Solutions to practice problems for Quiz 17
Math 2580
Spring 2016

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1. Calculate the derivative of the following vector-valued functions:
   (a) $\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle \quad \mathbf{r}'(t) = \langle 2t, 3t^2, 4t^3 \rangle$
   (b) $\mathbf{r}(t) = \langle \sin(t), e^{3t}, \cos(2t) \rangle \quad \mathbf{r}'(t) = \langle \cos(t), 3e^{3t}, -2\sin(2t) \rangle$.
   (c) $\mathbf{r}(t) = \langle \sin(t^2), \ln(t^2+1) \rangle \quad \mathbf{r}'(t) = \langle 2t\cos(t^2), \frac{2t}{t^2+1} \rangle$.

2. Calculate $\|\mathbf{r}'(t)\|$ for the vector-valued functions in problem 1. (Note that if $\mathbf{r}(t)$ is interpreted as position with respect to time, then $\mathbf{r}'(t)$ is velocity, and $\|\mathbf{r}'(t)\|$ is speed.
   (a) $\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 9t^4 + 16t^6}$
   (b) $\|\mathbf{r}'(t)\| = \sqrt{\cos^2(t) + 9e^{6t} + 4\sin^2(2t)}$
   (c) $\|\mathbf{r}'(t)\| = \sqrt{4t^2\cos^2(t^2) + \frac{4t^2}{(t^2+1)^2}}$.

3. Show that $\dfrac{d}{dt}\|\mathbf{r}(t)\|^2 = 2\mathbf{r}(t)\cdot r'(t)$.

   Using the product rule $\dfrac{d}{dt}(\mathbf{a}(t)\cdot\mathbf{b}(t)) = \mathbf{a}'(t)\cdot\mathbf{b}'(t) + \mathbf{a}(t)\cdot\mathbf{b}'(t)$, we have
   $$\frac{d}{dt}\|\mathbf{r}(t)\|^2 = \frac{d}{dt}(\mathbf{r}(t)\cdot\mathbf{r}(t)) = \mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 2\mathbf{r}(t)\cdot\mathbf{r}'(t).$$

4. Determine a vector-valued function $\mathbf{r}(t)$ and an interval $[a,b]$ that parameterize the line segment from $(1,2,0)$ to $(4,-3,2)$.

   Since we're given two points on the line, we can form the direction vector $\mathbf{v} = \langle 4-1, -3-2, 2-0 \rangle = \langle 3, -5, 2 \rangle$, and the line segmenet is given by
   $$\mathbf{r}(t) = \langle 1,2,0 \rangle + t\mathbf{v} = \langle 1,2,0 \rangle + t\langle 3,-5,2 \rangle = \langle 1+3t, 2-5t, 2t \rangle,$$
   with $t \in [0,1]$. (Note that since we took $P = (1,2,0)$ as our initial point, $\mathbf{r}(0) = \langle 1,2,0 \rangle$, and taking $\mathbf{v} = \overrightarrow{PQ}$, where $Q = (4,-2,3)$ is the final point, guarantees that adding $\mathbf{v}$ once to the initial point takes us to the final point, so $\mathbf{r}(1) = \langle 4,-3,2 \rangle$.)

5. Evaluate $\displaystyle\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot r'(t)\, dt$ for the vector field $\mathbf{F}$ and curve $\mathbf{r}$ given by

(a) $\mathbf{F}(x,y) = x^2\mathbf{i} - xy\mathbf{j}$, and $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j}$, $a = 0$, $b = \pi$.

We have $\mathbf{F}(\sin t, \cos t) = \sin^2(t)\mathbf{i} - (\sin t)(\cos t)\mathbf{j}$, and $\mathbf{r}'(t) = \langle \cos t, -\sin t\rangle$, so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \sin^2(t)\cos(t) + \sin^2(t)\cos(t) = 2\sin^2(t)\cos(t).$$

It follows that $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^\pi 2\sin^2(t)\cos(t)\, dt = \sin^2(\pi) - \sin^2(0) = 0.$

(b) $\mathbf{F}(x,y,z) = \langle xy^2, xyz, yz^2\rangle$, $\mathbf{r}(t) = \langle t, t^2, 4t\rangle$, $a = 0$, $t = 1$.

Since $\mathbf{F}(t, t^2, 4t) = \langle t(t^2)^2, t(t^2)(4t), t^2(4t)^2\rangle = \langle t^5, 4t^4, 16t^4\rangle$ and $\mathbf{r}'(t) = \langle 1, 2t, 4\rangle$, we have

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t^5 + 4t^4(2t) + 16t^4(4) = 9t^5 + 64t^4,$$

so $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 (9t^5 + 64t^4)\, dt = \frac{9}{6} + \frac{64}{5}.$