University of Lethbridge
Department of Mathematics and Computer Science
20^{th} March, 2015, 3:00 - 3:50 pm
MATH 3410 - Test #2

Last Name:____

First Name:____

Student Number:______________________________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

You must solve all problems on pages 2, 3, and 4, but you only need to do either page 5 or page 6. **Do not complete both page 5 and page 6.**

For grader's use only:

Page	Grade
2	/8
3	/8
4	/12
5/6	/12
Total	/40

1. Provide definitions for the following terms:

[2] (a) What it means for a linear map $T : V \to W$ to be **invertible**.

[2] (b) An **invariant subspace** for an operator $T : V \to V$.

[2] (c) What it means for a linear operator $T : V \to V$ to be **diagonalizable**.

[2] (d) The **eigenspace** $E(\lambda, T)$ of an operator $T : V \to V$ and scalar λ.

2. Short answer: provide a brief answer to the questions below. You do not have to explain your answers.

[1] (a) If V and W are finite-dimensional vector spaces, what is $\dim \mathcal{L}(V, W)$?

[3] (b) What is the matrix (with respect to the standard bases) of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x, y, z) = (2x - 3y + z, -x + 2y + 4z)?$$

[4] (c) If T is the operator on $\mathbb{R}^{2,1}$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -3 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and $p(x) = 2x^2 - 3x + 5$, determine the operator $p(T)$.

Please solve **both** problems on this page.

[6] 3. Let $S, T \in \mathcal{L}(V)$, where V is finite-dimensional. Prove that the operator ST is invertible if and only if S and T are invertible.

[6] 4. Suppose that $S, T \in \mathcal{L}(V)$ satisfy $ST = TS$. Prove that $\operatorname{null} S$ is invariant under T.

You may either solve both problems on this page, or leave it blank, and move on to the next page.

5. Let V be finite-dimensional, and let $P \in \mathcal{L}(V)$. Prove that if $P^2 = P$, then
[6] $V = \operatorname{null} P \oplus \operatorname{range} P$.

   Hint: $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$, so it suffices to show that $\operatorname{null} P \cap \operatorname{range} P = \{0\}$.

6. Suppose that $\dim V = n$, $T \in \mathcal{L}(V)$ has n distinct eigenvalues, and $S \in \mathcal{L}(V)$ has the same
[6] eigenvectors as T (but not necessarily the same eigenvalues). Prove that $ST = TS$.

If you solved the two problems on the previous page, then leave this page blank. If you skipped the last page, then please solve the following:

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the operator $T(x, y) = (5x - 2y, 7x - 4y)$.

[2] (a) Compute the matrix $\mathcal{M}(T)$ of T with respect to the standard basis of $\mathbb{R}^2$.

[4] (b) Find the eigenvalues of T.

[4] (c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of $\mathbb{R}^2$.

[2] (d) Is the operator T diagonalizable? Why or why not? If it is, give a matrix P such that $P^{-1}\mathcal{M}(T)P$ is diagonal. (You don't have to verify it's diagonal.)

Extra space for rough work or to complete a problem, as needed. Please do not remove this page. If there is work to be graded on this page, please indicate this next to the corresponding question.