MATH 1560 - Tutorial #10 Solutions

On the worksheet, you were provided with the following:

Intermediate Value Theorem (zero version): Suppose a function f is continuous on $[a, b]$, and either (a) $f(a) < 0$ and $f(b) > 0$, or (b) $f(a) > 0$ and $f(b) < 0$. Then there exists some real number $c \in (a, b)$ such that $f(c) = 0$.

Extra fun: Apply Newton's method to the equation $x^2 - a = 0$ (where $a > 0$) to derive the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

This formula represents the algorithm used by ancient Babylonians to compute $\sqrt{a}$.

In Newton's Method, we set $f(x) = x^2 - a$, so $f'(x) = 2x$. We then get

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{2x_n^2 - x_n^2 + a}{2x_n} = \frac{1}{2}\left(\frac{x_n^2 + a}{x_n}\right) = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right),$$

as required.

1. Consider the *Intermediate Value Theorem* (IVT), which is stated on the reverse of this page.

   (a) Use the IVT to show that the equation $3x^4 - 8x^3 + 2 = 0$ has a solution on the interval $[2, 3]$.

   We let $f(x) = 3x^4 - 8x^3 + 2$, which is a continuous function, since it's a polynomial. We then find $f(2) = -14 < 0$ and $f(3) = 29 > 0$. By the IVT as given on the worksheet, there must be some $c \in (2, 3)$ such that $f(c) = 0$.

   (b) Use Newton's method to find the solution, correct to six decimal places.

   We know our solution must lie between 2 and 3 by part (a), so as our first guess, we go half-way between, and take $x_1 = 2.5$. We have

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n^4 - 8x_n^3 + 2}{12x_n^3 - 24x_n^2} = \frac{9x_n^4 - 16x_n^3 - 2}{12x_n^3 - 24x_n^2}.$$

   We now go to the calculator/computer. We get the following values:

n	1	2	3	4	5	6
x_n	2.500000	2.655000	2.630725	2.630021	2.630020	2.630020

   We see that, up to 6 decimal places, $x_6 = x_5$, so our solution must be approximately $x_5 = 2.630020$.

2. Explain why Newton's Method doesn't work for finding a solution to the equation $x^3 - 3x + 6 = 0$ if the initial approximation is $x_1 = 1$.

   Given $f(x) = x^3 - 3x$ we have $f'(x) = 3x^2 - 3$, and $f'(1) = 0$. We cannot use the Newton's method formula since we would have to divide by zero.

   This makes sense since the formula computes the x-intercept of the tangent line at $(x_1, f(x_1))$, but in this case that would be the horizontal line $y = 4$, which never intersects the x-axis.

3. Apply Newton's Method to the equation $1/x - a = 0$ to derive the reciprocal algorithm $x_{n+1} = 2x_n - ax_n^2$.

   (This algorithm is used by computers to compute reciprocals without dividing.)

   We set $f(x) = \dfrac{1}{x} - a$ and apply the Newton's Method formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1/x_n - a}{-1/x_n^2} = x_n + x_n^2(1/x_n - a) = x_n + x_n - ax_n^2 = 2x_n - ax_n^2,$$

   as required.