Math 4310 Assignment #11 Solutions
University of Lethbridge, Fall 2014

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1. (a) Let $\gamma : [0,1] \to X$ be a path, and let $\rho : [0,1] \to [0,1]$ be any continuous function such that $\rho(0) = 0$ and $\rho(1) = 1$. Prove that the paths γ and $\gamma \circ \rho$ are homotopic. Hint: ρ is itself a path from 0 to 1 in $[0,1]$, and all such paths are homotopic in $[0,1]$.

   Let $I : [0,1] \to [0,1]$ denote the identity map $I(s) = s$. Given any $\rho : [0,1] \to [0,1]$ preserving the endpoints, the map

   $$F(s,t) = ts + (1-t)\rho(s)$$

   is a homotopy relative to $\{0,1\}$ between the maps I and ρ, so I and ρ are homotopic as paths in $[0,1]$. Given any path $\gamma : 0,1 \to X$, the maps $\gamma \circ \rho$ and $\gamma \circ I = \gamma$ are also paths, and since $\rho \underset{F}{\simeq} I$, we have $\gamma \circ \rho \underset{G}{\simeq} \gamma$, where $G : [0,1] \times [0,1] \to X$ is the homotopy

   $$G(s,t) = \gamma \circ F(s,t) = \gamma(ts + (1-t)\rho(s)).$$

   (b) Let α, β, and γ be loops based at a point $x_0 \in X$. Write down explicit formulas for $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$.

   Recalling that for two loops δ and ϵ (not intending to cause analysis flashbacks, but the first three Greek letters were already taken), we define

   $$\delta * \epsilon(s) = \begin{cases} \delta(2s), & 0 \leq s \leq 1/2 \\ \epsilon(2s-1), & 1/2 \leq s \leq 1 \end{cases},$$

we have

$$\alpha * (\beta * \gamma)(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ (\beta * \gamma)(2s-1), & 1/2 \le s \le 1 \end{cases}$$

$$= \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2(2s-1)), & 0 \le 2s-1 \le 1/2 \\ \gamma(2(2s-1)-1), & 1/2 \le 2s-1 \le 1 \end{cases}$$

$$= \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(4s-2), & 1/2 \le s \le 3/4 \\ \gamma(4s-3), & 3/4 \le s \le 1 \end{cases}$$

Similarly, we find that

$$(\alpha * \beta) * \gamma = \begin{cases} \alpha(4s), & 0 \le s \le 1/4 \\ \beta(4s-1), & 1/4 \le s \le 1/2 \\ \gamma(2s-1), & 1/2 \le s \le 1 \end{cases}.$$

(c) Prove that $[\alpha] * ([\beta] * [\gamma]) = ([\alpha] * [\beta]) * [\gamma]$.

Hint: use (a), and try the map $\rho(s) = \begin{cases} s/2 & \text{if } 0 \le s \le 1/2 \\ s - 1/4 & \text{if } 1/2 \le s \le 3/4 \\ 2s-1 & \text{if } 3/4 \le s \le 1 \end{cases}.$

With $\rho(s)$ as given, we note that

$$\text{If } 0 \le s \le 1/2, \text{ then } 0 \le \rho(s) = s/2 \le 1/4,$$
$$\text{if } 1/2 \le s \le 3/4, \text{ then } 1/4 \le \rho(s) = s - 1/4 \le 1/2,$$
$$\text{if } 3/4 \le s \le 1, \text{ then } 1/2 \le \rho(s) = 2s-1 \le 1.$$

Thus, $\rho : [0,1] \to [0,1]$ is a path from 0 to 1 in $[0,1]$ that maps $[0,1/2]$ to $[0,1/4]$, $[1/2,3/4]$ to $[1/4,1/2]$, and $[3/4,1]$ to $[1/2,1]$. Now we note that for any $s \in [0,1]$,

$$(\alpha * \beta) * \gamma(\rho(s)) = \begin{cases} \alpha(4\rho(s)), & 0 \le \rho(s) \le 1/4 \\ \beta(4\rho(s)-1), & 1/4 \le \rho(s) \le 1/2 \\ \gamma(2\rho(s)-1), & 1/2 \le \rho(s) \le 1 \end{cases}$$

$$= \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(4s-2), & 1/2 \le s \le 3/4 \\ \gamma(4s-3), & 3/4 \le s \le 1 \end{cases}$$

$$= \alpha * (\beta * \gamma)(s).$$

Since $(\alpha*\beta)*\gamma(\rho(s)) = \alpha*(\beta*\gamma)(s)$ for all $s \in [0,1]$, it follows that $[(\alpha * \beta) * \gamma] = [\alpha * (\beta * \gamma)]$ by part (a).

2. Let X be a space and let $\alpha, \beta : [0,1] \to X$ be two paths from x_0 to x_1, for two points $x_0, x_1 \in X$. These paths define isomorphisms $\varphi_\alpha, \varphi_\beta : \pi_1(X, x_0) \to \pi_1(X, x_1)$, but as noted in class, they may be different isomorphisms. Prove that the isomorphism φ_β is the composition of φ_α with the inner automorphism of $\pi_1(X, x_1)$ induced by the element $[\beta^{-1} * \alpha]$.

   We recall that for any path δ from x_0 to x_1, the isomorphism φ_δ is given by

$$\varphi_\delta([\gamma]) = \left[\delta^{-1} * \gamma * \delta\right].$$

   Note that the product on the right is given by concatenation of paths within the larger path groupoid $G \rightrightarrows X$ and not by the group multiplication in $\pi_1(X, x_1)$, since δ is a path from x_0 to x_1 and not a loop. Given two paths $\alpha, \beta : [0,1] \to X$ from x_0 to x_1, we see that $\beta^{-1} * \alpha$ is a loop based at x_1, since β^{-1} takes us from x_1 to x_0, and α takes us back to x_1. Note that the inverse of $[\beta^{-1} * \alpha]$ is given by $[\alpha^{-1} * \beta]$. Thus, given a loop γ based at x_0, we have

$$\varphi_\alpha([\gamma]) = \left[\alpha^{-1} * \gamma * \alpha\right],$$

   and

$$\begin{aligned}\left[\beta^{-1} * \alpha\right] * \varphi_\alpha([\gamma]) * \left[\beta^{-1} * \alpha\right]^{-1} &= \left[\beta^{-1} * (\alpha * \alpha^{-1}) * \gamma * (\alpha * \alpha^{-1}) * \beta\right] \\ &= \varphi_\beta([\alpha * \alpha^{-1}] * [\gamma] * [\alpha * \alpha^{-1}]) \\ &= \varphi_\beta([\gamma]).\end{aligned}$$

3. Prove that the two isomorphisms in the previous problem are the same if and only if $\pi_1(X, x_0)$ is Abelian.

   If $\pi_1(X, x_0)$ is Abelian, then so is $\pi_1(X, x_1)$, since the two groups are isomorphic. With $g = [\beta^{-1} * \alpha]$, we have, for any $[\gamma] \in \pi_1(X, x_0)$, that

$$\varphi_\beta([\gamma]) = g\varphi_\alpha([\gamma])g^{-1} = gg^{-1}\varphi_\alpha([\gamma]) = \varphi_\alpha([\gamma]).$$

   Conversely, suppose that all such isomorphisms $\varphi_\alpha, \varphi_\beta$ are equal, and let $g_1 = [\gamma_1], g_2 = [\gamma_2] \in \pi_1(X, x_0)$. Choose any path α from x_0 to x_1, and note that $\gamma_1 * \alpha$ is also a path from x_0 to x_1. (This is the path that follows γ_1 from x_0 back to x_0 and then α from x_0 to x_1. By assumption, $\varphi_\alpha = \varphi_{\gamma_1 * \alpha}$, which gives

$$\left[\alpha^{-1}\gamma_2\alpha\right] = \varphi_\alpha(g_2) = \varphi_{\gamma_1 * \alpha}(g_2) = \left[\alpha^{-1} * \gamma_1^{-1} * \gamma_2 * \gamma_1 * \alpha\right] = \varphi_\alpha(\left[\gamma_2^{-1} * \gamma_1 * \gamma_2\right]).$$

   Since φ_α is an isomorphism, it is a bijection, so

$$[\gamma_2] = [\gamma_1]^{-1} * [\gamma_2] * [\gamma_1],$$

   from which it follows that $\pi_1(X, x_0)$ is Abelian.

4. Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps $f : X \to Y$.

   (a) Let $I = [0, 1]$. Show that for any space X, $[X, I]$ contains a single element.

   Let X be a space and let $f, g : X \to I$ be continuous. Since I is convex, we have the homotopy
$$F(x, t) = tg(x) + (1 - t)f(x)$$
   between f and g. Since f and g were arbitary, $[X, I] = \{[f]\}$ for any $f : X \to I$.

   (b) Show that if Y is path connected, then the set $[I, Y]$ contains a single element.

   Let Y be a space and let $f, g : I \to Y$ be continuous maps. Suppose $f(0) = x_0$ and $g(0) = x_1$. Since Y is path connected, there exists a path $\gamma : [0, 1] \to Y$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Consider the map $F : [0, 1] \times [0, 1] \to Y$ given by
$$F(s, t) = \begin{cases} f((1 - 3t)s), & \text{if } 0 \leq t \leq 1/3 \\ \gamma(3t - 1), & \text{if } 1/3 \leq t \leq 2/3 \\ g((3t - 2)s), & \text{if } 2/3 \leq t \leq 1 \end{cases}.$$
   Then $F(s, 0) = f(s)$, $F(s, 1) = g(s)$, and F is continuous by the gluing lemma, since $F(s, 1/3) = f(0) = x_0 = \gamma(0)$ for all s, and $F(s, 2/3) = \gamma(1) = x_1 = g(0)$ for all s.

   Or, to put it another way, f and g are both homotopic to constant maps, and any two constant maps are homotopic, since Y is path-connected. Since homotopy of maps is an equivalence relation, f must be homotopic to g.

5. (**Do not submit**) A space X is called **contractible** if the identity map $i_X : X \to X$ is homotopic to a constant map. (If f is homotopic to a constant map, we say f is **nullhomotopic**.)

   (a) Show that I and $\mathbb{R}$ are contractible.

   With either $X = I$ or $X = \mathbb{R}$, define $F : X \times I \to X$ by $F(x, t) = (1 - t)x$. Then F is clearly continuous, $F(x, 0) = x$ is the identity map, and $F(x, 1) = 0$ is a constant map.

   (b) Show that a contractible space is path-connected.

   Suppose that X is contractible, and let $x_1, x_2 \in X$. If $I_X : X \to X$ denotes the identity map, let $F(x, t)$ be a homotopy between I_X and a constant map $g(x) = x_0$, for some x_0 in x, so $F(x, 0) = x$ for all $x \in X$, and $F(x, 1) = x_0$ for all x in X. Now define a path $\gamma : [0, 1] \to X$ by
$$\gamma(t) = \begin{cases} F(x_1, 2t), & 0 \leq t \leq 1/2 \\ F(x_2, 2 - 2t), & 1/2 \leq t \leq 1 \end{cases}.$$

Then γ is continuous by the gluing lemma, since $F(x_1, 2(1/2)) = F(x_1, 1) = x_0$ and $F(x_2, 2 - 2(1/2)) = F(x_2, 1) = x_0$, and $\gamma(0) = F(x_1, 0) = x_1$, and $\gamma(1) = F(x_2, 0) = x_2$. Thus, γ is a path from x_0 to x_1.

(c) Show that if Y is contractible, then for any set X, the set $[X, Y]$ has a single element.

Let $f, g : X \to Y$ be any two maps. Since Y is contractible, the identity map $I_Y : Y \to Y$ is nullhomotopic. We now basically repeat the argument from the previous problem: either argue that $f = I_Y \circ f$ must be homotopic to a constant map since I_Y is, and that the same is true of g or let $F : Y \times [0, 1] \to Y$ be the homotopy from I_Y to a constant map, and consider the map $G : X \times [0, 1] \to Y$ given by

$$G(x, t) = \begin{cases} F(f(x), 2t), & 0 \leq t \leq 1/2 \\ F(g(x), 2 - 2t), & 1/2 \leq t \leq 1 \end{cases}.$$

(d) Show that if X is contractible and Y is path-connected, then the set $[X, Y]$ has a single element.

The argument is the same as the one given in 4(b): Since I_X is homotopic to a constant map $c(x) = x_0$, $f = f \circ I_X$ is homotopic to the constant map $(f \circ c)(x) = f(x_0)$, and similarly g is homotopic to the constant map with value $g(x_0)$. Since Y is path-connected, a path in Y from $f(x_0)$ to $g(x_0)$ gives a homotopy between the constant maps with values $f(x_0)$ and $g(x_0)$, respectively.

6. Let $A \subseteq X$. Recall that a **retraction** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is a surjection.

Suppose $r : X \to A$ is a retraction map, and let $i : A \to X$ denote inclusion. Then $r \circ i : A \to A$ is the identity map on A, and thus the composition

$$\pi_1(A, a_0) \xrightarrow{i_*} \pi_1(X, a_0) \xrightarrow{r_*} \pi_1(A, a_0)$$

is equal to the identity map $I : \pi_1(A, a_0) \to \pi_1(A, a_0)$, since $r_* \circ i_* = (r \circ i)_* = (I_A)_*$. Since the identity map is a surjection, it follows that r_* is a surjection.

(For any functions $f : A \to B$ and $g : B \to C$ between arbitrary sets, if $g \circ f : A \to C$ is a surjection, then so is g, since if $c \in C$, there exists some $a \in A$ such that $(g \circ f)(a) = c$, but $(g \circ f)(a) = g(f(a))$, so setting $b = f(a)$ gives an element of B such that $g(b) = c$. Note that a similar argument guarantees that i_* is an injection.)