Name: Solutions

Choose **one** of the following two problems:

1. Prove the following proposition: For each integer a, if $a^2 = a$, then $a = 0$ or $a = 1$.

   Solution: Let a be any integer such that $a^2 = a$. We consider two cases: $a = 0$ or $a \neq 0$.[1]
   If $a = 0$, then we're done, since we want to show that either $a = 0$ or $a = 1$. Thus, if $a \neq 0$, it remains to show that we must have $a = 1$. But if $a \neq 0$, then we can divide by a, and

   $$a^2 = a \quad \Rightarrow \quad \frac{1}{a}(a^2) = \frac{1}{a}(a) \quad \Rightarrow \quad a = 1,$$

   which is what we needed to show.

2. Use cases based on congruence modulo 3 and properties of congruence[2] to prove that for each integer n, $n^3 \equiv n \pmod 3$.

   Solution: Let n be any integer. From the division algorithm, we know that we must have $n \equiv r \pmod 3$ for exactly one $r \in \{0, 1, 2\}$.
   Case 1: Suppose $n \equiv 0 \pmod 3$. Then $n^3 \equiv 0^3 \pmod 3$, but $0^3 = 0$, so $n^3 \equiv 0 \pmod 3$ and $0 \equiv n \pmod 3$, from which it follows that $n^3 \equiv n \pmod 3$.

   Case 2: Suppose $n \equiv 1 \pmod 3$. Then $n^3 \equiv 1^3 \pmod 3$, but $1^3 = 1$, so $n^3 \equiv 1 \pmod 3$ and $1 \equiv n \pmod 3$, and it follows that $n^3 \equiv n \pmod 3$.

   Case 3: Finally, suppose that $n \equiv 2 \pmod 3$. Then $n^3 \equiv 2^3 \pmod 3$, and $2^3 = 8 = 2 + 3(2)$, so $2^3 \equiv 2 \pmod 3$. Thus $n^3 \equiv 2 \pmod 3$ and $2 \equiv n \pmod 3$, and we have $n^3 \equiv n \pmod 3$, as required.

[1] Recall from the Law of the Excluded Middle that $P \vee \neg P$ is a tautology. Thus, in particular, either $a = 0$ is true, or its negation, $a \neq 0$ is true. This tautology is frequently used to obtain a proof by cases.

[2] In particular, you will want the property we proved on Tuesday: if $a \equiv b \pmod 3$ and $c \equiv d \pmod 3$, then $ac \equiv bd \pmod 3$.

Total: 10 points