Practice Problems for Quiz 6
Math 2000A

Quiz #6 will take place in class on Thursday, October 23rd.

Feel free to discuss the solutions to these quiz problems on Piazza. (As a reminder, this earns you participation credit.)

1. Determine if each of the assertions below is true or false. If the assertion is true, provide a proof. If not, provide a counterexample.
   (a) For all integers a, b, and c, with $a \neq 0$, if $a|b$ and $a|c$, then $a|(b-c)$.
   (b) For each $n \in \mathbb{Z}$, if n is odd, then n^3 is odd.
   (c) For all integers a, b, and c with $a \neq 0$, if $a|b$, then $a|(bc)$.
   (d) For every integer n, $4n^2 + 7n + 6$ is an odd integer.
   (e) For all integers a, b, and d with $d \neq 0$, if d divides both $a - b$ and $a + b$, then d divides a.
2. If x and y are integers, explain why $xy = 1$ implies either $x = 1$ or $x = -1$.
3. Is the following true or false? For all nonzero integers a and b, if $a|b$ and $b|a$, then $a = \pm b$.
4. Let a and b be integers. Prove that if $a \equiv 2 \pmod 3$ and $b \equiv 2 \pmod 3$, then $a+b \equiv 1 \pmod 3$ and $ab \equiv 1 \pmod 3$.
5. Are the following statements true or false? Justify your conclusions.
   (a) For each $a \in \mathbb{Z}$, if $a \equiv 2 \pmod 5$, then $a^2 \equiv 4 \pmod 5$.
   (b) For each $a \in \mathbb{Z}$, if $a^2 \equiv 4 \pmod 5$, then $a \equiv 2 \pmod 5$.
   (c) For each $a \in \mathbb{Z}$, $a \equiv 2 \pmod 5$ if and only if $a^2 \equiv 4 \pmod 5$.
6. Are the following statements true or false? Justify your conclusions. (For true statements, you might find that proof by contradiction is useful.)
   (a) For all integers a and b, if a is even and b is odd, then 4 does not divide $a^2 + b^2$.
   (b) For all integers a and b, if a is even and b is odd, then 6 does not divide $a^2 + b^2$.
   (c) For all integers a and b, if a is even and b is odd, then 4 does not divide $a^2 + 2b^2$.
   (d) For all integers a and b, if a is odd and b is odd, then 4 divides $a^2 + 3b^2$.

7. (a) Prove that for each $a \in \mathbb{Z}$, $a \not\equiv 0 \pmod 3$ if and only if $a^2 \equiv 1 \pmod 3$
   (b) Prove that for each $n \in \mathbb{N}$, $\sqrt{3n+2}$ is not a natural number.

8. Let A, B, and C be sets. Establish the following:
   (a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   (b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   (c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
   (d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$
   (e) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$
   (f) $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$
   (g) If $S \subseteq A$, then $S \times B \subseteq A \times B$
   (h) If $T \subseteq B$, then $A \times T \subseteq A \times B$

9. Let $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$.
   (a) Explain why $A \times B \neq B \times A$.
   (b) Explain why $(A \times B) \times C \neq A \times (B \times C)$.

10. Let $f : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^3 + 5x}{x}$, and let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = x^2 + 5$.
   (a) Calculate $f(2), f(-2), f(3)$, and $f(\sqrt{2})$.
   (b) Calculate $g(2), g(-2), g(3)$, and $g(\sqrt{2})$.
   (c) Is the function f equal to the function g? Explain.
   (d) If we let $h : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ be the function defined by $h(x) = x^2 + 5$, is the function h equal to f? Explain.