Math 2580 Assignment #1 Solutions
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1. Each of the equations below describes a quadric surface. Identify (as an ellipsoid, hyperboloid, etc.) and sketch each surface.

   (a) $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1.$

   This is an ellipsoid centred at the origin, with $-2 \leq x \leq 2$, $-3 \leq y \leq 3$, and $-1 \leq z \leq 1$. Anything that looks more or less like a rugby ball with the correct dimensions is acceptable.

   (b) $x^2 + z^2 = 1 - 2y^2$

   Rearranging gives the equation $x^2 + 2y^2 + z^2 = 1$, so this is another ellipsoid with $-1 \leq x, z \leq 1$ and $-1/\sqrt{2} \leq y \leq 1/\sqrt{2}$.

   (c) $z + y^2 = 2x^2$.

   This equation can be rewritten as $z = 2x^2 - y^2$, so this is a hyperbolic paraboloid (saddle surface). These guys are hard to draw, so any reasonable attempt is acceptable. The 2 in front of the x means it's squished a bit in the x direction, but not by any amount that can reasonably be reflected in your sketch.

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Compute f_x and f_y for $(x,y) \neq (0,0)$.

   For $(x,y) \neq (0,0)$ we have

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}\left(\frac{x^3y - xy^3}{x^2+y^2}\right) \\ &= \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}, \end{aligned}$$

and

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y}\left(\frac{x^3y - xy^3}{x^2+y^2}\right) \\ &= \frac{(x^3-3xy^2)(x^2+y^2) - 2y(x^3y - xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}. \end{aligned}$$

(b) Show that $f_x(0,0) = f_y(0,0) = 0$.

Using the limit definitions of f_x and f_y, we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and similarly, $f_y(0,0) = 0$.

(c) Show that $f_x(0,y) = -y$ when $y \neq 0$.

Plugging in $x = 0$ to our solution from part (a), we have

$$f_x(0,y) = \frac{-y^5}{(y^2)^2} = -y.$$

(d) What is $f_y(x,0)$ when $x \neq 0$?

Using the same argument as in part 2c, we have $f_y(x,0) = \dfrac{x^5}{x^4} = x$.

(e) Show that $f_{yx}(0,0) = 1$ and $f_{xy}(0,0) = -1$. (You'll need to use limits again.)

Since f_{yx} is the partial derivative of f_y with respect to x, using our result from part 2d we have

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

Similarly, f_{xy} is the partial derivative of f_x with respect to y, so using the result from 2c,

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1.$$

(f) Why does this not contradict the theorem about equality of mixed partials?

Clairaut's Theorem only guarantees equality of mixed second-order partial derivatives at a point if all second-order partial derivatives exist and are continuous on an open disc containing that point. Since $f_{xy}(0,0) \neq f_{yx}(0,0)$, it must be the case that these derivatives are not continuous at $(0,0)$. Checking this is a lot of work though: we'd have to compute $f_{xy}(x, y)$ and $f_{yx}(x, y)$ (and the other two second-order derivatives) for $(x, y) \neq (0, 0)$ and show that either the limit of these functions does not exist as $(x, y) \to (0, 0)$, or that the limits exist, but are not equal to -1 and 1, respectively.

If we believe Clairaut's Theorem to be true (and it is), then since the conclusion of the theorem failed in this case, it must be true that the hypothesis failed as well.