Math 4310 Assignment #10 Solutions
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Sean Fitzpatrick

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1. Let $f : S^1 \to \mathbb{R}$ be a continuous map, where $S^1 = \{(x, y)|x^2 + y^2 = 1\}$. Show that there exists a point $(x, y) \in S^1$ such that $f(x, y) = f(-x, -y)$. (Hint: S^1 is connected; in fact, it is path-connected.)

   Given $f : S^1 \to \mathbb{R}$, let $g(x, y) = f(x, y) - f(-x, -y)$. Choose a point $(x_0, y_0) \in S^1$ and let $\gamma : [0, 1] \to S^1$ be a path from (x_0, y_0) to $(-x_0, -y_0)$, and let $h = g \circ \gamma : [0, 1] \to \mathbb{R}$. Then $h(0) = g(\gamma(0)) = f(x_0, y_0) - f(-x_0, y_0)$. If $h(0) = 0$, we're done. If not, note that $h(1) = g(\gamma(1)) = f(-x_0, -y_0) - f(x_0, y_0) = -h(0)$. Then either $h(0) < 0$ and $h(1) > 0$ or $h(0) > 0$ and $h(1) < 0$; in either case, the Intermediate Value Theorem guarantees the existence of some $c \in (0, 1)$ such that $h(c) = 0$, and $(x, y) = \gamma(c)$ is the desired point.

   Alternative solution: suppose that no such point exists. Then the function $g(x, y) = \dfrac{f(x, y) - f(-x, -y)}{|f(x, y) - f(-x, -y)|}$ is defined on all of S^1, since $f(x, y) - f(-x, -y) \neq 0$, and since $g(-x, -y) = -g(x, y)$, g is a continuous surjection from S^1 to $\{-1, 1\}$. But this is impossible, since S^1 is connected.

2. Let X be a topological space. Prove that CX, the cone over X, is path-connected.

   (Recall that CX is the quotient of $X \times [0, 1]$ obtained by collapsing $X \times \{1\}$ to a single point.)

   Let $p : X \to CX$ denote the quotient map, and let $a \in CX$ denote the apex of the cone; that is, such that $p^{-1}(a) = X \times \{1\}$. It suffices to show that for any point $y \in CX$, there exists a path $\gamma : [0, 1] \to CX$ from y to a, since for any two points $y_1, y_2 \in CX$ with paths γ_1, γ_2 from y_1, y_2 to a, respectively, the path $\gamma_1 \star \gamma_2^{-1}$ given by

$$\gamma_1 \star \gamma_2^{-1}(s) = \begin{cases} \gamma_1(s) & \text{if } 0 \leq s \leq 1/2 \\ \gamma_2(2 - 2s) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

is a path from y_1 to y_2. So, let $y \in CX$. If $y = a$, we can take the constant path. If $y \neq a$, then $y = p(x,t)$ for some $x \in X$ and $t \in [0,1)$. Let $\alpha : [0,1] \to X \times [0,1]$ be the path given by

$$\alpha(s) = (x, t + s - st).$$

Then α is clearly continuous, since the map $f : [0,1] \to [0,1]$ given by $f(s) = t+(1-t)s$ is continuous, and α is the product of f and a constant map. Moreover, $\alpha(0) = (x,t)$ and $\alpha(1) = (x,1)$, so α is a path in $X \times [0,1]$ from (x,t) to $(x,1)$. It follows that the composition $\gamma = p \circ \alpha : [0,1] \to CX$ is a path from y to a.

3. (a) Let X be a connected topological space, and call a point $p \in X$ a *cut point* if $X \setminus \{p\}$ is not connected. Prove that the existence of a cut point is a topological property. (That is, if $f : X \to Y$ is a homeomorphism and X has a cut point p, $q = f(p)$ must be a cut point of Y.)

   We first prove the following more general result: if $f : X \to Y$ is a homeomorphism, and $p \in X$ is any point, then the function $g : X \setminus \{p\} \to Y \setminus \{f(p)\}$ given by $g(x) = f(x)$ for all $x \in X \setminus \{p\}$ is also a homeomorphism. (As usual we assume $X \setminus \{p\}$ and $Y \setminus \{f(p)\}$ are given the subspace topology.) To see this, we note that since f is a bijection that maps p to $f(p)$, g must also be a bijection. We know that g is continuous, since it's the restriction of a continuous function. Moreover, g^{-1} is continuous, since g^{-1} is just the restriction of f^{-1} to $Y \setminus \{f(p)\}$.
   Now, if X is connected and $f : X \to Y$ is a homeomorphism, then Y must be connected as well. If $p \in X$ is a cut point, then $X \setminus \{p\}$ is no longer connected, and since $X \setminus \{p\} \cong Y \setminus \{f(p)\}$, $Y \setminus \{f(p)\}$ must also no longer be connected, and thus $f(p)$ is a cut point of Y.

   Alternative solution: let $\{U, V\}$ be a separation of $X \setminus \{p\}$. Given a homeomorphism $f : X \to Y$, explain why $f(U)$ and $f(V)$ must give a separation of $Y \setminus \{f(p)\}$.

   (b) Prove that none of the intervals $[0,1]$, $(0,1)$, or $[0,1)$ can be homeomorphic.

   We know that $[0,1]$ cannot be homeomorphic to either of the other two intervals, since $[0,1]$ is compact and $(0,1)$ and $[0,1)$ are not. It remains to show that $(0,1)$ cannot be homeomorphic to $[0,1)$. To see this, note that both intervals are connected, and suppose that $f : (0,1) \to [0,1)$ is a bijection. Then there is some $x \in (0,1)$ such that $f(x) = 0$. Then $x \in (0,1)$ is a cut point, since $(0,1) \setminus \{x\} = (0,x) \cup (x,1)$ is no longer connected. However, $[0,1) \setminus \{0\} = (0,1)$ is still connected, so f cannot be a homeomorphism, by part (a).

   (c) Prove that the letters X and Y (viewed as subsets of $\mathbb{R}^2$ with the subspace topology) are not homeomorphic.

(Hint: extend your proof from (a) to show that if $f : X \to Y$ is a homeomorphism, p is a cut point of X, and $q = f(p)$, then $X \setminus \{p\}$ has the same number of connected components as $Y \setminus \{f(p)\}$.

First, we note that any homeomorphism $f : A \to B$ between topological spaces A and B determines a one-to-one correspondence between connected components, since f maps disjoint open subsets to disjoint open subsets. Thus, if $A \cong B$, then A and B have the same number of connected components.

Now let p denote the point at the center of the letter X. Removing p from X leaves us with four connected components (the four "legs" of X). If there was a homeomorphism f from X to Y, it would have to induce a homeomorphism from X with p removed to the space obtained by removing $f(p)$ from Y. However, it is clear that we can obtain at most three connected components by removing a point from Y, so X and Y cannot be homeomorphic.

4. Prove that any infinite subset of a compact space must have a limit point.

Let X be a compact space, and suppose that $A \subseteq X$ does not have a limit point. Then A is closed, since it contains its (non-existent) limit points. Thus, $X \setminus A$ is open. If we choose an open neighbourhood U_a of each $a \in A$, then the collection

$$\mathcal{A} = \{U_a | a \in A\} \cup \{X \setminus A\}$$

is an open cover of X. Moreover, since A does not have any limit points, we can choose each U_a such that $U_a \cap A = \{a\}$. Since X is compact, there must exist a finite subcover; that is, we must have

$$X = U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n} \cup (X \setminus A)$$

for some points $a_1, a_2, \ldots, a_n \in A$. It follows that $A \subseteq U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}$, so

$$\begin{aligned} A &\subseteq (U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}) \cap A \\ &= (U_{a_1} \cap A) \cup (U_{a_2} \cap A) \cup \cdots \cup (U_{a_n} \cap A) \\ &= \{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_n\} \\ &= \{a_1, a_2 \ldots, a_n\}. \end{aligned}$$

It follows that $A = \{a_1, a_2, \ldots, a_n\}$ is finite, and the result follows by taking the contrapositive.

5. A closed map $p : X \to Y$ is called a *perfect map* if p is a surjection and $p^{-1}(y)$ is a compact subset of X for every $y \in Y$. A quotient map $p : X \to Y$ is called a *proper map* if $p^{-1}(K)$ is compact whenever $K \subseteq Y$ is compact. Prove that any perfect map is proper.

(Hint: any open cover of $p^{-1}(K)$ is also an open cover of $p^{-1}(k)$ for each $k \in K$. If $p^{-1}(k) \subseteq U = U_1 \cup \cdots U_n$, then $F = X \setminus U$ is closed in X, and p is a closed map, so $p(F)$ is closed in Y, and thus $Y \setminus p(F)$ is an open neighbourhood of k in Y.)

Let $K \subseteq Y$ be compact, and assume that $p : X \to Y$ is perfect. We wish to show that $p^{-1}(K) \subseteq X$ is compact. Let $\mathcal{A}$ be an open cover for $p^{-1}(K)$. If $k \in K$, then $p^{-1}(k) \subseteq p^{-1}(K)$, so $\mathcal{A}$ is also an open cover of $p^{-1}(k)$. Since $p^{-1}(k)$ is compact, there exist finitely many sets $A_{1,k}, \ldots, A_{n_k,k}$ such that

$$p^{-1}(k) \subseteq A_k = A_{1,k} \cup \cdots \cup A_{n_k,k}.$$

Since A_k is open, $F_k = X \setminus A_k$ is closed. Since p is perfect, it's in particular a closed map, so $p(F_k)$ is a closed subset of Y. Since $p^{-1}(k) \cap F_k = \emptyset$, we must have $k \in U_k = Y \setminus p(F_k)$. The collection $\mathcal{U} = \{U_k | k \in K\}$ is then an open cover of K, and since K is compact, there exists a finite subcover $\{U_{k_1}, \ldots, U_{k_m}\}$ with

$$K \subseteq U = U_{k_1} \cup \cdots \cup U_{k_m}.$$

Since $K \subseteq U$, we have

$$p^{-1}(K) \subseteq p^{-1}(U) = p^{-1}(U_{k_1}) \cup \cdots \cup p^{-1}(U_{k_m}),$$

and for each $j \in \{1, \ldots, m\}$ we have

$$p^{-1}(U_{k_j}) = p^{-1}(Y \setminus p(F_{k_j})) = X \setminus p^{-1}(p(F_{k_j})).$$

Since $F_{k_j} \subseteq p^{-1}(p(F_{k_j}))$, we have $X \setminus p^{-1}(p(F_{k_j})) \subseteq X \setminus F_{k_j} = A_{k_j}$, and each A_{k_j} is a finite union of open sets. Thus, the collection

$$\{A_{1,k_1}, \ldots, A_{n_{k_1},k_1}, \ldots, A_{1,k_m}, \ldots, A_{n_{k_m},k_m}\}$$

is a finite subcover.