Solutions to Quiz 9 Practice Problems
Math 2580
Spring 2016

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February 9th, 2016

1. Calculate the partial derivatives of the function $f(x,y,z) = \cos(xy^2) + e^{3xyz}$ and $g(x,y,z) = x^{yz}$. (Be careful with the second one – what are you treating as a constant for each derivative? Should you be thinking of a power function or an exponential function?)

   For the first function, we have

   $$\begin{aligned} f_x(x,y,z) &= -y^2\sin(xy^2) + 3yze^{3xyz} \\ f_y(x,y,z) &= -2xy\sin(xy^2) + 3xze^{3xyz} \\ f_z(x,y,z) &= 3xye^{3xyz}. \end{aligned}$$

   For the second function, we have

   $$\begin{aligned} g_x(x,y,z) &= yzx^{yz-1} \\ g_y(x,y,z) &= zx^{yz}\ln x \\ g_z(x,y,z) &= yx^{yz}\ln x. \end{aligned}$$

2. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$ does not exist.

   If we let $(x,y) \to (0,0)$ along the x-axis, $y = 0$, and we get $\dfrac{x}{x+y} = \dfrac{x}{x} = 1$, for all $x \neq 0$. It follows that we get a limit of 1 as we approach along the x-axis. However, if we let $(x,y) \to (0,0)$ along the y-axis, then $x = 0$ and we get $\dfrac{x}{x+y} = 0$ for all $y \neq 0$, and thus, the limit is 0 as we approach along the y-axis. Since we get two different values along different paths, the limit does not exist.

3. Find the equation of the tangent plane to the graph $z = xy^2 - 3x^2 + 4xy$ at the point $(2,1,-2)$.

Letting $f(x,y) = xy^2 - 3x^2 + 4xy$, we have $f_x(x,y) = y^2 - 6x + 4y$ and $f_y(x,y) = 2xy + 4x$. If $x = 2$ and $y = 1$, this gives us $f_x(2,1) = -7$ and $f_y(2,1) = 12$. (Note also that $f(2,1) = -2$, so the point $(2,1,-2)$ is indeed on the graph. The equation of the tangent plane is therefore

$$z = -2 - 7(x-2) + 12(y-1).$$

4. Calculate $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ if $z = x^2 - 3xy - y^2$, where $x = 2u + 3v$ and $y = 3u - v$,

   (a) Using the Chain Rule (either via matrix multiplication or just writing out the patterns).

   If we let $g(u,v) = (2u + 3v, 3u - v)$, then

$$D_{(u,v)}g = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix},$$

   and if $f(x,y) = x^2 - 3xy - y^2$, then $D_{(x,y)}f = \begin{bmatrix} f_x(x,y) & f_y(x,y) \end{bmatrix} = \begin{bmatrix} 2x - 3y & -3x - 2y \end{bmatrix}$, so

$$D_{g(u,v)}f = \begin{bmatrix} 2(2u+3v) - 3(3u-v) & -3(2u+3v) - 2(3u-v) \end{bmatrix} = \begin{bmatrix} -5u + 9v & -12u - 7v \end{bmatrix}.$$

   The chain rule then gives us

$$\begin{aligned} \begin{bmatrix} \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{bmatrix} &= D_{(u,v)}(f \circ g) = D_{g(u,v)}f \cdot D_{(u,v)}g \\ &= \begin{bmatrix} -5u + 9v & -12u - 7v \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} (-5u+9v)(2) + (-12u-7v)(3) & (-5u+9v)(3) + (-12u-7v)(-1) \end{bmatrix} \\ &= \begin{bmatrix} -46u - 3v & -3u + 34v \end{bmatrix}. \end{aligned}$$

   Comparing coefficients of the first and last matrices, we have

$$\frac{\partial z}{\partial u} = -46u - 3v \quad \text{and} \quad \frac{\partial z}{\partial v} = -3u + 34v.$$

   (b) By first substituting the expressions for x and y in terms of u and v into the equation defining z.

   Letting $f(x,y) = x^2 - 3xy - y^2$, we have

$$\begin{aligned} z = f(2u+3v, 3u-v) &= (2u+3v)^2 - 3(2u+3v)(3u-v) - (3u-v)^2 \\ &= 4u^2 + 12uv + 9v^2 - 18u^2 - 21uv + 9v^2 - 9u^2 + 6uv - v^2 \\ &= -23u^2 - 3uv + 17v^2. \end{aligned}$$

   Thus, we have $\dfrac{\partial z}{\partial u} = -46u - 3v$, and $\dfrac{\partial z}{\partial v} = -3u + 34v$.

5. Let $f(x, y) = x^2 + y^2 - 3xy^3$. Compute

   (a) The gradient of f at the point $(a, b) = (1, 2)$.

   By definition, $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \langle 2x - 3y^3, 2y - 9xy^2\rangle$, so $\nabla f(1, 2) = \langle -22, -32\rangle$.

   (b) The directional derivative of f in the direction of $\vec{v} = \langle 1/2, \sqrt{3}/2\rangle$.

   By definition, $d_{\vec{v}} f(1, 2) = \nabla f(1, 2) \cdot \vec{v} = \langle -22, -32\rangle \cdot \langle 1/2, \sqrt{3}/2\rangle = -11 - 16\sqrt{3}$.

6. Find the equation of the tangent plane to the surface $xyz^2 = 4$ at the point $(1, 1, 2)$.

   Since we're dealing with a level surface $g(x, y, z) = 4$, with $g(x, y, z) = xyz^2$, we know that the normal vector is given by the gradient. We have

   $$\nabla g(x, y, z) = \langle yz^2, xz^2, 2xyz\rangle, \quad \text{so} \quad \nabla g(1, 1, 2) = \langle 4, 4, 4\rangle.$$

   The equation of the tangent plane is thus $4(x - 1) + 4(y - 1) + 4(z - 2) = 0$, or $x + y + z = 4$.

7. Find and classify the critical points of the function $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$. (You should find 4 critical points.)

   The gradient of f is given by $\nabla f(x, y) = \langle 6xy - 6x, 3x^2 + 3y^2 - 6y\rangle$. Any critical points occur when $\nabla f(x, y) = \langle 0, 0\rangle$, so we must have

   $$6xy - 6x = 6x(y - 1) = 0 \quad \text{and}$$
   $$3x^2 + 3y^2 - 6y = 0.$$

   The first equation tells that either $x = 0$ or $y = 1$. If $x = 0$, the second equation gives us $3y^2 - 6y = 3y(y - 2) = 0$, so $y = 0$ or $y = 2$. This gives us two critical points: $(0, 0)$ and $(0, 2)$. If $y = 1$, the second equation gives us $3x^2 - 3 = 0$, so $x^2 = 1$, giving us $x = \pm 1$, and two more critical points: $(1, 1)$ and $(-1, 1)$.

   To classify the critical points we compute the second derivatives of f. We have

   $$f_{xx}(x, y) = 6y - 6, \quad f_{xy}(x, y) = 6x = f_{yx}(x, y), \quad f_{yy}(x, y) = 6y - 6.$$

   (Oddly enough, we have $f_{xx}(x, y) = f_{yy}(x, y)$, which isn't usually the case, but it will make our lives easier).

   At $(0, 0)$ we have $A = f_{xx}(0, 0) = -6 = f_{yy}(0, 0) = C$ and $B = f_{xy}(0, 0) = 0$, so $D = AC - B^2 = 36$. Since $A < 0$ and $D > 0$, $(0, 0)$ is a local maximum.

   At $(0, 2)$ we have $A = f_{xx}(0, 2) = 6 = f_{yy}(0, 2) = C$ and $B = f_{xy}(0, 0) = 0$, so $D = AC - B^2 = 36$. Since $A > 0$ and $D > 0$, $(0, 2)$ is a local minimum.

   At $(1, 1)$ we have $A = f_{xx}(1, 1) = 0 = f_{yy}(1, 1) = C$ and $B = f_{xy}(1, 1) = 6$, so $D = AC - B^2 = -36$. Since $D < 0$, $(1, 1)$ is a saddle point.

   At $(-1, 1)$ we have $A = f_{xx}(-1, 1) = 0 = f_{yy}(-1, 1) = C$ and $B = f_{xy}(-1, 1) = -6$, so $D = AC - B^2 = -36$. Since $D < 0$, $(-1, 1)$ is a saddle point.