University of Lethbridge
Department of Mathematics and Computer Science
10th December 2014, 9:00 am - 12:00 pm

MATH 3500 - FINAL EXAM

Last Name:____

First Name:____

Student Number:_____________________________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

For full credit, you must complete **eight** of the nine problems on the exam, but you can attempt all nine problems. (Yes, a score above 100% is possible.)

For grader's use only:

Problem	Grade
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
Total	/80

1. Let S and T be nonempty subsets of $\mathbb{R}$ such that $s \leq t$ for all $s \in S$ and $t \in T$.

[1] (a) Explain why S is bounded above and T is bounded below.

[5] (b) Prove that $\sup S \leq \inf T$.

[2] (c) Give an example of sets S and T as above with $S \cap T \neq \emptyset$.

[2] (d) Give an example of sets S and T as above where $\sup S = \inf T$ but $S \cap T = \emptyset$.

[4] 2. (a) What is the set of all limit points of the set $A = \left\{ \dfrac{1}{n} + \dfrac{1}{m} : n, m \in \mathbb{N} \right\}$?
(A formal proof is not required.)

[6] (b) The closure $\overline{A}$ of a set A is defined to be the union of A and the set of limit points of A. Prove that $\overline{A} = A \cup \partial A$, where ∂A denotes the boundary of A.
(Hint: any limit point of A either belongs to A, or it doesn't.)

[4] 3. (a) Prove that the union of two open sets is open.

[3] (b) Prove that the intersection of two closed sets is closed.

[3] (c) Prove that the intersection of two compact sets is compact.

4. Let (x_n) be a sequence defined by $x_1 = 1$ and $x_{n+1} = \frac{1}{3}(x_n + 1)$ for $n \geq 1$.

[2] (a) Find x_2, x_3, and x_4.

[4] (b) Use induction to prove (x_n) is a decreasing sequence.

[2] (c) Argue that (x_n) is a bounded sequence. (You can use induction, but it isn't necessary.)

[2] (d) Explain why $\lim x_n$ exists, and find $\lim x_n$.

5. (a) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous at $c \in [a, b]$, and $f(c) > 0$. Prove that there
[5] exists an interval I with $c \in I \subseteq [a, b]$ such that $f(x) > 0$ for all $x \in I$.
(Hint: use the definition of continuity and choose $\epsilon > 0$ wisely.)

(b) Prove (using the definition) that $f(x) = 1/x$ is uniformly continuous on $[a, \infty)$ for any
[5] $a > 0$. Why do we know that f cannot be uniformly continuous on $(0, \infty)$?

6. (a) Prove that if $f : [a,b] \to \mathbb{R}$ is continuous, and $f(x) > 0$ for all $x \in [a,b]$, then $1/f$ is bounded on $[a,b]$. [5]

(b) Prove that for any continuous function $f : [0,1] \to [0,1]$ there exists a point $x \in [0,1]$ such that $f(x) = x$. [5]

[5] 7. (a) Suppose that f is differentiable on $\mathbb{R}$, and $f(0) = 0$, $f(1) = 1$, and $f(2) = 1$. Show that $f'(x) = 1/2$ for some $x \in (0, 2)$.

(b) For which values of a is the function $f(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

[5] differentiable at $x = 0$?

8. Let $L(x) = \int_1^x \frac{1}{t}\, dt$, for $x > 0$.

[2] (a) Calculate $L'(x)$.

[3] (b) Prove that L is an increasing function.

Since L is increasing, it is one-to-one. Define a function $E : \mathbb{R} \to (0, \infty)$ by $E = L^{-1}$.

[5] (c) Prove that $E(0) = 1$ and $E'(x) = E(x)$ for all $x \in \mathbb{R}$.
(Hint: use the Chain Rule, and the fact that $L(E(x)) = x$ for all $x \in \mathbb{R}$.)

9. Let f and g be bounded functions on $[a, b]$.

[5] (a) Prove that for any partition P of $[a, b]$, $U(f + g, P) \leq U(f, P) + U(g, P)$

[3] (b) Conclude that $U(f + g) \leq U(f) + U(g)$. ($U(f) = \inf\{U(f, P) : P \text{ is a partition}\}$.)

[2] (c) Give an example where the above inequality is strict.
(Hint: f and g won't be integrable, since in that case $U(f + g) = U(f) + U(g)$.)

 Total Points: 80

This page was left intentionally blank for rough work. Please do not remove.