Math 3410 Assignment #2
University of Lethbridge, Spring 2015

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Due date: Wednesday, February 11, by 5 pm.

Please provide solutions to the problems below, using the same guidelines as for Assignment #1:

1. Let U be a subspace of a vector space V, and let $S : U \to W$ be a linear transformation. Prove that the function $T : V \to W$ given by

$$Tv = \begin{cases} Sv, & \text{if } v \in U \\ 0, & \text{if } v \notin U \end{cases}$$

is **not** a linear transformation.

Hint: if $u \in U$ and $v \notin U$, can $u + v$ be an element of U? What about $-v$?

Solution: Let $u \in U$ be an element of U such that $Su \neq 0$, and choose some $v \in V$ with $v \notin U$. It follows that $u + v \notin U$, since otherwise we'd have that

$$v = 0 + v = (-u + u) + v = -u + (u + v) \in V,$$

since U is closed under addition. (And if $u \in U$, then we must have $-u \in U$ as well.) Since $u + v \notin U$, on the one hand we have

$$T(u + v) = 0,$$

by definition of T. On the other hand, since $u \in U$ and $v \notin U$, we have

$$T(u) + T(v) = S(u) + 0 = S(u) \neq 0.$$

Thus, T cannot be linear, since $T(u + v) \neq Tu + Tv$.

2. Suppose V is a finite-dimensional vector space, and let $U \subseteq V$ be a subspace. Prove that any linear transformation $S : U \to W$ can be extended to a linear transformation $T : V \to W$.

   Hint: any basis of U can be extended to a basis for V.

   Solution: Let $U \subseteq V$ be a subspace of V, and let $B_U = \{u_1, \ldots, u_k\}$ be a basis for U. We know that B_U can be extended to a basis

$$B_V = \{u_1, \ldots, u_k, v_1, \ldots, v_m\}$$

   of V. We know from class (or the textbook) that we can uniquely define a linear transformation by specifying its value on each basis vector. Thus, for example, we can define $T : V \to W$ by setting $Tu_i = Su_i$ for $i = 1, 2, \ldots, k$, and $Tv_i = 0$ for $i = 1, 2, \ldots, m$.

   Note: We don't have to define T to be zero on all the v_i, but it is a convenient choice. It also provides an opportunity to point out that this is the *correct* way to "extend by zero" from a subspace. Make sure you understand why the method in this question works, but the method in the first problem does not. Basically, we know that the extension by the vectors $v_1, \ldots, v_k$ defines a complementary subspace W such that $V = U \oplus W$. We can define T by setting $Tu = Su$ for all $u \in U$, and $Tw = 0$ for all $w \in W$. But this is **not** the same as defining $Tv = 0$ for all $v \notin U$, since if we take nonzero vectors $u \in U$ and $w \in W$, then $v = u + w$ belongs to *neither* U *nor* W.

3. Suppose that V is a finite-dimensional vector space, and $T : V \to W$ is a linear transformation. Prove that there exists a subspace $U \subseteq V$ such that:

   (a) $U \cap \operatorname{null} T = \{0\}$, and

   (b) $\operatorname{range} T = \{Tu : u \in U\}$.

   Solution: We know that $\operatorname{null} T$ is a subspace of V, and since V is finite dimensional, so is $\operatorname{null} T$. Thus, we can choose a basis $B_0 = \{x_1, \ldots, x_k\}$ of $\operatorname{null} T$. (Note that $Tx_i = 0$ for $i = 1, 2, \ldots, k$.)

   Now, since B_0 is a basis for a subspace of V, we can extend it to a basis of V. Let's say that $B_V = \{x_1, \ldots, x_k, y_1, \ldots, y_n\}$ is the extension. Since B_V is a basis for V, we know that any $v \in V$ can be written in the form

$$v = (a_1x_1 + \cdots + a_kx_k) + (b_1y_1 + \cdots + b_ny_n).$$

   Thus, if we define $U = \operatorname{span}\{y_1, \ldots, y_n\}$, it follows that

$$V = U \oplus \operatorname{null} T,$$

and thus $U \cap \operatorname{null} T = \{0\}$, and for any $v \in V$, we can write $v = x + y$, where $x \in \operatorname{null} T$ and $y \in U$. It follows that

$$Tv = T(x + y) = Tx + Ty = Ty,$$

and since v was arbitrary, any element of $\operatorname{range} T$ is of the form Ty, with $y \in U$.

4. Suppose V and W are finite-dimensional vector spaces.

   (a) Prove that there exists an injective (one-to-one) linear transformation $T : V \to W$ if and only if $\dim V \leq \dim W$.

   The Fundamental Theorem of Linear Maps tells us that

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

   If T is one-to-one, then $\dim \operatorname{null} T = 0$, so $\dim V = \dim \operatorname{range} T \leq \dim W$, since $\operatorname{range} T$ is a subspace of W. Conversely, suppose $\dim V \leq \dim W$, and choose bases $\{v_1, \ldots, v_k\}$ of V and $\{w_1, \ldots, w_n\}$ of W, with $k \leq n$. Let us define $T : V \to W$ by setting

$$Tv_1 = w_1,\ Tv_2 = w_2, \ldots, Tv_k = w_k,$$

   and extending by linearity. (As noted above, T is completely determined by its values on a basis of V.) Then T is an injection, as follows: suppose $v \in \operatorname{null} T$, and write $v = c_1 v_1 + \cdots + c_k v_k$ in terms of the given basis. Then

$$0 = Tv = T(c_1 v_1 + \cdots + c_k v_k) = c_1 T v_1 + \cdots + c_k T v_k = c_1 w_1 + \cdots + c_k w_k,$$

   and the vectors $w_1, \ldots, w_k$ are independent, since they're part of a basis. It follows that $c_1 = 0, \ldots, c_k = 0$, and thus $v = 0$, which implies that $\operatorname{null} T = \{0\}$ and that T is an injection.

   (b) Prove that there exists a surjective (onto) linear transformation $T : V \to W$ if and only if $\dim V \geq \dim W$.

   If T is onto, then $\operatorname{range} T = W$, so

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim \operatorname{null} T + \dim W \geq \dim W.$$

   Conversely, suppose $\dim V \geq \dim W$, and choose bases $\{v_1, \ldots, v_k\}$ and $\{2_1, \ldots, w_n\}$ of V and W respectively, with $k \geq n$. As above, we can define $T : V \to W$ in terms of the basis for V. We set

$$Tv_1 = w_1,\ Tv_2 = w_2, \ldots, Tv_n = w_n, \text{ and } Tv_j = 0 \text{ for } n + 1 \leq j \leq k.$$

   It follows that T is a surjection, since given any

$$w = c_1 w_1 + \cdots + c_n w_n \in W,$$

   we can let $v = c_1 v_1 + \cdots + c_n w_n$, and then $Tv = w$, as required.