University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, December 2008
DURATION: $2\frac{1}{2}$ hours

Second Year - Materials Science and Engineering

MAT294H1F - Calculus and Differential Equations

Calculator type: 3
Exam type: B

Examiner: Sean Fitzpatrick

Total: 100 marks

Family Name: ______________________________
(Please Print)

Given Name(s): ______________________________
(Please Print)

Please sign here: ______________________________

Student ID Number: ______________________________

No aids, electronic or otherwise, are permitted, with the exception of faculty-approved calculators. Partial credit will be given for partially correct work. Please read through the entire exam before beginning, and take note of how many points each question is worth.

The second page of the exam contains a list of formulas that may be helpful in the completion of the exam.

FOR MARKER'S USE ONLY	
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TOTAL:	/100

List of relevant formulas

Coordinate systems:

Cylindrical

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Spherical

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

Jacobian:

$$J_T(u,v) = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

Integrating factor:

$$u(x) = e^{\int P(x)\,dx}$$

Normal vector:

$$\vec{N}(u,v) = \frac{\partial(y,z)}{\partial(u,v)}\hat{\imath} + \frac{\partial(z,x)}{\partial(u,v)}\hat{\jmath} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}$$

Fundamental theorem for line integrals:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Stokes' theorem:

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\,dS = \oint_C \vec{F} \cdot d\vec{r}$$

Divergence theorem:

$$\iiint_T \nabla \cdot \vec{F}\,dV = \iint_S \vec{F} \cdot \vec{n}\,dS$$

Solution of the n^{th}-order system $\vec{r}\,'(t) = A\vec{r}(t)$:

$$\vec{r}(t) = c_1\vec{X}_1e^{\lambda_1 t} + c_2\vec{X}_2e^{\lambda_2 t} + \cdots + c_n\vec{X}_ne^{\lambda_n t}$$

1. Consider the function $f(x, y) = 4xy - 2x^4 - y^2$.

[3] (a) Calculate $\nabla f(x, y)$.

[3] (b) At which points (x, y) is the plane tangent to the surface $z = f(x, y)$ horizontal?

[3] (c) Which (if any) of the above points are local maxima?

[3] (d) Set up (but do not solve) the three equations that would let us find the maximum value of $f(x, y)$ subject to the constraint $2x^2 + 3y^2 = 6$.

2. Let R be the region in the first quadrant of the xy-plane bounded by the curves $y = x^2$, $y = 2x^2$, $x = y^2$ and $x = 4y^2$.

[2] (a) Sketch the region R.

[2] (b) Determine a transformation $(x, y) = T(u, v)$ such that R is the image under T of a rectangle of the form $a \le u \le b$, $c \le v \le d$, for some a, b, c, d.

[3] (c) Compute the Jacobian of the transformation T.

[3] (d) Evaluate the integral

$$\iint_R \left(\frac{y^2}{x^4} + \frac{x^2}{y^4} \right) dx\, dy.$$

3. Consider the following two first-order differential equations:

$$\text{(I) } (-2x + 3y)\,dx + x\,dy = 0 \qquad \text{(II)}(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0.$$

[4] (a) Classify each of the above equations as either sepearable, linear, or exact.

[4] (b) Find the general solution to equation (I).

[4] (c) Find the general solution to equation (II).

[6] (d) Evaluate the line integral $\int_C (3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy$, where C is the curve given by $(x(t), y(t)) = (2t^2, -3t)$, for $-1 \leq t \leq 1$.
You do **not** have to simplify your answer.

4. Consider the surface S given by $z = 9 - x^2 - y^2$, for $5 \leq z \leq 9$..

[4] (a) Describe S as a parametric surface $\vec{r}(u, v)$.

[5] (b) Using the parameterization from part (a), compute the normal vector $\vec{N}(u, v)$ to the surface S.

[6] (c) Compute the surface area of S.

5. Let $\vec{F}(x,y,z) = \langle yz,\ -xz,\ z^3\rangle$, and let S be the surface given by the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z=1$ and $z=3$.

[3] (a) Sketch the surface S.

[2] (b) Calculate $\nabla \times \vec{F}$.

[6] (c) Use Stokes' theorem to evaluate

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS.$$

Hint: This problem is easiest if you apply Stokes' theorem twice in a clever way to obtain two much simpler surface integrals. The boundary of S has two seperate parts so be sure to keep track of orientation!

[4] (d) In the special case that $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ and that S is a region R in the xy-plane, show that Stokes' theorem reduces to Green's theorem.

6. Let $\vec{F}(x, y, z) = ||\vec{r}||\vec{r}$, where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, and let S be the spherical surface $x^2 + y^2 + z^2 = 9$.

[6] (a) Evaluate the surface integral $\displaystyle\iint_S \vec{F} \cdot \vec{n}\, dS$ by direct computation.

[3] (b) Compute $\nabla \cdot \vec{F}$.

[6] (c) Evaluate $\displaystyle\iint_S \vec{F} \cdot \vec{n}\, dS$ using the divergence theorem.

7. Solve the following system of linear first order differential equations:

$$\vec{r}\,'(t) = (4x + 5y)\hat{\imath} + (-2x + 6y)\hat{\jmath}$$

Note: Your solution should involve complex numbers.
A correct final answer that still involves complex numbers is worth 10/15.
To score 15/15 you must give the correct answer in terms of *real numbers only.*

Extra space for rough work. Do **not** tear out this page.