Solutions to Quiz 19 Practice Problems
Math 2580
Spring 2016

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March 29th, 2016

1. Find a function f such that $\nabla f = \mathbf{F}$, and use this to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the given curve:

   (a) $\mathbf{F}(x,y) = \dfrac{y^2}{1+x^2}\mathbf{i} + 2y\arctan x\mathbf{j}$, C parameterized by $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}$, $0 \le t \le 1$.

   Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ and suppose $\mathbf{F} = \nabla f$ for some function f. If $f_x(x,y) = \dfrac{y^2}{1+x^2} = P(x,y)$, then we must have $f(x,y) = y^2\arctan x + g(y)$ for some function $g(y)$ of y only. Then on the one hand we have $f_y(x,y) = 2y\arctan x + g'(y)$, and on the other hand, $f_y(x,y) = 2y\arctan x = Q(x,y)$, so we can take $g(y) = 0$, and $f(x,y) = y^2\arctan x$.

   The endpoints of the curve are given by $\mathbf{r}(0) = \langle 0,0 \rangle$ and $\mathbf{r}(1) = \langle 1,2 \rangle$, so by the Fundamental Theorem of Calculus we have

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1,2) - f(0,0) = 4\arctan 1 - 0 = \pi.$$

   (b) $\mathbf{F}(x,y,z) = (2xz + y^2)\mathbf{i} + 2xy\mathbf{j} + (x^2 + 3z^2)\mathbf{k}$, C parameterized by $x = t^2, y = t + 1, z = 2t - 1$, $0 \le t \le 2$.

   Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and suppose $\mathbf{F} = \nabla f$ for some function f. Then we must have $f_x(x,y,z) = P(x,y,z) = 2xz + y^2$, which gives $f(x,y,z) = x^2z + xy^2 + g(y,z)$ for some function g that does not depend on x. Now we must have

   $$Q(x,y,z) = 2xy = f_y(x,y,z) = 2xy + g_y(y,z),$$

   which tells us that $g_y(y,z) = 0$, so $g(y,z) = g(z)$ depends only on z. Comparing z-components, we have

   $$R(x,y,z) = x^2 + 3z^2 = f_z(x,y,z) = x^2 + g'(z),$$

from which we find $g'(z) = 3z^2$, so $g(z) = z^3$ and $f(x, y, z) = x^2 z + xy^2 + z^3$. The endpoints of our curve are given by $(0^2, 0 + 1, 2(0) - 1) = (0, 1, -1)$ and $(2^2, 2 + 1, 2(2) - 1) = (4, 3, 3)$ so we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 3, 3) - f(0, 1, -1) = 16(3) + 4(9) + 27 - (-1)^3 = 110.$$

(c) $\mathbf{F}(x, y, z) = \langle e^y, xe^y, (z+1)e^z \rangle$, C parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $t \in [0, 1]$.

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and suppose $\mathbf{F} = \nabla f$ for some function f. Then we must have $f_x(x, y, z) = P(x, y, z) = e^y$, so $f(x, y, z) = xe^y + g(y, z)$ for some function g of y and z. Comparing y components, we have

$$Q(x, y, z) = xe^y = f_y(x, y, z) = xe^y + g_y(y, z),$$

so $g_y(y, z) = 0$, and $g(y, z) = g(z)$ depends on z only. Comparing z components,

$$R(x, y, z) = (z + 1)e^z = g'(z),$$

and since $(z + 1)e^z = \dfrac{d}{dz}(ze^z)$, we have $g(z) = ze^z$ and $f(x, y, z) = xe^y + ze^z$. Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1, 1) - f(0, 0, 0) = 2e.$$

2. Calculate the curl of the given vector field:

(a) $\mathbf{F}(x, y, z) = \langle 2xy, xz, y^2 z \rangle$

$$\nabla \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy & xz & y^2 z \end{vmatrix} = (2yz - x)\mathbf{i} + (z - 2x)\mathbf{k}.$$

(b) $\mathbf{F}(x, y, z) = \langle y \cos xy, x \cos xy, -\sin z \rangle$

$$\nabla \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y \cos xy & x \cos xy & -\sin z \end{vmatrix} = \langle 0, 0, 0 \rangle.$$

3. Verify that Green's Theorem holds for the following line integrals in the plane:

(a) $\int_C xy^2 \, dx + x^3 \, dy$, where C is the rectangle with corners at $(0, 0), (2, 0), (2, 3)$, and $(0, 3)$.

Let $C = C_1 + C_2 + C_3 + C_4$, as follows:

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C_1 is the bottom of the rectangle, with $y = 0$ and $x = 2t$, $t \in [0, 1]$, so along C_1, $y = 0$ and $dy = 0$, which means $xy^2 \, dx + x^3 \, dy = 0$.

C_2 is the right side of the rectangle, with $x = 2$ and $y = 3t$, $t \in [0, 1]$. Along C_1, $dx = 0$ since x is constant, so

$$xy^2 \, dx + x^3 \, dy = 0 + (2^3)(3 \, dt) = 24 \, dt.$$

C_3 is the top of the rectangle, with $y = 3$, $dy = 0$, and $x = 2 - 2t$, $t \in [0, 1]$, so $dx = -2 \, dt$ and

$$xy^2 \, dx + x^3 \, dy = (2 - 2t)(3)^2(-2 \, dt) = (36t - 36) \, dt.$$

C_4 is the left side of the rectangle, where $x = 0$ and $dx = 0$, so $xy^2 \, dx + x^3 \, dy = 0$. We thus have

$$\int_C xy^2 \, dx + x^3 \, dy = 0 + \int_0^1 24 \, dt + \int_0^1 (36t - 36) \, dt + 0 = 6.$$

On the other hand, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial x^3}{\partial x} - \frac{\partial xy^2}{\partial y} = 3x^2 - 2xy,$$

and

$$\int_0^2 \int_0^3 (3x^2 - 2xy) \, dy \, dx = \int_0^2 (9x^2 - 9x) \, dx 3(2^3) - \frac{9}{2}(2^2) = 24 - 18 = 6.$$

(b) $\int_C y \, dx - x \, dy$, where C is the unit circle.

To compute the integral directly, we parameterize C using $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$. Then

$$\int_C y \, dx - x \, dy = \int_0^{2\pi} \sin t(-\sin t \, dt) - (\cos t)(\cos t \, dt) = \int_0^{2\pi} (-1) \, dt = -2\pi.$$

Using Green's Theorem, we have $Q_x(x, y) - P_y(x, y) = -1 - 1 = -2$, so

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (-2) \, dA = -2A(D) = -2\pi.$$

(c) $\int_C x \, dx + y \, dy$, where C consists of the line segments from $(0, 1)$ to $(0, 0)$, and from $(0, 0)$ to $(1, 0)$, and the portion of the parabola $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$.

First, we compute the line integral directly: we have $C = C_1 + C_2 + C_3$, where C_1 is the portion of the curve along the y-axis, C_2 is along the x-axis, and C_3 is the

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parabolic arc. Along C_1 we have $x = 0$ and $dx = 0$, while $y = 1 - t$, $dy = -dt$, with $t \in [0, 1]$, so

$$\int_C {}_1 x\, dx + y\, dy = \int_0^1 (0 + (1 - t)(-dt)) = \int_0^1 (t - 1)\, dt = -\frac{1}{2}.$$

Along C_2 we have $x = t$, $dx = dt$, with $t \in [0, 1]$, and $y = 0$, $dy = 0$, so

$$\int_{C_2} x\, dx + y\, dy = \int_0^1 t\, dt = \frac{1}{2}.$$

Finally, along C_3, we take $x = 1 - t$, $y = 1 - (1 - t)^2 = 2t - t^2$, so $dx = -dt$ and $dy = (2 - 2t)\, dt$. This gives us

$$\int_{C_3} x\, dx + y\, dy = \int_0^1 (1-t)(-dt) + (2t - t^2)(2 - 2t)\, dt = \int_0^1 (2t^3 - 6t^2 + 5t - 1)\, dt = \frac{1}{2} - 2 + \frac{5}{2} - 1 = 0.$$

Thus, $\int_C x\, dx + y\, dy = -\frac{1}{2} + \frac{1}{2} + 0 = 0$.

(Note, for C_3, you can take $x = t$ and $y = 1 - t^2$, with $t \in [0, 1]$, as long as you're aware that this parameterization sends you along the parabola in the wrong direction, requiring you to change the sign of your answer at the end.)

Now, to compute the integral using Green's Theorem, we simply note that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial x} = 0 - 0 = 0$, so the double integral is automatically zero, in agreement with the line integral above.