Term Test 2 Review Sheet

November 7, 2014

The second term test takes place on Monday, the 10th of November. The test will cover Sections 4.4, 5.1-5.4, 6.1, and 6.2.

Section 4.4: Subsequences

Main definitions and results:

- Let (a_n) be a given sequence. A **subsequence** of (a_n) is a sequence of the form $b_k = a_{n_k}$, where (n_k) is a strictly increasing sequence of natural numbers. For example, given any sequence (a_n) we can form the subsequences (a_{2k}) and (a_{2k+1}) corresponding to whether n is even or odd, respectively.

- The **Bolzano-Weierstrass** theorem for sequences states that every *bounded* sequence has a convergent subsequence. (This is equivalent to the earlier version of the Bolzano-Weierstrass theorem, which stated that any bounded infinite subset of $\mathbb{R}$ has a limit point.)

- A **subsequential limit** of a sequence (a_n) is a limit of some convergent subsequence of (a_n). For example, the sequence $a_n = (-1)^n$ does not converge, but it has two subsequential limits: $1 = \lim(-1)^{2k}$, and $-1 = \lim(-1)^{2k+1}$.

- For any bounded sequence (a_n), let S denote the set of subsequential limits. We define

$$\limsup a_n = \sup S = \lim_{n\to\infty} (\sup\{a_1, \ldots, a_n\})$$
$$\liminf a_n = \inf S = \lim_{n\to\infty} (\inf\{a_1, \ldots, a_n\})\,.$$

- A sequence (a_n) converges if and only if $\limsup a_n = \liminf a_n$, which is if and only if every subsequence of (a_n) converges to the same value.

- Another important fact that we didn't discuss (and you're not responsible for) is that every sequence (a_n) has a monotone subsequence. If (a_n) is not bounded, it must admit a monotone subsequence that is not bounded. When (a_n) admits an unbounded increasing subsequence, it's conventional to define $\limsup a_n = \infty$, and if (a_n) admits an unbounded decreasing subsequence, we would define $\liminf a_n = -\infty$.

Exercises:

1. For each sequence (a_n), calculate the set S of subsequential limits, $\limsup a_n$, and $\liminf a_n$:

   (a) $(a_n) = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, \frac{1}{6}, \frac{6}{7}, \ldots\right)$

   (b) $a_n = \sin\left(\frac{n\pi}{6}\right)$

   (c) $(a_n) = \left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \ldots\right)$

2. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that the sequence $(f(1/n))$ has a convergent subsequence. (You'll need a result from Section 5.3 for this problem.)

Section 5.1: Limits

Main definitions and results:

- Let $f : D \to \mathbb{R}$ be a function, and let a be a limit point of D. We say that L is a **limit** of f as x approaches a if for every $\epsilon > 0$ there exists some $\delta > 0$ such that whenever $x \in D$ and $0 < |x - a| < \delta$ we have $|f(x) - L| < \epsilon$.

  Recall that a needs to be a limit point of D so that we can consider values of $f(x)$ for $x \in D$ arbitrarily close to, but not equal to, a.

- If $f : D \to \mathbb{R}$ has a limit at $x = a$, then this limit has to be unique. Thus, we can unambiguously talk about *the* limit of f as x approaches a, and write $\lim_{x \to a} f(x) = L$.

- In terms of sequences, $\lim_{x \to a} f(x) = L$ if and only if for every sequence (a_n) with $a_n \to a$ we have $f(a_n) \to L$.

- A corollary of the above is that if there exists a sequence (a_n) converging to a for which $f(a_n)$ does not converge, then $\lim_{x \to a} f(x)$ does not exist.

- The **limit laws** tell us how to take limits of sums, products, and quotients: if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

  - $\lim_{x \to a}(kf(x)) = kL$, for any $k \in \mathbb{R}$
  - $\lim_{x \to a}(f(x) + g(x)) = L + M$
  - $\lim_{x \to a}(f(x)g(x)) = LM$
  - $\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$, if $M \neq 0$.

Exercises:

1. Use the definition of the limit to verify the following:

(a) $\lim_{x\to 2}(x^2+1) = 5$

(b) $\lim_{x\to -1} \dfrac{x+1}{x-1} = 0$

(c) $\lim_{x\to 0} \dfrac{x^2}{|x|} = 0$

Section 5.2: Continuity

Main definitions and results:

- A function $f : D \to \mathbb{R}$ is *continuous* at a point $a \in D$ if for every $\epsilon > 0$ there exists some $\delta > 0$ such that if $x \in D$ and $|x-a| < \delta$, then $|f(x) - f(a)| < \epsilon$. If f is continuous at a for all $a \in D$, we say that f is continuous **on** D.

  Note: Unlike for limits, we require $a \in D$, but we do not require that a is a limit point of D. Thus, a function f is automatically continuous at every isolated point in its domain.

- Given $f : D \to \mathbb{R}$, if $a \in D$ is a limit point of D, then the following are equivalent:

  1. f is continuous at $x = a$.
  2. $\lim_{x\to a} f(x) = f(a)$
  3. For every sequence (a_n) with $a_n \to a$, we have $f(a_n) \to f(a)$.
  4. For every open neighbourhood V of $f(a)$, there exists an open neighbourhood U of a such that $f(U) \subseteq V$.

  (Note: the last item is equivalent to continuity even if a is not a limit point of D.)

- A function f is continuous on its domain D if and only if for every open subset V of $\mathbb{R}$, there exists an open subset U such that $f^{-1}(V) = D \cap U$.

- The sum, product, and quotient* of continuous functions is continuous. *Whenever the functio in the denominator is nonzero.

- If f is continuous at $x = a$ and g is continuous at $y = f(a)$, then $g \circ f$ is continuous at $x = a$.

Exercises:

1. Use the definition of continuity to prove that $f(x) = \frac{1}{x}$ is continuous at $x = 1$.

2. Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$.

   (a) Prove that f is continuous at 0.

   (b) Prove that f is discontinuous at every other point. (Hint: if $a \in \mathbb{Q}$, let (a_n) be a sequence of irrational numbers converging to a, and vice-versa.)

3. Prove that if f is continuous on (a,b) and $f(r) = 0$ for all $r \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in (a,b)$.

Section 5.3: Properties of continuous functions

Main definitions and results:

- If $f : D \to \mathbb{R}$ is continuous, and D is compact, then $f(D)$ is compact.
- Corollary: if f is continuous on a compact set D, then f is bounded on D.
- Corollary (**Extreme Value Theorem**): if f is continuous on a compact set D, (in particular if $D = [a,b]$) then there exist $x_1, x_2 \in D$ such that $f(x_1) = \min\{f(x)|x \in D\}$ and $f(x_2) = \max\{f(x)|x \in D\}$; that is, $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in D$.
- A function $f : D \to \mathbb{R}$ has the **intermediate value property** on D if for any $a, b \in D$ and $k \in \mathbb{R}$ such that $f(a) < k < f(b)$ (or $f(b) < k < f(a)$), there exists some $c \in D$ such that $f(c) = k$.
- (**Intermediate Value Theorem**) If $f : [a,b] \to \mathbb{R}$ is continuous, then f has the intermediate value property on $[a,b]$.

Exercises:

1. Prove that the equation $\cos x = x$ has a solution in $[0, \pi]$.
2. Let S be a set and let (x_n) be a sequence in S converging to some point $x \notin S$. Prove that there exists an unbounded continuous function defined on S.
3. Which of the following functions can't possibly be continuous? Why?
   (a) $f : (0,1) \to \mathbb{R}$ with $f((0,1)) = (-1,0) \cup (1,2)$
   (b) $f : [0,1] \to \mathbb{R}$ with $f([0,1]) = [-4,100]$
   (c) $f : [0,1] \to \mathbb{R}$ with $f([0,1]) = (0,1]$

Section 5.4: Uniform continuity

Main definitions and results:

- A function $f : D \to \mathbb{R}$ is **uniformly continuous** on D if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x, y \in D$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

  Note that the choice of δ depends only on ϵ and must work for all of D - it cannot depend on the values of a particular x and y. On the other hand, if f is merely continuous on D, then for every $\epsilon > 0$ **and** for every $x \in D$, there exists a $\delta > 0$ (depending now on ϵ *and* x) such that for any $y \in D$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
- If $f : D \to \mathbb{R}$ is continuous and D is compact, then f is uniformly continuous on D.

- If $f : D \to \mathbb{R}$ is uniformly continuous and (a_n) is a Cauchy sequence in D, then $(f(a_n))$ is a Cauchy sequence.
- A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous if and only if it can be extended to a continuous function on $[a, b]$.

Exercises:

1. Which of the following functions are continuous on the given set? Justify your answers.
   (a) $f(x) = x \sin(1/x)$ on $[1, 2]$
   (b) $f(x) = x \sin(1/x)$ on $(0, 1)$
   (c) $f(x) = x^2 + 4$ on $[0, 10]$
   (d) $f(x) = \dfrac{x+2}{x-1}$ on $(2, 3)$
   (e) $f(x) = \dfrac{x+2}{x-1}$ on $(1, 2)$
2. Use the defintion of uniform continuity to prove that $f(x) = x^2 + x - 3$ is uniformly continuous on $[1, 3]$.

Section 6.1: The derivative

Main definitions and results:

- Let $f : I \to \mathbb{R}$ be a function, where I is an interval, and let $a \in I$. We say that f is **differentiable** at a if the limit $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists. The value of this limit is called the **derivative** of f at a and denoted by $f'(a)$.
- By computing $f'(x)$ for each $x \in I$ where it exists, we get a new function f' defined on the set of all $x \in I$ such that f is differentiable.
- If f is differentiable at $x = a$, then f is continuous at $x = a$. The converse is **not** true.
- We have the constant, power, sum, product, quotient, and chain rules just as in Math 1560.
- **Fermat's theorem** tells us that if f has a maximum or minimum at a point c on the interior of an interval I (i.e. c is not an endpoint of I), then $f'(c) = 0$.
- **Darboux's theorem** tells us that if f is differentiable on I, then f' satisfies the intermediate value property on I. Thus, although f' is not guaranteed to be continuous, any discontinuity of f' cannot be a jump or removable discontinuity.

Exercises:

1. Use the definition of the derivative to determine whether or not f is differentiable at $x = 0$, where

(a) $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ x^2 - 1 & \text{if } x < 0 \end{cases}$

(b) $f(x) = \begin{cases} 3x + 1 & \text{if } x \geq 0 \\ 1 - x^2 & \text{if } x < 0 \end{cases}$

(c) $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

2. Prove that the derivative of an even function is an odd function.

Section 6.2: The Mean Value Theorem

Main definitions and results:

- **Rolle's Theorem** states the following: if f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$.
- The **Mean Value Theorem** states that if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- Consequences of the Mean Value Theorem include the following: if $f'(x) = 0$ for all $x \in I$, then f is constant on I. If $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some $C \in \mathbb{R}$. If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I.

Exercises:

1. Prove that $f(x) = x^5 + 2x$ has exactly one real root.
2. Recall that f is a *contraction mapping* if there exists some $c \in (0, 1)$ such that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in \mathbb{R}$. Prove that if $|f'(x)| < 1$ on $\mathbb{R}$, then f is a contraction mapping.
3. Let $f(x)$ and $g(x)$ be differentiable on $\mathbb{R}$. Show that if $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$, then $f(x) \leq g(x)$ for all $x \geq 0$.