University of California, Berkeley
Department of Mathematics
5^{th} October, 2012, 12:10-12:55 pm
MATH 53 - Test #1

Last Name:____

First Name:____

Discussion Section: ____________________________________

Name of GSI: ______________________________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

Page	Grade
1	/12
2	/12
3	/12
Total	/36

B

[4] 1. Find the equation of the tangent line to the curve C represented by the vector-valued function $\mathbf{r}(t) = \langle 4-3t, e^{t^2}, \ln(1+t)\rangle$ at the point $(4,1,0)$.

The point $(4,1,0)$ corresponds to $t=0$, and we have $\mathbf{r}'(t) = \langle -3, 2te^{t^2}, 1/(1+t)\rangle$, so the tangent vector to the curve at $(4,1,0)$ is $\mathbf{r}'(0) = \langle -3, 0, 1\rangle$. The equation of the tangent line is thus

$$\langle x,y,z\rangle = \langle 4,1,0\rangle + t\langle -3,0,1\rangle.$$

[5] 2. Find the area of the circle $r = 4\cos\theta$ using an integral in polar coordinates.

The given circle lies in the first and fourth quadrants, so we have $-\pi/2 \leq \theta \leq \pi/2$, and

$$\begin{aligned} A &= \int_{-\pi/2} \pi/2 \frac{1}{2}\left(16\cos^2\theta\right) d\theta \\ &= 4\int_{-\pi/2}^{\pi/2} (1+\cos 2\theta))\, d\theta \\ &= 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} \\ &= 4\pi. \end{aligned}$$

[3] 3. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Let $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ If we let $(x,y)\to(0,0)$ along the x-axis, we have $y=0$ and $f(x,0) = \frac{x^2}{x^2} = 1$, and so f appears to be approaching a limit of 1, while if we let $(x,y)\to(0,0)$ along the y-axis, where $x=0$, we have $f(0,y) = \dfrac{-y^2}{y^2} = -1$, and so f appears to be approaching a limit of $-1 \neq 1$. Thus, the limit cannot exist, since f approaches two different values along two different paths.

4. Consider the two lines in $\mathbb{R}^3$ given by

$$\mathbf{r}_1(t) = \langle 3, 2, 3\rangle + t\langle 3, 0, 2\rangle$$
$$\mathbf{r}_2(s) = \langle 0, 1, 2\rangle + s\langle 0, 1, -1\rangle.$$

[2] (a) Verify that the two lines intersect at the point $(0, 2, 1)$.

By direct computation we see that $\mathbf{r}_1(-1) = \langle 0, 2, 1\rangle$ and $\mathbf{r}_2(1) = \langle 0, 2, 1\rangle$, so the two lines indeed intersect at $(0, 2, 1)$.

[3] (b) Find the cosine of the angle between the two lines.

The direction vectors of the two lines are $\mathbf{v}_1 = \langle 3, 0, 2\rangle$ and $\mathbf{v}_2 = \langle 0, 1, -1\rangle$, and thus the angle between the two lines at their point of intersection is given by

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} = \frac{-2}{\sqrt{13}\sqrt{2}}.$$

[4] (c) Find the equation of the plane that contains the two lines.

We know that a point on the plane is $(0, 2, 1)$, and a normal vector to the plane is given by

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 0(-1) - 2(1), 2(0) - 3(-1), 3(1) - 0(0)\rangle = \langle -2, 3, 3\rangle.$$

The equation of the line is thus given by $-2x + 3(y-2) + 3(z-1) = 0$, or $-2x + 3y + 3z - 9 = 0$.

[3] (d) Find the distance between the point $P(1, -1, 2)$ and the plane from part (c).

The distance from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by $D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$, and thus,

$$D = \frac{|-2(1) + 3(-1) + 3(2) - 9|}{\sqrt{4 + 9 + 9}} = \frac{8}{\sqrt{22}}.$$

5. (a) Find the equation of the tangent plane to the surface $z = \sin(3+x^2-y^2)$ at the point
[5] $(1, 2, 0)$.

Letting $f(x,y) = \sin(3+x^2-y^2)$ we have $f_x(x,y) = 2x\cos(3+x^2-y^2)$, and $f_y(x,y) = -2y\cos(3+x^2-y^2)$. Thus, we have $f_x(1,2) = 2\cos(0) = 2$ and $f_y(1,2) = -4\cos(0) = -4$. Since $f(1,2) = 0$, the equation of the tangent plane is given by

$$z = 2(x-1) - 4(y-2).$$

(b) Show that the surface from part (a) has the horizontal tangent plane $z = 1$ at every
[2] point on the hyperbola $y^2 - x^2 = 3 - \dfrac{\pi}{2}$.

Whenever $x^2 - y^2 = \dfrac{\pi}{2} - 3$, we have $f(x,y) = \sin(\pi/2) = 1$ and $f_x(x,y) = f_y(x,y) = 0$, since $\cos(\pi/2) = 0$. The tangent plane at any such point is thus given by $z = 1+0+0 = 1$, which is clearly horizontal.

6. Use the chain rule to compute $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ if $f(x,y,z) = x^3yz^2$, where $x = 2u+v$, $y = u-3v$,
[5] and $z = uv$.

We have

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\\ &= 3x^2yz^2(2) + x^3z^2(1) + 2x^3yz(v)\\ &= 6(2u+v)^2(u-3v)u^2v^2 + (2u+v)^3u^2v^2 + 2(2u+v)^3(u-3v)uv^2,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}\\ &= 3x^2yz^2(1) + x^3z^2(-3) + 2x^3yz(u)\\ &= 3(2u+v)^2(u-3v)u^2v^2 - 3(2u+v)^3u^2v^2 + 2(2u+v)^3(u-3v)u^2v.\end{aligned}$$