University of Lethbridge
Department of Mathematics and Computer Science
13^{th} February, 2015, 3:00 - 3:50 pm

MATH 3410 - Test #1

Last Name: **Solutions**

First Name: **The**

Student Number:

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

Page	Grade
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3	/8
4	/10
Total	/30

1. True/False: For each of the statements below, state whether it is true or false, and give a **brief** explanation supporting your choice.

[3] (a) The set $U = \{(x, y, xy) \mid x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

False.

For example, $(1, 1, 1) \in U$, but $2(1, 1, 1) = (2, 2, 2) \notin U$, since $2 \cdot 2 = 4 \neq 2$. Thus, U is not closed under scalar multiplication, and therefore cannot be a subspace.

[3] (b) If a vector space V can be written as a direct sum $V = U \oplus W$, and for some $v \in V$ we have $v \notin U$, then $v \in W$.

False.

For example, take $V = \mathbb{R}^2$, $U = \text{span}\{(1, 0)\}$, and $W = \text{span}\{(0, 1)\}$. Then the vector $v = (1, 1)$ belongs to neither U nor W. (In general, any vector of the form $v = u + w$, where $u \in U$ and $w \in W$ are nonzero vectors, will do the job.)

[3] (c) For any subspace $U \subseteq V$, where V is finite-dimensional, there exists a subspace $W \subseteq V$ such that $V = U \oplus W$.

True.

Let $B_U = \{u_1, \ldots, u_k\}$ be any basis for U. As we know from class, B_U can be extended to a basis $B_V = \{u_1, \ldots, u_k, w_1, \ldots, w_m\}$, and letting $W = \text{span}\{w_1, \ldots, w_m\}$ provides the desired subspace.

[3] (d) If $T : V \to W$ is a linear transformation, and we know $\dim V = 4$ and $\dim W = 3$, then T cannot be one-to-one.

True.

Since $\operatorname{range} T \subseteq W$, we know that $\dim \operatorname{range} T \leq \dim W = 3$. Thus,

$$\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T \geq 4 - 3 = 1.$$

This shows that $\operatorname{null} T \neq \{0\}$, and therefore, T cannot be one-to-one.

Please provide a solution to **one** of the two problems on this page:

[8] 2. Suppose that the vectors v_1, v_2, v_3, v_4 form a basis for V. Prove that the vectors

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

also form a basis for V.

Solution: Since $\{v_1, v_2, v_3, v_4\}$ is a basis for V, we know that $\dim V = 4$. Since we are given four vectors, it suffices to show that *either* they're linearly indepenent, *or* they span.

To see that the vectors $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ are linearly indepenent, suppose that we have

$$c_1(v_1 + v_2) + c_2(v_2 + v_3) + c_3(v_3 + v_4) + c_4 v_4 = 0$$

for some scalars c_1, c_2, c_3, c_4. Then we have

$$0 = c_1 v_1 + (c_1 + c_2)v_2 + (c_2 + c_3)v_3 + (c_3 + c_4)v_4,$$

and since the vectors v_1, v_2, v_3, v_4 are linearly independent, we have

$$c_1 = 0, c_1 + c_2 = 0, c_2 + c_3 = 0, c_3 + c_4 = 0.$$

But putting $c_1 = 0$ into the second equation gives $c_2 = 0$, which in turn gives $c_3 = 0$ in the third equation, and then $c_4 = 0$ in the fourth equation. Thus, we've shown that the four vectors $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ are linearly independent, and therefore form a basis for V.

At this point you're done, but if you want to also show that the vectors $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ span V (or you did both) then note that we can write

$$\begin{aligned}
v_1 &= 1(v_1 + v_2) + (-1)(v_2 + v_3) + (1)(v_3 + v_4) + (-1)v_4 \\
v_2 &= 1(v_2 + v_3) + (-1)(v_3 + v_4) + (1)v_4 \\
v_3 &= 1(v_3 + v_4) + (-1)v_4 \\
v_4 &= v_4.
\end{aligned}$$

Since any vector in V can be written in terms of the vectors v_1, v_2, v_3, v_4, since these vectors form a basis and therefore span V, and each of these four basis vectors can be written in terms of the vectors $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$, it follows that these latter vectors span V. To see this directly, note that any $v \in V$ can be written as

$$\begin{aligned}
v &= c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 \\
&= c_1[(v_1 + v_2) + (-1)(v_2 + v_3) + (1)(v_3 + v_4) + (-1)v_4] \\
&\qquad + c_2[(v_2 + v_3) + (-1)(v_3 + v_4) + (1)v_4] \\
&\qquad\qquad + c_3[(v_3 + v_4) + (-1)v_4] + c_4 v_4 \\
&= c_1(v_1 + v_2) + (c_2 - c_1)(v_2 + v_3) + (c_3 - c_2 + c_1)(v_3 + v_4) + (c_4 - c_3 + c_2 - c_1)v_4
\end{aligned}$$

for some scalars c_1, c_2, c_3, c_4.

3. Determine whether or not the vector $v = (1, 3, -4)$ belongs to the span of the vectors
[8] $(2, 0, 1), (0, 3, -4)$, and $(4, -3, 9)$.

Solution: We need to determine whether or not there exist scalars $x, y, z \in \mathbb{R}$ such that

$$(1, 3, -4) = x(2, 0, 1) + y(0, 3, -4) + z(4, -3, 9) = (2x + 4z, 3y - 3z, x - 4y + 9z).$$

Since two vectors in $\mathbb{R}^3$ are equal if and only if their components are all equal, this yields the system of linear equations

$$\begin{array}{rcrcrcr} 2x & & & + & 4y & = & 1 \\ & & 3y & - & 3z & = & 3 \\ x & - & 4y & + & 9z & = & -4 \end{array}$$

We solve the system by setting up an augmented matrix and reducing, using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 1 \\ 0 & 3 & -3 & 3 \\ 1 & -4 & 9 & -4 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{3}R_2]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -4 & 9 & -4 \\ 0 & 1 & -1 & 1 \\ 2 & 0 & 4 & 1 \end{array}\right]$$
$$\xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -4 & 9 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 8 & -14 & 9 \end{array}\right]$$
$$\xrightarrow[R_3 \to R_3 - 8R_2]{R_1 \to R_1 + 4R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -6 & 1 \end{array}\right].$$

Since the rank of the coefficient matrix is equal to 3, which is also the number of variables, we know that we have a unique solution, which we can read off using back-substitution. The third row gives us $z = -1/6$; the second row yields $y - z = 1$, and plugging in $z = -1/6$ gives $y = 5/6$. Finally, the first row tells us $x + 5z = 0$, and substituting $z = -1/6$ gives us $x = 5/6$.

Thus, we have shown that

$$(1, 3, -4) = \frac{5}{6}(2, 0, 1) + \frac{5}{6}(0, 3, -4) - \frac{1}{6}(4, -3, 9),$$

which shows that $(1, 3, -4)$ is in the span of the given vectors.

Please provide a solution to **one** of the two problems on this page:

4. Suppose $T : V \to W$ is injective, and the vectors $v_1, \ldots, v_n$ are linearly independent in V.
[10] Prove that the vectors $Tv_1, \ldots, Tv_n$ are linearly independent in W.

Solution: Let $T : V \to W$ be an injective linear transformation, and let $v_1, \ldots, v_n$ be linearly independent vectors in V. Suppose that we have

$$c_1Tv_1 + c_2Tv_2 + \cdots + c_nTv_n = 0$$

for some scalars $c_1, c_2, \ldots, c_n$. It follows that

$$0 = c_1Tv_1 + c_2Tv_2 + \cdots + c_nTv_n = T(c_1v_1 + c_2v_2 + \cdots + c_nv_n),$$

which implies that $c_1v_1 + c_2v_2 + \cdots + c_nv_n \in \operatorname{null} T$. Since T is injective, $\operatorname{null} T = \{0\}$, and thus $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$. Since the vectors $v_1, v_2, \ldots, v_n$ are linearly independent, we must have $c_1 = c_2 = \cdots = c_n = 0$ as the only solution. Therefore, the vectors $Tv_1, Tv_2, \ldots, Tv_n$ are linearly independent.

5. Let $V = \mathbb{R}^{3,1} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$, and let $T : V \to V$ be the linear transformation

[10] given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \\ 4 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - y + 3z \\ -x + 4z \\ 4x - y - 5z \end{bmatrix}.$$

Determine the null space and range of T.

Solution: For both the null space and range, it therefore suffices to consider the system of equations

$$\begin{array}{rcrcrcl} 2x & - & y & + & 3z & = & a \\ -x & & & + & 4z & = & b \\ 4x & - & y & - & 5z & = & c \end{array}.$$

The null space corresponds to all solutions of the system with $a = b = c = 0$, while the range consists of all values of a, b, and c for which a solution exists. Reducing the corresponding augmented matrix, we find

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & a \\ -1 & 0 & 4 & b \\ 4 & -1 & -5 & c \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} -1 & 0 & 4 & b \\ 2 & -1 & 3 & a \\ 4 & -1 & -5 & c \end{array}\right] \xrightarrow[R_3 \to R_3 + 4R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} -1 & 0 & 4 & b \\ 0 & -1 & 11 & a + 2b \\ 0 & -1 & 11 & c + 4b \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to -R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -b \\ 0 & -1 & 11 & a + 2b \\ 0 & 0 & 0 & (c + 4b) - (a + 2b) \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -b \\ 0 & 1 & -11 & a + 2b \\ 0 & 0 & 0 & -a + 2b + c \end{array}\right].$$

From this, we see that to have $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ we must have $x - 4z = 0$ and $y - 11z = 0$.

Thus,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \operatorname{null} T \quad \Leftrightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4z \\ 11z \\ z \end{bmatrix} = z \begin{bmatrix} 4 \\ 11 \\ 1 \end{bmatrix},$$

so that $\operatorname{null} T = \operatorname{span} \left\{ \begin{bmatrix} 4 \\ 11 \\ 1 \end{bmatrix} \right\}$.

We also see that in order for the system to be consistent, we must have $-a + 2b + c = 0$, or $c = a - 2b$. Thus, the range of T consists of all vectors of the form

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ a - 2b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},$$

so that $\operatorname{range} T = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$.