1. Compute the inverse of the given matrix, if possible:

   (a) $A = \begin{bmatrix} 1 & -5 & 0 \\ -2 & 15 & 4 \\ 4 & -19 & 1 \end{bmatrix}$

   Using the algorithm given in class, we have:

$$\left[\begin{array}{ccc|ccc} 1 & -5 & 0 & 1 & 0 & 0 \\ -2 & 15 & 4 & 0 & 1 & 0 \\ 4 & -19 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-4R_1\to R_3]{R_2+2R_1\to R_2} \left[\begin{array}{ccc|ccc} 1 & -5 & 0 & 1 & 0 & 0 \\ 0 & 5 & 4 & 2 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 5 & 4 & 2 & 1 & 0 \end{array}\right]$$
$$\xrightarrow{R_3-5R_2\to R_3} \left[\begin{array}{ccc|ccc} 1 & -5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & -1 & 22 & 1 & -5 \end{array}\right]$$
$$\xrightarrow{(-1)R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & -5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & -22 & -1 & 5 \end{array}\right]$$
$$\xrightarrow{R_2-R_3\to R_2} \left[\begin{array}{ccc|ccc} 1 & -5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 18 & 1 & -4 \\ 0 & 0 & 1 & -22 & -1 & 5 \end{array}\right]$$
$$\xrightarrow{R_1+5R_2\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 91 & 5 & -20 \\ 0 & 1 & 0 & 18 & 1 & -4 \\ 0 & 0 & 1 & -22 & -1 & 5 \end{array}\right]$$

   We thus see that the reduced row-echelon form of A is equal to the identity matrix I_3, and we can conclude that the inverse of A is given by

$$A^{-1} = \begin{bmatrix} 91 & 5 & -20 \\ 18 & 1 & -4 \\ -2 & -1 & 5 \end{bmatrix}.$$

   To verify, we can check that

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & -5 & 0 \\ -2 & 15 & 4 \\ 4 & -19 & 1 \end{bmatrix} \begin{bmatrix} 91 & 5 & -20 \\ 18 & 1 & -4 \\ -2 & -1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 91-90+0 & 5-5+0 & 20-20+0 \\ -182+270-88 & -10+15-4 & 40-60+20 \\ 364-342-22 & 20-19-1 & -80+76+5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

   as expected.

(b) $B = \begin{bmatrix} 2 & 3 & 4 \\ -3 & 6 & 9 \\ -1 & 9 & 13 \end{bmatrix}$

Proceeding as above, we have:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ -3 & 6 & 9 & 0 & 1 & 0 \\ -1 & 9 & 13 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} -1 & 9 & 13 & 0 & 0 & 1 \\ -3 & 6 & 9 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(-1)R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & -9 & -13 & 0 & 0 & -1 \\ -3 & 6 & 9 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + 3R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -9 & -13 & 0 & 0 & -1 \\ 0 & -21 & -30 & 0 & 1 & 2 \\ 0 & 21 & 30 & 1 & 0 & -3 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & -9 & -13 & 0 & 0 & -1 \\ 0 & -21 & -30 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

At this point the algorithm stops: we can already see that $\text{rank}(B) = 2 < 3$, since the row of zeros will prevent us from obtaining a leading one in the third column. Since B cannot be carried to the identity matrix by elementary row operations, it is not invertible.

2. Simplify the expression $A^2(B^{-1}A)^{-1}(AB)^{-1}B$.

Recalling from class that $(AB)^{-1} = B^{-1}A^{-1}$ and that $(B^{-1})^{-1} = B$, we have

$$\begin{aligned} A^2(B^{-1}A)^{-1}(AB)^{-1}B &= A^2(A^{-1}(B^{-1})^{-1})(B^{-1}A^{-1})B \\ &= A(AA^{-1})(BB^{-1})(A^{-1}B) \\ &= A(I_n)(I_n)(A^{-1}B) \\ &= (AA^{-1})B = I_nB = B. \end{aligned}$$

3. Find an expression for A^{-1}, given that:

(a) $5A^3 = I$

Since $5A^3 = I$, it follows that $A(5A^2) = 5A(A^2) = 5A^3 = I$. Since $A(5A^2) = I$ and A^{-1} is the unique matrix X such that $AX = I$, we can conclude that $A^{-1} = 5A^2$.

(b) $A^2 - 2A + I = 0$

From the given equation we have

$$I = 2A - A^2 = A(2I - A).$$

As with the previous problem, since we have $A(2I - A) = I$, it follows that $A^{-1} = 2I - A$.

(c) A^2B is invertible, for some matrix B. (Give your answer in terms of $(A^2B)^{-1}$.)

Suppose that A^2B is invertible, and let $C = (A^2B)^{-1}$. By definition of the inverse, we have $(A^2B)C = I$, and thus $A(ABC) = I$, and once again it follows that

$$A^{-1} = ABC = AB(A^2B)^{-1}.$$

4. Write the matrix $A = \begin{bmatrix} 2 & -4 \\ -4 & 9 \end{bmatrix}$ as a product of elementary matrices.

We look for a sequence of elementary row operations that carry our matrix to the identity. We have

$$\begin{bmatrix} 2 & -4 \\ -4 & 9 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix} \xrightarrow{R_2+4R_1 \to R_2} \begin{bmatrix} 1 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{R_1+2R_2 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The inverses of these matrices are

$$E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

As discussed in class, we expect that $A^{-1} = E_3E_2E_1$ and $A = E_1^{-1}E_2^{-1}E_3^{-1}$, and indeed, we can verify that

$$E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 9 \end{bmatrix} = A,$$

and

$$E_3E_2E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & 2 \\ 2 & 1 \end{bmatrix} = A^{-1}.$$