The problems on this worksheet are for in-class practice during tutorial. You are free to collaborate and to ask for help. They don't count for course credit, but it's a good idea to make sure you know how to do everything before you leave tutorial – similar problems may show up on a test or assignment.

1. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^{\sqrt{x}} + C$, using the $u$-subsitution $u = \sqrt{x}$; $du = \dfrac{1}{2\sqrt{x}}\,dx$.

2. $\displaystyle\int \frac{\frac{1}{x}+1}{x^2}\,dx = -\int u\,du = -\frac{u^2}{2} = -\frac{(\frac{1}{x}+1)^2}{2} + C$, using the $u$-substitution $u = \frac{1}{x}+1$; $du = \frac{-1}{x^2}dx$.

3. The substitution for the last integral should have been clear. Note that the numerator can be written as $\frac{x+1}{x}$, so the whole integral can be re-written as $\displaystyle\int \frac{x+1}{x^3}\,dx$. Do you still want to do the integral by substitution, or is there a "better way"? Do your answers agree?

   Dividing term-by-term, we get $\displaystyle\int \frac{x+1}{x^3}\,dx = \int (x^{-2}+x^{-3})\,dx = -x^{-1} - \frac{1}{2}x^{-2} + C$, which is arguably easier. The answers look different, but note that
$$-\frac{1}{2}\left(\frac{1}{x}+1\right)^2 = -\frac{1}{2}\left(\frac{1}{x^2}+\frac{2}{x}+1\right) = -\frac{1}{2}x^{-2} - x^{-1} - \frac{1}{2},$$
   so the two answers are equal up to a constant (which is accounted for by the constant of integration).

4. $\displaystyle\int \tan^2(x)\sec^2(x)\,dx = \int u^2\,du = \frac{\tan^3(x)}{3} + C$, using the substitution $u = \tan(x)$; $du = \sec^2(x)\,dx$.

5. $\displaystyle\int \tan^2(x)\,dx = \int (\sec^2(x)-1)\,dx = \tan(x) - x + C.$

6. $\displaystyle\int_0^1 2x(1-x^2)^4\,dx = -\int_1^0 u^4\,du = \int_0^1 u^4\,du = \left.\frac{u^5}{5}\right|_0^1 = \frac{1}{5}$, using the subsitution $u = 1-x^2$, $du = -2x\,dx$, and noting that if $x=0$, then $u = 1-0^2 = 1$, and if $x=1$, then $u = 1-1^2 = 0$.

7. $\displaystyle\int x^3e^x\,dx$. This integral can be done using integration by parts directly, or by applying a reduction formula similar to the one on your assignment. If we do it directly, we have
$$\begin{aligned}\int x^3e^x\,dx &= x^3e^x - 3\int x^2e^x\,dx && \text{using } u=x^3, du=3x^2\,dx; dv=e^x\,dx, v=e^x\\ &= x^3e^x - 3\left(x^2e^x - 2\int xe^x\,dx\right) && \text{using } u=x^2, du=2x\,dx; dv=e^x\,dx, v=e^x\\ &= x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x\,dx\right) && \text{using } u=x, du=dx; dv=e^x\,dx, v=e^x\\ &= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C.\end{aligned}$$

8. $\int e^{2x}\sin(3x)\,dx$. This integral requires integration by parts twice, and collecting terms after the second step. Taking $u = \sin(3x)$ and $dv = e^{2x}\,dx$, we get

$$\begin{aligned}\int \sin(3x)e^{2x}\,dx &= \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\int \cos(3x)e^{2x}\,dx\\ &= \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\left(\frac{1}{2}e^{2x}\cos(3x) - \frac{3}{2}\int(-\sin(3x))e^{2x}\right)\,dx\\ &= \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x) - \frac{9}{4}\int \sin(3x)e^{2x}\,dx.\end{aligned}$$

Bringing the last integral over to the left-hand side, we have

$$\left(1+\frac{9}{4}\right)\int e^{2x}\sin(3x)\,dx = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x),$$

so dividing by $1+\frac{9}{4} = \frac{13}{4}$ and adding the constant of integration, we find

$$\int e^{2x}\sin(3x)\,dx = e^{2x}\left(\frac{2}{13}\sin(3x) - \frac{3}{13}\cos(3x)\right) + C.$$

9. $\int x\sec^2(x)\,dx = x\tan(x) - \int \tan(x)\,dx = x\tan(x) + \ln(|\cos(x)|) + C$ using integration by parts, with $u = x$ ($du = dx$) and $dv = \sec^2(x)\,dx$, so $v = \tan x$.
(Note that $\int \tan(x)\,dx = \ln(|\sec(x)|) = -\ln(|\cos(x)|)$.)

10. $\int x\sqrt{x-2}\,dx$. (Try this once using substitution, and again using integration by parts.)

If we let $u = x-2$, then $du = dx$ and $x = u+2$, so

$$\int x\sqrt{x-2}\,dx = \int (u+2)\sqrt{u}\,du = \int (u^{3/2} + 2u^{1/2})\,du = \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C.$$

If we use integration by parts with $u = x$ and $dv = \sqrt{x-2}\,dx$, then $du = dx$ and $v = \frac{2}{3}(x-2)^{3/2}$, so

$$\int x\sqrt{x-2}\,dx = \frac{2}{3}x(x-2)^{3/2} - \frac{2}{3}\int (x-2)^{3/2}\,dx = \frac{2}{3}x(x-2)^{3/2} - \frac{2}{3}\left(\frac{2}{5}\right)(x-2)^{5/2} + C.$$

Note that the two answers appear to be different. Are they? (They'd better not be!)

11. $\int e^{\ln x}\,dx$. (With a bit of work you can do this by substituting $u = \ln x$ and noting that $x = e^u$. Why is this a bad idea?)

Substitution is a bad idea here because $e^{\ln x} = x$, and you know how to do $\int x\,dx$.

12.

$$\begin{aligned}\int \sin^5(x)\cos^6(x)\,dx &= \int \sin(x)(1-\cos^2(x))^2\cos^6(x)\,dx\\ &= \int \sin(x)(\cos^6(x) - 2\cos^8(x) + \cos^{10}(x))\,dx,\end{aligned}$$

so letting $u = \cos(x)$, we have $du = \sin(x)\,dx$ and the integral becomes

$$\int (-u^6 + u^8 - u^{10})\,dx = -\frac{u^7}{7} + \frac{u^9}{9} - \frac{u^{11}}{11} + c = -\frac{1}{7}\cos^7(x) + \frac{1}{9}\cos^9(x) - \frac{1}{11}\cos^{11}(x) + C.$$

13. $\displaystyle\int \sin(x)\sin(2x)\,dx = \int \sin(x)(2\sin(x)\cos(x))\,dx = 2\int \sin^2(x)\cos(x)\,dx = \frac{2}{3}\sin^3(x) + C,$ using the $u$-substitution $u = \sin(x)$, $du = \sin(x)\,dx$.

(Could you do it by parts? Maybe. But why would you subject yourself to that when a quick application of a trig identity gives you a much easier integral?)

14. $\displaystyle\int \sec^3(x)\,dx$. This is another one of the "integrate by parts twice and rearrange" exercises. Notice that $\sec^3(x) = \sec^2(x)\sec(x)$, and since we know that $\sec^2(x)$ is the derivative of $\tan(x)$, we try integration by parts with $u = \sec(x)$ (so $du = \sec(x)\tan(x)\,dx$), and $dv = \sec^2(x)\,dx$ (so $v = \tan(x)$). This gives us

$$\begin{aligned}\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \tan(x)(\sec(x)\tan(x))dx\\ &= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)\,dx\\ &= \sec(x)\tan(x) - \int (\sec^2(x) - 1)\sec(x)\,dx\\ &= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx.\end{aligned}$$

At this point, we move the $\int \sec^3(x)\,dx$ from the right-hand side over to the left, giving us $2\int \sec^3(x)\,dx$ on the left. If we divide through by the 2, and remember that $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$, we get

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C.$$

15. $\int \sec^4(x)\,dx$. Not on the worksheet, but I meant to include it. We're raising the secant function to a higher power, which might make you think things will be harder, but for $\sec(x)$, even powers are easy, and odd powers are hard. We have

$$\int \sec^4(x)\,dx = \int (\tan^2(x) + 1)\sec^2(x)\,dx = \int (u^2 + 1)\,du = \frac{1}{3}\tan^3(x) + \tan(x) + C,$$

using the $u$-substitution $u = \tan(x)$.

16. $\int \sec^5(x)\,dx$. Just to drive home the point that odd powers are hard. We start out by writing $\sec^5(x) = \sec^3(x)\sec^2(x)$, and integrate by parts, with $u = \sec^3(x)$ (so $du = 3\sec^2(x)(\sec(x)\tan(x)\,dx = 3\sec^3(x)\tan(x)\,dx)$, and $dv = \sec^2(x)\,dx$ (so $v = \tan(x)$). This gives

$$\begin{aligned}\int \sec^5(x)\,dx &= \tan(x)\sec^3(x) - \int \tan^2(x)\sec^3(x)\,dx\\ &= \tan(x)\sec^3(x) - \int (\sec^2(x) - 1)\sec^3(x)\,dx\\ &= \tan(x)\sec^3(x) - \int \sec^5(x)\,dx + \int \sec^3(x)\,dx.\end{aligned}$$

At this point we see the reappearance of $\int \sec^5(x)\,dx$ on the right-hand side, with a minus sign, so we can move it over to the left, giving $2\int \sec^5(x)\,dx$. If we divide through by 2 and substitute in our answer for $\int \sec^3(x)\,dx$ above, we get

$$\int \sec^5(x)\,dx = \frac{1}{2}\tan(x)\sec^3(x) + \frac{1}{4}\tan(x)\sec(x) + \frac{1}{4}\ln|\tan(x) + \sec(x)| + C.$$

17. $\int \sqrt{9 - x^2}\,dx$. Here we use the trig substitution $x = 3\sin\theta$, which gives us $dx = 3\cos\theta\,d\theta$, and $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$, so $\sqrt{9 - x^2} = 3\cos\theta$. Thus,

$$\begin{aligned}\int \sqrt{9 - x^2}\,dx &= \int 9\cos^2\theta\,d\theta\\ &= \int \frac{9}{2}(1 + \cos(2\theta))\,d\theta\\ &= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C\\ &= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta.\end{aligned}$$

Now, we use the fact that $3\sin\theta = x$, so $\theta = \sin^{-1}(x/3)$, and $3\cos\theta = \sqrt{9 - x^2}$ to substitute back in terms of $x$, giving us

$$\int \sqrt{9 - x^2}\,dx = \frac{9}{2}\sin^{-1}\left(\frac{\theta}{3}\right) + \frac{1}{2}x\sqrt{9 - x^2} + C.$$

18. $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$. There are two options for this integral. We can either let $x = \sqrt{2}\tan\theta$, or $x = \sqrt{2}\sinh(t)$.

Taking the first option, we get $dx = \sqrt{2}\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+2} = \sqrt{2\tan^2\theta+2} = \sqrt{2\sec^2\theta} = \sqrt{2}\sec\theta,$$

so

$$\int \frac{8}{\sqrt{x^2+2}}\,dx = \int \frac{8\sqrt{2}\sec^2\theta}{\sqrt{2}\sec\theta}\,d\theta = \int 8\sec\theta\,d\theta = 8\ln|\sec\theta+\tan\theta| + C.$$

Now, $\tan\theta = x/\sqrt{2}$, and $\sec\theta = \sqrt{x^2+2}/\sqrt{2}$, so this becomes

$$\int \frac{8}{\sqrt{x^2+2}}\,dx = 8\ln\left|\frac{\sqrt{x^2+2}+x}{\sqrt{2}}\right| + C.$$

(Note: using log laws, you can get rid of the $\sqrt{2}$ in the denominator – you'll get a $-\ln\sqrt{2}$ term, which is a constant that can be absorbed into the constant of integration.)

If we take the second option, $x = \sqrt{2}\sinh(t)$, so $dx = \sqrt{2}\cosh(t)$, and

$$\sqrt{x^2+2} = \sqrt{2\sinh^2(t)+2} = \sqrt{2\cosh^2(t)} = \sqrt{2}\cosh(t).$$

Thus,

$$\int \frac{8}{\sqrt{x^2+2}}\,dx = \int \frac{8\sqrt{2}\cosh(t)}{\sqrt{2}}\cosh(t)\,dt = 8t + C = 8\sinh^{-1}(x/\sqrt{2}) + C.$$

**Exercise:** Can you show that the answers given by the two methods are equivalent? In other words, is it true that

$$\ln\left|\frac{\sqrt{x^2+2}+x}{\sqrt{2}}\right| = \sinh^{-1}(x/\sqrt{2}) + C$$

for some constant $C$? (If it is, then the derivative of either side should be equal.)

19. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\,dx$. Again there are two options: a trig substitution and a hyperbolic substitution. The trig substitution is to let $x = \sqrt{10}\sec\theta$, so that $dx = \sqrt{10}\sec\theta\tan\theta\,d\theta$, and

$$\sqrt{x^2-10} = \sqrt{10(\sec^2\theta-1)} = \sqrt{10\tan^2\theta} = \sqrt{10}\tan\theta.$$

Thus,

$$\int \frac{5x^2}{\sqrt{x^2-10}}\,dx = \int \frac{50\sec^2\theta}{\sqrt{10}\tan\theta}(\sqrt{10}\sec\theta\tan\theta)\,d\theta = \int 50\sec^3\theta\,d\theta.$$

Uh oh... the dreaded $\sec^3\theta$ integral. Luckily we have the answer sitting above on this worksheet, so we can plug it in, giving us

$$\int \frac{5x^2}{\sqrt{x^2-10}}\,dx = 25\sec\theta\tan\theta + 25\ln|\sec\theta+\tan\theta| + C,$$

and we note that $\sec\theta = x/\sqrt{10}$ and $\tan\theta = \sqrt{x^2-10}/\sqrt{10}$, so

$$\int \frac{5x^2}{\sqrt{x^2-10}}\,dx = \frac{5}{2}x\sqrt{x^2-10} + 25\ln|(x+\sqrt{x^2-10})/\sqrt{10}| + C.$$

If we use a hyperbolic substitution instead, we take $x = \sqrt{10}\cosh(t)$, so $dx = \sqrt{10}\sinh(t)$, and

$$\sqrt{x^2-10} = \sqrt{10(\cosh^2(t)-1)} = \sqrt{10\sinh^2(t)} = \sqrt{10}\sinh(t).$$

Thus,

$$\int \frac{5x^2}{\sqrt{x^2-10}}\,dx = \int \frac{50\cosh^2(t)}{\sqrt{10}\sinh(t)}(\sqrt{10})\sinh(t)\,dt = \int 50\cosh^2(t)\,dt.$$

Now we have to know how to integrate $\cosh^2(t)$. If we recall how $\cosh(t)$ is defined, we have

$$\cosh^2(t) = \left(\frac{e^t+e^{-t}}{2}\right)^2 = \frac{e^{2t}+e^{-2t}+2}{4}.$$

So you could simply write $\cosh^2(t)$ in terms of exponentials as above, and integrate term-by-term. The other option is to notice that there's an identity sitting there: $\dfrac{e^{2t}+e^{-2t}}{4} = \frac{1}{2}\cosh(2t)$, so

$$\int \cosh^2(t)\,dt = \int \left(\frac{1}{2}\cosh(2t) + \frac{1}{2}\right)\,dt = \frac{1}{4}\sinh(2t) + \frac{t}{2} + C.$$

Finally, we have to substitute back in terms of $x$. Would it surprise you to learn that $\sinh(2t) = 2\sinh(t)\cosh(t)$? Well, that turns out to be true. Since $\sinh(t) = \sqrt{x^2-10}/\sqrt{10}$ and $\cosh(t) = x/\sqrt{10}$, we get

$$\int \cosh^2(t)\,dt = \frac{5}{2}x\sqrt{x^2-10} + 25\cosh^{-1}(x/\sqrt{10}) + C.$$

The last thing you might be wondering is whether the two answers are the same. They certainly look different. It's a good exercise to see if you can show that

$$\ln(x+\sqrt{x^2-10}) = \cosh^{-1}(x/\sqrt{10}) \text{ (up to a constant)}$$

The easiest way to do that is to show that their derivatives are the same.