. In Section **??**, we'll learn how to graph these accurately *by hand*.




Our last example of the section uses the interplay between algebraic and graphical representations of a function to solve a real-world problem.

**Example 2.1.6.** The United States Postal Service mandates that when shipping parcels using 'Parcel Select' service, the sum of the length (the longest dimension) and the girth (the distance around the thickest part of the parcel perpendicular to the length) must not exceed 130 inches.[17] Suppose we wish to ship a rectangular box whose girth forms a square measuring $x$ inches per side as shown below.




It turns out[18] that the volume of a box, $V(x)$, measured in cubic inches, whose length plus girth is exactly 130 inches is given by the formula: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$.

[17] See here.

[18] We'll skip the explanation for now because we want to focus on just the different representations of the function. Rest assured, you'll be asked to construct this very model in Exercise **??** in Section **??**.

1. Find and interpret $V(5)$.

2. Make a table of values and use these along with a graphing utility to graph $y = V(x)$.

3. What is the largest volume box that can be shipped? What value of $x$ maximizes the volume? Round your answers to two decimal places.

**Solution.**

1. To find $V(5)$, we substitute $x = 5$ into the expression $V(x)$: $V(5) = (5)^2(130 - 4(5)) = 25(110) = 2750$. Our result means that when the length and width of the square measure 5 inches, the volume of the resulting box is 2750 cubic inches.[19]

2. The domain of $V$ is specified by the inequality $0 < x \leq 26$, so we can begin graphing $V$ by sampling $V$ at finitely many $x$-values in this interval to help us get a sense of the range of $V$. This, in turn, will help us determine an adequate viewing window on our graphing utility when the time comes.

   It seems natural to start with what's happening near $x = 0$. Even though the expression $x^2(130 - 4x)$ is defined when we substitute $x = 0$ (it reduces very quickly to 0), it would be incorrect to state $V(0) = 0$ because $x = 0$ is not in the domain of $V$. However, there is nothing stopping us from evaluating $V(x)$ at values $x$ 'very close' to $x = 0$. A table of such values is given below.

   | $x$ | $V(x)$ |
   |---|---|
   | 0.1 | 1.296 |
   | 0.01 | 0.012996 |
   | 0.001 | 0.000129996 |
   | $10^{-23}$ | $\approx 1.3 \times 10^{-44}$ |

   There is no such thing as a 'smallest' positive number,[20] so we will have points on the graph of $V$ to the right of $x = 0$ leading to the point $(0, 0)$. We indicate this behavior by putting a hole at $(0, 0)$.[21]

   Moving forward, we start with $x = 5$ and sample $V$ at steps of 5 in its domain. Our goal is to graph $y = V(x)$, so we plot our points $(x, V(x))$ using the domain as a guide to help us set the horizontal bounds (i.e., the bounds on $x$) and the sample values from the range to help us set the vertical bounds (i.e., the bounds on $y$). The right endpoint, $x = 26$, is included in the domain $0 < x \leq 26$ so we finish the graph by plotting the point $(26, V(26)) = (26, 17576)$. At the top of the next page on the left is the table of data and on the right is a graph produced with some help from desmos.

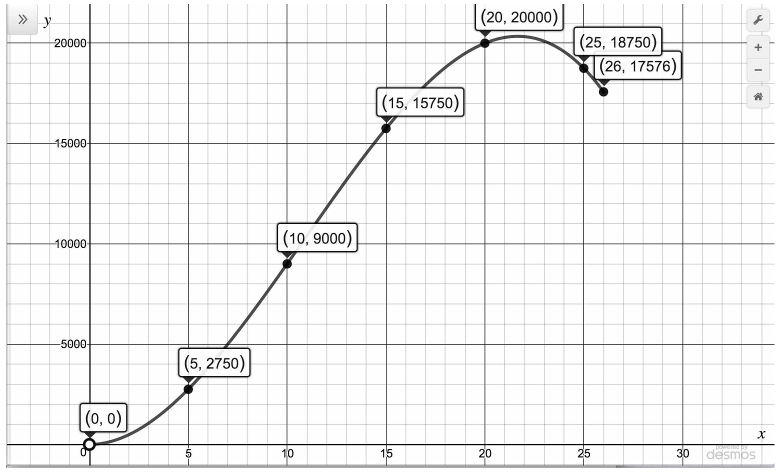
[19]Note that we have $V(5)$ and 25(110) in the same string of equality. The first set of parentheses is function notation and directs us to substitute 5 for $x$ in the expression $V(x)$ while the second indicates multiplying 25 by 110. Context is key!

[20]If $p$ is any positive real number, $0 < 0.5p < p$, so we can always find a smaller positive real number.

[21]What's really needed here is the precise definition of 'closeness' discussed in Calculus. This hand-waving will do for now.

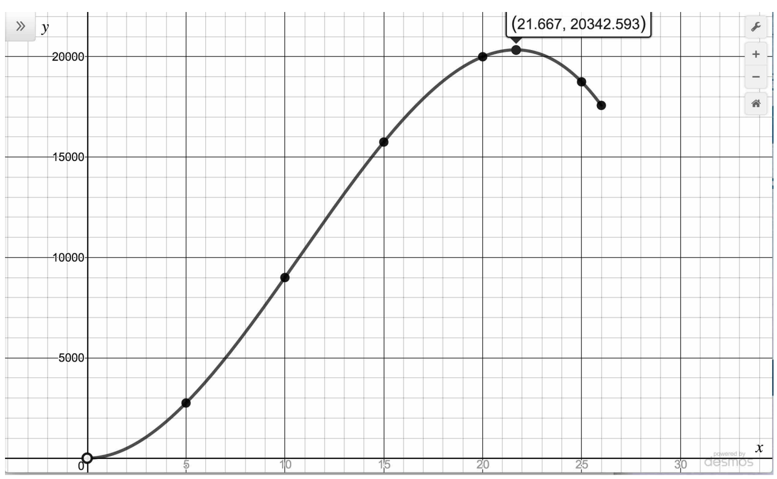
| $x$ | $V(x)$ | $(x, V(x))$ |
|---|---|---|
| $\approx 0$ | $\approx 0$ | hole at $(0, 0)$ |
| 5 | 2750 | $(5, 2750)$ |
| 10 | 9000 | $(10, 9000)$ |
| 15 | 15,750 | $(15, 15{,}750)$ |
| 20 | 20,000 | $(20, 20{,}000)$ |
| 25 | 18,750 | $(25, 18{,}750)$ |
| 26 | 17,576 | $(26, 17{,}576)$ |

Sampling $V$




The graph of $y = V(x)$

3. The largest volume in this case refers to the maximum of $V$. The biggest $y$-value in our table of data is 20,000 cubic inches which occurs at $x = 20$ inches, but the graph produced by the graphing utility indicates that there are points on the graph of $V$ with $y$-values (hence $V(x)$ values) greater than 20,000. Indeed, the graph continues to rise to the right of $x = 20$ and the graphing utility reports the maximum $y$-value to be $y \approx 20{,}342.593$ when $x \approx 21.667$. Rounding to two decimal places, we find the maximum volume obtainable under these conditions is about 20,342.59 cubic inches which occurs when the length and width of the square side of the box are approximately 21.67 inches.[22]




Finding the maximum volume using the graph of $y = V(x)$.

[22]We could also find the length of the box in this case as well. The sum of length and girth is 130 inches so the length is 130 minus the girth, or $130 - 4x \approx 130 - 4(21.67) = 43.32$ inches.

It is worth noting that while the function $V$ has a maximum, it did not have a minimum. Even though $V(x) > 0$ for all $x$ in its domain,[23] the presence of the hole at $(0, 0)$ means that 0 is not in the range of $V$. Hence, based on our model, we can never make a box with a 'smallest' volume.[24] □

Example **??** typifies the interplay between Algebra and Geometry which lies ahead. Both the algebraic description of $V$: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$, and the graph of $y = V(x)$ were useful in describing aspects of the physical situation at hand. Wherever possible, we'll use the algebraic representations of functions to *analytically* produce *exact* answers to certain problems and use the graphical descriptions to check the reasonableness of our answers.

That being said, we'll also encounter problems which we simply *cannot* answer analytically (such as determining the maximum volume in the previous example), so we will be forced to resort to using technology (specifically graphing technology) in order to find *approximate* solutions. The most important thing to keep in mind is that while technology may *suggest* a result, it is ultimately Mathematics that *proves* it.

We close this section with a summary of the different ways to represent functions.

[redacted]

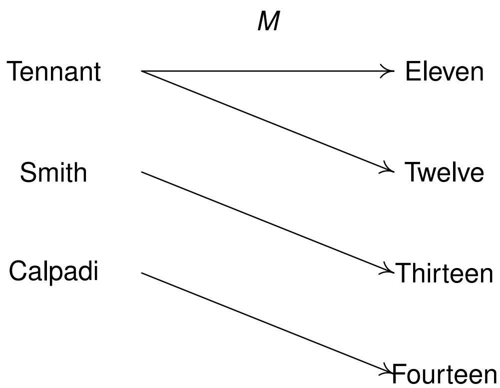
[23] said differently, the values of $V(x)$ are **bounded below** by 0.

[24] How realistic is this?

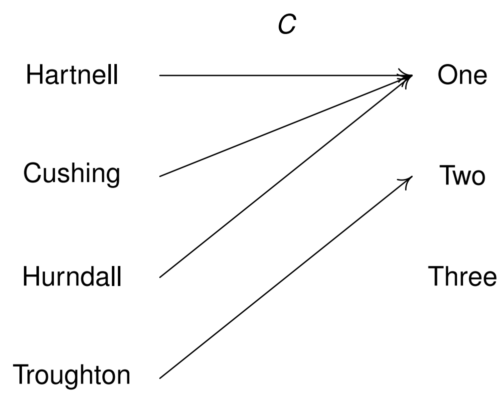
### 2.1.4 Exercises

In Exercises **??** - **??**, determine whether or not the mapping diagram represents a function. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

1.




2.




In Exercises **??** - **??**, determine whether or not the data in the given table represents $y$ as a function of $x$. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

3.

| x | y |
|---|---|
| $-3$ | 3 |
| $-2$ | 2 |
| $-1$ | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |

4.

| x | y |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 1 | $-1$ |
| 2 | 2 |
| 2 | $-2$ |
| 3 | 3 |
| 3 | $-3$ |

5. Suppose $W$ is the set of words in the English language and we set up a mapping from $W$ into the set of natural numbers $\mathbb{N}$ as follows: word $\to$ number of letters in the word. Explain why this mapping is a function. What would you need to know to determine the range of the function?

6. Suppose $L$ is the set of last names of all the people who have served or are currently serving as the President of the United States. Consider the mapping from $L$ into $\mathbb{N}$ as follows: last name $\to$ number of their presidency. For example, Washington $\to$ 1 and Obama $\to$ 44. Is this mapping a function? What if we use full names instead of just last names? (**HINT:** Research Grover Cleveland.)

7. Under what conditions would the time of day be a function of the outdoor temperature?

For the functions $f$ described in Exercises **??** - **??**, find $f(2)$ and find and simplify an expression for $f(x)$ that takes a real number $x$ and performs the following three steps in the order given:

8. (1) multiply by 2; (2) add 3; (3) divide by 4.
9. (1) add 3; (2) multiply by 2; (3) divide by 4.
10. (1) divide by 4; (2) add 3; (3) multiply by 2.
11. (1) multiply by 2; (2) add 3; (3) take the square root.
12. (1) add 3; (2) multiply by 2; (3) take the square root.
13. (1) add 3; (2) take the square root; (3) multiply by 2.

In Exercises **??** - **??**, use the given function $f$ to find and simplify the following:

- $f(3)$
- $f(-1)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

14. $f(x) = 2x + 1$
15. $f(x) = 3 - 4x$
16. $f(x) = 2 - x^2$
17. $f(x) = x^2 - 3x + 2$
18. $f(x) = 6$
19. $f(x) = 0$

In Exercises **??** - **??**, use the given function $f$ to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a)+f(2)$
- $f\left(\dfrac{2}{a}\right)$
- $\dfrac{f(a)}{2}$
- $f(a+h)$

20. $f(x) = 2x - 5$
21. $f(t) = 5 - 2t$
22. $f(w) = 2w^2 - 1$
23. $f(q) = 3q^2 + 3q - 2$
24. $f(r) = 117$
25. $f(z) = \dfrac{z}{2}$

In Exercises **??** - **??**, use the given function $f$ to find $f(0)$ and solve $f(x) = 0$

26. $f(x) = 2x - 1$
27. $f(x) = 3 - \frac{2}{5}x$
28. $f(x) = 2x^2 - 6$
29. $f(x) = x^2 - x - 12$

In Exercises **??** - **??**, determine whether or not the equation represents $y$ as a function of $x$.

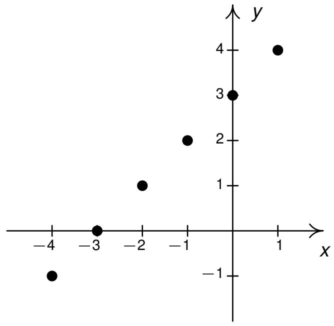
30. $y = x^3 - x$
31. $y = \sqrt{x-2}$
32. $x^3y = -4$
33. $x^2 - y^2 = 1$
34. $y = \dfrac{x}{x^2-9}$
35. $x = -6$
36. $x = y^2 + 4$
37. $y = x^2 + 4$
38. $x^2 + y^2 = 4$
39. $y = \sqrt{4 - x^2}$
40. $x^2 - y^2 = 4$
41. $x^3 + y^3 = 4$
42. $2x + 3y = 4$
43. $2xy = 4$
44. $x^2 = y^2$

Exercises **??** - **??** give a set of points in the $xy$-plane. Determine if $y$ is a function of $x$. If so, state the domain and range.

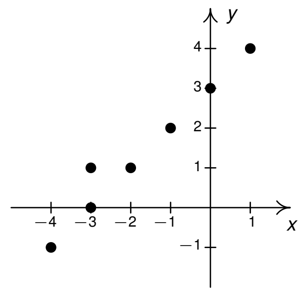
45. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$
46. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$
47. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$
48. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),...\}$
49. $\{(x,y) \mid x$ is an odd integer, and $y$ is an even integer$\}$
50. $\{(x,1) \mid x$ is an irrational number$\}$
51. $\{(1,0),(2,1),(4,2),(8,3),(16,4),(32,5),...\}$
52. $\{...(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9),...\}$
53. $\{(-2,y) \mid -3 < y < 4\}$
54. $\{(x,3) \mid -2 \leq x < 4\}$
55. $\{\left(x,x^2\right) \mid x$ is a real number$\}$
56. $\{\left(x^2,x\right) \mid x$ is a real number$\}$
57. The Vertical Line Test is a quick way to determine from a graph if the vertical axis variable is a function of the horizontal axis variable. If we are given a graph and asked to determine if the horizontal axis variable is a function of the vertical axis variable, we can use horizontal lines instead of vertical lines to check. Using Theorem **??** as a guide, formulate a 'Horizontal Line Test.' (We'll refer back to this exercise in Section **??**.)

In Exercises **??** - **??**, determine whether or not the graph suggests $y$ is a function of $x$. For the ones which do, state the domain and range.

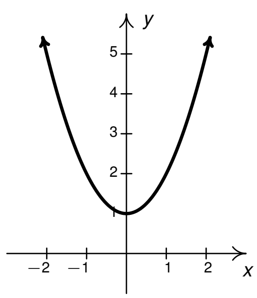
58.




59.




60.

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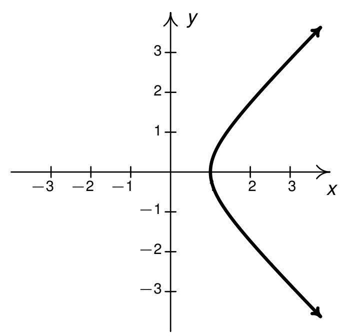
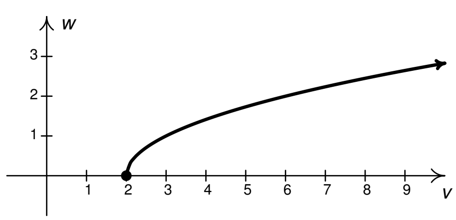


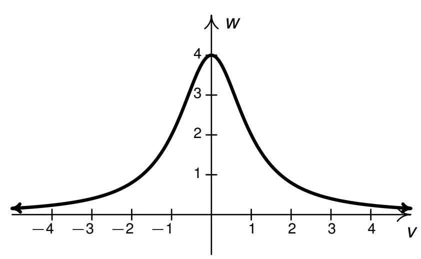

62. Determine which, if any, of the graphs in numbers **??** - **??** represent $x$ as a function of $y$. For the ones which do, state the domain and range. (Feel free to use Exercise **??**.)

In Exercises **??** - **??**, determine whether or not the graph suggests $w$ is a function of $v$ . For the ones which do, state the domain and range.

63.




64.



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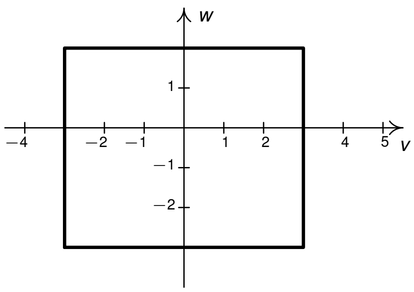
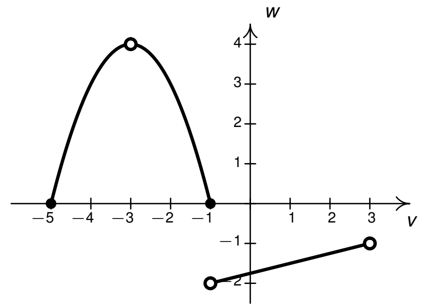


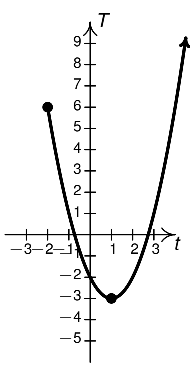

66.




67. Determine which, if any, of the graphs in numbers **??** - **??** represent $v$ as a function of $w$. For the ones which do, state the domain and range. (Feel free to use Exercise **??**.)

In Exercises **??** - **??**, determine whether or not the graph suggests $T$ is a function of $t$. For the ones which do, state the domain and range.

68.

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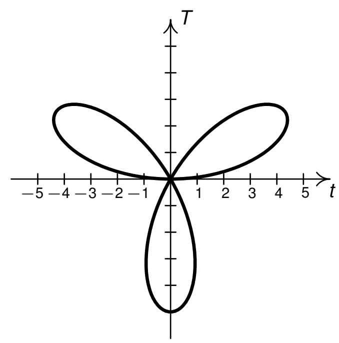
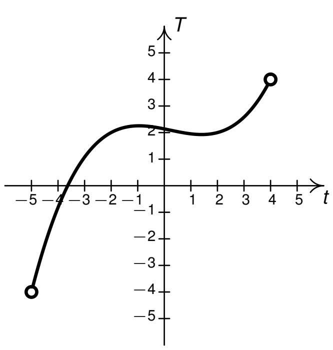
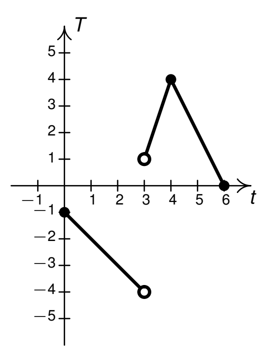



70.



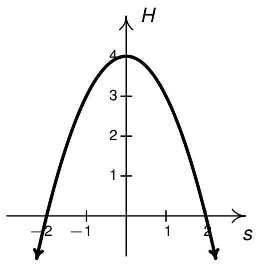

71.



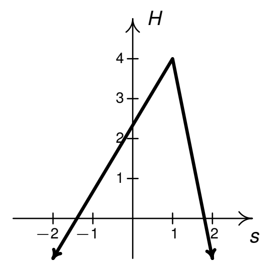

72. Determine which, if any, of the graphs in numbers **??** - **??** represent $t$ as a function of $T$. For the ones which do, state the domain and range. (Feel free to use Exercise **??**.)

In Exercises **??** - **??**, determine whether or not the graph suggests $H$ is a function of $s$. For the ones which do, state the domain and range.

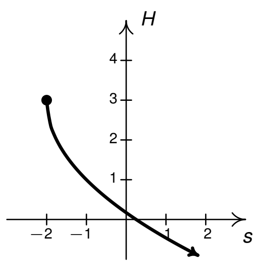
73.



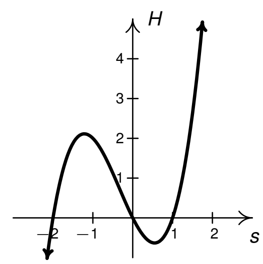

74.




75.



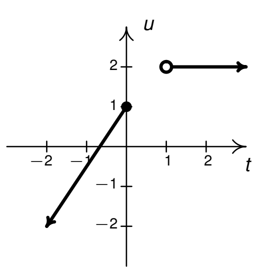

76.



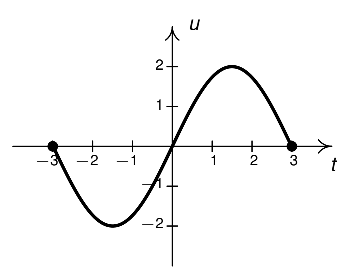

77. Determine which, if any, of the graphs in numbers **??** - **??** represent $s$ as a function of $H$. For the ones which do, state the domain and range. (Feel free to use Exercise **??**.)

In Exercises **??** - **??**, determine whether or not the graph suggests $u$ is a function of $t$. For the ones which do, state the domain and range.

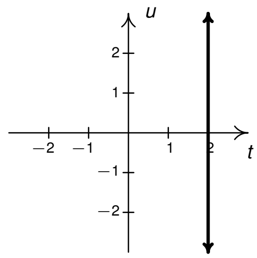
78.



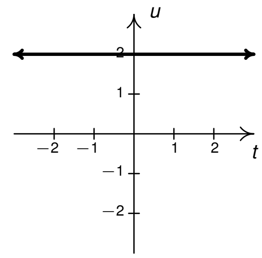

79.



80.

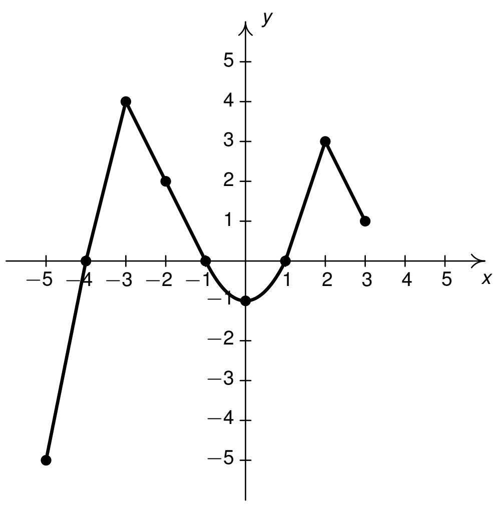



81.

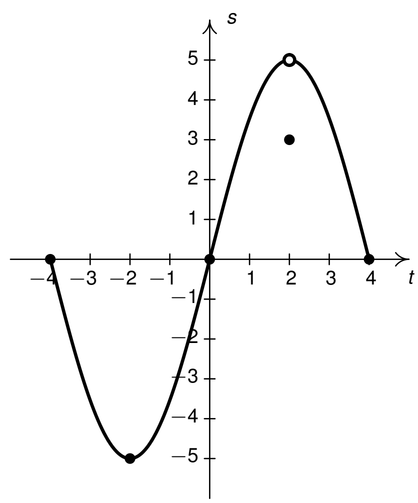



82. Determine which, if any, of the graphs in numbers **??** - **??** represent $t$ as a function of $u$. For the ones which do, state the domain and range. (Feel free to use Exercise **??**.)

In Exercises **??** - **??**, use the graphs of $f$ and $g$ below to find the indicated values.




The graph of $y = f(x)$.




The graph of $s = g(t)$.

83. $f(-2)$

84. $g(-2)$

85. $f(2)$

86. $g(2)$

87. $f(0)$

88. $g(0)$

89. Solve $f(x) = 0$.

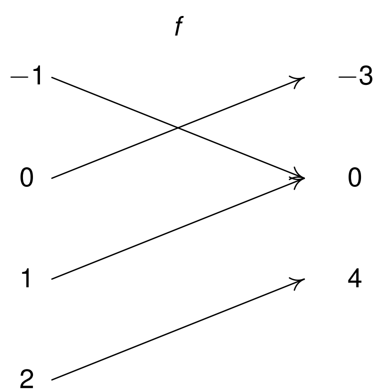
90. Solve $g(t) = 0$.

91. State the domain and range of $f$.

92. State the domain and range of $g$ .

In Exercises **??** - **??**, graph each function by making a table, plotting points, and using a graphing utility (if needed.) Use the independent variable as the horizontal axis label and the default '$y$' label for the vertical axis label. State the domain and range of each function.

93. $f(x) = 2 - x$
94. $g(t) = \dfrac{t-2}{3}$
95. $h(s) = s^2 + 1$
96. $f(x) = 4 - x^2$
97. $g(t) = 2$
98. $h(s) = s^3$
99. $f(x) = x(x-1)(x+2)$
100. $g(t) = \sqrt{t-2}$
101. $h(s) = \sqrt{5-s}$
102. $f(x) = 3 - 2\sqrt{x+2}$
103. $g(t) = \sqrt[3]{t}$
104. $h(s) = \dfrac{1}{s^2+1}$
105. Consider the function $f$ described below:




    (a) State the domain and range of $f$.
    (b) Find $f(0)$ and solve $f(x) = 0$.
    (c) Write $f$ as a set of ordered pairs.
    (d) Graph $f$.

106. Let $g = \{(-1,4), (0,2), (2,3), (3,4)\}$

    (a) State the domain and range of $g$.
    (b) Create a mapping diagram for $g$.
    (c) Find $g(0)$ and solve $g(x) = 0$.
    (d) Graph $g$.

107. Let $F = \{(t, t^2) \mid t \text{ is a real number}\}$. Find $F(4)$ and solve $F(x) = 4$.

    **HINT:** Elements of $F$ are of the form $(x, F(x))$.

108. Let $G = \{(2t, t+5) \mid t \text{ is a real number}\}$. Find $G(4)$ and solve $G(x) = 4$.

  **HINT:** Elements of $G$ are of the form $(x, G(x))$.

109. The area enclosed by a square, in square inches, is a function of the length of one of its sides $\ell$, when measured in inches. This function is represented by the formula $A(\ell) = \ell^2$ for $\ell > 0$. Find $A(3)$ and solve $A(\ell) = 36$. Interpret your answers to each. Why is $\ell$ restricted to $\ell > 0$?

110. The area enclosed by a circle, in square meters, is a function of its radius $r$, when measured in meters. This function is represented by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is $r$ restricted to $r > 0$?

111. The volume enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides $s$, when measured in centimeters. This function is represented by the formula $V(s) = s^3$ for $s > 0$. Find $V(5)$ and solve $V(s) = 27$. Interpret your answers to each. Why is $s$ restricted to $s > 0$?

112. The volume enclosed by a sphere, in cubic feet, is a function of the radius of the sphere $r$, when measured in feet. This function is represented by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is $r$ restricted to $r > 0$?

113. The height of an object dropped from the roof of an eight story building is modeled by the function: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground, in feet, $t$ seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is $t$ restricted to $0 \leq t \leq 2$?

114. The temperature in degrees Fahrenheit $t$ hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \leq t \leq 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.

115. The function $C(x) = x^2 - 10x + 27$ models the cost, in *hundreds* of dollars, to produce $x$ *thousand* pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.

116. Using data from the Bureau of Transportation Statistics, the average fuel economy in miles per gallon for passenger cars in the US can be modeled by $E(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where $t$ is the number of years since 1980. Use a calculator to find $E(0)$, $E(14)$ and $E(28)$. Round your answers to two decimal places and interpret your answers to each.

117. The perimeter of a square, in centimeters, is four times the length of one if its sides, also measured in centimeters. Represent the function $P$ which takes as its input the length of the side of a square in centimeters, $s$ and returns the perimeter of the square in inches, $P(s)$ using a formula.

118. The circumference of a circle, in feet, is $\pi$ times the diameter of the circle, also measured in feet. Represent the function $C$ which takes as its input the length of the diameter of a circle in feet, $D$ and returns the circumference of a circle in inches, $C(D)$ using a formula.

119. Suppose $A(P)$ gives the amount of money in a retirement account (in dollars) after 30 years as a function of the amount of the monthly payment (in dollars), $P$.

    (a) What does $A(50)$ mean?
    (b) What is the significance of the solution to the equation $A(P) = 250000$? .
    (c) Explain what each of the following expressions mean: $A(P + 50)$, $A(P) + 50$, and $A(P) + A(50)$.

120. Suppose $P(t)$ gives the chance of precipitation (in percent) $t$ hours after 8 AM.

    (a) Write an expression which gives the chance of precipitation at noon.
    (b) Write an inequality which determines when the chance of precipitation is more than 50%.

121. Explain why the graph in Exercise **??** suggests that not only is $v$ as a function of $w$ but also $w$ is a function of $v$. Suppose $w = f(v)$ and $v = g(w)$. That is, $f$ is the name of the function which takes $v$ values as inputs and returns $w$ values as outputs and $g$ is the name of the function which does vice-versa. Find the domain and range of $g$ and compare these to the domain and range of $f$.

122. Sketch the graph of a function with domain $(-\infty, 3) \cup [4, 5)$ with range $\{2\} \cup (5, \infty)$.

### 2.1.5 Answers

1. The mapping $M$ is not a function since 'Tennant' is matched with both 'Eleven' and 'Twelve.'

2. The mapping $C$ is a function since each input is matched with only one output. The domain of $C$ is $\{\text{Hartnell}, \text{Cushing}, \text{Hurndall}, \text{Troughton}\}$ and the range is $\{\text{One}, \text{Two}\}$. We can represent $C$ as the following set of ordered pairs: $\{(\text{Hartnell}, \text{One}), (\text{Cushing}, \text{One}), (\text{Hurndall}, \text{One}), (\text{Troughton}, \text{Two})\}$

3. In this case, $y$ is a function of $x$ since each $x$ is matched with only one $y$.

   The domain is $\{-3, -2, -1, 0, 1, 2, 3\}$ and the range is $\{0, 1, 2, 3\}$.

   As ordered pairs, this function is $\{(-3, 3), (-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2), (3, 3)\}$

4. In this case, $y$ is not a function of $x$ since there are $x$ values matched with more than one $y$ value. For instance, 1 is matched both to 1 and $-1$.

5. The mapping is a function since given any word, there is only one answer to 'how many letters are in the word?' For the range, we would need to know what the length of the longest word is and whether or not we could find words of all the lengths between 1 (the length of the word 'a') and it. See here.

6. Since Grover Cleveland was both the 22nd and 24th POTUS, neither mapping described in this exercise is a function.

7. The outdoor temperature could never be the same for more than two different times - so, for example, it could always be getting warmer or it could always be getting colder.

8. $f(2) = \frac{7}{4}$, $f(x) = \frac{2x+3}{4}$

9. $f(2) = \frac{5}{2}$, $f(x) = \frac{2(x+3)}{4} = \frac{x+3}{2}$

10. $f(2) = 7$, $f(x) = 2\left(\frac{x}{4} + 3\right) = \frac{1}{2}x + 6$

11. $f(2) = \sqrt{7}$, $f(x) = \sqrt{2x+3}$

12. $f(2) = \sqrt{10}$, $f(x) = \sqrt{2(x+3)} = \sqrt{2x+6}$

13. $f(2) = 2\sqrt{5}$, $f(x) = 2\sqrt{x+3}$

14. For $f(x) = 2x + 1$

    - $f(3) = 7$
    - $f(-1) = -1$
    - $f\left(\frac{3}{2}\right) = 4$
    - $f(4x) = 8x + 1$
    - $4f(x) = 8x + 4$
    - $f(-x) = -2x + 1$
    - $f(x-4) = 2x - 7$
    - $f(x) - 4 = 2x - 3$
    - $f\left(x^2\right) = 2x^2 + 1$

15. For $f(x) = 3 - 4x$

    - $f(3) = -9$
    - $f(-1) = 7$
    - $f\left(\frac{3}{2}\right) = -3$

- $f(4x) = 3 - 16x$
- $4f(x) = 12 - 16x$
- $f(-x) = 4x + 3$
- $f(x - 4) = 19 - 4x$
- $f(x) - 4 = -4x - 1$
- $f\left(x^2\right) = 3 - 4x^2$

16. For $f(x) = 2 - x^2$
    - $f(3) = -7$
    - $f(-1) = 1$
    - $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
    - $f(4x) = 2 - 16x^2$
    - $4f(x) = 8 - 4x^2$
    - $f(-x) = 2 - x^2$
    - $f(x - 4) = -x^2 + 8x - 14$
    - $f(x) - 4 = -x^2 - 2$
    - $f\left(x^2\right) = 2 - x^4$

17. For $f(x) = x^2 - 3x + 2$
    - $f(3) = 2$
    - $f(-1) = 6$
    - $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
    - $f(4x) = 16x^2 - 12x + 2$
    - $4f(x) = 4x^2 - 12x + 8$
    - $f(-x) = x^2 + 3x + 2$
    - $f(x - 4) = x^2 - 11x + 30$
    - $f(x) - 4 = x^2 - 3x - 2$
    - $f\left(x^2\right) = x^4 - 3x^2 + 2$

18. For $f(x) = 6$
    - $f(3) = 6$
    - $f(-1) = 6$
    - $f\left(\frac{3}{2}\right) = 6$
    - $f(4x) = 6$
    - $4f(x) = 24$
    - $f(-x) = 6$
    - $f(x - 4) = 6$
    - $f(x) - 4 = 2$
    - $f\left(x^2\right) = 6$

19. For $f(x) = 0$
    - $f(3) = 0$
    - $f(-1) = 0$
    - $f\left(\frac{3}{2}\right) = 0$
    - $f(4x) = 0$
    - $4f(x) = 0$
    - $f(-x) = 0$
    - $f(x - 4) = 0$
    - $f(x) - 4 = -4$
    - $f\left(x^2\right) = 0$

20. For $f(x) = 2x - 5$
    - $f(2) = -1$
    - $f(-2) = -9$
    - $f(2a) = 4a - 5$
    - $2f(a) = 4a - 10$
    - $f(a + 2) = 2a - 1$
    - $f(a) + f(2) = 2a - 6$

- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a+h) = 2a + 2h - 5$

21. For $f(x) = 5 - 2x$

- $f(2) = 1$
- $f(-2) = 9$
- $f(2a) = 5 - 4a$
- $2f(a) = 10 - 4a$
- $f(a+2) = 1 - 2a$
- $f(a) + f(2) = 6 - 2a$
- $f\left(\frac{2}{a}\right) = 5 - \frac{4}{a} = \frac{5a-4}{a}$
- $\frac{f(a)}{2} = \frac{5-2a}{2}$
- $f(a+h) = 5 - 2a - 2h$

22. For $f(x) = 2x^2 - 1$

- $f(2) = 7$
- $f(-2) = 7$
- $f(2a) = 8a^2 - 1$
- $2f(a) = 4a^2 - 2$
- $f(a+2) = 2a^2 + 8a + 7$
- $f(a) + f(2) = 2a^2 + 6$
- $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{2a^2-1}{2}$
- $f(a+h) = 2a^2+4ah+2h^2-1$

23. For $f(x) = 3x^2 + 3x - 2$

- $f(2) = 16$
- $f(-2) = 4$
- $f(2a) = 12a^2 + 6a - 2$
- $2f(a) = 6a^2 + 6a - 4$
- $f(a+2) = 3a^2 + 15a + 16$
- $f(a) + f(2) = 3a^2 + 3a + 14$
- $f\left(\frac{2}{a}\right) = \frac{12}{a^2} + \frac{6}{a} - 2 = \frac{12+6a-2a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{3a^2+3a-2}{2}$
- $f(a+h) = 3a^2 + 6ah + 3h^2 + 3a + 3h - 2$

24. For $f(x) = 117$

- $f(2) = 117$
- $f(-2) = 117$
- $f(2a) = 117$
- $2f(a) = 234$
- $f(a+2) = 117$
- $f(a) + f(2) = 234$
- $f\left(\frac{2}{a}\right) = 117$
- $\frac{f(a)}{2} = \frac{117}{2}$
- $f(a+h) = 117$

25. For $f(x) = \frac{x}{2}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = a$
- $2f(a) = a$
- $f(a+2) = \frac{a+2}{2}$
- $f(a) + f(2) = \frac{a}{2} + 1$
  $= \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = \frac{1}{a}$
- $\frac{f(a)}{2} = \frac{a}{4}$
- $f(a+h) = \frac{a+h}{2}$

26. For $f(x) = 2x - 1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$
27. For $f(x) = 3 - \frac{2}{5}x$, $f(0) = 3$ and $f(x) = 0$ when $x = \frac{15}{2}$
28. For $f(x) = 2x^2 - 6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$
29. For $f(x) = x^2 - x - 12$, $f(0) = -12$ and $f(x) = 0$ when $x = -3$ or $x = 4$
30. Function
31. Function
32. Function
33. Not a function
34. Function
35. Not a function
36. Not a function
37. Function
38. Not a function
39. Function
40. Not a function
41. Function
42. Function
43. Function
44. Not a function
45. Function
    domain = $\{-3, -2, -1, 0, 1, 2, 3\}$
    range = $\{0, 1, 4, 9\}$
46. Not a function
47. Function
    domain = $\{-7, -3, 3, 4, 5, 6\}$
    range = $\{0, 4, 5, 6, 9\}$
48. Function
    domain = $\{1, 4, 9, 16, 25, 36, ...\}$
    = $\{x \mid x$ is a perfect square$\}$
    range = $\{2, 4, 6, 8, 10, 12, ...\}$
    = $\{y \mid y$ is a positive even integer$\}$
49. Not a function
50. Function
    domain = $\{x \mid x$ is irrational$\}$
    range = $\{1\}$
51. Function
    domain = $\{x \mid 1, 2, 4, 8, ...\}$
    = $\{x \mid x = 2^n$ for some whole number $n\}$
    range = $\{0, 1, 2, 3, ...\}$
    = $\{y \mid y$ is any whole number$\}$
52. Function
    domain = $\{x \mid x$ is any integer$\}$
    range = $\{y \mid y$ is the square of an integer$\}$

53. Not a function

54. Function
domain = $\{x \mid -2 \leq x < 4\}$ = $[-2, 4)$,
range = $\{3\}$

55. Function
domain = $\{x \mid x \text{ is a real number}\}$ = $(-\infty, \infty)$
range = $\{y \mid y \geq 0\}$ = $[0, \infty)$

56. Not a function

57. **Horizontal Line Test:** A graph on the *xy*-plane represents $x$ as a function of $y$ if and only if no **horizontal** line intersects the graph more than once.

58. Function
domain = $\{-4, -3, -2, -1, 0, 1\}$
range = $\{-1, 0, 1, 2, 3, 4\}$

59. Not a function

60. Function
domain = $(-\infty, \infty)$
range = $[1, \infty)$

61. Not a function

62.
- Number **??** represents $x$ as a function of $y$.
  domain = $\{-1, 0, 1, 2, 3, 4\}$ and range = $\{-4, -3, -2, -1, 0, 1\}$
- Number **??** represents $x$ as a function of $y$.
  domain = $(-\infty, \infty)$ and range = $[1, \infty)$

63. Function
domain = $[2, \infty)$
range = $[0, \infty)$

64. Function
domain = $(-\infty, \infty)$
range = $(0, 4]$

65. Not a function

66. Function
domain = $[-5, -3) \cup (-3, 3)$
range = $(-2, -1) \cup [0, 4)$

67. Only number **??** represents $v$ as a function of $w$; domain = $[0, \infty)$ and range = $[2, \infty)$

68. Function
domain = $[-2, \infty)$
range = $[-3, \infty)$

69. Not a function

70. Function
domain = $(-5, 4)$
range = $(-4, 4)$

71. Function
domain = $[0, 3) \cup (3, 6]$
range = $(-4, -1] \cup [0, 4]$

72. None of numbers **??** - **??** represent $t$ as a function of $T$.

73. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

74. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

75. Function
domain = $[-2, \infty)$
range = $(-\infty, 3]$

76. Function
domain = $(-\infty, \infty)$
range = $(-\infty, \infty)$

77. Only number **??** represents $s$ as a function of $H$; domain = $(-\infty, 3]$ and range = $[-2, \infty)$

78. Function
domain = $(-\infty, 0] \cup (1, \infty)$
range = $(-\infty, 1] \cup \{2\}$

79. Function
domain = $[-3, 3]$
range = $[-2, 2]$

80. Not a function

81. Function
domain = $(-\infty, \infty)$
range = $\{2\}$

82. Only number **??** represents $t$ as a function of $u$; domain = $(-\infty, \infty)$ and range=$\{2\}$.

83. $f(-2) = 2$

84. $g(-2) = -5$

85. $f(2) = 3$

86. $g(2) = 3$

87. $f(0) = -1$

88. $g(0) = 0$

89. $x = -4, -1, 1$

90. $t = -4, 0, 4$

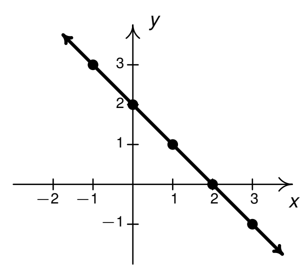
91. Domain: $[-5, 3]$, Range: $[-5, 4]$.

92. Domain: $[-4, 4]$, Range: $[-5, 5)$.

93. $f(x) = 2 - x$

Domain: $(-\infty, \infty)$

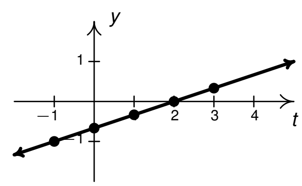
Range: $(-\infty, \infty)$




94. $g(t) = \dfrac{t-2}{3}$

Domain: $(-\infty, \infty)$

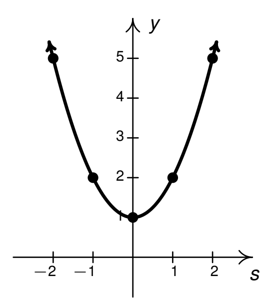
Range: $(-\infty, \infty)$



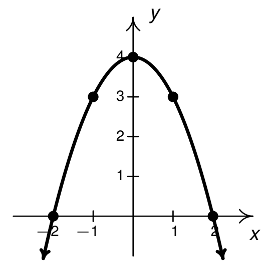

95. $h(s) = s^2 + 1$

Domain: $(-\infty, \infty)$

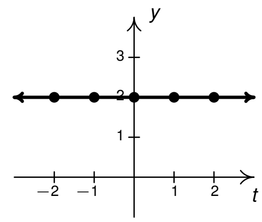
Range: $[1, \infty)$



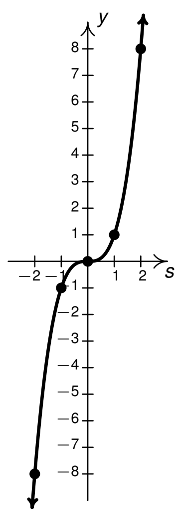
96. $f(x) = 4 - x^2$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 4]$



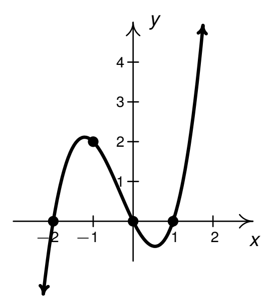
97. $g(t) = 2$
Domain: $(-\infty, \infty)$
Range: $\{2\}$



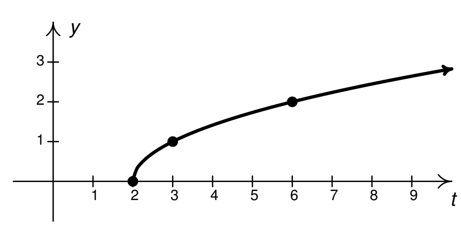
98. $h(s) = s^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



99. $f(x) = x(x-1)(x+2)$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



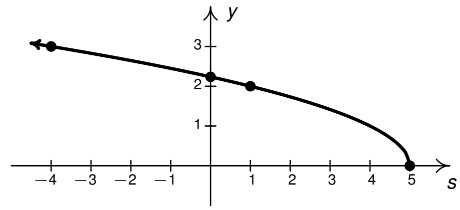
100. $g(t) = \sqrt{t-2}$
Domain: $[2, \infty)$
Domain: $[0, \infty)$



101. $h(s) = \sqrt{5 - s}$

Domain: $(-\infty, 5]$

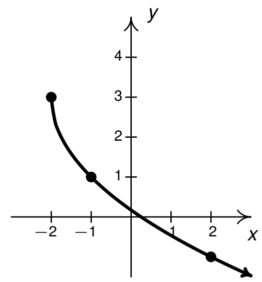
Range: $[0, \infty)$



102. $f(x) = 3 - 2\sqrt{x + 2}$

Domain: $[-2, \infty)$

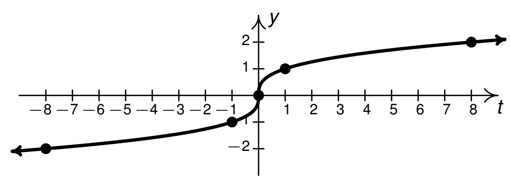
Range: $(-\infty, 3]$



103. $g(t) = \sqrt[3]{t}$

Domain: $(-\infty, \infty)$

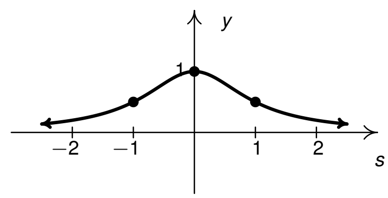
Range: $(-\infty, \infty)$



104. $h(s) = \dfrac{1}{s^2 + 1}$

Domain: $(-\infty, \infty)$

Range: $(0, 1]$

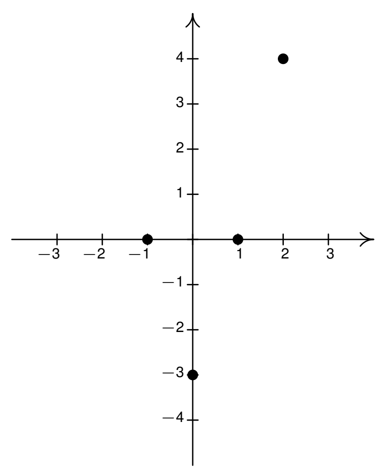


105. (a) domain = $\{-1, 0, 1, 2\}$, range = $\{-3, 0, 4\}$

(b) $f(0) = -3$, $f(x) = 0$ for $x = -1, 1$.

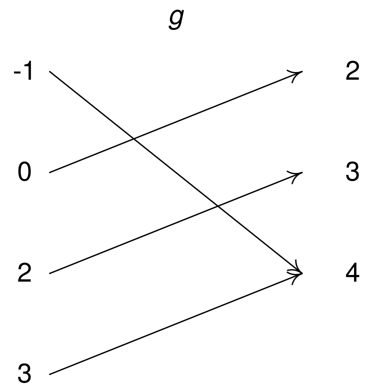
(c) $f = \{(-1, 0), (0, -3), (1, 0), (2, 4)\}$

(d)



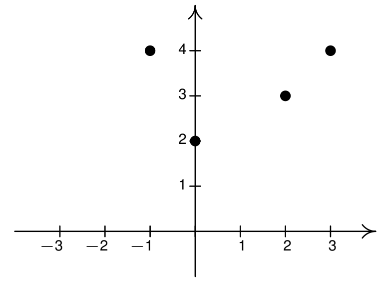
106. (a) domain = $\{-1, 0, 2, 3\}$, range = $\{2, 3, 4\}$

(b)




(c) Find $g(0) = 2$ and $g(x) = 0$ has no solutions.

(d)



107. $F(4) = 4^2 = 16$ (when $t = 4$), the solutions to $F(x) = 4$ are $x = \pm 2$ (when $t = \pm 2$).

108. $G(4) = 7$ (when $t = 2$), the solution to $G(t) = 4$ is $x = -2$ (when $t = -1$)

109. $A(3) = 9$, so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to $A(\ell) = 36$ are $\ell = \pm 6$. Since $\ell$ is restricted to $\ell > 0$, we only keep $\ell = 6$. This means for the area enclosed by the square to be 36 square inches, the length of the side needs to be 6 inches. Since $\ell$ represents a length, $\ell > 0$.

110. $A(2) = 4\pi$, so the area enclosed by a circle with radius 2 meters is $4\pi$ square meters. The solutions to $A(r) = 16\pi$ are $r = \pm 4$. Since $r$ is restricted to $r > 0$, we only keep $r = 4$. This means for the area enclosed by the circle to be $16\pi$ square meters, the radius needs to be 4 meters. Since $r$ represents a radius (length), $r > 0$.

111. $V(5) = 125$, so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to $V(s) = 27$ is $s = 3$. This means for the volume enclosed by the cube to be 27 cubic centimeters, the length of the side needs to 3 centimeters. Since $x$ represents a length, $x > 0$.

112. $V(3) = 36\pi$, so the volume enclosed by a sphere with radius 3 feet is $36\pi$ cubic feet. The solution to $V(r) = \frac{32\pi}{3}$ is $r = 2$. This means for the volume enclosed by the sphere to be $\frac{32\pi}{3}$ cubic feet, the radius needs to 2 feet. Since $r$ represents a radius (length), $r > 0$.

113. $h(0) = 64$, so at the moment the object is dropped off the building, the object is 64 feet off of the ground. The solutions to $h(t) = 0$ are $t = \pm 2$. Since we restrict $0 \leq t \leq 2$, we only keep $t = 2$. This means 2 seconds after the object is dropped off the building, it is 0 feet off the ground. Said differently, the object hits the ground after 2 seconds. The restriction $0 \leq t \leq 2$ restricts the time to be between the moment the object is released and the moment it hits the ground.

114. $T(0) = 3$, so at 6 AM (0 hours after 6 AM), it is $3^\circ$ Fahrenheit. $T(6) = 33$, so at noon (6 hours after 6 AM), the temperature is $33^\circ$ Fahrenheit. $T(12) = 27$, so at 6 PM (12 hours after 6 AM), it is $27^\circ$ Fahrenheit.

115. $C(0) = 27$, so to make 0 pens, it costs[25] \$2700. $C(2) = 11$, so to make 2000 pens, it costs \$1100. $C(5) = 2$, so to make 5000 pens, it costs \$2000.

116. $E(0) = 16.00$, so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon. $E(14) = 20.81$, so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon. $E(28) = 22.64$, so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.

117. $P(s) = 4s$, $s > 0$.

118. $C(D) = \pi D$, $D > 0$.

119. (a) The amount in the retirement account after 30 years if the monthly payment is \$50.
   (b) The solution to $A(P) = 250000$ is what the monthly payment needs to be in order to have \$250,000 in the retirement account after 30 years.
   (c) $A(P + 50)$ is how much is in the retirement account in 30 years if \$50 is added to the monthly payment $P$. $A(P) + 50$ represents the amount of money in the retirement account after 30 years if \$$P$ is invested each month plus an additional \$50. $A(P) + A(50)$ is the sum of money from two retirement accounts after 30 years: one with monthly payment \$$P$ and one with monthly payment \$50.

120. (a) Since noon is 4 hours after 8 AM, $P(4)$ gives the chance of precipitation at noon.
   (b) We would need to solve $P(t) \geq 50\%$ or $P(t) \geq 0.5$.

121. The graph in question passes the horizontal line test meaning for each $w$ there is only one $v$. The domain of $g$ is $[0, \infty)$ (which is the range of $f$) and the range of $g$ is $[2, \infty)$ which is the domain of $f$.

122. Answers vary.

[25] This is called the 'fixed' or 'start-up' cost. We'll revisit this concept in Example **??** in Section **??**.

## 2.2 Review of Linear Equations, their Graphs and Linear Functions

### 2.2.1 Review of Linear Equations

### 2.2.2 Linear Equations

The first equations we wish to review are **linear** equations as defined below.

[redacted]

One key point about Definition **??** is that the exponent on the unknown '$x$' in the equation is 1, that is $x = x^1$. Our main strategy for solving linear equations is summarized below.

[redacted]

We illustrate this process with a collection of examples below.

**Example 2.2.1.** Solve the following equations for the indicated variable. Check your answer.

1. Solve for $x$: $3x - 6 = 7x + 4$
2. Solve for $t$: $3 - 1.7t = \dfrac{t}{4}$
3. Solve for $a$: $\dfrac{1}{18}(7 - 4a) + 2 = \dfrac{a}{3} - \dfrac{4 - a}{12}$
4. Solve for $y$: $8y\sqrt{3} + 1 = 7 - \sqrt{12}(5 - y)$
5. Solve for $x$: $\dfrac{3x - 1}{2} = x\sqrt{50} + 4$
6. Solve for $y$: $x(4 - y) = 8y$

**Solution.**

1. The variable we are asked to solve for is $x$ so our first move is to gather all of the terms involving $x$ on one side and put the remaining terms on the other.[1]

$$\begin{array}{rcll}
3x - 6 & = & 7x + 4 & \\
(3x - 6) - 7x + 6 & = & (7x + 4) - 7x + 6 & \text{Subtract } 7x\text{, add } 6 \\
3x - 7x - 6 + 6 & = & 7x - 7x + 4 + 6 & \text{Rearrange terms} \\
-4x & = & 10 & 3x - 7x = (3 - 7)x = -4x \\
\dfrac{-4x}{-4} & = & \dfrac{10}{-4} & \text{Divide by the coefficient of } x \\
x & = & -\dfrac{5}{2} & \text{Reduce to lowest terms}
\end{array}$$

To check our answer, we substitute $x = -\frac{5}{2}$ into each side of the **orginial** equation to see the equation is satisfied. Sure enough, $3\left(-\frac{5}{2}\right) - 6 = -\frac{27}{2}$ and $7\left(-\frac{5}{2}\right) + 4 = -\frac{27}{2}$.

[1] In the margin notes, when we speak of operations, e.g.,'Subtract $7x$,' we mean to subtract $7x$ from **both** sides of the equation. The 'from both sides of the equation' is omitted in the interest of spacing.

2. In our next example, the unknown is $t$ and we not only have a fraction but also a decimal to wrangle. Fortunately, with equations we can multiply both sides to rid us of these computational obstacles:

$$
\begin{array}{rcll}
3-1.7t & = & \dfrac{t}{4} & \\
40(3-1.7t) & = & 40\left(\dfrac{t}{4}\right) & \text{Multiply by 40} \\
40(3)-40(1.7t) & = & \dfrac{40t}{4} & \text{Distribute} \\
120-68t & = & 10t & \\
(120-68t)+68t & = & 10t+68t & \text{Add } 68t \text{ to both sides} \\
120 & = & 78t & 68t+10t=(68+10)t=78t \\
\dfrac{120}{78} & = & \dfrac{78t}{78} & \text{Divide by the coefficient of } t \\
\dfrac{120}{78} & = & t & \\
\dfrac{20}{13} & = & t & \text{Reduce to lowest terms}
\end{array}
$$

To check, we again substitute $t=\frac{20}{13}$ into each side of the original equation. We find that $3-1.7\left(\frac{20}{13}\right) = 3-\left(\frac{17}{10}\right)\left(\frac{20}{13}\right) = \frac{5}{13}$ and $\frac{(20/13)}{4} = \frac{20}{13}\cdot\frac{1}{4} = \frac{5}{13}$ as well.

3. To solve this next equation, we begin once again by clearing fractions. The least common denominator here is 36:

$$
\begin{array}{rcll}
\dfrac{1}{18}(7-4a)+2 & = & \dfrac{a}{3}-\dfrac{4-a}{12} & \\
36\left(\dfrac{1}{18}(7-4a)+2\right) & = & 36\left(\dfrac{a}{3}-\dfrac{4-a}{12}\right) & \text{Multiply by 36} \\
\dfrac{36}{18}(7-4a)+(36)(2) & = & \dfrac{36a}{3}-\dfrac{36(4-a)}{12} & \text{Distribute} \\
2(7-4a)+72 & = & 12a-3(4-a) & \text{Distribute} \\
14-8a+72 & = & 12a-12+3a & \\
86-8a & = & 15a-12 & 12a+3a=(12+3)a=15a \\
(86-8a)+8a+12 & = & (15a-12)+8a+12 & \text{Add } 8a \text{ and } 12 \\
86+12-8a+8a & = & 15a+8a-12+12 & \text{Rearrange terms} \\
98 & = & 23a & 15a+8a=(15+8)a=23a \\
\dfrac{98}{23} & = & \dfrac{23a}{23} & \text{Divide by the coefficient of } a \\
\dfrac{98}{23} & = & a &
\end{array}
$$

The check, as usual, involves substituting $a=\frac{98}{23}$ into both sides of the original equation. The reader is encouraged to work through the (admittedly messy) arithmetic. Both sides work out to $\frac{199}{138}$.

4. The square roots may dishearten you but we treat them just like the real numbers they are. Our strategy is the same: get everything with the variable (in this case $y$) on one side, put everything else on the other and divide by the coefficient of the variable. We've added a few steps to the narrative that we would ordinarily omit just to help you see that this equation is indeed linear.

$$\begin{array}{rcll}
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5-y) & \\
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5)+\sqrt{12}y & \text{Distribute} \\
8y\sqrt{3}+1 & = & 7-(2\sqrt{3})5+(2\sqrt{3})y & \sqrt{12}=\sqrt{4\cdot 3}=2\sqrt{3} \\
8y\sqrt{3}+1 & = & 7-10\sqrt{3}+2y\sqrt{3} & \\
(8y\sqrt{3}+1)-1-2y\sqrt{3} & = & (7-10\sqrt{3}+2y\sqrt{3})-1-2y\sqrt{3} & \text{Subtract 1 and } 2y\sqrt{3} \\
8y\sqrt{3}-2y\sqrt{3}+1-1 & = & 7-1-10\sqrt{3}+2y\sqrt{3}-2y\sqrt{3} & \text{Rearrange terms} \\
(8\sqrt{3}-2\sqrt{3})y & = & 6-10\sqrt{3} & \\
6y\sqrt{3} & = & 6-10\sqrt{3} & \text{See note below} \\
\dfrac{6y\sqrt{3}}{6\sqrt{3}} & = & \dfrac{6-10\sqrt{3}}{6\sqrt{3}} & \text{Divide } 6\sqrt{3} \\
y & = & \dfrac{2\cdot\sqrt{3}\cdot\sqrt{3}-2\cdot 5\cdot\sqrt{3}}{2\cdot 3\cdot\sqrt{3}} & \\
y & = & \dfrac{\cancel{2}\cancel{\sqrt{3}}(\sqrt{3}-5)}{\cancel{2}\cdot 3\cdot\cancel{\sqrt{3}}} & \text{Factor and cancel} \\
y & = & \dfrac{\sqrt{3}-5}{3} &
\end{array}$$

In the list of computations above we marked the row $6y\sqrt{3} = 6 - 10\sqrt{3}$ with a note. That's because we wanted to draw your attention to this line without breaking the flow of the manipulations. The equation $6y\sqrt{3} = 6 - 10\sqrt{3}$ is in fact linear according to Definition **??**: the variable is $y$, the value of $A$ is $6\sqrt{3}$ and $B = 6 - 10\sqrt{3}$. Checking the solution, while not trivial, is good mental exercise. Each side works out to be $\frac{27-40\sqrt{3}}{3}$.

5. Proceeding as before, we simplify radicals and clear denominators. Once we gather all of the terms containing $x$ on one side and move the other terms to the other, we factor out $x$ to identify its

coefficient then divide to get our answer.

$$\begin{array}{rcll}
\dfrac{3x-1}{2} & = & x\sqrt{50}+4 & \\
\dfrac{3x-1}{2} & = & 5x\sqrt{2}+4 & \sqrt{50}=\sqrt{25\cdot 2} \\
2\left(\dfrac{3x-1}{2}\right) & = & 2\left(5x\sqrt{2}+4\right) & \text{Multiply by 2} \\
\dfrac{\cancel{2}\cdot(3x-1)}{\cancel{2}} & = & 2(5x\sqrt{2})+2\cdot 4 & \text{Distribute} \\
3x-1 & = & 10x\sqrt{2}+8 & \\
(3x-1)-10x\sqrt{2}+1 & = & (10x\sqrt{2}+8)-10x\sqrt{2}+1 & \text{Subtract } 10x\sqrt{2}\text{, add 1} \\
3x-10x\sqrt{2}-1+1 & = & 10x\sqrt{2}-10x\sqrt{2}+8+1 & \text{Rearrange terms} \\
3x-10x\sqrt{2} & = & 9 & \\
(3-10\sqrt{2})x & = & 9 & \text{Factor} \\
\dfrac{(3-10\sqrt{2})x}{3-10\sqrt{2}} & = & \dfrac{9}{3-10\sqrt{2}} & \text{Divide by the coefficient of } x \\
x & = & \dfrac{9}{3-10\sqrt{2}} &
\end{array}$$

The reader is encouraged to check this solution - it isn't as bad as it looks if you're careful! Each side works out to be $\dfrac{12+5\sqrt{2}}{3-10\sqrt{2}}$.

6. If we were instructed to solve our last equation for $x$, we'd be done in one step: divide both sides by $(4-y)$ - assuming $4-y\neq 0$, that is. Alas, we are instructed to solve for $y$, which means we have some more work to do.

$$\begin{array}{rcll}
x(4-y) & = & 8y & \\
4x-xy & = & 8y & \text{Distribute} \\
(4x-xy)+xy & = & 8y+xy & \text{Add } xy \\
4x & = & (8+x)y & \text{Factor}
\end{array}$$

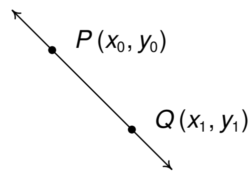
In order to finish the problem, we need to divide both sides of the equation by the coefficient of $y$ which in this case is $8+x$. This expression contains a variable so we need to stipulate that we may perform this division only if $8+x\neq 0$, or, in other words, $x\neq -8$. Hence, we write our solution as:

$$y=\frac{4x}{8+x},\quad \text{provided } x\neq -8$$

What happens if $x=-8$? Substituting $x=-8$ into the original equation gives $(-8)(4-y)=8y$ or $-32+8y=8y$. This reduces to $-32=0$, which is a contradiction. This means there is no solution when $x=-8$, so we've covered all the bases. Checking our answer requires some Algebra we haven't reviewed yet in this text, but the necessary skills *should* be lurking somewhere in the mathematical mists of your mind. The adventurous reader is invited to plug $y=\frac{4x}{8+x}$ into the original equation and show that both sides work out to $\frac{32x}{x+8}$. □

### 2.2.3 Review of Graphing Lines

In Section **??**, we concerned ourselves with the finite line segment between two points $P$ and $Q$. Specifically, we found its length (the distance between $P$ and $Q$) and its midpoint. In this section, our focus will be on the *entire* line, and ways to describe it algebraically. Consider the generic situation below.




To give a sense of the 'steepness' of the line, we recall that we can compute the **slope** of the line as follows. (Read the character $\Delta$ as 'change in'.)

[redacted]

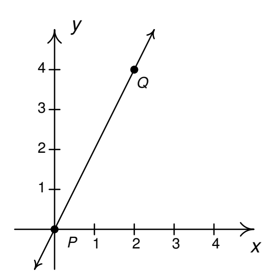
A couple of notes about Equation **??** are in order. First, don't ask why we use the letter '$m$' to represent slope. There are many explanations out there, but apparently no one really knows for sure.[1] Secondly, the stipulation $x_1 \neq x_0$ (or $\Delta x \neq 0$) ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically when the 'change in $x$' is 0; the anxious reader can skip along to the next example.

**Example 2.2.2.** Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

1. $P(0,0)$, $Q(2,4)$
2. $P(-1,2)$, $Q(3,4)$
3. $P(-2,3)$, $Q(2,-3)$
4. $P(-3,2)$, $Q(4,2)$
5. $P(2,3)$, $Q(2,-1)$
6. $P(2,3)$, $Q(2.1,-1)$

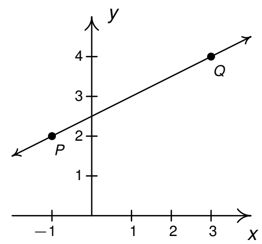
**Solution.** In each of these examples, we apply the slope formula, Equation **??**.

1. $$m = \frac{4-0}{2-0} = \frac{4}{2} = 2$$

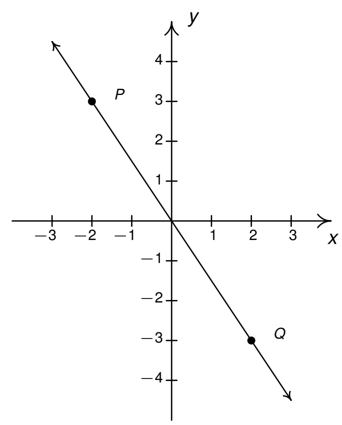



[1]See www.mathforum.org or www.mathworld.wolfram.com for discussions on this topic.

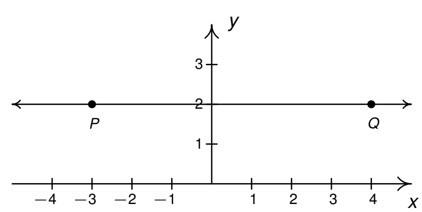
2. $m = \dfrac{4-2}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$



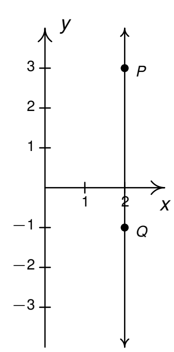

3. $m = \dfrac{-3-3}{2-(-2)} = \dfrac{-6}{4} = -\dfrac{3}{2}$



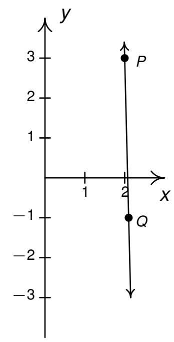

4. $m = \dfrac{2-2}{4-(-3)} = \dfrac{0}{7} = 0$




5. $m = \dfrac{-1-3}{2-2} = \dfrac{-4}{0}$, which is undefined



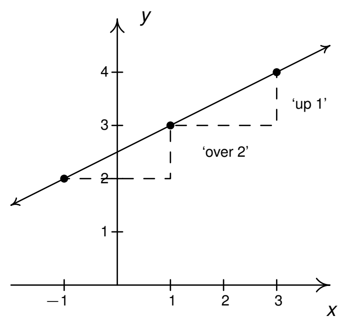
6. $m = \dfrac{-1-3}{2.1-2} = \dfrac{-4}{0.1} = -40$




□

A few comments about Example **??** are in order. First, if the slope is positive then the resulting line is said to be 'increasing', meaning as we move from left to right,[2] the $y$-values are getting larger.[3] Similarly, if the slope is negative, we say the line is 'decreasing', since as we move from left to right, the $y$-values are getting smaller. A slope of 0 results in a horizontal line which we say is 'constant', since the $y$-values here remain unchanged as we move from left to right, and an undefined slope results in a vertical line.[4]

Second, the larger the slope is in absolute value, the steeper the line. You may recall from Intermediate Algebra that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, if the slope works out to be $\frac{1}{2}$, we can interpret this as a 'rise' of 1 unit upward for every 'run' of 2 units to the right:




In this way, we may view the slope as 'the **rate of change** of $y$ with respect to $x$'. From the expression

$$m = \frac{\Delta y}{\Delta x}$$

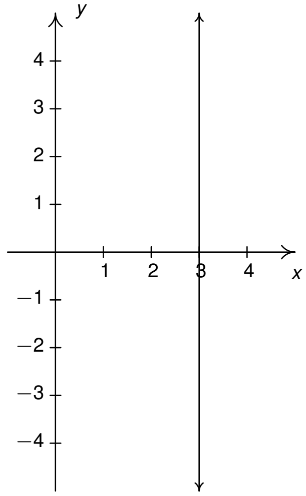
we get $\Delta y = m\Delta x$ so that the $y$-values change '$m$' times as fast as the $x$-values. We'll have more to say about this concept in Section **??** when we explore applications of linear functions; presently, we will keep our attention focused on the analytic geometry of lines. To that end, our next task is to find algebraic equations that describe lines and we start with a discussion of vertical and horizontal lines.

[2] That is, as we increase the $x$-values . . .

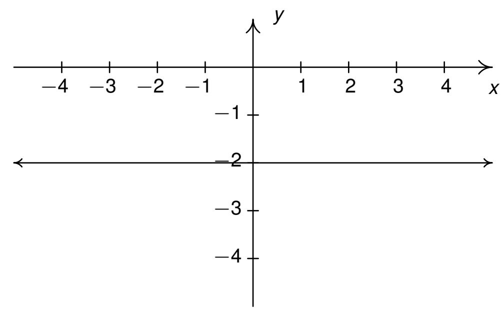
[3] We'll have more to say about this idea in Section **??**.

[4] Some authors use the unfortunate moniker 'no slope' when a slope is undefined. It's easy to confuse the notions of 'no slope' with 'slope of 0'. For this reason, we will describe slopes of vertical lines as 'undefined'.

Consider the two lines shown below: $V$ (for 'V'ertical Line) and $H$ (for 'H'orizontal Line).




The line $V$




The line $H$

All of the points on the line $V$ have an $x$-coordinate of 3. Conversely, any point with an $x$-coordinate of 3 lies on the line $V$. Said differently, the point $(x, y)$ lies on $V$ if and only if $x = 3$. Because of this, we say the equation $x = 3$ *describes* the line $V$, or, said differently, the *graph* of the equation $x = 3$ is the line $V$.

In Section **??**, we'll spend a great deal of time talking about graphing equations. For now, it suffices to know that a graph of an equation is a plot of all of the points which make the equation true. So to graph $x = 3$, we plot all of the points $(x, y)$ which satisfy $x = 3$ and this gives us our vertical line $V$.

Turning our attention to $H$, we note that every point on $H$ has a $y$-coordinate of $-2$, and vice-versa. Hence the equation $y = -2$ describes the line $H$, or the graph of the equation $y = -2$ is $H$. In general:

[redacted]

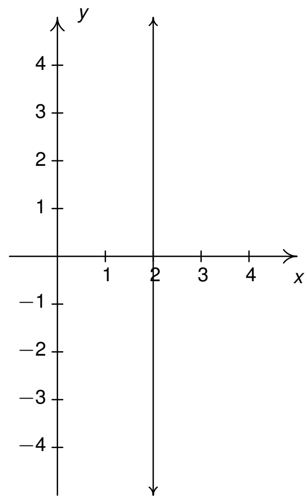
Of course, we may be working on axes which aren't labeled with the 'usual' $x$'s and $y$'s. In this case, we understand Equation **??** to say 'horizontal axis label $= a$' describes a *vertical* line through $(a, 0)$ and 'vertical axis label $= b$' describes a *horizontal* line through $(0, b)$.

**Example 2.2.3.**

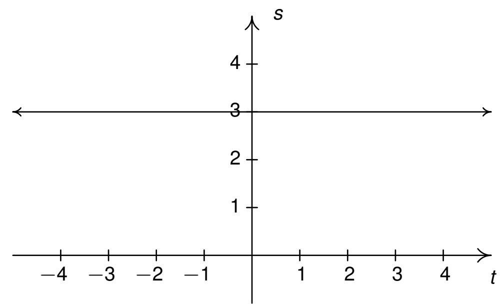
1. Graph the following equations in the $xy$-plane:

   (a) $y = 3$ (b) $x = -117$

2. Find the equation of each of the given lines.



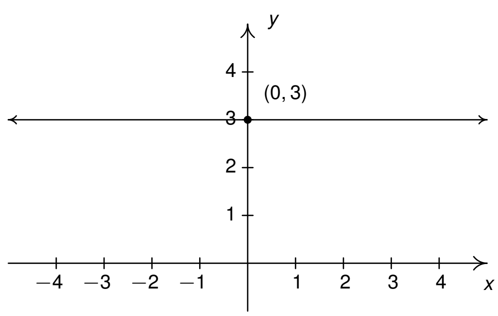

Line $L_1$



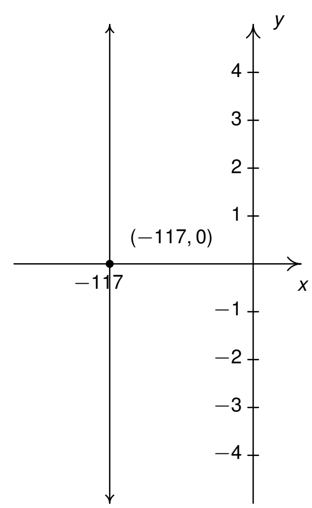

Line $L_2$

**Solution.**

1. Since we're in the familiar $xy$-plane, the graph of $y = 3$ is a horizontal line through $(0, 3)$, shown below on the left and the graph of $x = -117$ is a vertical line through $(-117, 0)$. We scale the $x$-axis differently than the $y$-axis to produce the graph below on the right.




The line $y = 3$




The line $x = -117$

2. Since $L_1$ is a vertical line through $(2, 0)$, and the horizontal axis is labeled with '$x$', the equation of $L_1$ is $x = 2$. Since $L_2$ is a horizontal line through $(0, 3)$ and the vertical axis is labeled as '$s$', the equation of this line is $s = 3$. □