Matrix multiplication and inverses

Math 1410 Linear Algebra

Row times column

1. $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Digression: summation notation

Last example: had product $AB = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.
Shorten the sum using summation notation:

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

In general, $\displaystyle\sum_{j=1}^{n} c_j = c_1 + c_2 + \cdots + c_n$. Examples:

1. $\displaystyle\sum_{i=1}^{6} i$
2. $\displaystyle\sum_{j=2}^{4} 2^j$

Matrix times column

2. $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Row times matrix

$$3. A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{bmatrix}$$

Definition of AB

Generalize from previous cases: to form the product AB, multiply the rows of A by the columns of B.

- Size matters: if A is size $m \times n$, B must be size $n \times p$.
- The product AB will be size $m \times p$.
- The (i, j)-entry of AB is given by multiplying the i^{th} row of A by the j^{th} column of B

Definition

The product AB of the $m \times n$ matrix A and the $n \times p$ matrix B is the matrix $AB = [c_{ij}]_{m\times p}$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}.$$

Basic examples

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -4 \\ 1 & 0 \\ -2 & 6 \end{bmatrix}$.

Which products are defined?

$$AB \quad BA \quad AC \quad CA \quad BC \quad CB$$

What size are they? What are their entries?

Example

Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix}$. Suppose we are given a matrix X such that $AX = B$.

(a) What size is X?

(b) What are the entries of X?

(c) Can you answer the problem if you're told instead that $XA = B$? What if $BX = A$?

Properties of matrix multiplication

- Multiplication is not commutative: $AB \neq BA$ in general.
- Given $A_{m\times n}$, $B_{n\times p}$, $C_{n\times p}$,
$$A(B+C) = AB + AC.$$
- Given $A_{m\times n}$, $B_{m\times n}$, $C_{n\times p}$,
$$(A+B)C = AC + BC.$$
- Given $A_{m\times n}$, $B_{n\times p}$ and $c \in \mathbb{R}$,
$$A(cB) = (cA)B = c(AB).$$
- Given $A_{m\times n}$, $B_{n\times p}$ and $C_{p\times q}$,
$$A(BC) = (AB)C.$$
- Given $A_{m\times n}$ and $B_{n\times p}$,
$$(AB)^T = B^T A^T.$$

Special matrices

- Zero matrix:

- Identity matrix:

Mutliplicative inverses of real numbers

Recall two basic facts about real numbers:

1. For any real number $a \in \mathbb{R}$, $1 \cdot a = a \cdot 1 = a$.
2. For any non-zero real number $a \in \mathbb{R}$, $a \neq 0$, the reciprocal $a^{-1} = \frac{1}{a}$ satisfies

$$\frac{1}{a} \cdot a = a \cdot \frac{1}{a} = 1.$$

Matrix inverses

Definition

Let A be an $n \times n$ square matrix. We say that A is invertible if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

We call B the inverse of A and write $B = A^{-1}$.

Example: the inverse of $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

Not all matrices are invertible

If $A = 0$, then $AB = 0$ for any matrix B, so A is not invertible.

What if $A \neq 0$? It's still not guaranteed. Consider $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$.

Cancellation

For numbers, if $ab = ac$ and $a \neq 0$, we have $b = c$. For matrices, this may not be the case.

Example

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}$, compute AB and AC.

However, if A is invertible and $AB = AC$, then we have $B = C$.

Properties of the inverse

- Inverses are unique: if A is invertible and there exist matrices B and C such that $AB = BA = I_n$ and $AC = CA = I_n$, then $B = C$.

- If A and B are invertible $n \times n$ matrices, then so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

More properties

If A is an invertible $n \times n$ matrix, then,

- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$.

Computing the inverse

Given an $n \times n$ matrix A, how do we
(a) Determine if A is invertible?
(b) Find the inverse of A?

To answer both, need to solve (if possible) $AB = I_n$.
Example: Given $A = \begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}$, let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Algorithm for finding A^{-1}

Given an $n \times n$ matrix A, we can find A^{-1} (if it exists) as follows:

1. Form the $n \times 2n$ augmented matrix $(A|I_n)$.
2. Use elementary row operations (as usual) to reduce $(A|I_n)$ to reduced row-echelon form.
3. If a row $\begin{bmatrix} 0 & 0 & 0 & | & a & b & c \end{bmatrix}$ appears, stop: A is not invertible.
4. If not, the RREF will be of the form $(I_n|A^{-1})$.

Consequence: An $n \times n$ matrix A is invertible if (and only if) $\operatorname{rank} A = n$.

Examples

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -4 \\ -1 & -6 & 17 \end{bmatrix}$$

Systems of equations

Can use inverses to solve system $AX = B$ of n equations in n variables. (So A is $n \times n$.)
If A is invertible, then $X = A^{-1}(AX) = A^{-1}B$.

Warning: This method is not useful in practice. The process of (a) finding A^{-1} and (b) computing $A^{-1}B$ takes roughly twice as much work as the Gaussian algorithm.