MATH 1410 - Tutorial #6 Solutions

Assigned problems:

1. For each system of equations below, write down the corresponding augmented matrix.

(a)
$$\begin{aligned} 2x \; - 3y + \; z &= 2 \\ 2y - 5z &= -3 \\ -3x \qquad + 2x &= 7 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 2 \\ 0 & 2 & -5 & -3 \\ -3 & 0 & 2 & 7 \end{array}\right]$$

(b)
$$\begin{aligned} x_1 \; + 4x_2 \qquad - 7x_4 &= 0 \\ -3x_1 - \; x_2 + 4x_3 \qquad &= 2 \\ 2x_2 - 4x_3 + \; x_4 &= -5 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & -7 & 0 \\ -3 & -1 & 4 & 0 & 2 \\ 0 & 2 & -4 & 1 & -5 \end{array}\right]$$

2. For each augmented matrix below, write down a corresponding system of equations using whatever variables you prefer.

(a) $\left[\begin{array}{ccc|c} 2 & -1 & 0 & 4 \\ -3 & 4 & 1 & -2 \\ 0 & 2 & 3 & -7 \end{array}\right]$

$$\begin{aligned} 2x \; - \; y \qquad &= 4 \\ -3x + 4y + \; z &= -2 \\ 2y + 3z &= -7 \end{aligned}$$

(b) $\left[\begin{array}{cccc|c} 3 & 2 & 0 & 1 & -5 \\ 0 & 4 & 2 & -7 & 2 \end{array}\right]$

$$\begin{aligned} 3x_1 + 2x_2 \qquad + \; x_4 &= -5 \\ 4x_2 + 2x_3 - 7x_4 &= 2 \end{aligned}$$

3. State whether or not the given augmented matrix is in reduced row-echelon form (RREF), and if not, why.

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

RREF

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right]$

Not RREF: the 4 in row 3 should be a 1.

$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Not RREF due to the 2 in row 1 above the leading 1 in row 2.

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Not in RREF: the 2 in row 2 needs to be a 1.

$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

Not in RREF: There are two non-zero entries above the leading 1 in row 3.

4. Suppose you want to perform Gaussian elimination on the augmented matrices below. For each matrix, what are the first two row operations you would perform, and why?

   (a) $\left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ -2 & 4 & 1 & 6 \\ 3 & 2 & -1 & 1 \end{array}\right]$

   $R_2 + 2R_1 \to R_2$ and $R_3 - 3R_1 \to R_3$, to create zeros in the first column below the leading 1.

   (b) $\left[\begin{array}{ccc|c} 2 & 4 & -8 & 10 \\ -1 & 2 & 4 & -5 \\ 0 & 1 & 5 & 2 \end{array}\right]$

   There are several reasonable options here. One is $\frac{1}{2}R_1 \to R_1$ to get a leading one in the first row, then $R_2 + R_1 \to R_2$ to create a zero below it. Another option would be $R_1 \leftrightarrow R_2$, followed by $-R_1 \to R_1$, to get a leading one with minimal arithmetic. Another would be $R_1 + R_2 \to R_1$ to create a leading one in the first row, followed by $R_2 - R_1 \to R_2$ to create a zero below it.

   (c) $\left[\begin{array}{ccc|c} 3 & 2 & -7 & 4 \\ 1 & 2 & -4 & 0 \\ 0 & -1 & 3 & 2 \end{array}\right]$

   $R_1 \leftrightarrow R_3$, to get a leading 1 in the first row without creating fractions, then $R_3 - 3R_1 \to R_3$ to create a zero below the leading 1.

5. For each matrix A and B below, write down the row operation that transforms A into B.

   (a) $A = \begin{bmatrix} 3 & -2 & 5 \\ 2 & 8 & -4 \\ 1 & -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$ $\qquad \frac{1}{2}R_2 \to R_2$

   (b) $A = \begin{bmatrix} 2 & 7 & -3 \\ 6 & 8 & 1 \\ 1 & 12 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 7 & -3 \\ 0 & -13 & 10 \\ 1 & 12 & -6 \end{bmatrix}$ $\qquad R_2 - 3R_1 \to R_2$

   (c) $A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 4 \\ -5 & 6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 6 & 0 \\ 1 & 3 & 4 \\ 4 & -2 & 3 \end{bmatrix}$ $\qquad R_1 \leftrightarrow R_3$

6. Write down the augmented matrix of the following system, and then use Gaussian elimination to solve the system.

$$\begin{aligned} x \; + 2y - \; z &= \;\; 4 \\ -x + \; y \; - 2z &= -1 \\ 2x \; + 6y - 3z &= \;\; 5 \end{aligned}$$

We have the following augmented matrix and elimination steps. We begin by eliminating all the non-zero entries below our first leading one.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ -1 & 1 & -2 & -1 \\ 2 & 6 & -3 & 5 \end{array}\right] \xrightarrow{R_2+R_1\to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 3 & -3 & 3 \\ 2 & 6 & -3 & 5 \end{array}\right] \xrightarrow{R_3-2R_1\to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 3 & -3 & 3 \\ 0 & 2 & -1 & -3 \end{array}\right]$$

Next, we can get our second leading one by dividing by 3 in the second row. We can then use that leading one to eliminate the non-zero entry below it:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 3 & -3 & 3 \\ 0 & 2 & -1 & -3 \end{array}\right] \xrightarrow{\frac{1}{3}R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & -3 \end{array}\right] \xrightarrow{R_3-2R_2\to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

At this point we've reached row-echelon form, and we have the option of solving by back-substitution. Row 3 tells us that $z = -5$. Row 2 says $y - z = 1$. Putting $z = -5$ into this equation, we get $y + 5 = 1$, so $y = -4$. row 1 says $x + 2y - z = 4$. Putting $y = -4$ and $z = -5$, we get $x - 8 + 5 = 4$, so $x = 7$.

Alternatively, we can continue with the augmented matrix, performing the "backward steps" to reach RREF:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -5 \end{array}\right] \xrightarrow{R_2+R_3\to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{array}\right] \xrightarrow{R_1+R_3\to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{array}\right] \xrightarrow{R_1-2R_2\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

From here we can directly read off the solution $x = 7, y = -4, z = -5$.

Of course, we can also confirm that our solution works by plugging these values into each of our original equations: $7+2(-4)-(-5) = 4$, $-7+(-4)+2(-5) = -1$, and $2(7)+6(-4)-3(-5) = 5$.

7. A system in variables x, y, z has an augmented matrix with RREF $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 6 \end{array}\right]$.

Write down the system of equations corresponding to this matrix. How would you describe the solution to the system?

(Hint: what geometric problem corresponds to a system of two equations in three variables?)

The first row corresponds to the equation $x-3z = 4$, and the second to the equation $y+2z = 6$. From Chapter 3, we recall that two equations in three variables represents the intersection of two planes, and we expect the solution to be a line. Indeed, we note that both equations can easily be solved, for x and y respectively, in terms of z. If we assign z to a parameter t, then we have

$$x = 4 + 3t, y = 6 - 2t, z = t,$$

which represents the parametric equations for a line through the point $(4, 6, 0)$ in the direction of the vector $\langle 3, -2, 1\rangle$.

Additional practice: (**do not submit**).

1. Use Gaussian elimination to find the reduced row-echelon form of the matrix:

   (a)

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 3 & -1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 4 & 0 \\ 0 & -5 & -1 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2 \to R_2}$$

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & \frac{1}{5} \end{bmatrix} \xrightarrow{R_1 - 4R_2 \to R_1} \begin{bmatrix} 1 & 0 & -\frac{4}{5} \\ 0 & 1 & \frac{1}{5} \end{bmatrix}$$

   (b) $\begin{bmatrix} 4 & 8 \\ -2 & -4 \end{bmatrix} \xrightarrow[-frac12R_2 \to R_2]{\frac{1}{4}R_1 \to R_1} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

   (c)

$$\begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 1 & 3 & 4 \\ -1 & 4 & 5 & 3 \end{bmatrix} \xrightarrow[R_3 + R_1 \to R_3]{R_2 + 2R_1 \to R_2} \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 7 & 7 & 2 \\ 0 & 7 & 7 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_2} \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 7 & 7 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{7}R_2 \to R_2}$$

$$\begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 1 & 1 & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 3R_2 \to R_1} \begin{bmatrix} 1 & 0 & -1 & -\frac{13}{7} \\ 0 & 1 & 1 & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Solve the system of equations:

   (a) $\begin{aligned} 2x - 3y &= 7 \\ -x + 2y &= 2 \end{aligned}$

   We set up the corresponding augmented matrix and reduce:

$$\left[\begin{array}{cc|c} 2 & -3 & 7 \\ -1 & 2 & 2 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{cc|c} 1 & -1 & 9 \\ -1 & 2 & 2 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 9 \\ 0 & 1 & 11 \end{array}\right].$$

   The second row gives us $y = 11$, and the first gives $x - y = 9$ Putting $y = 11$ in this equation gives us $x = 9 + 11 = 20$, so $x = 20$, $y = 11$ is the solution.

   (b) $\begin{aligned} x - 2y + 4z &= 2 \\ 2x - 3y + z &= -2 \\ -x + 2y - 2z &= 6 \end{aligned}$

   We have the following augmented matrix and "forward" reduction steps:

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 2 & -3 & 1 & -2 \\ -1 & 2 & -2 & 6 \end{array}\right] \xrightarrow[R_3 + R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -7 & -6 \\ 0 & 0 & 2 & 8 \end{array}\right] \xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -7 & -6 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

   Our matrix is now in row-echelon form. If we proceed by back substitution, we have

$$\begin{aligned} z &= 4 \\ y - 7z &= -6, \text{ so } y = -6 + 7(4) = 22. \\ x - 2y + 4z &= 2, \text{ so } x = 2 + 2(22) - 4(4) = 30. \end{aligned}$$

Alternatively, we can continue with the "backward" reduction steps for our augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -7 & -6 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow[R_2+7R_3 \to R_3]{R_1-4R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -14 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{R_1+2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Our matrix is now in reduced row-echelon form, and we can read off the solution $x = 30, y = 22, z = 4$.