Mount Allison University
Department of Mathematics and Computer Science
30^{th} November, 2009, 8:35-9:20 am

MATH2111 - Test #3

Last Name: SOLUTIONS

First Name: THE

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work. Be sure to show your work, and include all necessary justifications needed to support your arguments.

Some potentially useful results you may have forgotten from Calc II:

$$\int \cos\theta\, d\theta = \sin\theta + c,\ \int \sin\theta\, d\theta = -\cos\theta + c,\ \sin^2\theta + \cos^2\theta = 1$$

For grader's use only:

Q	Mark
1	/8
2	/10
3	/16
4	/16
Total	/50

[8] 1. Evaluate the following double integral:

$$\iint_D x^2 y\, dA, \text{ where } D = \{(x,y) | 0 \le x \le 1 \text{ and } x \le y \le 2-x\}$$

$$\iint_D x^2 y \, dA = \int_0^1 \int_x^{2-x} x^2 y \, dy\, dx \quad (3)$$

$$= \frac{1}{2}\int_0^1 x^2 \left(y^2 \Big|_x^{2-x} \right) dx \quad (1) \qquad = \frac{1}{2}\int (4x^2 - 4x^3)\, dx \quad (2)$$

$$= \frac{1}{2}\left(\frac{4}{3}x^3 - \frac{4}{4}x^4 \right)\Big|_0^1 \quad (1) \qquad = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \quad (1)$$

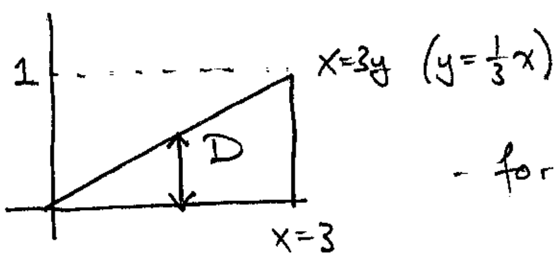
2. For the following double integral,

(a) Sketch the region D of integration.

[10] (b) Reverse the order of integration by describing D as a vertically simple (rather than horizontally simple) region.

(c) Evaluate the resulting integral.

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$$



(2)

- for fixed x, $0 \le y \le \frac{1}{3}x$
where $0 \le x \le 3$ (2)

$$\int_0^1 \int_{3y}^3 e^{x^2} dx\, dy = \int_0^3 \int_0^{\frac{1}{3}x} e^{x^2} dy\, dx \quad (1) \qquad = \int_0^3 \frac{1}{3} x e^{x^2} dx \quad (1)$$

$$= \frac{1}{6}\int_0^9 e^u du \quad (2) \qquad = \frac{1}{6} e^u \Big|_0^9 \quad (1) \qquad = \frac{e^9 - 1}{6} \quad (1)$$

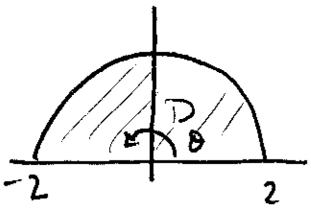
$\Big(u = x^2$
$\Rightarrow \frac{1}{2} du = x\,dx$
$x = 0 \Rightarrow u = 0$
$x = 3 \Rightarrow u = 9 \Big)$

3. Evaluate the integral by converting to polar coordinates:

(a)

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x+y)\,dy\,dx.$$

[8]



$0 \le y \le \sqrt{4-x^2}$

$y=\sqrt{4-x^2} \Rightarrow \underbrace{x^2+y^2}_{r^2}=4$, $r^2=4$

$\therefore\ 0 \le r \le 2$, $0 \le \theta \le \pi$ 3

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x+y)\,dy\,dx = \int_0^{\pi}\int_0^{2} (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$ 2

$$= \int_0^{\pi}\int_0^{2} r^2(\sin\theta+\cos\theta)\,d\theta = \int_0^{\pi} \frac{8}{3}(\cos\theta+\sin\theta)\,d\theta$$ 2

$$= \frac{8}{3}(\sin\theta - \cos\theta)\Big|_0^{\pi} = \frac{16}{3}$$ 1

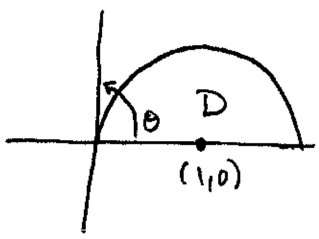
(Notice $\displaystyle\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}(x+y)\,dy\,dx = \underbrace{\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} x\,dy\,dx}_{=0} + \int_{-2}^{2}\int_0^{\sqrt{4-x^2}} y\,dy\,dx$ 1)

$=0$ since D symmetric about y-axis $\Rightarrow$ only y contributes.

(b)

$$\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$$

[8]



$0 \le y \le \sqrt{2x-x^2}$

$y=\sqrt{2x-x^2} \Rightarrow y^2 = 2x-x^2$

$\Rightarrow x^2+y^2-2x=0$

$\Rightarrow (x-1)^2+y^2=1$

$x^2+y^2=2x$

$\Rightarrow r^2 = 2r\cos\theta$

$\Rightarrow r=0$ (pt. (0,0))

or $r=2\cos\theta$ (circle)

Notice $0\le\theta\le\frac{\pi}{2}$

3

$$\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^{2\cos\theta} r\cdot r\,dr\,d\theta$$ 1

$$= \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{2\cos\theta} d\theta = \int_0^{\pi/2} \frac{8}{3}\cos^3\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}\cos\theta\cdot\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}\cos\theta\,(1-\sin^2\theta)\,d\theta$$ 2

($u=\sin\theta$, $du=\cos\theta\,d\theta$)

$\theta=0 \Rightarrow u=0$

$\theta=\pi/2 \Rightarrow u=1$

$$= \int_0^1 \frac{8}{3}(1-u^2)\,du$$ 1

$$= 8/3 - 8/9 = 16/9$$ 1

4. Given a differentiable function $f(x,y)$, we know that if $f(a,b)$ is a maximum or minimum value subject to a constraint $g(x,y)=c$, then

$$\nabla f(a,b) = \lambda \nabla g(a,b), \quad \text{for some } \lambda \in \mathbb{R}.$$

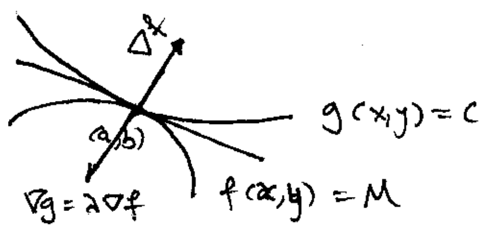
[4] (a) Explain the meaning of the above equation in geometic terms. In particular, if $f(a,b)=M$ is the maximum, what can you say about the curves $f(x,y)=M$ and $g(x,y)=c$ at the point (a,b)?

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle = \lambda \langle g_x(a,b), g_y(a,b)\rangle = \lambda \nabla g(a,b)$$

- means these two vectors point in the same direction (or opposite) (parallel) 2

- Since $\nabla g(a,b)$ is ~~tangent~~ the normal vector to $g(x,y)=c$ at (a,b), and $\nabla f(a,b)$ is normal to $f(x,y)=M$ at (a,b) $\Rightarrow$ both curves have the same tangent line at (a,b)

2



(b) Let $f(x,y)=x^2-y^2$.

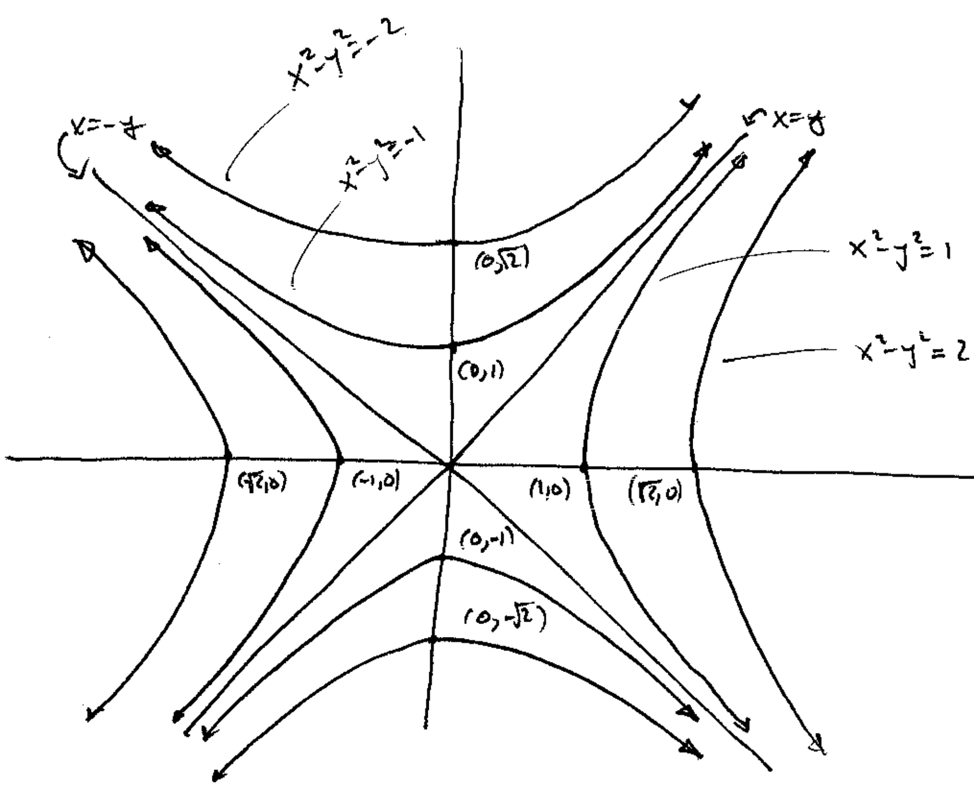
[3] i. Sketch the level curves $f(x,y)=c$, for $c=-2,-1,0,1,2$ on one set of axes.

$x^2-y^2=0 \Rightarrow x=\pm y$

$x^2-y^2=1$ hyperbola opening left/right

$x^2-y^2=-1$

$\Rightarrow y^2-x^2=1$ hyperbola opening up/down



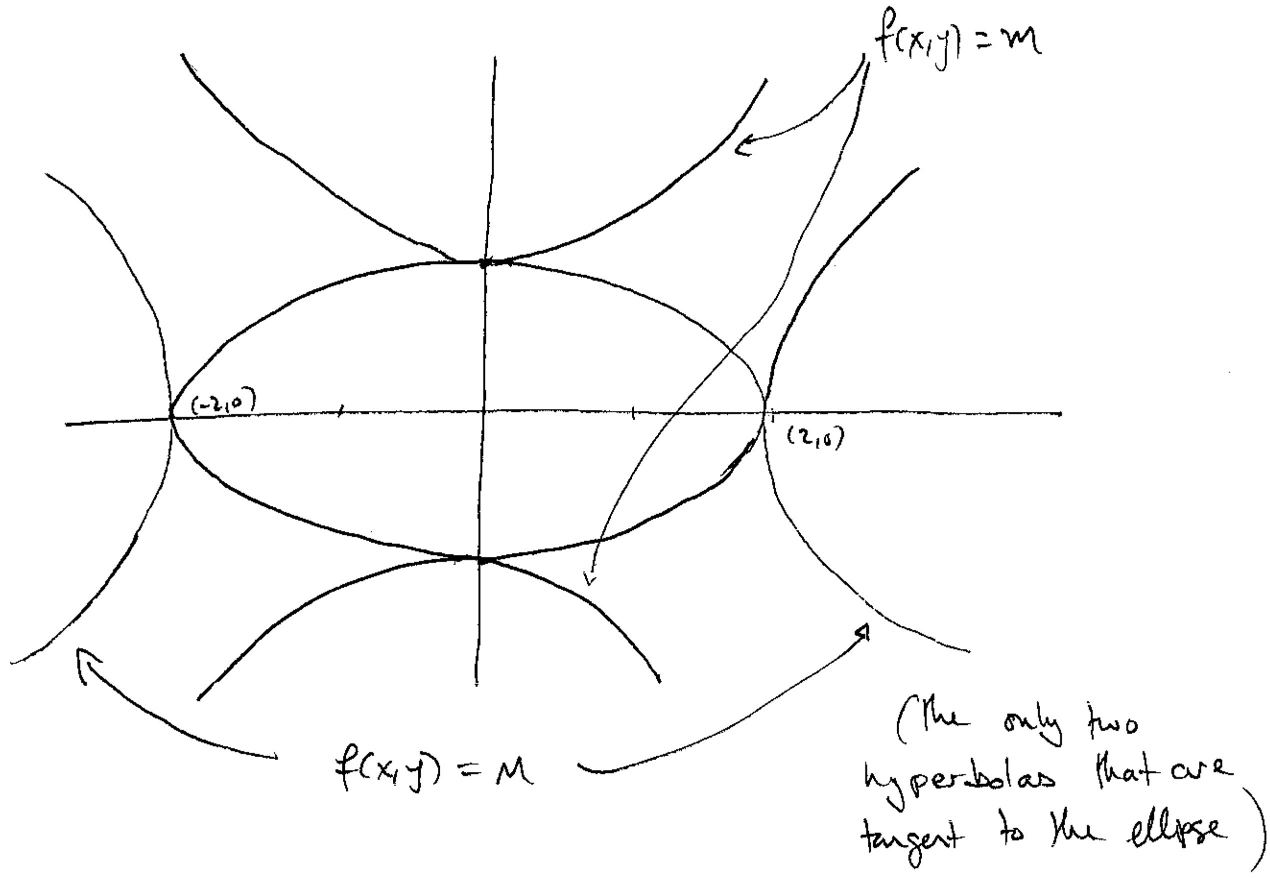
[1] ii. On a new set of axes, sketch the curve $x^2 + 4y^2 = 4$.

iii. On the same set of axes used in part (ii), sketch the level curves $f(x, y) = m$ and $f(x, y) = M$ corresponding to the minimum m and maximum M of
[4] $f(x, y)$ subject to the constraint $x^2 + 4y^2 = 4$.

Hint: no calculation is required. The geometric meaning (from part (a)) of the Lagrange multiplier condition should tell you where these curves have to intersect the constraint curve $x^2 + 4y^2 = 4$, and how they should intersect.

$x^2+4y^2=4$

$\Rightarrow \frac{x^2}{2^2}+\frac{y^2}{1^2}=1$



(c) Either by reading the answer off of your sketch above, or by explicitly solving the Lagrange multiplier equations, give the minimum value m and maximum value
[4] M of $f(x, y)$ subject to $x^2 + 4y^2 = 4$, and the points at which they occur.

- From the above, we see that there are two hyperbolas that intersect $x^2+4y^2=4$ and have a common tangent line with the ellipse at the point of intersection.

$x^2-y^2=4$ intersects at $(\pm 2, 0)$ $\Rightarrow$ abs. max.

$x^2-y^2=-1$ intersects at $(0, \pm 1)$ $\Rightarrow$ abs. min

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or// $\nabla f = \lambda \nabla g \Rightarrow$ $2x = \lambda(2x)$, $-2y = \lambda(8y)$, $x^2+4y^2=4$

$x \neq 0 \Rightarrow \lambda = 1 \Rightarrow -2y = 8y \Rightarrow y=0$

$y \neq 0 \Rightarrow \lambda = -1/4 \Rightarrow 2x = -\frac{1}{2}x \Rightarrow x = 0$

$x=0$ gives $\lambda = -1/4$ and $4y^2=4 \Rightarrow y = \pm 1$

$f(0,\pm 1) = 0^2 - (1)^2 = -1$

$y=0$ gives $\lambda = 1$ and $x^2=4 \Rightarrow x = \pm 2$

$f(\pm 2, 0) = 2^2 - 0^2 = 4$

$\therefore$ min $f(0,1) = f(0,-1) = -1$

max $f(2,0) = f(-2,0) = 4$