Mount Allison University
Department of Mathematics and Computer Science
5^{th} October, 2009, 8:35-9:20 am
MATH2111 - Test #1

Last Name: SOLUTIONS

First Name: THE

Student Number: ______

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

Q	Mark
1	/12
2	/10
3	/8
Total	/30

1. Determine whether the following series converge or diverge. In each case, state which test or theorem is being used.

[4] (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-2n}{n^3+3n+1}$

Using limit comparison with $b_n = \dfrac{1}{n}$, we see that

$$\lim_{n\to\infty} \frac{\frac{n^2-2n}{n^3+3n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-2n}{n^3+3n+1} = \lim_{n\to\infty} \frac{1-\frac{2}{n^2}}{1+\frac{3}{n^2}+\frac{1}{n^3}} = 1 \neq 0,$$

and since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the given series diverges as well.

[4] (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\ln n}{n^2}$

Here, we use the alternating series test. Letting $f(x) = \dfrac{\ln x}{x^2}$, we see that $f(x) > 0$ for $x > 1$, and $f'(x) = \dfrac{1 - 2\ln x}{x^3} < 0$ for $x > e^{1/2}$, so the sequence $a_n = f(n)$ is positive and decreasing for $n \geq 2$. Moreover,

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1}{2x^2} = 0,$$

using l'Hoptial's rule, so $\lim_{n\to\infty} a_n = 0$ as well. Therefore, the series converges by the alternating series test.
(It's also possible to show that the series converges absolutely but this is a bit trickier.)

[4] (c) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{3^{2n+1}}$

Here, we notice that

$$\frac{5^n}{3^{2n+1}} = \frac{5^n}{3\cdot(3^2)^n} = \frac{1}{3}\left(\frac{5}{9}\right)^n,$$

so the series is geometric with $r = 5/9 < 1$, and therefore, the series converges.
(You can also use either the ratio or root test to show $L = 5/9 < 1$.)

[4] 2. (a) Determine the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}(x+2)^n$$

Letting $a_n = \dfrac{(-1)^n(x+2)^n}{n2^n}$, the ratio test gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)2^{n+1}}\frac{n2^n}{(x+2)^n}\right| = \frac{|x+2|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x+2|}{2}.$$

Since we need this limit to be less than 1, we must have $|x+2| < 2$, so the radius of convergence is $R = 2$.

Now, $|x+2| < 2 \Leftrightarrow -2 < x+2 < 2 \Leftrightarrow -4 < x < 0$, so for the interval of convergence, we have to test the endpoints $x = -4$ and $x = 0$. When $x = -4$, we get $a_n = \dfrac{(-1)^n(-4+2)^n}{n2^n} = \dfrac{1}{n}$, so we get the harmonic series, which diverges. When $x = 0$, we similarly get $a_n = \dfrac{(-1)^n}{n}$, which gives the alternating harmonic series, which converges. Therefore, the interval of convergence is $(-4, 0]$.

(b) Find a power series representation for the functions below. Be sure to state the interval on which the representation is valid.

[3] (i) $f(x) = \dfrac{1}{4+x^2}$

We want to make use of the geometric series formula $\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$, so we re-write $f(x)$ as

$$f(x) = \frac{1}{4}\left(\frac{1}{1+x^2/4}\right) = \frac{1}{4}\left(\frac{1}{1-(-x^2/4)}\right).$$

Therefore, $r = -x^2/4$, and when $|x| < 2$, we have $|r| < 2^2/4 = 1$, so

$$f(x) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{-x^2}{4}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{2n}.$$

[3] (ii) $g(x) = \dfrac{x}{(1-x^2)^2}$ Hint: what is $\dfrac{d}{dx}\left(\dfrac{1}{1-x^2}\right)$?

We notice that

$$\frac{d}{dx}\left(\frac{1}{1-x^2}\right) = (-1)(1-x^2)^{-2}(-2x) = \frac{2x}{(1-x^2)^2} = 2g(x),$$

using the chain rule for derivatives. Therefore, when $|x| < 1$,

$$g(x) = \frac{1}{2}\frac{d}{dx}\frac{1}{1-x^2} = \frac{1}{2}\frac{d}{dx}\sum_{n=0}^{\infty} x^2n = \frac{1}{2}\sum_{n=1}^{\infty} 2nx^{2n-1} = \sum_{n=1}^{\infty} nx^{2n+1}.$$

3. Determine the limit of each of the following sequences, or show that it does not exist:

[2] (a) $a_n = \sqrt[n]{3}$

We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3^{1/n} = \lim_{n\to\infty} e^{\frac{ln3}{n}} = e^0 = 1.$$

[3] (b) $a_n = n \sin\left(\frac{1}{n}\right)$ Hint: let $\theta = \frac{1}{n}$

Following the hint, we make the substitution $\theta = \frac{1}{n}$. Since $n \to \infty$, it follows that $\theta \to 0^+$. Therefore,

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{\theta\to 0^+} \frac{1}{\theta} \sin\theta = 1,$$

since $\lim_{x\to 0} \frac{\sin x}{x} = 1$

[3] (c) $b_n = \frac{n^n}{n!}$

Intuitively, n^n grows faster than $n!$, so the sequence should diverge. To show this, note that

$$\frac{n^n}{n!} = \frac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} = \frac{n}{1}\frac{n}{2} \cdots \frac{n}{n-1}\frac{n}{n} > n,$$

and since $\lim_{n\to\infty} n = \infty$, the sequence diverges.

 Total Marks: 40