Math 1410 Assignment #3
University of Lethbridge, Spring 2017

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Due date: Thursday, March 2nd, by 4 pm.

Please review the **Guidelines for preparing your assignments** before submitting your work. You can find these guidelines, along with the required cover page, in the Assignments section on our Moodle site.

Assigned problems

1. For each of the following subsets S of $\mathbb{R}^3$ (viewed as the vector space of 3×1 column vectors), determine if S is a subspace. If S is a subspace, determine a set of vectors that spans S.

   (a) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, 3x - 4y + z = 2 \right\}$

   (b) $S = \left\{ \begin{bmatrix} 2u - 3v \\ u \\ v - 5u \end{bmatrix} \middle| \, u, v \in \mathbb{R} \right\}$

   (c) $S = \left\{ \vec{v} \in \mathbb{R}^3 \,|\, \vec{v} \cdot \vec{w} = 0 \right\}$, where $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

2. A set of vectors $\mathscr{A} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is called **orthogonal** if $\vec{v}_i \neq \vec{0}$ for each $i = 1, \ldots, k$, and if $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. In other words, $\mathscr{A}$ is a set of non-zero, mutually orthogonal vectors: each vector in the set is orthogonal to all the others.

   (a) Show that the set $\mathscr{A} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ is an orthogonal subset of $\mathbb{R}^4$.

   (b) Prove that any orthogonal set of vectors is linearly independent.

   Hint: Suppose you have a linear combination $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$. What do you get when you take the dot product of either side of this equation with $\vec{v}_1$? With $\vec{v}_2$? With $\vec{v}_i$?

   (c) Prove that if $\mathscr{A} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is an orthogonal set of vectors and $\vec{w}$ belongs to the span of $\mathscr{A}$, then

$$\vec{w} = \left(\frac{\vec{w} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{w} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 + \cdots + \left(\frac{\vec{w} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\right)\vec{v}_k.$$

   This is called the *Fourier decomposition theorem*.

   Hint: Saying that $\vec{w}$ belongs to the span of $\mathscr{A}$ means that there are scalars $a_1, \ldots, a_k$ such that $\vec{w} = a_1\vec{v}_1 + \cdots a_k\vec{v}_k$. By using appropriate dot products, as in part (b), determine the values of $a_1, \ldots, a_k$.

   (d) Let $\mathscr{A}$ be the orthogonal subset of $\mathbb{R}^4$ from part (a). Determine whether or not the following vectors belong to the span of $\mathscr{A}$:

$$\vec{a} = \begin{bmatrix} -4 \\ -7 \\ 5 \\ 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ -5 \\ 1 \end{bmatrix}.$$

   Hint: Use part (c). If a vector $\vec{w}$ does not belong to the span of $\mathscr{A}$, then

$$\vec{w} \neq \left(\frac{\vec{w} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{w} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 + \cdots + \left(\frac{\vec{w} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\right)\vec{v}_k.$$

3. In the previous problem, we saw that if $\mathscr{A}$ is an orthogonal set of vectors, and $\vec{w} \in \text{span}(\mathscr{A})$, then the $\vec{w}$ can be written in terms of the vectors in $\mathscr{A}$ using the Fourier decomposition theorem. If $\vec{w}$ is **not** in the span of $\mathscr{A}$, then the vector

$$\vec{v} = \left(\frac{\vec{w}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{w}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2 + \cdots + \left(\frac{\vec{w}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\right)\vec{v}_k.$$

is called the **orthogonal projection** of $\vec{w}$ onto the subspace $U = \text{span}(\mathscr{A})$, and dentoed by $\text{proj}_U(\vec{w})$. In more advanced linear algebra courses, one proves that $\text{proj}_U(\vec{w})$ is the element of U that is *closest* to $\vec{w}$, in the sense that $||\vec{w} - \text{proj}_U(\vec{w})||$ is as small as possible.

Consider the subspace $U \subseteq \mathbb{R}^3$ given by $U = \text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$. Note that U is a plane through the origin, and that the vectors $\vec{v} = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$ are orthogonal.

Determine the point Q on the plane U that is closest to the point $P = (3, -1, 4)$ (and the distance from P to Q):

(a) By computing the orthogonal projection of $\vec{p} = \begin{bmatrix}3\\-1\\4\end{bmatrix}$ onto U, as described above.

(b) Using the method described in Example 54 (and the discussion that follows) in Section 3.6 of the textbook.

Note: This method of orthogonal projection onto a subspace has a number of interesting applications to other areas of mathematics and to the sciences. In calculus, for an example, an infinite-dimensional version of this method is used to find a differentiable function that gives the "best approximation" to a badly-behaved function. The method of *least squares approximation* used to find a "best fit" curve for a data set is also a consequence of orthogonal projection.