Solutions for Quiz 3 Practice Problems
Math 2580
Spring 2016

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January 19th, 2016

1. Express the partial derivative $f_y(x,y,z)$ as a limit.

$$f_y(x,y,z) = \lim_{h\to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}.$$

(Note that this is consistent with the procedural description of holding the variables x and z constant while taking the derivative with respect to y.)

2. Calculate all four second-order partial derivatives for the function $f(x,y) = \cos(xy^2)$. Verify that the mixed second-order partial derivatives are equal.

We have $f_x(x,y) = -y^2\sin(xy^2)$ and $f_y(x,y) = -2xy\sin(xy^2)$, so

$$\begin{aligned}
f_{xx}(x,y) &= \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}(-y^2\sin(xy^2)) = -y^4\cos(xy^2),\\
f_{xy}(x,y) &= \frac{\partial^2 f}{\partial y\partial x}(x,y) = \frac{\partial}{\partial y}(-y^2\sin(xy^2)) = -2y\sin(xy^2) - 2xy^3\cos(xy^2),\\
f_{yx}(x,y) &= \frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial}{\partial x}(-2xy\sin(xy^2)) = -2y\sin(xy^2) - 2xy^3\cos(xy^2),\\
f_{yy}(x,y) &= \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}(-2xy\sin(xy^2)) = -2x\sin(xy^2) - 4x^2y^2\cos(xy^2).
\end{aligned}$$

We can easily see from the above that $f_{xy}(x,y) = f_{yx}(x,y)$.

Note: This is called **Clairault's Theorem**; it states that the mixed second-order partial derivatives are equal provided that all the second-order partial derivatives of f are continuous. We won't need this result (or second-order derivatives generally) until we discuss classification of critical points in another week or two, so I haven't mentioned it in class yet.

Note #2: On your assignment (due tomorrow) you don't need to show that the second-order derivatives are not continuous (it's a bit of a mess).

3. Give a convincing (but not rigorous – no $\epsilon - \delta$) argument that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + y^3}{x^2 + y^2} = 0.$$

The argument amounts to observing that $\dfrac{x^2y + y^3}{x^2 + y^2} = \dfrac{y(x^2 + y^2)}{x^2 + y^2} = y$, and pointing out that it's clear that if $(x, y) \to (0, 0)$, then in particular it must be true that $y \to 0$.

4. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$ does not exist.

It's easy to check that if we let $(x, y) \to (0, 0)$ along the x-axis or y-axis, the limit is zero. In fact, along any line $y = mx$, we have, for $x \neq 0$,

$$\frac{xy^3}{x^2 + y^6} = \frac{m^3x^4}{x^2 + m^6x^6} = \frac{m^3x^2}{1 + m^6x^4},$$

so as $x \to 0$, we get a limit of zero. However, along the curve $x = y^3$, we have, for $y \neq 0$,

$$\frac{xy^3}{x^2 + y^6} = \frac{y^3}{y^6 + y^6} = \frac{1}{2},$$

so if $(x, y) \to (0, 0)$ along this curve we get a limit of $\dfrac{1}{2} \neq 0$. Since we get different values when we approach the origin along different paths, the limit does not exist.

5. Find the equation of the tangent plane to the graph $z = x^3 + y^3 - 6xy$ at the point $(1, 2, -3)$.

In general, the tangent plane to $z = f(x, y)$ at the point $(a, b, f(a, b))$ is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We have $f(1, 2) = -3$, $f_x(x, y) = 3x^2 - 6y$, so $f_x(1, 2) = 3(1)^2 - 6(2) = -9$, and $f_y(x, y) = 3y^2 - 6x$, so $f_y(1, 2) = 3(2)^2 - 6(1) = 6$, and thus the equation of the plane is

$$z = -3 - 9(x - 1) + 6(y - 2).$$

If you must simplify, this becomes $9x - 6y + z = -6$.