Math 3500 Exercise Sheet

19 November, 2014

Given a function $f :\to \mathbb{R}$ that is n times differentiable, we can define the Taylor polynomials $P_{a,k,f}(x)$ for $0 \leq k \leq n$ by

$$\begin{aligned}
P_{a,0,f}(x) &= f(a) \\
P_{a,1,f}(x) &= f(a) + f'(a)(x-a) \\
P_{a,2,f}(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\
\vdots \quad & \quad \vdots \\
P_{a,k,f}(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k \\
\vdots \quad & \quad \vdots \\
P_{a,n,f}(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n
\end{aligned}$$

Each Taylor polynomial provides a succesively better approximation to the original function f on D; this is expressed by

Theorem 1. *If $f : D \to \mathbb{R}$ is n times differentiable at $x = a$, then*

$$\lim_{x\to a} \frac{f(x) - P_{k,a,f}(x)}{(x-a)^k} = 0$$

for each $k \in \{0, 1, \ldots, n\}$.

Exercise: (a) Verify the above theorem. (b) Prove the following:

Theorem 2. *Suppose that $f'(a) = \cdots = f^{(n-1)}(a) = 0$, and $f^n(a) \neq 0$.*

1. *If n is even and $f^{(n)}(a) > 0$, then f has a local minimum at $x = a$.*

2. *If n is even and $f^{(n)}(a) < 0$, then f has a local maximum at $x = a$.*

3. *If n is odd, f has neither a local maximum nor a local minimum at $x = a$.*

Note that the above result can be used in situations where the usual second derivative test fails. For example, we know that $f(x) = x^4$ has a local (and absolute) minimum at $x = 0$, but $f'(0) = f''(0) = 0$, so the second derivative test doesn't apply.

Here's a sketch of the steps involved: first, you can assume $f(a) = 0$ (otherwise, replace $f(x)$ by $g(x) = f(x) - f(a)$, whose graph is just a vertical shift of the graph of $f(x)$). Note the consequences of our assumptions on what the form of the Taylor polynomial for f is, and substitute this into Theorem 1 to conclude that if x is sufficiently close to a, then $\dfrac{f(x)}{(x-a)^n}$ has the same sign as $\dfrac{f^{(n)}(a)}{n!}$.

You might think that Theorem 2 settles the question of maxima and minima, but one can still run into problems. For example, the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

has a local minimum at $x = 0$ (graph it, or ask Wolfram Alpha to plot e^{-1/x^2} for you). However, it's possible to prove that $f^{(k)}(0) = 0$ for *all* $k \geq 0$, so Theorem 2 fails. (This function is an example of a "non-analytic smooth function": it has derivatives of all orders at every point, but it cannot be approximated by Taylor polynomials. There's a decent write-up of this phenomenon on Wikipedia. The existence of such functions turns out to be quite important.

Exercise: Prove the following:

Theorem 3. *Let P and Q be polynomials in $(x - a)$ of degree $\leq n$ and suppose that P and Q agree to order n at a. Then $P = Q$.*

As a result of this theorem, we have the result we came up with in class:

Theorem 4. *Let f be n times differentiable at $x = a$. Then $P = P_{n,a,f}$ is the* **unique** *polynomial that equals f up to order n at a. That is, if we let $R(x) = f(x) - P(x)$ denote the remainder upon subtracting P from f, then $P(x) = P_{n,a,f}$ if and only if*

$$\lim_{x \to a} \frac{R(x)}{(x-a)^n} = 0.$$

Exercise: Uniqueness is important, because it tells us that different ways of obtaining a polynomial approximation to a function will always lead to the same result. Consider the following:

(a) Use the definition of the Taylor polynomial to find the Taylor series for $f(x) = \cos x$ at $a = 0$.

(b) Attempt to use the definition of the Taylor polynomial to find the Taylor series for $g(x) = \arctan x$ at $a = 0$. Give up once you've gotten as far as $g'''(0)$. (Recall that $g'(x) = 1/(1 + x^2)$.)

(c) Use long division (with remainder) to show that

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}.$$

(d) Recall that $\arctan x = \displaystyle\int_0^x \frac{1}{1+t^2}\,dt$, by the Fundamental Theorem of Calculus. (Pretend that we're one week in the future and we've seen the FTC already.)

(e) Conclude that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}.$$

(f) Use the fact that

$$\left| \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \right| \le \left| \int_0^x t^{2n+2}\,dt \right| = \frac{|x|^{2n+3}}{2n+3}$$

to conclude that $\displaystyle\lim_{x\to 0} \frac{\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt}{x^{2n+1}} = 0$, and that the Taylor polynomial for arctan at 0 is therefore given by

$$P_{2n+1,0}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Now, we'll restate Taylor's Theorem:

Theorem 5. *Suppose $f, f', f'', \ldots, f^{(n+1)}$ are defined on (a,b) (where $a = -\infty$ or $b = \infty$ are allowed) and let $c \in (a,b)$. Then for each $x \neq c \in (a,b)$,*

$$f(x) = P_{n,a,f}(x) + R_{n,a,f}(x),$$

where

$$(1) R_{n,a,f}(x) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n(x-a), \text{ for some } t \text{ between } x \text{ and } c$$

(Cauchy's remainder formula),

$$(2) R_{n,a,f}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}, \text{ for some } t \text{ between } x \text{ and } c$$

(Lagrange's remainder formula), and if $f^{(n+1)}$ is integrable on $[a,x]$, (with $a \neq -\infty$), then

$$(3) R_{n,a,f}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$$

(integral remainder formula).

I'll give a proof of the first two remainder formulas. The idea is to view x as fixed, and $a = t$ as the variable. For each $t \in [a, x]$ we can write

$$f(x) = f(t) + f'(t)(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n + R_{n,t}(x)$$

If we take the derivative of both sides with respect to t, we get

$$0 = f'(t) + \left[-f'(t) + \frac{f''(t)}{1!}(x-t)\right] + \left[-\frac{f''(t)}{1!}(x-t) + \frac{f'''(t)}{2!}(x-t)^2\right]$$
$$+ \cdots + \left[\frac{-f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] + S'(t),$$

where $S(t) = R_{n,t,f}(x)$. Now a miracle happens: almost everything cancels out, and we're left with

$$S'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Now note that when $t = x$ we get $f(x) = f(x) + 0 + \cdots + 0 + S(x)$, so $S(x) = 0$, and $S(a) = R_{n,a,f}(x)$ is our desired remainder. Applying the Mean Value Theorem to $S(t)$ on $[a, x]$ tells us that there is some $t \in (a, x)$ such that

$$\frac{S(x) - S(a)}{x - a} = S'(t) = \frac{-f^{(n+1)}(t)}{n!}(x-t)^n.$$

Substituting $S(x) = 0, S(a) = R_{n,a,f}(x)$ and rearranging gives Cauchy's remainder formula.

Now we prove Lagrange's remainder formula (which ironically enough uses Cauchy's Mean Value Theorem): let $g(t) = (x-t)^{n+1}$. Note that $g(x) = 0$ and $g(a) = (x-a)^{n+1}$. By Cauchy's MVT, there exists some $t \in (a, x)$ such that

$$\frac{S(x) - S(a)}{g(x) - g(a)} = \frac{S'(t)}{g'(t)} = \frac{-\dfrac{f^{(n+1)}(t)}{n!}(x-t)^n}{-(n+1)(x-t)^n},$$

which gives $\dfrac{R_{n,a,f}(x)}{(x-a)^{n+1}} = \dfrac{f^{(n+1)}(t)}{(n+1)!}$, and rearranging gives Lagrange's formula.

Just for fun, let's end with a proof that Euler's constant e is irrational. For any n, we know that

$$e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n, \text{ where }) < R_n < \frac{3}{(n+1)!}.$$

If $e = a/b$ for some positive integers a and b, choose $n > \max\{b, 3\}$. Then

$$\frac{a}{b} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n, \text{ so } \frac{n!a}{b} = n! + n! + \cdots + 1 + n!R_n.$$

Every term in the second equation above is an integer, except possibly $n!R_n$, so it must be an integer as well. But $0 < R_n < 3/(n+1)!$, so

$$0 < n!R_n < \frac{3}{n+1} < \frac{3}{4} < 1,$$

which is impossible for an integer.