University of Lethbridge
Department of Mathematics and Computer Science
31^{st} October, 2016, 9:00 - 9:50 am

MATH 1410A - Test #2

Last Name: **Solutions**

First Name: **The**

Student Number: ______________________

Tutorial Section: ______________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, including intermediate steps. (You should show your work if you want to earn part marks.) Unless otherwise indicated, failure to justify your work may result in loss of marks, even for a correct answer.

No external aids are allowed, with the exception of a 5-function calculator.

For grader's use only:

Page	Grade
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3	/12
4	/10
5	/8
6	/8
Total	/50

A

1. Assume that A, B, and X are matrices of the same size.

[3] (a) Solve for X in terms of A and B, given that $2A - 3X = B$.

Adding $3X - B$ to both sides of the equation, we get $2A - B = 3X$. Multiplying both sides by $\frac{1}{3}$, we find that
$$X = \frac{2}{3}A - \frac{1}{3}B.$$

[3] (b) Determine the entries of the matrix X from part (a) if $A = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$.

Plugging the given values into our expression from part (a), we find
$$X = \frac{2}{3}\begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 4/3 & -2 \\ 0 & 8/3 \end{bmatrix} + \begin{bmatrix} -1/3 & -2/3 \\ 4/3 & -5/3 \end{bmatrix} = \begin{bmatrix} 1 & -8/3 \\ 4/3 & 1 \end{bmatrix}.$$

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a matrix transformation such that
$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 4 \end{bmatrix}.$$

[3] (a) Determine the matrix of T. (That is, find the matrix A such that $T(\vec{x}) = A\vec{x}$ for any vector $\vec{x}$ in $\mathbb{R}^3$.)

Since the values of T on the standard basis vectors determine the columns of A, in order, we have
$$A = \begin{bmatrix} 2 & -1 & -5 \\ -3 & 1 & 4 \end{bmatrix}.$$

[3] (b) Compute $T\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right)$.

Using the matrix from part (a), we have
$$T\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & -5 \\ -3 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}.$$

[4] 3. Determine vectors $\vec{u}$ and $\vec{v}$ such that $U = \text{span}\{\vec{u}, \vec{v}\}$, where U is the subspace

$$U = \left\{ \begin{bmatrix} 3a-2b \\ b-3a \\ 5a+4b \end{bmatrix} \middle| \; a, b \text{ are real numbers} \right\}.$$

(Recall that $\text{span}\{\vec{u}, \vec{v}\} = \{a\vec{u} + b\vec{v} \,|\, a, b \in \mathbb{R}\}$.)

If $\vec{w}$ is any vector in U, then for some real numbers a and b we have

$$\vec{w} = \begin{bmatrix} 3a-2b \\ b-3a \\ 5a+4b \end{bmatrix} = \begin{bmatrix} 3a \\ -3a \\ 5a \end{bmatrix} + \begin{bmatrix} -2b \\ b \\ 4b \end{bmatrix} = a \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}.$$

Thus, $\vec{w} = a\vec{u} + b\vec{v}$, where $\vec{u} = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$, so $U = \text{span}\left\{ \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \right\}$.

[4] 4. Verify that $x = -\dfrac{7}{2}, y = 2, z = \dfrac{1}{2}$ is a solution to the system

$$\begin{aligned} 2x + y \quad &= -5 \\ x \quad + 3z &= -2 \\ -y + 6z &= 1 \end{aligned}$$

Plugging in the given values, we have

$$\begin{aligned} 2(-7/2) + 2 = -7 + 2 &= -5, && \text{so the first equation works} \\ -7/2 + 3(1/2) = -7/2 + 3/2 = -4/2 &= -2, && \text{so the second equation works} \\ -2 + 6(1/2) = -2 + 3 &= 1, && \text{so the third equation works} \end{aligned}$$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(\vec{u}) = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} 0 \\ 6 \\ -3 \end{bmatrix}$ for

[4] some vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^2$. What is the value of $T(5\vec{u} - 3\vec{v})$?

Using the properties of lienar transformations, we have

$$T(5\vec{u} - 3\vec{v}) = 5T(\vec{u}) - 3T(\vec{v}) = 5 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -28 \\ 29 \end{bmatrix}.$$

[10] 6. Each of the matrices below is in row-echelon form, and represents a system of linear equations in the variables x, y, and z. If the system has no solution, explain why. If it does, determine the solution using either back substitution, or by finding the reduced row-echelon form of the matrix.

(a) $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Using back substitution, we have $z = 3$ from Row 3; Row 2 gives us $y - z = 2$, so $y = 2 + 3 = 5$, and from Row 1, we have $x - 2y + z = 4$, so $x = 4 + 2(5) - 3 = 11$.

Using row operations, we find

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_2 + R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right],$$

from which we can read off the solution $x = 11, y = 5, z = 3$ as before.

(b) $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Here, the third column contains no leading 1, so $z = t$ is a free parameter. The equation $y - 3z = 5$ from Row 2 then gives us $y = 5 + 3t$. From Row 1 we have the equation $x + 3y = 2$. Solving for x and plugging in $y = 5 + 3t$, we have $x = 2 - 3y = 2 - 3(5 + 3t) = 2 - 15 - 9t = -13 - 9t$.

Using row operations instead, we have

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 3R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 9 & -13 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We can then read off the solution $x = -13 - 9t, y = 5 + 3t, z = t$, where t is a free parameter, as before.

(c) $\left[\begin{array}{ccc|c} 1 & 5 & -4 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$

The last row of the augmented matrix corresponds to the equation $0x + 0y + 0z = 1$. Since $0x + 0y + 0z = 0 \neq 1$ for all possible values of x, y, and z, there is no solution to this system.

[8] 7. Compute the matrix products AB and BA, where $A = \begin{bmatrix} 3 & 1 & -2 \\ -4 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 3 & 2 \\ 5 & -3 \end{bmatrix}$.

We have

$$AB = \begin{bmatrix} 3 & 1 & -2 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -3+3-10 & 0+2+6 \\ 4+3+15 & 0+2-9 \end{bmatrix} = \begin{bmatrix} -10 & 8 \\ 22 & -7 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -1 & 0 \\ 3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ -4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -3+0 & -1+0 & 2+0 \\ 9-8 & 3+2 & -6+6 \\ 15+12 & 5-3 & -10-9 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 2 \\ 1 & 5 & 0 \\ 27 & 2 & -19 \end{bmatrix}$$

[8] 8. Solve the system
$$\begin{aligned} 3x - 4y - 5z &= 2 \\ x - 2y - z &= 4 \\ -2x + 2y + 4z &= 2 \end{aligned}$$

We set up the corresponding augmented matrix and reduce, as follows:

$$\left[\begin{array}{ccc|c} 3 & -4 & -5 & 2 \\ 1 & -2 & -1 & 4 \\ -2 & 2 & 4 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 3 & -4 & -5 & 2 \\ -2 & 2 & 4 & 2 \end{array}\right]$$
$$\xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 0 & 2 & -2 & -10 \\ -2 & 2 & 4 & 2 \end{array}\right]$$
$$\xrightarrow{R_3 + 2R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 0 & 2 & -2 & -10 \\ 0 & -2 & 2 & 10 \end{array}\right]$$
$$\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 0 & 2 & -2 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -6 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This last matrix is in reduced row-echelon form. We see that there are leading ones in the first and second columns, so x and y are basic variables, while z is a free variable, since there is no leading one in its column. Setting $z = t$, where t can be any real number, we have the solution

$$\begin{aligned} x &= -6 + 3t \\ y &= -5 + t \\ z &= t. \end{aligned}$$

 Total Points: 50