Math 2580 Assignment #8 Solutions
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1. Find the area of the portion of the hyperbolic paraboloid $z = xy$ that lies within the cylinder $x^2 + y^2 = 4$.

   Since the surface is a graph, we take x and y as parameters, and $\mathbf{r}(x, y) = \langle x, y, xy\rangle$, with $(x, y) \in D = \{(x, y) | x^2 + y^2 \leq 4\}$. We then have

$$\begin{aligned}
\mathbf{r}_x(x, y) &= \langle 1, 0, y\rangle \\
\mathbf{r}_y(x, y) &= \langle 0, 1, x\rangle \\
\mathbf{N}(x, y) &= \langle -y, -x, 1\rangle,
\end{aligned}$$

   so $\|\mathbf{N}(x, y)\| = \sqrt{x^2 + y^2 + 1}$ and the area is given by

$$\begin{aligned}
A &= \iint_S dS = \iint_D \|\mathbf{N}(x, y)\|\, dA = \iint_D \sqrt{x^2 + y^2 + 1}\, dA \\
&= \int_0^{2\pi} \int_0^2 \sqrt{r^2 + 1} r\, dr\, d\theta \\
&= \pi \int_1^5 u^{1/2}\, du = \frac{2\pi}{3}(5^{3/2} - 1).
\end{aligned}$$

   Note: We can set up the integral in x and y as above, and then convert to polar coordinates, but we can also use r and θ as parameters from the beginning. If we let $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2\sin\theta\cos\theta\rangle$. If you go ahead and compute $\mathbf{r}_r$ and $\mathbf{r}_\theta$ and take the cross product to get $\mathbf{N}$, you find

$$\mathbf{N}(r, \theta) = \langle -r^2\sin\theta, -r^2\cos\theta, r\rangle = r\langle -r\sin\theta, -r\cos\theta, 1\rangle = r\langle -y, x, 1\rangle$$

   so the normal vector for the parameters r and θ is equal to r times the normal vector we found above, and this r is exactly the one that appears above as the Jacobian of the polar coordinate transformation.

2. Evaluate the surface integral $\iint_S (x^2y + z^2)\, dS$, where S is the portion of the cylinder $x^2 + y^2 = 9$ that lies between the planes $z = 0$ and $z = 2$.

   Since our surface is a cylinder, a natural choice of parameters for the surface is to use cylindrical coordinates. We define

   $$\mathbf{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z\rangle,$$

   for $\theta \in [0, 2\pi]$ and $z \in [0, 2]$, and this gives us a parameterization of our cylinder. We then have

   $$\begin{aligned} \mathbf{r}_\theta(\theta, z) &= \langle -3\sin\theta, 3\cos\theta, 0\rangle \\ \mathbf{r}_z(\theta, z) &= \langle 0, 0, 1\rangle \\ \mathbf{N}(\theta, z) &= \langle 3\cos\theta, 3\sin\theta, 0\rangle. \end{aligned}$$

   (Note that the normal vector is indeed perpendicular to the cylinder at all points!) The length of the normal vector is simply $\|\mathbf{N}(\theta, z)\| = 3$, and in terms of this parameterization, we have

   $$x^2y + z^2 = 27\cos^2\theta\sin\theta + z^2,$$

   so

   $$\iint_S (x^2y + z^2)\, dS = \int_0^2 \int_0^{2\pi} (27\cos^2\theta\sin\theta + z^2)(3)\, d\theta\, dz = 2\pi \int_0^2 3z^2\, dz = 16\pi.$$

3. Evaluate the surface integral of the vector field $\mathbf{F}(x, y, z) = \langle y, z - y, x\rangle$ over the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

   This problem requires a bit of work, since our surface is a piecewise-smooth surface consisting of four planar triangles S_1, S_2, S_3, and S_4.

   - S_1 is the portion of the xy-plane given by $0 \le x \le 1$, $0 \le y \le 1 - x$, oriented upwards: With $\mathbf{r}(x, y) = \langle x, y, 0\rangle$ (note that $z = 0$ in the xy-plane) we find $\mathbf{N}_1(x, y) = \mathbf{i} \times \mathbf{j} = \mathbf{k}$.
   - S_2 is the portion of the yz-plane given by $0 \le y \le 1$, $0 \le z \le 1 - y$, oriented in the positive x-direction: With $\mathbf{r}(y, z) = \langle 0, y, z\rangle$, we find $\mathbf{N}_2(x, y) = \mathbf{j} \times \mathbf{k} = \mathbf{i}$.
   - S_3 is the portion of the xz-plane given by $0 \le z \le 1$, $0 \le x \le 1 - z$, oriented in the negative y-direction: With $\mathbf{r}(x, z) = \langle x, 0, z\rangle$, we find $\mathbf{N}_3(x, y) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$.
   - S_4 is diagonal face of the tetrahedron, which lies in the plane $x + y + z = 1$. If we view this as the graph $z = 1 - x - y$, with $0 \le x \le 1$ and $0 \le y \le 1 - x$ (the same region as S_1), we have $\mathbf{r}(x, y) = \langle x, y, 1 - x - y\rangle$, so $\mathbf{N}_4(x, y) = (\mathbf{i} - \mathbf{k}) \times (\mathbf{j} - \mathbf{k}) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

   Next, we compute the integral of $\mathbf{F}$ over each surface.

On S_1, we have $\mathbf{F}(x,y,0)\cdot\mathbf{N}_1(x,y) = \langle y, 0-y, x\rangle\cdot\mathbf{k} = x$, so

$$\iint_{S_1} \mathbf{F}\cdot\, d\mathbf{S} = \int_0^1\int_0^{1-x} x\,dy\,dx = \int_0^1 (x-x^2)\,dx = \frac{1}{6}.$$

On S_2, we have $\mathbf{F}(0,y,z)\cdot\mathbf{N}_2(y,z) = \langle y, z-y, 0\rangle\cdot\mathbf{i} = y$, so

$$\iint_{S_2} \mathbf{F}\cdot\, d\mathbf{S} = \int_0^1\int_0^{1-y} y\,dz\,dy = \int_0^1 (y-y^2)\,dy = \frac{1}{6}.$$

On S_3, we have $\mathbf{F}(x,0,z)\cdot\mathbf{N}_3(x,z) = \langle 0, z, x\rangle\cdot(-\mathbf{j}) = -z$, so

$$\iint_{S_3} \mathbf{F}\cdot\, d\mathbf{S} = \int_0^1\int_0^{1-z} (-z)\,dx\,dz = \int_0^1 (z^2-z)\,dz = -\frac{1}{6}.$$

On S_4, we have $\mathbf{F}(x,y,1-x-y)\cdot\mathbf{N}_4(x,y) = \langle y, 1-x-2y, x\rangle\cdot\langle 1,1,1\rangle = 1-y$, so

$$\iint_{S_4} \mathbf{F}\cdot\, d\mathbf{S} = \int_0^1\int_0^{1-y} (1-y)\,dx\,dy = \int_0^1 (1-2y+y^2)\,dy = \frac{1}{3}.$$

Now, the overall surface (the tetrahedron) is a closed surface, and by default we give it the outward orientation. To have an outward-pointing normal vector at all points, we need to reverse the orientations for S_1 and S_2 (we want S_1 oriented downwards, and S_2 oriented in the negative x-direction). The total surface is therefore

$$S = -S_1 - S_2 + S_3 + S_4,$$

so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_3}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = -\frac{1}{6}-\frac{1}{6}-\frac{1}{6}+\frac{1}{3} = -\frac{1}{6}.$$

Note: The point of this exercise was, of course, to do the whole thing by hand. Having done so, we can now appreciate the usefulness of the Divergence Theorem: since S is a closed surface, we have

$$\iint_S \mathbf{F}\cdot\, d\mathbf{S} = \iiint_E (\nabla\cdot\mathbf{F})\,dV,$$

where E is the region bounded by the tetrahedron, and $\nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(y) + \dfrac{\partial}{\partial y}(z-y) + \dfrac{\partial}{\partial z}(x) = -1$, and since the volume of a tetrahedron with height $h=1$ and base area $A = \frac{1}{2}(1)(1) = \frac{1}{2}$ is $V = \frac{1}{3}Ah = \frac{1}{6}$, we have

$$\iint_S \mathbf{F}\cdot\, d\mathbf{S} = \iiint_E (-1)\,dV = -V = -\frac{1}{6}.$$

4. Use Stokes' theorem to calculate the integral of $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ around the triangle C with vertices $(2, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 3)$.

The curve C is the boundary of the surface S given by the part of the plane $3x + 6y + 2z = 6$ that lies in the first octant. We can describe S as the graph $z = 3 - \frac{3}{2}x - 3y$, where (x, y) belongs to the region D in the (x, y)-plane bounded by the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 1)$. Thus, we parameterize S by

$$\mathbf{r}(x, y) = \langle x, y, 3 - \frac{3}{2}x - 3y\rangle,$$

with $(x, y) \in D$. As a Type 1 region, D is given by $0 \leq x \leq 2$, with $0 \leq y \leq 1 - \frac{1}{2}x$; as a Type 2 region, D is given by $0 \leq y \leq 1$, and $0 \leq x \leq 2 - 2y$. For this parameterization, we find that $\mathbf{N}(x, y) = \langle \frac{3}{2}, 3, 1\rangle$. (This is the normal vector we can read off from the equation of the plane, scaled by one half, since we divided by 2 to solve for z.)

Next, we compute

$$\nabla \times \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k},$$

so $\nabla \times \mathbf{F}(\mathbf{r}(x, y)) = \langle -y, \frac{3}{2}x + 3y - 3, -x\rangle$. Stokes' theorem then gives us

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
&= \int_0^1 \int_0^{2-2y} \langle -y, \frac{3}{2}x + 3y - 3, -x\rangle \cdot \langle \frac{3}{2}, 3, 1\rangle \, dx\, dy \\
&= \int_0^1 \int_0^{2-2y} \left(\frac{7}{2}x + \frac{15}{2}y - 9\right) dx\, dy \\
&= \int_0^1 \left(\frac{7}{4}(2 - 2y)^2 + \frac{15}{2}y(2 - 2y) - 9(2 - 2y)\right) dy \\
&= -\frac{25}{6}.
\end{aligned}$$

5. Calculate the line integral in Problem 4 directly to verify that Stokes' theorem holds in this case.

The curve C is piecewise smooth, consisting of three line segments C_1, C_2, and C_3. The line segment C_1 from $(2, 0, 0)$ to $(0, 1, 0)$ can be parameterized using

$$\mathbf{r}_1(t) = \langle 2 - 2t, t, 0\rangle, \quad t \in [0, 1].$$

The line segment C_2 from $(0, 1, 0)$ to $(0, 0, 3)$ can be parameterized using

$$\mathbf{r}_2(t) = \langle 0, 1 - t, 3t\rangle, \quad t \in [0, 1].$$

The line segmenet C_3 from $(0,0,3)$ to $(2,0,0)$ can be parameterized using

$$\mathbf{r}_3(t) = \langle 2t, 0, 3-3t\rangle, \quad t \in [0,1].$$

We then have

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot dr \\
&= \int_0^1 \langle (2-2t)t, 0, 0\rangle \cdot \langle -2, 1, 0\rangle \, dt \\
&\qquad + \int_0^1 \langle 0, (1-t)(3t), 0\rangle \cdot \langle 0, 1, 3\rangle \, dt \\
&\qquad\qquad + \int_0^1 \langle 0, 0, 2t(3-3t)\rangle \cdot \langle 2, 0, 3\rangle \, dt \\
&= \int_0^1 (4t^2 - 4t)\, dt + \int_0^1 (3t^2 - 3t)\, dt + \int_0^1 (18t^2 - 18t)\, dt \\
&= \int_0^1 (25t^2 - 25t)\, dt = -\frac{25}{6}.
\end{aligned}$$

6. Use Stokes' theorem to calculate the integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F}(x,y,z) = \mathbf{r} \times (\mathbf{i} + \mathbf{j})$, and S is the portion of the sphere $x^2 + y^2 + z^2 = 9$ where $x + y \geq 1$.

   Note: Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in the definition of $\mathbf{F}$. You can try to use Stokes' theorem directly, and integrate $\mathbf{F}$ around the boundary of S, but for best results, use Stokes' theorem indirectly: by applying Stokes' theorem twice, you can replace the original surface S by a simpler surface that shares the same boundary.

We first compute $\mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 1 & 1 & 0 \end{vmatrix} = \langle -z, z, x-y\rangle$, so

$$\nabla \times \mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -z & -z & x-y \end{vmatrix} = \langle -2, -2, 0\rangle = -2(\mathbf{i} + \mathbf{j}).$$

Now the curve of intersection C of the sphere $x^2 + y^2 + z^2 = 9$ and $x + y = 1$ is the boundary of the given surface S, but it is also the boundary of the portion of the plane $x + y = 1$ bounded by C. Let's call this portion S'. We'll show how to obtain the result in two ways: as a line integral around C, and also as a surface integral over S'. (We can replace S by S' since the two surfaces share the same oriented boundary.)

If we choose to integrate over S', then we treat the plane $x + y = 1$ as the graph $x = 1 - y$, and use the parameterization $\mathbf{r}(y,z) = \langle 1-y, y, z\rangle$. Determining the parameter domain D for y and z takes a bit of work. If we substitute $x = 1 - y$ into the equation of the sphere, we have

$$(1-y)^2 + y^2 + z^2 = 1 - 2y + y^2 + y^2 + z^2 = 2y^2 - 2y + z^2 + 1 = 9,$$

and completing the square in y gives us

$$2(y - \frac{1}{2})^2 + z^2 = \frac{17}{2}, \qquad \text{or} \qquad \frac{(y-1/2)^2}{17/4} + \frac{z^2}{17/2} = 1,$$

which we recognize as the equation of an ellipse in the (y, z)-plane. The desired curve C is obtained by letting $x = 1 - y$, where y and z satisfy the equation above.

If we want to compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, we need to parameterize C. If we let $a = \dfrac{\sqrt{17}}{2}$ and $b = \sqrt{\dfrac{17}{2}}$, then our ellipse is $\dfrac{(y-1/2)^2}{a^2} + \dfrac{z^2}{b^2} = 1$, which we can parametrize using $y = \dfrac{1}{2} + a\cos t$, $z = b\sin t$, with $t \in [0, 2\pi]$, so with $x = 1 - y$ we have

$$\mathbf{r}(t) = \langle \frac{1}{2} - a\cos t, \frac{1}{2} + a\cos t, b\sin t\rangle, \quad \mathbf{r}'(t) = \langle a\sin t, -a\sin t, b\cos t\rangle,$$

and $\mathbf{F}(\mathbf{r}(t)) = \langle -b\sin t, b\sin t, -2a\cos t\rangle$, so

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -b\sin t, b\sin t, -2a\cos t\rangle \cdot \langle a\sin t, -a\sin t, b\cos t\rangle \, dt \\
&= \int_0^{2\pi} (-ab\sin^2 t - ab\sin^2 t - 2ab\cos^2 t) \, dt \\
&= \int_0^{2\pi} (-2ab) \, dt = -4\pi ab,
\end{aligned}$$

and putting in our values of a and b gives us $-4\pi ab = -17\sqrt{2}\pi$ for the result.

If we want to compute the integral $\iint_{S'} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, we note that the parameter domain D for the parameterization $\mathbf{r}(y, z)$ above is given by the region in the (y, z) plane bounded by the above ellipse. The normal vector is $\mathbf{N}(y, z) = \mathbf{i} + \mathbf{j}$ (constant, since our surface lies in a plane), and we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S'} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D (-2(\mathbf{i} + \mathbf{j})) \cdot (\mathbf{i} + \mathbf{j}) \, dA = -4 \iint_D dA,$$

so the value of the integral is $-4A(D)$, where the area $A(D)$ is given by $A(D) = \pi ab = \pi\frac{17}{2\sqrt{2}}$, using the formula for the area of an ellipse, and again we obtain the result $-17\sqrt{2}\pi$.