Math 3500 Exercise Sheet

26 November, 2014

1. Consider the function $h : [0,1] \to \mathbb{R}$ given by

$$h(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 2 & \text{for } x = 1 \end{cases}.$$

   (a) Show that $L(f,P) = 1$ for every partition P of $[0,1]$.

   (b) Construct a partition P for which $U(f,P) < 1 + 1/10$.

   (c) Given $\epsilon > 0$, construct a partition P_ϵ for which $U(f,P_\epsilon) < 1 + \epsilon$.

2. Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.

   (a) If $|f|$ is integrable on $[a,b]$, then f is integrable on $[a,b]$.

   (b) Assume g is integrable and $g \ge 0$ on $[a,b]$. If $g(x) > 0$ for an infinite number of points $x \in [a,b]$, then $\int g > 0$.

   (c) If g is continuous on $[a,b]$ and $g \ge 0$ with $g(x_0) > 0$ for at least one point $x_0 \in [a,b]$, then $\int_a^b g > 0$.

   (d) If $\int_a^b f > 0$, there is an interval $[c,d] \subseteq [a,b]$ and a $\delta > 0$ such that $f(x) \ge \delta$ for all $x \in [c,d]$.

3. Consider the function $f(x) = x^n$ on $[0,1]$.

   (a) For any points x_{i-1} and x_i with $0 \le x_{i-1} < x_i < 1$, argue that

$$x_{i-1}^n \le \frac{x_{i-1}^n + x_{i-1}^{n-1}x_i + \cdots + x_{i-1}x_i^{n-1} + x_i^n}{n+1} \le x_i^n.$$

   (b) Argue that for any partition P of $[0,1]$, we have $L(f,P) \le \dfrac{1}{n+1} \le U(f,P)$.

   (c) Conclude that $\int_0^1 x^n\,dx = \dfrac{1}{n+1}$. (Note: why is it not necessary to prove $f(x) = x^n$ is integrable?)

4. Prove the Cauchy-Schwarz inequality for integrals: $\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$.

   Hint: let $\alpha = \int_a^b g^2$ and $\beta = -\int_a^b fg$, and consider $\int_a^b (\alpha f + \beta g)^2$.

Hint for 2(b): recall Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ in lowest terms, and } n > 0 \; . \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

It turns out that t is integrable on $[0,1]$ with $\int_0^1 t = 0$. To see this, note that $L(f,P) = 0$ for any partition P. Now consider the set $D_{\epsilon/2} = \{x \in [0,1] \,|\, t(x) \geq \epsilon/2\}$. Argue that this set is finite for all $\epsilon > 0$. From this observation it's possible to construct a partition P_ϵ such that $U(f, P_\epsilon) < \epsilon$. For details, see example 7.2.3 in the text.