Name: Solutions

The questions below are worth 5 points each, and the quiz is out of 10. You can either choose two, or solve all 3 for a maximum score of 15/10. Feel free to use the back of the page for extra space.

1. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

   We recall that for any operator $S \in \mathcal{L}(V)$, $\operatorname{null} S^* = (\operatorname{range} S)^\perp$ and $\operatorname{range} S^* = (\operatorname{null} S^*)^\perp$. With these preliminaries out of the way, we have:

$$\begin{aligned} \lambda \text{ is an eigenvalue of } T &\Leftrightarrow T - \lambda I \text{ is not invertible} \\ &\Leftrightarrow (T - \lambda I)^* \text{ is not invertible} \\ &\Leftrightarrow T^* - \overline{\lambda} I \text{ is not invertible} \\ &\Leftrightarrow \overline{\lambda} \text{ is an eigenvalue of } T^*. \end{aligned}$$

2. Suppopse $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

   Since S and T are self-adjoint, we have

$$(ST)^* = T^* S^* = TS,$$

   and from this it is clear that $ST = (ST)^*$ if and only if $ST = TS$.

3. Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

   Since 3 and 4 are eigenvalues of T, we can choose eigenvectors u_1, u_2 such that $Tu_1 = 3u_1$ and $Tu_2 = 4u_2$. By normalizing if necessary we may further assume that $\|u_1\| = \|u_2\| = 1$. Now, we let $v = u_1 + u_2$. Since T is normal, u_1 and u_2 are orthogonal, and therefore, by the Pythagorean Theorem, we have

$$\|v\|^2 = \|u_1 + u_2\|^2 = \|u_1\|^2 + \|u_2\|^2 = 1^2 + 1^2 = 2,$$

   and we can conclude that $\|v\| = \sqrt{2}$. Now, since any scalar multiples of orthogonal vectors are still orthogonal, we also have

$$\|Tv\|^2 = \|T(u_1 + u_2)\|^2 = \|Tu_1 + Tu_2\|^2 = \|3u_1 + 4u_2\|^2 = \|3u_1\|^2 + \|4u_2\|^2 = 3^2 + 4^2 = 5^2,$$

   and thus $\|Tv\| = 5$, as required.

Total: 10 points