Path Independence and Conservative Vector Fields

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Recall that in class we showed that for any gradient vector field $\mathbf{F} = \nabla f$, the line integral of $\mathbf{F}$ along a curve C depends only on the endpoints of C. Our argument was as follows: let C be a smooth oriented curve in $\mathbb{R}^n$ (for most purposes, $n = 2$ or $n = 3$, but this restriction is unnecessary), and let $\mathbf{r} : [a, b] \to \mathbb{R}^n$ be a parameterization of C, such that $\mathbf{r}(a) = P$ is the initial point of C, and $\mathbf{r}(b) = Q$ is the final point of C. By the definition of the line integral and the Fundamental Theorem of Calculus, we have

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t)))\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(Q) - f(P).$$

The answer therefore depends only on the endpoints. We call any vector field $\mathbf{F}$ with this property **conservative**. The above argument proves the following“

Theorem 1 (Fundamental Theorem for Line Integrals)**.** *If $\mathbf{F} = \nabla f$ for some function f, then $\mathbf{F}$ is conservative, and $\int_C \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P)$ for any smooth oriented curve C with initial point P and final point Q.*

The goal of this handout is to prove the converse:

Theorem 2. *Let D be an open[1] connected[2] If $\mathbf{F}$ is a continuous, conservative vector field on defined on D, then there exists a C^1 function $f : D \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.*

Proof. We prove the theorem by explicitly constructing the function f. (If we view Theorem 1 as the analogue of Part II of the Fundamental Theorem of Calculus, then this is the analogue of Part I.) Fix a point $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in D$, and define a function $f : D \to \mathbb{R}$ by

$$f(x_1, x_2, \ldots, x_n) = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is any curve contained in D with initial point $\mathbf{a}$ and final point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. (The assumption that D is an open, connected set guarantees that such a curve C exists.) Since $\mathbf{F}$ is conservative, the above integral depends only on $\mathbf{x}$, and not the curve C, so that f is indeed a well-defined function. (That is, the value of $f(\mathbf{x})$ is uniquely determined by $\mathbf{x}$.)

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[1] An open subset of $\mathbb{R}^n$ is one that does not contain its boundary points; for example, the set $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ consisting of all points inside the unit circle (the boundary of the set), but not the circle itself

[2] A connected subset of $\mathbb{R}^n$ is one that cannot be separated into two or more pieces, with “gaps” in between. For the purposes of this theorem, it's enough to know that any open, connected subset of $\mathbb{R}^n$ is **path-connected**, meaning that any two points in the set can be joined by a continuous curve.

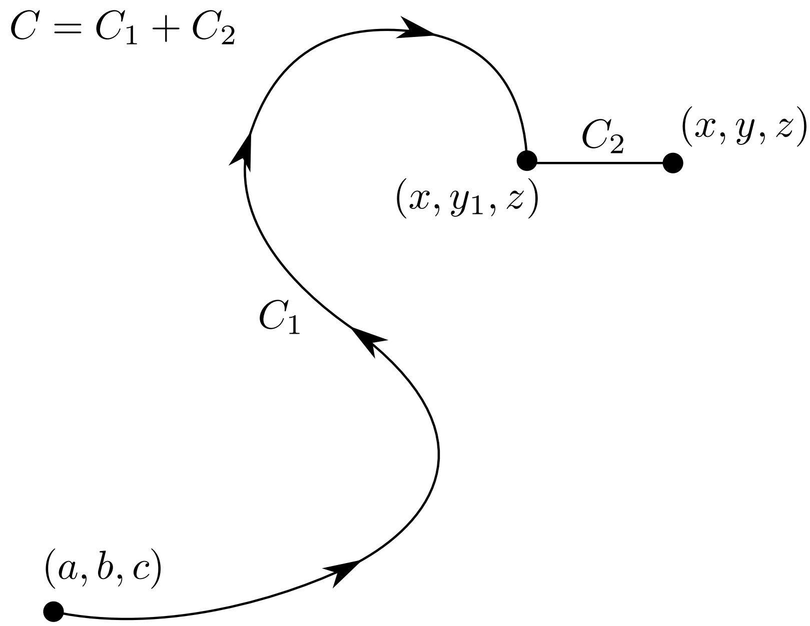

Figure 1: The curve C used to calculate $\dfrac{\partial f}{\partial y}$.

To simplify notation, we'll take $n = 3$, with $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, and initial point (a, b, c) and final point (x, y, z) for the curve C. The general proof is similar. We want to show that $\nabla f = \mathbf{F}$, so we begin by showing that $\dfrac{\partial f}{\partial x} = P$. Since C can be any curve with initial point (a, b, c) and final point (x, y, z), we choose a curve of the form $C = C_1 + C_2$, defined as follows:

First, choose a fixed value x_0 sufficiently close to x, such that the line segment from (x_0, y, z) to (x, y, z) lies entirely within D. We let C_1 be an arbitrary curve from (a, b, c) to (x_0, y, z), and we let C_2 be the line segment from (x_0, y, z) to (x, y, z). Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

The first integral does not depend on x, so we have

$$\frac{\partial}{\partial x} \int_C \mathbf{F} \cdot d\mathbf{r} = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

For the second integral, we parameterize C_2 using $r(t) = \langle t, y, z \rangle$, with $t \in [x_0, x]$. This gives us

$$\mathbf{F} \cdot d\mathbf{r} = P\,dx + Q\,dy + R\,dz = P(t, y, z)\,dt,$$

since $dt = dx$, and $dy = dz = 0$ (we're holding y and z constant). We then have

$$\frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \frac{\partial}{\partial x} \int_{x_0}^{x} P(t, y, z)\,dt = P(x, y, z),$$

by the Fundamental Theorem of Calculus. The proofs that $f_y = Q$ and $f_z = R$ are similar. $\square$

We also have the following result:

Theorem 3. *A vector field* $\mathbf{F}$ *is conservative if and only if* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *for every* ***closed*** *curve* C*.*

To see that this result is true, notice that by Theorem 2, if $\mathbf{F}$ is conservative (that is, if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path), then $\mathbf{F} = \nabla f$ for some function f, so if C is a closed curve parameterized by $\mathbf{r}(t)$, $t \in [a, b]$, we have $\mathbf{r}(a) = \mathbf{r}(b)$, and thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0.$$

On the other hand, suppose we know that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C, and let C_1 and C_2 be any two oriented curves from a point P to a point Q. Then the curve $-C_2$ is a curve from Q to P, and joining C_1 to $-C_2$ gives us the closed curve $C = C_1 - C_2$. Thus, we have

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r},$$

which implies that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Since C_1 and C_2 were arbitrary, we can conclude that $\mathbf{F}$ is conservative.