Solutions to Quiz 7 Practice Problems
Math 2580
Spring 2016

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1. Find the equation of the tangent plane to the surface $x^2 + 2y^2 + 3xz = 10$ at the point $(1, 2, \frac{1}{3})$.

   Since the surface is given as a level surface $f(x, y, z) = 10$, where $f(x, y, z) = x^2 + 2y^2 + 3xz$, we evalate the gradient of f at the given point to obtain the normal vector:

   $$\nabla f(x, y, z) = \langle 2x + 3z, 4y, 3x\rangle, \quad \text{so} \quad \nabla f(1, 2, \frac{1}{3}) = \langle 3, 8, 3\rangle.$$

   The equation of the tangent plane is therefore $3(x - 1) + 8(y - 2) + 3(z - \frac{1}{3}) = 0$.

2. On Assignment 3 you're asked to derive the following formula: if $y = g(x)$ is a function satisfying the relation $F(x, y) = C$ for some constant C, then

   $$\frac{dy}{dx} = g'(x) = -\frac{F_x(x, g(x))}{F_y(x, g(x))}.$$

   Use this result to find the slope of the tangent line to the curve $x^2 + y^4 = 5$ at the point $(2, 1)$.

   We first confirm that $2^1 + 1^4 = 5$, so the point $(2, 1)$ is indeed on the curve. With $F(x, y) = x^2 + y^4$, we have $F_x(x, y) = 2x$ and $F_y(x, y) = 4y^3$, so $F_x(2, 1) = 4$ and $F_y(2, 1) = 4$. The slope of the tangent line is therefore given by $m = g'(1) = -\dfrac{4}{4} = -1$, and the equation of the line is $y - 1 = -(x - 2)$.

3. Let $F(x, y, z) = xy^2 - x^2z + 2yz^2$, and suppose $z = g(x, y)$ satisfies the relation $F(x, y, z) = 1$. Use implicit differentiation to compute $g_x(1, 1)$ and $g_y(1, 1)$.

   Suppose that $xy^2 - x^2z + 2yz^2 = 1$ defines z as a function of x and y. If $x = 1$ and $y = 1$, the equation of our surface gives us $1 - z + 2z^2 = 1$, so $2z^2 - z = z(2z - 1) = 0$.

Thus, there are two points on the surface with $x = 1$ and $y = 1$: either $(1, 1, 0)$ or $(1, 1, 1/2)$. Let's write $z = g(x, y)$ for the implicit function satisfying $g(1, 1) = 0$, and $z = h(x, y)$ for the implicit function satisfying $h(1, 1) = 1/2$. Taking the derivative of both sides of the equation of the surface with respect to x, we have

$$y^2 - 2xz - x^2\frac{\partial z}{\partial x} + 4yz\frac{\partial z}{\partial x} = 0.$$

Solving for $\frac{\partial z}{\partial x}$, we have $\frac{\partial z}{\partial x} = \frac{2xz - y^2}{4yz - x^2}$. (Notice that we have $\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$.) For the point $(1, 1, 0)$ we have

$$g_x(1, 1) = \frac{2(1)(0) - 1^2}{4(1)(0) - 1^2} = \frac{-1}{-1} = 1,$$

and for the point $(1, 1, 1/2)$ we have

$$h_x(1, 1) = \frac{2(1)(1/2) - 1^2}{4(1)(1/2) - 1^2} = \frac{1 - 1}{1} = 0.$$

If we take the derivative of both sides of the equation of the surface with respect to y, we have

$$2xy - x^2\frac{\partial z}{\partial y} + 2z^2 + 4yz\frac{\partial z}{\partial y} = 0.$$

Solving for $\frac{\partial z}{\partial y}$, we have $\frac{\partial z}{\partial y} = \frac{-2xy - 2z^2}{4yz - x^2} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$. For the point $(1, 1, 0)$ we have

$$g_y(1, 1) = \frac{-2 - 0}{0 - 1} = \frac{-2}{-1} = 2,$$

and for the point $(1, 1, 1/2)$ we have

$$h_y(1, 1) = \frac{-2 - 2(1/4)}{4(1)(1/2) - 1^2} = \frac{-5/2}{1} = -\frac{5}{2}.$$

4. Suppose $\vec{n} = \langle a, b, c\rangle$ is the normal vector for the tangent plane at a point on surface in $\mathbb{R}^3$. What can you say about the values of a, b, and c if the plane is

   (a) Horizontal?

   If the tangent plane is horizontal, the normal vector must be vertical, so $\vec{n}$ must be a scalar multiple of $\mathbf{k} = \langle 0, 0, 1\rangle$. Thus we must have $a = b = 0$.

   (b) Vertical?

   If the tangent plane is vertical, the normal vector must be horizontal. This amounts to saying that $\vec{n}$ must be parallel to the xy-plane, which means we must have $c = 0$.

5. Find any points (a, b) at which the tangent plane to the surface $z = x^2 - 2x + y^2$ is horizontal.

Letting $f(x, y) = x^2 - 2x + y^2$, we know that the normal vector to the surface $z = f(x, y)$ when $(x, y) = (a, b)$ is given by $\vec{n} = \langle f_x(a, b), f_y(a, b), -1\rangle$. By the previous problem, the tangent plane will therefore be horizontal if $f_x(a, b) = f_y(a, b) = 0$. We have

$$f_x(x, y) = 2x - 2 = 2(x - 1) \quad \text{and} \quad f_y(x, y) = 2y,$$

from which we see that we must have $x = 1$ and $y = 0$. Thus, the tangent plane is horizontal at the point $(1, 0)$. (Technically we should say it's the point $(1, 0, -1)$, giving the z-coordinate, since we're talking about a point on a surface in $\mathbb{R}^3$ but often it's convenient to just give the values of the independent variables.)