Oriented curves and some basic topology

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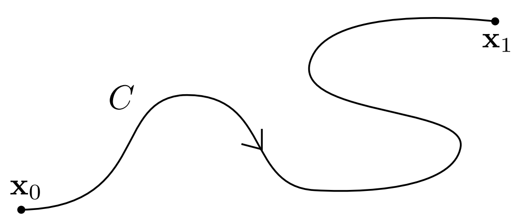
Abstract

We give a summary of definitions and properties related to oriented curves in $\mathbb{R}^n$, and then discuss the notion of connected and simply connected sets.

1 Oriented curves in $\mathbb{R}^n$

Let C be a bounded curve in $\mathbb{R}^n$, so that the boundary of C is either empty, or a two-point set $\partial C = \{\mathbf{x}_0, \mathbf{x}_1\}$ consisting of the endpoints of C. (If the boundary of C is empty, then C is known as a *closed curve*; we will discuss closed curves below. Assume that C has non-empty boundary. An *orientation* of C is a choice of initial point from ∂C; thus, it is clear that a curve can have two possible orientations. A curve C together with a choice of orientation is known as an *oriented curve*. The oriented curve with the opposite orientation is denoted by $-C$.

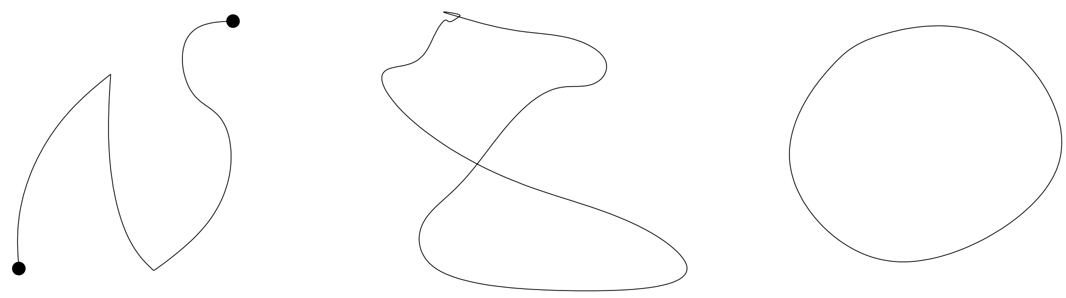


Suppose C is an oriented curve with initial point $\mathbf{x}_0$. A *parameterization* of C is a map $\mathbf{r} : [a, b] \to \mathbb{R}^n$ whose image is C, such that $\mathbf{r}(a) = \mathbf{x}_0$ and $\mathbf{r}(b) = x_1$. If $\mathbf{r}(t)$ is of class C^1 (that is, $\mathbf{r}'(t)$ exists and is continuous), and $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t \in (a, b)$, then we say that the oriented curve C is *smooth*.

Remark 1.1. The requirement that $\mathbf{r}'(t) \neq \mathbf{0}$ ensures that the curve C cannot have any corners or cusps. If we allow $\mathbf{r}'(t)$ to vanish, then it is possible for the curve C to have a corner while $\mathbf{r}$ remains C^1, by having $\mathbf{r}'$ approach the zero vector from either side of the corner. Note that this also means that a smooth curve cannot "double back" on itself: if $\mathbf{r}'(t_1) = \mathbf{x}_1$ for some $t_1 \neq b$, then we would have to have $\mathbf{r}'(t_1) = \mathbf{0}$. (A "particle" on the curve would have to stop in order to turn around and go back the way it came.)

By the above remark, a smooth oriented curve C cannot have corners or reverse direction. However, this is allowed for a *piecewise-smooth* oriented curve. Note that although a smooth curve cannot reverse direction, it can intersect itself at some finite number of points. When this happens, we say that the curve is *non-simple*.

Definition 1.2. *A continuous oriented curve* $\mathbf{r} : [a, b] \to \mathbb{R}^n$ *is called* **piecewise-smooth** *if there exist points* $a_0 = a < a_1 < \ldots < a_n = b$ *such that* $\mathbf{r}$ *is differentiable on* (a_i, a_{i+1}) *for each* i*, and* $\mathbf{r}'$ *is continuous on* $[a_i, a_{i+1}]$ *(which means that* $\lim\limits_{t \to a_i^+} \mathbf{r}'(t)$ *and* $\lim\limits_{t \to a_{i+1}^-} \mathbf{r}'(t)$ *exist). We say that* $\mathbf{r}$ *is* **closed** *if* $\mathbf{r}(a) = \mathbf{r}(b)$*, and* **simple** *if* $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ *for any* $a < t_1, t_2 < b$*.*



Piecewise-smooth and simple | Closed, not simple | Simple and closed

Note that $\mathbf{r}$ is a piecewise function, made up of C^1 functions $\mathbf{r}_i : [a_{i-1}, a_i]$. The image of each $\mathbf{r}_i$ is a smooth oriented curve C_i, and the curves C_i are such that the final point of C_i is equal to the initial point of C_{i+1}. To indicate that C is formed by joining together the smooth curves $C_1, \ldots, C_n$, we write $C = C_1 + \cdots + C_n$.

If two oriented smooth curves C_1 and C_2 have the same final point, then the oriented curve $-C_2$ has the same initial point as the final point of C_1, and we can form the piecewise-smooth oriented curve $C_1 + (-C_2)$, which we denote simply by $C_1 - C_2$. We can similarly form the piecewise-smooth curve $C_2 - C_1$, and the same argument holds if C_1 and C_2 have the same initial point.

Remark 1.3. Given a C^0 function f or vector field $\mathbf{F}$ defined along a piecewise smooth curve C as above, we define

$$\int_C f ds = \int_{C_1 + \cdots + C_n} f ds = \int_{C_1} f ds + \cdots + \int_{C_n} f ds, \text{ and}$$
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1 + \cdots + C_n} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot d\mathbf{r}.$$

Moreover, we have

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}, \quad \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

However, note that $\displaystyle\int_{-C} f ds = \int_C f ds$, so that line integrals of scalar functions do not depend on the orientation of the curve C.

As noted above, a smooth closed curve C is parameterized by a C^1 function $\mathbf{r}$ such that $\mathbf{r}(a) = \mathbf{r}(b)$, and in particular, C has no boundary points. Thus, the definition of orientation given above for non-closed curves does not apply. However, it is clear that there are nonetheless two distinct directions of motion along a closed curve, which define two opposite orientations. If C is a closed curve in $\mathbb{R}^2$, then there is a natural notion of positive and negative orientation corresponding to counter-clockwise and clockwise motion, respectively.[1] For closed curves in $\mathbb{R}^3$ (or higher dimensions) there is no natural notion of positive orientation. However, as we'll see when we get to Stokes' theorem, if C is the boundary of some surface S, then there is a notion of positive orientation of C relative to S. (Unlike in $\mathbb{R}^2$, the surface bounded by C is not unique.)

Remark 1.4. A curve in $\mathbb{R}^2$ that is closed and simple is also known as a *Jordan curve*. An important (and difficult to prove) result regarding Jordan curves is the *Jordan Curve Theorem*:

Theorem 1.5. *If* $\mathbf{r} : [a,b] \to \mathbb{R}^2$ *is a Jordan curve, then* $\mathbb{R}^2 \setminus \{\mathbf{r}(t) | t \in [a,b]\}$ *is not connected, and consists of two connected components, one of which is bounded (the "inside" of the curve).*

We discussed in class the fact that line integrals are independent of the choice of parameterization of a given curve C. If C is oriented and smooth, we should restrict ourselves to changes of parameter that preserve the chosen orientation as well as the smoothness of the curve.

Definition 1.6. *Let* $\mathbf{r} : [a,b] \to \mathbb{R}^2$ *be a piecewise-smooth curve. We say that a piecewise-smooth curve* $\tilde{\mathbf{r}} : [c,d] \to \mathbb{R}^2$ *is a* **reparameterization** *of* $\mathbf{r}$ *if there is a continuously differentiable function* $f : [a,b] \to [c,d]$ *with* $f'(t) > 0$*,* $f(a) = c$*, and* $f(b) = d$*, such that* $\mathbf{r}(t) = \tilde{\mathbf{r}}(f(t))$*.*

In other words, the image of both curves in $\mathbb{R}^2$ is the same, although the "speed" with which the curve is traced out may vary. Note that $\mathbf{r}'(t) = \tilde{\mathbf{r}}'(f(t))f'(t)$; since $f'(t) > 0$, this means that the tangent vectors to the two curves point in the same direction, but may have different lengths. For example, the unit circle can be parameterized by $\vec{r}(t) = \langle \cos t, \sin t \rangle$, with $t \in [0, 2\pi]$; a reparameterization of $\mathbf{r}$ is given by $\tilde{\mathbf{r}}(t) = \langle \cos 2t \sin 2t \rangle$, with $t \in [0, \pi]$ (here, $f(t) = t/2$). Note that our definition of a smooth curve required that $\mathbf{r}'(t) \neq \mathbf{0}$, which forces us to take $f(t) > 0$. We could also allow a change of parameter with $f'(t) < 0$; however, this reverses the orientation of the curve.

Remark 1.7. A consequence of our definition of smooth oriented curves C that are not closed is that they will only have finitely many self-intersections, and we can compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ using any parameterization compatible with the orientation. However, for closed curves, this need not be the case, as illustrated by the parametric curves $\mathbf{r}_1(t) = (\cos t, \sin t)$, $\mathbf{r}_2(t) = (\cos 2t, \sin 2t)$, with $t \in [0, 2\pi]$ in each case. (In the latter case every point on the circle is traced out twice.) However, if we restrict ourselves to *simple* oriented curves, whether closed or not, then it is possible to show that any two choices of parameterization of the same curve C are related by a reparameterization as defined above: viewing $C \subset \mathbb{R}^n$ as a set of points, suppose that C is the image of $\mathbf{r}(t) : [a,b] \to \mathbb{R}^n$ and also of $\tilde{\mathbf{r}}(u) : [c,d] \to \mathbb{R}^n$. Since C is simple, the functions $\mathbf{r}$ and $\tilde{\mathbf{r}}$ must be one-to-one on the respective open intervals (a,b) and (c,d) (but perhaps not on the closed intervals, allowing for closed curves, although it is clear that we must have $\mathbf{r}(a) = \tilde{\mathbf{r}}(c)$ and $\mathbf{r}(b) = \tilde{\mathbf{r}}(d)$). For each $t \in (a,b)$, let $\mathbf{x}_t = \mathbf{r}(t) \in C$. Since $\tilde{\mathbf{r}}$ is one-to-one, there is a unique $u_t \in (c,d)$ such that $\tilde{\mathbf{r}}(u_t) = \mathbf{x}_t$, and we can define a map $f : [a,b] \to [c,d]$ by $f(t) = u_t$. It is not hard to see that f is one-to-one; however, checking that f is actually C^1 takes a bit of work that we will omit here.

[1] In practice, C could be quite complicated, and it may not be clear which direction along C corresponds to "counter-clockwise" motion. Using the Jordan Curve Theorem we can define the "inside" of the curve, and define the positive orientation as that for which a person travelling along the curve would find the inside of the curve on his or her left.

2 Connected subsets of $\mathbb{R}^n$

Intuitively, a connected set is one that consists of a "single piece." For example, the connected subsets of $\mathbb{R}$ are intervals. Like many intuitive ideas, the precise definition needed to prove theorems about connected sets is much more technical.

Definition 2.1. *We say that two open sets $U, V \subseteq \mathbb{R}^n$ define a* **separation** *of a subset $A \subseteq \mathbb{R}^n$ of the complex plane if*

(a) $A \subseteq U \cup V$,

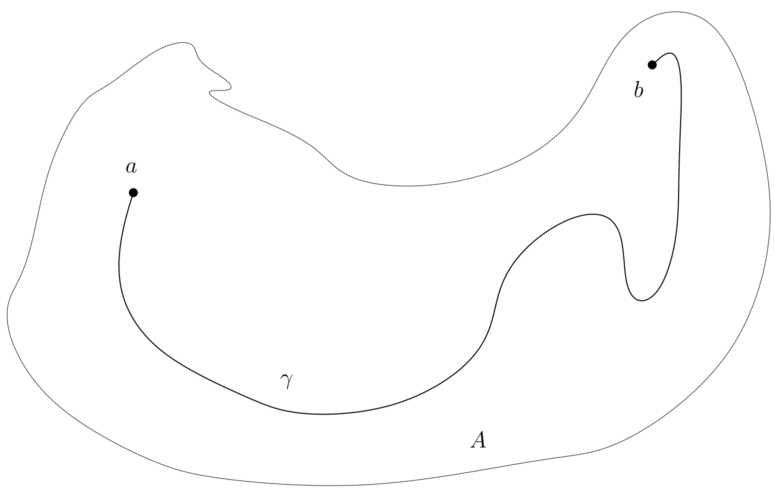
(b) $A \cap U \neq \emptyset$ *and* $A \cap V \neq \emptyset$,

(c) $(A \cap U) \cap (A \cap V) = \emptyset$.

If a separation of a set A exists, we say that A is **not connected***; otherwise, we say that A is* **connected***. We say that a subset $B \subseteq A$ is a* **connected component** *of A if B is connected, and maximal, in the sense that for any $a \in A \setminus B$, $B \cup \{a\}$ is not connected.*

The above definition of a connected set turns out to be a bit more general than a more intuitive notion of connectedness that we will find more useful in practice:

Definition 2.2. *We say that a set $A \subseteq \mathbb{R}^n$ is* **path-connected** *if for every $a, b \in \mathbb{R}^n$ there exists a continuous curve $\mathbf{r} : [0, 1] \to \mathbb{R}^n$ with $\mathbf{r}(0) = a$ and $\mathbf{r}(1) = b$.*



The requirement of path-connectedness is stronger than that of connectedness: one can show that every path-connected set is connected, but that the converse is not true. A common counterexample is the "topologist's sine curve," given as the union of the graph of $\sin(1/x)$ for $x > 0$ with the y-axis. However, if we require the set A to be open, then the two notions coincide:

Definition 2.3. *A* **domain** *in $\mathbb{R}^n$ is an open connected subset of $\mathbb{R}^n$.*

Proposition 2.4. *A domain is path-connected.*

While you may not have encountered the notion of connectedness before, it has one very familiar consequence:

Theorem 2.5 (The Intermediate Value Theorem)**.** *If a function f is continuous on a set C, and C is connected, then $f(C)$ is connected.*

Proof. If $f(C)$ were not connected, then there would be sets U and V that define a separation of $f(C)$, and the sets $f^{-1}(U)$ and $f^{-1}(V)$ would be a separation of C. $\square$

3 Homotopy, and simply-connected sets

3.1 Deformations of curves

When discussing problems such as independence of path for line integrals, it is often useful to be able to "deform" a given curve γ_1 into another, simpler curve γ_2 while leaving the value of the integral along such curves unaffected. The area of mathematics concerned with such deformations (and higher-dimensional analogues) is called *homotopy theory*, and is a branch of algebraic topology. Typically, we require such deformations to leave the endpoints of the curve unaffected.

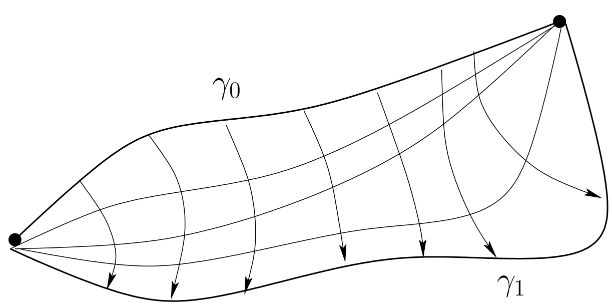
Definition 3.1. *Let $A \subset \mathbb{R}^n$ be an open connected set. Let $\gamma_0 : [0,1] \to A$ and $\gamma_1 : [0,1] \to A$ be two continuous curves such that $\gamma_0(0) = \gamma_1(0) = \mathbf{x}_0$ and $\gamma_0(1) = \gamma_1(1) = \mathbf{x}_1$. We say that γ_0 and γ_1 are* **homotopic** *if there exists a continuous function $H : [0,1] \times [0,1] \to A$, $H(s,t) = \gamma_s(t)$ (called a homotopy between γ_0 and γ_1) such that*

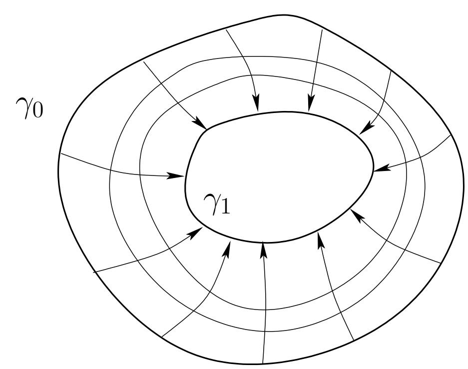
(a) $H(0,t) = \gamma_0(t)$ and $H(1,t) = \gamma_1(t)$,

(b) For each $s \in (0,1)$, $H(s,0) = \mathbf{x}_0$ and $H(s,1) = \mathbf{x}_1$,

(c) For each $s \in (0,1)$, $H(s,t)$ is a continuous curve contained in A.

In some cases we may require that each of the curves $\gamma_s(t)$ be piecewise-smooth, if γ_0 and γ_1 are. Homotopies between closed curves are similarly defined, with the exception that instead of requiring the endpoints to remain fixed, we require that $H(s,t)$ be a closed curve for each s: $H(s,0) = H(s,1)$ for all $s \in [0,1]$.



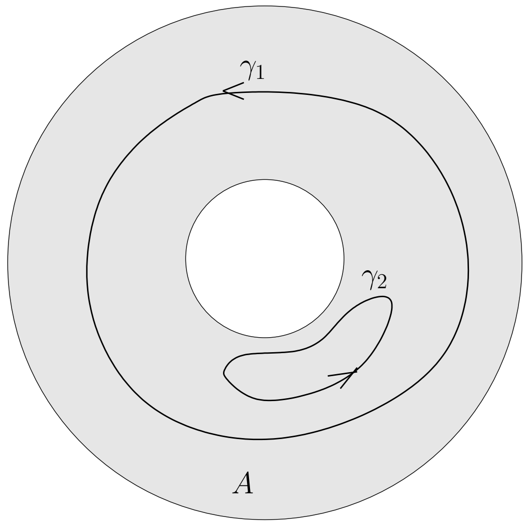


If a given simple closed curve γ_0 can be shrunk to a point (i.e. $\gamma_1(t) = H(1,t) = \mathbf{x}_0$ for all t, then we say that γ_0 is *contractible*, or *homotopic to a point.*

Definition 3.2. *A connected region $A \subseteq \mathbb{R}^n$ is called* **simply connected** *if any closed curve $\gamma : [0,1] \to A$ is contractible.*

Recall that for γ to be contractible, the image of each of the curves γ_s has to be contained within A. Thus, in the plane $\mathbb{R}^2$, a simply connected region is intuitively one that has no "holes": for example, if A is $\mathbb{R}^2$ minus the origin and γ is a closed curve that encircles the origin, then there is no way to shrink γ to a point without crossing over the origin, which does not belong to A. However, in $\mathbb{R}^3$ (and higher dimensions), simply removing a point does not result in a region that is simply connected, since a curve can now be "lifted over" the missing point.

Thus, as we will be able to prove using Stokes' theorem, if a vector field $\mathbf{F}$ in $\mathbb{R}^3$ is defined and C^1 except at perhaps finitely many points, then the condition $\nabla \times \mathbf{F} = \mathbf{0}$ is both necessary and sufficient for $\mathbf{F}$ to be the gradient of a function, while in $\mathbb{R}^2$, this is not the case.



The curve γ_2 is contractible within A, while γ_1 is not.

A region that is not simply connected is sometimes called *multiply connected*. The origin of this terminology seems to be that we can "connect" two points a and b in a path-connected region by some curve; if the region is simply connected then there is only one way to connect a to b "up to homotopy," while in a multiply connected region we can find two non-homotopic paths from a to b. (If γ_1 and γ_2 are both paths from a to b, then $\gamma_1 - \gamma_2$ is a closed curve, and γ_1 is homotopic to γ_2 if and only if $\gamma_1 - \gamma_2$ is contractible.)