Math 1410 Assignment #5 Solutions
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1. Let A be an $m \times n$ matrix. Let $\mathbb{R}^n$ and $\mathbb{R}^m$ denote the spaces of $n \times 1$ and $m \times 1$ column vectors, respectively. We define the *null space* of A by

$$\operatorname{null} A = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$

That is, $\operatorname{null} A$ is the set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$, which can also be thought of as the set of all solutions $\vec{x}$ to the homogeneous system of linear equations $A\vec{x} = \vec{0}$.

(a) Show that $\operatorname{null} A$ is a subspace of $\mathbb{R}^n$.

Let $\vec{0}$ denote the zero vector in $\mathbb{R}^n$. Since $A\vec{0} = \vec{0}$ for any $m \times n$ matrix A, we have that $\vec{0} \in \operatorname{null} A$.
If $\vec{x}, \vec{y} \in \operatorname{null} A$, then $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$; therefore, for any $a, b \in \mathbb{R}$,

$$A(a\vec{x} + b\vec{y}) = a(A\vec{x}) + b(A\vec{y}) = a\vec{0} = b\vec{0} = \vec{0},$$

so $a\vec{x} + b\vec{y} \in \operatorname{null} A$, and it follows that $\operatorname{null} A$ is a subspace of $\mathbb{R}^n$.

(b) If $A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{bmatrix}$, find a basis for $\operatorname{null} A$.

The reduced row-echelon form of A is given by

$$R = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the general solution to the equation $A\vec{x} = \vec{0}$ is given by

$$\vec{x} = \begin{bmatrix} 2s + t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

It follows that any element of null$A$ can be written as a linear combination of the vectors $\vec{u} = \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 1 & 0 & -2 & 1 \end{bmatrix}^T$, so $B = \{\vec{u}, \vec{v}\}$ is a basis for null$A$.

2. (a) Compute the norm of the complex numbers

$$\begin{aligned}\vec{u} &= \langle 1+i, 2-3i \rangle \\ \vec{z} &= \langle 1, 1-i, -2, i \rangle \\ \vec{w} &= \langle 1-i, 1+i, 1, 3-4i \rangle\end{aligned}$$

We have

$$\begin{aligned}\|\vec{u}\| &= \sqrt{(1^1+1^2)+(2^2+(-3)^2)} = \sqrt{15}. \\ \|\vec{v}\| &= \sqrt{(1^2+0^2)+(1^2+(-1)^2)+((-2)^2+0^2)+(0^2+1^2)} = \sqrt{8} \\ \|\vec{w}\| &= \sqrt{(1^2+(-1)^2)+(1^2+1^2)+(1^2+0^2)+(3^2+(-4)^2)} = \sqrt{30}.\end{aligned}$$

(b) Determine whether or not the following pairs of complex vectors are orthogonal:

$$\begin{aligned}\vec{z} = \langle 4, -3i, 2+i \rangle &\text{ and } \vec{w} = \langle i, 2, 2-4i \rangle \\ \vec{z} = \langle i, -i, 2+i \rangle &\text{ and } \vec{w} = \langle i, i, 2-i \rangle \\ \vec{z} = \langle 1, 1, i, i \rangle &\text{ and } \vec{w} = \langle 1, i, -i, 1 \rangle\end{aligned}$$

For the first pair, we have

$$\vec{z} \cdot \vec{w} = 4(-i) - 3i(2) + (2+i)(2+4i) = -4i - 6i + (4-4+8i+2i) = 0,$$

so this pair is orthogonal. For the second pair, we find

$$\vec{z} \cdot \vec{w} = i(-i) - i(-i) + (2+i)(2+i) = 1 - 1 + (4-1+2i+2i) = 3+4i \neq 0,$$

so this pair is not orthogonal. Finally, for the last pair, we get

$$\vec{z} \cdot \vec{w} = 1(1) + 1(-i) + i(i) + i(1) = 1 - i - 1 + i = 0,$$

so this pair is orthogonal.

3. In each case, decide whether the matrix A is diagonalizable. If so, find a matrix P such that $P^{-1}AP$ is diagonal:

(a) $A = \begin{bmatrix} 3 & 0 & 6 \\ 0 & -3 & 0 \\ 5 & 0 & 2 \end{bmatrix}$

We find that

$$\begin{aligned} c_A(x) &= \begin{vmatrix} x-3 & 0 & -6 \\ 0 & x+3 & 0 \\ -5 & 0 & x-2 \end{vmatrix} \\ &= (x+3)\begin{vmatrix} x-3 & -6 \\ -5 & x-2 \end{vmatrix} \\ &= (x+3)(x^2-5x-24) = (x+3)^2(x-8). \end{aligned}$$

Since the eigenvalue $\lambda = -3$ has multiplicity 2, we first compute the eigenspace $E(-3,A)$. We find that

$$A + 3I = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 0 \\ 5 & 0 & 5 \end{bmatrix},$$

which reduces to the row-echelon form $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, we have

$$E(-3,A) = \text{null}(A+3I) = \left\{ \begin{bmatrix} -t \\ s \\ t \end{bmatrix} \,|\, s,t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since $\dim E(-3,A) = 2$ matches the multiplicity of $\lambda = -3$, we know that we can diagonalize: we have the two independent eigenvectors $X_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$ and $X_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$. To find the matrix P, we need to also find an eigenvector corresponding to the eigenvalue $\lambda = 8$. Since

$$A - 8I = \begin{bmatrix} -5 & 0 & 6 \\ 0 & 5 & 0 \\ 5 & 0 & -6 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & -6/5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we see that $(A-8I)X = 0$ for $X = \begin{bmatrix} \frac{6}{5}t \\ 0 \\ t \end{bmatrix} = \frac{t}{5}\begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}$, an eigenvector for $\lambda = 8$ is given by $X_3 = \begin{bmatrix} 6 & 0 & 5 \end{bmatrix}^T$. Arranging our three eigenvectors as columns of a matrix, we find the matrix

$$P = \begin{bmatrix} -1 & 0 & 6 \\ 0 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}.$$

(b) $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

In this case,

$$c_A(x) = \begin{vmatrix} x-4 & 0 & 0 \\ 0 & x-2 & -2 \\ -2 & -3 & x-1 \end{vmatrix} = (x-4)^2(x+1),$$

so the eigenvalues of A are $\lambda = 4$, with multiplicity 2, and $\lambda = -1$, with multiplicity 1. We begin with the repeated eigenvalue: since

$$A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 2 & 3 & -3 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that $A - 4I$ has rank 2, and thus the dimension of null$(A - 4I)$ will be equal to 1, so $\dim E(4, A) = 1 < 2$, which means that A cannot be diagonalized.

4. (**Bonus**) Prove the following: Let A be any $n \times n$ matrix, and let $p(x)$ be a polynomial. Recall that if $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$, then

$$p(A) = a_0I_n + a_1A + a_2A^2 + \cdots + a_kA^k.$$

(a) Supppose that $\vec{x}$ is an eigenvector of A with eigenvalue λ. Prove that

$$p(A)\vec{x} = p(\lambda)\vec{x},$$

where $p(\lambda)$ denotes the scalar obtained by subsituting $x = \lambda$ in the polynomial $p(x)$.

Suppose that $A\vec{x} = \lambda\vec{x}$. It follows that

$$A^n\vec{x} = A^{n-1}(A\vec{x}) = A^{n-1}(\lambda\vec{x}) = \lambda A^{n-1}\vec{x} = \lambda^2A^{n-2}\vec{x} = \cdots = \lambda^n\vec{x},$$

and therefore

$$\begin{aligned} p(A)\vec{x} &= (a_0I_n + a_1A + \cdots + a_kA^k)\vec{x} \\ &= a_0\vec{x} + a_1(A\vec{x}) + \cdots + a_k(A^k\vec{x}) \\ &= a_0\vec{x} + a_1\lambda\vec{x} + \cdots + a_k\lambda^k\vec{x} \\ &= (a_0 + a_1\lambda + \cdots + a_k\lambda^k)\vec{x} = p(\lambda)\vec{x}, \end{aligned}$$

as required.

(b) Prove that if A is diagonalizable, then the Cayley-Hamilton theorem holds: we have $c_A(A) = 0$, where A is the characteristic polynomial of A.

If A can be diagonalized, then every vector $\vec{x}$ in $\mathbb{R}^n$ can be written as

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n,$$

where $c_1, \ldots, c_n$ are scalars, and $\vec{x}_1, \ldots, \vec{x}_n$ are eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively. Note that for each λ_i, we have $c_A(\lambda_i) = 0$. It follows that for any vector $\vec{x}$ in $\mathbb{R}^n$, we have

$$\begin{aligned} c_A(A)\vec{x} &= c_A(A)(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n) \\ &= c_1(c_A(A)\vec{x_1}) + \cdots + c_n(c_A(A)\vec{x}_n) \\ &= c_1(c_A(\lambda_1)\vec{x_1}) + \cdots + c_n(c_A(\lambda_n)\vec{x}_n) \\ &= c_1(0\vec{x}_1) + \cdots + c_n(0\vec{x}_n) = 0. \end{aligned}$$

Since $\vec{x}$ was arbitrary, it must be the case that $c_A(A) = 0$.