Name and student number: Solutions

[10] 1. Use mathematical induction to prove that for all $n \in \mathbb{N}$, $4^n - 1$ is divisible by 3.

Let $P(n)$ be the predicate $3|(4^n - 1)$. We wish to show that $P(n)$ holds for all $n \in \mathbb{N}$.

Base case: when $n = 1$, $4^1 - 1 = 3$, and $3|3$, so $P(1)$ is true.

Induction hypothesis: Suppose that $P(k)$ is true for some $k \geq 1$. That is, we assume that $3|(4^k - 1)$, which means that there exists some $p \in \mathbb{Z}$ such that $4^k - 1 = 3p$.

Inductive step: We wish to show that $4^{k+1} - 1$ is divisible by 3. Since $4^k = 1 + 3p$ by the induction hypothesis, we have

$$4^{k+1} = 4(4^k) = 4(1 + 3p) = 4 + 4(3p) = 1 + 3 + 3(4p) = 1 + 3(1 + 4p).$$

Thus, $4^{k+1} - 1 = 3(1 + 4p)$, and since $1 + 4p$ is an integer, $3|(4^{k+1} - 1)$, so $P(k + 1)$ is true. It follows that $3|(4^n - 1)$ for all $n \in \mathbb{N}$ by induction.

Note: It is not necessary to lable your steps as I have above when writing a proof by induction. I've done so in this case to make it easier to follow the proof for those who are not used to the method of induction. Anything that's clearly written using complete sentences is acceptable, as long as you remember to check the base case, and show that $P(k) \to P(k + 1)$.

Total: 10 points