Name: **Tutorial day and time:**

Select one *completed* problem for feedback:

1. Let A, B, and C be $n \times n$ matrices such that $\det(A) = 2$, $\det(B) = -1$, and $\det(C) = 3$. Evaluate $\det(A^3BC^TB^{-1})$.

$$|A^3BC^TB^{-1}| = |A^3||B||C^T||B^{-1}| = |A|^3|B||C|\left(\frac{1}{|B|}\right) = 2^3(4) = 24.$$

2. If A and B are 3×3 matrices such that $\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T)$, what are the values of $\det(A)$ and $\det(B)$?

Since A is 3×3, we have $\det(2A^{-1}) = 2^3\det(A^{-1}) = \frac{8}{\det(A)} = -4$, giving us $\det(A) = -2$. Using this, we find

$$\det(A^3(B^{-1})^T) = (\det(A))^3\det(B^{-1}) = \frac{(-2)^3}{\det(B)} = -4,$$

and solving for $\det(B)$ gives us $\det(B) = 2$.

3. The matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ has eigenvalues $\lambda = 2$ and $\lambda = 6$. Find the corresponding eigenvectors.

For the eigenvalue $\lambda = 2$, we have

$$A - 2I = \begin{bmatrix} 3-2 & 2 & 1 \\ 1 & 4-2 & 1 \\ 1 & 2 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that if $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution to $(A - 2I)\vec{x} = \vec{0}$, then $y = s$ and $z = t$ are free variables, while $x + 2y + z = 0$ gives $x = -2s - t$. Thus,

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\vec{u} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are both basic solutions to the homogeneous system $(A - 2I)\vec{x} = \vec{0}$, they must both be eigenvectors corresponding to the eigenvalue $\lambda = 2$. (You should verify this by computing $A\vec{u}$ and $A\vec{v}$.)

For the eigenvalue $\lambda = 6$, we have

$$A - 6I = \begin{bmatrix} 3-6 & 2 & 1 \\ 1 & 4-6 & 1 \\ 3 & 2 & 1-6 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 3 & 2 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, any solution $\vec{x} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to $(A - 6I)\vec{x} = \vec{0}$ must satisfy $x = y = z$, where $z = t$ is free.

Setting $t = 1$ gives us the eigenvector $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 6$.

4. Compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

We first compute the eigenvalues. We have

$\det(A - xI) = \begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = (1-x)(3-x) - 8 = x^2 - 4x + 3 - 8 = x^2 - 4x + 5 = (x+1)(x-5).$

Thus, $\det(A - xI) = 0$ for the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$.
For the eigenvalue $\lambda_1 = -1$, we have

$$A - \lambda_1 I = A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

so any solution $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ to $(A + \lambda_1 I)\vec{x} = \vec{0}$ must satisfy $x = -2y$, where y is free. Setting $y = 1$, we get the eigenvector $\vec{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ corresponding to $\lambda_1 = -1$.
For $\lambda_2 = 5$, we have

$$A - \lambda_2 I = A - 5I = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

so if $(A - 5I)\vec{x} = \vec{0}$, we must have $\vec{x} = \begin{bmatrix} t \\ t \end{bmatrix}$ for some free parameter t. Setting $t = 1$ gives us $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the eigenvector corresponding to $\lambda_2 = 5$.

5. Verify that the matrix $Z = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ has eigenvalues $\lambda_\pm = 2 \pm i$ with corresponding eigenvectors $\vec{x}_+ = \begin{bmatrix} 1+i \\ -2 \end{bmatrix}$, $\vec{x}_- = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$.

We compute

$$Z\vec{x}_+ = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ -2 \end{bmatrix} = \begin{bmatrix} 3+3i-2 \\ -2-2i-2 \end{bmatrix} = \begin{bmatrix} 1+3i \\ -4-2i \end{bmatrix}.$$

On the other hand,

$$\lambda_+\vec{x}_+ = (2+i) \begin{bmatrix} 1+i \\ -2 \end{bmatrix} = \begin{bmatrix} (2+i)(1+i) \\ (2+i)(-2) \end{bmatrix} = \begin{bmatrix} 1+3i \\ -4-2i \end{bmatrix},$$

so we have verified that $Z\vec{x}_+ = \lambda_+\vec{x}_+$. Similarly,

$$Z\vec{x}_- = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} 3-1-i \\ -2-1-i \end{bmatrix} = \begin{bmatrix} 2-i \\ -3-i \end{bmatrix} = (2-i) \begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \lambda_-\vec{x}_-,$$

as required.