The problems on this worksheet are for in-class practice during tutorial. You are free to collaborate and to ask for help. They don't count for course credit, but it's a good idea to make sure you know how to do everything before you leave tutorial – similar problems may show up on a test or assignment.

This week I've tried to make my best guess at what your test on Friday might look like.

1. Evaluate the following "immediate integrals":

   (a) $\displaystyle\int (2x+3)^4\,dx = \frac{1}{10}(2x+3)^5 + C$

   (b) $\displaystyle\int \frac{x^2+2x}{x^3+3x^2+5}\,dx = \frac{1}{3}\ln(x^3+3x^2+5) + C$

   (c) $\displaystyle\int \tan^5(x)\sec^2(x)\,dx = \frac{1}{6}\tan^6(x) + C$

   (d) $\displaystyle\int \frac{\ln(\sqrt{x+1})}{\sqrt{x+1}}\,dx = 2\sqrt{x+1}\ln\sqrt{x+1} - 2\sqrt{x+1} + C$
   (Okay, this one is only "immediate" if you remembered that $\int \ln u\,du = u\ln u - u + C$, which is obtained using integration by parts.)

   (e) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}(x/2) + C$

   (f) $\displaystyle\int \frac{x^3-4x^2}{\sqrt{x}}\,dx = \int (x^{5/2} - 4x^{3/2})\,dx = \frac{2}{7}x^{7/2} - \frac{8}{5}x^{5/2} + C$

   (g) $\displaystyle\int \frac{e^x+1}{e^x}\,dx == \int (1-e^{-x})\,dx = x + e^{-x} + C$

   (h) $\displaystyle\int \frac{\ln(x^3)}{x}\,dx = 3\int \frac{\ln(x)}{x}\,dx = 3(\ln(x))^2 + C$

   (i) $\displaystyle\int x(1-x^2)^5\,dx = -\frac{1}{12}(1-x^2)^6 + C$

   (j) $\displaystyle\int 3x^2\cos(x^3)e^{\sin(x^3)}\,dx = e^{\sin(x^3)} + C$

2. Evaluate the following integrals:

(a) $\displaystyle\int x\sec^2(x)\,dx = \int xd(\tan x) = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos(x)| + C$

(b) $\displaystyle\int e^{\sqrt{x}}\,dx$

First let $x = u^2$, so $dx = 2u\,du$, giving us

$$\begin{aligned}\int e^{\sqrt{x}}\,dx &= \int 2ue^u\,du = 2\int ud(e^u) = 2ue^2 - 2\int e^u\,du = 2ue^u - 2e^u + C\\ &= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.\end{aligned}$$

(c) $\displaystyle\int \cos(x)\cos(2x)\,dx = \int \cos(x)(1-2\sin^2 x)\,dx = \sin(x) - \frac{2}{3}\sin^3(x) + C.$

(d) $\displaystyle\int \tan^5(x)\sec^4(x)\,dx = \int \tan^5(x)(1+\tan^2(x))\sec^2(x)\,dx = \frac{1}{6}\tan^6(x) + \frac{1}{8}\tan^8(x) + C$

(e) $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$

Letting $x = \sqrt{2}\tan\theta$, we have $\sqrt{x^2+2} = \sqrt{2\sec^2\theta} = \sqrt{2}\sec\theta$ and $dx = \sqrt{2}\sec^2\theta\,d\theta$, so

$$\int \frac{8}{\sqrt{x^2+2}}\,dx = \int \frac{8\sqrt{2}\sec^2\theta}{\sqrt{2}\sec\theta}\,d\theta = 8\ln|\sec\theta + \tan\theta| + C = 8\ln|x + \sqrt{x^2+2}| + C.$$

(f) $\displaystyle\int \frac{\sqrt{5-x^2}}{x^2}\,dx$

Letting $x = \sqrt{5}\sin\theta$, so $\sqrt{5-x^2} = \sqrt{5}\cos\theta$ and $dx = \sqrt{5}\cos\theta\,d\theta$, we have

$$\begin{aligned}\int \frac{\sqrt{5-x^2}}{x^2}\,dx &= \int \frac{5\cos^2\theta}{5\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta = \int (\csc^2\theta - 1)\,d\theta\\ &= -\cot\theta - \theta + C = -\frac{\sqrt{5-x^2}}{x} - \sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c\end{aligned}$$

(g) $\displaystyle\int \frac{16x^2-2x}{(x+3)(2x-1)(x-1)}\,dx$

We look for a partial fraction decomposition

$$\frac{16x^2-2x}{(x+3)(2x-1)(x-1)} = \frac{A}{x+3} + \frac{B}{2x-1} + \frac{C}{x-1}.$$

Multiplying both sides of this decomposition by $x+3$ gives us

$$\frac{16x^2-2x}{(2x-1)(x-1)} = A + \frac{B(x+3)}{2x-1} + \frac{C(x+3)}{x-1}.$$

Plugging in $x = -3$ then gives $A = \dfrac{75}{14}$.

Multiplying both sides of the decomposition by $2x-1$ gives

$$\frac{16x^2-2x}{(x+3)(x-1)} = \frac{A(2x-1)}{x+3} + B + \frac{C(2x-1)}{x-1},$$

and plugging in $x = 1/2$ gives $B = \frac{12}{7}$.

Multiplying both sides of the decomposition by $x-1$ gives

$$\frac{16x^2-2x}{(x+3)(2x-1)} = \frac{A(x-1)}{x+3} + \frac{B(x-1)}{2x-1} + C,$$

and plugging in $x = 1$ gives $C = \frac{7}{2}$.

Putting everything together, we get

$$\begin{aligned}\int \frac{16x^2-2x}{(x+3)(2x-1)(x-1)}\,dx &= \frac{75}{14}\int \frac{1}{x+3}\,dx + \frac{12}{7}\int \frac{1}{2x-1}\,dx + \frac{7}{2}\int \frac{1}{x-1}\,dx \\ &= \frac{75}{14}\ln|x+3| + \frac{6}{7}\ln|2x-1| + \frac{7}{2}\ln|x-1| + C.\end{aligned}$$

(h) $\displaystyle\int \frac{2x+1}{x^3+x}\,dx$

This time we look for a decomposition $\dfrac{2x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$. Getting a common denominator on the right-hand side, we have

$$\frac{2x+1}{x^3+x} = \frac{Ax^2+A+Bx^2+Cx}{x^3+x}.$$

Comparing numerators, we have $0x^2+2x+1 = (A+B)x^2+Cx+A$. Constant terms must be equal, so $A = 1$, Coefficients of x must be equal, so $C = 2$. Coefficients of x^2 must be equal, so $A+B = 0$, giving $B = -A = -1$. Thus,

$$\int \frac{2x+1}{x^3+x}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + 2\tan^{-1}(x) + C.$$

These won't be on your test, but I thought I should give you a couple of practice problems involving improper integrals.

3. Evaluate the following improper integrals, or explain why they do not exist:

(a) $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$

Since the integrand isn't defined at $x = 0$, we have the improper integral

$$\int_0^4 x^{-1/2}\,dx = \lim_{a\to 0}\int_a^4 x^{-1/2}\,dx = \lim_{a\to 0}(2(\sqrt{4}-\sqrt{a})) = 4.$$

(b) $\displaystyle\int_1^\infty \frac{\ln(x)}{x^2}\,dx$

See Example 46 on Page 53 of the textbook.