Math 1560 Assignment #3 Solutions
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1. Show that the equation $6x^5 + 13x + 1 = 0$ has **exactly one** real number solution.

   Solution: Let $f(x) = 6x^5 + 13x + 1$. Since f is a polynomial function, it is continuous on $\mathbb{R}$. We notice that

   $$f(-1) = -6 - 13 + 1 = -18 < 0, \text{ and } f(0) = 0 + 0 + 1 = 1 > 0.$$

   Thus, by the Intermediate Value Theorem, there exists some number $c \in (-1, 0)$ such that $f(c) = 6c^5 + 13c + 1 = 0$, so there is at least one solution to the equation.

   Now, suppose there were two such solutions, say c_1 and c_2, with $c_1 < c_2$. Then we have that f is continuous on $[c_1, c_2]$, and differentiable on (c_1, c_2), and $f(c_1) = f(c_2) = 0$. It would then follow from Rolle's theorem that there is some number $a \in (c_1, c_2)$ such that $f'(a) = 0$.

   However, we find that
   $$f'(x) = 30x^4 + 13 \geq 13 > 0$$
   for all $x \in \mathbb{R}$. Thus, it is impossible for such a number a (with $f'(a) = 0$) to exist.

   If there were more than one solution, we would have a contradiction, and therefore, there must be exactly one solution to the equation, as required.

2. Show that for any real numbers a and b,

$$|\sin(a) - \sin(b)| \leq |a - b|.$$

Solution: Suppose $a < b$. (If $a = b$, the result holds, since $0 \leq 0$. If $a > b$, we can simply exchange the roles of a and b.)

Let $f(x) = \sin(x)$. We know that f is continuous on $[a, b]$ and differentiable on (a, b), so by the Mean Value Theorem there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}.$$

But we know that $f'(c) = \cos(c)$, and since $|\cos(c)| \leq 1$ for any number c, we have

$$1 \geq |\cos(c)| = \left| \frac{\sin(a) - \sin(b)}{a - b} \right| = \frac{|\sin(a) - \sin(b)|}{|a - b|}.$$

Multiplying both sides of the inequality by $|a - b|$, we obtain the desired result.

3. Sketch the graph of the function f with the following properties:

(i) f is continuous on $\mathbb{R}$

(ii) $f(0) = 1$, $f(1) = 0$, $f(-1) = 0$, and $f(2) = 1$.

(iii) $\lim_{x \to \infty} f(x) = 2$, and $\lim_{x \to -\infty} f(x) = -1$.

(iv) $f'(x) > 0$ on $(-\infty, 0) \cup (1, \infty)$, and $f'(x) < 0$ on $(0, 1)$.

(v) $f''(x) > 0$ on $(-\infty, 0) \cup (0, 2)$ and $f''(x) < 0$ on $(2, \infty)$.

Solution: The following is a representative (albeit imperfect) graph satisfying the required properties.