Math 3500 Exercise Sheet

15 October, 2014

This week is all about practice with ϵ-δ limit proofs. Let's recall the definition:
Definition: Let $f : D \to \mathbb{R}$ be a function, and let a be a limit point of $D \subseteq \mathbb{R}$. We say that L is a **limit** of f at a if for every $\epsilon > 0$ there exists some $\delta > 0$ such that for any $x \in D$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

We showed in class that limits are unique, so when the above definition is satisifed, we can say that L is *the* limit of f at a, and write $\lim\limits_{x \to a} f(x) = L$. Note that the condition $0 < |x - a| < \delta$ means that $x \in (a - \delta, a) \cup (a, a + \delta)$ (we're not allowing $x = a$), and we're requiring that $f(x)$ lands in the interval $(L - \epsilon, L + \epsilon)$.

Example: Prove that $\lim_{x\to 2} \left(\dfrac{x}{x+2}\right) = \dfrac{1}{2}$.

Our proof will begin with "Let $\epsilon > 0$ be given, and let $\delta = \ldots$", similarly to how we began our limit proofs for sequences. The trick is to figure out what δ should be. We begin with some rough work:

$$|f(x) - L| = \left|\frac{x}{x+2} - \frac{1}{2}\right| = \left|\frac{2x - (x+2)}{2(x+2)}\right| = \left|\frac{x-2}{2x+4}\right|.$$

Now, $|x - 2|$ is the thing we have control over: we can choose δ to be whatever we want, and then take $|x - 2| < \delta$. So we can make the above difference small by making $|x - 2|$ small, but we have to make sure that the denominator doesn't end up making things too big. We usually do this by trying a test value for δ, and to keep the arithmetic simple, a common choice is $\delta = 1$. If $|x - 2| < 1$, we get $-1 < x - 2 < 1$, so $1 < x < 3$. We need to deal with the $2x + 4$ in the denominator, so we note

$$1 < x < 3 \Rightarrow 2 < 2x < 6 \Rightarrow 6 < 2x + 4 < 10 \Rightarrow \frac{1}{10} < \frac{1}{2x+4} < \frac{1}{6}.$$

Note that shrinking δ will shrink the allowed range for x, which will in turn shrink the range of values for $1/(2x + 4)$. Thus, we can use this estimate by making sure that δ is no bigger than 1. Now, with $\delta \leq 1$ we can ensure that

$$\left|\frac{x-2}{2x+4}\right| = |x-2|\left|\frac{1}{2x+4}\right| < |x-2|\left(\frac{1}{6}\right),$$

and since we want this to be less than ϵ, and $|x - 2| < \delta$, we just need to make sure that δ is no bigger than 6ϵ. Since we need to guarantee simultaneously that $\delta \leq 1$ and $\delta \leq 6\epsilon$, this tells us that we should take $\delta = \min\{1, 6\epsilon\}$. With all this rough work done, we can assemble our proof:

Proof: Let $\epsilon > 0$ be given, and take $\delta = \min\{1, 6\epsilon\}$. Suppose that $|x-2| < \delta$. Since $\delta \leq 1$, we have $1 < x < 3$, and thus $6 < 2x+4 < 10$, which gives us

$$|f(x) - L| = \left|\frac{x}{x+2} - \frac{1}{2}\right| = \left|\frac{2x-(x+2)}{2(x+2)}\right| = \left|\frac{x-2}{2x+4}\right| < \frac{|x-2|}{6} < \frac{\delta}{6} \leq \frac{6\epsilon}{6} = \epsilon.$$

Remark: Since we require a to be a limit point of D in the defintion of the limit, we can find a sequence (a_n) with each $a_n \in D$, and $a_n \neq a$ for all $n \in \mathbb{N}$, such that $a_n \to a$. One can show (see Theorem 5.1.8 in the textbook) that

$$\lim_{x\to a} f(x) = L \text{ if and only if } f(a_n) \to L \text{ for any such sequence } (a_n).$$

A useful consequence of this fact is that if we can find a sequence (a_n) with $a_n \to a$ such that $f(a_n)$ does not converge, then f cannot have a limit at a. For example, if $f(x) = \sin(1/x)$ and $a_n = 2/(n\pi)$, then $a_n \to 0$ but $f(a_n) = \sin(n\pi/2)$, which gives the alternating sequence $(f(a_n) = (1, 0, -1, 0, 1, 0, -1, 0, \ldots)$ which does not converge. It follows that $\lim_{x\to 0}\sin(1/x)$ does not exist.

Problems

1. Show that $\lim\limits_{x\to a} k = k$ and $\lim\limits_{x\to a} x = a$ for any $a, k \in \mathbb{R}$.
2. Prove that each limit is correct using the definition of the limit:
   (a) $\lim\limits_{x\to 3} x^2 = 9$
   (b) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = 2$. (Hint: since we require $0 < |x-1|$, you know that $x \neq 1$, and this will let you simplify the function.)
   (c) $\lim\limits_{x\to -1}(x^2 - 2x + 1) = 4$.
   (d) $\lim\limits_{x\to -3} \dfrac{2x-1}{x+4} = -7$. (Note: you'll need a test value for δ as in the example above, but letting $\delta = 1$ will let $x+4$ get close to zero, preventing you from getting the bound that you need. How can you correct this?)
3. Prove the **limit laws**: Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be functions, and let a be a limit point of D. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then:
   (a) For any $k \in \mathbb{R}$, $\lim\limits_{x\to a}(kf(x)) = kL$.
   (b) $\lim\limits_{x\to a}(f(x) + g(x)) = L + M$
   (c) $\lim\limits_{x\to a}(f(x)g(x)) = LM$
   (d) If $g(x) \neq 0$ for all $x \in D$ and $M \neq 0$, then $\lim\limits_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$.