MATH 1560 - Tutorial #5 Solutions

Additional practice problems:

1. Evaluate the limits:

(a) $\lim_{x\to 0}\sqrt{4x^2-x+9}$

$=\sqrt{9}=3$

(b) $\lim_{x\to\infty}\frac{8x^5+3x+5}{4+5x^3+2x^5}$

$=\frac{8}{2}=4$

(c) $\lim_{x\to 3}\frac{x^2-9}{x-3}$

$=\lim_{x\to 3}(x+3)=6$

2. Compute the derivative:

(a) $\frac{d}{dx}(x^4+4x+9)=4x^3+4$

(b) $\frac{d}{dx}\sqrt{x^4+4}=\frac{2x^3}{\sqrt{x^4+4}}$

(c) $\frac{d}{dx}(5x^3e^x)$

$=5e^x(3x^2+x^3)$

3. Evaluate the immediate integral:

(a) $\int(4x^3+2x)\,dx=x^4+x^2+C$

(b) $\int\frac{1}{x}\,dx=\ln|x|+C$

(c) $\int 7e^{7x}\sin(e^{7x})\,dx=-\cos(e^{7x})+C$

(d) $\int\frac{\ln(x)}{x}\,dx=\frac{1}{2}(\ln(x))^2+C$

Assigned problems:
Test problems:

1. Compute the limit:

(a) $\lim_{x\to 2^+}\frac{(x^2-4)^2}{x-2}$

$=\lim_{x\to 2^+}(x-2)(x+2)^2$

$=0$

(b) $\lim_{n\to\infty}\left(\frac{1^3}{n^4}+\frac{2^3}{n^4}+\cdots+\frac{n^3}{n^4}\right)$

$=\lim_{n\to\infty}\frac{1}{n^4}(1^3+2^3+\cdots+n^3)$

$=\lim_{n\to\infty}\frac{1}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)$

$=\frac{1}{4}.$

2. Compute the derivative:

(a) $f(x)=\frac{\sin(x)}{e^x}$

$f'(x)=\frac{\cos(x)e^x-\sin(x)e^x}{e^{2x}}$

(b) $g(x)=\tan(5x^2)$

$g'(x)=10x\sec^2(5x^2)$

3. Compute $\dfrac{d}{dx}(x^x)$

   Two options: (A) write $y = x^x$, so $\ln(y) = \ln(x^x) = x\ln(x)$, and take the derivative of both sides with respect to x:

   $$\frac{1}{y}y' = \ln(x) + 1, \text{ so } y' = x^x(\ln(x) + 1).$$

   (B) Since $e^{\ln(u)} = u$ for any $u > 0$, $x^x = e^{\ln(x^x)} = e^{x\ln(x)}$. Thus,

   $$\frac{d}{dx}(x^x) = \frac{d}{dx}e^{x\ln(x)} = e^{x\ln(x)}(\ln(x) + 1) = x^x(\ln(x) + 1).$$

4. Evaluate the integral:

   (a) $\displaystyle\int 2x(x^2 + 4)^4\,dx = \frac{1}{5}(x^2 + 4)^5 + C$

   (b) $\displaystyle\int (\cos(2x) - sec^2(x))\,dx$
   $= \frac{1}{2}\sin(2x) - \tan(x) + C$

5. Compute $y' = \frac{dy}{dx}$, given:

   (a) $x^2 + y^2 = 25$, at the point $(-3, -4)$.

   We have $2x + 2y \cdot y' = 0$, so

   $$y' = -\frac{x}{y} = -\frac{3}{4}.$$

   (b) $y^3 + x^5 y = 2\ln(y) + \dfrac{x}{y}$ (don't solve for y')

   Taking the derivative of both sides with respect to x,

   $$3y^2 \cdot y' + 5x^4 y + x^5 y' = \frac{2}{y} \cdot y' + \frac{1}{y} - \frac{x}{y^2} \cdot y'.$$

6. Given $f(x) = x^2 e^x$, solve the equation $f'(x) = 0$.

   We have

   $$f'(x) = 2xe^x + x^2 e^x = x(2 + x)e^x.$$

   Since $e^x \neq 0$ for all x, we see that $f'(x) = 0$ if $x = 0$ or $x = -2$.

7. Given $f(x) = e^{-2x^2}$, solve the equation $f''(x) = 0$.

   We have $f'(x) = -4xe^{-2x^2}$, so

   $$f''(x) = -4e^{-2x^2} + 16x^2 e^{-2x^2} = -4(4x^2 - 1)e^{-2x^2} = -4(2x - 1)(2x + 1)e^{-2x^2}.$$

   Thus, $f''(x) = 0$ if $x = \pm\dfrac{1}{2}$.

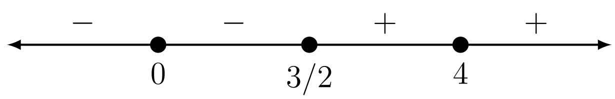
Additional problem:

8. Let $f(x) = x^3(x-4)^5$. Determine:

   (a) All values of x such that $f'(x) = 0$
   We find

   $$f'(x) = 3x^2(x-4)^5 + 5x^3(x-4)^4 = x^2(x-4)^4(3(x-4)+5x) = 4x^2(x-4)^2(2x-3).$$

   Thus, $f'(x) = 0$ for $x = 0$, $x = 4$, and $x = 3/2$. Marking these points on a sign diagram, we have

   (b) The intervals on which f is increasing or decreasing.

   From the sign diagram above, we see that f is increasing on $(3/2, \infty)$ and decreasing on $(-\infty, 3/2)$.

   (c) The coordinates of any local maxima or minima.

   On our sign diagram, the only place where f' changes sign is when $x = 3/2$, where it changes from negative to positive, so we have a local minimum at this point. We find that $f(3/2) = (\frac{3}{2})^3(-\frac{5}{2})^5 = -\frac{3^3 \cdot 5^5}{2^8}$, so our local minimum is at $(3/2, -3^3 \cdot 5^5/2^8)$.