Practice problems for change of variables, including Quizzes 14 and 15
Math 2580
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For Quiz 14 on Tuesday, you should be able to do the following:

1. Let D be the region bounded by the polar curve $r = 2\sin\theta$.

   (a) Find the area of D. (Caution: first identify the curve, and choose your limits of integration accordingly.

   (b) Find the volume of the solid that lies above the region D and below the graph of the function $f(x, y) = 4 - \sqrt{x^2 + y^2}$.

2. Evaluate $\displaystyle\iint_D e^{x^2+y^2}\, dA$, where D is the region bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

3. The volume of a solid T is given in cylindrical coordinates by $\displaystyle\int_0^{\pi/2}\int_0^2\int_0^{4-r^2} r\, dz\, dz\, d\theta$. Sketch the solid and compute its volume.

4. Evaluate the integral $\iiint_W \frac{1}{\sqrt{x^2+y^2}}\, dV$, where W is the region given by $0 \leq x \leq \sqrt{9 - y^2}$, $0 \leq y \leq 3$, $0 \leq z \leq \sqrt{9 - (x^2 - y^2)}$, using cylindrical coordinates.

5. Find the volume that the cylinder $r = a\cos\theta$ cuts out of the sphere $x^2 + y^2 + z^2 = a^2$, where $a > 0$.

6. Calculate the Jacobian of the transformation $T(u, v) = (u^2 v, uv^2)$.

For Quiz 15 on Thursday, you should be able to do the following:

1. The integral $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{\sqrt{8-x^2-y^2}} dz\,dy\,dx$ represents the volume of a solid. Sketch the solid, and compute its volume using either cylindrical or spherical coordinates, whichever you prefer.

2. Given the region R in the xy-plane bounded by the hyperbolas $y = 1/x$, $y = 4/x$, and the lines $y = x$, $y = 4x$, find equations for a transformation T that maps a rectangular region S in the uv-plane onto R, where the sides of S are parallel to the u- and v-axes.

3. Evaluate the integral $\iint_R (x+y)e^{x^2-y^2}\,dA$, where R is the rectangle defined by the inequalities $0 \le x - y \le 2$, $0 \le x + y \le 3$ by making an appropriate change of variables.

4. Evaluate the integral $\iint_D (6x - 4y)\,dA$, where D is the region bounded by the parallelogram with vertices $(0, 0)$, $(2, 3)$, $(4, 1)$, and $(6, 4)$.

5. Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to express the volume of the solid bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes as a triple integral in terms of u, v, and w. (You do not have to evaluate the integral. (Unless you want to.))

Extra fun: Let f be continuous on $[0, 1]$ and let R be the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Show that

$$\iint_R f(x+y)\,dA = \int_0^1 u f(u)\,du.$$

Additional practice problems:

1. Let D be the region in the first quadrant bounded by the curves $y = x$, $y = 2x$, $xy = 1$, and $xy = 2$.

   (a) Sketch the region, carefully noting all points of intersection of these curves.

   (b) Determine a change of variables that transforms a rectangle in the (u, v)-plane into the region D.

   (c) Use this change of variables to evaluate the integral $\displaystyle\iint_D \left(\frac{(x-y)^2}{x^2} - 1\right) dA$.

2. Repeat the problem above, if D is bounded by the curves $y = x^2$, $y = 2x^2$, $x = y^2$, and $x = 4y^2$, for the integral $\displaystyle\iint_D \left(\frac{x^2}{y^4} + \frac{y^2}{x^4}\right) dA$.

3. Using an appropriate change of variables, show that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.

4. Evaluate $\iint_D \sqrt{x^2+y^2}\, dA$, if D is bounded by the y-axis and the curve $y = \sqrt{4-x^2}$.

5. Evaluate the following integrals, either by converting to polar coordinates, or by reversing the order of integration:

   (a) $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\, dy\, dx.$

   (b) $\displaystyle\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\, dx\, dy.$

   (c) $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx.$