Solutions for Quiz 1 Practice
Math 2580
Spring 2016

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1. At what point does the line through the point $(1, 0, 3)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ cross the xy-plane?

   The vector equation of the line is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and the xy-plane is given by $z = 0$. Since $z = 3 + t$ for points on the line, setting $z = 0$ gives $t = -3$, and thus $x = 1 + (-3)(1) = -2$ and $y = 0 + (-3)(2) = -6$, so the point is $(-2, -6, 0)$.

   Quiz version: The line is through the point $(4, -2, 3)$ in the direction of $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, so the vector equation of the line is

   $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix},$$

   and the yz-plane is given by $x = 0$. At the point of intersection we have both $x = 0$ and $x = 4 + 2t$, so we must have $t = -2$, which gives us $y = -2 - 2(3) = -8$ and $z = 3 - 2(-4) = 11$. The point is therefore $(0, -8, 11)$.

2. Find the distance from the point $(1, 2, 0)$ to the plane $x - 2y + z = 4$.

   The given plane has normal vector $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, and the point $(4, 0, 0)$ lies on the plane. If we let

   $$\mathbf{v} = \begin{bmatrix} 1-4 \\ 2-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$$

   denote the vector from the point $(4, 0, 0)$ to the given point $(1, 2, 0)$, then the distance is given by the length of the projection of $\mathbf{v}$ onto $\mathbf{n}$:

   $$d = \|\mathrm{proj}_{\mathbf{n}}\, \mathbf{v}\| = \left\| \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} \right\| = \left| \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|} \right|.$$

We compute $\|\mathbf{n}\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$ and $\mathbf{v} \cdot \mathbf{n} = -3(1) + 2(-2) + 0(1) = -7$, so the distance is $d = \dfrac{7}{\sqrt{6}}$.

Alternative solution: The line through $(1, 2, 0)$ in the direction of $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is given by $x = 1 + t$, $y = 2 - 2t$, and $z = t$. At the point of intersection of this line and the plane $x - 2y + z = 4$ we must have

$$(1 + t) - 2(2 - 2t) + t = 4,$$

which gives $6t = 7$, so $t = 7/6$. The point Q on the plane closest to the the point $P = (1, 2, 0)$ is therefore $Q = (1 + 7/6, 2 - 2(7/6), 7/6) = (13/6, -1/3, 7/6)$, and the distance from P to Q is

$$\begin{aligned} d &= \sqrt{(13/6 - 1)^2 + (-1/3 - 2)^2 + (7/6 - 0)^2} \\ &= \sqrt{(7/6)^2 + (-7/3)^2 + (7/6)^2} \\ &= \sqrt{(7/6)^2(1 + 4 + 1)} \\ &= \frac{7}{6}(\sqrt{6}) = \frac{7}{\sqrt{6}}. \end{aligned}$$

3. Find the area of the triangle whose vertices are $(0, 1, 2)$, $(1, 1, 1)$, and $(2, 1, 0)$.

   As it turns out, the given three points are colinear (oops!), so the "triangle" is in fact a line segment, and therefore has zero area. If I hadn't messed up and given you three points all on the same line, the right approach to this problem would be to label the points as P, Q, R and compute the vectors $\mathbf{v} = \overrightarrow{PQ}$ and $\precapprox = \overrightarrow{PR}$ that make up two of the three sides of the triangle. The area of the triangle is then given by the formula

$$A = \|\mathbf{v} \times \mathbf{w}\|.$$

4. Determine the domain of the function $f(x, y) = \dfrac{x + y}{x^2 + y^2 - 1}$ and find the value $f(1, 2)$.

   The function is given by a rational expression in x and y, so it's defined as long as the denominator is nonzero, and this is the case as long as $x^2 + y^2 \neq 1$. The domain is therefore the set $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \neq 1\}$, and we have

$$f(1, 2) = \frac{1 + 2}{1^2 + 2^2 - 1} = \frac{3}{4}.$$

5. For a given function $f(x, y)$ of two variables and a value c in the range of f, what is the difference between the *level curve* $f(x, y) = c$ and the *section*[1] of the graph $z = f(x, y)$ corresponding to $z = c$? How are the two related?

[1]Sections are also known as *traces*

The level curves $f(x, y) = c$ are defined to be subsets of the *plane* $\mathbb{R}^2$ given by $\{(x, y)|f(x, y) = c\}$. On the other hand the section of a graph $z = f(x, y)$ in the plane $z = c$ is a subset of $\mathbb{R}^3$: it is the set of points $\{(x, y, c)|f(x, y) = c\}$. Since the two sets are subsets of different spaces, they are not the same. However, we see that there is a bijection between the two sets given by $f(x, y) = (x, y, c)$ for each point (x, y) such that $f(x, y) = c$. If we view the xy-plane ($z = 0$) as the standard copy of $\mathbb{R}^2$ sitting inside of $\mathbb{R}^3$, then the section of the graph is given by lifting the level curve from the plane $z = 0$ to the plane $z = c$.

6. The subset of $\mathbb{R}^2$ defined by the equation $x^2 + y^2 = 1$ is the unit circle. What does this equation define as a subset of $\mathbb{R}^3$?

   As a subset of $\mathbb{R}^3$, we have the set of points (x, y, z) such that $x^2+y^2 = 1$. For any fixed value $z = c$, we get a copy of the unit circle sitting inside the plane $z = c$. Taking all of the circles together, we get an infinite cylinder parallel to the z-axis that intersects the xy-plane in the unit circle.