Math 2580 Assignment #2 Solutions
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1. Let $r : \mathbb{R} \to \mathbb{R}^3$ be a smooth[1] curve given by $r(t) = (u(t), v(t), w(t))$, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable function given by

$$f(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)).$$

The composition $s(t) = (f \circ r)(t) = (x(r(t)), y(r(t)), z(r(t)))$ is then another curve in $\mathbb{R}^3$. Using the Chain Rule, show the following:

(a) If $r'(t)$ exists for all t, then $s'(t)$ exists for all t.

Suppose that $r'(t)$ is defined for all t. Then in particular $r(t)$ is defined for all t, and since f is continuously differentiable, the derivative matrix $D_{r(t)}f$ is defined at each point $r(t)$ on the curve. Since $s(t) = f(r(t))$, it follows from the Chain Rule that

$$s'(t) = \frac{d}{dt}(f(r(t))) = (D_{r(t)}f)r'(t)$$

is defined for all t.

(b) If $\vec{v}$ is tangent to the curve $r(t)$ at a point $\mathbf{u}_0 = (u_0, v_0, w_0) = r(t_0)$, then $D_{\mathbf{u}_0}f\vec{v}$ is tangent to the curve $s(t)$ at the point $\mathbf{x_0} = f(u_0, v_0, w_0) = s(t_0)$.

If $\vec{v}$ is tangent to the curve $r(t)$ at the point $r(t_0)$, then we must have $\vec{v} = cr'(t_0)$ for some scalar $c \in \mathbb{R}$. Using the Chain Rule result from part (a), it follows that

$$D_{r(t_0)}f \cdot \vec{v} = D_{r(t_0)}f(cr'(t_0)) = cD_{r(t_0)}f \cdot r'(t_0) = cs'(t_0),$$

so $D_{r(t_0)}f \cdot \vec{v}$ is a scalar multiple of $s'(t_0)$, and therefore tangent to the curve $s(t)$ at the point $s(t_0)$.

(c) **Bonus:** In order to say that the curve $s(t)$ is "smooth", we would need to also guarantee that $s'(t)$ is never zero. What condition on $D_{\mathbf{x}}f$ will guarantee this? (Hint: if $\vec{v}$ is a non-zero vector, how can you guarantee that $A\vec{v} \neq 0$ for an $m \times n$ matrix A?)

[1] For us, a curve will be *smooth* if $r'(t) = \langle u'(t), v'(t), w'(t)\rangle$ exists and is **non-zero** for all t.

Since $D_{\mathbf{x}}f$ is a 3×3 matrix in this case, we know that the only solution to the system of equations $D_{\mathbf{x}}f\vec{v} = \vec{0}$ is $\vec{v} = \vec{0}$ provide the matrix $D_{\mathbf{x}}f$ is invertible. Therefore a sufficent condition in this case is

$$\det(D_{\mathbf{x}}f) \neq 0.$$

Note: The answer is a bit simpler in this case because f was a function from $\mathbb{R}^3 \to \mathbb{R}^3$. In general, if f is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $m \neq n$, we need to be more careful. There are two cases to consider: if $n < m$, a sufficient condition is that $\text{rank}(D_{\mathbf{x}}f) = n$ for each point $\mathbf{x}$ along the curve. If $n > m$, it's impossible to guarantee that $s'(t) \neq 0$ in general: there will always be non-trivial solutions to the system of equations $A\vec{v} = \vec{0}$ when the matrix A has more columns than rows. The best we can ask for in this case is that the rank of the derivative matrix is equal to m. (The only way to avoid $s'(t) = 0$ in this case is to makes sure $r'(t)$ never belongs to the null space of the matrix $D_{r(t)}f$.)

2. Let $r(t) = (2\cos(t), 3\sin(t))$ be a curve in the plane, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = x^2 - 4xy^3$. The curve

$$s(t) = (2\cos(t), 3\sin(t), f(2\cos(t), 3\sin(t)))$$

is then a curve in $\mathbb{R}^3$ that lies on the surface $z = f(x, y)$.

(a) Explain why the claim above (that $s(t)$ defines a curve on the surface $z = f(x, y)$) is true.

Saying that the curve $s(t)$ lies on the surface is simply stating that every point on the curve must also be a point on the surface. If (x, y, z) is a point on the curve, then

$$(x, y, z) = s(t) = (x(t), y(t), z(t))$$

for some t, and requiring that (x, y, z) also lies on the surface $z = f(x, y)$ is simply the condition that $z(t) = f(x(t), y(t))$, and this is exactly what we're given.

(b) Show that the tangent vector to $s(t)$ when $t = 0$ lies in the tangent plane to the surface $z = f(x, y)$ at the point $(2, 0, 4)$.

The tangent vector the the curve $s(t)$ for any value of t is given by

$$s'(t) = (x'(t), y'(t), z'(t)),$$

where $x'(t) = -2\sin(t)$, $y'(t) = 3\cos(t)$, and by the Chain Rule,

$$z'(t) = f_x(x(t), y(t))y'(t) + f_y(x(t), y(t))y'(t) = (2x - 4y^3)(-2\sin(t)) - 12y^3(3\cos(t)).$$

When $t = 0$, $x(0) = 2$, $y(0) = 0$, $x'(t) = 0$, $y'(t) = 3$, and

$$z'(0) = (2(2) - 0)(0) - 0(3) = 0.$$

Thus, $s'(t) = \langle 0, 3, 0 \rangle$. On the other hand, the normal vector to the tangent plane is given by

$$\vec{n} = \langle f_x(2,0), f_y(2,0), -1 \rangle = \langle 4, 0, -1 \rangle,$$

and thus $\vec{n} \cdot s'(0) = \langle 4, 0, -1 \rangle \cdot \langle 0, 3, 0 \rangle = 0$, which shows that $s'(0)$ lies in the tangent plane to $z = f(x, y)$ at the point $s(0)$.

Note: the general case for this example is at the end of Section 15.3 in the Marsden and Weinstein text.