The lifting correspondence

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In class I stated the path/homotopy lifting lemma: Let $p : X \to B$ be a covering map, choose $x_0 \in X$ and let $\gamma : [0,1] \to B$ be a path with $\gamma(0) = b_0 = \gamma(x_0)$. Then there is a unique lift $\tilde{\gamma} : [0,1] \to X$ with $\gamma(0) = x_0$ such that $p \circ \tilde{\gamma} = \gamma$. Moreover, if $F : [0,1] \times [0,1] \to B$ is a continuous map with $F(0,0) = b_0$, then there is a unique lift $\tilde{F} : [0,1] \times [0,1] \to X$ such that $\tilde{F}(0,0) = x_0$.

Finally, if F is a homotopy between paths $\gamma_0, \gamma_1 : [0,1] \to B$ beginning at $b_0 \in B$ and $\tilde{\gamma}_0, \tilde{\gamma}_1$ the lifts of γ_0 and γ_1, then $\tilde{F}$ is a homotopy between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$.

Now, with $p : X \to B$ and $b_0 = p(x_0)$ as above, we consider the fundamental group $\pi_1(B, b_0)$. For each $[\gamma] \in \pi_1(B, b_0)$ we choose a representative γ and let $\tilde{\gamma}$ be the unique lift of γ to a path beginning at x_0. The **lifting correspondence** is the map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

given by $\phi([\gamma]) = \tilde{\gamma}(1)$. Since homotopic lifts in B lift to homotopic paths in X, the value of ϕ does not depend on the choice of representative γ. The map ϕ is probably most easily visualized in the example of the covering map $p : \mathbb{R} \to S^1$ given by $p(x) = e^{2\pi i x}$, with $b_0 = 1$. We have $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$, and we picture $\mathbb{R}$ as sitting inside of $\mathbb{R}^3$ as the image of the map $f : \mathbb{R} \to \mathbb{R}^3$ given by $f(t) = (\cos 2\pi t, \sin 2\pi t, t)$. If we take $x_0 = 0 \in \mathbb{R}$ (which would be the point $(1,0,0)$ on the spiral), a loop that wraps n times around the circle counter-clockwise will lift to a path beginning at $(1,0,0)$ and ending at $(1,0,n)$, while a loop that wraps n times around the circle clockwise will lift to a path beginning at $(1,0,0)$ and ending at $(1,0,-n)$. In particular, note that $\tilde{\gamma}$ need not be a loop: we can have $\tilde{\gamma}(1) = x_1$ where $x_1 \in p^{-1}(b)$ but $x_1 \neq x_0$. (That is, $p(x_1) = p(x_0) = b_0$.)

Theorem 1. *If X is path connected, then then map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ given by the lifting correspondence is surjective. If X is simply connected, then ϕ is bijective.*

Proof. If X is path connected and $x_1 \in p^{-1}(b_0)$, there exists a path $\tilde{\gamma}$ from x_0 to x_1, and $\gamma = p \circ \tilde{\gamma}$ is a loop in B based at b_0 with $\phi([\gamma]) = x_1$.

If X is simply connected and $\phi([\gamma_0]) = \phi([\gamma_1])$, then we have liftings $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ of γ_0 and γ_1 such that $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$. Since X is simply connected, there exists a homotopy $\tilde{F} : I \times I \to X$ with $\tilde{F}(s,0) = \tilde{\gamma}_0(s)$ and $\tilde{F}(s,1) = \tilde{\gamma}_1(s)$ for all $s \in I$, and then $F = p \circ \tilde{F}$ is a homotopy from γ_0 to γ_1 in B, so $[\gamma_0] = [\gamma_1]$, so ϕ is injective. □

Theorem 2. $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Proof. Let $\phi : \pi_1(S^1, 1) \to p^{-1}(1) = \mathbb{Z}$ be given by the lifting correspondence, with $p : \mathbb{R} \to \mathbb{Z}$ given by $p(x) = e^{2\pi i x}$ and $x_0 = 0 \in \mathbb{R}$. Since $\mathbb{R}$ is simply connected, ϕ is a bijection, so it remains to show that π is a group homomorphism. Given $[\alpha], [\beta] \in \pi_1(S^1, 1)$, choose representatives α and β and let $\tilde{\alpha}, \tilde{\beta}$ be their resepective liftings to paths in $\mathbb{R}$ beginning at 0. Suppose we have

$$\begin{aligned} \tilde{\alpha}(1) &= n = \phi([\alpha]) \\ \tilde{\beta}(1) &= m = \phi([\beta]). \end{aligned}$$

Let $\tilde{\gamma}$ be the path in $\mathbb{R}$ defined by $\tilde{\gamma}(s) = n + \tilde{\beta}(s)$. Since $\beta(\tilde{0}) = 0$, $\tilde{\gamma}$ is a path beginning at n and ending at $n + m$, and $p \circ \tilde{\gamma} = p \circ \tilde{\beta} = \beta$. Thus $\tilde{\alpha} * \tilde{\gamma}$ is defined, since $\tilde{\alpha}(1) = n = \tilde{\gamma}(0)$, and it is a lifting of $\alpha * \beta$ beginning at 0 and ending at $\tilde{\gamma}(1) = n + m$. Thus,

$$\phi([\alpha] * [\beta]) = n + m = \phi([\alpha]) + \phi([\beta]).$$ □