Solutions to Quiz 6 Practice Problems
Math 2580
Spring 2016

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January 28th, 2016

1. Show that $\nabla(1/r^2) = -2\mathbf{r}/r^4$ for $r \neq 0$, where $\mathbf{r} = \langle x, y, z\rangle$ is the position vector for the point (x, y, z), and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$.

   Our function is $f(x, y, z) = r^{-2} = (x^2 + y^2 + z^2)^{-1}$. The gradient vector is thus

$$\begin{aligned}
\nabla(1/r^2) &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \\
&= \left\langle -2x(x^2+y^2+z^2)^{-2}, -2y(x^2+y^2+z^2)^{-2}, -2z(x^2+y^2+z^2)^{-2}\right\rangle \\
&= \frac{-2}{(x^2+y^2+z^2)^2}\langle x, y, z\rangle \\
&= \frac{-2}{r^4}\mathbf{r}.
\end{aligned}$$

2. Verify the chain rule for the function $f(x, y, z) = e^{xyz}$ and curve $\mathbf{r}(t) = (6t, 3t^2, t^3)$.

   Using the chain rule, we have

$$\begin{aligned}
\frac{d}{dt} f(\mathbf{r}(t)) &= \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \\
&= \langle yze^{xyz}, xze^{xyz}, xye^{xyz}\rangle|_{(x,y,z)=\mathbf{r}(t)} \cdot \langle 6, 6t, 3t^2\rangle \\
&= (3t^2)(t^3)e^{6t(3t^2)(t^3)}(6) + 6t(t^3)e^{6t(3t^2)(t^3)}(6t) + 6t(3t^2)e^{6t(3t^2)(t^3)}(3t^2) \\
&= e^{18t^6}(18t^5 + 36t^5 + 54t^5) = 108t^5e^{18t^6}.
\end{aligned}$$

   If instead we first evaluate $f(x, y, z)$ at $\mathbf{r}(t)$, we get

$$f(\mathbf{r}(t)) = f(6t, 3t^3, t^3) = e^{6t(3t^2)(t^3)} = e^{18t^6},$$

   so $\dfrac{d}{dt} f(\mathbf{r}(t)) = \dfrac{d}{dt}(e^{18t^6}) = e^{18t^6}(6(18t^5)) = 108t^5e^{18t^6}$, as before.

3. Calculate the derivative of the function $f(x,y) = e^{x^2 \cos y}$ at the point $(1, \pi/2)$ in the direction of the vector $\mathbf{v} = \frac{1}{5}\langle 3, 4\rangle$.

The gradient of f is given by

$$\nabla f(x,y) = \left\langle \frac{\partial}{\partial x} e^{x^2 \cos y}, \frac{\partial}{\partial y} e^{x^2 \cos y} \right\rangle = \langle 2x \cos y e^{x^2 \cos y}, -x^2 \sin y e^{x^2 \cos y}\rangle,$$

so

$$\nabla f(1, \pi/2) = \langle 2\cos(\pi/2) e^{\cos(\pi/2)}, -1^2 \sin(\pi/2) e^{\cos(\pi/2)}\rangle = \langle 0, -1\rangle.$$

Thus,

$$d_{\mathbf{v}} f(1, \pi/2) = \nabla f(1, \pi/2) \cdot \mathbf{v} = \langle 0, -1\rangle \cdot \langle 3/5, 4/5\rangle = -\frac{4}{5}.$$

4. Determine the direction in which the function $f(x,y) = e^x \sin y$ is increasing fastest at the point $(1,1)$.

Since a function always increases fastest in the direction of its gradient vector, the desired direction is $\nabla f(1,1)$. We compute

$$\nabla f(x,y) = \langle e^x \sin y, e^x \cos y\rangle, \text{ so } \nabla f(1,1) = \langle e \sin 1, e \cos 1\rangle.$$

5. Find a unit normal vector to the surface $xyz = 8$ at the point $(2,2,2)$.

Letting $f(x,y,z) = xyz$, a normal vector to $xyz = 8$ is given by $\nabla f(2,2,2)$. We have

$$\nabla f(x,y,z) = \langle yz, xz, xy\rangle, \text{ so } \nabla f(2,2,2) = \langle 4, 4, 4\rangle.$$

A unit vector in this direction is then $\mathbf{n} = \dfrac{1}{\sqrt{3}}\langle 1, 1, 1\rangle$.

6. Find the equation of the tangent plane to the ellipsoid $x^2 + 2y^2 + 3z^2 = 9$ at the point $(2,1,1)$.

Letting $f(x,y,z) = x^2 + 2y^2 + 3z^2$, we have $\nabla f(x,y,z) = \langle 2x, 4y, 6z\rangle$, so a normal vector to the tangent plane at $(2,1,1)$ is $\nabla f(2,1,1) = \langle 4, 4, 6\rangle$. The equation of the tangent plane is therefore

$$4(x-2) + 4(y-1) + 6(z-1) = 0.$$