Name and student number: Solutions

1. Consider the assertion $\exists x \in \mathbb{R} : (\forall y \in \mathbb{R}, 2x - y = 3)$.

[3] (a) Is the statement true or false? Explain your answer.

Solution: The statement is false. Suppose such an x exists; let's say $x = a$. Then it would have to be true that $2a - y = 3$ for every $y \in \mathbb{R}$, but we must have $y = 2a - 3$, which is a unique number. (For example, if $a = 2$, then $y = 1$.) Since there is more than one real number, this is impossible.

[3] (b) Write the negation of this assertion in symbolic form.

Solution: Using the rules for negation with quantifiers, we have

$$\begin{aligned} \neg[\exists x \in \mathbb{R} : (\forall y \in \mathbb{R}, 2x - y = 3)] &\equiv \forall x \in \mathbb{R}, \neg(\forall y \in \mathbb{R}, 2x - y = 3) \\ &\equiv \forall x \in \mathbb{R}, (\exists y \in \mathbb{R} : \neg(2x - y = 3)) \\ &\equiv \forall x \in \mathbb{R}, (\exists y \in \mathbb{R} : 2x - y \neq 3). \end{aligned}$$

From this, we can also see that the statement in (a) is false, for its negation is certainly true: given any real number x, we can choose y such that $2x - y \neq 3$; indeed, any $y \neq 2x - 3$ will do the job.

Total: 10 points

[4] 2. Consider the assertion

For all natural numbers $n, n^2 + 1$ is prime.

(Note: a number $n \in \mathbb{N}$ is prime if it cannot be written in the form $n = a \cdot b$, where a and b are natural numbers other than 1 and n. For example, 5 is prime but 6 is not, since $6 = 2 \cdot 3$.)

Calculate $n^2 + 1$ for $n = 1, 2, 3, 4$. Based on this evidence, is the above assertion true or false? Explain your answer.

Solution: We compute $n^2 + 1$ as follows:

$$\begin{aligned} 1^2 + 1 &= 1 \\ 2^2 + 1 &= 5 \\ 3^2 + 1 &= 10 \\ 4^2 + 1 &= 17 \end{aligned}$$

From the above, we see that when $n = 3$, we have $3^2 + 1 = 10 = 2 \cdot 5$, so $3^2 + 1$ is not prime, and thus it cannot be true that $n^2 + 1$ is prime for all natural numbers n, since 3 is a natural number, and $3^2 + 1$ is not prime.