1. For each system of equations below, write down the corresponding augmented matrix.

(a)
$$\begin{aligned} 2x - 3y + z &= 2 \\ 2y - 5z &= -3 \\ -3x + 2z &= 7 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 2 \\ 0 & 2 & -5 & -3 \\ -3 & 0 & 2 & 7 \end{array}\right]$$

(b)
$$\begin{aligned} x_1 + 4x_2 - 7x_4 &= 0 \\ -3x_1 - x_2 + 4x_3 &= 2 \\ 2x_2 - 4x_3 + x_4 &= -5 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & -7 & 0 \\ -3 & -1 & 4 & 0 & 2 \\ 0 & 2 & -4 & 1 & -5 \end{array}\right]$$

2. For each augmented matrix below, write down a corresponding system of equations using whatever variables you prefer.

(a) $\left[\begin{array}{ccc|c} 2 & -1 & 0 & 4 \\ -3 & 4 & 1 & -2 \\ 0 & 2 & 3 & -7 \end{array}\right]$

$$\begin{aligned} 2x - y &= 4 \\ -3x + 4y + z &= -2 \\ 2y + 3z &= -7 \end{aligned}$$

(b) $\left[\begin{array}{cccc|c} 3 & 2 & 0 & 1 & -5 \\ 0 & 4 & 2 & -7 & 2 \end{array}\right]$

$$\begin{aligned} 3x_1 + 2x_2 \qquad x_4 &= -5 \\ 4x_2 + 2x_3 - 7x_4 &= 2 \end{aligned}$$

3. State whether or not the given augmented matrix is in reduced row-echelon form, and if not, why.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \left[\begin{array}{cccc|c} 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

From left to right: The first matrix is in RREF.
The second is not in REF due the 4 in the last row.
The third matrix is in REF, but not RREF, due to the 2 above the leading 1 in the second column.
The fourth matrix is not in REF due to the 2 in the second row.
The last matrix is in REF, but not RREF, due to the non-zero entries above the leading 1 in the fourth column.

4. The reduced row-echelon form of a system of equations in the variables x, y, and z is given. State the solution (if any) to the system.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Unique solution:
$x = 5$
$y = -3$
$z = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinitely many solutions:
$x = 1 + 2t$
$y = t - 3$
$z = t$ is a free parameter

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

There is no solution, since $0x + 0y + 0z = 0 \neq 1$ for all x, y, z.

5. Solve the following system of equations:

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & 3 \\ x & + & 2y & - & z & = & 4 \\ -x & + & y & - & 2z & = & -1 \end{array}$$

We form the corresponding augmented matrix, and reduce, as follow:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 1 & 2 & -1 & 4 \\ -1 & 1 & -2 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & -1 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -5 \\ -1 & 1 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -5 \\ 0 & 3 & -3 & 3 \end{array}\right] \xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -3 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This last augmented matrix is in reduced row-echelon form. Since there is no leading 1 in the z column, we conclude that z is a free variable. Setting $z = t$, where t can be any real number, we have the solution

$$\begin{aligned} x &= 2 - t \\ y &= 1 + t \\ z &= t. \end{aligned}$$

To verify our solution, note that

$$\begin{aligned} 2x - y + 3z &= 2(2 - t) - (1 + t) + 3t = 4 - 2t - 1 - t + 3t = 3 \\ x + 2y - z &= 2 - t + 2(1 + t) - t = 2 - t + 2 + 2t - t = 4 \\ -x + y - 2z &= -(2 - t) + (1 + t) - 2t = -2 + t + 1 + t - 2t = -1. \end{aligned}$$