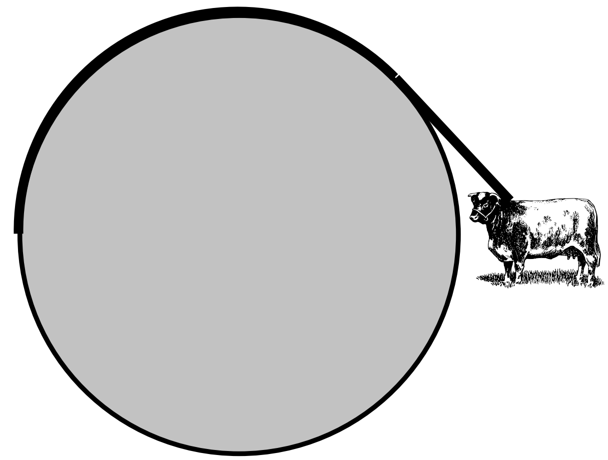
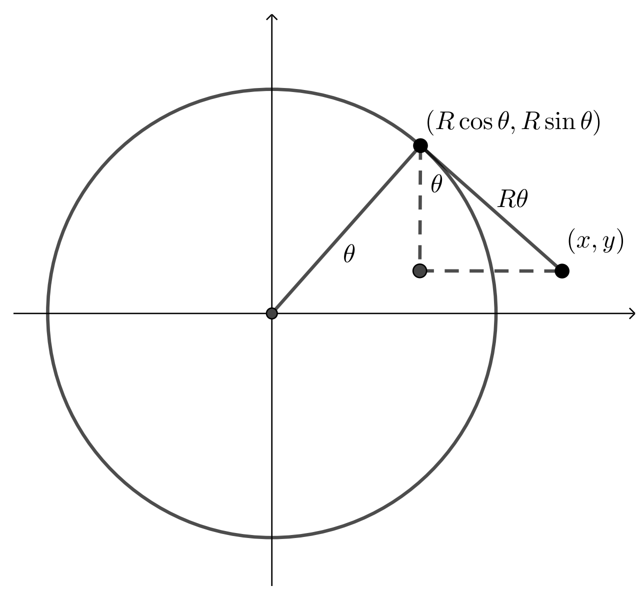
MATH 2565 - Tutorial #11 Solutions

Extra fun: A cow is tied to a silo or radius R by a rope just long enough to reach the opposite side of the silo. Find the grazing area available to the cow.



Solution: First, we need to determine the parametric equations for the path walked by the cow if she keeps the rope tight. (The resulting curve is part of the *involute* of the circle.)

Referring to the diagram on the right, note that the angle at the top of the right-angled triangle through the points $(R\cos\theta, R\sin\theta)$ and the point (x, y) we're looking for is also θ. (This can be worked out using the fact that the angles of a triangle sum to π.)



The amount of rope unwound through an angle of θ is $R\theta$, so this is the length of the hypotenuse of our triangle.

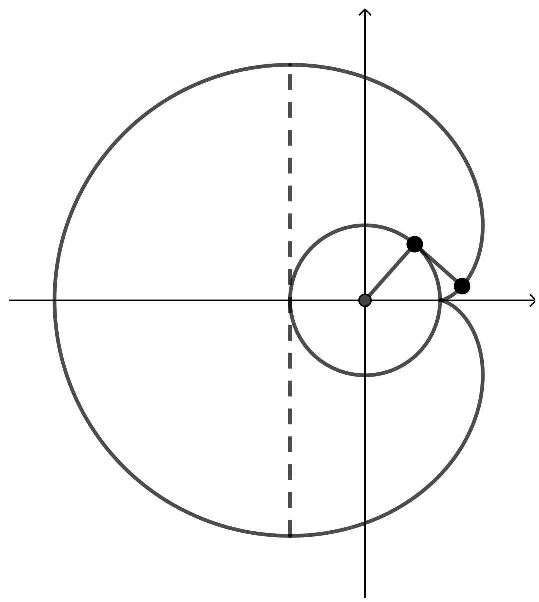
The opposite side thus has length $(R\theta)\sin\theta$, and adding this to the x value of the point on the circle, we find that the x coordinate of the point we're looking for is therefore:

$$x = R\cos\theta + R\theta\sin\theta = R(\cos\theta + \theta\sin\theta).$$

Similarly, the y coordinate is obtained by subtracting the length of the adjacent side from the y value of the point on the circle, so

$$y = R\sin\theta - R\theta\cos\theta = R(\sin\theta - \theta\cos\theta).$$

Now, to compute the area, we need to realize that there are three parts to the boundary. First, the involute described above, for $0 \leq \theta \leq \pi$. Once we reach $\theta = \pi$, we have used all the rope. If the cow continues towards the left, it traces out the left half of a circle of radius πR, centred at the point $(-R, 0)$. Finally, once the cow reaches the bottom of the circle and continues below the silo, the rope begins to wrap around the bottom, and we trace out the same involute, for $-\pi \leq \theta \leq 0$.



By symmetry, it's enough to compute the top half of the area, and then double it. The total area is given by

$$A = 2(A_1 - A_2 + A_3),$$

where A_1 is the area under the involute, for $0 \leq \theta \leq \pi$, A_2 is the area of half the silo, and A_3 is the area of a quarter circle of radius πR.

We immediately get:

$$A_2 = \frac{1}{2}\pi R^2 \quad \text{and} \quad A_3 = \frac{1}{4}\pi(\pi R)^2 = \frac{1}{4}\pi^3 R^2,$$

while

$$\begin{aligned} A_1 &= -\int_0^\pi y(\theta)x'(\theta)\,d\theta \\ &= -\int_0^\pi R(\sin\theta - \theta\cos\theta) \cdot R\theta\cos\theta\,d\theta \\ &= -R^2\int_0^\pi (\theta\sin\theta\cos\theta - \theta^2\cos^2\theta)\,d\theta \\ &= \frac{\pi R^2}{2} + \frac{\pi^3 R^2}{6}. \end{aligned}$$

The total area is therefore $A = \dfrac{\pi^3 R^2}{3} + \dfrac{\pi^3 R^2}{2} = \dfrac{5\pi^3 R^2}{6}$.

1. Eliminate the parameter to obtain an equation for the curve involving only x and y:

   (a) $x = \sec t$, $y = \tan t$
   Since $\tan^2(t) + 1 = \sec^2(t)$, we get $y^2 + 1 = x^2$, or $x^2 - y^2 = 1$. (The unit hyperbola.)

   (b) $x = 4 \sin t + 1$, $y = 3 \cos t - 2$ (Hint: first solve for $\cos t$ and $\sin t$.)
   Since $\cos^2 t + \sin^2 t = 1$, we have $\left(\frac{x-1}{4}\right)^2 + \left(\frac{y+2}{3}\right)^2 = 1$ (an ellipse).

   (c) $x = \frac{1}{t+1}$, $y = \frac{3t+5}{t+1}$. (Hint: try doing long division on the expression for y.)
   We have $y = \frac{3t+5}{t+1} = 3 + \frac{2}{t+1} = 3 + 2x$, so $y = 3 + 2x$.

2. Find any points of self-intersection for the following curves:

   (a) $x = t^3 - t - 3, y = t^2 - 3$
   A point of of self-intersection occurs whenever we have $(x(t_1), y(t_1)) = (x(t_2), y(t_2))$ for some $t_1 \neq t_2$. Looking at the y coordinate, we have $t_1^2 - 3 = t_2^2 - 3$ if and only if $t_1 = \pm t_2$. Since we want $t_1 \neq t_2$, we must have $t_2 = -t_1$.
   Now, we apply this to the x coordinate. We must have $x(t_1) = x(t_2) = x(-t_1)$. Writing t for t_1, we need to solve $x(-t) = x(t)$. This gives us

$$\begin{aligned} t^3 - t - 3 &= (-t)^3 - (-t) - 3 \\ t^3 - t - 3 &= -t^3 + t - 3 \\ 2t^3 - 2t &= 0 \\ 2t(t-1)(t+1) &= 0, \end{aligned}$$

   so $t = 0$, $t = 1$ and $t = -1$ are possibilities. We find that $(x(0), y(0)) = (-3, -3)$, while

$$(x(1), y(1)) = (-3, -2) = (x(-1), y(-1)),$$

   so $(-3, -2)$ is the point of intersection, for $t = \pm 1$.

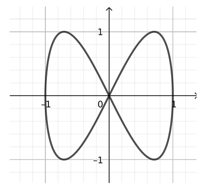
   (b) $x = \cos(t), y = \sin(2t), t \in [0, 2\pi]$
   Since we must have $x(t_1) = x(t_2)$ with $t_1, t_2 \in [0, 2\pi]$, we conclude that $t_2 = 2\pi - t_1$. Equating y coordinates, we get

$$\sin(2t_1) = \sin(2t_2) = \sin(2(2\pi - t_1)) = \sin(4\pi - 2t_1) = \sin(-2t_1) = -\sin(2t_1),$$

   The only way we can have $\sin(2t_1) = -\sin(2t_1)$ is if $\sin(2t_1) = 0$. Thus, we must have $2t_1 = 0, \pi, 2\pi, 3\pi, \ldots$, so for $t \in [0, \pi]$, we have $t = 0, \pi/2, \pi, 3\pi/2$, and 2π as possibilities. We get the following values:

t	0	$\pi/2$	π	$3\pi/2$	2π
$(x(t), y(t))$	$(1, 0)$	$(0, 0)$	$(-1, 0)$	$(0, 0)$	$(1, 0)$

   The point $(1, 0)$ appears twice, but it is not a self-intersection: this simply represents the fact that this is a *closed curve*: it begins and ends at the same point. The only point of self-intersection is in fact $(0, 0)$, which occurs when $t = \pi/2$ and $t = 3\pi/2$. Plotting the curve confirms this:



3. Find the length of the parametric curve:

   (a) $x = -3\sin(2t)$, $y = 3\cos(2t)$, $t \in [0, \pi]$.
   We have

$$L = \int_0^\pi \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^\pi \sqrt{36\cos^2(2t) + 36\sin^2(2t)}\,dt = \int_0^\pi 6\,dt = 6\pi.$$

   (b) $x = e^{t/10}\cos t, y = e^{t/10}\sin t$, $t \in [0, 2\pi]$.
   We first compute

$$\begin{aligned}
x'(t) &= \frac{1}{10}e^{t/10}\cos(t) - e^{t/10}\sin(t) = e^{t/10}(\frac{1}{10}\cos(t) - \sin(t)) \\
y'(t) &= \frac{1}{10}e^{t/10}\sin(t) + e^{t/10}\cos(t) = e^{t/10}(\frac{1}{10}\sin(t) + \cos(t)) \\
x'(t)^2 &= e^{2t/10}(\frac{1}{100}\cos^2(t) - \frac{2}{10}\cos(t)\sin(t) + \sin^2(t)) \\
y'(t)^2 &= e^{2t/10}(\frac{1}{100}\sin^2(t) + \frac{2}{10}\cos(t)\sin(t) + \cos^2(t).
\end{aligned}$$

   Adding $x'(t)^2 + y'(t)^2$, we see that the cross-terms cancel, and since $\sin^2(t) + \cos^2(t) = 1$, we get

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\frac{101}{100}e^{2t/10}} = \frac{\sqrt{101}}{10}e^{t/10}.$$

   Thus,

$$L = \int_0^{2\pi} \sqrt{\frac{101}{100}}e^{t/10}\,dt = \sqrt{101}(e^{\pi/5} - 1).$$

4. Find the area enclosed by the loop of the "teardrop" curve $x = t(t^2 - 1), y = t^2 - 1$. (See Figure 10.34 in the text.)

   We first note that $x = t(t-1)(t+1)$, so $x = 0$ for $t = 0, 1, -1$, while $y = 0$ for $t = 1, -1$. It follows (referring to the figure in the text) that the loop begins at $(0, 0)$ when $t = -1$, and ends at $(0, 0)$ when $t = 1$. We check that $x > 0$ for $-1 < t < 0$ and $x < 0$ for $0 < t < 1$, which tells us that the loop is traversed in the clockwise direction. The area is thus given by

$$\begin{aligned}A &= \int_{-1}^{1} y\,dx = \int_{-1}^{1} (t^2-1)(3t^2-1)\,dt \text{ (Note that } x(t)=t^3-t, \text{ so } x'(t)=3t^2-1.)\\ &= 2\int_0^1 (3t^4-4t^2+1)\,dt\\ &= 2\left(\frac{3}{5}-\frac{4}{3}+1\right) = \frac{8}{15}.\end{aligned}$$

Note: For closed curves, it's always the case that the area is given by $\int_a^b y(t)x'(t)\,dt$ for clockwise orientation, and $-\int_a^b y(t)x'(t)\,dt$ for counterclockwise orientation. Feel free to ask me if you want to know why it's not necessary to split the area up into pieces.

5. For each curve below, find the equation of the tangent line at the given value of t. Also: find all points where the tangent line is horizontal or vertical.

   (a) $x = t^2-1$, $y = t^3-t$, $t=1$.
   The slope of the tangent line is

   $$m = \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2-1}{2t}.$$

   When $t=1$ we get $x=0, y=0, m=1$, so our tangent line is $y=x$.
   The tangent line is horizontal when $y'(t)=0$, so $t=\pm 1/\sqrt{3}$, and vertical when $x'(t)=0$, so $t=0$.

   (Additional care is required if $x'(t)$ and $y'(t)$ are simultaneously zero, but that's not the case here.)

   (b) $x=\cos(t), y=\sin(2t), t=\pi/4$
   We have $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = -\dfrac{2\cos(2t)}{\sin(t)}$. When $t=\pi/4$ we have $x=1/\sqrt{2}$, $y=1$, and $m=0$, so our line is simply $y=1$.
   The above is one horizontal tangent; we see that more generally $y'(t)=0$ when $t = \pi/4+k\pi/2$, where k can be any integer.
   We get a vertical tangent when $x'(t)=\sin(t)=0$, which occurs when $t=k\pi$, for any integer k.