Math 3500 Assignment #4 Solutions
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1. Give an example of each of the following, or argue that such a request is impossible:

   (a) A sequence that does not contain 0 or 1 as a term, but contains subsequences converging to both 0 and 1.

   Solution: One such example would be the sequence (a_n) given by $a_n = \frac{1}{n}$, when n is even, and $a_n = 1 + \frac{1}{n}$ when n is odd. (Note: you don't want to use $1/n$ when n is odd or you'll end up with $a_1 = 1$.

   (b) A monotone sequence that diverges but has a convergent subsequence.

   Solution: This is impossible. Suppose (a_n) diverges, but (a_{n_k}) is a convergent subsequence. Then we know that (a_{n_k}) is bounded, since this is true of every convergent sequence, so there exists some $M > 0$ such that $|a_{n-k}| \leq M$ for all $k \in \mathbb{N}$. Now, suppose (a_n) is increasing. (If (a_n) is decreasing the proof is similar.) For some $N \in \mathbb{N}$ we have $a_n \geq 0$ for all $n \geq N$, or else 0 is an upper bound for (a_n) and (a_n) would converge. But then for each $k \geq N$ we have $N \leq k \leq n_k$, and since (a_n) is monotone, $0 \leq a_k \leq a_{n_k} \leq M$, and again (a_n) is bounded (by the maximum of M and $a_1, \ldots, a_N$), and thus converges by the Monotone Convergence Theorem, contradicting the assumption that (a_n) diverges.

   (c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, \ldots\}$.

   Solution: Consider the sequence

   $$1, 1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, \ldots.$$

   Then for each $n \in \mathbb{N}$, the constant sequence $1/n, 1/n, 1/n, \ldots$ appears as a subsequence.

(d) An unbounded sequence with a convergent subsequence.

Solution: One example is the sequence (a_n) where $a_n = n$ when n is even, and $a_n = 0$ when n is odd.

2. Prove that $\lim_{x\to 2} \dfrac{2x+1}{x+2} = \dfrac{5}{4}$ using the $\epsilon - \delta$ definition of the limit.

Solution: Let $\epsilon > 0$ be given, and let $\delta = \min\{1, 4\epsilon\}$. If $0 < |x-2| < \delta$, then in particular $0 < |x-2| < 1$, which gives $-1 < x-2 < 1$, so $3 < x+2 < 5$, and therefore $\dfrac{3}{x+2} < 1$. Thus, we have

$$\left|\frac{2x+1}{x+2} - \frac{5}{4}\right| = \left|\frac{8x+4-(5x+10)}{4(x+2)}\right| = \frac{3}{|x+2|}\frac{|x-2|}{4} \leq \frac{|x-2|}{4} < \frac{\delta}{4} \leq \frac{4\epsilon}{4} = \epsilon.$$

3. Suppose that f_1 and f_2 are functions for which $\lim_{x\to a^+} f_1(x) = L_1$ and $\lim_{x\to a^+} f_2(x) = L_2$ both exist.

(a) Show that if there exists an interval (a,b) such that $f_1(x) \leq f_2(x)$ for all $x \in (a,b)$, then $L_1 \leq L_2$.

Solution: We prove the contrapositive: if $L_1 > L_2$, then there exists some $x > a$ for which $f_1(x) > f_2(x)$.
Noting that $L_1 - L_2 > 0$, since $\lim_{x\to a^+} f_1(x) = L_1$ and $\lim_{x\to a^+} f_2(x) = L_2$, we can find some $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} &\text{If } a < x < a+\delta_1, \text{ then } |f_1(x) - L_1| < (L_1 - L_2)/2, \text{ and} \\ &\text{if } a < x < a+\delta_2, \text{ then } |f_2(x) - L_2| < (L_1 - L_2)/2. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, and suppose $x \in (a, a+\delta)$. Then

$$|f_1(x) - L_1| < \frac{L_1 - L_2}{2} \Rightarrow \frac{L_1 + L_2}{2} < f_1(x) < \frac{3L_1 - L_2}{2}$$

and

$$|f_2(x) - L_2| < \frac{L_1 - L_2}{2} \Rightarrow \frac{3L_2 - L_1}{2} < f_2(x) < \frac{L_1 + L_2}{2}.$$

Thus $f_2(x) < \dfrac{L_1 + L_2}{2} < f_1(x)$, so $f_1(x) > f_2(x)$ for $x \in (a, a+\delta)$, which is what we wanted to show.

(b) Suppose that we in fact have that $f_1(x) < f_2(x)$ for all $x \in (a, b)$. Can we conclude that $L_1 < L_2$?

Solution: No. For example, if $f_1(x) = 0$ and $f_2(x) = x$ for $x \in \mathbb{R}$, then $f_1(x) < f_2(x)$ for all $x \in (0, 1)$, but

$$\lim_{x \to 0^+} f_1(x) = 0 = \lim_{x \to 0^+} f_2(x).$$

4. Let $g : A \to \mathbb{R}$ be a given function and suppose that f is a bounded function defined on A. (That is, there exists some constant $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in A$.) Let a be a limit point of A, and show that if $\lim\limits_{x \to a} g(x) = 0$, then $\lim\limits_{x \to a}(f(x)g(x)) = 0$ as well.

Solution: Suppose that $|f(x)| \leq M$ for all $x \in A$, for some $M > 0$, and that $\lim_{x \to a} g(x) = 0$. Given any $\epsilon > 0$, there exists some $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|g(x)| < \epsilon / M$, and thus

$$|f(x)g(x)| = |f(x)||f(x)| \leq M|g(x)| < M(\epsilon/M) = \epsilon.$$

Thus, $\lim_{x \to a}(f(x)g(x)) = 0$.