Math 1560 Assignment #2 Solutions
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1. Consider the function
$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

   Show that f is differentiable at $x = 0$, and find $f'(0)$.

   Solution: By definition of the derivative, we have
$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^3 \sin(\frac{1}{h}) - 0}{h} && \text{(Note that } f(0) = 0\text{)} \\ &= \lim_{h\to 0} h^2 \sin\left(\frac{1}{h}\right). && \text{(Since } \tfrac{h^3}{h} = h^2 \text{ for } h \neq 0\text{)} \end{aligned}$$

   Since the range of the sine function is $[-1, 1]$, we have
$$-1 \leq \sin\left(\frac{1}{h}\right) \leq 1$$

   for any $h \neq 0$. Since $h^2 > 0$ for all $h \neq 0$, we can multiply across the inequality, giving us
$$-h^2 \leq h^2 \sin\left(\frac{1}{h}\right) \leq h^2.$$

   Since $\lim_{h\to 0}(-h^2) = \lim_{h\to 0}(h^2) = 0$, it follows from the Squeeze Theorem that
$$f'(0) = \lim_{h\to 0} h^2 \sin\left(\frac{1}{h}\right) = 0.$$

2. Let f and g be differentiable functions, and let $h(x) = f(x)g(x)$. We know from the product rule that
$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

(a) Compute $h''(x)$ (in terms of f and g and their derivatives) and simplify.

Solution: To compute $h''(x)$ we take the derivative of $h'(x)$:

$$\begin{aligned} h''(x) &= \frac{d}{dx}(h'(x)) = \frac{d}{dx}(f'(x)g(x) + f(x)g'(x)) \\ &= (f''(x)g(x) + f'(x)g'(x)) + (f'(x)g'(x) + f(x)g''(x)) \\ &= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x). \end{aligned}$$

(b) Compute $h'''(x)$ (in terms of f and g and their derivatives) and simplify.

Solution: To compute $h'''(x)$, we take the derivative of our result for $h''(x)$ above:

$$\begin{aligned} h'''(x) &= \frac{d}{dx}(h''(x)) = \frac{d}{dx}(f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)) \\ &= (f'''(x)g(x) + f''(x)g'(x)) + 2(f''(x)g'(x) + f'(x)g''(x)) \\ &\qquad + (f'(x)g''(x) + f(x)g'''(x)) \\ &= f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x). \end{aligned}$$

(c) (Do not submit an answer to this part) Can you guess a general product rule formula for the n^{th} derivative of $h(x)$?

Solution: (In case you were curious, but not curious enough to look it up) If you know the binomial formula, you might notice a familiar pattern:

$$\begin{aligned} (a+b)^1 &= a + b \\ (a+b)^2 &= a^2 + 2ab + b^2 \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ &\vdots \quad \vdots \end{aligned}$$

$$\begin{aligned} (fg)'(x) &= f'(x)g(x) + f(x)g'(x) \\ (fg)''(x) &= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) \\ (fg)'''(x) &= f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x) \\ &\vdots \quad \vdots \end{aligned}$$

Notice how the coefficients are the same in each case, for $n = 1, 2, 3$: the binomial coefficients $\binom{n}{k}$ appearing in Pascal's Triangle. It turns out that this pattern does indeed continue:

$$(a+b)^n = a^n + na^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k,$$

and

$$
\begin{aligned}
(fg)^{(n)}(x) = f^{(n)}(x)g(x) + nf^{(n-1)}(x)g(x) + \binom{n}{k} f^{(n-2}(x)g''(x) + \cdots \\
+ nf'(x)g^{(n-1)}(x) + f(x)g^{(n)}(x) \\
= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x),
\end{aligned}
$$

where $f^{(n)}(x)$ denotes the n^{th} derivative of f (and $f^{(0)}(x) = f(x)$).
This result is often known as Leibniz's Rule or Leibniz's Identity.

3. Two curves are said to be *orthogonal* if, at each point of intersection, they meet at a right angle. Show that the ellipse $3x^2 + 2y^2 = 5$ and the curve $y^3 = x^2$ are orthogonal.

   Hint: The curves intersect at the points $(1, 1)$ and $(-1, 1)$.

   Solution: For the ellipse, we find

$$
\frac{d}{dx}(3x^2 + 2y^2) = \frac{d}{dx}(5)
$$
$$
6x + 4y\frac{dy}{dx} = 0
$$

   using implicit differentiation. Solving for $\frac{dy}{dx}$, we find $\frac{dy}{dx} = -\frac{3x}{2y}$.

   For the second curve, implicit differentiation gives us

$$
3y^2\frac{dy}{dx} = 2x, \text{ so } \frac{dy}{dx} = \frac{2x}{3y^2}.
$$

   According to the hint, we need to check two points of intersection. At $(1, 1)$, we put $x = 1, y = 1$ into the expression for $\frac{dy}{dx}$ for each curve. For the ellipse,

$$
m_1 = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = -\frac{3(1)}{2(1)} = -\frac{3}{2}.
$$

   For the second curve,

$$
m_2 = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \frac{2(1)}{3(1)^2} = \frac{2}{3}.
$$

   Since $m_1 m_2 = -1$, the two curves intersect orthogonally at $(1, 1)$. At the point of intersection $(-1, 1)$, we put $x = -1$ and $y = 1$, giving us

$$
m_1 = -\frac{3(-1)}{2(1)} = \frac{3}{2} \text{ and } m_2 = \frac{2(-1)}{3(1)^2} = -\frac{2}{3},
$$

   and again we see that $m_1 m_2 = -1$, as required.