Math 3410 Assignment #4 Solutions
University of Lethbridge, Spring 2015

Sean Fitzpatrick

March 20, 2015

1. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

   Suppose that λ is an eigenvalue of ST; that is $ST(v) = \lambda v$ for some $v \neq 0$. We consider two cases:

   (a) $\lambda \neq 0$. Note that it follows that $Tv \neq 0$, since otherwise we'd have $S(Tv) = (ST)v = 0$, but both λ and v are nonzero, so $\lambda v \neq 0$. Moreover,
   $$TS(Tv) = T[(ST)v] = T(\lambda v) = \lambda T v,$$
   and since $Tv \neq 0$, it follows that λ is an eigenvalue of TS.

   (b) $\lambda = 0$. If 0 is an eigenvalue of ST, then ST is not invertible. Since V is finite-dimensional, it follows that either S is not invertible or T is not invertible. In either case, we can conclude that TS is not invertible either, and therefore 0 is an eigenvalue of TS.

   This shows that any eigenvalue of ST is an eigenvalue of TS, and similarly, we can show that any eigenvalue of TS is an eigenvalue of ST.

2. Suppose that V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$ with $v \neq 0$. Let p be a nonzero polynomial of smallest degree such that $p(T)v = 0$. Prove that every zero of p is an eigenvalue of T.

   Suppose $v \neq 0$, and let p be a polynomial such that $p(T)v = 0$. Suppose that there exists some $a \in \mathbb{F}$ such that a is not an eigenvalue of T and $p(a) = 0$. Since $p(a) = 0$, it follows that there exists a polynomial q with $\deg q = \deg p - 1$ such that $p(z) = (z - a)q(z)$, and thus,
   $$p(T)v = (T - aI)q(T)v = 0.$$
   Since a is not an eigenvalue of T, we know that $T - aI$ is invertible, and applying $(T - aI)^{-1}$ to both sides of the equation above, we obtain $q(T)v = 0$, and thus p is not the polynomial of least degree such that $p(T)v = 0$, since $\deg q < \deg p$. Since we've proved the contrapositive of the given statement, the result follows.

3. Recall that the *Fibonacci sequence* $(F_1, F_2, \ldots)$ is defined recursively by $F_1 = 1, F_2 = 1$, and

$$F_{n+2} = F_n + F_{n+1} \quad \text{for} \quad n \geq 1.$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (y, x + y)$.

(a) Show that $T^n(0, 1) = (F_n, F_{n+1})$ for each positive integer n.

We proceed by induction on $n \in \mathbb{N}$: for $n = 1$, we have

$$T^1(0, 1) = T(0, 1) = (1, 1) = (F_1, F_2).$$

If we have that $T^n(0, 1) = (F_n, F_{n+1})$ for some $n \geq 1$, then

$$T^{n+1}(0, 1) = T(T^n(0, 1)) = T(F_n, F_{n+1}) = (F_{n+1}, F_n + F_{n+1}) = (F_{n+1}, F_{n+2}).$$

(b) Find the eigenvalues of T.

Since $T(1, 0) = (0, 1)$ and $T(0, 1) = (1, 1)$, the matrix of T with respect to the standard basis is

$$A = \mathcal{M}(T) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and we know that the eigenvalues of A are equal to the eigenvalues of T. Moreover, we know that $A - \lambda I_2$ is not invertible if and only if

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 1,$$

and using the quadratic formula, we see that the eigevalues of A (and thus T) are given by

$$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}.$$

(c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

Suppose $X_\pm = \begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector for $\lambda_\pm$. Then we must have

$$(A - \lambda_\pm)X_\pm = \begin{bmatrix} -\lambda_\pm & 1 \\ 1 & 1 - \lambda_\pm \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and one possible solution to this homogeneous system is to take $X_\pm = \begin{bmatrix} 1 \\ \lambda_\pm \end{bmatrix}$.

Note: Multiplying $X_\pm$ by the first row of $A - \lambda_\pm I$ gives $-\lambda_\pm x + y = 0$, which tells us that we can take $y = \lambda_\pm x$, and of course we can set x to any value; in this case we took $x = 1$. It's not immediately obvious that the second row gives the same result; however, we note the following identities, which can easily be verified computationally by substituting the values of λ_+ and λ_-:

- $\lambda_+\lambda_- = -1$
- $\lambda_+ + \lambda_- = 1$
- $\lambda_+ - \lambda_- = \sqrt{5}$

The second identity tells us that $1 - \lambda_\pm = \lambda_\mp$, and the first tells that mutliplying the first row by $\lambda_\mp$ yields the row $\begin{bmatrix} 1 & \lambda_\mp \end{bmatrix}$.
The corresponding eigenvectors of T are $v_+ = (1, \lambda_+)$ and $v_- = (1, \lambda_-)$.

(d) Use the solution to part (c) to compute $T^n(0,1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each positive integer n.

First, we note that by part (a), $T^n(0,1) = (F_n, F_{n+1})$. We also know that since $T(1,0) = (0,1)$, we have

$$T^n(1,0) = T^{n-1}(T(1,0)) = T^{n-1}(0,1) = (F_{n-1}, F_n).$$

It follows that the matrix of T^n with respect to the standard basis is

$$\mathcal{M}(T^n) = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.$$

On the other hand, we know that $\mathcal{M}(T^n) = (\mathcal{M}(T))^n = A^n$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ as in part (b).
We now let $P = \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix}$ be the change of basis matrix whose columns are the eigenvectors of A. We note that

$$\det P = \lambda_- - \lambda_+ = -\sqrt{5},$$

by the third identity above, and thus

$$P^{-1} = \frac{-1}{\sqrt{5}}\begin{bmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -\lambda_- & 1 \\ \lambda_+ & -1 \end{bmatrix}.$$

Now, since $P^{-1}AP = D = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$, and $D^n = \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix}$, we have that

$$\begin{aligned}
\begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} = A^n &= (PDP^{-1})^n \\
&= PD^nP^{-1} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} \begin{bmatrix} -\lambda_- & 1 \\ \lambda_+ & -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \begin{bmatrix} -\lambda_+^n\lambda_- & \lambda_+^n \\ \lambda_-^n\lambda_+ & -\lambda_-^n \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \begin{bmatrix} \lambda_+^{n-1} & \lambda_+^n \\ -\lambda_-^{n-1} & -\lambda_-^n \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_+^{n-1} - \lambda_-^{n-1} & \lambda_+^n - \lambda_-^n \\ \lambda_+^n - \lambda_-^n & \lambda_+^{n+1} - \lambda_-^{n+1} \end{bmatrix}.
\end{aligned}$$

(e) Use part (d) to conclude that for each positive integer n, the Fibonacci number F_n is the integer that is closest to

$$\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n.$$

We note that $\lambda_+ \approx 1.62$, while $\lambda_- \approx -0.62$. From part (d), we have

$$\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - F_n = \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

The difference between F_n and $\frac{\lambda_+^n}{\sqrt{5}}$ is thus $\frac{\lambda_-^n}{\sqrt{5}}$, and we note that

$$\frac{\lambda_-}{\sqrt{5}} \approx -0.276, \frac{\lambda_-^2}{\sqrt{5}} \approx 0.171, \frac{\lambda_-^3}{\sqrt{5}} \approx -0.106, \frac{\lambda_-^4}{\sqrt{5}} \approx 0.065,$$

and that $\left| \frac{\lambda_-^{n+1}}{\sqrt{5}} \right| < \left| \frac{\lambda_-^n}{\sqrt{5}} \right|$ for all $n \in \mathbb{N}$. Since this difference is always less than 0.5 in absolute value, we can conclude that F_n is the closest integer to $\lambda_+^n/\sqrt{5}$.