Math 3500 Exercise Sheet

10 September, 2014

We will work on some of the following exercises in class. Those not done in class are recommended as homework problems.

1. Find the least upper bound and greatest lower bound of the following sets (if they exist):

   (a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

   (b) $\left\{\frac{1}{n} : n \in \mathbb{Z} \text{ and } n \neq 0\right\}$

   (c) $\{x : x = 0 \text{ or } x = 1/n \text{ for some } n \in \mathbb{N}\}$

   (d) $\{x : 0 \leq x \leq \sqrt{2} \text{ and } x \in \mathbb{Q}\}$

   (e) $\{x : x^2 + x + 1 \geq 0\}$

   (f) $\{x : x^2 + x - 1 < 0\}$

   (g) $\{x < 0 \text{ and } x^2 + x - 1 < 0\}$

   (h) $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$

2. (a) Suppose that $y - x > 1$. Prove that there is an integer k such that $x < k < y$. *Hint:* let l be the largest integer satisfying $l \leq x$ and consider $l + 1$.

   (b) Suppose $x < y$. Prove that there is a rational number $r \in \mathbb{Q}$ such that $x < r < y$. *Hint:* If $1/n < y - x$, then $ny - nx > 1$. *Hint* (to the two hints given so far): why are these questions being asked in the context of least upper bounds?

   (c) Suppose that $r < s$ are rational numbers. Prove that there is an irrational number between r and s. *Hint:* start by finding an irrational number between 0 and 1.

   (d) Suppose that $x < y$. Prove that there is an irrational number between x and y. *Hint:* no more work is needed at this point – your result should follow from parts (b) and (c).

3. Suppose $\alpha > 0$ Prove that every number x can be written uniquely in the form $x = k\alpha + y$, where k is an integer, and $0 \leq y < \alpha$.

4. Suppose that A and B are two nonempty sets of numbers such that $x \leq y$ for all $x \in A$ and $y \in B$.
   (a) Prove that $\sup A \leq y$ for all $y \in B$.
   (b) Prove that $\sup A \leq \inf B$.

5. (a) Consider a sequence of closed intervals $I_1 = [a_1, b_1], I_2 = [a_2, b_2], \ldots$. Suppose that $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all n. (Notice that this means $I_{n+1} \subseteq I_n$ for each n – it might help to draw a picture.) Prove that there is a point x which belongs to I_n for all $n \in \mathbb{N}$. (In other words, $x \in \bigcap_{n\in\mathbb{N}} I_n$.)
   (b) Show that this conclusion is false if we consider open intervals instead of closed intervals.

6. (a) Let $A = \{x : x < \alpha\}$. Prove the following (none of them are hard):
      i. If $x \in A$, and $y < x$, then $y \in A$.
      ii. $A \neq \emptyset$
      iii. $A \neq \mathbb{R}$
      iv. If $x \in A$, then there is some number $x' \in A$ such that $x < x'$.
   (b) Suppose, conversely, that A satisfies (i)-(iv) above. Prove that $A = \{x : x < \sup A\}$.

7. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that if $\sup S \in S$, then $\sup S = \max S$. *Hint:* your proof should be very short.

8. (a) Prove that $\inf S \leq \sup S$. (Again, your proof should be short.)
   (b) What can you say about S if $\inf S = \sup S$?

9. Let S and T be nonempty bounded subsets of $\mathbb{R}$.
   (a) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.
   (b) Prove that $\sup S \cup T = \max\{\sup S, \sup T\}$. (For (b) you're not assuming $S \subseteq T$.)

10. Prove that if $a > 0$ is any real number, then there exists $n \in \mathbb{N}$ such that $1/n < a < n$.