Math 1560 Assignment #1 Solutions
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1. Evaluate the following limits:

   (a) $\lim_{x\to\infty}(\sqrt{9x^2+x}-3x)$

   This limit has the indeterminate form $\infty-\infty$, so we investigate algebraically:

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{9x^2+x}-3x) &= \lim_{x\to\infty}(\sqrt{9x^2+x}-3x)\cdot\frac{(\sqrt{9x^2+x}+3x)}{(\sqrt{9x^2+x}+3x)}\\ &= \lim_{x\to\infty}\frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x}\\ &= \lim_{x\to\infty}\frac{x}{\sqrt{x^2(9+1/x)}+3x}\end{aligned}$$

   At this point, we note that since $x\to\infty$, $x>0$, and thus $\sqrt{x^2}=x$, allowing us to factor out an x from the denominator. Proceeding, we find:

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{9x^2+x}-3x) &= \lim_{x\to\infty}\frac{x\cdot 1}{x(\sqrt{9+1/x}+3)}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}\\ &= \frac{1}{\sqrt{9}+3}=\frac{1}{6},\end{aligned}$$

   since $1/x\to 0$ as $x\to\infty$.

   (b) $\lim_{x\to-\infty}(\sqrt{9x^2+x}-3x)$

   In this case, $x\to-\infty$, so in particular, x is negative. If we look at the two parts of our function, the square root is always positive, and for $x<0$, $-3x>0$, so when x is negative, both parts of the function are positive, and both approach ∞ as $x\to\infty$.

   Thus, there is no indeterminate form in this case: both halves of the function are approaching $+\infty$, so

$$\lim_{x\to-\infty}(\sqrt{9x^2+x}-3x)=\infty.$$

2. (a) Show that the function $g(x) = |x|$ is continuous on $\mathbb{R}$.

We recall that the absolute value function is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

At any point $a > 0$, we see that $g(x)$ is continuous at a, since $g(x) = x$ for $x > a$, and any polynomial function is continuous. Similarly, g is continuous at a for any $a < 0$.

It remains to be shown that g is continuous at 0. From the definition we see that $g(0) = |0| = 0$; we need to show that the limit of g as $x \to 0$ is also 0. We consider left- and right-hand limits:

$$\begin{aligned} \lim_{x\to 0^-} |x| &= \lim_{x\to 0^-} (-x) && \text{(Since } x < 0\text{)} \\ &= 0 && \text{(By direct subsitution)} \\ \lim_{x\to 0^+} |x| &= \lim_{x\to 0^+} (x) && \text{(Since } x > 0\text{)} \\ &= 0 && \text{(By direct subsitution)} \end{aligned}$$

Since the left- and right-hand limits both equal zero, we have that

$$\lim_{x\to 0} |x| = 0 = |0|,$$

and thus our function is continuous at 0, and therefore on all of $\mathbb{R}$.

(b) Show that if f is a continuous function, then so is $|f|$.

Let f be a continuous function, and let $g(x) = |x|$. Then $g \circ f = |f|$, since $g(f(x)) = |f(x)|$ for any x in the domain of f. It follows that $|f|$ is continuous, since it is the composition of two continuous functions.

(c) Give a counterexample showing that the converse to part (b) is false. That is, find a function f such that $|f|$ is continuous, but f is not.

Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0. \end{cases}$$

We immediately see that f has a jump discontinuity at 0, since $\lim_{x\to 0^-} f(x) = -1$, but $\lim_{x\to 0^+} f(x) = 1$.

However, since $|-1| = 1$, we see that for any $x \in \mathbb{R}$, $|f(x)| = 1$, so $|f|$ is a constant function, and we know that every constant function is continuous.

3. Show that $\lim_{x \to 0} f(x) = 0$, where

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Let x be any real number. We know that $f(x) = 0$ or $f(x) = x^2$, depending on whether x is rational or irrational. Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, we see that

$$0 \leq f(x) \leq x^2$$

for any real number x. Since

$$\lim_{x \to 0}(0) = 0 = \lim_{x \to 0} x^2,$$

it follows from the Squeeze Theorem that $\lim_{x \to 0} f(x) = 0$.