1. If $z = 3 - 2i$ and $w = -5 + 4i$, compute:

   (a) $3z = 3(3 - 2i) = 9 - 6i$

   (b) $z - 2w = (3 - 2i) - 2(-5 + 4i) = 3 - 2i + 10 - 8i = 13 - 10i$

   (c) $2w - 3z = 2(-5 + 4i) - 3(3 - 2i) = -10 + 8i - 9 + 6i = -19 + 14i$

   (d) $zw = (3 - 2i)(-5 + 4i) = -15 + 12i + 10i - 8i^2 = (-15 + 8) + i(12 + 10) = -7 + 22i$

   (e) $\overline{z} = 3 + 2i$
   (The complex conjugate is defined by $\overline{x + iy} = x - iy$.)

   (f) $|w| = \sqrt{(-5 + 4i)(-5 - 4i)} = \sqrt{(-5)^2 + 4^2} = \sqrt{41}$
   (The complex modulus (norm) is defined by $|w| = \sqrt{w\overline{w}}$.)

   (g) $\dfrac{z^2}{w} = \dfrac{(3 - 2i)(3 - 2i)}{-5 + 4i} = \dfrac{(5 - 12i)(-5 - 4i)}{(-5 + 4i)(-5 - 4i)} = \dfrac{-25 - 48 + i(-20 + 60)}{(-5)^2 + 4^2} = -\dfrac{73}{41} + \dfrac{40}{41}i$

2. Solve for z in the following equations:

   (a) $z + (2 - 3i) = -5 + 4i$
   $$z = -5 + 4i - (2 - 3i) = -7 + 7i$$

   (b) $3z - 2i = (2 - i)(3 + 4i)$
   $$3z = (10 + 5i) + 2i = 10 + 7i, \quad \text{so} \quad z = \frac{10}{3} + i\frac{7}{3}.$$

   (c) $2iz = 1 + i$
   $$z = -\frac{i}{2}(2iz) = -\frac{i}{2}(1 + i) = \frac{1}{2} - \frac{i}{2}$$

   (d) $(3 + 2i)z - 1 + 3i = 4 + i$
   $$(3 + 2i)z = (4 + i) - (-1 + 3i) = 5 - 2i,$$
   so
   $$13z = (3 - 2i)[(3 + 2i)z] = (3 - 2i)(5 - 2i) = 11 - 16i,$$
   which gives $z = \dfrac{11}{13} - \dfrac{16}{13}i$.

3. Find the eigenvalues of the following matrices:
   $$A = \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 + i \\ 2 - i & 7 \end{bmatrix}$$
   For A, we have
   $$\det(A - xI) = \begin{vmatrix} 2 - x & 4 \\ -4 & 2 - x \end{vmatrix} = (2 - x)^2 + 16,$$
   so $(\lambda - 2)^2 = -16$, giving $\lambda - 2 = \pm\sqrt{-16} = \pm 4i$, so $\lambda = 2 \pm 4i$. (You can also expand the quadratic and use the quadratic formula.)

   For the matrix B, we have
   $$\det(B - xI) = \begin{vmatrix} 3 - x & 2 + i \\ 2 - i & 7 - x \end{vmatrix} = (3 - x)(7 - x) - (2 + i)(2 - i)$$
   $$= x^2 - 10x + 21 - 5 = x^2 - 10x + 16 = (x - 2)(x - 8),$$
   so $\lambda = 2$ or $\lambda = 8$.

4. Verify that $\begin{bmatrix}1\\i\end{bmatrix}$ and $\begin{bmatrix}i\\1\end{bmatrix}$ are eigenvectors for the matrix A in the previous problem, and that $\begin{bmatrix}2+i\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\2-i\end{bmatrix}$ are eigenvectors for the matrix B in the previous problem.

   We have the following:

$$\begin{bmatrix}2 & 4\\-4 & 2\end{bmatrix}\begin{bmatrix}1\\i\end{bmatrix} = \begin{bmatrix}2+4i\\-4+2i\end{bmatrix} = (2+4i)\begin{bmatrix}1\\i\end{bmatrix},$$

   so $\begin{bmatrix}1\\i\end{bmatrix}$ is an eigenvector of A with eigenvalue $2+4i$.

$$\begin{bmatrix}2 & 4\\-4 & 2\end{bmatrix}\begin{bmatrix}i\\1\end{bmatrix} = \begin{bmatrix}2i+4\\-4i+2\end{bmatrix} = (2-4i)\begin{bmatrix}i\\1\end{bmatrix},$$

   so $\begin{bmatrix}i\\1\end{bmatrix}$ is an eigenvector of A with eigenvalue $2-4i$.

$$\begin{bmatrix}3 & 2+i\\2-i & 7\end{bmatrix}\begin{bmatrix}2+i\\-1\end{bmatrix} = \begin{bmatrix}6+3i-2-i\\(2-i)(2+i)-7\end{bmatrix} = \begin{bmatrix}4+2i\\-2\end{bmatrix} = 2\begin{bmatrix}2+i\\-1\end{bmatrix},$$

   so $\begin{bmatrix}2+i\\-1\end{bmatrix}$ is an eigenvector of B with eigenvalue 2.

$$\begin{bmatrix}3 & 2+i\\2-i & 7\end{bmatrix}\begin{bmatrix}1\\2-i\end{bmatrix} = \begin{bmatrix}3+(2+i)(2-i)\\2-i+14-7i\end{bmatrix} = \begin{bmatrix}8\\16-8i\end{bmatrix} = 8\begin{bmatrix}1\\2-i\end{bmatrix},$$

   so $\begin{bmatrix}1\\2-i\end{bmatrix}$ is an eigenvector of B with eigenvalue 8.

5. (Bonus superfun challenge problem) Let $Z = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$.

   (a) Verify that Z has eigenvalues $\pm i$ and eigenvectors $\vec{v} = \begin{bmatrix}i\\-1\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}-1\\i\end{bmatrix}$.

   We check that

$$Z\vec{v} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}i\\-1\end{bmatrix} = \begin{bmatrix}-1\\-i\end{bmatrix} = i\begin{bmatrix}i\\-1\end{bmatrix},$$

   so $\vec{v}$ is an eigevector with eigenvalue i, and

$$Z\vec{w} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}-1\\i\end{bmatrix} = \begin{bmatrix}i\\1\end{bmatrix} = -i\begin{bmatrix}-1\\i\end{bmatrix},$$

   so $\vec{w}$ is an eigenvector with eigenvalue $-i$.

   (b) Show that $\langle\vec{v},\vec{w}\rangle = 0$, where $\langle\vec{v},\vec{w}\rangle = \vec{v}\cdot\overline{\vec{w}}$ is the complex version of the dot product. (The notation $\overline{\vec{w}}$ means take the complex conjugate of each entry in $\vec{w}$.)

   We have

$$\langle\vec{v},\vec{w}\rangle = \vec{v}\cdot\overline{\vec{w}} = \begin{bmatrix}i\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\-i\end{bmatrix} = -i+i = 0.$$

(c) A matrix U is called **unitary** if $U^*U = I$, where $U^* = (\overline{U})^T$ is the *Hermitian conjugate* of U, formed by taking the transpose of the complex conjugate of U.
Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}$. (Note that the columns of U are eigenvectors of Z.) Show that U is unitary and that $U^*ZU = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

We have

$$U^* = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix},$$

so

$$U^*U = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I,$$

so $U^* = U^{-1}$, showing that U is unitary. Finally, we have

$$\begin{aligned} U^*ZU &= \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \end{aligned}$$

as required.

(d) Compute Z^{423}.

Let $D = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}$ be the diagonal matrix whose diagonal entries are the eigenvalues of Z. Then we have $U^*ZU = D$, and since $U^* = U^{-1}$, we can solve for Z, giving us $Z = UDU^*$. Now,

$$Z^n = (UDU^*)^n = (UDU^*)(UDU^*)(UDU^*)\cdots(UDU^*) = UD^nU^*,$$

since all the U^*U products in the interior are equal to the identity. Since D is diagonal we can easily compute

$$D^{423} = \begin{bmatrix} i^{423} & 0 \\ 0 & (-i)^{423} \end{bmatrix} = \begin{bmatrix} i^{420}i^3 & 0 \\ 0 & (-i)^{420}(-i)^3 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

where we have used the fact that $i^4 = (i^2)^2 = (-1)^2 = 1$, so $i^{420} = (i^4)^{105} = 1^105 = 1$, and similarly $(-i)^{420} = 1$. We therefore have

$$Z^{423} = UD^{423}U^* = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$