Math 1410 Assignment #5 Solutions
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1. Suppose A and B are 4×4 matrices such that $\det(A) = 3$ and $\det(B) = -4$. Determine the values of

   (a) $\det(A^2B) = (\det(A))^2(\det(B)) = 3^2(-4) = -36$.

   (b) $\det(B^TBAB^{-1}) = \det(B^T)\det(B)\det(A)\det(B^{-1}) = \det(B)\det(B)\det(A)\left(\dfrac{1}{\det B}\right) = \det(B)\det(A) = -12$.

   (c) $\det(2AB^{-1}) = 2^4\det(AB^{-1}) = 2^4\det(A)\det(B^{-1}) = 2^4\left(\dfrac{\det(A)}{\det(B)}\right) = 16\left(\dfrac{3}{-4}\right) = -12$.

2. Suppose $\det(AB) = 0$. Must it be the case that $\det(A) = 0$ or $\det(B) = 0$? Prove this, or give a counterexample.

   Yes. If this were not the case, then we'd have $\det(A) \neq 0$ and $\det(B) \neq 0$, in which case we'd know that both A and B are invertible. However, we know that if A and B are invertible, then so is AB, which would imply that $\det(AB) \neq 0$, contradicting our assumption that $\det(AB) = 0$.

3. We say that an $n \times n$ matrix B is **similar** to an $n \times n$ matrix A if $B = P^{-1}AP$ for some invertible matrix P, and write $B \sim A$.

   (a) Show that if $B \sim A$, then $\operatorname{tr}(B) = \operatorname{tr}(A)$.

   Recall that $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ for any $n \times n$ matrices X and Y. If $B \sim A$, then $B = P^{-1}AP$ for some invertible matrix P. Therefore, (with $X = P^{-1}$ and $Y = AP$) we have
   $$\operatorname{tr}(B) = \operatorname{tr}(P^{-1}AP) = \operatorname{tr}(APP^{-1}) = \operatorname{tr}(AI) = \operatorname{tr}(A).$$

   (b) Show that if $B \sim A$, then $\det(B) = \det(A)$.
   We know that $\det(XY) = \det(X)\det(Y)$ for any $n \times n$ matrices X and Y. If $B \sim A$, then $B = P^{-1}AP$ for some invertible matrix P, and
   $$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \left(\frac{1}{\det(P)}\right)\det(A)\det(P) = \det(A).$$

(c) Suppose A is similar to a matrix $D = \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}$, and we know that $\operatorname{tr}(A) = 0$, and $\det(A) = 16$. What are the values of x and y?

By parts (a) and (b), we know that $\operatorname{tr}(A) = \operatorname{tr}(D)$, and by direct computation we have $\operatorname{tr}(D) = x + x + y = 2x + y$. Thus, $2x + y = 0$. Similarly, $x^2 y = \det(D) = \det(A) = 16$, so we have two equations:

$$2x + y = 0 \quad \text{and} \quad x^2 y = 16.$$

From the first equation we have $y = -2x$; substituting this into the second, we have $x^2 y = -2x^3 = 16$, so $x^3 = -8$, giving us $x = -2$. Since $y = -2x$, we have $y = 4$.

4. Let $\operatorname{adj}(A)$ denote the adjugate matrix of an $n \times n$ matrix A. Show that

$$\det(\operatorname{adj}(A)) = (\det(A))^{n-1}.$$

We know from the proof of the adjugate formula for the inverse that $A(\operatorname{adj}(A)) = \det(A) I_n$. This is true for any $n \times n$ matrix A, but let us first assume that A is invertible, so that $\det(A) \neq 0$. In this case, taking the determinant of both sides of the equation $A \operatorname{adj}(A) = \det(A) I_n$ gives us

$$\det(A) \det(\operatorname{adj}(A)) = \det(\det(A) I_n) = (\det(A))^n.$$

(On the right-hand side, $\det(A) I_n$ is diagonal, with each diagonal entry equal to $\det(A)$, and the determinant is given by multiplying the diagonal entries.)

Assuming that $\det(A) \neq 0$, we divide both sides by $\det(A)$, giving $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$, as required.

Now, what if $\det(A) = 0$? We claim that we must have $\det(\operatorname{adj}(A)) = 0$ as well. Notice that if $\det(A) = 0$, we have $A \operatorname{adj}(A) = \det(A) I_n = \mathbf{0}_n$, since $\det(A) = 0$. If $\det(\operatorname{adj}(A)) \neq 0$, then $\operatorname{adj}(A)$ is invertible, and we would have

$$A = (A \operatorname{adj}(A))(\operatorname{adj}(A))^{-1} = \mathbf{0}_n (\operatorname{adj}(A))^{-1}) = \mathbf{0}_n.$$

But this is impossible, because if $A = \mathbf{0}_n$, then $\operatorname{adj}(A) = \mathbf{0}_n$ as well, and the zero matrix is not invertible.

Thus, if $\det(A) = 0$, then A is not invertible and then neither is $\operatorname{adj}(A)$, giving us

$$\det(\operatorname{adj}(A)) = 0 = 0^{n-1} = \det(A)^{n-1}.$$