MATH 1565 - Tutorial #9 Solutions

1. Consider the function $f(x) = x^4 \ln(x)$.

[1] (a) State the domain of f

The domain of f is $(0, \infty)$ since $\ln(x)$ is undefined for $x \leq 0$.

(b) Determine any x-intercepts. (If you answered (a) correctly, you'll know why I didn't ask
[1] for a y-intercept).

Since $x \neq 0$ we have $f(x) = 0$ when $\ln(x) = 0$, giving us $x = 1$. Thus, $(1, 0)$ is the only intercept.

[3] (c) Determine $\lim\limits_{x\to 0^+} f(x)$. (Use l'Hospital's Rule.)

We have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^4 \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-4}}.$$

This is a limit of the form "∞/∞". Taking the derivative of the top and bottom, we have

$$\lim_{x\to 0^+} \frac{1/x}{-4x^{-5}} = \lim_{x\to 0^+} \frac{-1}{4} x^4 = 0.$$

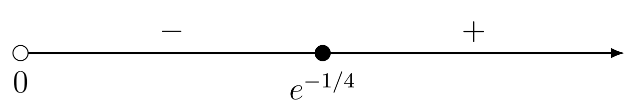
Since this limit exists, the original limit does as well, and $\lim\limits_{x\to 0^+} f(x) = 0$.

[2] (d) Compute $f'(x)$.

$$\begin{aligned} f'(x) &= 4x^3 \ln(x) + x^4(1/x) \\ &= 4x^3 \ln(x) + x^3 = x^3(4\ln(x) + 1). \end{aligned}$$

[2] (e) Construct a sign diagram for f'. On what intervals is f increasing/decreasing?

From above, since $x > 0$, we have $f'(x) = 0$ only if $4\ln(x)+1 = 0$, giving us $\ln(x) = -1/4$, so $x = e^{-1/4}$ is our only critical number. Since $\ln(x)$ is an increasing function, we see that $4\ln(x) + 1 < 0$ when $x < e^{-1/4}$, and $4\ln(x) + 1 > 0$ when $x > e^{-1/4}$. This gives the sign diagram



[1] (f) Classify any critical numbers found in part (e) as local maxima, minima, or neither.

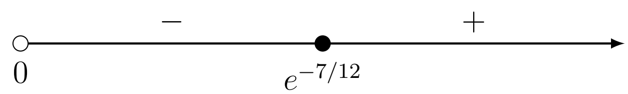
From the sign diagram above, the first derivative test tells us that we have a local minimum at $(e^{-1/4}, f(e^{-1/4}))$.

[2] (g) Compute $f''(x)$.

$$
\begin{aligned}
f''(x) &= 12x^2\ln(x) + 4x^3(1/x) + 3x^2 \\
&= 12x^2\ln(x) + 7x^2 = x^2(12\ln(x) + 7).
\end{aligned}
$$

[2] (h) Construct a sign diagram for f''. On what intervals is the graph of f concave up/down?

Using the same reasoning that we used to obtain the sign diagram for $f'(x)$, we see that $f''(x) = 0$ for $x = e^{-7/12}$, and we get the sign diagram



[1] (i) Determine any inflection points on the graph of f.

From the above sign diagram, we see that there is an inflection point at $(e^{-7/12}, f(e^{-7/12}))$.

[3] (j) Sketch the graph of f. Your graph should reflect your results in parts (a) - (i) above. Label any intercepts, critical points, and inflection points.

Our graph does not need to be to scale (and indeed, if we want to see the features noted above, we need to stretch the y scale). The precise location of the critical point and inflection point are not important, as long as we note that $e^{-7/12} < e^{-1/4}$ and $f(e^{-7/12}) > f(e^{-1/4})$, so the inflection point is above and to the left of the critical point. Using all the information above, we get the following graph: