Math 3500 Assignment #8
University of Lethbridge, Fall 2014

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Due date: Friday, November 21st, by 6 pm.

1. Let f be differentiable on some interval (c, ∞) and suppose that $\lim_{x\to\infty}[f(x)+f'(x)] = L$, where L is finite. Prove that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$.

   Hint: for all $x > c$, $f(x) = \dfrac{e^x f(x)}{e^x}$.

2. When we apply l'Hospital's rule to the limit $\lim_{x\to a} \dfrac{f(x)}{g(x)}$, we require that $g'(x) \neq 0$ near $x = a$. This exercise demonstrates the importance of that requirement: if l'Hospital's rule is applied carelessly, it's possible for the zeros of g' to cancel the zeros of f', leading to an incorrect result. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x + \cos x \sin x \quad g(x) = e^{\sin x}(x + \cos x \sin x).$$

   (a) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$

   (b) Show that $f'(x) = 2\cos^2 x$ and $g'(x) = e^{\sin x}\cos x[2\cos x + f(x)]$

   (c) Show $\dfrac{f'(x)}{g'(x)} = \dfrac{2e^{-\sin x}\cos x}{2\cos x + f(x)}$ if $\cos x \neq 0$ and $x > 3$.

   (d) Show that $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)} = 0$, and yet $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$ does not exist.

3. Find a Taylor polynomial that approximates $f(x) = e^x$ to within 0.2 on the interval $[-2, 2]$.

4. Show that if $x \in [0, 1]$, then

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \leq \ln(1 + x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}.$$