**Name:** **Tutorial time:**

**Problem you want feedback on:**

Please complete all problems below.

1. When you put the symbol "=" between two objects on the page, what are you saying about the relationship between those objects?

   An expression such as $a = b$ tells the reader that the objects $a$ and $b$ are **equal**. If an equal sign appears between two objectst that are not, in fact, equal, then you've written an incorrect statement, and will lose marks as a result. (Even if we "knew what you meant".)

2. Each of the augmented matrices below is in reduced row-echelon form. For each matrix, indicate the following:

   (a) The *rank* of the augmented matrix.

   (b) The number of variables in the corresponding system of equations.

   (c) The number of parameters needed to write down the general solution.

   (d) The general solution to the system, if any.

   i. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

   The rank is 2, and there are 3 variables (let's call them $x, y, z$), so there is $3-2=1$ parameter. The general solution is

$$\begin{aligned} x &= 4 + 2t \\ y &= -5 - 3t \\ z &= t, \end{aligned}$$

   where $t$ can be any real number.

   ii. $\left[\begin{array}{cccc|c} 1 & -3 & 0 & 4 & 2 \\ 0 & 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

   The rank is 2, and there are 4 variables (let's call them $w, x, y, z$), so there are $4-2=2$ parameters. The general solution is

$$\begin{aligned} w &= 2 + 3s - 4t \\ x &= s \\ y &= 7 + 3t \\ z &= t, \end{aligned}$$

   where $s$ and $t$ can be any real numbers.

iii. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$

The rank is 3, and there are 3 variables, but the third leading 1 appears in the constants column. There is therefore no solution.

iv. $\left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

The rank is 2, and there are 4 variables $(w, x, y, z)$, so there are $4 - 2 = 2$ parameters. The general solution is

$$\begin{aligned} w &= s \\ x &= 4 + t \\ y &= 2 \\ z &= t, \end{aligned}$$

where $s$ and $t$ can be any real numbers.

3. Determine the value(s) of $a$ such that the system of equations given by the augmented matrix below has no solution, one solution, or infinitely many solutions, if possible.

$$\left[\begin{array}{ccc|c} a & 1 & 2 & 1 \\ 2 & 1 & 7 & 3 \\ 1 & 1 & 2 & 1 \end{array}\right]$$

If we proceed with the standard Gaussian elimination algorithm, we first swap rows 1 and 3 to get a leading 1 in the upper left-hand corner, and then proceed to create zeros in the first column, as follows:

$$\left[\begin{array}{ccc|c} a & 1 & 2 & 1 \\ 2 & 1 & 7 & 3 \\ 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1 & 7 & 3 \\ a & 1 & 2 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - aR_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1-a & 2-2a & 1-a \end{array}\right].$$

At this point we notice that everything in the third row is a multiple of $1 - a$. If $a = 1$, then $1 - a = 0$, and we get a row of zeros. From here we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so there are infinitely many solutions of the form $x = 2 - 5t$, $y = -1 + 3t$, $z = t$, where $t$ can be any real number.

If $a \neq 1$, then $1 - a \neq 0$ and we can divide the third row by $1 - a$, giving us

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1-a & 2-2a & 1-a \end{array}\right] \xrightarrow{R_3 \to \frac{1}{1-a} R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 5 & 2 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{5} R_3]{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & \frac{2}{5} \end{array}\right],$$

giving us the unique solution $x = 0, y = \frac{1}{5}$, $z = \frac{2}{5}$.

4. Find the basic solutions to the homogeneous system of equations

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & - & 3x_4 & = & 0 \\ 2x_1 & + & x_2 & - & 4x_3 & + & x_4 & = & 0 \\ 3x_1 & - & x_2 & - & 3x_3 & - & 2x_4 & = & 0 \end{array}$$

We set up the augmented matrix and apply row operations, as follows:

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -3 & 0 \\ 2 & 1 & -4 & 1 & 0 \\ 3 & -1 & -3 & -2 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -3 & 0 \\ 0 & 5 & -6 & 7 & 0 \\ 0 & 5 & -6 & 7 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -3 & 0 \\ 0 & 5 & -6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{5} R_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -3 & 0 \\ 0 & 1 & -\frac{6}{5} & \frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{5} & -\frac{1}{5} & \\ 0 & 1 & -\frac{6}{5} & \frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

From here we can read off the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{5}s + \frac{1}{5}t \\ \frac{6}{5}s - \frac{7}{5}t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} \frac{7}{5} \\ \frac{6}{5} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ -\frac{7}{5} \\ 0 \\ 1 \end{bmatrix},$$

so the basic solutions are $\vec{v} = \begin{bmatrix} \frac{7}{5} \\ \frac{6}{5} \\ 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} \frac{1}{5} \\ -\frac{7}{5} \\ 0 \\ 1 \end{bmatrix}$.

*Note:* Many people prefer to pull out the scalar multiple of $\frac{1}{5}$ created to get the matrix into row-echelon form and use the vectors $\begin{bmatrix} 7 \\ 6 \\ 5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -7 \\ 0 \\ 5 \end{bmatrix}$ instead. Either answer is equivalent, since you can absorb the scalar multiple into the parameters.