University of California, Berkeley

FINAL EXAMINATION, Spring 2013
DURATION: 3 hours

Department of Mathematics

MATH 53 LEC 002 Multivariable Calculus

Examiner: Sean Fitzpatrick

Family Name: ______________________________
(Please Print)

Given Name(s): ______________________________
(Please Print)

Please sign here: ______________________________

Student ID Number: ______________________________

Discussion section: ______________________________

Name of GSI: ______________________________

No aids, electronic or otherwise, are permitted, with the exception of the formula sheet provided with your exam. Partial credit will be given for partially correct work. Please read through the entire exam before beginning, and take note of how many points each question is worth.

Good Luck!

FOR GRADER'S USE ONLY	
Problem 1:	/12
Problem 2:	/14
Problem 3:	/12
Problem 4:	/12
Problem 5:	/12
Problem 6:	/14
Problem 7:	/12
Problem 8:	/12
TOTAL:	/100

1. For parts (a)-(c), let $f(x, y, z) = x^3 \sin(yz)$.

   (a) Compute the gradient of f. [3]

   (b) Compute the directional derivative of f at the point $(2, 1, 0)$, in the direction of $\mathbf{v} = \langle -2, 1, 2 \rangle$. [3]

   (c) Compute $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ if $x = u^2$, $y = v^2$, and $z = u + v$. Your answer should be entirely in terms of u and v but does not have to be simplified. [6]

2. (a) What is the geometric meaning of the Lagrange multiplier equations $\nabla f(x, y) = \lambda \nabla g(x, y)$? [3]

   (b) Find the maximum and minimum of $f(x, y) = x^2 - y^2$ subject to the constraint $x^2/9 + y^2/4 = 1$. [7]

   (c) Show how your answer from (b) illustrates part (a) by sketching the constraint curve from (b), along with a contour plot of f that includes the level curves corresponding to the maximum and minimum values. [4]

3. Evaluate the following double integrals by either reversing the order of integration, or converting to polar coordinates.

   (a) $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx$ [6]

   (b) $\displaystyle\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy$ [6]

4. (a) Find the equation of the tangent line to the curve of intersection of the paraboloid $x = y^2 + z^2$ with the ellipsoid $x^2 + 4y^2 + z^2 = 9$ at the point $(2, 1, 1)$. [6]

   Hint: You don't need to find the curve itself in order to determine the tangent line.

   (b) Let C denote the set of points in the intersection of two smooth level surfaces $f(x, y, z) = c$ and $g(x, y, z) = d$. In general, C may not be a smooth curve. [6]

   What condition on f and g (or the corresponding surfaces) will guarantee that C is a smooth curve?

5. Evaluate the integral $\iint_D xy\,dA$, where D is the region in the first quadrant bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$, and the hyperbolas $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$. [12]

   Hint: Use an appropriate change of variables. You might find it especially convenient in this problem to use the fact that the Jacobians for a transformation and its inverse are related by $J_T(u,v) = \dfrac{1}{J_{T^{-1}}(x(u,v), y(u,v))}$.

6. Let $f(x, y, z)$ be a continuously differentiable function, and let $\mathbf{F}(x, y, z)$ be a continuously differentiable vector field.

   (a) Show that $\nabla(f^n) = nf^{n-1}\nabla f$. (Here f^n denotes f raised to the power of n.) [5]

   (b) Show that $\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}$. [5]

   (c) Show that $\nabla \cdot (\rho^n \mathbf{r}) = (n+3)\rho^n$, where $\mathbf{r}(x, y, z) = \langle x, y, z\rangle$ and $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2} = \|\mathbf{r}(x, y, z)\|$. [4]

7. Let E be the region in $\mathbb{R}^3$ bounded by the sphere S given by $x^2 + y^2 + z^2 = R^2$, and let $\mathbf{F}$ be the vector field defined in problem 6(c), for $n \geq 0$. Verify the Divergence Theorem by computing both $\iint_S \mathbf{F} \cdot d\mathbf{S}$ and $\iiint_E (\nabla \cdot \mathbf{F})\, dV$ and confirming that they're equal. [12]

8. A hot air balloon known as the TARDIS (for Tethered Aerial Release Developed In Style) has the shape of the surface S given by the part of ellipsoid $2x^2 + 2y^2 + z^2 = 9$ with $-1 \leq z \leq 3$. The hot gases that the balloon uses to fly have a velocity vector field given by $\mathbf{v} = \nabla \times \mathbf{F}$, where $\mathbf{F}(x, y, z) = \langle -y, x, xy + z^2 \rangle$. The rate at which the gases escape from the balloon is equal to the flux of $\mathbf{v}$ across the surface of the balloon, given by $\displaystyle\iint_S \mathbf{v} \cdot d\mathbf{S}$.

   Sketch the surface*, and then use Stokes' Theorem to calculate the rate at which the gases escape from the balloon. [12]

*If you can't figure out how to do the problem, I'll award 3/12 for a correct sketch of the surface S, together with the basket underneath and at least one person inside.