Name: Solutions

1. Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

[5]

Hint: Compute $(T-2I)(T-3I)(T-4I)v$, where v is an eigenvector with eigenvalue λ.

Suppose that λ is an eigenvalue of T. Thus, $Tv = \lambda v$ for some $v \neq 0$. Then we have that

$$(T-4I)v = Tv - 4v = \lambda v - 4v = (\lambda - 4)v.$$

Thus,

$$(T-3I)(T-4I)v = (T-3I)[(\lambda-4)v] = (\lambda-4)(T-3I)v = (\lambda-4)(\lambda-3)v,$$

and repeating this process one more time, we get

$$0 = (T-2I)(T-3I)(T-4I)v = (\lambda-4)(\lambda-3)(\lambda-2)v,$$

and since $v \neq 0$, we must have $(\lambda-4)(\lambda-3)(\lambda-2) = 0$. Since this is a product of scalars equal to zero, one of the terms in the product must be zero, and it follows that λ must equal one of 2, 3, or 4.

2. Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $E(\lambda, T) = E(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

[5]

Reminder: the eigenspace $E(\lambda, T)$ is defined to be $\operatorname{null}(T - \lambda I)$.

Suppose that $v \in E(\lambda, T)$ for some $\lambda \neq 0$. Then $(T-\lambda I)v = 0$, so we must have $Tv = \lambda v$. Since T is invertible and $\lambda \neq 0$, we have that

$$Tv = \lambda v \Rightarrow v = T^{-1}(\lambda v) = \lambda T^{-1}v \Rightarrow T^{-1}v = \frac{1}{\lambda}v.$$

Thus, $\left(T^{-1} - \dfrac{1}{\lambda}I\right)v = 0$, which shows that $v \in E(\frac{1}{\lambda}, T^{-1})$, and therefore $E(\lambda, T) \subseteq E(\frac{1}{\lambda}, T^{-1})$.

The proof that $E(\frac{1}{\lambda}, T^{-1}) \subseteq E(\lambda, T)$ is identical, and obtained by reversing the steps above, so the result follows.

Total: 10 points