Chapter 1

SEQUENCES AND THE BINOMIAL THEOREM

1.1 Sequences

In this section, we introduce *sequences* which are an important class of functions whose domains are, more or less, the set of natural numbers.[1] Before we get to far ahead of ourselves, let's look at what the term 'sequence' means mathematically. Informally, we can think of a sequence as an infinite list of numbers. For example, consider the sequence

$$\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}, -\frac{27}{16}, \ldots \tag{1}$$

As usual, the periods of ellipsis, ..., indicate that the proposed pattern continues forever. Each of the numbers in the list is called a *term*, and we call $\frac{1}{2}$ the 'first term', $-\frac{3}{4}$ the 'second term', $\frac{9}{8}$ the 'third term' and so forth. In numbering them this way, we are setting up a function, which we'll call 'a' per tradition, between the natural numbers and the terms in the sequence.

n	$a(n)$
1	$\frac{1}{2}$
2	$-\frac{3}{4}$
3	$\frac{9}{8}$
4	$-\frac{27}{16}$
$\vdots$	$\vdots$

In other words, $a(n)$ is the n^{th} term in the sequence. We formalize these ideas in our definition of a sequence and introduce some accompanying notation.

Definition 1.1. A **sequence** is a function a whose domain is the natural numbers. The value $a(n)$ is often written as a_n and is called the $\boldsymbol{n^{\text{th}}}$ **term** of the sequence. The sequence itself is usually denoted using the notation: a_n, $n \geq 1$ or the notation: $\{a_n\}_{n=1}^{\infty}$.

Applying the notation provided in Definition 1.1 to the sequence given (1), we have $a_1 = \frac{1}{2}$, $a_2 = -\frac{3}{4}$, $a_3 = \frac{9}{8}$. Suppose we wanted to know a_{117}, that is, the 117^{th} term in the sequence. While the pattern of the sequence is apparent, it would benefit us greatly to have an explicit formula for a_n. Unfortunately, there is no general algorithm that will produce a formula for every sequence, so any formulas we do develop will come from that greatest of teachers, experience. In other words, it is time for an example.

Example 1.1.1. Write the first four terms of the following sequences.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$

[1] Recall that this is the set $\mathbb{N} = \{1, 2, 3, \ldots\}$.

3. $\{2n-1\}_{n=1}^{\infty}$

4. $\left\{\dfrac{1+(-1)^i}{i}\right\}_{i=2}^{\infty}$

5. $a_1 = 7$, $a_{n+1} = 2 - a_n$, $n \geq 1$

6. $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$

Solution.

1. Since we are given $n \geq 1$, the first four terms of the sequence are a_1, a_2, a_3 and a_4. Since the notation a_1 means the same thing as $a(1)$, we obtain our first term by replacing every occurrence of n in the formula for a_n with $n = 1$ to get $a_1 = \frac{5^{1-1}}{3^1} = \frac{1}{3}$. Proceeding similarly, we get $a_2 = \frac{5^{2-1}}{3^2} = \frac{5}{9}$, $a_3 = \frac{5^{3-1}}{3^3} = \frac{25}{27}$ and $a_4 = \frac{5^{4-1}}{3^4} = \frac{125}{81}$.

2. For this sequence we have $k \geq 0$, so the first four terms are b_0, b_1, b_2 and b_3. Proceeding as before, replacing in this case the variable k with the appropriate whole number, beginning with 0, we get $b_0 = \frac{(-1)^0}{2(0)+1} = 1$, $b_1 = \frac{(-1)^1}{2(1)+1} = -\frac{1}{3}$, $b_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5}$ and $b_3 = \frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$. As a side-note, this sequence is called an *alternating* sequence since the signs alternate between '+' and '$-$.' The reader is encouraged to think what component of the formula is producing this effect.

3. The notation $\{2n-1\}_{n=1}^{\infty}$ means $a_n = 2n - 1$, $n \geq 1$. We get $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and $a_4 = 7$. In other words, we get the first four odd natural numbers. The reader is encouraged to examine whether or not this pattern continues indefinitely.

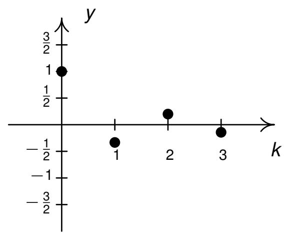
4. Here, we are using the letter i as a counter, not as the imaginary unit we saw in Section **??**. Proceeding as before, we set $a_i = \frac{1+(-1)^i}{i}$, $i \geq 2$. We find $a_2 = 1$, $a_3 = 0$, $a_4 = \frac{1}{2}$ and $a_5 = 0$.

5. To obtain the terms of this sequence, we start with $a_1 = 7$ and use the equation $a_{n+1} = 2 - a_n$ for $n \geq 1$ to generate successive terms. When $n = 1$, this equation becomes $a_{1+1} = 2 - a_1$ which simplifies to $a_2 = 2 - a_1 = 2 - 7 = -5$. When $n = 2$, the equation becomes $a_{2+1} = 2 - a_2$ so we get $a_3 = 2 - a_2 = 2 - (-5) = 7$. Finally, when $n = 3$, we get $a_{3+1} = 2 - a_3$ so $a_4 = 2 - a_3 = 2 - 7 = -5$.

6. As with the problem above, we are given a place to start with $f_0 = 1$ and given a formula to build other terms of the sequence. Substituting $n = 1$ into the equation $f_n = n \cdot f_{n-1}$, we get $f_1 = 1 \cdot f_0 = 1 \cdot 1 = 1$. Advancing to $n = 2$, we get $f_2 = 2 \cdot f_1 = 2 \cdot 1 = 2$. Finally, $f_3 = 3 \cdot f_2 = 3 \cdot 2 = 6$. $\square$

Some remarks about Example 1.1.1 are in order. We first note that since sequences are functions, we can graph them in the same way we graph functions. For example, if we wish to graph the sequence $\{b_k\}_{k=0}^{\infty}$ from Example 1.1.1, we graph the equation $y = b(k)$ for the values $k \geq 0$. That is, we plot the points $(k, b(k))$ for the values of k in the domain, $k = 0, 1, 2, \ldots$. The resulting collection of points is the graph of the sequence. Note that we do not connect the dots in a pleasing fashion as we are used to doing, because the domain is just the whole numbers in this case, not a collection of intervals of real numbers.[2]

[2] If you feel a sense of nostalgia, you should see Section **??**.



Graphing $y = b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$

Speaking of $\{b_k\}_{k=0}^{\infty}$, the astute and mathematically minded reader will correctly note that this technically isn't a sequence, since according to Definition 1.1, sequences are functions whose domains are the *natural* numbers, not the *whole* numbers, as is the case with $\{b_k\}_{k=0}^{\infty}$. In other words, to satisfy Definition 1.1, we need to shift the variable k so it starts at $k = 1$ instead of $k = 0$.

To see how we can do this, it helps to think of the problem graphically. What we want is to shift the graph of $y = b(k)$ to the right one unit, and thinking back to Section **??**, we can accomplish this by replacing k with $k - 1$ in the definition of $\{b_k\}_{k=0}^{\infty}$.

Specifically, let $c_k = b_{k-1}$ where $k - 1 \geq 0$. We get $c_k = \frac{(-1)^{k-1}}{2(k-1)+1} = \frac{(-1)^{k-1}}{2k-1}$, where now $k \geq 1$. We leave to the reader to verify that $\{c_k\}_{k=1}^{\infty}$ generates the same list of numbers as does $\{b_k\}_{k=0}^{\infty}$, but the former satisfies Definition 1.1, while the latter does not.

Like so many things in this text, we acknowledge that this point is pedantic and join the vast majority of authors who adopt a more relaxed view of Definition 1.1 to include any function which generates a list of numbers which can then be matched up with the natural numbers.[3]

One last note about Example 1.1.1 concerns the manner in which the sequences in numbers 5 and 6 are defined. We say these two sequences are described '*recursively*.' In each instance, an initial value of the sequence is given which is then followed by a *recursion equation* – a formula which enables us to use known terms of the sequence to determine other terms.

The terms of the sequence from number 6 is given notation and name: $f_n = n!$ is called *n-factorial*. Using the '!' notation, we can describe the factorial sequence as: $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$.

After $0! = 1$ the next four terms, written out in detail, are $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. From this, we see a more informal way of computing $n!$, which is $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! = 1$ as a special case. (We will study factorials in greater detail in Section 1.4.)[4]

While none of the sequences in Example 1.1.1 worked out to be the sequence in (1), they do give us some insight into what kinds of patterns to look for. Two patterns in particular are given in the next definition.

[3] We're basically talking about the 'countably infinite' subsets of the real number line when we do this.

[4] Another famous sequence, the Fibonacci Numbers are defined also recursively and are explored in the exercises.

Definition 1.2. Arithmetic and Geometric Sequences: Suppose $\{a_n\}_{n=k}^{\infty}$ is a sequence[a]

- If there is a number d so that $a_{n+1} = a_n + d$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **arithmetic sequence**. The number d is called the **common difference**.
- If there is a number r so that $a_{n+1} = ra_n$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called a **geometric sequence**. The number r is called the **common ratio**.

[a]Note that we have adjusted for the fact that not all 'sequences' begin at $n = 1$.

In English, an arithmetic sequence is one in which we proceed from one term to the next by always *adding* the fixed number d. If this sort of 'constant change' idea sounds familiar, it should. Indeed, arithmetic sequences are merely *linear* functions, something we will explore in more detail shortly. Note the name 'common difference' comes from a slight rewrite of the recursion equation from $a_{n+1} = a_n + d$ to $a_{n+1} - a_n = d$. That is, every pair of successive terms has the *same* or *common* difference, d.

Analogously, a geometric sequence is one in which we proceed from one term to the next by always *multiplying* by the same fixed number r. If this notion sounds familiar, it is because geometric sequences are, in fact, *exponential* functions. Again, we will explore this connection in more detail later. We note that if $a_n \neq 0$, we can rearrange the recursion equation to get $\frac{a_{n+1}}{a_n} = r$. Hence, every pair of successive terms has the *same* or *common* ratio, r.

Some sequences are arithmetic, some are geometric and some are neither as the next example illustrates.[5]

Example 1.1.2. Determine if the following sequences are arithmetic, geometric or neither. If arithmetic, find the common difference d; if geometric, find the common ratio r.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$
3. $\{2n-1\}_{n=1}^{\infty}$
4. $\dfrac{1}{2}, -\dfrac{3}{4}, \dfrac{9}{8}, -\dfrac{27}{16}, \ldots$

Solution. A good rule of thumb to keep in mind when working with sequences is "When in doubt, write it out!" Writing out the first several terms can help you identify the pattern of the sequence should one exist.

1. From Example 1.1.1, we know that the first four terms of this sequence are $\frac{1}{3}$, $\frac{5}{9}$, $\frac{25}{27}$ and $\frac{125}{81}$. To see if this is an arithmetic sequence, we look at the successive differences of terms. We find that $a_2 - a_1 = \frac{5}{9} - \frac{1}{3} = \frac{2}{9}$ and $a_3 - a_2 = \frac{25}{27} - \frac{5}{9} = \frac{10}{27}$. Since we get different numbers, there is no 'common difference' and we have established that the sequence is *not* arithmetic.

   To see if the sequence is geometric, we compute the ratios of successive terms. The first three ratios *suggest* the sequence is geometric:

$$\frac{a_2}{a_1} = \frac{\frac{5}{9}}{\frac{1}{3}} = \frac{5}{3}, \quad \frac{a_3}{a_2} = \frac{\frac{25}{27}}{\frac{5}{9}} = \frac{5}{3} \quad \text{and} \quad \frac{a_4}{a_3} = \frac{\frac{125}{81}}{\frac{25}{27}} = \frac{5}{3}$$

   To *prove* the sequence is geometric, however, we must show that $\frac{a_{n+1}}{a_n} = r$ for all n:

[5]Can a sequence be both arithmetic *and* geometric? See Exercise **??**.

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{5^{(n+1)-1}}{3^{n+1}}}{\dfrac{5^{n-1}}{3^n}} = \frac{5^n}{3^{n+1}} \cdot \frac{3^n}{5^{n-1}} = \frac{5}{3}$$

Hence, the sequence is geometric with common ratio $r = \frac{5}{3}$.

2. Again, we have Example 1.1.1 to thank for providing the first four terms of this sequence: 1, $-\frac{1}{3}$, $\frac{1}{5}$ and $-\frac{1}{7}$. We find $b_1 - b_0 = -\frac{4}{3}$ and $b_2 - b_1 = \frac{8}{15}$. Hence, the sequence is not arithmetic. To see if it is geometric, we compute $\frac{b_1}{b_0} = -\frac{1}{3}$ and $\frac{b_2}{b_1} = -\frac{3}{5}$. Since there is no 'common ratio,' we conclude the sequence is not geometric, either.

3. As we saw in Example 1.1.1, the sequence $\{2n-1\}_{n=1}^{\infty}$ generates the odd numbers: $1, 3, 5, 7, \ldots$. Computing the first few differences, we find $a_2 - a_1 = 2$, $a_3 - a_2 = 2$, and $a_4 - a_3 = 2$. This suggests that the sequence is arithmetic. To prove this is the case, we find

$$a_{n+1} - a_n = (2(n+1) - 1) - (2n - 1) = 2n + 2 - 1 - 2n + 1 = 2$$

This establishes that the sequence is arithmetic with common difference $d = 2$. To see if it is geometric, we compute $\frac{a_2}{a_1} = 3$ and $\frac{a_3}{a_2} = \frac{5}{3}$. Since these ratios are different, we conclude the sequence is not geometric.

4. We met our last sequence at the beginning of the section. Given that $a_2 - a_1 = -\frac{5}{4}$ and $a_3 - a_2 = \frac{15}{8}$, the sequence is not arithmetic. Computing the first few ratios, however, gives us $\frac{a_2}{a_1} = -\frac{3}{2}$, $\frac{a_3}{a_2} = -\frac{3}{2}$ and $\frac{a_4}{a_3} = -\frac{3}{2}$. Since these are the only terms given to us, we *assume* that the pattern of ratios continue in this fashion and conclude that the sequence is geometric. □

We are now one step away from determining an explicit formula for the sequence given in (1). We know that it is a geometric sequence and our next result gives us the explicit formula we require.

Equation 1.1. Formulas for Arithmetic and Geometric Sequences:

- An arithmetic sequence with first term $a_1 = a$ and common difference d is given by

$$a_n = a + (n-1)d, \quad n \geq 1$$

- A geometric sequence with first term $a_1 = a$ and common ratio $r \neq 0$ is given by

$$a_n = ar^{n-1}, \quad n \geq 1$$

An intuitive way to arrive at Equation 1.1 appeals to Definition 1.2 directly. Given an arithmetic sequence with first term a and common difference d, the way we get from one term to the next is by adding d. Hence, the terms of the sequence are: a, $a+d$, $a+2d$, $a+3d$, ... We see that to reach the nth term, we add d to a exactly $(n-1)$ times, which is exactly what the formula says.[6]

[6] We formalize this argument in Section 1.3.

Note if we rewrite the formula $a_n = a_1 + (n-1)d$ using traditional function notation as $a(n) = a(1) + d(n-1)$ we can see arithmetic sequences are linear functions.[7] Indeed, relabeling the function a as 'f' and the independent variable n as 'x,' we can make the identifications $x_0 = 1$, and $m = d$ so as to put the equation $a(n) = a(1) + d(n-1)$ into the form of Equation **??**:

$$\begin{array}{rcrcl} a(n) & = & a(1) & + & d(n-1) \\ f(x) & = & f(1) & + & m(x-1) \end{array}$$

Hence, arithmetic sequences are linear functions with slope d whose domains are the natural numbers.
The derivation of the formula for geometric series follows similarly. Here, we start with the first term a and go from one term to the next by multiplying by r. We get a, ar, ar^2, ar^3 and so forth. The nth term results from multiplying a by r exactly $(n-1)$ times.[8]
In the same way arithmetic sequences are linear functions, geometric sequences are exponential functions. Writing $a_n = a_1 r^{n-1}$ as $a(n) = a(1)r^{n-1}$, we can relabel a as f and n as x and make the identifications $x_0 = 1$ and $b = r$ to put the equation into the form described in Definition **??**:

$$\begin{array}{rcl} a(n) & = & a(1)r^{n-1} \\ f(x) & = & f(1)b^{x-1} \end{array}$$

So, geometric sequences are exponential functions with base r whose domains are the natural numbers.
With Equation 1.1 in place, we finally have the tools required to find an explicit formula for the nth term of the sequence given in (1). We know from Example 1.1.2 that it is geometric with common ratio $r = -\frac{3}{2}$. The first term is $a = \frac{1}{2}$ so by Equation 1.1 we get $a_n = ar^{n-1} = \frac{1}{2}\left(-\frac{3}{2}\right)^{n-1}$ for $n \geq 1$. After a touch of simplifying, we get $a_n = \frac{(-3)^{n-1}}{2^n}$ for $n \geq 1$. Note that we can easily check our answer by substituting in values of n and seeing that the formula generates the sequence given in (1). We leave this to the reader. In particular, the 117th term in the sequence is $a_{117} = \frac{1}{2}\left(-\frac{3}{2}\right)^{117-1} = \frac{3^{116}}{2^{117}}$.
Our next example gives us more practice finding patterns.

Example 1.1.3. Find an explicit formula for the n^{th} term of the following sequences.

1. $0.9, 0.09, 0.009, 0.0009, \ldots$
2. $\dfrac{2}{5}, 2, -\dfrac{2}{3}, -\dfrac{2}{7}, \ldots$
3. $1, -\dfrac{2}{7}, \dfrac{4}{13}, -\dfrac{8}{19}, \ldots$

Solution.

1. Although this sequence may seem strange, the reader can verify it is actually a geometric sequence with common ratio $r = 0.1 = \frac{1}{10}$. With $a = 0.9 = \frac{9}{10}$, we get $a_n = \frac{9}{10}\left(\frac{1}{10}\right)^{n-1}$ for $n \geq 0$. Simplifying, we get $a_n = \frac{9}{10^n}$, $n \geq 1$. There is more to this sequence than meets the eye and we shall return to this example in the next section.

[7] Note here a is a *function*, so the expressions $a(n)$ and $a(1)$ here represent the *outputs* from a. On the other hand, the expression $d(n-1)$ indicates *multiplication* of the real numbers d and $(n-1)$.

[8] We note here that the reason $r = 0$ is excluded from Equation 1.1 is to avoid an instance of 0^0 which is an indeterminant form. (See the remarks following Definition **??** in Section **??**.)

2. As the reader can verify, this sequence is neither arithmetic nor geometric. In an attempt to find a pattern, we rewrite the second term with a denominator to make all the terms appear as fractions and associate the '$-$' with the denominators so we have a constant numerator:

$$\frac{2}{5}, \frac{2}{1}, \frac{2}{-3}, -\frac{2}{-7}, \ldots$$

This tells us that we can tentatively sketch out the formula for the sequence as $a_n = \frac{2}{D_n}$ where D_n is the sequence of denominators.

The sequence of the denominators: $5, 1, -3, -7, \ldots$ is seen to be an arithmetic sequence with a common difference of -4. Using Equation 1.1 with $a = 5$ and $d = -4$, we get the nth denominator by the formula $D_n = 5 + (n-1)(-4) = 9 - 4n$ for $n \geq 1$. Hence, our final answer is $a_n = \frac{2}{9-4n}$, $n \geq 1$.

3. The sequence as given is neither arithmetic nor geometric, so we proceed as in the last problem to try to get patterns individually for the numerator and denominator. Letting C_n and D_n denote the sequence of numerators and denominators, respectively, so that $a_n = \frac{C_n}{D_n}$.

After some experimentation,[9] we choose to write the first term as a fraction and associate the negatives '$-$' with the numerators. This yields

$$\frac{1}{1}, \frac{-2}{7}, \frac{4}{13}, \frac{-8}{19}, \ldots$$

The numerators form the sequence $1, -2, 4, -8, \ldots$ which is geometric with $a = 1$ and $r = -2$, so we get $C_n = (-2)^{n-1}$, for $n \geq 1$.

The denominators $1, 7, 13, 19, \ldots$ form an arithmetic sequence with $a = 1$ and $d = 6$. Hence, we get $D_n = 1 + 6(n-1) = 6n - 5$, for $n \geq 1$.

Putting these two formulas together, we obtain our formula for $a_n = \frac{C_n}{D_n} = \frac{(-2)^{n-1}}{6n-5}$, for $n \geq 1$. We leave it to the reader to show that this checks out. $\square$

While the last problem in Example 1.1.3 was neither geometric nor arithmetic, it did resolve into a combination of these two kinds of sequences. If handed the sequence $2, 5, 10, 17, \ldots$, we would be hard-pressed to find a formula for a_n if we restrict our attention to these two archetypes. We said before that there is no general algorithm for finding the explicit formula for the nth term of a given sequence, and it is only through experience gained from evaluating sequences from explicit formulas that we learn to begin to recognize number patterns.

The pattern $1, 4, 9, 16, \ldots$ is rather recognizable as the squares, so the formula $a_n = n^2$, $n \geq 1$ may not be too hard to determine. With this in mind, it's possible to see $2, 5, 10, 17, \ldots$ as the sequence $1+1, 4+1, 9+1, 16+1, \ldots$, so that $a_n = n^2 + 1$, $n \geq 1$.

Of course, since we are given only a small *sample* of the sequence, we shouldn't be too disappointed to find out this isn't the *only* formula which generates this sequence. For example, consider the sequence defined by $b_n = -\frac{1}{4}n^4 + \frac{5}{2}n^3 - \frac{31}{4}n^2 + \frac{25}{2}n - 5$, $n \geq 1$. The reader is encouraged to verify that it also

[9] Here we take 'experimentation' to mean a frustrating guess-and-check session.

produces the terms $2, 5, 10, 17$. In fact, it can be shown that given any finite sample of a sequence, there are infinitely many explicit formulas all of which generate those same finite points. This means that there will be infinitely many correct answers to some of the exercises in this section.[10] Just because your answer doesn't match ours doesn't mean it's wrong. As always, when in doubt, write your answer out. As long as it produces the same terms in the same order as what the problem wants, your answer is correct.

Sequences play a major role in the Mathematics of Finance, as we have already seen with Equation **??** in Section **??**. Recall that if we invest P dollars at an annual percentage rate r and compound the interest n times per year, the formula for A_k, the amount in the account after k compounding periods, is $A_k = P\left(1+\frac{r}{n}\right)^k = \left[P\left(1+\frac{r}{n}\right)\right]\left(1+\frac{r}{n}\right)^{k-1}$, $k \geq 1$. We leave it to the reader to show this is a geometric sequence with first term $P\left(1+\frac{r}{n}\right)$ and common ratio $\left(1+\frac{r}{n}\right)$.
In retirement planning, it is seldom the case that an investor deposits a set amount of money into an account and waits for it to grow. Usually, additional payments of principal are made at regular intervals and the value of the investment grows accordingly. This kind of investment is called an *annuity* and will be discussed in the next section once we have developed more mathematical machinery that enables us to *add* sequences. For now, we invite you to gain some practice with sequence notation some of the more basic pattern recognition that goes along with it.

[10]For more on this, see When Every Answer is Correct: Why Sequences and Number Patterns Fail the Test.

Exercises on ‘re-indexing’ the sequence, some more applications.

1.2 Summation Notation

In the previous section, we introduced sequences. Each of the numbers in the sequence is called a 'term' which implies these numbers are meant to be added. To that end, we introduce the following notation which is used to describe the sum of (some of the) terms of a sequence.

Definition 1.3. Summation Notation: Given a sequence $\{a_n\}_{n=k}^{\infty}$ and numbers m and p satisfying $k \leq m \leq p$, the summation from m to p of the sequence $\{a_n\}$ is written

$$\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \ldots + a_p$$

The variable n is called the **index of summation**. The number m is called the **lower limit of summation** while the number p is called the **upper limit of summation**.

In English, Definition 1.3 is simply defining a short-hand notation for adding up the terms of the sequence $\{a_n\}_{n=k}^{\infty}$ from a_m through a_p. The symbol Σ is the capital Greek letter sigma and is shorthand for 'sum'. The lower and upper limits of the summation tells us which term to start with and which term to end with, respectively. For example, using the sequence $a_n = 2n - 1$ for $n \geq 1$, we can write $a_3 + a_4 + a_5 + a_6$ as

$$\begin{aligned}\sum_{n=3}^{6}(2n-1) &= (2(3)-1)+(2(4)-1)+(2(5)-1)+(2(6)-1)\\ &= 5+7+9+11\\ &= 32\end{aligned}$$

The index variable is considered a 'dummy variable' in the sense that it may be changed to any letter without affecting the value of the summation. For instance,

$$\sum_{n=3}^{6}(2n-1) = \sum_{k=3}^{6}(2k-1) = \sum_{j=3}^{6}(2j-1)$$

One place you may encounter summation notation is in mathematical definitions. For example, summation notation allows us to define polynomials as functions of the form

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

for real numbers a_k, $k = 0, 1, \ldots n$. The reader is invited to compare this with what is given in Definition **??**. Summation notation is particularly useful when talking about matrix operations. For example, we can write the product of the ith row R_i of a matrix $A = [a_{ij}]_{m\times n}$ and the j^{th} column C_j of a matrix $B = [b_{ij}]_{n\times r}$ as

$$Ri \cdot Cj = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Again, the reader is encouraged to write out the sum and compare it to Definition **??**. Our next example gives us practice with this new notation.

Example 1.2.1.

1. Find the following sums.

   (a) $\displaystyle\sum_{k=1}^{4} \frac{13}{100^k}$ (b) $\displaystyle\sum_{n=0}^{4} \frac{n!}{2}$ (c) $\displaystyle\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n$

2. Write the following sums using summation notation.

   (a) $1 + 3 + 5 + ... + 117$

   (b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - ... + \frac{1}{117}$

   (c) $0.9 + 0.09 + 0.009 + ... 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9$

Solution.

1. (a) We substitute $k = 1$ into the formula $\frac{13}{100^k}$ and add successive terms until we reach $k = 4$.

$$\begin{aligned}\sum_{k=1}^{4} \frac{13}{100^k} &= \frac{13}{100^1} + \frac{13}{100^2} + \frac{13}{100^3} + \frac{13}{100^4}\\ &= 0.13 + 0.0013 + 0.000013 + 0.00000013\\ &= 0.13131313\end{aligned}$$

   (b) Proceeding as in (a), we replace every occurrence of n with the values 0 through 4. We recall the factorials, $n!$ as defined in number Example 1.1.1, number 6 and get:

$$\begin{aligned}\sum_{n=0}^{4} \frac{n!}{2} &= \frac{0!}{2} + \frac{1!}{2} + \frac{2!}{2} + \frac{3!}{2} = \frac{4!}{2}\\ &= \frac{1}{2} + \frac{1}{2} + \frac{2\cdot 1}{2} + \frac{3\cdot 2\cdot 1}{2} + \frac{4\cdot 3\cdot 2\cdot 1}{2}\\ &= \frac{1}{2} + \frac{1}{2} + 1 + 3 + 12\\ &= 17\end{aligned}$$

   (c) We proceed as before, replacing the index n, but *not* the variable x, with the values 1 through 5 and adding the resulting terms.

$$\begin{aligned}\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n &= \frac{(-1)^{1+1}}{1}(x-1)^1 + \frac{(-1)^{2+1}}{2}(x-1)^2 + \frac{(-1)^{3+1}}{3}(x-1)^3\\ &\quad + \frac{(-1)^{1+4}}{4}(x-1)^4 + \frac{(-1)^{1+5}}{5}(x-1)^5\\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}\end{aligned}$$

2. The key to writing these sums with summation notation is to find the pattern of the terms. To that end, we make good use of the techniques presented in Section 1.1.

   (a) The terms of the sum 1, 3, 5, etc., form an arithmetic sequence with first term $a = 1$ and common difference $d = 2$. Using Equation 1.1, we get $a_n = 1 + (n-1)2 = 2n-1$, $n \geq 1$.

   At this stage, we have the formula for the terms, namely $2n-1$, and the lower limit of the summation, $n = 1$. To finish the problem, we need to determine the upper limit of the summation. In other words, we need to determine which value of n produces the term 117. Setting $a_n = 117$, we get $2n - 1 = 117$ or $n = 59$. Our final answer is

   $$1 + 3 + 5 + \ldots + 117 = \sum_{n=1}^{59} (2n-1)$$

   (b) We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives '$-$' with the numerators to get

   $$\frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \ldots + \frac{1}{117}$$

   The numerators, 1, -1, etc. can be described by the geometric sequence[1] $C_n = (-1)^{n-1}$ for $n \geq 1$, while the denominators are given by the arithmetic sequence[2] $D_n = n$ for $n \geq 1$. Hence, we get the formula $a_n = \frac{(-1)^{n-1}}{n}$ for our terms, and we find the lower and upper limits of summation to be $n = 1$ and $n = 117$, respectively. Thus

   $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots + \frac{1}{117} = \sum_{n=1}^{117} \frac{(-1)^{n-1}}{n}$$

   (c) Thanks to Example 1.1.3, we know that one formula for the n^{th} term is $a_n = \frac{9}{10^n}$ for $n \geq 1$. This gives us a formula for the summation as well as a lower limit of summation.

   To determine the upper limit of summation, we note that to produce the $n-1$ zeros to the right of the decimal point before the 9, we need a denominator of 10^n. Hence, n is the upper limit of summation.

   Since n is used in the limits of the summation, we need to choose a different letter for the index of summation.[3] We choose k and get

   $$0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0 \cdots 0}_{n-1 \text{ zeros}}9 = \sum_{k=1}^{n} \frac{9}{10^k}$$

$\square$

[1] This is indeed a geometric sequence with first term $a = 1$ and common ratio $r = -1$.
[2] It is an arithmetic sequence with first term $a = 1$ and common difference $d = 1$.
[3] To see why, try writing the summation using 'n' as the index.

The following theorem presents some general properties of summation notation.

Theorem 1.1. Properties of Summation Notation: Suppose $\{a_n\}$ and $\{b_n\}$ are sequences so that the following sums are defined.

- **Sum and Difference Property:** $\displaystyle\sum_{n=m}^{p}(a_n \pm b_n) = \sum_{n=m}^{p} a_n \pm \sum_{n=m}^{p} b_n$
- **Distributive Property:** $\displaystyle\sum_{n=m}^{p} c\, a_n = c\sum_{n=m}^{p} a_n$, for any real number c.
- **Additive Index Property:** $\displaystyle\sum_{n=m}^{j} a_n + \sum_{n=j+1}^{p} a_n = \sum_{n=m}^{p} a_n$, for any natural number $m \leq j < j+1 \leq p$.
- **Re-indexing:** $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$, for any integer r.

There is much to be learned by thinking about why the properties hold, so we leave the proof of these properties to the reader.[4]

Example 1.2.2.

1. If $\displaystyle\sum_{n=2}^{50}(a_n - 3b_n) = 17$ and $\displaystyle\sum_{n=2}^{50} a_n = 10$, find $\displaystyle\sum_{n=2}^{50} b_n$.
2. If $\displaystyle\sum_{n=1}^{20} a_n = -3$ and $\displaystyle\sum_{n=1}^{21} a_n = 7$, find a_{21}.
3. Rewrite the sum so the index starts at 0: $\displaystyle\sum_{n=2}^{437} n(n-1)x^{n-2}$

Solution.

1. Using the Sum and Difference Property along with the Distributive Property of Theorem 1.1, we get:

$$\sum_{n=2}^{50}(a_n - 3b_n) = \sum_{n=2}^{50} a_n - \sum_{n=2}^{50} 3b_n = \sum_{n=2}^{50} a_n - 3\sum_{n=2}^{50} b_n$$

Hence, $\displaystyle\sum_{n=2}^{50} a_n - 3\sum_{n=2}^{50} b_n = 17$. If $\displaystyle\sum_{n=2}^{50} a_n = 10$, then $\displaystyle 10 - 3\sum_{n=2}^{50} b_n = 17$ so $\displaystyle\sum_{n=2}^{50} b_n = -\frac{7}{3}$.

[4] To get started, remember the mantra "When in doubt, write it out!"

2. There are at least two ways to approach this problem. By definition, $\sum_{n=1}^{21} a_n = a_1 + a_2 + ... + a_{21}$. That is, we add up the first 21 terms of the sequence a_n. Similarly, $\sum_{n=1}^{20} a_n = a_1 + a_2 + ... + a_{20}$ means we add up the first 20 terms of the sequence. Hence, $a_{21} = \sum_{n=1}^{21} a_n - \sum_{n=1}^{20} a_n = 7 - (-3) = 10$.

   Alternatively, we can use the Additive Index Property:

$$\sum_{n=1}^{21} a_n = \sum_{n=1}^{20} a_n + \sum_{n=21}^{21} a_n = \sum_{n=1}^{20} a_n + a_{21},$$

   which gives $a_{21} = \sum_{n=1}^{21} a_n - \sum_{n=1}^{20} a_n = 7 - (-3) = 10$ as well.

3. To re-index $\sum_{n=2}^{437} n(n-1)x^{n-2}$ so n starts at 0, we follow the formula in Theorem 1.2.2 with $r = -2$:

$$\sum_{n=2}^{437} n(n-1)x^{n-2} = \sum_{n=2+(-2)}^{437+(-2)} (n-(-2))(n-(-2)-1)x^{n-(-2)-2} = \sum_{n=0}^{435}(n+2)(n+1)x^n.$$

   We leave it to the reader to check by writing out the first few, and last few, terms.

   Alternatively, to better see *why* the re-indexing works in this way, we can introduce a new counter, k. We want this new counter to start at $k = 0$ whereas the current counter starts at $n = 2$, so we want $k = n - 2$. When $n = 2$, $k = 0$, as required, and when $n = 437$, $k = 435$.

   Moreover, $n = k + 2$, so substituting this into the sum, we get

$$\sum_{n=2}^{437} n(n-1)x^{n-2} = \sum_{k=0}^{435}(k+2)((k+2)-1)x^{(k+2)-2} = \sum_{k=0}^{435}(k+2)(k+1)x^k,$$

   which is the same sum we had before, just with a different dummy variable. □

We now turn our attention to the sums involving arithmetic and geometric sequences. Given an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$, we let S denote the sum of the first n terms. To derive a formula for S, we write it out in two different ways

$$\begin{array}{ccccccccccc} S & = & a & + & (a+d) & + & ... & + & (a+(n-2)d) & + & (a+(n-1)d) \\ S & = & (a+(n-1)d) & + & (a+(n-2)d) & + & ... & + & (a+d) & + & a \end{array}$$

If we add these two equations and combine the terms which are aligned vertically, we get

$$2S = (2a + (n-1)d) + (2a + (n-1)d) + \ldots + (2a + (n-1)d) + (2a + (n-1)d)$$

The right hand side of this equation contains n terms, all of which are equal to $(2a + (n-1)d)$ so we get $2S = n(2a + (n-1)d)$. Dividing both sides of this equation by 2, we obtain the formula

$$S = \frac{n}{2}(2a + (n-1)d)$$

If we rewrite the quantity $2a + (n-1)d$ as $a + (a + (n-1)d) = a_1 + a_n$, we get the formula

$$S = n\left(\frac{a_1 + a_n}{2}\right)$$

A helpful way to remember this last formula is to recognize that we have expressed the sum as the product of the number of terms n and the *average* of the first and n^{th} terms.

To derive the formula for the geometric sum, we start with a geometric sequence $a_k = ar^{k-1}$, $k \geq 1$, and let S once again denote the sum of the first n terms. Comparing S and rS, we get

$$\begin{array}{rcccccccccccccc} S & = & a & + & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & & \\ rS & = & & & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \end{array}$$

Subtracting the second equation from the first forces all of the terms except a and ar^n to cancel out and we get $S - rS = a - ar^n$. Factoring, we get $S(1-r) = a(1-r^n)$. Assuming $r \neq 1$, we can divide both sides by the quantity $(1-r)$ to obtain

$$S = a\left(\frac{1-r^n}{1-r}\right)$$

If we distribute a through the numerator, we get $a - ar^n = a_1 - a_{n+1}$ which yields the formula

$$S = \frac{a_1 - a_{n+1}}{1-r}$$

In the case when $r = 1$, we get the formula

$$S = \underbrace{a + a + \ldots + a}_{n \text{ times}} = n\,a$$

Our results are summarized below.

Equation 1.2. Sums of Arithmetic and Geometric Sequences:

- The sum S of the first n terms of an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$ is

$$S = \sum_{k=1}^{n} a_k = n\left(\frac{a_1 + a_n}{2}\right) = \frac{n}{2}(2a + (n-1)d)$$

- The sum S of the first n terms of a geometric sequence $a_k = ar^{k-1}$ for $k \geq 1$ is

  1. $S = \sum\limits_{k=1}^{n} a_k = \dfrac{a_1 - a_{n+1}}{1-r} = a\left(\dfrac{1-r^n}{1-r}\right)$, if $r \neq 1$.
  2. $S = \sum\limits_{k=1}^{n} a_k = \sum\limits_{k=1}^{n} a = na$, if $r = 1$.

While we have made an honest effort to derive the formulas in Equation 1.2, formal proofs require the machinery in Section 1.3.

Example 1.2.3.

1. (a) Find the sum: $1 + 3 + 5 + ... + 117$

   (b) Find a formula for the sum $\sum\limits_{k=1}^{n} k$.

2. The classic wheat and chessboard problem asks the following question. Given a chessboard with its squares numbered 1 to 64, suppose on the first square was placed one grain of wheat, the second square, two grains, the third square, four grains, and so on, each square receiving twice the number of grains as its predecessor. How many total grains of wheat would end up on the chessboard?

Solution.

1. (a) Recognizing the terms of $1 + 3 + 5 + ... + 117$ as 1, 3, 5, and so on, we see we have an arithmetic sequence with $a = 1$ and $d = 2$. Using Equation 1.1, we get a formula for the terms $a_n = 1 + 2(n-1) = 2n - 1$ for $n \geq 1$. In order to use the formula in Equation 1.2, we need to determine the number of terms being added, n. Setting $2n - 1 = 117$, we find $n = 59$. Feeding in all of our data into Equation 1.2, we get $1 + 3 + 5 + ... + 117 = 59\left(\frac{1+117}{2}\right) = 3481$.

   (b) Applying the adage 'when in doubt, write it out,' we have $\sum\limits_{k=1}^{n} k = 1 + 2 + 3 + 4 + ... + n$. We see the terms here form an arithmetic sequence with $a = d = 1$. Moreover, we are adding exactly n terms, so Equation 1.2 gives $\sum\limits_{k=1}^{n} k = 1 + 2 + 3 + 4 + ... + n = \dfrac{n(n+1)}{2}$.
   As a side note, the special case: $1+2+3+...+100$ was allegedly given to Carl Friedrich Gauss while he was in elementary school. Instead of computing the sum in a brute force method, he

arrived at the answer by grouping 1 + 99 = 100, 2 + 98 = 100, etc. so that he had 50 groups of 100 with 50 left over for a total of 5050. This is the exact same methodology we used to prove the sum of the arithmetic sequence formula in Equation 1.2.

2. Since we are *doubling* the number of grains of wheat as we move from one square to the next, a geometric sequence with $r = 2$ describes the number of grains on each individual square.

   Since we start with one grain on the first square, the number of grains on the kth square is $a_k = (1)(2)^{k-1} = 2^{k-1}$ for $k \geq 1$.

   Adding up the number of grains on each square gives:

$$1 + 2 + ... + 2^{64-1} = 1 + 2 + ... 2^{63} = \frac{1 - 2^{64}}{1 - 2} = 2^{64} - 1 \approx 1.8 \times 10^{19},$$

   in accordance with Equation 1.2. (The weight of these grains would total approximately 2.6×10^{15} pounds which is approximately 15 times the entire biomass of the planet.) □

An important application of the geometric sum formula is the investment plan called an *annuity*. Annuities differ from the kind of investments we studied in Section **??** in that payments are deposited into the account on an on-going basis, and this complicates the mathematics a little.[5]

Suppose you have an account with annual interest rate r which is compounded n times per year. We let $i = \frac{r}{n}$ denote the interest rate per period. Suppose we wish to make ongoing deposits of P dollars at the *end* of each compounding period. Let A_k denote the amount in the account after k compounding periods. Then $A_1 = P$, because we have made our first deposit at the *end* of the first compounding period and no interest has been earned. During the second compounding period, we earn interest on A_1 so that our initial investment has grown to $A_1(1 + i) = P(1 + i)$ in accordance with Equation **??**. Adding our second payment at the end of the second period, we get

$$A_2 = A_1(1 + i) + P = P(1 + i) + P = P(1 + i)\left(1 + \frac{1}{1 + i}\right)$$

The reason for factoring out the $P(1 + i)$ will become apparent in short order. During the third compounding period, we earn interest on A_2 which then grows to $A_2(1 + i)$. We add our third payment at the end of the third compounding period to obtain

$$A_3 = A_2(1 + i) + P = P(1 + i)\left(1 + \frac{1}{1 + i}\right)(1 + i) + P = P(1 + i)^2\left(1 + \frac{1}{1 + i} + \frac{1}{(1 + i)^2}\right)$$

During the fourth compounding period, A_3 grows to $A_3(1 + i)$, and when we add the fourth payment, we factor out $P(1 + i)^3$ to get

$$A_4 = P(1 + i)^3\left(1 + \frac{1}{1 + i} + \frac{1}{(1 + i)^2} + \frac{1}{(1 + i)^3}\right)$$

This pattern continues so that at the end of the kth compounding, we get

[5] The reader may wish to re-read the discussion on compound interest in Section **??** before proceeding.

$$A_k = P(1+i)^{k-1}\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}}\right)$$

The sum in the parentheses above is the sum of the first k terms of a geometric sequence with $a = 1$ and $r = \frac{1}{1+i}$. Using Equation 1.2, we get

$$1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}} = 1\left(\frac{1-\dfrac{1}{(1+i)^k}}{1-\dfrac{1}{1+i}}\right) = \frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}$$

Hence, we get

$$A_k = P(1+i)^{k-1}\left(\frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}\right) = \frac{P\left((1+i)^k - 1\right)}{i}$$

If we let t be the number of years this investment strategy is followed, then $k = nt$, and we get the formula for the future value of an *ordinary annuity*.

Equation 1.3. Future Value of an Ordinary Annuity: Suppose an annuity offers an annual interest rate r compounded n times per year. Let $i = \frac{r}{n}$ be the interest rate per compounding period. If a deposit P is made at the end of each compounding period, the amount A in the account after t years is given by

$$A = \frac{P\left((1+i)^{nt} - 1\right)}{i}$$

The reader is encouraged to substitute $i = \frac{r}{n}$ into Equation 1.3 and simplify. Some familiar equations arise which are cause for pause and meditation. One last note: if the deposit P is made a the *beginning* of the compounding period instead of at the end, the annuity is called an *annuity-due*. We leave the derivation of the formula for the future value of an annuity-due as an exercise for the reader.

Example 1.2.4. An ordinary annuity offers a 6% annual interest rate, compounded monthly.

1. If monthly payments of \$50 are made, find the value of the annuity in 30 years.

2. How many years will it take for the annuity to grow to \$100,000?

Solution.

1. We have $r = 0.06$ and $n = 12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P = 50$ and $t = 30$,

$$A = \frac{50\left((1+0.005)^{(12)(30)} - 1\right)}{0.005} \approx 50225.75$$

Our final answer is \$50,225.75.

2. To find how long it will take for the annuity to grow to \$100,000, we set $A = 100000$ and solve for t. We isolate the exponential and take natural logs of both sides of the equation.

$$\begin{aligned}
100000 &= \frac{50\left((1+0.005)^{12t}-1\right)}{0.005} \\
10 &= (1.005)^{12t}-1 \\
(1.005)^{12t} &= 11 \\
\ln\left((1.005)^{12t}\right) &= \ln(11) \\
12t\ln(1.005) &= \ln(11) \\
t &= \tfrac{\ln(11)}{12\ln(1.005)} \approx 40.06
\end{aligned}$$

This means that it takes just over 40 years for the investment to grow to \$100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the annuity nearly doubles. This is a lesson worth remembering. □

1.2.1 Extensions to Calculus: Geometric Series

As defined in Section 1.1, sequences are an *infinite* list of numbers. So far in this section, we have concerned ourselves with adding only *finitely* many terms. In Calculus, *infinite* sums, called *series* are studied at great length. While we do not have the mathematical machinery to embark upon an exhaustive study here, we can nevertheless focus our attention on what is arguably one of the most prevalent and useful types of series, *Geometric Series*.

As a motivating example, consider the number $0.\overline{9}$. We can write this number as

$$0.\overline{9} = 0.9999... = 0.9 + 0.09 + 0.009 + 0.0009 + ...$$

From Example 1.2.1, we know we can write the sum of the first n of these terms as

$$0.\underbrace{9\cdots9}_{n \text{ nines}} = .9 + 0.09 + 0.009 + ...\,0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9 = \sum_{k=1}^{n}\frac{9}{10^k}$$

Using Equation 1.2, we have

$$\sum_{k=1}^{n}\frac{9}{10^k} = \frac{9}{10}\left(\frac{1-\dfrac{1}{10^{n+1}}}{1-\dfrac{1}{10}}\right) = 1-\frac{1}{10^{n+1}}$$

It stands to reason that $0.\overline{9}$ is the same value of $1-\frac{1}{10^{n+1}}$ as $n\to\infty$. Our knowledge of exponential expressions from Section **??** tells us that $\frac{1}{10^{n+1}}\to 0$ as $n\to\infty$, so $1-\frac{1}{10^{n+1}}\to 1$. We have just argued that $0.\overline{9} = 1$, which may shock some readers.[6]

[6] To make this more palatable, it is usually accepted that $0.\overline{3} = \frac{1}{3}$ so that $0.\overline{9} = 3\left(0.\overline{3}\right) = 3\left(\frac{1}{3}\right) = 1$.

Note that in this manner, any non-terminating decimal can be thought of as an infinite sum whose denominators are the powers of 10, so the phenomenon of adding up infinitely many terms and arriving at a finite number is not as foreign of a concept as it may appear.
The primary result concerning geometric series is below:

Theorem 1.2. Geometric Series: Given the sequence $a_k = ar^{k-1}$ for $k \geq 1$, where $|r| < 1$,

$$a + ar + ar^2 + ... = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the sum $a + ar + ar^2 + ...$ is not defined.

The justification of the result in Theorem 1.2 comes from taking the formula in Equation 1.2 for the sum of the first n terms of a geometric sequence and examining the formula as $n \to \infty$.
Assuming $|r| < 1$ means $-1 < r < 1$, so $r^n \to 0$ as $n \to \infty$. (It is essential to note n is a *natural* number especially when $r \leq 0$.) Hence as $n \to \infty$,

$$\sum_{k=1}^{n} ar^{k-1} = a\left(\frac{1-r^n}{1-r}\right) \to \frac{a}{1-r}$$

We'll explore what goes wrong when $|r| \geq 1$ in some of the exercises. For now, we put this theorem to good use in the following example.

Example 1.2.5.

1. Find the sum: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

2. Represent $4.2\overline{17}$ as a fraction in lowest terms.

Solution.

1. We recognize $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ as a geometric series with $a = r = \frac{1}{2}$. Using Theorem 1.2, we get

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

   The interested reader is invited to research this sum as it relates to Zeno's Dichotomy Paradox.

2. To use Theorem 1.2 as it applies to the repeating decimal $4.2\overline{17}$, we first need to rewrite this decimal in terms of a geometric series. Expanding $4.2\overline{17} = 4.2 + 0.017 + 0.00017 + 0.0000017 + ...$, we see the series $0.017 + 0.00017 + 0.0000017 + ...$ is geometric with $a = 0.017$ and $r = 0.01$. Hence, we can apply Theorem 1.2 to that part of the decimal to get:

$$0.017 + 0.00017 + 0.0000017 + ... = \frac{0.017}{1 - 0.01} = \frac{\frac{17}{1000}}{\frac{99}{100}} = \frac{17}{990}$$

Hence, $4.2\overline{17} = 4.2 + \dfrac{17}{990} = \dfrac{42}{10} + \dfrac{17}{990} = \dfrac{835}{198}$. □

We note that another popular method for converting repeating decimals to fractions goes something like this: let $x = 4.2\overline{17}$. Then, $100x = 421.7\overline{17}$. Hence, $99x = 100x - x = 421.7\overline{17} - 4.2\overline{17} = 417.5$. Hence, $x = \frac{417.5}{99} = \frac{835}{198}$. While this procedure results in the same (correct!) answer, the manipulations involved (such as the multiplication and subtraction) are actually using some of the properties listed in Theorem 1.1 extended to infinite sums.

1.2.2 Extensions to Calculus: Area

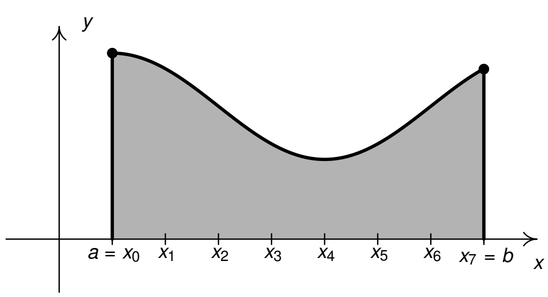
One of the (two) major geometric problems studied in Calculus is finding the area under a curve (more specifically, the area between the graph of a function and the x-axis.)[7] In this section, we explore how summation notation is used to help better formulate this problem, and, as with our study of Geometric Series, sneak a peak into Calculus itself.

Suppose we wish to determine the area between the graph of a continuous function $y = f(x)$ over the interval $[a, b]$ and the x-axis as shown below on the left. Since we don't know any area formulas for arbitrary regions, we stick to what we know - rectangles.

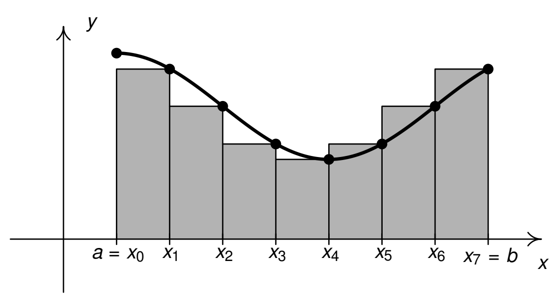
To keep things simple, we divide $[a, b]$ into n equal pieces (subintervals), and use the right-endpoints of each piece to determine the height of the rectangles.[8] We let x_k represent the right endpoint of the kth subinterval, so the height of the kth rectangle is $f(x_k)$.

The width of the kth rectangle is the length of the kth subinterval. Since the interval itself is $b - a$ units long and we are dividing the interval into n *equal* pieces, each piece is $\frac{b-a}{n}$ units long. For brevity, we'll call this length 'Δx.'

Below on the right is a depiction of RS_7, a 'right endpoint sum' using 7 (equally spaced) subintervals.[9]



Area under the graph of $y = f(x)$



Visualizing RS_7, a 'right endpoint sum.'

The idea here is to approximate the area of the shaded region by the sum of the areas of the rectangles. In symbols:

$$\text{Area} \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_7)\Delta x = \sum_{k=1}^{7} f(x_k)\Delta x$$

[7]The area can actually represent a wide variety of things such as displacement, probability, or, as odd as it sounds, volume.

[8]In Calculus, you'll also use left endpoints and midpoints . . . .

[9]On intervals over which the function is *increasing*, we find the area of rectangles *overestimates* the area we want; on intervals over which the function is *decreasing*, we find the area of the rectangles *underestimates* the area we want.

Our ultimate goal is to find a formula for the area approximation as described above as a function of the number of rectangles n and look to see what happens as $n \to \infty$.
We first note that the right endpoints x_k, are terms in an arithmetic sequence: the first right endpoint, x_1 is Δx to the right of $a = x_0$, so $x_1 = x_0 + \Delta x$; the second right endpoint, x_2 is Δx units to the right of x_1, so $x_2 = x_1 + \Delta x$; the third right endpoint $x_3 = x_2 + \Delta x$ and so on. In general, $x_k = x_{k-1} + \Delta x$, proving the x_k are terms of an arithmetic sequence with common difference $d = \Delta x$. It follows that x_k, the kth right endpoint is $k\Delta x$ units to the right of $x_0 = a$, so that $x_k = a + k\Delta x$. We summarize the notation and formulas for right endpoint sums below.

Summary of Formulas for Right Endpoint Sums, RS_n

- Number of rectangles: n
- Width of each rectangle: $\Delta x = \dfrac{b-a}{n}$
- Right endpoint: $x_k = a + k\Delta x$
- Height of kth rectangle: $f(x_k)$
- Area $\approx RS_n$ = the sum of the area of the rectangles = $\displaystyle\sum_{k=1}^{n} f(x_k)\Delta x_k$

Below we summarize some common summation formulas we'll need when actually computing these sums. Formal proofs of these require the machinery of Section 1.3 and are found there.

Summation Formulas

- $\displaystyle\sum_{k=1}^{n} c = cn$
- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

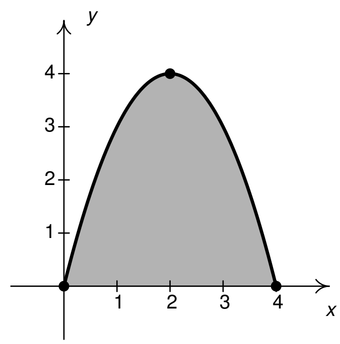
It is high time for an example.

Example 1.2.6. Consider $f(x) = 4x - x^2$ over the interval $[0, 4]$.

1. Graph f over this interval and shade the area between the graph of f and the x-axis.
2. Compute RS_n for $n = 4$ and $n = 8$. Interpret your results graphically.
3. Find a formula for RS_n in terms of n and determine the behavior of RS_n as $n \to \infty$.

Solution.

1. The graph of $f(x) = 4x - x^2$ is a parabola with intercepts $(0, 0)$ and $(4, 0)$ with a vertex at $(2, 4)$.

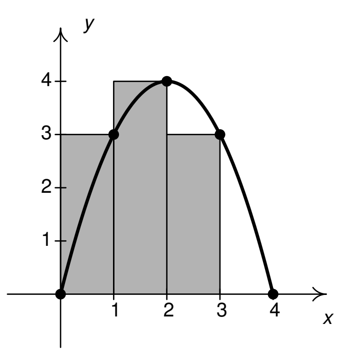


Area under the graph of $y = f(x)$

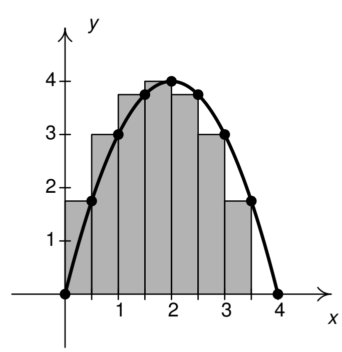
2. To find RS_4, we begin by chopping up the interval $[0, 4]$ into 4 equal pieces so each subinterval has length $\Delta x = \frac{4}{4} = 1$ unit. Our right endpoints are: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$. We find $f(1) = 3$, $f(2) = 4$, $f(3) = 3$, and $f(4) = 0$. Hence,

$$RS_4 = \sum_{k=1}^{4} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x = (3)(1) + (4)(1) + 3(1) + (0)(1) = 10.$$

Geometrically we have approximated the area under the graph of f to be 10 square units by the adding the areas of the shaded rectangles shaded below on the left. (Note that since $f(x_4) = f(4) = 0$, the fourth 'rectangle' has 0 height.)



Visualizing RS_4



Visualizing RS_8

To find RS_8, we divide the interval $[0, 4]$ into 8 equal pieces, so each has length $\Delta x = \frac{4}{8} = 0.5$ units. This produces the right endpoints: $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$, $x_5 = 2.5$, $x_6 = 3$, $x_7 = 3.5$, $x_8 = 4$. In addition to the function values we used to compute RS_4, we need $f(0.5) = 1.75$, $f(1.5) = 3.75$, $f(2.5) = 3.75$, and $f(3.5) = 1.75$. Hence,

$$\begin{aligned}
RS_8 &= \sum_{k=1}^{8} f(x_k)\Delta x \\
&= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
&\quad + f(x_5)\Delta x + f(x_6)\Delta x + f(x_7)\Delta x + f(x_8)\Delta x \\
&= (1.75)(0.5) + (3)(0.5) + (3.75)(0.5) + (4)(0.5) \\
&\quad + (3.75)(0.5) + (3)(0.5) + (1.75)(0.5) + (0)(0.5) \\
&= 10.5
\end{aligned}$$

Hence, the area under the graph f is approximately 10.5 square units as approximated by the sum of the rectangles above on the right. (Again, since $f(x_8) = f(4) = 0$, the eighth 'rectangle' has 0 height.)

3. To find a formula for RS_n, we imagine dividing the interval $[0, 4]$ into n equal pieces each of length $\Delta x = \frac{4}{n}$. We have n right endpoints, $x_1, x_2, \ldots x_n$ where $x_k = 0 + k\Delta x = \frac{4k}{n}$. Since $f(x) = 4x - x^2$,

$$f(x_k) = 4x_k - x_k^2 = 4\left(\frac{4k}{n}\right) - \left(\frac{4k}{n}\right)^2 = \frac{16k}{n} - \frac{16k^2}{n^2}.$$

Hence,

$$\begin{aligned}
RS_n &= \sum_{k=1}^{n} f(x_k)\Delta x \\
&= \sum_{k=1}^{n} \left[\frac{16k}{n} - \frac{16k^2}{n^2}\right]\left(\frac{4}{n}\right) \\
&= \sum_{k=1}^{n} \left[\frac{64k}{n^2} - \frac{64k^2}{n^3}\right] && \text{Distribute the } \frac{4}{n}. \\
&= \sum_{k=1}^{n} \frac{64k}{n^2} - \sum_{k=1}^{n} \frac{64k^2}{n^3} && \text{Sum and Difference Property} \\
&= \frac{64}{n^2}\sum_{k=1}^{n} k - \frac{64}{n^3}\sum_{k=1}^{n} k^2 && \text{Distributive Property[10]} \\
&= \frac{64}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) && \text{Summation Formulas} \\
&= \frac{32(n+1)}{n} - \frac{32(n+1)(2n+1)}{3n^2} = \frac{32n^2 - 32}{3n^2}
\end{aligned}$$

[10] Note: the counter here is 'k,' not 'n,' so as far as k is concerned, 'n' is a constant so we can factor it out of the summation.

Note we can partially check our answer at thus point by substituting $n = 4$ and $n = 8$ to our formula to RS_n to see if we recover our answers from above. We get $RS_4 = \frac{32(4)^2-32}{3(4^2)} = \frac{480}{48} = 10$ and $RS_8 = \frac{32(8)^2-32}{3(8)^2} = \frac{2016}{192} = 10.5$, as required.

Viewing $RS_n = \frac{32n^2-32}{3n^2}$ as a rational function in the variable n, we may use the techniques from Chapter **??** to get as $n \to \infty$, $RS_n \approx \frac{32n^2}{3n^2} = \frac{32}{3}$. Hence, as we use more and more rectangles,[11] the sum total of the area of those rectangles approaches $\frac{32}{3}$ square units. In Calculus, we more or less *define* the area under f to be $\frac{32}{3}$ square units. □

It is worth noting that, as with other examples in the text, Example 1.2.6 is more or less lifted straight out of a Calculus lecture. That being said, the vast majority of the mechanics here involve precalculus notions. (The only Calculus bit is when we let $n \to \infty$.) In general, the machinations in Calculus amount to applying one new idea to the mechanics of precalculus.

[11] even though they become skinnier and skinner and hence, *individually* have smaller and smaller areas . . .

1.3 Mathematical Induction

The Chinese philosopher Confucius is credited with the saying, "A journey of a thousand miles begins with a single step." In many ways, this is the central theme of this section. Here we introduce a method of proof, Mathematical Induction, which allows us to *prove* many of the formulas we have merely *motivated* in Sections 1.1 and 1.2 by starting with just a single step. A good example is the formula for arithmetic sequences we touted in Equation 1.1. Arithmetic sequences are defined recursively, starting with $a_1 = a$ and then $a_{n+1} = a_n + d$ for $n \geq 1$. This tells us that we start the sequence with a and we go from one term to the next by successively adding d. In symbols,

$$a, a + d, a + 2d, a + 3d, a + 4d + \ldots$$

The pattern *suggested* here is that to reach the nth term, we start with a and add d to it exactly $n - 1$ times, leading to the formula $a_n = a + (n - 1)d$ for $n \geq 1$. In order to *prove* this is the case, we have:

The Principle of Mathematical Induction (PMI):
Suppose $P(n)$ is a sentence involving the natural number n.

IF

1. $P(1)$ is true **and**
2. whenever $P(k)$ is true, it follows that $P(k + 1)$ is also true

THEN the sentence $P(n)$ is true for all natural numbers n.

The Principle of Mathematical Induction, or PMI for short, is exactly that - a principle.[1] It is a property of the natural numbers we either choose to accept or reject. The notation which is used here, '$P(n)$,' acts just like function notation. For example, if $P(n)$ is the sentence (formula) '$n^2 + 1 = 3$', then $P(1)$ would be '$1^2 + 1 = 3$', which is false. In this case, the construction $P(k + 1)$ would be '$(k + 1)^2 + 1 = 3$'.

In English, the PMI says that if we want to prove that a formula works for all natural numbers n, we start by showing it is true for $n = 1$ (the '*base step*') and then show that *if* it is true for a generic natural number k, *then* it must be true for the next natural number, $k + 1$ (the '*inductive step*'). In essence, by showing that $P(k + 1)$ must always be true when $P(k)$ is true, we are showing that the formula $P(1)$ can be used to get the formula $P(2)$, which in turn can be used to derive the formula $P(3)$, which in turn can be used to establish the formula $P(4)$, and so on, for all natural numbers n.

One might liken Mathematical Induction to a repetitive process like climbing stairs.[2] If you are sure that (1) you can get on the stairs (the base case) and (2) you can climb from any one step to the next step (the inductive step), then presumably you can climb the entire staircase.[3] We get some more practice with induction in the following example.

[1]Another word for this you may have seen is 'axiom.'
[2]Falling dominoes is the most widely used metaphor in the mainstream College Algebra books.
[3]This is how Carl climbed the stairs in the Cologne Cathedral. Well, that, and encouragement from Kai.

Example 1.3.1. Prove the following assertions using the Principle of Mathematical Induction.

1. If $a_1 = 4$ and $a_{n+1} = -\dfrac{a_n}{2}$ for $n \geq 1$, then prove $a_n = (-1)^{n-1}2^{3-n}$ for $n \geq 1$.
2. $1 + 3 + 5 + ... + (2n - 1) = n^2$
3. $3^n > 100n$ for $n > 5$.

Solution.

1. To prove $a_n = (-1)^{n-1}2^{3-n}$ for $n \geq 1$ by induction, we first identify the sentence $P(n)$ as the equation $a_n = (-1)^{n-1}2^{3-n}$. The sentence $P(1)$ is the equation $a_1 = (-1)^{1-1}2^{3-1}$ or, after simplifying, $a_1 = 4$, which we are told is true.

   Next, we *assume* the sentence $P(k)$ is true, that is, $a_k = (-1)^{k-1}2^{3-k}$ (this is called the '*induction hypothesis*') and must use this to *deduce* $P(k+1)$ is true. That is, we need to use the fact that $a_k = (-1)^{k-1}2^{3-k}$ to show $a_{k+1} = (-1)^{(k+1)-1}2^{3-(k+1)}$ or, after simplifying, $a_{k+1} = (-1)^k2^{2-k}$.

   We are told $a_{k+1} = -\frac{a_k}{2}$ and we are assuming $a_k = (-1)^{k-1}2^{3-k}$, so we put these together to get

   $$a_{k+1} = -\frac{a_k}{2} = -\frac{(-1)^{k-1}2^{3-k}}{2} = (-1)^1\frac{(-1)^{k-1}2^{3-k}}{2^1} = (-1)^{k-1+1}2^{3-k+1} = (-1)^k2^{2-k},$$

   as required. Hence, by induction, $a_n = (-1)^{n-1}2^{3-n}$ for $n \geq 1$.

   We take a moment and recognize the sequence here, as described, is a geometric sequence with $a = 4$ and $r = -\frac{1}{2}$. Using Equation 1.1 we arrive at the explicit formula for $a_n = 4\left(-\frac{1}{2}\right)^{n-1}$ for $n \geq 1$ which we leave to the reader to show reduces to $a_n = (-1)^{n-1}2^{3-n}$. (Note: You'll be asked to prove Equation 1.1 in Exercise **??**.)

2. As above, our first step is to identify the sentence $P(n)$ which is the equation $1+3+5+...+(2n-1) = n^2$ which is more precisely written using summation notation: $\displaystyle\sum_{j=1}^{n}(2j-1) = n^2$. (Note we use '$j$' as our dummy variable here since 'n' is already used and we usually reserve 'k' for the induction variable.) The sentence $P(1)$ is $\displaystyle\sum_{j=1}^{1}(2j-1) = 1^2$ which reduces to $2(1) - 1 = 1$ which is true. Next, we assume $P(k)$ is true, $\displaystyle\sum_{j=1}^{k}(2j-1) = k^2$, and use it to show $P(k+1)$ is true: $\displaystyle\sum_{j=1}^{k+1}(2j-1) = (k+1)^2$. We have:

   $$\underbrace{\sum_{j=1}^{k+1}(2j-1)}_{\text{adding } k+1 \text{ terms}} = \underbrace{\sum_{j=1}^{k}(2j-1)}_{\text{adding the first } k \text{ terms}} + \underbrace{(2(k+1)-1)}_{\text{adding the first } k+1 \text{ term}} = \underbrace{k^2}_{P(k)} \underbrace{+2k+1}_{\text{simplify}} = \underbrace{(k+1)^2}_{\text{factor}},$$

   as required. Hence, by induction, $1 + 3 + 5 + ... + (2n - 1) = n^2$ for all natural numbers $n \geq 0$.

As with the first example, this problem, too, can be shown using a previous result. The sequence being added in the equation $1 + 3 + 5 + ... + (2n - 1) = n^2$ is arithmetic, so Equation 1.2 applies to give the sum as $\frac{n}{2}(1 + (2n - 1)) = n^2$. We'll prove Equation 1.2 for arithmetic sequences in the next example. We leave the case for geometric sequences to the reader in Exercise **??**.

3. The first wrinkle we encounter in this problem is that we are asked to prove this formula for $n > 5$ instead of $n \geq 1$. Since n is a natural number, this means our base step occurs at $n = 6$. We can still use the PMI in this case, but our conclusion will be that the formula is valid for all $n \geq 6$.

   We let $P(n)$ be the inequality $3^n > 100n$, and check that $P(6)$ is true. Comparing $3^6 = 729$ and $100(6) = 600$, we see $3^6 > 100(6)$ as required.

   Next, we assume that $P(k)$ is true, that is we assume $3^k > 100k$. We need to show that $P(k + 1)$ is true, that is, we need to show $3^{k+1} > 100(k + 1)$. Since $3^{k+1} = 3 \cdot 3^k$, the induction hypothesis gives $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k$.

   To complete the proof, we need to show $300k > 100(k + 1)$ for $k \geq 6$. Solving $300k > 100(k + 1)$ we get $k > \frac{1}{2}$. Since $k \geq 6$, we know this is true.

   Putting all of this together, we have $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k > 100(k+1)$, and hence $P(k+1)$ is true. By induction, $3^n > 100n$ for all $n \geq 6$. $\square$

One of the things that may seem troubling about proving statements by induction is the induction hypothesis: that is, assuming that $P(k)$ is true. After all, isn't that what we are trying to prove? When we assume $P(k)$ is true, we are doing so with the *express purpose* of showing that $P(k + 1)$ follows. That is, we are interested in showing *how* we go 'from one step to the next.'

As mentioned at the beginning of this section, induction is the formal way to prove many the formulas we've used in Sections 1.1 and 1.2. Indeed, now that we have some experience using the PMI to prove formulas, we return to proving the formula for an arithmetic sequence.

Recall we define an arithmetic sequence recursively as: $a_1 = a$ and $a_{n+1} = a_n + d$ for $n \geq 1$. We need to prove $a_n = a + (n - 1)d$ for $n \geq 1$. Identifying $P(n)$ as the formula $a_n = a + (n - 1)d$, we see $P(1)$ is $a_1 = a + (1 - 1)d = a$, which is true.

Next, we assume $P(k)$ is true, that is, $a_k = a+(k-1)d$ and use this to show $P(k+1)$, or $a_{k+1} = a+((k+1)-1)d$ or $a_{k+1} = a + kd$ is true. We know $a_{k+1} = a_k + d$ from the definition of arithmetic sequence, hence

$$a_{k+1} = a_k + d = a + (k - 1)d + d = a + kd,$$

as required. Hence, $a_n = a + (n - 1)d$, for all natural numbers $n \geq 1$.

We conclude this section with three more proofs by induction.

Example 1.3.2. Prove the following assertions using the Principle of Mathematical Induction.

1. The sum formula for arithmetic sequences: $\displaystyle\sum_{j=1}^{n}(a+(j-1)d)=\frac{n}{2}(2a+(n-1)d)$.
2. For a complex number z, $(\overline{z})^n=\overline{z^n}$ for $n \geq 1$.
3. Let A be an $n \times n$ matrix and let A' be the matrix obtained by replacing a row R of A with cR for some real number c. Use the definition of determinant to show $\det(A')=c\det(A)$.

Solution.

1. We set $P(n)$ to be the equation we are asked to prove, namely $\displaystyle\sum_{j=1}^{n}(a+(j-1)d)=\frac{n}{2}(2a+(n-1)d)$. The statement $P(1)$, $\displaystyle\sum_{j=1}^{1}(a+(j-1)d)=\frac{1}{2}(2a+(1-1)d)$, reduces to $a+(0)d=\frac{1}{2}(2a)$ or $a=a$, which is true. Next we assume $P(k)$ is true, that is, we assume $\displaystyle\sum_{j=1}^{k}(a+(j-1)d)=\frac{k}{2}(2a+(k-1)d)$ and use this to show $P(k+1)$ is true: $\displaystyle\sum_{j=1}^{k+1}(a+(j-1)d)=\frac{k+1}{2}(2a+(k+1-1)d)=\frac{k+1}{2}(2a+kd)$:

$$\underbrace{\sum_{j=1}^{k+1}(a+(j-1)d)}_{\text{adding } k+1 \text{ terms}}=\underbrace{\sum_{j=1}^{k}(a+(j-1)d)}_{\text{adding the first } k \text{ terms}}+\underbrace{(a+((k+1)-1)d)}_{\text{adding the first } k+1 \text{ term}}=\underbrace{\frac{k}{2}(2a+(k-1)d)}_{P(k)}+\underbrace{a+kd}_{\text{simplify}}.$$

   We leave it to the reader (see Exercise **??**) to show that, indeed,

$$\frac{k}{2}(2a+(k-1)d)+a+kd=\frac{k+1}{2}(2a+d),$$

   which completes the proof that $P(k+1)$ is true. By induction, $\displaystyle\sum_{j=1}^{n}(a+(j-1)d)=\frac{n}{2}(2a+(n-1)d)$ for all natural numbers n.

2. We let $P(n)$ be the equation $(\overline{z})^n=\overline{z^n}$. The base case $P(1)$ is $(\overline{z})^1=\overline{z^1}$ reduces to $\overline{z}=\overline{z}$ which is true. We now assume $P(k)$ is true, that is, we assume $(\overline{z})^k=\overline{z^k}$ and use this to show that $P(k+1)$ is true, namely $(\overline{z})^{k+1}=\overline{z^{k+1}}$.

   Since $(\overline{z})^{k+1}=(\overline{z})^k\,\overline{z}$, we can use the induction hypothesis to write $(\overline{z})^k=\overline{z^k}$. Hence,

$$(\overline{z})^{k+1}=(\overline{z})^k\,\overline{z}=\overline{z^k}\,\overline{z}=\overline{z^k z}=\overline{z^{k+1}},$$

where the second-to-last equality is courtesy of the product rule for conjugates[4] This shows $P(k+1)$ is true and hence, by induction, $(\overline{z})^n = \overline{z^n}$ for all natural numbers n.

3. To prove this determinant property, we use induction on n, where we take $P(n)$ to be that the property we wish to prove is true for all $n \times n$ matrices. For the base case, we note that if A is a 1×1 matrix, then $A = [a]$ so $A' = [ca]$. By definition, $\det(A) = a$ and $\det(A') = ca$ so we have $\det(A') = c\det(A)$.

   Now suppose that the property we wish to prove is true for all $k \times k$ matrices. Let A be a $(k+1)\times(k+1)$ matrix. We have two cases, depending on if the row R being replaced is the first row of A.

   CASE 1: The row R being replaced is the first row of A. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p}$$

   where the $1p$ cofactor of A' is $C'_{1p} = (-1)^{(1+p)} \det\left(A'_{1p}\right)$ and A'_{1p} is the $k \times k$ matrix obtained by deleting the 1st row and pth column of A'.[5]

   Since the first row of A' is c times the first row of A, we have $a'_{1p} = c\, a_{1p}$. In addition, since the remaining rows of A' are identical to those of A, $A'_{1p} = A_{1p}$. (To obtain these matrices, the first row of A' is removed.) Hence $\det\left(A'_{1p}\right) = \det\left(A_{1p}\right)$, so that $C'_{1p} = C_{1p}$. As a result, we get

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p} = \sum_{p=1}^{n} c\, a_{1p} C_{1p} = c \sum_{p=1}^{n} a_{1p} C_{1p} = c\det(A),$$

   as required. Hence, $P(k+1)$ is true in this case, which means the result is true in this case for all natural numbers $n \geq 1$. (You'll note that we did not use the induction hypothesis at all in this case. It is possible to restructure the proof so that induction is only used where it is needed. While mathematically more elegant, it is less intuitive.)

   CASE 2: The row R being replaced is the not the first row of A. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p},$$

   where in this case, $a'_{1p} = a_{1p}$, since the first rows of A and A' are the same. The matrices A'_{1p} and A_{1p}, on the other hand, are different but in a very predictable way $-$ the row in A'_{1p} which corresponds to the row cR in A' is exactly c times the row in A_{1p} which corresponds to the row R in A.

   This means A'_{1p} and A_{1p} are $k \times k$ matrices which satisfy the induction hypothesis. Hence, we know $\det\left(A'_{1p}\right) = c\det\left(A_{1p}\right)$ and $C'_{1p} = c\, C_{1p}$. We get

[4] See Exercise **??** in Section **??**: $\overline{z}\,\overline{w} = \overline{zw}$.

[5] See Section **??** for a review of this notation.

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p} = \sum_{p=1}^{n} a_{1p} c\, C_{1p} = c \sum_{p=1}^{n} a_{1p} C_{1p} = c \det(A),$$

which establishes $P(k+1)$ to be true. Hence by induction, we have shown that the result holds in this case for $n \geq 1$ and we are done. □

While we have used the Principle of Mathematical Induction to prove some of the formulas we have merely motivated in the text, our main use of this result comes in Section 1.4 to prove the celebrated Binomial Theorem. The ardent Mathematics student will no doubt see the PMI in many courses yet to come. Sometimes it is explicitly stated and sometimes it remains hidden in the background. If ever you see a property stated as being true 'for all natural numbers n', it's a solid bet that the formal proof requires the Principle of Mathematical Induction.

1.4 The Binomial Theorem

In this section, we aim to prove the celebrated *Binomial Theorem*. Simply stated, the Binomial Theorem is a formula for the expansion of quantities $(a+b)^n$ for natural numbers n. In High School Algebra, you probably have seen specific instances of the formula, namely

$$\begin{aligned}(a+b)^1 &= a+b\\ (a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

If we wanted the expansion for $(a+b)^4$ we would write $(a+b)^4 = (a+b)(a+b)^3$ and use the formula that we have for $(a+b)^3$ to get $(a+b)^4 = (a+b)\left(a^3+3a^2b+3ab^2+b^3\right) = a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Generalizing this a bit, we see that if we have a formula for $(a+b)^k$, we can obtain a formula for $(a+b)^{k+1}$ by rewriting the latter as $(a+b)^{k+1} = (a+b)(a+b)^k$. Clearly this means Mathematical Induction plays a major role in the proof of the Binomial Theorem.[1] Before we can state the theorem we need to revisit the sequence of factorials which were introduced in Example 1.1.1 number 6 in Section 1.1.

Definition 1.4. Factorials:
For a whole number n, ***n* factorial**, denoted $n!$, is the term f_n of the sequence:

$$f_0 = 1,\, f_n = n\cdot f_{n-1}, \quad n \geq 1.$$

Recall this means $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$. Hence, $1! = 1\cdot 0! = 1\cdot 1 = 1$, $2! = 2\cdot 1! = 2\cdot 1 = 2$, $3! = 3\cdot 2! = 3\cdot 2\cdot 1 = 6$ and $4! = 4\cdot 3! = 4\cdot 3\cdot 2\cdot 1 = 24$. Informally, $n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$ with $0! = 1$ as our 'base case.' Our first example familiarizes us with some of the basics of factorial computations.

Example 1.4.1.

1. Simplify the following expressions.

   (a) $\dfrac{3!\,2!}{0!}$ (b) $\dfrac{7!}{5!}$ (c) $\dfrac{1000!}{998!\,2!}$ (d) $\dfrac{(k+2)!}{(k-1)!}$, $k \geq 1$

2. Prove $n! > 3^n$ for all $n \geq 7$.

Solution.

1. We keep in mind the mantra, "When in doubt, write it out!" as we simplify the following.

   (a) Recall $0! = 1$, by definition, $3! = 3\cdot 2\cdot 1 = 6$ and $2! = 2\cdot 1 = 2$. Hence, $\frac{3!\,2!}{0!} = \frac{(6)(2)}{1} = 12$.

   (b) We have $7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5040$ and $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$ so $\frac{7!}{5!} = \frac{5040}{120} = 42$.

   While this is correct, we note that we could have saved ourselves some of time had we approached the problem as follows:

[1] It's pretty much the reason Section 1.3 is in the book.

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 = 42$$

In fact, should we want to fully exploit the recursive nature of the factorial, we can write

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$$

(c) Keeping in mind the lesson we learned from the previous problem, we have

$$\frac{1000!}{998!\,2!} = \frac{1000 \cdot 999 \cdot 998!}{998! \cdot 2!} = \frac{1000 \cdot 999 \cdot \cancel{998!}}{\cancel{998!} \cdot 2!} = \frac{999000}{2} = 499500$$

(d) This problem continues the theme which we have seen in the previous two problems. We first note that since $k + 2$ is larger than $k - 1$, $(k + 2)!$ contains all of the factors of $(k - 1)!$ and as a result we can get the $(k - 1)!$ to cancel from the denominator.

To see this, we begin by writing out $(k+2)!$ starting with $(k+2)$ and multiplying it by the numbers which precede it until we reach $(k - 1)$: $(k + 2)! = (k + 2)(k + 1)(k)(k - 1)!$. As a result, we have

$$\frac{(k+2)!}{(k-1)!} = \frac{(k+2)(k+1)(k)(k-1)!}{(k-1)!} = \frac{(k+2)(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}} = k(k+1)(k+2)$$

The stipulation $k \geq 1$ is there to ensure that all of the factorials involved are defined.

2. We proceed by induction and let $P(n)$ be the inequality $n! > 3^n$. The base case here is $n = 7$ and we see that $7! = 5040$ is larger than $3^7 = 2187$, so $P(7)$ is true.

   Next, we assume that $P(k)$ is true, that is, we assume $k! > 3^k$ and attempt to show $P(k + 1)$ follows. Using the properties of the factorial, we have $(k + 1)! = (k + 1)k!$ and since $k! > 3^k$, we have $(k + 1)! > (k + 1)3^k$. Since $k \geq 7$, $k + 1 \geq 8$, so $(k + 1)3^k \geq 8 \cdot 3^k > 3 \cdot 3^k = 3^{k+1}$.

   Putting all of this together, we have $(k + 1)! = (k + 1)k! > (k + 1)3^k > 3^{k+1}$ which shows $P(k + 1)$ is true. By the Principle of Mathematical Induction, we have $n! > 3^n$ for all $n \geq 7$. □

Of all of the mathematical functions we have discussed in the text, factorials grow most quickly. In Example 1.4.1 above, we proved that $n!$ overtakes 3^n at $n = 7$. 'Overtakes' may be too polite a word, since $n!$ thoroughly trounces 3^n for $n \geq 7$, as any reasonable set of data will show.

It can be shown that for any real number $x > 0$, not only does $n!$ eventually overtake x^n, but the ratio $\frac{x^n}{n!} \to 0$ as $n \to \infty$. (This is extremely important for Calculus.)

Applications of factorials in the wild often involve counting arrangements. For example, if you have fifty songs on your mp3 player and wish arrange these songs in a playlist in which the order of the songs matters, it turns out that there are 50! different possible playlists.

If you wish to select only ten of the songs to create a playlist, then there are $\frac{50!}{40!}$ such playlists. If, on the other hand, you just want to select ten song files out of the fifty to put on a flash memory card so that now the order no longer matters, there are $\frac{50!}{40!10!}$ ways to achieve this.[2]

While some of these ideas are explored in the Exercises, the authors encourage you to take courses such as Finite Mathematics, Discrete Mathematics and Statistics. We introduce these concepts here because this is how the factorials make their way into the Binomial Theorem, as our next definition indicates.

Definition 1.5. Binomial Coefficients: Given two whole numbers n and j with $n \geq j$, the binomial coefficient $\binom{\boldsymbol{n}}{\boldsymbol{j}}$ (read, 'n choose j') is the whole number given by

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

The name 'binomial coefficient' will be justified shortly. For now, we can physically interpret $\binom{n}{j}$ as the number of ways to select j items from n items where the order of the items selected is unimportant. For example, suppose you won two free tickets to a special screening of the latest Hollywood blockbuster and have five good friends each of whom would love to accompany you to the movies. There are $\binom{5}{2}$ ways to choose who goes with you. Applying Definition 1.5, we get

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10$$

So there are 10 different ways to distribute those two tickets among five friends. (Some will see it as 10 ways to decide which three friends have to stay home.) The reader is encouraged to verify this by actually taking the time to list all of the possibilities.

We now state and prove a theorem which is crucial to the proof of the Binomial Theorem.

Theorem 1.3. For natural numbers n and j with $n \geq j$,

$$\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$$

The proof of Theorem 1.3 is purely computational and uses the definition of binomial coefficients, the recursive property of factorials and common denominators.

[2]For reference,

$$\begin{aligned} 50! &= 30414093201713378043612608166064768844377641568960512000000000000, \\ \frac{50!}{40!} &= 37276043023296000, \quad \text{and} \\ \frac{50!}{40!10!} &= 10272278170 \end{aligned}$$

$$\begin{aligned}
\binom{n}{j-1}+\binom{n}{j} &= \frac{n!}{(j-1)!(n-(j-1))!}+\frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)!}+\frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)(n-j)!}+\frac{n!}{j(j-1)!(n-j)!} \\
&= \frac{n!\,j}{j(j-1)!(n-j+1)(n-j)!}+\frac{n!(n-j+1)}{j(j-1)!(n-j+1)(n-j)!} \\
&= \frac{n!\,j}{j!(n-j+1)!}+\frac{n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,j+n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,(j+(n-j+1))}{j!(n-j+1)!} \\
&= \frac{(n+1)n!}{j!(n+1-j))!} \\
&= \frac{(n+1)!}{j!((n+1)-j))!} \\
&= \binom{n+1}{j}\ \checkmark
\end{aligned}$$

We are now in position to state and prove the Binomial Theorem where we see that binomial coefficients are just that - coefficients in the binomial expansion.

Theorem 1.4. Binomial Theorem: For nonzero real numbers a and b,

$$(a+b)^n=\sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j$$

for all natural numbers n.

To get a feel of what this theorem is saying and how it really isn't as hard to remember as it may first appear, let's consider the specific case of $n = 4$. According to the theorem, we have

$$\begin{aligned}
(a+b)^4 &= \sum_{j=0}^{4} \binom{4}{j} a^{4-j} b^j \\
&= \binom{4}{0} a^{4-0} b^0 + \binom{4}{1} a^{4-1} b^1 + \binom{4}{2} a^{4-2} b^2 + \binom{4}{3} a^{4-3} b^3 + \binom{4}{4} a^{4-4} b^4 \\
&= \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} ab^3 + \binom{4}{4} b^4
\end{aligned}$$

We forgo the simplification of the coefficients in order to note the pattern in the expansion. First note that in each term, the total of the exponents is 4 which matched the exponent of the binomial $(a+b)^4$. The exponent on a begins at 4 and decreases by one as we move from one term to the next while the exponent on b starts at 0 and increases by one each time.

Also note that the binomial coefficients themselves have a pattern. The upper number, 4, matches the exponent on the binomial $(a+b)^4$ whereas the lower number changes from term to term and matches the exponent of b in that term.

This is no coincidence and corresponds to the kind of counting we discussed earlier. If we think of obtaining $(a+b)^4$ by multiplying $(a+b)(a+b)(a+b)(a+b)$, our answer is the sum of all possible products with exactly four factors - some a, some b. If we wish to count, for instance, the number of ways we obtain 1 factor of b out of a total of 4 possible factors, thereby forcing the remaining 3 factors to be a, the answer is $\binom{4}{1}$. Hence, the term $\binom{4}{1} a^3 b$ is in the expansion. The other terms which appear cover the remaining cases.

While the foregoing discussion gives an indication as to *why* the theorem is true, a formal proof requires Mathematical Induction.[3]

To prove the Binomial Theorem, we let $P(n)$ be the expansion formula given in the statement of the theorem and we note that $P(1)$ is true since

$$\sum_{j=0}^{1} \binom{1}{j} a^{1-j} b^j = \binom{1}{0} a^{1-0} b^0 + \binom{1}{1} a^{1-1} b^1 = a+b = (a+b)^1.$$

Now we assume that $P(k)$ is true. That is, we assume that we can expand $(a+b)^k$ using the formula given in Theorem 1.4 and attempt to show that $P(k+1)$ is true.

[3]and a fair amount of tenacity and attention to detail.

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b)\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\
&= a\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j + b\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\
&= \sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1}
\end{aligned}$$

Our goal is to combine as many of the terms as possible within the two summations.

As the counter j in the first summation runs from 0 through k, we get terms involving a^{k+1}, $a^k b$, $a^{k-1}b^2$, $\ldots$, ab^k. In the second summation, we get terms involving $a^k b$, $a^{k-1}b^2$, $\ldots$, ab^k, b^{k+1}. In other words, apart from the first term in the first summation and the last term in the second summation, we have terms common to both summations.

Our next move is to 'kick out' the terms which we cannot combine and rewrite the summations so that we can combine them. To that end, we note

$$\sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j$$

and

$$\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1} = \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}$$

so that

$$(a+b)^{k+1} = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}$$

We now wish to write

$$\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1}$$

as a single summation. The wrinkle is that the first summation starts with $j = 1$, while the second starts with $j = 0$. Even though the sums produce terms with the same powers of a and b, they do so for different values of j. To resolve this, we need to shift the index on the second summation so that the index j starts at $j = 1$ instead of $j = 0$ and we make use of Theorem 1.1 in the process.

$$\begin{aligned}\sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} &= \sum_{j=0+1}^{k-1+1}\binom{k}{j-1}a^{k-(j-1)}b^{(j-1)+1}\\ &= \sum_{j=1}^{k}\binom{k}{j-1}a^{k+1-j}b^{j}\end{aligned}$$

We can now combine our two sums using Theorem 1.1 and simplify using Theorem 1.3

$$\begin{aligned}\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^{j}+\sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} &= \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^{j}+\sum_{j=1}^{k}\binom{k}{j-1}a^{k+1-j}b^{j}\\ &= \sum_{j=1}^{k}\left[\binom{k}{j}+\binom{k}{j-1}\right]a^{k+1-j}b^{j}\\ &= \sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^{j}\end{aligned}$$

Using this and the fact that $\binom{k+1}{0}=1$ and $\binom{k+1}{k+1}=1$, we get

$$\begin{aligned}(a+b)^{k+1} &= a^{k+1}+\sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^{j}+b^{k+1}\\ &= \binom{k+1}{0}a^{k+1}b^{0}+\sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^{j}+\binom{k+1}{k+1}a^{0}b^{k+1}\\ &= \sum_{j=0}^{k+1}\binom{k+1}{j}a^{(k+1)-j}b^{j}\end{aligned}$$

which shows that $P(k+1)$ is true. Hence, by induction, we have established that the Binomial Theorem holds for all natural numbers n.

Example 1.4.2. Use the Binomial Theorem to find the following.

1. $(x-2)^4$
2. 2.1^3
3. The term containing x^3 in the expansion $(2x+y)^5$

Solution.

1. Since $(x-2)^4 = (x+(-2))^4$, we identify $a = x$, $b = -2$ and $n = 4$ and obtain

$$
\begin{aligned}
(x-2)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j}(-2)^j \\
&= \binom{4}{0} x^{4-0}(-2)^0 + \binom{4}{1} x^{4-1}(-2)^1 + \binom{4}{2} x^{4-2}(-2)^2 + \binom{4}{3} x^{4-3}(-2)^3 + \binom{4}{4} x^{4-4}(-2)^4 \\
&= x^4 - 8x^3 + 24x^2 - 32x + 16
\end{aligned}
$$

2. At first this problem seem misplaced, but we can write $2.1^3 = (2+0.1)^3$. Identifying $a = 2$, $b = 0.1$ and $n = 3$, we get

$$
\begin{aligned}
(2+0.1)^3 &= \sum_{j=0}^{3} \binom{3}{j} 2^{3-j}(0.1)^j \\
&= \binom{3}{0} 2^{3-0}(0.1)^0 + \binom{3}{1} 2^{3-1}(0.1)^1 + \binom{3}{2} 2^{3-2}(0.1)^2 + \binom{3}{3} 2^{3-3}(0.1)^3 \\
&= 8 + 1.2 + 0.06 + 0.001 \\
&= 9.261
\end{aligned}
$$

3. Identifying $a = 2x$, $b = y$ and $n = 5$, the Binomial Theorem gives

$$
(2x+y)^5 = \sum_{j=0}^{5} \binom{5}{j} (2x)^{5-j} y^j
$$

Since we are concerned with only the term containing x^3, there is no need to expand the entire sum. The exponents on each term must add to 5 and if the exponent on x is 3, the exponent on y must be 2. Plucking out the term $j = 2$, we get

$$
\binom{5}{2} (2x)^{5-2} y^2 = 10(2x)^3 y^2 = 80x^3y^2
$$

$\square$

An important application of binomial coefficients is computing probabilities using the eponymous *binomial distribution*. Suppose an experiment has a probability p of 'success' and a probability of $1-p$ of 'failure.'[4] For instance, suppose we roll a 'fair' six-sided die. Let us say a 'success' is rolling a four. Then the probability here of a success is $p = \frac{1}{6}$ while the probability of failure here, or *not* rolling a four, is $1 - \frac{1}{6} = \frac{5}{6}$. If we run this experiment n times, then the probability of *exactly* j successes is given by $\binom{n}{j} p^j (1-p)^{n-j}$.

[4] In other words, there are just two possible outcomes: success or failure, and the fact these probabilities add to 1 means one or the other, but not both, will happen. This situation is called a *Bernoulli Trial*.

Here, the binomial coefficient counts the number of ways we can produce j successes out of n trials. The 'bi' in 'binomial' comes from the fact that each trial produces one of two outcomes: a 'success' (with a probability of p) or 'failure' (with probability $1 - p$).

So, for instance, if we roll the fair die 5 times, the probability we get *exactly* 2 fours is:

$$\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{5-2} = \frac{625}{3888} \approx 16\%$$

Moreover, the probability $\binom{n}{j}p^j(1-p)^{n-j}$ is the jth term in the binomial expansion of $((1-p)+p)^n = 1^n = 1$. That is,

$$1 = 1^n = ((1-p)+p)^n = \sum_{j=0}^{n}\binom{n}{j}(1-p)^{n-j}p^j$$

The fact that the *sum* of the probabilities of all the possibilities (0 successful trials up through n successful trials) is 1 can be loosely translated as the probability *something* will happen is 100%.

Suppose we wanted to compute the probability of rolling *at least* 2 fours on 5 rolls. To achieve this, we add the probabilities of obtaining exactly 2 fours, 3 fours, 4 fours, and 5 fours. That is, we get a partial sum of the binomial expansion:

$$\begin{aligned}
\sum_{j=2}^{5}\binom{5}{j}\left(\frac{5}{6}\right)^{5-j}\left(\frac{5}{6}\right)^{j} &= \underbrace{\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{5-2}}_{\text{probability of 2 fours}} + \underbrace{\binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{5-3}}_{\text{probability of 3 fours}} \\
&\quad + \underbrace{\binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{5-4}}_{\text{probability of 4 fours}} + \underbrace{\binom{5}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^{5-5}}_{\text{probability of 5 fours}} \\
&= \tfrac{736}{3888} \approx 20\%
\end{aligned}$$

We summarize the properties of the binomial distribution below.

Theorem 1.5. Binomial Distribution: If an experiment has a probability of success of p then the probability of *exactly* j successes in n independent Bernoulli Trials is:

$$\binom{n}{j}p^j(1-p)^{n-j}$$

for $0 \le j \le n$.

The probability of *at least* k successes in n independent Bernoulli Trials is:

$$\sum_{j=k}^{n}\binom{n}{j}p^j(1-p)^{n-j}$$

for $0 \le k \le n$.

We close this section with Pascal's Triangle, named in honor of the mathematician Blaise Pascal. Pascal's Triangle is obtained by arranging the binomial coefficients in the triangular fashion below.

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & & & \searrow\swarrow & & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & & & \searrow\swarrow & & \searrow\swarrow & & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
 & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}
$$

Since $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all whole numbers n, each row of Pascal's Triangle is bookended with 1.

To generate the numbers in the middle of the rows (from the third row onwards), we take advantage of the additive relationship expressed in Theorem 1.3. For instance,

$$\binom{1}{0} + \binom{1}{1} = \binom{2}{1}, \quad \binom{2}{0} + \binom{2}{1} = \binom{3}{1}, \quad \binom{2}{1} + \binom{2}{2} = \binom{3}{2}$$

and so forth. This relationship is indicated by the arrows in the array above.

With these two facts in hand, we can quickly generate Pascal's Triangle in the following way: we start with the first two rows, 1 and 1 1. Each successive row begins and ends with 1 and the middle numbers are generated using Theorem 1.3.

Below we attempt to demonstrate this building process to generate the first five rows of Pascal's Triangle.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ & & \searrow\swarrow & & \\ \boxed{1} & & \underline{1+1} & & \boxed{1} \end{array} \quad\longrightarrow\quad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ & & \searrow\swarrow & & \searrow\swarrow & & \\ \boxed{1} & & \underline{1+2} & & \underline{2+1} & & \boxed{1} \end{array} \quad\longrightarrow\quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \\ \boxed{1} & & \underline{1+3} & & \underline{3+3} & & \underline{3+1} & & \boxed{1} \end{array} \quad\longrightarrow\quad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

To see how we can use Pascal's Triangle to expedite the Binomial Theorem, suppose we wish to expand $(3x - y)^4$. The coefficients we need are $\binom{4}{j}$ for $j = 0, 1, 2, 3, 4$ and are the numbers which form the fifth row of Pascal's Triangle.

Since we know that the exponent of $(3x)$ in the first term is 4 and then decreases by one as we go from left to right while the exponent of $(-y)$ starts at 0 in the first term and then increases by one as we move from left to right, we quickly obtain

$$\begin{aligned} (3x - y)^4 &= (1)(3x)^4 + (4)(3x)^3(-y) + (6)(3x)^2(-y)^2 + 4(3x)(-y)^3 + 1(-y)^4 \\ &= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4 \end{aligned}$$

We would like to stress that Pascal's Triangle is a very quick method to expand an *entire* binomial. If only a term (or two or three) is required, then the Binomial Theorem is definitely the way to go.