MATH 2565 - Tutorial #1 Solutions

Additional practice (don't include your solutions here):

1. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

   $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^{\sqrt{x}} + C$, using the u-substitution $u = \sqrt{x}$; $du = \dfrac{1}{2\sqrt{x}}\,dx$.

2. $\displaystyle\int x\sqrt{x-2}\,dx$. (Try this once using substitution, and again using integration by parts.)

   If we let $u = x - 2$, then $du = dx$ and $x = u + 2$, so

   $$\int x\sqrt{x-2}\,dx = \int (u+2)\sqrt{u}\,du = \int (u^{3/2} + 2u^{1/2})\,du = \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C.$$

   If we use integration by parts with $u = x$ and $dv = \sqrt{x-2}\,dx$, then $du = dx$ and $v = \frac{2}{3}(x-2)^{3/2}$, so

   $$\int x\sqrt{x-2}\,dx = \frac{2}{3}x(x-2)^{3/2} - \frac{2}{3}\int (x-2)^{3/2}\,dx = \frac{2}{3}x(x-2)^{3/2} - \frac{2}{3}\left(\frac{2}{5}\right)(x-2)^{5/2} + C.$$

   Note that the two answers appear to be different. Are they? (They'd better not be!)

3. $\displaystyle\int e^{\ln x}\,dx$. (With a bit of work you can do this by substituting $u = \ln x$ and noting that $x = e^u$. Why is this a bad idea?)

   Substitution is a bad idea here because $e^{\ln x} = x$, and you know how to do $\int x\,dx$.

Evaluate the following integrals.

1. $\displaystyle\int_0^1 2x(1-x^2)^4\,dx$

   $\displaystyle\int_0^1 2x(1-x^2)^4\,dx = -\int_1^0 u^4\,du = \int_0^1 u^4\,du = \left.\frac{u^5}{5}\right|_0^1 = \frac{1}{5}$, using the subsitution $u = 1-x^2$, $du = -2x\,dx$, and noting that if $x=0$, then $u = 1-0^2 = 1$, and if $x=1$, then $u = 1-1^2 = 0$.

2. $\displaystyle\int \tan^2(x)\,dx$

   $\displaystyle\int \tan^2(x)\sec^2(x)\,dx = \int u^2\,du = \frac{\tan^3(x)}{3} + C$, using the substitution $u = \tan(x)$; $du = \sec^2(x)\,dx$.

3. $\displaystyle\int x^3 e^x\,dx$

   This integral can be done using integration by parts directly, or by applying a reduction formula. If we do it directly, we have

$$\begin{aligned}
\int x^3e^x\,dx &= x^3e^x - 3\int x^2e^x\,dx && \text{using } u = x^3, du = 3x^2\,dx; dv = e^x\,dx, v = e^x\\
&= x^3e^x - 3\left(x^2e^x - 2\int xe^x\,dx\right) && \text{using } u = x^2, du = 2x\,dx; dv = e^x\,dx, v = e^x\\
&= x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x\,dx\right) && \text{using } u = x, du = dx; dv = e^x\,dx, v = e^x\\
&= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C.
\end{aligned}$$

   As an additional exercise, see if you can come up with a general reduction formula for the integral $\displaystyle\int x^n e^x\,dx$

4. $\displaystyle\int e^{2x}\sin(3x)\,dx$

   This integral requires integration by parts twice, and collecting terms after the second step. Taking $u = \sin(3x)$ and $dv = e^{2x}\,dx$, we get

$$\begin{aligned}
\int \sin(3x)e^{2x}\,dx &= \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\int \cos(3x)e^{2x}\,dx\\
&= \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\left(\frac{1}{2}e^{2x}\cos(3x) - \frac{3}{2}\int(-\sin(3x))e^{2x}\right)dx\\
&= \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x) - \frac{9}{4}\int \sin(3x)e^{2x}\,dx.
\end{aligned}$$

Bringing the last integral over to the left-hand side, we have

$$\left(1+\frac{9}{4}\right)\int e^{2x}\sin(3x)\,dx = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x),$$

so dividing by $1+\frac{9}{4}=\frac{13}{4}$ and adding the constant of integration, we find

$$\int e^{2x}\sin(3x)\,dx = e^{2x}\left(\frac{2}{13}\sin(3x) - \frac{3}{13}\cos(3x)\right) + C.$$

5. $\displaystyle\int \sec^5(x)\,dx$

The integral for $\sec^3(x)$ was done in class, and this one's here just to drive home the point that odd powers are hard. We start out by writing $\sec^5(x) = \sec^3(x)\sec^2(x)$, and integrate by parts, with $u = \sec^3(x)$ (so $du = 3\sec^2(x)(\sec(x)\tan(x)\,dx = 3\sec^3(x)\tan(x)\,dx$), and $dv = \sec^2(x)\,dx$ (so $v = \tan(x)$). This gives

$$\begin{aligned}\int \sec^5(x)\,dx &= \tan(x)\sec^3(x) - \int \tan^2(x)\sec^3(x)\,dx \\ &= \tan(x)\sec^3(x) - \int (\sec^2(x)-1)\sec^3(x)\,dx \\ &= \tan(x)\sec^3(x) - \int \sec^5(x)\,dx + \int \sec^3(x)\,dx.\end{aligned}$$

At this point we see the reappearance of $\int \sec^5(x)\,dx$ on the right-hand side, with a minus sign, so we can move it over to the left, giving $2\int \sec^5(x)\,dx$. If we divide through by 2 and substitute in our answer for $\int \sec^3(x)\,dx$ above, we get

$$\int \sec^5(x)\,dx = \frac{1}{2}\tan(x)\sec^3(x) + \frac{1}{4}\tan(x)\sec(x) + \frac{1}{4}\ln|\tan(x)+\sec(x)| + C.$$

6. $\int \sec^6(x)\,dx$.

Not on the worksheet, but I thought I'd include it to point out that even powers a much, much easier. We're raising the secant function to a higher power, which might make you think things will be harder, but for $\sec(x)$, even powers are easy, and odd powers are hard. Since $\sec^2(x) = \tan^2(x)+1$, we have

$$\begin{aligned}\int \sec^6(x)\,dx &= \int (\tan^2(x)+1)^2\sec^2(x)\,dx = \int (u^2+1)^2\,du \\ &= \int (u^4+2u^2+1)\,du = \frac{1}{5}\tan^5(x) + \frac{2}{3}\tan^3(x) + \tan(x) + C,\end{aligned}$$

using the u-substitution $u = \tan(x)$.