Solutions to Quiz 24 Practice Problems
Math 2580
Spring 2016

Sean Fitzpatrick

April 12th, 2016

1. Use Stokes' theorem to evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = \langle yz, xz, xy \rangle$, and S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward.

   The boundary C of S is the circle $x^2 + y^2 = 4$ in the plane $z = 5$, oriented in the counter-clockwise direction, as seen from above. To use Stokes' theorem directly, we paramterize the circle using $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 5 \rangle$, with $t \in [0, 2\pi]$. We then have

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot \, d\mathbf{r} \\
&= \int_0^{2\pi} \langle 10\sin t, 10\cos t, 4\sin t\cos t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt \\
&= \int_0^{2\pi} -20\sin^2 t + 20\cos^2 t \rangle \, dt \\
&= 20 \int_0^{2\pi} \cos 2t \, dt = 0.
\end{aligned}$$

   Of course, if we actually compute the curl, we immediately have $\nabla \times \mathbf{F} = \mathbf{0}$, so the result above isn't particularly surprising. (If the curl had been nonzero, another option would be to integrate $\nabla \times \mathbf{F}$ over the disc D in the plane $z = 5$ given by $x^2 + y^2 \leq 4$.)

2. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, directly, where $\mathbf{F}(x,y,z) = \langle yz, xz, xy \rangle$, and S is the disk $x^2 + y^2 \leq 4$ in the plane $z = 5$.

   As noted above, $\nabla \times \mathbf{F} = \mathbf{0}$, so the answer is immediately 0.

3. How are Problems 1 and 2 related?

   This ended up being a bit boring, due to the curl being zero, but the point was that the both surfaces share the same boundary, so the integral of $\nabla \times \mathbf{F}$ over either surface should be the same.

4. Use Stokes' theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xy, 2x, 3y \rangle$, and C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$.

   Using Stokes' theorem, we need to compute the integral $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is any surface with boundary curve C. We first compute the curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & 2x & 3y \end{vmatrix} = \langle 3, 0, 2 - x \rangle.$$

   The simplest surface S to use is the portion of the plane $x + z = 1$ that lies within the cylinder $x^2 + y^2 = 9$. Treating the plane as the graph $z = 1 - x$, we can use the parameterization

$$\mathbf{r}(x, y) = \langle x, y, 1 - x \rangle, \quad x^2 + y^2 \leq 9.$$

   The normal vector in this case is found to be $\mathbf{N}(x, y) = \langle 1, 0, 1 \rangle$, so we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D \langle 3, 0, 2 - x \rangle \cdot \langle 1, 0, 1 \rangle \, dA = \iint_D (5 - x) \, dA,$$

   where D is the disc $x^2 + y^2 \leq 9$. By symmetry, the $-x$ term does not contribute to the integral, so the result is simply 5 times the area of the disc, or 45π.

5. Use Stokes' theorem to show that if $\mathbf{F}$ is C^1 vector field defined on all of $\mathbb{R}^3$ such that $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

   If $\mathbf{F}$ is C^1 on $\mathbb{R}^3$, then $\nabla \times \mathbf{F}$ is defined and continuous on all of $\mathbb{R}^3$. If C is any simple, closed curve, we can choose a surface S with $\partial S = C$, and then by Stokes' theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0.$$

   By a previous result, since the integral of $\mathbf{F}$ around any closed curve is 0, we can conclude that $\mathbf{F}$ is conservative.

6. Use the Divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the vector field $\mathbf{F}(x, y, z) = \langle x^4, -x^3 z^2, 4xy^2 z \rangle$, where S is the boundary of the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = x + 2$.

   We compute $\nabla \cdot \mathbf{F}(x, y, z) = \dfrac{\partial}{\partial x}(x^4) + \dfrac{\partial}{\partial y}(-x^3 z^2) + \dfrac{\partial}{\partial z}(4xy^2 z) = 4x^3 + 4xy^2 = 4x(x^2 + y^2)$. The region E bounded by S is given in cylindrical coordinates by $0 \leq z \leq 2 + r\cos\theta$, where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. The Divergence theorem thus gives us

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\nabla \cdot \mathbf{F}) dV = \int_0^{2\pi} \int_0^1 \int_0^{2 + r\cos\theta} 4r\cos\theta (r^2) r \, dz \, dr \, d\theta = \frac{16\pi}{5}.$$

7. Use the Divergence theorem to evaluate $\iint_S (2x + 2y + z^2)\, dS$, where S is the sphere $x^2 + y^2 + z^2 = 1$.

On the unit sphere, we know that $\mathbf{n} = \langle x, y, z\rangle$ is the unit normal vector. Note also that

$$2x + 2y + z^2 = \langle 2, 2, z\rangle \cdot \langle x, y, z\rangle = \mathbf{F} \cdot \mathbf{n},$$

where $\mathbf{F}(x, y, z) = \langle 2, 2, z\rangle$. The Divergence theorem thus gives us

$$\iint_S (2x + 2y + z^2)\, dS = \iiint_E (\nabla \cdot \mathbf{F})\, dV = \iiint_E (1)\, dV = \frac{4\pi}{3},$$

where E is the ball $x^2 + y^2 + z^2 \leq 1$, which has volume $\frac{4}{3}\pi(1)^3$.