Math 1410 Assignment #4
University of Lethbridge, Spring 2015

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Due date: Wednesday, March 25th, by 5 pm.

Assignment #4 should be prepared according to the guidelines as outlined on Assignments 1 and 3.

Assigned problems

1. Let A be an $m \times n$ matrix. Note that each column of A is of size $m \times 1$, and therefore a vector in $\mathbb{R}^m$. Recall that for a vector $\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ in $\mathbb{R}^n$, we have

$$A\vec{x} = x_1 C_1 + x_2 C_2 + \cdots + x_n C_n,$$

where $C_1, C_2, \ldots, C_n$ denote the columns of A. Show that the following are true:

   (a) The columns $C_1, C_2, \ldots, C_n$ are linearly independent if and only if[1] the only solution to the homogeneous equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

   (b) The non-homogeneous equation $A\vec{x} = \vec{y}$ has a solution if and only if $\vec{y} \in \operatorname{span}\{C_1, C_2, \ldots, C_n\}$.

2. Find the point of intersection (if any) of the following pairs of lines:

   (a)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

[1] For an "if and only if" statement you need to prove two things: if the columns are independent, then the only solution to the homogeneous system is the trivial solution, **and** if the only solution to the homogeneous system is the trivial solution, then the columns are linearly independent. For both directions, it's mostly a matter of looking up the definition of linear independence in your notes and writing it down in this context.

(b)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

3. (a) Show that $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is perpendicular to the line $ax + by + c = 0$.

   (b) Show that the shortest distance from the point $P_1 = (x_1, y_1)$ to the line is

$$\frac{|x_1 + y_1 + c|}{\sqrt{a^2 + b^2}}.$$

   Hint: Take any point P_0 on the line and project $\vec{u} = \overrightarrow{P_0P_1}$ onto $\vec{n}$. If you haven't drawn yourself a picture, you're probably doing it wrong.

   (c) Now, let L be a line in $\mathbb{R}^3$ through the point $P_0 = (x_0, y_0, z_0)$ with direction vector $\vec{d}$. Show that the shortest distance from a point $P_1 = (x_1, y_1, z_1)$ to the line is

$$\frac{\|\overrightarrow{P_0P_1} \times \vec{d}\|}{\|d\|}.$$

4. Find the shortest distance between the following pair of skew lines, and the points on each line that are closest together:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$