University of Lethbridge
Department of Mathematics and Computer Science
11$^{\text{th}}$ February, 2015, 10:00 - 10:50 am

MATH 1410A - Test #1

Last Name: **Solutions**

First Name: **The**

Student Number: ______

Tutorial Section: ______

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

No external aids are allowed, with the exception of a 5-function calculator.

For grader's use only:

Page	Grade
2	/10
3	/10
4	/10
5	/10
Total	/40

A

1. SHORT ANSWER: For each of the questions below, please provide a short (one line) answer.

[2] (a) What is the rank of a matrix?

Solution: The number of leading ones in the row-echelon form of the matrix.

[2] (b) What does it mean to say that an $n \times n$ matrix A is invertible?

Solution: There exists another $n \times n$ matrix B such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix.

[2] (c) The matrix $E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix. If A is any other 3×3 matrix, what elementary row operation would let us obtain EA from A?

Solution: Subtracting 3 times Row 3 from Row 1.

[2] (d) Do the values $x = 3$, $y = -2$, $z = 4$ provide a solution to the system of equations below? Why or why not?

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 5 \\ 2x & + & 4y & - & z & = & -6 \\ -3x & - & 5y & + & z & = & 3 \end{array}$$

Solution: They do not, since in the third equation we have $-3(3)-5(-2)+4 = 5 \neq 3$.

[2] (e) Identify the matrices below as symmetric, antisymmetric, or neither:

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$$

Solution: The matrices, from left to right, are:
Neither, Anti-symmetric, and symmetric.

[10] 2. Find the general solution to the following system of linear equations:

$$\begin{array}{ccccccccc} x & - & 2y & & & + & w & = & 0 \\ & & -y & + & 2z & - & w & = & 1 \\ x & - & 3y & + & 2z & & & = & 1 \end{array}$$

Solution: We set-up and reduce the augmented matrix of the system as follows:

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 1 \\ 1 & -3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 1 \\ 0 & -1 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & -1 \\ 0 & -1 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 4 & -1 & -2 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

From the reduced row-echelon form of the augmented matrix, we see that x and y are leading variables, and we assign the non-leading variables to parameters. With $z = s$ and $w = t$, where s and t can be any real numbers, we have the general solution

$$\begin{aligned} x &= -2 - 4s + t \\ y &= 1 + 2s - t \\ z &= s \\ w &= t. \end{aligned}$$

3. Suppose A, B, and X are 2×2 matrices.

[3] (a) Given that $3A + X^T = B$, solve for X in terms of A and B.

Solution: By subtracting $3A$ from both sides and then taking the transpose, we have

$$X = (B - 3A)^T = B^T - 3A^T.$$

[3] (b) If $A = \begin{bmatrix} 1 & -2 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$ and X is as in part (a), determine the entries of X.

Solution: Plugging the given values for A and B into the expression above, we have

$$X = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} - 3 \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -9 \\ 8 & -3 \end{bmatrix}$$

[4] 4. Suppose that A, B, C, and D are $n \times n$ matrices, with A, B, and C *invertible*. Given that

$$AB^{-1}XBC^T = AD,$$

solve for X in terms of A, B, C, and D.

Solution: We have

$$X = BA^{-1}(AB^{-1}XBC^T)(C^T)^{-1}B^{-1} = BA^{-1}(AD)(C^T)^{-1}B^{-1} = BD(C^T)^{-1}B^{-1}.$$

5. Let $A = \begin{bmatrix} 2 & -4 \\ -4 & 9 \end{bmatrix}$.

[5] (a) Find A^{-1}.

Solution: Using the augmented matrix algorithm for the inverse, we have

$$\left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ -4 & 9 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$
$$\xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & -2 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$
$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{9}{2} & 2 \\ 0 & 1 & 2 & 1 \end{array}\right],$$

so $A^{-1} = \begin{bmatrix} \frac{9}{2} & 2 \\ 2 & 1 \end{bmatrix}$.

[3] (b) Write A^{-1} as a product of elementary matrices.

Solution: We know that $A^{-1} = E_3E_2E_1$, where E_1, E_2, E_3 are the elementary matrices corresponding to the three row operations above, in the order they were performed. Therefore,

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

[2] (c) Write A as a product of elementary matrices.

Solution: Since $A^{-1} = E_3E_2E_1$, we have

$$A = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

 Total Points: 40