Math 2000 Tutorial Worksheet

October 9, 2015

This week's tutorial will focus on proofs by contradiction and proofs by cases. Please discuss the following problems with your classmates:

1. (Section 3.3 #3) Consider the following statement:

   $$\text{For each positive real number } r, \text{ if } r^2 = 18, \text{ then } r \text{ is irrational.}$$

   (a) If you were setting up a proof by contradiction for this statement, what would you assume? Carefully write down all conditions that you would assume.

   (b) Complete a proof by contradiction for this statement.

   (Hint: you might be tempted to mimic the proof for $\sqrt{2}$ and try to prove by cases that if $18 \mid a^2$ then $18 \mid a$. This is a bad idea. Instead, think about how 18 factors as a product of primes.)

2. (Section 3.3 #4 (partial)) Prove that $\sqrt[3]{2}$ is irrational.

   (Hint: on last week's worksheet, you were asked to show that n is odd if and only if n^3 is odd. Note that taking the contrapositive of both directions in this biconditional allows you to replace 'odd' by 'even'.)

3. (Section 3.3 #8)

   (a) Prove that for each real number x, either $(x + \sqrt{2})$ is irrational or $(-x + \sqrt{2})$ is irrational.

   (b) Generalize the proposition in Part (a) to use any irrational number in place of $\sqrt{2}$, and prove your proposition.

   (Hint: you're in Section 3.3, so you know you're looking for a proof by contradiction. Begin by carefully taking the negation of the given proposition.)

4. (Section 3.4 #4) Prove that if u is an odd integer, then the equation $x^2 + x - u = 0$ has no integer solutions.

   (Hint: try a proof by contradiction. That's right – a proof by cases within a proof by contradiction. It's exciting stuff.)

5. (Section 3.4 #7) Is the following proposition true or false? Justify your conclusion with a proof or counterexample:

$$\text{For each integer } n, \text{ if } n \text{ is odd, then } 8 \mid (n^2 - 1).$$

6. (Section 3.5 #10) Any set of three *consecutive* integers can be written in the form $\{m, m+1, m+2\}$, where m is an integer.

   (a) Explain why we can also represent three consecutive integers as $k-1, k, k+1$, where k is an integer.

   (b) Proposition 3.27 in Section 3.5 states that if n is an integer, then 3 divides $n^3 - n$. (See the text for a proof.) Explain why it follows from this result that the product of any three consecutive integers is divisible by 3.

   (c) Prove that the product of three consectutive integers is divisible by 6.

7. (Section 3.5 #13)

   (a) Prove that for each integer a, if $a \not\equiv 0 \pmod 7$, then $a^2 \not\equiv 0 \pmod 7$.

   (b) Prove that for each integer a, if 7 divides a^2, then 7 divides a.

   (c) Prove that $\sqrt{7}$ is irrational.

   (Yes, I went there. Use a proof by cases for (a), then understand why (b) is equivalent to (a), and finally, mimic the proofs we did in class for $\sqrt{2}$ and $\sqrt{3}$.)

8. (Section 3.5 #20) Are the following statements true or false? Either prove that the statement is true, or provide a counterexample to show it is false.

   (a) For all integers a and b, if $a \cdot b \equiv 0 \pmod 6$, then $a \equiv 0 \pmod 6$ or $b \equiv 0 \pmod 6$.

   (b) For all integers a and b, if $a \cdot b \equiv 0 \pmod 8$, then $a \equiv 0 \pmod 8$ or $b \equiv 0 \pmod 8$.

   (c) For all integers a and b, if $a \cdot b \equiv 1 \pmod 6$, then $a \equiv 1 \pmod 6$ or $b \equiv 1 \pmod 6$.

   (d) For all integers a and b, if $a \cdot b \equiv 7 \pmod{12}$, then $a \equiv 1 \pmod{12}$ or $a \equiv 7 \pmod{12}$.