Math 1410 Assignment #1 Solutions
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1. You empty the change in your pockets to discover pennies, nickels, and dimes totalling \$1.05. If there are 17 coins in total, how many of each coin do you have?

   Let x be the number of pennies, y the number of nickels, and z the number of dimes. We'll assume that we discovered at least one of each coin, so $x, y, z \geq 1$. (Otherwise there are lots of options; for example, 105 pennies, 21 nickels, 10 dimes and a nickel, etc.) Since there are 17 coins in total,

   $$x + y + z = 17.$$

   Since the total value is \$1.05, or 105 cents, we have

   $$x + 5y + 10z = 105.$$

   Using an augmented matrix to solve the system, we have

   $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 17 \\ 1 & 5 & 10 & 105 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 17 \\ 0 & 4 & 9 & 88 \end{array}\right]$$
   $$\xrightarrow{R_2 \to \frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 17 \\ 0 & 1 & \frac{9}{4} & 22 \end{array}\right]$$
   $$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{4} & -5 \\ 0 & 1 & \frac{9}{4} & 22 \end{array}\right]$$

   This leaves us with the solution

   $$\begin{aligned} x &= -5 + \frac{5}{4}t \\ y &= 22 - \frac{9}{4}t \\ z &= t \end{aligned}$$

   with $z = t$, the number of dimes, as a parameter. However, we can't take t to be any real number, since x, y, z are positive integers. Since $y = 22 - \frac{9}{4}t$, we see that t must be a multiple of 4, or else y would have a fractional value. This gives us the possibilities

$z = t = 4$ and $y = 22 - 9 = 13$, or $z = t = 8$ and $y = 22 - 18 = 4$. (If $t = 12$ or more, y would become negative.)

Now looking at $x = -5 + \frac{5}{4}t$, we see that $t = 4$ would give $x = 0$, so if we want at least one penny, then we have to take $t = 8$, giving us $x = -5 + 10 = 5$. Thus, we can conclude that there are 5 pennies, 4 nickels, and 8 dimes.

2. We know that every homogeneous system of linear equations has a solution (the trivial solution). The main theorem on homogeneous systems states the following:

> If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution. (In fact, it will have infinitely many solutions.)

Using this theorem,

(a) Show that there is a line through any pair of points (x_1, y_1) and (x_2, y_2) in the plane.

Let the two points be (x_1, y_1) and (x_2, y_2). In order for us to have a line through these points, we have to be able to find real numbers a, b, c (not all zero) such that

$$\begin{aligned} ax_1 + by_1 + c = 0 &\text{ and} \\ ax_2 + by_2 + c = 0&. \end{aligned}$$

Since this is a homogeneous system of two equations in the three variables a, b and c, we know that there must exist a nontrivial solution, since we have three variables, and the rank of the augmented matrix

$$\left[\begin{array}{ccc|c} x_1 & y_1 & 1 & 0 \\ x_2 & y_2 & 1 & 0 \end{array}\right]$$

is at most two, so there is at least $3 - 2 = 1$ parameter.

(b) Show that there is a plane through any three points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) in space.

The argument is identical to the one given in part (a). The three points give us a system of three equations

$$\begin{aligned} ax_1 + by_1 + cz_1 + d &= 0 \\ ax_2 + by_2 + cz_2 + d &= 0 \\ ax_3 + by_3 + cz_3 + d &= 0 \end{aligned}$$

in the variables a, b, c, d which determine the equation of the plane. Since this is a homogeneous system with more variables than equations, we know that a nontrivial solutions is guaranteed, and thus we can find an equation of the plane through the given three points.

3. Let $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$. Show that if

$$rA + sB + tC = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

then we must have $r = s = t = 0$.

Using the rules for scalar multiplication and addition of matrices, we have

$$\begin{aligned} rA + sB + tC &= r\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} + s\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} + t\begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} r & r & -r \end{bmatrix} + \begin{bmatrix} 0 & s & 2s \end{bmatrix} + \begin{bmatrix} 3t & 0 & t \end{bmatrix} \\ &= \begin{bmatrix} r+3t & r+s & -r+2s+t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Since two matrices are equal if and only if each corresponding entry is equal, we obtain the system of equations

$$\begin{array}{rcrcrcl} r & & & + & 3t & = & 0 \\ r & + & s & & & = & 0 \\ -r & + & 2s & + & t & = & 0 \end{array}$$

Reducing the corresponding augmented matrix to row-echelon form, we have

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{10}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

and using back substitution we can conclude that the only solution to our system of equations is $r = 0$, $s = 0$, and $t = 0$, as required.

4. In each of the following, either explain why the statement is true, or give an example showing that it is false:

   (a) If A is an $m \times n$ matrix where $m < n$, then $AX = B$ has a solution for every column B.

   This is false. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and the column $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We see that A is 2×3, and $2 < 3$, but there is no X such that $AX = B$, since for $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we would have to have

$$\begin{aligned} &x_1 + x_2 + x_3 = 0 \text{ and} \\ &x_1 + x_2 + x_3 = 1, \end{aligned}$$

but this is impossible, since we cannot have $x_1 + x_2 + x_3$ equal to 0 and 1 simultaneously.

(b) If $AX = B$ has a solution for some column B, then it has a solution for every column B.

This is false. For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then it's easy to check that $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ provides a solution. However, if $B = \begin{bmatrix} 0 & 1 \end{bmatrix}$, then no solution is possible, using the same reasoning as in part (a).

(c) If X_1 and X_2 are solutions to $AX = B$, then $X_1 - X_2$ is a solution to $AX = 0$.

This is true. Suppose that $AX_1 = B$ and $AX_2 = B$. Then we have

$$A(X_1 - X_2) = AX_1 - AX_2 = B - B = 0.$$

(d) If $AB = AC$ and $A \neq 0$, then $B = C$.

This is false in general. (We can only conclude $B = C$ if A is invertible.) Using an example from class, if $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}$, then $B \neq C$, but $AC = BC = \begin{bmatrix} 5 & 7 \\ 10 & 14 \end{bmatrix}$.

(e) If $A \neq 0$, then $A^2 \neq 0$.

This is false in general. For example, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A \neq 0$, but

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$