Math 3500 Assignment #7 Solutions
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1. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at exactly one point. (It might help to recall that we've seen an example of a function that is continuous at only one point.)

   Let $f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$. For any $a \neq 0$, let (a_n) be a sequence converging to a. If $a \in \mathbb{Q}$, we can take each a_n irrational, and then $a^2 = f(a) = f(\lim a_n) \neq 0$, but $\lim f(a_n) = 0$. Similarly, if $a \notin \mathbb{Q}$, we can take each a_n rational, and then $0 = f(a) = f(\lim a_n)$, but $\lim f(a_n) = \lim a_n^2 = a^2 \neq 0$. It follows that f is not continuous at any $a \neq 0$, so f cannot be differentiable at any $a \neq 0$.

   However, we claim that $f'(0)$ exists. To see this, note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \begin{cases} x, & \text{if } x \in \mathbb{Q}, x \neq 0 \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}.$$

   Thus, for each $x \neq 0$, $\left|\frac{f(x)}{x}\right| \leq |x|$, and thus $f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = 0$ exists.

2. (**Do not submit**) Let $f_a(x) = \begin{cases} x^a, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$, where a is some real number.

   (a) For which values of a is f continuous at 0?

   For $a > 0$ we have $\lim_{x \to 0^+} x^a = 0$, so f_a is continuous at 0. If $a = 0$, then $f_a(x) = 1$ for $x > 0$, so f cannot be continuous at 0, and if $a < 0$, then $\lim_{x \to 0^+} f_a(x) = \infty$, so f cannot be continuous at 0.

   (b) For which values of a is f differentiable at 0? In these cases, is f' continuous?

We only need to consider $a > 0$ since we know that f is not continuous at 0 for $a \leq 0$. The derivative is given by

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} g(x),$$

where $g(x) = x^{a-1}$ for $x > 0$, and $g(x) = 0$ for $x < 0$. For this limit to exist, we need $a > 1$, in which case we have $f'(0) = 0$, using the same argument as in part (a). Now, for all values of a we have $f'(x) = 0$ for $x < 0$, and $f'(x) = ax^{a-1}$, from which we see that f' is indeed continuous for $a > 1$, since $ax^{a-1} \to 0$ as $x \to 0^+$.

(c) For which values of a is f twice differentiable at 0?

For $a > 1$ we have $f'(x) = \begin{cases} ax^{a-1}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$. The same argument used in parts (a) and (b) tells us that f will be twice differentiable at 0 if $a > 2$.

3. Prove Leibniz's rule: for any $n \in \mathbb{N}$, $(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a)$, provided that f and g are both n times differentiable at a. (The notation $h^{(n)}$ indicates the n^{th} derivative of h, so $h^{(0)} = h, h^{(1)} = h', h^{(2)} = h''$, etc.)

When $n = 1$, we have $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$, by the product rule, so the result holds in this case. Suppose that for some $n \geq 1$ we have

$$\begin{aligned} (fg)^{(n)}(a) &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a) \\ &= f^{(n)}(a)g(a) + nf^{(n-1)}(a)g'(a) + \cdots + nf'(a)g^{(n-1)}(a) + f(a)g^{(n)}(a). \end{aligned}$$

Then we have

$$
\begin{aligned}
(fg)^{(n+1)}(a) &= ((fg)^{(n)})'(a) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}\right)'(a) \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)}(a) g^{(n-k+1)}(a) + f^{(k+1)}(a) g^{(n-k)}(a)\right) \\
&= f(a)g^{(n+1)}(a) + f'(a)g^{(n)}(a) \\
&\qquad + \sum_{k=1}^{n-1} \binom{n}{k} (f^{(k)}(a) g^{(n-k+1)}(a) + f^{(k+1)}(a) g^{(n-k)}(a)) \\
&\qquad\quad + f^{(n+1)}(a)g(a) + f^{(n)}(a)g'(a) \\
&= f(a)g^{(n+1)}(a) + \left(\sum_{k=1}^{n-1} \binom{n}{k} (f^{(k)}(a) g^{(n-k+1)}(a) + f^{(n)}(a)g'(a)\right) \\
&\qquad + \left(\sum_{k=1}^{n-1} f^{(k+1)}(a) g^{(n-k)}(a)) + f'(a)g^{(n)}(a)\right) + f^{(n+1)}(a)g(a) \\
&= f(a)g^{(n+1)}(a) + \sum_{k=1}^{n} \binom{n}{k} f^{(k)}(a) g^{(n+1-k)}(a) \\
&\qquad + \sum_{k=0}^{n-1} \binom{n}{k} f^{(k+1)}(a) g^{(n-k)}(a) + f^{(n+1)}(a)g(a) \\
&= f(a)g^{(n+1)}(a) + \sum_{k=1}^{n} \binom{n}{k} f^{(k)}(a) g^{(n+1-k)}(a) \\
&\qquad + \sum_{k=1}^{n} \binom{n}{k-1} f^{(k)}(a) g^{(n+1-k)}(a) + f^{(n+1)}(a)g(a) \\
&= f(a)g^{(n+1)}(a) + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1}\right) f^{(k)}(a) g^{(n+1-k)}(a) + f^{(n+1)}(a)g(a) \\
&= f(a)g^{(n+1)}(a) + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)}(a) g^{(n+1-k)}(a) + f^{(n+1)}(a)g(a) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(a) g^{(n+1-k)}(a).
\end{aligned}
$$

Thus, the result holds for all $n \in \mathbb{N}$ by induction.

4. A function $f : A \to \mathbb{R}$ is called a **Lipschitz** function if there exists some $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all $x, y \in A$.

(a) Prove that any Lipschitz function is uniformly continuous on its domain.

Suppose that $\left|\frac{f(x)-f(y)}{x-y}\right| \leq M$ for all $x, y \in A$, for some $M > 0$. Given any $\epsilon > 0$, take $\delta = \epsilon/M$. Then whenever $x, y \in A$ and $|x-y| < \delta$, we have

$$|f(x) - f(y)| \leq M|x-y| < M\delta = \epsilon.$$

Thus, f is uniformly continuous on A.

(b) Prove that if f is differentiable on a closed interval $[a, b]$ and f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

Suppose f' is continuous on $[a, b]$. Then by the Extreme Value Theorem, f' is bounded on $[a, b]$, so there exists some $M > 0$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$. Now, choose any $x, y \in [a, b]$ with $x < y$. Then f is differentiable on $[x, y]$ and thus continuous on $[x, y]$, so by the Mean Value Theorem there exists some $c \in (a, b)$ such that

$$\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq M.$$

Thus, f is Lipschitz on $[a, b]$.

5. Prove that if f is differentiable on an interval I and $f'(x) \neq 1$ for all $x \in I$, then f has at most one fixed point on I (that is, there is at most one $x_0 \in I$ such that $f(x_0) = x_0$).

Suppose f has more than one fixed point; say $f(x_1) = x_1$ and $f(x_2) = x_2$ for some $x_1, x_2 \in I$ with $x_1 < x_2$. Since f is differentiable on I, f is continuous on $[x, y] \subseteq I$ and differentiable on (x, y). Then by the Mean Value Theorem there must exist some $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1,$$

and the result follows by taking the contrapositive.

6. Let f be defined on $\mathbb{R}$ and suppose that $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f must be a constant function.

If $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$, then for any $a \in \mathbb{R}$ we have

$$\left|\frac{f(x)-f(a)}{x-a}\right| \leq |x-a|,$$

from which it follows that $f'(a) = 0$, since for any $\epsilon > 0$, if $|x - a| < \delta = \epsilon$, we have

$$\left| \frac{f(x) - f(a)}{x - a} - 0 \right| \leq |x - a| < \epsilon.$$

Since $f'(x) = 0$ for all $x \in \mathbb{R}$, it follows that f must be constant.

7. (**Do not submit**) Recall that a function $f : (a, b) \to \mathbb{R}$ is *increasing* on (a, b) if $f(x) \leq f(y)$ whenever $x < y$ in (a, b).

   (a) Show that if f is differentiable on (a, b) then f is increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

   If f is differentiable on (a, b), and $f'(x) \geq 0$ on (a, b), then by the Mean Value Theorem, for any $x, y \in (a, b)$ with $x < y$ there exists some $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x) \geq 0,$$

   so $f(x) \leq f(y)$, and f is increasing. Conversely, suppose that f is differentiable and increasing on (a, b). Then for all $x, y \in (a, b)$ with $x \neq y$ we have $\dfrac{f(x) - f(y)}{x - y} > 0$, since $x < y$ if and only if $f(x) < f(y)$. It follows that $f'(x) \geq 0$.

   (b) Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$.

   For $x \neq 0$ we can compute $g'(x)$ using rules of differentiation. We have:

$$g'(x) = \frac{1}{2} + 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

   At $x = 0$ we find $g'(0)$ using the definition of the derivative:

$$g'(0) = \lim_{x \to 0} \frac{x/2 + x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \left(\frac{1}{2} + x \sin\left(\frac{1}{x}\right)\right) = \frac{1}{2} > 0.$$

   (c) Show that g is *not* increasing on any open interval containing 0.

   For any $x > 0$ we can find some $n \in \mathbb{N}$ such that $0 < \dfrac{2}{(4n+1)\pi} < \dfrac{2}{(4n-1)\pi} < x$. We then have

$$g\left(\frac{2}{(4n+1)\pi}\right) = \frac{4n\pi + \pi + 4}{(4n+1)^2} > \frac{4n\pi - \pi - 4}{(4n-1)^2} = g\left(\frac{2}{(4n-1)\pi}\right),$$

so g cannot be increasing on $(0, x)$, and thus not on any open interval containing 0. (Of course one still must verify that the inequality above is valid. It is, but it's a bit of a mess to check.)

(d) Why do your results from (b) and (c) not contradict your result in part (a)?

In part (a) we only showed that $g'(0) > 0$ at the *point* 0. To guarantee that g is increasing, we'd need to show that $g'(x) > 0$ on an *interval* containing 0.