Math 3500 Assignment #9
University of Lethbridge, Fall 2014

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Due date: Friday, November 28th, by 6 pm.

This is the last regular assignment for the course. But don't forget that you have an essay assignment to submit by the last lecture!

1. Let f be a bounded function on $[a, b]$, let $\mathcal{P}$ denote the set of all partitions of $[a, b]$, and let $P \in \mathcal{P}$ be an arbitrary partition of $[a, b]$.
   (a) Prove that $U(f) \geq L(f, P)$, where $U(f) = \inf\{U(f, P) | P \in \mathcal{P}\}$.
   (b) Prove that $U(f) \geq L(f)$, where $L(f) = \sup\{L(f, P) | P \in \mathcal{P}\}$.

2. Let f be a bounded function on $[a, b]$.
   (a) Prove that f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying
   $$\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0.$$
   (b) For each n, let P_n denote the uniform partition of $[0, 1]$ into n equal subintervals of length $1/n$, and let $f(x) = x$. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ in terms of n.

   Hint: recall the summation formula $1 + 2 + \cdots + n = n(n + 1)/2$.
   (c) Use the results from (a) and (b) to prove that $f(x) = x$ is integrable on $[0, 1]$.

3. Let $f : [a, b] \to \mathbb{R}$ be bounded and increasing. Show that f is integrable on $[a, b]$.

   Hint: Use a uniform partition and the results of the previous problem. You should find that for an increasing function, the sum $U(f, P_n) - L(f, P_n)$ simplifies considerably.

4. Define the function $H(x) = \int_1^x \frac{1}{t}\,dt$, where $x > 0$.

   (a) What is the value of $H(1)$? What is $H'(x)$ for any $x > 0$?

   (b) Show that if $0 < x < y$, then $H(x) < H(y)$; that is, that H is strictly increasing on $(0, \infty)$.

   (c) Show that $H(cx) = H(c) + H(x)$ for any $c > 0$.

   Hint: it's possible to prove this using u-substitution, but a more efficient/clever approach is to treat c as a constant and consider the derivative of $g(x) = H(cx)$ with respect to x using the Chain Rule. (Of course via the FTC the Chain Rule and u-substitution are really just two sides of the same coin.) Keep in mind that two functions with the same derivative must differ by a constant.

   (d) Use a similar argument to show that $H(x^a) = aH(x)$.

   Note: One often writes the function $H(x)$ as $\ln(x)$, and refers to this function as the natural logarithm. Parts (c) and (d) then tell us that $\ln(xy) = \ln x + \ln y$ and $\ln(x^y) = y \ln x$. Since H is strictly increasing on $(0, \infty)$, it is one-to-one and therefore has a well-defined inverse function, which is usually denoted by $H^{-1}(x) = e^x$.

5. (**Bonus**) Define a bounded function f on $[0, 1]$ by $f(x) = \begin{cases} 1, & \text{if } x = 1/n \\ 0, & \text{otherwise} \end{cases}$.

   Prove that f is integrable on $[0, 1]$.