MATH 2565 - Tutorial #6 Solutions

Assigned problems:

1. Find the area of the surface obtained by revolving $y = x^2$, for $x \in [0,1]$, about the y-axis.

   We're revolving a function of x about the y-axis, so we use the formula $S = 2\pi \int_a^b x\sqrt{1+f'(x)^2}\,dx$. With $f(x) = x^2$ we have $f'(x) = 2x$, so

$$S = 2\pi \int_0^1 x\sqrt{1+4x^2}\,dx = \frac{\pi}{4}\int_1^5 \sqrt{u}\,du = \frac{\pi}{6}(5^{3/2}-1),$$

   using the substitution $u = 1+4x^2$, so $du = 8x\,dx$, and when $x = 0$, $u = 1$, and when $x = 4$, $u = 5$.

   Alternatively, one could write $x = \sqrt{y}$, so $\dfrac{dx}{dy} = \frac{1}{2\sqrt{y}}$, and

$$S = 2\pi \int_0^1 \sqrt{y}\sqrt{\frac{1}{4y}+1}\,dy = 2\pi \int_0^1 \sqrt{y+\frac{1}{4}}\,dy = \frac{4\pi}{3}\left(\left(\frac{5}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right).$$

   With a little bit of work, you can confirm that this result is equal to the one above.

2. Find the area of the surface obtained by revolving $x = 1+2y^2$, $1 \le y \le 2$, about the x-axis.

   Since we have x given as a function of y and we're revolving about the x-axis, we use the formula $S = \int_c^d y\sqrt{1+g'(y)^2}\,dy$. Here, $g(y) = 1+2y^2$, so $g'(y) = 4y$. Thus,

$$S = 2\pi \int_1^2 y\sqrt{1+16y^2}\,dy = \frac{\pi}{16}\int_{17}^{65} \sqrt{u}\,dy = \frac{\pi}{24}(65^{3/2} - 17^{3/2}).$$

   If for some reason you'd rather integrate with respect to x, we have $y = \sqrt{\dfrac{x-1}{2}}$ and $\dfrac{dy}{dx} = \dfrac{1}{2}\sqrt{\dfrac{2}{x-1}}$, so

$$S = 2\pi \int_3^9 \sqrt{\frac{x-1}{2}}\sqrt{1+\frac{1}{2(x-1)}}\,dx = \int_3^9 \frac{1}{\sqrt{2}}\sqrt{x-\frac{1}{4}}\,dx,$$

   and presumably working this out gives the same answer as above.

Additional practice (don't include your solutions here):

1. Find the area of the surface obtained by revolving $y = \sqrt{x}$, for $x \in [0,1]$, about the x-axis.

   Since we're revolving about the x-axis and y is given as a function of x, we use the formula $S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$. With $f(x) = \sqrt{x}$, we have

   $$1 + f'(x)^2 = 1 + \left(\frac{1}{2\sqrt{x}}\right)^2 = \frac{4x+1}{4x}.$$

   The surface area is thus

   $$S = 2\pi \int_0^1 \sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx = \pi \int_0^1 \sqrt{4x+1}\,dx = \frac{\pi}{4}\cdot\frac{2}{3}(4x+1)^{3/2}\bigg|_0^1 = \frac{\pi}{6}(5^{3/2}-1).$$

2. Verify that $x = Ce^{-t} + De^{2t}$ is a solution to $x'' - x' - 2x = 0$.

   We have

   $$\begin{aligned} x(t) &= Ce^{-t} + De^{2t} \\ x'(t) &= -Ce^{-t} + 2De^{2t} \\ x''(t) &= Ce^{-t} + 4De^{2t}, \end{aligned}$$

   so $x'' - x' - 2x = e^{-t}(C - (-C) - 2C) + e^{2t}(4D - 2D - 2D) = 0$, as required.

3. Find the solution from Problem 2 that satisfies $x(0) = 3$ and $x'(0) = -2$.

   Setting $x(0) = 3$ gives us $C + D = 3$. Setting $x'(0) = -2$ gives us $-C + 2D = -2$. We have two equations in the unknowns C and D, which can easily be solved to give us $C = \dfrac{8}{3}$ and $D = \dfrac{1}{3}$.

4. Solve $y' = y^3$ when $y(0) = 1$. (Hint: $\dfrac{1}{y'} = \dfrac{dx}{dy}$.)

   There are two ways to solve this differential equation. The first follows the hint:

   We first note that $y(x) = 0$ is a solution. If we assume that $y \neq 0$, we can write

   $$\frac{1}{y'} = \frac{dx}{dy} = \frac{1}{y^3} = y^{-3}.$$

   Here we're assuming that $y = f(x)$, where f has an inverse, so we can write $x = f^{-1}(y)$. (This may not be globally true, but it is true on any open interval that does not contain a critical point of f.)

If $\frac{dx}{dy} = y^{-3}$, then taking the antiderivative gives us $x = -\frac{1}{2y^2} + C$, so $y^2 = \frac{1}{2C - 2x}$. This leaves us with the problem of whether to take the positive or negative square root to solve for y, but the initial condition $y(0) = 1 > 0$ tells us that we must take the postitive square root. Applying the initial condition gives us

$$1^1 = 1 = \frac{1}{2C},$$

so $C = \frac{1}{2}$, and thus $y = \frac{1}{\sqrt{1-2x}}$.

The other approach is to treat the equation as a separable equation. From $\frac{dy}{dx} = y^3$ we have $\frac{dy}{y^3} = dx$, and integrating both sides gives us $-\frac{1}{y^2} = x + C$. The remainder of the solution is as above.

5. Solve $\frac{dx}{dt} = x \sin(t)$ for $x(0) = 1$.

We have a separable differential equation, which can be written as $\frac{dx}{x} = \sin t \, dt$. Integrating both sides gives us $\ln x = -\cos t + C$. We can solve now for x as a function of t but it's convenient to first apply the initial condition: when $t = 0$ we have $x = 1$, so

$$\ln(1) = 0 = -\cos(0) + C,$$

which gives us $C = 1$. Thus $\ln x = 1 - \cos t$, so $x = e^{1-\cos t}$.