Solutions Quiz 18 Practice Problems
Math 2580
Spring 2016

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If you can answer the following problems, you should be well-prepared for Quiz 18:

1. Evaluate the integral of the given vector field along the given curve:

   (a) $\mathbf{F}(x,y,z) = \langle x, y, z\rangle$, $\mathbf{r}(t) = \langle \cos t, \sin t, t^2\rangle$, $t \in [0, 2\pi]$.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t, t^2)\cdot \mathbf{r}'(t)\,dt\\ &= \int_0^{2\pi} \langle \cos t, \sin t, t^2\rangle \cdot \langle -\sin t, \cos t, 2t\rangle\,dt\\ &= \int_0^{2\pi} 2t^3\,dt = 8\pi^4.\end{aligned}$$

   (b) $\mathbf{F}(x,y,z) = \langle yz, xz, xy\rangle$, $\mathbf{r}(t) = \langle 3t, t^2, t^3\rangle$, $t \in [0, 1]$.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{1} \mathbf{F}(3t, t^2, t^3\rangle\cdot \mathbf{r}'(t)\,dt\\ &= \int_0^{1} \langle t^5, 3t^4, 3t^3\rangle \cdot \langle 3, 2t, 3t^2\rangle\,dt\\ &= \int_0^{1} 18t^5\,dt = 3.\end{aligned}$$

   (c) $\mathbf{F}(x,y,z) = (x^2+x)\mathbf{i} + \dfrac{x-y}{x+y}\mathbf{j} + (z-z^3)\mathbf{k}$, $\mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j} + t^2\mathbf{k}$, $t \in [0,1]$.

   This line integral is actually undefined: in the y-component of $\mathbf{F}$, notice that $x+y = 3t+(t-1) = 4t-1$ is in the denominator, and $4t-1 = 0$ for $t = \frac{1}{4} \in [0,1]$.

2. Evaluate the integral of the given function (scalar field) along the given curve:

   (a) $f(x, y, z) = xy^3$, $x = 4\sin t$, $y = 4\cos t$, $z = 3t$, $0 \le t \le \pi/2$.

   We have $\mathbf{r}'(t) = \langle 4\cos t, -4\sin t, 3\rangle$, so $\|\mathbf{r}'(t)\| = \sqrt{16\cos^2 t + 16\sin^2 t + 9} = 5$, and
$$f(\mathbf{r}(t)) = f(4\sin t, 4\cos t, 3t) = 4\sin t(4\cos t)^3 = 256\sin t\cos^3 t,$$
   so
$$\int_C f\,ds = \int_0^{2\pi} f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\;dt = \int_0^{2\pi} (256\sin t\cos^3 t)(5)\,dt = 0.$$

   (b) $f(x, y, z) = xe^{yz}$, along the line segment from $(0, 0, 0)$ to $(1, 2, 3)$.
   First, we parameterize the line segment using $\mathbf{r}(t) = \langle t, 2t, 3t\rangle$, with $t \in [0, 1]$, so $\|\mathbf{r}'(t)\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$. Thus
$$\int_C f\,ds = \int_0^1 f(t, 2t, 3t)\,\|\mathbf{r}'(t)\|\;dt = \int_0^1 te^{6t^2}(\sqrt{14})\,dt = \frac{\sqrt{14}}{12}(e^6 - 1).$$

3. Determine if the given vector field is conservative. If it is, find a function f such that $\mathbf{F} = \nabla f$:

   (a) $\mathbf{F}(x, y) = (6x + 5y)\mathbf{i} + (5x + 4y)\mathbf{j}$

   For this and the remaining problems, we use the fact that if the domain of $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is all of $\mathbb{R}^2$, then a necessary and sufficient condition for $\mathbf{F}$ to be conservative is $P_y(x, y) = Q_x(x, y)$. For this problem, we have $P_y(x, y) = 5 = Q_x(x, y)$, so $\mathbf{F}$ must be conservative. If $\mathbf{F} = \nabla f$ for some function f, then we must have $f_x(x, y) = P(x, y) = 6x + 5y$, so $f(x, y) = 3x^2 + 5xy + g(y)$ for (possibly) some function g of y only. Now, on the one hand we have
$$f_y(x, y) = \frac{\partial}{\partial y}(3x^2 + 5xy + g(y)) = 5x + g'(y),$$
   while on the other hand, $f_y(x, y) = Q(x, y) = 5x + 4y$. Comparing these two, we see that we must have $g'(y) = 4y$, so we can take $g(y) = 2y^2$, and thus
$$f(x, y) = 3x^2 + 5xy + 2y^2.$$
   (We could also add a constant, but this is unnecessary: note that the question asked for *a* function, not *all* functions.)

   (b) $\mathbf{F}(x, y) = (x^3 + 4xy)\mathbf{i} + (4xy - y^3)\mathbf{j}$

   We have $P_y(x, y) = 4x$ and $Q_x(x, y) = 4y$. Since $P_y \ne Q_x$, the vector field $\mathbf{F}$ cannot be conservative.

(c) $\mathbf{F}(x,y) = e^y\mathbf{i} + xe^y\mathbf{j}$.

We have $P_y(x,y) = e^y = Q_x(x,y)$, so $\mathbf{F}$ is a conservative vector field. If $\mathbf{F} = \nabla f$ for some function f, then we must have

$$f_x(x,y) = P(x,y) = e^y, \quad \text{so} \quad f(x,y) = xe^y + g(y)$$

for some function $g(y)$ of y only. Then we have $f_y(x,y) = xe^y + g'(y) = xe^y = Q(x,y)$, which tells us that $g'(y) = 0$, so we can take $g(y) = 0$ and $f(x,y) = xe^y$.

(d) $\mathbf{F}(x,y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$

We have $P_y(x,y) = e^x + \cos y = Q_x(x,y)$, so $\mathbf{F}$ is conservative. If $\mathbf{F} = \nabla f$ for some function f, then we must have

$$f_x(x,y) = e^x + \cos y, \quad \text{so} \quad f(x,y) = e^x + x\cos y + g(y)$$

for some function y, and comparing to $Q(x,y) = e^x + x\cos y = f_y(x,y)$, we see that we can take $g(y) = 0$, and $f(x,y) = e^x + x\cos y$.