MATH 1565 - Tutorial #11 Solutions

1. Calculate the following Taylor polynomials:

[4] (a) For $f(x) = e^{x^2}$, degree 4, about $x = 0$.
We have $f(0) = e^0 = 1$, and

$$\begin{aligned} f'(x) &= 2xe^{x^2} & f'(0) &= 0 \\ f''(x) &= 2e^{x^2} + 4x^2e^{x^2} & f''(0) &= 2 \\ f^{(3)}(x) &= 12xe^{x^2} + 8x^3e^{x^2} & f^{(3)}(0) &= 0 \\ f^{(4)}(x) &= 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2} & f^{(4)}(0) &= 12 \end{aligned}$$

Thus,

$$p_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 = 1 + x^2 + \frac{1}{2}x^4.$$

[2] (b) For $g(u) = e^u$, degree 2, about $u = 0$.
Since $g(u) = g'(u) = g''(u) = e^u$, we have $g(0) = g'(0) = g''(0) = 1$, and

$$p_2(u) = g(0) + g'(0)u + \frac{g''(0)}{2!}u^2 = 1 + u + \frac{1}{2}u^2.$$

(What happens if you put $u = x^2$ in your answer for part (b)?)

You didn't have to answer this part, but putting $u = x^2$ in your answer for (b) gives you the answer for (a), suggesting that the answer, for those of you who asked, is "Yes, there is an easier way to do this."

2. Calculate the following antiderivatives:

[3] (a) The antiderivative F of $f(x) = \dfrac{1}{1+x^2}$ such that $F(1) = \pi$.

We have $F(x) = \arctan(x) + C$ for some C. This gives us $\pi = F(1) = \arctan(1) + C = \frac{\pi}{4} + C$, so $C = \frac{3\pi}{4}$, and thus

$$F(x) = \arctan(x) + \frac{3\pi}{4}.$$

[3] (b) $\displaystyle\int (x^3 - 3\sqrt{x} + 4)\,dx = \frac{1}{4}x^4 - 2x^{3/2} + 4x + C.$

3. Estimate the area under $f(x) = 4 - 3x^2$, for $0 \leq x \leq 1$, using 3 rectangles and:

[3] (a) Left endpoints.

We have $\Delta x = \frac{1-0}{3} = \frac{1}{3}$, so our points are $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}$, and $x_3 = 1$. Our left endpoints are x_0, x_1, x_2, so we have

$$A \approx (f(0) + f(1/3) + f(2/3))(1/3) = (4 + 11/3 + 8/3)(1/3) = 31/9 \approx 3.44.$$

[3] (b) Right endpoints.

Using the data from above, our right endpoints are x_1, x_2, x_3, and

$$A \approx (f(1/3) + f(2/3) + f(1))(1/3) = (11/3 + 8/3 + 1)(1/3) = 22/9 \approx 2.44.$$

4. Given that

$$\int_1^4 f(x)\,dx = 4, \int_1^6 f(x)\,dx = 7, \int_1^4 g(x)\,dx = -3, \text{ and } \int_4^6 g(x)\,dx = 1,$$

compute:

[2] (a) $\displaystyle\int_4^6 f(x)\,dx = \int_1^6 f(x)\,dx - \int_1^4 f(x)\,dx = 7 - 4 = 3$

[2] (b) $\displaystyle\int_1^6 (f(x) + g(x))\,dx$

Since $\displaystyle\int_1^6 g(x)\,dx = \int_1^4 g(x)\,dx + \int_4^6 g(x)\,dx = -3 + 1 = -2$, we have

$$\int_1^6 (f(x) + g(x))\,dx = \int_1^6 f(x)\,dx + \int_1^6 g(x)\,dx = 7 + (-2) = 5.$$