MATH 1410 ASSIGNMENT #3 SOLUTIONS
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(1) An $n \times n$ matrix A is called **idempotent** if $A^2 = A$, where $A^2 = AA$.

(a) Show that the following matrices are idempotent:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

For each matrix A, we simply form the product $A^2 = A \cdot A$ as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+0(0) & 1(0)+0(1) \\ 0(1)+1(0) & 0(0)+1(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1(1)+1(0) & 1(1)+1(0) \\ 0(1)+0(0) & 0(1)+0(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \frac{1}{2}\left(\frac{1}{2}\right)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(b) Let I denote the $n \times n$ identity matrix. Show that if A is idempotent, then so is $I - A$, and that $A(I - A) = 0$.

Suppose that A is idempotent; that is, that $A^2 = A$. Then

$$(I-A)^2 = (I-A)(I-A) = I^2 - IA - AI + A^2 = I - A - A + A = I - A,$$

so $I - A$ is idempotent, and

$$A(I-A) = AI - A^2 = A - A = 0.$$

(c) Show that if A is an $n \times n$ idempotent matirx and B is any other $n \times n$ matrix, then

$$C = A + BA - ABA$$

is an idempotent matrix.

We have

$$\begin{aligned}
C^2 &= (A+BA-ABA)(A+BA-ABA) \\
&= A(A) + ABA - A(ABA) + (BA)(A) + (BA)(BA) - (BA)(ABA) - (ABA)(A) - (ABA)(BA) + (ABA)(ABA) \\
&= A^2 + ABA - A^2(BA) + B(A^2) + BABA - B(A^2)(BA) - AB(A^2) - ABABA + AB(A^2)(BA) \\
&= A + (ABA - ABA) + BA + (BABA - BABA) - ABA + (-ABABA + ABABA) \\
&= A + BA - ABA,
\end{aligned}$$

as required.

(2) Determine the matrix A such the matrix transformation $T\left(\begin{bmatrix} x\\ y\end{bmatrix}\right)=A\begin{bmatrix} x\\ y\end{bmatrix}$ perfoms the following transformations of the Cartesian plane, in order:
- First, a vertical reflection across the x-axis.
- Second, a horizontal reflection across the y-axis.
- Third, a counter-clockwise rotation through an angle of 90°.

Let $T_1(\vec{x})=A_1\vec{x}$, $T_2(\vec{x})=A_2\vec{x}$, $T_3(\vec{x})=A_3\vec{x}$ denote the three given transformations, in order.

From the textbook, we have:
$$A_1=\begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix},\quad A_2=\begin{bmatrix} -1 & 0\\ 0 & 1\end{bmatrix},\quad \text{and } A_3=\begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix},$$
and to perform the three transformations in the given order for an arbitrary vector $\vec{x}$ in $\mathbb{R}^2$, we must proceed as follows:

First, compute $\vec{x}_1=T_1(\vec{x})=A_1\vec{x}$.

Second, compute $\vec{x}_2=T_2(\vec{x}_1)=T_2(T_1(\vec{x}))=A_2(A_1\vec{x})$.

Third, compute $\vec{x}_3=T_3(\vec{x}_2)=T_3(T_2(T_1(\vec{x})))=A_3(A_2(A_1\vec{x}))=(A_3A_2A_1)\vec{x}$.

The vector $\vec{x}_3$ is our desired result from performing the three transformations on the vector $\vec{x}$. Thus, our overall transformation must be $T(\vec{x})=A\vec{x}$, where $A=A_3A_2A_1$, and we compute
$$A=A_3A_2A_1=\begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix} -1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix} -1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}.$$
(This, by the way, is the matrix for a *clockwise* rotation by 90°. Feel free to convince yourself that performing the three given transformations in order does indeed result in a clockwise rotation by 90°.)

(3) In each of the following, either explain why the statement is true, or give an example showing that it is false:

(a) If A is an $m\times n$ matrix where $m<n$, then $AX=B$ has a solution for every column B.

This statement is false. Consider the matrices $A=\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix} 1\\ 1\end{bmatrix}$. We see that A is a 2×3 matrix, and $2<3$, but clearly there does not exist a vector $X=\begin{bmatrix} x\\ y\\ z\end{bmatrix}$ such that $AX=B$, since $AX=\begin{bmatrix} 0\\ 0\end{bmatrix}$ for any vector X.

(b) If $AX=B$ has a solution for some column B, then it has a solution for every column B.

This statement is also false. Consider the matrix $A=\begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}$. The equation $AX=B$ has a solution for the column $B=\begin{bmatrix} 1\\ 0\end{bmatrix}$, since if $AX=\begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} x\\ 0\end{bmatrix}=\begin{bmatrix} 1\\ 0\end{bmatrix}=B$, then we have the solution $X=\begin{bmatrix} 1\\ 0\end{bmatrix}$. However, with the same matrix A, the equation

$AX = B$ does not have a solution for the column $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, since there are no values of x and y for which $\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) If X_1 and X_2 are solutions to $AX = B$, then $X_1 - X_2$ is a solution to $AX = 0$.

This statement is true. Suppose that X_1 and X_2 are solutions to $AX = B$; that is, that $AX_1 = B$ and $AX_2 = B$. Then

$$A(X_1 - X_2) = AX_1 - AX_2 = B - B = 0,$$

which shows that $X_1 - X_2$ is a solution to $AX = 0$.

(d) If $AB = AC$ and $A \neq 0$, then $B = C$.

This statement is false. Consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. We have

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = AC,$$

but clearly, $B \neq C$, since $1 \neq 2$.

(e) If $A \neq 0$, then $A^2 \neq 0$.

This statement is false. Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We have $A \neq 0$, since the $(1,2)$-entry of A is nonzero, but

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$