Math 2580 Assignment #7
University of Lethbridge, Spring 2016

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April 6, 2016

Due date: Thursday, April 7th, by 3 pm.

Please provide solutions to the problems below, using the following guidelines:

- Your submitted assignment should be a **good copy** – figure out the problems first, and then write down organized solutions to each problem.
- You should include a **cover page** with the following information: the course number and title, the assignment number, your name, and a list of any resources you used or people you worked with.
- Since you have plenty of time to work on the problems, assignment solutions will be held to a higher standard than on a test. Your explanations should be clear enough that any of your classmates can understand your solutions.
- Group work is permitted, but copying is not. If you're not sure what the difference is, feel free to ask. If you get help solving a problem, you should (a) make sure you completely understand the solution, and (b) re-write the solution for your good copy by yourself, in your own words.
- Assignments can be submitted in class, or in the designated drop box on the 5th floor of University Hall, across from the Math Department office.
- Late assignments will not be accepted without prior permission.

Assigned problems

1. Verify Green's Theorem holds for the integral $\int_C (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy$, where C is the unit circle.

2. A vector field $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ in $\mathbb{R}^2$ can be viewed as a special case of a vector field in $\mathbb{R}^3$ that does not depend on z, with z-component equal to zero. With this identification,

   (a) Show that $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$.

   (b) Use part (a) to show that Green's Theorem can be written in the vector form
   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA,$$
   where $D \subseteq \mathbb{R}^2$ is a region to which Green's Theorem applies, and $C = \partial D$ is the positively-oriented boundary of D.

   (c) Show that Green's Theorem implies the *Divergence Theorem in the Plane*:
   Let $D \subseteq \mathbb{R}^2$ be a region to which Green's Theorem applies, and let $C = \partial D$ be its positively-oriented boundary. Let $\mathbf{n}$ denote the outward-pointing unit normal vector to C: if $\mathbf{r} : [a,b] \to \mathbb{R}^2$, $\mathbf{r}(t) = (x(t), y(t))$ defines a positively-oriented parameterization of C, then $\mathbf{n}$ is given by
   $$\mathbf{n} = \frac{y'(t)\mathbf{i} - x'(t)\mathbf{j}}{\sqrt{(x'(t))^2 + (y'(t))^2}}. \quad \text{(Verify this.)}$$
   If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a C^1 vector field on D, then
   $$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D (\nabla \cdot \mathbf{F})\, dA.$$

3. The **Laplacian** is a differential operator $\Delta = \nabla^2$ that acts on functions $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, defined by
   $$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$
   A C^2 function f is called **harmonic** if $\Delta f = 0$. Harmonic functions are important in many areas of Engineering and Physics, such as heat transfer, electrodynamics, fluid flow, robotics[1], etc.

   (a) Determine whether or not the functions $f(x,y) = e^x \sin y$, $g(x,y) = x^3 + y^3$, $h(x,y) = x^3 - 3xy^2$ are harmonic.

   (b) Prove that for any harmonic function f defined on a region D for which Green's Theorem holds, we have
   $$\int_{\partial D} \frac{\partial f}{\partial y}\, dx - \frac{\partial f}{\partial x}\, dy = 0.$$

[1] According to the internet.