1. Consider the integrals $\int_1^4 x^2\,dx$, $\int_1^4 2x\,dx$, and $\int_1^4 (x^2-2x)\,dx$

   (a) Approximate the value of each integral using 6 rectangles, and left endpoints.

   With 6 rectangles, our partition has $\Delta x = \frac{1}{2}$, and is given by $P = \{1, 1.5, 2, 2.5, 3, 3.5, 4\}$. We have

$$\int_1^4 x^2\,dx \approx \left(1^2 + (1.5)^2 + 2^2 + (2.5)^2 + 3^2 + (3.5)^2\right)\Delta x = 17.375,$$

   and

$$\int_1^4 2x\,dx \approx (2(1) + 2(1.5) + 2(2) + 2(2.5) + 2(3) + 2(3.5))\,\Delta x = 13.5$$

   using left endpoints. If you used right endpoints instead, then we drop the $f(1)$ terms above and replace it by $f(4)$, giving 24.875 for the first integral, and 16.5 for the second.
   Since $\int_1^4 (x^2-2x)\,dx = \int_1^4 x^2\,dx - \int_1^4 2x\,dx$, we can approximate the last rectangle by subtracting our two approximations. Thus,

$$\int_1^4 (x^2-2x)\,dx = 3.875$$

   using left endpoints, and 8.325 using right endpoints.

   (b) Find an expression (in terms of n) for the value of each integral using n rectangles, and right endpoints.

   With n rectangles, we have $\Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}$, and $x_i = x_0 + i\Delta x = 1 + \dfrac{3i}{n}$.
   For the first integral, using $c_i = x_i$, we have

$$f(x_i) = \left(1 + \frac{3i}{n}\right)^2 = 1 + \frac{6i}{n} + \frac{9i^2}{n^2}.$$

   We thus have

$$\int_1^4 x^2\,dx \approx \sum_{i=1}^n \frac{3}{n}\left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right).$$

   If we use the formulas $\displaystyle\sum_{i=1}^n 1 = n, \sum_{i=1}^n i = \frac{n(n+1)}{2}, \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$, we get

$$\begin{aligned}
\int_1^4 x^2\,dx &\approx \frac{3}{n}\sum_{i=1}^n 1 + \frac{18}{n^2}\sum_{i=1}^n i + \frac{27}{n^3}\sum_{i=1}^n i^2 \\
&= \frac{3}{n}(n) + \frac{18}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 3 + 9\left(\frac{n+1}{n}\right) + \frac{9}{2}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = 21 + \frac{27}{2n} + \frac{3}{2n^2}.
\end{aligned}$$

For the second integral, we similarly have

$$\int_1^4 2x\,dx \approx \sum_{i=1}^n 2\left(1+\frac{3i}{n}\right)\frac{3}{n} = \sum_{i=1}^n \left(\frac{6}{n}+\frac{18i}{n^2}\right)$$
$$= \frac{6}{n}(n) + \frac{18}{n^2}\left(\frac{n(n+1)}{2}\right) = 6 + 9\left(\frac{n+1}{n}\right) = 15 + \frac{9}{n}.$$

Since $\int_1^4 (x^2-2x)\,dx = \int_1^4 x^2\,dx - \int_1^4 2x\,dx$, we have

$$\int_1^4 (x^2-2x)\,dx \approx 21 + \frac{27}{2n} + \frac{3}{2n^2} - \left(15+\frac{9}{n}\right) = 6 + \frac{9}{2n} + \frac{3}{2n^2}.$$

2. Compute the derivatives of the following functions:

   (a) $f(x) = \int_2^x \frac{2t^2}{t^3+4t}\,dt$

   By direct application of the Fundamental Theorem of Calculus, $f'(x) = \dfrac{2x^2}{x^3+4x}$.

   (b) $g(x) = \int_x^4 \sin(t^2)\,dt$

   Since $g(x) = -\int_4^x \sin(t^2)\,dt$, the FTC gives us $g'(x) = -\sin(x^2)$.

   (c) $h(x) = \int_x^{\sin(x)} e^{t^2}\,dt$

   Using properties of integrals, we have

$$h(x) = \int_0^{\sin(x)} e^{t^2}\,dt + \int_x^0 e^{t^2}\,dt = \int_0^{\sin(x)} e^{t^2}\,dt - \int_0^x e^{t^2}\,dt.$$

   Using the FTC (plus the Chain Rule on the first term), we have

$$h'(x) = e^{\sin^2(x)}\cos(x) - e^{x^2}.$$

3. Evaluate the integral $\int_0^1 \left(\frac{1}{1+x^2} - 2x + 5e^x\right)\,dx$

We first determine that the function

$$F(x) = \arctan(x) - x^2 + 5e^x$$

is an antiderivative of the integrand $\frac{1}{1+x^2} - 2x + 5e^x$. It follows from the second part of the FTC that

$$\int_0^1 \left(\frac{1}{1+x^2} - 2x + 5e^x\right)\,dx = F(1) - F(0) = \arctan(1) - 1^2 + 5e^1 - (\arctan(0) - 0^2 + 5e^0)$$
$$= \frac{\pi}{4} + 5e - 6.$$