MATH 1560 - Tutorial #9 Solutions

Additional Practice:

1. Compute the derivative:

   (a) $\frac{d}{dx}(e^x \cos(x)) = e^x(\cos(x) - \sin(x))$ (product rule)

   (b) $\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$

   (c) $\frac{d}{dx}(1 + x^2)^{12} = 24x(1 + x^2)^{11}$ (chain rule)

   (d) $\frac{d}{dx}\frac{e^x}{x} = \frac{e^x(x - 1)}{x^2}$ (quotient rule)

   (e) $\frac{d}{dx} \ln(\sin(x)) = \frac{\cos(x)}{\sin(x)} = \cot(x)$ (chain rule)

   (f) $\frac{d}{dx} 2\sin^4(x) = 8\sin^3(x)\cos(x)$ (chain rule)

2. Evaluate the immediate integral:

   (a) $\int (3x^2 + 1 + \frac{1}{x} + \frac{1}{x^2})\, dx = x^3 + x + \ln|x| - \frac{1}{x} + C$

   (b) $\int x(x^2 + 5)^4\, dx = \frac{1}{10}(x^2 + 5)^5 + C$

3. Given $y(x) = \pi x(50 - x)$, solve $y'(x) = 0$.

   $y(x) = (50x - x^2)$, so $y'(x) = \pi(50 - 2x) = 2\pi(25 - x)$, and $y'(x) = 0$ for $x = 25$.

4. Given $D(x) = \sqrt{5x^2 + 20x + 25}$, solve $D'(x) = 0$

   Since $D'(x) = \frac{10x + 20}{2\sqrt{5x^2 + 20x + 25}} = \frac{5(x + 2)}{\sqrt{5x^2 + 20x + 25}}$, we get $D'(x) = 0$ when $x = -2$.

1. Compute the derivative:

   (a) $\dfrac{d}{dx}\sin(1/x) = -\dfrac{1}{x^2}\cos(1/x)$ (chain rule)

   (b) $\dfrac{d}{dx}\ln(A + Bx^4) = \dfrac{4Bx^3}{A + Bx^4}$ (chain rule)

   (c) $\dfrac{d}{dx}\sqrt{1 + x^4} = \dfrac{2x^3}{\sqrt{1 + x^4}}$ (chain rule)

   (d) $\dfrac{d}{dx}\ln[f(x)g(x)] = \dfrac{d}{dx}(\ln(f(x)) + \ln(g(x)) = \dfrac{f'(x)}{f(x)} + \dfrac{g'(x)}{g(x)}$ (chain rule)

2. Evaluate the integral:

   (a) $\displaystyle\int 10\cos(x)\sin^4(x)\,dx = 2\sin^5(x) + C$

   (b) $\displaystyle\int 6x\sqrt{x^2 + 7}\,dx = 2(x^2 + 7)^{3/2} + C$

   (c) $\displaystyle\int \frac{2x + \cos(x)}{x^2 + \sin(x)}\,dx = \ln|x^2 + \sin(x)| + C$

   (d) $\displaystyle\int \frac{dx}{x\ln^3|x|} = -\frac{1}{2\ln^2|x|} + C$

3. Given $V(r) = \pi H\left((r^2 - \frac{r^3}{R}\right)$, with H and R constants, solve $V'(r) = 0$.

   Since $V'(r) = \pi H\left(2r - \dfrac{3r^2}{R}\right) = \dfrac{\pi H}{R}r(2R - 3r)$, we have $V'(r) = 0$ for $r = 0$ or $R = \dfrac{2R}{3}$.

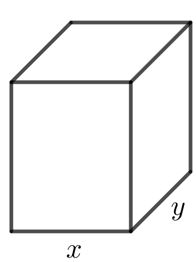
4. Given $y(x) = e^{-x^2}$, solve $y''(x) = 0$.

   We have $y'(x) = -2xe^{-x^2}$, so

   $$y''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 2e^{-x^2}(2x^2 - 1) = 2e^{-x^2}(\sqrt{2}x - 1)(\sqrt{2}x + 1),$$

   so $y''(x) = 0$ for $x = \pm\dfrac{1}{\sqrt{2}}$.

5. Find the minimum possible cost for a square based, 12 litre box, with lid, if the base material costs \$0.20 per square centimetre, and the sides and lid cost \$0.10 per square centimetre. (Recall that 1 litre = 1000 cubic centimetres.)



Let x be the length of one side of the base, and y the height, as shown. Then the volume of the box is $V = x^2y = 12000$. The cost of the base is given by $0.2x^2$, the lid costs $0.1x^2$, and the four sides cost $4(0.1)xy = 0.4xy$, giving a total cost of $C = 0.3x^2 + 0.4xy$.

Using the volume constraint to write $y = 12000/x^2$, we get

$$C(x) = 0.3x^2 + \frac{0.4x(12000)}{x^2} = 0.3x^2 + \frac{4800}{x}$$
$$C'(x) = 0.6x - \frac{4800}{x^2} = \frac{0.6x^3 - 4800}{x^2} = \frac{0.6(x^3 - 8000)}{x^2},$$

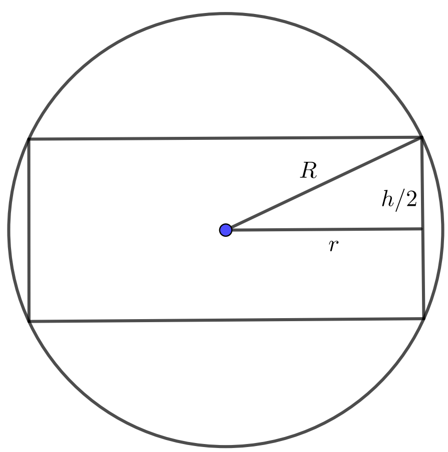
so $C'(x) = 0$ when $x^3 = 8000$, or $x = 20$. We can see that $C'(x) < 0$ for $x < 20$ and $C'(x) > 0$ for $x > 20$ so this is a local minimum. (And the global minimum: it's the only critical point, and $C(x)$ becomes very large if x is either very small or very large.)

When $x = 20$, we find that $y = 30$, and the total cost is

$$C(20) = (\$0.3)(20)^2 + \frac{\$4800}{20} = \$3.60.$$

6. Find the maximum possible volume of a circular cylinder that can be put inside a sphere of radius R.

   Recall that the volume of a cylinder of radius R and height H is $\pi R^2 H$.



The diagram to the left shows a cross-section of the situation: a cylinder of height h and radius r sits inside a sphere of radius R. The cylinder must be centred with respect to the sphere, or we could shift it over, and increase its size.

We can see that the values r and h must satisfy $r^2 + (h/2)^2 = R^2$.

Now, the volume of the cylinder is given by $V = \pi r^2 h$, and since $r^2 = R^2 - \frac{h^2}{4}$, we get

$$V(h) = \pi \left(R^2 h - \frac{h^3}{4} \right)$$
$$V'(h) = \pi \left(R^2 - \frac{3}{4}h^2 \right) = \pi \left(R - \frac{\sqrt{3}}{2}h \right) \left(R + \frac{\sqrt{3}}{2}h \right),$$

so $V'(h) = 0$ when $h = \pm\frac{2R}{\sqrt{3}}$. The negative root is rejected since we must have $h > 0$, and we can also check that this is the critical number which gives the desired maximum. The maximum volume is therefore

$$V(2R/\sqrt{3}) = \frac{4\pi R^3}{3\sqrt{3}}.$$