Name: Solutions

Use mathematical induction to prove that for each natural number n,

$$2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n + 1)}{2}.$$

Hint: Before you begin, you might want some rough work on the side where you plug both $n = k$ and $n = k + 1$ into the equation above. That way, when you get to the induction step, you'll know (a) what to assume, and (b) what you need to prove.

Solution: Let $P(n)$ represent the predicate

$$2 + 5 + \cdots + (3n - 1) = \frac{n(3n + 1)}{2}. \tag{1}$$

When $n = 1$, we see that $\frac{1(3(1) + 1)}{2} = 2$, which shows that $P(1)$ (the base case) is true.

Assume that for some $k \geq 1$, we know that $P(k)$ is true; that is, that

$$2 + 5 + \cdots + (3k - 1) = \frac{k(3k + 1)}{2}. \tag{2}$$

We wish to show that $P(k) \to P(k + 1)$. Note that if we set $n = k + 1$ in (1), we obtain

$$2 + 5 + \cdots + (3k + 2) = \frac{(k + 1)(3k + 4)}{2}. \tag{3}$$

Thus, we need to show that (3) can be obtained from (2). To see this, note that if we add $3k + 2 = 3(k + 1) - 1$ to both sides of (2), then we obtained

$$\begin{aligned}
2 + 5 + \cdots + (3k - 1) + (3k + 2) &= \frac{k(3k + 1)}{2} + (3k + 2) \\
&= \frac{3k^2 + k + 2(3k + 2)}{2} \\
&= \frac{3k^2 + 7k + 4}{2} \\
&= \frac{(k + 1)(3k + 4)}{2},
\end{aligned}$$

which is what we needed to show. Thus, since $P(1)$ is true and $P(k) \to P(k + 1)$ for all $k \geq 1$, it follows that $P(n)$ is true for all $n \in \mathbb{N}$, by induction.

Total: 10 points