Math 3410 Assignment #3 Solutions
University of Lethbridge, Spring 2015

Sean Fitzpatrick

March 6, 2015

1. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_5(\mathbb{R})$ be the linear transformation given by

$$(Tp)(x) = (3 - 2x + x^2)p(x).$$

(a) Compute the matrix of T with respect to the standard bases $\{1, x, x^2, x^3\}$ of $\mathcal{P}_3(\mathbb{R})$ and $\{1, x, x^2, x^3, x^4, x^5\}$ of $\mathcal{P}_5(\mathbb{R})$.

Solution: Let $p_i(x) = x^i$, $i = 0, 1, 2, 3$ denote the standard basis elemenets of $\mathcal{P}_3(\mathbb{R})$, and let $q_j(x) = x^j$, $j = 0, 1, 2, 3, 4, 5$ denote the standard basis elements of $\mathcal{P}_5(\mathbb{R})$. We then compute

$$\begin{aligned}
(Tp_0)(x) &= (3 - 2x + x^2)(1) = 3q_0 - 2q_1 + q_2\\
(Tp_1)(x) &= (3 - 2x + x^2)(x) = 3q_1 - 2q_2 + q_3\\
(Tp_2)(x) &= (3 - 2x + x^2)(x^2) = 3q_2 - 2q_3 + q_4\\
(Tp_3)(x) &= (3 - 2x + x^2)(x^3) = 3q_3 - 2q_4 + q_5.
\end{aligned}$$

It follows that the matrix of T is given by

$$\mathcal{M}(T) = \begin{array}{c} \\ q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{array} \begin{array}{c} \begin{array}{cccc} p_0 & p_1 & p_2 & p_3 \end{array} \\ \begin{pmatrix} 3 & 0 & 0 & 0\\ -2 & 3 & 0 & 0\\ 1 & -2 & 3 & 0\\ 0 & 1 & -2 & 3\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

(b) Find the null space and range of T.

Solution: There are two ways to proceed. First we'll show how to solve the problem using the direct approach. Suppose $q(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 +$

b^5x^5 belongs to range T. Then there must exist scalars a_0, a_1, a_2, a_3 such that

$$\mathcal{M}(T)\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}.$$

Finding the a_i corresponds to reducing an augmented matrix as follows:

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & b_0 \\ -2 & 3 & 0 & 0 & b_1 \\ 1 & -2 & 3 & 0 & b_2 \\ 0 & 1 & -2 & 3 & b_3 \\ 0 & 0 & 1 & -2 & b_4 \\ 0 & 0 & 0 & 1 & b_5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{3}b_0 \\ 0 & 1 & 0 & 0 & \frac{1}{3}b_1 + \frac{2}{9}b_0 \\ 0 & 0 & 1 & 0 & \frac{1}{3}b_2 + \frac{2}{9}b_1 + \frac{1}{27}b_0 \\ 0 & 0 & 0 & 1 & \frac{1}{3}b_3 + \frac{2}{9}b_2 + \frac{1}{27}b_1 - \frac{4}{81}b_0 \\ 0 & 0 & 0 & 0 & b_4 + \frac{2}{3}b_3 + \frac{1}{9}b_2 - \frac{4}{27}b_1 - \frac{11}{81}b_0 \\ 0 & 0 & 0 & 0 & b_5 - \frac{1}{3}b_3 - \frac{2}{9}b_2 - \frac{1}{27}b_1 + \frac{4}{81}b_0 \end{array}\right].$$

From this we see that null $T = \{0\}$ (there is a leading one in every column, so if a solution exists, it must be unique; this also follows from the fact that if $p(x)q(x) = 0$ for two polynomials p and q, then $p(x) = 0$ or $q(x) = 0$). We also see that a solution exists if and only if

$$\begin{aligned} b_4 + \frac{2}{3}b_3 + \frac{1}{9}b_2 - \frac{4}{27}b_1 - \frac{11}{81}b_0 = 0 \text{ and} \\ b_5 - \frac{1}{3}b_3 - \frac{2}{9}b_2 - \frac{1}{27}b_1 + \frac{4}{81}b_0 = 0, \end{aligned}$$

so the range consists of all polynomials $b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5$ such that $b_4 = \frac{11}{81}b_0 + \frac{4}{27}b_1 - \frac{1}{9}b_2 - \frac{2}{3}b_3$ and $b_5 = -\frac{4}{81}b_0 + \frac{1}{27}b_1 + \frac{2}{9}b_2 + \frac{1}{3}b_3$.

At this point, you're probably thinking, "there must be a better way!" It turns out that there is. Using the handout posted on Moodle, the null space and range of T correspond to the null space and column space of $\mathcal{M}(T)$. In that handout, we have the following theorem:

> A basis for the column space of a matrix A consists of those columns of A for which there is a leading one in the corresponding column of the row-echelon form of A.

(Note that we don't take the columns with leading ones in the REF of A - we take the corresponding columns in the original matrix of A.) Since every column of the row-echelon form of $\mathcal{M}(T)$ contains a leading one, it follows that the columns of $\mathcal{M}(T)$ correspond to a basis of range T. Thus, we can also conclude that

$$\operatorname{range} T = \operatorname{span}\{3 - 2x + x^2, 3x - 2x^2 + x^3, 3x^2 - 2x^3 + x^4, 3x^3 - 2x^4 + x^5\}.$$

Of course, we could also conclude this immediately, since these basis vectors are just the image of the standard basis of $\mathcal{P}_3(\mathbb{R})$ under T, and since $T = \{0\}$, we know two things:

i. $\dim \operatorname{range} T = \dim \mathcal{P}_3(\mathbb{R}) = 4$.

ii. T is injective, and $\{p_0, p_1, p_2, p_3\}$ are basis vectors and therefore linearly independent in $\mathcal{P}_3(\mathbb{R})$, so Tp_0, Tp_1, Tp_2, Tp_3 are linearly independent in $\mathcal{P}_5(\mathbb{R})$, and therefore form a basis for $\operatorname{range} T$.

2. Let T_1 and T_2 be linear maps from V ot W.

(a) Suppose that W is finite-dimensional. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible linear operator $S : W \to W$ such that $T_1 = ST_2$.

Solution: Suppose there exists an invertible linear operator $S : W \to W$ such that $T_1 = ST_2$. If $v \in \operatorname{null} T_1$, then $0 = T_1 v = (ST_2)v = S(T_2 v)$. Since S is invertible, we have $\operatorname{null} S = \{0\}$, and thus $T_2 v = 0$, so $v \in \operatorname{null} T_2$. Similarly, if $v \in \operatorname{null} T_2$, then $T_1 v = (ST_2)v = S(T_2 v) = S(0) = 0$, so $v \in \operatorname{null} T_1$.

Conversely, suppose that $\operatorname{null} T_1 = \operatorname{null} T_2$, and that $\dim W = n$. Following the hint, let $\{w_1, \ldots, w_m\}$ be a basis for $\operatorname{range} T_1 \subseteq W$, and choose vectors $v_1, \ldots, v_m \in V$ such that $T_1 v_1 = w_1, \ldots, T_1 v_m = w_m$.

Now, consider the vectors $u_1 = T_2 v_1, \ldots, u_m = T_2 v_m$. We want to show that the vectors $u_1, \ldots, u_m$ are linearly independent. To that end, suppose that

$$c_1 u_1 + \cdots + c_m u_m = 0$$

for some scalars $c_1, \ldots, c_m$. Then we have

$$0 = c_1 u_1 + \cdots + c_m u_m = c_1 T_2 v_1 + \cdots + c_m T_2 u_m = T_2(c_1 v_1 + \cdots + c_m v_m).$$

It follows that $c_1 v_1 + \cdots + c_m v_m \in \operatorname{null} T_2$, but $\operatorname{null} T_2 = \operatorname{null} T_1$, which implies that

$$T_1(c_1 v_1 + \cdots + c_m v_m) = c_1 T v_1 + \cdots + c_m T v_m = c_1 w_1 + \cdots + c_m w_m = 0.$$

But the vectors $w_1, \ldots, w_m$ are linearly independent, so we must have $c_1 = 0, \ldots, c_m = 0$. Thus, the vectors $u_1, \ldots, u_m$ are linearly independent in $\operatorname{range} T_2$, which implies that $\dim \operatorname{range} T_2 \geq m = \dim \operatorname{range} T_1$. (Any basis of $\operatorname{range} T_2$ is a spanning set, and therefore contains at least as many vectors as a linearly independent set.)

Repeating the above argument with the roles of T_1 and T_2 exchanged, we see that $\dim \operatorname{range} T_1 \geq \dim \operatorname{range} T_2$, and thus $\dim \operatorname{range} T_1 = \dim \operatorname{range} T_2$. It follows that $\operatorname{range} T_1$ is isomorphic to $\operatorname{range} T_2$.

From the above argument, we see that $\dim \operatorname{range} T_1 = \dim \operatorname{range} T_2$, but it also follows that if $v_1, \ldots, v_m$ are vectors in V such that the vectors $w_1 = T_1 v_1, \ldots, w_m = T_1 v_m$ form a basis for $\operatorname{range} T_1$, then $u_1 = T_2 v_1, \ldots, u_m = T_2 v_m$ form a basis for $\operatorname{range} T_2$. Now note that since $\operatorname{range} T_1, \operatorname{range} T_2$ are subspaces of W, we can extend these bases to obtain two bases

$$B_1 = \{w_1, \ldots, w_m, x_1, \ldots, x_k\}$$
$$B_2 = \{u_1, \ldots, u_m, y_1, \ldots, y_k\}$$

of W. We now define a map $S : W \to W$ by defining it on the basis B_2 by

$$Su_1 = w_1, \ldots, Su_m = w_m, Sy_1 = x_1, \ldots, Sy_k = x_k.$$

The map S is invertible, since it takes a basis to a basis. Moreover, for any $v \in V$, we have scalars $c_1, \ldots, c_m$ such that

$$T_2 v = c_1 T_2 v_1 + \cdots + c_m T_2 v_m = c_1 u_1 + \cdots + c_m u_m.$$

Also note that if we write $v = v' + v''$, where $v' = c_1 v_1 + \cdots + c_m v_m$ and $v'' \in \operatorname{null} T_2 = \operatorname{null} T_1$, then

$$\begin{aligned}
(ST_2)v &= S(T_2 v) = S(c_1 u_1 + \cdots + c_m u_m) \\
&= c_1 S u_1 + \cdots + c_m S u_m \\
&= c_1 w_1 + \cdots + c_m w_m \\
&= c_1 T_1 v_1 + \cdots + c_m T_1 v_m \\
&= T_1 v' \\
&= T_1(v' + v'') \\
&= T_1 v.
\end{aligned}$$

(b) Suppose that V is finite-dimensional. Prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ if and only if there exists an invertible linear operator $S : V \to V$ such that $T_1 = T_2 S$.

Solution: Suppose that there exists an invertible linear operator $S : V \to V$ such that $T_1 = T_2 S$. If $w \in \operatorname{range} T_1$, then there exists $v \in V$ such that $T_1 v = w$. But then

$$w = T_1 v = (T_2 S) v = T_2(Sv),$$

which shows that $w \in \operatorname{range} T_2$. If $w \in \operatorname{range} T_2$, then $w = T_2 v$ for some $v \in V$. If we let $v' = S^{-1} v$, then

$$T_1 v' = (T_2 S) v' = (T_2 S)(S^{-1} v) = T_2(S S^{-1} v) = T_2 v = w,$$

so $w \in \operatorname{range} T_1$. Thus, $\operatorname{range} T_1 = \operatorname{range} T_2$.

Conversely, suppose that $\operatorname{range} T_1 = \operatorname{range} T_2$. Since V is finite-dimensional, we have that

$$\dim \operatorname{null} T_1 = \dim V - \dim \operatorname{range} T_1 = \dim V - \dim \operatorname{range} T_2 = \dim \operatorname{null} T_2.$$

Moreover, we know that $\operatorname{range} T_1 = \operatorname{range} T_2$ is finite-dimensional. Let $w_1, \ldots, w_k$ be a basis for this subspace of W, and choose[1] vectors $u_1, \ldots, u_k \in V$ and $v_1, \ldots, v_k \in V$ such that

$$T_1 u_1 = w_1 = T_2 v_1, \ldots, T_1 u_k = w_k = T_2 v_k.$$

[1]The vectors are not uniquely defined. For each w_j, any two vectors u_j, u_j' such that $T_1 u_j = T_1 u_j' = w_j$ differ by an element of $\operatorname{null} T_1$.

Recall (from an earlier exercise) that the sets $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ are linearly independent. If we let $U = \operatorname{span}\{u_1, \dots, u_k\}$, then $V = \operatorname{null} T_1 \oplus U$.
(If this is not clear, note that for any $v \in V$, we have

$$T_1 v = c_1 w_1 + \cdots + c_k w_k = c_1 T_1 u_1 + \cdots + c_m T_1 u_m = T_1(c_1 u_1 + \cdots + c_m u_m).$$

Thus, $T_1 v = T_1 v'$, where $v' = c_1 u_1 + \cdots + c_m u_m \in U$. If $v'' = v - v'$, then $T_1 v'' = T_1 v - T_1 v' = 0$, so $v'' \in \operatorname{null} T_1$, and since the vectors $T_1 u_i$ form a basis for $\operatorname{range} T_1$, we see that $\operatorname{null} T_1 \cap U = \{0\}$.)
If we let $\{x_1, \dots, x_m\}$ be a basis for $\operatorname{null} T_1$, then

$$B_1 = \{u_1, \dots, u_m, x_1, \dots, x_k\}$$

is a basis of V. Similarly, if $\{y_1, \dots, y_m\}$ denotes a basis for $\operatorname{null} T_2$ (and recall from above that $\dim \operatorname{null} T_2 = \dim \operatorname{null} T_1$), then

$$B_2 = \{v_1, \dots, v_m, y_1, \dots, y_k\}$$

is also a basis of V. We now define a linear map $S : V \to V$ in terms of the basis B_1 by

$$Su_1 = v_1, \dots, Su_m = v_m, Sx_1 = y_1, \dots, Sx_k = y_k.$$

Then S is invertible, since it takes a basis to a basis. Given any $v \in V$, write

$$v = a_1 u_1 + \cdots + a_k u_k + b_1 x_1 + \cdots + b_m x_m.$$

Then we have

$$\begin{aligned}
T_1 v &= T_1(a_1 u_1 + \cdots + a_k u_k) + T_1(b_1 x_1 + \cdots + b_m x_m) \\
&= a_1 w_1 + \cdots + a_k w_k + 0 \\
&= a_1 T_2 v_1 + \cdots + a_k T_2 w_k + T_2(b_1 y_1 + \cdots + b_m y_m) \\
&= T_2(a_1 S u_1 + \cdots + a_k S u_k) + T_2(b_1 S x_1 + \cdots + b_m S x_m) \\
&= T_2(S(a_1 u_1 + \cdots + a_k u_k + b_1 x_1 + \cdots + b_m x_m)) \\
&= T_2(Sv).
\end{aligned}$$

3. Let $U \subseteq \mathbb{F}^\infty$ denote the vector space of sequences

with "finite support": for each $x = (x_n) \in U$ there exists a natural number N_x such that $x_i = 0$ for all $i \geq N_x$. Thus, each $x \in U$ looks like

$$x = (x_1, x_2, x_3, \dots, x_{N_x}, 0, 0, \dots).$$

Prove that the vector space U is isomorphic to the vector space $\mathcal{P}(\mathbb{F})$ of all polynomials (of arbitrary degree) with coefficients in $\mathbb{F}$.

Solution: We define a map $T : \mathcal{P}(\mathbb{R}) \to U$ as follows: for any element

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

of $\mathcal{P}(\mathbb{R})$, we define

$$T(p(x)) = (a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots).$$

This clearly defines a map from $\mathcal{P}(\mathbb{R})$ to U. It is linear, since if $q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$,[2] then

$$\begin{aligned}
T(p(x) + q(x)) &= T((a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n + b_{n+1}x^{n+1} + \cdots + b_m x^m) \\
&= (a_0 + b_0, a_1 + b_1, \ldots, a_n + b_n, b_{n+1}, \ldots, b_m, 0, 0, \ldots) \\
&= (a_0, a_1, \ldots, a_n, 0, 0, \ldots) + (b_0, b_1, \ldots, b_m, 0, 0, \ldots) \\
&= T(p(x)) + T(q(x)),
\end{aligned}$$

and for any scalar c,

$$\begin{aligned}
T(cp(x)) &= T(ca_0 + ca_1 x + \cdots + ca_n x^n) \\
&= (ca_0, ca_1, \ldots, ca_n, 0, 0, \ldots) \\
&= c(a_0, a_1, \ldots, a_n, 0, 0, \ldots) \\
&= cT(p(x)).
\end{aligned}$$

Thus $T : \mathcal{P}(\mathbb{R}) \to U$ is a linear map. It's clear that T is injective, since if

$$T(p(x)) = (0, 0, 0, \ldots),$$

we must have $p(x) = 0$ (since all the coefficients are zero), and T is surjective, since given any element $u = (a_0, a_1, \ldots, a_N, 0, 0, \ldots) \in U$, we can let

$$p(x) = a_0 + a_1 x + \cdots + a_N x^N,$$

and then $T(p(x)) = u$.

[2] Assume without loss of generality that $n \leq m$. If $m > n$ we can always reverse the roles of p and q.