**Name:** **Tutorial time:**

1. Determine the Maclaurin polynomial $p_5$ (degree 5 Taylor polynomial, about $x = 0$) for the following functions:

   (a) $f(x) = \tan(x)$

   We compute the derivatives of $f$ at zero, as follows:

$$\begin{array}{ll}
f(x) = \tan(x) & f(0) = 0 \\
f'(x) = \sec^2(x) & f'(0) = 1 \\
f''(x) = 2\sec^2(x)\tan(x) & f''(0) = 0 \\
f^{(3)}(x) = 4\sec^2(x)\tan^2(x) + 2\sec^4(x) & f^{(3)}(0) = 2 \\
f^{(4)}(x) = 8\sec^2(x)\tan^3(x) + 16\sec^4(x)\tan(x) & f^{(4)}(0) = 0 \\
f^{(5)}(x) = 16\sec^2(x)\tan^4(x) + 88\sec^4(x)\tan^2(x) + 16\sec^6(x) & f^{(5)}(0) = 16
\end{array}$$

   Therefore,

$$\begin{aligned}
p_5(x) &= 0 + 1(x) + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5 \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5.
\end{aligned}$$

   (b) $g(x) = e^x \sin(x)$

   Our derivatives are given as follows:

$$\begin{array}{ll}
g(x) = e^x\sin(x) & g(0) = 0 \\
g'(x) = e^x\sin(x) + e^x\cos(x) & g'(0) = 1 \\
g''(x) = 2e^x\cos(x) & g''(0) = 2 \\
g^{(3)}(x) = 2e^x\cos(x) - 2e^x\sin(x) & g^{(3)}(0) = 2 \\
g^{(4)}(x) = -4e^x\sin(x) & g^{(4)}(0) = 0 \\
g^{(5)}(x) = -4e^x\sin(x) - 4e^x\cos(x) & g^{(5)}(0) = -4
\end{array}$$

   Our polynomial is therefore given by

$$\begin{aligned}
p_5(x) &= 0 + 1(x) + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 - \frac{4}{5!}x^5 \\
&= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5.
\end{aligned}$$

2. Determine the degree 4 Taylor polynomial for $f(x) = \cos(x)$ about $a = \pi/3$.

   We have

$$\begin{aligned} f(\pi/3) &= \cos(\pi/3) = \frac{1}{2} \\ f'(\pi/3) &= -\sin(\pi/3) = -\frac{\sqrt{3}}{2} \\ f''(\pi/3) &= -\cos(\pi/3) = -\frac{1}{2} \\ f'''(\pi/3) &= \sin(\pi/3) = \frac{\sqrt{3}}{2} \\ f''''(\pi/3) &= \cos(\pi/3) = \frac{1}{2}. \end{aligned}$$

   Our Taylor polynomial is therefore

$$p_4(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4.$$

3. Find a function $f(x)$ satisfying the given conditions:

   (a) $f'(x) = \dfrac{1}{1+x^2}$, and $f(0) = \dfrac{\pi}{4}$.

   We know that $\dfrac{d}{dx}(\arctan(x)) = \dfrac{1}{1+x^2}$, so we have $f(x) = \arctan(x) + C$ for some constant $C$. Since we must have $f(0) = 0 + C = \frac{\pi}{4}$, we have $C = \frac{\pi}{4}$, and thus

$$f(x) = \arctan(x) + \frac{\pi}{4}.$$

   (b) $f''(x) = 6x + 4$, $f(0) = 3$, and $f'(0) = -2$.

   Since $f''(x)$ is the derivative of $f'(x)$, we find that $f'(x)$ is given by the antiderivative

$$f'(x) = 3x^2 + 4x + C,$$

   for some constant $C$. The requirement that $f'(0) = -2$ gives us $C = -2$, so $f'(x) = 3x^2 + 4x - 2$. Taking the antiderivative again, we find

$$f(x) = x^3 + 2x^2 - 2x + D,$$

   for some constant $D$. Since we have to have $f(0) = 3$, it follows that $D = 3$, and thus $f(x) = x^3 + 2x^2 - 2x + 3$.