MATH 2565 - Tutorial #7 Solutions

Assigned problems:

1. Evaluate the indefinite integral:

   (a) $\int e^{\sqrt{x}}\,dx$ (Hint: try a substitution first.)

   First let $x = u^2$, so $dx = 2u\,du$, giving us

$$\begin{aligned}\int e^{\sqrt{x}}\,dx &= \int 2ue^u\,du = 2\int u d(e^u) = 2ue^2 - 2\int e^u\,du = 2ue^u - 2e^u + C\\ &= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.\end{aligned}$$

   (b) $\int \cos(x)\cos(2x)\,dx = \int \cos(x)(1-2\sin^2 x)\,dx = \sin(x) - \frac{2}{3}\sin^3(x) + C.$

   Alternatively, one could use the identity

$$\cos(ax)\cos(bx) = \frac{1}{2}\left(\cos(ax+bx) + \cos(ax-bx)\right)$$

   to write $\cos(x)\cos(2x) = \frac{1}{2}\cos(3x) + \frac{1}{2}\cos(x)$, giving

$$\int \cos(x)\cos(2x)\,dx = \frac{1}{2}\int (\cos(3x) + \cos(x))\,dx = \frac{1}{6}\sin(3x) + \frac{1}{2}\sin(x) + C.$$

   (c) $\int \frac{\sqrt{5-x^2}}{x^2}\,dx$

   Letting $x = \sqrt{5}\sin\theta$, so $\sqrt{5-x^2} = \sqrt{5}\cos\theta$ and $dx = \sqrt{5}\cos\theta\,d\theta$, we have

$$\begin{aligned}\int \frac{\sqrt{5-x^2}}{x^2}\,dx &= \int \frac{5\cos^2\theta}{5\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta = \int (\csc^2\theta - 1)\,d\theta\\ &= -\cot\theta - \theta + C = -\frac{\sqrt{5-x^2}}{x} - \sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c\end{aligned}$$

   (d) $\int \frac{16x^2 - 2x}{(x+3)(2x-1)(x-1)}\,dx$

   We look for a partial fraction decomposition

$$\frac{16x^2-2x}{(x+3)(2x-1)(x-1)} = \frac{A}{x+3} + \frac{B}{2x-1} + \frac{C}{x-1}.$$

Multiplying both sides of this decomposition by $x+3$ gives us

$$\frac{16x^2-2x}{(2x-1)(x-1)} = A + \frac{B(x+3)}{2x-1} + \frac{C(x+3)}{x-1}.$$

Plugging in $x=-3$ then gives $A = \dfrac{75}{14}$.
Multiplying both sides of the decomposition by $2x-1$ gives

$$\frac{16x^2-2x}{(x+3)(x-1)} = \frac{A(2x-1)}{x+3} + B + \frac{C(2x-1)}{x-1},$$

and plugging in $x = 1/2$ gives $B = \dfrac{12}{7}$.
Multiplying both sides of the decomposition by $x-1$ gives

$$\frac{16x^2-2x}{(x+3)(2x-1)} = \frac{A(x-1)}{x+3} + \frac{B(x-1)}{2x-1} + C,$$

and plugging in $x=1$ gives $C = \dfrac{7}{2}$.
Putting everything together, we get

$$\begin{aligned}\int \frac{16x^2-2x}{(x+3)(2x-1)(x-1)}\,dx &= \frac{75}{14}\int \frac{1}{x+3}\,dx + \frac{12}{7}\int \frac{1}{2x-1}\,dx + \frac{7}{2}\int \frac{1}{x-1}\,dx \\ &= \frac{75}{14}\ln|x+3| + \frac{6}{7}\ln|2x-1| + \frac{7}{2}\ln|x-1| + C.\end{aligned}$$

2. Evaluate the improper integral, or explain why it does not exist:

   (a) $\displaystyle\int_0^\infty e^{4-3x}\,dx \lim_{t\to\infty}\int_0^t e^{4-3x}\,dx = \lim_{t\to\infty}\frac{1}{3}(e^4 - e^{4-3t}) = e^4.$

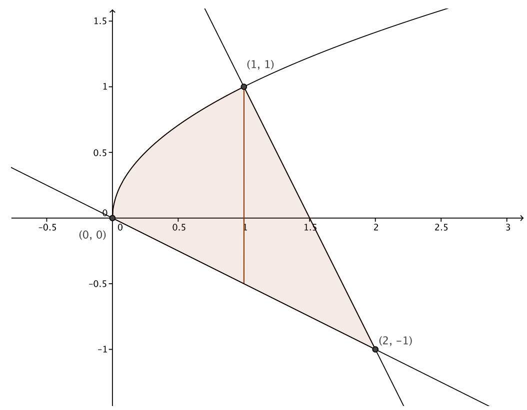
   (b) $\displaystyle\int_{-\infty}^\infty \frac{1}{4+x^2}\,dx = \lim_{s\to\infty}\int_{-s}^0 \frac{1}{4+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{4+x^2}\,dx.$

   Now we recall that $\displaystyle\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\tan^{-1}(x/2)$, $\tan^{-1}(0)=0$, and $\lim_{x\to\pm\infty}\tan^{-1}x = \pm\frac{\pi}{2}$. It follows that $\lim_{x\to\infty}\tan^{-1}(\pm x/2) = \pm\frac{\pi}{2}$, since $x/2\to\infty$ if $x\to\infty$. Thus, we get

$$\int_{-\infty}^\infty \frac{1}{4+x^2}\,dx = -\frac{1}{2}\lim_{s\to\infty}\tan^{-1}(-s/2) + \frac{1}{2}\lim_{t\to\infty}\tan^{-1}(t/2) = \frac{\pi}{2}.$$

3. Find the area between the curves $y = \sqrt{x}$, $y = -2x + 3$, and $y = -\frac{1}{2}x$.

   We begin by sketching the region.



   We can see from the sketch that it's necessary to break up the area into two regions.

   For $0 \le x \le 1$, the upper curve is $y = \sqrt{x}$ and the lower curve is $y = -\frac{1}{2}x$, giving us the area

$$A_1 = \int_0^1 \left(\sqrt{x} + \frac{1}{2}x\right) dx = \frac{11}{12}.$$

   For $1 \le x \le 2$, the upper curve changes to $y = 3 - 2x$, giving the area

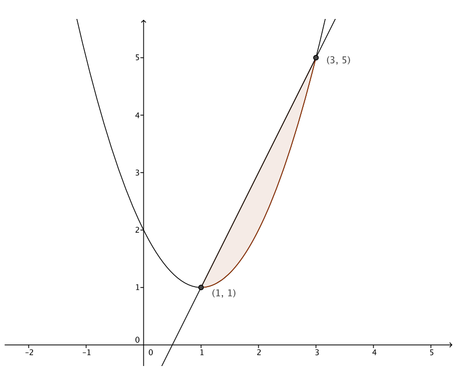
$$A_2 = \int_1^2 \left(3 - 2x + \frac{1}{2}x\right) dx = \frac{3}{4}.$$

   The total area is therefore $A = A_1 + A_2 = \frac{5}{3}$.

4. Find the volume of the solid of revolution:

   (a) Generated by revolving the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the line $y = 1$.

   We first sketch the region:



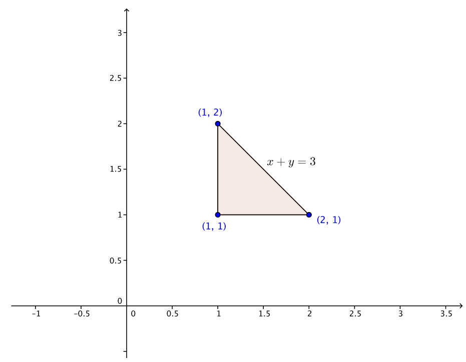
   Since the axis of rotation is vertical this time, we want to use shells in order to integrate with respect to x. The radius of each shell is $r(x) = x - 1$, and the height is given by the difference in the y-values of the two curves: $h(x) = (2x - 1) - (x^2 - 2x + 2) = -x^2 + 4x - 3$.

Thus, we have

$$V = 2\pi \int_1^3 (x-1)(-x^2+4x-3)\,dx = \frac{8\pi}{3}.$$

(b) Generated by revolving the triangle with vertices $(1,1), (1,2)$, and $(2,1)$ about the x-axis.

In the additional practice, we found that the volume obtained by revolving this region about the x-axis was $4\pi/3$. The symmetry of the region suggests that we should get the same answer when revolving about the y-axis. Let's confirm.



Since the axis is vertical, if we use shells, the integral is with respect to x, with $r = x$ and $h = (3-x) - 1 = 2 - x$, for $1 \le x \le 2$. This gives

$$V = 2\pi \int_1^2 x(2-x)\,dx = 2\pi \left(x^2 - \frac{1}{3}x^3\right)\Big|_1^2 = 2\pi\left(\left(4-\frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right) = \frac{4\pi}{3},$$

as expected. If you used washers instead, the integral is with respect to y, with $r_{\text{in}} = 1$ and $r_{\text{out}} = 3 - y$, so again we get

$$V = \pi \int_1^2 (3-y)^2 - 1^2)\,dy = \pi\left(-\frac{1}{3}(3-y)^3 - y\right)\Big|_1^2 = \pi\left(\left(-\frac{1}{3} - 2\right) - \left(-\frac{8}{3} - 1\right)\right) = \frac{4\pi}{3}.$$

5. Find the area of the surface generated by revolving the the curve $y = x^2$, for $0 \le x \le 1$, about the y-axis.

Since we're revolving about the y-axis, we use the formula $S = 2\pi \int_a^b x\sqrt{1+(y')^2}\,dy$, giving us

$$S = 2\pi \int_0^1 x\sqrt{1+(2x)^2}\,dx = \frac{\pi}{6}(5\sqrt{5}-1).$$

Additional practice (don't include your solutions here):

1. Evaluate the indefinite integral:

   (a) $\displaystyle\int x\sec^2(x)\,dx = \int xd(\tan x) = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos(x)| + C$

   (b) $\displaystyle\int \tan^5(x)\sec^4(x)\,dx = \int \tan^5(x)(1+\tan^2(x))\sec^2(x)\,dx = \frac{1}{6}\tan^6(x) + \frac{1}{8}\tan^8(x) + C$

   (c) $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$

   Letting $x = \sqrt{2}\tan\theta$, we have $\sqrt{x^2+2} = \sqrt{2\sec^2\theta} = \sqrt{2}\sec\theta$ and $dx = \sqrt{2}\sec^2\theta\,d\theta$, so

   $$\int \frac{8}{\sqrt{x^2+2}}\,dx = \int \frac{8\sqrt{2}\sec^2\theta}{\sqrt{2}\sec\theta}\,d\theta = 8\ln|\sec\theta + \tan\theta| + C = 8\ln|x + \sqrt{x^2+2}| + C.$$

   (d) $\displaystyle\int \frac{2x+1}{x^3+x}\,dx$

   This time we look for a decomposition $\dfrac{2x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$. Getting a common denominator on the right-hand side, we have

   $$\frac{2x+1}{x^3+x} = \frac{Ax^2 + A + Bx^2 + Cx}{x^3+x}.$$

   Comparing numerators, we have $0x^2 + 2x + 1 = (A+B)x^2 + Cx + A$. Constant terms must be equal, so $A = 1$, Coefficients of x must be equal, so $C = 2$. Coefficients of x^2 must be equal, so $A + B = 0$, giving $B = -A = -1$. Thus,

   $$\int \frac{2x+1}{x^3+x}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + 2\tan^{-1}(x) + C.$$

2. Evaluate the improper integral, or explain why it doesn't exist:

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \int_0^{\infty} \frac{x}{1+x^2}\,dx.$

   This integral diverges, since both of the two integrals on the right-hand side above diverge. Note that $\displaystyle\int \frac{1}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)$, and as $x \to \pm\infty$, $\ln(1+x^2) \to \infty$.

   (b) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$

   Using integration by parts, $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x}$. We thus have

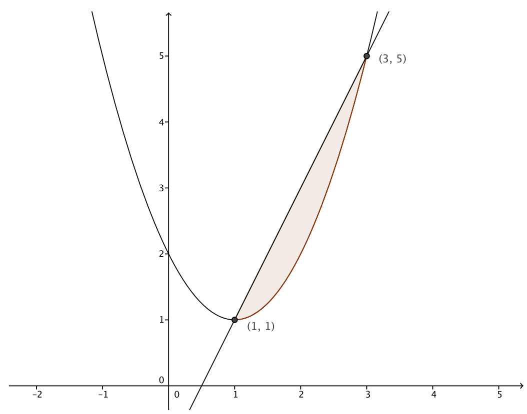
   $$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x^2}\,dx &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\,dx \\ &= \lim_{t\to\infty} \left(1 - \frac{1}{t} - \frac{\ln t}{t}\right) = 1,\end{aligned}$$

where we have used the limits $\lim_{t\to\infty} \frac{1}{t} = 0$ and (using L'Hospital's rule for the indeterminate form ∞/∞)

$$\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{1/t}{1} = 0.$$

3. Find the volume of the solid of revolution:

   (a) Generated by revolving the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.



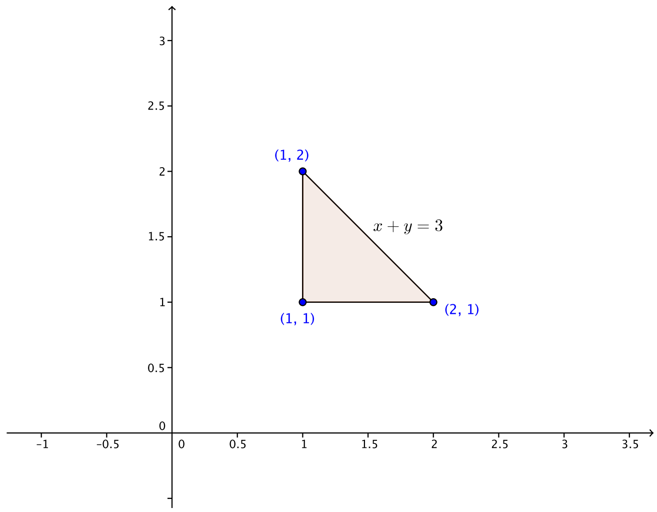
The region for parts (a) and (b) is shown above. Since we're revolving about the x-axis for part (a), the washer method is more convenient, as it involves an integral with respect to x. (The shell method would require us to solve for x as a function of y.)

The outer radius for our washer is given by the value of the y-coordinate of the curve that is furthest from the x-axis, so we have $r_{out} = 2x - 1$. The inner radius is given by the closer of the two curves, so $r_{in} = x^2 - 2x + 2$. Putting these into the formula for volume by washers gives us

$$\begin{aligned} V &= \pi \int_1^3 \left[(2x-1)^2 - (x^2 - 2x + 2)^2\right] dx \\ &= \pi \int_1^3 (-x^4 + 4x^3 - 4x^2 + 4x - 3)\, dx = \frac{104\pi}{15}. \end{aligned}$$

(I don't promise that I avoided computational errors on this one!)

   (b) Generated by revolving the triangle with vertices $(1, 1)$, $(1, 2)$, and $(2, 1)$ about the y-axis.



The region to be revolved is shown above. If we choose to use washers, then we write the equation of the hypotenuse of the triangle as $x = 3 - y$, since we integrate with respect to y for a vertical axis. The inner radius is simply 1 (the vertical side of the triangle), so we have (as seen in tutorial)

$$V = \pi \int_1^2 [(3-y)^2 - 1]\, dy = \frac{4\pi}{3}.$$

If we choose to use shells instead, then the radius of the shell is x, and the height is $(3-x) - 1 = 2 - x$, giving us

$$V = 2\pi \int_1^2 x(2-x)\, dx = \frac{4\pi}{3}.$$

4. Find the length of the curve $y = 2x^{3/2} - \dfrac{1}{6}\sqrt{x}$, for $0 \le x \le 9$.

   Since $y' = 3\sqrt{x} - 1/(12\sqrt{x})$, we have

$$1 + (y')^2 = 1 + \left(3\sqrt{x} - \frac{1}{12\sqrt{x}}\right)^2 = 1 + 9x - \frac{1}{2} + \frac{1}{144x} = 9x + \frac{1}{2} + \frac{1}{144x} = \left(3\sqrt{x} + \frac{1}{12\sqrt{x}}\right)^2.$$

   Thus, the length is

$$L = \int_0^9 \sqrt{1 + (y')^2}\, dx = \int_0^9 \left(3x^{1/2} + \frac{1}{12}x^{-1/2}\right) dx = \frac{107}{2}.$$