MATH 1560 - Tutorial #3 Solutions

Additional practice problems:

1. Compute the derivatives of the following functions using the product rule:

   (a) $f(x) = x^2\cos(x)$, $f'(x) = 2x\cos(x) - x^2\sin(x)$

   (b) $g(x) = \sec(x)\tan(x)$, $g'(x) = (\sec(x)\tan(x))\tan(x) + \sec(x)(\sec^2(x)) = \sec(x)\tan^2(x) + \sec^3(x)$.

   (c) $h(x) = \sqrt{x}(x^2+1)$, $h'(x) = \frac{1}{2}x^{-1/2}(x^2+1) + \sqrt{x}(2x)$.
   Alternatively, since $h(x) = x^{1/2}(x^2+1) = x^{5/2} + x^{1/2}$, we have $h'(x) = \frac{5}{2}x^{3/2} + \frac{1}{2}x^{-1/2}$. These answers agree, since

$$\frac{1}{2}x^{-1/2}(x^2+1) + \sqrt{x}(2x) = \frac{1}{2}x^{3/2} + \frac{1}{2}x^{-1/2} + 2x^{3/2} = \frac{5}{2}x^{3/2} + \frac{1}{2}x^{-1/2}.$$

2. Compute the derivatives of the following functions using the quotient rule:

   (a) $f(x) = \dfrac{\sin(x)}{x^2+1}$, $f'(x) = \dfrac{\cos(x)(x^2+1) - 2x\sin(x)}{(x^2+1)^2}$.

   (b) $g(x) = \dfrac{x^3 - 2x^2 + 5x}{x^4 + e^x}$, $g'(x) = \dfrac{(3x^2 - 4x + 5)(x^4 + e^x) - (x^3 - 2x^2 + 5x)(4x^3 + e^x)}{(x^4 + e^x)^2}$.

   (c) $h(x) = \dfrac{x^8 + \sqrt[3]{x}}{x^3}$, $h'(x) = \frac{(8x^7 + \frac{1}{3}x^{-2/3})(x^3) - 3x^2(x^8 + \sqrt[3]{x})}{x^6}$.
   Alternatively, we can divide both terms in the numerator by x^3, giving us $h(x) = x^5 + x^{-8/3}$ (note $\sqrt[3]{x} = x^{1/3}$ and $1/3 - 3 = 1/3 - 9/3 = -8/3$), so $h'(x) = 5x^4 - \frac{8}{3}x^{-11/3}$. I'll leave it as an exercise to confirm that result is equivalent to the one above.

Assigned problems

1. Let $f(x) = \dfrac{1}{\sqrt{x}}$. Compute $f'(1)$ **using the definition of the derivative**.

   (Note: in the definition, one can set $x = 1$ at the very beginning, or at the end. Which is going to be less work?)

   We have

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} && \text{(definition of the derivative)}\\
&= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{1+h}} - 1}{h} && \text{(putting in the function; note } f(1)=1\text{)}\\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1-\sqrt{1+h}}{\sqrt{1+h}}\right) && \text{(common denominator)}\\
&= \lim_{h\to 0} \frac{1-(1+h)}{h\sqrt{1+h}(1+\sqrt{1+h})} && \text{(rationalizing the numerator)}\\
&= \lim_{h\to 0} \frac{-h}{h\sqrt{1+h}(1+\sqrt{1+h})} && \text{(simplifying the numerator)}\\
&= \lim_{h\to 0} \frac{-1}{\sqrt{1+h}(1+\sqrt{1+h})} && \text{(cancelling } h \text{ top and bottom)}\\
&= \frac{-1}{\sqrt{1}(1+\sqrt{1})} = -\frac{1}{2}
\end{aligned}
$$

   If you decided to do everything in terms of x (perhaps because it would be good practice in case a future question asked for $f'(x)$ instead of $f'(1)$, your work would look like the following:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}}{h}\\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right)\\
&= \lim_{h\to 0} \frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\\
&= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\\
&= \lim_{h\to 0} \frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\\
&= \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x}+\sqrt{x})} = -\frac{1}{2(\sqrt{x})^3}.
\end{aligned}
$$

If we now wanted to know the value of $f'(1)$, we would have $f'(1) = -\dfrac{1}{2(\sqrt{1})^3} = -\dfrac{1}{2}$.

2. Compute the derivative of $f(x) = 6\sqrt{x^3} - 7\sin(x) + 4e^x + \pi$. Noting that $\sqrt{x^3} = x^{3/2}$, we have

$$f'(x) = 9x^{1/2} - 7\cos(x) + 4e^x.$$

3. Compute the derivative of $f(x) = x^3(x-1)^2$ by (a) using the product rule, and (b) first multiplying everything out. Confirm that your answers agree. Which method do you prefer? (What if you need to find where $f'(x) = 0$?)

   (a) Using the product rule,

   $$f'(x) = 3x^2(x-1)^2 + 2x^3(x-1).$$

   (b) Multiplying things out,

   $$f(x) = x^3(x^2 - 2x + 1) = x^5 - 2x^4 + x^3, \text{ so } f'(x) = 5x^4 - 8x^3 + 3x^2.$$

   Both methods take about the same amount of work. However, if we want to solve the equation $f'(x) = 0$, we need to first factor $f'(x)$. In the first case, we can easily identify the common factors x^2 and $x - 1$, so

   $$f'(x) = x^2(x-1)(3(x-1) + 2x) = x^2(x-1)(5x-3),$$

   and we find that $f'(x) = 0$ for $x = 0, 1$, or $3/5$. In the second case, the common factor of x^2 is present; factoring it out gives us $f'(x) = x^2(5x^2 - 8x + 3)$, and it remains to factor the quadratic. Of course, we know from the above that it must factor as $5x^2 - 8x + 3 = (5x-3)(x-1)$, but that might not be so obvious otherwise.

4. Compute the derivative of $g(x) = \dfrac{x^7 - 3x^5 + 2x}{x^3}$ by
(a) using the quotient rule, and (b) simplifying first. Which method do you prefer?

   (a) Applying the quotient rule directly,

   $$g'(x) = \frac{(7x^6 - 15x^4 + 2)x^3 - (x^7 - 3x^5 + 2x)(3x^2)}{x^6}.$$

   (b) If we first divide each term by x^3, we get

   $$g(x) = x^4 - 3x^2 + 2x^{-2}, \text{ so } g'(x) = 4x^3 - 6x - 4x^{-3}.$$

   (Personally, I think I'd rather do the second approach. I'll leave it to you to simplify the answer in part (a) and confirm that it matches the one in part (b).)

5. For the following functions, find all values of x such that $f'(x) = 0$:

(a) $f(x) = x^5 - 15x^3$

We have

$$f'(x) = 5x^4 - 45x^2 = 5x^2(x^2 - 9) = 5x^2(x-3)(x+3),$$

so $f'(x) = 0$ for $x = 0, 3$ and -3.

(b) $f(x) = x^{5/3} - 5x^{2/3}$

Since

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2),$$

we have $f'(x) = 0$ when $x = 2$. Note that $x^{-1/3} \cdot x = x^{1-1/3} = x^{2/3}$, which is how we factored out the $x^{-1/3}$ above. Don't like this factoring approach? You could also proceed as follows:

$$\begin{aligned}
f'(x) &= \frac{5}{3}\left(x^{2/3} - \frac{2}{x^{1/3}}\right) && \text{(since } x^{-1/3} = 1/x^{1/3}\text{)} \\
&= \frac{5}{3}\left(\frac{x^{2/3} \cdot x^{1/3} - 2}{x^{1/3}}\right) && \text{(common denominator)} \\
&= \frac{5}{3}\left(\frac{x-2}{x^{1/3}}\right). && \text{(simplifying exponents)}
\end{aligned}$$

(c) $f(x) = \dfrac{1 - x^2}{1 + x^2}$

Using the quotient rule and simplifying,

$$f'(x) = \frac{-2x(1+x^2) - 2x(1-x^2)}{(1+x^2)^2} = \frac{-4x}{(1+x^2)^2},$$

so $f'(x) = 0$ when $x = 0$.