FACULTY OF APPLIED SCIENCE AND ENGINEERING
University of Toronto

MAT294H1Y
Calculus and Differential Equations

Term Test #1
Duration: 110 minutes

NO AIDS ALLOWED. **Total: 50 marks**

Family Name: SOLUTIONS
(Please Print)

Given Name(s): THE
(Please Print)

Please sign here: ______________________

Student ID Number: ______________________

You may not use calculators, cell phones, or PDAs during the test. Partial credit will be given for partially correct work. Please read through the entire test before starting, and take note of how many points each question is worth.

FOR MARKER'S USE ONLY	
Problem 1:	/10
Problem 2:	/15
Problem 3:	/8
Problem 4:	/9
Problem 5:	/8
TOTAL:	/50

1. (a) Calculate the first-order partial derivatives of the following functions:

[2] (i) $f(x, y, z) = z\sin(x - y)$

$$f_x(x, y, z) = z\cos(x - y)$$
$$f_y(x, y, z) = -z\cos(x - y)$$
$$f_z(x, y, z) = \sin(x - y)$$

[2] (ii) $f(x, y) = x^2ye^{xy}$

$$f_x(x, y) = 2xye^{xy} + x^2y^2e^{xy}$$
$$f_y(x, y) = x^2e^{xy} + x^3ye^{xy}$$

[2] (iii) $f(x, y, z) = \dfrac{\ln z}{xy}$

$$f_x(x, y, z) = -\frac{\ln z}{x^2y}$$
$$f_y(x, y, z) = -\frac{\ln z}{xy^2}$$
$$f_z(x, y, z) = \frac{1}{xyz}$$

[4] (b) Find all second-order partial derivatives of the function

$$f(x, y) = x^2\tan y + y\ln x.$$

We have first-order partial derivatives

$$f_x(x, y) = 2x\tan y + \frac{y}{x} \quad \text{and} \quad f_y(x, y) = x^2\sec^2 y + \ln x,$$

so

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(2x\tan y + \frac{y}{x}) = 2\tan y - \frac{y}{x^2},$$
$$f_{yy}(x, y) = \frac{\partial}{\partial y}(x^2\sec^2 y + \ln x) = 2x^2\sec^2 y\tan y,$$
$$f_{xy}(x, y) = \frac{\partial}{\partial y}(2x\tan y + \frac{y}{x}) = 2x\sec^2 y + \frac{1}{x} = f_{yx}(x, y)$$

2. Let $f(x, y) = x^3 - 3xy - y^3$.

[3] (a) Locate all critical points of f.

The partial derivatives of f are

$$f_x(x, y) = 3x^2 - 3y \quad \text{and} \quad f_y(x, y) = -3x - 3y^2,$$

and the critical points of f will be those points where both f_x and f_y are zero. We find:

$$f_x(x, y) = 0 \text{ and } f_y(x, y) = 0 \Rightarrow y = x^2 \Rightarrow (x^2)^2 + x = 0,$$

which gives $x = 0$, (and $y = 0$) or $x = -1$ (and $y = (-1)^2 = 1$), so there are two critical points: $(0, 0)$ and $(-1, 1)$.

(b) Classify any critcal points found in part (a) as local maxima, local minima, or
[4] saddle points.

We have $f_{xx}(x, y) = 6x$, $f_{xy}(x, y) = -3$ and $f_{yy}(x, y) = -6y$.
At $(0, 0)$, this gives $A = 0$, $B = -3$, $C = 0$ and $\Delta = 0 - (-3)^2 = -9$. Since $\Delta < 0$, $(0, 0)$ is a saddle point.
At $(-1, 1)$, this gives $A = -6$, $B = -3$, $C = -6$ and $\Delta = 36 - 9 = 27$. Since $\Delta > 0$ and $A < 0$, $(-1, 1)$ is a local maximum.

(c) Find the maximum and minimum of $f(x, y)$ subject to the constraint $x+2y-1 = 0$
[5] (**For $-1 \leq x \leq 1$).

We let $g(x, y) = x + 2y - 1$. The Lagrange multiplier equations $\nabla f(x, y) = \lambda \nabla g(x, y)$, and $g(x, y) = 0$ become:

(i) $f_x = \lambda g_x \Rightarrow\ 3x^2 - 3y = \lambda(1)$

(ii) $f_y = \lambda g_y \Rightarrow\ -3x - 3y^2 = \lambda(2)$

(iii) $g(x, y) = 0 \Rightarrow\ x + 2y = 1$

Comparing (i) and (ii) we see $f_y = 2\lambda = 2f_x$, so

$$-3x - 3y^2 = 2(3x^2 - 3y).$$

If we use (iii) to solve for $x = 1 - 2y$ and plug this into the above equation, we get

$$6(1 - 2y)^2 - 6y + 3(1 - 2y) + 3y^2 = 27y^2 - 36y + 9 = 9(3y - 1)(y - 1) = 0$$

This gives two points at which a max/min can occur: $y = 1/3$ and $x = 1-2(1/3) = 1/3$, or $y = 1$ and $x = -1$. $x = -1$ is one of our "end points"; the other end point is $(1, 0)$.

We find $f(1/3, 1/3) = -1/3$ is the minimum, and $f(-1, 1) = f(1, 0) = 1$ is the maximum.

** If you don't apply the restriction $-1 \leq x \leq 1$, then there technically is no maximum or minimum - the two points found above are "local" max/min points: $f(x, y)$ can be arbitrarily large (positive or negative).

(d) Find the absolute maximum and minumum of $f(x, y)$ on the region bounded by
[3] the coordinate axes and the line $x + 2y = 1$.

Hint: The only thing you still have to check is the value of $f(x, y)$ along the axes.

At the critical points, we have $f(0, 0) = 0$ and $f(-1, 1) = 1$, but only $(0, 0)$ lies inside of the region.

Along the line $x + 2y = 1$ we have the minimum $f(1/3, 1/3) = -1/3$ found above.

Along the y-axis, $f(0, y) = -y^3$ has maximum $f(0, 0) = 0$ and minimum $f(0, 1/2) = -1/8$.

Along the x-axis, $f(x, 0) = x^3$ has maximum $f(1, 0) = 1$ and minimum $f(0, 0) = 0$.

Thus, the absolute maximum is $f(1, 0) = 1$ and the absolute minimum is $f(1/3, 1/3) = -1/3$.

3. Let $f(x, y) = \sqrt{x^2 + y^2}$.

[2] (a) Find $\nabla f(x, y)$.

$$\nabla f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\,\hat{\imath} + \frac{y}{\sqrt{x^2 + y^2}}\,\hat{\jmath}.$$

(b) Find the equation of the tangent plane to the surface $z = \sqrt{x^2 + y^2}$ at the point

[3] $(3, 4)$.

The general equation is $z - z_0 = \nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle$.
Here we have $x_0 = 3$, $y_0 = 4$, and $z_0 = \sqrt{3^2 + 4^2} = 5$. Therefore, the equation of the plane is

$$\begin{aligned} z - 5 &= \nabla f(3, 4) \cdot \langle x - 3, y - 4 \rangle \\ &= \langle 3/5, 4, 5 \rangle \cdot \langle x - 3, y - 4 \rangle \\ &= \frac{3}{5}(x - 3) + \frac{4}{5}(y - 4). \end{aligned}$$

[3] (c) Use the differential df to approximate the value of $\sqrt{(2.97)^2 + (3.04)^2}$.
Note: $3/5 = 0.6$ and $4/5 = 0.8$.
We have the approximation $f(\vec{x} + \Delta\vec{x}) \approx f(\vec{x}) + df(\vec{x})$, where $df(\vec{x}) = \nabla f(\vec{x}) \cdot \Delta\vec{x}$.
Here, $\vec{x} = 3\hat{\imath} + 4\hat{\jmath}$, $\Delta\vec{x} = -0.03\hat{\imath} + 0.04\hat{\jmath}$, and $\nabla f(\vec{x}) = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$.
Therefore, we get:

$$\begin{aligned} \sqrt{(2.97)^2 + (3.04)^2} &= f(3 - 0.03, 4 + 0.04) \\ &\approx f(3, 4) + \nabla f(3, 4) \cdot \langle -0.03, 0.04 \rangle \\ &\approx 5 + 0.6(-0.03) + 0.8(0.04) = 5.014. \end{aligned}$$

4. Let $f(x,y,z) = \ln(x+y+z)$ and let $\vec{r}(t) = \cos^2 t\,\hat{\imath} + \sin^2 t\,\hat{\jmath} + t^2\,\hat{k}$.

[2] (a) Write the chain rule formula for the derivative $\dfrac{d}{dt}(f(\vec{r}(t)))$.

$$\frac{d}{dt}(f(\vec{r}(t))) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t),$$

or

$$\frac{d}{dt}(f(x(t),y(t),z(t)) = f_x(x(t),y(t),z(t))x\,'(t) + f_y(x(t),y(t),z(t))y\,'(t) + f_z(x(t),y(t),z(t))z\,'(t),$$

or

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

[2] (b) Use your formula from (a) to evaluate $\dfrac{d}{dt}(f(\vec{r}(t)))$ when $t = \pi/4$.

When $t = \pi/4$, $\vec{r}(t) = \frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{\pi^2}{16}\hat{k}$.

$\nabla f(x,y,z) = \dfrac{1}{x+y+z}(\hat{\imath}+\hat{\jmath}+\hat{k})$, so $\nabla f(\vec{r}(\pi/4)) = \dfrac{\hat{\imath}+\hat{\jmath}+\hat{k}}{1+\frac{\pi^2}{16}}$.

We have $\vec{r}\,'(t) = -2\cos t\sin t\hat{\imath} + 2\cos t\sin t\hat{\jmath} + 2t\hat{k}$, so $\vec{r}\,'(\pi/4) = -\hat{\imath} + \hat{\jmath} + \frac{\pi}{2}\hat{k}$.

Thus,

$$\frac{d}{dt}f(\vec{r}(t))|_{t=\pi/4} = \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{1+\frac{\pi^2}{16}} \cdot (-\hat{\imath}+\hat{\jmath}+\frac{\pi}{2}\hat{k}) = \frac{\pi/2}{1+\pi^2/16}.$$

[2] (c) Verify your calculation in part (b) by first making the substitutions $x = \cos^2 t$, $y = \sin^2 t$ and $z = t^2$ in f and then differentiating with respect to t.

If we let $g(t) = f(x(t), y(t), z(t))$, then we get

$$g(t) = \ln(\cos^2 t + \sin^2 t + t^2) = \ln(1 + t^2) \quad \text{and} \quad g'(t) = \frac{2t}{1 + t^2}.$$

Thus,

$$\frac{d}{dt} f(\vec{r}(t))|_{t=\pi/4} = g'(\pi/4) = \frac{\pi/2}{1 + \pi^2/16}.$$

[3] (d) Find the derivative of f in the direction of the curve $\vec{r}(t)$ at the point $(1/2, 1/2, \pi^2/16)$.

Hint: You've done most of the work for this problem already!

Differentiating in the direction of the curve means taking the derivative at the point $\vec{r}(\pi/4)$ in the direction of $\vec{r}\,'(\pi/4)$.

The unit vector is thus

$$\hat{u} = \frac{\vec{r}\,'(\pi/4)}{||\vec{r}\,'(\pi/4)||},$$

and the directional derivative we want is

$$D_{\hat{u}} f(1/2, 1/2, \pi^2/16) = \nabla f(\vec{r}(\pi/4)) \cdot \frac{\vec{r}\,'(\pi/4)}{||\vec{r}\,'(\pi/4)||},$$

which is just $\frac{d}{dt} f(\vec{r}(t))|_{t=\pi/4}$ divided by $||\vec{r}\,'(\pi/4)|| = \sqrt{2 + \pi^2/4}$.

[3] 5. (a) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist by making the substitution $y = mx$, where m can be any real number.

If we substitute $y = mx$ into the limit, then we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2},$$

which simplifies to

$$\lim_{x\to 0} \frac{1 - m^2}{1 + m^2} = \frac{1 - m^2}{1 + m^2}.$$

Since this result depends on the choice of m, (for example, $m = 1$ gives 0, while $m = 0$ gives 1) the value of the limit is not the same along all paths to the origin, and therefore the limit does not exist.

(b) Recall that $f(\vec{x})$ is differentiable at $\vec{a}$ if and only if

$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a})\cdot\vec{h}}{||\vec{h}||} = 0.$$

Show that the statement, "f is continuous at any point at which it is differentiable" holds for functions of more than one variable.
Hint: Note that $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = \lim\limits_{\vec{h}\to\vec{0}} f(\vec{a}+\vec{h})$.

We have

$$\begin{aligned}
\lim_{\vec{x}\to\vec{a}} f(\vec{x}) &= \lim_{\vec{h}\to\vec{0}} f(\vec{a}+\vec{h}) \\
&= \lim_{\vec{h}\to\vec{0}} \left(f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a})\cdot\vec{h} + (f(\vec{a}) + \nabla f(\vec{a})\cdot\vec{h}) \right) \\
&= \lim_{\vec{h}\to\vec{0}} \left(\frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a})\cdot\vec{h}}{||\vec{h}||} ||\vec{h}|| + (f(\vec{a}) + \nabla f(\vec{a})\cdot\vec{h}) \right) \\
&= 0 + f(\vec{a}) + \nabla(\vec{a})\cdot\vec{0} = f(\vec{a}),
\end{aligned}$$

where in the last line we used the fact that f was differentiable at $\vec{a}$.
Since we've shown that $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$, f is continuous at $\vec{a}$.

Extra space for rough work. Do **not** tear out this page.