University of Toronto at Mississauga

Mid-Term Make-up Exam

MAT232HF
Calculus of Several Variables

Instructor: Sean Fitzpatrick
Duration: 110 minutes

NO AIDS ALLOWED. **Total: 60 marks**

Family Name: ______________________________
(Please Print)

Given Name(s): ______________________________
(Please Print)

Please sign here: ______________________________

Student ID Number: ______________________________

You may not use calculators, cell phones, or PDAs during the exam. Partial credit will be given for partially correct work. Please read through the entire test before starting, and take note of how many points each question is worth. Please put a box around your solutions so that the grader may find them easily.

FOR MARKER'S USE ONLY	
Problem 1:	/10
Problem 2:	/10
Problem 3:	/10
Problem 4:	/15
Problem 5:	/8
Problem 6:	/7
TOTAL:	/60

[5] 1. (a) Obtain the equation of the ellipse with foci at $(\pm 3, 0)$ and major axis of length 10, and then sketch the graph.

Since the foci are at $(\pm 3, 0)$ the major axis is along the x-axis, and the vertices along that axis are $(\pm 5, 0)$ since the centre must be at $(0, 0)$. The relationship $5^2 = 3^2 + a^2$, where a is the length of the semi-minor axis gives $a = 4$, so the other two vertices are at $(0, \pm 4)$, and the equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The graph then follows.

[5] (b) Eliminate the parameter to sketch the parametric curve $x(t) = \cos 2t$, $y(t) = \sin t$, $t \in [-\pi, \pi]$. Then, describe the motion of a particle moving according to these equations as t changes.

Since $\cos 2t = 1 - 2\sin^2 t$ we get $x = 1 - 2y^2$, which is a parabola with vertex $(1, 0)$, opening to the left.

As t varies from $-\pi$ to π the particle starts at the vertex $(1, 0)$, moves along the parabola to the point $(-1, -1)$ (at $t = -\pi/2$), then back to the vertex (at $t = 0$), out to the point $(-1, 1)$ ($t = \pi/2$), and then back again to the vertex at $t = \pi$.

2. Recall the identity $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.

[4] (a) Show that

$$(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{w} \cdot \vec{u})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}$$

Hint: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

We have

$$\begin{aligned}(\vec{u} \times \vec{v}) \times \vec{w} &= -\vec{w} \times (\vec{u} \times \vec{v}) \\ &= -(\vec{w} \cdot \vec{v})\vec{u} + (\vec{w} \cdot \vec{u})\vec{v} \\ &= (\vec{w} \cdot \vec{u})\vec{v} - (\vec{w} \cdot \vec{v})\vec{u}.\end{aligned}$$

(b) Let $\vec{a}$ be a given non-zero vector, and $\hat{u}$ a unit vector. Show that $\vec{a}$ can be written as $\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$, where $\vec{a}_{\parallel} = (\vec{a} \cdot \hat{u})\hat{u}$ is parallel to $\hat{u}$ and $\vec{a}_{\perp} = (\hat{u} \times \vec{a}) \times \hat{u}$
[6] is perpendicular to $\vec{a}$.

We have

$$\begin{aligned}\vec{a}_{\perp} &= (\hat{u} \times \vec{a}) \times \hat{u} \\ &= (\hat{u} \cdot \hat{u})\vec{a} - (\hat{u} \cdot \vec{a})\hat{u} \\ &= \vec{a} - \vec{a}_{\parallel}.\end{aligned}$$

3. In each case, determine whether or not the given line L and plane P are parallel, or intersect. If they intersect, find the point of intersection.

[5] (a) $L:\ x = 7 - 4t,\ y = 3 + 6t,\ z = 9 + 5t$, and $P:\ 4x + y + 2z = 17$.

The direction vector for the line is $\vec{v} =< -4, 6, 5 >$, while the normal vector for the plane is $\vec{n} =< 4, 1, 2 >$. Since $\vec{v} \cdot \vec{n} = -4 \cdot 4 + 6 \cdot 1 + 5 \cdot 2 = 0$, the vector is parallel to the plane. ($\vec{n}$ is perpendicular to all directions parallel to the plane.)

[5] (b) $L:\ x = 3 + 3t,\ y = 6 - 5t,\ z = 2 + 3t$, and $P:\ 3x + 2y - 4z = 1$.

In this case we find $\vec{v} \cdot \vec{n} =< 3, -5, 3 > \cdot < 3, 2, -4 >= -13 \neq 0$, so the line is not parallel to the plane.
To find the point of intersection, we need to find the value of t such that $x = 3+3t$, $y = 6 - 5t$, and $z = 2 + 3t$ satisfy the equation $3x + 2y - 4z = 1$ of the plane. Subsituting, we find

$$1 = 3(3 + 3t) + 2(6 - 5t) - 4(2 + 3t) = 9 + 12 - 8 + t(9 - 10 - 12) = 13 - 13t,$$

so we must have $t = 12/13$, giving the point of intersection $(75/13, 18/13, 62/13)$.

[8] 4. (a) Find all first-order partial derivatives of the following functions:

(i) $f(x, y) = \dfrac{x \sin y}{y \cos x}$.

$$f_x(x, y) = \frac{\sin y}{y \cos x} - \frac{x \sin y}{y \cos^2 x}(-\sin x)$$
$$f_y(x, y) = \frac{x}{\cos y} - \frac{x \sin y}{y^2 \cos x}$$

(ii) $g(x, y, z) = \ln(\frac{x}{y}) - ye^{xz}$.

$$g_x(x, y, z) = \frac{y}{x} - yze^{xz}$$
$$g_y(x, y, z) = \frac{y}{x}(-\frac{x}{y^2}) - e^{xz}$$
$$g_z(x, y, z) = -xye^{xz}$$

(iii) $h(x, y, z) = xy + yz + zx$.

$$h_x(x, y, z) = y + z$$
$$h_y(x, y, z) = x + z$$
$$h_z(x, y, z) = y + x$$

[7] (b) Given $u(x, y) = \dfrac{xy}{x + y}$, show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$$

I think I will skip typing out the details; I think it's a reasonably straight-forward calculation; any correct derivatives should get marks even if they don't manage to get 0 in the end.

5. Consider the elliptic paraboloid $x^2 + \frac{y^2}{b^2} = z$.

[2] (a) Describe the trace of this paraboloid in the plane $z = 1$.

The trace in the plane $z = 1$ is the ellipse $x^2 + \frac{y^2}{b^2} = 1$.

[3] (b) What happens to this trace as $b \to \infty$?

As b gets larger and larger, the above equation describes an ellipse with major axis (for $b > 1$) along the y axis getting increasingly longer. n In the limit, we are left with just a pair of lines: $x = 1$ and $x = -1$.

[3] (c) What happens to the paraboloid as $b \to \infty$?

For other positive values of z, the trace is still an ellipse, and in the limit, we get the lines $x = \pm\sqrt{z}$. At $z = 0$ we get the single line $x = 0$, and for $z < 0$ we get nothing. Thus the surface becomes the parabolic cylinder $z = x^2$.

[7] 6. Show that the limit

$$\lim_{(x,y)\to(1,1)} \frac{x - y^4}{x^3 - y^4}$$

does not exist.

The limit as $y \to 1$ along the line $x = 1$ is equal to 1, while the limit as $x \to 1$ along the line $y = 1$ is $1/3$. Thus, the limit cannot exist, as we do not get the same result along these two paths.

Extra space for rough work. Do **not** tear out this page.