University of California, Berkeley
Department of Mathematics
12^{th} April, 2013, 12:10-12:55 pm
MATH 53 - Test #3

Last Name: Solutions

First Name: The

Student Number: ______

What is your discussion section number (201-215)? ______ [1]

What is the name of your GSI? ______ [1]

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

There is a list of potentially useful formulas available on the last page of the exam.

For grader's use only:

Page	Grade
1	/2
2	/14
3	/12
4	/12
Total	/40

B

1. Evaluate the integral $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} \sqrt{4-x^2}\,dx\,dy.$ [6]

   (You can do it without reversing the order of integration, but it's not recommended.)

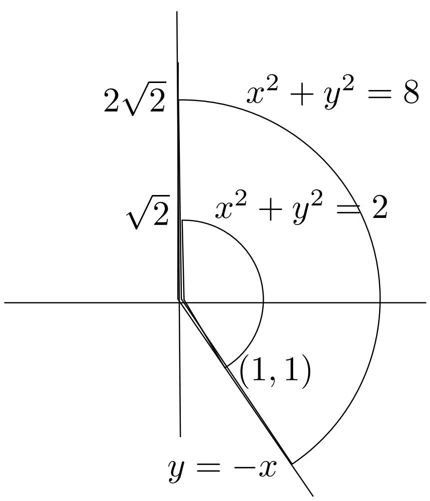
   The region of integration is the right half of the disk $x^2+y^2\leq 4$. As a Type I region it's given by $0\leq x\leq 2$ with $-\sqrt{4-x^2}\leq y\leq\sqrt{4-x^2}$, so

$$\begin{aligned}\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} \sqrt{4-x^2}\,dx\,dy &= \int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx\\ &= \int_0^2 2(4-x^2)\,dx\\ &= 2(8-8/3) = 32/3.\end{aligned}$$

2. The integral below computes the area of a region. Sketch the area, and compute it by converting to polar coordinates. [6]

$$\int_{-2}^{-1}\int_{-y}^{\sqrt{8-y^2}} dx\,dy + \int_{-1}^{\sqrt{2}}\int_{\sqrt{2-y^2}}^{\sqrt{8-y^2}} dx\,dy + \int_{\sqrt{2}}^{2\sqrt{2}}\int_{0}^{\sqrt{8-y^2}} dx\,dy$$

   The region given by the above integral lies to the right of the y-axis between the circles $x^2+y^2=2$ and $x^2+y^2=8$ and the line $y=-x$.



   In polar coordinates this region is given by $-\pi/4\leq\theta\leq\pi/2$ and $\sqrt{2}\leq r\leq 2\sqrt{2}$, so we have

$$\begin{aligned}A &= \int_{-\pi/4}^{\pi/2}\int_{\sqrt{2}}^{2\sqrt{2}} r\,dr\,d\theta\\ &= \frac{3\pi}{4}\left(\frac{8-2}{2}\right)\\ &= \frac{9\pi}{4}.\end{aligned}$$

3. Evaluate the integral $\int_0^4 \int_{-\sqrt{4x-x^2}}^0 \sqrt{x^2+y^2}\,dy\,dx$ by converting to polar coordinates. [6]

The half-circle $y = -\sqrt{4x-x^2}$, or $x^2+y^2 = 4x$ (which becomes $(x-2)^2+y^2 = 1$ after completing the square) is given in polar coordinates by $r^2 = 4r\cos\theta$, or $r = 4\cos\theta$. Since the region of integration is in the fourth quadrant, we have $-\pi/2 \le \theta \le 0$, so

$$\begin{aligned}
\int_0^4 \int_{-\sqrt{4x-x^2}}^0 \sqrt{x^2+y^2}\,dy\,dx &= \int_{-\pi/2}^0 \int_0^{4\cos\theta} r^2\,dr\,d\theta \\
&= \frac{64}{3}\int_{-\pi/2}^0 \cos^3\theta\,d\theta \\
&= \frac{64}{3}\int_{-\pi/2}^0 (1-\sin^2\theta)\cos\theta\,d\theta \\
&= \frac{64}{3}\int_{-1}^0 (1-u^2)\,du \\
&= \frac{64}{3}\left(1-\frac{1}{3}\right) \\
&= \frac{128}{9}.
\end{aligned}$$

4. Find the centroid (geometric center) of the triangle with vertices $(0,0)$, $(1,-3)$, and $(1,3)$. [8]

The triangle is given as a Type I region by $0 \le x \le 1$ with $-3x \le y \le 3x$. The triangle has base width 6 and height 1, so its area is $A = \frac{1}{2}(6)(1) = 3$. Since the region is symmetric about the x-axis we have

$$\overline{y} = \frac{1}{A}\iint_D y\,dA = 0$$

by symmetry, since $f(x,y) = y$ is an odd function of y. The x-coordinate of the centroid is given by

$$\begin{aligned}
\overline{x} &= \frac{1}{A}\iint_D x\,dA \\
&= \frac{1}{3}\int_0^1 \int_{-3x}^{3x} x\,dx\,dy \\
&= \frac{1}{3}\int_0^1 6x^2\,dy \\
&= \frac{1}{3}(2)(1^3) \\
&= \frac{2}{3}.
\end{aligned}$$

Thus, the centroid of the triangle is at $\left(\frac{2}{3}, 0\right)$.

5. Set up, but do not evaluate, the integral $\iiint_E x^2\cos(yz)\,dV$, where E is the tetrahedron with vertices $(1,0,0)$, $(0,3,0)$, and $(0,0,6)$. [6]

The tetrahedron is bounded by the coordinate planes and the plane passing through the three points other than the origin. This plane intersects the coordinate planes in the lines $3x+y=3$, $z=0$ (or $6x+2y=6$), $6x+z=6$, $y=0$, and $2y+z=6$, $x=0$. Thus, the equation of the plane must be $6x+2y+z=6$, so we can describe the region by $0\le z\le 6-6x-2y$ with (x,y) in the triangle bounded by $x=0$, $y=0$, and $3x+y=3$, which we can write as $0\le y\le 3-3x$, with $0\le x\le 1$. Thus, we have

$$\iiint_E x^2\cos(yz)\,dV = \int_0^1\int_0^{3-3x}\int_0^{6-6x-2y} x^2\cos(yz)\,dz\,dy\,dx.$$

6. Let $E\subseteq\mathbb{R}^3$ be the region bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2=2z$. Express the volume of E as a triple integral in **both** cylindrical and spherical coordinates. You do not have to compute the volume. [6]

The sphere $x^2+y^2+z^2=2z$, or $x^2+y^2+(z-1)^2=1$, intersects the cone $z=\sqrt{x^2+y^2}$ when $z^2+z^2=2z$, which gives $z=0$ or $z=1$. The intersection at $z=0$ is where the base of the cone meets the bottom of the sphere, while the intersection at $z=1$ is the circle $z=1=x^2+y^2$. The region thus consists of the top half of the sphere, lying on top of the portion of the cone between $z=0$ and $z=1$.

In cylindrical coordinates, we have $0\le\theta\le 2\pi$ and $0\le r\le 1$, since the region projects down to the xy-plane onto the disk $x^2+y^2\le 1$. The equation of the cone is simply $z=r$, while the sphere becomes $(z-1)^2=1-r^2$, so $z=1+\sqrt{1-r^2}$ for the top half of the sphere (positive root). Thus, the volume is given in cylindrical coordinates by

$$V=\int_0^{2\pi}\int_0^1\int_r^{1+\sqrt{1-r^2}} r\,dz\,dr\,d\theta.$$

In spherical coordinates the cone is given by $\varphi=\pi/4$, so the interior of the cone corresponds to $0\le\varphi\le\pi/4$, and the sphere is given by $\rho^2=2\rho\cos\varphi$, or $\rho=2\cos\varphi$ (since $\rho=0$ can be obtained by setting $\varphi=\pi/2$). Since the region lies inside the sphere, we have $0\le\rho\le 2\cos\varphi$, and since the region is symmetric about the z-axis we have $0\le\theta\le 2\pi$. Thus, we have

$$V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

 Total Points: 40