Practice Problems for Quiz 12
Math 2000A

Quiz #12 will take place in class on Thursday, December 4th. It's the final quiz, unless you count the final exam (which is like a quiz, but lasts longer). As usual, solving the problems on this sheet will significantly improve your chances of getting a high score on the quiz.

Note to help session tutors: It's 100% OK for you to help my students solve these questions.

1. For which $n \in \mathbb{N}$ is it true that $n^2 < 2^n$? Justify your conclusion.

2. Define a sequence $a_1, a_2, a_3, \ldots$ of real numbers *recursively* by defining $a_1 = a$, and for each $n \in \mathbb{N}$,
$$a_{n+1} = r \cdot a_n,$$
where r is some fixed real number. Thus, $a_1 = a, a_2 = r \cdot a, a_3 = r \cdot a_2 = r \cdot (r \cdot a)$, etc. This sequence is known as a *geometric* sequence. Use mathematical induction to prove that $a_n = a \cdot r^{n-1}$ for all $n \in \mathbb{N}$.

3. Define a sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1, a_2 = 1$, and for each $n \in \mathbb{N}$,
$$a_{n+2} = \frac{1}{2}\left(a_{n+1} + \frac{2}{a_n}\right).$$
   (a) Calculate a_3 through to a_5. If you're feeling adventurous, calculate a_6.
   (b) Use the strong form of mathematical induction to prove that $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$.

4. Let $A = \{a, b, c\}$ and let $R = \{(a,a), (a,c), (b,b), (b,c), (c,a), (c,b)\}$ define a relation on A. Determine whether the following statements are true or false. Explain your answer.
   (a) For each $x \in A$, $x\,R\,x$.
   (b) For every $x, y \in A$, if $x\,R\,y$, then $y\,R\,x$.
   (c) For every $x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$.
   (d) The relation R defines a function from A to A.

5. Define a relation on $\mathbb{R}$ by $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}$.
   (a) What are the domain and range of R?
   (b) Is the relation R a function from R to R? Explain.

6. Let $A = \{a, b\}$, and consider the relations $R_1 = \{(a, a), (b, b)\}$ and $R_2 = \{(a, a), (a, b)\}$. Show that R_1 is an equivalence relation but R_2 is not. Is R_2 transitive?

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 4$, and define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $f(x) = f(y)$.

   (a) Is $\sim$ an equivalence relation? Justify your answer.

   (b) Determine the set of all real numbers x such that $x \sim 5$.

8. Define a relation $\sim$ on $\mathbb{Q}$ by $a \sim b$ if and only if $a - b \in \mathbb{Z}$. The *equivalence class* of an element $a \in \mathbb{Q}$ is the set $[a] = \{b \in \mathbb{Q} : b \sim a\}$.

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Prove that $\left[\frac{3}{5}\right] = \left\{\frac{3}{5} + m : m \in \mathbb{Z}\right\}$.

   (c) If $a \in \mathbb{Z}$, what is $[a]$?

   (d) Prove that for $a \in \mathbb{Z}$, there is a bijection from $[a]$ to $\left[\frac{3}{5}\right]$.