Math 4310 Assignment #7
University of Lethbridge, Fall 2014

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Due date: Friday, October 24th, by 5 pm.

Please submit solutions to the following problems (except where indicated). As usual, all the textbook exercises are recommended as practice problems if they don't appear as an assigned problem below.

1. Let A and B be subsets of a topological space X. Suppose that A is connected, and that B is both open and closed in X. Prove that if $A \cap B \neq \emptyset$, then $A \subseteq B$.

   [Hint: if $A \not\subseteq B$, consider $U = A \cap B$ and $V = A \cap B^c$.]

2. Show that if X and Y are connected topological spaces, then $X \times Y$ is connected.

   [Hint: suppose $f : X \times Y \to \{0, 1\}$ is continuous and nonconstant. Then there are points $(x_0, y_0), (x_1, y_1) \in X \times Y$ with $f(x_0, y_0) = 0$ and $f(x_1, y_1) = 1$. Note that either $f(x_0, y_1) = 0$ or $f(x_0, y_1) = 1$. In the first case, consider the map $i_{y_1} : X \to X \times Y$ given by $i_{y_1}(x) = (x, y_1)$. In the second case, consider i_{y_0}.]

3. Prove that a topological space X is connected if and only if $\partial A \neq \emptyset$ for every proper nonempty subset $A \subseteq X$.

   [Hint: you might find it easier to prove the contrapositive in both directions, and you proved a result on an earlier assignment that will be helpful.]

4. Prove that if A and B are path-connected subsets of a topological space X and $A \cap B \neq \emptyset$, then $A \cup B$ is path-connected. Conclude that for any finite collection $\{A_1, \ldots, A_n\}$ of path connected subsets of X, with $A_i \cap A_j \neq \emptyset$, $\bigcup_{i=1}^n A_i$ is path-connected.

5. Give an example to show that the intersection of two connected subspaces need not be connected. (Consider $\mathbb{R}^2$.)

6. Prove that the space $\mathcal{C}[0, 1]$ of all continuous real-valued functions on $[0, 1]$, equipped with the sup-norm metric (d_∞) is path-connected.

   [Hint: you can show the space is in fact convex.]