Matrix Algebra

Math 1410 Linear Algebra

Matrices

An $m \times n$ matrix (or simply *matrix*, if we do not need to specify the size) is a rectangular array of (real) numbers consisting of m rows and n columns. Examples of matrices include:

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & -2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -5.2 & \pi \\ 1.234 & 0 \\ \sqrt{2} & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The number in row i and column j is referred to as the (i, j)-entry of the matrix.

General notation

In order to talk about matrices in general, we use the notation a_{ij} for the (i,j)-entry of a matrix, and write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

to indicate the matrix as a whole.

Other notation: $A = [a_{ij}]$ or $A = [a_{ij}]_{m \times n}$.
Note:

- Size $m \times n$ means m rows, n columns.
- Entry a_{ij} lies in row i and column j.

Matrix equality

Definition

Two matrices A and B are equal, denoted by $A = B$, if

1. Both A and B have the same size.
2. The corresponding entires in each matrix are equal.

In other words:

$$A = [a_{ij}]_{m \times n} = [b_{ij}]_{k \times l} = B$$

means $m = k$, $n = l$, and $a_{ij} = b_{ij}$ for all i and j.

Matrix addition

We can add two matrices A and B, provided they are the same size. The **sum** $A + B$ is formed by adding the corresponding entries:

$$\text{If } A = [a_{ij}] \text{ and } B = [b_{ij}], \text{ then } A + B = [a_{ij} + b_{ij}].$$

Example

Consider $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 0 & 1 \\ 0 & -3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$

Example

If $\begin{bmatrix} a & b & c \end{bmatrix} + \begin{bmatrix} c & a & b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$, what are a, b, and c?

Properties of matrix addition

For any matrices A, B, C of the same size, we have

$$A + B = B + A \text{ (commutative law)}$$
$$A + (B + C) = (A + B) + C \text{ (associative law)}$$

The $m \times n$ matrix with every entry equal to zero ($a_{ij} = 0$ for all i, j) is called the zero matrix, and denoted by 0 (or 0_{mn} to specify the size). For any other $m \times n$ matrix A we have

$$A + 0 = 0 + A = A.$$

Given a matrix $A = [a_{ij}]$, we define its negative by $-A = [-a_{ij}]$. Notice that for any matrix A we have

$$A + (-A) = 0.$$

Using the negative we can define the difference by $A - B = A + (-B) = [a_{ij} - b_{ij}]$.

Example 1

If $A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \\ 4 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -3 \\ 2 & 5 \\ 0 & 4 \end{bmatrix}$, compute $A - B$, $A + C$, and $A + B - C$.

Example 2

Find the matrix X such that

$$\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} + X = \begin{bmatrix} 5 & -7 \\ 0 & -3 \end{bmatrix}.$$

Scalar multiplication

Given a matrix A and a number k, the scalar multiple kA is defined as follows:

$$\text{If } A = [a_{ij}], \text{ then } kA = [ka_{ij}].$$

In other words, to multiply A by k, we multiply every entry of A by k.

Note: Here, "number" (or scalar) usually will mean *real number*. Later on we'll also encounter complex scalars.

Properties of scalar multiplication

Note that kA is always the same size as A. If either $k = 0$ or $A = 0$, we have $kA = 0$. That is,

$$0A = 0 \text{ and } k0 = 0.$$

The converse is also true: if $kA = 0$, then $k = 0$ or $A = 0$. We also have:

$$\begin{align} k(A + B) &= kA + kB \tag{1} \\ (h + k)A &= hA + kA \tag{2} \\ h(kA) &= (hk)A \tag{3} \\ 1A &= A \tag{4} \end{align}$$

Here, (1) and (2) are referred to as distributive properties. Property (3) tells us that scalar multiplication is associative.

Transpose

If a matrix A is the coefficient matrix for a system of linear equations, we have a clear distinction between rows and columns: each row corresponds to an *equation*, while each column corresponds to a variable.
In many other cases, given a result about the rows of matrix, there is an analogous result about the columns. The transpose is an operation on a matrix that exchanges rows and columns:

Definition

Let $A = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix. The **transpose** of A, denoted A^T, is the $n \times m$ matrix obtained by exchanging rows and columns. That is, if $A^T = [b_{ij}]_{n \times m}$, then $b_{ij} = a_{ji}$ for all i and j.

Examples

Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 0 & 3 \end{bmatrix}$

Properties of the transpose

Let A and B be $m \times n$ matrices, and let k be a scalar. Then

1. $(A^T)^T = A$
2. $(kA)^T = kA^T$
3. $(A + B)^T = A^T + B^T$.

Definition

We say that an $n \times n$ matrix A is symmetric if $A^T = A$, and antisymmetric if $A^T = -A$.

Matrix multiplication - row times column

Let $R = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ be a $1 \times n$ row matrix, and let $C = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$ be an $n \times 1$ column matrix. We define the product (sometimes called a *dot product*) to be the number

$$RC = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Matrix multiplication - matrix times column

The product AX of an $m \times n$ matrix and an $n \times 1$ column X is an $m \times 1$ column.

Two ways to compute AX:

Matrix multiplication - row times matrix

- $1 \times n$ row $R = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$
- $n \times p$ matrix $B = [B_1 | B_2 | \cdots | B_p]$

($B_1, B_2, \ldots, B_p$ are the $(n \times 1)$ columns of B.)

$RB =$

Matrix multiplication - general case

Extend from the previous examples: to form the product AB, multiply the rows of A by the columns of B.
When is the product of matrices A and B defined?

Examples

“Practical” example

Suppose corn and barley are grown as feed crops on a farm that raises cows and pigs. Three chemical fertilizers are used on the feed crops. The amount of each chemical absorbed by the feed (in milligrams), and the amount of feed consumed by each animal, are given by the following tables:

$$\begin{array}{cc} & \begin{array}{cc} \text{Corn} & \text{Barley} \end{array} \\ \begin{array}{c} \text{Chemical 1} \\ \text{Chemical 2} \\ \text{Chemical 3} \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 2 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} \text{Cows} & \text{Pigs} \end{array} \\ \begin{array}{c} \text{Corn} \\ \text{Barley} \end{array} & \begin{pmatrix} 27 & 15 \\ 15 & 5 \end{pmatrix} \end{array}$$

How much of each chemical is consumed by each animal?