Change of Variables

Since we don't have enough time in class to go over all the details, (and I don't have time to type them) here are some hand-written notes on the change of variables formula.

Note: please see also the handout on properties of transformations, and the one with two change of variables examples.

We will mainly encounter transformations in the following context:

we want to integrate a function f over a closed, bounded region $B \subseteq \mathbb{R}^n$ $(n = 1, 2, 3, \dots)$ where either f or B makes the integral too difficult to compute by hand, so we find a region $A \subseteq \mathbb{R}^n$ and a function $T: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ so that

$$B = T(A) = \{\vec{x} = T(\vec{u}) \mid \vec{u} \in A\},$$

and (a) T is C^1

(b) T is one-to-one (it's onto if $B = T(A)$)

(c) $J_T(\vec{u}) \neq 0$, where J_T is the Jacobian of T.

((b) and (c) can fail on the boundary of A.)

The change of variables formula then tells us

$$\int_B f(\vec{x})\, d\vec{x} = \int_A f(T(\vec{u}))\, |J_T(\vec{u})|\, d\vec{u},$$

where ideally either A is simpler than B, or $f(T(\vec{u}))|J_T(\vec{u})|$ is simpler than $f(\vec{x})$.

(Here $\int_B f(\vec{x})d\vec{x}$ denotes, $\int_a^b f(x)dx$, $\iint_B f(x,y)dA$, $\iiint_B f(x,y,z)dV$, etc, depending on dimension)

Remark: Another, possibly more practical, reason for considering a change of variables is the following: two scientists performing the same experiment or calculation may choose to work in different coordinate systems, and want to be able to compare their resuts.

So, what is going on when we do a transformation?

Consider $n=1$ first. We have a function f, an interval $B=[a,b]$, and the integral $\int_a^b f(x)\,dx$.

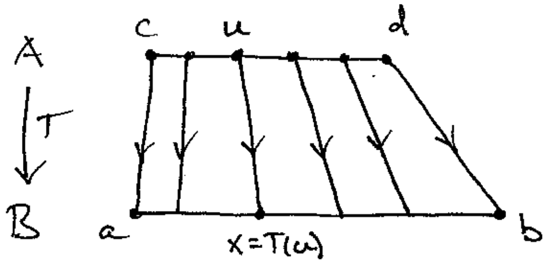
Here, a transformation is just a function

$$T:[c,d]\to[a,b].$$

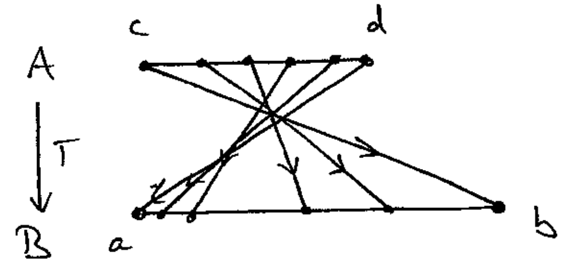
For $n=1$, condition (b) is unnecessary: if $J_T(u)=T'(u)\neq 0$ for all $u\in(c,d)$, and $T'(u)$ is continuous (T is C^1), then either $T'(u)>0$ on (c,d) (increasing)

or $T'(u)<0$ on (c,d) (decreasing)

so T is one-to-one.



$T'(u)>0$



$T'(u)<0$

We can think of T as "transforming" the interval $[c,d]$ into the interval $[a,b]$, as above. Note that if $T'(u)<0$, the interval gets "flipped over". If $T'(u)<0$, we have

$$|T'(u)| = -T'(u) = \frac{d}{du}\left(K-T(u)\right),$$ for some constant K,

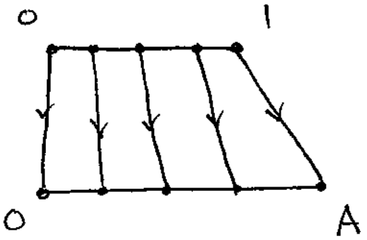
so we could always have $T'(u)>0$ (possibly with a new interval $[c,d]$).

The change of variables formula here is

$$\int_a^b f(x)\,dx = \int_c^d f(T(u))|T'(u)|\,du$$

You should think of the relationship $dx = T'(u)\,du$ as telling you how much each part of the interval $[c,d]$ gets stretched when it's transformed into $[a,b]$

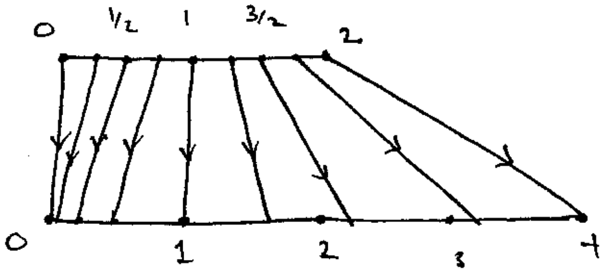
eg: $x = Au = T(u)$, $A>0$ a constant



- transformation is linear

Each piece Δu gets stretched by the same amount : $\Delta x = A\,\Delta u$ everywhere.

eg: $x = u^2 = T(u)$, $u \in [0,2]$



- non-linear transformation

$\Delta u_i = u_i - u_{i-1}$ stretched to

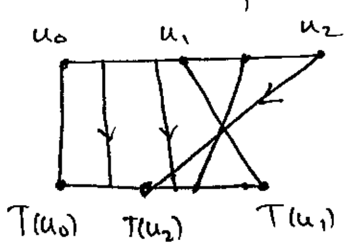
$$\Delta x_i = u_i^2 - u_{i-1}^2 = (u_i + u_{i-1})\,\Delta u_i$$

$$(u_{i-1} < u_i^* < u_i) \quad = 2u_i^*\,\Delta u_i$$

- each piece Δu_i gets streched by $2u_i^* = T'(u_i^*)$ (or shrunk, when $0 < T'(u) < 1$)

<u>Note</u>: If we had $T'(u) \geq 0$ for $u_0 \leq u \leq u_1$
and $T'(u) \leq 0$ for $u_1 \leq u \leq u_2$, with $T'(u_1) = 0$,

T would fold $[u_0, u_2]$ over on itself!



- This is bad behaviour : the points between $T(u_2)$ and $T(u_1)$ get counted twice.
- requiring $T'(u) \neq 0$ avoids this.

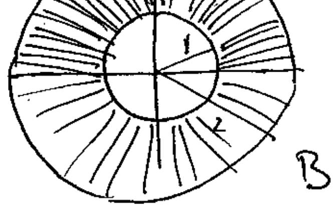
Now, consider $n = 2$. Here, $J_T(u,v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$,

and requiring $J_T(u,v) \neq 0$ no longer guarantees T is one-to-one.

eg: $A = \underset{r}{[1,2]} \times \underset{\theta}{[0, 3\pi]}$, $T(r,\theta) = (r\cos\theta, r\sin\theta)$

We have $J_T(r,\theta) = r > 0$, but the image of T

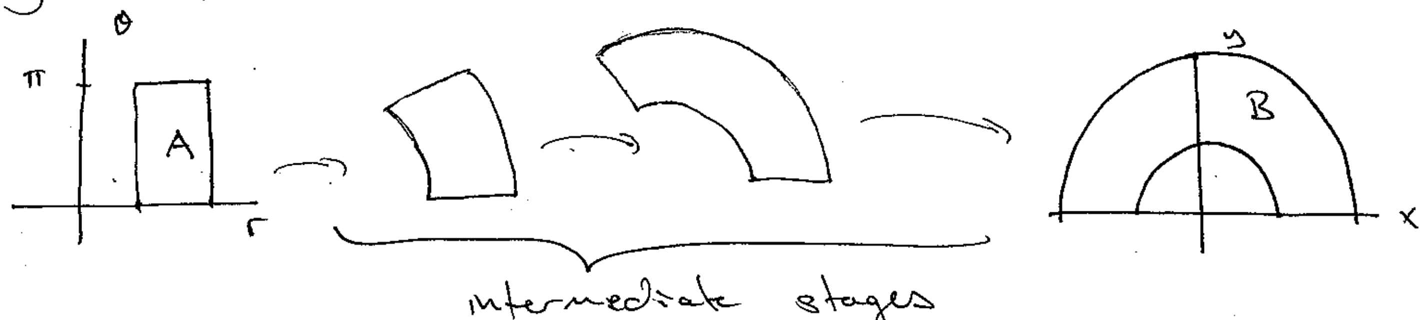
is



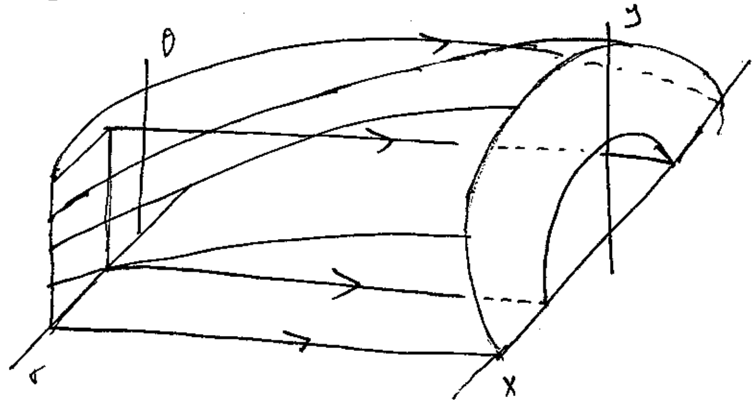
, with the upper half of the annulus counted twice.

If $J_T(u,v) > 0$, can think of the transformation as a movie

eg: $J_T(r,\theta) = (r\cos\theta, r\sin\theta)$, $1 \le r \le 2$, $0 \le \theta \le \pi$

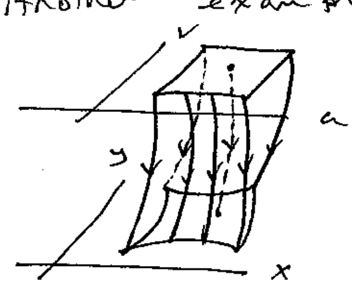


or as a 3-D solid with A at one end, and B at the other!



(This isn't so easy to picture, however!)

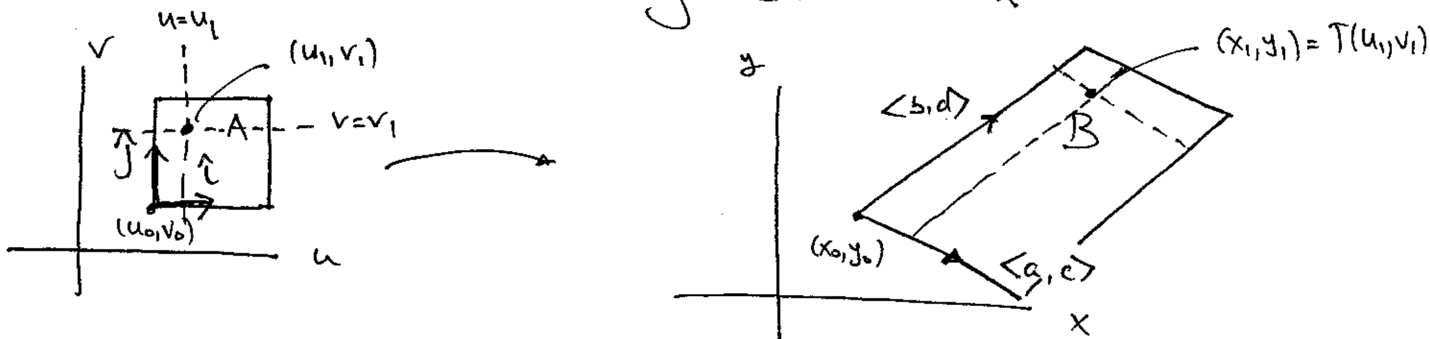
Another example:



However, if $J_T(u,v) < 0$, T "reverses orientation": as it transforms A into B, it flips it over (just like in the $n=1$ case). When this happens the picture becomes much harder to draw!

When $(x,y) = T(u,v)$, we think of u,v as "curvilinear coordinates". In the linear case things are simple :

$$x = au + bv + k$$
$$y = cu + dv + l$$



Here $(x_0,y_0) = T(u_0,v_0) = (au_0+bv_0,\ cu_0+dv_0)$

Note $\hat{\imath} = \langle (u_0+1)-u_0,\ v_0-v_0\rangle$

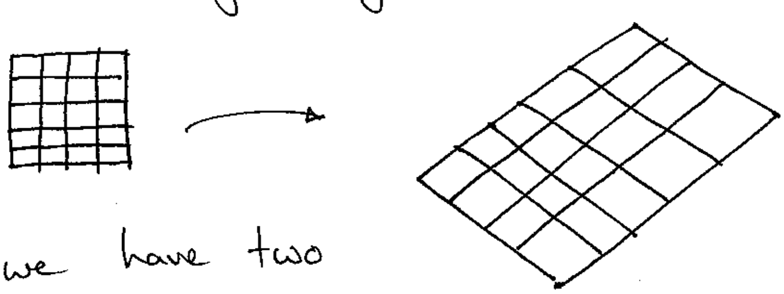
becomes $\vec{v} = \langle (a(u_0+1)+bv_0)-(au_0+bv_0),\ (c(u_0+1)+dv_0)-(cu_0+dv_0)\rangle$

$= \langle a, c\rangle$

and similarly $\hat{j}$ becomes $\langle b,d\rangle$.

(If you know matrix multiplication, $\begin{bmatrix} a \\ c\end{bmatrix} = \begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} 1 \\ 0\end{bmatrix}$ and $\begin{bmatrix} b \\ d\end{bmatrix} = \begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} 0 \\ 1\end{bmatrix}$.)

Note that a grid system in A becomes a grid system in B



- Here we have two observations :

1. Each rectangle becomes a parallelogram, with the <u>same scale factor</u> for each one: the overall scale factor $\det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = ad-bc$
2. At each point the vectors $\langle a,c\rangle$, $\langle b,d\rangle$ define a coordinate system.

In the non-linear case, just like for $n=1$, the scale factor varies from point to point.

Given $T(u,v) = (x(u,v), y(u,v))$, we set

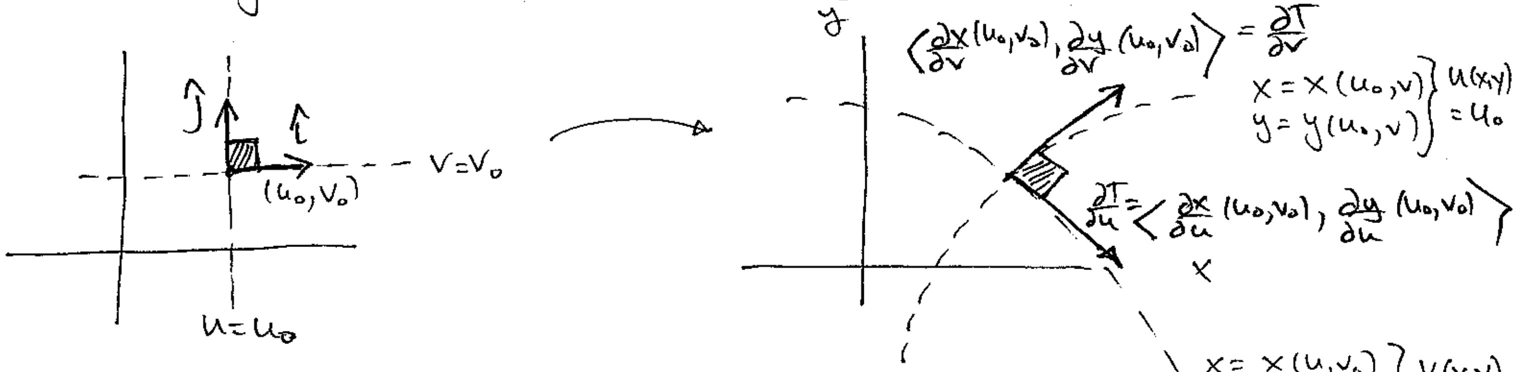
$$DT(u_0,v_0) = \begin{pmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \end{pmatrix}.$$

Since T is C^1, the corresponding linear transformation,

$$x = x_u(u_0,v_0)\,u + x_v(u_0,v_0)\,v + k$$
$$y = y_u(u_0,v_0)\,u + y_v(u_0,v_0)\,v + l$$

is the <u>best linear approximation</u> to T at (u_0,v_0).

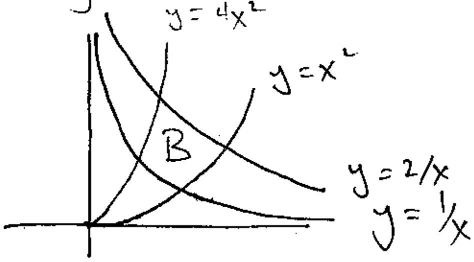
Also, the columns of $DT(u_0,v_0)$ give the tangent vectors in the xy-plane corresponding to $\hat{\imath}, \hat{\jmath}$ at (u_0,v_0):



– If $J_T(u_0,v_0) > 0$, the vectors $\frac{\partial T}{\partial u}, \frac{\partial T}{\partial v}$ have the same orientation as $\hat{\imath}, \hat{\jmath}$. If $J_T(u_0,v_0) < 0$, it's reversed.

– A small rectangle spanned by $\Delta u\,\hat{\imath}$, and $\Delta v\,\hat{\jmath}$, becomes a small parallelogram spanned by $\Delta u \frac{\partial T}{\partial u}(u_0,v_0)$, $\Delta v \frac{\partial T}{\partial v}(u_0,v_0)$, with area $|\det(DT(u_0,v_0))|\Delta u \Delta v = |J_T(u_0,v_0)|\Delta u \Delta v$, and for $\Delta u, \Delta v$ small enough, this is a good approximation to the area bounded by the level curves $u=u_0$, $u=u_0+\Delta u$, $v=v_0$, $v=v_0+\Delta v$, and this gives the change of variables formula.

eg: $B \subseteq \mathbb{R}^2$ bounded by $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = x^2$, $y = 4x^2$



Two families of curves:

$$y = \frac{m}{x}, \qquad y = nx^2$$
$$1 \le m \le 2 \qquad\qquad 1 \le n \le 4$$

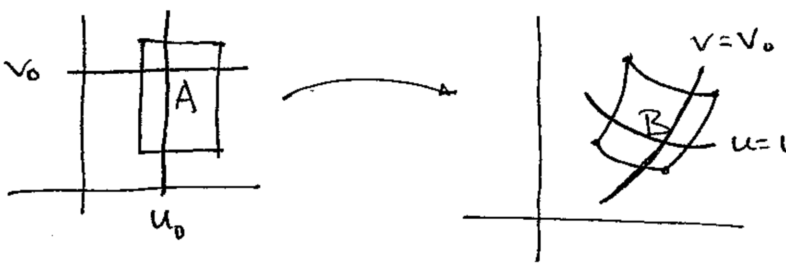
Let $u(x,y) = xy = m$, $\quad v(x,y) = \frac{y}{x^2} = n$

- Gives us $T^{-1}(x,y) \Rightarrow$ solve for x, y

$$y = vx^2 \Rightarrow u = xy = vx^3 \Rightarrow x = \left(\frac{u}{v}\right)^{1/3} = u^{1/3} v^{-1/3}$$

$$y = v \cdot u^{2/3} v^{-2/3} = u^{2/3} v^{1/3}$$

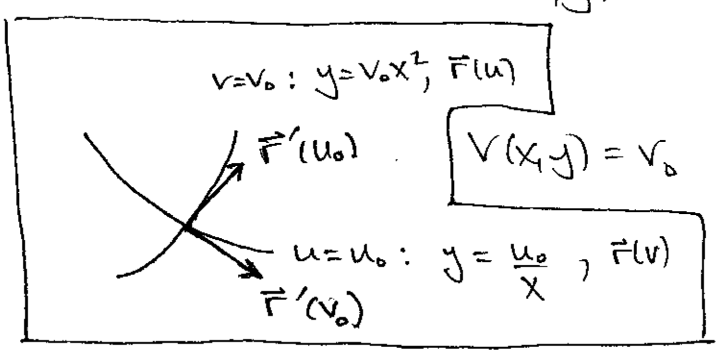
$$\therefore T(u,v) = \left(u^{1/3} v^{-1/3},\ u^{2/3} v^{1/3}\right),$$
$$1 \le u \le 2, \quad 1 \le v \le 4$$



Note: transformation dictated by curves defining the boundary of B.

Note level curves $u(x,y) = u_0 \longleftrightarrow \vec{r}(v) = \langle u_0^{1/3} v^{-1/3}, u_0^{2/3} v^{1/3} \rangle$

$$\vec{r}\,'(v_0) = \left\langle -\tfrac{1}{3} u_0^{1/3} v_0^{-4/3}, \tfrac{1}{3} u_0^{2/3} v_0^{-2/3} \right\rangle$$



$V(x,y) = v_0 \longleftrightarrow \vec{r}(u) = \langle u^{1/3} v_0^{-1/3}, u^{2/3} v_0^{1/3} \rangle$

$$\vec{r}\,'(u_0) = \left\langle \tfrac{1}{3} u_0^{-2/3} v_0^{-1/3}, \tfrac{2}{3} u_0^{-1/3} v_0^{1/3} \right\rangle$$

$$DT(u_0,v_0) = \begin{pmatrix} \frac{1}{3} u_0^{-2/3} v_0^{-1/3} & -\frac{1}{3} u_0^{1/3} v_0^{-4/3} \\ \frac{2}{3} u_0^{-1/3} v_0^{1/3} & \frac{1}{3} u_0^{2/3} v_0^{-2/3} \end{pmatrix}, \quad J_T(u_0,v_0) = \det(DT(u_0,v_0))$$
$$= \frac{1}{9} v_0^{-1} - \left(-\frac{2}{9} v_0^{-1}\right)$$
$$= \frac{1}{3v_0}$$

$$\uparrow \qquad\qquad \uparrow$$
$$\frac{\partial T}{\partial u} = \vec{r}\,'(u_0) \qquad \frac{\partial T}{\partial v} = \vec{r}\,'(v_0)$$

Notice: $J_{T^{-1}}(x,y) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$
$= y\left(\frac{1}{x^2}\right) - x\left(-\frac{2y}{x^3}\right) = \frac{3y}{x^2} = 3v$