Math 1410 Assignment #4 Solutions
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1. Let A be an $m \times n$ matrix. Note that each column of A is of size $m \times 1$, and therefore a vector in $\mathbb{R}^m$. Recall that for a vector $\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ in $\mathbb{R}^n$, we have

$$A\vec{x} = x_1C_1 + x_2C_2 + \cdots + x_nC_n,$$

where $C_1, C_2, \ldots, C_n$ denote the columns of A. Show that the following are true:

(a) The columns $C_1, C_2, \ldots, C_n$ are linearly independent if and only if the only solution to the homogeneous equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

Proof: Suppose that the columns $C_1, \ldots, C_n$ of A are linearly independent, and suppose that $A\vec{x} = \vec{0}$, where $\vec{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$. Then we have that

$$\vec{0} = A\vec{x} = x_1C_1 + \cdots + x_nC_n,$$

and since the columns are linearly independent, it follows that $x_i = 0$ for all $i = 1, \ldots, n$, and therefore $\vec{x} = \vec{0}$. Since $\vec{x}$ was arbitrary, this must be the only solution. Conversely, suppose that the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, and suppose that

$$x_1C_1 + x_2C_2 + \cdots + x_nC_n = \vec{0}$$

for some scalars $x_1, \ldots, x_n$. Then we have that $A\vec{x} = \vec{0}$, where $\vec{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$, and therefore we must have $\vec{x} = \vec{0}$, which implies that $x_i = 0$ for all $i = 1, \ldots, n$. Thus, the columns $C_1, \ldots, C_n$ are linearly independent.

(b) The non-homogeneous equation $A\vec{x} = \vec{y}$ has a solution if and only if $\vec{y} \in \operatorname{span}\{C_1, C_2, \ldots, C_n\}$.

Proof: The proof is similar to the one given above: if we let $\vec{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$, then the equation $A\vec{x} = \vec{y}$ is equivalent to the equation

$$x_1C_1 + x_2C_2 + \cdots + x_nC_n = \vec{y}.$$

Therefore, given $\vec{y} \in \mathbb{R}^m$, if $A\vec{x} = \vec{y}$ has a solution $\vec{x}$, then we can find scalars $x_1, \ldots, x_n$ such that $x_1 C_1 + \cdots + x_n C_n = \vec{y}$, and thus $\vec{y}$ is in the span of the columns $C_1, \ldots, C_n$.
Conversely, if $\vec{y} \in \text{span}\{C_1, \ldots, C_n\}$, then there exist scalars $x_1, \ldots, x_n$ such that $x_1 C_1 + \cdots + x_n C_n = \vec{y}$, and thus $A\vec{x} = \vec{y}$ has a solution.

2. Find the point of intersection (if any) of the following pairs of lines:

(a)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Solution: If (x, y, z) is a point of intersection of the two lines, then we would have to have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix},$$

which gives us the system of equations

$$\begin{aligned} 3 + t &= 1 + 2s \\ -1 + t &= 1 \\ 2 - t &= -2 + 3s. \end{aligned}$$

The second equation requires that $t = 2$. Putting $t = 2$ in the first equation gives us $s = 2$, but in the third equation, it gives us $s = 2/3$. Thus, there is no solution to the system, and the two lines do not intersect.

(b)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

Solution: If (x, y, z) is a point of intersection for the two lines, then we must have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix},$$

which gives us the system of equations

$$\begin{aligned} 4 + t &= 2 \\ -1 &= -7 - 2s \\ 5 + t &= 12 + 3s. \end{aligned}$$

The first equation tells us that $t = -2$, and the second requires $s = -3$. If we plug these values into the third equation, we get $5 - 2 = 3$ on the left, and $12 + 3(-3) = 3$ on the right. Since $t = -2$ and $s = -3$ satisfies all three equations, the lines intersect. Using the equation for either line, we see that the point of intersection is $(2, -1, 3)$, since

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix},$$

3. (a) Show that $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is perpendicular to the line $ax + by + c = 0$.

Solution: There are two ways to solve the problem. (Well, there are more than two, but these are the two I'm going to show you.)

Option 1: Let (x_0, y_0) and (x_2, y_2) be two points on the line. Then we know that (i) $ax_0 + by_0 = -c$, and $ax_2 + by_2 = -c$, since both points are on the line, and (ii) the vector $\vec{v} = \langle x_2 - x_0, y_2 - y_0 \rangle$ is parallel to the line. Since

$$\langle a, b \rangle \cdot \langle x_2 - x_0, y_2 - y_0 \rangle = a(x_2 - x_0) + b(y_2 - y_0) = (ax_2 + by_2) - (ax_0 + by_0) = -c + c = 0,$$

it follows that $\vec{n}$ is perpendicular to $\vec{v}$, and thus to the line.

Option 2: If we view the equation $ax + by + c = 0$ as a system of one linear equation in two unknowns, the general solution is given by setting $y = t$, where t is a parameter, and thus $x = -c/a - b/at$. The vector form of this solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c/a \\ 0 \end{bmatrix} + t \begin{bmatrix} -b/a \\ 1 \end{bmatrix},$$

so $\vec{v} = \begin{bmatrix} -b/a \\ 1 \end{bmatrix}$ is parallel to the line, and $\vec{n} \cdot \vec{v} = a(-b/a) + b(1) = 0$.

(b) Show that the shortest distance from the point $P_1 = (x_1, y_1)$ to the line is

$$\frac{|x_1 + y_1 + c|}{\sqrt{a^2 + b^2}}.$$

Hint: Take any point P_0 on the line and project $\vec{u} = \overrightarrow{P_0P_1}$ onto $\vec{n}$. If you haven't drawn yourself a picture, you're probably doing it wrong.

Solution: Let $P_0 = (x_0, y_0)$ be any point on the line, and let $\vec{v} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0 \rangle$. The distance from P_1 to the line is then

$$\|\mathrm{proj}_{\vec{n}}\, \vec{v}\| = \left| \frac{\vec{n} \cdot \vec{v}}{\|\vec{n}\|^2} \right| \|\vec{n}\| = \frac{|a(x_1 - x_0) - b(y_1 - y_0)|}{a^2 + b^2} \sqrt{a^2 + b^2} = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}},$$

since $-(ax_0 + by_0) = c$.

(c) Now, let L be a line in $\mathbb{R}^3$ through the point $P_0 = (x_0, y_0, z_0)$ with direction vector $\vec{d}$. Show that the shortest distance from a point $P_1 = (x_1, y_1, z_1)$ to the line is

$$\frac{\|\overrightarrow{P_0P_1} \times \vec{d}\|}{\|d\|}.$$

Solution: Let $\vec{v} = \overrightarrow{P_0P_1}$, and notice that the vectors $\vec{v}$ and $\vec{d}$ span a parallelogram whose area is given by $A = \|\vec{v} \times \vec{d}\|$. Moreover, length of the base of the parallelogram is $b = \|\vec{d}\|$, and the height h of the parallelogram is precisely the distance from the point P_1 to the line. Since the area of the parallelogram is also given by $A = bh$, we can equate the two areas and solve for h, and this gives the formula above.

4. Find the shortest distance between the following pair of skew lines, and the points on each line that are closest together:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Suppose that Q_1 and Q_2 are the points on L_1 and L_2, respectively, that are closest together, where L_1 denotes the first line (with parameter t), and L_2 denotes the second line (with parameter s). We note that since the lines are skew (which you should verify), they lie in parallel planes. Indeed, if we let $\vec{d}_1 = \langle 1,1,1\rangle$ be the direction vector of the first line, and $\vec{d}_2 = \langle 3,1,0\rangle$ be the direction vector of the second line, and take $\vec{n} = \vec{d}_1 \times \vec{d}_2$, then L_1 lies in the plane $\vec{n}\cdot\langle x-1, y+1, z\rangle = 0$, and L_2 lies in the plane $\vec{n}\cdot\langle x-2, y+1, z-3\rangle = 0$.

We make two observations: first, the distance between the two lines is equal to the distance between the two planes. Second, this distance is equal to the distance between the points Q_1 and Q_2, and the vector $\overrightarrow{Q_1Q_2}$ must be parallel to $\vec{n}$, and thus orthogonal to $\vec{d}_1$ and $\vec{d}_2$. We have $Q_1 = (1+t, -1+t, t)$ for some $t \in \mathbb{R}$, and $Q_2 = (2+3s, -1+s, 3)$ for some $s \in \mathbb{R}$. Thus,

$$\overrightarrow{Q_1Q_2} = \langle 1+3s-t, s-t, 3-t\rangle,$$

and since $\vec{d}_1 \cdot \overrightarrow{Q_1Q_2} = 0$ and $\vec{d}_2\overrightarrow{Q_1Q_2} = 0$, we must have

$$\begin{aligned}
1(1+3s-t)+1(s-t)+1(3-t) &= 4s-3t+4 = 0,\\
3(1+3s-5)+1(s-t)+0(3-t) &= 10s-4t+3 = 0.
\end{aligned}$$

This gives us a system of two equations in the two variables s and t. The solution is easily found to be $s = 1/2$ and $t = 2$, which gives us the points $Q_1 = (3,1,2)$ and $Q_2 = (7/2, -1/2, 3)$. Thus, the distance between the two lines is

$$d(Q_1,Q_2) = \sqrt{(7/2-2)^2 + (-1/2-1)^2 + (3-1)^2} = \frac{\sqrt{14}}{2}.$$

To verify that we've correctly found the two closest points, we note that the distance between the two planes can also be computed as follows: we know that $P_1 = (1,-1,0)$ lies on the first plane, and $P_2 = (2,-1,3)$ lies on the second plane. Therefore, the distance between the two planes is given by the length of the projection of $\vec{v} = \overrightarrow{P_1P_2}$ onto the normal vector

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 1 & 1\\ 3 & 1 & 0\end{vmatrix} = -\hat{\imath} + 3\hat{\jmath} - 2\hat{k} = \langle -1, 3, -2\rangle.$$

We have $\vec{n}\cdot\vec{v} = -7$ and $\|\vec{n}\| = \sqrt{14}$, and thus the distance between the lines is equal to

$$\|\mathrm{proj}_{\vec{n}}\,\vec{v} = \left|\frac{\vec{v}\cdot\vec{n}}{\|\vec{n}\|^2}\right| \|\vec{n}\| = \frac{|-7|}{14}\sqrt{14} = \frac{\sqrt{14}}{2},$$

which agrees with our previous calculation.