Math 4310 Assignment #10
University of Lethbridge, Fall 2014

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Due date: Friday, November 14th, by 5 pm.

This assignment will be a review assignment, in advance of the second term test, which takes place on Monday, the 17th. This test will cover everything since the previous test: in particular, you're responsible for Chapters 12 (connected spaces), 13 (compact spaces), and 15 (quotient spaces).

Please submit solutions to the following problems (except where indicated). As usual, all the textbook exercises are recommended as practice problems if they don't appear as an assigned problem below.

1. Let $f : S^1 \to \mathbb{R}$ be a continuous map, where $S^1 = \{(x,y) | x^2 + y^2 = 1\}$. Show that there exists a point $(x,y) \in S^1$ such that $f(x,y) = f(-x,-y)$. (Hint: S^1 is connected; in fact, it is path-connected.)

2. Let X be a topological space. Prove that CX, the cone over X, is path-connected.

   (Recall that CX is the quotient of $X \times [0,1]$ obtained by collapsing $X \times \{1\}$ to a single point.)

3. (a) Let X be a connected topological space, and call a point $p \in X$ a *cut point* if $X \setminus \{p\}$ is not connected. Prove that the existence of a cut point is a topological property. (That is, if $f : X \to Y$ is a homeomorphism and X has a cut point p, $q = f(p)$ must be a cut point of Y.)

   (b) Prove that none of the intervals $[0,1]$, $(0,1)$, or $[0,1)$ can be homeomorphic.

   (c) Prove that the letters X and Y (viewed as subsets of $\mathbb{R}^2$ with the subspace topology) are not homeomorphic.

   (Hint: extend your proof from (a) to show that if $f : X \to Y$ is a homeomorphism, p is a cut point of X, and $q = f(p)$, then $X \setminus \{p\}$ has the same number of connected components as $Y \setminus \{f(p)\}$.

4. Prove that any infinite subset of a compact space must have a limit point.

5. A closed map $p : X \to Y$ is called a *perfect map* if p is a surjection and $p^{-1}(y)$ is a compact subset of X for every $y \in Y$. A quotient map $p : X \to Y$ is called a *proper map* if $p^{-1}(K)$ is compact whenever $K \subseteq Y$ is compact. Prove that any perfect map is proper.

   (Hint: any open cover of $p^{-1}(K)$ is also an open cover of $p^{-1}(k)$ for each $k \in K$. If $p^{-1}(k) \subseteq U = U_1 \cup \cdots U_n$, then $F = X \setminus U$ is closed in X, and p is a closed map, so $p(F)$ is closed in Y, and thus $Y \setminus p(F)$ is an open neighbourhood of k in Y.)