Math 1560 Assignment #5 Solutions
University of Lethbridge, Fall 2017

Sean Fitzpatrick

November 29, 2017

1. Let f and g be two functions with 3 or more continuous derivatives at $x = a$.

   Let $p(x)$ and $q(x)$ be the degree 3 Taylor polynomials at $x = a$ for f and g, respectively.

   (a) Compute the product $p(x)q(x)$.

   We have

$$\begin{aligned}
p(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3\\
q(x) &= g(a) + g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + \frac{g'''(a)}{6}(x-a)^3
\end{aligned}$$

   This gives us

$$\begin{aligned}
p(x)q(x) &= \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3\right)\\
&\quad \times \left(g(a) + g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + \frac{g'''(a)}{6}(x-a)^3\right)\\
&= f(a)g(a) + (f'(a)g(a) + f(a)g'(a))(x-a)\\
&\quad + \left(\frac{f''(a)}{2}g(a) + f'(a)g'(a) + f(a)\frac{g''(a)}{2}\right)(x-a)^2\\
&\quad + \left(\frac{f'''(a)}{6}g(a) + \frac{f''(a)}{2}g'(a) + f'(a)\frac{g''(a)}{2} + f(a)\frac{g'''(a)}{6}\right)(x-a)^3\\
&\quad + \left(\frac{f'''(a)}{6}g'(a) + \frac{f''(a)}{2}\cdot\frac{g''(a)}{2} + f'(a)\frac{g'''(a)}{6}\right)(x-a)^4\\
&\quad + \left(\frac{f'''(a)}{6}\cdot\frac{g''(a)}{2} + \frac{f''(a)}{2}\frac{g'''(a)}{6}\right)(x-a)^5 + \frac{f'''(a)}{6}\cdot\frac{g'''(a)}{6}(x-a)^6.
\end{aligned}$$

   (b) Compute the degree 3 Taylor polynomial at $x = a$ for fg. (You may use your results from Assignment 2 here.)

From Assignment 2, we had

$$\begin{aligned}(fg)'(a) &= f'(a)g(a) + f(a)g'(a)\\ (fg)''(a) &= f''(a)g(a) + 2f'(a)g'(a) + f(a)g''(a)\\ (fg)'''(a) &= f'''(a)g(a) + 3f''(a)g'(a) + 3f'(a)g''(a) + f(a)g'''(a)\end{aligned}$$

The degree 3 Taylor polynomial for $f(x)g(x)$ at $x = a$ is therefore

$$\begin{aligned}P_3(x) = f(a)g(a) &+ (f'(a)g(a) + f(a)g'(a))(x-a)\\ &+ \frac{(f''(a)g(a) + 2f'(a)g'(a) + f(a)g''(a))}{2}(x-a)^2\\ &+ \frac{(f'''(a)g(a) + 3f''(a)g'(a) + 3f'(a)g''(a) + f(a)g'''(a))}{6}(x-a)^3.\end{aligned}$$

(c) How do your answers in parts (a) and (b) compare?

Since

$$\frac{1}{2}(f''(a)g(a) + 2f'(a)g'(a) + f(a)g''(a)) = \frac{f''(a)}{2}g(a) + f'(a)g'(a) + f(a)\frac{g''(a)}{2}$$

and

$$\begin{aligned}&\frac{1}{6}(f'''(a)g(a) + 3f''(a)g'(a) + 3f'(a)g''(a) + f(a)g'''(a)) =\\ &\qquad\qquad \frac{f'''(a)}{6}g(a) + \frac{f''(a)}{2}g'(a) + f'(a)\frac{g''(a)}{2} + f(a)\frac{g'''(a)}{6},\end{aligned}$$

we see that, up to degree three, the two polynomials agree. (That is, if we drop the terms involving $(x-a)^k$ for $k = 4, 5, 6$ from the product $p(x)q(x)$, the results are the same.)

2. Consider the function $f(x) = 3x^2 - 2x$, for $x \in [1, 3]$.

(a) Determine the partition points $x_0, x_1, \ldots, x_{10}$ for a uniform partition of $[1, 3]$ into 10 subintervals.

We have $\Delta x = \frac{3-1}{10} = \frac{1}{5} = 0.2$. Our partition is therefore given by

$$\begin{aligned}&x_0 = 1, x_1 = 1.2, x_2 = 1.4, x_3 = 1.6, x_4 = 1.8, x_5 = 2,\\ &\qquad\qquad x_6 = 2.2, x_7 = 2.4, x_8 = 2.6, x_9 = 2.8, \text{ and } x_{10} = 3.\end{aligned}$$

(b) Using the "Riemann sum calculator #2" (available in the section for November 21 - 23 on Moodle), estimate the value of $\int_1^3 f(x)\,dx$ using your partition from part (a) and:

(i) left endpoints: $\int_1^3 f(x)\,dx \approx 16.04$

(ii) right endpoints: $\int_1^3 f(x)\,dx \approx 20.04$

(c) Estimate the value of $\int_1^3 f(x)\,dx$ using your partition from part (a) and the midpoint of each subinterval.

We choose $c_i = \frac{x_{i-1}+x_i}{2}$, for $i = 1, \ldots, 10$, and evaluate, as follows:

i	1	2	3	4	5	6	7	8	9	10
c_i	1.1	1.3	1.5	1.7	1.9	2.1	2.3	2.5	2.7	2.9
$f(c_i)$	1.43	2.47	3.75	5.27	7.03	9.03	11.27	13.75	16.47	19.43

Our approximation is given by

$$\int_1^3 f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\Delta x = 0.2\sum_{i=1}^{n} f(c_i),$$

since $\Delta x = 0.2$ is common to each term and can be factored out. Thus, we need to add up the 10 values $f(c_1), \ldots, f(c_{10})$ and then multiply by 0.2. (All of this is most easily done using a spreadsheet.) We find that

$$\int_1^3 f(x)\,dx \approx 17.98.$$

(d) Estimate the value of $\int_1^3 f(x)\,dx$ using your partition from part (a) and the trapezoid rule.

The trapezoid rule was given to us in the problem as

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right] \Delta x_i$$

where in our case, $n = 10$. Using properties of summation, we find

$$\sum_{i=1}^{n} \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right] \Delta x_i = \frac{1}{2}\left(\sum_{i=1}^{n} f(x_{i-1})\Delta x_i + \sum_{i=1}^{n} f(x_i)\Delta x_i\right).$$

But this is just the average of the sum obtained using left endpoints and the sum obtained using right endpoints. Since we already have these values in part (b), we get the approximation

$$\int_1^3 f(x)\,dx \approx \frac{16.04 + 20.04}{2} = 18.04.$$

3. Let $f(x) = 3x^2 - 2x$, for $x \in [1, 3]$.

(a) Let n be any positive integer, and let $x_0, x_1, \ldots, x_n$ be the points of a uniform partition of $[1, 3]$ into n subintervals. Determine an expression for x_i in terms of i, where $1 \le i \le n$.

We have $\Delta x = \frac{3-1}{n} = \frac{2}{n}$. Since $\Delta x = x_i - x_{i-1}$ for each i, we have $x_i = x_{i-1} + \Delta x$, for $i = 1, \ldots, n$. This gives us

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= x_0 + \Delta x = 1 + \frac{2}{n} \\
x_2 &= x_1 + \Delta x = 1 + \frac{2}{n} + \frac{2}{n} = 1 + \frac{2(2)}{n} \\
x_3 &= x_2 + \Delta x = 1 + \frac{2(2)}{n} = \frac{2}{n} = 1 + \frac{3(2)}{n} \\
\cdots
\end{aligned}$$

Recognizing the pattern, we see that in order to reach x_i, we add $\Delta x = \frac{2}{n}$ to $x_0 = 1$ i times, giving us

$$x_i = 1 + \frac{2i}{n}.$$

(b) Write out the Riemann sum $R(f, P)$ for $f(x)$ on $[1, 3]$ using your partition from part (a) and **right** endpoints.

Our Riemann sum formula is given by

$$R(f, P) = \sum_{i=1}^{n} f(c_i)\Delta x_i,$$

where in our case $\Delta x_i = \frac{2}{n}$ for each i, and since we're taking right endpoints for $c_i \in [x_{i-1}, x_i]$, we have $c_i = x_i = 1 + \frac{2i}{n}$. This gives us

$$f(c_i) = f\left(1 + \frac{2i}{n}\right) = 3\left(1 + \frac{2i}{n}\right)^2 - 2\left(1 + \frac{2i}{n}\right) = \frac{12i^2}{n^2} + \frac{8i}{n} + 1,$$

so

$$R(f, P) = \sum_{i=1}^{n} \left(\frac{12i^2}{n^2} + \frac{8i}{n} + 1\right)\frac{2}{n}.$$

(c) Determine the value of $\int_1^3 f(x)\,dx$ by computing $\lim_{n\to\infty} R(f, P)$.

We first re-write our Riemann sum in three parts using properties of summation:

$$\begin{aligned}
R(f,P) &= \sum_{i=1}^{n}\left(\frac{12i^2}{n^2}+\frac{8i}{n}+1\right)\frac{2}{n} \\
&= \sum_{i=1}^{n}\frac{12i^2}{n^2}\cdot\frac{2}{n}+\sum_{i=1}^{n}\frac{8i}{n}\cdot\frac{2}{n}+\sum_{i=1}^{n}1\cdot\frac{2}{n} \\
&= \frac{24}{n^3}\sum_{i=1}^{n}i^2+\frac{16}{n^2}\sum_{i=1}^{n}i+\frac{2}{n}\sum_{i=1}^{n}1.
\end{aligned}$$

Using the summation formulas

$$\sum_{i=1}^{n}1=n, \sum_{i=1}^{n}i=\frac{n(n+1)}{2}, \text{ and } \sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$$

given in the textbook, we find

$$\begin{aligned}
R(f,P) &= \frac{24}{n^3}\sum_{i=1}^{n}i^2+\frac{16}{n^2}\sum_{i=1}^{n}i+\frac{2}{n}\sum_{i=1}^{n}1 \\
&= \frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{16}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{2}{n}(n) \\
&= 4\left(\frac{n(n+1)(2n+1)}{n^3}\right)+8\left(\frac{n(n+1)}{n^2}\right)+2.
\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{n^3}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)=1\cdot 2=2$$

and

$$\lim_{n\to\infty}\frac{n(n+1)}{n^2}=\lim_{n\to\infty}\frac{n+1}{n}=1,$$

we find that

$$\int_1^3 f(x)\,dx=\lim_{n\to\infty}R(f,P)=4(2)+8(1)+2=18.$$

(d) Which of your estimates in Q2 was closest to the exact value in part (c)?

Our approximation was 17.98 using midpoints, and 18.04 using the Trapezoid rule. In this case, the midpoint approximation is the closest to the exact value.

(Note: this is not always the case. In fact, in most cases the Trapezoid rule is more reliable.)