MATH 1410 - Tutorial #4 Solutions

Additional practice:

1. Find the equation of the line through the point $(1, -2, 4)$ that is parallel to the line

$$\langle x, y, z \rangle = \langle 3 - 4t, 2 + t, 5 \rangle.$$

The given line has direction vector $\langle -4, 1, 0 \rangle$. Our line should have the same direction vector but pass through $(1, -2, 4)$, so the equation is

$$\langle x, y, z \rangle = \langle 1 - 4t, -2 + t, 4 \rangle.$$

2. Find the area of the parallelogram with vertices $(1, 2, 3), (4, 0, 7), (0, 3, 5)$, and $(3, 1, 9)$.

Labelling the vertices P, Q, R, S, respectively, we find

$$\overrightarrow{PQ} = \langle 3, -2, 4 \rangle \text{ and } \overrightarrow{PR} = \langle -1, 1, 2 \rangle.$$

However, we need to check to make sure that the vertices Q and R are adjacent to P. Since we have $\overrightarrow{RS} = \langle 3, -2, 4 \rangle = \overrightarrow{PQ}$, we know that the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ represent opposite sides of the parallelogram, and S is the vertex not adjacent to P.

We now compute

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} \mathbf{k} = \langle -8, -10, 1 \rangle,$$

so the area is given by

$$A = \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{8^2 + 10^2 + 1^2} = \sqrt{165}.$$

3. Prove *Lagrange's identity*: for any vectors $\vec{a}, \vec{b}$ in $\mathbb{R}^3$, $\|\vec{a}\|^2 \|\vec{b}\|^2 = \|\vec{a} \times \vec{b}\|^2 + (\vec{a} \cdot \vec{b})^2$.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$. Then

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

and

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

so

$$\|\vec{a} \times \vec{b}\|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$
$$= a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 - 2(a_1 a_2 b_1 b_2 + a_1 a_3 b_1 b_3 + a_2 a_3 b_2 b_3),$$

and

$$(\vec{a} \cdot \vec{b})^2 = (a_1b_1 + a_2b_2 + a_3b_3)^2$$
$$= a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2(a_1b_1a_2b_2 + a_1b_1a_3b_3 + a_2b_2a_3b_3).$$

Adding these together, we see that second terms in each of the above equations cancel, giving us

$$\|\vec{a} \times \vec{b}\|^2 + (\vec{a} \cdot \vec{b})^2 = a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2$$
$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$
$$= \|\vec{a}\|^2 + \|\vec{b}\|^2,$$

as required.

4. Prove that if $\vec{a}$ is parallel to $\vec{b}$, then $\vec{a} \times \vec{b} = \vec{0}$. You may use the properties of the cross product given in Theorem 17 on page 80 of the textbook.

If $\vec{a}$ is parallel to $\vec{b}$, then $\vec{b} = k\vec{a}$ for some scalar k. (If we consider the zero vector to be parallel to any vector, we could include this as a degenerate case, but if one of the two vectors is zero, so is the cross product.) Using properties of the cross product, we then have

$$\vec{a} \times \vec{b} = \vec{a} \times (k\vec{a}) = k(\vec{a} \times \vec{a}) = k\vec{0} = \vec{0},$$

since the cross product of any vector with itself is zero.

5. Determine the point of intersection (if any) of the lines

$$\ell_1(s) = \langle 5, 3, 0 \rangle + s\langle 3, 1, -2 \rangle \quad \text{and} \quad \ell_2(t) = \langle -2, 4, 4 \rangle + t\langle 1, -3, 0 \rangle$$

If (x, y, z) is a point of intersection, then we must have

$$x = 5 + 3s = -2 + t$$
$$y = 3 + s = 4 - 3t$$
$$z = -2s = 4,$$

since the point is on both lines, and must therefore satisfy both equations. Looking at the z coordinate, we must have $s = -2$. Putting this value in for the y coordinate gives us $3 - 2 = 1 = 4 - 3t$, so $-3t = 1 - 4 = -3$, giving $t = 1$. We now check whether these values both work for the x coordinate: $5 + 3(-2) = -1$, and $-2 + 1 = -1$, so putting $s = -2$ in the first line gives us the point $(-1, 1, 4)$, and putting $t = 1$ into the second line gives the same point, so the point $(-1, 1, 4)$ must be the point of intersection.

Assigned problems:

1. Find the area of the triangle with vertices $P = (2, 0, -1)$, $Q = (-3, 4, 2)$, and $R = (0, -3, 1)$.

   Consider the vectors

   $$\vec{u} = \overrightarrow{PQ} = \langle -5, 4, 3 \rangle \text{ and}$$
   $$\vec{v} = \overrightarrow{PR} = \langle -2, -3, 2 \rangle.$$

   We know that the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ is given by $\|\vec{u} \times \vec{v}\|$, and the given triangle is exactly half of this parallelogram. Since

   $$\vec{u} \times \vec{v} = \begin{vmatrix} 4 & 3 \\ -3 & 2 \end{vmatrix} \hat{\imath} - \begin{vmatrix} -5 & 3 \\ -2 & 2 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} -5 & 4 \\ -2 & -3 \end{vmatrix} \hat{k}$$

   $$= (4(2) - 3(-3))\hat{\imath} - (-5(2) - 3(-2))\hat{\jmath} + (-5(-3) - 4(-2))\hat{k}$$
   $$= 17\hat{\imath} + 4\hat{\jmath} + 23\hat{k} = \langle 17, 4, 23 \rangle,$$

   we have

   $$A = \frac{1}{2}\|\vec{u} \times \vec{v}\| = \frac{1}{2}\sqrt{(17)^2 + 4^2 + (23)^2}.$$

2. Find the equation of the line that passes through the points $P = (2, -1, 4)$ and $Q = (-1, 3, 2)$.

   We know that a vector in the direction of the line is given by

   $$\vec{v} = \overrightarrow{PQ} = \langle -3, 4, -2 \rangle.$$

   Choosing $P = (2, -1, 4)$ as our reference point on the line, we have the vector equation

   $$\langle x, y, z \rangle = \langle 2, -1, 4 \rangle + t\langle -3, 4, -2 \rangle = \langle 2 - 3t, -1 + 4t, 4 - 2t \rangle$$

   as the equation of the line.

3. Show that for any two vectors $\vec{a}, \vec{b}$ in $\mathbb{R}^3$, $\vec{a}$ is orthogonal to $\vec{a} \times \vec{b}$.

   Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$. Then

   $$\vec{a} \cdot (\vec{a} \times \vec{b}) = \langle a_1, a_2, a_3 \rangle \cdot \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$
   $$= a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1)$$
   $$= (a_1 a_2 b_3 - a_1 a_3 b_2) + (a_2 a_3 b_1 - a_2 a_1 b_3) + (a_3 a_1 b_2 - a_3 a_2 b_1)$$
   $$= (a_1 a_2 b_3 - a_2 a_1 b_3) + (a_2 a_3 b_1 - a_3 a_2 b_1) + (a_3 a_1 b_2 - a_1 a_3 b_2)$$
   $$= 0 + 0 + 0 = 0.$$

   Since the dot product is zero, the two vectors are orthogonal.

4. Find the shortest distance from the point $P = (1, 3, -2)$ to the line through the point $P_0 = (2, 0, -1)$ in the direction of $\vec{v} = \langle 1, -1, 0 \rangle$. Also find the point P_1 on the line that is closest to P. **Include a diagram.**

We label a generic diagram as shown to the right, with the points P_0, P_1 on the line labelled, as well as the point P not on the line. From the diagram, we can see that the projection of the vector $\overrightarrow{P_0 P}$ onto the line (which is the same as the projection of $\overrightarrow{P_0 P}$ onto the vector $\vec{v}$, since $\vec{v}$ is parallel to the line) gives us the vector $\overrightarrow{P_0 P_1}$: we have $\overrightarrow{P_0 P_1} = \text{proj}_{\vec{v}} \overrightarrow{P_0 P}$.
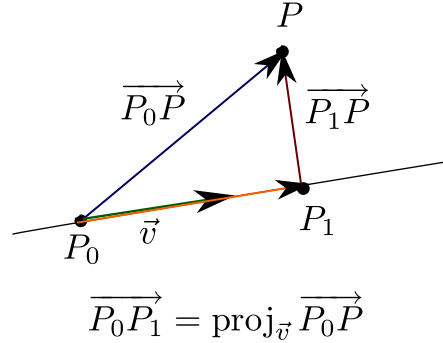
$$\overrightarrow{P_0 P_1} = \text{proj}_{\vec{v}} \overrightarrow{P_0 P}$$

We're given $\vec{v} = \langle 1, -1, 0 \rangle$, and we compute $\overrightarrow{P_0 P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \langle 1, 3, -2 \rangle - \langle 2, 0, -1 \rangle = \langle -1, 3, -1 \rangle$. Since $\vec{v} \cdot \overrightarrow{P_0 P} = 1(-1) + (-1)(3) + (0)(-1) = -4$ and $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v} = 1^2 + (-1)^2 + 0^2 = 2$, we have

$$\overrightarrow{P_0 P_1} = \text{proj}_{\vec{v}} \overrightarrow{P_0 P} = \left(\frac{\vec{v} \cdot \overrightarrow{P_0 P}}{\|\vec{v}\|^2} \right) \vec{v} = \frac{-4}{2} \langle 1, -1, 0 \rangle = \langle -2, 2, 0 \rangle.$$

Since $\overrightarrow{P_0 P_1} = \overrightarrow{OP_1} - \overrightarrow{OP_0}$, we have

$$\overrightarrow{OP_1} = \overrightarrow{OP_0} + \overrightarrow{P_0 P_1} = \langle 2, 0, -1 \rangle + \langle -2, 2, 0 \rangle = \langle 0, 2, -1 \rangle,$$

and thus $P_1 = (0, 2, -1)$. Finally, since P_1 is the closest point on our line to the point P (as per the diagram above), the distance from the point P to the line is the same as the distance from P to P_1. Thus,

$$d = d(P, P_1) = \sqrt{(1 - 0)^2 + (3 - 2)^2 + (-2 + (-1))^2} = \sqrt{1 + 1 + 1} = \sqrt{3}.$$

Note: if we wanted only the distance but didn't need to find the point P_1, we can notice (from – guess what? – the diagram!) that the distance from the point P to the line is given by the length of the vector $\overrightarrow{P_1 P}$, and that

$$\overrightarrow{P_1 P} = \overrightarrow{P_0 P} - \overrightarrow{P_0 P_1} = \langle -1, 3, -1 \rangle - \langle -2, 2, 0 \rangle = \langle 1, 1, -1 \rangle,$$

and thus $d = \|\overrightarrow{P_1 P}\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$.

5. Find the distance between the skew lines

$$\ell_1(s) = \langle x, y, z \rangle = \langle 1, 2, 1 \rangle + s\langle 2, -1, 1 \rangle$$
$$\ell_2(t) = \langle x, y, z \rangle = \langle 3, 3, 3 \rangle + t\langle 4, 2, -1 \rangle$$

using the method of Example 47 in the text.

Letting $P_1 = (1, 2, 1)$ and $P_2 = (3, 3, 3)$ be the given points on the lines ℓ_1 and ℓ_2, respectively, we have the vector

$$\vec{w} = \overrightarrow{P_1 P_2} = \langle 2, 1, 2 \rangle,$$

which has its tail on ℓ_1 and head on ℓ_2. To compute the distance from ℓ_1 to ℓ_2, we need to determine the component of $\vec{w}$ that is perpendicular to both lines. We are given the direction vectors $\vec{v}_1 = \langle 2, -1, 1 \rangle$ and $\vec{v}_2 = \langle 4, 2, -1 \rangle$ for the two lines; since these vectors are parallel to their respective lines, the cross product

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix} = \langle -1, 6, 8 \rangle$$

must be perpendicular to both lines. (The notation $\vec{n}$ is used to indicate the fact that the two skew lines lie in parallel planes, both of which have the normal vector $\vec{n}$. Finding the distance between the two lines is thus reduced to finding the distance between the corresponding parallel planes.)

The desired distance is now given by the length of the projection of $\vec{w}$ onto $\vec{n}$. We have

$$d = \|\text{proj}_{\vec{n}} \vec{w}\| = \frac{|\vec{n} \cdot \vec{w}|}{\|\vec{n}\|} = \frac{-2 + 6 + 16}{\sqrt{1^2 + 6^2 + 8^2}} = \frac{20}{\sqrt{101}}.$$