1. Solve the following separable differential equations:

   (a) $xy' = y + 2x^2y$, where $y(1) = 1$.

   We have $xy' = y(1 + 2x^2)$, and rearranging gives $\frac{dy}{y} = \left(\frac{1}{x} + 2x\right) dx$. Integrating this, we have $\ln y = \ln x + x^2 + C$. Applying the initial condition $y(1) = 1$, we have $0 = 0 + 1 + C$, so $C = -1$. Solving for y, our final solution is

   $$y = xe^{x^2-1}.$$

   Note: this can also be solved as a linear equation: $\frac{dy}{dx} - \left(\frac{1}{x} + 2x\right) y = 0$, and of course the result is the same.

   (b) $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$, where $y(0) = 1$. (Give an implicit solution.)

   Rearranging this equation gives us $(y^2 + 1)\, dy = (x^2 + 1)\, dx$, and integrating both sides, we have

   $$\frac{1}{3}y^3 + y = \frac{1}{3}x^3 + x + C.$$

   The initial condition $y(0) = 1$ gives is $\frac{1}{3} + 1 = C$, so $C = \frac{4}{3}$. The solution is thus given implicitly by $y^3 + 3y = x^3 + 3x + 4$.

   (c) $(4y + yx^2)\, dy - (2x + xy^2)\, dx = 0$

   We have $y(4 + x^2)\, dy = x(2 + y^2)\, dy$, so $\frac{y}{2+y^2}\, dy = \frac{x}{4+x^2}\, dx$. This gives us $\ln(2 + y^2) = \ln(4 + x^2) + C = \ln(k(4 + x^2))$, where $k = e^C$. Thus, our solution is

   $$y^2 = k(4 + x^2) - 2.$$

   The value of k (and choice of positive or negative square root) will depend on the initial condition, which was not given.

2. Solve the following linear differential equations. State an interval on which the general solution is defined.

   (a) $\frac{dy}{dx} + y = e^{3x}$

   Here the coefficient of y is $f(x) = 1$, so the integrating factor is simply $I = e^x$. This gives us

   $$e^x y' + e^x y = \frac{d}{dx}(e^x y) = e^x e^{3x} = e^{4x},$$

   so $e^x y = \frac{1}{4}e^{4x} + C$, and thus $y = \frac{1}{4}e^{3x} + Ce^{-x}$. This solution is valid for all real numbers x.

(b) $(1+x^2)\,dy + (xy + x^3 + x)\,dx = 0$

We first rearrange the equation to put it into standard form. Dividing by $(1+x)^2\,dx$, we get

$$\frac{dy}{dx} + \left(\frac{x}{1+x^2}\right) y = -\frac{x^3+x}{x^2+1} = -x,$$

so the equation is linear, and the coefficient of y is $f(x) = \dfrac{x}{x^2+1}$. The integrating factor is therefore

$$I = \exp\left(\int \frac{x}{x^2+1}\,dx\right) = \exp\left(\frac{1}{2}\ln(x^2+1)\right) = \exp(\ln(x^2+1)^{1/2}) = \sqrt{1+x^2}.$$

Multiplying the equation by $\sqrt{1+x^2}$, we get

$$\frac{d}{dx}(\sqrt{1+x^2}y) = \sqrt{1+x^2}\frac{dy}{dx} + \frac{x}{\sqrt{1+x^2}}y = -x\sqrt{1+x^2}.$$

Integrating both sides gives

$$\sqrt{1+x^2}y = -\int x\sqrt{1+x^2}\,dx = -\frac{1}{3}(1+x^2)^{3/2} + C,$$

so $y = -\dfrac{1}{3}(1+x^2) + C(1+x^2)^{-1/2}$. This solution is valid for all real x, since $1+x^2 \neq 0$ for all x.

Note: It's a useful exercise for this (or any differential equation) to verify that this is indeed a solution. I'll leave it for you to check this.

(c) $(1-x^3)\dfrac{dy}{dx} = 3x^2y$

Note that this equation is also separable, so you should try solving it as a separable equation to verify that the result is the same. We'll treat it as a linear equation, however. Dividing by $1-x^3$, we have

$$\frac{dy}{dx} - \frac{3x^2}{1-x^3} = 0.$$

The coefficient of y is $f(x) = -\dfrac{3x^2}{1-x^3}$, and since this is undefined when $x=1$, our solution will need to be restricted to either the interval $(1,\infty)$ or $(-\infty,1)$. (The choice depends on the initial condition, which is not provided.)

Since $\int f(x)\,dx = \ln(x^3-1)$, our integrating factor is $I(x) = x^3-1$. Multiplying by $I(x)$ gives us

$$(x^3-1)\frac{dy}{dx} + 3x^2y = \frac{d}{dx}((x^3-1)y) = 0,$$

so $(x^3-1)y = C$, and thus $y = \dfrac{C}{x^3-1}$.

(d) $(x^2 + x)\,dy + (xy + x^3 + x)\,dx = 0$

Dividing by $(x^2 + x)\,dx$ and rearranging, we get the equation

$$\frac{dy}{dx} + \frac{1}{x+1}y = -x,$$

so the equation is linear, with $f(x) = \dfrac{1}{x+1}$ as the coefficient of y. The integrating factor is therefore $I(x) = \exp\left(\int \dfrac{dx}{x+1}\right) = \exp(\ln(x+1)) = x + 1$.
(Note: we divided by $x^2+x = x(x+1)$, which is undefined when $x = 0$ and $x = -1$, so our solution will have to be for one of the intervals $(-\infty, -1)$, $(-1, 0)$, or $(0, \infty)$. On the first interval, since $x + 1 < 0$, $\ln(x + 1)$ is not defined, and our integrating factor would have to be $-(x + 1)$ instead. However, this change amounts to multiplying the entire differential equation by -1, which doesn't make any difference.)
Multiplying the equation by $x + 1$, we have

$$(x+1)\frac{dy}{dx} + y = \frac{d}{dx}((x+1)y) = -x(x+1) = -x^2 - x,$$

so $(x+1)y = -\dfrac{1}{3}x^3 - \dfrac{1}{2}x^2 + C$, giving us $y = -\dfrac{2x^3 + 3x^2}{6x + 6} + \dfrac{C}{x+1}$.

(e) $\cos x\dfrac{dy}{dx} + y\sin x = 1$

Dividing by $\cos x$, we obtain $\dfrac{dy}{dx} + \tan xy = \sec x$, so the coefficient of y is $f(x) = \tan x$. The integrating factor is therefore

$$I(x) = \exp\left(\int \tan x\,dx\right) = \exp(\ln|\sec x|) = |\sec x|$$

Again, we note that the absolute value is unnecessary, and take $I(x) = \sec x$ as our integrating factor. We must restrict our solution to an interval where $\tan x$ (and $\sec x$) is defined; the interval $(-\pi/2, \pi/2)$ is the natural choice.
Multiplying the equation by $\sec x$, we get

$$\sec x\frac{dy}{dx} + (\sec x\tan x)y = \frac{d}{dx}(y\sec x) = \sec^2 x.$$

Thus, we find $y\sec x = \tan x + C$, and multiplying both sides by $\cos x$, we get $y = \sin x + C\cos x$.

3. Determine whether the sequence converges or diverges. If it converges, give the limit.

(a) $a_n = (-1)^n\dfrac{n}{n^2+1}$

We have $a_n = (-1)^n\dfrac{1/n}{1 + 1/n^2}$, and since $1/n^k \to 0$ as $n \to \infty$ for any $k > 0$ the sequence $\{a_n\}$ converges to zero.

(b) $a_n = (-1)^n \dfrac{2n+1}{3n+4}$

In this case $a_n = (-1)^n \dfrac{2+1/n}{3+4/n}$, so for n very large, $a_{2n} \approx \dfrac{2}{3}$, and $a_{2n+1} \approx -\dfrac{2}{3}$. Since limits of sequences have to be unique, and there is no way for this sequence to approach a single limiting value, the sequence diverges.

(c) $a_n = \dfrac{n-1}{n} - \dfrac{n}{n-1}$

If we get a common denominator and simplify, we have

$$a_n = \frac{1-2n}{n^2-n} = \frac{1/n^2 - 2/n}{1-1/n},$$

so the sequence converges, with $\lim\limits_{n\to\infty} a_n = \dfrac{0-0}{1-0} = 0.$

(d) $a_n = \dfrac{4n}{\sqrt{9n^2+4}}$

Since $a_n = \dfrac{4n}{n\sqrt{9+4/n^2}} = \dfrac{4}{\sqrt{9+4/n^2}}$ and $4/n^2 \to 0$ as $n \to \infty$, the sequence converges, with $\lim a_n = \dfrac{4}{3}$.

(e) $a_n = 1 - \dfrac{1}{n}$

Since $\dfrac{1}{n} \to 0$ as $n \to \infty$, the sequence converges, and $\lim a_n = 1$.

(f) The sequence $\{a_n\}$ defined by $a_1 = 1$ and $a_{n+1} = 3 - \dfrac{1}{a_n}$ for all $n \geq 1$. (You may assume that the sequence converges. If you want to actually *show* that it converges, feel free to ask me how.

The reason the sequence converges is that the sequence is both increasing and bounded above, so the *Monotone Convergence Theorem* applies. Showing that the sequence is increasing and bounded above requires the method of *Proof by Mathematical Induction*, which is usually covered in Math 2000. Since this course is not a prerequisite, we can't expect you to be able to rigorously establish these facts.

Assuming that the limit exists, let $a = \lim a_n$, and note that $\lim a_{n+1} = \lim a_n$. Thus, we have

$$a = \lim a_{n+1} = \lim \left(3 - \frac{1}{a_n}\right) = 3 - \frac{1}{\lim a_n} = 3 - \frac{1}{a},$$

using the limit laws for sequences. The above expression for a can be rearranged to give us $\dfrac{a^2-3a+1}{a} = 0$. We know that $a \neq 0$, since the sequence is increasing and $a_1 = 1$, so $a_n > 1$ for all n. Using the quadratic formula for the numerator gives us

$a = \dfrac{3 \pm \sqrt{5}}{2}$. We know that $2 < \sqrt{5} < 3$, so $0 < \dfrac{3 - \sqrt{5}}{2} < \dfrac{1}{2}$, and we again reject this possibility since we know $a > 1$. Thus, it must be the case that $a = \dfrac{3 + \sqrt{5}}{2}$.

(g) The sequence $\{a_n\}$ defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for all $n \geq 1$. (Again, you may assume that the sequence converges.)

As with the previous problem, we assume that $a = \lim a_n$ exists. Taking the limit of both sides of the recursion formula $a_{n+1} = \sqrt{2 + a_n}$, we have

$$a = \lim a_{n+1} = \lim \sqrt{2 + a_n} = \sqrt{2 + \lim a_n} = \sqrt{2 + a}.$$

Squaring both sides of $a = \sqrt{2 + a}$, we have $a^2 = 2 + a$, or $a^2 - a - 2 = 0$, giving us $a = 2$ or $a = -1$. Since $a_n > 0$ for all n (the sequence is defined using the *positive* square root), we must have $a = 2$.