MATH 1410 - Tutorial #3 Supplement Solutions
Wednesday, January 31

1. Suppose you are given vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and asked to compute vectors $\vec{w}_{\|}$ and $\vec{w}_{\perp}$ such that $\vec{w}_{\|}$ is parallel to $\vec{v}$, $\vec{w}_{\perp}$ is orthogonal to $\vec{v}$, and

$$\vec{w}_{\|} + \vec{w}_{\perp} = \vec{w}.$$

It is a good idea, in such problems, to construct a diagram to illustrate the problem. Which of the following are true statements about that diagram?

- *False:* It should be drawn as tiny as possible so it doesn't take up too much room.
- *True:* It should be drawn reasonably large, so that it's easy to read.
- *False (this just makes the diagram harder to read)*: The vectors should be plotted in three dimensions with accurate magnitudes and directions.
- *True (the diagram lets you set up your equations, and you can deal with the values there):* Magnitudes and directions are unimportant.
- *True:* Vectors shouldn't be drawn at right angles unless you know they're orthogonal.
- *True:* The diagram should be a simple two-dimensional schematic.
- *True:* All vectors should be clearly labelled.
- *True (or at least, it'll get you started):* The diagram can be used to establish any calculations that are required.

Having decided on which of the above are important, find the vectors $\vec{w}_{\|}$ and $\vec{w}_{\perp}$, given $\vec{v} = \langle 2, -1, 2\rangle$ and $\vec{w} = \langle 4, -3, 1\rangle$.

Your diagram should look much like the one below, with $\vec{v}$ corresponding to the side labelled $\vec{b}$, and $\vec{w}$ corresponding to $\vec{a}$. The vector $\vec{w}_{\|}$ is in the direction of $\vec{a}$, with length equal to d as indicated.

We have

$$\vec{w}_{\|} = \mathrm{proj}_{\vec{v}}\,\vec{w} = \left(\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \frac{13}{9}\langle 2, -1, 2\rangle.$$

From here, we find

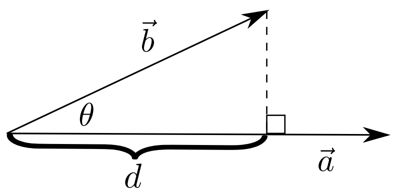
$$\vec{w}_{\perp} = \vec{w} - \vec{w}_{\|} = \langle 4, -3, 1\rangle - \langle 26/9, -13/9, 26/9\rangle = \left\langle \frac{10}{9}, -\frac{14}{9}, -\frac{17}{9}\right\rangle.$$

To make sure we didn't make any mistakes, we can compute

$$\vec{v}\cdot\vec{w}_{\perp} = \frac{20}{9} + \frac{14}{9} - \frac{34}{9} = 0,$$

as expected.

2. Explain, using the diagram below, why the distance d as indicated is given by $d = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}$.



This is basically an exercise in right-triangle trig: d is the length of the side adjacent to the angle, so $\cos\theta = \dfrac{d}{\|\vec{b}\|}$, or $d = \|\vec{b}\|\cos\theta$. On the other hand, we know that $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$, so

$$d = \|\vec{b}\|\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|},$$

as required.

3. Show that

$$(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 - \|\vec{v}\|^2$$

for any vectors $\vec{u}$, $\vec{v}$ in $\mathbb{R}^3$.

Since the dot product satisfies the distributive property, we can multiply out the left-hand side as follows:

$$\begin{aligned}(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} \\ &= \|\vec{u}\|^2 - \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{v} - \|\vec{v}\|^2 \\ &= \|\vec{u}\|^2 - \|\vec{v}\|^2.\end{aligned}$$

Note that we've used the properties $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$ and $\vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$ to simplify the above.