Last Name: _____ Solutions _____

First Name: _____ The _____

Student Number: _____

Discussion Section: _____

Name of GSI: _____

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

There is a list of potentially useful formulas available on the last page of the exam.

For grader's use only:

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2	/13
3	/13
Total	/40

A

1. (a) Describe the motion of a particle whose position $(x(t), y(t))$ at time $t \in [0, 2\pi]$ is given
[6] by $x = \cos t$, $y = \sin^2 t$. (In particular, what is the Cartesian equation of the curve?)

We notice that $y = \sin^2 t = 1 - \cos^2 t = 1 - x^2$, so that the particle travels along the
parabola $y = 1 - x^2$, for $-1 \le x \le 1$ (since $-1 \le \cos t \le 1$ for all t).

At time $t = 0$ the particle begins at the point $(1, 0)$, and then travels up towards the
vertex $(0, 1)$, which it reaches at time $t = \pi/2$. It then continues to the other end of
the parabola at $(-1, 0)$, which it reaches at time $t = \pi$, before turning around and
travelling across the parabola again, returning to $(0, 1)$ when $t = 3\pi/2$, and getting
back to the starting point $(1, 0)$ when $t = 2\pi$.

(b) Set up, but do not evaluate, the integral which computes the length of the curve from
[4] part (a). How does this compare to the distance travelled by the particle?

We saw that the full length of the curve is travelled in the interval $0 \le t \le \pi$, and
thus, we have

$$L = \int_0^\pi \sqrt{x'(t)^2 + y'(t)^2} = \int_0^\pi \sqrt{\sin^2 t + 4\cos^2 t \sin^2 t}\, dt.$$

This length is half of the total distance travelled by the particle.

2. Find the equation of the tangent line to the curve represented by the vector-valued function
[4] $\mathbf{r}(t) = \langle t^5, t^4, t^3 \rangle$ at the point $(1, 1, 1)$.

First, we note that the point $(1, 1, 1)$ corresponds to $\mathbf{r}(1) = \langle 1, 1, 1 \rangle$. The tangent vector
at an arbitrary point is given by $\mathbf{r}'(t) = \langle 5t^4, 4t^3, 3t^2 \rangle$, and thus in particular, we have
$\mathbf{r}'(1) = \langle 5, 4, 3 \rangle$. The equation of the tangent line is therefore

$$\mathbf{r}(s) = \langle 1, 1, 1 \rangle + s\langle 5, 4, 3 \rangle,$$

or $x = 1 + 5s$, $y = 1 + 4s$, $z = 1 + 3s$. (Either parametric or vector form is sufficient;
the parameter s is used for the line instead of t to avoid confusion, but using t again is
acceptable.)

[8] 3. (a) Find the equation of the line of intersection of the planes given by the equations

$$x - 2y + 3z = -2$$
$$2x + y - 4z = 6.$$

Multiplying the first equation by -2 gives us $-2x + 4y - 6z = 4$; adding this to the second gives $5y - 10z = 10$. We can thus solve for y in terms of z, giving $y = 2z + 2$. Plugging this back into the first equation gives us

$$x = -2 + 2y - 3z = -2 + 4z + 4 - 3z = z + 2.$$

Thus, if we let $z = t$ be our parameter, we obtain the parametric equations $x = 2 + t$, $y = 2 + 2t$, $z = t$ for the line.

[2] (b) What is the cosine of the angle of intersection of the two planes in part (a)?

The two planes have normal vectors $\mathbf{n_1} = \langle 1, -2, 3 \rangle$ and $\mathbf{n_2} = \langle 2, 1, -4 \rangle$. The angle between the planes is therefore given by

$$\cos \theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{\|\mathbf{n_1}\|\|\mathbf{n_2}\|} = \frac{2 - 2 - 12}{\sqrt{1 + 4 + 9}\sqrt{4 + 1 + 16}} = \frac{-12}{\sqrt{14}\sqrt{21}}.$$

(You can simplify the answer further, but it's not necessary.)

[3] (c) What is the distance from the plane $x - 2y + 3z = -2$ to the point $(3, -1, 4)$?

Assuming that you didn't bother to fill your brain with unnecessary distance formulas, we proceed as follows: we know that the point $(2, 2, 0)$ is on the plane from part (a), and thus a vector from the plane to the point is $\mathbf{a} = \langle 3 - 2, -1 - 2, 4 - 0 \rangle = \langle 1, -3, 4 \rangle$. The distance is then given by the length of the normal component of this vector; that is,

$$d = |\text{comp}_{\mathbf{n_1}} \mathbf{a}| = \frac{|\mathbf{a} \cdot \mathbf{n_1}|}{\|\mathbf{n_1}\|} = \frac{|1 + 6 + 12|}{\sqrt{1 + 4 + 9}} = \frac{19}{\sqrt{14}}.$$

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[7]   4. Find the area of the triangle ΔPQR, for points $P(0,0,0)$, $Q(1,2,-1)$, $R(-2,3,2)$.

The area of the triangle is given by $A = \dfrac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$. We have $\overrightarrow{PQ} = \langle 1, 2, -1 \rangle$ and $\overrightarrow{PR} = \langle -2, 3, 2 \rangle$, which gives us

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 3 & 2 \end{vmatrix} = \langle 7, 0, 7 \rangle.$$

The area is thus $A = \dfrac{1}{2}\|\langle 7, 0, 7 \rangle\| = \dfrac{7}{\sqrt{2}}$.

[6]   5. Let $\mathbf{a}$, $\mathbf{b}, \mathbf{c}$ be nonzero vectors in $\mathbb{R}^3$. For each of the following, prove the statement, or give an example showing that the statement is false:

(a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

(c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

For part (a), let $\mathbf{a} = \hat{\imath}$, $\mathbf{b} = \hat{\jmath}$, and $\mathbf{c} = \hat{k}$. Then $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = 0$, but $\hat{\jmath} \neq \hat{k}$, so the statement is false.

For part (b), let $\mathbf{a} = \hat{\imath}$, $\mathbf{b} = \hat{\imath}$, and $\mathbf{c} = 2\,\hat{\imath}$. Then $\mathbf{c} \neq \mathbf{b}$, but $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} = \mathbf{0}$, so the statement is false.

For part (c), suppose that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$. Since the cross product is distributive, this implies that $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = 0$, which tells us that $\mathbf{b} - \mathbf{c} = k\mathbf{a}$ for some scalar k. (Either $\mathbf{b} = \mathbf{c}$, so $k = 0$, or $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$.) If we also have that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then it follows that

$$0 = \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (k\mathbf{a}) = k\|\mathbf{a}\|^2.$$

Since $\mathbf{a} \neq \mathbf{0}$, we must have $k = 0$, and thus $\mathbf{b} = \mathbf{c}$.

List of potentially useful formulas and facts:

In the following, assume $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ are constant vectors in $\mathbb{R}^3$, and $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a vector-valued function with domain $[a, b]$.

- Length of a vector: $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

- Dot product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos \theta$

- Cross product: $\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_3, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$; $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin \theta$.

- Projections: $\text{proj}_{\mathbf{a}} \, \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} \left(\dfrac{\mathbf{a}}{\|\mathbf{a}\|} \right)$, $\text{comp}_{\mathbf{a}} \, \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$.

- Planes: $ax + by + cz = d$, or $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$.

- Lines: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$, or $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$.

- Quadric surfaces: there aren't any on the midterm, so you can relax and play with some vectors.

- Parametric area: $A = -\displaystyle\int_a^b y(t) x'(t) \, dt$ for a positively-oriented curve.

- Tangent vectors: For each $t_0 \in [a, b]$, $\mathbf{r}(t)$ has tangent vector $\mathbf{r}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$.

- Parametric length: $L = \displaystyle\int_a^b \|\mathbf{r}'(t)\| \, dt$.

List of basic facts I hope were in fact entirely unnecessary to include:

- $\sin^2 \theta + \cos^2 \theta = 1$

- $\dfrac{d}{dt}(t^n) = nt^{n-1}$, $\dfrac{d}{dt} \sin t = \cos t$, $\dfrac{d}{dt} \cos t = -\sin t$

- $\dfrac{d}{dt}(f(t)g(t)) = f'(t)g(t) + f(t)g'(t)$