MATH 1560 - Tutorial #7 Solutions

Additional practice:

1. A person standing on a ledge 75 m above the ground throws a ball upward at 20 m/s. Assume that acceleration due to gravity is -10 m/s^2.

   (a) How high (from the ground) does the ball get?

   Let $v(t)$ denote the velocity as a function of time. The maximum height will be reached when $v(t) = 0$. Since $v(0) = 20$, we have $v(t) = 20 - 10t$, so $v(2) = 0$. The ball therefore reaches its maximum height after 2 seconds.

   We now need $y(t)$, the height as a function of t. Since $v(t) = y'(t)$ and $y(0) = 75$, we get $y(t) = 75 + 20t - 5t^2$, so $y(2) = 75 + 40 - 20 = 95$ m above the ground (20 m above the ledge).

   (b) How many seconds does it take before the ball hits the ground?

   The ball hits the ground when

   $$x(t) = 0 = 75 + 20t - 5t^2.$$

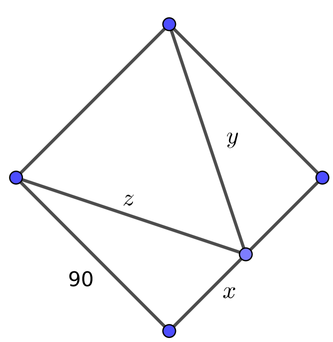
   The quadratic formula gives

   $$t = \frac{-20 \pm \sqrt{20^2 - 4(-5)(75)}}{2(-5)} = 2 \pm \sqrt{19}.$$

   Since $2 - \sqrt{19} < 0$, we must take the positive square root, giving us $t = 2 + \sqrt{19} \approx 6.36$ seconds.

2. A baseball diamond is a square with sides of length 90 feet. A baseball player hits the ball, and runs toward first base at a speed of 24 ft/s.

   (a) At what rate is his distance from second base decreasing when he is halfway to first base?



   A diagram of the situation is shown on the left. If we let x denote the distance from home plate to the player, then $\frac{dx}{dt} = 24$, and the distance from the player to first base is $90 - x$.

Using the Pythagorean theorem, the distance y from the player to second base satisfies

$$y(t)^2 = (90 - x(t))^2 + 90^2,$$

so $2y \cdot y'(t) = -2(90 - x)x'(t)$. When the player is half way to first base, we have $x = 45$, so $90 - x = 45$, and

$$y^2 = 90^2 + 45^2 = (2 \cdot 45)^2 + 45^2 = 4(45^2) + 45^2 = 5(45^2),$$

so $y = 45\sqrt{5}$, and

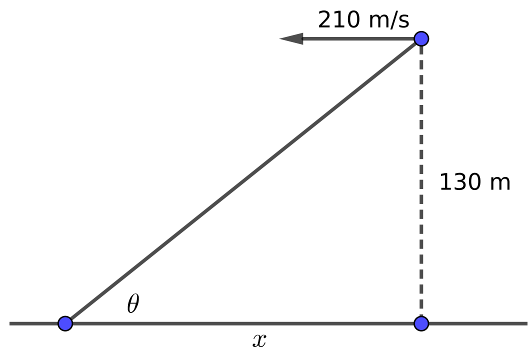
$$y'(t) = -\frac{45}{45\sqrt{5}}(24) = -\frac{24}{\sqrt{5}}.$$

Thus, his distance from second is decreasing at a rate of $24/\sqrt{5}$ feet per second.

(b) At what rate is his distance from third base increasing at the same moment?

Referring to the diagram above, we see that $z^2 = x^2 + 90^2$, so $2z\frac{dz}{dt} = 2x\frac{dx}{dt}$. When $x = 45$ we get $z = 45\sqrt{5}$, as above, and $\frac{dz}{dt} = \frac{24}{\sqrt{5}}$, so the distance from the player to third base is increasing at a rate of $24/\sqrt{5}$ feet per second.

1. An aircraft is flying away from a viewer on the ground at a speed of 210 m/s at an elevation of 130 m. The person on the ground is watching the plane through a set of binoculars. Let θ denote the viewing angle, measured from the ground, in radians. At what rate is θ changing when the plane is 500 m away (as measured along the ground)?



A sketch of the situation is given above. From the diagram, we see that $\tan(\theta(t)) = \dfrac{130}{x(t)}$, or equivalently (and more conveniently, for derivatives), $\cot(\theta(t)) = \dfrac{x(t)}{130}$.

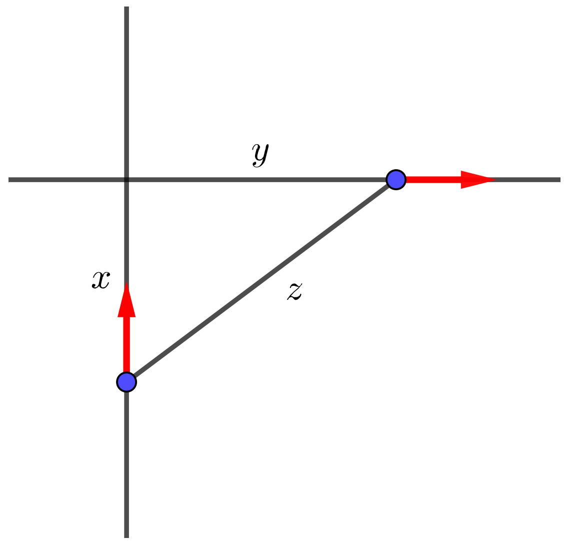
Taking derivatives of both sides with respect to t, $-\csc^2(\theta(t)) \cdot \dfrac{d\theta(t)}{dt} = \dfrac{1}{130}\dfrac{dx(t)}{dt}$, so $\dfrac{d\theta(t)}{dt} = -\dfrac{\sin^2\theta}{130}\dfrac{dx(t)}{dt}$.

Now, when $x = 500$, the straight-line distance from the viewer to the plane is $\sqrt{130^2+500^2}$, and $\sin\theta = \dfrac{130}{\sqrt{130^2+500^2}}$. Since the plane is moving away from the viewer, x is increasing, so $\dfrac{dx(t)}{dt} = +210$, and we get

$$\frac{d\theta(t)}{dt} = -\frac{1}{130}\left(\frac{130}{\sqrt{130^2+500^2}}\right)^2(210) = -\frac{130(210)}{130^2+500^2} \text{ radians per second.}$$

2. Suppose a police officer is 300 metres south of an intersection, and travelling north at 80 km/h. At the same time, a vehicle is 400 metres east of the same intersection, travelling east (away from the intersection).

   If the officer's radar gun registers a speed of 32 km/h, how fast is the car travelling?



A diagram of the situation is given above. The Pythagorean theorem gives us $z(t)^2 = x(t)^2 + y(t)^2$, so

$$2z(t) \cdot z'(t) = 2x(t) \cdot x'(t) + 2y(t) \cdot y'(t).$$

Since the police car is moving *toward* the intersection x is decreasing, so $\dfrac{dx}{dt} = -80$.

At the time that the speed is recorded on radar, $x = 300$ m $= 0.3$ km, and $y = 400$ m $=$ 0.4 km.

This gives us $z = 0.5$ km when the speed is registered, and the radar gun gives $\dfrac{dz}{dt} = 32$.

Thus, we have

$$0.4\frac{dy}{dt} = 0.5\frac{dz}{dt} - 0.3\frac{dx}{dt} = 0.5(32) - 0.3(-80) = 16 + 24 = 40,$$

so $\dfrac{dy}{dt} = \dfrac{40}{0.4} = 100$ km/h.