University of Lethbridge
Department of Mathematics and Computer Science
20^{th} March, 2015, 10:00 - 10:50 am

MATH 1410A - Test #2

Last Name:____

First Name:____

Student Number:____________________________________

Tutorial Section: ___________________________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

No external aids are allowed, with the exception of a 5-function calculator.

For grader's use only:

Page	Grade
2	/10
3	/10
4	/10
5	/5
6	/5
Total	/40

A

1. SHORT ANSWER: For each of the questions below, please provide a short (one line) answer.

[2] (a) Suppose $\det A = 4$ and the matrix B is obtained from A by first multiplying the first row of A by 5, and then exchanging rows 1 and 3. What is $\det B$?

$\det B = -5 \det A = -20$.

[2] (b) Let A and B be 3×3 matrices. If $\det A = 2$ and $\det B = -3$, what is the value of $\det(2A^2B^TA^{-1})$?

$$\det(2A^2B^TA^{-1}) = 2^3|A|^2|B|\left(\frac{1}{|A|}\right) = 8|A||B| = -48.$$

[2] (c) Calculate the dot product of the vectors $\vec{u} = \begin{bmatrix}1 & -2 & 4\end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix}4 & -2 & 3\end{bmatrix}^T$

$\vec{u} \cdot \vec{v} = 1(4) - 2(-2) + 4(3) = 20.$

[2] (d) Let $\vec{v}$ be a vector in $\mathbb{R}^n$. If $\|\vec{v}\| = 3$ and $\vec{w} = -4\vec{v}$, what is $\|\vec{w}\|$?

$\|\vec{w}\| = \|-4\vec{v}\| = |-4|\|\vec{v}\| = 4(3) = 12.$

[2] (e) Find $\vec{x}$, given that $\vec{u} = \begin{bmatrix}2 & -1 & 3\end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix}-4 & 7 & 5\end{bmatrix}^T$, and $3\vec{u} - 2\vec{x} = \vec{v}$.

$\vec{x} = \frac{1}{2}(3\vec{u} - \vec{v}) = \frac{3}{2}\vec{u} - \frac{1}{2}\vec{v} = \begin{bmatrix}5 & -5 & 2\end{bmatrix}$

2. Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 4 \\ -3 & 0 & 1 \end{bmatrix}$.

[5] (a) Compute $\det A$.

By adding 2 times the second column to the first and then using cofactor expansion along the first row, we get

$$\det A = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 3 & 4 \\ -3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 6 & 3 & 4 \\ -3 & 0 & 1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 6 & 4 \\ -3 & 1 \end{vmatrix} = 6 + 12 = 18.$$

[5] (b) Given the system of equations $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 4 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, use Cramer's rule to find the value of y, if possible.

Cramer's rule tells us that $y = \dfrac{|A_2|}{|A|}$, where A_2 is obtained from A by replacing the second column of A by the right-hand side of the system. This gives us

$$\det A_2 = \begin{vmatrix} 2 & 2 & 0 \\ 0 & -1 & 4 \\ -3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & 4 \\ 3 & 6 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & 4 \\ 6 & 1 \end{vmatrix} = 2(-1-24) = -50.$$

Thus, $y = \dfrac{-50}{18} = -\dfrac{25}{9}$.

3. (a) Find a vector equation of the line in $\mathbb{R}^3$ that passes through the points $P = (2, -3, 1)$
[4] and $Q = (4, 1, -2)$.

A direction vector is $\vec{d} = \overrightarrow{PQ} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$, so an equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}.$$

(b) Let L_1 be the line through $P_1 = (2, 0, -1)$ with direction vector $\vec{d_1} = \begin{bmatrix} -1 & 3 & 2 \end{bmatrix}^T$, and let L_2 be the line through $P_2 = (8, 6, 7)$ with direction vector $\vec{d_2} = \begin{bmatrix} -4 & 0 & -2 \end{bmatrix}^T$.
[6] Determine the point of intersection of L_1 and L_2, if any.

For L_1 we have $x = 2-s, y = 3s, z = -1+2s$, and for L_2 we have $x = 8-4t, y = 6, z = 7-2t$. If the lines intersect, we must have $2-s = 8-4t$, $3s = 6$, and $-1+2s = 7-2t$. The second of these equations gives us $s = 2$. Putting this into the first equation gives $t = 2$ as well. In the third equation, we verify that $-1 + 2(2) = 3 - 7 - 2(2)$, so the two lines intersect.

Putting $s = 2$ into the parametric equations for L_1 gives $x = 0, y = 6, z = 3$, so the point of intersection is $(0, 6, 3)$.

4. Find the shortest distance from the point $P = (3, 2, -1)$ to the line L given by the vector
[5] equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

We have the point $P_0 = (2, 1, 3)$ on the line, and the direction vector $\vec{d} = \langle 3, -1, -2 \rangle$. If Q is the point on the line closest to P, then we have

$$\text{proj}_{\vec{d}} \overrightarrow{P_0P} = \overrightarrow{P_0Q},$$

and the shortest distance is given by $\|\overrightarrow{QP}\| = \|\overrightarrow{P_0P} - \overrightarrow{P_0Q}\|$. We find

$$\overrightarrow{P_0P} = \langle 1, 1, -4 \rangle,$$

so

$$\overrightarrow{P_0Q} = \frac{\vec{d} \cdot \overrightarrow{P_0P}}{\|\vec{d}\|^2} \vec{d} = \frac{10}{14} \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 15/7 \\ -5/7 \\ -10/7 \end{bmatrix}.$$

Thus $\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q} = \begin{bmatrix} -8/7 \\ 12/7 \\ 18/7 \end{bmatrix}$, which gives the distance

$$\|\overrightarrow{QP}\| = \sqrt{(-8/7)^2 + (12/7)^2 + (18/7)^2}.$$

5. Consider the triangle in $\mathbb{R}^3$ with vertices $P = (2, 0, -3)$, $Q = (5, -2, 1)$, and $R = (7, 5, 3)$.

[3] (a) Show that the triangle is a right-angled triangle.

The vectors $\overrightarrow{QP} = \langle -3, 2, -4 \rangle$ and $\overrightarrow{QR} = \langle 2, 7, 2 \rangle$ make up two of the three sides of the triangle, and

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = -6 + 14 - 8 = 0$$

which shows that these two sides are perpendicular, and thus the triangle is a right-angled triangle.

[2] (b) Compute the lengths of the three sides of the triangle and verify that the Pythagorean theorem ($a^2 + b^2 = c^2$) holds.

We have $\|QP\|^2 = (-3)^2 + 2^2 + (-4)^2 = 29$ and $\|QR\|^2 = 2^2 + 7^2 + 2^2 = 57$, so $\|QP\|^2 + \|QR\|^2 = 86$.

The remaining side is given by $\overrightarrow{PR} = \langle 5, 5, 6 \rangle$, and we see that

$$\|\overrightarrow{PR}\|^2 = 5^2 + 5^2 + 6^2 = 86$$

as well, so the Pythagorean theorem holds.

 Total Points: 40