Eigenvalues and Eigenvectors

Math 1410 Linear Algebra

Matrix transformations

Let A be an $m \times n$ matrix. If X is a vector in $\mathbb{R}^n$ (an $n \times 1$ matrix) then we know that $AX = Y$ defines a vector in $\mathbb{R}^m$ (an $m \times 1$ matrix.

For any such matrix A, we get a function $f_A : \mathbb{R}^n \to \mathbb{R}^m$, called a matrix transformation, or linear transformation.

If A is a square $n \times n$ matrix, we get a function f_A that transforms each vector in $\mathbb{R}^n$ to another vector in $\mathbb{R}^n$.

For example, for $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, the function f_A rotates a each vector $X = \begin{bmatrix} x & y \end{bmatrix}^T$ by an angle of θ.

Multiplication operators

For any scalar λ, we can consider the function $f_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_\lambda(X) = \lambda X$.

This is the operator of scalar multiplication: if $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, then

$$f_\lambda(X) = \begin{bmatrix} \lambda x_1 & \lambda x_2 & \cdots & \lambda x_n \end{bmatrix}^T.$$

We note that f_λ is also a matrix transformation: $f_\lambda = f_A$, where

$$A = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} = \lambda I_n,$$

where I_n is the $n \times n$ identity matrix.

Invariant directions for matrix transformations

For the matrix $A = \lambda I_n$, we note that for every vector X in $\mathbb{R}^n$, AX is parallel to X. For other matrices, it's possible that there are no vectors with this property: for example, if A is the rotation matrix in $\mathbb{R}^2$ (unless θ is a multiple of 2π).

We'll be interested in the case where there are some vectors such that AX is parallel to X. For example, the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $f(x, y) = (x + y, y)$ is an example of a shear transformation. It can be represented as the matrix transformation

$$f_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Eigenvalues and eigenvectors

Definition

For any $n \times n$ matrix A, we say that a scalar λ is an eigenvalue for A if there exists a non-zero vector X, called an eigenvector, such that

$$AX = \lambda X$$

Example

Let $A = \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$. Then we have

$$\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

so $\lambda = 1$ and $\lambda = 6$ are eigenvalues of A, with corresponding eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively.

Eigenvectors and characteristic directions

Theorem

Suppose X is an eigenvector of an $n \times n$ matrix A, with eigenvalue λ. Then for any scalar $k \neq 0$, kX is also an eigenvector of A, corresponding to the same eigenvalue.

Finding eigenvalues and eigenvectors

Let A be an $n \times n$ matrix, and suppose λ is an eigenvalue of A. Then there exists a non-zero vector X such that $AX = \lambda X$. That is, $X \neq 0$ and

$$AX - \lambda X = (A - \lambda I_n)X = 0.$$

We note that $A - \lambda I_n$ is an $n \times n$ matrix, and that X is a non-trivial solution to the homogeneous system $(A - \lambda I_n)X = 0$. This means that $A - \lambda I_n$ cannot be invertible. Since a matrix is invertible if and only if its determinant is non-zero, we arrive at:

Theorem

A scalar $\lambda \in \mathbb{C}$ *is an eigenvalue of* A *if and only if* $\det(A - \lambda I_n) = 0$

The characteristic polynomial

Definition

For any $n \times n$ matrix A, we define its characteristic polynomial by

$$c_A(x) = \det(xI_n - A).$$

Note: The eigenvalues of A are precisely the zeros of the characteristic polynomial of A.

Example

Find the characteristic polynomial of $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 3 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$.

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ 5 & 1 & 3 \end{bmatrix}$.

Similar matrices

Definition

We say that two matrices A and B are similar if there exists an invertible matrix P such that $B = P^{-1}AP$.

Theorem

If A and B are similar matrices, then they have the same eigenvalues.

Diagonal and upper-triangular matrices

Theorem

If A is a triangular matrix, then the eigenvalues of A are given by the entries on the main diagonal of A.

Note: In particular, this is true if A is diagonal, in which case the standard unit basis vectors are eigenvectors for A.

Examples

Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Diagonalization

Definition

We say that an $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix; that is, if there exists an invertible matrix P such that $D = P^{-1}AP$ is diagonal.

Note: since similar matrices have the same eigenvalues, we must have

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Theorem

An $n \times n$ matrix A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Example

Determine whether or not the matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{bmatrix}$ can be diagonalized.

Case of distinct eigenvalues

Theorem

If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of a matrix A, then the corresponding eigenvectors $X_1, \ldots, X_m$ are linearly independent.

Fact: Any set of n linearly independent vectors forms a basis of $\mathbb{R}^n$.

Repeated eigenvalues

In general a matrix A will have characteristic polynomial

$$c_A(x) = (x-\lambda_1)^{k_1}(x-\lambda_2)^{k_2}\cdots(x-\lambda_m)^{k_m},$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues and $k_1, \ldots, k_m$ are the multiplicities of the eigenvalues.

Definition

Given an eigenvalue λ of a matrix A, we define the eigenspace $E(\lambda, A)$ of A with respect to λ by

$$E(\lambda, A) = \{X \mid (A - \lambda I_n)X = 0\}.$$

Note: we always have $1 \leq \dim E(\lambda_j, A) \leq k_j$ for each j.
A matrix A is diagonalizable if and only if $\dim E(\lambda_j, A) = k_j$ for each $j = 1, 2, \ldots, k$.

Example

Determine whether or not the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & 3 \\ -1 & 2 & 2 \end{bmatrix}$ is diagonalizable.

Example

Determine whether or not the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ -2 & -2 & 2 \\ -5 & -10 & 7 \end{bmatrix}$ is diagonalizable.

Powers of matrices

Suppose we wanted to find A^7, where A was the matrix from the last slide. Finding this by hand would take a very long time. (For large matrices and high powers, even a computer will take a long time.)

However, we know that $A = PDP^{-1}$, where $P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$.

Polynomials of matrices

Suppose $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial and we want to compute $p(A)$, where A is diagonalizable.

Symmetric matrices

Recall: an $n \times n$ matrix A is symmetric if $A^T = A$.

Theorem

Suppose A is a symmetric matrix. If X_1 and X_2 are eigenvalues of A corresponding to eigenvalues $\lambda_1 \neq \lambda_2$, then $X_1 \cdot X_2 = 0$.

Theorem

If A is an $n \times n$ symmetric matrix, then there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Example

Given $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, find an orthogonal matrix P such that $P^T AP$ is diagonal.

Example

Sketch the curve defined by the equation $3x^2 + 2xy + 3y^2 = 1$.