1. Calculate $\lim_{n\to\infty} a_n$ to show that the series $\sum a_n$ diverges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n(n+2)}$

We have $a_n = \frac{3n^2}{n^2+2n}$, and

$$\lim_{n\to\infty} \frac{3n^2}{n^2+2n} = \lim_{n\to\infty} \frac{3}{1+2/n} = \frac{3}{1+0} = 3 \neq 0,$$

so the series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

Here $a_n = \frac{n!}{10^n}$, and intuitively we expect that $a_n \to \infty$ since $a_{n+1} = \dfrac{n+1}{10} a_n$, and for $n \geq 10$, $\dfrac{n+1}{10} > 1$. One way to see this precisely is to notice that for $n > 20$, we have

$$a_n = a_{20}\left(\frac{21}{10}\right)\left(\frac{22}{20}\right)\cdots\left(\frac{n}{10}\right) > a_{20}(2^n)(2^n)\cdots(2^n) = a_{20}\cdot 2^{n-20}.$$

Since a_{20} is a constant and $\lim_{n\to\infty} 2^{n-20} = \infty$, we see that $a_n \to \infty$, and since the sequence diverges, the series certainly does.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}+1}$

We have $a_n = \dfrac{2^n}{2^{n+1}+1} = \dfrac{1}{2+2^{-n}}$, so $\displaystyle\lim_{n\to\infty} a_n = \frac{1}{2+0} = \frac{1}{2} \neq 0$, and thus the series diverges.

2. Determine if the series diverges or converges. (Each series is a p-series, or geometric, or there is an argument involving basic properties of series. See Key Idea 17 on page 126 of the textbook for additional guidance.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

This is a p-series with $p = 5 > 1$, so the series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2}$

We have

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2} = \sum_{n=1} \infty \left(\frac{\sqrt{n}}{n^2} + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + \sum_{n=1}^{\infty} \frac{1}{n^2},$$

giving us the sum of two p-series, with $p = 3/2 > 1$ and $p = 2 > 1$, respectively. Since both of these series converge, the original series converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n}$

We have
$$\sum_{n=1}^{\infty} \frac{3^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n,$$
so this is a geometric series with $r = 3/5 < 1$, which converges. Indeed, in this case we can even say what it converges to:
$$\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \frac{3}{5}\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{3}{5}\left(\frac{1}{1-3/5}\right) = \frac{3}{2}.$$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{6^n}$

This is once again a geometric series, with $r = 7/6 > 1$, so it diverges.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n!}$

We have $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n!} = 10\sum_{n=1}^{\infty} \frac{1}{n!} = 10\left(\left(\sum_{n=0}^{\infty} \frac{1}{n!}\right) - 1\right) = 10e - 10.$

Here, we've used the fact that $\sum_{n=0}^{\infty} \frac{1}{n!} = e$ (from Key Idea 17 in the text) and that
$$\sum_{n=0} \infty \frac{1}{n!} = \frac{1}{0!} + \sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}$$

(f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n!} + \frac{1}{n}\right)$

We can write the above series as the sum of two series, the second of which is the harmonic series, $\sum \frac{1}{n}$. Since we know that the harmonic series is divergent, the series diverges.

3. Determine if each series converges or diverges. If it converges, determine the value it converges to.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$. (Geometric)

This is geometric, with $r = 1/4 < 1$, so the series converges to $\dfrac{1}{1-1/4} = \dfrac{4}{3}.$

(b) $\sum_{n=1}^{\infty} e^{-n}$. (Geometric?)

This is geometric, with $r = \frac{1}{e} < 1$. (Notice that $r^n = \frac{1}{e^n} = e^{-n}$.) We know that

$$\sum_{n=0}^{\infty} e^{-n} = \frac{1}{1-1/e} = \frac{e}{e-1}$$

using the formula for the sum of a geometric series. Since our series starts at $n = 1$ instead of $n = 0$, we have to subtract the value of $e^{-0} = 1$, giving us

$$\sum_{n=1}^{\infty} e^{-n} = \frac{e}{e-1} - 1 = \frac{1}{e-1}.$$

(c) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (Telescoping)

Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we see that the series is telescoping. The N^{th} partial sum is

$$s_N = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) = 1 - \frac{1}{N+1},$$

so the series converges to $\lim_{N\to\infty} s_N = 1$.

(d) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ (Telescoping?)

Recall that $\ln\left(\frac{n}{n+1}\right) = \ln n - \ln(n+1)$ using the properties of logarithms, so the N^{th} partial sum is

$$s_N = (\ln(1) - \ln(2)) + (\ln(2) - \ln(3)) + \cdots + (\ln N - \ln(N+1)) = -\ln(N+1),$$

so the series is telescoping, but it diverges, since $\lim_{N\to\infty} s_N = -\infty$.

4. Use the integral test to determine if the series converges:

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

We compare to the integral

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} (\ln(\ln b) - \ln(\ln 2)) = \infty$$

which diverges, so the series diverges as well.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

We compare to the integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{\ln x}\bigg|_2^b\right) = \frac{1}{\ln 2}.$$

Since the improper integral converges, so does the series.

5. Use direct comparison to determine if the series converges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n + n^2 - n}$

Since $n^2 \geq n$ for $n \geq 1$, we have $n^2 - n \geq 0$, so $4^n + n^2 - n \geq 4^n > 0$, which shows that

$$\frac{1}{4^n + n^2 - n} \leq \frac{1}{4^n}$$

for all $n \geq 1$. Since the series $\sum \frac{1}{4^n}$ converges (it's geometric with $r = 1/4 < 1$), the original series converges as well, by the comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} - 2}$

Since $\sqrt{n} - 2 < \sqrt{n}$, it follows that $\frac{1}{\sqrt{n}-2} > \frac{1}{\sqrt{n}}$, and since $\sum \frac{1}{\sqrt{n}}$ diverges (p-series with $p = 1/2 < 1$), the original series diverges, by the comparison test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 \ln n}$

Recall that ln is an increasing function, and since $3 > e$, we know that $\ln n \geq \ln 3 \geq \ln e = 1$ for all $n \geq 3$. It follows that $\frac{1}{n^2 \ln n} \leq \frac{1}{n^2}$ for all $n \geq 3$, and since $\sum \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$), the original series converges, by the comparison test.

6. Use the Limit Comparison Test to determine if the series converges. (Be sure to state what series you're using for comparison.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n - n^2}$

Notice that direct comparison with the geometric series $\sum 1/4^n$ doesn't work as easily as in the previous problem, since the terms in this series are *larger* than those of the geometric series. But they're not "too much" larger, which is the right setting for the limit comparison test.

With $a_n = \dfrac{1}{4^n - n^2}$ and $b^n = \dfrac{1}{4^n}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4^n}{4^n - n^2} = \lim_{n\to\infty} \frac{1}{1 - n^2/4^n} = 1.$$

Since the limit of a_n/b_n is finite and nonzero, and $\sum b_n$ converges (geometric series with $r = 1/4 < 1$), we can conclude that $\sum a_n$ converges as well, by the limit comparison test.

(In the above, we used the fact that $\lim_{n\to\infty} \dfrac{n^2}{4^n} = 0$, which can be easily verified using l'Hospital's rule for the corresponding functions of x:

$$\lim_{x\to\infty} \frac{x^2}{4^x} = \lim_{x\to\infty} \frac{2x}{4^x \ln 4} = \lim_{x\to\infty} \frac{2}{4^x (\ln 4)^2} = 0.$$

If this was already clear to you since you're aware that exponential functions always go to infinity faster than any polynomial, there's no need to verify this limit.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

We let $a_n = \dfrac{1}{\sqrt{n^2+n}}$ and take $b_n = \dfrac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}} = 1.$$

Again, we get a finite, nonzero limit, but since $\sum b_n$ diverges (harmonic series), we conclude that $\sum a_n$ diverges as well, by the limit comparison test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-5}$

Let $a_n = \dfrac{n+5}{n^3-5}$, and let $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b^n} = \lim_{n\to\infty} \frac{n^2(n+5)}{n^3-5} = \lim_{n\to\infty} \frac{1+5/n}{1-5/n^3} = 1.$$

Since the above limit is finite and nonzero, and since $\sum b_n$ converges (p-series with $p = 2 > 1$), we know that $\sum a_n$ converges, by the limit comparison test.