Some solutions from Section 3.D.

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Problem 3: Suppose V is finite dimensional, $U \subseteq V$ is a subspace, and $S \in \mathcal{U}, \mathcal{V}$. Show that there exists an invertible operator $T \in \mathcal{L}(V)$ such that $T|_U = S$ if and only if S is injective. (Here $T|_U$ denotes the restriction of T to U. In other words, $Tu = Su$ for all $u \in U$.)

Solution: First, note that if $U = V$, we can take $T = S$ and there is nothing to prove, so we will assume that U is a proper subspace of V. If $T : V \to V$ is invertible, then in particular T is injective. Thus, if $S = T|_U$, then whenever $Su_1 = Su_2$ for some $u_1, u_2 \in U$, we have $Tu_1 = Su_1 = Su_2 = Tu_2$, and since T is injective, $u_1 = u_2$, which shows that S is injective.

Conversely, suppose that $S : U \to V$ is injective, and let $\{u_1, \ldots, u_k\}$ be a basis for U. We can extend this to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_l\}$ of V. We now note that since S is injective, the set $\{Su_1, \ldots, Su_k\}$ is linearly independent, and therefore forms a basis for range S. We extend this to a basis $\{Su_1, \ldots, Su_k, w_1, \ldots, w_l\}$ of V, and define $T : V \to V$ by

$$Tu_1 = Su_1, \ldots, Tu_k = Su_k, Tv_1 = w_1, \ldots, Tv_l = w_l.$$

Then T is invertible, since it takes a basis to a basis, and since T agrees with S on a basis for U, we must have $Tu = Su$ for all $u \in U$.

Problem 7: Suppose V and W are finite-dimensional and let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) \,|\, Tv = 0\}.$$

Part (a) asks us to show that E is a subspace of $\mathcal{L}(V, W)$. Checking this is straightforward using the subspace test: it's clear that the zero transformation $0 : V \to W$ given by $0v = 0$ for all $v \in V$ is an element, and if $T_1 v = T_2 v = 0$, then $(T_1 + T_2)v = T_1 v + T_2 v = 0 + 0 = 0$, so $T_1 + T_2 \in E$, and for any scalar c, if $T \in E$, then $(cT)v = c(Tv) = c0 = 0$, so $cT \in V$.

Part (b) asks us what the dimension of E is, given that $v \neq 0$. We first have to recall that $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$ (see the text – this follows from the fact that the map $T \to \mathcal{M}(T)$ that sends a linear map to its matrix in $\mathbb{F}^{m,n}$ is an isomorphism, and the space of $m \times n$ matrices is mn-dimensional).

We claim that $\dim E = (\dim V - 1)(\dim W) = \dim \mathcal{L}(V, W) - \dim W$. There are two ways to see this. The first way is as follows: since $v \neq 0$, the set $\{v\}$ is a basis for $\operatorname{span}\{v\}$.

Thus, we can extend this to a basis $\{v, v_2, \ldots, v_n\}$ of V. Let $U \subseteq V$ be the subspace $U = \operatorname{span}\{v_2, \ldots, v_n\}$; note that $\dim U = \dim V - 1$. Now, consider the map

$$\varphi : E \to \mathcal{L}(U, W)$$

given by

$$(\varphi T)(c_2v_2 + \cdots c_nv_n) = T(c_2v_2 + \cdots + c_nv_n).$$

We claim this is an isomorphism. First, if $\varphi T = 0$, then for any $w \in V$ we have

$$w = c_1v + c_2v_2 + \cdots + c_nv_n$$

for scalars $c_1, \ldots, c_n$, and thus

$$Tw = c_1Tv + T(c_2v_2 + \cdots + c_nv_n) = 0 + (\varphi T)(c_2v_2 + \cdots + c_nv_n) = 0.$$

Since $w \in V$ was arbitrary, $T = 0$. This shows that $\operatorname{null} \varphi = \{0\}$, so φ is injective. Now, if $S : U \to W$ is any linear map, we can define $T : V \to W$ by

$$T(c_1v + c_2v_2 + \cdots + c_nv_n) = 0 + (\varphi T)(c_2v_2 + \cdots + c_nv_n) = S(c_2v_2 + \cdots + c_nv_n),$$

which shows that φ is surjective, and thus an isomorphism. Since $\dim \mathcal{L}(U, W) = (\dim V - 1)(\dim W)$, the result follows.

Another way to see this is to construct a basis $\{v, v_2, \ldots, v_n\}$ for V as above, and notice that with respect to this basis, the matrix of any $T \in E$ is going to be of the form

$$\mathcal{M}(T) = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

and then note that the dimension of the space of all $m \times n$ matrices with first column equal to zero is $m(n-1) = mn - m$.

Note: when we were playing around with this in the help session we noted that if our vector v was (say) $v = (1, 2)$ for a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, and $Tv = 0$, then we'd have

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which shows that our matrix must be of the form

$$\begin{bmatrix} 2a & -a \\ 2b & -b \\ 2c & -c \end{bmatrix},$$

so there are three parameters a, b, c, giving $\dim E = 3 = 2(3) - 3$ in this case. But you might be wondering where the column of zeros is. It's not there because the above matrix gives the matrix of T with respect to the *standard basis* $\{(1, 0), (0, 1)\}$ of T. If we instead used a basis such as $\{(1,2),(2,1)\}$ that contains the given vector v as the first basis element and computed the matrix of a given $T \in E$, then we'd get our column of zeros.

Problem 8: Suppose V is finite-dimensional and $T : V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

Solution: Recall from problem 3.B #12 (which was on the second assignment) that we can choose a subspace $U \subseteq V$ such that $V = \operatorname{null} T \oplus U$, and that $\operatorname{range} T = \{Tu : u \in U\} = \operatorname{range} T|_U = W$. Choosing such a subspace U, we know that $T|_U$ is still a surjection, and $T|_U$ is also injective, since if $Tu = 0$ for some $u \in U$ then $u \in U \cap \operatorname{null} T = \{0\}$, and thus $u = 0$.

Problem 9: Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if S and T are both invertible.

Solution: If S and T are both invertible, then we know that ST is invertible by problem 1 from 3.D (see also Quiz 5). Conversely, suppose that ST is invertible. Then ST must be a bijection. It follows that T must be an injection and S must be a surjection. (Recall from class on February 27th, or from Math 2000, that for *any* functions f and g, if $f \circ g$ is injective, then g is injective, and if $f \circ g$ is surjective, then g is surjective.)

But since S and T are operators on a finite-dimensional space, we know that being either injective or surjective is equivalent to being bijective, and thus invertible, so both S and T are invertible.

Problem 10: Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

Solution: We will prove that if $ST = I$, then $TS = I$. The converse follows by exchanging the roles of S and T. Assuming that $ST = I$, we note that since I is surjective, so is S, and thus S is bijective, since V is finite-dimensional. Thus, S^{-1} exists, and

$$TS = (S^{-1}S)TS = S^{-1}(ST)S = S^{-1}IS = S^{-1}S = I.$$

Problem 11: Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ such that $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Solution: Suppose that $STU = S(TU) = I$. Since I is a bijection, we can conclude that TU is an injection, but since $TU \in \mathcal{L}(V)$ and V is finite-dimensional, TU is a bijection, and in particular a surjection, which implies that T is surjective and thus invertible. Similar arguments show that S and U must also be invertible. Applying S^{-1} on the left to both sides of $STU = I$, we have $TU = S^{-1}$, and if we apply U^{-1} on the right to both sides of this equation, we get $T = S^{-1}U^{-1}$. Taking the inverse of both sides, we obtain

$$T^{-1} = (S^{-1}U^{-1})^{-1} = (U^{-1})^{-1}(S^{-1})^{-1} = US,$$

as required.