Rank and Homogeneous systems

Math 1410 Linear Algebra

Notation

Given a system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\quad \vdots \qquad\qquad\quad &\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

denote

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Structure of a solution

- With notation as on the previous page, write $(A|B)$ for the augmented matrix of our system.
- Use row operations to reduce $(A|B)$ to an augmented matrix $(A'|B')$ in (reduced) row echelon form.
- Cases:
  1. The matrix $(A'|B')$ has a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$.
  2. Every column in A' contains a leading 1.
  3. A has n columns and A' has k leading 1s, with $k < n$.
- Note: The number of leading 1s is equal to the number of non-zero rows.

Rank

Definition

The rank of a matrix A is the number of leading ones in the row-echelon form of A.

Theorem

Let $(A|B)$ denote the augmented matrix of a system of m linear equations in n variables. Then:

1. *If* $\operatorname{rank} A < \operatorname{rank}(A|B)$, *then the system is inconsistent.*
2. *If* $\operatorname{rank} A = \operatorname{rank}(A|B) = n$, *then the system has a unique solution.*
3. *If* $\operatorname{rank} A = \operatorname{rank}(A|B) = r < n$, *then the system has infinitely many solutions, with $n - r$ parameters.*

Note: $\operatorname{rank} A \leq \min\{m, n\}$. If our system is consistent and $n > m$, then we will have infinitely many solutions.

Examples

In each case, find rank A, rank$(A|B)$, and the solution of the corresponding system of equations:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & 0 & 1 & 4 \\ 0 & -8 & 14 & 8 \end{array}\right] \tag{1}$$

$$\left[\begin{array}{cccc|c} -2 & 8 & 4 & 0 & 6 \\ 0 & 3 & -2 & 1 & -5 \\ -1 & 0 & 2 & 6 & 7 \\ 0 & 0 & 1 & -1 & 4 \end{array}\right] \tag{2}$$

Column vectors

A column vector is an object of the form $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$. For example, we could have

$$A = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{or } C = \begin{bmatrix} 7 \\ -5 \\ 3.72 \\ 0 \end{bmatrix}.$$

We say two column vectors are equal if each corresponding entry is equal. Instead of writing $x_1 = 2, x_2 = -4, x_3 = 1$, we can write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}.$$

Algebra of column vectors

We allow two operations on column vectors: addition, and scalar multiplication.

$$\text{Addition: } X+Y=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}+\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}=\begin{bmatrix} x_1+y_2 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

$$\text{Scalar multiplication: } cX=c\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

Examples

Vector form of a system

Given a system of m equations in n variables, let A, X, and B be as before. Let

$$A_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, A_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, A_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

denote the columns of A. If we define $AX = x_1A_1 + x_2A_2 + \cdots + x_nA_n$, then we can write our system compactly as

$$AX = B.$$

Example

Re-write the system of equations as a single matrix equation:

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 1 \\ 2x_1 & - & 4x_2 & + & x_3 & & & = & 5 \\ x_1 & - & 2x_2 & + & 2x_3 & - & 3x_4 & = & 4 \end{array}$$

Solutions in vector form

Solving our system: row-reducing gives

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Leading variables:
Parameters:
Solution:
Vector form:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Homogeneous systems

A homogeneous system of equations is a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

That is, all the constants b_i on the right-hand side are zero.
Matrix form: $AX = 0$.
Note: homogeneous systems are always consistent. (Why?)

Homogeneous system: relationship among the variables.
$\Rightarrow$ unique solution not desirable.

Example

Solve the homogeneous system

$$\begin{array}{rcrcrcrcr} x & + & 3y & - & 2z & + & w & = & 0 \\ 2x & - & y & + & 5z & & & = & 0 \\ & & 14y & - & 2z & + & w & = & 0 \end{array}$$

Linear combinations

Definition

We say that a column vector Y is a linear combination of column vectors $X_1, X_2, \ldots, X_k$ if Y can be written in the form

$$Y = c_1X_1 + c_2X_2 + \cdots c_kX_k$$

for constants $c_1, c_2, \ldots, c_k$.

Given a homogeneous system $AX = 0$ in n variables, with $\operatorname{rank} A = r$, we will have parameters $t_1, t_2, \ldots, t_k$, where $k = n - r$. The general solution is then of the form

$$X_h = t_1X_1 + t_2X_2 + \cdots + t_kX_k.$$

Note: in Lyryx the vectors $X_1, \ldots X_k$ are referred to as basic solutions.

General solution - nonhomogeneous case

Let's return to the case of a general system $AX = B$.
Suppose:

1. The system is consistent.
2. We have the general solution X_h to the homogeneous system $AX = 0$.
3. We have a particular solution (no parameters) X_p to $AX = B$.

Then the general solution to $AX = B$ is given by $X = X_h + X_p$.

Example

Let's return to an earlier example: the system

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 1 \\ 2x_1 & - & 4x_2 & + & x_3 & & & = & 5 \\ x_1 & - & 2x_2 & + & 2x_3 & - & 3x_4 & = & 4 \end{array}$$