Math 4310 Assignment #8 Solutions
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1. Prove that any finite subset of a topological space is compact.

   Solution: Let X be a topological space and let $A = \{x_1, \ldots, x_n\}$ be a finite subset of X. Given any open cover $\mathcal{A}$ of A, we know that for each $i = 1, \ldots, n$ there exists $A_i \in \mathcal{A}$ such that $x_i \in A_i$. It follows that $A \subseteq A_1 \cup \cdots \cup A_n$, so $\{A_i : i = 1, \ldots, n\}$ is a finite subcover.

2. Let X be a set and let $\mathcal{T}_1, \mathcal{T}_2$ be two topologies on X, such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

   (a) Prove that if $(X, \mathcal{T}_2)$ is compact, then $(X, \mathcal{T}_1)$ is compact.

   Solution: Suppose $(X, \mathcal{T}_2)$ is compact and let $\mathcal{A}$ be an open cover of $(X, \mathcal{T}_1)$. Then $\mathcal{A}$ is a collection of open sets in the topology $\mathcal{T}_1$ such that $X = \bigcup_{A \in \mathcal{A}} A$. But any open set in $\mathcal{T}_1$ is also an open set in the topology $\mathcal{T}_2$, since $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Thus, $\mathcal{A}$ is an open cover of $(X, \mathcal{T}_2)$. Since this space is compact, there must be a finite subcover $\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$ such that $X = A_1 \cup \cdots \cup A_n$, and since each A_i was originally chosen to be open in $\mathcal{T}_1$, we have our finite subcover and thus $(X, \mathcal{T}_1)$ must be compact.

   (b) Prove that if $(X, \mathcal{T}_1)$ is Hausdorff and $(X, \mathcal{T}_2)$ is compact, then $\mathcal{T}_1 = \mathcal{T}_2$.

   Solution: Suppose $(X, \mathcal{T}_1)$ is Hausdorff, and $(X, \mathcal{T}_2)$ is compact, where $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Consider the identity map $I_X : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ given by $I_X(x) = x$ for all $x \in X$. For any $U \in \mathcal{T}_1$ we have $I_X^{-1}(U) = U$, and since $\mathcal{T}_1 \subseteq \mathcal{T}_2$, U is open in $(X, \mathcal{T}_2)$. Thus, I_X is a continuous bijection. Since any continuous bijection from a compact space to a Hausdorff space is a homeomorphism, we must have that $\mathcal{T}_1 = \mathcal{T}_2$.
   (Recall the proof of this fact: if $F \subseteq (X, \mathcal{T}_2)$ is closed, then it is compact, since $(X, \mathcal{T}_2)$ is compact, and thus $I_X(U) = U$ is a compact subset of $(X, \mathcal{T}_1)$, since I_X is continuous and the continuous image of a compact set is compact. But $(X, \mathcal{T}_1)$ is Hausdorff, and compact subsets of a Hausdorff space are closed. Thus I_X sends closed sets to closed sets, which shows that the inverse map $I_X^{-1} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is also continuous.)

3. Prove that if $\{A_\alpha\}$ is any collection of compact subsets of a Hausdorff space X, then $\bigcap_\alpha A_\alpha$ is compact.

   Solution: Let $\{A_\alpha\}$ be a collection of compact subsets of a Hausdorff space X. Since X is Hausdorff, can conclude that A_α is closed for each α, since A_α is compact. Thus, $\bigcap_\alpha A_\alpha$ is closed, since the intersection of closed sets is closed. Since $\bigcap A_\alpha \subseteq A_\beta$ for any β, $\bigcap A_\alpha$ is a closed subset of a compact set, and is therefore compact.

4. Prove that if Y is compact, then the projection $\pi_X : X \times Y \to X$ is a closed map.

   Solution: I'll present two proofs. The first uses the Tube Lemma: Suppose $F \subseteq X \times Y$ is closed. We will show that $X \setminus \pi_X(F)$ is closed. Choose any $x_0 \in X \setminus \pi_X(F)$, and consider the slice $S_0 = \{x_0\} \times Y$. Since $x_0 \notin \pi_X(F)$, we must have $(x_0, y) \notin F$ for all $y \in Y$, so $S_0 \cap F = \emptyset$. Thus $N = X \times Y \setminus F$ is an open neighbourhood of the slice S_0, so by the Tube Lemma there exists a neighbourhood W of x_0 such that $W \times Y \subseteq N$. Since $W \times Y \cap F = \emptyset$, it follows that $W \cap \pi_X(F) = \emptyset$, so W is the desired open neighbourhood of x_0.

   The second proof uses the compactness of Y directly. Suppose $F \subseteq X \times Y$ is closed, and $x_0 \in X \setminus \pi_X(F)$. Then, for each $y \in Y$, $(x_0, y) \notin F$, so $(x_0, y) \in X \times Y \setminus F$, which is open, so there exists a basic open set $U_y \times V_y$ such that $(x_0, y) \in U_y \times V_y \subseteq X \times Y \setminus F$. Since the open sets V_y cover Y and Y is compact, there exist finitely many sets $V_{y_1}, \ldots, V_{y_n}$ such that $Y = V_{y_1} \cup \cdots \cup V_{y_n}$. Now let $U = U_{y_1} \cap \cdots \cap U_{y_n}$. Since $x_0 \in U_{y_i}$ for $i = 1, \ldots, n$ and U is open, since it's the intersection of finitely many open sets, we see that U is a neighbourhood of x_0. Moreover, $U \cap \pi_X(F) = \emptyset$ since if $x \in U$ then $(x, y) \notin F$, for each $y \in Y$, so $x \notin \pi_X(F)$. (Note that this is essentially the proof of the Tube Lemma.)

5. Prove the following theorem: Let Y be a compact Hausdorff space, and let $f : X \to Y$ be a map. Then f is continuous if and only if the graph of f, $\Gamma_f = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$.

   Hint: If Γ_f is closed and V is a neighbourhood of $f(x_0)$ in Y, then the intersection of Γ_f and $X \times (Y \setminus V)$ is closed. Now apply the previous problem.

   Solution: First, suppose that f is continuous, and choose a point $(x, y) \in X \times Y \setminus \Gamma_f$. Then $y \neq f(x)$, and since Y is Hausdorff, there exist open neighbourhoods U of y and V of $f(x)$ in Y with $U \cap V = \emptyset$. Since f is continuous, $W = f^{-1}(V)$ is open in X and $x' \in W$ satisfies $f(x') \in f(f^{-1}(V)) \subseteq V$. If $(x', y') \in W \times U$ we must have $y' \neq f(x')$ since $y' \in U$ and $f(x') \in V$, and U and V are disjoint. Thus, $W \times U$ is a neighbourhood of (x, y) contained in the complement of Γ_f. It follows that Γ_f is closed, since its complement is open.

   Conversely, suppose that Γ_f is closed in $X \times Y$, where Y is a compact Hausdorff space. Fix a point $x_0 \in X$, and let V be a neighbourhood of $f(x_0) \in Y$. Since

$X \times (Y \setminus V)$ is closed, so is $F = (X \times (Y \setminus V)) \cap \Gamma_f = \{(x, f(x)) : f(x) \notin V\}$. Since Y is compact, $\pi_X(F) = \{x \in X | f(x) \notin V\}$ is closed, by the previous problem. Thus, $f^{-1}(V) = \{x \in X | f(x) \in V\}$ is open, since it's the complement of a closed set. Thus, f is continuous.

Note: For the first direction, it's tempting to say that if f is continuous, then the map $x \mapsto (x, f(x))$ defines a homeomorphism of X onto the subspace $\Gamma_f \subseteq X \times Y$, (Proposition 10.18 in the text) and it follows that Γ_f is closed. Technically this is true, but in fact we've only proved that Γ_f is closed *as a subset of itself*. It's not guaranteed that it will be closed as a subset of the ambient space $X \times Y$.