Math 1410 Assignment #3
University of Lethbridge, Spring 2015

Sean Fitzpatrick

February 24, 2015

Due date: Wednesday, March 11th, by 5 pm.

For instructions on completing this assignment, please see Assignment #1, but please keep the following additional points in mind:

- You don't have to type your work. (If you think your handwriting is too illegible to read, you can still type, but it's generally more work than it's worth to type out your math homework.)
- If you do your work in a spiral notebook and tear out the pages, cut off the tattered edges before submitting. (This is pretty much the most annoying thing for a grader to deal with. If you get a bunch of assignments like this in a pile, the tattered bits get all tangled up.)
- There's a stapler right next to the assignment drop box. Please use it.
- On the topic of staples, using industrial-sized staples of death that result in sharp pointy bits poking out of your assignment is a bad idea.
- Plastic covers aren't really necessary. (They mostly just get in the way.)

Assigned problems

1. Given a polynomial $p(x) = a + bx + cx^2 + dx^3 + x^4$, the matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & -c & -d \end{bmatrix}$$

is called the *companion matrix* of $p(x)$. Show that $\det(xI_4 - C) = p(x)$.

2. If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ is a polynomial of degree k (the degree of $p(x)$ is the highest power of x, so we're assuming that $a_k \neq 0$). Given any such polynomial $p(x)$ and any $n \times n$ (square) matrix A, it's possible to plug A into the polynomial to obtain a *new* matrix, denoted $p(A)$, given by

$$p(A) = a_0I_n + a_1A + a_2A^2 + \cdots + a_kA^k.$$

For example, if $p(x) = 2 - 3x + x^2$, then $p(A) = 2I_n - 3A + A^2$.

   (a) If $p(x) = 3 - 4x + 2x^2$ and $A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$, compute $p(A)$.

   (b) The *characteristic polynomial* of an $n \times n$ matrix A is defined by

$$c_A(x) = \det(xI_n - A).$$

   The *Cayley-Hamilton Theorem* is a famous theorem in linear algebra which states that for any $n \times n$ matrix A, $c_A(A) = 0$ (where the zero on the right is the zero matrix).

   Verify that the Cayley-Hamilton Theorem is true for $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$.

Bonus opportunity: Prove the Cayley-Hamilton Theorem for the $n = 2$ case. That is, show that the theorem holds for a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

3. In each case, either explain why the statement is true (in general), or give an example showing that it is false:

   (a) If $\|\vec{v} - \vec{w}\| = 0$, then $\vec{v} = \vec{w}$.

   (b) If $\vec{v} = -\vec{v}$, then $\vec{v} = \vec{0}$.

   (c) If $\|\vec{v}\| = \|\vec{w}\|$, then $\vec{v} = \vec{w}$.

   (d) If $\|\vec{v}\| = \|\vec{w}\|$, then $\vec{v} = \pm\vec{w}$.

   (e) $\|\vec{v} + \vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$.

4. Let $\vec{u} = \begin{bmatrix} 3 & -1 & 0 \end{bmatrix}^T$, $\vec{v} = \begin{bmatrix} 4 & 0 & 1 \end{bmatrix}^T$, and $\vec{w} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. In each case, either find scalars a, b, c such that $\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$, or explain why no such scalars exist:

   (a) $\vec{x} = \begin{bmatrix} 5 & 1 & 2 \end{bmatrix}^T$

   (b) $\vec{x} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$.

Note: for problems 1 and 2 above, xI_n denotes the identity matrix multiplied by the (variable) scalar x. For example, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$xI_2 - A = x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x - a & -b \\ -c & x - d \end{bmatrix}.$$