Name: Solutions

Solve **one** of the following two questions:

1. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p(6) = 0\}$.

[2] (a) Find a basis for U. (Hint: note that $p(x) = x - 6$ is an element of U.)

Consider the polynomials $p_1(x) = x - 6$, $p_2(x) = x^2 - 6x$, $p_3(x) = x^3 - 6x^2$, $p_4(x) = x^4 - 6x^3$. Since $x - 6$ is a factor of each polynomial, we see that they are all elements of U. Moreover, if

$$\begin{aligned} 0 &= c_1p_1(x) + c_2p_2(x) + c_3p_3(x) + c_4p_4(x) \\ &= c_1(x-6) + c_2(x^2-6x) + c_3(x^3-6x^2) + c_4(x^4-6x^3) \\ &= -6c_1 + (c_1 - 6c_2)x + (c_2 - 6c_3)x^2 + (c_3 - 6c_4)x^3 + c_4x^4, \end{aligned}$$

then we must have $c_1 = c_2 = c_3 = c_4 = 0$, so the set $B = \{p_1, p_2, p_3, p_4\}$ is linearly independent. Since U is a proper subspace of $\mathcal{P}_4(\mathbb{R})$ (since, for example, the polynomial $q(x) = 1$ does not belong to U), we must have $\dim U \leq 4$, and since the set B contains four linearly independent vectors, it must be a basis for U.

[4] (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.

As noted above, the polynomial $q(x) = 1$ does not belong to U, since $q(6) = 1 \neq 0$. It follows that q is not in the span of B, and therefore the set $B' = \{p_1, p_2, p_3, p_4, q\}$ is linearly independent. Since $\dim \mathcal{P}_4(\mathbb{R}) = 5$, B' must be a basis.

[4] (c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $U \oplus W = \mathcal{P}_4(\mathbb{R})$.

Let $W = \operatorname{span}\{q\}$, where $q(x) = 1$ as above. We must have $\mathcal{P}_4(\mathbb{R}) = U + W$, since any polynomial $p(x) \in \mathcal{P}_4(\mathbb{R})$ can be written as

$$p(x) = c_1p_1(x) + c_2p_2(x) + c_3p_3(x) + c_4p_4(x) + c_5q(x) = u(x) + w(x),$$

where $u(x) = c_1p_1(x) + c_2p_2(x) + c_3p_3(x) + c_4p_4(x) \in U$ and $w(x) = c_5q(x) \in W$, since B' is a basis. Since W is the subspace of constant polynomials, and the only constant polynomial that is equal to zero when $x = 6$ is the zero polynomial, we must have $U \cap W = \{0\}$, and the result follows.

Total: 10 points

2. Suppose U and W are subspaces of V such that $V = U \oplus W$. Show that if $\{u_1, \ldots, u_m\}$ is
[10] a basis for U, and $\{w_1, \ldots, w_k\}$ is a basis for W, then

$$\{u_1, \ldots, u_m, w_1, \ldots, w_k\}$$

is a basis for V.

Let $v \in V$ be any vector. Since $V = U \oplus W$, we have in particular that $V = U + W$, so there exist vectors $u \in U$ and $w \in W$ such that

$$v = u + w.$$

Since $\{u_1, \ldots, u_m\}$ is a basis for U, there exist scalars $a_1, \ldots, a_m$ such that

$$u = a_1u_1 + \cdots + a_mu_m,$$

and similarly, there exist scalars $b_1, \ldots, b_k$ such that

$$w = b_1w_1 + \cdots + b_kw_k.$$

It follows that v can be written as

$$v = a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_kw_k.$$

Since $v \in V$ was arbitary, we can conclude that the set $\{u_1, \ldots, u_m, w_1, \ldots, w_k\}$ spans V. It remains to show that the set is independent. Suppose we have

$$a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_kw_k = 0$$

for some scalars $a_1, \ldots, a_m, b_1, \ldots, b_k$. Then $0 = u + w$, where $u = a_1u_1 + \cdots + a_mu_m \in U$ and $w = b_1w_1 + \cdots + b_kw_k \in W$. Since $V = U \oplus W$ is a direct sum, we know that the only way to write $0 = u + w$ with $u \in U$ and $w \in W$ is if $u = w = 0$.

But if $u = 0$, then $a_1 = \cdots = a_m = 0$, since $\{u_1, \ldots, u_m\}$ is a basis for U. Similarly, since $w = 0$, we must have $b_1 = \cdots = b_k = 0$. Thus, the only linear combination equal to zero is the trivial combination, so the set is linearly independent, and therefore a basis for V.

Total: 10 points