The problems on this worksheet are for in-class practice during tutorial. You are free to collaborate and to ask for help. They don't count for course credit, but it's a good idea to make sure you know how to do everything before you leave tutorial – similar problems may show up on a test or assignment.

1. Eliminate the parameter to obtain an equation for the curve involving only x and y:

   (a) $x = \sec t$, $y = \tan t$

   Since $\tan^2 t + 1 = \sec^2 t$, we immediately get $y^2 + 1 = x^2$, or $x^2 - y^2 = 1$, which is the standard unit hyperbola.

   (b) $x = 4\sin t + 1$, $y = 3\cos t - 2$ (Hint: first solve for $\cos t$ and $\sin t$.)

   We have $\sin t = \dfrac{x-1}{4}$ and $\cos t = \dfrac{y+2}{3}$, giving us the equation

   $$\frac{(x-1)^2}{4^2} + \frac{(y+2)^2}{3} = 1,$$

   which is an ellipse centred at $(1, -2)$, with semimajor axis of length 4, and semiminor axis of length 3.

   (c) $x = \dfrac{1}{t+1}$, $y = \dfrac{3t+5}{t+1}$. (Hint: try doing long division on the expression for y.)

   It's possible to solve for t in terms of x in the first equation and substitute into the second, but it's faster to notice that

   $$y = \frac{3t+5}{t+1} = 3 + 2\left(\frac{1}{t+1}\right) = 3 + 2x.$$

   Thus, we have the line $y = 3 + 2x$. Note however that for the given parameterization, we have $x \neq 0$ for all t, so there is a hole in the line at the point $(0, 3)$.

   (d) $x = \cosh t, y = \sinh t$

   Thanks to the identity $\cosh^2 t - \sinh^2 t = 1$, we immediately get the equation $x^2 - y^2 = 1$ of the standard unit hyperbola. Note however that $\cosh t > 0$ for all $t \in \mathbb{R}$, so this parameterization only gives us the right half of the hyperbola. (The left half is given by $x = -\cosh t, y = \sinh t$.)

2. Find the length of the parametric curve:

   (a) $x = -3\sin(2t)$, $y = 3\cos(2t)$, $t \in [0, \pi]$.

   For a parametric curve we have $dx = x'(t)\,dt$ and $dy = y'(t)\,dt$, which gives $ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt$, and the arc length is given by $s = \int_0^\pi ds$ as usual.
   In this case $x'(t) = -6\cos(2t)$ and $y'(t) = -6\sin(2t)$, so $x'(t)^2 + y'(t)^2 = 36$. Thus, we have

   $$s = \int_0^\pi \sqrt{36}\,dt = 6\pi.$$

(b) $x = e^{t/10}\cos t, y = e^{t/10}\sin t$, $t \in [0, 2\pi]$.

We use the same procedure as the previous problem. We have

$$(x'(t))^2 = \left(\frac{1}{10}e^{t/10}\cos t - e^{t/10}\sin t\right)^2 = e^{t/5}\left(\frac{1}{100}\cos^2 t - \frac{1}{5}\cos t\sin t + \sin^2 t\right)$$
$$(y'(t))^2 = \left(\frac{1}{10}e^{t/10}\sin t + e^{t/10}\cos t\right)^2 = e^{t/5}\left(\frac{1}{100}\sin^2 t + \frac{1}{5}\cos t\sin t + \cos^2 t\right),$$

so $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{e^{t/5}\left(\frac{1}{100}+1\right)} = \sqrt{101}\dfrac{e^{t/10}}{10}$. Thus,

$$s = \int_0^{2\pi} \sqrt{101}\frac{e^{t/10}}{10}\,dt = \sqrt{101}(e^{\pi/5} - 1).$$

3. Find the area enclosed by the astroid $x = \cos^3 t, y = \sin^3 t$, $t \in [0, 2\pi]$. (There is some work involved here to evaluate the integral.)

For the given parameterization, the astroid is traced out in a counterclockwise direction. It follows that the area is given by

$$\begin{aligned}
A &= -\int_0^{2\pi} y\,dx = -\int_0^{2\pi} \sin^3 t(-3\cos^2 t\sin t)\,dt \\
&= 3\int_0^{2\pi} \sin^4 t\cos^2 t\,dt \\
&= 3\int_0^{2\pi} \left(\frac{1-\cos(2t)}{2}\right)^2\left(\frac{1+\cos(2t)}{2}\right)\,dt \\
&= \frac{3}{8}\int_0^{2\pi} (1 - \cos(2t) - \cos^2(2t) + \cos^3(2t))\,dt \\
&= \frac{3}{8}\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(4t) - \cos(2t)\sin^2(2t)\right)\,dt \\
&= \frac{3\pi}{8}.
\end{aligned}$$

4. Find the area enclosed by the loop of the "teardrop" curve $x = t(t^2 - 1), y = t^2 - 1$. (See Figure 5.34 in the text.)

We first note that $x = t(t-1)(t+1)$, so $x = 0$ for $t = 0, 1, -1$, while $y = 0$ for $t = 1, -1$. It follows (referring to the figure) that the loop begins at $(0, 0)$ when $t = -1$, and ends at $(0, 0)$ when $t = 1$. We check that $x > 0$ for $-1 < t < 0$ and $x < 0$ for $0 < t < 1$, which tells us that the loop is traversed in the clockwise direction. The area is thus given by

$$\begin{aligned}
A &= \int_{-1}^{1} y\,dx = \int_{-1}^{1} (t^2-1)(3t^2-1)\,dt \text{ (Note that } x(t) = t^3 - t\text{, so } x'(t) = 3t^2 - 1.) \\
&= 2\int_0^1 (3t^4 - 4t^2 + 1)\,dt \\
&= 2\left(\frac{3}{5} - \frac{4}{3} + 1\right) = \frac{8}{15}.
\end{aligned}$$

5. Verify that $x = Ce^{-t} + De^{2t}$ is a solution to $x'' - x' - 2x = 0$.

   We have

$$\begin{aligned} x(t) &= Ce^{-t} + De^{2t} \\ x'(t) &= -Ce^{-t} + 2De^{2t} \\ x''(t) &= Ce^{-t} + 4De^{2t}, \end{aligned}$$

   so $x'' - x' - 2x = e^{-t}(C - (-C) - 2C) + e^{2t}(4D - 2D - 2D) = 0$, as required.

6. Find the solution from Problem 5 that satisfies $x(0) = 3$ and $x'(0) = -2$.

   Setting $x(0) = 3$ gives us $C + D = 3$. Setting $x'(0) = -2$ gives us $-C + 2D = -2$. We have two equations in the unknowns C and D, which can easily be solved to give us $C = \dfrac{8}{3}$ and $D = \dfrac{1}{3}$.

7. Solve $y' = y^3$ when $y(0) = 1$. (Hint: $\dfrac{1}{y'} = \dfrac{dx}{dy}$.)

   There are two ways to solve this differential equation. The first follows the hint:

   We first note that $y(x) = 0$ is a solution. If we assume that $y \neq 0$, we can write

$$\frac{1}{y'} = \frac{dx}{dy} = \frac{1}{y^3} = y^{-3}.$$

   Here we're assuming that $y = f(x)$, where f has an inverse, so we can write $x = f^{-1}(y)$. (This may not be globally true, but it is true on any open interval that does not contain a critical point of f.)

   If $\dfrac{dx}{dy} = y^{-3}$, then taking the antiderivative gives us $x = -\dfrac{1}{2y^2} + C$, so $y^2 = \dfrac{1}{2C - 2x}$. This leaves us with the problem of whether to take the positive or negative square root to solve for y, but the initial condition $y(0) = 1 > 0$ tells us that we must take the postitive square root. Applying the initial condition gives us

$$1^1 = 1 = \frac{1}{2C},$$

   so $C = \frac{1}{2}$, and thus $y = \dfrac{1}{\sqrt{1 - 2x}}$.

   The other approach is to treat the equation as a separable equation. From $\dfrac{dy}{dx} = y^3$ we have $\dfrac{dy}{y^3} = dx$, and integrating both sides gives us $-\dfrac{1}{y^2} = x + C$. The remainder of the solution is as above.

8. Solve $\dfrac{dx}{dt} = x\sin(t)$ for $x(0) = 1$.

   We have a separable differential equation, which can be written as $\dfrac{dx}{x} = \sin t\, dt$. Integrating both sides gives us $\ln x = -\cos t + C$. We can solve now for x as a function of t but it's convenient to first apply the initial condition: when $t = 0$ we have $x = 1$, so

   $$\ln(1) = 0 = -\cos(0) + C,$$

   which gives us $C = 1$. Thus $\ln x = 1 - \cos t$, so $x = e^{1-\cos t}$.