Math 3410 Assignment #1 Solutions
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1. Let $V = M_{n\times n}(\mathbb{R})$ denote the space of $n \times n$ matrices.

   (a) Let $E \in V$ be a matrix such that $E^2 = E$, and let $U = \{A \in V : AE = A\}$ and $W = \{B \in V : BE = 0\}$. Show U and W are subspaces of V, and that $V = U \oplus W$.
   Hint: Observe that $XE \in U$ for any matrix $X \in V$.

   Let $E \in V$ be such that $E^2 = E$, and let $U = \{A \in V : AE = A\}$. Then U is a subspace of V, since

   i. $0E = 0$, so $0 \in U$.

   ii. If $A_1, A_2 \in U$ (so $A_1E = A_1$ and $A_2E = A_2$), then
   $$(A_1 + A_2)E = A_1E + A_2E = A_1 + A_2,$$
   so $A_1 + A_2 \in U$.

   iii. If $A \in U$ and $c \in \mathbb{R}$, then $(cA)E = c(AE) = cA$, so $cA \in U$.

   The proof that W is a subspace is almost identical: it's clear that $0 \in W$, and if $B_1E = B_2E = 0$, then $(B_1 + B_2)E = 0$ and $(cB_1)E = 0$ for any $B_1, B_2 \in E$.
   Given any matrix $X \in V$, write
   $$X = XE + (X - XE).$$
   Then $XE \in U$, since $(XE)E = XE^2 = XE$, and $X - XE \in W$, since $(X - XE)E = XE - XE^2 = XE - XE = 0$. This tells us that $U + W = V$. Finally we note that $U \cap W = \{0\}$, since if $A \in U$ and $A \in W$, then we have $A = AE = 0$. It follows that $V = U \oplus W$.

   (b) Let U and W denote the subspaces of symmetric and skew-symmetric matrices, respectively. (That is $U = \{A \in V : A^T = A\}$, and $V = \{B \in V : B^T = -B\}$.) Show that $V = U \oplus W$.
   Hint: First show that for any matrix $X \in V$, $X + X^T \in U$ and $X - X^T \in W$.

   Since $0^T = 0 = -0$, we see that $0 \in U$ and $0 \in W$. If $A, B \in U$, then $(A + B)^T = A^T + B^T = A + B$, so $A + B \in U$. Similarly, if $A, B \in W$, then $(A + B)^T =$

$A^T + B^T = -A - B = -(A+B)$, so $A + B \in W$. Finally, given $A \in U$ and $B \in W$ and any $c \in \mathbb{R}$, we have $(cA)^T = cA^T = cA$ and $(cB)^T = cB^T = c(-B) = -(cB)$, so $cA \in U$ and $cB \in W$. It follows that U and W are subspaces.

Now, given any $X \in V$, we can write X as

$$X = \frac{1}{2}(X + X^T) + \frac{1}{2}(X - X^T),$$

and since

$$[\frac{1}{2}(X + X^T)]^T = \frac{1}{2}(X^T + (X^T)^T) = \frac{1}{2}(X + X^T)$$

and

$$[\frac{1}{2}(X - X^T)]^T = \frac{1}{2}(X^T - (X^T)^T) = \frac{1}{2}(X^T - X) = -\frac{1}{2}(X - X^T),$$

we see that $\frac{1}{2}(X + X^T) \in U$ and $\frac{1}{2}(X - X^T) \in W$, so $V = U + W$. Now, if $A \in U \cap W$, then we have $A = A^T = -A$, from which we get $2A = 0$ and thus $A = 0$. Therefore $U \cap W = \{0\}$, and we can conclude that $V = U \oplus W$.

2. Let U and W be subspaces of a vector space V. Prove that $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq V$.

   Bonus: For a 10% bonus, prove that the union of three subspaces is a subspace if and only if one of the subspaces contains the other two. (This is 1.C.13 from the text; it comes with the warning that it's more difficult than the case of two subspaces. I'm not sure how much more difficult – I haven't tried to solve it.)

   Let $U, W \subseteq V$ be subspaces. If $U \subseteq W$, then $U \cup W = W$, and thus $U \cup W$ is a subspace. Similarly if $W \subseteq U$ then $U \cup W = U$ is a subspace.

   Conversely, suppose that neither subspace is a subset of the other. Then there is some $u \in U$ such that $u \notin W$, and there is some $w \in W$ such that $w \notin U$. Note that we have $u, w \in U + W$, and consider $u + w$. We know that $u + w \notin U$, since otherwise, using the fact that $-u \in U$ (since U is a subspace), we have

$$-u + (u + w) = (-u + u) + w = 0 + w = w \in U.$$

   However, $w \notin U$ by assumption, so $u + w \notin U$. Similarly, we must hav $u + w \notin W$. It follows that $u + w \notin U \cup W$, and thus $U \cup W$ cannot be a subspace.

3. Let U be the subspace of $V = \mathbb{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in V : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

(a) Find a basis for U.

Substituting $z_2 = 6z_1$ and $z_3 = -2z_4 - 3z_5$, we see that an arbitrary element of U is of the form

$$v = (z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5) = z_1(1, 6, 0, 0, 0) + z_4(0, 0, -2, 1, 0) + z_5(0, 0, -3, 0, 1).$$

Thus, the set $B = \{(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)\}$ spans U, and it is linearly independent: if

$$a(1, 6, 0, 0, 0) + b(0, 0, -2, 1, 0) + c(0, 0, -3, 0, 1) = (0, 0, 0, 0, 0),$$

we have $a = 0$ (comparing z_1 components), $b = 0$ (comparing z_4 components), and $c = 0$ (comparing z_5 components). Thus, B is a basis for U.

(b) Extend your basis in part (a) to a basis for V.

We need to find two independent vectors that are not in the span of B. It's clear that one such vector is $(1, 0, 0, 0, 0)$. Now note that any vector of the form $(z, w, 0, 0, 0)$ is in the span of $(1, 0, 0, 0, 0)$ and $(1, 6, 0, 0, 0)$, but no vector $(0, 0, a, b, c)$ with any of $a, b, c \neq 0$ is. Thus, it suffices to find a vector $(0, 0, a, b, c)$ not in the span of $(0, 0, -2, 1, 0)$ and $(0, 0, -3, 0, 1)$, and one such vector is $(0, 0, 1, 0, 0)$.

To verify that this is a basis, we can either check that it spans (in which case it's a minimal spanning set) or that it is linearly independent (in which case it's a maximal independent set). Let's see that it spans. Given

$$\begin{aligned} v &= (z_1, z_2, z_3, z_4, z_5) \\ &= a(1, 6, 0, 0, 0) + b(1, 0, 0, 0, 0) + c(0, 0, 1, 0, 0) + d(0, 0, -2, 1, 0) + e(0, 0, -3, 0, 1), \end{aligned}$$

we must take $a = z_2/6$, in which case $b = z_1 - z_2/6$, $d = z_4$, $e = z_5$, and thus $c = z_3 + 2z_4 + 2z_5$.

(c) Find a subspace $W \subseteq V$ such that $V = U \oplus W$.

By our previous construction, we can take $W = \text{span}\{(1, 0, 0, 0, 0), (0, 0, 1, 0, 0)\}$, and we immediately have $V = U + W$. If $v \in U \cap W$, then $v = (s, 0, t, 0, 0) = (a, 6a, -2b - 3c, b, c)$. Comparing the 4th and 5th components gives $b = c = 0$, and comparing the second components gives $a = 0$ (and thus $s = t = 0$ as well). Thus $U \cap W = \{0\}$, so $V = U \oplus W$.

4. Prove or give a counterexample: if $\{v_1, v_2, v_3, v_4\}$ is a basis for V, and U is a subspace of V such that $v_1, v_2 \in U$, but $v_3 \notin U$ and $v_4 \notin U$, then $\{v_1, v_2\}$ is a basis for U.

This is false. Consider $\mathbb{R}^4$ with the standard basis. Let U be the span of $v_1 = (1, 0, 0, 0)$, $v_2 = (0, 1, 0, 0)$, and $w = v_3 + v_4 = (0, 0, 1, 1)$. Thus,

$$\begin{aligned} U &= \{x(1, 0, 0, 0) + y(0, 1, 0, 0) + z(0, 0, 1, 1) \mid x, y, z \in \mathbb{R}\} \\ &= \{(x, y, z, z) \mid x, y, z \in \mathbb{R}\}, \end{aligned}$$

from which it is clear that $v_3, v_4 \notin U$. However, $\{v_1, v_2\}$ cannot not a basis for U, since it contains only two vectors, and $\dim U = 3$.

5. Prove that if U and W are both 4-dimensional subspaces of $\mathbb{C}^6$, then $U \cap W$ contains at least two linearly independent vectors.

   We have the dimension formula

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

   Since $U + W \subseteq \mathbb{C}^6$ we have $\dim(U + W) \leq 6$. Thus,

$$\dim(U \cap W) = \dim U + \dim W - \dim(U + W) \geq 4 + 4 - 6 = 2.$$

   Therefore, the dimension of $U \cap W$ is at least 2, and the result follows.