MATH 1410 ASSIGNMENT #5 SOLUTIONS
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SEAN FITZPATRICK

(1) Recall that an $n \times n$ matrix A is **idempotent** if $A^2 = A$. Show that:

(a) The identity matrix I is the only invertible idempotent matrix.

Solution: First, we note that I is invertible and idempotent (since $I \cdot I = I$). Suppose now that A is idempotent, so that $A^2 = A$. If, in addition, we assume that A is invertible, then we can multiply both sides of $A^2 = A$ on the left by A^{-1}, giving us:

$$\begin{aligned} A^{-1}(AA) &= A^{-1}(A) \\ (A^{-1}A)A &= I_n \\ I_n A &= I_n \\ A &= I_n, \end{aligned}$$

so A is necessarily equal to the identity matrix.

(b) A matrix A is idempotent if and only if $I - 2A$ is self-inverse.

Solution: First, we note that $(I-2A)(I-2A) = I - 4A + 4A^2$. Now, if we assume that A is idempotent, then $A^2 = A$, and thus

$$(I-2A)(I-2A) = I - 4A + 4A = I,$$

which shows that $(I-2A)^{-1} = I - 2A$, so A is self-inverse. Conversely, if we assume that A is self-inverse, then $(I-2A)(I-2A) = I$, and thus $I = I - 4A + 4A^2$. Cancelling I from both sides we get $0 = -4A + 4A^2$, and thus $4A = 4A^2$. Multiplying both sides by $\frac{1}{4}$ gives us $A^2 = A$, and thus A is idempotent, as required.

(c) If A is idempotent, then $I - kA$ is invertible for any $k \neq 1$, and

$$(I - kA)^{-1} = I + \left(\frac{k}{1-k}\right)A.$$

Solution: Recall that for any matrix X, if we can find a matrix Y such that $XY = I$, then we know that X is invertible, and that $Y = X^{-1}$. Thus, it suffices to show that if $A^2 = A$, then $(I - kA)\left(\left(\frac{k}{1-k}\right)A\right) = I$. Thus, let us assume $A^2 = A$. We then have

$$\begin{aligned}
(I-kA)\left(I+\left(\frac{k}{1-k}\right)A\right) &= I+\left(\frac{k}{1-k}\right)A - kA + \left(\frac{k^2}{1-k}\right)A^2 \\
&= I+\left(\frac{k}{1-k}\right)A - kA + \left(\frac{k^2}{1-k}\right)A \qquad \text{(since } A^2 = A\text{)} \\
&= I+\left(\frac{k}{1-k} - k + \frac{k^2}{1-k}\right)A \\
&= I+\left(\frac{k-k(1-k)+k^2}{1-k}\right)A \\
&= I+\left(\frac{0}{1-k}\right)A \\
&= I+0 = I,
\end{aligned}$$

which is what we needed to show.

(2) Recall that an $n \times n$ matrix A is **symmetric** if $A^T = A$, and **antisymmetric** if $A^T = -A$.

(a) Show that $B + B^T$ is symmetric for **any** $n \times n$ matrix B.

Solution: Using the properties of the transpose, we have

$$(B+B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T,$$

so $B + B^T$ is symmetric.

(b) Show that $B - B^T$ is antisymmetric for **any** $n \times n$ matrix B.

Solution: As above, we have

$$(B-B^T)^T = B^T - (B^T)^T = B^T - B = -(B - B^T),$$

so $B - B^T$ is antisymmetric.

(c) Given any $n \times n$ matrix B, find a symmetric matrix U and an antisymmetric matrix V such that $B = U + V$.

Let $U = \frac{1}{2}(B + B^T)$ and let $V = \frac{1}{2}(B - B^T)$. Since we know that $(kA)^T = kA^T$ for any matrix A, we have

$$U^T = \left[\frac{1}{2}(B+B^T)\right]^T = \frac{1}{2}(B+B^T)^T = \frac{1}{2}(B+B^T) = U,$$

and

$$V^T = \left[\frac{1}{2}(B-B^T)\right]^T = \frac{1}{2}(B-B^T)^T = \frac{1}{2}(-(B-B^T)) = -\frac{1}{2}(B-B^T) = -V,$$

so U is symmetric, V is antisymmetric, and

$$U+V = \frac{1}{2}(B+B^T) + \frac{1}{2}(B-B^T) = \frac{1}{2}B + \frac{1}{2}B + \frac{1}{2}B^T - \frac{1}{2}B^T = B,$$

as required.

(3) What can be said about the determinant of A if:

(a) A is idempotent.

Solution: If A is idempotent, then $A^2 = A$, and thus

$$\det(A) = \det(A^2) = (\det(A))^2.$$

Therefore, if $x = \det(A)$, we must have $x = x^2$, so $x^2 - x = x(x-1) = 0$, and thus $\det(A)$ must equal either 0 or 1.

(b) A is self-inverse.

Solution: If A is self-inverse, then $A^2 = I$, and thus

$$1 = \det(I) = \det(A^2) = \det(A)^2,$$

which tells us that $\det(A) = \pm 1$.

(c) A is antisymmetric.

Solution: If A is antisymmetric, then $A^T = -A = (-1)A$. We then have

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A).$$

If A is an $n \times n$ matrix where n is odd, then this gives us $\det(A) = -\det(A)$, and thus $\det(A) = 0$. However, if n is even, we get the equation $\det(A) = \det(A)$, which tells us nothing. So when n is even, nothing can be said about the determinant.

(4) Determine all values of k such that the following matrices are invertible:

$$A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix} \quad B = \begin{bmatrix} k & k & 0 \\ k^2 & 4 & k^2 \\ 0 & k & k \end{bmatrix}$$

Solution: For the matrix A, we expand along the first column, giving us

$$\begin{aligned} \det(A) &= k \begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} + k \begin{vmatrix} -k & 3 \\ k+1 & 1 \end{vmatrix} \\ &= k(k^2 - 1 + 8) + k(-k - 3k - 3) \\ &= k(k^2 - 4k + 4) \\ &= k(k-2)^2. \end{aligned}$$

Thus, we see that $\det(A) = 0$ for $k = 0, 2$, and therefore, A is invertible for $k \neq 0, 2$.

For the matrix B, we expand along the first row, giving us

$$\begin{aligned} \det(B) &= k \begin{vmatrix} 4 & k^2 \\ k & k \end{vmatrix} - k \begin{vmatrix} k^2 & k^2 \\ 0 & k \end{vmatrix} \\ &= k(4k - k^3) - k(k^3) \\ &= k(4k - 2k^3) = 2k^2(2 - k^2). \end{aligned}$$

Thus, $\det(B) = 0$ if $k = 0$ or $k = \pm\sqrt{2}$, and thus B is invertible for all values of $k \neq 0, \sqrt{2}, -\sqrt{2}$.