1. (a) Suppose $k \leq l$ and let $\mathbb{N}_k = \{1, 2, \ldots, k\}$ and let $\mathbb{N}_l = \{1, 2, \ldots, l\}$. Show that the
[2] function $f : \mathbb{N}_k \to \mathbb{N}_l$ given by $f(n) = n$ for $n = 1, 2, \ldots, k$ is one-to-one.

For any $n, m \in \mathbb{N}_k$, if $f(n) = f(m)$ then $n = m$, since $n = f(n) = f(m) = m$. Thus, f is one-to-one.

[1] (b) Is the function f in part (a) necessarily onto? Explain.

If $l > k$, then f will not be onto, since the numbers $k + 1, \ldots, l$ will not be in the range of f.

(c) Suppose $k \leq l$, and suppose A and B are sets, with $|A| = k$ and $|B| = l$. Let $g : A \to \mathbb{N}_k$ and $h : B \to \mathbb{N}_l$ be bijections. (The bijections g and h exist, by the definition of the cardinality of finite sets.) Use part (a) and the bijections g and h to
[2] construct a one-to-one function from A to B.

Given the bijections $g : A \to \mathbb{N}_k$ and $h : B \to \mathbb{N}_l$, note that $h^{-1} : \mathbb{N}_l \to B$ is also a bijection, and let $f : \mathbb{N}_k \to \mathbb{N}_l$ be the function from part (a). Since f, g and h^{-1} are all one-to-one, the composition $k = h^{-1} \circ f \circ g : A \to B$ is one-to-one. That is, $k : A \to B$ is given by the diagram below:

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "g"] & B \arrow[d, "h"] \\ \mathbb{N}_k \arrow[r, "f"] & \mathbb{N}_l \end{tikzcd}$$

2. Suppose every student at a university has three initials, say F.M.L. for First, Middle, Last. How many students must the university have to guarantee that two students have the same
[5] initials?

Hint: Let A be the set of university students, and let B be the set of letters in the alphabet, so $|B| = 26$. The set of all possible initials can be identified with $B \times B \times B$, since an ordered triple of letters (b_1, b_2, b_3) corresponds to a set of initials. If $f : A \to B \times B \times B$ is the function that assigns each student to their initials, how big must the cardinality of A be to guarantee that f cannot be one-to-one?

Since $|B| = 26$, $|B \times B \times B| = 26^3$, by the multiplication principle. By the pigeonhole principle, if $|A| > |B \times B \times B|$, then there are no one-to-one functions $A \to B \times B \times B$, which in this case implies that there is no way to assign each student to a unique set of initials. Thus, if $|A| \geq 26^3 + 1$, then at least two students have the same set of initials.

Note: For the record, I would never expect anyone to actually multiply out 26^3. It's perfectly fine to leave this as is.

Total: 10 points