Math 4310 Assignment #7 Solutions
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1. Let A and B be subsets of a topological space X. Suppose that A is connected, and that B is both open and closed in X. Prove that if $A \cap B \neq \emptyset$, then $A \subseteq B$.

   [Hint: if $A \nsubseteq B$, consider $U = A \cap B$ and $V = A \cap B^c$.]

   Solution: Suppose A is connected and $B \subseteq X$ is both open and closed, with $A \cap B \neq \emptyset$. Since B is open in X, $U = A \cap B \neq \emptyset$ is open in A. Since B is closed, $V = A \cap B^c$ is also open in A, and $U \cap V = \emptyset$. Since A is connected, we must have $V = \emptyset$, or else $\{U, V\}$ would be a separation of A. Thus $A \cap B^c = \emptyset$, which is equivalent to $A \subseteq B$.

2. Show that if X and Y are connected topological spaces, then $X \times Y$ is connected.

   [Hint: suppose $f : X \times Y \to \{0, 1\}$ is continuous and nonconstant. Then there are points $(x_0, y_0), (x_1, y_1) \in X \times Y$ with $f(x_0, y_0) = 0$ and $f(x_1, y_1) = 1$. Note that either $f(x_0, y_1) = 0$ or $f(x_0, y_1) = 1$. In the first case, consider the map $i_{y_1} : X \to X \times Y$ given by $i_{y_1}(x) = (x, y_1)$. In the second case, consider i_{y_0}.]

   Solution: Suppose $f : X \times Y \to \{0, 1\}$ is continuous and nonconstant. Then there exist points $(x_0, y_0), (x_1, y_1) \in X \times Y$ with $f(x_0, y_0) = 0$ and $f(x_1, y_1) = 1$. Now consider the point $(x_0, y_1) \in X \times Y$. If $f(x_0, y_1) = 0$, define $g : X \to \{0, 1\}$ by $g = f \circ \iota_{y_1}$, where $\iota_{y_1} : X \to X \times Y$ is given by $\iota_{y_1} = (x, y_1)$. Since ι_{y_1} is continuous and f is continuous, it follows that $g : X \to \{0, 1\}$ is continuous, and $g(x_0) = 0$, while $g(x_1) = 1$. But this is impossible, since X is connnected.

   Similarly, if $f(x_0, y_1) = 1$, the function $h = f \circ \iota_{x_0} : Y \to \{0, 1\}$, where $\iota_{x_0} : Y \to X \times Y$ is given by $\iota_{x_0}(y) = (x_0, y)$, is a continuous, nonconstant function, and this is also impossible, since Y is connected. Thus, $X \times Y$ must be connected.

3. Prove that a topological space X is connected if and only if $\partial A \neq \emptyset$ for every proper nonempty subset $A \subseteq X$.

   [Hint: you might find it easier to prove the contrapositive in both directions, and you proved a result on an earlier assignment that will be helpful.]

Solution: One option is to recall that for any proper nonempty subset $A \subseteq X$ we can write X as the disjoint union $X = \overset{\circ}{A} \cup \partial A \cup (X \setminus A)^\circ$. If $\partial A = \emptyset$ then $\{\overset{\circ}{A}, (X \setminus A)^\circ\}$ is a separation of X. Conversely, if X is not connected and $\{A, B\}$ is a separation of X, then for any $x \in X$, either $x \in A$ and there is a neighbourhood of x that doesn't intersect B, or vice versa.

Another option is to note that for any $A \subseteq X$ we have $\overset{\circ}{A} \subseteq A \subseteq \overline{A}$, and $\partial A = \overline{A} \setminus \overset{\circ}{A}$. Thus X is not connected if and only if there exists a proper nonempty subset $A \subseteq X$ that is both open and closed, and A is both open and closed in X if and only if $A = \overset{\circ}{A}$ and $A = \overline{A}$, which is if and only if $\partial A = \emptyset$.

4. Prove that if A and B are path-connected subsets of a topological space X and $A \cap B \neq \emptyset$, then $A \cup B$ is path-connected. Conclude that for any finite collection $\{A_1, \ldots, A_n\}$ of path connected subsets of X, with $A_i \cap A_j \neq \emptyset$, $\bigcup_{i=1}^n A_i$ is path-connected.

   Solution: Suppose that A and B are path-connected, and that $A \cap B \neq \emptyset$. Choose any points $x, yin A \cup B$. If x and y both belong to A or both belong to B, then we can find a path from x to y, since A and B are path-connected. Now suppose, without loss of generality, that $x \in A$ and $y \in B$. (If $x \in B$ and $y \in A$ we can simply re-label x and y.) Since $A \cap B \neq \emptyset$, choose some $z \in A \cap B$. Since $z \in A$ there exists a path $\gamma_1 : [0, 1] \to A$ such that $\gamma_1(0) = x$ and $\gamma_1(1) = z$. Since $z \in B$, there exists a path $\gamma_2 : [0, 1] \to B$ such that $\gamma_2(0) = z$ and $\gamma_2(1) = y$. Then the path $\gamma : [0, 1] \to A \cup B$ given by

$$\gamma(t) = \begin{cases} \gamma_1(2t), & \text{if } 0 \leq t \leq 1/2 \\ \gamma_2(2t-1), & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

   satisfies $\gamma(0) = x$ and $\gamma(1) = y$, and γ is continuous since γ_1 and γ_2 are continuous, and $\gamma_1(1) = \gamma_2(0) = z$.

5. Give an example to show that the intersection of two connected subspaces need not be connected. (Consider $\mathbb{R}^2$.)

   Solution: There are many examples. One option is to note that the graph of any continuous function from $\mathbb{R}$ to $\mathbb{R}$ is connected, so in particular, the graphs $A = \{(x, x^2) : x \in \mathbb{R}\}$ and $B = \{(x, 1) : x \in \mathbb{R}\}$ are connected, but the intersection is $A \cap B = \{(-1, 1), (1, 1)\}$, which is clearly not connected.

6. Prove that the space $\mathcal{C}[0, 1]$ of all continuous real-valued functions on $[0, 1]$, equipped with the sup-norm metric (d_∞) is path-connected.

   [Hint: you can show the space is in fact convex.]

   Solution: Let $f, g \in \mathcal{C}[0, 1]$, and define $\gamma : [0, 1] \to \mathcal{C}[0, 1]$ by

$$\gamma(t) = tg + (1-t)f.$$

It's clear that for each $t \in [0,1]$ we have $\gamma(t) \in \mathcal{C}[0,1]$, since any linear combination of continuous functions is continuous. It remains to check that γ is a continuous map. Given $\epsilon > 0$ and $f, g \in \mathcal{C}[0,1]$ with $f \neq g$, choose $\delta = \epsilon/(d_\infty(f,g))$. (We need $f \neq g$ so that $d_\infty(f,g) \neq 0$. If $f = g$ we can take γ to be the constant path, which is clearly continuous.) If $|t - t_0| < \delta$, then we have

$$d_\infty(\gamma(t), \gamma(t_0)) = \|(tg+(1-t)f)-(t_0g+(1-t_0)f)\|_\infty = \|(t-t_0)(g-f)\|_\infty = |t-t_0| d_\infty(f,g) < \epsilon.$$