Name: Solutions

Solve **one** of the following two questions:

1. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, v_2, \ldots, v_m$ are vectors in V such that the vectors $Tv_1, Tv_2, \ldots, Tv_m$ are linearly independent in W. Prove that the vectors $v_1, \ldots, v_m$ are linearly independent in V.

   Solution: Suppose that the vectors $Tv_1, \ldots, Tv_m$ are linearly independent. We want to show that the vectors $v_1, \ldots, v_m$ are linearly independent. Thus, we suppose that

   $$c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0$$

   for some scalars $c_1, c_2, \ldots, c_m \in \mathbb{F}$. We need to show that we're forced to take each of these scalars equal to zero. Since T is a linear transformation, we know that $T(0) = 0$. Therefore, we have

   $$\begin{aligned} 0 &= T(0) \\ &= T(c_1v_1 + c_2v_2 + \cdots + c_mv_m) && \text{(since } c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0\text{)} \\ &= c_1Tv_1 + c_2Tv_2 + \cdots + c_mTv_m && \text{(since } T \text{ is a linear map)} \end{aligned}$$

   But this means we have a linear combination of the vectors $Tv_1, \ldots, Tv_m$ equal to zero, and these vectors were assumed to be linearly independent. Thus, the only possibility is that each of the scalars is zero: $c_1 = 0, c_2 = 0, \ldots, c_m = 0$. But this is what we needed to show. Therefore, the vectors $v_1, \ldots, v_m$ are linearly independent.

2. Suppose that the vectors $v_1, \ldots, v_m$ span the vector space V, and that $T : V \to W$ is a linear transformation. Prove that the vectors $Tv_1, \ldots, Tv_m$ span $\operatorname{range} T$.

   Solution: Suppose that $V = \operatorname{span}\{v_1, \ldots, v_m\}$, and that $w \in \operatorname{range} T$. By definition, if $w \in \operatorname{range} T$, then there is some $v \in V$ such that $Tv = w$. Since we know that $V = \operatorname{span}\{v_1, \ldots, v_m\}$ and $v \in V$, it follows that there exist scalars $c_1, \ldots, c_m \in \mathbb{F}$ such that

   $$v = c_1v_1 + c_2v_2 + \cdots + c_mv_m.$$

   Using the linearity of T, this implies that

   $$\begin{aligned} w &= Tv \\ &= T(c_1v_1 + c_2v_2 + \cdots + c_mv_m) \\ &= c_1Tv_1 + c_2Tv_2 + \cdots + c_mTv_m. \end{aligned}$$

   But this means that any $w \in \operatorname{range} T$ can be written as a linear combination of the vectors $Tv_1, Tv_2, \ldots, Tv_m$, which is exactly the definition of what it means to say that the vectors $Tv_1, Tv_2, \ldots, Tv_m$ span the range of T, so we're done.

Total: 10 points