MATH 1560 - Tutorial #11 Solutions

1. Evaluate the definite integral:

   (a) $\displaystyle\int_0^{\pi/2} \cos(x)\,dx = \sin(x)|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1.$

   (b) $\displaystyle\int_0^{2} (x^3 - 2x + 3)\,dx = \left.\frac{x^4}{4} - x^2 + 3x\right|_0^2 = 4 - 4 + 6 - 0 + 0 - 0 = 6.$

   (c) $\displaystyle\int_0^{3} x\sqrt{1+x}\,dx$

   Here, we let $u = 1 + x$, so $du = dx$, and when $x = 0$, $u = 1$, while when $x = 1$, $u = 4$. Notice also that $x = u - 1$, so we get

$$\begin{aligned}\int_0^3 x\sqrt{1+x}\,dx &= \int_1^4 (u-1)\sqrt{u}\,du = \int_1^4 (u^{3/2} - u^{1/2})\,du \\ &= \left.\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right|_1^4 = \left(\frac{2}{5}(32) - \frac{2}{3}(8)\right) - \left(\frac{2}{5}(1) - \frac{2}{3}(1)\right) = \frac{116}{15}.\end{aligned}$$

   (d) $\displaystyle\int_0^{1} x^2\sin(x^3)\,dx = \left.-\frac{1}{3}\cos(x^3)\right|_0^1 = \frac{1-\cos(1)}{3}.$

   The above treats the antiderivative as an "immediate integral" and then applies the Fundamental Theorem of Calculus. If you prefer to go through the process of substitution, we proceed as follows:

   First, we identify $u = x^3$ (since this is the "inside" function of a composition). Next, compute $du = 3x^2\,dx$, so that $x^2\,dx = \frac{1}{3}\,du$. Finally, note that $0^3 = 0$ and $1^3 = 1$, so the limits of integration for u are (conveniently) the same as the limits for x. Thus, we have

$$\int_0^1 x^2\sin(x^3)\,dx = \int_0^1 \sin(u)\cdot\frac{1}{3}\,du = -\frac{\cos(u)}{3}|_0^1 = \frac{1-\cos(1)}{3},$$

   as before.

2. Evaluate the integral $\int_0^2 |2x-2|\,dx$.

   If you choose to graph $y = |2x-2|$ for $0 \le x \le 2$, you will find that you get two triangles, each with base 1 and height 2, so each has area 1, giving a total area under the curve of 2.

   To confirm analytically, note that $|2x-2| = \begin{cases} 2x-2, & \text{if } x \ge 1 \\ 2-2x, & \text{if } x < 1 \end{cases}$. Thus,

$$\begin{aligned}
\int_0^2 |2x-2|\,dx &= \int_0^1 |2x-2|\,dx + \int_1^2 |2x-2|\,dx \\
&= \int_0^1 (2-2x)\,dx + \int_1^2 (2x-2)\,dx \\
&= [2x-x^2]_0^1 + [x^2-2x]_1^2 \\
&= (2-1-(0-0)) + (4-4-(1-2)) = 1+1 = 2.
\end{aligned}$$

   In the first line above, we used a property of definite integrals to split the integral in two. In the next line, we used the fact that $2x-2 \le 0$ on $[0,1]$ and $2x-2 \ge 0$ on $[1,2]$ to eliminate the absolute values.

3. Find the area between the curves $y = 2 - x^2$ and $y = x^2$.

   Plotting the curves (which you should do, even though I haven't here... computer plots take time) we find that $y = 2 - x^2$ lies above $y = x^2$ for $-1 \le x \le 1$, where $x = \pm 1$ are the two points where the curves intersect. The area is thus

$$A = \int_{-1}^1 (2 - x^2 - x^2)\,dx = \int_{-1}^1 (2-2x^2)\,dx = 2x - \frac{2}{3}x^3\Big|_{-1}^1 = \frac{8}{3}.$$

   (Note: if you noticed from the graphs that the area is symmetric about the y-axis, or that the function to be integrated is even, you can apply a shortcut: $\int_{-1}^1 (2-2x^2)\,dx = 2\int_0^1 (2-2x^2)\,dx$.

4. Calculate the indicated Taylor polynomial:

   (a) Degree 5, for $f(x) = \cos(x)$, about $x = \pi/3$.
   Our general polynomial is

$$P_5(x) = f(\pi/3) + f'(\pi/3)(x - \pi/3) + \frac{f''(\pi/3)}{2!}(x - \pi/3)^2 + \frac{f'''(\pi/3)}{3!}(x - \pi/3)^3 + \frac{f^{(4)}(\pi/3)}{4!}(x - \pi/3)^4 + \frac{f^{(5)}(\pi/3)}{5!}(x - \pi/3)^5.$$

   Next, we compute the needed derivatives and their values at $\pi/3$:

n	0	1	2	3	4	5
$f^{(n)}(x)$	$\cos(x)$	$-\sin(x)$	$-\cos(x)$	$\sin(x)$	$\cos(x)$	$-\sin(x)$
$f^{(n)}(\pi/3)$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

   Putting in these values, we get

$$P_5(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}(x-\pi/3) - \frac{1}{2 \cdot 2!}(x-\pi/3)^3 + \frac{\sqrt{3}}{2 \cdot 3!}(x-\pi/3)^3 + \frac{1}{2 \cdot 4!}(x-\pi/3)^4 - \frac{\sqrt{3}}{2 \cdot 5!}(x-\pi/3)^5.$$

   (b) Degree 2, for $f(x) = \sec(x)$, about $x = 0$.

Proceeding as above, we have $f'(x) = \sec(x)\tan(x)$ and $f''(x) = \sec(x)\tan^2(x) + \sec^3(x)$, so $f(0) = 1$, $f'(0) = 0$, and $f''(0) = 1$, giving us

$$P_2(x) = 1 + \frac{1}{2}x^2.$$

5. Use the degree 3 Maclaruin polynomial for $f(x) = \sin(x)$ to approximate the value of $\sin(1)$.
   The desired polynomial is $P_3(x) = x - \frac{1}{3!}x^3$, so our approximation is

$$\sin(1) \approx P_3(1) = 1 - \frac{1}{6}(1^3) = \frac{5}{6} \approx 0.833333333.$$

   The calculator gives us $\sin(1) \approx 0.841470985$. Just for fun, if we add one more term, we get

$$P_5(1) = 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} \approx 0.841666667,$$

   which isn't that far off.