Complex Numbers

Math 1410 Linear Algebra

The complex number system

We define the set of complex numbers, denoted $\mathbb{C}$, as

$$\mathbb{C} = \{x + iy \,|\, x, y \in \mathbb{R}\},$$

where i denotes a (non-real) number with the property that $i^2 = -1$.

- Complex numbers date back to 16th-century Italy, and Cardano's *Ars Magna* (1545).
- Used (relucantly) by Bombelli to solve equations in 1572.
- Largely ignored as nonsense for 250 years. (Some dabbling by Euler around 1770.)
- Acceptance follows geometric interpretation by Gauss, Argand, and others at the end of the 18th century.
- Most development of the subject (by Cauchy, Riemann, et al) took place between 1814 and 1851.

Bombelli and the cubic

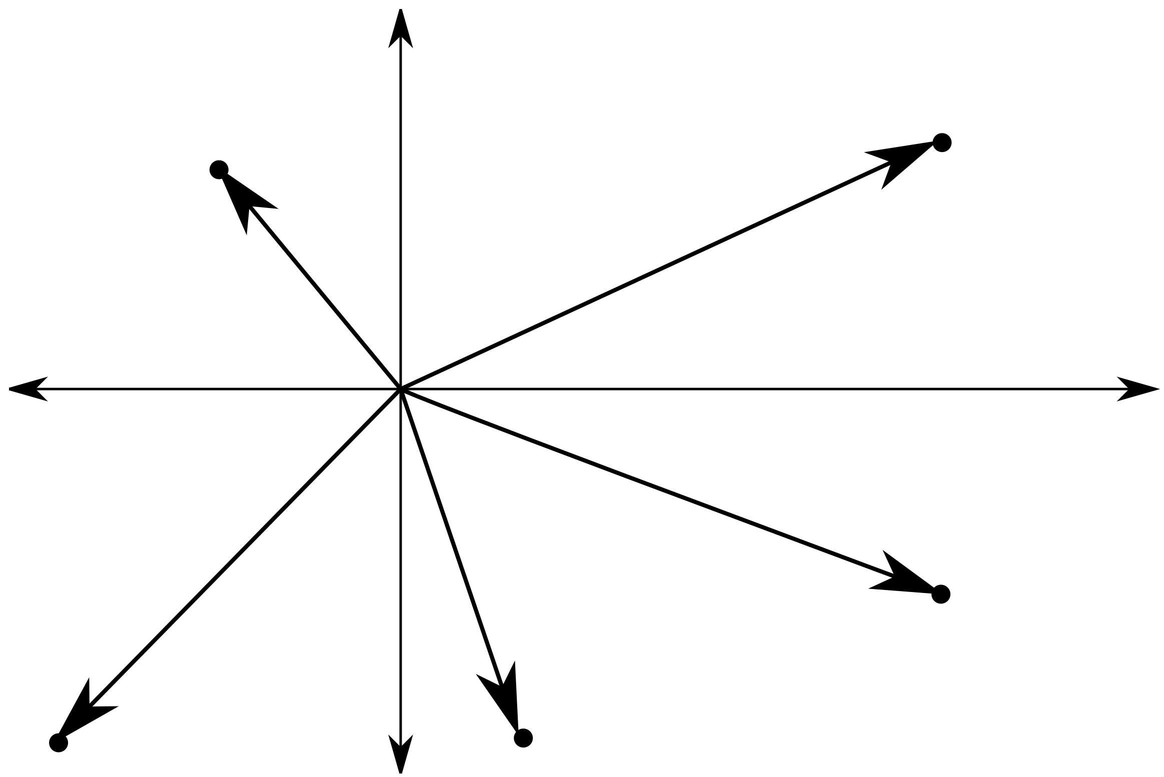
- Many texts (incorrectly) assert that complex numbers arose out of the need to solve quadratic equations like $x^2 + 1 = 0$.
- This is historically false: geometrically no solutions were expected.
- First compelling reason was the cubic equation $x^3 = 3px + 2q$.
- Cubic formula due to Cardano:

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

- Result is a real number even if there are negative numbers under the square roots.

The Argand Plane

Geometrically, we identify $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$. This visualization is usually called the Argand plane or Gauss plane, after the mathematicians who introducded this point of view.



Addition of complex numbers

Addition of complex numbers is the same as the addition of geometric vectors in $\mathbb{R}^2$:
If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then we define

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Examples:

Properties of addition

Addition in $\mathbb{C}$ follows the same rules as addition in $\mathbb{R}$ (or $\mathbb{R}^2$):

- $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in \mathbb{C}$.
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$
- $0 + z = z + 0 = z$ for all $z \in \mathbb{C}$, where $0 = 0 + i0$.
- Given $z = x + iy$, if we define $-z = -x - iy$, then $z + (-z) = -z + z = 0$.

Multiplication of complex numbers

A big difference between $\mathbb{C}$ and $\mathbb{R}^2$ (algebraically) is that we can multiply complex numbers. Given $z = x + iy$ and $w = u + iv$, zw is computed as a product of binomials, where we remember that $i^2 = -1$:

$$zw = (x+iy)(u+iv) = xu+ixv+iyu+i^2yv = (xu-yv)+i(xv+yu)$$

Examples:

Multiplicative inverses

Given $z \in \mathbb{C}$ with $z \neq 0$, can we find a complex number z^{-1} (or $1/z$) such that $zz^{-1} = 1$?
Say $z = x + iy$ and $w = u + iv$ satisfy $zw = 1$. Then

$$zw = (xu - yv) + i(xv + yu) = 1 = 1 + i0,$$

which gives a system of equations in u and v:

$$xu - yv = 1 \quad \text{and} \quad xv + yu = 0.$$

Solving gives $u = \dfrac{x}{x^2 + y^2}$ and $v = \dfrac{-y}{x^2 + y^2}$, which suggests:

$$\frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

Properties of complex multiplication

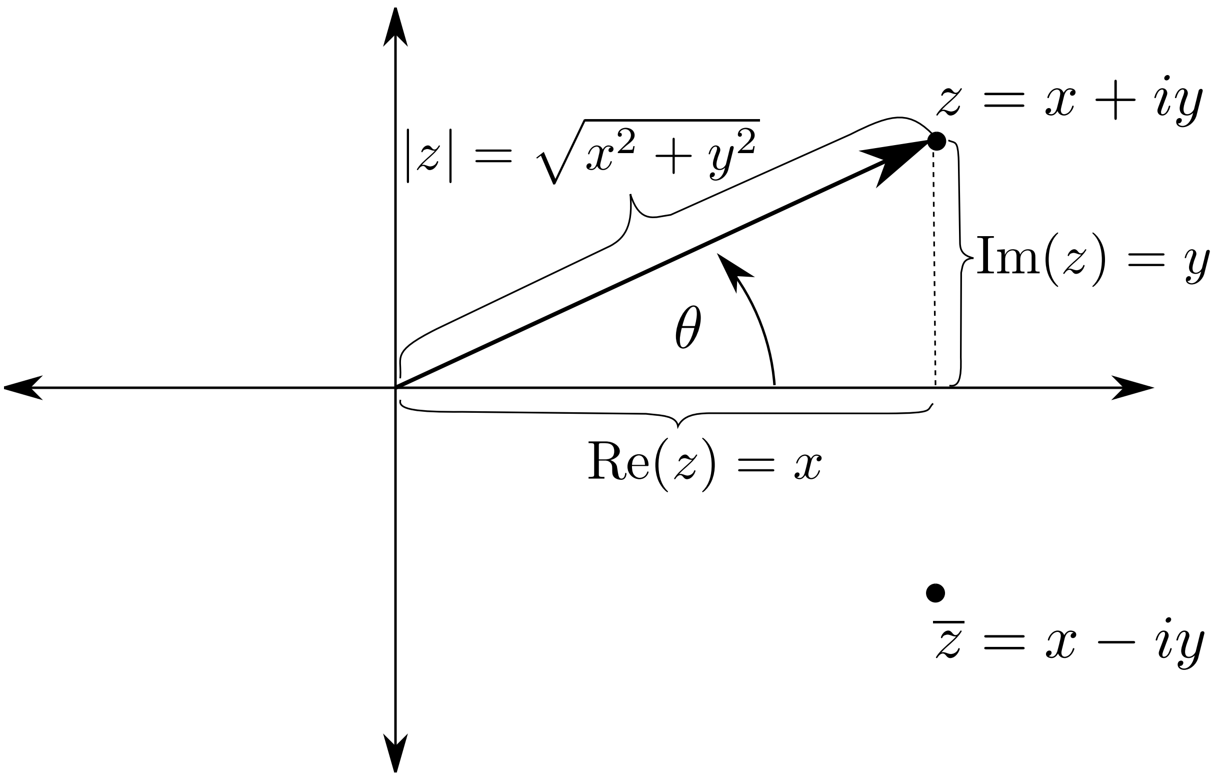
The results on the previous slide tell us that every nonzero complex number has a multiplicative inverse. One can check that the following properties all hold:

- $z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in \mathbb{C}$.
- $z_1(z_2 z_3) = (z_1 z_2) z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$.
- $1z = z1 = z$ for all $z \in \mathbb{C}$, where $1 = 1 + i0$.
- For all $z \neq 0$, there exists $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = z^{-1}z = 1$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$

Elements of complex numbers

For any $z = x + iy \in \mathbb{C}$, we define the following:

- The real part of z, $\operatorname{Re} z = x$.
- The imaginary part of z, $\operatorname{Im} z = y$.
- The complex conjugate of z, $\bar{z} = x - iy$.
- The modulus of z, $|z| = \sqrt{x^2 + y^2}$.
- The argument of z, $\arg z = \theta$, where $\tan \theta = \frac{y}{z}$.



Division in $\mathbb{C}$

We saw above that for every non-zero $z \in \mathbb{C}$, we can define $z^{-1} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$ such that $zz^{-1} = 1$.
In principle, this allows us to define division, but the result is hard to remember. Instead, we note the following:

Theorem

For all $z \in \mathbb{C}$, $z\bar{z} = |z|^2$.

Proof.

$\square$

How does it help? $|z|^2 = x^2 + y^2$ is a real number, and we know how to divide by real numbers.

Examples

A matrix model for $\mathbb{C}$

If you find the idea of defining a number i such that $i^2 = -1$, consider the following:

Let V denote the set of all 2×2 matrices. Let $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$, and recall that this matrix satisfies $A\mathbf{1} = \mathbf{1}A = A$ for all $A \in V$.

Now, let $U \subseteq V$ denote the subset

$$U = \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}.$$

Notice that if $A \in U$, then $A = x\mathbf{1} + y\mathbf{i}$, where $\mathbf{i} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The polar form of a complex number

Given $z = x + iy$, we have $|z| = \sqrt{x^2 + y^2}$, and if $\theta = \arg z = \tan^{-1}(y/x)$, then basic trigonometry tells us

$$\begin{aligned} x &= |z| \cos\theta \\ y &= |z| \sin\theta \end{aligned}$$

Thus, if we let $r = |z|$, we can write any $z \in \mathbb{C}$ in polar form as

$$z = r\cos\theta + ir\sin\theta.$$

Note: For the above to be well-defined, we have to define a "branch" of the argument: there are infinitely many values of θ that work. We'll require $\theta \in (-\pi, \pi]$. (Some texts take $\theta \in [0, 2\pi)$.)

Euler's Formula

Euler's identity is one of the most remarkable formulas in mathematics: we define the complex exponential $e^{i\theta}$ by

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This is usually taken as a definition, although there are several motivations for it. In particular, using the angle addition trigonometric identities, we find that

$$\begin{aligned}
e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\
&= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
&= e^{i\alpha}e^{i\beta}
\end{aligned}$$

Another famous result, called de Moivre's theorem, asserts that for all natural numbers n,

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta}.$$

Examples

1. Find the values of $e^{i\pi/4}$, $e^{i\pi/2}$, $e^{i\pi}$ and $e^{i2\pi/3}$.

2. Show that $e^{i\theta} = e^{i(\theta+2\pi k)}$ for all integers k.

Example

Establish trigonometric identities for $\cos 3\theta$ and $\sin 3\theta$.

Polar form, again

If we introduce polar coordinates $r = \sqrt{x^2 + y^2}$ and $\theta \in (-\pi, \pi]$ such that $\tan\theta = y/x$, we can write

$$z = |z|\cos\theta + i|z|\sin\theta = re^{i\theta},$$

with the help of Euler's theorem. This form of a complex number can be very convenient. For one thing, we will see that it makes finding roots of complex numbers much easier. For another, it gives us a geometric interpretation of complex multiplication.

Example

Compute $(1+i)^5$.

Roots of complex numbers

Polar coordinates make it easy to solve equations of the form $z^n = w$.
Given $w = u + iv$, re-write in polar form: $w = ae^{i\phi}$. If $z = re^{i\theta}$, then $z^n = w$ becomes

$$r^n e^{in\theta} = ae^{i\phi} = ae^{i(\phi+2\pi k)},\ k = 0, 1, 2, \ldots.$$

Solutions: $r = a^{1/n}$, and $\theta = \dfrac{\phi}{n} + \dfrac{2k}{n}\pi,\ k = 0, 1, 2, \ldots$

Examples

Solve the following equations: $z^6 = 1$, and $z^4 = 16i$.

The Fundamental Theorem of Algebra

Complex numbers don't just allow us to solve quadratic or cubic equations involving real numbers. In fact, we have the following remarkable theorem:

Theorem

Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *be any polynomial with complex coefficients* $a_0, a_1, \ldots, a_n \in \mathbb{C}$*,* $n \geq 1$*. Then p has a root.*

Consequence: every polynomial – even with complex coefficients – can be completely factored over $\mathbb{C}$.