Math 4310 Assignment #1
University of Lethbridge, Fall 2014

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Due date: Friday, September 12, by 5 pm.

Practice Problems (do not submit)

From Sutherland: 3.1, 3.3, 3.5, 3.6, 3.9, 4.1, 4.4, 4.9, 4.13, all from Chapter 5.

Other problems:

1. Let $f : A \to B$ be given and let $\{X_\alpha\}_{\alpha\in I}$ be an indexed family of subsets of A. Prove:
   (a) $f(\bigcup_{\alpha\in I} X_\alpha) = \bigcup_{\alpha\in I} f(X_\alpha)$
   (b) $f(\bigcap_{\alpha\in I} X_\alpha) \subseteq \bigcap_{\alpha\in I} f(X_\alpha)$
   (c) If $f : A \to B$ is one-to-one, then $f(\bigcap_{\alpha\in I} X_\alpha) = \bigcap_{\alpha\in I} f(X_\alpha)$

2. Let $f : A \to B$ be given and let $\{Y_\alpha\}_{\alpha\in I}$ be an indexed family of subsets of B. Prove:
   (a) $f^{-1}(\bigcup_{\alpha\in I} Y_\alpha) = \bigcup_{\alpha\in I} f^{-1}(Y_\alpha)$
   (b) $f^{-1}(\bigcap_{\alpha\in I} Y_\alpha) = \bigcap_{\alpha\in I} f^{-1}(Y_\alpha)$
   (c) If X is a subset of B, then $f^{-1}(X^c) = (f^{-1}(X))^c$, where $X^c = B \setminus X$ denotes the complement of X in B (and similarly $(f^{-1}(X))^c$ denotes the complement of $f^{-1}(X)$ in A).
   (d) If X is a subset of A and Y is a subset of B, then $f(X \cap f^{-1}(Y)) = f(X) \cap Y$.

3. Let A be the set of all functions $f : [a, b] \to \mathbb{R}$ that are continuous on $[a, b]$. Let B be the subset of A consisting of all the functions possessing a continuous derivative on $[a, b]$. Let C be the subset of B consisting of all functions whose value at a is 0. Let $d : B \to A$ be the correspondence that associates with each function in B its derivative. Is the function d invertible?

   To each $f \in A$, let $h(f)$ be the function defined by $(h(f))(x) = \int_a^x f(t)\,dt$, for $x \in [a, b]$. Verify that $h : A \to C$. Find the function $g : C \to A$ such that these two functions are inverses of each other.

4. Let m, n be positive integers. Let X be a set with m distinct elements and Y a set with n distinct elements. How many distinct functions are there from X to Y? Let A be a subset of X with r distinct elements, with $0 \leq r \leq m$, and $f : A \to Y$. In how many distinct ways can we extend f to a function defined on all of X?

5. Prove that $(\mathbb{R}^n, d'')$ is a metric space, where the function d'' is defined by the correspondence

$$d''(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|,$$

for $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots y_n) \in \mathbb{R}^n$. In $(\mathbb{R}^2, d'')$ determine the shape and position of the set of points $\mathbf{x}$ such that $d(\mathbf{x}, \mathbf{y}) \leq 1$ for a given point $\mathbf{y} \in \mathbb{R}^2$.

6. Let (X_i, d_i), (Y_i, d'_i), $i = 1, \ldots, n$ be metric spaces. Let $f_i : X_i \to Y_i$ be continuous functions with respect to the given metrics. Let $X = \prod_{i=1}^{n} X_i$ and $Y = \prod_{i=1}^{n} Y_i$, and make X and Y into metric spaces as in the previous problem. Prove that the function $F : X \to Y$ defined by

$$F(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n))$$

is continuous.

7. Let $\delta(x, y)$ be a real-valued function on $X \times X$ for some set X, and suppose that for all $x, y, z \in X$ we have

   (i) $\delta(x, y) \geq 0$ and $\delta(x, y) = 0$ if and only if $x = y$.

   (ii) $\delta(x, y) \leq \delta(x, z) + \delta(y, z)$.

   Deduce that, in addition, $\delta(x, y) = \delta(y, x)$, and thus that $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$.

8. If d is a metric on a set X, show that $d_1(x, y) = \dfrac{d(x, y)}{1 + dx, y}$ and $d_2(x, y) = \min\{d(x, y), 1\}$ are also metrics on X.

Assigned problems (to be submitted)

1. Let A be a set and let $E \subseteq A$. The function $\chi_E : A \to \{0, 1\}$ defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$$

   is called the *characteristic function* of E. Let E and F be subsets of A.

   (a) Show that $\chi_{E \cap F} = \chi_E \cdot \chi_F$, where $\chi_E \cdot \chi_F(x) = \chi_E(x)\chi_F(x)$.

   (b) Show that $\chi_{E \cup F} = \chi_E + \chi_F - \chi_{E \cap F}$.

   (c) Find a similar expression for $\chi_{E \cup F \cup G}$

2. Let X denote the set of all continuous functions $f : [a,b] \to \mathbb{R}$, and let X' denote the set of all bounded functions $f : [a,b] \to \mathbb{R}$. (Note: by the Extreme Value Theorem, $X \subseteq X'$.)

   (a) For $f, g \in X$, define $d(f,g) = \int_a^b |f(t) - g(t)|\, dt$. Using appropriate theorems from calculus, prove that (X, d) is a metric space.

   (b) For $f, g \in X'$, define $d'(f,g) = \sup_{x\in[a,b]}(|f(x) - g(x)|)$. Prove that (X', d') is a metric space.

   (c) Since $X \subseteq X'$, the metric d' defines a metric on X by restriction. Compare the two metrics d and d' on X.

3. Given metric spaces $X_1, \dots, X_n$ with metrics $d_1, \dots, d_n$ respectively, let $X = \prod_{i=1}^n X_i$. Then the function $d : X \times X \to \mathbb{R}$ defined by

$$d(\mathbf{x}, \mathbf{y}) = \max_{1\le i\le n}\{d_i(x_i, y_i)\}.$$

   makes (X, d) into a metric space. Let d continue to denote the metric on $\mathbb{R}^n$ defined by this result, where each d_i is the usual absolute value metric on $\mathbb{R}$, let d' be the standard Euclidean metric on $\mathbb{R}^n$, and let d'' be the metric defined in problem #5 from the practice problems above. Prove that for each pair of points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &\le d'(\mathbf{x}, \mathbf{y}) \le \sqrt{n} \cdot d(\mathbf{x}, \mathbf{y}), \\ d(\mathbf{x}, \mathbf{y}) &\le d''(\mathbf{x}, \mathbf{y}) \le n \cdot d(\mathbf{x}, \mathbf{y}). \end{aligned}$$

4. Define a function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} \|x - y\|, & \text{if } \mathbf{x} = c\mathbf{y} \text{ for some } c \in \mathbb{R} \\ \|\mathbf{x}\| + \|y\| & \text{otherwise} \end{cases}$$

   Verify that d defines a metric on $\mathbb{R}^2$, and describe the open balls $B_a^d(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^2 \mid d(\mathbf{x}, \mathbf{y}) < a\}$ with respect to this metric. (Hint: if you get stuck, this metic is known as the "Paris Metro" metric. This should let you find help online – but cite your sources! Once you figure it out, take a look at a map of the Paris subway system to understand where the name comes from.)