University of California, Berkeley
Department of Mathematics
5^{th} October, 2012, 12:10-12:55 pm
MATH 53 - Test #1

Last Name:____

First Name:____

Discussion Section: ___________________________________

Name of GSI: ______________________________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

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Total	/36

A

[4] 1. Find the equation of the tangent line to the curve C represented by the vector-valued function $\mathbf{r}(t) = \langle e^{2t}, t^2, \sin t\rangle$ at the point $(1, 0, 0)$.

The point $(1, 0, 0)$ corresponds to the parameter value $t = 0$, and $\mathbf{r}'(t) = \langle 2e^{2t}, 2t, \cos t\rangle$, so the tangent vector at $(1, 0, 0)$ is $\mathbf{r}'(0) = \langle 2, 0, 1\rangle$. The tangent line is thus the line through $(1, 0, 0)$ in the direction of $\langle 2, 0, 1\rangle$, so the equation of the line is

$$\langle x, y, z\rangle = \langle 1, 0, 0\rangle + t\langle 2, 0, 1\rangle.$$

[5] 2. Find the area of one loop of the 4-leaved rose $r = \cos 2\theta$.

The right-hand loop of the 4-leaved rose corresponds to the angles $-\frac{\pi}{4} \leq \theta\frac{\pi}{4}$, so the area is

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos^2 2\theta d\theta \\ &= \frac{1}{4}\int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta))d\theta \\ &= \frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} \\ &= \frac{\pi}{8}. \end{aligned}$$

[3] 3. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ does not exist.

If we evaluate the limit by letting $(x, y) \to (0, 0)$ along one of the coordinate axes, we immediately get 0 for the limit. However, if we let $(x, y) \to (0, 0)$ along the line $x = y$, we get $\dfrac{xx}{x^2 + x^2} = \frac{1}{2}$, so we get a limit of $1/2 \neq 0$. Since we get different values along different paths, the limit does not exist.

4. Consider the two lines in $\mathbb{R}^3$ given by

$$\begin{aligned}\mathbf{r}_1(t) &= \langle 2, 0, -2\rangle + t\langle 1, 0, -2\rangle \\ \mathbf{r}_2(s) &= \langle -2, 1, 0\rangle + s\langle 3, -1, 0\rangle.\end{aligned}$$

[2] (a) Verify that the two lines intersect at the point $(1, 0, 0)$.

We see by direct computation that

$$\mathbf{r}_1(-1) = \langle 2, 0, -2\rangle - \langle 1, 0, -2\rangle = \langle 1, 0, 0\rangle$$

and

$$\mathbf{r}_2(1) = \langle -2, 1, 0\rangle + \langle 3, -1, 0\rangle = \langle 1, 0, 0\rangle.$$

[3] (b) Find the cosine of the angle between the two lines.

The directions of the two lines are given by the vectors $\mathbf{v}_1 = \langle 1, 0, -2\rangle$ and $\mathbf{v}_2 = \langle 3, -1, 0\rangle$, so the angle between the lines at their point of intersection is given by

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} = \frac{3+0+0}{\sqrt{5}\sqrt{10}} = \frac{3}{\sqrt{50}}.$$

[4] (c) Find the equation of the plane that contains the two lines.

We know that the two lines intersect at the point $(1, 0, 0)$, which must therefore lie on the plane, and a normal vector is given by

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 0(0) - (-2)(-1), -2(3) - 1(0), 1(-1) - 0(3)\rangle = \langle -2, -6, -1\rangle.$$

The equation of the plane is therefore $-2(x-1) - 6y - z = 0$, or $2x + 6y + z - 2 = 0$.

[3] (d) Find the distance between the point $P(2, -1, 3)$ and the plane from part (c).

The distance from a point $P(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$ is given by $D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$. Thus,

$$D = \frac{|2(2) + 6(-1) + 3 - 2|}{\sqrt{2^2 + 6^2 + 1^2}} = \frac{1}{\sqrt{41}}.$$

 Total Points: 36

[5] 5. (a) Find the equation of the tangent plane to the surface $z = \sqrt{x^2 + y^3}$ at the point $(1, 2, 3)$.

Letting $f(x, y) = \sqrt{x^2 + y^2}$, we have $f_x(x, y) = \dfrac{x}{\sqrt{x^2 + y^3}}$ and $f_y(x, y) = \dfrac{3y^2}{2\sqrt{x^2 + y^3}}$, so the partial derivatives of f at the point $(1, 2)$ are $f_1(1, 2) = \dfrac{1}{3}$ and $f_2(1, 2) = \dfrac{3(2^2)}{2(3)} = 2$. Thus, the equation of the tangent plane to $z = f(x, y)$ at $(1, 2, 3)$ is

$$z = 3 + \frac{1}{3}(x - 1) + 2(y - 2).$$

[2] (b) Use the result from part (a) to approximate the value of $\sqrt{(1.03)^2 + (2.05)^3}$.

Since the point $(1.03, 2.05)$ is close to the point $(1, 2)$, we use the linear approximation $f(x, y) \approx L(x, y)$ (where $z = L(x, y)$ is the equation of the tangent plane) to compute

$$\sqrt{(1.03)^2 + (2.05)^3} \approx 3 + \frac{1}{3}(0.03) + 2(0.05) = 3 + 0.01 + 0.1 = 3.11.$$

[5] 6. Use the chain rule to compute $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ if $f(x, y, z) = x^2 y + y^2 z^3$, where $x = v^2$, $y = u^2$, and $z = u^2 v^2$.

We have

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \\
&= 2xy(0) + (x^2 + 2yz^3)(2u) + 3y^2 z^2(2uv^2) \\
&= 2uv^4 + 4u^8 v^6 + 6u^9 v^6,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} \\
&= 2xy(2v) + (x^2 + 2yz^3)(0) + 3y^2 z^2(2u^2 v) \\
&= 4u^2 v^3 + 6u^{10} v^5.
\end{aligned}$$