[4] 1. Find the absolute maximum and minimum of $f(x) = x + \dfrac{4}{x}$ on $[1, 4]$.

We first check that the end point values are given by $f(1) = 1 + 4/1 = 5$ and $f(4) = 4 + 4/4 = 5$.

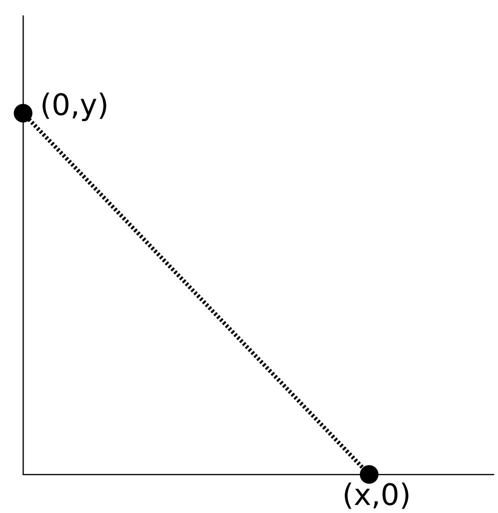
The derivative of f is given by

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = \frac{(x-2)(x+2)}{x^2},$$

so $f'(x) = 0$ for $x = 2$ and $x = -2$. Of these values, only 2 is in the interval $[1, 4]$, so we check the critical value $f(2) = 2 + 4/2 = 4$.

Comparing values, we see that the absolute maximum value is given by $5 = f(1) = f(4)$, and the absolute minimum value is $4 = f(2)$.

2. A 5 metre long ladder is leaning against a vertical wall. If the base of the ladder is being pulled away from the wall at a rate of 1/3 m/s, how fast is the top of the ladder sliding
[4] down the wall when it is 3 m from the ground?



Referring to the diagram on the left, since our ladder has a length of 5, we must have

$$x^2 + y^2 = 5^2.$$

Since the ladder is being pulled away from the wall, x is increasing, so $\frac{dx}{dt} = \frac{1}{3}$. Differentiating both sides of the above equation with respect to t, we find that

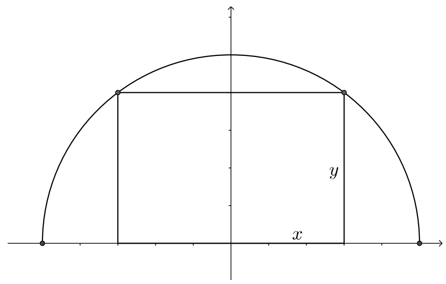
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

When the top of the ladder is 3 m from the ground, we have $y = 3$, and thus $x^2 + 3^2 = 5^2$, giving us $x = 4$. Solving for $\dfrac{dy}{dt}$, we find

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{4}{3}\left(\frac{1}{3}\right) = -\frac{4}{9},$$

so the ladder is sliding down the wall at a rate of 4/9 m/s.

[4] 3. Find the area of the largest rectangle that can be inscribed in a *semicircle* of radius R, if one side of the rectangle must lie along the diameter of the semicircle.



We draw our rectangle as shown on the left. Note that if the rectangle is not symmetric about the y-axis, then we can increase the area by lengthening one side of the base, so our rectangle is as shown, with height y and base length $2x$. Since (x, y) is a point on the circle, we have $x^2 + y^2 = R^2$. Solving for y gives us $y = \sqrt{R^2 - x^2}$.

The area of our rectangle is thus given by $A(x) = 2x\sqrt{R^2 - x^2}$, with $0 \leq x \leq R$. We note that $A(0) = A(R) = 0$, so the maximum must occur at a critical point. We have

$$\begin{aligned} A'(x) &= 2\sqrt{R^2 - x^2} + 2x\left(\frac{-x}{\sqrt{R^2 - x^2}}\right) \\ &= \frac{2(R^2 - x^2) - 2x^2}{\sqrt{R^2 - x^2}} = \frac{2R^2 - 4x^2}{\sqrt{R^2 - x^2}}. \end{aligned}$$

It follows that $A'(x) = 0$ for $2R^2 - 4x^2 = 0$, giving us $x^2 = \dfrac{R^2}{2}$, so $x = \dfrac{R}{\sqrt{2}}$ (positive root since $x > 0$). For this value of x, we have

$$y = \sqrt{R^2 - R^2/2} = \sqrt{R^2/2} = \frac{R}{\sqrt{2}}.$$

The area of our rectangle is therefore

$$A = 2\left(\frac{R}{\sqrt{2}}\right)^2 = R^2.$$

[4] 4. Use a linear approximation to estimate the value of $\sqrt{9.2}$.

We use $f(x) = \sqrt{x}$ as our function, and approximate near $a = 9$. We have

$$l(x) = f(9) + f'(9)(x - 9) = \sqrt{9} + \frac{1}{2\sqrt{9}}(x - 9) = 3 + \frac{1}{6}(x - 9).$$

Thus, our approximation is

$$\sqrt{9.2} \approx l(9.2) = 3 + \frac{1}{6}(9.2 - 9) = 3 + \frac{1}{6}(0.2) = 3.0333.$$

(This is a pretty good estimate compared to the calculator value of 3.03315.