Math 1410 Assignment #3 Solutions
University of Lethbridge, Spring 2017

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1. For each of the following subsets S of $\mathbb{R}^3$ (viewed as the vector space of 3×1 column vectors), determine if S is a subspace. If S is a subspace, determine a set of vectors that spans S.

   (a) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; 3x - 4y + z = 2 \right\}$

   The set S is not a subspace, since it does not contain the zero vector:

   $$3(0) - 4(0) + 0 = 0 \neq 2.$$

   (b) $S = \left\{ \begin{bmatrix} 2u - 3v \\ u \\ v - 5u \end{bmatrix} \middle| \; u, v \in \mathbb{R} \right\}$

   We note that for a general element $\vec{w} \in S$ we have

   $$\vec{w} = \begin{bmatrix} 2u - 3v \\ u \\ v - 5u \end{bmatrix} = \begin{bmatrix} 2u \\ u \\ -5u \end{bmatrix} + \begin{bmatrix} -3v \\ 0 \\ v \end{bmatrix} = u \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + v \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

   Letting $\vec{a} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, we have that $\vec{w} \in S$ if and only if $\vec{w} = u\vec{a} + v\vec{b}$ for scalars u and v, which is equivalent to stating that $\vec{w} \in \operatorname{span}\{\vec{a}, \vec{b}\}$.
   It follows that $S = \operatorname{span}\{\vec{a}, \vec{b}\}$, and thus S is a subspace.

(c) $S = \{\vec{v} \in \mathbb{R}^3 \,|\, \vec{v} \cdot \vec{w} = 0\}$, where $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

The set S is a subspace, which we can demonstrate in one of two ways. Using the definition, we check that S is non-empty, since $\vec{0} \cdot \vec{w} = 0$, and thus $\vec{0} \in S$. If $\vec{v}_1, \vec{v}_2$ are any two elements of S, then by definition of S, we have $\vec{v}_1 \cdot \vec{w} = 0$ and $\vec{v}_2 \cdot \vec{w} = 0$. Therefore,

$$(\vec{v}_1 + \vec{v}_2) \cdot \vec{w} = \vec{v}_1 \cdot \vec{w} + \vec{v}_2 \cdot \vec{w} = 0 + 0 = 0,$$

which shows that $\vec{v}_1 + \vec{v}_2 \in S$, and thus S is closed under addition. Similarly, if $c \in \mathbb{R}$ is any scalar, then with $\vec{v}_1 \in S$ as above, we have

$$(c\vec{v}_1) \cdot \vec{w} = c(\vec{v}_1 \cdot \vec{w}) = c(0) = 0,$$

so $c\vec{v}_1 \in S$, showing that S is closed under scalar multiplication. Therefore, by our definition of subspace, S is a subspace.

Alternatively, note that for any vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ we have $\vec{v} \cdot \vec{w} = 3a - b + 2c$, and thus $\vec{v} \in S$ if and only if $3a - b + 2c = 0$. Solving for b, we see that for any $\vec{v} \in S$, we have $b = 3a + 2c$, and thus

$$\vec{v} = \begin{bmatrix} a \\ 3a + 2c \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix},$$

showing that $S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$, from which it follows that S is a subspace.

2. A set of vectors $\mathscr{A} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is called **orthogonal** if $\vec{v}_i \neq \vec{0}$ for each $i = 1, \ldots, k$, and if $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. In other words, $\mathscr{A}$ is a set of non-zero, mutually orthogonal vectors: each vector in the set is orthogonal to all the others.

(a) Show that the set $\mathscr{A} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ is an orthogonal subset of $\mathbb{R}^4$.

It is clear that none of the vectors in $\mathscr{A}$ is the zero vector. We check that

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ -1 \\ -2 \end{bmatrix} = 1(4) - 2(1) + 0(-1) + 1(-2)4 - 2 - 2 = 0,$$

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} = 1(1) - 2(1) + 0(3) + 1(1) = 1 - 2 + 1 = 0, \quad \text{and}$$

$$\begin{bmatrix} 4 \\ 1 \\ -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} = 4(1) + 1(1) - 1(3) - 2(1) = 4 + 1 - 3 - 2 = 0.$$

Since all dot products of different vectors in $\mathscr{A}$ are zero, the set $\mathscr{A}$ is orthogonal.

(b) Prove that any orthogonal set of vectors is linearly independent.

Suppose $\mathscr{A} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthgonal set of vectors, and suppose that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0} \tag{1}$$

for some scalars $c_1, c_2, \ldots, c_k$. To show that $\mathscr{A}$ is linearly independent, we need to show that each of these scalars must equal to zero. If we take the dot product of both sides of (1) with respect to the vector $\vec{v}_i$, for some $i \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned} 0 &= \vec{v}_i \cdot \vec{0} \\ &= \vec{v}_i \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) \\ &= c_1(\vec{v}_i \cdot \vec{v}_1) + c_2(\vec{v}_i \cdot \vec{v}_2) + \cdots + c_k(\vec{v}_i \cdot \vec{v}_k). \end{aligned}$$

But we know that $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$, so the above reduces to simply

$$0 = c_i(\vec{v}_i \cdot \vec{v}_i).$$

We also know that $\vec{v}_i \cdot \vec{v}_i = ||\vec{v}_i||^2 \neq 0$, since by assumption none of the vectors in $\mathscr{A}$ are the zero vector. It follows that $c_i = 0$, and this is true for each $i = 1, 2, \ldots, k$, which is what we needed to show.

(c) Prove that if $\mathscr{A} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is an orthogonal set of vectors and $\vec{w}$ belongs to the span of $\mathscr{A}$, then

$$\vec{w} = \left(\frac{\vec{w} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{w} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 + \cdots + \left(\frac{\vec{w} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\right)\vec{v}_k.$$

This is called the *Fourier decomposition theorem*.

Suppose that $\vec{w} \in \operatorname{span}(\mathscr{A})$. Then, by the definition of the span of a set of vectors, there exist scalars $a_1, a_2, \ldots, a_k$ such that

$$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k. \tag{2}$$

As with the solution to part (b), we take the dot product of both sides of (2) with the vector $\vec{v}_i$, for some choice of $i \in \{1, 2, \ldots, k\}$, giving us

$$\vec{v}_i \cdot \vec{w} = a_i(\vec{v}_i \cdot \vec{v}_i),$$

where we have simplified the right-hand side as in part (b), using the fact that $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$. Since $\vec{v}_i \cdot \vec{v}_i = \|\vec{v}_i\|^2 \neq 0$, we can solve for a_i, giving us $a_i = \dfrac{\vec{v}_i \cdot \vec{w}}{\vec{v}_i \cdot \vec{v}_i}$ for each $i = 1, 2 \ldots, k$. Subsituting these values into (2), we obtain our result.

(d) Let $\mathscr{A}$ be the orthogonal subset of $\mathbb{R}^4$ from part (a). Determine whether or not the following vectors belong to the span of $\mathscr{A}$:

$$\vec{a} = \begin{bmatrix} -4 \\ -7 \\ 5 \\ 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ -5 \\ 1 \end{bmatrix}.$$

We use the result from part (c). We compute the right-hand side of (2) for each vector, and compare to the original vector. We compute the following:

$$\begin{aligned}
\vec{v}_1 \cdot \vec{v}_1 &= 1^2 + (-2)^2 + 0^2 + 1^2 = 6 \\
\vec{v}_2 \cdot \vec{v}_2 &= 4^2 + 1^2 + (-1)^2 + (-2)^2 = 22 \\
\vec{v}_3 \cdot \vec{v}_3 &= 1^2 + 1^2 + 3^2 + 1^2 = 12.
\end{aligned}$$

For the vector $\vec{a}$, we have

$$\begin{aligned}
\vec{a} \cdot \vec{v}_1 &= -4(1) - 7(-2) + 5(0) + 8(1) = 18 \\
\vec{a} \cdot \vec{v}_2 &= -4(4) - 7(1) + 5(-1) + 8(-2) = -44 \\
\vec{a} \cdot \vec{v}_3 &= -4(1) - 7(1) + 5(3) + 8(1) = 12.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\left(\frac{\vec{a} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{a} \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_2}\right)\vec{v}_2 + \left(\frac{\vec{a} \cdot \vec{v}_3}{\vec{v}_1 \cdot \vec{v}_3}\right)\vec{v}_3 &= \frac{18}{6}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \frac{-44}{22}\begin{bmatrix} 4 \\ 1 \\ -1 \\ -2 \end{bmatrix} + \frac{12}{12}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -6 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -8 \\ -2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 5 \\ 8 \end{bmatrix} = \vec{a}
\end{aligned}$$

This shows that $\vec{a} = 3\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3$, so $\vec{a} \in \operatorname{span}(\mathscr{A})$.

For the vector $\vec{b}$, we have

$$\begin{aligned}
\vec{b}\cdot\vec{v}_1 &= 2(1)+3(-2)-5(0)+1(1) = -3\\
\vec{b}\cdot\vec{v}_2 &= 2(4)+3(1)-5(-1)+1(-2) = 14\\
\vec{b}\cdot\vec{v}_3 &= 2(1)+3(1)-5(3)+1(1) = -9.
\end{aligned}$$

This gives us

$$\begin{aligned}
\left(\frac{\vec{b}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{b}\cdot\vec{v}_2}{\vec{v}_1\cdot\vec{v}_2}\right)\vec{v}_2 + \left(\frac{\vec{b}\cdot\vec{v}_3}{\vec{v}_1\cdot\vec{v}_3}\right)\vec{v}_3 &= \frac{-3}{6}\begin{bmatrix}1\\-2\\0\\1\end{bmatrix} + \frac{-14}{22}\begin{bmatrix}4\\1\\-1\\-2\end{bmatrix} + \frac{-9}{12}\begin{bmatrix}1\\1\\3\\1\end{bmatrix}\\
&= \begin{bmatrix}-167/44\\-17/44\\71/44\\1/44\end{bmatrix} \neq \vec{b}
\end{aligned}$$

This tells us that $\vec{b} \notin \operatorname{span}(\mathscr{A})$, since equation (2) is not satisfied.

3. In the previous problem, we saw that if $\mathscr{A}$ is an orthogonal set of vectors, and $\vec{w} \in \operatorname{span}(\mathscr{A})$, then the $\vec{w}$ can be written in terms of the vectors in $\mathscr{A}$ using the Fourier decomposition theorem. If $\vec{w}$ is **not** in the span of $\mathscr{A}$, then the vector

$$\vec{v} = \left(\frac{\vec{w}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{w}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2 + \cdots + \left(\frac{\vec{w}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\right)\vec{v}_k. \tag{3}$$

is called the **orthogonal projection** of $\vec{w}$ onto the subspace $U = \operatorname{span}(\mathscr{A})$, and dentoed by $\operatorname{proj}_U(\vec{w})$. In more advanced linear algebra courses, one proves that $\operatorname{proj}_U(\vec{w})$ is the element of U that is *closest* to $\vec{w}$, in the sense that $\|\vec{w} - \operatorname{proj}_U(\vec{w})\|$ is as small as possible.

Consider the subspace $U \subseteq \mathbb{R}^3$ given by $U = \operatorname{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$. Note that U is a plane through the origin, and that the vectors $\vec{v} = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$ are orthogonal.

Determine the point Q on the plane U that is closest to the point $P = (3, -1, 4)$ (and the distance from P to Q):

(a) By computing the orthogonal projection of $\vec{p} = \begin{bmatrix}3\\-1\\4\end{bmatrix}$ onto U, as described above.

We will compute the projection $\vec{q} = \text{proj}_U \vec{p}$ using equation (3) above. We have

$$\begin{aligned}\vec{q} &= \left(\frac{\vec{p}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} + \left(\frac{\vec{p}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}\\ &= \frac{3(1)-1(2)+4(0)}{1^2+2^2+0^2}\begin{bmatrix}1\\2\\0\end{bmatrix} + \frac{3(2)-1(-1)+4(1)}{2^2+(-1)^2+1^2}\begin{bmatrix}2\\-1\\1\end{bmatrix}\\ &= \frac{1}{5}\begin{bmatrix}1\\2\\0\end{bmatrix} + \frac{11}{6}\begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}58/15\\-43/30\\11/6\end{bmatrix}.\end{aligned}$$

The point on the plane closest to P is therefore $Q = \left(\frac{58}{15}, -\frac{43}{30}, \frac{11}{6}\right)$, and the distance is

$$d(P,Q) = \sqrt{\left(3-\frac{58}{15}\right)^2 + \left(-1+\frac{43}{30}\right)^2 + \left(4-\frac{11}{6}\right)^2} = \sqrt{(-13/15)^2+(13/30)^2+(13/6)^2}$$

If we want to simplify this, note that

$$\left(\frac{-13}{15}\right)^2+\left(\frac{13}{30}\right)^2+\left(\frac{13}{6}\right)^2 = \left(-2\cdot\frac{13}{30}\right)^2+\left(\frac{13}{30}\right)^2+\left(5\cdot\frac{13}{30}\right)^2 = \left(\frac{13}{30}\right)^2((-2)^2+1^2+5^2).$$

Thus, $d(P,Q) = \frac{13}{30}\sqrt{(-2)^2+1^2+5^2} = \frac{13}{30}\sqrt{30} = \frac{13}{\sqrt{30}}$.

(b) Using the method described in Example 54 (and the discussion that follows) in Section 3.6 of the textbook.

We first compute the normal vector

$$\vec{n} = \vec{v}\times\vec{w} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 2 & 0\\ 2 & -1 & 1\end{vmatrix} = \begin{bmatrix}2\\-1\\-5\end{bmatrix}.$$

We next compute the projection of $\vec{p}$ onto $\vec{n}$. We find

$$\text{proj}_{\vec{n}}\vec{p} = \left(\frac{\vec{p}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\right)\vec{n} = -\frac{13}{30}\begin{bmatrix}2\\-1\\-5\end{bmatrix}.$$

According to Example 54 in the textbook, we have $\text{proj}_{\vec{n}}\vec{p} = \overrightarrow{QP}$, where Q is the point on the plane closest to P. This tells us that the distance from the point P to the plane is

$$d(P,Q) = \|\text{proj}_{\vec{n}}\vec{p}\| = \left\|-\frac{13}{30}\begin{bmatrix}2\\-1\\5\end{bmatrix}\right\| = \frac{13}{30}\left\|\begin{bmatrix}2\\-1\\-5\end{bmatrix}\right\| = \frac{13}{30}\sqrt{30} = \frac{13}{\sqrt{30}},$$

the same as before.
The point P can be found using the fact that $\overrightarrow{QP} = \vec{p} - \vec{q}$, so

$$\vec{q} = \vec{p} - \overrightarrow{QP} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} - \left(-\frac{13}{30}\right) \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 58/15 \\ -43/30 \\ 11/6 \end{bmatrix},$$

which also agrees with our previous answer.