University of Lethbridge
Department of Mathematics and Computer Science
29th March, 2015, 3:05 - 4:20 pm

Math 2000B - Midterm

Last Name:____

First Name:____

Student Number:______________________________________

Record your answers below each question in the space provided. **Left-hand pages may be used as scrap paper for rough work.** If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

The value of each problem is indicated in the left-hand margins. The value of a problem does not always indicate the amount of work required to do the problem.

Outside aids, including, but not limited to, cheat sheets, smart phones, laptops, spy cameras, drones, and telepathic communication, are not permitted. You can keep a calculator with you if it makes you feel better.

For grader's use only:

Page	Grade
2	/10
3	/10
4	/8
5	/12
6	/10
7	/10
Total	/60

1. For each conditional statement below, identify (i) the hypothesis, (ii) the conclusion, and (iii) whether it is true or false.

[2] (a) If $3 + 4 = 8$, then $42 > 15$.

[2] (b) If $1 + 1 = 2$, then $5 - 3 = 7$.

[2] (c) If $5 \geq 5$, then $3 - 7 > 0$.

2. For each predicate below, add (i) a universal set, and (ii) appropriate quantifier(s) such that the resulting quantified statement is true.
(For example, given the predicate $2x - 4 = 6$, you could form the true statement $(\exists x \in \mathbb{Z})(2x - 4 = 6)$.)

[2] (a) $x^2 + 1 > 0$.

[2] (b) $2m - 3n = 4$.

3. For each of the problems below, provide a definition or example, as requested.

[2] (a) Define the **truth set** of a predicate $P(x)$.

[2] (b) Give an example of a tautology.

[2] (c) Define what it means for an integer a to be **congruent** to an integer b, modulo n.

[2] (d) Give four examples of integers a such that $a \equiv 3 \pmod 7$.

[2] (e) What is the **contrapositive** of a conditional statement $P \to Q$?

[3] 4. A set of real numbers A is defined to be *pronghornian*[1] if for each $a \in A$ there exists some element $b \in A$ such that $a^2 + b$ is even and $a^2 \equiv b^3 \pmod{2015}$. Complete the following sentence:

A set of real numbers A is **not** pronghornian if...

[5] 5. Prove the following logical equivalence using previously established logical equivalences: $(P \wedge Q) \rightarrow R \equiv (P \rightarrow R) \vee (Q \rightarrow R)$.

[1] Yes, I just made that up.

6. Determine whether the following statements are true or false. If a statement is true, give a **direct** proof of the statement. If it is false, provide a counterexample to support your claim.

[4] (a) For any integers a and b, if $a \equiv 3 \pmod 5$ and $b \equiv 2 \pmod 5$, then $ab \equiv 1 \pmod 6$.

[4] (b) For any integers a and b, if $ab \equiv 0 \pmod{12}$, then $a \equiv 0 \pmod{12}$ or $b \equiv 0 \pmod{12}$.

[4] (c) For any integers a, b, and c, if $a|b$ and $a|c$, then $a|(2b - 3c)$.

[5] 7. Given that $x \in \mathbb{R}$ is **irrational**, prove the following:

For every real number y, either $x + y$ is irrational, or $x - y$ is irrational.

Hint: Use proof by contradition, and take care to correctly form the negation of the given statement.

[5] 8. Use proof by cases to prove that for each integer n, $n^3 \equiv n \pmod 3$.

[5] 9. Use mathematical induction to prove that $4 \mid (5^n - 1)$ for each natural number n.

[5] 10. For which natural numbers n is it true that $2^n > (n+1)^2$? Support your claim with a proof by induction.

List of basic equivalences

1. Equivalences involving conditional statements
   (a) $P \to Q \equiv \neg P \vee Q$
   (b) $\neg(P \to Q) \equiv P \wedge \neg Q$
   (c) $P \to Q \equiv \neg Q \to \neg P$
2. Commutative properties
   (a) $P \vee Q \equiv Q \vee P$
   (b) $P \wedge Q \equiv Q \wedge P$
3. Associative properties
   (a) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
   (b) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
4. Distributive properties
   (a) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
   (b) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
5. Idempotent laws
   (a) $P \vee P \equiv P$
   (b) $P \wedge P \equiv P$
6. De Morgan's Laws
   (a) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
   (b) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
7. Law of the Excluded Middle
   (a) $P \vee \neg P \equiv T$
   (b) $P \wedge \neg P \equiv F$
8. Effect of Tautologies and Contradictions
   (a) $P \vee T \equiv T$
   (b) $P \wedge T \equiv P$
   (c) $P \vee F \equiv P$
   (d) $P \wedge F \equiv F$