*University of Lethbridge*
Department of Mathematics and Computer Science
18th December, 2015, 9:00 am - 12:00 pm
**Math 2000 A&B - FINAL EXAM**

Last Name:________________________________________

First Name:________________________________________

Student Number:____________________________________

Record your answers below each question in the space provided. **Left-hand pages may be used as scrap paper for rough work.** If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

No external aids are permitted.

For grader's use only:

| Page | Grade |
|---|---|
| 2 | /10 |
| 3 | /9 |
| 4 | /9 |
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| 6 | /10 |
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| 8 | /10 |
| 9 | /10 |
| 10 | /10 |
| 11 | /11 |
| Total | /100 |

1. Give a precise definition of each of the terms written in **bold** below.

[2] (a) The **Cartesian product** of two sets $A$ and $B$.

[2] (b) The **range** of a function $f : A \to B$.

[2] (c) The **pre-image** of a subset $Y \subseteq B$ with respect to a function $f : A \to B$.

[2] (d) The **equivalence class** of an object $a \in A$, with respect to an equivalence relation $\sim$ on $A$.

[2] (e) What it means for a set $A$ to be **finite**.

2. Let $A = \{1, 3, 4, 6, 8, 9\}$, $B = \{2, 3, 5, 6, 7, 9\}$, and $C = \{2, 3, 5, 8\}$ be subsets of $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

[1] (a) Calculate $A \cup B$.

[1] (b) Calculate $A \cap B$.

[1] (c) Calculate $A \setminus B$.

[3] (d) Verify that $(A \cap B)^c = A^c \cup B^c$.

[3] (e) Verify that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

3. Let $A, B$, and $C$ be subsets of some universal set $U$. Prove or disprove each of the following:

[3] (a) $A \cap B \subseteq A$.

[3] (b) If $A \cup C \subseteq B \cup C$, then $A \subseteq B$.

[3] (c) If $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

4. Let $A$, $B$, and $C$ be sets.

[3] (a) Show that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

[4] (b) Suppose that $A \cup C = B \cup C$ **and** $A \cap C = B \cap C$. Prove that $A = B$.
**Hint:** It's enough to prove that $A \subseteq B$. The proof that $B \subseteq A$ is identical.

[4] (c) Prove that if $X \subseteq A$, then $X \times B \subseteq A \times B$.

5. Let $A = \{1, 2, 3, 4, 5\}$ and let $B = \{s, t, u, v\}$, and define $f : A \to B$ by

$$f(1) = u,\ f(2) = s,\ f(3) = u,\ f(4) = t,\ f(5) = v.$$

[2] (a) Draw an arrow diagram that represents the function $f$.

[2] (b) Is the function $f$ an injection? Why or why not?

[2] (c) Is the function $f$ a surjection? Why or why not?

6. Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, and define $g : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $g(x) = 3x + 2 \pmod 5$.
[4] Verify that $g$ is a bijection, and determine the function $g^{-1}$.
**Note:** You don't need to find a formula for $g^{-1}$ but it'll be extra nifty if you do.

7. Let $f : A \to B$ and $g : B \to C$ be functions such that $g \circ f : A \to C$ is a surjection.

[3] (a) Either prove that $g$ must be a surjection, or give an example where it is not.

[3] (b) Either prove that $f$ must be a surjection, or give an example where it is not.

8. Prove that if $f : A \to B$ and $g : B \to C$ are both injections, then $g \circ f : A \to C$ is an
[4] injection.

9. Let $A = \{1, 2, 3, 4\}$, and consider the following relations on $A$:

$$\begin{aligned} R &= \{(1,1),(2,4),(3,3),(4,1)\} \\ S &= \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\} \\ T &= \{(1,2),(1,3),(2,2),(2,3),(3,2),(3,3),(4,4)\} \end{aligned}$$

[3] (a) Which of the above relations define a function? Which do not? Justify your answer.

[3] (b) Which of the above relations are reflexive? Which are not? Justify your answer.

[3] (c) Which of the above relations are symmetric? Which are not? Justify your answer.

[1] (d) Which one is an equivalence relation? (No justification needed.)

10. Let $A = \{1, 2, 3, 4, 5\}$. Draw a directed graph that represents an equivalence relation on $A$ with two distinct equivalence classes. [4]

11. For any $m \in \mathbb{Z}$, let $[m] = \{n \in \mathbb{Z} \,|\, n \equiv m \pmod 3\}$ denote the congruence class of $m$ modulo 3. Use mathematical induction to prove that $[10^n] = [1]$ for all natural numbers $n$. [6]

[4] 12. Give an example of two sets $A$ and $B$ such that $A \subseteq B$, and $A \approx B$.
Demonstrate the equivalence by giving an explicit bijection $f : A \to B$.

13. Suppose $A$ and $B$ are disjoint finite sets, with cardinalities $|A| = 3$ and $|B| = 7$.
[3] What are the cardinalities of $A \cup B$ and $A \times B$?

[3] 14. Give three examples of countably infinite sets, and three examples of uncountable sets.

[4] 15. Let $B$ be a countable set. Prove that any subset $A \subseteq B$ is countable.
**Hint:** You need to show there is an injection $f : A \to \mathbb{N}$.

[3] 16. Prove that if $A$ is uncountable and $A \subseteq B$, then $B$ is uncountable.

[4] 17. Let $g : \mathbb{N} \to A$ be a surjection. For each $a \in A$, let $n_a \in \mathbb{N}$ be the least element of the set $g^{-1}(a)$, and define $f : A \to \mathbb{N}$ by $f(a) = n_a$. Show that $g \circ f = I_A$, and explain why this implies that $f$ is an injection.

**Bonus (3 points)**: On the next page, explain why the above definition of $f$ makes sense.

**Extra space for rough work. Do not remove unless there is nothing on this page you want to be graded.**