Week 2: Solving systems of equations

Math 1410 Linear Algebra

Systems of linear equations - Examples

- One equation, two variables: $2x - 3y = 6$
- Two equations, two variables:

$$\begin{aligned} x - 3y &= 4 \\ 3x + y &= 6 \end{aligned}$$

- One equation, three variables: $x + 2y - 3z = 6$
- Two equations, three variables:

$$\begin{aligned} -x + y + 4z &= 8 \\ 2x - y + z &= 6 \end{aligned}$$

- etc.

Systems of linear equations - Definitions

Definition

A **system** of m linear equations in n variables is a collection of equations of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\quad \vdots \qquad\qquad & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

If we can find at least one solution, the system is **consistent**.
If there is no solution to the system, we say that the system is **inconsistent**.

2 equations, 2 variables

Example

Solve the system

$$\begin{aligned} 3x - 2y &= 4 \\ -2x + 5y &= -2 \end{aligned}$$

Verifying a solution

Previous example: the solution to the system

$$\begin{aligned} 3x - 2y &= 4 \\ -2x + 5y &= -2 \end{aligned}$$

is $x = \dfrac{16}{11}, y = \dfrac{2}{11}$.

2 equations, 2 variables, no solution

Example

Solve the system

$$\begin{aligned} x - 2y &= 4 \\ -2x + 4y &= 0 \end{aligned}$$

2 equations, 3 variables

Example

Find all solutions to the system

$$
\begin{aligned}
2x - 4y - z &= -6 \\
3x + y + 2z &= 4
\end{aligned}
$$

Elementary operations

For more than two variables/equations, a systematic approach is needed. There are three basic manipulations we use to attempt to solve a system. These are the elementary operations.

Definition

The **elementary operations** on a system of linear equations are as follows:

1. Change the places of any two equations.
2. Multiply both sides of any equation by a (non-zero) constant.
3. Add any multiple of one equation to another.

3 equations, 3 variables

Example

Solve the system

$$\begin{aligned} x + 3y - 2z &= 4 \\ x - y \phantom{{}-2z} &= -2 \\ 3x - 4y + z &= 0 \end{aligned}$$

The augmented matrix

Solving the system

$$\begin{aligned} x + 3y - 2z &= 4 \\ x - y &= -2 \\ 3x - 4y + z &= 0 \end{aligned}$$

gets messy - there are lots of variables to keep track of. An **augmented** matrix is a way to keep track of the same information, without writing down the variables. The above system is represented by the matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 1 & -1 & 0 & -2 \\ 3 & -4 & 1 & 0 \end{array}\right]$$

Row operations

Each elementary operation for our system of equations corresponds to an **elementary row operation** for the resulting augmented matrix:

- Exchange the order of two equations $\rightsquigarrow$ Interchange two rows.
- Multiply both sides of an equation by a constant $\rightsquigarrow$ Multiply a row by a constant.
- Add a multiple of one equation to another $\rightsquigarrow$ Add a multiple of one row to another.

Each operation produces a new matrix that represents an equivalent system of equations. If we simplify the matrix, we simplify the system of equations.

Row operations by example

- Exchange Row 1 and Row 2:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 1 & -1 & 0 & -2 \\ 3 & -4 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 1 & 3 & -2 & 4 \\ 3 & -4 & 1 & 0 \end{array}\right]$$

- Multiply Row 3 by $\frac{1}{3}$:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 1 & -1 & 0 & -2 \\ 3 & -4 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{3} R_3} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 1 & -1 & 0 & -2 \\ 1 & -\frac{4}{3} & \frac{1}{3} & 0 \end{array}\right]$$

- Add -1 times Row 1 to Row 2:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 1 & -1 & 0 & -2 \\ 3 & -4 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 0 & -4 & 2 & -6 \\ 3 & -4 & 1 & 0 \end{array}\right]$$

Example Revisited 1

Applying elementary operations to our system

$$\begin{aligned} x + 3y - 2z &= 4 \\ x - y \phantom{{}-2z} &= -2 \\ 3x - 4y + z &= 0 \end{aligned}$$

gave us the simpler system

$$\begin{aligned} x - y \phantom{{}-z} &= -2 \\ + y - z &= -6 \\ + z &= 15. \end{aligned}$$

Note that we can reduce further to

$$\begin{aligned} x \phantom{{}+y+z} &= 7 \\ y \phantom{{}+z} &= 9 \\ z &= 15 \end{aligned}$$

Example Revisited 2

Let's work instead with the augmented matrix using row operations:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 1 & -1 & 0 & -2 \\ 3 & -4 & 1 & 0 \end{array}\right]$$

Another Example

Use Gaussian elimination to solve the system

$$\begin{aligned} 2x + y + 5z + w &= 4 \\ x + 3y \phantom{{}+5z} - 2w &= 0 \\ 2y - z + w &= 1 \end{aligned}$$

Terminology

- An (augmented) matrix is in row echelon form if:
  1. Any rows of zeros are at the bottom
  2. The first entry in any non-zero row is a 1 (the "leading 1")
  3. Every leading 1 is to the right of any leading 1s above it.
- An (augmented) matrix that is in row-echelon form is in reduced row-echelon form if every leading 1 is the only non-zero entry in its column.
- A column containing a leading 1 is called a pivot column. The corresponding variable is called a pivot, or basic variable.
- Variables corresponding to columns without a leading 1 are called free variables.

Examples

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & -14 \\ 0 & 1 & -5 & 3 \end{array}\right] \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian elimination: Computer version

Gaussian elimination is an algorithm for reducing an (augmented) matrix to (reduced) row-echelon form.

1. If the matrix contains only zeros, stop.
2. If not, find the first column containing a non-zero entry. (Call this entry a.)
3. Move the row containing a to the top of the matrix.
4. Multiply the row by $1/a$ to create a leading 1.
5. By subtracting multiples of the first row from the rows below it, make every entry below the leading 1 zero.
6. Repeat steps 1-5 for the matrix consisting of all remaining rows.

Gaussian elimination: Human version

When solving by hand it's convenient to avoid introducing fractions until we have to.

Modify the algorithm: proceed as before, but:

1. Find the first non-zero column. If it has a 1 or -1, move that row to the top. (Multiply by -1 if necessary.)
2. If none of the entries are a 1 or -1, check to see if subtracting a multiple of one row from another will create a 1.

Example

Given $\left[\begin{array}{ccc|c} -2 & 5 & 0 & -3 \\ 7 & 0 & 2 & 1 \end{array}\right]$, consider adding 3 times row 1 to row 2. This gives the equivalent matrix

$$\left[\begin{array}{ccc|c} -2 & 5 & 0 & -3 \\ 1 & 15 & 2 & -8 \end{array}\right].$$

Examples and Exercises

Solve the following systems of equations:

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 1 \\ 2x_1 & - & 4x_2 & + & x_3 & & & = & 5 \\ x_1 & - & 2x_2 & + & 2x_3 & - & 3x_4 & = & 4 \end{array} \tag{1}$$

$$\begin{array}{rcrcrcr} x & & & + & 10z & = & 5 \\ 3x & + & y & - & 4z & = & -1 \\ 4x & + & y & + & 6z & = & 1 \end{array} \tag{2}$$

$$\begin{array}{rcrcrcr} 3x & + & 4y & + & z & = & 1 \\ 2x & + & 3y & + & & + & 0 \\ 4x & + & 3y & - & z & = & -2 \end{array} \tag{3}$$

An "application"

Example

Determine the values of a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the points $(-1, 8)$, $(0, 4)$, and $(2, 2)$.

More notation

Given a system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

denote

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Structure of a solution

- With notation as on the previous page, write $(A|B)$ for the augmented matrix of our system.
- Use row operations to reduce $(A|B)$ to an augmented matrix $(A'|B')$ in (reduced) row echelon form.
- Cases:
  1. The matrix $(A'|B')$ has a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$.
  2. Every column in A' contains a leading 1.
  3. A has n columns and A' has k leading 1s, with $k < n$.
- Note: The number of leading 1s is equal to the number of non-zero rows.

A slightly harder problem

Example

Find a condition on the numbers a, b, and c such that the following system of linear equations is consistent. When that condition is satisfied, find all solutions (in terms of a, b, and c).

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & x_3 & = & a \\ -x_1 & - & 2x_2 & + & x_3 & = & b \\ 3x_1 & + & 7x_2 & - & x_3 & = & c \end{array}$$

Vector form of a solution

A column vector is an object of the form $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$. For example, we could have

$$A = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{or } C = \begin{bmatrix} 7 \\ -5 \\ 3.72 \\ 0 \end{bmatrix}.$$

We say two column vectors are equal if each corresponding entry is equal. Instead of writing $x_1 = 2, x_2 = -4, x_3 = 1$, we can write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}.$$

Vector form, with parameters

We saw earlier that the solution to

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 1 \\ 2x_1 & - & 4x_2 & + & x_3 & & & = & 5 \\ x_1 & - & 2x_2 & + & 2x_3 & - & 3x_4 & = & 4 \end{array}$$

was $x_1 = 2 + 2s - t, x_2 = s, x_3 = 1 + 2t, x_4 = t$. In vector form we write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$