Math 2580 Assignment #7 Solutions
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1. Verify Green's Theorem holds for the integral $\displaystyle\int_C (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy$, where C is the unit circle.

   If we wish to compute the integral directly, we can use the parameterization $x = \cos t$, $y = \sin t$, with $t \in [0, 2\pi]$. Then

   $$\begin{aligned}(2x^3 - y^3)\,dx &= (2\cos^3 t + \sin^3 t)(-\sin t)\,dt = (-2\cos^3 t\sin t + \sin^4 t)\,dt\\ (x^3 + y^3)\,dy &= (\cos^3 t + \sin^3 t)(\cos t)\,dt = (\cos^4 t + \sin^3 t\cos t)\,dt,\end{aligned}$$

   giving us the line integral

   $$\int_C (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy = \int_0^{2\pi} (\cos^4 t + \sin^4 t - 2\cos^3 t\sin t + \sin^3 t\cos t)\,dt = \frac{3\pi}{2}.$$

   To compute the last integral above, note that the last two terms in the integral do not contribute: they integrate to powers of $\cos t$ and $\sin t$, respectively, and the limits of integration are 0 and 2π. For the first two terms, note that

   $$\cos^4 t + \sin^4 t = \left(\frac{1 + \cos 2t}{2}\right)^2 + \left(\frac{1 - \cos 2t}{2}\right)^2 = \frac{1}{4}(2 + 2\cos^2 2t) = \frac{1}{4}(3 + \cos 4t),$$

   and there is no contribution from the $\cos 4t$ term, leaving us with $\dfrac{3}{4}(2\pi) = \dfrac{3\pi}{2}$.

   To compute the integral using Green's theorem, we note that C is the boundary of the disc D given by $x^2 + y^2 \leq 1$, so

   $$\begin{aligned}\int_C (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy &= \iint_D \left(\frac{\partial}{\partial x}(x^3 + y^3) - \frac{\partial}{\partial x}(2x^3 - y^3)\right)dA\\ &= \iint_D (3x^2 + 3y^2)\,dA = \int_0^{2\pi}\int_0^1 3r^2 \cdot r\,dr\,d\theta = \frac{3\pi}{2},\end{aligned}$$

   as above.

2. A vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in $\mathbb{R}^2$ can be viewed as a special case of a vector field in $\mathbb{R}^3$ that does not depend on z, with z-component equal to zero. With this identification,

   (a) Show that $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$.
   Using the determinant formula for curl, we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k},$$

   and taking the dot product of this with $\mathbf{k}$ gives the result, since $\mathbf{k} \cdot \mathbf{k} = 1$.

   (b) Use part (a) to show that Green's Theorem can be written in the vector form

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA,$$

   where $D \subseteq \mathbb{R}^2$ is a region to which Green's Theorem applies, and $C = \partial D$ is the positively-oriented boundary of D.

   We have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA,$$

   as required.

   (c) Show that Green's Theorem implies the *Divergence Theorem in the Plane*:
   Let $D \subseteq \mathbb{R}^2$ be a region to which Green's Theorem applies, and let $C = \partial D$ be its positively-oriented boundary. Let $\mathbf{n}$ denote the outward-pointing unit normal vector to C: if $\mathbf{r} : [a, b] \to \mathbb{R}^2$, $\mathbf{r}(t) = (x(t), y(t))$ defines a positively-oriented parameterization of C, then $\mathbf{n}$ is given by

$$\mathbf{n} = \frac{y'(t)\mathbf{i} - x'(t)\mathbf{j}}{\sqrt{(x'(t))^2 + (y'(t))^2}}. \quad \text{(Verify this.)}$$

   If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a C^1 vector field on D, then

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D (\nabla \cdot \mathbf{F}) \, dA.$$

Let us write $\mathbf{N}(t) = y'(t)\mathbf{i} - x'(t)\mathbf{j}$ for the non-unit normal vector, noting that $\mathbf{N}$ and $\mathbf{n}$ point in the same direction. We can see that $\mathbf{N}$ must be normal to the curve, since

$$\mathbf{N}(t) \cdot \mathbf{r}'(t) = y'(x)x'(t) - x'(t)y'(t) = 0,$$

so $\mathbf{N}$ is orthogonal to the tangent vector $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$, and if we treat $\mathbf{N}$ and $\mathbf{r}$ as vectors in the xy-plane of $\mathbb{R}^3$, we see that

$$\mathbf{N}(t) \times \mathbf{r}'(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ y'(t) & -x'(t) & 0 \\ x'(t) & y'(t) & 0 \end{vmatrix} = (x'(t)^2 + y'(t)^2)\mathbf{k}$$

points in the positive z-direction. This tells us that $\mathbf{N}$ is always to the right of $\mathbf{r}'$ (this is easiest to see with a picture, if you use the right-hand rule for the direction of the cross product), and therefore $\mathbf{N}$ is the outward-pointing normal vector, and $\|\mathbf{N}(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$, so $\mathbf{n}$ is the unit normal vector.

Now, noting that $\|\mathbf{N}(t)\| = \sqrt{x'(t)^2 + y'(t)^2} = \|\mathbf{r}'(t)\|$, we have

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{n}\, ds &= \int_a^b \langle P(\mathbf{r}(t)), Q(\mathbf{r}(t))\rangle \cdot \frac{1}{\|\mathbf{N}(t)\|} \langle y'(t), -x'(t)\rangle \|\mathbf{r}'(t)\|\, dt \\ &= \int_a^b (P(\mathbf{r}(t))y'(t) - Q(\mathbf{r}(t))x'(t))\, dt \\ &= \int_C -Q\, dx + P\, dy \\ &= \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial - Q}{\partial y} \right) dA \\ &= \iint_D (\nabla \cdot \mathbf{F})\, dA. \end{aligned}$$

3. The **Laplacian** is a differential operator $\Delta = \nabla^2$ that acts on functions $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, defined by

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

A C^2 function f is called **harmonic** if $\Delta f = 0$. Harmonic functions are important in many areas of Engineering and Physics, such as heat transfer, electrodynamics, fluid flow, robotics[1], etc.

(a) Determine whether or not the functions $f(x, y) = e^x \sin y$, $g(x, y) = x^3 + y^3$, $h(x, y) = x^3 - 3xy^2$ are harmonic.

We have

$f_{xx}(x, y) = e^x \sin y$ and $f_{yy}(x, y) = -e^x \sin y$, so $f_{xx}(x, y) + f_{yy}(x, y) = \Delta f(x, y) = 0$,

showing that f is harmonic. For g we have

$$\Delta g(x, y) = g_{xx}(x, y) + g_{yy}(x, y) = 6x + 6y \neq 0,$$

[1] According to the internet.

so g is not harmonic. For h we have

$$\Delta h(x,y) = h_{xx}(x,y) + h_{yy}(x,y) = \frac{\partial}{\partial x}(3x^2 - 3y^2) + \frac{\partial}{\partial y}(-6xy) = 6x - 6x = 0,$$

so h is harmonic.

(b) Prove that for any harmonic function f defined on a region D for which Green's Theorem holds, we have

$$\int_{\partial D} \frac{\partial f}{\partial y}\, dx - \frac{\partial f}{\partial x}\, dy = 0.$$

Note that any harmonic function f is C^2 by definition, so the partial derivatives f_x and f_y are C^1, and therefore we can apply Green's theorem to obtain

$$\int_{\partial D} \frac{\partial f}{\partial y}\, dx - \frac{\partial f}{\partial x}\, dy = \iint_D \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} \right) dA = \iint_D (-\Delta f)\, dA = \iint_D (0)\, dA = 0.$$