1. Let $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$. Calculate the following:

   (a) $\vec{u} + \frac{1}{2}\vec{v}$

$$\vec{u} + \frac{1}{2}\vec{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 2-2 \\ -3+0 \\ 1+3/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 5/2 \end{bmatrix}.$$

   (b) $\vec{u} \cdot \vec{v}$

$$\vec{u} \cdot \vec{v} = 2(-4) + (-3)(0) + 1(3) = -8 + 0 + 3 = -5.$$

   (c) $\|\vec{v}\|$

$$\|\vec{v}\| = \sqrt{(-4)^2 + 0^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5.$$

   (d) Find a unit vector in the direction of $\vec{v}$.
   Let $\vec{w}$ be the desired unit vector. Then $\vec{w} = c\vec{v}$ for some $c > 0$, since $\vec{w}$ is in the same direction as $\vec{v}$, and we want $\|\vec{w}\| = 1$, so

$$1 = \|\vec{w}\| = \|c\vec{v}\| = c\|\vec{v}\| = c(5),$$

   using part (c). Thus $c = 1/5$, so $\vec{w} = \frac{1}{5}\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 0 \\ 3/5 \end{bmatrix}$.

   Note: The above solution assumes that you haven't seen the rule for forming a unit vector, and are working things out from scratch. If you look at the details above, you can see that in general, our unit vector is $\vec{w} = \dfrac{1}{\|\vec{v}\|}\vec{v}$. If you know this result, you can just use it directly.

2. Let $P = (2, -1, 3)$ and $Q = (0, 3, -2)$. Find the coordinates of the point R that is $\frac{1}{5}$ of the way from P to Q.

   Since the point R is one fifth of the way from P to Q, it follows that $\overrightarrow{PR} = \dfrac{1}{5}\overrightarrow{PQ}$. (The vector from P to R is one fifth as long as the vector from P to Q.) Thus,

$$\overrightarrow{PR} = \frac{1}{5}\begin{bmatrix} 0-2 \\ 3-(-1) \\ -2-3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ -1 \end{bmatrix}.$$

Now, to determine the coordinates of the point R, it suffices to find the components of the position vector $\overrightarrow{OR}$, where O is the origin. Since $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$ (to get from O to R, we can first travel from O to P, and then from P to R; it may help to draw a diagram) we have

$$\overrightarrow{OR} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2/5 \\ 4/5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-2/5 \\ -1+4/5 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 8/5 \\ -1/5 \\ 2 \end{bmatrix}.$$

Thus, the point R is given by $R = (8/5, -1/5, 2)$.

3. Find the parametric equations of the line that passes through the points $P = (3, -1, 4)$ and $Q = (1, 0, 2)$.

To get the equation of a line in $\mathbb{R}^3$, we need a point on the line, and we need a direction vector. We're given two points on the line, so we can choose either one of them. Let's take $P = (3, -1, 4)$. Since the line passes through both P and Q, the vector $\overrightarrow{PQ}$ must be parallel to the line. Thus,

$$\vec{v} = \overrightarrow{PQ} = \begin{bmatrix} 1-3 \\ 0-(-1) \\ 2-4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

is a direction vector for the line. It follows that the vector equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix},$$

and the corresponding parametric equations are

$$x = 3 - 2t, \quad y = -1 + t, \quad z = 4 - 2t.$$

4. Let L be the line given by the parametric equations

$$\begin{aligned} x &= 2 - t \\ y &= -3 + 2t \\ z &= 1 + t \end{aligned}$$

Determine a point P on the line L such that the distance from P to $(2, -3, 1)$ is equal to 3.

There are several valid approaches to this problem. For ease of reference, let us write $Q = (2, -3, 1)$ for the given point, and note that Q is a point on the line: it corresponds to setting $t = 0$ in the parametric equations for the line.

The first approach uses the distance formula. We let $P = (2 - t, -3 + 2t, 1 + t)$ be our point on the line. (Note that any point on the line must satisfy the parametric equations for the line.) The distance from P to Q is then given by

$$\begin{aligned} d &= \sqrt{(2-(2-t))^2+(-3-(-3+2t))^2+(1-(1+t))^2} \\ &= \sqrt{t^2+(-2t)^2+(-t)^2} \\ &= \sqrt{t^2+4t^2+t^2} \\ &= \sqrt{6t^2} = \sqrt{6}|t|. \end{aligned}$$

Since we want $d = 3$, we must have $\sqrt{6}|t| = 3$, so $|t| = 3/\sqrt{6}$, and thus $t = \pm 3/\sqrt{6}$. We choose the positive solution $t = 3/\sqrt{6}$, and pluggin this into the parametric equations for the line gives us the point

$$P = \left(2 - \frac{3}{\sqrt{6}}, -3 + \frac{6}{\sqrt{6}}, 1 + \frac{3}{\sqrt{6}}\right).$$

The other two approaches involve vectors. From the parametric equations of the line, we can read off (by looking at the numbers multiplying t) the direction vector $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. We notice that $\|\vec{v}\| = \sqrt{(-1)^2+2^2+1^2} = \sqrt{6}$, so a unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{1}{\sqrt{6}}\vec{v} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

Note that the position vector of any point P on the line can be written as $\overrightarrow{OP} = \overrightarrow{OQ} + t\vec{u}$; in particular, the point $Q = (2, -3, 1)$ corresponds to setting $t = 0$. We want P to be on the line a distance of 3 units away from Q, which means that we have to have

$$\overrightarrow{OP} = \overrightarrow{OQ} + 3\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 2-3/\sqrt{6} \\ -3+6/\sqrt{6} \\ 1+3/\sqrt{6} \end{bmatrix},$$

and converting $\overrightarrow{OP}$ to the corresponding point P, we see that we get the same answer as above.

(If the vector equation above isn't clear, note the following: we want to start at Q, and move along the line, which means moving in the direction of the vector $\vec{v}$. The unit vector $\vec{u}$ points in the same direction points in the same direction as $\vec{v}$, and has length 1. Since we want to move a distance of 3 units, we add the vector $3\vec{u}$, which has length 3.)

The last approach is a minor variation on the one above. We want to move from the point Q in the direction of $\vec{v}$ a distance of 3 units. That means we need to add a scalar multiple $t\vec{v}$ of $\vec{v}$ to $\overrightarrow{OQ}$ to obtain $\overrightarrow{OP}$. We want $\|t\vec{v}\| = 3$, so

$$3 = \|t\vec{v}\| = |t|\|\vec{v}\| = |t|\sqrt{6},$$

giving us $|t| = 3/\sqrt{6}$ and $t = \pm 3/\sqrt{6}$, the same as above.

A last note on this problem: the two approaches using vectors were feasible because of the fact that the point Q is on the given line. One could just as easily ask for a point P on the line that is a distance of 3 away from a point Q that is **not** on the line. In this case, the first approach is the most reasonable way to do the problem.