Practice Problems for Quiz 11
Math 2000A

Quiz #11 will take place in class on Thursday, November 27th. As usual, solving the problems on this sheet will significantly improve your chances of getting a high score on the quiz.

Note to help session tutors: It's 100% OK for you to help my students solve these questions.

1. Define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by
$$f(m,n) = 2^{m-1}(2n-1), \qquad \text{for all } (m,n) \in \mathbb{N} \times \mathbb{N}.$$
   (a) Prove that f is one-to-one.
   Hint: If $f(m,n) = f(k,l)$, then there are three cases to consider: $m > k, m < k$, and $m = k$. In the first two cases, divide both sides by the smaller power of 2 to obtain a contradiction.

   (b) Prove that f is onto. You may use without proof the fact that any $n \in \mathbb{N}$ can be written in the form $n = 2^k m$, where m is an odd natural number and $k \geq 0$ is an integer.

2. The previous problem shows that $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$. Use this fact to give an alternative proof that if A and B are countably infinite, then $A \times B$ is countably infinite.

3. Let C be the set of all infinite sequences consisting of 0s and 1s. Thus, $(0,1,0,1,0,1,\ldots)$ and $(1,1,1,1,\ldots)$ are elements of C, but $(0,1,2,0,1,2,\ldots)$ is not (since there is a 2 in the sequence). Show that C is uncountable.

   Hint: Suppose C is countable. Then there is a bijection $f : \mathbb{N} \to C$, so we can list the elements of C according to
$$\begin{aligned} f(1) &= (a_{11}, a_{12}, a_{13}, \ldots) \\ f(2) &= (a_{21}, a_{22}, a_{23}, \ldots) \\ f(3) &= (a_{31}, a_{32}, a_{33}, \ldots) \\ \vdots \quad & \quad \vdots \end{aligned}$$
   Now consider the sequence $(b_1, b_2, b_3, \ldots)$, where $b_i = 0$, if $a_{ii} = 1$, and $b_i = 1$, if $a_{ii} = 0$. Argue that this sequence cannot appear anywhere on the list, so that the function f cannot be onto, and therefore is not a bijection.

4. Use mathematical induction to prove that for each $n \in \mathbb{N}$,
$$1 + 5 + 9 + \cdots + (4n-3) = n(2n-1).$$

5. Use mathematical induction to prove each of the following:
   (a) For each $n \in \mathbb{N}$, 3 divides $4^n - 1$.
   (b) For each $n \in \mathbb{N}$, 6 divides $n^3 - n$.
6. Use mathematical induction to prove that the sum of the cubes of three consecutive natural numbers is divisible by 9.