Math 4310 Assignment #2 Solutions
University of Lethbridge, Fall 2014

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1. Since any ϵ-neighbourhood in a metric space X is open in X, we know that the union of any collection of such neighbourhoods is an open set in X. Prove that this is in fact the most general type of open set. That is, prove that any open subset $U \subseteq X$ of a metric space X is a union of ϵ-neighbourhoods.

   Solution: Let $U \subseteq X$ be open in X. Then for any $x \in U$ there exists some $\epsilon_x > 0$ such that $N_{\epsilon_x}(x) \subseteq U$. We claim that U can be written as the union

$$U = \bigcup_{x \in U} N_{\epsilon_x}(x),$$

   since if $x \in U$, then $x \in N_{\epsilon_x}(x) \subseteq \bigcup N_{\epsilon_x}(x)$, and since each neighbourhood $N_{\epsilon_x}(x)$ is contained in U, their union must be a subset of U as well.

2. Let (X, d) be a metric space. Prove that $d : X \times X \to \mathbb{R}$ is continuous with respect to the product metric d_1 on $X \times X$. (See the text if you need a reminder on how d_1 is defined.)

   Solution: We equip $Y = X \times X$ with the metric d_1 given by

$$d_1((x, y), (a, b)) = d(x, a) + d(y, b),$$

   for any points $(x, y), (a, b) \in Y$. We want to show $d : Y \to \mathbb{R}$ is continuous with respect to d_1 and the standard metric on $\mathbb{R}$. Let $\epsilon > 0$ be given, and take $\delta = \epsilon$. If $d_1((x, y), (a, b)) < \delta$, then we have

$$\begin{aligned} d(x, y) - d(a, b) &\leq d(x, a) + d(a, y) - d(a, b) \\ &\leq d(x, a) + d(a, b) - d(b, y) - d(a, b) \\ &= d(x, a) + d(y, b). \end{aligned}$$

   Similarly, we can show that $d(a, b) - d(x, y) \leq d(x, a) + d(y, b)$. Thus, we have that

$$|d(x, y) - d(a, b)| \leq d(x, a) + d(y, b) = d_1((x, y), (a, b)) < \delta = \epsilon.$$

3. (Do not hand in) Suppose that in a metric space X we have that $N_a(x) = N_b(y)$ for some $x, y \in X$ and $a, b \in \mathbb{R}$. Can we conclude that $a = b$ and $x = y$?

   Solution: Consider the discrete metric on $\mathbb{R}$. Then we have, for example, that $N_2(0) = N_3(1) = \mathbb{R}$, but $2 \neq 3$ and $0 \neq 1$.

4. (Do not hand in) Prove that any finite subset of a metric space X is closed in X.

   Solution: Since the union of any finite collection of closed sets is closed, it suffices to prove that $A = \{x\}$ is closed for any $x \in X$. To that end, we need to show that if y is a point of closure of A, then $y \in A$. If $y \neq x$ then we can take $\epsilon = |y - x|/2$, and then $N_\epsilon(y) \cap A = \emptyset$. Thus, the only point of closure of A is x itself, and $x \in A$, so A is closed.

5. Prove that the Cantor set is a closed subset of $\mathbb{R}$ with respect to the standard metric on $\mathbb{R}$. (See Problem 6.5 in the text, or type 'Cantor set' into Google and follow the first link for a definition.)

   Solution: We saw in class that $C = \bigcap C_n$, where each C_n is the union of 2^n closed intervals of length 3^{-n}. Since any finite union of closed sets is closed, it follows that each C_n is closed. But then C is the interesection of a collection of closed sets, which implies that C is closed.

6. (Do not hand in) Let $\mathcal{C}[0,1]$ be the space of continuous functions on $[0,1]$, equipped with the sup-norm metric (d_∞). For any subset $A \subseteq [0,1]$, show that the set $Y = \{f \in \mathcal{C}[0,1] : f(a) = 0 \text{ for all } a \in A\}$ is a closed subset of $\mathcal{C}[0,1]$.

   I'm short on time, so I'll skip this one, but I'm happy to discuss it during office hours or on Piazza if anyone wants the solution.

7. Prove that a map $f : X \to Y$ of metric spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets $A \subseteq X$, where $\overline{B}$ denotes the closure of B.

   Solution: I'm actually going to give two proofs for this one, mainly to illustrate the fact that it's a good idea to know several different characterizations of what it means for a function to be continuous. First, let's work with the $\epsilon - \delta$ definition of continuity:

   Proof: Let (X, d) and (Y, d') be metric spaces, and let $f : X \to Y$ be (d, d')-continuous. We want to show that $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset $A \subseteq X$. Let $y \in f(\overline{A})$. Then $y = f(x)$ for some $x \in \overline{A}$. Now, let $\epsilon > 0$ be given. We need to show that $N_\epsilon(y)$ contains an element of $f(A)$. Since f is continuous, there exists some $\delta > 0$ such that $f(N_\delta(x)) \subseteq N_\epsilon(y)$. Since x is a point of closure[1] of A, there exists some $a \in A$ such that $a \in N_\delta(x)$, which implies that $f(a) \in N_\epsilon(y)$, which is what we needed to show.

[1]Here is an example of where the definition of a point of closure is more convenient than defining $\overline{A}$ as the union of A and its limit points: we'd otherwise have to consider the cases $x \in A$ and $x \notin A$ separately.

Now suppose that f is not continuous at some point $a \in X$. Then there exists some $\epsilon > 0$ such that for all $\delta > 0$, there is some $x \in X$ such that $d_X(x, a) < \delta$ but $d_Y(f(x), f(a)) > \epsilon$. In particular, for each $n \in \mathbb{N}$, if we take $\delta = 1/n$, then there is some $x_n \in X$ with $d_X(x_n, a) < 1/n$, but $d_Y(f(x), f(a)) > \epsilon$. Let $A = \{x_n\}$ be the set of all points x_n so defined. Then $a \in \overline{A}$, since every neighbourhood of a contains one of the points x_n, so $f(a) \in f(\overline{A})$. However, we have that $f(a) \notin \overline{f(A)}$, since we have already established the existence of an $\epsilon > 0$ such that $d_Y(f(a), f(x_n)) > \epsilon$ for all x_n, and thus, $N_\epsilon(f(a))$ contains no points of $f(A) = \{f(x_n) \,|\, n \in \mathbb{N}\}$. Therefore, there exists a subset $A \subseteq X$ such that $f(\overline{A}) \nsubseteq \overline{f(A)}$.

(Note: this is closely related to the fact that a function is continuous if any only if $\lim f(a_n) = f(a)$ whenever $a_n \to a$ is a convergent sequence in a metric space X. This observation alone is not quite enough however. If you start with an arbitrary sequence, rather than the tailor-made one constructed above, you're almost guaranteed to get stuck. For example, assuming continuity (and thus sequential continuity, that the limit of $f(a_n)$ is $f(a)$) and taking $A = \{a_n\}$, you can show that every neighbourhood of $f(a)$ contains a point of $f(A)$ — infinitely many points, in fact — but this is not enough. Why? Because you can easily construct a function where $f(a_{2k}) \to f(a)$, but $f(a_{2k+1}) = 100$ (or some other fixed and unhelpful value), so each neighbourhood of $f(a)$ can contain infinitely many points of $f(A)$ and still not contain all points $f(a_n)$ for *all* $n \geq N$ for some N: there might always be the occasional point that jumps out. But still, we've got a situation where working with sequences was helpful.)

Let's now proceed with Proof #2. We will use the following facts: for any $A \subseteq X$ we have $A \subseteq f^{-1}(f(A))$ and for any $B \subseteq Y$, we have $f(f^{-1}(B)) \subseteq B$;[2] we'll also use the fact that a function $f : X \to Y$ is continuous if and only if whenever $B \subseteq Y$ is *closed* in Y, $f^{-1}(B)$ is closed in X. (As an exercise, you should verify that this is a corollary of the fact that the inverse image of any open set is open for a continuous function.)

Proof: Suppose that f is continuous. Since $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is closed. Since $f(A) \subseteq \overline{f(A)}$, we have that

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}).$$

Since $f^{-1}(\overline{f(A)}$ is closed and $\overline{A}$ is the smallest closed set containing A,[3] we have that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$, and thus, $f(\overline{A}) \subseteq \overline{f(A)}$.[4]

Conversely, suppose that we know that $f(\overline{A}) \subseteq \overline{f(A)}$ for each $A \subseteq X$. Let $B \subset Y$ be closed in Y, and let $A = f^{-1}(B)$. Then, since $f(\overline{A}) \subseteq \overline{f(A)}$, we have

$$\overline{A} \subseteq f^{-1}(f(\overline{A})) \subseteq f^{-1}(\overline{f(A)}) = f^{-1}(\overline{B}) = f^{-1}(B) = A,$$

[2]The first inclusion is an equality provided that f is an injection (one-to-one), and the second is an equality if f is a surjection (onto)

[3]I believe this is proved in the text. If not, as an exercise, you can prove that $\overline{A} = \bigcap F$, where the intersection is taken over all closed subsets containing A, then then explain why it follows that if $A \subseteq F$ and F is closed, then $\overline{A} \subseteq F$.

[4]If $a \in f^{-1}(B)$ then by definition, $f(a) \in f(B)$, so $A \subseteq f^{-1}(B)$ implies that $f(A) \subseteq B$.

since B is closed, so $\overline{B} = B$. But then we have $\overline{A} \subseteq A$, and $A \subseteq \overline{A}$ by definition, so $A = \overline{A}$, and A is closed, and thus f is continuous.

(So the second proof was... shorter. I'd say easier as well, at least for me – it took awhile to think up the set A for the proof of the converse. The second approach illustrates two useful lessons: the usefulness of having multiple formulations of continuity, and the usefulness of being comfortable with the set gymnastics involved with direct and inverse images.)

8. (Do not hand in) Let A be a nonempty subset of a metric space (X, d). For $x \in X$, define

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

   (a) Prove that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

   (b) Show that if $y \in X$ is another point of X, then $d(x, A) \leq d(x, y) + d(y, A)$.

   (c) Prove that $x \to d(x, A)$ defines a continuous map $X \to \mathbb{R}$.

   OK, I was totally going to include solutions for this one and then I spent all my time coming up with a probably unnecessary second proof for the last question. In any case, the main one we need is 8(a), and I proved this in class. Well, I proved (ok explained; it maybe doesn't count as a proof if I say it out loud rather than writing it down) in class that $d(x, A) = 0$ if and only if for all $\epsilon > 0$ there exists some $a \in A$ with $d(x, a) < \epsilon$, and this latter condition is the same thing as requiring x to be a limit point of A.

9. Let A be a nonempty subset of a metric space X. Prove that a point $x \in X$ belongs to the boundary ∂A of A if and only if $d(x, A) = d(x, X \setminus A) = 0$, where $d(x, A)$ is the distance from a point to a set defined in the previous problem.

   Solution: From 8(a) we know that $d(x, A) = 0$ and $d(x, X \setminus A) = 0$ if and only if x is a point of closure of both A and $X \setminus A$, which means that every neighbourhood of x contains points of both A and $X \setminus A$. But this is exactly the definition of a boundary point.

10. (Do not hand in, unless you really want to) For a subset A of a metric space X, prove:

   (a) $\mathring{A} = A \setminus \partial A = \overline{A} \setminus \partial A$

   (b) $\overline{X \setminus A} = X \setminus \mathring{A}$

   (c) $\partial A = \overline{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$

   (d) ∂A is closed in X.

   Nobody handed it in and I don't really want to write the solutions, at least not now – I'd rather be sleeping. I'm happy to solve any of them on request, however.