Name and student number: Solutions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. If $A = [0, 2]$ and $B = [-1, 1]$, compute:

[2] (a) $f(A \cap B)$

We have $A \cap B = [0, 2] \cap [-1, 1] = [0, 1]$, and

$$f([0, 1]) = \{f(x) | x \in [0, 1]\} = \{x^2 | 0 \leq x \leq 1\} = [0, 1].$$

[2] (b) $f(A) \cap f(B)$

We have $f(A) = \{x^2 | 0 \leq x \leq 2\} = [0, 4]$ and $f(B) = \{x^2 | -1 \leq x \leq 1\} = [0, 1]$, so

$$f(A) \cap f(B) = [0, 4] \cap [0, 1] = [0, 1].$$

[2] (c) $f^{-1}(f(A))$

From (b) we have that $f(A) = [0, 4]$, so

$$f^{-1}(f(A)) = f^{-1}([0, 4]) = \{x \in \mathbb{R} | 0 \leq x^2 \leq 4\} = [-2, 2].$$

Total: 10 points

2. Let $f : A \to B$ be a given function. We know that for any subset $C \subseteq A$, $C \subseteq f^{-1}(f(C))$, since if $x \in C$, then $f(x) \in f(C) = \{f(c) | c \in C\}$, and thus

$$x \in f^{-1}(f(C)) = \{x \in A | f(x) \in f(C)\}.$$

[4] Prove that if f is one-to-one, then the reverse inclusion holds; that is, $f^{-1}(f(C)) \subseteq C$.

Hint: Suppose $x \in f^{-1}(f(C))$, and let $y = f(x)$. If $y \in f(C)$, that doesn't immediately guarantee that $x \in C$, but it does guarantee that there is some $c \in C$ such that $f(c) = y$.

Proof: Suppose $x \in f^{-1}(f(C)) = \{x \in A | f(x) \in C\}$. Then $f(x) \in f(C)$, so there exists some $c \in C$ such that $f(c) = f(x)$. Since f is one-to-one, it follows that $c = x$ and thus $x \in C$.

Since $x \in C$ whenever $x \in f^{-1}(f(C))$, it follows that $f^{-1}(f(C)) \subseteq C$.