1. In each case, give an elementary matrix E such that $EA = B$:

   (a) $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

   B is obtained from A using the row operation $R_2 \to R_2 - R_1$, so $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

   B is obtained from A using the row operation $R_1 \to -1R_1$, so $E = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}$

   B is obtained from A using the row operation $R_1 \leftrightarrow R_2$, so $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   (d) $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$

   B is obtained from A using the row operation $R_1 \to R_1 - R_2$, so $E = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

   (e) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

   B is obtained from A using the row operation $R_2 \to -1R_2$, so $E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2. Find an invertible matrix U such that UA is in row-echelon form, where $A = \begin{bmatrix} 3 & 5 & 0 \\ 3 & 7 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. How do you know U is invertible?

   We reduce A to row-echelon form using elementary row operations, while simultaneously performing the same operations on the identity:

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 0 & 1 & 0 & 0 \\ 3 & 7 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 3 & 7 & 1 & 0 & 1 & 0 \\ 3 & 5 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -3 \\ 3 & 5 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -3 \\ 0 & -1 & -3 & 1 & 0 & -3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -3 \\ 0 & 0 & -5 & 1 & 1 & -6 \end{array}\right]$$

$$\xrightarrow{R_3 \to -\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & -3 \\ 0 & 0 & 1 & -\frac{1}{5} & -\frac{1}{5} & \frac{6}{5} \end{array}\right].$$

If we represent the five row operations above by E_1, E_2, E_3, E_4, E_5, then the last augmented matrix above can be written as

$$[\ E_5E_4E_3E_2E_1A \mid E_5E_4E_3E_2E_1\ ] = [\ UA \mid U\ ],$$

where $U = E_5E_4E_3E_2E_1$, and UA is in row-echelon form.

Thus, we may take $U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ -\frac{1}{5} & -\frac{1}{5} & \frac{6}{5} \end{bmatrix}$. We know that U is invertible since it can be expressed as a product of elementary matrices, and every elementary matrix is invertible.

Note: Since you were only asked to get A into row-echelon form and not reduced-row echelon form, your answer may have been different from the one above. (Only the *reduced* row-echelon form is unique.)

3. Write the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$ as a product of elementary matrices.

We begin by reducing A to reduced row-echelon form, noting that it will only be possible to write A as a product of elementary matrices if A is invertible. (This is why it's a good idea to verify that A^{-1} exists while doing this problem.) We have:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{R_1 \to R_1 - 2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the reduced row-echelon form of A is the identity matrix, we know that A is invertible. Moreover, if we write

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

for the four elementary matrices corresponding to the four row operations above, then we have

$$E_4(E_3(E_2(E_1A))) = (E_4E_3E_2E_1)A = I,$$

which implies that $E_4E_3E_2E_1 = A^{-1}$, since we know A is invertible, and thus A^{-1} is the unique matrix such that $A^{-1}A = I$. This gives us

$$A = (A^{-1})^{-1} = (E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1},$$

so to write A as a product of elementary matrices, we need to find the inverses of the elementary matrices above. We have

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note: Since different people may choose to do different row operations, or do them in a different order, the answer above is not unique: it depends on your choices of row operations.

4. Prove that if A is invertible, so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

Suppose that A is invertible. Then A^{-1} exists, and we know that $AA^{-1} = A^{-1}A = I$. To show that A^T is invertible, we need to show that there exists some matrix B such that $A^T B = I$ and $BA^T = I$. It then follows from the uniqueness of the inverse that $B = (A^T)^{-1}$. We will show that $B = (A^{-1})^T$. (Note that this choice of B is defined: we are assuming that A^{-1} exists, and it's always possible to take the transpose of a matrix.) Using properties of the transpose, we have

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I, \text{ and } (A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

Thus, it must be the case that $(A^{-1})^T = (A^T)^{-1}$.

5. Prove that if A is invertible and $a \in \mathbb{R}$ is a scalar, with $a \neq 0$, then aA is invertible.

The argument is the same as the one above: to show that aA is invertible, we need to find a matrix B such that $(aA)B = I = B(aA)$. We'll show that $B = \frac{1}{a}A^{-1}$ does the job.

Suppose that A is an invertible matrix and that a is a nonzero scalar. Using properties of matrix multiplication, we have

$$(aA)\left(\frac{1}{a}A^{-1}\right) = \left(a \cdot \frac{1}{a}\right)(AA^{-1}) = 1(I) = I,$$

and

$$\left(\frac{1}{a}A^{-1}\right)(aA) = \left(\frac{1}{a} \cdot a\right)(A^{-1}A) = 1(I) = I.$$

Thus, it must be the case that aA is invertible, and $(aA)^{-1} = \frac{1}{a}A^{-1}$.