MATH 1410 - Tutorial #9 Solutions

1. Let $A = \begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & -5 \\ 1 & 1 & 9 \end{bmatrix}$.

   (a) Compute A^{-1}.

   We form the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$ and proceed to RREF, as follows:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ -2 & 3 & -5 & - & 1 & 0 \\ 1 & 1 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_1\to R_3]{R_2+2R_1\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2\to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & -1 & -5 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & -1 \end{array}\right]$$

$$\xrightarrow[R_2-3R_3\to R_2]{R_1-4R_3\to R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -19 & -8 & 4 \\ 0 & 1 & 0 & -13 & -5 & 3 \\ 0 & 0 & 1 & 5 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_1+R_2\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -32 & -13 & 7 \\ 0 & 1 & 0 & -13 & -5 & 3 \\ 0 & 0 & 1 & 5 & 2 & -1 \end{array}\right]$$

   Since we now have identity matrix on the left, we can conclude that $A^{-1} = \begin{bmatrix} -32 & -13 & 7 \\ -13 & -5 & 3 \\ 5 & 2 & -1 \end{bmatrix}$.

   (b) Solve for X, if $AX + \begin{bmatrix} 2 & -3 \\ -1 & -5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & -3 \\ -2 & 1 \end{bmatrix}$

   Since $AX = \begin{bmatrix} 1 & 0 \\ 4 & -3 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -1 & -5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 5 & 2 \\ -6 & -1 \end{bmatrix}$, we can multiply by A^{-1} (as determined above) to get

$$X = \begin{bmatrix} -32 & -13 & 7 \\ -13 & -5 & 3 \\ 5 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \\ -6 & -1 \end{bmatrix} = \begin{bmatrix} -75 & -129 \\ -30 & -52 \\ 11 & 20 \end{bmatrix}.$$

2. For each equation given below, explain why A is invertible, and determine an expression (in terms of A) for A^{-1}.

   Hint: Recall that A is invertible provided there exists a matrix B such that $AB = I$, in which case $B = A^{-1}$.

   (a) $A^5 = I$

   Since $A(A^4) = A^5 = I$, we can conclude that A is invertible, and $A^{-1} = A^4$.

   (b) $A^4 = 9I$

   We have $\frac{1}{9}A^4 = I$, so $A\left(\frac{1}{9}A^3\right) = \frac{1}{9}A(A^3) = \frac{1}{9}A^4 = I$.
   Therefore, A is invertible, and $A^{-1} = \frac{1}{9}A^3$.

   (c) $A^2 - 5A + 6I = 0$

   Notice that $6I = 5A - A^2$, so

   $$I = \frac{1}{6}(5A - A^2) = \frac{5}{6}A - \frac{1}{6}A^2 = (\frac{5}{6}I - \frac{1}{6}A)A.$$

   Thus, A is invertible, and $A^{-1} = \frac{5}{6}I - \frac{1}{6}A$.
   Note: when factoring an expression such as $5A - A^2$, we must be careful. First instinct suggests that we can write $5A - A^2 = A(5 - A)$, but the expression $5 - A$ is undefined: we cannot subtract a matrix from a number. The correct factorization is $5A - A^2 = A(5I - A)$. Adding the identity matrix ensures that everything is defined, and does not affect the product: $A(5I) = 5(AI) = 5A$.

3. Show that the matrix $A = \begin{bmatrix} -3 & -3 \\ 10 & 8 \end{bmatrix}$ satisfies the equation $A^2 - 5A + 6I = 0$. Using your result from problem 2(c), determine A^{-1}.

   We find $A^2 = \begin{bmatrix} -3 & -3 \\ 10 & 8 \end{bmatrix}\begin{bmatrix} -3 & -3 \\ 10 & 8 \end{bmatrix} = \begin{bmatrix} -21 & -15 \\ 50 & 34 \end{bmatrix}$, so

   $$A^2 - 5A + 6I = \begin{bmatrix} -21 & -15 \\ 50 & 34 \end{bmatrix} + \begin{bmatrix} 15 & 15 \\ -50 & -40 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

   as required. It follows from 2(c) that

   $$A^{-1} = \frac{5}{6}I - \frac{1}{6}A = \begin{bmatrix} 5/6 & 0 \\ 0 & 5/6 \end{bmatrix} + \begin{bmatrix} 1/2 & 1/2 \\ -5/3 & 4/3 \end{bmatrix} = \begin{bmatrix} 4/3 & 1/2 \\ -5/3 & -1/2 \end{bmatrix}.$$

4. For each problem below, assume T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

(a) Given vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$ such that $T(\vec{u}) = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$, determine the value of $T(4\vec{u} - 3\vec{v})$.

$$T(4\vec{u} - 3\vec{v}) = 4T(\vec{u}) - 3T(\vec{v}) = 4\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 1 \end{bmatrix}.$$

(b) Determine the matrix of T, if $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

It is a general fact (see Theorem 35 on p. 266 of the textbook) that the if A is the matrix of T, then the columns of A are given by the values of T on the standard unit vectors. In this case, $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ gives the first column, and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ the second, so

$$A = [T] = \begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 0 & 2 \end{bmatrix}.$$

(c) *For fun*: find the matrix of the transformation T in part(a), if $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Hint: First determine how to write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in terms of $\vec{u}$ and $\vec{v}$.

First, notice that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\vec{u} - \frac{1}{3}\vec{v}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\vec{u} + \frac{2}{3}\vec{v}$. Thus,

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T(\frac{1}{3}\vec{u} - \frac{1}{3}\vec{v}) = \frac{1}{3}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -1 \\ -1/3 \end{bmatrix},$$

and

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) T(\frac{1}{3}\vec{u} + \frac{2}{3}\vec{v}) = \frac{1}{3}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 14/3 \end{bmatrix},$$

so $[T] = \begin{bmatrix} 4/3 & 1/3 \\ -1 & 1 \\ -1/3 & 14/3 \end{bmatrix}$. (Feel free to confirm the values of $T(\vec{u})$ and $T(\vec{v})$ using this matrix.)