University of Lethbridge
Department of Mathematics and Computer Science
29th April, 2015, 2:00-5:00 pm

Math 2000B - FINAL EXAM

Last Name:____

First Name:____

Student Number:_____________________________________

Record your answers below each question in the space provided. **Left-hand pages may be used as scrap paper for rough work.** If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

No external aids are permitted.

For grader's use only:

Page	Grade
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7	/10
8	/10
9	/10
10	/10
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Total	/100

1. Write the converse and the contrapositive of each of the following statements:

[2] (a) If a is an odd integer, then $3a$ is an odd integer.

[3] (b) If $ab = 0$, then $a = 0$ or $b = 0$.

2. Write the negation of each of the following statements. Use of symbolic notation is allowed but not required.

[2] (a) For all integers m, m^2 is even.

[3] (b) For all integers m, there exists an integer n such that $2m + 3n = 5$.

3. Let $A = \{1, 3, 6, 7\}$, $B = \{2, 3, 4, 8, 9\}$, and $C = \{1, 5, 7, 8\}$ be subsets of the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Verify the following by explicitly computing the sets involved:

[3] (a) $A^c \cup B^c = (A \cap B)^c$.

[3] (b) $(A \cup B) \setminus B = A \setminus (A \cap B)$

[4] (c) $A \cap (B \cup B) = (A \cap B) \cup (A \cap C)$.

4. For each $n \in \mathbb{N}$, let $A_n = \{k \in \mathbb{Z} : -n \leq k \leq n\}$.

[2] (a) Compute $\displaystyle\bigcup_{n=1}^{4} A_n$.

[2] (b) Compute $\displaystyle\bigcap_{n=3}^{6} A_n$.

[2] (c) What are $\displaystyle\bigcup_{n=1}^{\infty} A_n$ and $\displaystyle\bigcap_{n=1}^{\infty} A_n$?

[4] (d) Let $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ be any indexed collection of sets (where the index set I is non-empty). Prove that if $B \subseteq A_\alpha$ for all $\alpha \in I$, then $B \subseteq \bigcap_{\alpha \in I} A_\alpha$.

5. Let $f : A \to B$ be a function.

[2] (a) Given an element $b \in B$, define the preimage $f^{-1}(b)$.

[1] (b) Given a subset $C \subseteq B$, define the preimage $f^{-1}(C)$.

6. Let $f : A \to B$ and $g : B \to C$ be functions.

[5] (a) Prove that if $g \circ f : A \to C$ is an injection, then f is an injection.

[2] (b) Give an example to show that g need not be an injection.

7. Determine if each of the following statements is true. If it is true, give a direct proof of the statement. If it is false, give a counterexample.

[3] (a) For all integers a, b, c, with $a \neq 0$, if $a|b$ then $a|(bc)$.

[2] (b) For all integers a and b with $a \neq 0$, if $6|(ab)$, then $6|a$ or $6|b$.

[5] (c) The function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ given by $f(x) = x^2 + 3 \pmod 5$ is a bijection, where $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$.

[4] 8. Prove the following biconditional statement: For any integer n, n is odd if and only if n^2 is odd.

[6] 9. Use mathematical induction to prove that for all natural numbers n,

$$1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1).$$

[5] 10. Prove that for any real numbers x and y, if x is rational and y is irrational, then $x+y$ is irrational. (Hint: proof by contradiction.)

[5] 11. Prove that $8|(n^2-1)$ for all **odd** natural numbers n.

(*Hint:* a proof by induction is possible, but a proof by cases is easier.)

12. Let $A = \{1, 2, 3\}$. For each of the relations $R \subseteq A \times A$ on A below, determine if R is a function, an equivalence relation, or neither.

[3] (a) $R = \{(1, 1), (2, 1), (3, 1), (2, 3), (3, 3)\}$

[3] (b) $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

[3] (c) $R = \{(1, 1), (2, 3), (3, 3)\}$

13. Give an example of a relation R on a nonempty set A such that R is both a function and
[1] an equivalence relation, or explain why such a relation cannot exist.

14. Determine the addition and multiplication tables for modular arithmetic on $\mathbb{Z}_3 = \{[0], [1], [2]\}$.
[4]

15. (a) Prove the following: For all $[a], [b] \in \mathbb{Z}_3$, if $[a]^2 \oplus [b]^2 = [0]$, then $[a] = [0]$ and $[b] = [0]$.
[4]

(b) Explain why it follows from Part (a) that for all $a, b \in \mathbb{Z}$, if 3 divides $a^2 + b^2$, then 3 divides a and 3 divides b.
[2]

[1] 16. (a) Define what it means for two sets to be **equivalent**.

[1] (b) Define what it means for a set to be **finite**.

[4] (c) Suppose $f : A \to B$ is an injection. Prove that there exists a subset $C \subseteq B$ such that A is equivalent to C.

[4] (d) For each subset of $\mathbb{R}$ below, indicate whether the set is finite, countably infinite, or uncountable:

- $A = \{x \in \mathbb{R} : x^2 = 1\}$.
- $B = \{x \in \mathbb{R} : x^2 \leq 1\}$.
- $C = \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.
- $\mathbb{Q}$ (the set of rational numbers).

Extra space for rough work. Do not remove unless there is nothing on this page you want to be graded.