University of Lethbridge
Department of Mathematics and Computer Science

MATH 1565 - Tutorial #7 Solutions

[5] 1. Find the equation of the tangent line at the point $(1, 1)$ for the curve

$$(x^2 + y^2)^2 = 4xy.$$

Solution: Taking the derivative of both sides with respect to x, we obtain:

$$2(x^2 + y^2)(2x + 2y\frac{dy}{dx}) = 4y + 4x\frac{dy}{dx}$$

Since we are only interested in the value of $\frac{dy}{dx}$ at the point $(1, 1)$, we set $x = 1$ and $y = 1$ in the above, obtaining

$$4(2 + 2\frac{dy}{dx}) = 4 + 4\frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we find that our slope is

$$m = \frac{4 - 8}{8 - 4} = -1.$$

The equation of the line is therefore $y - 1 = -1(x - 1)$, or $y = -x + 2$.

If you solved first for $\frac{dy}{dx}$ in terms of x and y, you should have found (after cancelling a lot of 4s)

$$\frac{dy}{dx} = \frac{y - x^3 - xy^2}{y^3 + x^2y - x}.$$

Putting $x = 1$ and $y = 1$ at this stage still returns a slope of $m = -1$.

[2] 2. The function $f(x) = \dfrac{1}{x^2 + 1}$, with domain $[0, \infty)$, is one-to-one. Compute the value of $(f^{-1})'(1/2)$.

Hint: It is not necessary to find $f^{-1}(x)$. Note that $f(1) = 1/2$.

Solution: We use the formula $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$, with $f(x) = \dfrac{1}{x^2 + 1} = (x^2 + 1)^{-1}$.

We find

$$f'(x) = -1(x^2 + 1)^{-2}(2x) = \frac{-2x}{(x^2 + 1)^2}.$$

Since $f(1) = \frac{1}{2}$, we have $f^{-1}(1/2) = 1$, and thus

$$f'(f^{-1}(1/2)) = f'(1) = \frac{-2(1)}{(1^2 + 1)^2} = -\frac{1}{2}.$$

We conclude that $(f^{-1})'(1/2) = \dfrac{1}{f'(1)} = -2.$

[3] 3. Compute the derivative of $f(x) = \tan^{-1}(x^3)$, and $g(x) = \cosh^{-1}(x)$. (For $g(x)$, see handout.)

For the derivative of $f(x)$, we use the known result $\frac{d}{du}(\tan^{-1}(u)) = \frac{1}{1+u^2}$ and the Chain Rule to obtain

$$f'(x) = \frac{1}{1+(x^3)^2}\frac{d}{dx}(x^3) = \frac{3x^2}{1+x^6}.$$

For the derivative of $g(x)$, there are two possible approaches. For the first approach, we let $y = g(x) = \cosh^{-1}(x)$, so that $\cosh(y) = x$. Taking the derivative of both sides of the equation $\cosh(y) = x$ with respect to x, we obtain

$$\sinh(y)\frac{dy}{dx} = 1, \quad \text{so} \quad g'(x) = \frac{dy}{dx} = \frac{1}{\sinh(y)}.$$

To express $g'(x)$ in terms of x, we note that

$$\cosh^2(y) - \sinh^2(y) = 1,$$

so $\sinh^2(y) = \cosh^2(y) - 1 = x^2 - 1$. This gives us $\sinh(y) = \sqrt{x^2-1}$, so we find

$$g'(x) = \frac{1}{\sqrt{x^2-1}}.$$

Note: when we define $g(x) = \cosh^{-1}(x)$, we start with the function $h(x) = \cosh(x)$, restricted to $x \geq 0$, since cosh is not one-to-one if we also allow $x < 0$. This gives us a domain of $[0, \infty)$ and range of $[1, \infty)$ for h, so the domain of $g(x)$ is $[1, \infty)$, and the range is $[0, \infty)$.

Since $y = \cosh^{-1}(x) \geq 0$, we have $\sinh(y) \geq 0$, since $\sinh(t) \geq 0$ when $t \geq 0$. This is why we can take the positive square root above.

The alternative approach is to first derive (or look up) an explicit expression for $\cosh^{-1}(x)$ in terms of known functions.

If $y = \cosh{}^{-}1(x)$, then $x = \cosh(y) = \frac{e^y + e^{-y}}{2}$.

Multiplying this equation by $2e^y$ and rearranging, we obtain

$$(e^y)^2 - 2xe^y + 1 = 0,$$

which is quadratic in e^y. The quadratic formula gives us

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}.$$

Now, recall that given the domain and range of $g(x)$, we have $x \geq 1$ and $y \geq 0$, so $e^y \geq 1$. It's easy to check that the negative square root gives values between 0 and 1, so we take the positive square root, and then solve for y, giving us

$$y = g(x) = \ln(x + \sqrt{x^2-1}).$$

We can then find $g'(x)$ using the chain rule:

$$\begin{aligned}
g'(x) &= \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{1}{2}(x^2-1)^{-1/2}(2x)\right)\\
&= \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{x}{\sqrt{x^2-1}}\right)\\
&= \frac{1}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)\\
&= \frac{1}{\sqrt{x^2-1}},
\end{aligned}$$

as before.