Math 4310 Assignment #11
University of Lethbridge, Fall 2014

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Due date: Friday, November 21st, by 5 pm.

Please submit solutions to the following problems (except where indicated). As usual, all the textbook exercises are recommended as practice problems if they don't appear as an assigned problem below.

1. (a) Let $\gamma : [0,1] \to X$ be a path, and let $\rho : [0,1] \to [0,1]$ be any continuous function such that $\rho(0) = 0$ and $\rho(1) = 1$. Prove that the paths γ and $\gamma \circ \rho$ are homotopic.
   Hint: ρ is itself a path from 0 to 1 in $[0,1]$, and all such paths are homotopic in $[0,1]$.

   (b) Let α, β, and γ be loops based at a point $x_0 \in X$. Write down explicit formulas for $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$.

   (c) Prove that $[\alpha] * ([\beta] * [\gamma]) = ([\alpha] * [\beta]) * [\gamma]$.
   Hint: use (a), and try the map $\rho(s) = \begin{cases} s/2 & \text{if } 0 \le s \le 1/2 \\ s - 1/4 & \text{if } 1/2 \le s \le 3/4 \\ 2s - 1 & \text{if } 3/4 \le s \le 1 \end{cases}$.
   Note: A similar approach can be used to verify the other group axioms as well. For example, let $e_{x_0} : [0,1] \to X$ be the constant map $e_{x_0}(s) = x_0$ for all $s \in [0,1]$, and let $\gamma : [0,1] \to X$ be a loop based at x_0. Let $e_0 : [0,1] \to [0,1]$ be the constant map $e_0(t) = 0$ for all $t \in [0,1]$, and let $i : [0,1] \to [0,1]$ be the identity map. Then $e_0 * i : [0,1] \to [0,1]$ is homotopic to i, and

$$\gamma \circ (e_0 * i) = (\gamma \circ e_0) * (\gamma \circ i) = e_{x_0} * \gamma.$$

   If F is a homotopy from $e_0 * i$ to i, then $\gamma \circ F$ is a homotopy from $e_{x_0} * \gamma$ to $\gamma \circ i = \gamma$. Thus, $[e_{x_0}] * [\gamma] = [\gamma]$.
   (Here we used the fact that for any paths α and β in a space Y and a map $f : Y \to X$, $f \circ (\alpha * \beta) = (f \circ \alpha) * (f \circ \beta)$ since both are equal to the map given by $f(\alpha(2s))$ for $s \in [0, 1/2]$ and $f(\beta(2s - 1))$ for $s \in [1/2, 1]$, and as mentioned in class, if F is a homotopy from α to β in Y, then $G = f \circ F$ is a homotopy from $f \circ \alpha$ to $f \circ \beta$ in X.)

2. Let X be a space and let $\alpha, \beta : [0,1] \to X$ be two paths from x_0 to x_1, for two points $x_0, x_1 \in X$. These paths define isomorphisms $\varphi_\alpha, \varphi_\beta : \pi_1(X, x_0) \to \pi_1(X, x_1)$, but as noted in class, they may be different isomorphisms. Prove that the isomorphism φ_β is the composition of φ_α with the inner automorphism of $\pi_1(X, x_1)$ induced by the element $[\beta^{-1} * \alpha]$.

3. Prove that the two isomorphisms in the previous problem are the same if and only if $\pi_1(X, x_0)$ is Abelian.

4. Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps $f : X \to Y$.

   (a) Let $I = [0,1]$. Show that for any space X, $[X, I]$ contains a single element.

   (b) Show that if Y is path connected, then the set $[I, Y]$ contains a single element.

5. (**Do not submit**) A space X is called **contractible** if the identity map $i_X : X \to X$ is homotopic to a constant map. (If f is homotopic to a constant map, we say f is **nullhomotopic**.)

   (a) Show that I and $\mathbb{R}$ are contractible.

   (b) Show that a contractible space is path-connected.

   (c) Show that if Y is contractible, then for any set X, the set $[X, Y]$ has a single element.

   (d) Show that if X is contractible and Y is path-connected, then the set $[X, Y]$ has a single element.

6. Let $A \subseteq X$. Recall that a **retraction** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

   is a surjection.