Elementary Matrices and Inverses

Math 1410 Linear Algebra

Matrix Inverses

Recall:

- The inverse of an $n \times n$ matrix A satisfies

$$AA^{-1} = A^{-1}A = I_n.$$

- Inverse only possible for square matrices.
- Inverse of A exists if and only if $\operatorname{rank} A = n$.

Algorithm:

1. Begin with $[A|I_n]$.
2. Apply elementary row operations until A is in RREF.
3. If $\operatorname{rank} A = n$, result will be $[I_n|A^{-1}]$.
4. If $\operatorname{rank} A < n$, there will be a row of zeros on the left, and A^{-1} does not exist.

Example 1

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}$$

Example 2

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 2 \\ -1 & -2 & 2 \end{bmatrix}$$

Example 3

$$A = \begin{bmatrix} 2 & 0 & 7 \\ -1 & 3 & 5 \\ 0 & 12 & 34 \end{bmatrix}$$

A system of equations

Solve the system

$$\begin{array}{rcrcrcr} x & + & 3y & - & 2z & = & 3 \\ 2x & & & + & 2z & = & -2 \\ -x & - & 2y & + & 2z & = & 4 \end{array}$$

Elementary matrices

Definition

An elementary matrix is an $n \times n$ matrix obtained from I_n by a single elementary row operation.

Example

Type 1: use row operation $R_i \leftrightarrow R_j$

. (These are also called permutation matrices)

Type 2: use row operation $R_i \to kR_i$

Here, $k \neq 0$ is a scalar.

Type 3: use row operation $R_i \to R_i + kR_j$

Inverse of an elementary matrix

From the above examples, we see:

> *Multiplying A on the left by an elementary matrix has the same effect as performing the corresponding elementary row operation on A.*

Now, given an elementary matrix E, can we find a matrix F such that $EF = FE = I_n$? Yes, and it's another elementary matrix!

Reason: every elementary row operation is reversible.

Examples

Row-echelon form revisited

Let A be an $m \times n$ matrix with RREF R. We know we can obtain R from A be a series of elementary row operations; call them $RO_1, RO_2, \ldots, RO_k$
For each operation there is a corresponding elementary matrix $E_1, E_2, \ldots, E_k$. We get:

$$\begin{tikzcd} A \arrow[r, "RO_1"] & A_1 \arrow[r, "RO_2"] \arrow[d, equal] & A_2 \arrow[r, "RO_3"] \arrow[d, equal] & \cdots \arrow[r, "RO_k"] & R \arrow[d, equal] \\ & E_1 A & E_2(E_1 A) & \cdots & UA \end{tikzcd}$$

where $U = E_k E_{k-1} \cdots E_2 E_1$ is a product of elementary matrices. Note also that $A = U^{-1}R$, where

$$U^{-1} = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}.$$

Elementary matrices and inverses

Now, suppose that A is invertible. Then we know we can perform a series of elementary row operations as follows:

$$A \to A_1 \to A_2 \to \cdots \to A_{k-1} \to A_k = I_n$$

with corresponding elementary matrices $E_1, E_2, \ldots, E_k$. Then:

$$\begin{aligned} A_1 &= E_1 A \\ A_2 &= E_2 A_1 = (E_2 E_1) A \\ A_3 &= E_3 A_2 = (E_3 E_2 E_1) A \\ \vdots \quad & \quad \vdots \\ A_k &= E_k A_{k-1} = (E_k E_{k-1} \cdots E_2 E_1) A \end{aligned}$$

In other words, $(E_k \cdots E_1)A = I_n$, which suggests $A^{-1} = E_k \cdots E_1$.

Example

Consider $A = \begin{bmatrix} 1 & -3 \\ 3 & 2 \end{bmatrix}$

Example

Consider $B = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 2 & 1 \\ 0 & 3 & -2 \end{bmatrix}$.

Row-echelon form, one more time

Theorem

Let A and B be any matrices (not necessarily square) such that $AB = I_n$. Then the RREF of A does not have a row of zeros.

Proof.

Let R be the RREF of A. Then $R = U^{-1}A$ as above. Now consider

$R(BU)$

□

Uniqueness of inverses, again

Recall that A^{-1} must satisfy $AA^{-1} = A^{-1}A = I_n$. As long as A is a square matrix, checking one of these is enough:

Theorem

If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that $AB = I_n$, then $BA = I_n$, and thus A and B are invertible, with $B = A^{-1}$.

Proof.

Since $AB = I$, the RREF R of A does not have a row of zeros, so $R = U^{-1}A = I$

$\square$