Math 3500 Exercise Sheet

1 October, 2014

This week we'll go over the material for the first term test by section. The first two sections in Chapter 3 are mainly review, so we'll start with 3.3, the section on the real number system.

Section 3.3: The Real Numbers

Main definitions and results:

- A set $A \subseteq \mathbb{R}$ is **bounded above** if there exists some $k \in \mathbb{R}$ such that $a \leq k$ for all $a \in A$. Any such real number k is called an **upper bound** for A. We say that s is the **least upper bound**, or **supremum** of A, and write $s = \sup A$, if s is an upper bound for A, and if t is any other upper bound for A, then $s \leq t$.

  Similarly, a set $A \subseteq \mathbb{R}$ is **bounded below** if there exists some $k \in \mathbb{R}$ such that $a \geq k$ for all $a \in A$. Any such real number k is called a **lower bound** for A. We say that s is the **greatest lower bound**, or **infimum** of A, and write $s = \inf A$, if s is a lower bound for A, and if t is any other lower bound for A, then $s \geq t$.

- The **completeness axiom** for $\mathbb{R}$ states that any nonempty set $A \subseteq \mathbb{R}$ that is bounded above has a least upper bound.

- The **Archimedean property** for $\mathbb{R}$ is the fact that the set $\mathbb{N}$ of natural numbers has no upper bound in $\mathbb{R}$. Equivalently, given any real number $x > 0$, there exists some $n \in \mathbb{N}$ such that $1/n < x$.

- The fact that the set $\mathbb{Q}$ of rational numbers is **dense** in $\mathbb{R}$ is the statement that given any two real numbers $x, y \in \mathbb{R}$ with $x < y$, there exists some $q \in \mathbb{Q}$ such that $x < q < y$.

Exercises:

1. Show that $\mathbb{Q}$ has the Archimedean property. (Given any rational number a/b can you find a natural number larger than it?)

2. Prove that any nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.

3. For each of the following sets, decide if it is bounded above/below. If so, find the supremum/infimum:

$$A = \{n \in \mathbb{N} : n^2 < 10\} \quad B = \{x \in \mathbb{Q} : |x| < 2\} \quad C = \{x \in \mathbb{R} : x^2 < 2x\}$$

4. Given subsets $A, B \subseteq \mathbb{R}$, let $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Show that $\sup(A + B) = \sup A + \sup B$.

Remark: We showed that the Archimedean property for $\mathbb{R}$ follows from the completeness axiom, but it is not equivalent to it. In fact, it's possible to show that $\mathbb{Q}$ is Archimedean, but we know that $\mathbb{Q}$ is not complete.

Section 3.4: Topology of $\mathbb{R}$

Main definitions and results:

- For any $\epsilon > 0$, the ϵ-**neighbourhood** of x is the set

$$N_\epsilon(x) = \{y \in \mathbb{R} : |x - y| < \epsilon\} = (x - \epsilon, x + \epsilon).$$

- A point $x \in \mathbb{R}$ is an **interior point** of a set $A \subseteq \mathbb{R}$ if there exists some $\epsilon > 0$ such that $N_\epsilon(x) \subseteq A$. The set of all interior points is called the **interior** of A, and is often denoted by $\overset{\circ}{A}$.

- A point $x \in \mathbb{R}$ is a **boundary point** of a set $A \subseteq \mathbb{R}$ if for every $\epsilon > 0$, both $N_\epsilon(x) \cap A$ and $N_\epsilon(x) \cap (\mathbb{R} \setminus A)$ are nonempty. The set of all boundary points of A is called the **boundary** of A and denoted by $\mathrm{bd}A$ or by ∂A.

- A point $x \in \mathbb{R}$ is a **limit point** (or accumulation point) of a set $A \subseteq \mathbb{R}$ if for every $\epsilon > 0$, $N_\epsilon(x)$ contains some point $a \in A$, with $a \neq x$. The set of all limit points of A is sometimes denoted by A'.

- A point $a \in A$ is called an **isolated point** of A if $a \neq A'$.

- A set $A \subseteq \mathbb{R}$ is **open** if every $a \in A$ is an interior point of A.

- A set $A \subseteq \mathbb{R}$ is **closed** if $\mathbb{R} \setminus A$ is open, or equivalently, if $\partial A \subseteq A$.

- The **closure** of a set $A \subseteq \mathbb{R}$ is denoted $\overline{A}$ and defined by $\overline{A} = A \cup A'$. (It's also possible to show that $\overline{A} = A \cup \partial A$.

Exercises:

1. Decide whether the following sets are open, closed, neither, or both:

$$\text{(i) } \mathbb{Q} \quad \text{(ii) } \mathbb{N} \quad \text{(iii) } \{x : x^2 > 0\} \quad \text{(iv) } \bigcap_{n=1}^{\infty} (0, 1/n)$$

2. Determine the interior, boundary, and limit points for the sets in problem 1.

3. If S is a nonempty bounded set of real numbers and $m = \sup S$, is m a boundary point of S? Why or why not?

4. If S has both a maximum and a minimum, is S necessarily closed? Why or why not?

Section 3.5: Compact sets

Main definitions and results:

- An **open cover** of a set $A \subseteq \mathbb{R}$ is a collection of open sets $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ such that $A \subseteq \bigcup_{\alpha \in I} U_\alpha$. A **finite subcover** of an open cover of A is a finite set $\{U_1, \ldots, U_n\}$ with $U_j \in \mathcal{U}$ for $1 \leq j \leq n$ such that $A \subseteq U_1 \cup \cdots \cup U_n$.
- A set $A \subseteq \mathbb{R}$ is **compact** if every open cover of A admits a finite subcover.
- The **Heine-Borel theorem** states that a set $A \subseteq \mathbb{R}$ is compact if and only if A is closed and bounded.
- The **Bolzano-Weierstrass theorem** states that any bounded, infinite subset of $\mathbb{R}$ has at least one limit point.
- The **nested intervals theorem** states that if $\{A_n : n \in \mathbb{N}\}$ is any collection of closed intervals such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

Exercises:

1. Prove that the intersection of any collection of compact sets is compact.

2. Prove that if a set $A \subseteq \mathbb{R}$ is compact, then every infinite subset $B \subseteq A$ has a limit point contained in A.

3. Prove that if A is compact, then $\sup A$ and $\inf A$ exist, and are elements of A.

Section 4.1: Convergence of sequences

Main definitions and results:

1. A **sequence** is a function $f : \mathbb{N} \to \mathbb{R}$. We usually denote $f(n)$ by a_n and the sequence by (a_n).

2. A sequence (a_n) **converges** to $a \in \mathbb{R}$, if for every $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for every $n \geq N$. We call a the **limit** of the sequence, and write $\lim a_n = a$ or $a_n \to a$.

3. If a sequence (a_n) converges, then it is bounded. (That is, the set $\{a_n : n \in \mathbb{N}\}$ is bounded.)

4. If $\lim a_n$ exists, then it is unique.

Exercises:

1. Determine whether or not the following sequences converge. If a sequence does converge, find the limit and prove your result using the definition of convergence. If it does not converge, explain why.

$$a_n = \frac{\sin n}{n} \quad b_n = (-1)^n(1 - 1/n) \quad c_n = \frac{2n+1}{3n-1}$$

2. Show that if $\lim a_n = 0$ and (b_n) is a bounded sequence, then $\lim(a_n b_n) = 0$.

3. Prove that a point $x \in \mathbb{R}$ is a limit point of a set A if and only if there exists a sequence (a_n) of points in $A \setminus \{a\}$ with $\lim a_n = a$.

4. Prove that a set $A \subseteq \mathbb{R}$ is closed if and only if for any sequence (a_n) with $a_n \in A$ for all $N \in \mathbb{N}$ converges to some $a \in \mathbb{R}$, then $a \in A$.

Section 4.2: Limit theorems

Main definitions and results:

- Suppose that (a_n) and (b_n) are convergent sequences with $a_n \to a$ and $b_n \to b$. Then

$$\begin{aligned}
&\lim(a_n + b_n) = a + b \\
&\lim(ka_n) = ka, \text{ for any } k \in \mathbb{R} \\
&\lim(a_n b_n) = ab \\
&\lim(a_n/b_n) = a/b, \text{ provided that } b_n \neq 0 \text{ for all } n \in \mathbb{N} \text{ and } b \neq 0.
\end{aligned}$$

- If $a_n \leq b_n$ for all $n \in \mathbb{N}$ and $a_n \to a$ and $b_n \to b$, then $a \leq b$.

- A sequnce (a_n) **diverges to** ∞ if for any $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $a_n \geq M$ for all $n \geq \mathbb{N}$.

Exercises:

1. Prove that $\displaystyle\lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$.

2. Let (a_n) be a sequence of positive numbers. Prove that $\lim a_n = \infty$ if and only if $\lim(1/a_n) = 0$.

Section 4.3: Monotone and Cauchy sequences

Main definitions and results:

- A sequence (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, and **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. If a sequence is either increasing or decreasing, we say that it is a **monotone** sequence.

- The **monotone convergence theorem** states that any monotone sequence converges if and only if it is bouded.

- A sequence (a_n) is a **Cauchy sequence** if for every $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|a_n - a_m| < \epsilon$.

- Any convergent sequence is a Cauchy sequence.

- Any Cauchy sequence is bounded.

- Any Cacuhy sequence converges.

Exercises:

1. Prove that each sequence is monotone and bounded. Then find the limit.

   (a) $a_1 = 1$ and $a_{n+1} = \frac{1}{5}(a_n + 7)$ for all $n \geq 1$

   (b) $a_1 = 2$ and $a_{n+1} = \frac{1}{4}(2a_n + 7)$ for all $n \geq 1$

   (c) $a_1 = 5$ and $a_{n+1} = \sqrt{4a_n + 1}$ for all $n \geq 1$

2. A sequence (a_n) is called *contractive* if there exists some constant $k \in (0, 1)$ such that $|a_{n+2} - a_{n+1}| \leq k|a_{n+1} - a_n|$ for all $n \in \mathbb{N}$. Prove that any contractive sequence is a Cauchy sequence.