Solutions to Quiz 4 Practice Problems
Math 2580
Spring 2016

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1. Find the equation of the tangent plane to the graph of f at the point $(2, -1, f(2, -1))$, if $f(x, y) = x^2 + 4y^2$.

   We have $f_x(x, y) = 2x$, so $f_x(2, -1) = 4$, and $f_y(x, y) = 8y$, so $f_y(2, -1) = -8$. We note that $f(2, -1) = 4 + 4 = 8$, so the equation of the tangent plane is

   $$\begin{aligned} z &= f(2, -1) + f_x(2, -1)(x - 2) + f_y(2, -1)(y + 1) \\ &= 8 + 4(x - 2) - 8(y + 1) = 4x - 8y - 8. \end{aligned}$$

2. Let $f(x, y) = x^2y + xy^3$. Find a normal vector to the graph $z = f(x, y)$ at the point $(1, 1, 2)$.

   A normal vector to $z = f(x, y)$ at (a, b) is given by $\mathbf{n} = \begin{bmatrix} f_x(a, b) & f_y(a, b) & -1 \end{bmatrix}^T$. We have $f_x(x, y) = 2xy + y^3$ and $f_y(x, y) = x^2 + 3xy^2$, so $f_x(1, 1) = 3$ and $f_y(1, 1) = 4$. One possible normal vector is therefore $\mathbf{n} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$. (Any nonzero scalar multiple of this vector would also do.)

3. Use a linear approximation to the function $f(x, y) = x^3 + y^3 - 6xy$ to give an approximate value for
   $$(0.99)^3 + (2.01)^3 - 6(0.99)(2.01).$$

   The point $(0.99, 2.01)$ is close to the point $(1, 2)$, and we know that $f(x, y) \approx L_{(1,2)}(x, y)$, where $f(x, y) = x^3 + y^3 - 6xy$ and $L_{(1,2)}$ is the linear approximation of f near the point $(1, 2)$, given by

   $$L_{(1,2)} = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2).$$

We have $f(1,2) = 1^3 + 2^3 - 6(1)(2) = -3$, $f_x(x,y) = 3x^2 - 6y$, so $f_x(1,2) = -9$, and $f_y(x,y) = 3y^2 - 6x$, so $f_y(1,2) = 6$. Thus,

$$f(0.99, 1.01) \approx -3 - 9(0.99 - 1) + 6(2.01 - 2) = -3 + 0.09 + 0.06 = -2.85.$$

(The actual value is -2.8485.)

4. Verify[1] the chain rule for the function $f(x,y,z) = x + y^2 + z^3$ and curve $\mathbf{r}(t) = (\cos t, \sin t, t)$.

   According to the chain rule,

$$\begin{aligned} \frac{d}{dt} f(\mathbf{r}(t)) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= (1)(-\sin t) + 2y(\cos(t)) + 3z^2(1) \\ &= -\sin t + 2\sin(t)\cos(t) + 3t^2, \end{aligned}$$

   where in the last line we've substituted in the values of x, y, z in terms of t. On the other hand, if we substitute first, we have

$$f(\mathbf{r}(t)) = \cos(t) + (\sin(t))^2 + (t)^3,$$

   so

$$\frac{d}{dt}(f(\mathbf{r}(t)) = \frac{d}{dt}(\cos(t) + \sin^2(t) + t^3) = -\sin(t) + 2\sin(t)\cos(t) + 3t^2,$$

   which agrees with our answer from above.

5. Express your chain rule formula from the previous problem as a product of two derivative matrices. (One will be a row vector, and one will be a column vector.)

   The derivative matrix of f at a point (x,y,z) is given by

$$D_{(x,y,z)} f = \begin{bmatrix} f_x(x,y,z) & f_y(x,y,z) & f_z(x,y,z) \end{bmatrix} = \begin{bmatrix} 1 & 2y & 3z^2 \end{bmatrix},$$

   so if $(x,y,z) = \mathbf{r}(t) = (\cos t, \sin t, t)$, we have

$$D_{\mathbf{r}(t)} f = \begin{bmatrix} 1 & 2\sin t & 3t^2 \end{bmatrix}$$

   The derivative matrix of $\mathbf{r}(t)$ at a point t is given by

$$D_t \mathbf{r} = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}.$$

[1] That is, calculate $\frac{d}{dt}(f(\mathbf{r}(t))$ first by using the chain rule, and then by explicitly substituting in the parameterization and differentiating with respect to t, and verify that the two answers are the same

Thus, according to the general chain rule,

$$D_t(f \circ \mathbf{r}) = D_{\mathbf{r}(t)} f \, D_t \mathbf{r} = \begin{bmatrix} 1 & 2\sin t & 3t^2 \end{bmatrix} \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix} = -\sin t + 2\sin t \cos t + 3t^2,$$

as before.

6. Find the derivative matrix for the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(u, v) = (u \sin v, e^{uv})$, and evaluate it at the point $(0, 1)$.

   (That is, compute $\dfrac{\partial(x, y)}{\partial(u, v)}$ if $x = u \sin v$ and $y = e^{uv}$.)

   The derivative matrix for f is given by $D_{(u,v)} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$, where

$$\begin{aligned} x_u &= \frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u \sin v) = \sin v \\ x_v &= \frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(u \sin v) = u \cos v \\ y_u &= \frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(e^{uv}) = ve^{uv} \\ y_v &= \frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(e^{uv}) = ue^{uv} \end{aligned}$$

   Evaluating everything at the point $(0, 1)$ gives us

$$D_{(0,1)} f = \begin{bmatrix} \sin(1) & 0 \\ 1 & 0 \end{bmatrix}.$$