Math 3500 Assignment #5 Solutions
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1. (The Contraction Mapping Theorem) Let f be a function defined on all of $\mathbb{R}$, with the property that there exists some c with $0 < c < 1$, such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq c|x - y|.$$

   (a) Prove that f is continuous on $\mathbb{R}$.

   Solution: Let $\epsilon > 0$ be given and take $\delta = \epsilon$. Then, if $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq c|x - y| < |x - y| < \delta = \epsilon,$$

   since $0 < c < 1$.

   (b) Choose any point $x \in \mathbb{R}$, and consider the sequence $(x, f(x), f(f(x)), \ldots)$. (That is, the sequence is defined recursively by $x_1 = x$ and $x_{n+1}) = f(x_n)$ for $n \geq 1$.) Show that this sequence is a Cauchy sequence.

   Solution: Choose $x \in \mathbb{R}$, and let $K = |f(x) - x| = |x_2 - x_1|$. We claim that for any $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq c^{n-1}K$. The proof is by induction: for $n = 1$, we have $|x_2 - x_1| = c^0 K$, so the result holds. If $|x_{k+1} - x_k| \leq c^{k-1}K$ for some $k \geq 1$, then

$$|x_{k+2} - x_{k+1}| = |f(x_{k+1}) - f(x_k)| \leq c|x_{k+1} - x_k| \leq c(c^{k-1}K) = c^k K,$$

   so the result holds for all $n \in \mathbb{N}$ by induction.

   Now, let $\epsilon > 0$ be given, and choose $N \in \mathbb{N}$ sufficiently large that $c^{N-1} < \dfrac{(1-c)\epsilon}{K}$. (This is possible since $\lim c^N = 0$.) Then, if $m, n \geq N$, we have (assuming without loss of generality that $m = n + k$ for some $k \geq 1$)

$$\begin{aligned}
|x_m - x_n| &= |x_{n+k} - x_n| \\
&\leq |x_{n+k} - x_{n+k-1}| + |x_{n+k-1} - x_{n+k-2}| + \cdots + |x_{n+1} - x_n| \\
&\leq c^{n-k-2}K + c^{n-k-3}K + \cdots + c^{n-1}K \\
&= Kc^{n-1}(1 + c + \cdots + c^{k-1}) \\
&< Kc^{n-1}(1 + c + c^2 + \cdots) \\
&= \frac{Kc^{n-1}}{1-c} \leq \frac{Kc^{N-1}}{1-c} < \epsilon.
\end{aligned}$$

Thus, (x_n) is Cauchy. (Note: we used the result proved above, together with the fact that $\sum_{j=0}^{k-1} c^j < \sum_{j=0}^{\infty} c^j = \frac{1}{1-c}$.)

(c) Since the sequence in part (b) is Cauchy, it converges. Let $y = \lim x_n$, and prove that y is a *fixed point* of f. That is, prove that $f(y) = y$.

Solution: Since f is continuous and (x_n) converges to y, we have

$$f(y) = f(\lim x_n) = \lim f(x_n) = \lim x_{n+1} = y.$$

(d) Show that $y = \lim x_n$ is the **unique** fixed point of f.

Solution: Suppose $f(y_1) = y_1$ and $f(y_2) = y_2$ for some $y_1, y_2 \in \mathbb{R}$. Then

$$|y_1 - y_2| = |f(y_1) - f(y_2)| \leq c|y_1 - y_2|.$$

Thus, we must have $|y_1 - y_2| = 0$ and $y_1 = y_2$, or else we would have $c \geq 1$, which is not possible since $0 < c < 1$.

(e) Prove that if $z \in \mathbb{R}$ is any arbirary point, then the sequence $(z, f(z), f(f(z)), \ldots)$ still converges to y.

Solution: If we choose some other point z, then the sequence $(f^n(z))$ will converge as above to some limit w, and the same argument already given would show that $f(w) = w$. Since we proved that the fixed point of f is unique, we must have $w = y$.

2. (**Do not submit**) Let f and g be functions defined on some domain $D \subseteq \mathbb{R}$, and suppose that both f and g are continuous at $a \in D$.

(a) Show that $|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$ is continuous at a.

Solution: Let $\epsilon > 0$ be given. Since f is continuous at a, there exists some $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. If $f(a) \neq 0$ then we can choose δ small enough that $f(x)$ and $f(a)$ have the same sign when $|x - a| < \delta$, and continuity of $|f|$ follows from the continuity of f (and $-f$).
If $f(a) = 0$, then

$$||f(x)| - |f(a)|| = |f(x)| = |f(x) - f(a)| < \epsilon.$$

(b) Let $\max(f,g)(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } g(x) \geq f(x) \end{cases}$. Show that

$$\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|.$$

Solution: Choose some $x \in D$. If $f(x) \geq g(x)$, then

$$\frac{1}{2}(f(x)+g(x)) + \frac{1}{2}|f(x)-g(x)| = \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}(f(x)-g(x)) = f(x)$$

If $f(x) < g(x)$, then

$$\frac{1}{2}(f(x)+g(x)) + \frac{1}{2}|f(x)-g(x)| = \frac{1}{2}(f(x)-g(x)) - \frac{1}{2}(f(x)-g(x)) = g(x).$$

(c) Similarly define $\min(f,g)$ and show that $\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$.

Solution: I'll leave this as an easy exercise.

(d) Show that $\max(f,g)$ and $\min(f,g)$ are continuous. (Hint: an ϵ-δ proof is not required.)

Solution: Since f and g are continuous, so are $f+g$ and $f-g$. Since $f-g$ is continuous, so is $|f-g|$ by part (a). The result now follows by parts (b) and (c).

3. Let $g(x) = \sqrt[3]{x}$.

(a) Show that g is continuous at $c = 0$.

Solution: Let $\epsilon > 0$ be given and choose $\delta = \epsilon^3$. Then if $|x| < 0$ we have

$$|g(x) - g(0)| = |x^{1/3}| = |x|^{1/3} < \delta^{1/3} = \epsilon.$$

(b) Prove that g is continuous at any point $c \neq 0$. (You might find the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ helpful.)

Solution: Choose $c \in \mathbb{R}$ with $c \neq 0$, and let $\epsilon > 0$ be given. Let $\delta = \min\left\{\frac{|c|}{2}, c^{2/3}\epsilon\right\}$. Suppose that $|x-c| < \delta$. Notice that since $|x-c| < |c|/2$, $x \neq 0$ and x and c must have the same sign:

$$|x-c| < |c|/2 \quad \Leftrightarrow \quad c - |c|/2 < x < c + |c|/2,$$

so if $c < 0$, $3c/2 < x < c/2$, and if $c > 0$, $c/2 < x < 3c/2$. Thus $xc > 0$, so $x^{2/3} + x^{1/3}c^{1/3}c^{2/3} > c^{2/3} > 0$, and we have

$$|x^{1/3} - c^{1/3}| = \frac{|x-c|}{x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}} < \frac{|x-c|}{c^{2/3}} < \frac{\delta}{c^{2/3}} \leq \epsilon.$$

4. (**Do not submit**) Explain why any function with domain $\mathbb{Z} \subseteq \mathbb{R}$ is necessarily continuous at every point in its domain.

   Solution: If the domain of f is $\mathbb{Z}$ then every point in the domain is an isolated point. If we take $\delta = 1/2$ then $a \in \mathbb{Z}$ and $|x-a| < \delta$ implies that $x = a$, so $|f(x)-f(a)| = 0 < \epsilon$ for any $\epsilon > 0$.

5. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$, and let $K = \{x \in \mathbb{R} : h(x) = 0\}$. Prove that K is a closed subset of $\mathbb{R}$.

   Solution: We'll give two different proofs of this result:

   Option 1: The complement of K is $K^c = \{x \in \mathbb{R} : h(x) \neq 0\} = h^{-1}((-\infty, 0) \cup (0, \infty))$. Since $(-\infty, 0) \cup (0, \infty) \subseteq \mathbb{R}$ is open and h is continuous, K^c is open and thus K is closed. (Note that when the domain is all of $\mathbb{R}$, "relatively open" is the same thing as open.)

   Option 2: We know that K is closed if and only if K contains all of its limit points. Thus, let $a \in \mathbb{R}$ be a limit point of K (if any exist). Then there exists a sequence (x_n) in K such that $x_n \to a$, and since h is continuous on $\mathbb{R}$, we have

   $$f(a) = f(\lim x_n) = \lim f(x_n) = \lim 0 = 0.$$

   Since $f(a) = 0$, $a \in K$, and thus K contains its limit points.

6. Show that if f is continuous on $[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$, then $1/f$ is bounded on $[a, b]$.

   Solution: Since f is continuous on $[a, b]$, by the Extreme Value Theorem there exists some $y \in [a, b]$ such that $f(y) \leq f(x)$ for all $x \in [a, b]$. Since $f(y) > 0$ by assumption, we have $1/f(x) \leq 1/f(y)$ for all $x \in [a, b]$, and thus $1/f$ is bounded on $[a, b]$.

7. (**Do not submit**) Prove that $\cos x = x$ for some $x \in (0, \pi/2)$.

   Solution: Consider the function $f(x) = \cos x - x$. We know that f is continuous, since it's the difference of two continuous functions. Since $f(0) = \cos 0 - 0 = 1 > 0$ and $f(\pi/2) = \cos(\pi/2) - \pi/2 = -\pi/2 < 0$, by the Intermediate Value Theorem there exists some $c \in (0, \pi/2)$ such that $f(c) = 0$, which gives $\cos c = c$, as required.

8. (**Do not submit**) We say that a function f satisfies the *intermediate value property* if it satisfies the conclusion of the Intermediate Value Theorem. Show that the function given by $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$ has the property, even though it is not continuous at $x = 0$.

Solution: Let $x, y \in \mathbb{R}$. If x and y are both positive or both negative, then the result follows from the IVT since f is continuous on $(0, \infty)$ and $(\infty, 0)$. If $x < 0$ and $y > 0$ it's easy to see that f takes on every value between -1 and 1 on (x, y), since it oscillates infinitely often as we approach 0. In fact, choose $n \in \mathbb{N}$ such that $x < \dfrac{-1}{\pi/2 + 2n\pi}$ and $\dfrac{1}{\pi/2 + 2n\pi} > y$. Then f takes on every value between -1 and 1 on $\left[\dfrac{-1}{\pi/2 + 2n\pi}, \dfrac{-1}{3\pi/2 + 2n\pi}\right]$ and on $\left[\dfrac{1}{3\pi/2 + 2n\pi}, \dfrac{1}{\pi/2 + 2n\pi}\right]$.