Term Test 2 Review

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The second term test for Math 3410 takes place on Friday, March 20th. The test covers sections 3.C, 3.D, and Chapter 5. (In other words, the material on the matrix of a linear map, invertible linear maps and isomorphisms, plus everything on eigenvalues and eigenvectors. In terms of format, you can expect something along the following lines:

- Short answer: This time the short answer questions will be *really* short answers – basically, quick tests of factual knowledge.
- Definitions: I **will** ask you for one or more definitions.
- Computational questions: I'll include one computational question, involving finding either the matrix of a linear map, or the eigenvalues and eigenvectors of an operator.
- Theoretical questions: These will be problems similar to some of the suggested homework problems from the textbook. Basically, short proofs involving the definitions and theorems we've covered.
- Choices: I'll probably give you something like four different questions and ask you to do three of them. One of the questions will be computational and the other three will be short proofs, so you can't avoid proofs entirely on this test.

Things you should know

Previous knowledge

The test doesn't directly cover Chapters 1 and 2, or sections 3.A and 3.B, but that doesn't mean you can forget about the material that was covered. In particular, many of the questions from Chapter 5 still involve direct sums, subspaces, linear independence, span, linear transformations, null spaces, and ranges. So you might want to review that stuff.

In particular, I hope that by now, everyone knows that in problems involving linear transformations and a basis (or independence, span, etc.) that the defining property

$$T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1Tv_1 + c_2Tv_2 + \cdots + c_nTv_n$$

of a linear transformation might come in handy.

Definitions

You should know the definitions of the following:

- The matrix of a linear map $T : V \to W$ with respect to bases B_V of V and B_W of W is defined.
- Linear operators.
- What it means for a linear map to be invertible.
- Invariant subspaces for an operator.
- Powers T^n of an operator.
- Polynomials $p(T)$ of an operator.
- Eigenvalues.
- Eigenvectors.
- Eigenspaces.
- What it means for an operator to be diagonalizable.

Theorems and other results

You should know the following results:

- The fact that the map $\mathcal{M} : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$ (where $\mathbb{F}^{m,n}$ is the space of $m \times n$ matrices with entries in a field $\mathbb{F}$) is **linear**, and that it is an **isomorphism**, and that $\mathcal{M}(T_1T_2) = \mathcal{M}(T_1)\mathcal{M}(T_2)$ for any composable linear maps T_1, and T_2.
- In particular, you should know what the previous result implies about the matrix of $p(T)$ if T is an operator on a finite-dimensional vector space V.
- Inverses are unique.
- A linear map is invertible if and only if it is bijective.
- For linear operators on a finite-dimensional space, either injectivity or surjectivity implies bijectivity.
- (From Math 2000) That if a composition of operators ST is injective, then T is injective, and if ST is surjective, then S is surjective.
- Vector spaces of equal (finite) dimension are isomorphic.
- That a linear map between finite-dimensional vector spaces is invertible if and only if it (bijectively) maps a basis to a basis.
- How to find the polynomial of an operator.

- That polynomials of operators can be factored.
- You should know several conditions equivalent to the statement "λ is an eigenvalue of T".
- You should know how to find the eigenvalues and eigenvectors of an operator. In particular, you should know:
  - How to construct the matrix of an operator T with respect to the standard basis of a vector space V.
  - How to find the eigenvalues and eigenvectors of that matrix.
  - That the eigenvalues of $\mathcal{M}(T)$ are equal to the eigenvalues of T.
  - How to convert the eigenvectors of $\mathcal{M}(T)$ (in column-vector form) into the eigenvectors of T.
  - If T is diagonalizable, how to construct the change of basis matrix P (whose columns are the eigenvectors of $\mathcal{M}(T)$.
  - That $P^{-1}\mathcal{M}(T)P$ will be a diagonal matrix, and that the diagonal entries are the eigenvalues of T.
  - That if T cannot be diagonalized, it can still be put into upper-triangular form, with eigenvalues on the main diagonal.
- What the invertibility (or not) of T tells you about its eigenvalues.
- That eigenvectors corresponding to distinct eigenvalues are linearly independent.
- That, as a result of the above, there are at most $\dim V$ distinct eigenvalues for any operator on T.
- And, as a result, if $\lambda_1 \neq \lambda_2$, then $E(\lambda_1, T) \cap E(\lambda_2, T) = \{0\}$.
- Also, as a result, that the sum of the dimensions of eigenspaces for distinct eigenvalues must be less than the dimension of V.
- Finally, you should know several different conditions equivalent to the statement "T is diagonalizable."