University of Lethbridge
Department of Mathematics and Computer Science
14^{th} March, 2017, 1:40 - 2:55 pm

MATH 1410A - Test #2

Last Name: **Solutions**

First Name: **The**

Student Number: ____________________

Tutorial Time: ____________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

To earn partial credit, you must show your work. Correct answers without adequate justification in most cases do not receive full marks.

No external aids are allowed, with the exception of a 5-function calculator.

For grader's use only:

Problem	Grade
1	/8
2	/9
3	/9
4	/8
5	/5
6	/6
7	/5
Total	/50

1. Complete the following definitions:

[2] (a) The **null space** of an $m \times n$ matrix A is the set null(A) defined by

null(A) $=\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$.

[2] (b) A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is **linearly dependent** if:

Either: one of the vectors $\vec{v}_i$ can be written as a linear combination of the others

or: there exist scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$, not all equal to zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}.$$

[2] (c) A set $S \subseteq \mathbb{R}^n$ is a **subspace** if:

the following three conditions hold:

i. S is non-empty (also acceptable: $\vec{0} \in S$)
ii. S is closed under addition (if $\vec{x}, \vec{y} \in S$, then $\vec{x} + \vec{y} \in S$)
iii. S is closed under scalar multiplication (if $\vec{x} \in S$ and $c \in \mathbb{R}$, then $c\vec{x} \in S$).

[2] (d) The **span** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is the set:

span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}$

2. Perform the computations as indicated:

[3] (a) Simplify the following linear combination (write it as a single vector):

$$4\begin{bmatrix}2\\-1\\3\end{bmatrix} - 2\begin{bmatrix}1\\0\\3\end{bmatrix} + 3\begin{bmatrix}0\\5\\-2\end{bmatrix} = \begin{bmatrix}6\\11\\0\end{bmatrix}$$

[3] (b) Compute $T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right)$ for the matrix transformation $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

$$T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right) = \begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}14\\-9\end{bmatrix}$$

[3] (c) Verify that $x = 2, y = -3, z = 1$ is a solution to the system $\begin{aligned} 2x - y + 3z &= 10\\ -x + 2y + 5z &= -3\\ 5x + 2y - 4z &= 0 \end{aligned}$

We check that

$$\begin{aligned} 2(2) - (-3) + 3(1) &= 4 + 3 + 3 = 10,\\ -(2) + 2(-3) + 5(1) &= -2 - 6 + 5 = -3, \quad \text{and}\\ 5(2) + 2(-3) - 4(1) &= 10 - 6 - 4 = 0. \end{aligned}$$

Since the given values satisfy all three equations, this is a solution.

[9] 3. Each of the matrices below is in row-echelon form, and represents a system of linear equations in the variables x, y, and z. If the system has no solution, explain why. If it does, determine the solution using either back substitution, or by finding the reduced row-echelon form of the matrix.

(a) $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Using back-substitution: from Row 3 we have $z = 0$. Row 2 gives us the equation $y - z = 2$. Putting $z = 0$ yields $y = 2$. Finally, Row 1 gives us the equation $x - 2y + z = 4$. Putting $y = 2$ and $z = 0$ in this equation, we have $x - 4 = 4$, so $x = 8$. Our solution is therefore

$$x = 8, \quad y = 2, \quad z = 0.$$

(b) $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Here, we see that x and y are leading variables, while z is free. To more easily solve for x and y in terms of z, we proceed to reduced row-echelon form. To create a zero above the leading 1 in the second column, we perform the row operation $R_1 - 3R_2 \to R_1$, giving us the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & -13 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix is in reduced row echelon form, and we read off the solution

$$x = -13 - 9z, \quad y = 5 + 3z, \quad z \text{ is free.}$$

(c) $\left[\begin{array}{ccc|c} 1 & 5 & -4 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Here, the third row corresponds to the equation $0x + 0y + 0z = 1$, which asserts that $0 = 1$, regardless of the values of x, y, and z. Since it is impossible to satisfy this condition, there is no solution to the system.

4. Given the matrices $A = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$, compute:

[4] (a) AB

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -3 & 4 \\ -1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2(5) - 3(1) + 4(0) & 2(-2) - 3(-1) + 4(4) \\ -1(5) + 0(1) + 5(0) & -1(-2) + 0(-1) + 5(4) \end{bmatrix} \\ &= \begin{bmatrix} 7 & 15 \\ -5 & 22 \end{bmatrix}. \end{aligned}$$

[4] (b) BA

$$\begin{aligned} BA &= \begin{bmatrix} 5 & -2 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 & 4 \\ -1 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5(2) - 2(-1) & 5(-3) - 2(0) & 5(4) - 2(5) \\ 1(2) - 1(-1) & 1(-3) - 1(0) & 1(4) - 1(5) \\ 0(2) + 4(-1) & 0(-3) + 4(0) & 0(4) + 4(5) \end{bmatrix} \\ &= \begin{bmatrix} 12 & -15 & 10 \\ 3 & -3 & -1 \\ -4 & 0 & 20 \end{bmatrix}. \end{aligned}$$

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a matrix transformation such that

$$T\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ -1 \\ -4 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} -9 \\ 1 \\ 7 \end{bmatrix}.$$

[3] (a) What is the value of $T\left(3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\right)$?

$$\begin{aligned} T\left(3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\right) &= 2T\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\right) \\ &= 3\begin{bmatrix} 9 \\ -1 \\ -4 \end{bmatrix} + 2\begin{bmatrix} -9 \\ 1 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix}. \end{aligned}$$

(b) Given that $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$, and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, deter-
[2] mine a matrix A such that $T(\vec{x}) = A\vec{x}$ for any vector $\vec{x} \in \mathbb{R}^3$.

Since the columns of A are given by $T(\hat{\imath})$, $T(\hat{\jmath})$, and $T(\hat{k})$ respectively, we can immediately conclude that

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 2 \\ 0 & 2 & -1 \end{bmatrix}.$$

[6] 6. Solve the following system of linear equations, if possible:

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 3 \\ -3x_1 & - & 6x_2 & + & 2x_3 & - & x_4 & = & -7 \\ 2x_1 & + & 4x_2 & - & x_3 & & & = & 4 \end{array}$$

We set up the corresponding augmented matrix and reduce, as follows:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ -3 & -6 & 2 & -1 & -7 \\ 2 & 4 & -1 & 0 & 4 \end{array}\right] \xrightarrow[R_3-2R_1\to R_3]{R_2+3R_1\to R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 1 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{R_3+R_2\to R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1\to R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1+R_2\to R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This last matrix is in reduced row-echelon form. From here, we see that the variables x_2 and x_4 are free; solving for x_1 and x_3 using rows 1 and 2, respectively, we have

$$\begin{aligned} &x_1 = 1 - 2x_2 + x_4 \\ &x_2 \text{ is free} \\ &x_3 = -2 + 2x_4 \\ &x_4 \text{ is free} \end{aligned}$$

If we (optionally) want to verify our solution, we can check that

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= (1 - 2x_2 + x_4) + 2x_2 - (-2 + 2x_4) + x_4 = 3 \\ -3x_1 - 6x_2 + 2x_3 - x_4 &= -3(1 - 2x_2 + x_4) - 6x_2 + 2(-2 + 2x_4) + x_4 = -7 \\ 2x_1 + 4x_2 - x_3 &= 2(1 - 2x_2 + x_4) + 4x_2 - (-2 + 2x_4) = 4, \end{aligned}$$

as required.

[5] 7. Solve **one** of the following two problems.

(a) Let A be an $m \times n$ matrix and consider the set $\text{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$. Prove that $\text{null}(A)$ is a subspace of $\mathbb{R}^n$.

We note that $\text{null}(A)$ is non-empty, since $A\vec{0} = \vec{0}$, giving us $\vec{0} \in \text{null}(A)$.
Now, suppose that $\vec{x} \in \text{null}(A)$ and $\vec{y} \in \text{null}(A)$, so that $A\vec{x} = A\vec{y} = \vec{0}$. Then we have

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0},$$

showing that $\vec{x} + \vec{y} \in \text{null}(A)$. Since $\vec{x}$ and $\vec{y}$ were arbirtrary elements of $\text{null}(A)$, we see that $\text{null}(A)$ is closed under addition.
Finally, let $c \in \mathbb{R}$ be any scalar, and choose $\vec{x} \in \text{null}(A)$ as above. Then

$$A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0},$$

showing that $c\vec{x} \in \text{null}(A)$, and thus $\text{null}(A)$ is closed under scalar multiplication. It follows from the definition of a subspace of $\mathbb{R}^n$ that $\text{null}(A)$ is a subspace.

(b) Determine whether or not the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$ are linearly independent.

The given vectors are linearly independent if the only scalars c_1, c_2, c_3 such that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ are $c_1 = 0, c_2 = 0, c_3 = 0$. We have

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} c_1 + 2c_2 + c_3 \\ -3c_1 - 5c_2 - 5c_3 \\ 2c_1 + 4c_2 + 2c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

giving us the homogeneous system of equations $\begin{array}{rcrcrl} c_1 & + & 2c_2 & + & c_3 & = 0 \\ -3c_1 & - & 5c_2 & - & 5c_3 & = 0 \\ 2c_1 & + & 4c_2 & + & 2c_3 & = 0 \end{array}$. We solve as usual by reducing the corresponding augmented matrix, as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -3 & -5 & -5 & 0 \\ 2 & 4 & 2 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The matrix on the right is already in row-echelon form, and we can see that the variable c_3 is free, meaning that there exist infinitely many non-trivial solutions to the system. Indeed, we have

$$\begin{aligned} c_1 &= -2c_2 - c_3 = -5c_3 \\ c_2 &= 2c_3 \\ c_3 &\text{ is free} \end{aligned}$$

Choosing any non-zero value for c_3 gives us a non-trivial linear combination; for example, setting $c_3 = 1$ gives $c_1 = -4$ and $c_2 = 3$, and we can verify that $-5\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3\vec{0}$.