MATH 1410 - Tutorial #11 Solutions

1. Let $A = \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$. Compute $A\vec{u}_i$ for $i = 1, 2, 3, 4$, where:

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

Which of the above were eigenvectors? What are the eigenvalues of A?

We compute each product as follows:

$$\begin{aligned}
A\vec{u}_1 &= \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 21 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ for any scalar } \lambda. \\
A\vec{u}_2 &= \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} 4 \\ -2 \end{bmatrix} \\
A\vec{u}_3 &= \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\
A\vec{u}_4 &= \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & -3 \end{bmatrix} = \begin{bmatrix} 12 \\ -12 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ -3 \end{bmatrix}
\end{aligned}$$

From the above, we can conclude that $\vec{u}_2, \vec{u}_3, \vec{u}_4$ are eigenvectors, and that $\lambda = 3, 4$ are eigenvalues. Since A is 2×2, these must be all the eigenvalues. (Note that $\vec{u}_3$ and $\vec{u}_4$ are parallel vectors.)

2. Verify that the matrix $Z = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$ with corresponding eigenvectors $\vec{x}_1 = \begin{bmatrix} 1+i \\ -2 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$.

We compute $Z\vec{x}_i$ and $\lambda_i \vec{x}_i$ for $i = 1, 2$ to confirm:

$$\begin{aligned}
Z\vec{x}_1 &= \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ -2 \end{bmatrix} = \begin{bmatrix} 3+3i-2 \\ -2-2i-2 \end{bmatrix} = \begin{bmatrix} 1+3i \\ -4-2i \end{bmatrix} \\
\lambda_1 \vec{x}_1 &= (2+i) \begin{bmatrix} 1+i \\ -2 \end{bmatrix} = \begin{bmatrix} 2-1+2i+i \\ -4-2i \end{bmatrix} = \begin{bmatrix} 1+3i \\ -4-2i \end{bmatrix} = Z\vec{x}_1 \\
Z\vec{x}_2 &= \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} 3-1-i \\ -2-1-i \end{bmatrix} = \begin{bmatrix} 2-i \\ -3-i \end{bmatrix} \\
\lambda_2 \vec{x}_2 &= (2-i) \begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} 2-i \\ -2-1-2i+i \end{bmatrix} = \begin{bmatrix} 2-i \\ -3-i \end{bmatrix} = Z\vec{x}_2
\end{aligned}$$

3. The matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ has characteristic polynomial $c_A(\lambda) = -(\lambda-2)^2(\lambda-6)$. Find the eiqenvalues of A, and the corresponding eigenvectors.

Since the eigenvalues of A are the roots of the characteristic polynomial, we see that the eigenvalues are $\lambda_1 = 2$ (with multiplicity 2) and $\lambda_2 = 6$.

For $\lambda_1 = 2$, we get

$$A - 2I = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3-R_1\to R_3]{R_2-R_1\to R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, for $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to be a solution to $(A-2I)\vec{v} = \vec{0}$, we must have $x = -2y - z$, where y and z are free variables. Thus,

$$\vec{v} = \begin{bmatrix} -2y - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

giving us two independent eigenvectors: $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 6$, we get

$$A - 6I = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 1 \\ -3 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow[R_3-R_1\to R_3]{R_2+3R_1\to R_3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{bmatrix}$$
$$\xrightarrow{R_3+R_2\to R_3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2\to R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+2R_2\to R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution to $(A-6I)\vec{w} = \vec{0}$, we must therefore have $x = z$ and $y = z$, so $\vec{w} = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This gives us the eigenvector $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_2 = 6$.

4. Compute the eigenvalues of the matrix $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 4 & 3 \end{bmatrix}$

   The characteristic polynomial is

$$\begin{aligned} c_A(\lambda) &= \begin{vmatrix} 3-\lambda & -1 & 2 \\ 0 & 3-\lambda & 1 \\ 0 & 4 & 3-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda)((3-\lambda)^2 - 4) = (3-\lambda)((3-\lambda)+2)((3-\lambda)-2) \\ &= (3-\lambda)(5-\lambda)(1-\lambda). \end{aligned}$$

   The eigenvalues of A are therefore $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 5$.

5. Compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

   The characteristic polynomial is

$$c_A(\lambda) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1),$$

   so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 5$.
   For $\lambda_1 = -1$, we get

$$A - (-1)I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

   For $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ to solve $(A+I)\vec{v} = \vec{0}$ we must have $x = -2y$, so $\vec{v} = \begin{bmatrix} -2y \\ y \end{bmatrix} = -y \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, giving us the eigenvector $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ corresponding to $\lambda_1 = -1$.
   For $\lambda_2 = 5$, we get

$$A - 5I = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

   For $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ to solve $(A-5I)\vec{v} = \vec{0}$ we must have $x = y$, so $\vec{v} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, giving us the eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_2 = 5$.