University of California, Berkeley
Department of Mathematics
5^{th} October, 2012, 12:10-12:55 pm
MATH 53 - Test #1

Last Name:____

First Name:____

Discussion Section: ____________________________________

Name of GSI: ______________________________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

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C

[4] 1. Find the equation of the tangent line to the curve C represented by the vector-valued function $\mathbf{r}(t) = \langle t^3, \sin(\pi t), 2t+1\rangle$ at the point $(1,0,3)$.

The point $(1,0,3)$ corresponds to $t=1$, and we have $\mathbf{r}'(t) = \langle 3t^2, \pi\cos\pi t, 2\rangle$, so the tangent vector to the curve at $(1,0,3)$ is $\mathbf{r}'(1) = \langle 3, -\pi, 2\rangle$. The equation of the tangent line is therefore

$$\langle x, y, z\rangle = \langle 1, 0, 3\rangle + t\langle 3, -\pi, 2\rangle.$$

[5] 2. Find the area of the cardioid $r = 1+\cos\theta$.

The cardioid is traced out for $0 \le \theta \le 2\pi$, so its area is given by

$$\begin{aligned}
A &= \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta \\
&= \int_0^{2\pi} \left(\frac{3}{4} + \cos\theta + \frac{1}{4}\cos 2\theta\right) d\theta \\
&= \left[\frac{3}{4}\theta + \sin\theta + \frac{1}{8}\sin 2\theta\right]_0^{2\pi} \\
&= \frac{3\pi}{2}.
\end{aligned}$$

[3] 3. Evaluate $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^3-y^3}{x-y}$, or explain why it does not exist.

Since $x^3 - y^3 = (x-y)(x^2+xy+y^2)$, we have

$$\lim_{(x,y)\to(1,1)} \frac{x^3-y^3}{x-y} = \lim_{(x,y)\to(1,1)} (x^2+xy+y^2) = 3$$

by direct substitution, since polynomial functions are continuous.

4. Consider the two lines in $\mathbb{R}^3$ given by

$$\mathbf{r}_1(t) = \langle 1, 6, 1\rangle + t\langle 0, 4, -2\rangle$$
$$\mathbf{r}_2(s) = \langle 0, 3, 0\rangle + s\langle 1, -1, 3\rangle.$$

[2] (a) Verify that the two lines intersect at the point $(1, 2, 3)$.

Since $\mathbf{r}_1(-1) = \langle 1, 2, 3\rangle$ and $\mathbf{r}_2(1) - \langle 1, 2, 3\rangle$, the two curves intersect at $(1, 2, 3)$ by direct computation.

[3] (b) Find the cosine of the angle between the two lines.

The direction vectors for the two lines are $\mathbf{v}_1 = \langle 0, 4, -2\rangle$ and $\mathbf{v}_2 = \langle 1, -1, 3\rangle$, so the angle between the two lines at their point of intersection is given by

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} = \frac{0(1) + 4(-1) - 2(3)}{\sqrt{16+4}\sqrt{1+1+9}} = \frac{10}{\sqrt{20}\sqrt{11}}.$$

[4] (c) Find the equation of the plane that contains the two lines.

We know that the point $(1, 2, 3)$ lies on the plane, and a normal vector is given by

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4(3) - (-2)(-1), -2(1) - 0(3), 0(-1) - 4(1)\rangle = \langle 10, -2, -4\rangle.$$

The equation of the plane is therefore $10(x-1) - 2(y-2) - 4(z-3) = 0$, or $10x - 2y - 4z + 6 = 0$.

[3] (d) Find the distance between the point $P(3, 4, -2)$ and the plane from part (c).

The distance between a point $P(x_1, y_1, z_1)$ and the plane $ax + by + cz + d = 0$ is given by $D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$, so the distance is equal to

$$D = \frac{|10(3) - 2(4) - 4(-2) + 6|}{\sqrt{100 + 4 + 16}} = \frac{36}{\sqrt{120}}.$$

5. (a) Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + 2y^2 + z^2}$ at the point $(4, 5, 6)$. (Note: $4^2 + 2(5^2) + 6^2 = 100 = 10^2$.)

[5]

The partial derivatives of f are given by

$$f_x(x,y,z) = \frac{x}{\sqrt{x^2+2y^2+z^2}},\ f_y(x,y,z) = \frac{2y}{\sqrt{x^2+2y^2+z^2}},\ f_z(x,y,z) = \frac{z}{\sqrt{x^2+2y^2+z^2}},$$

so at the point $(4, 5, 6)$ we have $f_x(4,5,6) = \frac{4}{10} = \frac{2}{5}$, $f_y(4,5,6) = \frac{10}{10} = 1$, and $f_z(4,5,6) = \frac{6}{10} = \frac{3}{5}$, so the function $L(x, y, z)$ that gives the desired linear approximation is

$$\begin{aligned} L(x,y,z) &= f(4,5,6) + f_x(4,5,6)(x-4) + f_y(4,5,6)(y-5) + f_z(4,5,6)(z-6) \\ &= 10 + \frac{2}{5}(x-4) + (y-5) + \frac{3}{5}(z-6). \end{aligned}$$

(b) Use your result from part (a) to approximate the value of $\sqrt{(4.1)^2 + 2(4.95)^2 + (6.03)^2}$

[2]

Since $(4.1, 4.95, 6.03)$ is close to $(4, 5, 6)$, we have $f(4.1, 4.95, 6.03) \approx L(4.1, 4.95, 6.03)$, and thus,

$$\begin{aligned} \sqrt{(4.1)^2 + 2(4.95)^2 + (6.03)^2} &\approx 10 + \frac{2}{5}(0.1) + 1(-0.05) + \frac{3}{5}(0.03) \\ &= 10 + 0.04 - 0.05 + 0.018 = 10.008. \end{aligned}$$

6. Use the chain rule to compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ if $f(x, y, z) = xe^{y^2 z}$, where $x = 2uv$, $y = u^2 - v^2$, and $z = u$.

[5]

We have

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \\ &= e^{y^2 z}(2v) + 2xyze^{y^2 z}(2u) + xy^2 e^{y^2 z}(1) \\ &= 2ve^{(u^2-v^2)^2 u} + 8u^3 v(u^2 - v^2)e^{(u^2-v^2)^2 u} + 2uv(u^2-v^2)^2 e^{(u^2-v^2)^2 u}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} \\ &= e^{y^2 z}(2u) + 2xyze^{y^2 z}(-2v) + xy^2 e^{y^2 z}(0) \\ &= 2ue^{(u^2-v^2)^2 u} - 8u^2 v^2 (u^2 - v^2)e^{(u^2-v^2)^2 u}. \end{aligned}$$