University of Lethbridge
Department of Mathematics and Computer Science
24th April, 2017, 9:00 am - 12:00 pm
MATH 1410 - FINAL EXAM

Last Name:____

First Name:____

Student Number:____

Lecture Section (circle): **A** (1:40 - 2:55 pm) **B** (10:50 am - 12:05 pm)

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page. Additional scrap paper may be requested if needed.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

No external aids are permitted, with the exception of a basic (non-scientific and non-graphing) calculator.

For grader's use only:

Page	Grade
2	/10
3	/14
4	/12
5	/10
6	/10
7	/9
8	/9
9	/8
10	/8
11	/10
Total	/100

1. DEFINITIONS. (2 points each) Give the definition of the term in bold by completing the sentence.

   (a) Two vectors $\vec{v}$ and $\vec{w}$ are **orthogonal** if:

   (b) A **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is any expression of the form:

   (c) The **trace** of an $n \times n$ matrix A is given by:

   (d) An $n \times n$ matrix A is **invertible** if:

   (e) We say that an $n \times n$ matrix B is **similar** to an $n \times n$ matrix A if:

2. Given the complex numbers $z = 3 + 5i$ and $w = -2 + 3i$, compute:

[2] (a) $z + \overline{w}$

[2] (b) z^2

[3] (c) $\dfrac{z}{w}$

3. Given the vectors $\vec{v} = \langle 2, -1, 5 \rangle$ and $\vec{w} = \langle -3, 0, 2 \rangle$, compute:

[2] (a) $2\vec{v} - 3\vec{w}$

[2] (b) $\vec{v} \cdot \vec{w}$

[3] (c) $\vec{v} \times \vec{w}$

4. SHORT ANSWER – More calculations. You don't have to show work, but it can earn you part marks if you do.

[3] (a) Compute the determinant of the matrix $A = \begin{bmatrix} 1-i & 1+2i \\ 2-4i & 3+3i \end{bmatrix}$.

[3] (b) Assume A and B are 3×3 matrices. Simplify the matrix product $(A^TBC)^{-1}(C^TA^{-1})^T$.

[3] (c) Compute the matrix product $\begin{bmatrix} 3 & -5 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & -6 \end{bmatrix}$.

[3] (d) Determine if $x = 2, y = -1, z = 3$ is a solution to the system $\begin{array}{rcrcrcl} x & - & 3y & + & 2z & = & 5 \\ -2x & + & y & + & z & = & 1 \\ x & + & 3y & + & z & = & 2 \end{array}$.

5. SHORT(-ISH) ANSWSER — Even more calculations:

[4] (a) Determine the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$

[4] (b) Show that $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors of the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$.

[2] (c) Compute $T\left(\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right)$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a matrix transformation such that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

6. Given the matrix $A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$:

[3] (a) Find the eigenvalues of A.

[6] (b) For each eigenvalue found in part (a), determine a corresponding eigenvector.

[1] (c) Determine a matrix P such that $P^{-1}AP$ is a diagonal matrix, or explain why no such P exists. You do not have to verify that your matrix P works (unless you want to).

[6] 7. Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{bmatrix}$$

8. Suppose that the matrix $B = \begin{bmatrix} 3 & 7 & -2 \\ 0 & -2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ is obtained from a matrix A using the following

[3] row operations (A_1, A_2, A_3 represent intermediate steps):

$$A \xrightarrow{R_2+3R_3\to R_2} A_1 \xrightarrow{\frac{1}{3}R_2\to R_2} A_2 \xrightarrow{R_1\leftrightarrow R_3} A_3 \xrightarrow{R1-2R_2\to R_1} A_4 = B.$$

What is the determinant of A?

[6] 9. Compute the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 6 \\ -1 & 2 & -2 \end{bmatrix}$.

[3] 10. For which values of x is the matrix $\begin{bmatrix} x & -2 & x^2 \\ -x & 2 & 3x \\ -1 & 2x & 2 \end{bmatrix}$ invertible?

[6] 11. (a) Determine the solution to the system

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 1 \\ 2x_1 & - & 4x_2 & + & x_3 & & & = & 5 \\ -x_1 & + & 2x_2 & - & 2x_3 & + & 3x_4 & = & -4 \end{array}$$

Write your answer in vector form.

(b) Let $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ -1 & 2 & -2 & 3 \end{bmatrix}$ be the coefficient matrix from part (a). Determine vectors

[2] $\vec{v}$ and $\vec{w}$ such that $\text{null}(A) = \text{span}\{\vec{v}, \vec{w}\}$.

Hint: $\vec{v}$ and $\vec{w}$ are solutions to the homogeneous system corresponding to the system in part (a).

12. (a) Find an equation of the plane containing the points $P = (0, 1, 2)$, $Q = (-1, 2, 1)$, and $R = (2, 0, 3)$. [4]

(b) Determine the line of intersection of the planes $x+2y-3z = 2$ and $-2x-3y+4z = 1$. [4]

13. An $n \times n$ matrix is called **idempotent** if $A^2 = A$.

[3] (a) Show that $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ are all idempotent.

[3] (b) Show that if an idempotent matrix is invertible, then it must be the identity matrix.

[4] (c) Prove that if an $n \times n$ matrix A is idempotent, then $I - 2A$ is equal to its own inverse, where I denotes the $n \times n$ identity matrix.

[5] (d) **Bonus** (complete on next page): Prove that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$.

Scrap page: Extra space for the 13(b) bonus problem, or for rough work.

Note: The bonus problem will be graded strictly. Any solution involving an example, or that refers to the individual entries of the matrix, will receive no credit. Your solution, should you choose to provide one, should involve little more than the definitions of *idempotent* and *eigenvalue*.

Additional problem for those who ignored my disclaimer about there being no solutions to past exams:

[0] 14. Rick Astley is never gonna:

(a) Give you up.

(b) Let you down.

(c) Desert you.

(d) All of the above.