Math 3500 Assignment #2 Solutions
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1. Let S and T be nonempty bounded subsets of $\mathbb{R}$.

   (a) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

   Solution: Suppose that S and T are nonempty subsets of $\mathbb{R}$. If $a = \inf T$, then $a \leq s$ for all $s \in S$, since if $s \in S$, then $s \in T$ and a is a lower bound for T. But this means that $\inf S \geq \inf T = a$, since $\inf S$ is the *greatest* lower bound. Similarly, the supremum of T is an upper bound for S, since $S \subseteq T$, so $\sup S \leq \sup T$, since $\sup S$ is the least upper bound of S. Finally, since S is nonempty, we can take any $s \in S$, and then (as we saw in class), we have $\inf S \leq s \leq \sup S$, by definition of the infimum and supremum. The result now follows by the transitivity of the order relation on $\mathbb{R}$.

   (b) Prove that $\sup S \cup T = \max\{\sup S, \sup T\}$.
   (Do not assume that $S \subseteq T$ for part (b).)

   Solution: Let $a = \sup S$ and $b = \sup T$. Let $c = max\{a, b\}$, so that $c \geq a$ and $c \geq b$. If $x \in S \cup T$, then either $x \in S$, and $x \leq a \leq c$ or $x \in T$, and $s \leq b \leq c$. Therefore c is an upper bound for $S \cup T$.
   If d is an upper bound for $S \cup T$ then, since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, d is an upper bound for both S and T. Thus $d \geq a$ and $d \geq b$, since a and b are the least upper bounds for S and T, respectively. Thus $d \geq max\{a, b\} = c$. Since d was an arbitrary upper bound, we can conclude that $c = \sup(S \cup T)$.

2. Let $\mathcal{B}[a, b]$ denote the set of all bounded functions defined on the interval $[a, b]$. (That is, for each $f \in \mathcal{B}[a, b]$, there exist constants $k, l \in \mathbb{R}$ such that $k \leq f(x) \leq l$ for all $x \in [a, b]$.) The *norm* of a function $f \in \mathcal{B}[a, b]$ is defined by

$$\|f\| = \sup\{|f(x)| : x \in [a, b]\}.$$

   Prove that $\|f + g\| \leq \|f\| + \|g\|$ for any $f, g \in \mathcal{B}[a, b]$.

   Solution: By the triangle inequality, for any $x \in [a, b]$ we have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|,$$

since $|f(x)| \le \|f\|$ and $|g(x)| \le \|g\|$ for all $x \in [a,b]$. Thus, $\|f\| + \|g\|$ is an upper bound for $\{|f(x)+g(x)| \,|\, x \in [a,b]\}$. Since $|f+g|$ is defined to be the *least* upper bound of this set, we have

$$\|f+g\| \le \|f\| + \|g\|,$$

as required.

Note: In this problem it's important to distinguish between the *function* f and its *value* $f(x)$ at a particular $x \in [a,b]$. To establish that a particular fact holds for the function, you need to verify that it is true for *all* values of x.

3. Prove that if A is any nonempty open subset of $\mathbb{R}$, then $A \cap \mathbb{Q} \neq \emptyset$.

   Solution: Suppose $A \subseteq \mathbb{R}$ is open, and $A \neq \emptyset$. (Again, note that A being open does **not** imply that A is an interval!) Then there exists and $a \in A$ and $\epsilon > 0$ such that $N_\epsilon(a) = (a-\epsilon, a+\epsilon) \subseteq A$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that there exists a $q \in \mathbb{Q}$ with $a - \epsilon < q < a + \epsilon$. Therefore $q \in N_\epsilon(a) \subseteq A$, so $A \cap \mathbb{Q} \neq \emptyset$.

4. For any set $S \subseteq \mathbb{R}$, let $\overline{S}$ denote the intersection of all the closed sets containing S.

   (a) Prove that $\overline{S}$ is a closed subset of $\mathbb{R}$.

   Solution: Let $\mathcal{F} = \{F \subseteq \mathbb{R} | S \subseteq F \text{ and } F \text{ is closed}\}$. We know that the intersection of any family of closed subsets is closed (e.g. via Corollary 3.4.11 in the text). Therefore, $\overline{S} = \bigcap\limits_{F \in \mathcal{F}} F$ is closed.

   (b) Prove that $\overline{S}$ is the *smallest* closed set containing S. That is, show that $S \subseteq \overline{S}$, and if C is any closed set containing S, then $\overline{S} \subseteq C$.

   Solution: Let C be any closed set containing S. Then $C \in \mathcal{F}$, and we know that for any collection of sets $\mathcal{F}$, the intersection $\bigcap_{F \in \mathcal{F}} F$ is a subset of each set in the collection. Thus, $\overline{S} \subseteq C$, as required.

   (c) Prove that $\overline{S}$ is equal to the closure of S.

   Solution: Let $\operatorname{cl} S$ denote the closure of S. Since $\operatorname{cl} S$ is closed (e.g. via Theorem 3.4.17 in the text), and $S \subseteq \operatorname{cl} S$, we know that $\overline{S} \subseteq \operatorname{cl} S$, by part (b). Now, we need to show that $\operatorname{cl} S \subseteq \overline{S}$. If $x \in \operatorname{cl} S$, then either $x \in S$, in which case we have $x \in \overline{S}$, since $S \subseteq \overline{S}$, or x is a limit point of S. If we know that $x \in F$ for all $F \in \mathcal{F}$, then we'd have $x \in \overline{S}$ and we'd be done. Thus, it suffices to prove the following lemma:

   Lemma: If x is a limit point of a set S, then $x \in F$ for any closed set F with $S \subseteq F$.

   Proof: Suppose x is a limit point of S, and $S \subseteq F$, with F closed. Suppose that $x \notin F$. Then $x \in F^c$, the complement of F, which is open, since F is closed. Thus, there exists $\epsilon > 0$ such that $N_\epsilon(x) \subseteq F^c$. But $F^c \subseteq S^c$, since $S \subseteq F$, which means that $N_\epsilon(x) \subseteq S^c$, or $N_\epsilon(x) \cap S = \emptyset$. Since this contradicts the assumption that x is a limit point of S, it must be the case that $x \in F$.

(d) Prove that if S is bounded, then $\overline{S}$ is bounded as well.

Solution: If S is bounded, then $S \subseteq [a, b]$ for some $a, b \in \mathbb{R}$. But then $[a, b]$ is a closed set containing S, so $[a, b] \in \mathcal{F}$ and thus $\overline{S} \subseteq [a, b]$ by part (b).

5. The Nested Intervals Theorem (from the September 10th worksheet, and also mentioned on Piazza) states that if $\{A_n : n \in \mathbb{N}\}$ is a collection of closed bounded intervals (of the form $[a, b]$), and we have $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, then the intersection $\bigcap A_n$ is nonempty.

   Show that the intervals A_n need to be **both** closed and bounded by giving examples where the theorem fails (that is, where $\bigcap A_n = \emptyset$), if

   (a) The intervals A_n are closed, but not bounded.

   Solution: Consider the intervals $A_n = [n, \infty)$, for $n \in \mathbb{N}$. Each A_n is closed, since $\partial A_n = \{n\} \subseteq A_n$, and the A_n are not bounded. Moreover, we have that

$$\bigcap_{n=1}^{\infty} A_n = \emptyset,$$

   since for any $x \in \mathbb{R}$ there exists $N \in \mathbb{N}$ with $N > x$ (by the Archimedean property of $\mathbb{R}$), and thus $x \notin A_N$, so x cannot be in the intersection.

   (b) The intervals A_n are bounded, but not closed.

   Solution: This time we let $A_n = (0, 1/n)$, for $n \in \mathbb{N}$. Each A_n is bounded, since we have $A_n \subseteq [0, 1]$ for all n, but none of the A_n are closed, since $0 \in \partial A_n$ for all n, but $0 \notin A_n$. We then have that

$$\bigcap_{n=1}^{\infty} A_n = \emptyset,$$

   since any $x \in \mathbb{R}$ with $x \leq 0$ belongs to none of the A_n, and if $x > 0$, then there exists $N \in \mathbb{N}$ such that $1/N < x$, by the Archimedean property of $\mathbb{R}$, and hence $x \notin A_N$, and thus $x \notin \bigcap A_N$.

   Note: Pointing out that A_n is open is **not** the same as saying that it's not closed! There are sets which are both open and closed (i.e. $\mathbb{R}$ and $\emptyset$), and many sets which are neither open nor closed (e.g. $[0, 1)$).

6. An important theorem regarding compact sets is that if $S \subseteq \mathbb{R}$ is compact, and T is a closed subset of S, then T is compact. Prove this fact using:

   (a) The definition of compactness.

   Solution: We will use the same argument obtained during our class discussion: Let $S \subseteq \mathbb{R}$ be compact, and suppose $T \subseteq S$, with T closed in $\mathbb{R}$. Let $\{G_\alpha\}_{\alpha \in I}$ be an open cover of T, for some index set I. We need to show that there are finitely many $\alpha_1, \ldots, \alpha_k \in I$ such that

$$T \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_k}.$$

Since T is closed, $\mathbb{R} \setminus T$ is open. Since $T \cup (\mathbb{R} \setminus T) = \mathbb{R}$ and $T \subseteq \bigcup G_\alpha$, it follows that $S \subseteq (\bigcup G_\alpha) \cup (R \setminus T)$, so that $\{G_\alpha\}_{\alpha \in I} \cup \{\mathbb{R} \setminus T\}$ is an open cover of S. Since S is compact, this open cover must admit a finite subcover. Thus, we have

$$S \subseteq G_{\alpha_1} \cup \cdots G_{\alpha_k} \cup (\mathbb{R} \setminus T)$$

for some $\alpha_1, \ldots, \alpha_k \in I$. Since $T \cap (\mathbb{R} \setminus T) = \emptyset$, we must have $T \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_k}$, which is what we needed to show.

(b) The Heine-Borel theorem.

Solution: Let $S \subseteq \mathbb{R}$ be compact, and suppose $T \subseteq S$, with T closed in $\mathbb{R}$. Since S is compact, it is closed and bounded, by the Heine-Borel theorem. Since $T \subseteq S$, T must be bounded as well. (Any upper bound for S will be an upper bound for T, and likewise for lower bounds.) Since T is also assumed to be closed, we must have that T is compact, by the Heine-Borel theorem.