Please complete all problems below.

1. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ -1 & 2 \\ 0 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$.

   Compute each of the following, or explain why they're not defined:

   (a) $2A - 3B^T$. (B^T denotes the transpose of B. Ask if you don't know what that is.)

$$\begin{aligned} 2A - 3B^T &= 2\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -2 \end{bmatrix} - 3\begin{bmatrix} 3 & 5 \\ -1 & 2 \\ 0 & -2 \end{bmatrix}^T \\ &= \begin{bmatrix} 2 & -4 & 6 \\ 0 & 8 & -4 \end{bmatrix} - \begin{bmatrix} 9 & -3 & 0 \\ 15 & 6 & -6 \end{bmatrix} = \begin{bmatrix} -7 & -1 & 6 \\ -15 & 2 & 2 \end{bmatrix}. \end{aligned}$$

   (b) $2A - 3C$.

   Not defined, since you can't add or subtract matrices of different sizes: $2A$ is a 2×3 matrix, and $3C$ is a 2×2 matrix.

   (c) AB

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & 2 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1(3) - 2(-1) + 3(0) & 1(5) - 2(2) + 3(-2) \\ 0(3) + 4(-1) - 2(0) & 0(5) + 4(2) - 2(-2) \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -4 & 12 \end{bmatrix}. \end{aligned}$$

   (d) BA

$$\begin{aligned} BA &= \begin{bmatrix} 3 & 5 \\ -1 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3(1) + 5(0) & 3(-2) + 5(4) & 3(3) + 5(-2) \\ -1(1) + 2(0) & -1(-2) + 2(4) & -1(3) + 2(-2) \\ 0(1) - 2(0) & 0(-2) - 2(4) & 0(3) - 2(-2) \end{bmatrix} = \begin{bmatrix} 3 & 14 & -1 \\ -1 & 10 & -7 \\ 0 & -8 & 4 \end{bmatrix}. \end{aligned}$$

   (e) $AB + C$

   We calculated AB above. Using that result, we have

$$AB + C = \begin{bmatrix} 5 & -5 \\ -4 & 12 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 7 & -9 \\ -7 & 18 \end{bmatrix}.$$

   (f) $BA + C$

   This is undefined, since BA is a 3×3 matrix and C is a 2×2 matrix, and you can't add matrices of different sizes.

2. Compute the inverses of the following matrices, if possible:

   (a) $A = \begin{bmatrix} 1 & 3 \\ -4 & -2 \end{bmatrix}$

   Our algorithm for finding the inverse is to use Gaussian elimination to convert the augmented matrix $[A|I]$ to $[I|A^{-1}]$, if possible. We proceed as follows:

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -4 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 4R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 10 & 4 & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{10}R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & \frac{1}{10} \end{array}\right]$$
$$\xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{5} & -\frac{3}{10} \\ 0 & 1 & \frac{2}{5} & \frac{1}{10} \end{array}\right].$$

   Therefore, we conclude that $A^{-1} = \begin{bmatrix} -\frac{1}{5} & -\frac{3}{10} \\ \frac{2}{5} & \frac{1}{10} \end{bmatrix}$.

   (b) $B = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 2 & 0 & 9 \end{bmatrix}$

   We use the same algorithm as above, but this time for a 3×3 matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & 0 \\ 2 & 0 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$
$$\xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - 4R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 0 & -4 \\ 0 & -3 & 0 & 4 & 1 & -2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 0 & -4 \\ 0 & 1 & 0 & -\frac{4}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right].$$

   Thus, $A^{-1} = \begin{bmatrix} 9 & 0 & -4 \\ -\frac{4}{3} & -\frac{1}{3} & \frac{2}{3} \\ -2 & 0 & 1 \end{bmatrix}$.

3. Solve the following systems. (Hint: use your answer from 2(a))

   (a)

$$\begin{array}{rcrcr} x & + & 3y & = & 3 \\ -4x & - & 2y & = & -7 \end{array}$$

   In matrix form, we have $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$, so

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & -\frac{3}{10} \\ \frac{2}{5} & \frac{1}{10} \end{bmatrix}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} \frac{15}{10} \\ \frac{5}{10} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}.$$

(b)

$$\begin{array}{rcrcr} x & + & 3y & = & 2a \\ -4x & - & 2y & = & -3b \end{array}$$

As above, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 2a \\ -3b \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & -\frac{3}{10} \\ \frac{2}{5} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 2a \\ -3b \end{bmatrix} = \begin{bmatrix} -\frac{2a}{5} + \frac{9b}{10} \\ \frac{4a}{5} - \frac{3b}{10} \end{bmatrix}.$$

Possibly useful note: the question "Can the vector V be written as a linear combination of the vectors A, B, C?" is equivalent to the question "Do there exist scalars x, y, z such that $xA + yB + zC = V$?" This latter question can be turned into a system of equations in the variables x, y, z.

Similarly, the question "Given the vectors A, B, C, D, can any one of these vectors be written as a linear combination of the others?" is equivalent to the question, "Do there exist scalars w, x, y, z, *not all equal to zero*, such that $wA + xB + yC + zD = 0$?" (See if you can figure out why these two questions are the same.) This latter question can be turned into a homogeneous system of equations, and the answer is "yes" if this system has a non-trivial solution.

Note on this note: There is one question on the Lyryx Lab where you need to know how to answer the first question. In Chapter 6 we will briefly encounter the concepts of linear independence and span, which are best understood in terms of the note above.