# Math 3500 Exercise Sheet

29 October, 2014

This week's problems involve uniform continuity. First, recall that a function $f : D \to \mathbb{R}$ is **continuous** on $D$ if for every $\epsilon > 0$ and for every $y \in D$ there exists a $\delta > 0$ such that whenever $x \in D$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Note that in this definition our choice of $\delta$ depends on both $\epsilon$ and the point $y \in D$. A function $f : D \to \mathbb{R}$ is **uniformly continuous** on $D$ if for every $\epsilon > 0$, whenever $x, y \in D$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Here, the same $\delta$ has to work on all of $D$, whereas for regular continuity we can choose a different $\delta$ at each point.

As practice, make sure that it's clear to you that uniform continuity implies continuity. We proved in class that if $D$ is *compact*, then continuity implies uniform continuity, so the two definitions coincide in this case.

1. Show that if $f : D \to \mathbb{R}$ is uniformly continuous and $(x_n)$ is a Cauchy sequence in $D$, then $f(x_n)$ is a Cauchy sequence.

2. (a) Suppose that $f$ is uniformly continuous on $(a, b)$, and $(x_n)$ is a sequence in $(a, b)$ such that $x_n \to a$. Show that $\lim f(x_n)$ exists.

   (b) Suppose $(x_n)$ and $(y_n)$ are two sequences in $(a, b)$ that converge to $a$. Let $(z_n)$ be the sequence given by $(x_1, y_1, x_2, y_2, \ldots)$. Explain why $\lim f(z_n) = \lim f(x_n) = \lim f(y_n)$.

   Hint: if a sequence $(a_n)$ converges to some limit $L$, then any subsequence must also converge to $L$.

   (c) Explain why (a) and (b) guarantee that $\lim_{x \to a^+} f(x)$ exists.

   (d) Conclude that if $f$ is uniformly continuous on $(a, b)$, then there exists a continuous function $\tilde{f}$ on $[a, b]$ such that $\tilde{f}(x) = f(x)$ for all $x \in [a, b]$. (Such a function $\tilde{f}$ is called an **extension** of $f$ from $(a, b)$ to $[a, b]$.)

   (e) Finally, notice that we've proved the following theorem: a function $f$ is uniformly continuous on $(a, b)$ if and only if it can be extended to a continuous function on $[a, b]$.

3. Decide whether or not the following functions are uniformly continuous on the given interval. You may use any of the theorems mentioned or proved above on this worksheet.

   (a) $f(x) = x^{17} \sin x - e^x \cos(3x)$ on $[0, \pi]$

(b) $f(x) = x^3$ on $[0, 1]$

(c) $f(x) = x^3$ on $(0, 1)$

(d) $f(x) = x^3$ on $\mathbb{R}$

(e) $f(x) = 1/x^3$ on $(0, 1]$

(f) $f(x) = \sin(1/x^2)$ on $(0, 1]$

(g) $f(x) = x^2 \sin(1/x^2)$ on $(0, 1]$

4. Use the $\epsilon - \delta$ definition of uniform continuity to prove that the following functions are uniformly continuous on the given interval:

   (a) $f(x) = 3x + 1$ on $\mathbb{R}$

   (b) $f(x) = \dfrac{x}{x+1}$ on $[0, 2]$

5. Suppose $f$ is continuous on $[a, b]$, and let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a uniform partition of $[a, b]$.

   (a) Explain why, for each $i = 1, 2, \ldots, n$, there exist $m_i, M_i \in \mathbb{R}$ such that $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$.

   (b) Let $U_n = \displaystyle\sum_{i=1}^{n} M_i \Delta x$ and $L_n = \displaystyle\sum_{i=1}^{n} m_i \Delta x$, where $\Delta x = (b - a)/n$. Notice that for any $x_i^* \in [x_{i-1}, x_i]$, we have $L_n \leq \displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x \leq U_n$. Explain why it follows that if $\displaystyle\lim_{n\to\infty} (U_n - L_n) = 0$, then $f$ is intergrable on $[a, b]$ (pretend that we've defined this).

   (c) Given $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \dfrac{\epsilon}{b - a}$, since $f$ is uniformly continuous. Now, suppose we choose $N \in \mathbb{N}$ large enough that $1/N < \delta$. Show that if $n > N$, then $M_i - m_i < \dfrac{\epsilon}{b - a}$ for each $i = 1, 2, \ldots, n$. Conclude that $0 \leq U_n - L_n < \epsilon$.

   (d) Finally, explain why it follows that any continuous function defined on a closed interval is integrable.