Math 3500 Assignment #2
University of Lethbridge, Fall 2014

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Due date: Friday, September 19, by 6 pm.

Please submit solutions to the problems below. For this and the remaining assignments, you are responsible for attempting the exercises in your textbook as practice problems – I won't be providing a list.

Don't forget that you can use Piazza to discuss hints to assigned problems and share solutions to practice problems.

1. Let S and T be nonempty bounded subsets of $\mathbb{R}$.

   (a) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

   (b) Prove that $\sup S \cup T = \max\{\sup S, \sup T\}$.
   (Do not assume that $S \subseteq T$ for part (b).)

2. Let $\mathcal{B}[a, b]$ denote the set of all bounded functions defined on the interval $[a, b]$. (That is, for each $f \in \mathcal{B}[a, b]$, there exist constants $k, l \in \mathbb{R}$ such that $k \leq f(x) \leq l$ for all $x \in [a, b]$.) The *norm* of a function $f \in \mathcal{B}[a, b]$ is defined by

   $$\|f\| = \sup\{|f(x)| : x \in [a, b]\}.$$

   Prove that $\|f + g\| \leq \|f\| + \|g\|$ for any $f, g \in \mathcal{B}[a, b]$.

   Hint: we know that the absolute value function on $\mathbb{R}$ satisfies $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for all x. The remainder of the proof consists mainly of following the two parts of the definition of a least upper bound to the desired conclusion.

   Note: one can check that the set of all bounded functions on an interval has the structure of a vector space, and a norm on a vector space is a generalization of the notion of the length of a vector in $\mathbb{R}^n$. Now you'll be ready in case anyone ever asks you how long a function is.

3. Prove that if A is any nonempty open subset of $\mathbb{R}$, then $A \cap \mathbb{Q} \neq \emptyset$.

   Hint: use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Note: For problems 4-6, you only need to submit **two** of the three problems – you're allowed to skip one.

4. For any set $S \subseteq \mathbb{R}$, let $\overline{S}$ denote the intersection of all the closed sets containing S.
   (a) Prove that $\overline{S}$ is a closed subset of $\mathbb{R}$.
   (b) Prove that $\overline{S}$ is the *smallest* closed set containing S. That is, show that $S \subseteq \overline{S}$, and if C is any closed set containing S, then $\overline{S} \subseteq C$.
   (c) Prove that $\overline{S}$ is equal to the closure of S.
   (d) Prove that if S is bounded, then $\overline{S}$ is bounded as well.

5. The Nested Intervals Theorem (from the September 10th worksheet, and also mentioned on Piazza) states that if $\{A_n : n \in \mathbb{N}\}$ is a collection of closed bounded intervals (of the form $[a, b]$), and we have $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, then the intersection $\bigcap A_n$ is nonempty.

   Show that the intervals A_n need to be **both** closed and bounded by giving examples where the theorem fails (that is, where $\bigcap A_n = \emptyset$), if
   (a) The intervals A_n are closed, but not bounded.
   (b) The intervals A_n are bounded, but not closed.

6. An important theorem regarding compact sets is that if $S \subseteq \mathbb{R}$ is compact, and T is a closed subset of S, then T is compact. Prove this fact using:
   (a) The definition of compactness.
   (b) The Heine-Borel theorem.