Here are the solutions to the worksheet for the January 28th tutorial. I should probably point out that all of the problems from this worksheet come from Sections 1.4 and 1.5 in the textbook that I assembled for the class (the one you can download from Moodle or order from the Bookstore). If you're looking for more practice, there are plenty more where these came from.

There are probably a few mistakes and typos in these solutions. Please let me know if you find any.

1. $\displaystyle\int \frac{x^2-11}{x}\,dx = \int (x - 11x^{-1})\,dx = \frac{1}{2}x^2 - 11\ln|x| + C$

2. $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx$

   Letting $u = x^2-3$, we have $\frac{1}{2}du = x\,dx$, so
$$\int \frac{x}{\sqrt{x^2-3}}\,dx = \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} + C = \sqrt{x^2-3} + C.$$
   Alternatively, you can let $x = \sqrt{3}\sec\theta$, so $dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-3} = \sqrt{3}\sqrt{\sec^2\theta - 1} = \sqrt{3}\tan\theta$, and then
$$\int \frac{x}{\sqrt{x^2-3}}\,dx = \int \frac{\sqrt{3}\sec\theta}{\sqrt{3}\tan\theta}(\sqrt{3}\sec\theta\tan\theta)d\theta = \sqrt{3}\int \sec^2\theta\,d\theta = \sqrt{3}\tan\theta + C$$
   From our work above we see that $\sqrt{3}\tan\theta = \sqrt{x^2-3}$, and so we get the same answer as above.

   If you want yet another option, try letting $x = \sqrt{3}\cosh(t)$, so $dx = \sqrt{3}\sinh(t)\,dt$ and $\sqrt{x^2-3} = \sqrt{3}\sqrt{\cosh^2(t) - 1} = \sqrt{3}\sinh(t)$. With these substitutions, the integral becomes $\displaystyle\sqrt{3}\int \cosh(t)\,dt = \sqrt{3}\sinh(t) + C = \sqrt{x^2-3}$, as before.

3. $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

   Letting $x = \sin\theta$, $dx = \cos\theta\,d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$, so we get
$$\begin{aligned}\int x^2\sqrt{1-x^2}\,dx &= \int \sin^2\theta\cos\theta(\cos\theta)\,d\theta = \int \sin^2\theta\cos^2\theta\,d\theta\\ &= \int \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)d\theta\\ &= \frac{1}{4}\int (1-\cos^2(2\theta))\,d\theta = \frac{1}{4}\int \sin^2(2\theta)\,d\theta\\ &= \frac{1}{4}\int\left(\frac{1-\cos(4\theta)}{2}\right)d\theta\\ &= \frac{1}{8}\left(\theta - \frac{1}{4}\sin(4\theta)\right) + C\\ &= \frac{1}{8}\sin^{-1}(x) - \frac{1}{32}\sin(4\sin^{-1}x) + C.\end{aligned}$$

If you want to simplify that last term, note that $\sin\theta = x$ and $\cos\theta = \sqrt{1-x^2}$, and

$$\sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = 4\sin(\theta)\cos(\theta)(\cos^2(\theta)-\sin^2(\theta)) = 4\sin(\theta)\cos^3(\theta)-4\sin^3(\theta)\cos(\theta),$$

so

$$\frac{1}{32}\sin(4\sin^{-1}x) = \frac{1}{8}(x(1-x^2)^{3/2} - x^3(1-x^2)^{1/2}) = \frac{1}{8}x(1-2x^2)\sqrt{1-x^2}.$$

4. $\displaystyle\int \frac{1}{(x^2+4x+13)^2}\,dx$

Completing the square, we have $x^2+4x+13 = x^2+4x+4+9 = (x+2)^2+3^2$, suggesting that we try letting $x+2 = 3\tan\theta$. This gives us $dx = 3\sec^2\theta\,d\theta$, and

$$x^2+4x+13 = (x+2)^2+3^3 = 3^2\tan^2\theta + 3^2 = 3^2(\tan^2\theta+1) = 9\sec^2\theta.$$

Substituting everything into the integral, we get

$$\begin{aligned}\int \frac{1}{(x^2+4x+13)^2}\,dx &= \int \frac{1}{81\sec^4\theta}(3\sec^2\theta)\,d\theta\\ &= \frac{1}{27}\int \cos^2\theta\,d\theta\\ &= \frac{1}{54}\int (1+\cos(2\theta))\,d\theta\\ &= \frac{\theta}{54} + \frac{1}{108}\sin(2\theta) + C\\ &= \frac{\theta}{54} + \frac{1}{54}\sin\theta\cos\theta + C.\end{aligned}$$

To get everything back in terms of x, we note that $\tan\theta = \frac{x+2}{3}$. If we have a right-angled triangle with sides of length $x+2$ (opposite θ) and 3 (adjacent θ), then the hypotenuse has length $\sqrt{(x+2)^2+3^2} = \sqrt{x^2+4x+13}$, and we get $\sin\theta = \dfrac{x+2}{\sqrt{x^2+4x+13}}$ and $\cos\theta = \dfrac{3}{\sqrt{x^2+4x+13}}$. Plugging all of this in, we get the final answer

$$\int \frac{1}{(x^2+4x+13)^2}\,dx = \frac{1}{54}\tan^{-1}\left(\frac{x+2}{3}\right) + \frac{1}{18}\frac{x+2}{x^2+4x+13}.$$

5. $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}\,dx$

Again, we make a tangent substitution: $x = 2\tan\theta$, so $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{x^2+4} = 2\sec\theta$, giving us

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2+4}}\,dx &= \int \frac{4\tan^2\theta}{2\sec\theta}(2\sec^2\theta)\,d\theta\\ &= 4\int \tan^2\theta\sec\theta\,d\theta = 4\int(\sec^2\theta-1)\sec\theta\,d\theta\\ &= 4\int\sec^3\theta\,d\theta - 4\int\sec\theta\,d\theta.\end{aligned}$$

From class, you know that $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$, and from last week's worksheet (but also from class, I believe), you know that

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| + C,$$

so

$$\begin{aligned}4\int \sec^3\theta\,d\theta - 4\int \sec\theta\,d\theta &= 4\left(\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| - \ln|\sec(\theta)+\tan(\theta)|\right) + C\\ &= 2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| + C\end{aligned}$$

From the substitution work above, we know that $\tan\theta = \frac{x}{2}$, and that $\sec\theta = \frac{1}{2}\sqrt{x^2+4}$. Putting everything together, we get

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2+4}}\,dx &= 2\left(\left(\frac{1}{2}\sqrt{x^2+4}\right)\left(\frac{x}{2}\right) - \ln\left|\frac{x}{2} + \frac{1}{2}\sqrt{x^2+4}\right|\right) + C\\ &= \frac{1}{2}x\sqrt{x^2+4} - 2\ln|x + \sqrt{x^2+4}| + C,\end{aligned}$$

where in the last line, I've used the fact that $\ln(u/2) = \ln(u) - \ln(2)$, and absorbed the constant $-\ln(2)$ into the constant of integration.

6. $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx$

We look for a partial fraction decomposition

$$\frac{7x+7}{x^2+3x-10} = \frac{7x+7}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5} = \frac{A(x+5)+B(x-2)}{(x-2)(x+5)}$$

Since the denominators of the first and last terms of the above inequality are equal, the numerators must be equal as well:

$$7x+7 = A(x+5) + B(x-2).$$

Since this equality holds for all values of x, it holds in particular when $x=2$ and $x=-5$. Putting $x=2$ gives us $7(2)+7 = A(7)+B(0)$, so $7A = 21$ and thus $A = 3$. Putting $x=-5$ gives us $7(-5)+7 - A(0) + B(-7)$, so $-7B = -28$, and thus $B = 4$. Returning to the integral, we thus have

$$\begin{aligned}\int \frac{7x+7}{x^2+3x-10}\,dx &= 3\int \frac{1}{x-2}\,dx + 4\int \frac{1}{x+5}\,dx\\ &= 3\ln|x-2| + 4\ln|x+5| + C = \ln|(x-2)^3(x+5)^4| + C.\end{aligned}$$

7. $\displaystyle\int \frac{7x-2}{x^2+x}\,dx$

Using partial fractions, if

$$\frac{7x-2}{x^2+x} = \frac{7x-2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)},$$

then we must have $A(x+1)+Bx = 7x-2$. When $x=0$ we get $A=-2$, and when $x=-1$ we get $-B=-9$, so $B=9$. Thus, we have

$$\int \frac{7x-2}{x^2+x}\,dx = -2\int \frac{1}{x}\,dx + 9\int \frac{1}{x+1}\,dx = -2\ln|x| + 9\ln|x+1| + C = \ln\left|\frac{(x+1)^9}{x^2}\right| + C.$$

8. $\displaystyle\int \frac{x+7}{(x+5)^2}\,dx$

Again we use partial fractions. Because of the repeated root in the denominator, we write

$$\frac{x+7}{(x+5)^2}\,dx = \frac{A}{x+5} + \frac{B}{(x+5)^2} = \frac{A(x+5)+B}{(x+5)^2},$$

and equating numerators gives us $x+7 = A(x+5)+B$. Putting $x=-5$ immediately gives us $B=2$, and plugging this back in, we have $x+7 = Ax+5A+2$, so we must have $A=1$. Thus,

$$\int \frac{x+7}{(x+5)^2}\,dx = \int \frac{1}{x+5}\,dx + 2\int (x+5)^{-2}\,dx = \ln|x+5| - 2(x+5)^{-1} + C.$$

9. $\displaystyle\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx$

Our partial fraction decomposition in this case takes the form

$$\frac{9x^2+11x+7}{x(x+1)^2}\,dx = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2},$$

so $A(x+1)^2 + Bx(x+1) + Cx = 9x^2+11x+7$. Putting $x=0$ gives us $A=7$ immediately, and putting $x=-1$ gives us $-C = 9-11+7 = 5$, so $C=-5$. This leaves us with $7(x+1)^2 + Bx(x+1) - 5x = 9x^2+11x+7$. To find B, we try $x=1$, which gives us $7(4)+2B-5 = 9+11+7$, so $2B = 27-19 = 8$, giving us $B=4$. Putting everything into the integral, we have

$$\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx = \int \left(\frac{7}{x} + \frac{4}{x+1} - \frac{5}{(x+1)^2}\right)dx = 7\ln|x| + 4\ln|x+1| + \frac{5}{x+1} + C.$$

10. $\displaystyle\int \frac{x^3}{x^2-x-20}\,dx$

Since the degree of the numerator is not less than that of the denominator, we first perform long division:

$$\begin{array}{r|l} & \quad\quad\quad\quad\; x \;\; +1 \\ \hline x^2-x-20 & x^3 \\ & -x^3 + x^2 + 20x \\ \cline{2-2} & \quad\quad x^2 + 20x \\ & \quad -x^2 \quad + x + 20 \\ \cline{2-2} & \quad\quad\quad\quad 21x + 20 \end{array}$$

This tells us that we can write $\dfrac{x^3}{x^2-x-20} = x+1+\dfrac{21x+20}{x^2-x-20}$, and it remains to perform a partial fraction decomposition on the last term:

$$\frac{21x+20}{(x-5)(x+4)} = \frac{A}{x-5} + \frac{B}{x+4} = \frac{A(x+4)+B(x-5)}{(x+4)(x-5)},$$

giving us $21x+20 = A(x+4)+B(x-5)$. If $x=5$, we get $125 = 9A$, so $A = \frac{125}{9}$. If $x=-4$, we get $-64=-9B$, so $B = -\frac{64}{9}$. Thus, we have

$$\begin{aligned}\int \frac{x^3}{x^2-x-20}\,dx &= \int \left(x+1+\frac{125}{9(x-5)} - \frac{64}{9(x+4)}\right) dx\\ &= \frac{1}{2}x^2+x+\frac{125}{9}\ln|x-5| - \frac{64}{9}\ln|x+4| + C.\end{aligned}$$

11. $\displaystyle\int \frac{1}{x^3+2x^2+3x}\,dx$

Factoring the denominator, we have

$$x^3+2x^2+3x = x(x^2+2x+3),$$

where $x^2+2x+3 = (x+1)^2+2$ is an irreducible quadratic. Our partial fraction decomposition is thus

$$\frac{1}{x^3+2x^2+3x} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+3} = \frac{A(x^2+2x+3)+(Bx+C)x}{x(x^2+2x+3)}.$$

Equating numerators gives us $1 = A(x^2+2x+3)+(Bx+C)x$. Setting $x=0$ gives us $1=3A$, so $A=\frac{1}{3}$. Since x^2+2x+3 has no real roots, there isn't any x value we can plug in to make the A term vanish. Instead, we put $x=1$, giving us $1 = \frac{1}{3}(1+2+3)+(B+C)(1)$, so $B+C = 1-2 = -1$. Putting $x=-2$ gives us $1 = \frac{1}{3}(4-4+3)+(-2B+C)(-2)$, so $4B-2C+1 = 1$, which simplifies to $2B-C=0$. (If you're wondering why I chose $x=-2$, it was so x^2+2x+3 would be a multiple of 3, allowing me to avoid fractions.)

We're left with the equations $B+C=-1$ and $2B-C=0$. Adding the two equations gives us $3B=-1$, so $B=-\frac{1}{3}$, and thus $C=2B=-\frac{2}{3}$, so

$$\frac{1}{x^3+2x^2+x} = \frac{1}{3}\left(\frac{1}{x} - \frac{x+2}{x^2+2x+3}\right) = \frac{1}{3}\left(\frac{1}{x} - \frac{x+1}{x^2+2x+3} - \frac{1}{x^2+2x+3}\right).$$

Why did we break up the second fraction into two pieces? Well, for the first piece, if we let $u = x^2+2x+3$, then $du = 2(x+1)\,dx$, so

$$\int \frac{x+1}{x^2+2x+3}\,dx = \frac{1}{2}\ln(x^2+2x+3)+C.$$

For the second piece, writing $x^2+2x+3 = (x+1)^2+2$, we can let $x+1 = \sqrt{2}\tan\theta$, so $dx = \sqrt{2}\sec^2\theta\,d\theta$ and $(x+1)^2+2 = 2\sec^2\theta$, so

$$\int \frac{1}{x^2+2x+3}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right)+C.$$

Altogether, we have

$$\int \frac{1}{x^3+2x^2+x}\,dx = \frac{1}{3}\ln|x| - \frac{1}{6}\ln(x^2+2x+3) - \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right) + C.$$

12. $\displaystyle\int \frac{2x^2+2x+1}{(x+1)(x^2+9)}\,dx$

Once more with partial fractions: if

$$\frac{2x^2+2x+1}{(x+1)(x^2+9)}\,dx = \frac{A}{x+1} + \frac{Bx+C}{x^2+9} = \frac{A(x^2+9)+(Bx+C)(x+1)}{(x+1)(x^2+9)},$$

then equating numerators gives us $2x^2+2x+1 = A(x^2+9)+(Bx+C)(x+1)$. Putting $x=-1$, we get $1 = A(10)$, so $A = 1/10$. Putting $x = 0$, we get $1 = 9A + C$, so $C = 1 - 9/10 = 1/10$. Finally, putting $x = 1$ gives us $5 = 10A + 2(B + C)$, so $2(B + C) = 5 - 10(1/10) = 4$, which simplifies to $B + C = 2$. Since $C = 1/10$, this gives us $B = 19/10$. Therefore, we have

$$\begin{aligned}\int \frac{2x^2+2x+1}{(x+1)(x^2+9)}\,dx &= \int \left(\frac{1}{10x} + \frac{19x}{10(x^2+9)} + \frac{1}{10(x^2+9)}\right)dx \\ &= \frac{1}{10}\ln|x| + \frac{19}{20}\ln(x^2+9) + \frac{1}{30}\tan^{-1}\left(\frac{x}{3}\right) + C.\end{aligned}$$