1. Calculate the degree 3 Taylor polynomials, centred at $a = 0$, for the following functions:

[3] (a) $f(x) = \ln(x+1)$

We have $f(x) = \ln(x+1)$, $f'(x) = (x+1)^{-1}$, $f''(x) = -1(x+1)^{-2}$, and $f'''(x) = 2(x+1)^{-3}$. Thus,

$$\begin{aligned} p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= 0 + x - \frac{1}{2}x^2 + \frac{2}{6}x^3 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3. \end{aligned}$$

[3] (b) $g(x) = e^{2x}$

We note that, using the Chain Rule, $\frac{d}{dx}(e^{2x}) = e^{2x}\frac{d}{dx}(2x) = 2e^{2x}$. Using this result repeatedly, we find

$$g(x) = e^{2x}, g'(x) = 2e^{2x}, g''(x) = 4e^{2x}, \text{ and } g'''(x) = 8e^{2x}.$$

Our Taylor polynomial is therefore

$$\begin{aligned} p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3. \end{aligned}$$

[2] 2. Let $f(x) = 2x^3 + \sin(x) - \frac{1}{\sqrt{1-x^2}}$.
Determine the antiderivative $F(x)$ of $f(x)$ such that $F(0) = 7$.

The most general antiderivative is given by $F(x) = \frac{2x^4}{4} - \cos(x) - \arcsin(x) + C$. The requirement that $F(0) = 7$ gives us

$$F(0) = 7 = \frac{0^4}{2} - \cos(0) - \arcsin(0) + C = -1 + C,$$

so we must have $C = 8$, and thus,

$$F(x) = \frac{x^4}{2} - \cos(x) - \arcsin(x) + 8.$$

[3] 3. Estimate the area under the graph of $f(x) = \dfrac{x}{x^2+1}$ between $x = 1$ and $x = 4$ using 3 rectangles of equal width, if the height of each rectangle is computed using the left endpoint of each interval.

With 3 rectangles, we have $\Delta x = \dfrac{4-1}{3} = 1$. Our partition points are thus $x_0 = 1, x_1 = 2, x_2 = 3$, and $x_3 = 4$, and the first three of these are our left endpoints. This gives us the approximation

$$\begin{aligned}\sum_{i=1}^{3} f(x_{i-1})\Delta x &= f(1)(1) + f(2)(1) + f(3)(1)\\ &= \frac{1}{1^2+1} + \frac{2}{2^2+1} + \frac{3}{3^2+1}\\ &= \frac{1}{2} + \frac{2}{5} + \frac{3}{10} = \frac{12}{10} = \frac{6}{5} = 1.2.\end{aligned}$$

[2] 4. Compute the derivative of $f(x) = x\int_1^x \sin(t^3+1)\,dt$.

Using the product rule and Part I of the Fundamental Theorem of Calculus, we have

$$f'(x) = (1)\int_1^x \sin(t^3+1)\,dt + x\sin(x^3+1).$$

[3] 5. Evaluate the integral $\displaystyle\int_0^4 \left(2x + \frac{1}{\sqrt{x}}\right) dx$.

An antiderivative of $f(x) = 2x + x^{-1/2}$ is given by $F(x) = x^2 + \frac{x^{1/2}}{1/2} = x^2 + 2\sqrt{x}$, so by Part II of the Fundamental Theorem of Calculus, we have

$$\int_0^4 f(x)\,dx = F(4) - F(0) = 4^2 + 2\sqrt{4} - (0^2 + 2\sqrt{0}) = 20.$$