Name and student number: Solutions

1. Let $A = \{a, b, c, d\}$, $B = \{a, b, c\}$, and $C = \{s, t, u, v\}$.

   (a) Create a function $f : A \to C$ whose range is the set $\{u, v\}$, or explain why it is not
[2] possible to do so.

   Solution: Such a function is given by $f(a) = u, f(b) = u, f(c) = u, f(d) = v$. (There are many other possibilities, of course.)

   (b) Create a function $f : B \to C$ whose range is the entire set C, or explain why it is not
[2] possible to do so.

   Solution: This is not possible. The set $\{f(a), f(b), f(c)\}$ can contain at most three elements of C, and C contains 4 elements. To obtain the entire set C we would have to assign some element of B to more than one value, and then f would not be a function.

2. In each part, you're given sets A and B, and a function $f : A \to B$. Determine which functions are one-to-one.

[1] (a) $A = \{1, 2, 3\}, B = \{1, 2, 3, 4\}$, and $f(1) = 3, f(2) = 2, f(3) = 1$.

   Solution: This function is one-to-one by inspection: we can see that no value $f(x)$ appears twice.

[1] (b) $A = B = \{1, 2, 3, 4\}$, and $f(1) = 2, f(2) = 1, f(3) = 2, f(4) = 1$.

   Solution: Since $f(1) = f(3) = 2$ but $1 \neq 3$, f is not one-to-one.

[2] (c) $A = B = \mathbb{Z}$, and $f(m) = -m$.

   Solution: Suppose that $f(m) = f(n)$ for some $m, n \in \mathbb{Z}$. Then we have $-m = -n$, which after multiplying by -1 gives $m = n$. Thus, f is one-to-one.

[2] (d) $A = B = \mathbb{N}$, and $f(n) = n - 1$ if n is even, and $f(n) = n + 1$, if n is odd.

   Solution: We note that for each $k \in \mathbb{N}$, $f(2k) = 2k - 1$ and $f(2k - 1) = 2k$, so f interchanges each consecutive pair of natural numbers: $f(1) = 2$ and $f(2) = 1$, $f(3) = 4$ and $f(4) = 3$, etc. It follows that f is one-to-one.
   (Another way to see this is to note that for each $n \in \mathbb{N}$, $f(f(n)) = n$, since if n is even, then $n - 1$ is odd, and $f(f(n)) = f(n - 1) = n - 1 + 1 = n$, with a similar result if n is odd. If $f(n_1) = f(n_2)$ for some $n_1, n_2 \in \mathbb{N}$, then since f is a function, $n_1 = f(f(n_1)) = f(f(n_2)) = n_2$.)

Total: 10 points