Math 3410 Assignment #6 Solutions
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1. Suppose $T \in \mathcal{L}(V)$ is normal. Prove that $\operatorname{null} T^k = \operatorname{null} T$ for every positive integer k.

   We know that $\operatorname{null} T \subseteq \operatorname{null} T^k$ for any operator T, normal or not, since if $Tv = 0$, then $T^n v = T^{n-1}(Tv) = T^{n-1}(0) = 0$.

   We will now show that $v \in \operatorname{null} T^k \Rightarrow v \in \operatorname{null} T^{k-1}$ for any $k \geq 2$. Applying this result inductively will show that

   $$\operatorname{null} T^k \subseteq \operatorname{null} T^{k-1} \subseteq \cdots \subseteq \operatorname{null} T.$$

   Let $k \geq 2$ be an integer, and suppose that $v \in \operatorname{null} T^k$, so $T^k v = 0$. We know that for a normal operator $\|Tu\| = \|T^* u\|$ for any $u \in V$, so in particular we have

   $$0 = \|T^k v\| = \|T(T^{k-1}v)\| = \|T^*(T^{k-1}v)\|,$$

   so $T^*(T^{k-1}v) = 0$. This implies that $T^{k-1}v \in \operatorname{null} T^*$, and since $T^{k-1}v = T(T^{k-2}v)$, we have $T^{k-1}v \in \operatorname{range} T$ as well. But $\operatorname{null} T^* = (\operatorname{range} T)^\perp$, so we have

   $$T^{k-1}v \in \operatorname{null} T^* \cap \operatorname{range} T = \{0\},$$

   which shows that $T^{k-1}v = 0$, so $\operatorname{null} T^k \subseteq \operatorname{null} T^{k-1}$, and the result follows.

2. Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

   Suppose that T is normal and that $T^9 = T^8$. By the complex spectral theorem, there exists an orthonormal basis $B = \{e_1, \ldots, e_n\}$ of eigenvectors of T: we have $Te_j = \lambda_j e_j$ for some $\lambda_j \in \mathbb{C}$, for all $j = 1, \ldots, n$. Thus, for each j we have

   $$\lambda_j^9 e_j = T^9 e_j = T^8 e_j = \lambda_j^8,$$

   and thus $\lambda_j^9 = \lambda_j^8$, which implies that $\lambda_j = 0$ or $\lambda_j = 1$. Since the only eigenvalues of T are 0 and 1, which are real, T must be self-adjoint. (To see this, note that since T is normal,

   $$T^* e_j = \overline{\lambda_j} e_j = \lambda_j e_j = T e_j$$

for all $j = 1, \ldots, n$, which implies that $T^* = T$. Finally, since $0^2 = 0$ and $1^2 = 1$, we have

$$T^2 e_j = \lambda_j^2 e_j = \lambda_j e_j = T e_j$$

for all $j = 1, \ldots, n$, from which it follows that $T^2 = T$.

Note: Using results from Chapter 8, we could also argue as follows:

Since $T^9 = T^8$, we have $T^9 - T^8 = T^8(T - I) = 0$. Thus, $p(T) = 0$, where $p(z) = z^8(z-1)$, which implies that $p(z)$ is a multiple of the minimal polynomial of T, and thus the only possible eigenvalues of T are 0 and 1.

3. Suppose $T \in \mathcal{L}(V)$, m is a positive integer, and $v \in V$ is such that $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that the vectors $v, Tv, T^2v, \ldots, T^{m-1}v$ are linearly independent.

Suppose that we have

$$c_0 v + c_1 T v + \cdots + c_{m-1} T^{m-1} v = 0$$

for some $c_0, c_1, \ldots, c_{m-1} \in \mathbb{F}$, with T, v as above. Applying T^{m-1} to both sides of the above equation gives

$$c_0 T^{m-1} v + c_1 T^m v + \cdots c_{m-1} T^{2m-2} v = 0.$$

Since $T^m v = T^{m+1} v = \cdots = T^{2m-2} v = 0$, this gives us $c_0 T^{m-1} v = 0$. Since we're assuming that $T^{m-1} v \neq 0$, we must have $c_0 = 0$, leaving us with

$$c_1 T v + c_2 T^2 v + \cdots + c_{m-1} T^{m-1} v = 0.$$

Proceeding as above, we can apply T^{m-2} to both sides of the equation to obtain $c_1 = 0$, and so on, eventually showing that all of the c_j must be zero, from which the result follows.

4. Determine all possible Jordan Canonical Forms for a linear transformation with characteristic polynomial $(x-2)^3(x-3)^2$. Find the corresponding minimal polynomial for each JCF.

The characteristic polynomial is of degree 5, which tells us that the matrix of T must be a 5×5 matrix, with eigenvalues $\lambda = 2$, of multiplicity 3, and $\lambda = 3$, of multiplicity 2. This tells us that the JFC of T will have three 2s on the main diagonal, and two 3s. We now recall that the minimal polynomial must be of the form $m_T(x) = (x-2)^k(x-3)^l$, where $1 \leq k \leq 3, 1 \leq l \leq 2$, and the powers k and l tell us the size of the largest Jordan block for the respective eigenvalues. We thus obtain the following six possible

Jordan Canonical Forms, with their respective minimal polynomials:

Jordan Canonical Form	Minimal Polynomial
$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$	$m_t(x) = (x-2)(x-3)$
$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$	$m_t(x) = (x-2)^2(x-3)$
$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$	$m_t(x) = (x-2)^3(x-3)$
$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$	$m_t(x) = (x-2)(x-3)^2$
$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$	$m_t(x) = (x-2)^2(x-3)^2$
$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$	$m_t(x) = (x-2)^3(x-3)^2$

Note that for the second and fifth entries above, it is also acceptable to have a 2×2 Jordan block for $\lambda = 2$ followed by a 1×1 block, rather than 1×1 followed by 2×2, as above.

Alternate Quiz Problem: For the matrix below, find the characteristic and minimal polynomials, a Jordan basis, and the Jordan Canonical Form:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix}$$

The characteristic polynomial is given by

$$\begin{aligned} c_A(x) = \det(xI_4 - A) &= \begin{vmatrix} x-1 & -1 & -1 & -1 \\ 0 & x-2 & -2 & 0 \\ 0 & 0 & x-2 & 0 \\ 1 & -1 & 0 & x-3 \end{vmatrix} \\ &= (x-2) \begin{vmatrix} x-1 & -1 & -1 \\ 0 & x-2 & 0 \\ 1 & -1 & x-3 \end{vmatrix} \quad \text{(expanding along row 3)} \\ &= (x-2)^2 \begin{vmatrix} x-1 & -1 \\ 1 & x-3 \end{vmatrix} \quad \text{(expanding along row 2)} \\ &= (x-2)^2(x^2-4x+4) = (x-2)^4. \end{aligned}$$

Thus, A has the single eigenvalue $\lambda = 2$, with multiplicity 4.

If we wanted to, we could find immediately find the minimal polynomial of A by noting that

$$A - 2I = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \neq 0, (A-2I)^2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \neq 0, \text{ while } (A-2I)^3 = 0,$$

so $m_A(x) = (x-2)^3$. We thus expect a 3×3 Jordan block in the JCF of A, which suggests that we should find two eigenvectors and two generalized eigenvectors. Let's confirm that this is the case. We begin with eigenvalues:

$$A - 2I = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \text{ has RREF } \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

from which we see that the general solution to $(A-2I)X = 0$ is

$$X = \begin{bmatrix} s+t \\ s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so $E(2,A) = \operatorname{span}(X_1, X_2)$, where $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

We now look for generalized eigenvectors. We expect to find generalized eigenvectors $Y \in \operatorname{null}(A-2I)^2$ and $Z \in \operatorname{null}(A-2I)^3$, since $m_A(x) = (x-2)^3$. To obtain the proper Jordan Canonical Form, we want $Y \in \operatorname{null}(A-2I)^2$ to satisfy $(A-2I)Y = X_2$. (Also, one can check that the system $(A-2I)Y = X_1$ is inconsistent. The system $(A-2I)Y = X_2$ has augmented matrix

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 \end{array}\right] \text{ which has RREF } \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

and this gives us the general solution

$$Y = \begin{bmatrix} -1+s+t \\ s \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + sX_1 + tX_2.$$

Ordinarily at this state we would set $s = t = 0$ to obtain Y, but we will find that doing so leads to an inconsistent system at the next step. Instead, we write $Y = \begin{bmatrix} -1+u+v \\ u \\ 0 \\ v \end{bmatrix}$, with u and v to be determined.

Finally, we want to find $Z \in \operatorname{null}(Z-2I)^3$ such that $(A-2I)^2 Z = Y$. This leads to the augmented matrix

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & 1 & -1+u+v \\ 0 & 0 & 2 & 0 & u \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & v \end{array}\right] \text{ which has RREF } \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1-u/2-v \\ 0 & 0 & 1 & 0 & u/2 \\ 0 & 0 & 0 & 0 & 1-u/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The third row in the reduced row-echelon form above tells us that for a consistent system, we must set $u = 2$, while we're free to set v to any value we like. For convenience, we set $v = 0$. The values $u = 2$ and $v = 0$ then give us $Y = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, and in the augmented matrix above, plugging in $u = 2$ and $v = 0$ gives the general solution

$$Z = \begin{bmatrix} s+t \\ s \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + sX_1 + tX_2.$$

Setting $s = t = 0$ gives us a particular value for Z, and the Jordan basis

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

and the Jordan Canonical Form of A with respect to this basis is

$$\mathrm{JCF}(A) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$