Math 3500 Assignment #8
University of Lethbridge, Fall 2014

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Due date: Friday, November 21st, by 6 pm.

1. Let f be differentiable on some interval (c, ∞) and suppose that $\lim_{x\to\infty} [f(x)+f'(x)] = L$, where L is finite. Prove that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$.

   Hint: for all $x > c$, $f(x) = \dfrac{e^x f(x)}{e^x}$.

   First, we note that if $\lim_{x\to\infty} f(x) = 0$, then the same is true for $f'(x)$. Otherwise, suppose that $\lim_{x\to\infty} f(x) = L > 0$ (the case $L < 0$ is similar). Then there exists $N > 0$ such that for all $x > N$, we have $f'(x) > L/2$. Choose some $x_0 > N$ and let $y_0 = f(x_0)$, and $g(x) = \frac{L}{2}(x - x_0) + y_0$. Then $f(x_0) = g(x_0)$ and $f'(x) > g'(x) = L/2$ for all $x \geq x_0$, which implies that $f(x) \geq g(x)$ for all $x \geq x_0$ (apply the Mean Value Theorem to $f(x) - g(x)$ on $[x_0, x]$). Since $g(x) \to \infty$ as $x \to \infty$, the same must be true of $f(x)$.

   Having established this fact, suppose that $\lim_{x\to\infty}(f(x)+f'(x)) = L$. We now consider $\lim_{x\to\infty} f(x)$; note that if this limit is zero, then so is $\lim_{x\to\infty} f'(x)$ and we're done. If not, then then since both $f(x)e^x \to \infty$ and $e^x \to \infty$ as $x \to \infty$, by l'Hospital's rule we have

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{f(x)e^x}{e^x} = \lim_{x\to\infty} \frac{f'(x)e^x + f(x)e^x}{e^x} = \lim_{x\to\infty}(f(x) + f'(x)),$$

   from which the result follows.

2. When we apply l'Hospital's rule to the limit $\lim_{x\to a} \dfrac{f(x)}{g(x)}$, we require that $g'(x) \neq 0$ near $x = a$. This exercise demonstrates the importance of that requirement: if l'Hospital's rule is applied carelessly, it's possible for the zeros of g' to cancel the zeros of f', leading to an incorrect result. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by

   $$f(x) = x + \cos x \sin x \quad g(x) = e^{\sin x}(x + \cos x \sin x).$$

(a) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$

Since $-1 \leq \cos x \sin x \leq 1$ for all $x \in \mathbb{R}$, given any $N > 0$ we can let $M = N + 1$, and then whenever $x > M$ we have $f(x) = x + \cos x \sin x \geq x - 1 > M - 1 = N$, and thus, $\lim_{x\to\infty} f(x) = \infty$.
Similarly, since $1/e \leq e^{\sin x} \leq e$ for all $x \in \mathbb{R}$, given any $N > 0$ we can choose $M = eN + 1$, and then whenever $x > M$ we have

$$g(x) = e^{\sin x}(x + \cos x \sin x) \geq \frac{1}{e}(x-1) > \frac{1}{e}(M-1) = N,$$

so $\lim_{x\to\infty} g(x) = \infty$.

(b) Show that $f'(x) = 2\cos^2 x$ and $g'(x) = e^{\sin x} \cos x[2\cos x + f(x)]$

This follows from basic rules of differentiation:

$$\begin{aligned}
&f'(x) = 1 - \sin^2 x + \cos^2 x = 2\cos^2 x, \text{ and} \\
&g'(x) = e^{\sin x} \cos x(x + \sin x \cos x) + e^{\sin x}(2\cos^2 x) = e^{\sin x} \cos x(f(x) + 2\cos x).
\end{aligned}$$

(c) Show $\dfrac{f'(x)}{g'(x)} = \dfrac{2e^{-\sin x}\cos x}{2\cos x + f(x)}$ if $\cos x \neq 0$ and $x > 3$.

Whenever $x > 3$, $f(x) = x + \sin x \cos x > 3 - 1 = 2$, so $f(x) + 2\cos x > 2 - 2 = 0$. Thus, when $\cos x \neq 0$, $g'(x) \neq 0$ and we get

$$\frac{f'(x)}{g'(x)} = \frac{2\cos^2 x}{e^{\sin x}\cos x(f(x) + 2\cos x)} = \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)},$$

as required.

(d) Show that $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)} = 0$, and yet $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$ does not exist.

The first limit follows from the fact that $2e^{-\sin x}\cos x$ is bounded and $f(x) \to \infty$ as $x \to \infty$. The second limit does not exist since $\dfrac{f(x)}{g(x)} = e^{\sin x}$, and if we consider the sequence $x_n = \pi(2n + 1/2)$, $n = 1, 2, \ldots$, then $x_n \to \infty$ and $e^{\sin x_n} = e$ for all n, while if we take the sequence $y_n = \pi(2n + 3/2)$, $n - 1, 2, \ldots$, then $y_n \to \infty$ and $e^{\sin y_n} = 1/e \neq e$ for all n.
Note: if you're worried about the fact that $\cos x = 0$ for $x = \pi/2 + n\pi$, for all $n \in \mathbb{N}$, you can check that the limit of $f'(x)/g'(x)$ at each such point is zero, so one can redefine f'/g' to be equal to zero at each such point. (This falls under the general adage that removable discontinuities don't affect a limit.)

3. Find a Taylor polynomial that approximates $f(x) = e^x$ to within 0.2 on the interval $[-2, 2]$.

We have $f(x) = 1 + x + \dfrac{x}{2} + \cdots + \frac{x^n}{n!} + R_n(x)$, where by the Lagrange form of the remainder, there exists some t between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1}.$$

Since $f^{(n+1)}(t) = e^t$ and we must have $x, t \in [-2, 2]$, we have $|f^{(n+1)}(t)| \leq e^2$ and $|x| \leq 2$. It follows that $|R_n(x)| \leq \dfrac{e^2 2^{n+1}}{(n+1)!}$. We then compute as follows (rounded to two decimal places):

n	1	2	3	4	5	6
$\dfrac{e^2 2^{n+1}}{(n+1)!}$	14.78	9.85	4.93	1.97	0.66	0.18

Thus, when $n = 6$ we have $|R_6(x)| < 0.2$ for all $x \in [-2, 2]$, so the polynomial

$$P_{6,0,f}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

will suffice.

4. Show that if $x \in [0, 1]$, then

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}.$$

Using long division, we have

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \frac{(-1)^{n+1} t^{n+1}}{1+t}$$

for each $n = 1, 2, \ldots$. Thus, we have

$$\ln(1+x) = \int_0^1 \frac{1}{1+t}\, dt = x - \frac{x^2}{2} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + (-1)^{n+1} \int_0^x \frac{t^{n+1}}{1+t}\, dt.$$

Thus, $x - x^2/2 + x^3/3 - x^4/4 = \ln(1+x) - (-1)^4 \displaystyle\int_0^x \frac{t^4}{1+t}\, dt \leq \ln(1+x)$, since $\dfrac{t^4}{1+t} \geq 0$ on $[0, 1]$. Similarly, $x - x^2 x + x^3/3 = \ln(1+x) - (-1)^3 \displaystyle\int_0^x \frac{t^3}{1+t}\, dt \geq \ln(1+x)$.