MATH 1410 - Tutorial #2 Solutions

Additional practice

1. Solve for z, if $(2+3i) = 4 + \frac{(2-i)z}{4z}$.

   If this problem seemed off to you, you're right! I didn't mean to include the z in the numerator on the right. Notice that $z \neq 0$, since we can't have zero in the denominator. But if $z \neq 0$ can cancel, giving the (false) equation

   $$2 + 3i = 4 + (2 - i).$$

   Since this is impossible, there is no solution.

2. Solve for z, if $(1-3i)z + 2i\overline{z} = 4$.

   We let $z = a + ib$, so $\overline{z} = a - ib$. Substituting, expanding, and simplifying, we have

   $$\begin{aligned}(1-3i)(a+ib) + 2i(a-ib) &= 4 \\ a - 3b(-1) + bi - 3ai + 2ai - 2b(-1) &= 4 \\ (a+5b) + i(-a+b) &= 4 = 4 + 0i.\end{aligned}$$

   Comparing real and imaginary parts, $a + 5b = 4$, and $-a + b = 0$. The latter tells us that $a = b$, so $b + 5b = 6b = 4$, giving $a = b = \frac{2}{3}$.

   Thus, $z = \frac{2}{3} + i\frac{2}{3}$.

3. Compute the magnitude $\|\vec{v}\|$ of the vector $\vec{v} = \langle 2, -3, 1\rangle$. Then find a unit vector $\vec{u}$ in the direction of $\vec{v}$

   By definition, the magnitude of $\vec{v} = \langle a, b, c\rangle$ is given by $\|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}$. Thus,

   $$\|\vec{v}\| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}.$$

   For any nonzero vector $\vec{v}$, an unit vector in the same direction is always given by $\vec{u} = \dfrac{1}{\|\vec{v}\|}\vec{v}$, so

   $$\vec{u} = \frac{1}{\sqrt{14}}\langle 2, -3, 1\rangle = \left\langle \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle.$$

Assigned questions:

1. Given $z = 2 - 2i$ and $w = \sqrt{3} + i$, compute the following.
   Answers can be left in either rectangular or polar form.

   (a) $2z - 3\overline{w} = 2(2-2i) - 3(\sqrt{3} - i) = (4-4i) + (-3\sqrt{3} + 3i) = (4 - 3\sqrt{3}) - i$

   For the next two, we note that $|z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ and $|w| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$. Thus

$$z = 2 - 2i = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}}\right)\right) = 2\sqrt{2}e^{-i\pi/4}$$
$$w = \sqrt{3} + i = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 2e^{i\pi/6}.$$

   (b) $\dfrac{z}{w^3} = zw^{-3} = (2\sqrt{2}e^{-i\pi/4})(2e^{i\pi/6})^{-3} = (2\sqrt{2}\cdot 2^{-3})e^{-i\pi/4 - i\pi/2} = \frac{\sqrt{2}}{4}e^{-3i\pi/4}$.

   If you used $7\pi/4$ instead of $-\pi/4$ for the argument of z, your answer will be $\dfrac{\sqrt{2}}{4}e^{5i\pi/4}$. Since this is a known angle on the unit circle, you have the option of converting back to rectangular:

$$\frac{z}{w^3} = \frac{\sqrt{2}}{4}e^{-3i\pi/4} = \frac{\sqrt{2}}{4}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -\frac{1}{4} - \frac{1}{4}i.$$

   Note that there are some other options for solving this problem:
   Blended: leave $z = 2 - 2i$, and note that

$$\frac{1}{w^3} = \frac{1}{8e^{i\pi/2}} = \frac{1}{8i} = -\frac{1}{8}i,$$

   so $\dfrac{z}{w^3} = (2-2i)\left(-\dfrac{1}{8}i\right) = -\frac{1}{4} - \frac{1}{4}i$.
   Rectangular: Using $\frac{z}{w^3} = \frac{z(\overline{w})^3}{w^3(\overline{w})^3}$, and noting that

$$w^3(\overline{w})^3 = (w\overline{w})^3 = ((\sqrt{3}+i)(\sqrt{3}-i))^3 = 4^3 = 64,$$

   you can compute

$$\frac{z}{w^3} = \frac{1}{64}(2-2i)(\sqrt{3}-i)(\sqrt{3}-i)(\sqrt{3}-i).$$

   (But I'm not sure why you'd want to.)

   (c) All 3 cube roots of w.
   Suppose $z = w^{1/3}$, so that $z^3 = w$. If $z = re^{i\theta}$, then $z^3 = r^3e^{i(3\theta)} = 2e^{i\pi/6} = w$. Equating the two polar coordinates, we get $r^3 = 2$, so $r = \sqrt[3]{2} = 2^{1/3}$, and

$$3\theta = \pi/6, \text{ or } 13\pi/6, \text{ or } 25\pi/6,$$

where we've obtained the other two angles by adding $2\pi = 12\pi/6$ once and then twice. Dividing by 3, we can solve for θ, giving us the three cube roots

$$w_0 = 2^{1/3}e^{i\pi/18}, w_1 = 2^{1/3}e^{13i\pi/18}, w_2 = 2^{1/3}e^{25i\pi/18}.$$

The roots are given with arguments in $[0, 2\pi)$, since $2\pi = 36\pi/18$. If the question specified that the arguments should be in $(-\pi, \pi]$, then you'd want to replace $25\pi/18$ with $-11\pi/18 = (\pi/6 - (2\pi))/3$.

An alternative approach: since $w = 2e^{i(\pi/6+2\pi k)}$, where $k = 0, 1, 2, \ldots$, we have

$$w^{1/3} = (2e^{i(\pi/6+2pik)})^{1/3} = 2^{1/3}e^{i(\pi/18+k\cdot 2\pi/3)}.$$

Putting $k = 0, 1, 2$ generates the same answers as above. ($k = -1$ can replace $k = 2$ for angles in $(-\pi, \pi]$.)

2. Compute the vector $\overrightarrow{AB}$, where $A = (2, -1, 3)$ and $B = (-4, 5, 2)$.

By definition, the components of $\overrightarrow{AB}$ are obtained by subtracting the coordinates of the tail A from corresponding coordinates of the tip B (tip-minus-tail). Thus,

$$\overrightarrow{AB} = \langle -4 - 2, 5 - (-1), 2 - 3\rangle = \langle -6, 6, -1\rangle.$$

3. Given $\vec{v} = \langle 2, -1, 4\rangle$ and $\vec{w} = \langle -1, 3, 0\rangle$, compute:

(a) $2\vec{v} - 3\vec{w}$.

Recall that we add vectors by adding corresponding components, and we multiply by a scalar by multiplying each component by that scalar. Thus,

$$2\vec{v} - 3\vec{w} = 2\langle 2, -1, 4\rangle - 3\langle -1, 3, 0\rangle = \langle 4, -2, 8\rangle + \langle 3, -9, 0\rangle = \langle 7, -11, 8\rangle.$$

(b) $\|\vec{v}\|$

Using the formula for $\|\vec{v}\|$ given in the solutions to the practice problems above, we have

$$\|\vec{v}\| = \sqrt{(2^2 + (-1)^2 + 4^2} = \sqrt{4 + 1 + 16} = \sqrt{21}.$$

(c) A vector in the same direction as $\vec{v}$, but three times as long.

Recall that when we multiply by a positive scalar c, $c\vec{v}$ points in the same direction as $\vec{v}$, while the length is changed according to the rule $\|c\vec{v}\| = c\|\vec{v}\|$. Thus, the desired vector is given by

$$3\vec{v} = 3\langle 2, -1, 4\rangle = \langle 6, -3, 12\rangle.$$