Math 3500 Exercise Sheet

5 November, 2014

This week we'll look at applications of the Mean Value Theorem, as well as the interpretation of the derivative as a linear approximation.

The Mean Value Theorem

Recall that the Mean Value Theorem states the following: suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

1. Prove that if $f'(x) = 0$ for all x in some interval I, then f is constant on I.

2. Using the previous problem, show that $f'(x) = g'(x)$ on some interval I if and only $f(x) = g(x) + C$ for some constant C, for all $x \in I$.

3. Recall that a function f is **increasing** on an interval I if for any $x, y \in I$ with $x < y$ we have $f(x) \leq f(y)$. Prove that if $f'(x) \geq 0$ on I, then f is increasing on I. Similarly show that if $f'(x) \leq 0$ on I, then f is decreasing on I.

   Note: The converse to this result is not necessarily true: it's possible to have an increasing or decreasing function f on an interval I that is not differentiable on all of I.

4. Prove *Cauchy's Mean Value Theorem*: if f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that

   $$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

   In the case that $g'(x) \neq 0$ on $[a, b]$ we can write $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$. We'll need this result to prove l'Hospital's rule.

   Hint: consider $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$.

Derivatives and linear approximations

Given a differentiable function $f : I \to \mathbb{R}$, we know that the tangent line to the graph $y = f(x)$ at $x = a$ is given by

$$y = f(a) + f'(a)(x - a).$$

The function $l(x) = f(a) + f'(a)(x - a)$ whose graph gives the tangent line is a *linear* function (it's of the form $l(x) = Ax + b$).

5. Let $R(x) = f(x) - l(x)$, and prove that $\lim_{x \to a} \frac{R(x)}{|x - a|} = 0$. The function $R(x)$ is the *remainder* once we subtract $l(x)$ from $f(x)$. This tells us that the difference between $f(x)$ and $l(x)$ goes to zero *faster* than $|x - a|$ as $x \to a$. (One says that $R(x)$ is **sublinear** near $x = a$.)

6. Let $g(x) = Ax + b$ be any other linear function. Prove that if $f(x) - g(x)$ is sublinear near $x = a$, then $g(x) = l(x)$.

   Hint: First explain why you must have $g(a) = f(a)$.

7. Given $f : (a, b) \to \mathbb{R}$ and $x \in (a, b)$, choose Δx small enough that $x + \Delta x \in (a, b)$. We define the **increment** of f from x to $x + \Delta x$ by

   $$\Delta f = f(x + \Delta x) - f(x),$$

   and we define the **differential** of f at x (with increment Δx) by

   $$df = f'(x)\Delta x$$

   (a) One often writes the differential of f as $df = f'(x)dx$. Explain why we can write $dx = \Delta x$. (Consider the identity function.)

   (b) Explain why the approximation $\Delta f \approx df$ is valid. (Note that each measures the difference in y values for nearby points on a particular graph.)

   (c) This approximation is very useful in applications. For example, it can be used to estimate the error in a calculuated quantity due to errors in measurement. For a particular example, use the differential to approximate the possible error in calculating the area of a square, if we measure each side to have a length of 10 cm with a possible error in measurement of 1 mm.

Derivatives and montonic functions

A function $f : I \to \mathbb{R}$ is **monotonic** if it is either increasing or decreasing on I. We say that f is **strictly increasing (decreasing)** if for all $x, y \in I$ with $x < y$ we have $f(x) < f(y)$ $(f(x) > f(y))$.

8. Prove that if $f'(x) \neq 0$ on I, then f is either strictly increasing or strictly decreasing. (Hint: use Darboux's Theorem.) It's possible to prove that if f is monotonic on an interval $I = (a, b)$, then $\lim_{t\to x^+} f(t)$ and $\lim_{t\to x^-} f(t)$ exist at each $x \in I$. More precisely,

$$\sup\{f(t) : a < t < x\} = \lim_{t\to x^-} f(t) \leq f(x) \leq \lim_{t\to x^+} f(t) = \inf\{f(t) : x < t < b\}.$$

   Moreover, if $a < x < y < b$,then $\lim_{t\to x^+} f(t) \leq \lim_{t\to y^-} f(t)$. In particular, this tells us that a monotonic function can only have jump discontinuities. Proving this is not too difficult but the proof is a bit long and wouldn't leave us with time to work on other problems.

9. (Bonus fun not really related to derivatives) Prove that if f is monotonic, then the set of discontinuities of f is at most countable. (Hint: if f has a jump discontinuity at $x = a$ then there is a rational number $r(a)$ such that $\lim_{x\to a^-} f(x) < r(a) < \lim_{x\to a^+} f(x)$.)

10. Prove that if $f : (a, b) \to \mathbb{R}$ is continuous and strictly increasing, then f is one-to-one, the range of f is an interval (c, d), and $f^{-1} : (c, d) \to (a, b)$ is also continuous.

11. Prove that if $f'(x) \neq 0$ on some interval I, then f is one-to-one, f^{-1} is differentiable on $f(I)$, and

$$f^{-1}(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(x))}$$

   for all $y = f(x) \in f(I)$.

   Hint: the previous problems tell us that f must be either strictly increasing or strictly decreasing, and that f^{-1} is continuous. Define $x = f^{-1}(y)$ and $\Delta x = f^{-1}(y+\Delta y)-x = \Delta f^{-1}$. Check that $\Delta y = \Delta f = f(x + \Delta x) - f(x)$. Note that since f and f^{-1} are one-to-one, $\Delta x = 0$ if and only if $\Delta y = 0$, and since f and f^{-1} are continuous, $\Delta x \to 0$ if and only if $\Delta y \to 0$.