MATH 1560 - Test #6 Solutions
Examiner: Sean Fitzpatrick

1. Compute the following antiderivatives:

[3] (a) The antiderivative F of $f(x) = 2x + \sec^2(x)$ such that $F(0) = 4$.

We have $F(x) = x^2 + \tan(x) + C$ for some C. Since $F(0) = 4$,

$$4 = F(0) = 0^2 + \tan(0) + C = C$$

so $C = 4$ and $F(x) = x^2 + \tan(x) + 4$.

[3] (b)

$$\begin{aligned}\int (3x^2 + 2\sqrt{x} - 5)\,dx &= \int (3x^2 + 2x^{1/2} - 5)\,dx \\ &= 3\left(\frac{1}{3}x^3\right) + 2\left(\frac{2}{3}x^{3/2}\right) - 5x + C \\ &= x^3 + \frac{4}{3}x^{3/2} - 5x + C\end{aligned}$$

[3] (c) $\displaystyle\int \left(\cos(x) - \frac{1}{\sqrt{1-x^2}}\right)dx = \sin(x) - \arcsin(x) + C$

[3] (d) $\displaystyle\int x^3 e^{x^4+2}\,dx$

Letting $u = x^4 + 2$, we have $du = 4x^3\,dx$, so $\frac{1}{4}\,du = x^3\,dx$ and

$$\int x^3 e^{x^4+2}\,dx = \int e^u \cdot \frac{1}{4}\,du = \frac{1}{4}e^u + C = \frac{1}{4}e^{x^4+2} + C.$$

2. Use Part I of the Fundamental Theorem of Calculus to compute the derivatives of the following functions:

[2] (a) $F(x) = \int_2^x \sin(t^2 + 3t)\,dt$

By direct application of FTC I, since our integrand is $f(t) = \sin(t^2 + 3t)$, we have

$$F'(x) = f(x) = \sin(x^2 + 3x).$$

[3] (b) $G(x) = \int_x^{x^2} \sqrt{t^4 + 1}\,dt$

First, we re-write $G(x)$ using properties of integrals:

$$G(x) = \int_x^0 \sqrt{t^4+1}\,dt + \int_0^{x^2} \sqrt{t^4+1}\,dt = -\int_0^x \sqrt{t^4+1}\,dt + \int_0^{x^2} \sqrt{t^4+1}\,dt.$$

For the second integral, we use the fact that combining the Chain Rule with FTC I gives us

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x).$$

Thus, we find

$$G'(x) = -\sqrt{x^4+1} + \sqrt{(x^2)^4+1}(2x) = 2x\sqrt{x^8+1} - \sqrt{x^4+1}.$$

3. Use Part II of the Fundamental Theorem of Calculus to evaluate the following definite integrals:

[3] (a)

$$\begin{aligned}\int_0^1 \left(3x^2 - 2x + 4\right)\,dx &= x^3 - x^2 + 4x\Big|_0^1 \\ &= 1^3 - 1^2 + 4(1) - (0^3 - 0^2 + 4(0)) = 4.\end{aligned}$$

[4] (b) $\int_0^{\pi/2} \cos(x)\sin^3(x)\,dx$

Letting $u = \sin(x)$, we have $du = \cos(x)\,dx$, and when $x = 0$, $u = \sin(0) = 0$, and when $x = \pi/2$, $u = \sin(0) = 1$. Thus,

$$\int_0^{\pi/2} \cos(x)\sin^3(x)\,dx = \int_0^1 u^3\,du = \frac{1}{4}u^4\Big|_0^1 = \frac{1}{4}.$$

 Total Points: 30

Extra group questions!

[3] 4. Evaluate the integral $\int_0^2 |2x-2|\,dx$.

Suggestion: either use properties of integrals to simplify, or sketch the graph and evaluate by interpreting the result as an area.

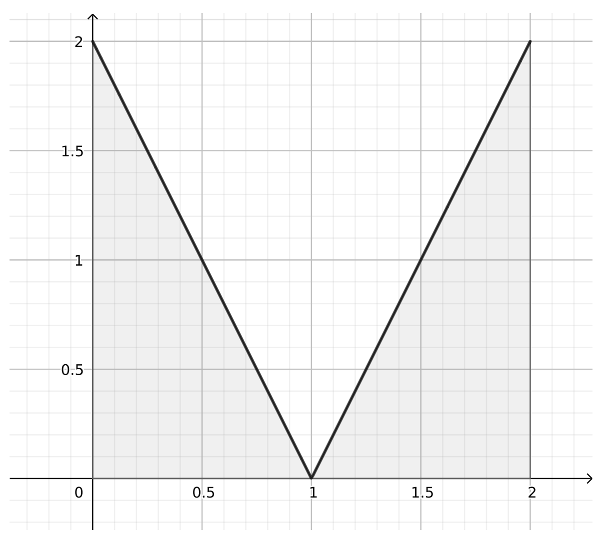
Since

$$|2x-2| = \begin{cases} 2x-2 & \text{if } 2x-2 \geq 0 \\ -(2x-2) & \text{if } 2x-2 < 0 \end{cases} = \begin{cases} 2x-2 & \text{if } x \geq 1 \\ 2-2x & \text{if } x < 1, \end{cases}$$

we have

$$\begin{aligned} \int_0^2 |2x-2|\,dx &= \int_0^1 |2x-2|\,dx + \int_1^2 |2x-2|\,dx \\ &= \int_0^1 (2-2x)\,dx + \int_1^2 (2x-2)\,dx \\ &= \left(2x - x^2\Big|_0^1\right) + \left(x^2 - 2x\Big|_1^2\right) \\ &= (2(1) - 1^2 - (0-0)) + (2^2 - 2(2) - (1^2 - 2(1))) \\ &= 1 + 1 = 2. \end{aligned}$$

Alternatively, from the sketch of the graph (shown on the right), we see that the area consists of the area of two triangles, each with base 1 and height 2. Thus, the integral is equal to the area



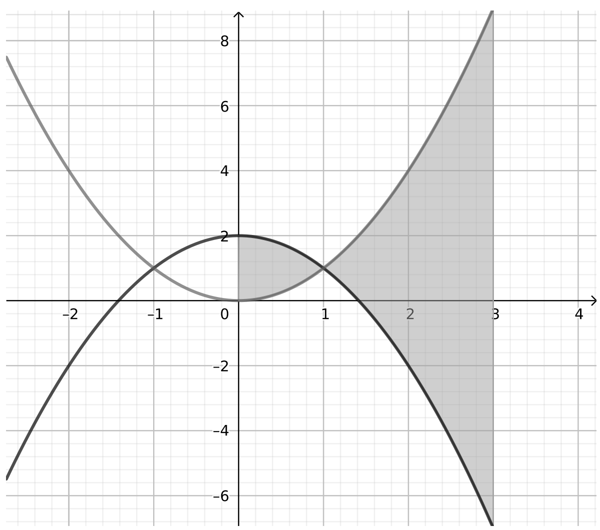
$$A = \frac{1}{2}(1)(2) + \frac{1}{2}(1)(2) = 2.$$

[3] 5. Find the area between the curves $y = 2 - x^2$ and $y = x^2$ for $0 \le x \le 3$.

First, we note that if $2 - x^2 = x^2$, then $2 = 2x^2$, so the curves intersect when $x^2 = 1$, or $x = \pm 1$. For $0 \le x \le 3$, this gives us the point of intersection when $x = 1$.

We note that for $0 \le x \le 1$, $2 - x^2 \ge x^2$, while for $1 \le x \le 3$, $x^2 \ge 2 - x^2$. Our area is therefore

$$\begin{aligned} A = \int_0^3 |(2-x^2) - x^2|\,dx &= \int_0^1 (2 - 2x^2)\,dx + \int_1^3 (2x^2 - 2)\,dx \\ &= \left(2x - \frac{2}{3}x^3\Big|_0^1\right) + \left(\frac{2}{3}x^3 - 2x\Big|_1^3\right) \\ &= \left(2(1) - \frac{2}{3}(1) - (0-0)\right) + \left(\frac{2}{3}(27) - 2(3) - (\frac{2}{3}(1) - 2(1)\right) \\ &= 2 - \frac{2}{3} + 18 - 6 - \frac{2}{3} + 2 = 16 - \frac{4}{3} = \frac{44}{3}. \end{aligned}$$



 Total Points: 30