University of Lethbridge
Department of Mathematics and Computer Science
24th April, 2015, 9:00 am - 12:00 pm

MATH 1410 - FINAL EXAM

Last Name:____

First Name:____

Student Number:____________________________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

Page	Grade
2	/10
3	/8
4	/8
5	/8
6	/8
7	/10
8	/10
9	/10
10	/8
11	/10
12	/10
Total	/100

1. DEFINITIONS (2 points each):

   (a) What is the **rank** of a matrix?

   (b) What does it mean to say that a system of linear equations is **consistent**?

   (c) What does it mean to say that two vectors are **orthogonal**?

   (d) What does it mean to say that a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is **linearly independent**?

   (e) What is an **eigenvalue** of an $n \times n$ matrix A?

2. SHORT ANSWER – Calculations (2 points each): You do not have to show your work.

   (a) If $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 \\ 0 & 5 \end{bmatrix}$, what is the matrix $3A - 2B$?

   (b) Given $z = 2 - 3i$ and $w = 1 + 2i$, compute $z + w$ and zw.

   (c) If $\vec{v} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T$ and $\vec{w} = \begin{bmatrix} -2 & 4 & 1 \end{bmatrix}^T$, compute $\vec{v} \times \vec{w}$.

   (d) Determine whether $x = 2$, $y = 3$, $z = -1$ is a solution to the system

$$\begin{aligned} 2x - 3y + z &= -6 \\ -3x + y + 2z &= 5 \end{aligned}$$

3. SHORT ANSWER – More calculations (2 points each): You do not have to show your work.

   (a) Given $\vec{a} = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix}^T$ and $\vec{b} = \begin{bmatrix} -1 & 4 & 5 \end{bmatrix}^T$, determine the vector $\vec{x}$ such that $3\vec{a} + 2\vec{x} = 4\vec{b}$.

   (b) Let A and B be 3×3 matrices such that $\det A = -3$ and $\det B = 4$. What is the determinant of the matrix $C = 2A^3B^TA^{-1}$?

   (c) What is the projection of the vector $\vec{v} = \begin{bmatrix} -2 & 5 & 3 \end{bmatrix}^T$ onto the vector $\vec{w} = \begin{bmatrix} 4 & -3 & 0 \end{bmatrix}^T$?

   (d) Is the vector $\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ an eigenvector of the matrix $A = \begin{bmatrix} 7 & 2 \\ 3 & 6 \end{bmatrix}$?

4. SHORT ANSWER – A little more theory this time (2 points each):

   (a) Suppose $AX = B$ is the matrix form of a consistent system of 6 equations in 7 variables, such that the coefficient matrix A has rank 4. How many parameters will there be in the general solution to the system?

   (b) Suppose the matrix A is not invertible. Can there possibly exist a matrix B such that AB is invertible? Why or why not?

   (c) Suppose $\vec{a}$ and $\vec{b}$ are vectors in $\mathbb{R}^n$ for which it is known that $\vec{a} \cdot \vec{b} = 4$ and $(\vec{a} + \vec{b}) \cdot \vec{b} = 13$. What is $\|\vec{b}\|$?

   (d) Suppose $\lambda = 0$ is an eigenvalue of A. Can A be invertible? Why or why not?

[8] 5. Compute the determinant of the matrix

$$A = \begin{bmatrix} 4 & -1 & 3 & -1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & -1 & 1 \end{bmatrix}$$

6. Let $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ -2 & 2 & 15 \end{bmatrix}$

[5] (a) Compute the inverse of A.

[3] (b) Write A^{-1} as a product of elementary matrices.

[2] (c) Write A as a product of elementary matrices.

[5] 7. (a) Find the equation of the plane containing the points $P = (0, 1, 2)$, $Q = (-1, 0, 4)$, and $R = (2, 1, 3)$.

[2] (b) Find the area of the triangle whose corners are given by the three points P, Q, R in part (a).

[3] (c) Find the shortest distance from the point $P_0 = (4, -2, 7)$ to the plane in part (a).

[5] 8. Find point of intersection of the line $\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T + t \begin{bmatrix} -2 & 3 & 1 \end{bmatrix}^T$ and the plane given by the equation $x - 3y + 2z = 4$.

[5] 9. Find the line of intersection of the planes $x + y + z = 7$ and $2x - y + 4z = 3$.

[4] 10. Given $z = \sqrt{3} - i$, find the value of z^7. Express your answer in the form $x + iy$.

[4] 11. Find all complex solutions to the equation $z^3 = -1$. Express your answers in the form $x + iy$.

[10] 12. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(You do not have to compute P^{-1} or compute the product $P^{-1}AP$ to show it equals D.)

13. Let A be an $n \times n$ matrix.

[3] (a) Show that if B is similar to A, then $\det A = \det B$.

[4] (b) Suppose $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ are eigenvectors of A that all correspond to the same eigenvalue λ. Show that $\vec{v} \in \operatorname{span}\{\vec{u}_1, \ldots, \vec{u}_k\}$, then $\vec{v}$ is also an eigenvector corresponding to λ.

[3] (c) Let A be an $n \times n$ matrix of real numbers. Show that if $\lambda \in \mathbb{C}$ is an eigenvalue of A, then so is $\overline{\lambda}$.

Extra page for rough work. Do not remove if you want anything on this page to be graded.