MATH 2565 - Tutorial #9 Solutions

1. Use the ratio or root test to determine whether the series is convergent or divergent. (If the test is inconclusive, or impractical, determine converge with another test.)

   (a) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{5}\right)^n$

   The ratio test gives us

   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(3/5)^{n+1}}{n(3/5)^n}\right| = \frac{3}{5}\lim_{n\to\infty}\frac{n+1}{n} = \frac{3}{5} < 1,$$

   so the series converges.

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$

   Using the ratio test again (note that $2(n+1) = 2n+2$),

   $$\lim_{n\to\infty}\left|\frac{(2n+2)!}{((n+1)!)^2}\cdot\frac{(n!)^2}{(2n)!}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)(2n)!}{((n+1)n!)^2}\cdot\frac{(n!)^2}{(2n)!} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2} = 4 > 1,$$

   so the series diverges.

   (c) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

   Using the root test, we have

   $$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{1}{n}-\frac{1}{n^2}\right)^{2}n\right|} = \lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right) = 0 < 1,$$

   so the series converges.

   (d) $\displaystyle\sum_{n=1}^{\infty}\frac{5^n+n^4}{7^n+n^2}$

   Here, we first split the series into two parts:

   $$\sum_{n=1}^{\infty}\frac{5^n+n^4}{7^n+n^2} = \sum_{n=1}^{\infty}\frac{5^n}{7^n+n^2} + \sum_{n=1}^{\infty}\frac{n^4}{7^n+n^2}.$$

   Since $\displaystyle\frac{5^n}{7^n+n^2} < \frac{5^n}{7^n}$ for all $n \geq 1$ and $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{7^n}$ is a convergent geometric series, the first part of the series converges by comparison.

Now, we note that $\frac{n^4}{7^n+n^2} < \frac{n^4}{7^n}$ for all $n \geq 1$. For the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{7^n}$, the ratio test gives us

$$\lim_{n\to\infty} \left| \frac{(n+1)^4}{7^{n+1}} \cdot \frac{7^n}{n^4} \right| = \frac{1}{7} \lim_{n\to\infty} \frac{(n+1)^4}{n^4} = \frac{1}{7} < 1,$$

so this series converges, and thus, the second part of the series converges by comparison. Since our series is the sum of two convergent series, it converges.

(e) $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

Here, the root test gives us

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\left(1 + \frac{1}{n}\right)^{n^2} \right)^{1/n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1,$$

so the series diverges.

Note: If you didn't remember this last limit as a result from Math 1565, note that

$$\lim_{x\to\infty} (1 + 1/x)^x = \lim_{x\to\infty} e^{x\ln(1+1/x)} = e^{\lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x}} = e^{\lim_{x\to\infty} \frac{1}{1+1/x}} = e^1 = e,$$

using L'Hospital's Rule.

2. Determine if the series converges conditionally, or absolutely, or not at all:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{1 + n\sqrt{n}}$

We note that for $a_n = \dfrac{\sin(n\pi/3)}{1 + n\sqrt{n}}$ we have

$$|a_n| = \frac{|\sin(n\pi/3)|}{1 + n\sqrt{n}} \leq \frac{1}{1 + n^{3/2}} < \frac{1}{n^{3/2}},$$

so the series converges absolutely by comparison with the convergent p-series $\sum_{n=1}^{\infty} 1/n^{3/2}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

The series converges by the Alternating Series Test, since the terms are of the form $(-1)^n a_n$, where $a_n = 1/\sqrt{n}$, and $\{a_n\}$ is a positive, decreasing sequence with $\lim_{n\to\infty} a_n = 0$.

However, the series does not converge absolutely, since $\left| \dfrac{(-1)^n}{\sqrt{n}} \right| = \dfrac{1}{\sqrt{n}}$ is a p-series with $p = 1/2 < 1$, which diverges.

(c) $\sum_{n=1}^{\infty} n \cos(\pi n) \sin(1/n)$

This series does not converge. To see this, note that $\cos(\pi n) = (-1)^n$, while

$$\lim_{n\to\infty} n \sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1.$$

The series thus fails the divergence test, since $\lim_{n\to\infty} n \cos(\pi n) \sin(1/n)$ does not exist (for large n the terms alternate between values close to 1 and close to -1).

3. One can show that $\pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$. What is the least value of N such that the partial sum $S_N = \sum_{n=1}^{N} \frac{4(-1)^n}{2n+1}$ approximates the value of π, correct to 3 decimal places?

According to the Alternating Series Approximation Theorem, if a series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges to some limit L, then for each N, $\left|\sum_{n=1}^{N} -L\right| < a_{N+1}$. Thus, it suffices to determine the least N such that $a_N = \frac{4}{2N+1} < 0.0005$. For this, we must have

$$N > \frac{1}{2}\left(\frac{4}{0.0005} - 1\right) = \frac{7999}{2}.$$

Rounding to the nearest integer, we see that we need $N = 4000$ terms.

From this, we can conclude that this is not a very effective method for computing π. (A method referred to at least once as Gottfried's slow jam.)

4. For each series below, indicate whether it converges or diverges. Also indicate which convergence test you used, and why.

(a) $\sum_{m=1}^{\infty} me^{-m}$

We apply the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n} = \frac{1}{e} < 1,$$

so the series converges.

(b) $\sum_{n=1}^{\infty} \frac{2(n^2+2)^{2018}}{3(n^3+n+3)^{2018}}$

Noticing that for large n our terms are approximately $\dfrac{2(n^2)^{2018}}{3(n^3)^{2018}} = \dfrac{2}{3n^{2018}}$, we use limit comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2018}}$. We have

$$\lim_{n\to\infty} \frac{\dfrac{2(n^2+2)^{2018}}{3(n^3+n+3)^{2018}}}{\dfrac{1}{n^{2018}}} = \frac{2}{3} \lim_{n\to\infty} \frac{n^{2018}(n^2+2)^{2018}}{(n^3+n+3)^{2018}} = \frac{2}{3} \lim_{n\to\infty} \left(\frac{n^3+2n}{n^3+n+2}\right)^{2018} = \frac{2}{3} \cdot 1^{2018} = \frac{2}{3}.$$

Since this limit is positive and finite, we can conclude that our series converges by the limit comparison test.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^k}{k^{4k}}$

Here we try the root test, due to the powers in both the numerator and denominator. (It's less messy than the ratio test in this case.) We have

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{k!}{k^4} = \lim_{k\to\infty} \frac{k(k-1)(k-2)(k-3)}{k^4} \cdot (k-4)! = \infty,$$

so the series diverges. (One could also have verified that the series fails the divergence test.)