Practice for Quiz 13
Math 2580
Spring 2016

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If you can answer the following problems, you should be well-prepared for Quiz 13:

1. Evaluate $\displaystyle\iint_D (x^2+y^2)^{3/2}\,dA$, where D is the disc $x^2+y^2\leq 4$, using polar coordinates.

   In polar coordinates we have $(x^2+y^2)^{3/2} = (r^2)^{3/2} = r^3$, and the disc is given by $0\leq r\leq 2$, and $0\leq\theta\leq 2\pi$. Therefore,
$$\iint_D (x^2+y^2)^{3/2}\,dA = \int_0^{2\pi}\int_0^2 r^3\cdot r\,dr\,d\theta = 2\pi\left(\frac{2^4}{4}\right) = 8\pi.$$

2. Evaluate $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)\,dy\,dx$ by converting to polar coordinates.

   The bounds $-1\leq x\leq 1$, $-\sqrt{1-x^2}\leq y\leq\sqrt{1-x^2}$ describe the unit disc, so we have $0\leq r\leq 1$ and $0\leq\theta\leq 2\pi$, and thus
$$\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)\,dy\,dx = \int_0^{2\pi}\int_0^1 \sin(r^2)r\,dr\,d\theta = \pi(1-\cos(1)).$$

3. Evaluate $\displaystyle\iiint_W (x^2+y^2+z^2)^{5/2}\,dV$, where W is the ball $x^2+y^2+z^2\leq 1$.

   This integral is most easily done in spherical coordinates: the unit ball is given by $0\leq\rho\leq 1$, with $0\leq\phi\leq\pi$ and $0\leq\theta\leq 2\pi$. Since $\rho^2 = x^2+y^2+z^2$, we have
$$\iiint_W (x^2+y^2+z^2)^{5/2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^5\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 4\pi\left[\frac{\rho^8}{8}\right]_0^1 = \frac{\pi}{2}.$$

   Note: the integral $\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = 4\pi$ comes up often enough in spherical coordinates that you just end up remembering the value.

4. Evaluate $\iiint_W \frac{1}{(x^2+y^2+z^2)^{3/2}}\,dV$, where W is the solid bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, where $a,b>0$.

   In spherical coordinates we have $a \le \rho \le b$, with $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$, so we get

$$\iiint_W \frac{1}{(x^2+y^2+z^2)^{3/2}}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{1}{\rho^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 4\pi\ln\left(\frac{b}{a}\right).$$

5. Find the volume of the region enclosed by the cones $z=\sqrt{x^2+y^2}$ and $z=1-2\sqrt{x^2+y^2}$.

   This volume is most easily computed in cylindrical coordinates. The cones $z=r$ and $z=1-2r$ intersect when $3r=1$, or $r=\frac{1}{3}$. We thus have $0 \le r \le \frac{1}{3}$ and $0 \le \theta \le 2\pi$, so the volume is given by

$$V = \int_0^{2\pi}\int_0^{1/3}[(1-2r)-r]r\,dr\,d\theta = \frac{\pi}{27}.$$

6. Find the average of $f(x,y)=e^{x+y}$ over the triangle D with vertices at $(0,0)$, $(0,1)$, and $(1,0)$.

   We treat the region D as a Type 1 region, with $0 \le y \le 1-x$, and $0 \le x \le 1$. The area of the region is $A=\frac{1}{2}$, since it's a triangle with base 1 and height 1. The average of f is thus

$$f_{av} = \frac{1}{A}\iint_D e^{x+y}\,dA = 2\int_0^1\int_0^{1-x} e^x e^y\,dy\,dx = 2\int_0^1 e^x(e^{1-x}-1)\,dx = 2\int_0^1 (e-e^{-x})\,dx = 4e-2.$$