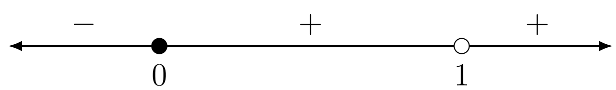
1. Sketch the graph of $f(x) = \dfrac{x}{x^2 - 2x + 1}$. Your solution should include the following:

   - The domain of f, and all intercepts and asymptotes.
   - The critical points of f, including the location of any local maxima or minima.
   - The intervals on which f is increasing or decreasing.
   - The inflection points of f, and the intervals on which f is concave up or concave down.

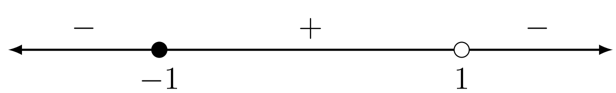
Since $f(x) = \frac{x}{(x-1)^2}$, we see that there is a vertical asymptote at $x = 1$, and $f(0) = 0$ is the only x-intercept (which happens to also be the y-intercept). The sign diagram for f is given by



The first derivative of f is

$$f'(x) = \frac{(1)(x-1)^2 - x(2)(x-1)}{((x-1)^2)^2} = \frac{(x-1)((x-1)-2x)}{(x-1)^4} = \frac{-(x+1)}{(x-1)^3}.$$

We see that $f'(-1) = 0$, so $x = -1$ is the only critical point. (Note $x = 1$ is not in the domain of f. The sign diagram for f' is given by

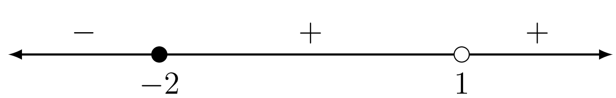


From the sign diagram, we see that $(-1, f(-1)) = (-1, -1/4)$ is a local minimum, and that f is increasing on $(-1, 1)$, and decreasing on $(-\infty, -1) \cup (1, \infty)$.

The second derivative of f is

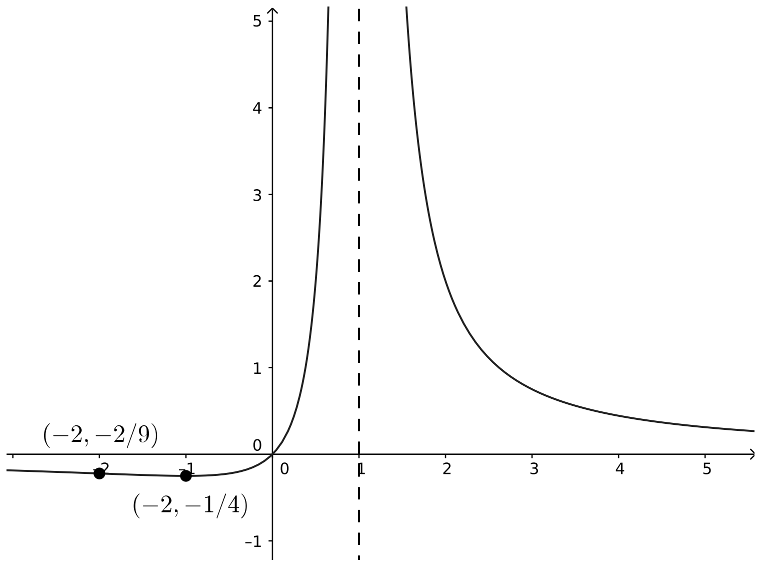
$$f''(x) = \frac{d}{dx}\left(\frac{-(x+1)}{(x-1)^3}\right) = -\frac{(x-1)^3 - (x+1)(3(x-1)^2)}{(x-1)^6} = -\frac{(x-1) - 3(x+1)}{(x-1)^4} = \frac{2(x+2)}{(x-1)^4}.$$

We see that $f''(-2) = 0$, and this is the only possible inflection point. The sign diagram for f'' is given by



We see that $(-2, f(-2)) = (-2, -2/9)$ is an inflection point, and that the graph of f is concave up on $(-2, 1) \cup (1, \infty)$, and concave down on $(-\infty, -2)$.

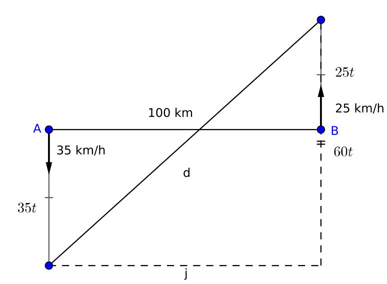
The graph of f is given below:



2. At noon, ship A ('The Valiant') is 100 km west of ship B ('Excelsior'). Ship A is sailing south at 35 km/h, and ship B is steaming north at 25 km/h. How fast is the distance between the ships changing at 4:00 pm?

A diagram of the situation is shown to the right. After t hours, Ship A will be $35t$ km south of its original position, and Ship B will be $25t$ km north of its original position. The distance z between the ships is the length of the hypotenuse of the large right-angled triangle with base length 100 km, and height given by $35t + 25t = 60t$. The distance between the ships therefore satisfies the equation

$$z^2 = 100^2 + (60t)^2.$$



Differentiating both sides of the equation $z^2 = 10000 + 3600t^2$ with respect to t, we obtain:

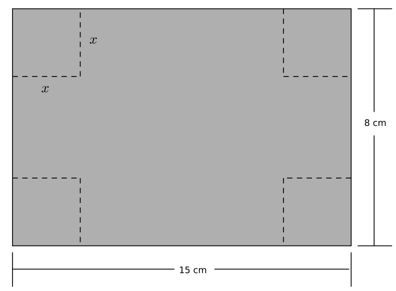
$$2z\frac{dz}{dt} = 7200t,$$

so $\frac{dz}{dt} = \frac{3600t}{z}$. At 4:00 pm, we have $t = 4$, and $z^2 = 10000 + (3600)(4^2) = 67600 = 260^2$. Thus, we find

$$\frac{dz}{dt} = \frac{3600(4)}{260} \approx 55.4 \text{ km/h}.$$

3. An 8-foot-tall fence runs parallel to a tall building, at a distance of 4 feet from the building. What is the length of the shortest ladder that will reach from the ground, over the fence, to the wall of the building?

The diagram to the right shows our piece of cardboard, and the squares to be removed. The volume of our box is given by $V = l{\cdot}w{\cdot}h$, where l is the length, w is the width, and h is the height. We immediately see that $h = x$, while the length and width are each obtained by subtracting $2x$ (the amount being cut out) from the original length and width of the cardboard. Therefore, we have $l = 15 - 2x$, and $w = 8 - 2x$.



The volume of the box is therefore

$$V(x) = x(15 - 2x)(8 - 2x) = 120x - 46x^2 + 4x^3.$$

We note that we must have $0 \leq x \leq 4$, since x cannot be negative (it's a length), and it cannot be more than half the width. As we might expect, $V(0) = 0$ and $V(4) = 0$, so the maximum volume will be obtained at some critical point, where $V'(x) = 0$. We find

$$V'(x) = 120 - 92x + 12x^2,$$

and with the help of a calculator, we find that $V'(x) = 0$ for $x = 6$ or $x = 5/3$. Since $x = 6$ is outside the domain for V, we must have $x = 5/3$. We can quickly check that $V''(x) = 24x - 92$, so $V''(5/3) = 40 - 92 = -52 < 0$, telling us that the critical point at $x = 5/3$ is, indeed, a maximum.

Thus, we should remove squares of length $5/3$ cm from each corner.