University of Lethbridge
Department of Mathematics and Computer Science
23rd April, 2015, 9:00 am - 12:00 pm

MATH 3410 - FINAL EXAM

Last Name:____

First Name:____

Student Number:____

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

Problem	Grade
Definitions	/8
Short Answer I	/9
Short Answer II	/9
Short Answer III	/6
Question 3 #1	/6
Question 3 #2	/6
Question 3 #3	/6
Quesiton 3 #4	/6
Question 4 #1	/8
Question 4 #2	/8
Question 4 #3	/8
Total	/80

1. Define the following terms:

[2] (a) A **subspace** of a vector space.

[2] (b) A **linearly independent** set of vectors.

[2] (c) The **orthogonal complement** of a subspace $U \subseteq V$.

[2] (d) A **generalized eigenvector** of an operator $T \in \mathcal{L}(V)$.

2. Short-answer problems.

[3] (a) Determine the matrix (with respect to the standard basis) of the linear operator $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}))$ given by $(Tp)(x) = 3p(x) - 2p'(x)$.

[3] (b) Given that $B = \{(1, 0, 3, 0), (0, 2, 0, -4)\}$ is an orthogonal basis of a subspace $U \subseteq \mathbb{R}^4$, find the orthogonal projection of $v = (3, 1, -1, 2)$ onto U.

[3] (c) Is the set $U = \{(x, y, z) \in \mathbb{R}^3 : xy = z\}$ a subspace of $\mathbb{R}^3$? Why or why not?

[3] (d) Suppose U and W are both 5-dimensional subspaces of $\mathbb{R}^9$. Is it possible to have $U \cap W = \{0\}$? Explain.

[3] (e) Is it possible to have a set of six linearly independent polynomials of degree four or less? Why or why not?

[3] (f) Suppose $T \in \mathcal{L}(\mathbb{R}^5)$ and you know that $\dim E(8, T) = 4$. Explain why at least one of $T - 2I$ or $T - 6I$ has to be invertible.

[3] (g) Suppose that T is a normal operator on V, and you have vectors $v, w \in V$ such that $\|v\| = \|w\| = 2$, $Tv = 3v$, and $Tw = 4w$. Show that $\|T(v+w)\| = 10$.

[3] (h) Give an example of an operator on $\mathbb{R}^4$ whose characteristic polynomial is equal to $(x-1)(x-5)^3$ and whose minimal polynomial is $(x-1)(x-5)^2$.

3. Solve **four** of the following five problems on the pages that follow.

[6] (a) Suppose that V is finite-dimensional and $T \in \mathcal{L}(V)$ is a self-adjoint operator whose only eigenvalues are 2 and 3. Prove that $T^2 - 5T + 6I = 0$.

[6] (b) Suppose $T \in \mathcal{L}(V, W)$ and $\{v_1, \ldots, v_m\}$ is a set of vectors in V such that the vectors $Tv_1, \ldots, Tv_m$ are linearly independent in W. Prove that the vectors $v_1, \ldots, v_m$ are linearly independent in V.

[6] (c) Suppose $T \in \mathcal{L}(V)$ and v is an eigenvector of T with eigenvalue λ. Prove that for any polynomial $p \in \mathcal{P}(\mathbb{F})$, $p(T)v = p(\lambda)v$.

[6] (d) Let V be a real inner product space. Prove that for all $u, v \in V$,

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

[6] (e) Let $T \in \mathcal{L}(V)$ and let U be a subspace of V. Prove that if U is invariant under T, then $U^\perp$ is invariant under T^*.

Space for first problem from Question 3

Space for second problem from Question 3

Space for third problem from Question 3

Space for fourth problem from Question 3

4. **There are six problems below**. Three are theoretical, and three are computational. You may choose any **three** of these problems to solve. They are worth 8 points each.

   (a) Suppose that V and W are finite-dimensional. Prove that there exists an injective linear map $T : V \to W$ if and only if $\dim V \leq \dim W$.

   (b) Suppose that V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct nonzero eigenvalues of T. Prove that

   $$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \operatorname{range} T.$$

   (c) Suppose $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ is invertible. Prove that T and $S^{-1}TS$ have the same eigenvalues with the same multiplicities.

   (d) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

   $$T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + 4x_4, 3x_1 + 2x_3 - x_4, -x_1 - 4x_2 - 2x_3 + 9x_4).$$

   i. Compute the matrix of T with respect to the standard bases of $\mathbb{R}^4$ and $\mathbb{R}^3$.
   ii. Determine a basis for both $\operatorname{null} T$ and $\operatorname{range} T$.

   (e) Let U be the subspace of $\mathbb{R}^4$ defined by

   $$U = \{(x_1, x_2, x_3, x_4) : x_1 = x_3 \text{ and } x_2 + 2x_3 - 3x_4 = 0\}.$$

   i. Determine a basis for U.
   ii. Find a basis for the orthogonal complement $U^\perp$.
   iii. Find an orthonormal basis for $U^\perp$.

   (f) Let $T \in \mathcal{L}(\mathbb{R}^3)$ be the operator whose matrix with respect to the standard basis is given by

   $$\mathcal{M}(T) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}.$$

   Find a Jordan basis for T, the characteristic and minimal polynomials of T, and the Jordan Canonical Form of T.

Space for your first problem from Question 4

Space for your second problem from Question 4

Space for your third problem from Question 4

5. **Bonus problems:** If you have time left and you'd like some bonus points, I'll give a 5% bonus for correct solutions to each of the following:

   (a) Let U be a subspace of a finite-dimensional vector space V. Prove that $P_{U^\perp} = 1 - P_U$, where P_U and $P_{U^\perp}$ denote orthogonal projection onto U and $U^\perp$, respectively.

   (b) Suppose $T \in \mathcal{L}(V)$ and u, v are eigenvectors of T such that $u+v$ is also an eigenvector. Prove that u and v are eigenvectors corresponding to the same eigenvalue. Deduce from this that if every nonzero $v \in V$ is an eigenvector, then T is a scalar multiple of the identity.

Extra space for rough work. Do not remove this page if you want it to be graded.