Math 4310 Assignment #2
University of Lethbridge, Fall 2014

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Due date: Friday, September 19, by 5 pm.

Please submit solutions to the following problems (except where indicated). As usual, all the textbook exercises are recommended as practice problems if they don't appear as an assigned problem below.

1. Since any ϵ-neighbourhood in a metric space X is open in X, we know that the union of any collection of such neighbourhoods is an open set in X. Prove that this is in fact the most general type of open set. That is, prove that any open subset $U \subseteq X$ of a metric space X is a union of ϵ-neighbourhoods.

   Hint: If you're unsure of how to start, you might want to look at the online supplements for the textbook. (There's a link on Moodle to the website for the textbook.) There's a proof of this fact for the case $X = \mathbb{R}$ in the supplement to Chapter 5.

2. Let (X, d) be a metric space. Prove that $d : X \times X \to \mathbb{R}$ is continuous with respect to the product metric d_1 on $X \times X$. (See the text if you need a reminder on how d_1 is defined.)

3. (Do not hand in) Suppose that in a metric space X we have that $N_a(x) = N_b(y)$ for some $x, y \in X$ and $a, b \in \mathbb{R}$. Can we conclude that $a = b$ and $x = y$?

   Hint: For some metrics, yes. What metric should you go to if you're looking for a counterexample?

4. (Do not hand in) Prove that any finite subset of a metric space X is closed in X.

   Hint: Why is it enough to prove that a set containing one element is closed?

5. Prove that the Cantor set is a closed subset of $\mathbb{R}$ with respect to the standard metric on $\mathbb{R}$. (See Problem 6.5 in the text, or type 'Cantor set' into Google and follow the first link for a definition.)

6. (Do not hand in) Let $\mathcal{C}[0, 1]$ be the space of continuous functions on $[0, 1]$, equipped with the sup-norm metric (d_∞). For any subset $A \subseteq [0, 1]$, show that the set $Y = \{f \in \mathcal{C}[0, 1] : f(a) = 0 \text{ for all } a \in A\}$ is a closed subset of $\mathcal{C}[0, 1]$.

7. Prove that a map $f : X \to Y$ of metric spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets $A \subseteq X$, where $\overline{B}$ denotes the closure of B.

8. (Do not hand in) Let A be a nonempty subset of a metric space (X, d). For $x \in X$, define
$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$
   (a) Prove that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

   (b) Show that if $y \in X$ is another point of X, then $d(x, A) \leq d(x, y) + d(y, A)$.

   (c) Prove that $x \to d(x, A)$ defines a continuous map $X \to \mathbb{R}$.

9. Let A be a nonempty subset of a metric space X. Prove that a point $x \in X$ belongs to the boundary ∂A of A if and only if $d(x, A) = d(x, X \setminus A) = 0$, where $d(x, A)$ is the distance from a point to a set defined in the previous problem.

10. (Do not hand in, unless you really want to) For a subset A of a metric space X, prove:

   (a) $\mathring{A} = A \setminus \partial A = \overline{A} \setminus \partial A$

   (b) $\overline{X \setminus A} = X \setminus \mathring{A}$

   (c) $\partial A = \overline{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$

   (d) ∂A is closed in X.