1. Solve the system of linear equations below by (a) forming the corresponding augmented matrix, and (b) using Gaussian elimination (row operations) to reduce it to row-echelon form.

$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 4 \\ -x & + & y & - & 2z & = & -2 \\ 2x & - & 4y & + & 3z & = & 9 \end{array}$$

The augmented matrix is given by $[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 1 & -2 & -2 \\ 2 & -4 & 3 & 9 \end{array}\right]$. We proceed using row operations to simplify this matrix. Since we already have a leading 1 in the upper left-hand corner, we begin by adding Row 1 to Row 2, and then subtracting 2 times Row 1 from Row 3, to create zeros below this leading one:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 1 & -2 & -2 \\ 2 & -4 & 3 & 9 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

At this point, our augmented matrix is in row-echelon form: each row begins with a leading 1, and each leading 1 has only 0s below it. If we like, we can now solve using back substitution: the third row tells us $z = 1$; the second row tells us $y + z = -2$, and plugging $z = 1$ into this gives us $y + 1 = -2$, so $y = -3$. The first row tells us that $x - 2y + z = 4$, and putting $y = -3$, $z = 1$ into this equation gives us $x - 2(-3) + 1 = 4$, or $x + 7 = 4$, so $x = -3$. This gives us the solution

$$x = -3, \quad y = -3, \quad , z = 1.$$

We can verify that our solution works: $-3 - 2(-3) + 1 = -3 + 6 + 1 = 4$, so it satisfies the first equation. $-(-3) + (-3) - 2(1) = -2$, so it satisfies the second equation. $2(-3) - 4(-3) + 3(1) = -6 + 12 + 3 = 9$, so it satisfies the third equation, as well.

If you prefer not to do back substitution, you can proceed to reduced row-echelon form. Since the third row now has two 0s and a 1, we can use the 1 to eliminate entries above it in the third column, without changing anything in the first two columns. Once that's done, our last step will be to eliminate the -2 at the top of the second column:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

This is the reduced row-echelon form of our augmented matrix, and from here we can immediately read off our solution of $x = -3, y = -3, z = 1$.

2. Consider the following system of two equations in three variables:

$$\begin{array}{rcrcrl} x & + & 2y & - & 3z & = 6 \\ 2x & + & 5y & - & 4z & = -3 \end{array}$$

(a) What geometric object in $\mathbb{R}^3$ (3-dimensional space) is defined by each of the individual equations? (A point? A line? A plane? Something else?)

Each of these equations defines a plane in $\mathbb{R}^3$.

(b) Find a one-parameter family of solutions to the system of equations above. (It should only take one row operation for row-echelon form, two for reduced row-echelon form.)

We probably don't need to use an augmented matrix here, but just for fun, we have

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & 5 & -4 & -3 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & 2 & -15 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 36 \\ 0 & 1 & 2 & -15 \end{array}\right]$$

Since the z-column has no leading one, we set $z = t$ as our parameter. The first row of the reduced row-echelon form then gives us $x - 7z = 36$, so $x = 36 + 7t$, and the second row gives us $y + 2z = -15$, so $y = -15 - 2t$. Our solution is therefore

$$x = 36 + 7t, \quad y = -15 - 2t, \quad z = t.$$

(c) Write your solution to part (b) in vector form (as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \ldots$). What geometric object in $\mathbb{R}^3$ does your solution represent? Does this seem reasonable?
(Note that any point (x, y, z) that satisfies **both** of the original equations belongs to the **intersection** of the objects defined by those equations.

In vector form, we have $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 36 \\ -15 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}$, which is the vector equation of a line through the point $(36, -15, 0)$ in the direction of the vector $\vec{v} = \begin{bmatrix} 7 & -2 & 1 \end{bmatrix}^T$. Since each of the original equations describes a plane, any solution to the system must be a point that lies on both planes (since it satisfies both equations). The set of all solutions is therefore the intersection of the two planes, and it makes sense that the intersection of two planes in $\mathbb{R}^3$ is a line.

(d) Suppose it had turned out that the system of equations had no solutions. How would you explain this visually?

The only way there would be no solution is if the left-hand side of the second equation was a multiple of the first, but this was not true of the right-hand side. Since the coefficients of x, y, and z determine the normal vector, this would mean that the normal vector for the second plane was a scalar multiple of the first, and the two planes would therefore be parallel. A system of two equations in three unknowns is therefore represented by a pair of parallel planes, and since parallel planes don't intersect, there should not be any solutions to the system.