Name: Solutions

Solve **both** of the following two questions:

[5] 1. Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that range S is invariant under T.

Solution: Let $w \in \operatorname{range} S$. Then $w = Sv$ for some $v \in V$. Thus, assuming that $ST = TS$, we have

$$Tw = T(Sv) = S(Tv) \in \operatorname{range} S,$$

which shows that range S is invariant under T.

2. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by $T(x, y) = (-3y, x)$.

[3] (a) What are the eigenvalues of T?

Solution: λ is an eigenvalue of T provided that $T(x, y) = (-3y, x) = \lambda(x, y)$ for some nonzero vector $v = (x, y)$. Thus, we must have $-3y = \lambda x$ and $x = \lambda y$, which gives us the system

$$\begin{aligned} \lambda x + 3y &= 0 \\ x - \lambda y &= 0 \end{aligned}$$

To have a non-trivial solution the second equation must be a multiple of the first equation. If we multiply the second equation by λ we have $\lambda x - \lambda^2 y = 0$, which tells us that we must have

$$\lambda^2 = -3.$$

However, this is impossible if λ is a real number, so T does not have any real eigenvalues.

[2] (b) Does your answer change if we view T as an operator on $\mathbb{C}^2$ rather than $\mathbb{R}^2$?

Solution: Yes – over the complex numbers, the argument above shows that we have the eigenvalues $\lambda = \pm i\sqrt{3}$.

Total: 10 points