Practice Problems for Quiz 8
Math 2000A

Quiz #8 will take place in class on Thursday, November 6th.

As usual, solving the problems on this sheet will significantly improve your chances of getting a high score on the quiz.

Note to help session tutors: It's 100% OK for you to help my students solve these questions.

1. Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. Let $r : \mathbb{Z} \to \mathbb{Z}_5$ be the remainder function defined by the division algorithm: for any $n \in \mathbb{Z}$, there exist unique integers $q(n)$ and $r(n)$ such that $n = 5q(n) + r(n)$ and $r(n) \in \mathbb{Z}_5$. For each of the following functions $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ below, determine if the function is a bijection. For those functions that are bijections, determine the inverse of the function. (It suffices to define f^{-1} by giving its value on each element of $\mathbb{Z}_5$. You don't need to find a formula.)

   (a) $f(x) = r(3x + 1)$

   (b) $f(x) = r(2x^2)$

   (c) $f(x) = r(1 + x^3)$

   Note: There's nothing special about the number 5 here. We could look at the remainder function for any natural number. A set with 5 elements just happens to be a reasonable size to work with. Also, it helps that 5 is prime. (We'll see why later in the course.)

2. Find the composition $g \circ f$ for each of the following pairs of functions:

   (a) $f : \mathbb{Z} \to \mathbb{N}$; $f(m) = m^2 + 1$, $g : \mathbb{N} \to (0, \infty)$; $g(n) = 1/n$.

   (b) $f : \mathbb{R} \to (0, 1)$; $f(x) = 1/(1 + x^2)$, $g : (0, 1) \to (0, 1)$; $g(x) = 1 - x$.

   (c) $f : \mathbb{R} \to [1, \infty)$; $f(x) = x^2 + 1$, $g : [1, \infty) \to [0, \infty)$; $g(x) = \sqrt{x - 1}$.

   (d) $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$; $f(1) = 4, f(2) = 1, f(3) = 2, f(4) = 3$,
   $g : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$; $g(1) = 3, g(2) = 4, g(3) = 1, g(4) = 2$.

3. Construct an example of sets A, B, and C, and functions $f : A \to B$ and $g : B \to C$ such that both f and $g \circ f$ are one-to-one, but g is not.

4. Construct an example of sets A, B, and C, and functions $f : A \to B$ and $g : B \to C$ such that both g and $g \circ f$ are onto, but f is not.

5. We discussed in class that if $f : A \to B$ and $g : B \to C$ are both one-to-one, then so is $g \circ f$, and that if they're both onto, then so is $g \circ f$. We also briefly looked at an example (with $A = \{1\}$ having one element, $B = \{a, b, c\}$ having three elements (or any nubmer greater than one), and $C = \{2\}$ having one element), that illustrated that it's possible to have $g \circ f$ both one-to-one and onto even if these properties fail for one of f or g. However, at least one of the two functions has to have the given property, as the following results show:

   (a) Prove that if $g \circ f : A \to C$ is one-to-one, then so is $f : A \to B$.
   Hint: If $f(x_1) = f(x_2)$, what can you say about $g \circ f(x_1)$ and $g \circ f(x_2)$?

   (b) Prove that if $g \circ f : A \to C$ is onto, then so is $g : B \to C$.

6. Let $f : A \to B$ be one-to-one and onto. Prove that $(f^{-1})^{-1} = f$.

7. Let $f : A \to B$ and $g : B \to A$, and let $I_A : A \to A$ and $I_B : B \to B$ denote the identity functions on A and B, respectively.

   (a) Show that if $g \circ f = I_A$, then f is one-to-one.

   (b) Show that if $f \circ g = I_B$, then f is onto.

8. Suppose you can find functions $f : A \to B$ and $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$. Show that f must be both one-to-one and onto, and that $g = f^{-1}$.

9. Let $S = \{a, b, c, d\}$. Define $f : S \to S$ by defining f to be the following set of ordered pairs:
$$f = \{(a, c), (b, b), (c, d), (d, a)\}.$$

   (a) Draw an arrow diagram to represent the function f. Is f a bijection?

   (b) Write the inverse of f as a set of ordered pairs. Is f^{-1} function? Explain.

   (c) Draw an arrow diagram for f^{-1} using the arrow diagram from part (a).

   (d) Compute $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$ for each $x \in S$. What theorem does this illustrate?

10. Consider the function $f(x) = (2x - 1)^3$.

   (a) Find functions $g(x)$ and $h(x)$ such that $f = g \circ h$.

   (b) Show that the functions g and h you found in part (a) are bijections, and find their inverses.

   (c) Explain how your results from part (b) can be used to solve the equation
$$(2x - 1)^3 = 27,$$
   and then solve for x.