Name and student number: Soltions

[4] 1. (a) Prove that for each $a \in \mathbb{Z}$, $a \not\equiv 0 \pmod 3$ if and only if $a^2 \equiv 1 \pmod 3$.

Solution: (Version 1: Using theorems from class) First, suppose that $a \equiv 0 \pmod 3$. Then $a^2 \equiv 0 \pmod 3$ since $0^2 = 0$, so in particular, $a^2 \not\equiv 1 \pmod 3$.

Now, suppose that $a \not\equiv 0 \pmod 3$. Then either $a \equiv 1 \pmod 3$ or $a \equiv 2 \pmod 3$. In the first case, we have $a^2 \equiv 1 \pmod 3$ since $1^2 = 1$. Similarly, if $a \equiv 2 \pmod 3$, then $a^2 \equiv 4 \equiv 1 \pmod 3$. In either case, we see that $a^2 \equiv 1 \pmod 3$.

(Version 2: Using the definition of congruence) First, suppose that $a \equiv 0 \pmod 3$. Then $a = 3k$ for some $k \in \mathbb{Z}$, so $a^2 = 9k^2 = 3(3k^2)$, which shows that $a^2 \equiv 0 \pmod 3$ and thus $a^2 \not\equiv 0 \pmod 3$.

Now, suppose that $a \not\equiv 0 \pmod 3$. Then either $a \equiv 1 \pmod 3$ or $a \equiv 2 \pmod 3$. If $a \equiv 1 \pmod 3$, then $a = 3k+1$ for some $k \in \mathbb{Z}$, so $a^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$, which shows that $a^2 \equiv 1 \pmod 3$. If $a \equiv 2 \pmod 3$, then $a = 3k+2$ for some $k \in \mathbb{Z}$, so $a^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, so that $a^2 \equiv 1 \pmod 3$. In either case, we see that $a^2 \equiv 1 \pmod 3$.

[2] (b) Prove that for each $n \in \mathbb{N}$, $\sqrt{3n+2}$ is not a natural number.

Solution: Let $n \in \mathbb{N}$ be arbitrary, and suppose to the contrary that $k = \sqrt{3n+2}$ is an integer. If this is the case, then $k^2 = 3n + 2$, so $k^2 \equiv 2 \pmod n$. But we saw in part (a) that this is impossible: for any integer k, either $k^2 \equiv 0 \pmod 3$ or $k^2 \equiv 1 \pmod 3$. Thus, k cannot be an integer.

[4] 2. Let A and B be sets. Prove that if $S \subseteq A$, then $S \times B \subseteq A \times B$.

Solution: Suppose $S \subseteq A$, and suppose that (a, b) is an arbitrary element of $S \times B$. Then $a \in S$ and $b \in B$. Since $S \subseteq A$, it follows that $a \in A$. Since $a \in A$ and $b \in B$, we know that $(a, b) \in A \times B$. Since (a, b) was arbitrary, it follows that $S \times B \subseteq A \times B$ by the definition of subset.

Total: 10 points