A triple integral example

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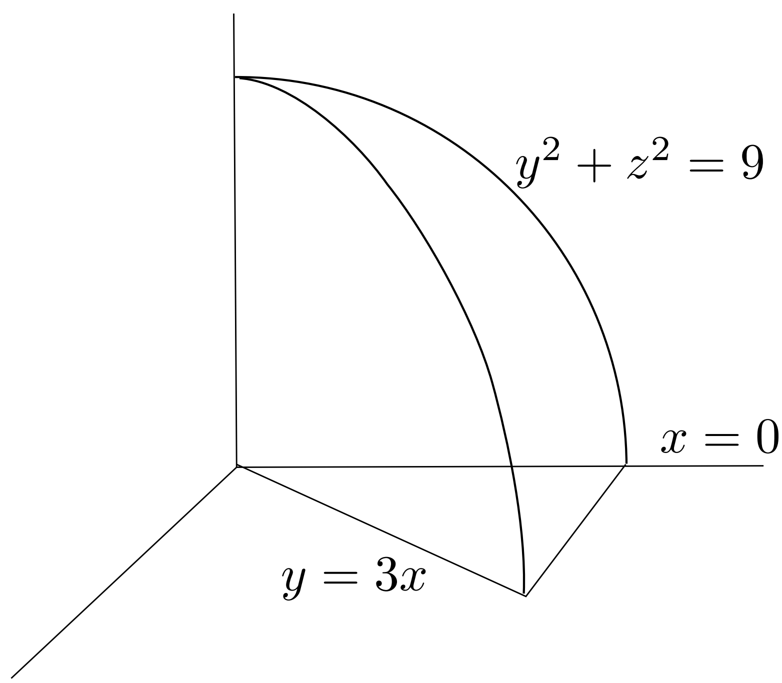
Since my first example from today's class was mostly effective in demonstrating the difficulty of setting up a triple integral, and less so on how to actually get it done, here it is again: we wish to evaluate the integral

$$\iiint_E z\,dV,$$

where E is the region in the first octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $y=3x$, and $z=0$. The sketch of the region is as follows:



Note that the region of integration projects to a region in the xy-plane bounded by $x=0$ and $y=3x$. To close off this region, we note that since we must have $y^2+z^2\leq 9$, it follows that $y\leq 3$, so we have the triangle bounded by $x=0$, $y=3x$, and $y=3$. The region of integration then lies between the plane $z=0$ and the cylinder $z=\sqrt{9-y^2}$, and above our triangle in the xy-plane, which is given by the inequalities $0 \leq x \leq 1$ and $3x \leq y \leq 3$. The iterated integral is therefore

$$\begin{aligned}
\iiint_E z\, dV &= \int_0^1 \int_0^{3x} \int_0^{\sqrt{9-y^2}} z\, dz\, dy\, dx \\
&= \int_0^1 \int_{3x}^3 \frac{1}{2}(9-y^2)\, dy\, dz \\
&= \int_0^1 \frac{1}{2}\left[9y - \frac{1}{3}y^3\right]_{3x}^3 dx \\
&= \int_0^1 \left(\frac{27}{2} - \frac{9}{2} - \frac{27}{2}x + \frac{9}{2}x^3\right) dx \\
&= \frac{9}{8}x^4 - \frac{27}{4}x^2 + \frac{18}{2}x\bigg|_0^1 \\
&= 27/4.
\end{aligned}$$

If we wanted to treat our triangle as a Type II region, given by $0 \leq y \leq 3$ and $0 \leq x \leq y/3$, we would instead have

$$\iiint_E z\, dV = \int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z\, dz\, dx\, dy,$$

and you can check that the result is the same. The other orders of integration are less natural, but still possible. If we wanted to integrate first with respect to x, we would note that $0 \leq x \leq y/3$, with $y^2 + z^2 \leq 9$ giving our region of integration in the yz-plane. If we wrote this region as $0 \leq y \leq \sqrt{9-z^2}$ with $0 \leq z \leq 3$, we would have the integral

$$\iiint_E z\, dV = \int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{y/3} z\, dx\, dy\, dz,$$

with a similar integral if we reversed the order of y and z. Finally, if we wanted to integrate first with respect to y, we would have to note that for a given x and z, y runs from $y = 3x$ to $y = \sqrt{9-z^2}$. To determine the region of integration in the xz-plane, we note that when the cylinder $y^2 + z^2 = 9$ intersects the line $y = 3x$, we have $(3x)^2 + z^2 = 9$, or $9x^2 + z^2 = 9$. Our region is thus bounded by $x = 0$, $z = 0$ and the ellipse $9x^2 + z^2 = 9$. If we write this as $0 \leq x \leq 1$ with $0 \leq z \leq 3\sqrt{1-x^2}$, then we have

$$\iiint_E z\, dV = \int_0^1 \int_0^{3\sqrt{1-x^2}} \int_{3x}^{\sqrt{9-z^2}} z\, dy\, dz\, dx,$$

with a similar integral for the order $dy\, dx\, dz$.