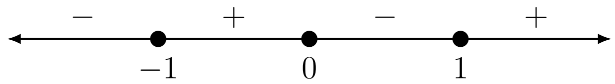
1. For each of the functions below, use the sign diagram of its derivative to find and classify any critical points.

   (a) $f(x) = 2x^4 - 4x^2 + 6$

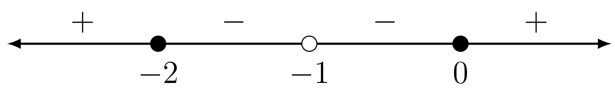
   Since $f'(x) = 8x^3 - 8x = 8x(x^2 - 1) = 8x(x+1)(x-1)$, we get the sign diagram



   Using the first derivative test, we see that f has a local minimum at $x = -1$, a local maximum at $x = 0$, and a local minimum at $x = 1$.

   (b) $g(x) = \dfrac{x^2}{1+x}$

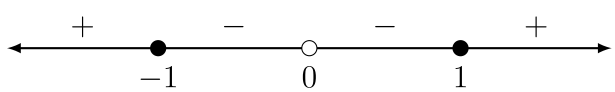
   Since $g'(x) = \dfrac{2x(1+x) - x^2(1)}{(1+x)^2} = \dfrac{x(x+2)}{(x+1)^2}$, we get the sign diagram



   We note that $x = -1$ is not a critical point, since it is not in the domain of f. Since f' changes from positive to negative at $x = -2$, this is a local maximum, and since f' changes from negative to positive at $x = 0$, this is a local minimum.

   (c) $h(x) = x^{7/3} - 7x^{1/3}$

   We find $h'(x) = \frac{7}{3}x^{4/3} - \frac{7}{3}x^{-2/3} = \frac{7}{3}x^{-2/3}(x^2 - 1) = \frac{7}{3x^{2/3}}(x-1)(x+1)$. The sign diagram of h is given by

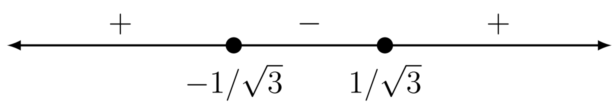


   We note that although $h'(0)$ is undefined, $h(0) = 0$ is defined, so $x = 0$ is a critical point. However, it is neither a maximum nor a minimum, since h' has the same sign on both sides of zero. (The graph of h has a vertical tangent at this point.) There are two other critical points: a local maximum at $x = -1$, and a local minimum at $x = 1$.

2. For each function below, determine the intervals on which it is increasing/decreasing and concave up/concave down.
(Note: these are the same functions as in the previous problem.)

(a) $f(x) = 2x^4 - 4x^2 + 6$

The intervals of increase and decrease can be obtained directly from the sign diagrams in the previous problem. Since f is increasing wherever $f'(x) > 0$, we see that f is increasing on $(-1, 0) \cup (1, \infty)$, and similarly, f is decreasing wherever $f'(x) < 0$, which is on $(-\infty, -1) \cup (0, 1)$.
For concavity, we compute $f''(x) = 24x^2 - 8 = 8(3x^2 - 1) = 8(\sqrt{3}x - 1)(\sqrt{3}x + 1)$, so $f''(x) = 0$ when $x = \pm\dfrac{1}{\sqrt{3}}$. The sign diagram for f'' is



From the sign diagram, we see that f is concave up for $x \in (-\infty, -1/\sqrt{3}) \cup (1/\sqrt{3}, \infty)$, and concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$.

(b) $g(x) = \dfrac{x^2}{1+x}$

From the sign diagram for g', we see that g is increasing on $(-\infty, -2) \cup (0, \infty)$, and decreasing on $(-2, -1,) \cup (-1, 0)$. (Note that we must exclude $x = -1$ from the interval of decrease, since there is a vertical asymptote there.)
The second derivative is given by

$$g''(x) = \frac{d}{dx}\left(\frac{x^2 + 2x}{(x+1)^2}\right) = \frac{(2x+2)(x+1)^2 - (x^2+2x)(2(x+1))}{(x+1)^4} = \frac{2}{(x+1)^3}$$

We see that g'' is undefined at $x = -1$ (the vertical asymptote), and $g''(x) > 0$ (so g is concave up) on $(-1, \infty)$, while $g''(x) < 0$ (so g is concave down) on $(-\infty, -1)$.

(c) $h(x) = x^{7/3} - 7x^{1/3}$

The sign diagram for h' shows that h is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$. (Note that h is defined at $x = 0$, even if h' is not. It's a fairly arbitrary choice as to whether or not to include 0 here.) For concavity, we compute

$$h''(x) = \frac{d}{dx}\left(\frac{7}{3}x^{4/3} - \frac{7}{3}x^{-2/3}\right) = \frac{28}{9}x^{1/3} + \frac{14}{9}x^{-5/3} = \frac{14}{9x^{5/3}}(x^2 + 1).$$

We see that $h''(x)$ is undefined at $x = 0$ and that h'' has no zeros. We can conclude that the graph of h is concave up for $x > 0$, and concave down for $x < 0$.