Math 1410 Assignment #4 Solutions
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1. Determine the null space and column space of the matrix $A = \begin{bmatrix} 2 & -3 & 1 & 4 \\ -1 & 2 & 2 & -3 \\ 1 & 0 & 8 & -1 \end{bmatrix}$.

   We begin by computing the reduced row-echelon form of A. We have:

$$\begin{bmatrix} 2 & -3 & 1 & 4 \\ -1 & 2 & 2 & -3 \\ 1 & 0 & 8 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 8 & -1 \\ -1 & 2 & 2 & -3 \\ 2 & -3 & 1 & 4 \end{bmatrix}$$
$$\xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 2 & 10 & -4 \\ 0 & -3 & -15 & 6 \end{bmatrix}$$
$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 1 & 5 & -2 \\ 0 & -3 & -15 & 6 \end{bmatrix}$$
$$\xrightarrow{R_3 + 3R_2 \to R_3} \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   This last matrix is in reduced row-echelon form. The null space is the set of all vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $A\vec{x} = \vec{0}$; this matrix equation corresponds to a system of equations with augmented matrix $\left[\begin{array}{cccc|c} 2 & -3 & 1 & 4 & 0 \\ -1 & 2 & 2 & -3 & 0 \\ 1 & 0 & 8 & -1 & 0 \end{array}\right]$ which, by the above, reduces to

$$\left[\begin{array}{cccc|c} 1 & 0 & 8 & -1 & 0 \\ 0 & 1 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This tells us that x_3 and x_4 are free variables, while $x_1+8x_3-x_4=0$ and $x_2+5x_3-2x_4=0$, so $x_1=-8x_3+x_4$ and $x_2=-5x_3+2x_4$. Thus, any solution $\vec{x}$ to $A\vec{x}=\vec{0}$ satisfies

$$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} -8x_3+x_4 \\ -5x_3+2x_4 \\ x_3 \\ x_4 \end{bmatrix}=x_3\begin{bmatrix} -8 \\ -5 \\ 1 \\ 0 \end{bmatrix}+x_4\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

It follows that $\operatorname{null}(A)=\operatorname{span}\left\{\begin{bmatrix} -8 \\ -5 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}$.

According to Theorem 30 in the textbook, the column space is generated by the columns of A that correspond to columns with leading ones in the reduced row-echelon form of A. Since there are leading ones in columns 1 and 2, we have that

$$\operatorname{col}(A)=\operatorname{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}\right\}.$$

2. Factor the matrix $A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \\ -1 & 2 & 4 \end{bmatrix}$ as a product of elementary matrices.

We reduce the matrix A to reduced row-echelon form, keeping track of the row operations. For each row operation on the left, we write down the corresponding elementary matrix on the right, along with its inverse. We have

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \\ -1 & 2 & 4 \end{bmatrix} \xrightarrow{R_2-2R_1\to R_2} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ -1 & 2 & 4 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3+R_1\to R_3} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 7 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}R_3\to R_3} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\xrightarrow{R_2+5R_3\to R_2} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1-3R_3\to R_1} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1+2R_2\to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_6^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since performing a row operation is the same as multiplying on the left by the corresponding elementary matrix, we have

$$I = E_6(E_5(E_4(E_3(E_2(E_1A))))) = (E_6E_5E_4E_2E_2E_1)A.$$

It follows that $A^{-1} = E_6E_5E_4E_3E_2E_1$, and thus

$$\begin{aligned} A &= (A^{-1})^{-1} = (E_6E_5E_4E_2E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

3. For each statement below, either prove the statement or give a counterexample showing that it is false.

   (a) If A and B are both invertible, then $A+B$ is invertible.

   The statement is false. For example the matrix $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ is invertible, with $A^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and the matrix $B = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ is invertible, with inverse $B^{-1} = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}$. However, $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is not invertible.

(b) If $AB = I$, then $AB = BA$.

This is false, since there is no assumption made on the sizes of A and B. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; however, $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq AB$.
If, on the other hand, we make the additional assumption that A and B are both of size $n \times n$, then the statement is true. If A and B are square matrices of the same size and $AB = I$, then by the Invertible Matrix Theorem, A is invertible, and by the uniqueness of the inverse, we must have $B = A^{-1}$. It then follows from the definition of the inverse that $BA = A^{-1}A = I$.

(c) If $AB = B$ for some matrix $B \neq 0$, then A is invertible.

This is false. For example, the matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not invertible, and the matrix $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is non-zero, but

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B.$$

(d) If A^3 is invertible, then A is invertible.

This is true. Suppose that A^3 is invertible. Then there exists a matrix B (the inverse of A^3) such that $A^3B = I$. But since $A^3 = A(A^2)$, we have

$$A(A^2B) = (A(A^2))B = A^3B = I.$$

It follows from the Invertible Matrix Theorem that A is invertible and from the uniqueness of the inverse that $A^{-1} = A^2B$.

4. Let A be a non-zero $n \times n$ matrix, and let I be the $n \times n$ identity matrix.

(a) Show that if $A^2 = 0$, then $(I - A)^{-1} = I + A$.

By the uniqueness of the inverse and the Invertible Matrix Theorem, it suffices to show that $(I - A)(I + A) = I$. We have

$$(I - A)(I + A) = I(I) + I(A) - A(I) - A(A) = I + A - A - A^2 = I,$$

since $A^2 = 0$ by assumption.

(b) Show that if $A^3 = 0$, then $(I - A)^{-1} = I + A + A^2$.

Similarly, it suffices to show that $(I - A)(I + A + A^2) = I$. We have

$$(I - A)(I + A + A^2) = I(I) + I(A) + I(A^2) - A(I) - A(A) - A(A^2) = I + A + A^2 - A - A^2 - A^3 = 0,$$

since we're assuming that $A^3 = 0$.

(c) Find the inverse of $B = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

We notice that $B = I - A$, where $A = \begin{bmatrix} 0 & -3 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$. We compute

$$A^2 = \begin{bmatrix} 0 & -3 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -3 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$A^3 = A^2(A) = \begin{bmatrix} 0 & 0 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -3 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Since $B = I - A$ where $A^3 = 0$, it follows from part (b) that

$$B = I + A + A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 14 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) Given that $A \neq 0, A^2 \neq 0, \ldots, A^{n-1} \neq 0$ but $A^n = 0$, determine a formula for $(I - A)^{-1}$, and show that your answer is correct.

Following the pattern above, we conjecture that $(I - A)^{-1} = I + A + A^2 + \cdots + A^{n-1}$. To confirm that this is correct, we compute

$$\begin{aligned} (I - A)(I + A + A^2 + \cdots + A^{n-1}) &= I + A + A^2 + \cdots + A^{n-1} - (A + A^2 + A^3 + \cdots + A^n) \\ &= I + (A - A) + (A^2 - A^2) + \cdots + (A^{n-1} - A^{n-1}) + A^n \\ &= I, \end{aligned}$$

since we're assuming $A^n = 0$.