Math 3500 Assignment #6 Solutions
University of Lethbridge, Fall 2014

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1. (**Do not submit**) Let $f : D \to \mathbb{R}$ be continuous. For each of the following, prove the result, or give a counterexample.

   (a) If D is open, then $f(D)$ is open.

   Let $D = (0, 1)$ and define $f(x) = 0$ for all $x \in D$. Then D is open but $f(D) = \{0\}$ is closed.

   (b) If D is closed, then $f(D)$ is closed.

   Let $D = \mathbb{N}$, and define $f(x) = 1/x$. Then D is closed (every $n \in \mathbb{N}$ is isolated, so $\mathbb{N}$ has no limit points) but $f(D) = \{1/n : n \in \mathbb{N}\}$ is not closed, since 0 is a limit point of $f(D)$ and $0 \notin f(D)$.

   (c) If D is not open, then $f(D)$ is not open.

   Let $D = [-1, 0) \cup (0, 2)$ and define $f(x) = 1/x^2$. Then D is not open, since $-1 \in D$ is not interior an point, but $f(D) = [1, \infty) \cup (1/4, \infty) = (1/4, \infty)$ is open. (Here f has a discontinuity at 0, but $0 \notin D$, so f is continuous on D.)

   (d) If D is not closed, then $f(D)$ is not closed.

   Let $D = (-\sqrt{3}, \sqrt{3})$, and define $f(x) = x^3 - 3x$. Then D is not closed, since it does not contain the limit points ± 3, but $f(D) = [-2, 2]$ is closed. (Here we have $f(\pm\sqrt{3}) = 0$ and $f(0) = 0$, and f has an absolute maximum at $(-1, 2)$, and an absolute miniumum at $(1, -2)$.)

   (e) If D is not compact, then $f(D)$ is not compact.

   Let D be any noncompact subset of $\mathbb{R}$, for example, $D = \mathbb{R}$, and let f be a constant function, say $f(x) = 0$ for all x. Then $f(D)$ is compact, since it consists of a single point.

(f) If D is unbounded, then $f(D)$ is unbounded.

Let $D = \mathbb{R}$ and take $f(x) = e^{-x^2}$. Then D is not bounded, but $0 < f(x) < 1$ for all $x \in \mathbb{R}$, so f is bounded.

(g) If D is finite, then $f(D)$ is finite.

This is true for any function, continuous or not: if $D = \{x_1, \ldots, x_n\}$, then $f(D) = \{f(x_1), \ldots, f(x_n)\}$, so the cardinality of $f(D)$ is less than or equal to the cardinality of D (with equality if f is one-to-one).

(h) If D is infinite, then $f(D)$ is infinite.

Let $D = \mathbb{R}$ and take $f(x) = 0$.

(i) If D is an interval, then $f(D)$ is an interval.

This is true. Choose any points $u, v \in f(D)$, with $u < v$. Then $u = f(x)$ and $v = f(y)$ for some $x, y \in D$, and by the Intermediate Value Theorem, for any $w \in \mathbb{R}$ such that $u < w < v$, there exists some z between x and y ($z \in (x, y)$ if $x < y$ or $z \in (y, x)$ if $y < x$) such that $f(z) = w$. It follows that $f(D)$ is an interval.

(j) If D is an interval that is not open, then $f(D)$ is an interval that is not open.

Let $D = [0, \infty)$, and let $f(x) = x \sin x$. Then D is an interval that is not open, since $0 \in D$ is not an interior point, but $f(D) = \mathbb{R}$, which is open.

(Note: this is problem 5.3.3 in the text, and there's a hint in the back.)

2. (a) Let $a \in \mathbb{R}$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x - a|$. Prove that f is continuous.

Let $f(x) = |x - a|$, and let $\epsilon > 0$ be given. Let $\delta = \epsilon$ and suppose that $|x - y| < \delta$. Then we have

$$|f(x) - f(y)| = ||x - a| - |y - a|| \leq |x - a - (y - a)| = |x - y| < \delta = \epsilon,$$

using the inequality $||u| - |v|| \leq |u - v|$ for all $u, v \in \mathbb{R}$.

(b) Let K be a nonempty compact subset of $\mathbb{R}$ and let $a \in \mathbb{R}$. We define the distance from a to K by

$$d(a, K) = \inf\{|x - a| : x \in K\}.$$

(The infimum exists since $\{|x - a| : x \in K\}$ is bounded below by zero.) Prove that there exists a point $b \in K$ that is *closest* to a, in the sense that $|b - a| = d(a, K)$.

Let $K \subseteq \mathbb{R}$ be compact, and let $a \in \mathbb{R}$. Define $f : K \to \mathbb{R}$ by $f(x) = |x - a|$. Then f is continuous, by part (a), so f has an absolute miniumum by the Extreme Value Theorem, since K is compact. That is, there exists some $b \in K$ such that $|b - a| = f(b) \leq f(x) = |x - a|$ for all $b \in K$. Since $f(b)$ is the minimum of the set $\{|x - a| : a \in K\}$, it must be the infimum, and thus $f(b) = d(a, K)$, as required.

3. Prove that if $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \in \mathbb{Q}$ for all $x \in [a, b]$, then f is constant.

   Since $f : [a, b] \to \mathbb{R}$ is continuous, it satisfies the Intermediate Value Theorem on $[a, b]$. If there exist $x, y \in [a, b]$ with $f(x) < f(y)$, then we can find an irrational number $z \in \mathbb{R}$ with $f(x) < z < f(y)$, and then we would have to have $z = f(c)$ for some c between x and y. But this is impossible, since $f(x) \in \mathbb{Q}$ for all $x \in [a, b]$. Thus, f must have the same value at every point, which is to say that f is constant.

4. Suppose f is continuous on $[0, 2]$, and $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ with $|y - x| = 1$ and $f(x) = f(y)$.

   Hint: Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.

   Let f be a continuous function on $[0, 2]$ with $f(0) = f(2)$, and let $g(x) = f(x+1) - f(x)$, with $x \in [0, 1]$. Then g is continuous on $[0, 1]$ (the function $h(x) : [0, 1] \to [1, 2]$ given by $h(x) = x + 1$ is continuous, so $f \circ h(x) = f(x + 1)$ is continous on $[0, 1]$ since it's the composition of continuous functions, and thus g is the difference of two continous functions.

   Let $a = f(0) = f(2)$, and let $b = f(1)$. If $a = b$, we're done, since we can take $x = 0$ and $y = 1$. If not, we note that

   $$g(0) = f(1) - f(0) = b - a,$$

   and

   $$g(1) = f(2) - f(1) = a - b = -(b - a).$$

   Since we're assuming that $b - a \neq 0$ we must have either $g(0) < 0 < g(1)$ or $g(1) < 0 < g(0)$. Thus, there exists some $c \in [0, 1]$ such that $g(c) = 0$, by the intermediate value theorem. But then we have $0 = g(c) = f(c+1) - f(c)$, so we can take $x = c$ and $y = c + 1$.

5. Prove that each of the following functions is uniformly continuous on the specified set using the ϵ-δ definition of uniform continuity:

   (a) $f(x) = x^2$ on $[0, 3]$

   Let $\epsilon > 0$ be given, and let $\delta = \epsilon/6$. Then for any $x, y \in [0, 3]$ we have $0 \leq x+y \leq 6$, so if $|x - y| < \delta$, then

   $$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| < \delta \cdot 6 = \epsilon.$$

(b) $g(x) = \dfrac{1}{x}$ on $[\frac{1}{2}, \infty)$

Let $\epsilon > 0$ be given, and let $\delta = 4\epsilon$. For $x, y \geq 1/2$, we have $1/x, 1/y \leq 2$, so if $|x - y| < \delta$, then

$$|g(x) - g(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| < \frac{\delta}{2 \cdot 2} = \epsilon.$$

6. (**Do not submit**) Prove that if f is uniformly continuous on a bounded set $D \subseteq \mathbb{R}$, then f is bounded on D.

Hint: If f is not bounded on D, you can find some sequence (a_n) in D with $|f(a_n)| \geq n$ for all $n \in \mathbb{N}$. But since D is bounded, (a_n) is a bounded sequence and therefore has a convergent subsequence. We also know that if f is uniformly continuous and (x_n) is a Cauchy sequence, then $(f(x_n))$ is also a Cauchy sequence.

Following the hint, suppose that f is not bounded on D. Then for each $n \in \mathbb{N}$ there exists some $a_n \in D$ such that $|f(a_n)| \geq n$. Let (a_n) be the resulting sequence. Since D is bounded, and $a_n \in D$ for all $n \in \mathbb{N}$, (a_n) is bounded, so there must be a convergent subsequence (a_{n_k}). Since this subsequence converges, it must be a Cauchy sequence, and since f is uniformly continuous on D, it follows that $(f(a_{n_k}))$ is a Cauchy sequence. But then it must be the case that $(f(a_{n_k}))$ is a bounded sequence, which contradicts the assumption that $|f(a_{n_k})| \geq n_k \geq k$ for all $k \in \mathbb{N}$. Thus, f must be bounded on D.

7. Prove that $f(x) = \sin x$ is uniformly continous on $\mathbb{R}$.

Hint: First use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Let $\epsilon > 0$ be given, and let $\delta = \epsilon$. Choose any $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Assume without loss of generality that x, y. Since $f(x) = \sin x$ is continuous on $[x, y]$ and differentiable on (x, y), there exists some $c \in (x, y)$ such that

$$f'(c) = \cos c = \frac{\sin y - \sin x}{y - x} \quad \text{and thus} \quad \left|\frac{\sin y - \sin x}{y - x}\right| = |\cos c| \leq 1.$$

It follows that

$$|f(x) - f(y)| = |\sin x - \sin y| \leq |x - y| < \delta = \epsilon.$$