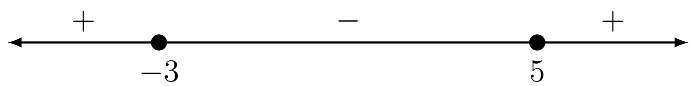
1. Solve the following inequalities:

   (a) $x^2 - 2x \geq 15$

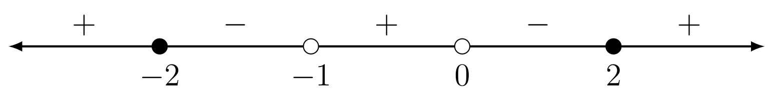
   We have
   $$x^2 - 2x \geq 15 \Leftrightarrow x^2 - 2x - 15 \geq 0 \Leftrightarrow (x+3)(x-5) \geq 0.$$
   The sign diagram for this inequality is given by

   From the sign diagram we see that $(x+3)(x-5) \geq 0$ for $x \in (-\infty, -3] \cup [5, \infty)$.

   (b) $1 + \dfrac{3}{x+1} \leq \dfrac{4}{x}$

   Rearranging, we have
   $$1+\frac{3}{x+1} \leq \frac{4}{x} \Leftrightarrow \frac{x(x+1)+3x-4(x+1)}{x(x+1)} \leq 0 \Leftrightarrow \frac{x^2-4}{x(x+1)} \leq 0 \Leftrightarrow \frac{(x-2)(x+2)}{x(x+1)} \leq 0.$$
   To solve the inequality, we construct the sign diagram for the left-hand side. We find:

   From the sign diagram, we see that the solution to the inequality is given by $x \in [-2, -1) \cup (0, 2]$.

   (Note that we do not include 0 or -1 since the expression $\dfrac{(x-2)(x+2)}{x(x+1)}$ is not defined at these points. This is indicated on the sign diagram by using hollow dots.)

2. Give a one-sentence explanation (in words) why the following are true:

   (a) $\lim_{x \to a} b = b$ for any real numbers a and b.

   Consider the constant *function* $f(x) = b$. To say that $f(x)$ has limit b as $x \to a$ is to say that we can make the value of $f(x)$ as close to b as we want by taking x sufficiently close to a. But the value of $f(x)$ is *equal* to b for *all* x, so this condition is satisfied automatically.

   (b) $\lim_{x \to a} x = a$ for any real number x.

   If we consider the function $g(x) = x$, we are saying that we can make the value of $g(x)$ as close to a as we want by making x sufficiently close to a. But since $g(x) = x$, making $g(x)$ close to a is the same thing as making x close to a.

3. Using properties of limits and the facts given in Problem #2, show that for any *polynomial* $p(x)$, and any real number a, we have $\lim_{x\to a} p(x) = p(a)$.

   Let $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be our polynomial function. We wish to show that
   $$\lim_{x\to a} p(x) = c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = p(a).$$
   From Problem 2(a) we know $\lim_{x\to a}(c_0) = c_0$, and using the product rule for limits with 2(a) and 2(b), we have
   $$\lim_{x\to a}(c_1 x) = (\lim_{x\to a}(c_1))(\lim_{x\to a}(x)) = c_1 a.$$
   Similarly, for $k = 2, 3, \ldots, n$, the power rule gives us $\lim_{x\to a}(c_k x^k) = c_k \left(\lim_{x\to a}(x)\right)^k = c_k a^k$. Using this, together with the sum rule, we have
   $$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0)\\ &= \lim_{x\to a}(c_n x^n) + \lim_{x\to a}(c_{n-1}x^{n-1}) + \cdots + \lim_{x\to a}(c_1 x) + \lim_{x\to a}(c_0)\\ &= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = p(a),\end{aligned}$$
   which is what we wanted to show.

4. Evaluate each of the following limits, or explain it does not exist.

   (a) $\lim_{x\to 3} \dfrac{x^2-9}{x^2-5x+6} = \lim_{x\to 3} \dfrac{(x-3)(x+3)}{(x-3)(x-2)} = \lim_{x\to 3} \dfrac{x+3}{x-2} = \dfrac{3+3}{3-2} = 6.$

   (b) $\lim_{x\to 2} \dfrac{x^2+4}{x-2}$ does not exist. Notice that as $x \to 2$ the denominator is approaching $2-2=0$, while the numerator is approaching $2^2+4=8$. Thus, as x approaches 2, the value of $\dfrac{x^2+4}{x-2}$ increases without bound.

   Also acceptable: $\lim_{x\to 2^-} \dfrac{x^2+4}{x-2} = -\infty$, while $\lim_{x\to 2^+} \dfrac{x^2+4}{x-2} = +\infty$. Since the left and right hand limits do not agree, the limit does not exist. (We didn't discuss this approach in class, however, so I wasn't really expecting it.)

(c) $\displaystyle\lim_{x\to 0}\frac{\sin(3x)}{\tan(5x)}$

We employ a bit of algebraic manipulation to make use of known limits from class. Recall that $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$; this result holds in particular for $\theta=3x$ and $\theta=5x$ if x is approaching 0. We have

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(3x)}{\tan(5x)} &= \lim_{x\to 0}\frac{\sin(3x)}{\sin(5x)/\cos(5x)} = \lim_{x\to 0}\frac{\sin(3x)}{\sin(5x)}(\cos(5x))\\ &= \lim_{x\to 0}\frac{3}{5}\left(\frac{\sin(3x)}{3x}\right)\left(\frac{5x}{\sin(5x)}\right)(\cos(5x))\\ &= \frac{3}{5}(1)(1)(1) = \frac{3}{5}.\end{aligned}$$

Notice that the expression in the second line is equal to that at the end of the first: the additional terms all cancel out. Note also that we've made use of the fact that

$$\lim_{x\to 0}\frac{5x}{\sin(5x)} = \lim_{x\to 0}\frac{1}{\frac{\sin(5x)}{5x}} = \frac{1}{\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{5x}} = \frac{1}{1} = 1.$$