Math 1410 Assignment #1 Solutions
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1. Prove the *distributive property* for complex arithmetic. That is, prove that for any complex numbers u, v, w, we have
$$u(v+w) = uv + uw.$$

Solution: Let $u = a + ib$, $v = c + id$, and $w = e + if$ be arbitrary complex numbers, where $a, b, c, d, e, f \in \mathbb{R}$. Then we have

$$\begin{aligned}
u(v+w) &= (a+ib)[(c+id)+(e+if)] \\
&= (a+ib)[(c+e)+i(d+f)] \\
&= [a(c+e)-b(d+f)]+i[(b(c+e)+a(d+f)] && \text{(using } i^2=-1\text{)} \\
&= (ac+ae-bd-bf)+i(bc+be+ad+af) \\
&= [(ac-bd)+i(bc+ad)]+[(ae-bf)+i(be+af)] && \text{(rearranging)} \\
&= [(a+ib)(c+id)]+[(a+ib)(e+if)] \\
& && \text{(reversing the definition of complex multiplication)} \\
&= uv+uw,
\end{aligned}$$

as required.

2. Recall that the complex conjugate of $z \in \mathbb{C}$ is denoted by $\overline{z}$, and the modulus of z is denoted by $|z|$. Show that:

   (a) $|\overline{z}| = |z|$

   (b) $|z| = \sqrt{z\overline{z}}$

   (c) $\operatorname{Re}(z) = \dfrac{z+\overline{z}}{2}$ and $\operatorname{Im}(z) = \dfrac{z-\overline{z}}{2i}$, where $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ denote the real and imaginary parts of z, respectively.

Solution: Let $z = x + iy$, where $x, y \in \mathbb{R}$, be any complex number. Then:

(a) Since $\overline{z} = x - iy = x + i(-y)$ by definition of the complex conjugate, the modulus formula gives us
$$|\overline{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |z|.$$

(b) Using the rules for complex multiplication, we have

$$z\bar{z} = (x+iy)(x-iy) = x^2 - ixy + ixy - i^2y^2 = x^2 + y^2.$$

It follows that $|z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$.

(c) By definition, with $z = x+iy$ we have $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$. We can then verify that

$$\frac{z+\bar{z}}{2} = \frac{(x+iy)+(x-iy)}{2} = \frac{2x}{2} = x = \mathrm{Re}(z),$$

and

$$\frac{z-\bar{z}}{2i} = \frac{(x+iy)-(x-iy)}{2i} = \frac{2iy}{2i} = y = \mathrm{Im}(z).$$

3. Convert $z = -1+\sqrt{3}i$ to polar form, and compute the value of $z^6 = (-1+\sqrt{3}i)^6$. Express your answer in rectangular form.

Solution: We compute $|z| = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$, and thus

$$z = 2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right) = 2e^{i(2\pi/3)}.$$

It follows that

$$z^6 = (2e^{i(2\pi/3)})^6 = 2^6e^{i\cdot 6(2\pi/3)} = 2^6e^{i(4\pi)} = 64(\cos(4\pi)+i\sin(4\pi)) = 64.$$

4. Let $\vec{v} = \langle 3,-1,4\rangle$ and $\vec{w} = \langle -2,5,1\rangle$ be two vectors in $\mathbb{R}^3$. Find the coordinates of:

(a) The point P, one half of the way from the tip of $\vec{v}$ to the tip of $\vec{w}$.

Solution: Referring to the diagram below, we see that the vector $\vec{v}-\vec{w}$ (when drawn with its tail at the tip of $\vec{w}$), points all the way from the tip of $\vec{w}$ to the tip of $\vec{v}$. We want the point P (as marked) that is half-way between these two points; this can be achieved by adding one half of the vector $\vec{v}-\vec{w}$ to $\vec{w}$. Thus, the position vector for P is given by

$$\begin{aligned}\overrightarrow{OP} &= \vec{w} + \frac{1}{2}(\vec{v}-\vec{w})\\ &= \frac{1}{2}\vec{v} + \frac{1}{2}\vec{w} = \frac{1}{2}(\vec{v}+\vec{w})\\ &= \frac{1}{2}(\langle 3,-1,4\rangle + \langle -2,5,1\rangle)\\ &= \frac{1}{2}\langle 1,4,5\rangle = \left\langle \frac{1}{2}, 2, \frac{5}{2}\right\rangle.\end{aligned}$$

(b) The point Q, one third of the way from the tip of $\vec{v}+\vec{w}$ to the tip of $\vec{v}-\vec{w}$.

Solution: Again referring to our diagram, we see that if we draw the vector $\vec{v}-\vec{w}$ with its tail at the origin, then the vector $\vec{w}+\vec{w}=2\vec{w}$ points all the way from the tip of $\vec{v}-\vec{w}$ to the tip of $\vec{v}+\vec{w}$. We want the point Q that is one-third of the way *from* the tip of $\vec{v}+\vec{w}$ *to* the tip of $\vec{v}-\vec{w}$. If we are starting at the tip of $\vec{v}+\vec{w}$ and ending at the tip of $\vec{v}-\vec{w}$, then we are moving opposite to the vector $2\vec{w}$, so we need to use $-2\vec{w}$. However, we only want to go one third of the way, which tells us that we should add the vector $\frac{1}{3}(-2\vec{w})=-\frac{2}{3}\vec{w}$ to the vector $\vec{v}+\vec{w}$, according to the tip-to-tail rule for vector addition. Thus, we have

$$\begin{aligned}\overrightarrow{OQ} &= (\vec{v}+\vec{w})+\left(-\frac{2}{3}\vec{w}\right)\\ &= \vec{v}+\frac{1}{3}\vec{w}\\ &= \langle 3,-1,4\rangle+\frac{1}{3}\langle -2,5,1\rangle\\ &= \langle 3,-1,4\rangle+\left\langle -\frac{2}{3},\frac{5}{3},\frac{1}{3}\right\rangle\\ &= \left\langle \frac{7}{3},\frac{2}{3},\frac{13}{3}\right\rangle.\end{aligned}$$