1. For the following matrices, find (i) the characteristic polynomial, (ii) the eigenvalues of the matrix, and (iii) the corresponding eigenvectors.

   (a) $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & -4 \\ 0 & -1 & -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

2. For each of the matrices in problem 1, find an invertible matrix P such that $P^{-1}AP$ is diagonal, or explain why no such P exists. (You don't have to compute P^{-1}, unless you want to make sure you did everything correctly.)

3. An $n \times n$ matrix A is called **orthogonal** if $A^T = A^{-1}$ (that is, $A^T A = I$).

   (a) Show that the matrix $A = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$ is orthogonal.

   (b) Prove that a matrix A is orthogonal if and only if the columns of A form an orthonormal set of vectors. (That is, the columns $C_1, \ldots, C_n$ of A satisfy $\|C_i\| = 1$ for each $i = 1, \ldots, n$, and $C_i \cdot C_j = 0$ for each $i \neq j$.)

4. (**Bonus fun:**) An $n \times n$ matrix A is called *symmetric* if $A^T = A$. An important theorem in linear algebra, called the *Spectral Theorem*, guarantees that every symmetric matrix A can be "orthongonally diagonalized", meaning that there exists an **orthogonal** matrix P such that $P^T AP = D$ is diagonal. (Note from the previous problem that $P^T = P^{-1}$.)

   (a) Show that the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ is symmetric.

   (b) It is possible to prove that eigenvectors corresponding to **distinct** eigenvalues of a symmetric matrix are orthogonal. Use this fact to find an orthogonal matrix P such that $P^T AP$ is diagonal.

   (c) Show that the matrix C from problem 1(c) is also symmetric. You should have found that C had only two eigenvalues: $\lambda = 2$, and $\lambda = -1$ (which is repeated). Letting X denote the eigenevector corresponding to $\lambda = 2$, and letting Y_1, Y_2 denote the eigenevectors corresponding to $\lambda = -1$, show that X is orthogonal to both Y_1 and Y_2.

   (d) Chances are the two eigenvectors Y_1 and Y_2 corresponding to $\lambda = -1$ were not themselves orthogonal. Can you replace Y_2 by an eigenvector for $\lambda = -1$ that **is** orthogonal to Y_1? (*Hint:* projection.)

Name:

Tutorial time:

Please submit **one** *completed* solution from the worksheet for feedback.

Note: Your solution needs to contain enough detail for it to be clear what problem you're trying to solve!