University of Lethbridge
Department of Mathematics and Computer Science
10^{th} November 2014, 2:00-2:50 pm
MATH 3500 - Test #2

Last Name:**Solutions**

First Name:**The**

Student Number:

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

Page	Grade
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3	/10
4	/10
5	/12
Total	/40

1. Let (a_n) be the sequence defined by $a_n = \cos\left(\frac{n\pi}{3}\right)$ for $n = 1, 2, 3, \ldots$.

[2] (a) Recall that $|\cos x| \leq 1$ for all $x \in \mathbb{R}$. Explain why this guarantees that (a_n) has a covergent subsequence.

The fact that $|\cos x| \leq 1$ tells us that (a_n) is bounded, and the Bolzano-Weierstrass theorem guarantees that every bounded sequence has a convergent subsequence.

[4] (b) Find the set of subsequential limits of (a_n). (That is find the set of limits of convergent subsequences. It might help to recall that $\cos(\pi/3) = 1/2$.)

We have $(a_n) = (1/2, -1/2, -1, -1/2, 1/2, 1, 1/2, -1/2, -1, -1/2, 1/2, 1, \ldots)$, so

$$S = \{1, -1, 1/2, -1/2\}.$$

[2] (c) What are the values of $\limsup a_n$ and $\liminf a_n$?

We have

$$\limsup a_n = \sup S = 1,$$

and

$$\liminf a_n = \inf S = -1.$$

[5] 2. Use the $\epsilon - \delta$ definition of the limit to prove that $\lim_{x \to 2} \frac{x+1}{2x-1} = 1$.

Let $\epsilon > 0$ be given, and take $\delta = \min\{1, \epsilon\}$. If $0 < |x-2| < \delta$, then we have

$$|x-2| < 1 \Rightarrow -1 < x-2 < 1 \Rightarrow 1 < x < 3 \Rightarrow 1 < 2x-1 < 5,$$

so $\frac{1}{5} < \frac{1}{2x-1} < 1$, and thus

$$|f(x) - L| = \left| \frac{x+1}{2x-1} - 1 \right| = \left| \frac{2-x}{2x-1} \right| < |x-2| < \delta \leq \epsilon.$$

3. Use the $\epsilon - \delta$ definition of uniform continuity to prove that $f(x) = \frac{x}{x+1}$ is continuous on
[5] $[0, 2]$.

Let $\epsilon > 0$ be given, and take $\delta = \epsilon$. Notice that if $x \in [0,2]$, then $1 \leq x+1 \leq 3$, so $0 < \frac{1}{x+1} \leq 1$.

If $|x-y| < \delta$, then we have

$$|f(x) - f(y)| = \left| \frac{x}{x+1} - \frac{y}{y+1} \right| = \left| \frac{x-y}{(x+1)(y+1)} \right| < |x-y|(1)(1) < \delta = \epsilon.$$

4. Decide whether or not the given function f is uniformly continuous on its domain D. Justify your answer in each case with a suitable theorem.

[2] (a) $f(x) = x^2 + 2x$ on $D = [0, 3]$

Since f is a polynomial it is continuous on D, and since D is compact, f is uniformly continuous on D.

[2] (b) $f(x) = 1/x^2$ on $D = (0, 1]$

Since $\lim_{x\to 0^+} f(x) = \infty$, f is not bounded on D and thus cannot be uniformly continuous on D.
Also acceptable is to note that since $f(x) \to \infty$ as $x \to 0^+$, f cannot be extended to a continuous function on $[0, 1]$.

[2] (c) $f(x) = \dfrac{1}{x}\sin^2 x$ on $D = (0, \pi)$.

Since $\dfrac{\sin^2 \pi}{\pi} = 0$ and

$$\lim_{x\to 0^+} \frac{\sin^2 x}{x} = \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)(\sin x) = (1)(0) = 0,$$

we can extend f to the continuous function $\tilde{f}$ on $[0, \pi]$ given by $\tilde{f}(x) = f(x)$ for $x \in (0, \pi)$ and $\tilde{f}(0) = \tilde{f}(\pi) = 0$. Thus, f is uniformly continuous on $(0, \pi)$.

5. Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function.

[2] (a) Is it possible for the range of f to equal $[0, 1] \cup [2, 3]$? Why or why not?

This is not possible, since if the range of f contains 1 and 2, then by the Intermediate Value Theorem it must contain every $x \in (1, 2)$, but the interval $(1, 2)$ is not contained in the range of f.

[2] (b) Is it possible for the range of f to equal either $(0, 1)$ or $[1, \infty)$? Why or why not?

This is not possible, since if f is continuous, then $f([a, b])$ must be compact, but neither $(0, 1)$ nor $[1, \infty)$ are compact.

6. Let $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$.

[4] (a) Show that f is discontinuous at every point $a \neq 0$. (Hint: if a is rational/irrational, consider a sequence of irrational/rational numbers converging to a.)

Suppose $a \in \mathbb{Q}$ and $a \neq 0$. Let (a_n) be a sequence of irrational numbers converging to a. Then $a^2 = f(a) = f(\lim a_n)$, but $\lim f(a_n) = \lim 0 = 0 \neq a^2$, so f cannot be continuous at a.

Similarly, if $a \notin \mathbb{Q}$, let (a_n) be a sequence of rational numbers converging to a. Then $0 = f(a) = f(\lim a_n)$, but $\lim f(a_n) = \lim a_n^2 = a^2 \neq 0$, and again f cannot be continuous at a.

[4] (b) Prove that f is differentiable (and hence continuous) at $x = 0$.

We note that for any $x \neq 0$, we have

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x)}{x}\right| \leq |x|,$$

since either $f(x) = 0$ or $f(x) = x^2$. In either case, given $\epsilon > 0$, if we choose $\delta = \epsilon$, then whenever $0 < |x| < \delta$ we have $|f(x)/x| < \epsilon$, and thus $f'(0) = \lim_{x\to 0} \dfrac{f(x)}{x} = 0$ exists.

7. Prove that if $f'(x) \neq 0$ for all $x \in (a, b)$, then f is either strictly increasing or strictly
[4] decreasing on (a, b). (Caution: f' is not guaranteed to be continuous.)

If $f'(x) \neq 0$ on (a, b), then we must either have $f'(x) > 0$ for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$, since if f' is both positive and negative on (a, b), then Darboux's theorem would guarantee the existence of some $c \in (a, b)$ such that $f'(c) = 0$.

If $f'(x) > 0$ on (a, b), then the Mean Value Theorem guarantees that f is strictly increasing on (a, b), as discussed in class; for if $x, y \in (a, b)$ with $x < y$, then there exists some $c \in (a, b)$ such that

$$f(x) - f(y) = f'(c)(x - y) < 0,$$

since $f'(c) > 0$ and $x - y < 0$, and thus $f(x) < f(y)$. A similar argument shows that if $f'(x) < 0$ on (a, b), then f is strictly decreasing on (a, b).