MATH 1410 - Tutorial #3 Solutions

Additional practice: (**do not submit**).

1. Given $\vec{v} = \langle 3, -4, 1\rangle$ and $\vec{w} = \langle -2, 1, 5\rangle$, compute:

   (a) $4\vec{v} - 3\vec{w} = \langle 12, -16, 4\rangle + \langle 6, -3, -15\rangle = \langle 18, -19, -11\rangle$

   (b) The vector $\vec{x}$ such that $-3\vec{v} + 5\vec{x} = 2\vec{w}$
   Adding $3\vec{v}$ to both sides, $5\vec{x} = 3\vec{v} + 2\vec{w}$, so

$$\vec{x} = \frac{3}{5}\vec{v} + \frac{2}{5}\vec{w} = \langle \frac{9}{5}, -\frac{12}{5}, \frac{3}{5}\rangle + \langle -\frac{4}{5}, \frac{2}{5}, 2\rangle = \langle 1, -2, \frac{13}{5}\rangle.$$

   (c) $\vec{v} \cdot (3\vec{w})$, $(3\vec{v}) \cdot \vec{w}$, and $3(\vec{v} \cdot \vec{w})$
   All three are equal to

$$3(3(-2) - 4(1) + 1(5)) = 3(-5) = -15.$$

   (d) $\text{proj}_{\vec{v}}\,\vec{w}$ and $\text{proj}_{\vec{w}}\,\vec{v}$

$$\text{proj}_{\vec{v}}\,\vec{w} = \left(\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = -\frac{5}{26}\langle 3, -4, 1\rangle.$$

$$\text{proj}_{\vec{w}}\,\vec{v} = \left(\frac{\vec{w}\cdot\vec{v}}{\vec{w}\cdot\vec{w}}\right)\vec{w} = -\frac{5}{30}\langle -2, 1, 5\rangle.$$

   (e) Vectors $\vec{w}_{\parallel}$ and $\vec{w}_{\perp}$ such that $\vec{w}_{\parallel}$ is parallel to $\vec{v}$, $\vec{w}_{\perp}$ is orthogonal to $\vec{v}$, and $\vec{w}_{\parallel} + \vec{w}_{\perp} = \vec{w}$.
   The vector $\vec{w}_{\parallel}$ is given by $\vec{w}_{\parallel} = \text{proj}_{\vec{v}}\,\vec{w} = -\frac{5}{26}\langle 3, -4, 1\rangle$.
   The vector $\vec{w}_{\perp}$ is given by

$$\vec{w}_{\perp} = \vec{w} - \vec{w}_{\parallel} = \langle -2, 1, 5\rangle - \left\langle \frac{-15}{26}, \frac{20}{26}, -\frac{5}{26}\right\rangle = \left\langle -\frac{37}{26}, \frac{6}{26}, \frac{135}{26}\right\rangle.$$

Assigned problems:

1. Let $\vec{v} = \langle 4, 3\rangle$ and $\vec{w} = \langle 2, 3\rangle$.

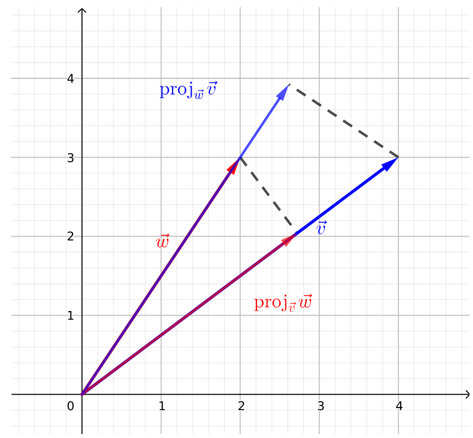
   (a) Compute $\text{proj}_{\vec{v}}\,\vec{w}$ and $\text{proj}_{\vec{w}}\,\vec{v}$.

   (b) Sketch $\vec{v}$, $\vec{w}$, $\text{proj}_{\vec{v}}\,\vec{w}$, and $\text{proj}_{\vec{w}}\,\vec{v}$ on one set of coordinate axes.

$$\begin{aligned}
\text{proj}_{\vec{v}}\,\vec{w} &= \left(\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\right)\vec{v} \\
&= \frac{8+9}{16+9}\langle 4, 3\rangle \\
&= \frac{17}{25}\langle 4, 3\rangle = \left\langle \frac{68}{25}, \frac{51}{25}\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\text{proj}_{\vec{w}}\,\vec{v} &= \left(\frac{\vec{w}\cdot\vec{v}}{\vec{w}\cdot\vec{w}}\right)\vec{w} \\
&= \frac{9+8}{4+9}\langle 2, 3\rangle \\
&= \frac{17}{13}\langle 2, 3\rangle = \left\langle \frac{34}{13}, \frac{51}{13}\right\rangle.
\end{aligned}$$



2. Show that for **any** vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^2$,

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}.$$

Then, illustrate the result with an example.

Let $\vec{u} = \langle u_1, u_2\rangle$, $\vec{v} = \langle v_1, v_2\rangle$, and $\vec{w} = \langle w_1, w_2\rangle$ for some real numbers $u_1, u_2, v_1, v_2, w_1, w_2$. Then

$$\begin{aligned}
\vec{u}\cdot(\vec{v}+\vec{w}) &= \langle u_1, u_2\rangle \cdot (\langle v_1, v_2\rangle + \langle w_1, w_2\rangle) && \text{(subsituting expressions)} \\
&= \langle u_1, u_2\rangle \cdot \langle v_1 + w_1, v_2 + w_2\rangle && \text{(definition of vector addition)} \\
&= u_1(v_1 + w_1) + u_2(v_2 + w_2) && \text{(definition of dot product)} \\
&= (u_1v_1 + u_1w_1) + (u_2v_2 + u_2w_2) && \text{(distributive property of } \mathbb{R}) \\
&= (u_1v_1 + u_2v_2) + (u_1w_1 + u_2w_2) && \text{(changing order of addition)} \\
&= \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}, && \text{(definition of dot product)}
\end{aligned}$$

as required.

For example, if $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle 3, -1\rangle$, and $\vec{w} = \langle -2, 4\rangle$, then

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \langle 1, 2\rangle \cdot (\langle 3, -1\rangle + \langle -2, 4\rangle) = \langle 1, 2\rangle \cdot \langle 1, 3\rangle = 1 + 6 = 7,$$

while

$$\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} = \langle 1, 2\rangle \cdot \langle 3, -1\rangle + \langle 1, 2\rangle \cdot \langle -2, 4\rangle = (3-2) + (-2+8) = 7,$$

so the two sides agree, as they must.