MATH 2565 - Tutorial #2 Solutions

Additional practice problems:

1.

$$\begin{aligned}\int \sin^5(x)\cos^6(x)\,dx &= \int \sin(x)(1-\cos^2(x))^2\cos^6(x)\,dx\\ &= \int \sin(x)(\cos^6(x) - 2\cos^8(x) + \cos^{10}(x))\,dx,\end{aligned}$$

so letting $u = \cos(x)$, we have $du = \sin(x)\,dx$ and the integral becomes

$$\int(-u^6 + u^8 - u^{10})\,dx = -\frac{u^7}{7} + \frac{u^9}{9} - \frac{u^{11}}{11} + c = -\frac{1}{7}\cos^7(x) + \frac{1}{9}\cos^9(x) - \frac{1}{11}\cos^{11}(x) + C.$$

2. $\int \sin(x)\sin(2x)\,dx = \int \sin(x)(2\sin(x)\cos(x))\,dx = 2\int \sin^2(x)\cos(x)\,dx = \frac{2}{3}\sin^3(x) + C$, using the u-substitution $u = \sin(x)$, $du = \sin(x)\,dx$.

   (Could you do it by parts? Maybe. But why would you subject yourself to that when a quick application of a trig identity gives you a much easier integral?)

3. $\int \sqrt{9-x^2}\,dx$

   Here we use the trig substitution $x = 3\sin\theta$, which gives us $dx = 3\cos\theta\,d\theta$, and $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$, so $\sqrt{9-x^2} = 3\cos\theta$. Thus,

$$\begin{aligned}\int \sqrt{9-x^2}\,dx &= \int 9\cos^2\theta\,d\theta\\ &= \int \frac{9}{2}(1+\cos(2\theta))\,d\theta\\ &= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C\\ &= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta.\end{aligned}$$

   Now, we use the fact that $3\sin\theta = x$, so $\theta = \sin^{-1}(x/3)$, and $3\cos\theta = \sqrt{9-x^2}$ to substitute back in terms of x, giving us

$$\int \sqrt{9-x^2}\,dx = \frac{9}{2}\sin^{-1}\left(\frac{\theta}{3}\right) + \frac{1}{2}x\sqrt{9-x^2} + C.$$

4. $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$

There are two options for this integral. We can either let $x = \sqrt{2}\tan\theta$, or $x = \sqrt{2}\sinh(t)$. Taking the first option, we get $dx = \sqrt{2}\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+2} = \sqrt{2\tan^2\theta + 2} = \sqrt{2\sec^2\theta} = \sqrt{2}\sec\theta,$$

so

$$\int \frac{8}{\sqrt{x^2+2}}\,dx = \int \frac{8\sqrt{2}\sec^2\theta}{\sqrt{2}\sec\theta}\,d\theta = \int 8\sec\theta\,d\theta = 8\ln|\sec\theta + \tan\theta| + C.$$

Now, $\tan\theta = x/\sqrt{2}$, and $\sec\theta = \sqrt{x^2+2}/\sqrt{2}$, so this becomes

$$\int \frac{8}{\sqrt{x^2+2}}\,dx = 8\ln\left|\frac{\sqrt{x^2+2}+x}{\sqrt{2}}\right| + C.$$

(Note: using log laws, you can get rid of the $\sqrt{2}$ in the denominator – you'll get a $-\ln\sqrt{2}$ term, which is a constant that can be absorbed into the constant of integration.)

If we take the second option, $x = \sqrt{2}\sinh(t)$, so $dx = \sqrt{2}\cosh(t)$, and

$$\sqrt{x^2+2} = \sqrt{2\sinh^2(t)+2} = \sqrt{2\cosh^2(t)} = \sqrt{2}\cosh(t).$$

Thus,

$$\int \frac{8}{\sqrt{x^2+2}}\,dx = \int \frac{8\sqrt{2}\cosh(t)}{\sqrt{2}}\cosh(t)\,dt = 8t + C = 8\sinh^{-1}(x/\sqrt{2}) + C.$$

Exercise: Can you show that the answers given by the two methods are equivalent? In other words, is it true that

$$\ln\left|\frac{\sqrt{x^2+2}+x}{\sqrt{2}}\right| = \sinh^{-1}(x/\sqrt{2}) + C$$

for some constant C? (If it is, then the derivative of either side should be equal.)

5. $\displaystyle\int \frac{1-\tan^2(x)}{\sec^2(x)}\,dx$

Since

$$\frac{1-\tan^2(x)}{\sec^2(x)} = \cos^2(x)(1-\tan^2(x)) = \cos^2(x) - \sin^2(x) = \cos(2x),$$

we have

$$\int \frac{1-\tan^2(x)}{\sec^2(x)}\,dx = \int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C.$$

6. $\displaystyle\int \frac{dx}{\cos(x)-1}\,dx$

First we manipulate the integrand:

$$\frac{1}{\cos(x)-1} = \frac{cos(x)+1}{\cos^2(x)-1} = -\frac{\cos(x)+1}{\sin^2(x)} = -\frac{\cos(x)}{\sin^2(x)} - \csc^2(x).$$

Having written the integrand as a sum of two terms, we split up the integral into two pieces:

$$\int \frac{1-\tan^2(x)}{\sec^2(x)}\,dx = -\int \frac{\cos(x)}{\sin^2(x)}\,dx - \int \csc^2(x)\,dx.$$

The first integral can be evaluated using the substitution $u = \sin(x)$, $du = \cos(x)\,dx$, resulting in $\int \frac{-1}{u^2}\,du = \frac{1}{u} + C$; the second integral is immediate. We get

$$\int \frac{dx}{\cos(x)-1}\,dx = \frac{1}{\sin(x)} + \cot(x) + C.$$

Assigned problems: Evaluate the following integrals.

1. $\displaystyle\int \tan^4(x)\sec^6(x)\,dx$

We employ the identity

$$\sec^4(x) = (1+\tan^2(x))^2 = 1 + 2\tan^2(x) + \tan^4(x)$$

to obtain

$$\int \tan^4(x)\sec^6(x)\,dx = \int (\tan^4(x) + 2\tan^6(x) + \tan^8(x))\sec^2(x)\,dx$$

$$\frac{1}{5}\tan^5(x) + \frac{2}{7}\tan^7(x) + \frac{1}{9}\tan^9(x) + C.$$

2. $\displaystyle\int \tan^3(x)\sec^5(x)\,dx$

This time we use $\tan^2(x) = \sec^2(x) - 1$ to get

$$\tan^3(x)\sec^5(x) = (\tan^2(x)\sec^4(x))\sec(x)\tan(x) = (\sec^6(x) - \sec^4(x))\sec(x)\tan(x),$$

so

$$\int \tan^3(x)\sec^5(x)\,dx = \frac{1}{7}\sec^7(x) - \frac{1}{5}\sec^5(x) + C.$$

3. $\displaystyle\int \sin(8x)\cos(5x)\,dx$

Here we need the product-to-sum identity

$$\sin(8x)\cos(5x) = \frac{1}{2}[\sin(8x-5x) + \sin(8x+5x)] = \frac{1}{2}(\sin(3x) + \sin(13x)),$$

giving us

$$\int \sin(8x)\cos(5x)\,dx = -\frac{1}{6}\cos(6x) - \frac{1}{26}\cos(13x) + C.$$

4. $\displaystyle\int_0^{\pi/6} \sqrt{1+\cos(2x)}\,dx$

Recalling that $\cos(2x) = 2\cos^2(x) - 1$, we get

$$\int_0^{\pi/6} \sqrt{1+\cos(2x)}\,dx = \int_0^{\pi/6} \sqrt{2\cos^2(x)}\,dx = \sqrt{2}\int_0^{\pi/6} \cos(x)\,dx = \sqrt{2}/2.$$

(Note that $\cos(x) \geq 0$ on $[0, \pi/6]$, so $\sqrt{\cos^2(x)} = \cos(x)$ – there is no need to worry about the absolute value.)

5. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\,dx$, using a secant substitution.

As suggested, we let $x = \sqrt{10}\sec\theta$, so that $dx = \sqrt{10}\sec\theta\tan\theta\,d\theta$, and

$$\sqrt{x^2-10} = \sqrt{10(\sec^2\theta - 1)} = \sqrt{10\tan^2\theta} = \sqrt{10}\tan\theta.$$

Thus,

$$\int \frac{5x^2}{\sqrt{x^2-10}}\,dx = \int \frac{50\sec^2\theta}{\sqrt{10}\tan\theta}(\sqrt{10}\sec\theta\tan\theta)\,d\theta = \int 50\sec^3\theta\,d\theta.$$

Uh oh... the dreaded $\sec^3\theta$ integral. Luckily we have that answer in our notes, so we can plug it in, giving us

$$\int \frac{5x^2}{\sqrt{x^2-10}}\,dx = 25\sec\theta\tan\theta + 25\ln|\sec\theta + \tan\theta| + C,$$

and we note that $\sec\theta = x/\sqrt{10}$ and $\tan\theta = \sqrt{x^2-10}/\sqrt{10}$, so

$$\int \frac{5x^2}{\sqrt{x^2-10}}\,dx = \frac{5}{2}x\sqrt{x^2-10} + 25\ln|(x+\sqrt{x^2-10})/\sqrt{10}| + C.$$

6. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\,dx$, using a hyperbolic substitution.

If we use a hyperbolic substitution instead, we take $x = \sqrt{10}\cosh(t)$, so $dx = \sqrt{10}\sinh(t)$, and

$$\sqrt{x^2-10} = \sqrt{10(\cosh^2(t)-1)} = \sqrt{10\sinh^2(t)} = \sqrt{10}\sinh(t).$$

Thus,

$$\int \frac{5x^2}{\sqrt{x^2-10}}\,dx = \int \frac{50\cosh^2(t)}{\sqrt{10}\sinh(t)}(\sqrt{10})\sinh(t)\,dt = \int 50\cosh^2(t)\,dt.$$

Now we have to know how to integrate $\cosh^2(t)$. If we recall how $\cosh(t)$ is defined, we have

$$\cosh^2(t) = \left(\frac{e^t + e^{-t}}{2}\right)^2 = \frac{e^{2t} + e^{-2t} + 2}{4}.$$

So you could simply write $\cosh^2(t)$ in terms of exponentials as above, and integrate term-by-term. The other option is to notice that there's an identity sitting there: $\dfrac{e^{2t}+e^{-2t}}{4} = \frac{1}{2}\cosh(2t)$, so

$$\int \cosh^2(t)\,dt = \int \left(\frac{1}{2}\cosh(2t) + \frac{1}{2}\right)\,dt = \frac{1}{4}\sinh(2t) + \frac{t}{2} + C.$$

Finally, we have to substitute back in terms of x. Would it surprise you to learn that $\sinh(2t) = 2\sinh(t)\cosh(t)$? Well, that turns out to be true. Since $\sinh(t) = \sqrt{x^2-10}/\sqrt{10}$ and $\cosh(t) = x/\sqrt{10}$, we get

$$\int \cosh^2(t)\,dt = \frac{5}{2}x\sqrt{x^2-10} + 25\cosh^{-1}(x/\sqrt{10}) + C.$$

The last thing you might be wondering is whether the two answers are the same. They certainly look different. It's a good exercise to see if you can show that

$$\ln(x+\sqrt{x^2-10}) = \cosh^{-1}(x/\sqrt{10}) \text{ (up to a constant)}$$

The easiest way to do that is to show that their derivatives are the same.

Discussion problem (no submission required):
Prove the following formulas, where m and n are integers:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

Suppose a function f can be written as a *finite Fourier series*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n \sin(nx) + \sum_{m=1}^{M} b_m \cos(mx).$$

Show that the coefficients a_n $(n = 0, \ldots, N)$ and b_m $(m = 1, \ldots, M)$ are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, (i = 1, \ldots, N), b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx.$$

If you want to know how to do this problem, you can come "discuss" it with me during office hours.