Name: Solutions

Solve **one** of the following two questions:

1. Let $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ be invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible, and show that $(ST)^{1-} = T^{-1}S^{-1}$.

   Solution: By defintion, $ST : U \to W$ is invertible if there exists a linear map $R : W \to U$ such that $(ST)R$ is the identity on W, and $R(ST)$ is the identity on U. If such a map R exists, then we can furthermore conclude that $R = (ST)^{-1}$.

   We claim that $R = T^{-1}S^{-}$. To see this, note that

   $$(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = S(I_V)S^{-1} = SS^{-1} = I_W,$$

   where I_W denotes the identity on W, and I_V denotes the identity on V. (Note that we can use either $SI_V = S$ or $I_VS^{-1} = S^{-1}$ above.) Similarly, we have

   $$(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}(I_V)T = T^{-1}T = I_U.$$

   This proves both that ST is invertible, and that $(ST)^{-1} = T^{-1}S^{-1}$.

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be given by

   $$T(w, x, y, z) = (3w - 2x + z, x + 3y - 4z, w - x + y + z).$$

   Compute the matrix of T with respect to the bases

   $$\begin{aligned} B_4 &= \{(1, 0, 2, 0), (0, 3, 0, 1), (1, -2, 0, 0), (0, 0, -1, 1)\} \text{ of } \mathbb{R}^4, \text{ and} \\ B_3 &= \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \text{ of } \mathbb{R}^3. \end{aligned}$$

   Solution: We calculate the value of T on the basis B_4 as follows:

   $$\begin{aligned} T(1, 0, 2, 0) &= (3(1) - 2(0) + 0, 0 + 3(2) - 4(0), 1 - 0 + 2 + 0) = (3, 6, 3) \\ T(0, 3, 0, 1) &= (3(0) - 2(3) + 1, 3 + 3(0) - 4(1), 0 - 3 + 0 + 1) = (-5, -1, -2) \\ T(1, -2, 0, 0) &= (3(1) - 2(-2) + 0, -2 + 3(0) - 4(0), 1 - (-2) + 0 + 0) = (7, -2, 3) \\ T(0, 0, -1, 1) &= (3(0) - 2(0) + 1, 0 + 3(-1) - 4(1), 0 - 0 + (-1) + 1) = (1, -7, 0). \end{aligned}$$

   Therefore, the matrix of T with respect to the bases B_4 and B_3 is given by

   $$\mathcal{M}(T) = \begin{bmatrix} 3 & -5 & 7 & 1 \\ 6 & -1 & -2 & -7 \\ 3 & -2 & 3 & 0 \end{bmatrix}.$$

Total: 10 points

Note: In question 2, I asked you only to find the matrix $\mathcal{M}(T)$. You didn't have to verify that it was correct or use it to find the null space and range of T or anything like that.

If you did want to verify that your matrix was correct, there's a bit of work involved, since you'd have to find the matrix of $(w,x,y,z) \in \mathbb{R}^4$ with respect to the given basis. Supposing that you wanted to do this (noting again that this was **not** necessary), you'd set

$$(w,x,y,z) = a(1,0,2,0)+b(0,3,0,1)+c(1,-2,0,0)+d(0,0,-1,1) = (a+c, 3b-2c, 2a-d, b+d)$$

for some scalars a,b,c,d. This gives you a system of 4 equations in the 4 variables a,b,c,d. If you solve it, you find

$$\begin{aligned}
a &= -\frac{1}{2}w + \frac{1}{2}x + \frac{3}{4}y - \frac{3}{2}z\\
b &= w - \frac{1}{2}y + z\\
c &= \frac{3}{2}w - \frac{1}{2}x - \frac{3}{4}y + \frac{3}{2}z\\
d &= -w + \frac{1}{2}y.
\end{aligned}$$

Thus,

$$\mathcal{M}(w,x,y,z) = \begin{bmatrix} -\frac{1}{2}w + \frac{1}{2}x + \frac{3}{4}y - \frac{3}{2}z \\ w - \frac{1}{2}y + z \\ \frac{3}{2}w - \frac{1}{2}x - \frac{3}{4}y + \frac{3}{2}z \\ -w + \frac{1}{2}y \end{bmatrix},$$

and you can verify that

$$\mathcal{M}(T)\mathcal{M}(w,x,y,z) = \begin{bmatrix} 3 & -5 & 7 & 1 \\ 6 & -1 & -2 & -7 \\ 3 & -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}w + \frac{1}{2}x + \frac{3}{4}y - \frac{3}{2}z \\ w - \frac{1}{2}y + z \\ \frac{3}{2}w - \frac{1}{2}x - \frac{3}{4}y + \frac{3}{2}z \\ -w + \frac{1}{2}y \end{bmatrix} = \begin{bmatrix} 3w-2x+z \\ x+3y-4z \\ 2-x+y+z \end{bmatrix},$$

which shows you that you got the right matrix (since the right-hand side is the matrix of $T(w,x,y,z)$ with respect to the standard basis of $\mathbb{R}^3$, but it's really more trouble than it's worth.

One thing to be careful of, with respect to the above, is that if you did want to do a quick check to verify things, the elements of $\mathbb{R}^{4,1}$ corresponding to the basis vectors in B_r are *not* $\begin{bmatrix}1\\0\\2\\0\end{bmatrix}$, $\begin{bmatrix}0\\3\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}$, and $\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$ – they're the vectors $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, since

$$(1,0,2,0) = 1(1,0,2,0) + 0(0,3,0,1) + 0(1,-2,0,0) + 0(0,0,-1,1),$$

and so on.

Finally, if you really did want to know about the null space and range, you can reduce the matrix $\mathcal{M}(T)$ to reduced row-echelon form, which gives

$$\mathcal{M}(T) \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{18} \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & 0 & 1 & \frac{3}{2}. \end{bmatrix}$$

Total: 10 points

We can then see that the $\mathcal{M}(T)\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ if (setting $d = t$ to be our parameter)

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = t \begin{bmatrix} 7/18 \\ -5/3 \\ -3/2 \\ 1 \end{bmatrix}$$

If we replace t with the parameter $s = t/18$, then we can use the vector $\begin{bmatrix} 7 \\ -30 \\ -27 \\ 18 \end{bmatrix}$ as the basis for the null space of $\mathcal{M}(T)$. But we want the null space of the original linear transformation T. Converting back, we conclude that the vector

$$7(1,0,2,0) - 30(0,3,0,1) - 27(1,-2,0,0) + 18(0,0,-1,1) = (-20,-36,-4,-12)$$

forms a basis for the null space of T. Simplifying slightly, we can multiply the above by the scalar $-1/2$, so if our computations are correct, we should have

$$\operatorname{null} T = \operatorname{span}\{(10,18,2,6)\}.$$

Let's check to see if we're right:

$$T(10,18,2,6) = (0,0,0),$$

which is what we should expect. Finally, a basis for the column space of $\mathcal{M}(T)$ is give by the first three columns of $\mathcal{M}(T)$, since these are the columns containing the leading ones in the row-echelon form. Therefore,

$$\operatorname{col} \mathcal{M}(T) = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} \right\},$$

so $\operatorname{range} T$ is given by the span of the corresponding vectors in $\mathbb{R}^3$:

$$\operatorname{range} T = \operatorname{span}\{(3,6,3),(-5,-1,-2),(7,-2,3)\}.$$

Of course, these are three linearly independent vectors in $\mathbb{R}^3$, so we can conclude that $\operatorname{range} T = \mathbb{R}^3$, which tells us that T is a surjection. (Of course, we could have also come to this conclusion with much less work by noting that $\dim \operatorname{null} T = 1$, so

$$\dim \operatorname{range} T = \dim \mathbb{R}^4 - \dim \operatorname{null} T = 4 - 1 = 3,$$

and if $\operatorname{range} T$ is a 3-dimensional subspace of $\mathbb{R}^3$, we must have $\operatorname{range} T = \mathbb{R}^3$.

Total: 10 points