Math 1410 Assignment #3 Solutions
University of Lethbridge, Spring 2015

Sean Fitzpatrick

March 12, 2015

1. Given a polynomial $p(x) = a + bx + cx^2 + dx^3 + x^4$, the matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & -c & -d \end{bmatrix}$$

is called the *companion matrix* of $p(x)$. Show that $\det(xI_4 - C) = p(x)$.

Solution: We have that

$$xI_4 - C = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & -c & -d \end{bmatrix} = \begin{bmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a & b & c & x+d \end{bmatrix}.$$

Thus, using a cofactor expansion along the first row, we have

$$\begin{aligned}
\det(xI_4 - C) &= x\begin{vmatrix} x & -1 & 0 \\ 0 & x & -1 \\ b & c & x+d \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 & 0 \\ 0 & x & -1 \\ a & c & x+d \end{vmatrix} \\
&= x^2\begin{vmatrix} x & -1 \\ c & x+d \end{vmatrix} + x\begin{vmatrix} 0 & -1 \\ b & x+d \end{vmatrix} + a\begin{vmatrix} -1 & 0 \\ x & -1 \end{vmatrix} \\
&= x^2(x^2 + dx + c) + xb + a \\
&= a + bx + cx^2 + dx^3 + x^4 = p(x).
\end{aligned}$$

(You could also simplify the determinant using row/column operations if you prefer.)

2. If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ is a polynomial of degree k (the degree of $p(x)$ is the highest power of x, so we're assuming that $a_k \neq 0$). Given any such polynomial $p(x)$ and any $n \times n$ (square) matrix A, it's possible to plug A into the polynomial to obtain a *new* matrix, denoted $p(A)$, given by

$$p(A) = a_0I_n + a_1A + a_2A^2 + \cdots + a_kA^k.$$

For example, if $p(x) = 2 - 3x + x^2$, then $p(A) = 2I_n - 3A + A^2$.

(a) If $p(x) = 3 - 4x + 2x^2$ and $A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$, compute $p(A)$.

Solution: Since $A^2 = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -9 \\ 0 & 4 \end{bmatrix}$, we have

$$p(A) = 3I - 4A + 2A^2 = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & -9 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 0 & 3 \end{bmatrix}.$$

(b) The *characteristic polynomial* of an $n \times n$ matrix A is defined by

$$c_A(x) = \det(xI_n - A).$$

The *Cayley-Hamilton Theorem* is a famous theorem in linear algebra which states that for any $n \times n$ matrix A, $c_A(A) = 0$ (where the zero on the right is the zero matrix). Verify that the Cayley-Hamilton Theorem is true for $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$.

Solution: For $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ we have

$$xI_2 - A = \begin{bmatrix} x-3 & -2 \\ -1 & x+1 \end{bmatrix},$$

so

$$c_A(x) = \det(xI_2 - A) = (x-3)(x+1) - 2 = x^2 - 2x - 5.$$

We check that $A^2 = \begin{bmatrix} 11 & 4 \\ 2 & 3 \end{bmatrix}$; thus,

$$c_A(A) = A^2 - 2A - 5I_2 = \begin{bmatrix} 11 & 4 \\ 2 & 3 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

as required.

Bonus opportunity: Prove the Cayley-Hamilton Theorem for the $n = 2$ case. That is, show that the theorem holds for a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$c_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = (x-a)(x-d) - bc = x^2 - (a+d)x + (ad - bc).$$

Note that $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix}$, so we have

$$\begin{aligned} c_A(x) &= A^2 - (a+d)A + (ad-bc)I_2 \\ &= \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+dc & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which shows that the theorem is true for any 2×2 matrix.

3. In each case, either explain why the statement is true (in general), or give an example showing that it is false:

   (a) If $\|\vec{v} - \vec{w}\| = 0$, then $\vec{v} = \vec{w}$.

   Solution: This is true. Let $\vec{v} = \langle v_1, \ldots, v_n\rangle$ and $\vec{w} = \langle w_1, \ldots, w_n\rangle$. Then

   $$\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n\rangle,$$

   so if

   $$\|\vec{v} - \vec{w}\| = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2 + \cdots + (v_n - w_n)^2} = 0,$$

   then we must have $v_1 - w_1 = 0, v_2 - w_2 = 0, \ldots, v_n - w_n = 0$, since the only value of x for which $\sqrt{x} = 0$ is $x = 0$, and a sum of squares is zero if and only if each one of the squares is zero (since the square of a real number can't be negative). Thus, $v_i = w_i$ for $i = 1, 2, \ldots, n$, which implies that $\vec{v} = \vec{w}$.

   (b) If $\vec{v} = -\vec{v}$, then $\vec{v} = \vec{0}$.

   Solution: This is true. Given $\vec{v} = -\vec{v}$, we can add $\vec{v}$ to both sides of the equation, giving us $2\vec{v} = \vec{0}$. If we now multiply both sides by $\frac{1}{2}$, we're left with $\vec{v} = \vec{0}$.

   (c) If $\|\vec{v}\| = \|\vec{w}\|$, then $\vec{v} = \vec{w}$.

   Solution: This is false. For example if $\vec{v} = \langle 1, 0\rangle$ and $\vec{w} = \langle 0, 1\rangle$, then $\vec{v} \neq \vec{w}$, but $\|\vec{v}\| = \|\vec{w}\| = 1$.

   (d) If $\|\vec{v}\| = \|\vec{w}\|$, then $\vec{v} = \pm\vec{w}$.

   Solution: This is also false, and the previous counterexample can be applied here as well.

   (e) $\|\vec{v} + \vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$.

   Solution: This is false. If we take our two vectors in the solution to part (c), then we see that $\vec{v} + \vec{w} = \langle 1, 1\rangle$, so $\|\vec{v} + \vec{w}\| = \sqrt{2}$, while $\|\vec{v}\| + \|\vec{w}\| = 1 + 1 = 2$.

4. Let $\vec{u} = \begin{bmatrix} 3 & -1 & 0 \end{bmatrix}^T, \vec{v} = \begin{bmatrix} 4 & 0 & 1 \end{bmatrix}^T$, and $\vec{w} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. In each case, either find scalars a, b, c such that $\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$, or explain why no such scalars exist:

   (a) $\vec{x} = \begin{bmatrix} 5 & 1 & 2 \end{bmatrix}^T$

   If $\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$, we obtain the vector equation

$$a \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$$

   which leads to the system of equations

$$\begin{array}{rcrcrcr} 3a & + & 4b & + & c & = & 5 \\ -a & & & + & c & = & 1 \\ & + & b & + & c & = & 2 \end{array}$$

   The general solution to this system (found, as usual, by setting up and reducing the augmented matrix of the system) is given by

$$\begin{aligned} a &= -1 + t \\ b &= 2 - t \\ c &= t, \end{aligned}$$

   where t can be any real number. In particular, we see that for $t = 0$, we have $a = -1$, $b = 2$, and $c = 0$ and we verify that

$$-\vec{u} + 2\vec{v} + 0\vec{w} = (-1) \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = \vec{x},$$

   as required.

   (b) $\vec{x} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$.

   The setup here is the same as in part (a), and yields the system of equations

$$\begin{array}{rcrcrcr} 3a & + & 4b & + & c & = & 1 \\ -a & & & + & c & = & 2 \\ & + & b & + & c & = & 1 \end{array}.$$

   (Note that the only change is to the constants on the right-hand side given by the vector $\vec{x}$.) This time, if we set up and reduce our augmented matrix, we end up with the row-echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7/4 \\ 0 & 0 & 0 & -3/4 \end{array}\right],$$

   and the last row tells us that the system must be inconsistent, since $0 = -3/4$ is impossible. Thus, in this case there can be no values of a, b and c that satisfy the given vector equation.