University of Lethbridge
Department of Mathematics and Computer Science
17^{th} November, 2014, 5:00-5:50 pm
Math 4310 - Term Test II

Last Name: **SOLUTIONS**
First Name: **THE**

Student Number:

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

Page	Grade
2	/12
3	/8
4	/10
5	/10
Total	/40

[6] 1. Let X be a topological space. List as many conditions as possible equivalent to the statement "X is not connected." You don't need to justify your responses.

(Note: It is possible to earn bonus points on this problem. I will give 2 points per correct response, up to 6 points, plus 1 point for any additional correct responses.)

Here I was looking for answers that (a) didn't depend on X being a particular type of space, and (b) were *equivalent* in the sense of "X is not connected if and only if..." and were more interesting than "X is homeomorphic to Y and Y is not connected." Possible responses included:

- There exists a continuous surjection $f : X \to \{0, 1\}$.
- There exist nonempty, open, disjoint sets $U, V \subseteq X$ such that $X = U \cup V$.
- There exist nonempty disjoint sets $A, B \subseteq X$ such that $X = A \cup B$, and $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.
- There exists a proper nonempty subset $A \subseteq X$ that is both open and closed in X.
- There exists a proper nonempty subset $A \subseteq X$ with $\partial A = \emptyset$.

[6] 2. Let X be a subspace of $\mathbb{R}^n$ (with the Euclidean topology). List as many conditions as possible equivalent to the statement "X is compact." The same scoring rules apply as in Problem #1.

With the same expectations as the previous problem, X is a compact subspace of $\mathbb{R}^n$ if and only if

- Every open cover of X admits a finite subcover.
- X is closed and bounded.
- Any infinite subset of X has a limit point.
- Any sequence in X has a convergent subsequence.

(Note that except for the first point, the other three depend on X being a subspace of Euclidean space.)

3. Let X and Y be topological spaces, where X is compact and Y is Hausdorff, and let $f : X \to Y$ be continuous.

[5] (a) Prove that if $f : X \to Y$ is a surjection, then f is a quotient map.

It suffices to prove that f is a closed map, since if $f^{-1}(A)$ is closed in X for some subset $A \subseteq Y$, then since f is a surjection we have that $f(f^{-1}(A)) = A$ is closed in Y. (If f was not a surjection we would only have $f(f^{-1}(A)) \subseteq A$.) It follows that $f^{-1}(A)$ is closed if and only if A is closed, and by taking complements this is equivalent to the requirement that $f^{-1}(A)$ is open if and only if A is open.

Now, suppose that $F \subseteq X$ is closed. Since X is compact, F must also be compact. Since f is continuous, $f(F)$ is a compact subset of Y. Since Y is Hausdorff, $f(F)$ is closed. Thus f takes closed sets to closed sets, and therefore f is a quotient map.

[3] (b) Prove that if $f : X \to Y$ is a bijection, then f is a homeomorphism.

Since f is a continuous bijection, it remains to show that f^{-1} is continuous. From part (a) we know that f is a closed map. Thus, if $A \subseteq X$ is closed, then $(f^{-1})^{-1}(A) = f(A)$ is closed in Y, so f^{-1} is continuous.

4. Prove that if $f : X \to Y$ is a continuous surjection and X is path-connected, then Y is
[5] path-connected.

Suppose X is path connected and let $f : X \to Y$ be a continuous surjection. Choose any points $y_0, y_1 \in Y$. Since f is onto, there exist points x_0, x_1 in X with $f(x_0) = y_0$ and $f(x_1) = y_1$. Since X is path-connected, there exists a path $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, and then $f \circ \gamma : [0,1] \to Y$ is path in Y with $f \circ \gamma(0) = f(x_0) = y_0$ and $f \circ \gamma(1) = f(x_1) = y_1$. Thus, Y is path-connected.

5. Suppose that $X = A \cup B$, with A, B nonempty, disjoint, open subsets of X. (That is, $\{A, B\}$ is a separation of X.) Prove that a map $f : X \to Y$ is continuous if and only if the
[5] restrictions $f|_A : A \to Y$ and $f|_B : B \to Y$ are continuous.

If $f : X \to Y$ is continuous, then the restrictions $f|_A$ and $f|_B$ are continuous, since restrictions are always continuous. (Recall that if $i_A : A \to X$ and $i_B : B \to X$ denote the inclusion maps, which are always continuous in the subspace topology, then $f|_A = f \circ i_A$ and $f|_B = f \circ i_B$.)

Now suppose that $f|_A : A \to Y$ and $f|_B : B \to Y$ are continuous. There are two approaches that work well here. One is to note that for any open subset $U \subseteq Y$,

$$(f|_A)^{-1}(U) = (f \circ i_A)^{-1}(U) = i_A^{-1}(f^{-1}(U)) = A \cap f^{-1}(U),$$

which is open in X, since A and $f^{-1}(U)$ are open, and similarly $(f|_B)^{-1}(U) = B \cap f^{-1}(U)$ is open in X, so

$$(A \cap f^{-1}(U)) \cup (B \cap f^{-1}(U))(A \cup B) \cap f^{-1}(U) = X \cap f^{-1}(U) = f^{-1}(U)$$

is open in X, and thus f is continuous. An alternative proof is to note that $f = (f|_A) \cup (f|_B) : A \cup B \to Y$ and apply the gluing lemma, which holds trivially since $A \cap B = \emptyset$.

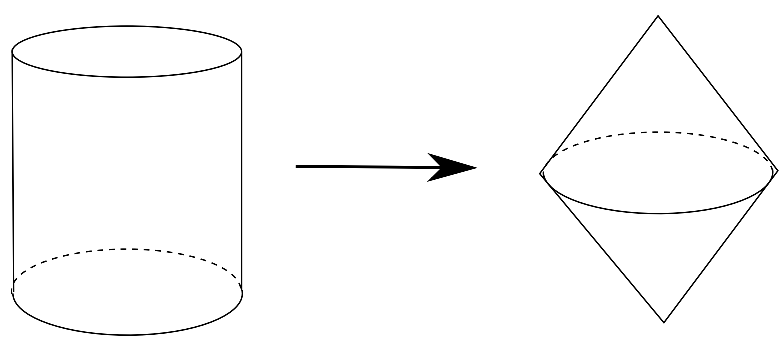
6. Suppose $\{F_n | n \in \mathbb{N}\}$ is a family of nonempty closed subsets of a topological space X, such that $F_{n+1} \subseteq F_n$ for each $n \in \mathbb{N}$. Prove that if X is compact, then $\bigcap_{n\in\mathbb{N}} F_n$ is nonempty. [6]

   Hint: To get a contradiction, suppose $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$, and take complements.

   Suppose that $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. Then $X = \emptyset^c = \left(\bigcap_{n\in\mathbb{N}} F_n\right)^c = \bigcup_{n\in\mathbb{N}} F_n^c$, so the collection $\{F_n^c = X \setminus F_n : n \in \mathbb{N}\}$ is an open cover of X. Since X is compact, there exists some finite subcover, so we can write
   $$X = F_{n_1}^c \cup F_{n_2}^c \cup \cdots \cup F_{n_k}^c,$$
   where without loss of generality we assume $n_1 < n_2 < \cdots < n_k$. Since $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$, it follows that $F_m \subseteq F_n$ for all $m > n$, and thus $F_n^c \subseteq F_m^c$ for all $n < m$. Therefore we have $F_{n_1}^c \subseteq F_{n_2}^c \subseteq \cdots \subseteq F_{n_k}^c$, and thus $X = F_{n_k}^c$. But this implies that $F_{n_k} = \emptyset$, which is a contradiction, since all of the sets F_n were assumed to be nonempty.

7. Consider the cylinder $X = S^1 \times [-1, 1]$, and let $\sim$ be the equivalence relation on X defined by the partition $\mathcal{P}$ consisting of the sets $S^1 \times \{-1\}$, $S^1 \times \{1\}$, and the single point sets $\{(x, t)\}$, where $x \in S^1$ and $t \in (-1, 1)$. Describe the quotient space $X/\sim$. (A formal proof is not required.) [4]

   The quotient space corresponding to the given partition is obtained by collapsing the sets $X \times 1$ and $X \times -1$ to a single point, and leaving all other points unaffected. If we collapsed only the set $X \times 1$ to a point, we'd have a space homeomorphic to the cone CS^1, since $[0, 1]$ is homeomorphic to $[-1, 1]$. In this case we also collapse the bottom of the cone to a point, so the result is a "double cone". This space is known as the *suspension* of S^1; the suspension of any topological space is defined in the same manner. In the particular case of S^1 the suspension is also homeomorphic to the sphere S^2.



 Total Points: 40