Solutions to Quiz 11 Practice Problems
Math 2580
Spring 2016

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1. Evaluate the following iterated integrals:

   (a) $\displaystyle\int_0^2\int_0^3 x^2y^3\,dx\,dy = \int_0^2\left(\frac{1}{3}x^3y^3\Big|_0^3\right)\,dy = \int_0^2 9y^3\,dy = \frac{9}{4}y^4\Big|_0^2 = 36.$

   (b) $\displaystyle\int_{-1}^1\int_0^2 x^2y\sin(y^2)\,dx\,dy = \int_0^2 x^2\left(\int_{-1}^1 y\sin(y^2)\,dy\right)\,dx = 0$, since we're integrating a function that is odd (with respect to y) between symmetric limits.

   (c) $\displaystyle\int_{-1}^1\int_0^3 y^5e^{xy^3}\,dx\,dy$

   Recall that when we perform the integral with respect to x we're treating y as a constant. We thus have

   $$\int_0^3 y^5e^{xy^3}\,dx = y^5\left(\frac{1}{y^3}e^{xy^3}\right)\Big|_0^3 = y^2(e^{3y^3}-1).$$

   Putting this into the original integral gives us

   $$\int_{-1}^1\int_0^3 y^5e^{xy^3}\,dx\,dy = \int_{-1}^1(y^2e^{3y^3}-y^2)\,dy = \frac{1}{9}e^{y^3}-\frac{1}{3}y^3\Big|_{-1}^1 = \frac{e^3}{9}-\frac{e^{-3}}{9}-\frac{2}{3}.$$

   (d) $\displaystyle\int_0^\pi\int_{-\pi/2}^{\pi/2}\sin(x+y)\,dx\,dy$

   $$\begin{aligned}\int_0^\pi\int_{-\pi/2}^{\pi/2}\sin(x+y)\,dx\,dy &= \int_0^\pi\left(-\cos(x+y)|_{x=-\pi/2}^{x=\pi/2}\right)\,dy\\ &= \int_0^\pi(-\cos(y+\pi/2)+\cos(y-\pi/2))\,dy\\ &= -\sin(y+\pi/2)+\sin(y-\pi/2)|_0^\pi\\ &= -\sin(3\pi/2)+\sin(\pi/2)+\sin(\pi/2)-\sin(-\pi/2) = 4.\end{aligned}$$

2. Evaluate the following integrals over the given region D:

   (a) $\iint_D x^3y^2\,dA$, where $D = \{(x,y)|0 \le x \le 2, -x \le y \le x\}$.

   Since the region is Type 1, we integrate first with resepct to y. Thus,

$$\begin{aligned}\iint_D x^3y^2\,dA &= \int_0^2 \int_{-x}^{x} x^3y^2\,dy\,dx \\ &= \int_0^2 \left(\frac{1}{3}x^3y^3\Big|_{-x}^{x}\right)\,dx \\ &= \int_0^2 \frac{2}{3}x^6\,dx = \frac{2}{21}x^7\Big|_0^2 \\ &= \frac{256}{21}.\end{aligned}$$

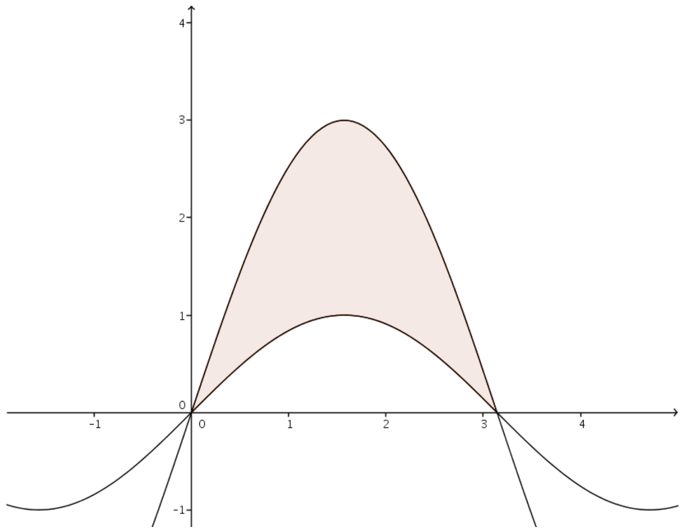
   (b) $\iint_D (1-\sin(\pi x))\,dA$, where D is the region bounded by the lines $y = x$, $y = 0$, and $x = 1$.

   The integral is of both Type 1 and Type 2, so we can integrate in either order. However, if we integrate first with respect to y, we'll be left with the integral of $x - x\sin(\pi x)$ with respect to x, requiring us to integrate by parts. We therefore integrate first with respect to y, giving us

$$\begin{aligned}\iint_D (1-\sin(\pi x))\,dA &= \int_0^1 \int_y^1 (1-\sin(\pi x))\,dx\,dy \\ &= \int_0^1 \left(x + \frac{1}{\pi}\cos(\pi x)\Big|_y^1\right)\,dy \\ &= \int_0^1 \left(1 - y - \frac{1}{\pi} - \frac{1}{\pi}\cos(\pi y)\right)\,dy \\ &= (1-\frac{1}{\pi})y - \frac{1}{2}y^2 - \frac{1}{\pi^2}\sin(\pi y)\Big|_0^1 \\ &= \frac{1}{2} - \frac{1}{\pi}.\end{aligned}$$

3. For each of the iterated integrals below, (i) sketch the region of itegration, and (ii) evaluate the integral.

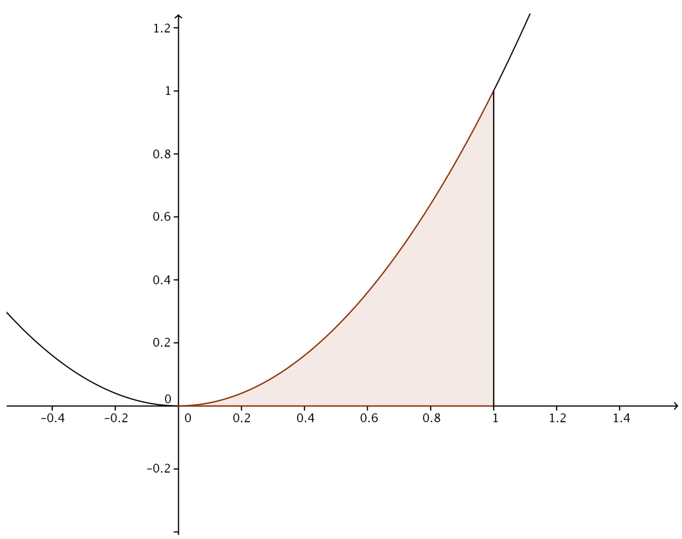
   (a) $\int_0^{\pi} \int_{\sin x}^{3\sin x} x(1+y)\,dy\,dx.$



We evaluate the iterated integral, using a trig identity and integration by parts to handle the integral with respect to x:

$$\begin{aligned}
\int_0^{\pi}\int_{\sin x}^{3\sin x} x(1+y)\,dy\,dx &= \int_0^{\pi}\left(x\left(y+\frac{1}{2}y^2\right)\Big|_{\sin x}^{3\sin x}\right)\,dx \\
&= \int_0^{\pi} x(2\sin x + 4\sin^2 x)\,dx \\
&= \int_0^{\pi} x(2\sin x + 2 - 2\cos(2x))\,dx \\
&= -2x\cos x + 2\sin x + x^2 - x\sin(2x) + \frac{1}{2}\cos(2x)\Big|_0^{\pi} \\
&= \pi^2 + 2\pi.
\end{aligned}$$

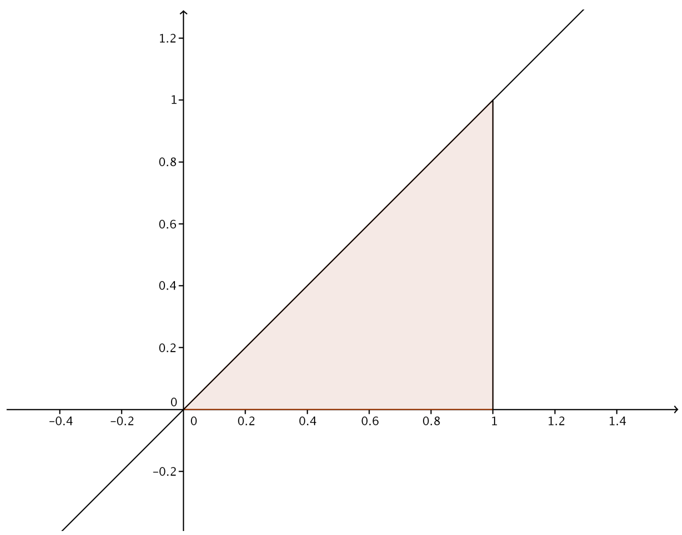
(b) $\displaystyle\int_0^1\int_0^{x^2}(x^2+xy-y^2)\,dy\,dx.$



$$\begin{aligned}\int_0^1 \int_0^{x^2} (x^2 + xy - y^2)\, dy\, dx &= \int_0^1 \left(x^2 y + \frac{1}{2} x y^2 - \frac{1}{3} y^3 \right)\Bigg|_0^{x^2} dx \\ &= \int_0^1 \left(x^4 + \frac{1}{2} x^5 - \frac{1}{3} x^6 \right) dx \\ &= \frac{1}{5} + \frac{1}{12} - \frac{1}{21} = \frac{33}{140}.\end{aligned}$$

4. For each of the iterated integrals below, (i) sketch the region of integration, (ii) change the order of integration, and (iii) evaluate the integral.

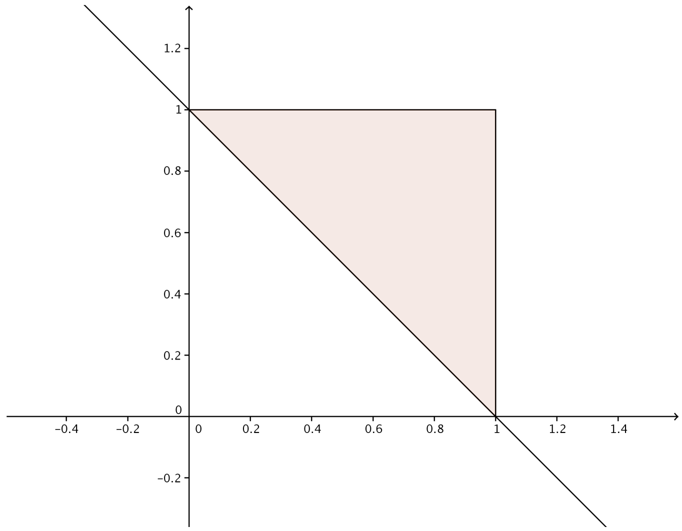
   (a) $\displaystyle\int_0^1 \int_0^x xy\, dy\, dx$



   The region is given as the Type 1 region $0 \le x \le 1$, $0 \le y \le x$. We can express it as the Type 2 region $0 \le y \le 1$, $y \le x \le 1$, giving us

$$\begin{aligned}\int_0^1 \int_0^x xy\, dy\, dx &= \int_0^1 \int_y^1 xy\, dx\, dy \\ &= \int_0^1 \left(\frac{1}{2} x^2 y \Bigg|_y^1 \right) dy \\ &= \frac{1}{2} \int_0^1 (y - y^3)\, dy \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{8}.\end{aligned}$$

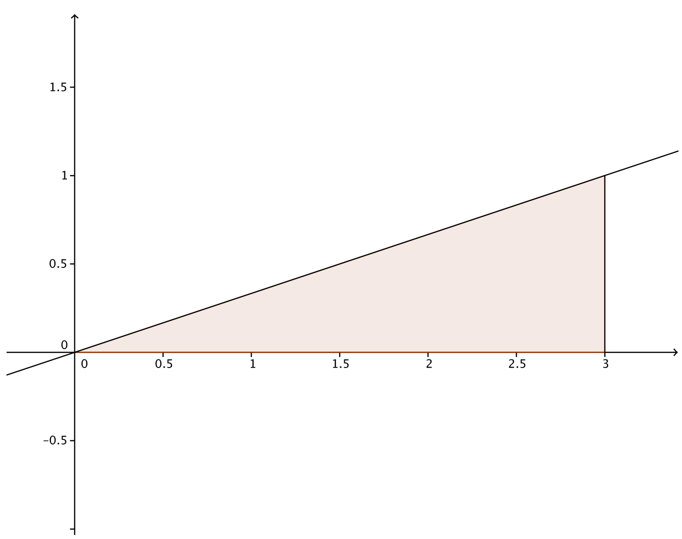
   (b) $\displaystyle\int_0^1 \int_{1-y}^1 (x + y^2)\, dx\, dy$



The Type 2 region $0 \leq 1 \leq 1$, $1 - y \leq x \leq 1$ can be written as the Type 2 region $0 \leq x \leq 1$, $1 - x \leq y \leq 1$, giving us

$$\begin{aligned}
\int_0^1 \int_{1-y}^1 (x + y^2)\, dx\, dy &= \int_0^1 \int_{1-x}^1 (x + y^2)\, dy\, dx \\
&= \int_0^1 \left(xy + \frac{1}{3} y^3 \right) \Big|_{1-y}^1 \, dx\, dy \\
&= \int_0^1 \left(x - x(1-x) + \frac{1}{3} - \frac{1}{3}(1-x)^3 \right) \, dx \\
&= \int_0^1 \left(x + \frac{1}{3} x^3 \right) \, dx \\
&= \frac{1}{2} + \frac{1}{12} = \frac{7}{12}.
\end{aligned}$$

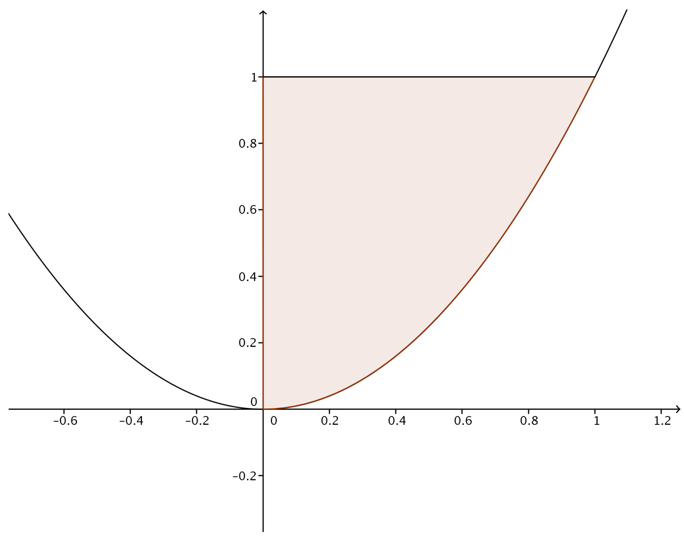
(c) $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$



We're given the Type 2 region $0 \le y \le 1$, $3y \le x \le 3$, but this is not very useful, since we don't have an antiderivative for e^{x^2}. Changing the order of integration, we can write $0 \le x \le 3$, $0 \le y \le \frac{1}{3}x$, giving us

$$\begin{aligned}\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy &= \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx\\ &= \int_0^3 \frac{1}{3} x e^{x^2}\,dx\\ &= \frac{1}{6} e^{x^2}\Big|_0^3 = \frac{e^9 - 1}{6}.\end{aligned}$$

(d) $\displaystyle \int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\,dy\,dx$



We're given the Type 1 region $0 \le x \le 1$, $x^2 \le y \le 1$. As a Type 2 region, it's given by $0 \le y \le 1$, $0 \le x \le \sqrt{y}$. Thus,

$$\begin{aligned}\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\,dy\,dx &= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\,dx\,dy\\ &= \int_0^1 \left(\frac{1}{4} x^4 \sin(y^3) \Big|_{x=0}^{x=\sqrt{y}} \right) dy\\ &= \int_0^1 \frac{1}{4} y^2 \sin(y^3)\,dy\\ &= -\frac{1}{12} \cos(y^3)\Big|_0^1\\ &= \frac{1 - \cos(1)}{12}.\end{aligned}$$