Math 2580 Assignment #3 Solutions
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1. In class, I mentioned the fact that if we want to find the equation of the tangent line to a level curve $f(x, y) = c$ at a point (a, b) on the curve (so $f(a, b) = c$), there are two ways to do it:

   - Using implicit differentiation, as in Calculus I: take the derivative of both sides with respect to x, assuming that the equation defines y implicitly as a function of x ($y = g(x)$), let's say.
   - Using the gradient: since $\nabla f(a, b)$ is a normal vector for the tangent line, we have

   $$0 = \nabla f(a, b) \cdot \langle x - a, y - b\rangle = f_x(a, b)(x - a) + f_y(a, b)(y - b). \tag{1}$$

   (a) Verify that both above methods give the same equation for the tangent line to the curve $x^2y + xy^2 = 6$ at the point $(2, 1)$.

   Using implicit differentiation à la 1560, we get

   $$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0,$$

   so $\dfrac{dy}{dx} = \dfrac{-y^2 - 2xy}{x^2 + 2xy}$. Plugging in $x = 2$ and $y = 1$ gives $m = -\dfrac{5}{8}$ for the slope of the tangent line, so the equation of the tangent line is

   $$y - 1 = -\frac{5}{8}(x - 2).$$

   Now, if we let $f(x, y) = x^2y + xy^2$, then $f_x(x, y) = 2xy + y^2$, so $f_x(2, 1) = 5$, and $f_y(x, y) = x^2 + 2xy$, giving $f_y(2, 1) = 8$. Using equation (1), we get the tangent line

   $$5(x - 2) + 8(y - 1) = 0,$$

   and if we subtract $8(y-1)$ from both sides and divide by 8, we obtain our previous result.

   (b) Confirm that the two methods are equivalent, as follows:
   The Implicit Function Theorem for a function $f : \mathbb{R}^2 \to \mathbb{R}$ states the following:

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. At any point (a, b) such that $f_y(a, b) \neq 0$, the equation $f(x, y) = c$ defines y implicitly as a function g of x for all x in some interval[1] centred at $x = a$, and

$$\frac{dy}{dx} = g'(x) = -\frac{f_x(x, y)}{f_y(x, y)} \tag{2}$$

for all x in this interval.

Assuming that you can prove that the equation $f(x, y) = c$ defines y as a function of x for x near a, if $f_y(a, b) \neq 0$, show that Equation (2) is true.

Suppose that $f(x, y) = c$ implicitly defines $y = g(x)$ such that $f(x, g(x)) = c$. Letting $r(x) = (x, g(x))$ and applying the Chain Rule to $f(r(x)) = c$, we have

$$0 = \frac{d}{dx}(f(r(x))) = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = f_x(x, y)(1) + f_y(x, y)g'(x).$$

Solving for $g'(x)$, we obtain equation (2). Thus, we see that in general, the first method will give us the tangent line

$$y - b = -\frac{f_x(a, b)}{f_y(a, b)}(x - a),$$

which is just a rearrangement of equation (1).

2. Now consider a continuously differentiable function $F(x, y, z)$, and suppose (a, b, c) is a point on the level surface $F(x, y, z) = k$. We discussed in class that one way to get the tangent plane to the surface at (a, b, c) is to use the gradient: the vector $\nabla F(a, b, c)$ is normal to the surface at (a, b, c), so

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c\rangle = 0$$

gives the equation of the tangent plane. On the other hand, we could try generalizing the method of implicit differentiation above. Suppose that the equation $F(x, y, z) = k$ defines z implicitly as a function of x and y. That is, assume there exists a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $z = g(x, y)$ satisfies

$$F(x, y, g(x, y)) = k$$

for all points (x, y) near the point (a, b).

(a) Using the Chain Rule, show that if $F_z(a, b, c) \neq 0$, then at the point (a, b, c),

$$\frac{\partial z}{\partial x} = g_x(a, b) = -\frac{F_x(a, b, c)}{F_z(a, b, c)} \quad \text{and} \quad \frac{\partial z}{\partial y} = g_y(a, b) = -\frac{F_y(a, b, c)}{F_z(a, b, c)}.$$

[1] Don't worry too much about the "in some interval" part. The argument is as follows: since $f_y(x, y)$ is continuous, if $f_y(a, b) \neq 0$, then $f_y(x, y) \neq 0$ for all (x, y) in some disk centred at (a, b). (The function can't suddenly jump to zero.)

Let us suppose that $F_z(a,b,c) \neq 0$ and that the equation $F(x,y,z) = k$ implicitly defines $z = g(x,y)$ for (x,y) near (a,b). Let us consider the function $r(u,v) = (u,v,g(u,v))$ (we're defining $x = u$, $y = v$, and $z = g(u,v)$; you can equally well take $r(x,y) = (x,y,g(x,y))$ but this will help avoid some confusion), which is chosen such that $F(r(u,v)) = k$ for all values of (u,v) near (a,b). Applying the Chain Rule gives us the derivatives

$$0 = \frac{\partial}{\partial u}(F(r(u,v))) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u}$$
$$0 = \frac{\partial}{\partial v}(F(r(u,v))) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial v}.$$

Now, we note that since $x = u$, $y = v$, and $z = g(u,v)$, we have $\frac{\partial x}{\partial u} = 1$, $\frac{\partial y}{\partial u} = 0$, $\frac{\partial x}{\partial v} = 0$, and $\frac{\partial y}{\partial v} = 1$, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} = g_x(x,y) \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial y} = g_y(x,y).$$

Plugging everything in, we have $F_x(x,y,z) + F_z(x,y,z)g_x(x,y) = 0$ and $F_y(x,y,z) + F_z(x,y,z)g_y(x,y) = 0$. Since we're assuming $F_z(a,b,c) \neq 0$ we can solve these equations for $g_x(a,b)$ and $g_y(a,b)$ respectively, giving us our result.

(b) Suppose $F(x,y,z) = k$ implicitly defines $z = g(x,y)$ near a point (a,b,c). Then near this point, we've expressed our level surface as a graph. It might not be possible to do this for the entire surface (there might, for example, be points where F_z equals zero), but at least it works locally. This puts us in a position to calculate the normal vector to the surface at (a,b,c) in two ways:

i. Using the gradient vector $\nabla F(a,b,c)$, where we describe our surface via the equation $F(x,y,z) = k$.

ii. Using the result $\vec{n} = \langle g_x(a,b), g_y(a,b), -1\rangle$ that we obtained for graphs, where we describe our surface as the graph $z = g(x,y)$.

Use your result from part (a) to show that these two vectors are parallel.

The first method gives us the normal vector $\vec{N} = \langle F_x(a,b,c), F_y(a,b,c), F_z(a,b,c)\rangle$. The second method gives us the vector

$$\vec{n} = \langle g_x(a,b), g_y(a,b), -1\rangle = \left\langle -\frac{F_x(a,b,c)}{F_z(a,b,c)}, -\frac{F_y(a,b,c)}{F_z(a,b,c)}, -1 \right\rangle = -\frac{1}{F_z(a,b,c)}\vec{N}.$$

Since $\vec{N} = -F_z(a,b,c)\vec{n}$, the two vectors are scalar multiples of each other, and therefore parallel.