Name: **Tutorial time:**

1. Evaluate the following limits:

   (a)

$$\begin{aligned}\lim_{x\to 2}\frac{2-\sqrt{x+2}}{x-2} &= \lim_{x\to 2}\frac{(2-\sqrt{x+2})(2+\sqrt{x+2})}{(x-2)(2+\sqrt{x+2})}\\ &= \lim_{x\to 2}\frac{2-(x+2)}{(x-2)(2+\sqrt{x+2})}\\ &= \lim_{x\to 2}\frac{(-1)}{2+\sqrt{x+2}} = \frac{-1}{2+\sqrt{4}} = -\frac{1}{4}.\end{aligned}$$

   (b)

$$\begin{aligned}\lim_{\theta\to 0}\frac{\tan\theta}{\sin\theta+2\theta} &= \lim_{\theta\to 0}\frac{\frac{1}{\theta}(\tan\theta)}{\frac{1}{\theta}(\sin\theta+2\theta)}\\ &= \lim_{\theta\to 0}\frac{\left(\frac{\sin\theta}{\theta}\right)\frac{1}{\cos\theta}}{\frac{\sin\theta}{\theta}+2}\\ &= \frac{\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\frac{1}{\lim_{\theta\to 0}\cos\theta}}{\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)+2}\\ &= \frac{1(1)}{1+2} = \frac{1}{3}.\end{aligned}$$

   (c) $\lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right)$ (Hint: squeeze theorem)

   Since $-1 \le \cos(1/x^2) \le 1$ for all $x \neq 0$, it follows that

$$-x^2 \le x^2\cos(1/x^2) \le x^2$$

   for all $x \neq 0$. Since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0}(x^2) = 0$, it follows from the squeeze theorem that $\lim_{x\to 0} x^2\cos(1/x^2) = 0$.

   (d) $\lim_{x\to 2^+}\frac{x^2-9}{x^2-4}$.

   Since the denominator is zero at $x = 2$ but the numerator is not, we must have a vertical asymptote at $x = 2$. We note that $\frac{x^2-9}{x^2-4} = \frac{(x-3)(x+3)}{(x-2)(x+2)}$ is negative for $2 < x < 3$. (Verify using a sign diagram.) It follows that

$$\lim_{x\to 2^+}\frac{x^2-9}{x^2-4} = -\infty.$$

2. Let $f(x) = \dfrac{x^2 - 4}{x^2 - 4x + 3}$.

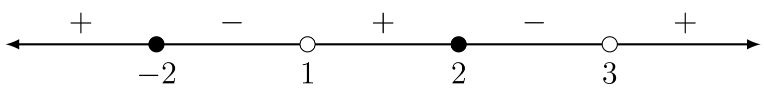
   (a) What is the horizontal asymptote for the graph $y = f(x)$?

   The highest power of x top and bottom is 2, and both x^2 terms have a coefficient of 1, so there is a horizontal asymptote at $y = 1$.

   (b) What are the vertical asymptotes for the graph $y = f(x)$?

   The denominator $x^2 - 4x + 3 = (x-1)(x-3)$ is equal zero for $x = 1$ and $x = 3$. Since the numerator is non-zero at both of these values, both $x = 1$ and $x = 3$ are vertical asymptotes.

   (c) What are the left and right-hand limits of $f(x)$ at each vertical asymptote?
   From the sign diagram



we can read off the desired limits:

$$\lim_{x\to 1^-} f(x) = -\infty, \ \lim_{x\to 1^+} f(x) = \infty, \ \lim_{x\to 3^-} f(x) = -\infty, \ \text{and} \ \lim_{x\to 3^+} f(x) = \infty.$$

3. Find and classify the discontinuities of $f(x) = \begin{cases} \frac{x^2+2x+1}{x+1}, & \text{if } x \leq 0 \\ \frac{1}{x-2}, & \text{if } x > 0 \end{cases}$

We note that $\frac{x^2+2x+1}{x+1} = \frac{(x+1)^2}{x+1} = x + 1$ for $x \neq -1$. Thus $f(-1)$ is undefined, but the limit at -1 exists, so there is a removable discontinuity at $x = -1$. Since $\lim_{x\to 0^-} f(x) = \frac{0^2+2(0)+1}{0+1} = 1$ but $\lim_{x\to 0^+} f(x) = \frac{1}{0-2} = -\frac{1}{2}$, there is a jump discontinuity at $x = 0$. Finally, we can easily see that there is a vertical asymptote at $x = 2$, so there is an infinite discontinuity at this point.

4. Using the **definition** of the derivative, find the equation of the tangent line to $y = x^2 + 1$ at the point $(1, 2)$.

By definition, for $f(x) = x^2 + 1$ we have:

$$f'(1) = \lim_{h\to 0} \frac{((1+h)^2 + 1) - (1^2 + 1)}{h} = \lim_{h\to 0} \frac{1 + 2h + h^2 + 1 - 2}{h} = \lim_{x\to 0}(2 + h) = 2.$$

Since $f'(1)$ gives the slope of the tangent line at $x = 1$, we have the equation $y - 2 = 2(x-1)$, or $y = 2x$, for the tangent line.