Solutions to Quiz 5 Practice Problems
Math 2580
Spring 2016

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1. Let $f(x, y) = xy$ and suppose $x = g(t)$ and $y = h(t)$. Show that applying the chain rule to the derivative $\dfrac{d}{dt}f(g(t), h(t)) = \dfrac{d}{dt}(g(t)h(t))$ produces the product rule for derivatives in one variable.

   We have $f(g(t), h(t)) = g(t)h(t)$, so on the one hand $\dfrac{d}{dt}f(g(t), h(t)) = \dfrac{d}{dt}(g(t)h(t))$. On the other hand,
   $$\frac{d}{dt}(f(g(t), h(t)) = f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t) = h(t)g'(t) + g(t)h'(t),$$
   since $\dfrac{\partial f}{\partial x} = y$ and $\dfrac{\partial f}{\partial y} = x$.

2. Suppose an insect is flying through a room along the path
   $$r(t) = (e^t, t^2, \sin t),$$
   and that the temperature in the room is given by $T(x, y, z) = \sin(x)\cos(y)\sqrt{z}$. Find the rate $\dfrac{dT}{dt}$ at which the temperature experienced by the insect changes as it flies through the room.

   Using the chain rule, we have
   $$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial x}x'(t) + \frac{\partial T}{\partial y}y'(t) + \frac{\partial T}{\partial z}z'(t)\\ &= (\cos(x)\cos(y)\sqrt{z})e^t + (-\sin(x)\sin(y)\sqrt{z})(2t) + \left(\frac{\sin(x)\cos(y)}{2\sqrt{z}}\right)(\cos(t))\\ &= \cos(e^t)\cos(t^2)\sqrt{\sin t}e^t - 2t\sin(e^t)\sin(t^2)\sqrt{\sin t} + \frac{\sin(e^t)\cos(t^2)\cos(t)}{2\sqrt{\sin t}}.\end{aligned}$$

   (In this case, Chain Rule is probably easier than first plugging in the functions of t and using multiple applications of the product rule.)

3. Consider the function $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, where $D = \{(u,v) | v \geq 1\}$, given by

$$F(u,v) = \left(\sqrt[3]{uv}, \sqrt[3]{\frac{u}{v^2}} \right).$$

Calculate the derivative matrix $D_{(u,v)} f$ at a general point $(u,v) \in D$.

Using the definition of $D_{(u,v)} f$, we have

$$D_{(u,v)} f = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u^{1/3} v^{1/3}}{\partial u} & \dfrac{\partial u^{1/3} v^{1/3}}{\partial u} \\ \\ \dfrac{\partial u^{1/3} v^{-2/3}}{\partial u} & \dfrac{\partial u^{1/3} v^{-2/3}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} u^{-2/3} v^{1/3} & \frac{1}{3} u^{1/3} v^{-2/3} \\ \frac{1}{3} u^{-2/3} v^{-2/3} & -\frac{2}{3} u^{1/3} v^{-5/3} \end{bmatrix}.$$

4. Let $g(x,y) = xy^2 \cos(xy)$, where $x = \sqrt[3]{uv}$ and $y = \sqrt[3]{\frac{u}{v^2}}$. Compute $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$ using the Chain Rule.

I'll give a solution using the product of derivative matrices, but you can also just write out the chain rule patterns in this case.
The derivative matrix for g is given by

$$D_{(x,y)} g = \begin{bmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} y^2 \cos(xy) - xy^3 \sin(xy) & 2xy \cos(xy) - x^2 y^2 \sin(xy) \end{bmatrix},$$

so

$$\begin{aligned} \begin{bmatrix} \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{bmatrix} &= \begin{bmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \\ &= \begin{bmatrix} y^2 \cos(xy) - xy^3 \sin(xy) & 2xy \cos(xy) - x^2 y^2 \sin(xy) \end{bmatrix} \begin{bmatrix} \frac{1}{3} u^{-2/3} v^{1/3} & \frac{1}{3} u^{1/3} v^{-2/3} \\ \frac{1}{3} u^{-2/3} v^{-2/3} & -\frac{2}{3} u^{1/3} v^{-5/3} \end{bmatrix}. \end{aligned}$$

This product gives a 1×2 row matrix whose first entry is

$$\frac{\partial g}{\partial u} = (y^2 \cos(xy) - xy^3 \sin(xy))(\frac{1}{3} u^{-2/3} v^{1/3}) + (2xy \cos(xy) - x^2 y^2 \sin(xy))(\frac{1}{3} u^{-2/3} v^{-2/3}),$$

and whose second entry is

$$\frac{\partial g}{\partial v} = (y^2 \cos(xy) - xy^3 \sin(xy))(\frac{1}{3} u^{1/3} v^{-2/3}) + (2xy \cos(xy) - x^2 y^2 \sin(xy))(-\frac{2}{3} u^{1/3} v^{-5/3}).$$

As a final step we should really write x and y in terms of u and v but this is tedious and not particularly enlightening.

5. What is the **gradient** of a continuously differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$? How is the gradient of f related to the derivative $D_{(x,y,z)}f$?

   The gradient of f is a vector field whose value at (x, y, z) is the vector $\nabla f(x, y, z) = \begin{bmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{bmatrix}$, and this is simply the transpose of the corresponding row matrix given by $D_{(x,y,z)}f$.

6. Calculate the gradient of the function $f(x, y) = 3x^2 - 4xy$ at the point $(1, 2)$.

   We have $f_x(x, y) = 6x - 4y$ and $f_y(x, y) = -4x$, so $f_x(1, 2) = 6 - 8 = -2$ and $f_y(x, y) = -4$. Thus, $\nabla f(1, 2) = \begin{bmatrix} f_x(1,2) \\ f_y(1,2) \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$.