1. Determine the rank of each of the following matrices:

(a) $A = \begin{bmatrix} 2 & -3 & 1 & 4 \\ -1 & 3 & 5 & -7 \\ 1 & 0 & 6 & -3 \end{bmatrix}$

The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 0 & 6 & -3 \\ 0 & 1 & 11/3 & -10/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are two leading ones, we have $\text{rank}(A) = 2$.

(b) $B = \begin{bmatrix} 2 & 6 \\ 5 & -3 \\ 3 & 2 \end{bmatrix}$

The reduced row-echelon form of B is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since there are two leading ones, we have $\text{rank}(B) = 2$.

2. Determine the basic solutions of the homogeneous system of equations

$$\begin{array}{rcrcrcrcl} 2x_1 & - & 3x_2 & & & - & 4x_4 & = & 0 \\ -x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 0 \\ 3x_1 & - & 4x_2 & - & x_3 & - & 5x_4 & = & 0 \end{array}$$

The system can be written in the form $A\vec{x} = \vec{0}$, where $A = \begin{bmatrix} 2 & -3 & 0 & -4 \\ -1 & 2 & -3 & 3 \\ 3 & -4 & -1 & -5 \end{bmatrix}$. The reduced row-echelon form of the augmented matrix $[\ A \mid 0\ ]$ is

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & \end{array}\right]$$

From the reduced row-echelon form, we can see that $x_3 = s$ and $x_4 = t$ are free parameters, while $x_1 - 3x_3 + x_4 = 0$ and $x_2 - 2x_3 + 2x_4 = 0$. Solving for x_1 and x_2, we have $x_1 = 3x_3 - x_4 = 3s - t$, and $x_4 = 2x_3 - 2x_4 = 2s - 2t$. Writing our solution in vector form, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s - t \\ 2s - 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix},$$

so our basic solutions are $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

3. Determine whether or not the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

are linearly independent.

We recall that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent provided that the only solution to the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

is $x_1 = 0, x_2 = 0, x_3 = 0$. Our vector equation can be re-written as a system of linear equations with augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & 4 & 5 & 0 \end{array}\right].$$

The reduced row-echelon form of this matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$, which shows that we have the unique solution $x_1 = 0, x_2 = 0, x_3 = 0$, and thus, our vectors are linearly independent.

4. Determine whether or not $\vec{w} \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \\ -8 \\ 6 \end{bmatrix}.$$

The vector $\vec{w}$ belongs to the span of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ provided that there exist scalars x_1, x_2, x_3 such that

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{w}.$$

This vector equation leads to a system of linear equations with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & -4 & -8 \\ -1 & -3 & 1 & 6 \end{array}\right].$$

The reduced row-echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -26 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since a solution exists, we can conclude that $\vec{w} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Indeed, we have the unique solution $x_1 = -26, x_2 = 3, x_3 = -11$, and we can verify that

$$-26\vec{v}_1 + 3\vec{v}_2 - 11\vec{v}_3 = -26\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix} - 11\begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -8 \\ 6 \end{bmatrix} = \vec{w}.$$

Note for Problem 2: Since the null space of A is defined to be the set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$, we see that

$$\operatorname{null}(A) = \left\{ \begin{bmatrix} 3s - t \\ 2s - 2t \\ s \\ t \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2\},$$

and thus, the set $\{\vec{v}_1, \vec{v}_2\}$ forms a basis for the null space of A. (The vectors $\vec{v}_1$ and $\vec{v}_2$ are not multiples of each other, so this set is clearly linearly independent, in addition to spanning the null space.)

According to Theorem 29 in the textbook, a basis for the column space of A is given by the columns of A in which we find leading ones in the RREF of A. Since we had leading ones in the first and second columns, it follows that

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \right\}$$

is a basis for the column space of A. (Recall that the column space of A is defined to be the span of the columns of A. The above result tells us that the first two columns of A are sufficient to generate this span. Consequently, the third and fourth columns must both be linear combinations of the first two. Recall also that if $\vec{y} = A\vec{x}$ for some vector $\vec{x}$, then $\vec{y}$ can be written as a linear combination of the columns of A; that is, the column space of A determines the *range* of the matrix transformation $T(\vec{x}) = A\vec{x}$.)