Determinants

Math 1410 Linear Algebra

Introduction

A determinant is a number that can be associated to any $n \times n$ matrix.

- Definition is recursive: first define 2×2 determinants, then 3×3 in terms of 2×2, 4×4 in terms of 3×3, etc.
- Original use: determining if a system of n equations in n variables has a unique solution.
- Historically, they pre-date matrices. (By about 2200 years!)
- Applications include solving systems of equations, volumes in 2, 3 and higher dimensions, differential equations, change of variables in multiple integrals, etc.

Determinants: the 2×2 case

We begin with 2×2 determinants.

$$\text{General } 2 \times 2 \text{ matrix } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Definition

The determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

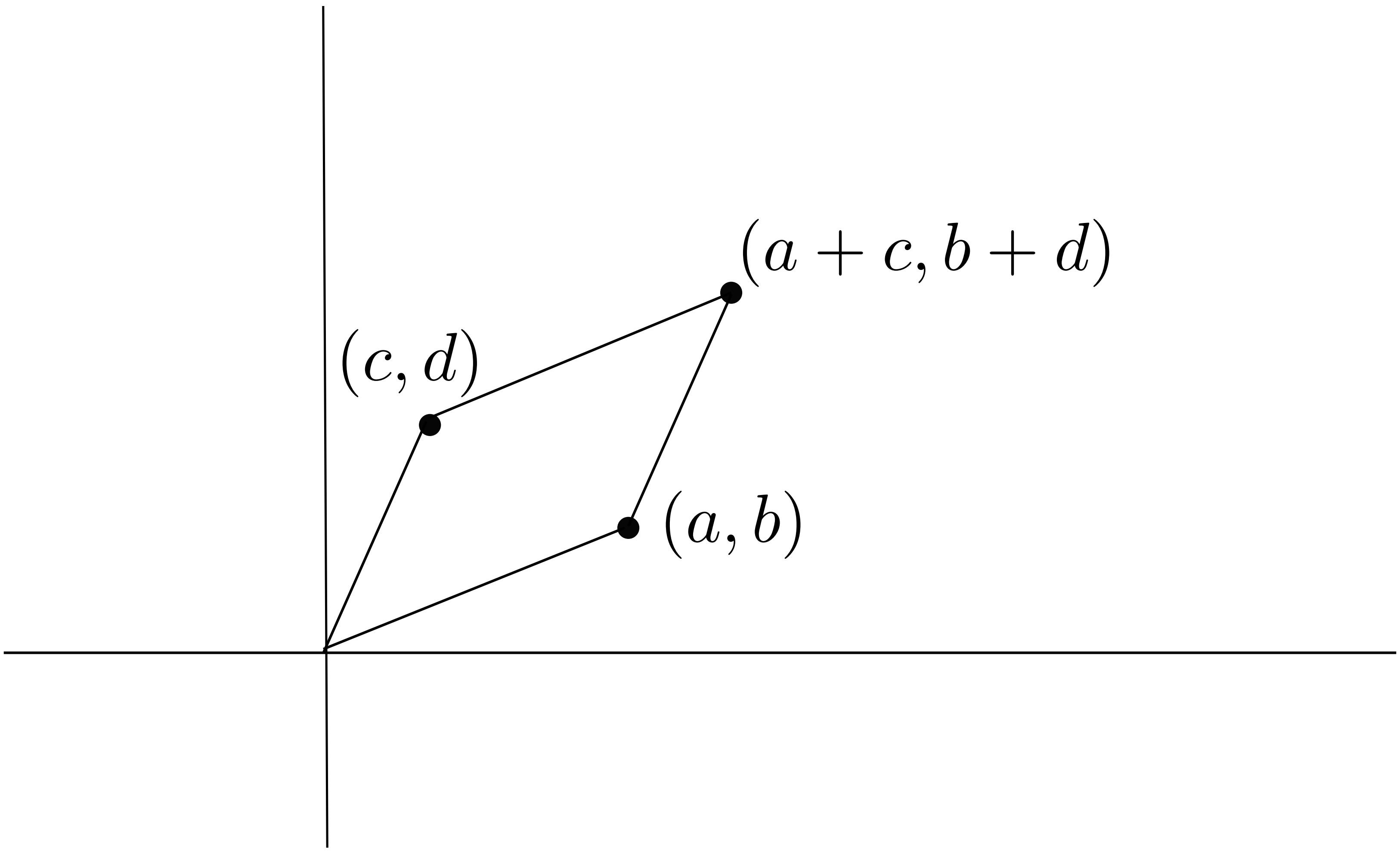
$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Note: we write either $\det A$ or $|A|$ to denote the determinant of the matrix A.

Examples

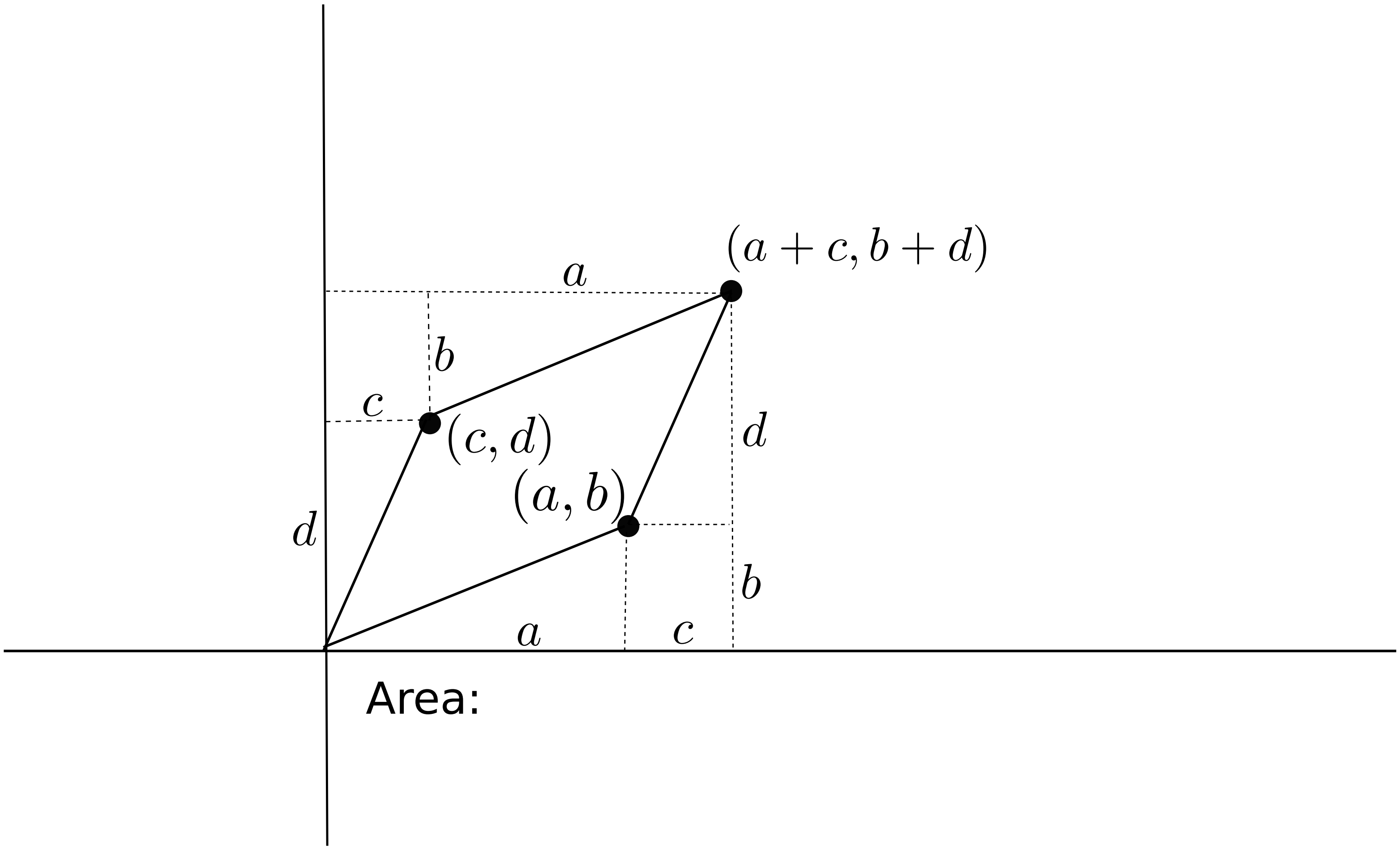
2×2 determinants and area

The area of the parallelogram is given by $ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$



Where does that formula come from?

Look at the area again:



Area:

Minors

In general, the (i, j)-minor of an $n \times n$ matrix A is the *determinant* of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A. It is denoted by $\text{minor}(A)_{ij}$

So, far we only know 2×2 determinants, so let $n = 3$. Deleting a row and a column leaves us with a 2×2 matrix, and then we take the determinant.

$$\begin{bmatrix} a_{11} & a_{12} & \cancel{a_{13}} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} \to \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Above: computing $\text{minor}(A)_{23}$ for a 3×3 matrix A.

Example

Cofactors

Definition

The (i, j)-cofactor of an $n \times n$ matrix A is denoted by $\operatorname{cof}(A)_{ij}$ and defined by

$$\operatorname{cof}(A)_{ij} = (-1)^{i+j} \operatorname{minor}(A)_{ij}.$$

Note: the only difference between the cofactor and corresponding minor is a sign factor.

$$(-1)^{i+j} = \begin{cases} +1, & \text{if } i+j \text{ is even} \\ -1, & \text{if } i+j \text{ is odd} \end{cases}$$

$$\Rightarrow \text{ sign pattern: } \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Example

The Laplace expansion

The Laplace expansion tells us how to write any 3×3 determinant in terms of 2×2 determinants.

Definition

Let A be a 3×3 matrix. We define $\det A$ via Laplace expansion along the first row of A as follows:

$$\begin{aligned} \det A &= a_{11} \operatorname{cof}(A)_{11} + a_{12} \operatorname{cof}(A)_{12} + a_{13} \operatorname{cof}(A)_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

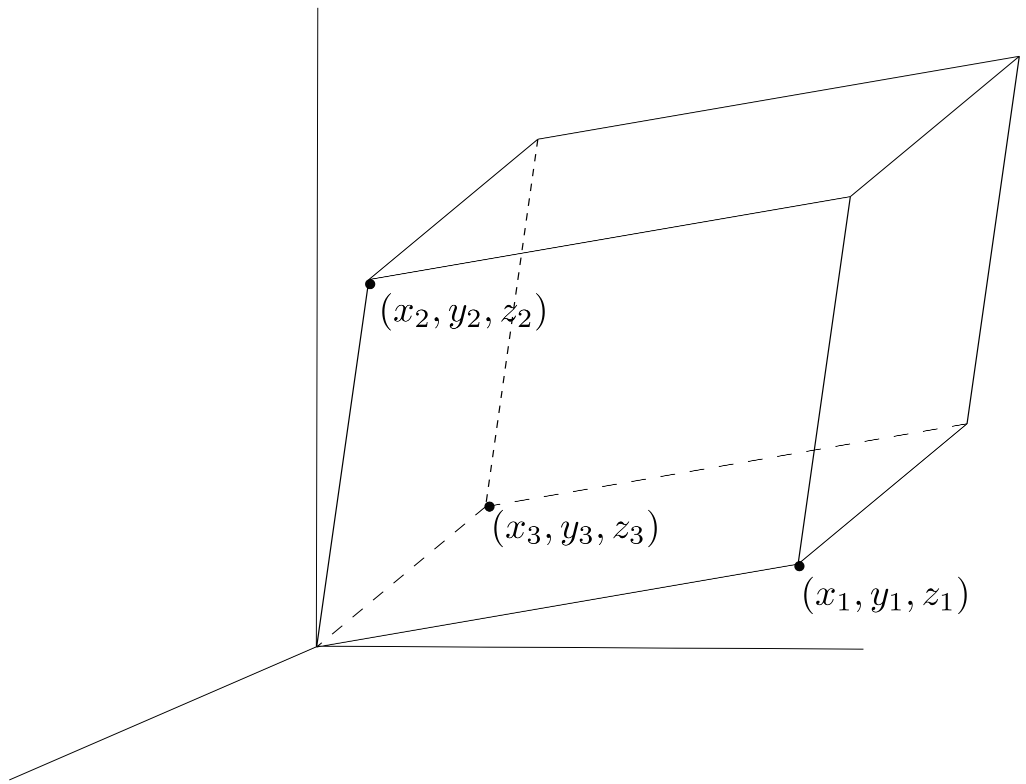
Note: Expanding along any row or column gives the same value for $\det A$.

Tip: With the above in mind, pick a row or column with zeros in it!

Example

Volume

Just as 2×2 determinants give area, 3×3 determinants give volume:



The volume of the solid shown is $V = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$.

Example

4×4 and higher determinants

Take the same definition: given an $n \times n$ matrix A,

$$\det A = \sum_{i=1}^{n} a_{1i} \operatorname{cof}(A)_{1i}.$$

Keep expanding in terms of smaller and smaller determinants until you reach size 2×2.

Why the Laplace expansion?

One important use of determinants is in deciding whether or not a given matrix is invertible. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we row-reduce as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{a} R_1} \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \xrightarrow{R_2 \to -cR_1} \begin{bmatrix} 1 & b/a \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$$

We know there's an inverse as long as we don't get a row of zeros on the bottom, so we need $ad - bc = \det A \neq 0$

A similar argument can be applied to a 3×3 matrix

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, with a bit more work. If you want to see where the 3×3 determinant comes from, try reducing this general matrix to REF and seeing what condition guarantees that there is no row of zeros.

Triangular matrices

- The main diagonal of an $n \times n$ matrix consists of the entries $a_{11}, a_{22}, \dots, a_{nn}$.
- A matrix A is upper triangular if all entries *below* the main diagonal (those a_{ij} with $i > j$) are zero.
- A matrix A is lower triangular if all entries *above* the main diagonal (those a_{ij} with $i < j$) are zero.
- A matrix A is triangular if it is either upper or lower triangular.

Theorem

If A is an $n \times n$ triangular matrix, then

$$\det A = a_{11}a_{22}\cdots a_{nn}.$$

Examples

Properties of Determinants

Theorem (Effect of row operations)

Let A be an $n \times n$ matrix. Then:

1. *If B is obtained from A by exchanging any two rows ($R_i \leftrightarrow R_j$), then $\det B = -\det A$.*
2. *If B is obtained from A by multiplying a row by a constant k ($R_i \to kR_i$), then $\det B = k \det A$.*
3. *If B is obtained from A by adding a multiple of one row to another ($R_i \to R_i + kR_j$), then $\det B = \det A$.*

Corollary

1. *If A has a row of zeros, then $\det A = 0$.*
2. *If one row of A is a multiple of another row, then $\det A = 0$.*

Examples

Proofs

We'll look at how these properties work in the 2×2 case. The general argument requires proof by mathematical induction (covered in Math 2000).

Determinants via row operations

Knowing the effect of row operations on determinants means that we can use them to simplify our determinants. Main principles to follow:

1. Try to reduce the determinant to triangular form. (Determinants of triangular matrices are easy.)
2. Try to stick to Type 3 row operations (adding a multiple of one row to another), since they don't change the value of the determinant.
3. If you do use Type 1 or Type 2 row operations, keep track of the changes.

Example

Scalar Multiplication

- Let A be an $n \times n$ matrix.
- Let B be obtained from A by multiplying row i of A by a nonzero scalar k.
- We know that $\det B = k \det A$.
- What can we conclude about $\det(kA)$?

(Recall that kA is formed by multiplying all rows of A by k.)

Determinants of Elementary Matrices

What are the determinants of the three types of elementary matrix? Recall that:

1. $\det I_n = 1$.
2. An elementary matrix E is obtained from I_n via a single row operation.

Determinant of a product

Theorem

Let A and B be any $n \times n$ matrices. Then

$$\det(AB) = \det A \det B.$$

Example

Consider $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 7 & -2 \end{bmatrix}$.

Proof that $\det(AB) = \det A \det B$

Two cases:

1. A is not invertible, so $A = E_k \cdots E_2 E_1 R$, where R (REF) has a row of zeros.
   Therefore RB has a row of zeros, so $\det(RB) = 0$.
   Now note $\det A$, $\det(AB)$ are obtained from $\det R$, $\det(RB)$ by elementary row operations, so $\det A = 0$ and $\det(AB) = 0$.
2. A is invertible, so $A = E_k \cdots E_2 E_1$ is a product of elementary matrices.
   Note A obtained from I_n via elementary row operations.
   Also AB obtained from B via the same row operations.

Determinant of an invertible matrix

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det A \neq 0$, and if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

Proof.

$\square$

Example

Determinants and transpose

Theorem

For any $n \times n$ matrix A, $\det A = \det A^T$*.*

Consequence: we can also simplify a determinant using column operations. (Why?)

Example

Example

The adjugate matrix

Recall: For any $n \times n$ matrix A, the (i,j)-cofactor of A is given by

$$\mathrm{cof}(A)_{ij} = (-1)^{i+j}\,\mathrm{minor}(A)_{ij}.$$

If we replace every entry a_{ij} in A by the corresponding cofactor, we obtain the cofactor matrix of A:

$$\mathrm{cof}(A) = [\mathrm{cof}(A)_{ij}]_{n\times n}.$$

The adjugate of A is the *transpose* of the cofactor matrix:

$$\mathrm{adj}(A) = \mathrm{cof}(A)^T.$$

(The adjugate is sometimes referred to as the *adjoint* of A, but this has another meaning.)

Example

A determinant formula for A^{-1}

Theorem

Let A be an $n \times n$ matrix such that $\det A \neq 0$. Then A is invertible, and

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A).$$

Remark: For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, this theorem provides a formula some of you may have seen:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example

Sketch of the proof

Let A be an $n \times n$ matrix with $\det A \neq 0$. It suffices to show that

$$A\,\mathrm{adj}(A) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \mathrm{cof}(A)_{11} & \cdots & \mathrm{cof}(A)_{n1} \\ \mathrm{cof}(A)_{12} & \cdots & \mathrm{cof}(A)_{n2} \\ \vdots & \ddots & \vdots \\ \mathrm{cof}\, A_{1n} & \cdots & \mathrm{cof}(A)_{nn} \end{bmatrix} = \det(A) I_n$$

When to use the determinant formula

Take any 4×4 matrix of integers, and compare finding A^{-1} using row operations to the determinant formula. The old method is much less work.

Why use the new formula? There are a couple of cases where it works better:

1. Matrices with non-integer entries. (Especially decimals or irrational numbers.) This is generally done with a calculator or computer.
2. Matrices with entries that are functions.

Example

The spherical coordinate transformation for calculus in three variables is given by $T(\rho, \theta, \phi) = (x, y, z)$, where

$$\begin{aligned} x &= \rho \cos\theta \sin\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\phi \end{aligned}$$

The derivative of this transformation is given by

$$DT = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \sin\phi & -\rho \sin\theta \sin\phi & \rho \cos\theta \cos\phi \\ \sin\theta \sin\phi & \rho \cos\theta \sin\phi & \rho \sin\theta \cos\phi \\ \cos\phi & 0 & -\rho \sin\phi \end{bmatrix}$$

For what values of ρ, θ, and ϕ is the derivative matrix invertible?

Cramer's Rule

Cramer's Rule applies to systems of n equations in n variables of the form

$$AX = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = B,$$

where A is invertible. We have

$$X = A^{-1}B = \frac{1}{\det A}\,\mathrm{adj}(A)B,$$

which means x_j is given by $\dfrac{1}{\det A}(\text{row } j \text{ of } \mathrm{adj}(A))B.$

Cramer's Rule, again

Given a system $AX = B$ of n equations in n variables with A invertible, let A_i be the matrix obtained by replacing column i of A by B. Then for each $i = 1, \dots, n$,

$$x_i = \frac{\det A_i}{\det A}.$$

Caution: Once again, this result isn't really useful for systems with integer coefficients. (Compared to our previous method.) Its main use is when the coefficients are non-integers or functions.

Example