University of Lethbridge
Department of Mathematics and Computer Science
10th December, 2014, 2:00 - 5:00 pm
Math 2000A - FINAL EXAM

Last Name:____

First Name:____

Student Number:______________________________________

Record your answers below each question in the space provided. **Left-hand pages may be used as scrap paper for rough work.** If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

The value of each problem is indicated in the left-hand margins. The value of a problem does not always indicate the amount of work required to do the problem.

No external aids are permitted, including calculators, cell phones, cheat sheets, laptops, spy cameras, drones, and secret radio communications devices.

For grader's use only:

Page	Grade		Page	Grade
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3	/8		8	/8
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5	/12		10	/10
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Total:				/100

[5] 1. Establish the following logical equivalence (justify each step):

$$P \to (Q \vee R) \equiv (P \wedge \neg Q) \to R$$

2. Given a function $f : A \to B$, we say that f is *onto* if for every $b \in B$ there exists some $a \in A$ such that $f(a) = b$.

[2] (a) Re-write this definition using symbolic logic and the quantifier symbols $\forall$ and $\exists$.

[3] (b) Using symbolic logic, define what it means to say that f is **not** onto.

[3] 3. What is the contrapositive of the conditional statement "If $a^2 = 25$, then $a = 5$ or $a = -5$"?

4. Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and let $A, B, C \subseteq U$ be given by

$$A = \{2, 5, 7, 8\}, \quad B = \{1, 3, 4, 9\}, \quad C = \{0, 2, 4\}.$$

[2] (a) What is the power set of C?

[2] (b) What is $A \cup (B \cap C)$?

[2] (c) What is $A \setminus C$?

[2] (d) What is the complement of $A \cup B \cup C$?

5. Determine if the statements below are true or false, and support your answer with a proof or counterexample.

[4] (a) For any subsets A, B, C of some universal set U,

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

[4] (b) For any subsets A, B, C of some universal set U,

$$\text{if } A \cap C \subseteq B \cap C, \text{ then } A \subseteq B.$$

6. Determine if the statements below are true or false, and support your answer with a proof or counterexample.

[4] (a) For all $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$, then $a|(b-c)$.

[4] (b) For all $a, b \in \mathbb{Z}$, if $ab \equiv 0 \pmod 6$, then $a \equiv 0 \pmod 6$ or $b \equiv 0 \pmod 6$.

[4] (c) For each $a \in \mathbb{Z}$, if 3 does not divide a, then 3 divides $2a^2 + 1$.

7. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 5x + 3$.

[3] (a) Is f one-to-one?

[3] (b) Is f onto?

[3] (c) Does your answer to either (a) or (b) change if f is a function from $\mathbb{Q}$ to $\mathbb{Q}$? Explain.

8. (a) If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 2x + 1$ and $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = \dfrac{1}{x^2+1}$,
[4] determine formulas for the compositions $f \circ g$ and $g \circ f$.

(b) Construct an example of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f : A \to C$
[2] is onto, but f is not.

[4] (c) Given $f : A \to B$ and $g : B \to C$, prove that if $g \circ f : A \to C$ is onto, then so is g.

9. Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $B = \{r, s, t, u, v\}$. Define $f : A \to B$ by

$$f(1) = s,\ f(2) = t,\ f(3) = r,\ f(4) = t,\ f(5) = u,\ f(6) = s.$$

[2] (a) If $C = \{2, 4, 6\}$, what is $f(C)$?

[2] (b) If $D = \{r, s, v\}$, what is $f^{-1}(D)$?

[5] 10. Prove that for any function $f : A \to B$ and any subsets $C, D \subseteq B$, we have

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D).$$

[4] 11. (a) If $k \geq l$, show that there exists an onto function $f : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, l\}$.

[4] (b) Using part (a), show that if A and B are finite sets with $|A| \geq |B|$, then there exists an onto function $g : A \to B$.

[4] 12. Give four examples of countably infinite sets.

[10] 13. Use mathematical induction to prove the following statement:

$$\forall n \in \mathbb{N}, \quad 2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n + 1)}{2}.$$

14. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, and define a relation $\sim$ on A by

$$a \sim b \text{ if and only if } a^2 \equiv b^2 \pmod 9.$$

[5] (a) Prove that $\sim$ is an equivalence relation on A.

[3] (b) Determine all of the distinct equivalence classes determined by $\sim$.

[2] 15. For which $[x] \in \mathbb{Z}_4 = \{[0], [1], [2], [3]\}$ is it true that $([2] \odot [x]) \oplus [1] = [3]$?
(Hint: there are only four values of $[x]$ to check.)

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