Name: Solutions

Prove any **two** of the following three statements. (5 points each)

1. For all integers a, b, and c, with $a \neq 0$, if $a \mid b$ and $a \mid c$, then $a \mid (b - c)$.

   Solution: First, as a reminder/clarification: the notation $a \mid b$ represents a *statement*, not a number. If we write $a \mid b$ we are saying that a **divides** b, which means that we can write $b = ak$ for some integer k. You can't replace it with expressions such as $\dfrac{b}{a}$ (which would represent a number; namely, a fraction). If you're ever unsure, try putting some numbers in to see if the sentence makes sense.
   (We probably would not make a statement like "If $\frac{2}{3}$ and $\frac{7}{3}$, then $\frac{-5}{3}$.")

   The proof is as follows: suppose $a \mid b$ and $a \mid c$. Then there exist integers k and l such that $b = ak$ and $c = al$. It follows that

   $$b - c = ak - al = a(k - l).$$

   Since $k - l \in \mathbb{Z}$, it follows that $a \mid b - c$.

2. For any integer n, if n is an odd integer, then n^3 is an odd integer.

   There are two methods here: the "brute force" method, and the "rely on previous knowledge" method. First, the brute force method:

   Suppose n is an odd integer. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. It follows that

   $$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k + 3k) + 1,$$

   which is of the form $n^3 = 2l + 1$, where l is the integer $4k^3 + 6k + 3k$), so n^3 is odd.

   Note: Be careful with the expression $(2k+1)^3$. I noticed a few people with the result $8k^3 + 1$ as the quizzes were handed in. Don't forget that there are cross terms:

   $$(2k+1)^3 = (2k+1)(2k+1)(2k+1).$$

   (I saw at least 5 or 6 like this, so you're not alone if you made this mistake.) If you've encountered the binomial formula before, you can raise a binomial to the third (or higher) power reasonably quickly:

   $$(a+b)^n = a^n + na^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + nab^{n-1} + b^n,$$

   where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ are the "binomial coefficients". If you haven't seen these before (and/or have no idea what $n!$ means), don't worry about it. If you've ever encountered Pascal's Triangle, the binomial coefficients are the numbers you see there.

Now, here is the "rely on previous knowledge" method:

From the textbook, we know that the product of two odd integers is odd. In particular, this means that if n is odd, then $n^2 = n \cdot n$ is odd. Since n is odd and n^2 is odd, it follows that

$$n^3 = n(n^2)$$

is odd.

If you don't want to quote the textbook, you could also write something like the following:

Lemma: the product of two odd integers is an odd integer.

Proof: suppose x and y are odd integers. Then there exist integers $k, l \in \mathbb{Z}$ such that $x = 2k + 1$ and $y = 2l + 1$, so

$$xy = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$$

is odd.

From here, you could proceed as above to conclude that n and n^2 are both odd, so n^3 must be odd as well.

(A "lemma" is a small theorem needed to prove a bigger theorem.)

3. For each integer a, if $4 \mid (a - 1)$, then $4 \mid (a^2 - 1)$.

   Let a be an integer, and suppose that $4 \mid (a - 1)$. Then there exists some $k \in \mathbb{Z}$ such that $a - 1 = 4k$. It follows that

$$a^2 - 1 = (a - 1)(a + 1) = 4k(a + 1) = 4[k(a + 1)].$$

   Thus, $4 \mid (a^2 - 1)$.

Total: 10 points