University of California, Berkeley
Department of Mathematics
15^{th} March, 2013, 12:10-12:55 pm
MATH 53 - Test #2

Last Name: Solutions

First Name: The

Student Number: ______

Discussion Section: ______

Name of GSI: ______

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

There is a list of potentially useful formulas available on the last page of the exam.

For grader's use only:

Page	Grade
1	/15
2	/15
3	/10
Total	/40

B

1. Let $f(x,y) = y^2 e^{xy}$.

[4] (a) Find the linearization of f at the point $(0,1)$.

The partial derivatives of f are $f_x(x,y) = y^3 e^{xy}$ and $f_y(x,y) = 2ye^{xy} + xy^2e^{xy}$, so the linearization of f at $(0,1)$ is given by

$$\begin{aligned} L(x,y) &= f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1) \\ &= 1 + x + 2(y-1) = x + 2y - 1. \end{aligned}$$

[3] (b) Find the derivative of f in the direction of $\mathbf{v} = \langle 3, -4 \rangle$ at the point $(0,1)$.

The directional derivative is given by

$$D_{\mathbf{v}} f(0,1) = \frac{\nabla f(0,1) \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 1,2\rangle \cdot \langle 3,-4\rangle}{\sqrt{3^2 + (-4)^2}} = -1.$$

[5] (c) If $x(t) = 2 - 2t$ and $y(t) = t^2$, use the chain rule to find the tangent vector to the curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ when $t = 1$, where $z(t) = f(x(t), y(t))$.

We have $x'(t) = -2$, $y'(t) = 2t$ and

$$z'(t) = \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

When $t = 1$, $x(1) = 0$, $y(1) = 1$, $x'(1) = -2$, and $y'(1) = 2$. Thus, $z'(1) = f_x(0,1)x'(1) + f_y(0,1)y'(1) = 1(-2) + 2(2) = 2$, so $\mathbf{r}'(1) = \langle -2, 2, 2\rangle$.

[3] (d) Verify that the tangent vector found in part (c) is tangent to the surface $z = f(x,y)$ at the point $(0,1,1)$.

The tangent plane is given by $z = L(x,y) = x + 2y - 1$, so a normal vector is $\mathbf{n} = \langle 1, 2, -1\rangle$. Since $\mathbf{n} \cdot \mathbf{r}'(1) = \langle 1,2,-1\rangle \cdot \langle -2,2,2\rangle = -2 + 4 - 2 = 0$, $\mathbf{r}'(1)$ must lie in the tangent plane at $(0,1,1)$.

2. Let $f(x,y) = 8x^3 + 12xy - y^3$.

[8] (a) Find and classify the critical points of f.

The gradient of f is given by

$$\nabla f(x,y) = \langle 24x^2 + 12y, 12x - 3y^2 \rangle,$$

which vanishes when $y = -2x^2$ and $4x = y^2$. Plugging the first equation into the second gives $4x = 4x^4$, so $x = 0$ or $x = 1$. This gives us the critical points $(0,0)$ and $(1,-2)$.

The second derivatives of f are given by $f_{xx}(x,y) = 48x$, $f_{xy}(x,y) = 12$, and $f_{yy}(x,y) = -6y$. Thus, we have

- At $(0,0)$, $A = 0$, $B = 12$, $C = 0$, and $D = AC - B^2 = -144 < 0$, so $(0,0)$ is a saddle point.
- At $(1,-2)$, $A = 48 > 0$, $B = 12$, $C = 12$, and $D = AC - B^2 = 4(12)^2 - 12^2 > 0$, so $(1,-2)$ is a local minimum.

[7] (b) Find the absolute maximum and minimum of f on the set D given by the triangular region with vertices at $(0,0)$, $(1,0)$, and $(1,-2)$.

From part (a) we have the critical values $f(0,0) = 0$ and $f(1,-2) = -8$, which occur at points in D, since $(0,0)$ and $(1,-2)$ are two of the three vertices of the triangle. The value of f at the remaining vertex is $f(1,0) = 8$.

Since there are no critical points on the interior of D, it remains to check for any boundary extrema. The boundary consists of three lines: $y = 0$, $x = 1$, and $y = -2x$. Since we've already checked the end points of these lines (the vertices of the triangle), we just have to check for critical points of the restriction of f to each line.

For $y = 0$ we get $f(x,0) = 8x^3$, which has its only critical point when $x = 0$, which gives the point $(0,0)$, which we've already checked. For $x = 1$ we get $g(y) = f(1,y) = 8 + 12y - y^3$. We have $g'(y) = 12 - 3y^2$, so g has critical points when $y = \pm 2$, but $(1,2)$ is not in D, and $(1,-2)$ we've already checked.

Finally, if $y = -2x$ we have $h(x) = f(x,-2x) = 16x^3 - 24x^2$, so $h'(x) = 48x - 48x^2$, and h has critical points $x = 0$ and $x = 1$, corresponding to the points $(0,0)$ and $(1,-2)$, which we've already checked. Having exhausted all possibilities, we conclude that the absolute maximum is $f(1,0) = 8$, and the absolute minimum is $f(1,-2) = -8$.

3. (a) Define what it means for a function $f(x, y, z)$ to be *continuous* at a point (a, b, c) in
[2] its domain.

A function $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is continuous at $(a, b, c) \in D$ if

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c).$$

(b) Define what it means for a function $f(x, y, z)$ to be *differentiable* at a point (a, b, c) in
[3] its domain.

A function $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $(a, b, c) \in D$ if

$$\lim_{(x,y,z)\to(a,b,c)} \frac{f(x, y, z) - f(a, b, c) - \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle}{\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}} = 0.$$

[5] (c) Show that if f is differentiable at a point (a, b, c), then it is continuous at (a, b, c).
Hint: You can show this using only the above two definitions and the limit laws.

Let $\mathbf{x} = \langle x, y, z\rangle$ be the position vector for (x, y, z), and similarly define $\mathbf{a} = \langle a, b, c\rangle$. Suppose that f is differentiable at $\mathbf{a}$. Then we have

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) &= \lim_{\mathbf{x}\to\mathbf{a}} \left(f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + f(\mathbf{a})\right) \\
&= \lim_{\mathbf{x}\to\mathbf{a}} \left[\left(\frac{f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|}\right)(\|\mathbf{x} - \mathbf{a}\|)\right] \\
&\qquad + \lim_{\mathbf{x}\to\mathbf{a}} \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{a}) \\
&= 0(0) + \nabla f(\mathbf{a}) \cdot \mathbf{0} + f(\mathbf{a}) \\
&= f(\mathbf{a}).
\end{aligned}$$

Thus, f is continuous at $\mathbf{a}$.

 Total Points: 40