Math 3500 Assignment #3 Solutions
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1. Determine whether or not each of the following sequences (x_n) converge. For those that do, find $\lim x_n$, and prove that $x_n \to x$ using the **definition** of convergence (i.e. no limit theorems allowed). For those that do not converge, explain why there is no limit.

   (a) $x_n = \dfrac{(-1)^n}{n}$

   Solution: The sequence converges to zero. To see this, let $\epsilon > 0$ be given, and choose $N \in \mathbb{N}$ such that $1/n < \epsilon$ for all $n \geq N$. (We know this is possible since $\mathbb{R}$ is Archimedean.) Then whenever $n \geq N$ we have

   $$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n} < \epsilon.$$

   (b) $x_n = (-1)^n(1 - 1/n)$

   Solution: Let $\epsilon = 1$, and note that for any $n > 2$, $\dfrac{1}{n} + \dfrac{1}{n+1} < \dfrac{1}{2} + \dfrac{1}{2} = 1$. Thus, for $n > 2$ we have

   $$|x_{n+1} - x_n| = \left|1 - \frac{1}{n} + 1 - \frac{1}{n+1}\right| = \left|2 - \frac{1}{n} - \frac{1}{n+1}\right| > 2 - 1 = 1 = \epsilon.$$

   It follows that (x_n) is not a Cauchy sequence, and thus cannot converge.

   Note: Another acceptable solution is to note that $\{x_n : n \in \mathbb{N}\}$ has both 1 and -1 as limit points, and explain why it follows that the sequence cannot converge.

   (c) $x_n = \dfrac{3n+1}{2n+5}$

   Solution: Let $\epsilon > 0$ be given, and choose $N \in \mathbb{N}$ sufficiently large that $\dfrac{1}{n} < \dfrac{4\epsilon}{13}$ for all $n \geq N$. Then we have that

   $$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{(6n+2) - (6n+15)}{2(2n+5)}\right| = \left|\frac{-13}{4n+10}\right| < \frac{13}{4n} < \epsilon.$$

(d) $x_n = \dfrac{n^2 - n}{n^3 + 1}$

Solution: Let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ such that $1/n < \epsilon$ for all $n \geq N$. Then

$$\left|\frac{n^2 - n}{n^3 + 1}\right| < \left|\frac{n^2}{n^3}\right| = \frac{1}{n} < \epsilon.$$

2. Prove that the Squeeze Theorem holds for sequences: if $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and $a_n \to L$ and $c_n \to L$, then $b_n \to L$.

Solution: Let $\epsilon > 0$ be given. Since $a_n \to L$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow |a_n - L| < \epsilon$, and since $c_n \to L$ there exits $N_2 \in \mathbb{N}$ such that $n \geq N_2 \Rightarrow |c_n - L| < \epsilon$. Let $N = \max N_1, N_2$ and suppose that $n \geq N$. Then we have

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon,$$

and thus that $L - \epsilon < b_n < L + \epsilon$, or equivalently, $|b_n - L| < \epsilon$. Thus, $b_n \to L$ as well.

Note: You might be tempted to argue that since $a_n \leq b_n$ for all n, then $\lim a_n \leq \lim b_n$, and similarly that $\lim b_n \leq \lim c_n$. But this only works if you know in advance that $\lim b_n$ exists!

3. Prove that the convergence of (a_n) implies the convergence of $(|a_n|)$. Is the converse true? Why or why not?

Solution: Suppose that $a_n \to a$. We claim that $|a_n| \to |a|$. To see this, let $\epsilon > 0$ be given, and choose $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N$. Then (using an inequality from the first assignment) we have

$$||a_n| - |a|| \leq |a_n - a| < \epsilon.$$

The converse is not true, however. For example, if $a_n = (-1)^n$, then $|a_n| = 1$, and the sequence is constant and therefore converges. However, we know that (a_n) does not converge.

4. (a) Let (s_n) be a sequence such that $|s_{n+1} - s_n| < 2^{-n}$ for all $n \geq 1$. Prove that (s_n) is a Cauchy sequence and hence convergent.

Solution: Let (s_n) be such a sequence, and let $\epsilon > 0$ be given. Since $2^n > n$ for all $n \in \mathbb{N}$ we know that there exists some $N \in \mathbb{N}$ such that $2^{-n+1} < \epsilon$ for all $n \geq N$. Suppose that $m, n \geq N$, and without loss of generality, suppose that $m > n$. Then we can write $m = n + k$ for some $k \geq 1$, and

$$\begin{aligned}
|a_m - a_n| &= |a_{n+k} - a_n| \\
&= |a_{n+k} - a_{n+k-1} + a_{n+k-1} - a_{n+k-2} + \cdots + a_{n+1} - a_n| \\
&\leq |a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + \cdots + |a_{n+1} - a_n| \\
&< 2^{-(n+k-1)} + 2^{-(n+k-2)} + \cdots + 2^{-n} \\
&= 2^{-n}(1 + 2^{-1} + \cdots + 2^{-k+1}) \\
&< 2^{-n}(2) = 2^{-n+1} < \epsilon.
\end{aligned}$$

(b) Is the result in part (a) true if we only assume that $|s_{n+1} - s_n| < 1/n$ for all $n \geq 1$? (If not, can you find a counterexample?)

Solution: The result does not necessarily hold. For example, take

$$s_n = 1 + 1/2 + 1/3 + \cdots + 1/n.$$

Then $|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}$, but (s_n) does not converge, and therefore cannot be Cauchy.

5. Consider the following two definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \in A$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $A = \{1\}$?

Solution: The sequence is frequently in $\{1\}$, since for any $N \in \mathbb{N}$, either N or $N+1$ is even, so either $(-1)^N = 1$ or $(-1)^{N+1}$ is even. It is not eventually in $\{1\}$ since there are also infinitely many n for which $(-1)^n = -1$.

(b) Which definition is stronger? Does eventually imply frequently? Does frequently imply eventually?

Solution: The first definition is stronger. If $a_n \in A$ for all $n \geq N$, then there certainly exists some $n \geq N$ for which $a_n \in A$.

(c) Given $a \in \mathbb{R}$ and $A = N_\epsilon(a) = (a - \epsilon, a + \epsilon)$, which of the above two definitions could we use to give an alternative definition of convergence? Explain your answer.

Solution: The first definition could be used: if we say that (a_n) converges if for every $\epsilon > 0$, (a_n) is eventually in $N_\epsilon(a)$, then we are claiming that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$, which is the usual definition of convergence.

(d) Suppose an infinite number of terms in a sequence (x_n) are equal to 3. Can we conclude that (x_n) is eventually in the set $A = (2.9, 3.1)$? Is it frequently in A?

Solution: We can only conclude that the sequence is frequently in A: For any $N \in \mathbb{N}$ there must exist an $n \geq N$ such that $x_n = 3 \in A$, or else x_n would equal 3 only a finite number (at most N) of times. But as the example $x_n = 3(-1)^n$ shows, it's possible to have $x_n = 3$ an infinite number of times without it being true that (x_n) is eventually in A.

(e) Let (x_n) be a sequence and let $X = \{x_n \,|\, n \in \mathbb{N}\}$. Suppose that x is a limit (accumulation) point of X. Can we conclude that $\lim x_n = x$? (If $A = N_\epsilon(x)$, is (x_n) frequently in A, or eventually in A?)

Solution: If x is a limit point of X, then we can conclude that any neighbourhood of x contains infinitely many points of X, so we can conclude that (x_n) is frequently in A, as in the previous part. However, this does not guarantee that (x_n) is eventually in A: it could be the case that $x_n \in A$ only when n is even, for example.

6. Consider the sequence (a_n) defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$ for all $n \geq 1$. (Thus, (a_n) is the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$.)

   (a) Prove that (a_n) is an increasing sequence. (This is most easily done by induction.)

   Solution: We wish to show that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$. Since $a_2 = \sqrt{2\sqrt{2}} > \sqrt{2}$, the result is true when $n = 1$. Now, suppose that for some $k \geq 1$ we have $a_{k+1} \geq a_k$. Then

   $$a_{k+2} = \sqrt{2a_{k+1}} \geq \sqrt{2a_k} = a_{k+1},$$

   and thus (a_n) is increasing by induction.

   (b) Prove that (a_n) converges.

   Solution: We first show that $a_n \leq 2$ for all $n \in \mathbb{N}$. Certainly $a_1 = \sqrt{2} < 2$, and if $a_k \leq 2$ for some $k \geq 1$, then

   $$a_{k+1} = \sqrt{2a_k} \leq \sqrt{2 \cdot 2} = 2.$$

   Thus, it follows that $a_n \leq 2$, by induction. We now note that since (a_n) is bounded above by 2, and is increasing (by part (a)), it follows that (a_n) converges, by the Monotone Convergence Theorem.

   (c) Prove that if (a_n) converges, then $\lim a_n^2 = (\lim a_n)^2$. (Use known limit theorems.)

   Solution: This follows from the rule for products, with $b_n = a_n$: we have $\lim(a_n \cdot a_n) = a \cdot a = a^2$.

   (d) Explain why, if (a_n) converges, then $\lim a_{n+1} = \lim a_n$.

   Solution: In the definition of convergence we have that $|a_n - a| < \epsilon$ for all $n \geq N$. If (a_n) converges, then for any $\epsilon > 0$, since $n + 1 > n$, we have in particular that $|a_{n+1} - a| < \epsilon$ whenever $N \in \mathbb{N}$ is such that $n \geq N$ implies $|a_n - a|\epsilon$.

   (e) Find the limit of the sequence (a_n).

   Solution: From part (b) we know that $a = \lim a_n$ exists. Combining parts (c) and (d), it follows that

   $$a^2 = \lim a_n^2 = \lim(a_{n+1})^2 = \lim(\sqrt{2a_n})^2 = \lim 2a_n = 2a.$$

   This shows that $a^2 = 2a$, so either $a = 0$ or $a = 2$. Since $a_n > 0$ for all n (this follows easily by induction) we must have $a > 0$, and thus $a = 2$.

Note: your solutions for parts (c) and (d) should work for an arbitrary convergent sequence. Part (e) refers to the sequence defined at the start of the problem.