1. Find the area of the triangle with vertices $P = (2, 0, -1)$, $Q = (-3, 4, 2)$, and $R = (0, -3, 1)$.

Consider the vectors

$$\vec{u} = \overrightarrow{PQ} = \langle -5, 4, 3\rangle \text{ and}$$
$$\vec{v} = \overrightarrow{PR} = \langle -2, -3, 2\rangle.$$

We know that the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ is given by $\|\vec{u} \times \vec{v}\|$, and the given triangle is exactly half of this parallelogram. Since

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} 4 & 3 \\ -3 & 2 \end{vmatrix} \hat{\imath} - \begin{vmatrix} -5 & 3 \\ -2 & 2 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} -5 & 4 \\ -2 & -3 \end{vmatrix} \hat{k} \\ &= (4(2) - 3(-3))\hat{\imath} - (-5(2) - 3(-2))\hat{\jmath} + (-5(-3) - 4(-2))\hat{k} \\ &= 17\hat{\imath} + 4\hat{\jmath} + 23\hat{k} = \langle 17, 4, 23\rangle,\end{aligned}$$

we have

$$A = \frac{1}{2}\|\vec{u} \times \vec{v}\| = \frac{1}{2}\sqrt{(17)^2 + 4^2 + (23)^2}.$$

2. Find the point of intersection (if any) of the line $\langle x, y, z\rangle = \langle 1, -2, 3\rangle + t\langle 3, 5, -1\rangle$ with the plane $x - 2y + 3z = -6$

If (x, y, z) is a point that lies on both the line and the plane, then we know that (on the one hand)

$$x = 1 + 3t, \quad y = -2 + 5t, \quad \text{and } z = 3 - t, \tag{1}$$

since (x, y, z) lies on the line, and (on the other hand)

$$x - 2y + 3z = -6, \tag{2}$$

since (x, y, z) lies on the plane. Substituting (1) into (2), we get

$$(1 + 3t) - 2(-2 + 5t) + 3(3 - t) = -6,$$

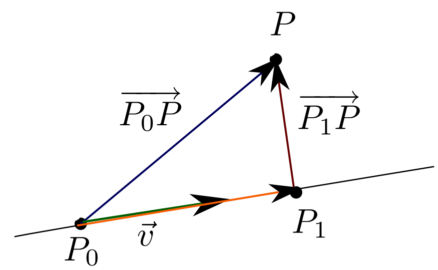
which simplifies to $-10t + 14 = -6$, so $-10t = -20$, and thus $t = 2$. Plugging this value for t into (1), we get

$$x = 1 + 3(2) = 7, \quad y = -2 + 5(2) = 8, \quad \text{and } z = 3 - 2 = 1.$$

Thus, the point of intersection is $(7, 8, 1)$. We can verify that this point is indeed on the plane, since $7 - 2(8) + 3(1) = -6$.

3. Find the shortest distance from the point $P = (1, 3, -2)$ to the line through the point $P_0 = (2, 0, -1)$ in the direction of $\vec{v} = \langle 1, -1, 0\rangle$. Also find the point P_1 on the line that is closest to P. **Include a diagram.**

We label a generic diagram as shown to the right, with the points P_0, P_1 on the line labelled, as well as the point P not on the line. From the diagram, we can see that the projection of the vector $\overrightarrow{P_0P}$ onto the line (which is the same as the projection of $\overrightarrow{P_0P}$ onto the vector $\vec{v}$, since $\vec{v}$ is parallel to the line) gives us the vector $\overrightarrow{P_0P_1}$: we have $\overrightarrow{P_0P_1} = \text{proj}_{\vec{v}}\,\overrightarrow{P_0P}$.



$\overrightarrow{P_0P_1} = \text{proj}_{\vec{v}}\,\overrightarrow{P_0P}$

We're given $\vec{v} = \langle 1, -1, 0\rangle$, and we compute $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \langle 1, 3, -2\rangle - \langle 2, 0, -1\rangle = \langle -1, 3, -1\rangle$. Since $\vec{v}\cdot\overrightarrow{P_0P} = 1(-1) + (-1)(3) + (0)(-1) = -4$ and $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v} = 1^2 + (-1)^2 + 0^2 = 2$, we have

$$\overrightarrow{P_0P_1} = \text{proj}_{\vec{v}}\,\overrightarrow{P_0P} = \left(\frac{\vec{v}\cdot\overrightarrow{P_0P}}{\|\vec{v}\|^2}\right)\vec{v} = \frac{-4}{2}\langle 1, -1, 0\rangle = \langle -2, 2, 0\rangle.$$

Since $\overrightarrow{P_0P_1} = \overrightarrow{OP_1} - \overrightarrow{OP_0}$, we have

$$\overrightarrow{OP_1} = \overrightarrow{OP_0} + \overrightarrow{P_0P_1} = \langle 2, 0, -1\rangle + \langle -2, 2, 0\rangle = \langle 0, 2, -1\rangle,$$

and thus $P_1 = (0, 2, -1)$. Finally, since P_1 is the closest point on our line to the point P (as per the diagram above), the distance from the point P to the line is the same as the distance from P to P_1. Thus,

$$d = d(P, P_1) = \sqrt{(1-0)^2 + (3-2)^2 + (-2+(-1))^2} = \sqrt{1+1+1} = \sqrt{3}.$$

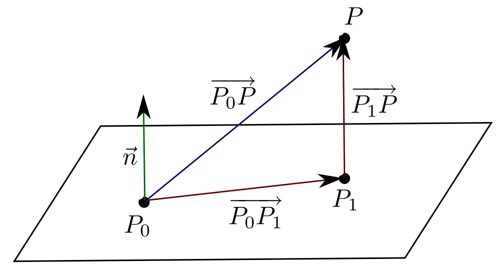
Note: if we wanted only the distance but didn't need to find the point P_1, we can notice (from – guess what? – the diagram!) that the distance from the point P to the line is given by the length of the vector $\overrightarrow{P_1P}$, and that

$$\overrightarrow{P_1P} = \overrightarrow{P_0P} - \overrightarrow{P_0P_1} = \langle -1, 3, -1\rangle - \langle -2, 2, 0\rangle = \langle 1, 1, -1\rangle,$$

and thus $d = \|\overrightarrow{P_1P}\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$.

4. Find the shortest distance from the point $P = (2, 8, 5)$ to the plane given by the equation $x - 2y - 2z = 1$. Also find the point P_1 on the plane that is closest to P.
Hint: Begin by finding any point P_0 that lies on the plane. **Include a diagram.**

We'll give two solutions. The first one uses the hint, along with vectors and projections, as with the previous problem. We first choose a point on the plane $x - 2y - 2z = 1$. If we set $y = z = 0$ in this equation, we're left with $x = 1$, so we can take $P_0 = (1, 0, 0)$.



Now, referring to the diagram above, we see that the desired distance is given by the length of the vector $\overrightarrow{P_1P}$, where P_1 is the point on the plane closest to P. Moreover, this vector is the projection of the vector $\overrightarrow{P_0P}$ onto the normal vector $\vec{n}$: $\overrightarrow{P_1P} = \text{proj}_{\vec{n}}\,\overrightarrow{P_0P}$. (Your answer will not depend on the point P_0 that you choose. Changing P_0 will change the vectors $\overrightarrow{P_0}$ and $\overrightarrow{P_0P_1}$, but it will not change the vector $\overrightarrow{P_1P}$.)

Recalling that for a general plane $ax+by+cz = d$, the normal vector is given by $\vec{n} = \langle a, b, c\rangle$, we conclude from the equation $x - 2y - 2z = 1$ that our normal vector is $\vec{n} = \langle 1, -2, -2\rangle$. Since we chose $P_0 = (1, 0, 0)$, we have

$$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \langle 2, 8, 5\rangle - \langle 1, 0, 0\rangle = \langle 1, 8, 5\rangle.$$

Since $\vec{n} \cdot \overrightarrow{P_0P} = 1(1) - 2(8) - 2(5) = -25$ and $\|\vec{n}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$, we have

$$\overrightarrow{P_1P} = \text{proj}_{\vec{n}}\,\overrightarrow{P_0P} = \left(\frac{\vec{n} \cdot \overrightarrow{P_0P}}{\|\vec{n}\|^2}\right)\vec{n} = \left(\frac{-25}{9}\right)\langle 1, -2, -2\rangle = \langle -25/9, 50/9, 50/9\rangle.$$

The distance from P to the plane is therefore

$$d = \|\overrightarrow{P_1P}\| = \left\|\left(\frac{-25}{9}\right)\langle 1, -2, -2\rangle\right\| = \frac{25}{9}\,\|\langle 1, -2, -2\rangle\| = \frac{25}{9}(3) = \frac{25}{3}.$$

To find the point P_1, we note that $\overrightarrow{P_1P} = \overrightarrow{OP} - \overrightarrow{OP_1}$, so

$$\begin{aligned}\overrightarrow{OP_1} = \overrightarrow{OP} - \overrightarrow{P_1P} &= \langle 2, 8, 5\rangle - \left\langle -\frac{25}{9}, \frac{50}{9}, \frac{50}{9}\right\rangle \\ &= \left\langle 2 + \frac{25}{9}, 8 - \frac{50}{9}, 5 - \frac{50}{9}\right\rangle = \left\langle \frac{43}{9}, \frac{22}{9}, -\frac{5}{9}\right\rangle,\end{aligned}$$

so $P_1 = \left(\frac{43}{9}, \frac{22}{9}, -\frac{5}{9}\right)$.

The second solution is to turn Problem 4 in to Problem 2. Referring again to the diagram above, if we construct the line L that passes through the point P in the direction of the normal vector $\vec{n}$, then the point P_1 we're looking for is exactly the point where L intersects the plane $x - 2y - 2z = 1$. As above, we have $\vec{n} = \langle 1, -2, -2\rangle$, so the line L is given by the vector equation

$$\langle x, y, z\rangle = \langle 2, 8, 5\rangle + t\langle 1, -2, -2\rangle.$$

Substituting $x = 2 + t$, $y = 8 - 2t$, and $z = 5 - 2t$ into the equation $x - 2y - 2z = 1$ of the plane, we have

$$(2 + t) - 2(8 - 2t) - 2(5 - 2t) = 9t - 24 = 1,$$

so $9t = 25$, and thus $t = \dfrac{25}{9}$. Putting this value for t back into the equations of our normal line through P, we get

$$P_1 = \left(2 + \frac{25}{9}, 8 - \frac{50}{9}, 5 - \frac{50}{9}\right) = \left(\frac{43}{9}, \frac{22}{9}, -\frac{5}{9}\right),$$

which is the same result we found using the other method. The distance from the point P to the plane is then the same as the distance from P to P_1, so using the distance formula we get

$$\begin{aligned}
d = d(P_1, P) &= \sqrt{\left(2 + \frac{25}{9} - 2\right)^2 + \left(8 - \frac{50}{9} - 8\right)^2 + \left(5 - \frac{50}{9} - 5\right)^2}\\
&= \sqrt{\left(\frac{25}{9}\right)^2 + \left(-\frac{50}{9}\right)^2 + \left(-\frac{50}{9}\right)^2}\\
&= \sqrt{\left(\frac{25}{9}\right)^2 (1^2 + (-2)^2 + (-2)^2)}\\
&= \frac{25}{9}\sqrt{1^2 + (-2)^2 + (-2)^2} = \frac{25}{9}(3) = \frac{25}{3},
\end{aligned}$$

which is the same distance as above.