# Math 3500 Assignment #9 Solutions
# University of Lethbridge, Fall 2014

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December 1, 2014

1. Let $f$ be a bounded function on $[a, b]$, let $\mathcal{P}$ denote the set of all partitions of $[a, b]$, and let $P \in \mathcal{P}$ be an arbitrary partition of $[a, b]$.

   (a) Prove that $U(f) \geq L(f, P)$, where $U(f) = \inf\{U(f, P)|P \in \mathcal{P}\}$.

   We know that for any partition $P'$, $L(f, P) \leq U(f, P')$. Thus, $L(f, P)$ is a lower bound for $\{U(f, P) : P \in \mathcal{P}\}$. Since $U(f)$ is the *greatest* lower bound, we have $U(f) \geq L(f, p)$.

   (b) Prove that $U(f) \geq L(f)$, where $L(f) = \sup\{L(f, P)|P \in \mathcal{P}\}$.

   Since the partition $P$ in part (a) was arbitrary, it follows that $U(f)$ is an upper bound for $\{L(f, P) : P \in \mathcal{P}\}$. Since $L(f)$ is the *least* upper bound, we have $L(f) \leq U(f)$.

2. Let $f$ be a bounded function on $[a, b]$.

   (a) Prove that $f$ is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying
   $$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0.$$

   If $f$ integrable, then for each $n \in \mathbb{N}$, taking $\epsilon = 1/n$ there must exist a partition $P_n$ such that $0 \leq U(f, P_n) - L(f, P_n) < 1/n$. Since $1/n \to 0$ as $n \to \infty$, it follows that $U(f, P_n) - L(f, P_n) \to 0$ as well.

   Conversely, suppose there exists a sequence of partitions $(P_n)$ such that $a_n = U(f, P_n) - L(f, P_n) \to 0$ as $n \to \infty$. Then for any $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n < \epsilon$, and thus for any partition $P_n$ with $n \geq N$, we have $U(f, P_n) - L(f, P_n) < \epsilon$, so $f$ must be integrable.

(b) For each $n$, let $P_n$ denote the uniform partition of $[0,1]$ into $n$ equal subintervals of length $1/n$, and let $f(x) = x$. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ in terms of $n$.

Our partition is given by $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\}$. Since $f$ is increasing on $[0,1]$, on each subinterval $[(i-1)/n, i/n]$ we have $m_i = (i-1)/n$ and $M_i = i/n$. It follows that

$$L(f, P_n) = \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right) = \frac{1}{n^2}\left(\frac{n(n+1)}{2} - n\right) = \frac{n-1}{2n},$$

and

$$U(f, P_n) = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{n+1}{2n}.$$

(c) Use the results from (a) and (b) to prove that $f(x) = x$ is integrable on $[0,1]$.

For each $n$ we have that $U(f, P_n) - L(f, P_n) = \frac{n+1}{2n} - \frac{n-1}{2n} = \frac{1}{n}$. Thus, for any $\epsilon > 0$ we can choose $n$ such that $1/n < \epsilon$, and the result follows.

3. Let $f : [a,b] \to \mathbb{R}$ be bounded and increasing. Show that $f$ is integrable on $[a,b]$.

Let $P_n = \left\{a, a + \dfrac{b-a}{n}, a + \dfrac{2(b-a)}{n}, \ldots, a + \dfrac{n(b-a)}{n} = b\right\}$ denote the uniform partition of $[a,b]$ into $n$ subintervals of length $\Delta x = \dfrac{b-a}{n}$. Since $f$ is increasing on $[a,b]$, for each $i = 1, \ldots, n$ we have $m_i = f(x_{i-1})$ and $M_i = f(x_i)$, where $x_i = a + \dfrac{i(b-a)}{n}$. Given $\epsilon > 0$, choose $n$ sufficiently large that $(f(b) - f(a))\dfrac{(b-a)}{n} < \epsilon$. Then we have

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\left(\frac{b-a}{n}\right) \\
&= (f(b) - f(a))\frac{(b-a)}{n} \quad \text{(telescoping sum)} \\
&< \epsilon.
\end{aligned}$$

4. Define the function $H(x) = \displaystyle\int_1^x \frac{1}{t}\, dt$, where $x > 0$.

   (a) What is the value of $H(1)$? What is $H'(x)$ for any $x > 0$?

   By definition, $H(1) = \int_1^1 \frac{1}{t}\, dt = 0$. By the Fundamental Theorem of Calculus, $H'(x) = 1/x$ for all $x > 0$.

   (b) Show that if $0 < x < y$, then $H(x) < H(y)$; that is, that $H$ is strictly increasing on $(0, \infty)$.

   Since $H'(x) = 1/x > 0$ on $(0, \infty)$, the result follows from the Mean Value Theorem: $H(y) - H(x) = \dfrac{1}{c}(y - x) > 0$ for some $c \in (x, y)$.

   (c) Show that $H(cx) = H(c) + H(x)$ for any $c > 0$.

   Let $g(x) = H(cx)$. Then by the Chain Rule we have

   $$g'(x) = H'(cx) \cdot c = \frac{1}{cx} \cdot c = \frac{1}{x} = H'(x).$$

   Since $g'(x) = H'(x)$ we must have $g(x) = H(x) + k$ for some $k \in \mathbb{R}$, for all $x > 0$. Setting $x = 1$, we have $k = g(1) = H(c)$, since $H(1) = 0$. Thus, $H(cx) = H(x) + H(c)$.

   (d) Use a similar argument to show that $H(x^a) = aH(x)$.

   If we let $f(x) = H(x^a)$, then

   $$f'(x) = H'(x^a)(ax^{a-1}) = \frac{ax^{a-1}}{x^a} = \frac{a}{x} = aH'(x).$$

   Thus, we must have $f(x) = aH(x) + k$ for some $k \in \mathbb{R}$, for all $x > 0$. Since $H(1) = 0$, we find $k = f(1) = H(1^a) = 0$, and the result follows.

Note: One often writes the function $H(x)$ as $\ln(x)$, and refers to this function as the natural logarithm. Parts (c) and (d) then tell us that $\ln(xy) = \ln x + \ln y$ and $\ln(x^y) = y \ln x$. Since $H$ is strictly increasing on $(0, \infty)$, it is one-to-one and therefore has a well-defined inverse function, which is usually denoted by $H^{-1}(x) = e^x$.

5. (**Bonus**) Define a bounded function $f$ on $[0, 1]$ by $f(x) = \begin{cases} 1, & \text{if } x = 1/n \\ 0, & \text{otherwise} \end{cases}$.

   Prove that $f$ is integrable on $[0, 1]$.

Let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ such that $1/N < \epsilon/2$. Since $f$ has finitely many discontinuities on $[1/n, 1]$, it is integrable on $[1/N, 1]$, and thus there exists a partition $P'$ of $[1/N, 1]$ with $U(f, P') - L(f, P') < \epsilon/2$. On $[0, 1/N]$ we take the partition $P'' = \{0, 1/N\}$. Then $U(f, P'') = (1)\left(\frac{1}{N} - 0\right) = \frac{1}{N}$, and $L(f, P'') = 0$, since $0 \leq f(x) \leq 1$ on $[0, 1/N]$. It follows that $U(f, P'') - L(f, P'') = 1/N < \epsilon/2$.

Therefore, taking the partition $P = P' \cup P''$ of $[0, 1]$, we have $U(f, P) - L(f, P) < \epsilon/2 + \epsilon/2 = \epsilon.$, so $f$ is integrable.