Solutions to Quiz 12 Practice
Math 2580
Spring 2016

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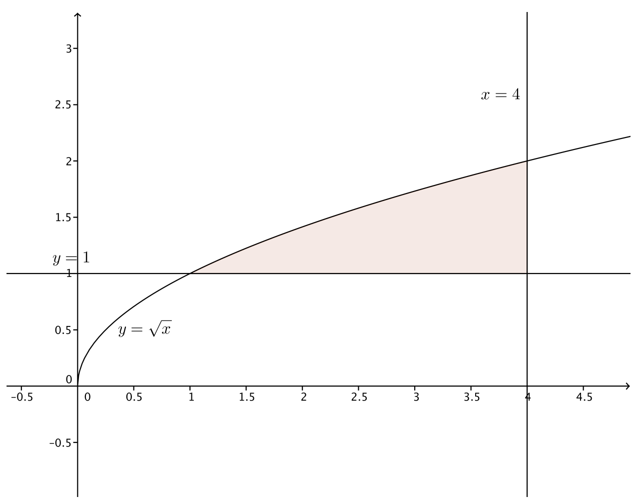
1. For the following double integral, sketch the region of integration, change the order of integration, and evaluate:
$$\int_1^4 \int_1^{\sqrt{x}} (x^2 + y^2)\, dy\, dx.$$
Our region of integration is bounded above by $y = \sqrt{x}$ and below by $y = 1$, for $1 \leq x \leq 4$, which gives us the following Type 1 region:



Reversing the order, we have the Type 2 region $y^2 \leq x \leq 4$, for $1 \leq y \leq 2$. Thus, we have

$$\begin{aligned}
\int_1^4 \int_1^{\sqrt{x}} (x^2+y^2)\,dy\,dx &= \int_1^2 \int_{y^2}^4 (x^2+y^2)\,dx\,dy \\
&= \int_1^2 \left(\frac{1}{3}x^3 + xy^2 \Big|_{x=y^2}^{x=4} \right) dy \\
&= \int_1^2 \left(\frac{64}{3} - \frac{y^6}{3} + 4y^2 - y^4 \right) dy \\
&= \frac{64y}{3} - \frac{y^7}{21} + \frac{4y^3}{3} - \frac{y^5}{5} \Big|_1^2 \\
&= \frac{64}{3}(2-1) - \frac{1}{21}(2^7-1) + \frac{4}{3}(2^3-1) - \frac{1}{5}(2^5-1).
\end{aligned}$$

2. Evaluate the integral $\iiint_B x^2\,dV$, where $B = [0,1]\times[-1,1]\times[0,1]$.

We have

$$\iiint_B x^2\,dV = \int_0^1 \int_{-1}^1 \int_0^1 x^2\,dx\,dy\,dz = \int_0^1 \int_{-1}^1 \frac{1}{3}\,dy\,dz = \int_0^1 \frac{2}{3}\,dz = \frac{2}{3}.$$

3. Write the integral $\iiint_W f(x,y,z)\,dV$, where W is the region between the cone $z = \sqrt{x^2+y^2}$ and the paraboloid $z = x^2+y^2$, as an interated integral. (Start by describing W using inequalities of the form $g_1(x,y) \le z \le g_2(x,y)$, where $(x,y) \in D$, and then describe D as either a Type 1 or Type 2 region.)

The cone and parabola intersect when $\sqrt{x^2+y^2} = x^2+y^2$. Letting $r = \sqrt{x^2+y^2}$, this requires $r = r^2$, so $r = 0$ or $r = 1$. (Notice that the region of integration can be obtained by taking the area above the parabola $z = y^2$ and below the line $z = y$, and revolving about the z-axis.) So the two surfaces intersect at the origin, and again along the circle $x^2+y^2 = 1$ in the plane $z = 1$.

We can thus describe our region via the inequalities $x^2+y^2 \le z \le \sqrt{x^2+y^2}$, where $x^2+y^2 \le 1$. We can describe the disc $x^2+y^2 \le 1$ as the Type 1 region $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. Thus,

$$\iiint_W f(x,y,z)\,dV = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} f(x,y,z)\,dz\,dy\,dx.$$

4. Evaluate the integral $\iiint_W z\,dV$, where W is the region bounded by the cylinder $x^2+y^2 = 4$ and the planes $z = 0$ and $z = 1$.

Here, we have $0 \le z \le 1$, where $x^2 + y^2 \le 4$. Carrying out the integral over z, we have

$$\iiint_W z\,dV = \iint_D (\int_0^1 z\,dz)\,dA = \iint_D \frac{1}{2}\,dA = \frac{A(D)}{2},$$

where D is the disc $x^2 + y^2 \le 4$. We know that this disc has area $A(D) = \pi(2)^2 = 4\pi$, so the result of the integral must be 2π.

5. Describe the surfaces given in cylindrical coordinates by (i) $r = 3$, (ii) $\theta = \pi/4$, and (iii), $z = 2$.

   (i) This is the cylinder $x^2 + y^2 = 9$.

   (ii) This is the portion of the plane $x = y$ for which $x, y \ge 0$. (It is a vertical plane obtained by extending horizontally from the z-axis along the line $x = y$ for each z-value.

   (iii) This is just the horizontal plane $z = 2$, as usual.

6. Express the surface $z = x^2 + y^2$ in spherical coordinates.

   In spherical coordinates we have $x = \rho\cos\theta\sin\varphi$ and $y = \rho\sin\theta\sin\varphi$, so

$$x^2 + y^2 = \rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi = \rho^2\sin^2\varphi,$$

   and $z = \rho\cos\varphi$. Equating the two gives $\rho\cos\varphi = \rho^2\sin^2\varphi$, which we can simplify to $\rho = \cot\varphi\csc\varphi$. (Note that this last equation still has $\rho = 0$ as a solution, so we haven't lost the solution $\rho = 0$ by cancelling ρ from both sides.)