University of Lethbridge
Department of Mathematics and Computer Science
15^{th} October, 2014, 5:00-5:50 pm
Math 4310 - Term Test I

Last Name: **SOLUTIONS**

First Name: **THE**

Student Number: ______________________

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

There is a list of potentially useful formulas available on the last page of the exam.

For grader's use only:

Page	Grade
2	/12
3	/8
4	/9
5	/6
Total	/35

1. For each of the following, give an example, or explain why no such example exists:

[3] (a) A subset of a topological space that is both open and closed.

Solution: Let $X = \mathbb{R}$ with the Euclidean topology. Then $\emptyset \in \mathbb{R}$ is both open and closed (as it is in any topological space).

[3] (b) A continuous function $f : X \to Y$, if X is equipped with the indiscrete topology.

Solution: We know that constant functions are continuous in any topology. To see this, note that if $f(x) = a$ for all $x \in \mathbb{R}$, for some $a \in Y$, then for any open subset $U \subseteq Y$ (in fact, for any subset), $f^{-1}(U) = X$ if $a \in U$, and $f^{-1}(U) = \emptyset$ if $a \notin U$. Thus, the topology $\{\emptyset, X\}$ is sufficient for f to be continuous.

[3] (c) An interior point that is not a limit point.

Solution: Let X have the discrete topology, let $x \in X$, and let $A = \{x\}$. Then x is in the interior of A, since A itself is an open neighbourhood of x contained in A. However, x cannot be a limit point of A since every neighbourhood of x would have to contain some $a \in A$ with $a \neq x$, and this is impossible if $A = \{x\}$.

Note: I'm pretty sure this is the only topology in which an interior point can fail to be a limit point.

[3] (d) A metric space that is not Hausdorff.

Solution: No such space can exist. Given any two points $x \neq y$ in a metric space (X, d), let $\epsilon = d(x, y)/2$. It follows that the open neighbourhoods $N_\epsilon(x)$ and $N_\epsilon(y)$ are disjoint, since if not, there exists some $z \in N_\epsilon(x)$ with $d(z, y) < \epsilon$. But then we have

$$d(x, y) \leq d(x, z) + d(z, y) < \epsilon + \epsilon = d(x, y),$$

and this is impossible.

2. Let $X = l^1(\mathbb{R}) = \left\{ \sum_{n=1}^{\infty} a_n \,\middle|\, \sum_{n=1}^{\infty} |a_n| < \infty \right\}$ be the space of absolutely convergent sequences

[8] of real numbers. Prove that the function $d : X \times X \to \mathbb{R}$ given by

$$d\left(\sum a_n, \sum b_n\right) = \sum_{n=1}^{\infty} |a_n - b_n|$$

is well-defined (i.e. that $d(x, y)$ is finite for all $x, y \in X$) and makes X into a metric space.

Solution: Let $x = \sum a_n$, $y = \sum b_n$ and $z = \sum c_n$ be absolutely convergent series. Since $\sum |a_n|$ and $\sum |b_n|$ converge and $|a_n - b_n| \leq |a_n| + |b_n|$ for all $n \in \mathbb{N}$, we see that $d(x, y) = \sum |a_n - b_n|$ converges by comparison.

Thus, we obtain a well-defined function $d : X \times X \to \mathbb{R}$. We now verify that d is a metric:

- Since $|a_n - b_n| \geq 0$ for all $n \in \mathbb{N}$, it follows that $d(x, y) \geq 0$ for all $x, y \in X$, and if $\sum_{n=1} \infty |a_n - b_n| = 0$ then we must have $|a_n - b_n| = 0$ for all $n \in \mathbb{N}$, and thus $x = y$.
- Since $|a_n - b_n| = |b_n - a_n|$ for all $n \in \mathbb{N}$, it follows that $d(x, y) = d(y, x)$ for all $x, y \in X$.
- Since $|a_n - b_n| = |a_n - c_n + c_n - b_n| \leq |a_n - c_n| + |c_n - b_n|$ for all $n \in \mathbb{N}$, it follows that for any $N \in \mathbb{N}$ we have

$$\sum_{i=1}^{N} |a_i - b_i| \leq \sum_{i=1}^{N} |a_i - c_i| + \sum_{i=1}^{N} |c_i - b_i| \leq \sum_{i=1}^{\infty} |a_i - c_i| + \sum_{i=1}^{\infty} |c_i - b_i|.$$

  Thus, $d(x, z) + d(z, y)$ is an upper bound for the increasing sequence $s_N = \sum_{i=1}^{N} |a_i - b_i|$, and since the limit of this sequence is $d(x, y)$, it follows that $d(x, y) \leq d(x, z) + d(z, y)$.

3. (a) Define what it means for a set $\mathcal{B}$ of subsets of a set X to be a **basis** for a topology on
[3] X.
(Either of the two definitions we discussed is acceptable.)

Solution: A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a **basis** for a topology on X if

i. $X \subseteq \bigcup_{B \in \mathcal{B}} B$

ii. For any $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Alternatively, if we are given a topology $\mathcal{T}_X$ for X, then a basis for $\mathcal{T}_X$ is a collection $\mathcal{B} \subseteq \mathcal{T}_X$ such that any $U \subseteq T_X$ can be written as a union of basic open subsets $B \in \mathcal{B}$.

(b) Let X and Y be topological spaces, and let $\mathcal{B}$ be a basis for the topology on Y. Prove that a function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X for every
[6] $U \in \mathcal{B}$.

Solution: If f is continuous and $U \in \mathcal{B}$, then U is open in Y, so $f^{-1}(U)$ is open in X. Conversely, suppose that $f^{-1}(U)$ is open for all $U \in \mathcal{B}$, and let V be any open subset of Y. Then there exists a collection $\{B_\alpha : \alpha \in I\} \subseteq \mathcal{B}$ such that $V = \bigcup_{\alpha \in I} B_\alpha$.

(Using the first definition above, we declare $V \subseteq Y$ to be open if for each $y \in Y$ there exists some $B_y \in \mathcal{B}$ with $y \in B_y \subseteq V$, and it follows that we can write $V = \bigcup_{y \in V} B_y$, but with either definition you can just state without justification that V is a union of basic open sets.)

It follows that $f^{-1}(V) f^{-1}\left(\bigcup_{\alpha \in I} B_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(B_\alpha)$ is a union of open subsets of X, and therefore is open.

[6] 4. Solve **one** of the following two problems:

(a) Let X, Y, and Z be topological spaces, and equip $X \times Y$ with the product topology. Show that a map $f : Z \to X \times Y$ is continuous if and only if the maps $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are continuous.
(Hint: one direction is easy. For the other, use 3(b).)

Solution: If f is continuous, then so are $\pi_X \circ f$ and $\pi_Y \circ f$, since they are the composition of continuous functions. Conversely, suppose that $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous. We wish to show that f is continuous. By 3(b), it suffices to prove that $f^{-1}(U \times V)$ is open in Z whenever U is open in X and V is open in Y. Letting U and V be open subsets of X and Y, respectively, we have

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}((U \times Y) \cap (X \times V)) \\ &= f^{-1}(U \times Y) \cap f^{-1}(X \times V) \\ &= f^{-1}(\pi_x^{-1}(U)) \cap f^{-1}(\pi_Y^{-1}(V)) \\ &= (\pi_X \circ f)^{-1}(U) \cap (\pi_Y \circ f)^{-1}(V), \end{aligned}$$

which is open in Z, since it's the intersection of open sets, due to the assumption that $pi_x \circ f$ and $\pi_Y \circ f$ are continuous.

(b) Given a topological space X, let X_0 denote the space with the same underlying set as X, but with the cofinite topology. Show that the identity map $I : X \to X_0$ (given by $I(x) = x$) is continuous if and only if X is a T_1 space.
Hint: X is T_1 if and only if finite point sets are closed.

Solution: Since I is the identity map, we have $f^{-1}(A) = A$ for any $A \subseteq X_0$. (Note $X = X_0$ as sets.) Suppose I is continuous. Then $I^{-1}(F) = F$ is closed in X whenever F is closed in X_0. But the closed sets of X_0 are the finite subsets, so every finite subset of X must be closed. Thus, X is T_1. Conversely, if X is T_1 and $F \subseteq X_0$ is closed, then F is finite, and $I^{-1}(F) = F$ is finite and therefore closed, so that I must be continuous.

 Total Points: 35