MATH 2565 - Tutorial #3 Solutions

Additional practice problems:

1. $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx$

   Letting $u = x^2 - 3$, we have $\frac{1}{2}du = x\,dx$, so

$$\int \frac{x}{\sqrt{x^2-3}}\,dx = \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} + C = \sqrt{x^2-3} + C.$$

   Alternatively, you can let $x = \sqrt{3}\sec\theta$, so $dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-3} = \sqrt{3}\sqrt{\sec^2\theta - 1} = \sqrt{3}\tan\theta$, and then

$$\int \frac{x}{\sqrt{x^2-3}}\,dx = \int \frac{\sqrt{3}\sec\theta}{\sqrt{3}\tan\theta}(\sqrt{3}\sec\theta\tan\theta)d\theta = \sqrt{3}\int \sec^2\theta\,d\theta = \sqrt{3}\tan\theta + C$$

   From our work above we see that $\sqrt{3}\tan\theta = \sqrt{x^2-3}$, and so we get the same answer as above.

   If you want yet another option, try letting $x = \sqrt{3}\cosh(t)$, so $dx = \sqrt{3}\sinh(t)\,dt$ and $\sqrt{x^2-3} = \sqrt{3}\sqrt{\cosh^2(t)-1} = \sqrt{3}\sinh(t)$. With these substitutions, the integral becomes $\sqrt{3}\displaystyle\int \cosh(t)\,dt = \sqrt{3}\sinh(t) + C = \sqrt{x^2-3}$, as before.

2. $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}\,dx$

   Seeing the pattern $x^2 + a^2$, we make a tangent substitution: $x = 2\tan\theta$, so $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{x^2+4} = 2\sec\theta$, giving us

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2+4}}\,dx &= \int \frac{4\tan^2\theta}{2\sec\theta}(2\sec^2\theta)\,d\theta \\ &= 4\int \tan^2\theta\sec\theta\,d\theta = 4\int(\sec^2\theta - 1)\sec\theta\,d\theta \\ &= 4\int \sec^3\theta\,d\theta - 4\int\sec\theta\,d\theta.\end{aligned}$$

   From class, you know that $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$, and you know that

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)| + C,$$

   so

$$\begin{aligned}4\int \sec^3\theta\,d\theta - 4\int\sec\theta\,d\theta &= 4\left(\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| - \ln|\sec(\theta)+\tan(\theta)|\right) + C \\ &= 2\sec\theta\tan\theta - 2\ln|\sec\theta + \tan\theta| + C\end{aligned}$$

From the substitution work above, we know that $\tan\theta = \frac{x}{2}$, and that $\sec\theta = \frac{1}{2}\sqrt{x^2+4}$. Putting everything together, we get

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2+4}}\,dx &= 2\left(\left(\frac{1}{2}\sqrt{x^2+4}\right)\left(\frac{x}{2}\right) - \ln\left|\frac{x}{2}+\frac{1}{2}\sqrt{x^2+4}\right|\right) + C \\ &= \frac{1}{2}x\sqrt{x^2+4} - 2\ln|x+\sqrt{x^2+4}| + C,\end{aligned}$$

where in the last line, I've used the fact that $\ln(u/2) = \ln(u) - \ln(2)$, and absorbed the constant $-\ln(2)$ into the constant of integration.

3. $\displaystyle\int \frac{7x-2}{x^2+x}\,dx$

Using partial fractions, if

$$\frac{7x-2}{x^2+x} = \frac{7x-2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)},$$

then we must have $A(x+1)+Bx = 7x-2$. When $x=0$ we get $A=-2$, and when $x=-1$ we get $-B=-9$, so $B=9$. Thus, we have

$$\int \frac{7x-2}{x^2+x}\,dx = -2\int \frac{1}{x}\,dx + 9\int \frac{1}{x+1}\,dx = -2\ln|x| + 9\ln|x+1| + C = \ln\left|\frac{(x+1)^9}{x^2}\right| + C.$$

4. $\displaystyle\int \frac{1}{x^3+2x^2+3x}\,dx$

Factoring the denominator, we have

$$x^3+2x^2+3x = x(x^2+2x+3),$$

where $x^2+2x+3 = (x+1)^2+2$ is an irreducible quadratic. Our partial fraction decomposition is thus

$$\frac{1}{x^3+2x^2+3x} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+3} = \frac{A(x^2+2x+3)+(Bx+C)x}{x(x^2+2x+3)}.$$

Equating numerators gives us $1 = A(x^2+2x+3)+(Bx+C)x$. Setting $x=0$ gives us $1=3A$, so $A=\frac{1}{3}$. Since x^2+2x+3 has no real roots, there isn't any x value we can plug in to make the A term vanish. Instead, we put $x=1$, giving us $1 = \frac{1}{3}(1+2+3)+(B+C)(1)$, so $B+C = 1-2 = -1$. Putting $x=-2$ gives us $1 = \frac{1}{3}(4-4+3)+(-2B+C)(-2)$, so $4B-2C+1 = 1$, which simplifies to $2B-C=0$. (If you're wondering why I chose $x=-2$, it was so x^2+2x+3 would be a multiple of 3, allowing me to avoid fractions.)

We're left with the equations $B+C=-1$ and $2B-C=0$. Adding the two equations gives us $3B=-1$, so $B=-\frac{1}{3}$, and thus $C=2B=-\frac{2}{3}$, so

$$\frac{1}{x^3+2x^2+x} = \frac{1}{3}\left(\frac{1}{x} - \frac{x+2}{x^2+2x+3}\right) = \frac{1}{3}\left(\frac{1}{x} - \frac{x+1}{x^2+2x+3} - \frac{1}{x^2+2x+3}\right).$$

Why did we break up the second fraction into two pieces? Well, for the first piece, if we let $u = x^2 + 2x + 3$, then $du = 2(x+1)\,dx$, so

$$\int \frac{x+1}{x^2+2x+3}\,dx = \frac{1}{2}\ln(x^2+2x+3) + C.$$

For the second piece, writing $x^2 + 2x + 3 = (x+1)^2 + 2$, we can let $x + 1 = \sqrt{2}\tan\theta$, so $dx = \sqrt{2}\sec^2\theta\,d\theta$ and $(x+1)^2 + 2 = 2\sec^2\theta$, so

$$\int \frac{1}{x^2+2x+3}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right) + C.$$

Altogether, we have

$$\int \frac{1}{x^3+2x^2+x}\,dx = \frac{1}{3}\ln|x| - \frac{1}{6}\ln(x^2+2x+3) - \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right) + C.$$

5. $\displaystyle\int \frac{x+7}{(x+5)^2}\,dx$

Again we use partial fractions. Because of the repeated root in the denominator, we write

$$\frac{x+7}{(x+5)^2}\,dx = \frac{A}{x+5} + \frac{B}{(x+5)^2} = \frac{A(x+5)+B}{(x+5)^2},$$

and equating numerators gives us $x + 7 = A(x+5) + B$. Putting $x = -5$ immediately gives us $B = 2$, and plugging this back in, we have $x + 7 = Ax + 5A + 2$, so we must have $A = 1$. Thus,

$$\int \frac{x+7}{(x+5)^2}\,dx = \int \frac{1}{x+5}\,dx + 2\int (x+5)^{-2}\,dx = \ln|x+5| - 2(x+5)^{-1} + C.$$

6. $\displaystyle\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx$

Our partial fraction decomposition in this case takes the form

$$\frac{9x^2+11x+7}{x(x+1)^2}\,dx = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2},$$

so $A(x+1)^2 + Bx(x+1) + Cx = 9x^2 + 11x + 7$. Putting $x = 0$ gives us $A = 7$ immediately, and putting $x = -1$ gives us $-C = 9 - 11 + 7 = 5$, so $C = -5$. This leaves us with $7(x+1)^2 + Bx(x+1) - 5x = 9x^2 + 11x + 7$. To find B, we try $x = 1$, which gives us $7(4) + 2B - 5 = 9 + 11 + 7$, so $2B = 27 - 19 = 8$, giving us $B = 4$. Putting everything into the integral, we have

$$\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx = \int \left(\frac{7}{x} + \frac{4}{x+1} - \frac{5}{(x+1)^2}\right)dx = 7\ln|x| + 4\ln|x+1| + \frac{5}{x+1} + C.$$

Assigned problems:

1. $\int x^2\sqrt{1-x^2}\,dx$

   Letting $x=\sin\theta$, $dx=\cos\theta\,d\theta$ and $\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\cos\theta$, so we get

$$\begin{aligned}\int x^2\sqrt{1-x^2}\,dx &= \int \sin^2\theta\cos\theta(\cos\theta)\,d\theta = \int \sin^2\theta\cos^2\theta\,d\theta\\ &= \int\left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)d\theta\\ &= \frac{1}{4}\int(1-\cos^2(2\theta))\,d\theta = \frac{1}{4}\int\sin^2(2\theta)\,d\theta\\ &= \frac{1}{4}\int\left(\frac{1-\cos(4\theta)}{2}\right)d\theta\\ &= \frac{1}{8}\left(\theta-\frac{1}{4}\sin(4\theta)\right)+C\\ &= \frac{1}{8}\sin^{-1}(x)-\frac{1}{32}\sin(4\sin^{-1}x)+C.\end{aligned}$$

   If you want to simplify that last term, note that $\sin\theta=x$ and $\cos\theta=\sqrt{1-x^2}$, and

   $\sin(4\theta)=2\sin(2\theta)\cos(2\theta)=4\sin(\theta)\cos(\theta)(\cos^2(\theta)-\sin^2(\theta))=4\sin(\theta)\cos^3(\theta)-4\sin^3(\theta)\cos(\theta)$,

   so

$$\frac{1}{32}\sin(4\sin^{-1}x)=\frac{1}{8}(x(1-x^2)^{3/2}-x^3(1-x^2)^{1/2})=\frac{1}{8}x(1-2x^2)\sqrt{1-x^2}.$$

2. $\int \frac{1}{(x^2+4x+13)^2}\,dx$

   Completing the square, we have $x^2+4x+13=x^2+4x+4+9=(x+2)^2+3^2$, suggesting that we try letting $x+2=3\tan\theta$. This gives us $dx=3\sec^2\theta\,d\theta$, and

$$x^2+4x+13=(x+2)^2+3^3=3^2\tan^2\theta+3^2=3^2(\tan^2\theta+1)=9\sec^2\theta.$$

   Substituting everything into the integral, we get

$$\begin{aligned}\int\frac{1}{(x^2+4x+13)^2}\,dx &= \int\frac{1}{81\sec^4\theta}(3\sec^2\theta)\,d\theta\\ &= \frac{1}{27}\int\cos^2\theta\,d\theta\\ &= \frac{1}{54}\int(1+\cos(2\theta))\,d\theta\\ &= \frac{\theta}{54}+\frac{1}{108}\sin(2\theta)+C\\ &= \frac{\theta}{54}+\frac{1}{54}\sin\theta\cos\theta+C.\end{aligned}$$

To get everything back in terms of x, we note that $\tan\theta = \frac{x+2}{3}$. If we have a right-angled triangle with sides of length $x+2$ (opposite θ) and 3 (adjacent θ), then the hypotenuse has length $\sqrt{(x+2)^2+3^2} = \sqrt{x^2+4x+13}$, and we get $\sin\theta = \dfrac{x+2}{\sqrt{x^2+4x+13}}$ and $\cos\theta = \dfrac{3}{\sqrt{x^2+4x+13}}$. Plugging all of this in, we get the final answer

$$\int \frac{1}{(x^2+4x+13)^2}\,dx = \frac{1}{54}\tan^{-1}\left(\frac{x+2}{3}\right) + \frac{1}{18}\frac{x+2}{x^2+4x+13}.$$

3. $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx$

   We look for a partial fraction decomposition

$$\frac{7x+7}{x^2+3x-10} = \frac{7x+7}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5} = \frac{A(x+5)+B(x-2)}{(x-2)(x+5)}$$

   Since the denominators of the first and last terms of the above inequality are equal, the numerators must be equal as well:

$$7x+7 = A(x+5)+B(x-2).$$

   Since this equality holds for all values of x, it holds in particular when $x=2$ and $x=-5$. Putting $x=2$ gives us $7(2)+7 = A(7)+B(0)$, so $7A = 21$ and thus $A=3$. Putting $x=-5$ gives us $7(-5)+7 - A(0)+B(-7)$, so $-7B = -28$, and thus $B=4$. Returning to the integral, we thus have

$$\begin{aligned}\int \frac{7x+7}{x^2+3x-10}\,dx &= 3\int \frac{1}{x-2}\,dx + 4\int \frac{1}{x+5}\,dx \\ &= 3\ln|x-2| + 4\ln|x+5| + C = \ln|(x-2)^3(x+5)^4| + C.\end{aligned}$$

4. $\displaystyle\int \frac{x^3}{x^2-x-20}\,dx$ (First do long division.)

   Since the degree of the numerator is not less than that of the denominator, we first perform long division:

$$\begin{array}{r|l} & x \quad +1 \\ \hline x^2-x-20 & x^3 \\ & -x^3+x^2+20x \\ \hline & \quad x^2+20x \\ & \quad -x^2 \quad +x+20 \\ \hline & \qquad 21x+20 \end{array}$$

   This tells us that we can write $\dfrac{x^3}{x^2-x-20} = x+1+\dfrac{21x+20}{x^2-x-20}$, and it remains to perform a partial fraction decomposition on the last term:

$$\frac{21x+20}{(x-5)(x+4)} = \frac{A}{x-5} + \frac{B}{x+4} = \frac{A(x+4)+B(x-5)}{(x+4)(x-5)},$$

giving us $21x+20 = A(x+4)+B(x-5)$. If $x=5$, we get $125 = 9A$, so $A = \frac{125}{9}$. If $x=-4$, we get $-64 = -9B$, so $B = \frac{64}{9}$. Thus, we have

$$\begin{aligned}\int \frac{x^3}{x^2-x-20}\,dx &= \int \left(x+1+\frac{125}{9(x-5)}+\frac{64}{9(x+4)}\right)dx\\ &= \frac{1}{2}x^2+x+\frac{125}{9}\ln|x-5|+\frac{64}{9}\ln|x+4|+C.\end{aligned}$$

5. $\displaystyle\int \frac{2x^2+2x+1}{(x+1)(x^2+9)}\,dx$

Once more with partial fractions: if

$$\frac{2x^2+2x+1}{(x+1)(x^2+9)}\,dx = \frac{A}{x+1}+\frac{Bx+C}{x^2+9} = \frac{A(x^2+9)+(Bx+C)(x+1)}{(x+1)(x^2+9)},$$

then equating numerators gives us $2x^2+2x+1 = A(x^2+9)+(Bx+C)(x+1)$. Putting $x=-1$, we get $1 = A(10)$, so $A = 1/10$. Putting $x=0$, we get $1 = 9A+C$, so $C = 1-9/10 = 1/10$. Finally, putting $x=1$ gives us $5 = 10A+2(B+C)$, so $2(B+C) = 5-10(1/10) = 4$, which simplifies to $B+C=2$. Since $C = 1/10$, this gives us $B = 19/10$. Therefore, we have

$$\begin{aligned}\int \frac{2x^2+2x+1}{(x+1)(x^2+9)}\,dx &= \int \left(\frac{1}{10x}+\frac{19x}{10(x^2+9)}+\frac{1}{10(x^2+9)}\right)dx\\ &= \frac{1}{10}\ln|x|+\frac{19}{20}\ln(x^2+9)+\frac{1}{30}\tan^{-1}\left(\frac{x}{3}\right)+C.\end{aligned}$$

6. $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$

Here we see that we have both a square root and a cube root. The least common multiple of 2 and 3 being 6, we attempt the rationalizing substitution $x = u^6$, so $dx = 6u^5\,du$, and $\sqrt{x} = \sqrt{u^6} = u^3$, while $\sqrt[3]{x} = \sqrt[3]{x^6} = u^2$.[1]

Making these substitutions, we find

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \int \frac{6u^5}{u^3+u^2}\,du = 6\int \frac{u^3}{u+1}\,du.$$

Using long division, we find that

$$\frac{u^3}{u+1} = u^2-u+1-\frac{1}{u+1},$$

[1] In case you are concerned about the fact that $\sqrt{u^6} = |u|^3$ in general (you probably weren't, but just in case): generally, to have a well-defined substitution, one must define $x = f(u)$ where f is a one-to-one function. (When we do trig substitution, we officially are working with the restricted trig functions that are used when we define the inverse trig functions.) If the substitution $x = u^6$ is to be one-to-one, we implicitly have the restriction $u \geq 0$, even if we don't state it.

so

$$\begin{aligned}
\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx &= 6\int \frac{u^3}{u+1}\,du \\
&= 6\int \left(u^2-u+1-\frac{1}{u+1}\right)\,du \\
&= 2u^3-3u^2+6u-6\ln(u+1)+C \\
&= 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(\sqrt[6]{x}+1)+C.
\end{aligned}$$

7. $\displaystyle\int_0^{\pi/2} \frac{\cos(x)}{2-\cos(x)}\,dx$

This one is borrowed from Dr. Kaminski's handout, and it's a bit of a workout, so hold onto your hats.

We use the "tangent half-angle" substitution $t=\tan(x/2)$, which yields

$$\cos(x)=\frac{1-t^2}{1+t^2} \quad \text{and} \quad dx=\frac{2}{1+t^2}\,dt.$$

Notice that when $x=0$, $t=\tan(0/2)=0$, and when $x=\pi/2$, $t=\tan(\pi/4)=1$. Thus,

$$\int_0^{\pi/2}\frac{\cos(x)}{2-\cos(x)}\,dx=\int_0^1\frac{\frac{1-t^2}{1+t^2}}{2-\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt.$$

Cleaning up this mess, we find

$$\int_0^1\frac{\frac{1-t^2}{1+t^2}}{2-\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt=\int_0^1\frac{2-2t^2}{(1+3t^2)(1+t^2)}\,dt.$$

The remaining integral requires partial fractions, and it's lots of fun, because there are two irreducible quadratics. Writing

$$\frac{2-2t^2}{(1+3t^2)(1+t^2)}=\frac{At+B}{1+3t^2}+\frac{Ct+D}{1+t^2},$$

and then re-writing the right-hand side over a common denominator, we can equate numerators, giving us

$$2-2t^2=t^3(A+3C)+t^2(B+2D)+t(A+C)+(B+D).$$

Comparing coefficients of odd powers, we find $A+3C=0$ and $A+C=0$, which is only possible if $A=C=0$. Comparing powers of even coefficients, we find $B+3D=-2$ and $B+D=2$. Solving these two equations gives us $B=4$ and $D=-2$.

Let's put in those values. We get

$$\int_0^1\frac{2-2t^2}{(1+3t^2)(1+t^2)}\,dt=\int_0^1\left(\frac{4}{1+3t^2}-\frac{2}{1+t^2}\right)\,dt.$$

Both of these terms produce arctangent integrals. The second is direct; the first, with a bit of work, produces

$$\int \frac{1}{1+3t^2}\,dt = \frac{1}{\sqrt{3}}\arctan\sqrt{3}t + C.$$

Now you're probably thinking about how you're going to substitute this back in terms of x but worry not! We had the foresight to adjust the limits of integration when we substituted, so all that remains is to apply the Fundamental Theorem of Calculus:

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos(x)}{2-\cos(x)}\,dx &= \int_0^1 \left(\frac{4}{1+3t^2} - \frac{2}{1+t^2}\right) dt && \text{(by all our work above)}\\
&= \left(\frac{4}{\sqrt{3}}\arctan(\sqrt{3}x) - 2\arctan(x)\right)\Bigg|_0^1\\
&= \frac{4}{\sqrt{3}}(\arctan(\sqrt{3}) - \arctan(0)) - 2(\arctan(1) - \arctan(0))\\
&= \frac{4}{\sqrt{3}}\left(\frac{\pi}{3}\right) - 2\left(\frac{\pi}{4}\right)\\
&= \pi\left(\frac{4}{3\sqrt{3}} - \frac{1}{2}\right).
\end{aligned}$$