MATH 2565 - Tutorial #10

1. Find the radius and interval of convergence for the following power series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} x^n$

Using the ratio test, we need

$$\lim_{n\to\infty} \left| \frac{(-1)^n/((n+1)5^{n+1})}{(-1)^{n-1}/(n5^n)} \right| = \lim_{n\to\infty} \frac{n}{5(n+1)} |x| = \frac{1}{5} |x| < 1,$$

so we must have $|x| < 5$, giving us 5 as the radius of convergence.

Now we check the endpoints: when $x = 5$, we get $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, which converges by the Alternating Series Test. When $x = -5$, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{-1}{n},$$

which diverges (since it's a multiple of the harmonic series). The interval of convergence is therefore $(-5, 5]$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$

Applying the ratio test, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2n-1}{2n+1} \cdot \frac{|x-1|}{2} = \frac{|x-1|}{2}.$$

We need this limit to be less than 1, so $|x-1| < 2$, giving us a radius of convergence equal to 2. Note that

$$|x-1| < 2 \Leftrightarrow -2 < x-1 < 2 \Leftrightarrow -1 < x < 3.$$

When $x = -1$, we get $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{(2n-1)2^n} = \sum_{n=1}^{\infty} \frac{1}{2n-1}$, which diverges (by limit comparison with the harmonic series).

When $x = 3$, we get $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n-1)2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$, which converges, by the Alternating Series Test. The interval of convergence is therefore $(-1, 3]$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2x^n}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)} = \sum_{n=1}^{\infty}\frac{n^2x^n}{2^n\cdot n!}$.

The ratio test gives us

$$\lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}}{2^{n+1}(n+1)!}\cdot\frac{2^n\cdot n!}{n^2x^n}\right| = \lim_{n\to\infty}\frac{(n+1)^2\cdot n!}{n^2(n+1)n!}\cdot\frac{|x|}{2} = \frac{|x|}{2}\lim_{n\to\infty}\frac{n+1}{n^2} = 0.$$

Since this limit is equal to $0<1$ for all values of x, the radius of convergence is infinite, and the interval is $(-\infty,\infty)$.

2. Let p and q be real numbers with $p<q$. Find a power series whose radius of convergence is:

(a) $[p,q]$ (b) (p,q) (c) $[p,q)$ (d) $(p,q]$

For each of these, let $a=\frac{p+q}{2}$ be the midpoint of the interval, and let $r=\frac{q-p}{2}$ be the radius of the interval.

Note that when $x=p$,

$$x-a=p-a=\frac{2p}{2}-\frac{p+q}{2}=\frac{p-q}{2}=-r,$$

and when $x=q$,

$$x-a=q-a=\frac{2q}{2}-\frac{p+q}{2}=\frac{q-p}{2}=r.$$

For a series with interval of convergence $[p,q]$, we take

$$\sum_{n=1}^{\infty}\frac{(x-a)^n}{n^2r^n}.$$

Notice that

$$\lim_{n\to\infty}\left|\frac{(x-a)^{n+1}}{(n+1)^2r^{n+1}}\cdot\frac{n^2r^n}{(x-a)^n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}\frac{|x-a|}{r} = \frac{|x-a|}{r},$$

so the radius is r, as required. When $x=p$, we get $\displaystyle\sum_{n=1}^{\infty}\frac{(-r)^n}{n^2r^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, and when $x=q$ we similarly get $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. Both of these series converge, so the interval of convergence is $[p,q]$, as required.

To get a series with interval of convergence (p,q), we go to a geometric series, and take $\displaystyle\sum_{n=1}^{\infty}\frac{(x-a)^n}{r^n}$.

Thinking back to the examples in problem 1, we can work out that the series $\displaystyle\sum_{n=1}^{\infty}\frac{(x-a)^n}{nr^n}$ will have interval of convergence $[p,q)$, while $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(x-a)^n}{nr^n}$ will have interval of convergence $(p,q]$.

3. Given that $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, can we conclude that each of the following series is convergent?

   (a) $\displaystyle\sum_{n=0}^{\infty} c_n(-2)^n$

   Yes, this series converges. Knowing that $\sum c_n 4^n$ converges tells us that the power series $\sum c_n x^n$ has a radius of convergence of at least 4, and an interval of convergence that is at least $(-4, 4]$, which includes $x = -2$.

   (b) $\displaystyle\sum_{n=0}^{\infty} c_n(-4)^n$

   We cannot tell if this series converges. We know that the power series converges for all x in $(-4, 4]$, be we can't determine convergence at $x = -4$ without more information.

4. Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series?

   (a) $\sum_{n=0}^{\infty} c_n$ (b) $\sum_{n=0}^{\infty} c_n 8^n$ (c) $\sum_{n=0}^{\infty} c_n(-3)^n$

   The information given tells us that the radius of convergence of our power series is at least 4, but no more than 6. Thus, the series will converge for $|x| < 4$, diverge for $|x| > 6$, and for $4 \leq |x| \leq 6$, we cannot draw any conclusion.

   The first series has $x = 1$, so it converges. The second has $x = 8$, so it diverges. The third has $x = -3$, so it converges.

5. Recall that $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$, for $|x| < 1$.

   (a) Find a power series representation for $g(x) = (1 + x)^{-2}$. What is the radius of convergence?

   Since $\frac{d}{dx}(1 + x)^{-1} = -(1 + x)^{-2}$, we have

   $$g(x) = -\frac{d}{dx}f(x) = -\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{d}{dx}(x^n) = \sum_{n=0}^{\infty}(-1)^{n+1}nx^{n-1}.$$

   This representation is valid for all x with $|x| < 1$, so the radius of convergence is 1.

   (b) Find a power series representation for $h(x) = \dfrac{x^2}{(1 + x)^3}$.

   We have

   $$h(x) = x^2(1 + x)^{-3} = x^2\left(-\frac{1}{2}\frac{d}{dx}(1 + x)^{-2}\right) = -\frac{1}{2}x^2 g(x).$$

   Thus, for $|x| < 1$, we have

   $$h(x) = -\frac{1}{2}x^2\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^{n+1}nx^{n-1} = \frac{1}{2}x^2\sum_{n=0}^{\infty}(-1)^n n(n-1)x^{n-2} = \sum_{n=0}^{\infty}\frac{1}{2}(-1)^n n(n-1)x^n.$$

   (Note that we could start the sum at $n = 2$ here, since the first two terms vanish.)

6. Find a power series representation for the function:

   (a) $f(x) = x^2 \arctan(x^3)$
   Since $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$, we have

$$f(x) = x^2 \arctan(x) = x^2 \sum_{n=0}^{\infty}(-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+5}}{2n+1}.$$

   (b) $g(x) = \left(\frac{x}{2-x}\right)^3$
   First we note that

$$\frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1-x/2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}},$$

   for $|x| < 2$.
   Next, $\frac{d^2}{dx^2}\frac{1}{2-x} = \frac{2}{(2-x)^3}$, so

$$\frac{1}{(2-x)^3} = \frac{1}{2}\frac{d^2}{dx^2}\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}} = \sum_{n=0}^{\infty}\frac{n(n-1)x^{n-2}}{2^{n+2}}.$$

   Finally, we get

$$g(x) = \left(\frac{x}{2-x}\right)^3 = x^3\left(\frac{1}{(2-x)^3}\right) = x^3\sum_{n=0}^{\infty}\frac{n(n-1)x^{n-2}}{2^{n+2}} = \sum_{n=2}^{\infty}\frac{n(n-1)x^{n+1}}{2^{n+2}}.$$

7. Express the antiderivative as a power series;

   (a) $\int \frac{t}{1+t^3}\,dt$

   Since $\frac{t}{1+t^3} = t\sum_{n=0}^{\infty}(-t^3)^n = \sum_{n=0}^{\infty}(-1)^n t^{3n+1}$, we get

$$\int \frac{t}{1+t^3}\,dt = \int\left(\sum_{n=0}^{\infty}(-1)^n t^{3n+1}\right)dt = \sum_{n=0}^{\infty}(-1)^n\int t^{3n+1}\,dt = \sum_{n=1}^{\infty}(-1)^n\frac{t^{3n+2}}{3n+2}.$$

   (b) $\int \frac{\arctan(x)}{x}\,dx$
   We have

$$\frac{\arctan(x)}{x} = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2n+1},$$

   so

$$\int \frac{\arctan(x)}{x}\,dx = \int\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2n+1}\right)dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2}.$$