1. Let $A = \begin{bmatrix} 1 & 5 & -3 \\ 2 & 3 & 4 \\ -2 & -7 & 3 \end{bmatrix}$

(a) Compute $\det A$ using cofactor (Laplace) expansion along the row or column of your choice.

Expanding along the first row, we have

$$\begin{aligned} \det A &= 1(+1)\begin{vmatrix} 3 & 4 \\ -7 & 3 \end{vmatrix} + 5(-1)\begin{vmatrix} 2 & 4 \\ -2 & 3 \end{vmatrix} + (-3)(+1)\begin{vmatrix} 2 & 3 \\ -2 & -7 \end{vmatrix} \\ &= 1(9+28) - 5(6+8) - 3(-14+6) = 37 - 70 + 24 = -9. \end{aligned}$$

(b) Compute $\det A$ by first using row operations to reduce A to triangular form. (Keep in mind that some row operations effect the value of $\det A$.)

We proceed as follows:

$$\begin{aligned} \begin{vmatrix} 1 & 5 & -3 \\ 2 & 3 & 4 \\ -2 & -7 & 3 \end{vmatrix} &= \begin{vmatrix} 1 & 5 & -3 \\ 0 & -7 & 10 \\ 0 & 3 & -3 \end{vmatrix} && \text{(Using } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 + 2R_1\text{)} \\ &= \begin{vmatrix} 1 & 5 & -3 \\ 0 & -1 & 4 \\ 0 & 3 & -3 \end{vmatrix} && \text{(Using } R_2 \to R_2 + 2R_3\text{)} \\ &= \begin{vmatrix} 1 & 5 & -3 \\ 0 & -1 & 4 \\ 0 & 0 & 9 \end{vmatrix} && \text{(Using } R_3 \to R_3 + 3R_2\text{)} \\ &= 1(-1)(9) = -9. \end{aligned}$$

Note that the above used only "Type 3" row operations, which do not affect the determinant. One option you might have tried is to get rid of the factor of 3 in the third row after the first equality. You need to remember that the row operation $R_3 \to \dfrac{1}{3}R_3$ affects the value of the determinant: the determinant of the resulting matrix is one third the determinant of the original matrix. One way to keep track of this is to think of "factoring out" the 3, as follows:

$$\begin{vmatrix} 1 & 5 & -3 \\ 0 & -7 & 10 \\ 0 & 3 & -3 \end{vmatrix} = 3\begin{vmatrix} 1 & 5 & -3 \\ 0 & -7 & 10 \\ 0 & 1 & -1 \end{vmatrix}.$$

Note also that I did one more row operation than necessary in order to avoid fractions. From the right-hand side of the first line, I could have used the row operation $R_3 \to R_3 + \frac{3}{7}R_2$ to get (in one step)

$$\begin{vmatrix} 1 & 5 & -3 \\ 0 & -7 & 10 \\ 0 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -3 \\ 0 & -7 & 10 \\ 0 & 0 & \frac{9}{7} \end{vmatrix} = 1(-7)\left(\frac{9}{7}\right) = -9.$$

(c) Use Cramer's rule to find the value of x in the solution to the following system of equations:

$$\begin{array}{rcrcrcr} x & + & 5y & - & 3z & = & 2 \\ 2x & + & 3y & + & 4z & = & -1 \\ -2x & - & 7y & + & 3z & = & 0 \end{array}$$

Cramer's rule states that $x = \dfrac{\det A_x}{\det A}$, where $A_x = \begin{bmatrix} 2 & 5 & -3 \\ -1 & 3 & 4 \\ 0 & -7 & 3 \end{bmatrix}$ is the matrix obtained by replacing the x column in the coefficient matrix A by the column $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ of constants from the right-hand sides of the equations. Noting that the coefficient matrix A is the same one from part (a) — now that I've fixed the typo — we know that $\det A = -9$. We then compute

$$\det A_x = \begin{vmatrix} 2 & 5 & -3 \\ -1 & 3 & 4 \\ 0 & -7 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 11 & 5 \\ -1 & 3 & 4 \\ 0 & -7 & 3 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 11 & 5 \\ -7 & 3 \end{vmatrix} = 1(33+35) = 68,$$

where we've used the row operation $R_1 \to R_1 + R_2$ to create a zero in the upper left-hand corner, and then expanded along the third column. We thus have $x = -\dfrac{68}{9}$.

2. Let A be a 3×3 matrix such that $\det A = 4$. Compute the determinant of the following matrices:

(a) $B = EA$, where E is the elementary matrix $E = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since EA is obtained from A by multiplying Row 1 by 3, we know that $\det(EA) = 3\det A = 12$.

Alternatively, $\det(EA) = \det E \cdot \det A = 3 \cdot 4 = 12$.

(b) The matrix C obtained by switching rows 2 and 3 of A.

Since swapping two rows changes the sign of the determinant, $\det C = -\det A = -4$.

(c) The matrix $2A$.

Multiplying A by 2 multiplies each of the three rows of A by 2. Since multiplying a single row of A by 2 multiplies the determinant by 2, and we're doing this three times, we have $\det(2A) = 2 \cdot 2 \cdot 2 \cdot \det A = 2^3 \det A = 8 \cdot 4 = 32$.

3. With the help of your classmates, come up with as many answers as possible to fill in the blank below:

   An $n \times n$ matrix A is invertible if and only if ________________________.

   Some possible answers include:

   - $AB = I$ for some $n \times n$ matrix B.
   - $\det A \neq 0$
   - The system of equations $AX = B$ has a unique solution.
   - The rank of A is n.
   - There are no rows of zeros in the row-echelon form of A.
   - A is a product of elementary matrices.

   There are a few more possibilities, but most of them involve language we haven't gotten to yet.

4. In each case, either prove the statement, or give an example showing that it is false:

   (a) $\det(A + B) = \det A + \det B$.

   If we let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $\det A = \det B = 1$, so $\det A + \det B = 2$. On the other hand, $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $\det(A + B) = 0$. Since $2 \neq 0$, this statement is false in general.

   (b) If $\det A = 0$, then A has two equal rows.

   This is false. For example, the rows of $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ are not equal, but $\det A = 2 - 2 = 0$. (The converse is true however: if A has two equal rows, then $\det A = 0$.)

   (c) For any 2×2 matrix A, $\det(A^T) = \det A$.

   This is true. A general 2×2 matrix can be written as $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, in which case $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, and we can easily verify that $\det A = \det A^T = ad - bc$.

   (d) $\det(-A) = -\det A$

   This is false in general. For example, in our example for part (a), we had $B = -A$, but $\det B = \det A$.
   What is true is that $\det(-A) = (-1)^n \det A$, if A is an $n \times n$ matrix, so the result is false when n is even, and true when n is odd.

   (e) If $\det A \neq 0$ and $AB = AC$, then $B = C$.

   This is true. If $\det A \neq 0$, then we know A is invertible, so if $AB = BC$ and $\det A \neq 0$, then we can multiply both sides of this equation on the left by A^{-1} to obtain $B = C$.

5. What can be said about $\det A$ if:

   (a) $A^2 = A$

   Since $A^2 = A$, we know that $\det(A^2) = \det(A)$. Since $\det(AB) = \det A \cdot \det B$ for any $n \times n$ matrices A and B, we know that

   $$\det(A^2) = \det(AA) = \det A \cdot \det A = (\det A)^2.$$

   Thus, if we let $x = \det A$, we must have $x^2 = x$, or $x^2 - x = x(x-1) = 0$, so $\det A = 0$ or $\det A = 1$.

   (b) $A^2 = I$

   Since $\det I = 1$, we have $\det(A^2) = (\det A)^2 = 1$, which implies that $\det A = \pm 1$.

   (c) $PA = P$, where P is invertible.

   If $PA = P$, then $\det(PA) = \det P$. But $\det(PA) = \det P \cdot \det A$, so we have $\det P \cdot \det A = \det P$. Since P is invertible, we know that $\det P \neq 0$, so we can divide both sides of the last equation by $\det P$, giving us $\det A = 1$.