Math 3500 Assignment #6
University of Lethbridge, Fall 2014

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1. (**Do not submit**) Let $f : D \to \mathbb{R}$ be continuous. For each of the following, prove the result, or give a counterexample.

   (a) If D is open, then $f(D)$ is open.

   (b) If D is closed, then $f(D)$ is closed.

   (c) If D is not open, then $f(D)$ is not open.

   (d) If D is not closed, then $f(D)$ is not closed.

   (e) If D is not compact, then $f(D)$ is not compact.

   (f) If D is unbounded, then $f(D)$ is unbounded.

   (g) If D is finite, then $f(D)$ is finite.

   (h) If D is infinite, then $f(D)$ is infinite.

   (i) If D is an interval, then $f(D)$ is an interval.

   (j) If D is an interval that is not open, then $f(D)$ is an interval that is not open.

   (Note: this is problem 5.3.3 in the text, and there's a hint in the back.)

2. (a) Let $a \in \mathbb{R}$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x - a|$. Prove that f is continuous.

   (b) Let K be a nonempty compact subset of $\mathbb{R}$ and let $a \in \mathbb{R}$. We define the distance from a to K by
   $$d(a, K) = \inf\{|x - a| : x \in K\}.$$
   (The infimum exists since $\{|x-a| : x \in K\}$ is bounded below by zero.) Prove that there exists a point $b \in K$ that is *closest* to a, in the sense that $|b - a| = d(a, K)$.

3. Prove that if $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \in \mathbb{Q}$ for all $x \in [a, b]$, then f is constant.

4. Suppose f is continuous on $[0, 2]$, and $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ with $|y - x| = 1$ and $f(x) = f(y)$.

   Hint: Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.

5. Prove that each of the following functions is uniformly continuous on the specified set using the ϵ-δ definition of uniform continuity:

   (a) $f(x) = x^2$ on $[0, 3]$

   (b) $g(x) = \dfrac{1}{x}$ on $[\frac{1}{2}, \infty)$

6. (**Do not submit**) Prove that if f is uniformly continuous on a bounded set $D \subseteq \mathbb{R}$, then f is bounded on D.

   Hint: If f is not bounded on D, you can find some sequence (a_n) in D with $|f(a_n)| \geq n$ for all $n \in \mathbb{N}$. But since D is bounded, (a_n) is a bounded sequence and therefore has a convergent subsequence. We also know that if f is uniformly continuous and (x_n) is a Cauchy sequence, then $(f(x_n))$ is also a Cauchy sequence.

7. Prove that $f(x) = \sin x$ is uniformly continous on $\mathbb{R}$.

   Hint: First use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.