Name:

1. Calculate the Jacobian of the transformation $T(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

   We have $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. To simplify the computation of the 3×3 determinant below, we recall that (i) we can perform the cofactor expansion along any row or column, and (ii) if any row or column has a common factor, it can be removed from the determinant. For example, $\begin{vmatrix} ax & y \\ az & w\end{vmatrix} = a\begin{vmatrix} x & y \\ z & w\end{vmatrix}$. Our Jacobian is given as follows:

$$\begin{aligned}
J_T(\rho,\phi,\theta) &= \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} \\
&= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= \rho^2\cos^2\phi\sin\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} + \rho^2\sin^3\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \\
&= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi)(\cos^2\theta + \sin^2\theta) \\
&= \rho^2\sin\phi.
\end{aligned}$$

2. Surprise bonus review problem!!
   Find the area of the parallelogram with vertices $(1,1), (3,2), (2,4), (4,5)$.

   Hint: choose two adjacent sides, represent them as vectors, and fit these into a 2×2 determinant.

   The points $(3,2)$ and $(2,4)$ are adjacent to the point $(1,1)$ in the parallelogram, with $(4,5)$ being the point opposite to $(1,1)$. The vectors $\vec{v} = \begin{bmatrix} 3-1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2-1 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ thus give us two adjacent sides of the parallelogram. The area of the parallelogram is therefore

$$A = |\det(\vec{v}|\vec{w})| = \left|\det\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\right| = |-5| = 5.$$