Practice for Quiz 8
Math 2580
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If you can answer the following problems, you should be well-prepared for Quiz 8:

1. Find the critical points of the following functions:

   (a) $f(x, y) = x^2y - xy^2$

   We have $\nabla f(x, y) = \langle 2xy - y^2, x^2 - 2xy\rangle$, and critical points occur when $\nabla f(x, y) = \langle 0, 0\rangle$. This gives us the equations $2xy - y^2 = 0$ and $x^2 - 2xy = 0$. The first equation can be written as $y(2x - y) = 0$, so we need to have either $y = 0$ or $y = 2x$. If $y = 0$, the second equation forces us to have $x = 0$, so $(0, 0)$ is a critical point. Note that we could similar show that if $x = 0$, then $y = 0$, so $y = 0$ if and only if $x = 0$. Thus, if we assume $y \neq 0$, then $x \neq 0$ as well, letting us divide the second equation by x, leaving us with $x - 2y = 0$. We now have the pair of equations $y = 2x$ and $x = 2y$, but these can hold only if $x = y = 0$. It follows that $(0, 0)$ is the only critical point.

   (b) $f(x, y) = x^2 + y^2 - 2xy$

   We have $\nabla f(x, y) = \langle 2x - 2y, 2y - 2x\rangle$, and setting $\nabla f(x, y) = \langle 0, 0\rangle$ produces the single condition $x = y$. (The two equations that result are the same equation.) It follows that any point along the line $y = x$ is a critical point.

   (c) $f(x, y) = 2(x^2 + y^2)e^{-x^2-y^2}$

   Here we find

   $$\begin{aligned}\nabla f(x, y) &= \langle 4xe^{-x^2-y^2} - 4x(x^2 + y^2)e^{-x^2-y^2}, 4ye^{-x^2-y^2} - 4y(x^2 + y^2)e^{-x^2-y^2}\rangle\\ &= 4e^{-x^2-y^2}\langle x(1 - (x^2 + y^2)), y(1 - (x^2 + y^2))\rangle\\ &= 4e^{-x^2-y^2}(1 - (x^2 + y^2))\langle x, y\rangle.\end{aligned}$$

   Thus, we see that there are two ways to have $\nabla f(x, y) = \langle 0, 0\rangle$: either $\langle x, y\rangle = \langle 0, 0\rangle$, giving us the critical point $(0, 0)$, or $x^2 + y^2 = 1$, so that every point on the unit circle is also a critical point.

2. Consider the function $f(x, y) = xy + 5y$, defined on the disc $D = \{(x, y)|x^2 + y^2 \leq 4\}$.

   (a) Find any critical points of f that are contained within D.

   Since $\nabla f(x, y) = \langle y, x + 5\rangle$, the only critical point is when $y = 0$ and $x = -5$, so $(-5, 0)$ is a critical point, but it is not within the disc D.

   (b) Recall that the circle $x^2+y^2 = 4$ can be parameterized using $r(t) = (2\cos t, 2\sin t)$, with $t \in [0, 2\pi]$. Find any critical points of the one-variable function $g(t) = f(2\cos(t), 2\sin(t))$ on the interval $[0, 2\pi]$. (This is a Calc I question.)

   We have $g(t) = 4\cos(t)\sin(t)+10\sin(t)$, so $g'(t) = 4\cos^2(t)-4\sin^2(t)+10\cos(t) = 8\cos^2(t) + 10\cos(t) - 4$. (In the last step we used the identity $\sin^2(t) = 1 - \cos^(t)$ to get everything in terms of $\cos(t)$.) Setting $g'(t) = 0$ gives us the quadratic equation $8u^2 + 10u - 4 = 0$ in $u = \cos(t)$. This gives us the (admittedly horrible) result

$$\cos(t) = \frac{-5 \pm \sqrt{57}}{8}.$$

   Only the solution $\dfrac{\sqrt{57} - 5}{8}$ lies between -1 and 1, so we have the points $t \in [0, 2\pi]$ such that $\cos(t) = \frac{\sqrt{57}-5}{8}$.

   (c) Using your answers from (a) and (b), determine the absolute maximum and minimum of f on the disc D.

   Using the results above, we check that

$$\sin^2(t) = 1 - \cos^2(t) = \frac{5\sqrt{57} - 9}{32},$$

   so $\sin(t) = \pm\dfrac{\sqrt{5\sqrt{57} - 9}}{4\sqrt{2}}$, giving us two points on the circle $x^2 + y^2 = 4$ (of the form $(2\cos(t), 2\sin(t))$) to check:

$$(x_0, \pm y_0) = \left(\frac{\sqrt{57} - 5}{4}, \frac{\sqrt{5\sqrt{57} - 9}}{2\sqrt{2}}\right) \quad \text{and} \quad \left(\frac{\sqrt{57} - 5}{4}, -\frac{\sqrt{5\sqrt{57} - 9}}{2\sqrt{2}}\right).$$

   We should also check the point $(2, 0)$ corresponding to when $t = 0$ or $t = 2\pi$. We have $f(2, 0) = 0$, and for the two points we found above, we get the values (you might want to check my arithmetic, though)

$$f(x_0, \pm y_0) = \pm\frac{\sqrt{5\sqrt{57} - 9}}{8\sqrt{2}}(\sqrt{57} + 15).$$

   The negative value will be the absolute minimum, and the positive value will be the absolute maximum.
   Note: This was a textbook question. I didn't realize the numbers would be quite so... gross.

3. In the diagram on below, I've plotted several level curves for the function $f(x, y) = x^2y - xy^2$, along with the parabola $y = x^2$. The marked point (a, b) is the intersection of the curve $x^2y - xy^2 = 1$ (in yellow), with the parabola $y = x^2$. Suppose we want to find the maximum value of $f(x, y)$ subject to the constraint $y = x^2$.

   (a) Explain why the maximum cannot occur at the point (a, b).

   We can see that there are points on the constraint curve $y = x^2$ on either side of the point (a, b) that intersect a level curve of the form $x^2y - xy^2 = c$. The value of c one one side of (a, b) is going to be greater than the value of c for the curve passing through (a, b), while the value of c on the other side will be higher. Thus, c is neither a maximum nor a minimum value of $f(x, y)$ subject to the constraint $y = x^2$.

   (b) Indicate a point on the graph where the maximum *might* occur.

   One possibility is the point (c, d) as shown, where one of the level curves appears to be tangent to the parabola.