Solutions to Quiz 20 Practice Problems
Math 2580
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1. Use Green's Theorem to evaluate the given line integral. Assume the orientation of the curve is positive, unless otherwise indicated.

   (a) $\int_C x^2y^2\,dx + 4xy^3\,dy$, where C is the triangle with vertices $(0,0)$, $(1,3)$, and $(0,3)$.

   The curve C bounds the triangular region D described by $3x \leq y \leq 1$, for $0 \leq x \leq 1$, so by Green's Theorem we have

$$\int_C x^2y^2\,dx + 4xy^3\,dy = \iint_D \left(\frac{\partial 4xy^3}{\partial x} - \frac{\partial x^2y^2}{\partial y}\right) dA = \int_0^1 \int_{3x}^1 (4y^3 - 2x^2y)\,dy\,dx$$
$$= \int_0^1 (1 - x^2 - 72x^4)\,dx = 1 - \frac{1}{3} - \frac{72}{5}.$$

   (b) $\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$, where C is the boundary of the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

   Here, our region is described best in polar coordinates as $1 \leq r \leq 2$, with $0 \leq \theta \leq 2\pi$. We have

$$\frac{\partial x^4 + 2x^2y^2}{\partial x} - \frac{\partial xe^{-2x}}{\partial y} = 4x^3 + 4xy^2 = 4x(x^2+y^2) = 4r^3\cos\theta$$

   in polar coordinates. Thus, by Green's Theorem we have

$$\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy = \int_0^{2\pi} \int_1^2 4r^4\cos\theta\,dr\,d\theta = 0.$$

   Note: the fact that this integral is zero tells us that the integral around the circle $x^2+y^2=1$ is equal to the integral around the circle $x^2+y^2=4$, since C consists of two circles, with positive orientation for the outer circle, and negative orientation for the inner circle.

(c) $\int_C y^3\,dx - x^3\,dy$, where C is the circle $x^2+y^2=4$.

We have $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = -3x^2-3y^2 = -3r^2$, and C bounds the region given in polar coordinates by $0 \le r \le 2$, with $0 \le \theta \le 2\pi$. Thus, we have

$$\int_C y^3\,dx - x^3\,dy = \int_0^{2\pi}\int_0^2 (-3r^2)r\,dr\,d\theta = -\frac{3\pi}{2}.$$

(d) $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \langle y^2\cos x, x^2+2y\sin x\rangle$ and C is the trianglular path from $(0,0)$ to $(2,6)$ to $(2,0)$, and back to $(0,0)$.

First, we note that the given path is the negatively-oriented boundary of the region given by $0 \le x \le 2$, $0 \le y \le 3x$, so we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = -\int_0^2\int_0^{3x}(2x)\,dy\,dx = -\int_0^2 6x^2\,dx = -16.$$

2. Find a normal vector to the given parameterized surface at the given point:

(a) $x = 2u$, $y = u^2+v$, $z = v^2$, at the point $(0,1,1)$.

With $\mathbf{r}(u,v) = \langle 2u, u^2+v, v^2\rangle$, we have $\mathbf{r}_u(u,v) = \langle 2, 2u, 0\rangle$ and $\mathbf{r}_v(u,v) = \langle 0,1,2v\rangle$. At the point $(0,1,1)$ we have $2u=0$, so $u=0$, and comparing y-coordinates, $0^2+v=1$, which gives $v=1$ (and we can check that $1^1=1$ works for the z-coordinate).
Since $\mathbf{r}_u(0,1) = \langle 2,0,0\rangle$ and $\mathbf{r}_v(u,v) = \langle 0,1,2\rangle$, we have

$$\mathbf{N}(0,1) = \mathbf{r}_u(0,1)\times\mathbf{r}_v(0,1) = \langle 2,0,0\rangle\times\langle 0,1,2\rangle = \langle 0,-4,2\rangle.$$

(b) $x = u^2-v^2$, $y = u+v$, $z = u^2+4v$, at the point $(-\frac{1}{4}, \frac{1}{2}, 2)$.

We have $\mathbf{r}(u,v) = \langle u^2-v^2, u+v, u^2+4v\rangle$, so $\mathbf{r}_u(u,v) = \langle 2u, 1, 2u\rangle$ and $\mathbf{r}_v(u,v) = \langle -2v, 1, 4\rangle$. We now need to determine the values of u and v that correspond to the point $(-1/4, 1/2, 2)$. It's possible to guess the answer, but if you want to proceed systematically, note that if we take the difference of the z and x coordinates, we have

$$(u^2+4v)-(u^2-v^2) = v^2+4v = 2-\left(-\frac{1}{4}\right) = \frac{9}{4},$$

so $v^2+4v-\frac{9}{4} = (v-\frac{1}{2})(v+\frac{9}{2}) = 0$, giving either $v=\frac{1}{2}$ or $v=-\frac{9}{2}$. The first solution works in all three coordinates if we take $u=0$, and you can check that $v=9/2$ does not lead to consistent values for u in all three coordinates. Thus $u=0$ and $v=1/2$, giving us

$$\mathbf{r}_u(0,1/2) = 0,1,0\rangle, \mathbf{r}_v(0,1/2) = \langle -1,1,4\rangle, \mathbf{N}(0,1/2) = \langle 4,0,1\rangle.$$