University of Toronto at Mississauga

Mid-Term Exam

MAT232HF
Calculus of Several Variables

Instructor: Sean Fitzpatrick
Duration: 110 minutes

NO AIDS ALLOWED. **Total: 60 marks**

Family Name: SOLUTIONS
(Please Print)

Given Name(s): THE
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Please sign here: ______

Student ID Number: ______

You may not use calculators, cell phones, or PDAs during the exam. Partial credit will be given for partially correct work. Please read through the entire test before starting, and take note of how many points each question is worth. Please put a box around your solutions so that the grader may find them easily.

FOR MARKER'S USE ONLY	
Problem 1:	/11
Problem 2:	/10
Problem 3:	/8
Problem 4:	/15
Problem 5:	/10
Problem 6:	/6
TOTAL:	/60

[3] 1. (a) Sketch the conic section $4y^2 - 9x^2 - 18x - 8y = 41$

Solution: We first complete the square in both x and y, obtaining

$$4(y-1)^2 - 9(x+1)^2 = 36.$$

Dividing through by 36, we obtain the standard form of a hyperbola,

$$\frac{-(x-1)^2}{2^2} + \frac{(y-1)^2}{3^2} = 1.$$

We see that the hyperbola has centre $(-1, 1)$, and asymptotes $y-1 = \pm 3/2(x+1)$, and that $y = 1$ is impossible, while $x = -1$ gives $y - 1 = \pm 3$, so the hyperbola opens vertically, with vertices $(-1, 4)$ and $(-1, -2)$.
Your sketch should convey this information, and look roughly like a hyperbola.

[4] (b) Sketch the parametric curve $x(t) = 3 + 2\cos t$, $y(t) = 5 - 3\sin t$, $t \in [0, 2\pi]$.

Solution: We sketch the curve by first eliminating the parameter t. Since we know that $\sin^2 t + \cos^2 t = 1$, we solve for $\sin t$ and $\cos t$, getting

$$\cos t = \frac{x-3}{2}, \quad \sin t = \frac{5-y}{3}.$$

Thus, we get that

$$1 = \sin^2 t + \cos^2 t = \left(\frac{y-5}{3}\right)^2 + \left(\frac{x-3}{2}\right)^2,$$

or $\frac{(x-3)^2}{2^2} + \frac{(y-5)^2)}{3^2} = 1$, which is the equation of an ellipse, with centre $(3, 5)$, and vertices $(3, 2), (3, 8), (5, 5), (1, 5)$.
Since $\sin t$ and $\cos t$ both have period 2π, we see that we get the entire ellipse for $t \in [0, 2\pi]$.

[4] (c) Identify the traces of the surface $9x^2+4z^2-36y^2-36=0$ in each of the co-ordinate planes. Then, sketch the surface.

Solution: The trace in the yz-plane ($x=0$) is the curve $4z^2-36y^2-36=0$, or $\frac{z^2}{3^2}-y^2=1$, which is a hyperbola.

Similarly, the trace in the xz-plane ($y=0$) is the ellipse $\frac{x^2}{2^2}+\frac{z^2}{3^2}=1$, and the trace in the xy-plane ($z=0$) is the hyperbola $\frac{x^2}{3^2}-y^2=1$.

The surface is the hyperboloid of one sheet given by $\frac{x^2}{4}-y^2+\frac{z^2}{9}=1$ which has its axis in the y direction (that is, it opens outward along the y-axis) and centre at the origin. Your sketch should look roughly like figure 12.7.17 of the textbook, but turned on its side to line up with the y-axis.

2. Let $\vec{u}$ and $\vec{v}$ be non-zero vectors such that $|\vec{u}+\vec{v}| = |\vec{u}-\vec{v}|$.

[2] (a) What can you conclude about the parallelogram spanned by $\vec{u}$ and $\vec{v}$?

Solution: If we sketch the parallelogram in question, we see that its diagonals are the vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$. Since the lengths of these two vectors are equal, we have a parallelogram whose diagonals are of equal length; that is, a rectangle.

[3] (b) Show that $\vec{u}\cdot\vec{v} = 0$.

Solution: Recall that $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$, where θ is the angle between the two vectors. Since these vectors form the sides of a rectangle, we must have $\theta = \pi/2$, so that $\cos\theta = 0$. Thus, $\vec{u}\cdot\vec{v} = 0$.

Alternatively, this can be shown algebraically, using techniques similar to those used in part (c) below.

[5] (c) Prove the paralleogram law: $|\vec{u}-\vec{v}|^2 + |\vec{u}+\vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2$.

Solution: Recall that for any vector $\vec{a}$, we have that $|\vec{a}|^2 = \vec{a}\cdot\vec{a}$. Thus, we have:

$$\begin{aligned}
|\vec{u}-\vec{v}|^2 + |\vec{u}+\vec{v}|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) + (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= (\vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}) + (\vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}) \\
&= |\vec{u}|^2 - 2\vec{u}\cdot\vec{v} + |\vec{v}|^2 + |\vec{u}|^2 + 2\vec{u}\cdot\vec{v} + |\vec{v}|^2 \\
&= 2|\vec{u}|^2 + 2|\vec{v}|^2.
\end{aligned}$$

3. Determine whether or not the following sets of points lie on the same line. If they do, give the vector equation of the line. If not, give the equation of the plane containing them.

[4] (a) $P = (0, -2, 4)$, $Q = (1, -3, 5)$, $R = (4, -6, 8)$

Solution: We see that $\overrightarrow{PQ} =< 1 - 0, -3 - (-2), 5 - 4 >=< 1, -1, 1 >$, while similarly $\overrightarrow{QR} =< 3, -3, 3 >= 3\overrightarrow{PQ}$. Thus the two vectors are parallel, and so the points P, Q, R must all lie on the same line.

To get the equation of the line we need a point and a vector in the direction of the line. The point P and the vector $\overrightarrow{PQ}$ fit that description, so the equation of the line is

$$\vec{r} =< x, y, z >=< 0, -2, 4 > +t < 1, -1, 1 > .$$

[4] (b) $P = (1, 1, 1)$, $Q = (3, -2, 3)$, $R = (3, 4, 6)$.

Solution: In this case, we find $\overrightarrow{PQ} =< 2, -3, 2 >$, and $\overrightarrow{QR} =< 0, 6, 3 >$, and these two vectors are clearly not parallel. Thus the points do not lie on the same line, and therefore define a plane.

We can give the equation of the plane using a point on the plane (P will do) and a normal vector. The obvious candidate for normal vector is

$$\begin{aligned}\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -3 & 2 \\ 0 & 6 & 3 \end{vmatrix} \\ &= \hat{\imath}(-3 \cdot 3 - 2 \cdot 6) - \hat{\jmath}(2 \cdot 3 - 2 \cdot 0) + \hat{k}(2 \cdot 6 - (-3) \cdot 0) \\ &= -21\hat{\imath} - 6\hat{\jmath} + 12\hat{k}.\end{aligned}$$

Thus, the equation of the plane is

$$-21(x - 1) - 6(y - 1) + 12(z - 1) = 0.$$

[8] 4. (a) Find all first-order partial derivatives of the following functions:

(i) $f(x, y) = e^2 e^{xy}$.

Solution: We have

$$f_x(x, y) = \frac{\partial}{\partial x}(e^2 e^{xy}) = ye^2 e^{xy},$$

and

$$f_y(x, y) = \frac{\partial}{\partial y}(e^2 e^{xy}) = xe^2 e^{xy}.$$

(ii) $h(x, y, z) = x^2 y^3 z^4$.

Solution: We have

$$h_x(x, y, z) = 2xy^3 z^4,$$

$$h_y(x, y, z) = 3x^2 y^2 z^4,$$

and

$$h_z(x, y, z) = 4x^2 y^3 z^3.$$

(iii) $k(x, y, z) = z \sin(x - y)$. **Solution:** We have

$$k_x(x, y, z) = z \cos(x - y),$$

$$k_y(x, y, z) = -z \cos(x - y),$$

and

$$k_z(x, y, z) = \sin(x - y).$$

[4] (b) Verify that $f_{xy} = f_{yx}$ for $f(x, y) = xye^{-xy}$.

Solution: The first partial derivatives are

$$f_x(x, y) = ye^{-xy} - xy^2e^{-xy},$$

and

$$f_y(x, y) = xe^{-xy} - x^2ye^{-xy}.$$

The mixed partials are therefore

$$\begin{aligned} f_{xy}(x, y) &= \frac{\partial}{\partial y} f_x(x, y) \\ &= e^{-xy} - xye^{-xy} - 2xye^{-xy} + x^2y^2e^{-xy}, \end{aligned}$$

while

$$\begin{aligned} f_{yx}(x, y) &= \frac{\partial}{\partial x} f_y(x, y) \\ &= e^{-xy} - xye^{-xy} - 2xye^{-xy} + x^2y^2e^{-xy} \\ &= f_{xy}(x, y). \end{aligned}$$

[3] (c) Can there exist a continuous function $f(x, y)$ such that $f_x(x, y) = \cos^2(xy)$ and $f_y(x, y) = \sin^2(xy)$? Why?

Solution: We see that both first-order partial derivatives of f are continuous, and all of the second-order partial derivatives would be as well. However, we can check that we would have $f_{xy}(x, y) = -2y\cos(xy)\sin(xy)$, while $f_{yx}(x, y) = 2x\sin(xy)\cos(xy)$.

Therefore, no such continuous function f can exist, for if f were a continuous function with continuous first and second order partial derivatives, then its mixed partial derivatives f_{xy} and f_{yx} would have to be the same.

[6] 5. Find and classify the critical points of the function

$$f(x,y) = 2x^3 + y^3 - 3x^2 - 12x - 3y$$

Solution: The first partial derivatives of $f(x,y)$ are

$$f_x(x,y) = 6x^2 - 6x - 12 \quad \text{and} \quad f_y(x,y) = 3y^2 - 3,$$

and since these are continuous everywhere, the critical points of f will be those points where both derivatives are zero:

If $f_x(x,y) = 6(x^2 - x - 2) = 6(x-2)(x+1) = 0$, then $x = 2$, or $x = -1$, and if $f_y(x,y) = 3(y^2 - 1) = 3(y+1)(y-1) = 0$, then $y = 1$ or $y = -1$, so the critical points of $f(x,y)$ are $(2,1)$, $(2,-1)$, $(-1,1)$ and $(-1,-1)$.

The second order partial derivatives of $f(x,y)$ are

$$f_{xx}(x,y) = 12x - 6, \quad f_{xy}(x,y) = f_{yx}(x,y) = 0 \quad \text{and} \quad f_{yy}(x,y) = 6y.$$

At the point $(2,1)$ we have $A = f_{xx}(2,1) = 18$, $B = f_{xy}(2,1) = 0$, $C = f_{yy}(x,y)(2,1) = 6$, and $\Delta = AC - B^2 = 108$

Since $A > 0$ and $\Delta > 0$, $f(x,y)$ has a local minimum at $(2,1)$.

Similarly, at the point $(2,-1)$ we have $A = 18$, $B = 0$, $C = -6$, and $\Delta = -108$. Since $\Delta < 0$ the critical point is neither a local maximum nor a local minimum.

At $(-1,1)$ we have $A = -18$, $B = 0$, $C = 6$ and $\Delta = -108$. Since $\Delta < 0$, the critical point is neither a local maximum nor a local minimum.

Finally, at $(-1,-1)$ we have $A = -18$, $B = 0$, $C = -6$ and $\Delta = 108$. Since $A < 0$ and $\Delta > 0$, $f(x,y)$ has a local maximum at the point $(-1,-1)$.

6. Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}}$$

as (x, y) approaches the origin along:

[1] (a) the x-axis.

Solution: Along the x-axis we have $y = 0$, so the limit becomes

$$\lim_{x\to 0} \frac{x\cdot 0}{(0^2+y^2)^{3/2}} = 0.$$

[2] (b) the line $y = mx$.

Solution: Substituting $y = mx$, and noting that if $x \to 0$, so does y, the limit becomes

$$\lim_{x\to 0} \frac{x(mx)^2}{(x^2+(mx)^2)^{3/2}} = \lim_{x\to 0} \frac{m^2x^3}{((1+m^2)x^2)^{3/2}} = \frac{m^2}{(1+m^2)^{3/2}}.$$

[3] (c) the path $\vec{r}(t) = \frac{1}{t}\hat{\imath} + \frac{\sin t}{t}\hat{\jmath}$, $t > 0$.

Solution: We need the path defined by $\vec{r}(t)$ to approach the origin $(0, 0)$; that is, for $x(t) \to 0$ and $y(t) \to 0$. For the given functions, this happens as $t \to \infty$. Thus, along the given path our limit becomes

$$\lim_{t\to\infty} \frac{(\frac{1}{t})(\frac{\sin t}{t})^2}{\left((\frac{1}{t})^2+(\frac{\sin t}{t})^2\right)^{3/2}} = \lim_{t\to\infty} \frac{\sin^2 t}{(1+\sin^2 t)^{3/2}},$$

and this limit does not exist, since there arbitrarily large values of t at which the expression is equal to either 0 or $\frac{1}{2\sqrt{2}}$.

Extra space for rough work. Do **not** tear out this page.