Practice Problems for Quiz 9
Math 2000A

Quiz #9 will take place in class on Thursday, November 13th. There's no class on the 11th (Remembrance Day) so this quiz covers less material than usual.

As usual, solving the problems on this sheet will significantly improve your chances of getting a high score on the quiz.

Note to help session tutors: It's 100% OK for you to help my students solve these questions.

1. A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists some $T \in \mathbb{R}$, with $T > 0$, such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. The smallest such T is called the *period* of f. Prove that no periodic function can be one-to-one.

2. Let $f : S \to T$ be a function, and let A and B be subsets of S, and C and D subsets of T. For $x \in S$ and $y \in T$, carefully explain what it means to say that

   (a) $y \in f(A \cap B)$
   (b) $y \in f(A \cup B)$
   (c) $y \in f(A) \cap f(B)$
   (d) $y \in f(A) \cup f(B)$
   (e) $x \in f^{-1}(C \cap D)$
   (f) $x \in f^{-1}(C \cup D)$
   (g) $x \in f^{-1}(C) \cap f^{-1}(D)$
   (h) $x \in f^{-1}(C) \cup f^{-1}(D)$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = -2x + 1$. Let

   $$A = [2, 5] \quad B = [-1, 3] \quad C = [-2, 3] \quad D = [1, 4].$$

   Find each of the following:

   (a) $f(A)$
   (b) $f^{-1}(f(A))$
   (c) $f^{-1}(C)$
   (d) $f(f^{-1}(C))$
   (e) $f(A \cap B)$
   (f) $f(A) \cap f(B)$
   (g) $f^{-1}(C \cap D)$
   (h) $f^{-1}(C) \cap f^{-1}(D)$

4. Repeat Question 3 for $f(x) = x^2 + 3$.

5. Let $f : A \to B$ be a given function, and let $g : A \to f(A)$ be defined by $g(x) = f(x)$ for all $x \in A$. Prove that g is onto.

6. Let $f : A \to B$ be a function, and let $C \subseteq A$.

   (a) Prove that $C \subseteq f^{-1}(f(C))$.

   (b) Give an example where C is a *proper* subset of $f^{-1}(f(C))$.

   (c) Prove that if f is one-to-one, then $C = f^{-1}(f(C))$. (This fact might suggest an example for part (b))

7. Let $g : A \to B$ be a function, and let $D \subseteq B$.

   (a) Prove that $f(f^{-1}(D)) \subseteq D$.

   (b) Give an example where $f(f^{-1}(D))$ is a *proper* subset of D.

   (c) Prove that if f is onto, then $f(f^{-1}(D)) = D$.

8. Let $f : A \to B$ and let C and D be subsets of B. Prove the following, or give a counterexample to show it is false:

   (a) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

   (b) If $f^{-1}(C) \subseteq f^{-1}(D)$, then $C \subseteq D$.

9. Let $f : A \to B$ and let U and V be subsets of A. Prove the following, or give a counterexample to show it is false:

   (a) $f(U \cap V) \subseteq f(U) \cap f(V)$

   (b) $f(U) \cap f(V) \subseteq f(U \cap V)$

10. For Question 9, you should have found that (a) was true, but (b) was false. Does it affect your answer if you know that f is one-to-one?