Math 3500 Exercise Sheet

17 September, 2014

We will work on some of the following exercises in class. Those not done in class are recommended as homework problems.

1. For each of the subsets of $\mathbb{R}$ below, determine the following:

   (a) Is the set open? closed? compact?

   (b) What are the interior points? boundary points? accumulation points? isolated points?

   (c) What is the boundary? What is the closure?

   (i)$(0,1)$ (ii)$(0,1) \cup (1,2)$ (iii) $\left\{ \dfrac{n}{n+1} : n \in \mathbb{N} \right\}$ (iv)$\mathbb{R} \setminus \mathbb{Q}$ (v)$\{1,2,7\} \cup (7,10]$

2. Prove that if $\{K_\alpha\}$ is a collection of compact sets, then $\bigcap K_\alpha$ is compact.

3. Prove that a closed subset of a compact set is compact.

   Hint: Let K be compact and let $F \subseteq K$ be closed in $\mathbb{R}$. If $\{U_\alpha\}$ is any open cover of F, explain why $\{U_\alpha\} \cup \{\mathbb{R} \setminus F\}$ must be an open cover of K. Now use the fact that K is compact.

4. Prove that any closed interval $[a,b]$ is compact.

   Hint: Use proof by contradiction, and the following steps:

   (a) Let $I_0 = [a,b]$ and suppose there exists an open cover $\{U_\alpha\}$ of I_0 for which there is no finite subcover. Then divide the interval in half: at least one of the two intervals $[a,(a+b)/2]$ and $[(a+b)/2,b]$ cannot be covered by finitely many of the U_α (why?). Call this interval I_1.

   (b) Explain how to repeat the procedure in part (a) to obtain a sequence of intervals $I_1, I_2, I_3, \ldots$ such that each I_n cannot be covered by finitely many of the U_α.

   (c) Note that there must be some point $x_0 \in \mathbb{R}$ such that $x \in I_n$ for all $n \in \mathbb{N}$. (Sub-hint: Nested Intervals Theorem)

   (d) Since the collection $\{U_\alpha\}$ covers $[a,b]$ and $x_0 \in [a,b]$, we must have $x_0 \in U_\beta$ for some open set U_β. Explain why there must exist some $r > 0$ such that $(x_0 - r, x_0 + r)$ is contained in U_β.

(e) Notice that each interval I_n has length $(b-a)/2^n$. Explain why this means that we must have $I_n \subseteq U_\beta$ for some $n \in \mathbb{N}$.

(f) Explain why the result from part (e) results in a contradiction.

5. Prove that any closed and bounded subset of $\mathbb{R}$ is compact.

   Hint: Use the results from problems 3 and 4.

6. Prove that any compact subset of $\mathbb{R}$ is bounded.

7. Pove that any compact subset K of $\mathbb{R}$ is closed.

   Hint: Prove that the complement $K^c = \mathbb{R} \setminus K$ is open: if $p \in K$ and $q \notin K$, let N_p and N_q be neighbourhoods of p and q, respectively, each with radius less than $|p-q|/2$ (so that they don't overlap). Since $\{N_p\}_{p\in K}$ is an open cover of K, there exist finitely many points $p_1, \ldots, p_k \in K$ such that

   $$K \subseteq N_{p_1} \cup N_{p_2} \cup \cdots \cup N_{p_n},$$

   and such that each N_{p_k} has radius $\epsilon_k < |p_k - q|/2$, for $k = 1, \ldots, n$. Now, what can you say about the set

   $$U = N_{\epsilon_1}(q) \cap N_{\epsilon_2}(q) \cap \cdots \cap N_{\epsilon_n}(q)?$$

   Note: see the text for an alternative proof, using the fact that if K is not closed, then there must exist a limit point of K that does not belong to K.

8. Combining problems 5, 6, and 7, conclude that the *Heine-Borel Theorem* is true: a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

9. We have one big theorem left, the Bolzano-Weierstrass theorem. This theorem says that if $B \subseteq \mathbb{R}$ is a bounded, infinite subset, then B has a limit point. We definitely won't have time to get to this one, so here's a proof. (Another one is in the textbook.)

   Proof: Suppose that $B \subseteq \mathbb{R}$ is bounded and infinite. Since B is bounded, there exists an interval $[a, b]$ with $B \subseteq [a, b]$. By problem 4, $[a, b]$ is compact, so it suffices to prove:

   Lemma: an infinite subset B of a compact set K has a limit point in K.

   Proof of lemma: if no $k \in K$ is a limit point of B, then each $k \in K$ has a neighbourhood N_k that contains at most one point of B (the point k itself, if $k \in K$). Since B is infinite, no finite subcollection of $\{N_k\}$ can cover B, and since $B \subseteq K$, that means that $\{N_k\}_{k\in K}$ is an open cover of K with no finite subcover, which contradicts the assumption that K is compact.

10. There's a bit of room left, so here's a practice problem: define the distance from a point $x \in \mathbb{R}$ to a *set* $A \subseteq \mathbb{R}$ by $d(x, A) = \inf\{|x - a| : a \in A\}$. Prove that $x \in \partial A$ if and only if $d(x, A) = 0$ and $d(x, \mathbb{R} \setminus A) = 0$.