[3] 1. Evaluate the limit:

$$\begin{aligned}\lim_{x\to\infty}\frac{5+2x-3x^3}{5x^3-4x^2+7} &= \lim_{x\to\infty}\frac{x^3(5/x^3+2/x^2-3)}{x^3(5-4/x+7/x^3)}\\ &= \lim_{x\to\infty}\frac{5/x^3+2/x^2-3}{5-4/x+7/x^3}\\ &= \frac{0+0-3}{5+0+0} = -\frac{3}{5}.\end{aligned}$$

[3] 2. Is the function

$$f(x) = \begin{cases} 5x-x^2, & \text{if } x<2\\ 4x-2, & \text{if } x\geq 2\end{cases}$$

continuous at $x=2$? Why or why not?

By definition, f is continuous at 2 if $\lim\limits_{x\to 2} f(x) = f(2)$.

From the formula for $f(x)$ we see that $f(2) = 4(2)-2 = 6$. For the limit, we must consider left and right hand limits. For the left-hand limit, we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(5x-x^2) = 5(2)-2^2 = 6.$$

For the right-hand limit,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(4x-2) = 4(2)-2 = 6.$$

Since the left and right hand limits are equal, we can conclude that

$$\lim_{x\to 2} f(x) = 6 = f(2),$$

and thus f is continuous at 2.

3. Let $f(x) = \sqrt{x^2+1}$. Write down, but do not evaluate, a limit that computes $f'(0)$ *according*
[2] *to the definition of the derivative.*

The derivative at a point $x=a$ is defined by

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

In our case, $a=0$, and $f(x)=\sqrt{x^2+1}$, so $f(0+h) = f(h) = \sqrt{h^2+1}$, while $f(0) = \sqrt{0^2+1} = 1$. Thus, we have

$$f'(0) = \lim_{x\to 0}\frac{\sqrt{h^2+1}-1}{h}.$$

4. Compute $f'(x)$ for each function $f(x)$ below. You do **not** need to simplify your answers.

[2] (a) $f(x) = 4x^5 - 2x^3 + \sqrt{2}x - 3^4$

$$f'(x) = 20x^4 - 6x^2 + \sqrt{2}.$$

[2] (b) $f(x) = x^3 \sin(x)$

$$f'(x) = \frac{d}{dx}(x^3) \cdot \sin(x) + x^3 \frac{d}{dx}(\sin(x)) = 3x^2 \sin(x) + x^3 \cos(x).$$

[3] (c) $f(x) = \dfrac{x^3 - \sqrt{x}}{x^2}$

Notice that we can simplify the function first. Dividing each term in the numerator by x^2, we have

$$f(x) = \frac{x^3}{x^2} - \frac{x^{1/2}}{x^2} = x - x^{-3/2}.$$

Thus,

$$f'(x) = 1 + \frac{3}{2}x^{-5/2}.$$

If you didn't notice this, then the quotient rule gives

$$f'(x) = \frac{\frac{d}{dx}(x^3 - \sqrt{x}) \cdot x^2 - (x^3 - \sqrt{x})\frac{d}{dx}(x^2)}{(x^2)^2} = \frac{(3x^2 - \frac{1}{2}x^{-1/2})x^2 - (x^3 - \sqrt{x})(2x)}{x^4}.$$

[3] (d) $f(x) = e^{\sqrt{x^2+1}}$

We have $f(x) = g(h(k(x)))$ with $g(x) = e^x$, $h(x) = \sqrt{x}$, and $k(x) = x^2 + 1$. The Chain Rule gives us

$$\begin{aligned} f'(x) &= g'(h(k(x)))\frac{d}{dx}(h(k(x))) = g'(h(k(x)))h'(k(x))k'(x) \\ &= e^{\sqrt{x^2+1}}(\frac{1}{2}(x^2+1)^{-1/2})(2x). \end{aligned}$$

If you prefer the Leibniz notation, $y = e^u$, where $u = \sqrt{v}$ and $v = x^2 + 1$, and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} \\ &= e^u(\frac{1}{2}v^{-1/2})(2x) \\ &= e^{\sqrt{x^2+1}}(\frac{1}{2}(x^2+1)^{-1/2})(2x). \end{aligned}$$

[2] 5. (Extra group question!) Suppose f and g are continuous functions on an interval $[a, b]$, and you know that $f(a) < g(a)$, and $f(b) > g(b)$.

Show that there must be some number $c \in (a, b)$ such that $f(c) = g(c)$.

Hint: Apply the Intermediate Value Theorem to $h(x) = f(x) - g(x)$. Be sure to justify your work.

Following the hint, we let $h(x) = f(x) - g(x)$. Since h is the difference of two continuous functions, we know that h is continuous on $[a, b]$ as well. Since $f(a) < g(a)$, we see that

$$h(a) = f(a) - g(a) < 0,$$

while

$$h(b) = f(b) - g(b) > 0,$$

since $f(b) > g(b)$. Since h is continuous on $[a, b]$, and $h(a) < 0$ while $h(b) > 0$, it follows from the Intermediate Value Theorem that there must exist some $c \in (a, b)$ such that

$$h(c) = 0 = f(c) - g(c),$$

and thus $f(c) = g(c)$, as required.