1. For the matrices

$$A = \begin{bmatrix} 2 & -3 & 3 \\ 1 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & 2 \end{bmatrix},$$

determine which of the products $A^2, AB, AC, BA, B^2, BC, CA, CB, C^2$ are defined. Compute at least **three** of the products that are defined.

The following matrix products are defined:

$$AC = \begin{bmatrix} 2(2) - 3(-1) + 3(3) & 2(0) - 3(4) + 3(2) \\ 1(2) + 0(-1) + 5(3) & 1(0) + 0(4) + 5(2) \end{bmatrix} = \begin{bmatrix} 16 & -6 \\ 17 & 10 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2(2) + 0(1) & 2(-3) + 0(0) & 2(3) + 0(5) \\ 1(2) - 4(1) & 1(-3) - 4(0) & 1(3) - 4(5) \end{bmatrix} = \begin{bmatrix} 4 & -6 & 6 \\ -2 & -3 & -17 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 2(2) + 0(1) & 2(0) + 0(-4) \\ 1(2) - 4(1) & 1(0) - 4(-4) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2 & 16 \end{bmatrix}$$

$$CA = \begin{bmatrix} 2(2) + 0(1) & 2(-3) + 0(0) & 2(3) + 0(5) \\ -1(2) + 4(1) & -1(-3) + 4(0) & -1(3) + 4(5) \\ 3(2) + 2(1) & 3(-3) + 2(0) & 3(3) + 2(5) \end{bmatrix} = \begin{bmatrix} 4 & -6 & 6 \\ 2 & 3 & 17 \\ 8 & -9 & 19 \end{bmatrix}$$

$$CB = \begin{bmatrix} 2(2) + 0(1) & 2(0) + 0(-4) \\ -1(2) + 4(1) & -1(0) + 4(-4) \\ 3(2) + 2(1) & 3(0) + 2(-4) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & -16 \\ 8 & -8 \end{bmatrix}$$

2. Determine the matrix of the transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}.$$

Using the fact that any matrix transformation T maps the standard basis vectors to the corresponding columns of its matrix, we have $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & 7 & -1 \\ 1 & 3 & 5 & 4 \end{bmatrix}.$$

It's straightforward to verify our work by confirming that the values of T on the standard basis vectors are as given above.

3. Determine the matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that performs the following operations, in order: First, a horizontal stretch by a factor of 4. Second, a counter-clockwise rotation by $3\pi/4$. Third, a reflection across the x-axis.

The transformation is given as $T(\vec{x}) = A_3(A_2(A_1\vec{x}))$, where

$$A_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{performs the horizontal stretch}$$

$$A_2 = \begin{bmatrix} \cos(3\pi/4) & -\sin(3\pi/4) \\ \sin(3\pi/4) & \cos(3\pi/4) \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1\sqrt{2} \end{bmatrix} \qquad \text{perfoms the rotation}$$

$$A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{perfoms the reflection}$$

Thus, we have $T(\vec{x}) = A\vec{x}$, where

$$\begin{aligned} A = A_3A_2A_1 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2\sqrt{2} & -1/\sqrt{2} \\ 2\sqrt{2} & -1\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2\sqrt{2} & -1/\sqrt{2} \\ -2\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. \end{aligned}$$

Note: we can also determine T by tracking the standard basis vectors. We have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{horiz. stretch}} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \xrightarrow{\text{rotation}} \begin{bmatrix} -2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix} \xrightarrow{\text{reflection}} \begin{bmatrix} -2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{horiz. stretch}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{rotation}} \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \xrightarrow{\text{reflection}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

The two resulting vectors form the two columns of our matrix A, resulting in the same answer as before.

4. For fun: Find a 2×2 matrix A such that A^{12} is the identity matrix, but A^k is not for $1 \le k \le 11$. (Hint: rotation.)

Consider the rotation matrix $A = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$. Multiplying a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ on the left by A corresponds to a rotation through an angle of $\pi/6$. It follows that repeated multiplication corresponds to repeated rotation:

Multiplication by A^2 is rotation by $2(\pi/6) = \pi/3$, multiplication by A^3 is rotation by $3(\pi/6) = \pi/2$, and so on, up to multiplication by A^{11}, which corresponds to rotation by $11\pi/6$, and then A^{12}, which represents a rotation by $12\pi/6 = 2\pi$. But a rotation by 2π returns everything to where we started, and it's easy to check that the rotation matrix for $\theta = 2\pi$ is the identity matrix.