Math 2580 Assignment #2
University of Lethbridge, Spring 2016

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Due date: Thursday, January 28, by 3 pm.

Please provide solutions to the problems below, using the following guidelines:

- Your submitted assignment should be a **good copy** – figure out the problems first, and then write down organized solutions to each problem.
- You should include a **cover page** with the following information: the course number and title, the assignment number, your name, and a list of any resources you used or people you worked with.
- Since you have plenty of time to work on the problems, assignment solutions will be held to a higher standard than on a test. Your explanations should be clear enough that any of your classmates can understand your solutions.
- Group work is permitted, but copying is not. If you're not sure what the difference is, feel free to ask. If you get help solving a problem, you should (a) make sure you completely understand the solution, and (b) re-write the solution for your good copy by yourself, in your own words.
- Assignments can be submitted in class, or in the designated drop box on the 5th floor of University Hall, across from the Math Department office.
- Late assignments will not be accepted without prior permission.

Terminology

With functions of several variables there is a hierarchy of "well-behavedness". For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, continuity is defined as usual ($\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$). In class, we defined what it means for a function to be differentiable in terms of the existence of a linear approximation.

The desired linear approximation at a point $\mathbf{a} \in \mathbb{R}^n$ is given by $L_{\mathbf{a}}(\mathbf{x}) = A(\mathbf{x} - \mathbf{a}) + \mathbf{b}$, where $\mathbf{b}$ is a constant vector (point) in $\mathbb{R}^m$, and A is an $m \times n$ matrix, called the **Jacobian matrix** of f at $\mathbf{a}$. The Jacobian matrix is often viewed as "the" derivative in higher dimensions, and it is denoted by

$$A = D_{\mathbf{a}} f \quad \text{or} \quad A = \frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)}.$$

The Jacobian matrix is an $m \times n$ matrix defined as follows: If $f = (f_1, \ldots, f_m)$, where each f_i is a real-valued function of $x_1, \ldots, x_n$, then

$$D_{\mathbf{a}} f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \frac{\partial f_m}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

Our funtion f is then **differentiable** at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|f(\mathbf{x}) - L_{\mathbf{a}}(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{a}\|} = 0,$$

which means that the difference between f and the linear approximation shrinks to zero as $\mathbf{x}$ approaches $\mathbf{a}$ (and that it does so faster than the difference between $\mathbf{x}$ and $\mathbf{a}$).

The following are true (see the handout on differentiability that I posted):

- If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.
- If f is differentiable at $\mathbf{a}$, then all partial derivatives of f exist at $\mathbf{a}$.

Note however that if f is not differentiable, f might still be continuous, and f might have partial derivatives, but neither of these properties implies the other. Now, while mere existence of partial derivatives at a point doesn't tell us much (it doesn't even guarantee continuity of f), it turns out that requiring the partial derivatives to be *continuous* is a much stronger requirement.

Definition: We say that a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is **continuously differentiable** at $\mathbf{a}$ if all partial derivatives of f exist **and** are continuous on an open neighbourhood[1] of $\mathbf{a}$.

It is then a theorem (I'll post a proof on Moodle) that any continuously differentiable function is differentiable. So, while mere existence of partial derivatives is not enough to guarantee a good linear approximation, continuity of those partial derivatives is.

Continuously differentiable functions are sometimes referred to as C^1 functions, for brevity. This notation is part of a collection: a C^0 function is a function which is simply continuous. A C^2 function is one whose *second-order* partial derivatives exist and are continuous, a C^3 function has continuous third-order partial derivatives, and so on.

Assigned problems

1. Let $r : \mathbb{R} \to \mathbb{R}^3$ be a smooth[2] curve given by $r(t) = (u(t), v(t), w(t))$, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable function given by

$$f(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)).$$

The composition $s(t) = (f \circ r)(t) = (x(r(t)), y(r(t)), z(r(t)))$ is then another curve in $\mathbb{R}^3$. Using the Chain Rule, show the following:

   (a) If $r'(t)$ exists for all t, then $s'(t)$ exists for all t.

   (b) If $\vec{v}$ is tangent to the curve $r(t)$ at a point $\mathbf{u}_0 = (u_0, v_0, w_0) = r(t_0)$, then $D_{\mathbf{u}_0} f \vec{v}$ is tangent to the curve $s(t)$ at the point $\mathbf{x_0} = f(u_0, v_0, w_0) = s(t_0)$.
   (Hint: If $\vec{v}$ is tangent to the curve $r(t)$ at $r(t_0)$, then it must be a scalar multiple of $r'(t_0)$.)

   (c) **Bonus:** In order to say that the curve $s(t)$ is "smooth", we would need to also guarantee that $s'(t)$ is never zero. What condition on $D_{\mathbf{x}} f$ will guarantee this? (Hint: if $\vec{v}$ is a non-zero vector, how can you guarantee that $A\vec{v} \neq 0$ for an $m \times n$ matrix A?)

2. Let $r(t) = (2\cos(t), 3\sin(t))$ be a curve in the plane, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = x^2 - 4xy^3$. The curve

$$s(t) = (2\cos(t), 3\sin(t), f(2\cos(t), 3\sin(t)))$$

is then a curve in $\mathbb{R}^3$ that lies on the surface $z = f(x, y)$.

   (a) Explain why the claim above (that $s(t)$ defines a curve on the surface $z = f(x, y)$) is true.

   (b) Show that the tangent vector to $s(t)$ when $t = 0$ lies in the tangent plane to the surface $z = f(x, y)$ at the point $(2, 0, 4)$.

Note: the general case for this example is at the end of Section 15.3 in the Marsden and Weinstein text.

[1] An **open neighbourhood** of a point $\mathbf{a} \in \mathbb{R}^n$ is a set of the form $N_r(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\| < r\}$.

[2] For us, a curve will be *smooth* if $r'(t) = \langle u'(t), v'(t), w'(t) \rangle$ exists and is **non-zero** for all t.