Math 1565 Tutorial #6 Solutions

1. State the domain and range of $f(x) = \cos^{-1}(x)$.

   The domain of f is $[-1, 1]$. The range of f is $[0, \pi]$.

2. Evaluate the following limits:

   (a) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-16} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(x+4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{(x+4)(\sqrt{x}+2)} = \frac{1}{32}.$

   (b) $\lim_{\theta\to 0} \frac{\tan(5x)}{\sin(3x)} = \lim_{\theta\to 0} \frac{\sin(5x)}{5x} \frac{3x}{\sin(3x)} \frac{1}{\cos(5x)} \frac{5}{3} = (1)(1)\frac{1}{1} \cdot \frac{5}{3} = \frac{5}{3}.$

3. List all horizontal and vertical asymptotes for the function $f(x) = \dfrac{5x^3 - 4x + 6}{x^3 - 4x}$.

   Since the degree of the top and bottom are the same, the horizontal asymptote can be found by comparing coefficients of top powers: we find $y = 5$.

   The vertical asymptotes occur where the denominator is zero. Since $x^3 - 4x = x(x-2)(x+2)$ we have zeros at $x = 0, 2, -2$. None of these are also zeros for the numerator, so $x = 0$, $x = 2$, and $x = -2$ are vertical asymptotes.

4. Prove that there exists a real number x such that $\cos(x) = x$. (Hint: IVT)

   Consider $f(x) = \cos(x) - x$. This is a continuous function, since it's the sum of continuous functions. We also note that $f(0) = 1 - 0 = 1 > 0$, while $f(\pi/2) = 0 - \pi/2 = -\pi/2 < 0$.

   Thus, by the Intermediate Value Theorem, there must exist some $x \in (0, \pi/2)$ such that $f(x) = 0$, and the result follows.

5. Given $f(x) = \dfrac{1}{x-3}$, determine $f'(2)$ **using the definition of the derivative**.

We have

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(2+h)-3} - \frac{1}{2-3}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{h-1} + 1\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1+(h-1)}{h-1}\right) \\ &= \lim_{h\to 0} \frac{1}{h-1} = -1. \end{aligned}$$

6. Evaluate the derivatives of the following functions:

(a) $f(x) = 4x^{11} - 3x^{2/3} + 5\ln(x) + e^{\pi}$

$$f'(x) = 44x^{10} - 2x^{-1/3} + \frac{5}{x}.$$

(b) $g(x) = e^{3x}\cos(5x)$

$$g'(x) = 3e^{3x}\cos(5x) - 5e^{3x}\sin(5x).$$

(c) $h(x) = \dfrac{x^5 - 4x^3}{x^2}$

Since $h(x) = x^3 - 4x$, we have $h'(x) = 3x^2 - 4$. Or you can do it the hard way using the quotient rule.

(d) $k(x) = \sec(\ln(x^5 + 3x^2))$

$$k'(x) = \sec(\ln(x^5+3x^2))\tan(\ln(x^5+3x^2)) \cdot \frac{5x^4+6x}{x^5+3x^2}.$$