Math 3410 Assignment #3
University of Lethbridge, Spring 2015

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Due date: Thursday, March 5th, by 5 pm.

Please provide solutions to the problems below, using the same guidelines as for Assignment #1:

1. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_5(\mathbb{R})$ be the linear transformation given by

$$(Tp)(x) = (3 - 2x + x^2)p(x).$$

   (a) Compute the matrix of T with respect to the standard bases $\{1, x, x^2, x^3\}$ of $\mathcal{P}_3(\mathbb{R})$ and $\{1, x, x^2, x^3, x^4, x^5\}$ of $\mathcal{P}_5(\mathbb{R})$.

   (b) Find the null space and range of T.

2. Let T_1 and T_2 be linear maps from V ot W.

   (a) Suppose that W is finite-dimensional. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible linear operator $S : W \to W$ such that $T_1 = ST_2$.

   (b) Suppose that V is finite-dimensional. Prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ if and only if there exists an invertible linear operator $S : V \to V$ such that $T_1 = T_2S$.

   Hint: The 'only if' direction of part (a) is more difficult than it might seem at first. In particular, you are *not* told that V is finite-dimensional, so you can't make use of the fundamental theorem of linear maps. Instead, you might try the following:

   We know that $\operatorname{range} T_1$ is a subspace of W, which is finite-dimensional. Therefore, there exists a basis $\{w_1, w_2, \ldots, w_m\}$ of $\operatorname{range} T_1$, and by definition of range, there exist vectors $v_1, \ldots, v_m \in V$ such that $T_1 v_i = w_i$ for $i = 1, \ldots, m$ (and by a recent quiz problem, you know that the v_i are independent). Now use the assumption $\operatorname{null} T_1 = \operatorname{null} T_2$ to conclude that the vectors $T_2 v_1, \ldots, T_2 v_m$ are linearly independent.

   Once you've done this, explain why it follows that $\dim \operatorname{range} T_1 \leq \dim \operatorname{range} T_2$. Then repeat the argument to get the opposite inequality, which will let you conclude that $\dim \operatorname{range} T_1 = \dim \operatorname{range} T_2$. Finally, explain why this tells you that there exists an isomorphism between the two subspaces, and then figure out how to extend this to an invertible linear operator on W.

3. Let $U \subseteq \mathbb{F}^\infty$ denote the vector space of sequences

   with "finite support": for each $x = (x_n) \in U$ there exists a natural number N_x such that $x_i = 0$ for all $i \geq N_x$. Thus, each $x \in U$ looks like

   $$x = (x_1, x_2, x_3, \ldots, x_{N_x}, 0, 0, \ldots).$$

   Prove that the vector space U is isomorphic to the vector space $\mathcal{P}(\mathbb{F})$ of all polynomials (of arbitrary degree) with coefficients in $\mathbb{F}$.

   Hint: Both vector spaces are infinite-dimensional, which means that you can't make an argument based on dimension. Instead, you will need to figure out how to explicitly construct an isomorphism.