Math 1565 Tutorial 1 Solutions

1. **Using the definition** of $\sinh(x)$, show that

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y).$$

Starting on the right-hand side, using the definitions of $\sinh(t)$ and $\cosh(t)$, we have

$$\begin{aligned}\sinh(x)\cosh(y) + \cosh(x)\sinh(y) &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \frac{1}{4}\left(e^x e^y + e^x e^{-y} - e^{-x}e^y - e^{-x}e^{-y}\right) \\ &\quad + \frac{1}{4}\left(e^x e^y - e^x e^{-y} + e^{-x}e^y - e^{-x}e^{-y}\right) \\ &= \frac{1}{4}\left(2e^{x+y} - 2e^{-x-y}\right) \\ &= \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y),\end{aligned}$$

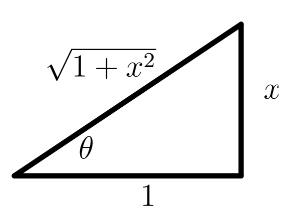
as required.

2. Show that $\sin(\tan^{-1}(x)) = \dfrac{x}{\sqrt{1+x^2}}$.

We let $\theta = \tan^{-1}(x)$, which gives us $\tan\theta = x$. We can now proceed in one of two ways:

(a) Geometric approach: we draw a little triangle, with one angle labelled by $\theta$:

Since $\tan\theta = x = \dfrac{x}{1}$ gives the ratio of the length of the side opposite $\theta$ to that adjacent to $\theta$, we can label two of the three sides as shown, and the Pythagorean theorem gives us the remaining side.




Since $\sin\theta$ gives the ratio of the lengths of the opposite side and hypotenuse, we find that

$$\sin(\tan^{-1}(x)) = \sin\theta = \frac{x}{\sqrt{1+x^2}}.$$

However, we note that our solution is only valid for $x \geq 0$, since we need $0 \leq \theta \leq \pi/2$ to draw the triangle. To address this, we note that both sides of our identity involve odd functions:

$$\sin(\tan^{-1}(-x)) = \sin(-\tan^{-1}(x)) = -\sin(\tan^{-1}(x))$$

since both the sin and $\tan^{-1}$ functions are odd, and

$$\frac{-x}{\sqrt{1+(-x)^2}} = -\frac{x}{\sqrt{1+x^2}}.$$

Analytic approach: From $\tan\theta = x$, we get $\sin\theta = x\cos\theta$. Now, since $\tan^2\theta + 1 = \sec^2\theta = 1/\cos^2\theta$, we get

$$\cos\theta = \pm\frac{1}{\sqrt{1+\tan^2\theta}} = \pm\frac{1}{\sqrt{1+x^2}}.$$

Since $\tan^{-1}(x) \in (-\pi/2, \pi/2)$ for all $x \in \mathbb{R}$ and $\cos\theta > 0$ for all $x$ in $(-\pi/2, \pi/2)$, we take the positive root and the result follows.