University of Lethbridge
Department of Mathematics and Computer Science
December 2014
MATH 3500 - Practice Exam

Last Name:____

First Name:____

Student Number:____________________________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For grader's use only:

Problem	Grade
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
Total	/100

[4] 1. (a) Find the supremum and infimum of the following sets, if they exist. If either one does not exist, explain why.

$$A = \left\{(-1)^n\left(1+\frac{1}{n}\right) : n \in \mathbb{N}\right\} \qquad B = \left\{\sin\frac{1}{x} : x \in (0,1]\right\}$$

(b) Let $D \subseteq \mathbb{R}$ be non-empty and suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions.

[4] i. Prove that $\sup[(f+g)(D)] \leq \sup[f(D)] + \sup[g(D)]$.

[2] ii. Give an example to show that the above inequality may be strict.

2. (a) For any non-empty subset $A \subseteq \mathbb{R}$, prove that $\partial A = \overline{A} \setminus A^\circ$, where ∂A denotes the boundary of A, A° denotes the interior of A and $\overline{A}$ denotes the closure of A. [5]

(b) Give an example of a set $A \subseteq \mathbb{R}$ that has a limit point but does not contain a limit point. [2]

(c) Prove that a closed subset of a compact set is compact. [3]

[10] 3. Prove that a point x is a limit point of a set $S \subseteq \mathbb{R}$ if and only if there exists a sequence (x_n) of points in $S \setminus \{x\}$ such that $\lim x_n = x$.

4. (a) Suppose $I_n = [a_n, b_n]$ is a nested sequence of intervals; that is, $I_{n+1} \subseteq I_n$ for all n.
[4] Prove that $\lim a_n$ and $\lim b_n$ both exist.

[6] (b) Prove that $\lim_{n\to\infty} \frac{n+1}{n} = 1$.

[5] 5. (a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^2 + 6 & \text{if } x \in \mathbb{Q} \\ 5x & \text{if } x \notin \mathbb{Q} \end{cases}$. Prove that f is continuous at $x = 2$.

[5] (b) Use the $\epsilon - \delta$ definition to prove that $f(x) = x^2 + 4$ is uniformly continuous on the interval $[1, 3]$.

[6] 6. (a) Prove that any polynomial of odd degree has at least one real root.

[4] (b) Prove that if f is uniformly continuous on an interval (a, b), then f is bounded on (a, b).

7. (a) [5] Suppose that f and g are differentiable functions such that $f(a) = g(a)$ and $f'(x) < g'(x)$ for all $x \geq a$. Prove that $f(x) < g(x)$ for all $x \geq a$.

(b) [5] Prove that if $n \geq 1$, then $(1 + x)^n > 1 + nx$ for $x > 0$.

[5] 8. (a) Let $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$. Prove that f is differentiable at $x = 0$ and give a formula for $f'(x)$ for all $x \in \mathbb{R}$.

[5] (b) Prove that $f(x) = x^3 + 3x$ has exactly one root.

[10] 9. Suppose that f is integrable on $[a, b]$ and that $[c, d] \subseteq [a, b]$. Prove that f is integrable on $[c, d]$.

10. Find a function g such that

[3] (a) $\displaystyle\int_0^x tg(t)\,dt = x + x^2$

[3] (b) $\displaystyle\int_0^{x^2} tg(t)\,dt = x + x^2$

[4] (c) Suppose that f is a differentiable function with $f(0) = 0$ and $0 < f' \leq 1$. Prove that for all $x \geq 0$ we have

$$\int_0^x f^3 \leq \left(\int_0^x f\right)^2.$$