Name and student number: Solutions

1. Define functions $f : \mathbb{R} \to [1, \infty)$ and $g : [1, \infty) \to \mathbb{R}$ by $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$, respectively.

[2] (a) Compute $f(g(x))$ for $x \geq 1$.

Solution: For any $x \geq 1$, we have

$$f(g(x)) = f(\sqrt{x-1}) = (\sqrt{x-1})^2 + 1 = x - 1 + 1 = x.$$

[2] (b) Compute $g(f(x))$ for $x \in \mathbb{R}$.

Solution: For any $x \in \mathbb{R}$, we have

$$g(f(x)) = g(x^2 + 1) = \sqrt{x^2 - 1 + 1} = \sqrt{x^2}.$$

Note however that $\sqrt{x^2} = x$ only for $x \geq 0$. If $x < 0$ we get $\sqrt{x^2} = -x$. (Since x is negative, $-x$ is positive.)

[1] (c) Is $g = f^{-1}$? Why or why not?

Solution: We note that f can't have an inverse, since f is not one-to-one. Also, g can't be the inverse of f, since inverses are bijections, and g is not onto, since $g(x) \geq 0$ for all $x \geq 1$. Either reason is valid. Another reason (although you may have missed it in part (b)) is that it's **not** true that $g(f(x)) = x$ for all $x \in \mathbb{R}$ – this is only valid for $x \geq 0$, as noted above. (If it were true that $g(f(x)) = x$ for all x, then we'd have to conclude that $g = f^{-1}$.)

2. Construct an example of functions $f : A \to B$ and $g : B \to C$ such that g and $g \circ f$ are
[3] onto, but f is not.

Solution: There are many possible examples. A simple example is to take $A = \{1, 2\}$, $B = \{a, b, c\}$, and $C = \{u, v\}$, and define $f : A \to B$ by $f(1) = a$ and $f(2) = b$ (so f is not onto, since c is not in the range of f), and define $g : B \to C$ by $g(a) = u$, $g(b) = v$, $g(c) = v$. Then g is onto, and $g \circ f(1) = g(f(1)) = g(a) = u$ and $g \circ f(2) = g(f(2)) = g(b) = v$, so $g \circ f$ is onto.

3. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f : A \to C$ is onto, then g
[2] onto.

Solution: Suppose that $g \circ f$ is onto, and let $c \in C$ be arbitrary. We need to find some $b \in B$ such that $g(b) = c$. Since $g \circ f : A \to C$ is onto, we know that there exists some $a \in A$ such that $g \circ f(a) = c$. If we let $b = f(a)$, then we have $g(b) = g(f(a)) = g \circ f(a) = c$. Thus, g is onto.

Total: 10 points