Name: Solutions

Solve the following **two** questions.

[5] 1. Suppose that $u, v \in V$ are such that $\|u\| = 3$, $\|u+v\| = 4$, and $\|u-v\| = 6$. What is the value of $\|v\|$?

We note that

$$\begin{aligned}\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\ &= \|u\|^2 + \langle u,v\rangle + \langle v,u\rangle + \|v\|^2 + \|u\|^2 - \langle u,v\rangle - \langle v,u\rangle + \|v\|^2 \\ &= 2\|u\|^2 + 2\|v\|^2.\end{aligned}$$

Subsitutiting $\|u\| = 3$, $\|u+v\| = 4$, and $\|u-v\| = 6$, we find

$$4^2 + 6^2 = 2(3^2) + 2\|v\|^2,$$

which gives $2\|v\|^2 = 16 + 36 - 18 = 34$, so $\|v\|^2 = 17$, and thus $\|v\| = \sqrt{17}$.

[5] 2. Prove that for all positive numbers $a, b, c, d \in \mathbb{R}$, we have

$$16 \leq (a+b+c+d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

Consider the vectors $u = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$ and $v = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right)$ in $\mathbb{R}^4$. We have $\langle u, v\rangle = 1+1+1+1 = 4$, $\|u\|^2 = a+b+c+d$, and $\|v\|^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$.
Since the Cauchy-Schwarz inequality guarantees that $\langle u, v\rangle^2 \leq \|u\|^2\|v\|^2$, the result follows.

[5] 3. (**Bonus**) Suppose that V is a real inner product space. Prove that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

for all $u, v \in V$.

For any $u, v \in V$, we have (noting that $\langle u, v\rangle = \langle v, u\rangle$, since V is a real inner product space)

$$\begin{aligned}\|u+v\|^2 - \|u-v\|^2 &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle - (\langle u,u\rangle - 2\langle u,v\rangle + \langle v,v\rangle) \\ &= 4\langle u,v\rangle,\end{aligned}$$

and the result follows upon dividing both sides of the equality by 4.

Total: 10 points