MATH 1410 - Tutorial #8 Solutions

1. For each matrix A and vector $\vec{b}$ below, solve the equation $A\vec{x} = \vec{b}$. Express your answer in terms of the vector $\vec{x}$.

   If there are infinitely many solutions, give your answer in the form $\vec{x} = \vec{x}_p + \vec{x}_h$, where $\vec{x}_p$ is a particular solution, and $\vec{x}_h$ is the general solution to the homogeneous system $A\vec{x} = \vec{0}$. (Express $\vec{x}_h$ in terms of basic solutions.)

   (a) $A = \begin{bmatrix} 1 & 0 & -4 \\ -2 & 1 & 4 \\ 1 & 0 & 6 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$.

   Setting up and reducing the corresponding augmented matrix, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 2 \\ -2 & 1 & 4 & -1 \\ 1 & 0 & 6 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 16/5 \\ 0 & 1 & 0 & 21/5 \\ 0 & 0 & 1 & 3/10 \end{array}\right].$$

   Thus, we have a unique solution ($\vec{x}_h = \vec{0}$), and

$$\vec{x} = \vec{x}_p = \begin{bmatrix} 16/5 \\ 32/5 \\ 3/10 \end{bmatrix}.$$

   (b) $A = \begin{bmatrix} 1 & 0 & 2 & -4 \\ 3 & 1 & 5 & -7 \\ -2 & -2 & -2 & -2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix}$

   Again, we set up and reduce our augmented matrix, getting:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 3 \\ 3 & 1 & 5 & -7 & 2 \\ -2 & -2 & -2 & -2 & 8 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 3 \\ 0 & 1 & -1 & 5 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

   Writing $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, we have $x_1 = 3 - 2x_3 + 4x_4$ and $x_2 = -7 + x_3 - 5x_4$, where x_3 and x_4 are free variables. Assigning parameters $x_3 = s$ and $x_4 = t$, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 - 2s + 4t \\ -7 + s - 5t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\vec{x}_p = \begin{bmatrix} 3 \\ -7 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{x}_h = s\vec{x}_1 + t\vec{x}_2$, where $\vec{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 4 \\ -5 \\ 0 \\ 1 \end{bmatrix}$ are the basic solutions to $A\vec{x} = \vec{0}$.

2. Consider the matrices

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 \\ 5 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 6 & 3 \end{bmatrix}.$$

For each of the 9 possible products ($A^2, AB, AC, BA, B^2, BC, CA, CB, C^2$), compute the product, or state why it is undefined.

A^2 is undefined: only square matrices can be multiplied by themselves.

AB is undefined: A is 2×3, B is 2×2, and $3 \neq 2$.

$$AC = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 22 & 18 \\ -15 & 10 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -12 & 16 \\ 15 & -1 & 13 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 4 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ 25 & -9 \end{bmatrix}$$

BC is undefined: B is 2×2, C is 3×2, and $2 \neq 3$.

$$CA = \begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 22 & 15 & -5 \\ -9 & -2 & -4 \\ 27 & 6 & 12 \end{bmatrix}$$

$$CB = \begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 2 \\ -13 & 3 \\ 39 & -9 \end{bmatrix}$$

C^2 is undefined: only square matrices can be multiplied by themselves.

3. Consider a system of equations, written in matrix form as $A\vec{x} = \vec{b}$. Prove that if there is more than one solution to the system (say, $\vec{x}_1$ and $\vec{x}_2$, with $\vec{x}_1 \neq \vec{x}_2$), then there are infinitely many solutions.

Suppose $\vec{x}_1$ and $\vec{x}_2$ are distinct solutions to $A\vec{x} = \vec{b}$. Then

$$A\vec{x}_1 = \vec{b}, A\vec{x}_2 = \vec{b}, \text{ and } \vec{x}_1 \neq \vec{x}_2.$$

It follows that $\vec{x}_1 - \vec{x}_2 \neq \vec{0}$ and

$$A(\vec{x_1} - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}.$$

Thus, $\vec{x}_h = \vec{x}_1 - \vec{x}_2$ is a non-zero solution to the homogeneous system $A\vec{x} = \vec{0}$.
Since $A(t\vec{x}_h) = t(A\vec{x}_h) = t\vec{0} = \vec{0}$ for any real number t, we find that

$$A(\vec{x}_1 + t\vec{x}_h) = A\vec{x}_1 + A(t\vec{x}_h) = \vec{b} + \vec{0} = \vec{b}.$$

Therefore, $\vec{x} = \vec{x}_1 + t(\vec{x}_1 - \vec{x}_2)$ is a solution to $A\vec{x} = \vec{b}$ for each real number t, and since there are infinitely many real numbers, we get infinitely many solutions.

4. For which values of k will the system $\begin{array}{rcrcrcr} x &+& y &+& kz &=& 1 \\ x &+& ky &+& z &=& 1 \\ kx &+& y &+& z &=& -2 \end{array}$ have:

(a) No solution? (b) A unique solution? (c) Infinitely many solutions?

Reducing our corresponding augmented matrix, we find

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{array}\right] \xrightarrow[R_3 - kR_1 \to R_1]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & -2-k \end{array}\right]$$

At this point, we notice that if $k = 1$, then we get the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right]$, and from the third row, we can conclude that if $k = 1$, there is no solution to the system.

If $k \neq 1$, then $k - 1 \neq 0$, so we can divide by $k - 1$. This lets us proceed as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & -2-k \end{array}\right] \xrightarrow[\frac{1}{1-k}R_3 \to R_3]{\frac{1}{k-1}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1+k & \frac{-2-k}{1-k} \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2+k & \frac{-2-k}{1-k} \end{array}\right]$$

Now, we notice that if $k = -2$, then we get the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, and since there is no leading 1 in the third column, we have infinitely many solutions.

If $k \neq -2$, then $k+2 \neq 0$, so we can divide by $k+2$. This lets us do one more row operation:

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2+k & \frac{-2-k}{1-k} \end{array}\right] \xrightarrow{\frac{1}{k+2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{-1}{1-k} \end{array}\right].$$

Since we have a leading one in each of the variable columns, we conclude that there is a unique solution to the system.

In conclusion, if $k = 1$, there is no solution. If $k = -2$, there are infinitely many solutions. For all other values of k, there is a unique solution.

Algebra notes: Notice that $1 - k = (-1)(k-1)$, which is why dividing $1-k$ by $k-1$ gave us -1. Also $1 - k^2 = (1-k)(1+k)$, which is why dividing $1-k^2$ by $1-k$ gave us $1+k$.