Name: Solutions

1. For each of the following, given an example[1] of functions $f : A \to B$ and $g : B \to C$ that satisfy the stated conditions, or explain why no such example is possible:

[3] (a) The function f is a surjection, but the function $g \circ f$ is not a surjection.

Finte example: take $A = B = \{1\}$, and $C = \{1, 2\}$. Define $f : A \to B$ by $f(1) = 1$ and $g : B \to C$ by $g(1) = 1$. Then f is a surjection, since range $f = \{1\} = B$. However, we have $g \circ f(1) = g(f(1)) = g(1) = 1$, so the range of $g \circ f$ is $\{1\} \neq C$, and thus $g \circ f$ is not a surjection.

Real-valued example (the one from the back of the textbook): take $A = B = C = \mathbb{R}$ and define $f(x) = x$ and $g(x) = x^2$. Then f is a surjection, but $g \circ f(x) = x^2$ is not, since, for example -1 is not in the range of $g \circ f$.

[3] (b) The function f is an injection, but the function $g \circ f$ is not an injection.

Finite example: take $A = B = \{1, 2\}$ and $C = \{1\}$. Define $f : A \to B$ by $f(1) = 1$ and $f(2) = 2$, and define $g : B \to C$ by $g(1) = g(2) = 1$. Then f is clearly an injection, but $g \circ f(1) = g \circ f(2) = 1$, so $g \circ f$ is not an injection.

Real-valued example: use the same one as above (also from the back of the textbook).

2. Let $f : A \to B$ and $g : B \to A$ be functions, and let $I_B : B \to B$ denote the identity
[4] function on B. Prove that if $f \circ g = I_B$, then f is a surjection.

Proof: Suppose $f \circ g = I_B$, and let $b \in B$ be arbitrary. We need to show that there exists some $a \in A$ such that $f(a) = b$. Since g is a function from B to A, we can set $a = g(b)$. Then

$$f(a) = f(g(b)) = f \circ g(b) = I_B(b) = b,$$

which is what we needed to show.

[1] Arrow diagrams are acceptable, as long as they clearly indicate (a) what the sets A, B, C are, and (b) how the functions are defined.

Total: 10 points