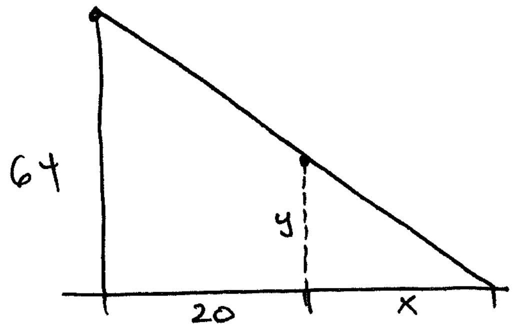
Q1



A diagram of the situation is shown on the left. Using similar triangles,

$$\frac{x}{y} = \frac{x+20}{64}.$$

This gives $\quad 64x = xy + 20y$ *

$$\Rightarrow x(64-y) = 20y \quad \Rightarrow x = \frac{20y}{64-y}.$$ **

$$\therefore \frac{dx}{dt} = \frac{dx}{dy}\cdot\frac{dy}{dt} = \left(\frac{20(64-y)-20y(-1)}{(64-y)^2}\right)\frac{dy}{dt}$$

$$= \frac{1280}{(64-y)^2}\cdot\frac{dy}{dt}.$$

We're given $y(t) = 64 - 16t^2$, so when $t=1$, $y = 64-16 = 48$, and

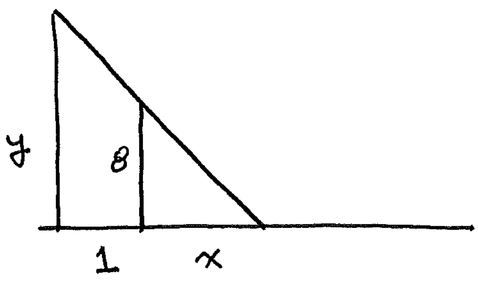
$$y'(t) = -32t \quad \Rightarrow \quad y'(1) = -32 \text{ ft/s}$$

$\therefore$ when $t=1$, $\dfrac{dx}{dt} = \dfrac{1280}{(64-48)^2}(-32) = \underline{\underline{-160 \text{ ft/s}}}.$

* One can use implicit differentiation immediately, giving
$64\frac{dx}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} + 20\frac{dy}{dt}$, and solve for $\frac{dx}{dt}$.

** One can also do everything explicitly. With $y = 64-16t^2$,
$$x = \frac{20(64-16t^2)}{64-(64-16t^2)} = \frac{1280-320t^2}{16t^2} = \frac{80}{t^2} - 20.$$

Q2



Referring to the diagram on the left, the length L satisfies

$$L^2 = y^2 + (x+1)^2,$$

using the Pythagorean Theorem.

Using similar triangles, we have $\frac{x}{8} = \frac{x+1}{y}$,

giving $y = \frac{8(x+1)}{x}$.

$\therefore$ we have $L^2 = \left(\frac{8(x+1)}{x}\right)^2 + (x+1)^2$.

Since L will be a minimum if and only if L^2 is a minimum, we consider the function

$$f(x) = (x+1)^2 + \frac{64(x+1)^2}{x^2}, \quad x \in (0, \infty).$$

$$= (x+1)^2\left(1 + \frac{64}{x^2}\right).$$

To minimize f, we look for critical points:

$$f'(x) = 2(x+1)\left(1+\frac{64}{x^2}\right) + (x+1)^2\left(-\frac{128}{x^3}\right)$$

$$= 2(x+1)\left[\frac{x^2+64}{x^2} - \frac{64(x+1)}{x^3}\right] = \frac{2(x+1)(x^3-64)}{x^3}.$$

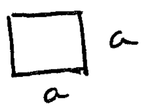
Since $x > 0$ we have $f'(x) = 0$ when $x^3 - 64 = 0$, or $x = 4$. We note $f'(x) < 0$ for $0 < x < 4$ and $f'(x) > 0$ for $x > 4$, so there is a local minimum at $x = 4$, as required.

The length is $L = \sqrt{f(4)} = \sqrt{5^2\left(1+\frac{64}{16}\right)} = 5\sqrt{5}$.

Q3

Let x be the length of the first piece; the second has length $10-x$.

Note $0 \le x \le 10$.

Square : 

(use the first piece of wire)

Area is $A_1 = a^2$.
Perimeter is $4a = x$

$\therefore A_1 = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$.

Circle : 

(use the second piece of wire)

Area is $A_2 = \pi b^2$.
Circumference is $2\pi b = 10 - x$

$\Rightarrow b = \frac{10-x}{2\pi}$,

so $A_2 = \frac{(10-x)^2}{4\pi}$.

$\therefore$ Our total area is

$$A(x) = \frac{x^2}{16} + \frac{(10-x)^2}{4\pi}, \text{ with } x \in [0,10].$$

Note $A(0) = \frac{10^2}{4\pi} = \frac{25}{\pi}$ and $A(10) = \frac{10^2}{16} = \frac{25}{4}$.

We have $A'(x) = \frac{2x}{16} + \frac{2(10-x)(-1)}{4\pi} = \frac{\pi x + 4x - 40}{8\pi}$.

$\therefore A'(x) = 0$ when $(\pi+4)x = 40$, or $x = \frac{40}{\pi+4}$,

so $10 - x = 10 - \frac{40}{\pi+4} = \frac{10\pi}{\pi+4}$.

$\therefore A\left(\frac{40}{\pi+4}\right) = \frac{1}{16}\cdot\frac{40^2}{(\pi+4)^2} + \frac{1}{4\pi}\cdot\frac{(10\pi)^2}{(\pi+4)^2} = \frac{25}{\pi+4}$.

Since $\frac{25}{\pi+4} < \frac{25}{4} < \frac{25}{\pi}$, we have:

(a) The maximum is $A(0) = 25/\pi$: use all the wire for the circle

(b) The minimum is $A\left(\frac{40}{\pi+4}\right) = \frac{25}{\pi+4}$: use $\frac{40}{\pi+4}$ cm for the square, $\frac{10\pi}{\pi+4}$ cm for the circle