**University of California, Berkeley**

FINAL EXAMINATION, Fall 2012
DURATION: 3 hours

Department of Mathematics

**MATH H53** Honors Multivariable Calculus

Examiner: Sean Fitzpatrick

**Total: 100 points**

Family Name: ______________________________
(Please Print)

Given Name(s): ______________________________
(Please Print)

Please sign here: ______________________________

Student ID Number: ______________________________

No aids, electronic or otherwise, are permitted, with the exception of the formula sheet provided with your exam. Partial credit will be given for partially correct work. Please read through the entire exam before beginning, and take note of how many points each question is worth.

**Good Luck!**

| FOR GRADER'S USE ONLY | |
|---|---|
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1. Consider the curve $C$ in $\mathbb{R}^2$ given by the vector-valued function $\mathbf{r}(t) = \langle t \sin t, t \cos t \rangle$, for $-\infty < t < \infty$.

[5] (a) Find the equations of the tangent lines to $C$ when $t = 0$, $\pi/2$ and $-\pi/2$.

[3] (b) If we restrict to $t \in [-\pi/2, \pi/2]$ we obtain a simple, closed curve. Sketch the curve using your results from part (a). Indicate the orientation of the curve.

(c) For the simple, closed curve in part (b), set up, but do not evaluate*, integrals for

[3] (i) The area enclosed by the curve.

[3] (ii) The arc length of the curve.

[4] (d) **Bonus**: Sketch the curve for $t \in [-\pi, \pi]$. What about $t \in [-2\pi, 2\pi]$? (Hint: swapping the functions for $x$ and $y$ gives you a polar curve.)

*You don't need to evaluate the integrals, but you should attempt to simplify the integrand.

2. Let $f(x, y) = x^3 + y^3 + 3xy - 27$.

[2] (a) Compute $\nabla f(x, y)$.

[6] (b) Find and classify all critical points of $f$.

[4] (c) Compute the derivative of $f$ at the point $(2, 4)$ in the direction of the *curve* given by $\mathbf{r}(t) = \langle 2t^2, 3t + 1\rangle$.

3. (a) Find the equation of the tangent plane to the surface $xyz^2 = 6$ at the point
[5] $(3, 2, 1)$.

(b) Find parametric equations for the tangent line to the curve of intersection of the
[7] paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

4. The integral $\int_{-2}^{2} \int_{0}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dz\, dy\, dx$ represents the volume of a solid.

[2] (a) Sketch the solid.

[6] (b) Re-write the integral using both cylindrical and spherical coordinates.

[4] (c) Find the volume using whichever one of the above integrals you prefer.

5. Evaluate the integral $\iint_D xy\, dA$, where $D$ is the region in the first quadrant bounded by the curves $y = x$, $y = 3x$, $xy = 1$, and $xy = 4$, using an appropriate change of variables.

[10]

6. Consider the vector field $\mathbf{F}(x, y, z) = \langle y^2, axy + z^3, byz^2 \rangle$, where $a$ and $b$ are constants.

   [8] (a) Find values of $a$ and $b$ such that $\mathbf{F}$ is a conservative vector field, and then find a potential function $f(x, y, z)$ such that $\nabla f(x, y, z) = \mathbf{F}(x, y, z)$.

   [4] (b) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is the line segment between the points $(1, 2, 0)$ and $(2, -1, 3)$ using the values of $a$ and $b$ found in part (a).

7. (a) Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle z, -x, -y\rangle$, and $S$ is the surface of the paraboloid $z = x^2 + y^2$, for $1 \leq z \leq 4$, oriented towards
[10] the $xy$-plane.

(b) Explain why we can conclude that $\mathbf{F}(x, y, z) = (x, -y, 3)$ is the curl of some other vector field $\mathbf{G}(x, y, z)$ (that is, $\mathbf{F} = \nabla \times \mathbf{G}$), and — without trying to find $\mathbf{G}$ —
[6] evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $S$ is the hemisphere $z = \sqrt{4 - x^2 - y^2}$.
(There is an easy way and a hard way.)

8. Prove *Gauss' Law*: Let $\mathbf{F}$ be the vector field $\mathbf{F} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}$, where $\mathbf{r}(x, y, z) = \langle x, y, z\rangle$. Let $E$ be any closed, bounded region in $\mathbb{R}^3$ with piecewise-smooth boundary $S$, oriented by the outward-pointing unit normal vector, such that $S$ does not pass through the origin. Then

[12]

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \begin{cases} 4\pi, & \text{if } 0 \in E \\ 0, & \text{if } 0 \neq E \end{cases},$$

Hint: Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ directly for the sphere $x^2 + y^2 + z^2 = a^2$. Then use the Divergence Theorem.