Math 4310 Assignment #9 Solutions
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1. Let $p : X \to Y$ be a quotient map, and let $A \subseteq X$ be a subspace. Show that the restricted map $q = p|_A : A \to p(A)$ need not be a quotient map. (Hint: consider the following example: $X = [0,1] \cup [2,3]$, $A = [0,1) \cup [2,3]$, and $p(x) = x$ for $x \in [0,1]$, and $p(x) = x - 1$ for $x \in [2,3]$.)

   Let $X = [0,1] \cup [2,3]$ and let $Y = [0,2]$. Define $p : X \to Y$ by
   $$p(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ x-1, & \text{if } x \in [2,3]. \end{cases}$$
   It's clear that p is a surjection, and p is continuous, since p is continuous on the two connected components of X. Moreover, since X is compact and Y is Hausdorff, p is a quotient map.

   Now let $A \subseteq X$ be given by $A = [0,1) \cup [2,3]$ with the subspace topology, and let $q : A \to p(A)$ be the restriction of p to A viewed as a surjection onto its image. Since restrictions of continuous maps are always continuous, q is a continuous surjection, but it is not a quotient map, since $p^{-1}([0,1]) = [2,3]$ is open in X (connected components are always open subsets), but $[0,1]$ is not an open subset of $[0,2]$.

2. With the same terminology as the previous problem, show that if either A is open in X and p is an open map, or A is closed in X and p is a closed map, then $p_A : A \to p(A)$ is a quotient map.

   Let $p : X \to Y$ be an open map, and let $A \subseteq X$ be open. Consider the restricted map $p_A : A \to p(A)$. Since p is continuous, its restriction p_A is continuous, and is a surjection by construction. Now, if $U \subseteq A$ is open in the subspace topology, then $U = V \cap A$ for some open subset $V \subseteq X$. Since A and V are open in X, so is U. Since p is an open map, $p_A(U) = p(U)$ is open in Y, and since $U \subseteq A$, $p(U) \subseteq p(A)$. Since A is open in X, $p(A)$ is open in Y, and thus $p(U) = p(U) \cap p(A)$ is open in $p(A)$. Thus, p_A is an open map, and therefore a quotient map.

   The proof when A is closed and p is a closed map is idenitical, with every instance of 'open' replaced by 'closed'.

3. Let X denote the quotient space obtained from $\mathbb{R}$ by identifying all of the integers to a single point.

   (a) Explain why X can be viewed as a countable union of circles that are all joined at a single point.

   To see this, note that $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[n, n+1]$, and that identifying the endpoints of the interval $[n, n+1]$ produces a copy of S^1. Thus, identifying the endpoints of *all* intervals to a single point produces one copy of S^1 for each $n \in \mathbb{Z}$, with all copies of S^1 joined at the single point in X corresponding to the set $\mathbb{Z}$ in the partition of $\mathbb{R}$ consisting of $\mathbb{Z}$ and the sets $\{x\}$ for $x \notin \mathbb{Z}$.

   Another way to think of it (although it doesn't quite work out exactly) is to consider the disjoint union

$$\tilde{\mathbb{R}} = \bigsqcup_{n\in\mathbb{Z}}[n, n+1] = \bigcup_{n\in\mathbb{Z}}[n, n+1] \times \{n\}$$

   and let $p : \tilde{\mathbb{R}} \to \mathbb{R}$ be the quotient map given by identifying $(n, n) \in [n, n+1] \times \{n\}$ with $(n, n-1) \in [n-1, n] \times \{n-1\}$. (That is we obtain $\mathbb{R}$ from $\tilde{\mathbb{R}}$ by gluing the disjoint union of intervals back together at their endpoints.

   Now, for each $n \in \mathbb{Z}$, we have a quotient map $p_n : [n, n+1] \to S^1$ given by identifying the endpoints of the interval. This allows us to define the map

$$\sqcup p_n : \tilde{\mathbb{R}} \to \bigsqcup_{n\in\mathbb{Z}} S^1$$

   given by applying the map p_n to $[n, n+1]$ for each $n \in \mathbb{Z}$. Now fix a point $x_0 \in S^1$ and define a quotient of $\bigsqcup S^1$ by identifying the points $(x_0, n) \in S^1 \times \{n\}$ for each $n \in \mathbb{N}$. The resulting space X' is then countably many copies of S^1 that have all been glued together at the point x_0. At this point we'd like to just claim that $X' = X$, but the details get messy, so let's just go with the first explanation.

   (b) Let Y be the union of the circles $(x - 1/n)^2 + y^2 = 1/n^2$, for $n \in \mathbb{N}$. (The space Y is called the "Hawaiian Earring".) Show that Y is *not* homeomorphic to X. (For a hint, see the first paragraph of the Wikipedia entry on the Hawaiian Earring.)

   We note that the space Y is compact. To see this, let $\mathcal{A}$ be any open cover of Y. (Since Y is a subspace of $\mathbb{R}^2$ it suffices to cover Y by open subsets of $\mathbb{R}^2$.) Some $A \in \mathcal{A}$ will have to contain the origin, and since A is open in $\mathbb{R}^2$, it contains an open disc D of radius $\epsilon > 0$. Choosing $N \in \mathbb{N}$ such that $1/N < \epsilon$, we note that all of the circles S^1_n given by $(x - 1/n)^2 + y^2 = 1/n^2$ for $n \geq N$ lie within the disc D and thus within A. It follows that $Y \setminus A$ consists of the union of the finitely many sets $S^1_n \setminus A$ for $n = 1, \dots, N-1$, and since A is open, and each circle S^1_n is closed, each $S^1_n \setminus A$ is closed and bounded, and therefore compact, and thus their union is compact. Thus, there exist finitely many sets $A_1, \dots, A_n \in \mathcal{A}$ that cover $Y \setminus A$, and thus $\{A_1, \dots, A_n, A\}$ is a finite subcover of Y.

Now, notice that X cannot be compact, since we can take an open cover of X as follows: choose an open neighbourhood of the point p corresponding to the integers whose preimage in $\mathbb{R}$ is of the form $\bigcup(n-1/4, n+1/4)$, together with the open intervals $(n, n+1)$. Then this is an open cover of X with no finite subcover.

Since Y is compact and X is not, X cannot be homeomorphic to Y.

4. Let $f : X \to X'$ be a continuous function and suppose that we have partitions $\mathcal{P}, \mathcal{P}'$ of X and X', respectively, such that if two points in X lie in the same member of $\mathcal{P}$, then $f(x)$ and $f(x')$ lie in the same member of $\mathcal{P}'$. If Y and Y' are the quotient spaces of X and X' corresponding to the given partitions, show that f induces a map $\tilde{f} : Y \to Y'$ and that if f is a quotient map, then so is $\tilde{f}$.

   Define a map $\tilde{f} : Y \to Y'$ by $\tilde{f}([x]) = [f(x)]$, where $[x] \in Y$ denotes the equivalence class of $x \in X$, and $[f(x)] \in Y'$ denotes the equivalence class of $f(x) \in X'$. By assumption, if $y \in [x]$, then $f(y) \in [f(x)]$, so $\tilde{f}$ does not depend on the choice of representative in $[x]$, and therefore is well-defined.

   Now, suppose that f is a quotient map, let $p : X \to Y$ and $p' : X \to Y'$ denote the quotient maps defined by the partitions $\mathcal{P}$ and $\mathcal{P}'$, and notice that $\tilde{f}$ is defined by the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & X' \arrow[d, "p'"] \\ Y \arrow[r, "\tilde{f}"] & Y' \end{tikzcd}$$

   since for any $x \in X$, $\tilde{f}(p(x)) = \tilde{f}([x]) = [f(x)] = p'(f(x))$. Now, we note that for any subset $U \subseteq Y'$, we have

$$p^{-1}(\tilde{f}^{-1}(U)) = (\tilde{f} \circ p)^{-1}(U) = (p' \circ f)^{-1}(U) = f^{-1}((p')^{-1}(U)). \tag{1}$$

   Thus, U is open in Y' if and only if $(p')^{-1}(U)$ is open in X', which is if and only if $f^{-1}((p')^{-1}(U))$ is open in X, which is if and only if $p^{-1}(\tilde{f}^{-1}(U))$ is open in X (by (1)), which is if and only if $\tilde{f}^{-1}(U)$ is open in Y. Therefore, $\tilde{f}$ is a quotient map.

5. (a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

   Let $p : X \to Y$ be given and suppose such a map f exists. Then p must be onto, since for any $y \in Y$ we have $p(f(y)) = I_Y(y) = y$. Moreover, if $p^{-1}(U)$ is open in X, then $f^{-1}(p^{-1}(U)) = (p \circ f)^{-1}(U) = I_Y^{-1}(U) = U$ is open in Y, and of course if U is open in Y then $p^{-1}(U)$ is open in X, since p is continuous. Thus, p is a quotient map.

(b) If $A \subseteq X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Show that any retraction map is a quotient map.

Given a retraction map $r : X \to A$, let $i : A \to X$ denote the inclusion map given by $i(a) = a$ for all $a \in A$. We know that any inclusion map is continuous in the subspace topology, and for any $a \in A$ we have $(r \circ i)(a) = r(i(a)) = r(a) = a$, so $r \circ i = I_A$, and thus r is a quotient map, by part (a).