University of Lethbridge

Department of Mathematics and Computer Science

20^{th} March, 2015, 3:00 - 3:50 pm

MATH 3410 - Test #2

Last Name: **Solutions**

First Name: **The**

Student Number: ______________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

You must solve all problems on pages 2, 3, and 4, but you only need to do either page 5 or page 6. **Do not complete both page 5 and page 6.**

For grader's use only:

Page	Grade
2	/8
3	/8
4	/12
5/6	/12
Total	/40

1. Provide definitions for the following terms:

[2] (a) What it means for a linear map $T : V \to W$ to be **invertible**.

An operator $T : V \to W$ is **invertible** if there exists a linear map $T^{-1} : W \to V$ such that $T^{-1}T = I_V$ and $TT^{-1} = I_W$, where I_V and I_W denote the identity operators on V and W, respectively.

[2] (b) An **invariant subspace** for an operator $T : V \to V$.

A subspace $U \subseteq V$ is an **invariant subspace** for an operator $T \in \mathcal{L}(V)$ if for all $u \in U$ we have $Tu \in U$.

[2] (c) What it means for a linear operator $T : V \to V$ to be **diagonalizable**.

An operator $T \in \mathcal{L}(V)$ is **diagonalizable** if there exists a basis B of V with respect to which the matrix of T is diagonal.

[2] (d) The **eigenspace** $E(\lambda, T)$ of an operator $T : V \to V$ and scalar λ.

For any $T \in \mathcal{L}(V)$ and any $\lambda \in \mathbb{F}$, we define the **eigenspace** $E(\lambda T)$ by $E(\lambda, T) = \text{null}(T - \lambda I)$, where I is the identity operator on V.

2. Short answer: provide a brief answer to the questions below. You do not have to explain your answers.

[1] (a) If V and W are finite-dimensional vector spaces, what is $\dim \mathcal{L}(V,W)$?

$$\dim \mathcal{L}(V,W) = \dim V \cdot \dim W.$$

[3] (b) What is the matrix (with respect to the standard bases) of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x,y,z) = (2x-3y+z, -x+2y+4z)?$$

Since $T(1,0,0) = (2,-1)$, $T(0,1,0) = (-3,2)$, and $T(0,0,1) = (1,4)$, we have

$$\mathcal{M}(T) = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 2 & 4 \end{bmatrix}.$$

[4] (c) If T is the operator on $\mathbb{R}^{2,1}$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -3 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and $p(x) = 2x^2 - 3x + 5$, determine the operator $p(T)$.

Let $A = \begin{bmatrix} -3 & -2 \\ 2 & 5 \end{bmatrix}$, so that $T(X) = AX$, and thus $p(T)X = p(A)X$. Since

$$A^2 = \begin{bmatrix} -3 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 4 & 21 \end{bmatrix},$$

we have

$$p(A) = 2A^2 - 3A + 5I_2 = 2\begin{bmatrix} 5 & -4 \\ 4 & 21 \end{bmatrix} - 3\begin{bmatrix} -3 & -2 \\ 2 & 5 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 24 & -2 \\ 2 & 32 \end{bmatrix},$$

and thus $p(T)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24 & -2 \\ 2 & 32 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Please solve **both** problems on this page.

3. Let $S, T \in \mathcal{L}(V)$, where V is finite-dimensional. Prove that the operator ST is invertible
[6] if and only if S and T are invertible.

If S and T are invertible, then ST is invertible, with inverse $T^{-1}S^{-1}$, since

$$ST(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = S(I_V S^{-1}) = SS^{-1} = I_V,$$

and similarly $(T^{-1}S^{-1})(ST) = I_V$.

Conversely, suppose that ST is invertible. Then ST is a bijection, and therefore S must be surjective and T must be injective. (As proved in Math 2000 and discussed in class and mentioned on the review sheet.) Since V is finite-dimensional, either injectivity or surjectivity implies bijectivity, and thus S and T are both bijections, and therefore invertible.

[6] 4. Suppose that $S, T \in \mathcal{L}(V)$ satisfy $ST = TS$. Prove that $\operatorname{null} S$ is invariant under T.

Suppose that $ST = TS$, and let $v \in \operatorname{null} S$. We need to show that $Tv \in \operatorname{null} S$; that is, we need to show that $S(Tv) = 0$. But we're assuming that $ST = TV$, so

$$S(Tv) = T(Sv) = T(0) = 0,$$

where we've used the fact that $Sv = 0$, since $v \in \operatorname{null} S$. Thus, $Tv \in \operatorname{null} S$, and since v was arbitrary, the result follows.

You may either solve both problems on this page, or leave it blank, and move on to the next page.

[6] 5. Let V be finite-dimensional, and let $P \in \mathcal{L}(V)$. Prove that if $P^2 = P$, then $V = \operatorname{null} P \oplus \operatorname{range} P$.

Hint: $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$, so it suffices to show that $\operatorname{null} P \cap \operatorname{range} P = \{0\}$.

Suppose that $P^2 = P$. By the hint, we need to show that $\operatorname{null} P \cap \operatorname{range} P = \{0\}$. Thus, suppose that $w \in \operatorname{null} P \cap \operatorname{range} P$. Then we have that $Pw = 0$, and $w = Pv$ for some $v \in V$. But we're assuming that $P^2 = P$, and thus

$$0 = Pw = P^2 v = Pv = w,$$

so we must have $w = 0$, and this completes the proof.

[6] 6. Suppose that $\dim V = n$, $T \in \mathcal{L}(V)$ has n distinct eigenvalues, and $S \in \mathcal{L}(V)$ has the same eigenvectors as T (but not necessarily the same eigenvalues). Prove that $ST = TS$.

Suppose that $T \in \mathcal{L}(V)$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, where $n = \dim V$. Since eigenvectors corresponding to distinct eigenvalues are linearly independent, it follows that we can choose a basis $B = \{v_1, \ldots, v_n\}$ of eigenvectors of T, where $Tv_i = \lambda_i v_i$ for $i = i, \ldots, n$.

Now suppose that the vectors in B are also eigenvectors for S. Thus, $Sv_i = \mu_i v_i$, for some scalars μ_i, for $i = 1, \ldots, n$. We wish to show that $ST = TS$; that is, that $S(Tv) = T(Sv)$ for all $v \in V$. Thus, let us choose an arbitrary element $v \in V$.

Since B is a basis, there exist unique scalars $c_1, \ldots, c_n$ such that $v = c_1 v_1 + \cdots + c_n v_n$, and thus

$$\begin{aligned}
S(Tv) &= S(T(c_1 v_1 + \cdots + c_n v_n)) \\
&= S(c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n) \\
&= c_1 \lambda_1 \mu_1 v_1 + \cdots + c_n \lambda_n \mu_n v_n \\
&= c_1 \mu_1 \lambda_1 v_1 + \cdots + c_n \mu_n \lambda_n v_n \\
&= T(c_1 \mu_1 v_1 + \cdots + c_n \mu_n v_n) \\
&= T(S(c_1 v_1 + \cdots + c_n v_n)) \\
&= T(Sv).
\end{aligned}$$

If you solved the two problems on the previous page, then leave this page blank. If you skipped the last page, then please solve the following:

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the operator $T(x, y) = (5x - 2y, 7x - 4y)$.

[2] (a) Compute the matrix $\mathcal{M}(T)$ of T with respect to the standard basis of $\mathbb{R}^2$.

$T(1, 0) = (5, 7)$ and $T(0, 1) = (-2, -4)$, so $\mathcal{M}(T) = \begin{bmatrix} 5 & -2 \\ 7 & -4 \end{bmatrix}$.

[4] (b) Find the eigenvalues of T.

Since λ is an eigenvalue of T if and only if $\mathcal{M}(T) - \lambda I_2$ is not invertible, we must have

$$0 = \begin{vmatrix} 5 - \lambda & -2 \\ 7 & -4\lambda \end{vmatrix} = (5 - \lambda)(-4 - \lambda) + 14 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

The eigenvalues of T are therefore $\lambda = 3$ and $\lambda = -2$.

[4] (c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of $\mathbb{R}^2$.

For $\lambda = 3$ we have $\mathcal{M}(T) - 3I_2 = \begin{bmatrix} 2 & -2 \\ 7 & -7 \end{bmatrix}$, and since $\begin{bmatrix} 2 & -2 \\ 7 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we have that $v_1 = (1, 1) \in \operatorname{null}(T - 3I)$.
For $\lambda = -2$ we have $\mathcal{M}(T) + 2I_2 = \begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix}$, and since $\begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, it follows that $v_2 = (2, 7) \in \operatorname{null}(T + 2I)$.
Since v_1 and v_2 are not parallel, they are linearly independent, and therefore form a basis of $\mathbb{R}^2$.

[2] (d) Is the operator T diagonalizable? Why or why not? If it is, give a matrix P such that $P^{-1}\mathcal{M}(T)P$ is diagonal. (You don't have to verify it's diagonal.)

Since we found a basis of eigenvectors of T in part (c), we know that T is diagonalizable. Since the columns of P are given by the column vectors for v_1 and v_2, we have

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 7 \end{bmatrix}$$

Although it was not required, you may wish to verify that

$$P^{-1}AP = \begin{bmatrix} 7/5 & -2/5 \\ -1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$

 Total Points: 40