Math 2580 Assignment #4
University of Lethbridge, Spring 2016

Sean Fitzpatrick

February 18, 2016

Due date: Thursday, February 25th, by 3 pm.

Please provide solutions to the problems below, using the following guidelines:

- Your submitted assignment should be a **good copy** – figure out the problems first, and then write down organized solutions to each problem.
- You should include a **cover page** with the following information: the course number and title, the assignment number, your name, and a list of any resources you used or people you worked with.
- Since you have plenty of time to work on the problems, assignment solutions will be held to a higher standard than on a test. Your explanations should be clear enough that any of your classmates can understand your solutions.
- Group work is permitted, but copying is not. If you're not sure what the difference is, feel free to ask. If you get help solving a problem, you should (a) make sure you completely understand the solution, and (b) re-write the solution for your good copy by yourself, in your own words.
- Assignments can be submitted in class, or in the designated drop box on the 5th floor of University Hall, across from the Math Department office.
- Late assignments will not be accepted without prior permission.

Assigned problems

1. Evaluate the integeral below, where $D = \{(x,y) | x^2 + y^2 \leq 1\}$ is the unit disc. Explain your result.
$$\iint_D (x^3 e^{x^2} + y^{1/3} \sin(y^4) + 3)\, dA.$$
Hint: as regions go, the unit disc is about as symmetric as they get.

2. The integral below expresses the integral of a function f over a region D as a sum of two iterated integrals. Sketch the region of integration, and express the integral as a single iterated integral with the order of integration reversed:
$$\iint_D f(x,y)\, dA = \int_0^1 \int_1^{x^2+1} f(x,y)\, dy\, dx + \int_1^3 \int_1^{\frac{1}{4}(x-3)^2+1} f(x,y)\, dy\, dx.$$

3. Prove that $2 \int_a^b \int_x^b f(x) f(y)\, dy\, dx = \left(\int_a^b f(x)\, dx \right)^2$.

Hint: Notice that $\left(\int_a^b f(x)\, dx \right)^2 = \iint_{[a,b]\times[a,b]} f(x) f(y)\, dA$.

4. Prove the following Mean Value Theorem for double integrals: suppose $D \subseteq \mathbb{R}^2$ is an elementary region (Type 1 or Type 2), and that $f : D \to \mathbb{R}$ is continuous. Then there exists a point $(x_0, y_0) \in D$ such that
$$\iint_D f(x,y)\, dA = f(x_0, y_0) A(D),$$
where $A(D)$ denotes the area of D.
(Note that $\frac{1}{A(D)} \iint_D f(x,y)\, dA$ gives the average value of f on D.)

You may use the following facts in your proof:

- The Extreme Value Theorem holds in $\mathbb{R}^2$: if $D \subseteq \mathbb{R}^2$ is closed and bounded[1] and $f : D \to \mathbb{R}$ is continuous, then there exist $m, M \in \mathbb{R}$ such that $m \leq f(x,y) \leq M$ for all $(x,y) \in D$; moreover, there exist points $(x_1, y_1), (x_2, y_2) \in D$ such that $f(x_1, y_1) = m$ and $f(x_2, y_2) = M$. (That is, f attains its minimum and maximum values on D.)
- The Intermediate Value Theorem holds in $\mathbb{R}^2$: Suppose $D \subseteq \mathbb{R}^2$ is connected[2] and $f : D \to \mathbb{R}$ is continuous. Then if $f(x_1, y_1) = a$ for some $(x_1, y_1) \in D$ and $f(x_2, y_2) = b$ for some $(x_2, y_2) \in D$, and c is any real number between a and b, then there exists some point $(x_0, y_0) \in D$ such that $f(x_0, y_0) = c$.

[1] A subset of $\mathbb{R}^n$ is *closed* if it contains its boundary. It is *bounded* if it can be contained within a disk of sufficiently large radius: it doesn't go off to infinity in any direction.

[2] A subset D of $\mathbb{R}^2$ is *connected* if it is impossible to find two non-intersecting disks that both contain part of D. In other words, D cannot be split into two pieces that do not touch. Note that in particular, every elementary region is connected.