1. Let $P = (1, 0, -2)$, $Q = (-3, 2, 4)$, and $R = (0, 5, -1)$ be points in $\mathbb{R}^3$.

   (a) Calculate the vectors $\vec{u} = \overrightarrow{PQ}$, $\vec{v} = \overrightarrow{QR}$, and $\vec{w} = \overrightarrow{PR}$.

   To find the components of a vector between two points, we subtract the coordinates of the initial point from the coordinates of the final point. Therefore,

$$\vec{u} = \begin{bmatrix} -3-1 \\ 2-0 \\ 4-(-2) \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 0-(-3) \\ 5-2 \\ -1-4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 0-1 \\ 5-0 \\ -1-(-2) \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}.$$

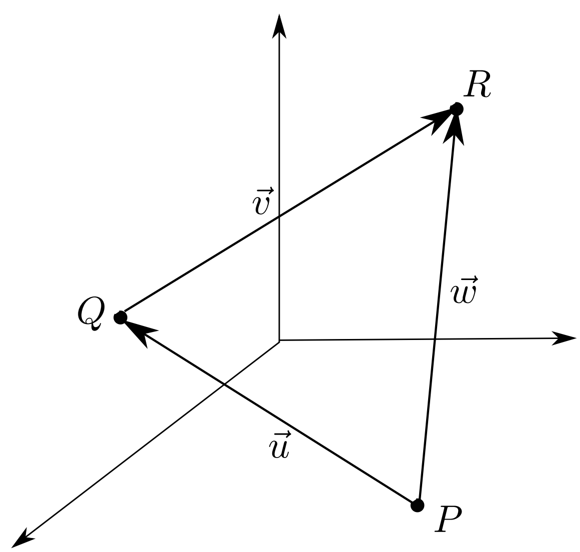
   (b) Show that $\vec{u} + \vec{v} = \vec{w}$.

   To add vectors we add the corresponding components; therefore,

$$\vec{u} + \vec{v} = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -4+3 \\ 2+3 \\ 6-5 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} = \vec{w}.$$

   (c) Explain, with a diagram, why your result in part (b) makes sense. (You do not have to accurately plot the points P, Q, R.)

   An inaccurate plot is given on the right. (The points P, Q, R don't reflect their true coordinates, but given these points, the vectors are drawn correctly.) It makes sense that $\vec{u} + \vec{v} = \vec{w}$, since $\vec{w}$ represents travelling directly from P to R, while the tip-to-tail rule for adding $\vec{u}$ and $\vec{v}$ can be thought of as travelling from P to R with a detour via the point Q.



   For an accurate (and interactive) plot of the three points and the corresponding vectors, see http://tube.geogebra.org/m/g1ivjhY4.

2. Let $\vec{a} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$.

   Find the vector $\vec{c}$ given by the linear combination $\vec{c} = 4\vec{a} - 3\vec{b}$.

   Using the definitions of addition and scalar multiplication of vectors, we have

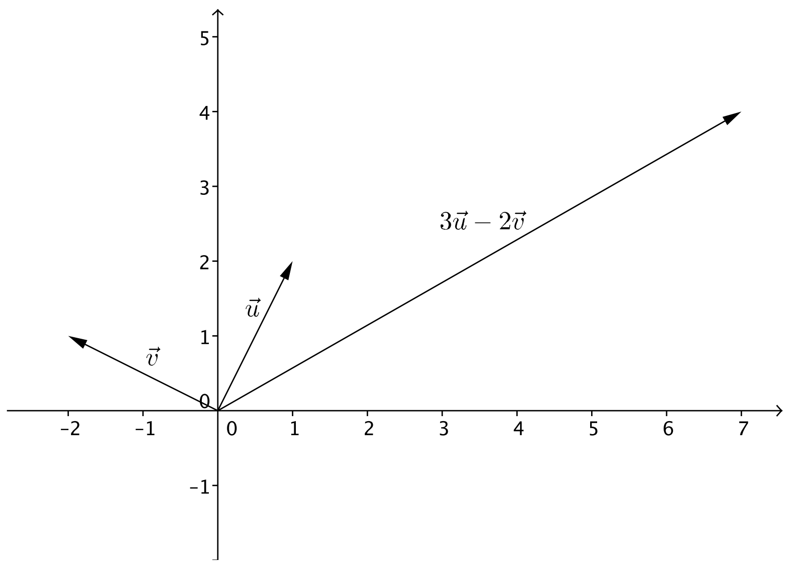
$$\vec{c} = 4\vec{a} - 3\vec{b} = 4\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} - 3\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ -28 \end{bmatrix} + \begin{bmatrix} 9 \\ -15 \\ -6 \end{bmatrix} = \begin{bmatrix} 4+9 \\ 16-15 \\ -28-6 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ -34 \end{bmatrix}.$$

Note: the calculation above *added* the vectors $4\vec{a}$ and $-3\vec{b}$ but you could equally well subtract the vectors $4\vec{a}$ and $3\vec{b}$.

3. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. Sketch the vectors $\vec{u}, \vec{v}$, and $3\vec{u} - 2\vec{v}$.

   We plot all three vectors in their standard positions (with tails at the origin). Note that

$$3\vec{u} - 2\vec{v} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$



4. Recall that the *absolute value* function $|x|$ is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

   (a) Calculate $|2|$, $|3.5|$, $|0|$, $|-5|$, and $|-7/4|$.

   Since $2 \geq 0$, the definition of $|x|$ gives $|2| = 2$. Simlarly, $3.5 \geq 0$ and $0 \geq 0$, so $|3.5| = 3.5$ and $|0| = 0$.
   Since $-5 < 0$, the definition of $|x|$ gives $|-5| = -(-5) = 5$, and similarly, $|-7/4| = -(-7/4) = 7/4$.

   (b) Explain in your own words what the effect of $|x|$ is on a real number x.

   If $x = 0$ or x is a positive real number, then $|x| = x$, so the absolute value function does nothing.
   If x is a negative real number, then $|x| = -x$, so the absolute value function switches the sign to give the corresponding positive number.

(c) Calculate $\sqrt{(2^2)}$, $\sqrt{(0)^2}$, $\sqrt{(-1)^2}$ and $\sqrt{(-2)^2}$.

We have $\sqrt{2^2} = \sqrt{4} = 2$, $\sqrt{0^2} = \sqrt{0} = 0$, $\sqrt{(-1)^2} = \sqrt{1} = 1$, and $\sqrt{(-2)^2} = \sqrt{4} = 2$.

(d) Explain why it's true that $\sqrt{x^2} = |x|$ for any real number x.

Generalizing from the examples above, we can see that whenever $x \geq 0$, $\sqrt{x^2} = x$, whereas if x is negative, the square root function (which is never negative) gives us back not x, but the corresponding positive number, which is $-x$. This is exactly the same as our description of the absolute value function in part (b) above.

(e) Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a vector in $\mathbb{R}^3$, and let $c \in \mathbb{R}$ be any scalar. Recall that $\|\vec{v}\|$ is defined by

$$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}.$$

Show that $\|c\vec{v}\| = |c|\|\vec{v}\|$. How is this related to the geometric interpretation of scalar multiplication?

By the algebraic definition of scalar multiplication, we have $c\vec{v} = c \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$, so using the definition of the magnitude of a vector given above,

$$\begin{aligned} \|c\vec{v}\| &= \sqrt{(cx)^2 + (cy)^2 + (cz)^2} \\ &= \sqrt{c^2x^2 + c^2y^2 + c^2z^2} \\ &= \sqrt{c^2(x^2 + y^2 + z^2)} \\ &= \sqrt{c^2}\sqrt{x^2 + y^2 + z^2} \\ &= |c|\|\vec{v}\|, \end{aligned}$$

which is what we needed to show. (In the last step, we used the definition of $\|\vec{v}\|$ given above, and the result from part (d).)

One final note of caution: a common algebraic error is to treat the square root function as if it were a linear function. Since $(x + y)^2 = x^2 + 2xy + y^2 \neq x^2 + y^2$, it's also true that $\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$. (If these were in fact equal, the Pythagorean Theorem would not be very interesting.) For example, $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. This is **not** equal to $\sqrt{3^2} + \sqrt{4^2} = 3 + 4 = 7$.