Traffic Flow

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Abstract

We discuss the modeling of traffic flow problems with nonlinear first-order partial differential equations, along with the existence and uniqueness of solutions to such problems. The method of characteristics is presented, along with applications of weak solutions, entropy conditions, shock waves, and rarefaction waves. This is used to model the scenarios of a stop light turning green from red, red from green, and a combination of the two.

1 Introduction

There are various ways that one can model traffic. One can look at the cars as discrete entities whose velocities are based on various factors: the road conditions, the distance to the nearest cars, and how safe a driver is. By forming a discrete model involving each individual car, we could potentially find a system of ordinary differential equations and use that to solve for the positions of each car. Another option, discussed in [1], is to take a continuum picture where we look at macroscopic variables such as densities instead of the positions of individual cars. This is similar to fluid or continuum mechanics, where one considers fluids or materials as smooth continuous objects rather than discrete atoms with space between them. Similarly to how fluid mechanics gives us the Navier-Stokes equations, we can use partial differential equations to model the car density. This comes in the form of a conservation law, a first-order PDE of the form

$$\rho_t + \nabla \cdot \mathbf{q}(\rho) = 0 \tag{1}$$

with some initial or boundary conditions. In this paper, we will limit the discussion to one spatial dimension and to initial value problems.

In order to create a continuum model of the traffic, we first take the traffic density $\rho(x,t)$ to be the density of cars, so that

$$N(a,b;t) = \int_a^b \rho \, dx \tag{2}$$

is the number of cars in the interval between a and b at time t. In the one-dimensional case, we can consider a single-lane road from $-\infty$ to ∞ with no entrances or exits- not exactly somewhere you'd want to be stuck in traffic. As in [2], we define the traffic flux $q(x,t)$ as the rate at which cars pass through a point, expressed in cars per unit time. Physically, $q(x,t) =$

$v(x,t)\rho(x,t)$, where $v(x,t)$ is the instantaneous velocity of a car at that point. Knowing that the change in the number of cars in an interval must equal the number of cars that exit and enter at its endpoints during that time (conservation of cars), we can equate the two rates and use the fundamental theorem of calculus (Stokes' theorem in higher dimensions) to get

$$\begin{aligned}\frac{d}{dt}N(a,b;t) &= -q(b,t) + q(a,t) \\ 0 &= \int_a^b \left[\rho_t + q_x\right] dx\end{aligned}$$

where $\rho_t \equiv \frac{\partial \rho}{\partial t}$, $q_x \equiv \frac{\partial q}{\partial t}$, etc.

In the case of our model, we will make the assumption in [1] that the velocity depends on the traffic density, meaning $v = v(\rho)$, and the traffic flux $q(\rho) = \rho v(\rho)$ can be considered to be a function $q : \mathbb{R} \to \mathbb{R}$. This is a reasonable approximation to make if we're assuming that all drivers drive similarly to each other and their speeds depend only on the distance to the nearest cars (which is captured in the density).

To proceed further, we must make some more technical assumptions about our functions. Let us take $\rho : \mathbb{R} \times [0,\infty) \to \mathbb{R}$, so that we have an initial value problem with $\rho = g$ on $\mathbb{R} \times \{t = 0\}$. On $\mathbb{R} \times [0,\infty)$, we will (for the time being) assume that ρ is in C^1, i.e. it is continuously differentiable. $q : \mathbb{R} \to \mathbb{R}$ is also taken to be C^1. Next, we prove a lemma:

Lemma 1.1. *Suppose that $\int_a^b f(x)dx = 0$ for all $a, b \in \mathbb{R}$, $a < b$, where $f : \mathbb{R} \to \mathbb{R}$ is C^0. Then $f(x) = 0$ for all $x \in \mathbb{R}$.*

Proof. First, suppose to the contrary that for some $x_0 \in \mathbb{R}$, $f(x_0) \neq 0$. Without loss of generality, take $f(x_0) > 0$ (in the negative case we instead consider $-f$ and come to the same result). Since f is continuous, there exists some $\delta > 0$ such that $|f(x) - f(x_0)| < f(x_0)/2$ for $x \in (x_0 - \delta, x_0 + \delta)$. For any x in this interval, either we have $f(x) \geq f(x_0) > 0$, or $f(x) < f(x_0)$, meaning $f(x_0) - f(x) < f(x_0)/2$, and so $f(x) > f(x_0)/2 > 0$. Thus $f(x) > 0$ in the interval. Setting $a = x_0 - \delta$ and $b = x_0 + \delta$, this implies that $\int_{x_0-\delta}^{x_0+\delta} f(x)dx > 0$, contradicting our original assumption that $\int_a^b f(x)dx = 0$. So, $f(x) = 0$ for all $x \in \mathbb{R}$. $\square$

Now, noting that for any given t, $\rho_t + q_x$ is continuous, our lemma says that the integrand is zero, thus giving us our PDE

$$\rho_t + q(\rho)_x = 0 \tag{3}$$

in $\mathbb{R} \times (0, \infty)$.

2 The Method of Characteristics

Our PDE can be written in the form

$$\rho_t + q'(\rho)\rho_x = 0 \tag{4}$$

which is a first-order quasi-linear PDE. The most general form for a first-order quasi-linear PDE with an initial condition is

$$a(x,t,\rho)\rho_x + b(x,t,\rho)\rho_t = c(x,t,\rho) \tag{5}$$

$$\rho(x,0) = g(x) \tag{6}$$

Note that this can be written in the form of a dot product,

$$\langle a,b,c\rangle \cdot \langle \rho_x, \rho_t, -1\rangle = 0,$$

where we are considering vector fields in $(x,t,u) \in \mathbb{R} \times [0,\infty) \times \mathbb{R}$. The graph of our solution will be a two-dimensional surface. As in [2], it can be defined either parametrically by $S = \{(x,t,u = \rho(x,t)) : (x,t) \in \mathbb{R} \times [0,\infty)\}$, or as the solution to $f(x,t,u) \equiv \rho(x,t) - u = 0$. (Similarly presented in [4, 6].) Note that the gradient of the latter can be written as $\nabla f = \langle \rho_x, \rho_t, -1\rangle$, which is the normal vector field to the surface. This means that our PDE says that the vector field $\langle a,b,c\rangle$ is tangent to the graph of the solution.

Consider now a curve C such that C is tangent to the vector field $\langle a,b,c\rangle$. Since the vector field is tangent to the graph, this means that if C is on the graph at one point, it will continue to stay on the graph. Given an initial value problem where we know the solution at $t = 0$, we can start a curve along the graph and use the tangency to the vector field to get first-order ordinary differential equations that continue the curve along the graph. These are called the characteristic curves of the PDE.

Note that in the homogeneous case, where $c = 0$, our tangent vector field will always lie flat in the x-t plane. Any curves which are tangent to the vector field will remain at constant u, and so along the characteristic curves, the solution remains constant.

Let us write $C = \{(x(s), t(s), u(s)) : s \in [0, \infty)\}$. The condition of tangency gives us the system of ODES

$$\frac{dx}{ds} = a(x, t, u), \quad \frac{dt}{ds} = b(x, t, u), \quad \frac{du}{ds} = c(x, t, u). \tag{7}$$

Assuming that at $s = 0$ the characteristic curve passes through x_0 at $t = 0$, the initial condition is

$$x(0) = x_0, \quad t(0) = 0, \quad u(0) = g(x_0). \tag{8}$$

As an example, consider the constant velocity transport equation

$$\rho_t + v\rho_x = 0. \tag{9}$$

Our characteristics are given by the solutions to

$$\frac{dx}{ds} = v, \quad \frac{dt}{ds} = 1, \quad \frac{du}{ds} = 0 \tag{10}$$

which are

$$(x, t, u) = (vs + x_0, s, g(x_0)). \tag{11}$$

In the x-t plane, these are straight lines with a slope of v. ρ is constant along these. Eliminating s and x_0, we find that our solution is

$$\rho(x, t) = g(x - vt). \tag{12}$$

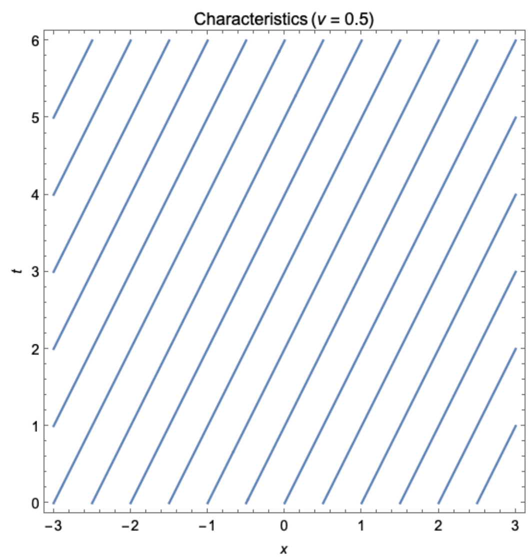

Figure 1: Characteristics for the constant velocity transport equation

The plot of the characteristics is shown in figure 1, showing that information travels to the right at a constant rate. If, for example, our initial function was $g(x) = \exp[-x^2]$, our solution would be $\rho(x) = \exp[-(x - vt)^2]$, shown in figure 2.

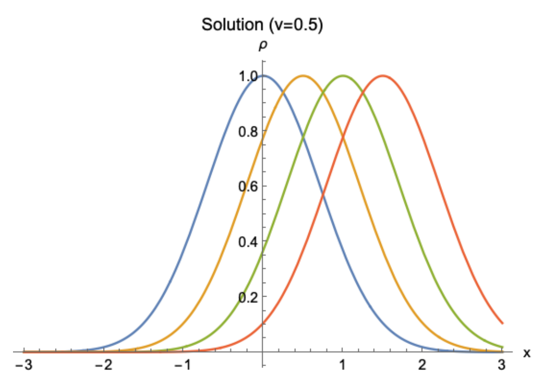

Figure 2: A solution to the constant velocity transport equation

Consider now the case we are studying, the conservation equation

$$\rho_t + q'(\rho)\rho_x = 0. \tag{13}$$

Our characteristics are given by

$$\frac{dx}{ds} = q'(u), \quad \frac{dt}{ds} = 1, \quad \frac{du}{ds} = 0. \tag{14}$$

The initial conditions give again $t = s$ and $u = g(x_0)$, so that $x = q'(g(x_0))t + x_0$. Our solution can then be written implicitly as

$$\rho = g(x - q'(\rho)t). \tag{15}$$

However, this is not guaranteed to be continuously differentiable, and so may not be a valid solution to the problem. We can see this by noting that

$$\rho_t = -\frac{q'(\rho)g'(x - q'(\rho)t)}{1 + q''(\rho)g'(x - q'(\rho))t} \tag{16}$$

$$\rho_x = \frac{g'(x - q'(\rho)t)}{1 + q''(\rho)g'(x - q'(\rho))t}. \tag{17}$$

So if we have

$$1 + q''(\rho)g'(x - q'(\rho))t = 0 \tag{18}$$

then we may have a singularity in the derivatives at this point. It's also possible that the implicit equation may have no solutions or multiple solutions at a point - this may happen if we have intersecting characteristics or regions without any characteristics. Additionally, we may desire to have a discontinuous initial value, such as a road with a constant density of cars behind a stop light that turns green at $t = 0$, or perhaps an initial value that isn't even a function, such as a Dirac delta.

3 Weak Solutions

When dealing with a partial differential equation where we desire solutions that may not be continuous or differentiable, we must generalize our notion of derivative. We call $\phi : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ a test function if it is smooth and has compact support (it is zero outside of a compact set in $\mathbb{R} \times [0, \infty)$), i.e., $\phi \in C_c^\infty(\mathbb{R} \times [0, \infty))$. Consider any function $\rho \in L^\infty(\mathbb{R} \times [0, \infty))$, meaning ρ is Lebesgue measurable and is essentially bounded (there is some constant $K \in \mathbb{R}$ such that $|\rho(x,t)| \leq K$ for all (x,t) except on a set of measure zero). Technically, ρ is an equivalence class of functions - for more details, refer to [7]. Now, if ρ were differentiable, say, ρ_x and ρ_t existed, we could multiply the PDE by ϕ and integrate by parts to find:

$$\begin{aligned}
0 &= \int_0^\infty \int_{-\infty}^\infty (\rho_t + q(\rho)_x)\phi \, dx \, dt \\
&= \int_{-\infty}^\infty \rho\phi|_{t=0}^\infty \, dx - \int_0^\infty \int_{-\infty}^\infty \rho\phi_t \, dx \, dt \\
&+ \int_0^\infty q(\rho)\phi \, |_{x=-\infty}^\infty \, dt - \int_0^\infty \int_{-\infty}^\infty q(\rho)\phi_x \, dx \, dt \\
&= \int_{-\infty}^\infty g(x)\phi(x,0) \, dx + \int_0^\infty \int_{-\infty}^\infty (\rho\phi_t + q(\rho)\phi_x) \, dx \, dt
\end{aligned}$$

where we have used ϕ being compact. The last line gives us an alternative to the PDE, as it involves no derivatives of ρ. Relaxing the condition on g so that $g \in L^\infty(\mathbb{R})$, the final line involves integrating g or ρ against test functions. If this holds for all test functions ϕ, we say that ρ is a weak solution of the continuity equation (defined in [2]).

Lemma 3.1. *Let $\Omega \in \mathbb{R}^n$ be some open set. Suppose that $\int_\Omega f\phi \, d^n x = 0$ for* *all* $\phi \in C_c^\infty(\Omega)$, $f \in C^0(\Omega)$. *Then* $f = 0$.

Proof. We prove this similarly to Lemma 1.1. First, suppose to the contrary that for some $x_0 \in \Omega$, $f(x_0) \neq 0$. Without loss of generality, take $f(x_0) > 0$. Since f is continuous, there is some bounded open ball N such that $f(x) > 0$ for all $x \in N$. By the existence of bump functions in topology from [5], there exists a positive test function ϕ that is 0 outside of N. For this ϕ, we then must have

$$\int_\Omega f\phi \, d^n x = \int_N f\phi \, d^n x > 0$$

however, as ϕ is a test function this is a contradiction. Thus $f = 0$ everywhere on Ω. $\square$

Theorem 3.2. *Suppose that* ρ *is a weak solution to the continuity equation, and additionally, it is differentiable in* $\mathbb{R} \times [0, \infty)$. *Suppose also that* g *is continuous. Then* ρ *is a (non-weak) solution to the continuity equation.*

Proof. As ρ is a weak solution, it satisfies

$$-\int_{-\infty}^{\infty} g(x)\phi(x,0)\, dx - \int_0^\infty \int_{-\infty}^{\infty} (\rho\phi_t + q(\rho)\phi_x)\, dx\, dt = 0.$$

for all test functions ϕ. From integration by parts, we also have that

$$\begin{aligned} 0 = &-\int_0^\infty \int_{-\infty}^{\infty} (\rho_t + q(\rho)_x)\phi\, dx\, dt - \int_{-\infty}^{\infty} \rho(x,0)\phi(x,0)\, dx \\ &- \int_0^\infty \int_{-\infty}^{\infty} (\rho\phi_t + q(\rho)\phi_x)\, dx\, dt \end{aligned}$$

and on subtracting the first equation from the second, we have

$$0 = \int_0^\infty \int_{-\infty}^\infty (\rho_t + q(\rho)_x)\phi \, dx \, dt + \int_{-\infty}^\infty (\rho(x,0) - g(x))\phi(x,0) \, dx$$

for all test functions ϕ. First, we show that the first term vanishes. If, to the contrary, there were some point (x_0, t_0), $t_0 > 0$, such that $\phi_t + q(\rho)_x \neq 0$, then we could choose a neighborhood N' around (x_0, t_0) where $\phi_t + q(\rho)_x$ was non-zero (using that it is continuous). Let N be the neighborhood $N \cap \mathbb{R} \times (t_0/2, \infty)$ so that it does not intersect the x-axis; the integrand will still be positive. We may choose a test function ϕ that is zero everywhere outside of N and positive inside of N. Thus, $(\rho_t + q(\rho)_x)\phi$ must be either positive or negative everywhere inside of N, and as it integrates to 0 outside N, the first integral must be either positive or negative, and thus non-zero. As we have chosen ϕ to be zero outside of N, and thus zero on the x-axis, the second integral vanishes. The total integral thus must be non-zero, giving us a contradiction. So, we must have $\phi_t + q(\rho)_x = 0$ at all points in the domain where $t_0 > 0$.

Now, since the first integral does not take place on the x-axis, it must be equal to zero as its integrand is zero. The second integral must then be equal to zero as well. By Lemma 3.1, this means that $\rho(x, 0) - g(x) = 0$, and so ρ satisfies the initial condition. Thus, any differentiable weak solution with a continuous initial condition is a non-weak solution as well. $\square$

4 Shock Waves

As a motivating example, consider Burgers' equation, the conservation equation with $q(\rho) = \frac{1}{2}\rho^2$:

$$\rho_t + \rho\rho_x = 0 \tag{19}$$

$$\rho(x, 0) = g(x) \tag{20}$$

This looks like the transport equation, but with a velocity of ρ. We can think of this as traveling waves of amplitude ρ, but where the velocity of the waves in the $+x$ direction is equal to the amplitude of the wave. Note that this velocity is different than the velocity in defining $q(\rho) = v(\rho)\rho$: that is the velocity of individual particles, the phase velocity, whereas here $q'(\rho) = \rho$ is the velocity of the wave, the velocity that information (i.e., the characteristics) travels at, the group velocity. If we wish to keep thinking of ρ as density, this could be air pressure waves. Our characteristics are solved by

$$\frac{dx}{ds} = u, \quad \frac{dt}{ds} = 1, \quad \frac{du}{ds} = 0 \tag{21}$$

which are

$$(x, t, u) = (g(x_0)s + x_0, s, g(x_0)). \tag{22}$$

Again, our characteristics are all straight lines in the x-t plane. ρ is constant along them, but their slopes are now whatever the initial value of ρ was at where they intersected the x-axis.

However, suppose that we have a case where $g(x)$ is decreasing. This can lead to the case where the higher-slope characteristics on the left can overtake and intersect with the lower-slope characteristics on the right. Consider the characteristics with the initial condition

$$g(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0. \end{cases} \tag{23}$$

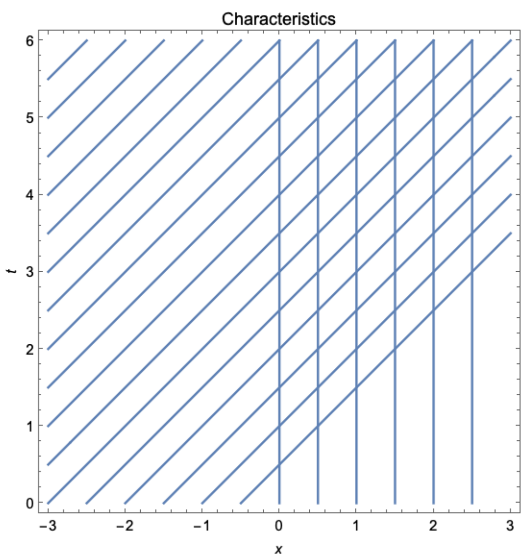

Figure 3: Characteristics for Burgers' equation with a decreasing discontinuity

The characteristics are shown in figure 3. In the region to the left of the intersection, we have $\rho = 1$ by extending the characteristics, and to the region of the right, we have $\rho = 0$. However, we have no way to decide between the two solutions within the intersection region. Note that our initial condition is discontinuous, so we need to expect that we have a weak solution. Even if we didn't have a discontinuity in the initial condition, we would have still found that, at some point, the characteristics intersect, causing a discontinuity.

Suppose that we have a solution to the PDE that is C^1, except for on some curve of discontinuity C, that we assume is also C^1. We take R to be an open region within $\mathbb{R} \times (0, \infty)$ (not intersecting the x-axis) that C intersects, and R_l and R_r to be the two halves that C divides R into, the left and right, respectively. As in [3], we will analyze the definition of weak solutions in these regions:

$$0 = \int_{-\infty}^{\infty} g(x)\phi(x,0)\,dx + \int_0^{\infty}\int_{-\infty}^{\infty} (\rho\phi_t + q(\rho)\phi_x)\,dx\,dt \tag{24}$$

Say that $\rho = \rho_l$ in R_l and $\rho = \rho_r$ in R_r. First, we note that in R_-, our original PDE holds for ρ_l. This is because if we take a test function ϕ with compact support in R_l only (it is zero outside of R_l), we have

$$\begin{aligned}
0 &= \int_0^{\infty}\int_{-\infty}^{\infty} (\rho\phi_t + q(\rho)\phi_x)\,dx\,dt \\
&= \iint_{R_l} (\rho_l\phi_t + q(\rho_l)\phi_x)\,dx\,dt \\
&= \int_0^{\infty}\int_{-\infty}^{\infty} (\rho_l\phi_t + q(\rho_l)\phi_x)\,dx\,dt \\
&= -\int_0^{\infty}\int_{-\infty}^{\infty} ((\rho_l)_t + q(\rho_l)_x)\phi\,dx\,dt
\end{aligned}$$

where we have used integration by parts, noting that u_l is C^1 in R_l, and that the integrand vanishes on the boundary. Since this holds for all test functions ϕ with compact support in R_l, then by Lemma 3.1 we have

$$(\rho_l)_t + q(\rho_l)_x = 0. \tag{25}$$

Similarly, this holds for ρ_r in R_r. Now, take ϕ to have compact support in all of R. We get

$$0 = \iint_{R_l} (\rho\phi_t + q(\rho)\phi_x)\, dx\, dt + \iint_{R_r} (\rho\phi_t + q(\rho)\phi_x)\, dx\, dt. \tag{26}$$

Consider taking a line integral around R_l. Letting $\mathbf{r}$ be the parametrization of C going counter-clockwise around R_l, $C = \{(x(t), t) : t \in [t_1, t_2]\}$, we find that

$$\begin{aligned}
\oint_{\partial R_l} (\rho_l, -q(\rho_l))\phi \cdot d\mathbf{r} &= \int_C (\rho_l, -q(\rho_l))\phi \cdot d\mathbf{r} \\
&= \iint_{R_l} ((\rho_l\phi)_t + (q(\rho_l)\phi)_x)\, dx\, dt \\
&= \iint_{R_l} ((\rho_l)_t\phi + q(\rho_l)_x\phi + \rho_l\phi_t + q(\rho_l)\phi_x)\, dx\, dt \\
&= \iint_{R_l} (\rho\phi_t + q(\rho)\phi_x)\, dx\, dt.
\end{aligned}$$

Similarly, but noting that we traverse C in the opposite direction, we have

$$-\int_C (\rho_r, -q(\rho_r))\phi \cdot d\mathbf{r} = \iint_{R_r} (\rho\phi_t + q(\rho)\phi_x)\, dx\, dt.$$

Putting it altogether, we then have

$$\begin{aligned}
0 &= \int_C (\rho_l - \rho_r, -q(\rho_l) + q(\rho_r))\phi \cdot d\mathbf{r} &(27)\\
&= \int_{t_1}^{t_2} \left(\frac{dx}{dt}(\rho_l - \rho_r) - (q(\rho_l) - q(\rho_r)) \right) \phi|_C\, dt. &(28)
\end{aligned}$$

Using Lemma 3.1 implies that the integrand is zero, as this is for any test function on the curve. Defining σ to be the speed of the curve of discontinuity, $\frac{dx}{dt}$, and $[[f]]$ to be the change in a quantity across the curve from left to right, we get the Rankine-Hugoniot condition [3]:

$$[[q(\rho)]] = \sigma[[\rho]]. \tag{29}$$

Returning to our example, the discontinuity curve is thus found from

$$\frac{dx}{dt} = \frac{q(1) - q(0)}{1 - 0} = \frac{1}{2}.$$

Since the discontinuity begins at $(0, 0)$, the curve given by $x = \frac{t}{2}$. The solution is 1 to the left of this curve and 0 to the right. This is the curve of a forward-traveling shock wave: the discontinuity moves forward in time. The characteristics are shown in figure 4.

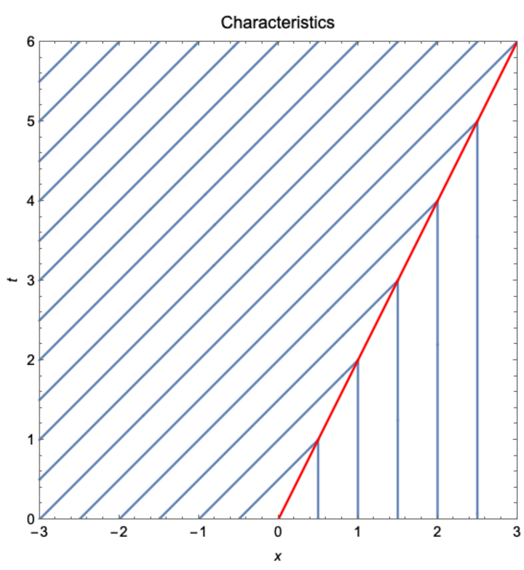

Figure 4: Characteristics for Burgers' equation with a shock (shown in red)

The solution to the example is then

$$\rho(x,t) = \begin{cases} 1 & x < \frac{t}{2} \\ 0 & x \geq \frac{t}{2} \end{cases} \tag{30}$$

5 The Entropy Condition

Consider now our previous example, but with the initial condition instead of

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases} \tag{31}$$

We now have a region with no characteristics (figure 5) - too little information rather than too much.

One option is to form a shock in the unknown region, shown in figure 6. If we decide it to be 0 on the left and 1 on the right, the Rankine-Hugoniot condition says that the shock is given by

$$\frac{dx}{dt} = \frac{q(0) - q(1)}{0 - 1} = \frac{1}{2}$$

and so again is $x = \frac{t}{2}$. The characteristics will flow from the shock.

However, we could have decided the characteristics coming from the shock to have been different slopes (figure 7). Had it been $\frac{1}{2}$ on the left and 1 on the right instead, we would have had $\sigma = \frac{3}{4}$, and $x = \frac{3}{4}t$. This would have created another region without characteristics, which we could have had be

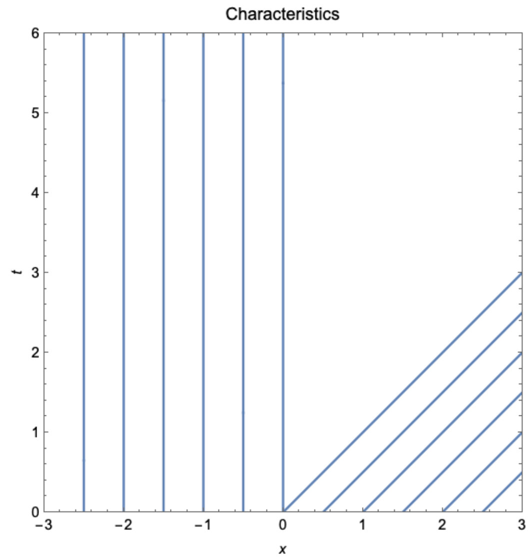

Figure 5: Characteristics for Burgers' equation with an increasing discontinuity

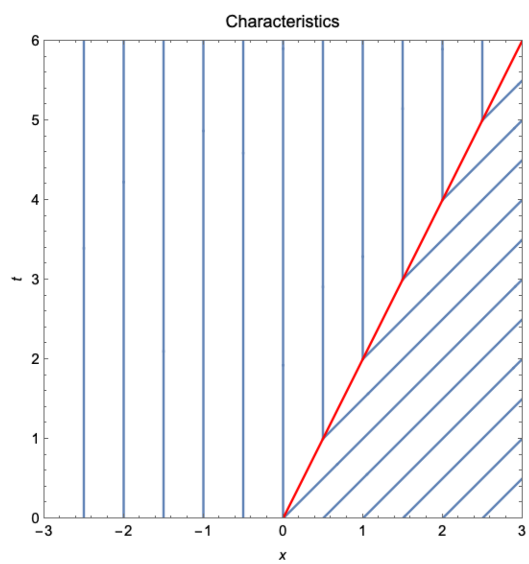

Figure 6: Adding in a shock

0 on the left and $\frac{1}{2}$ on the right, to create a second shock with $\sigma = \frac{1}{4}$ and $x = \frac{1}{4}t$.

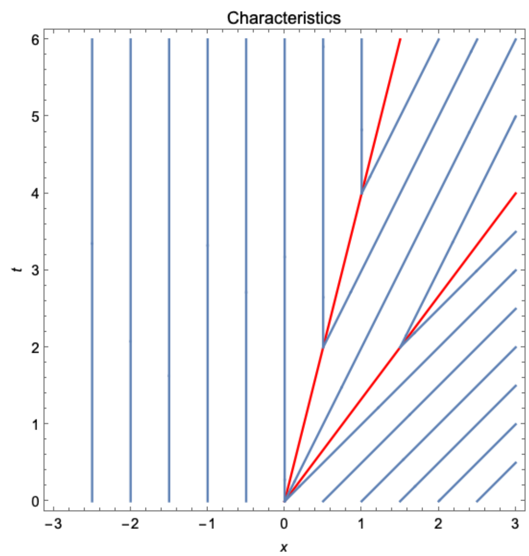

Figure 7: Adding in two shocks

This process can, in fact, be repeated ad infinitum until there is no curve of discontinuity at all when $t > 0$. This is done by creating a rarefaction wave, where the characteristics fan out from $(0, 0)$.

From the plot in figure 8, we can see that the characteristic in the region passing through a point (x, t) has slope $\frac{x}{t}$, and so this solution has

$$\rho(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 \leq x < t \\ 1 & x \geq t \end{cases} . \tag{32}$$

We see that we have a problem: we have a potentially infinite set of

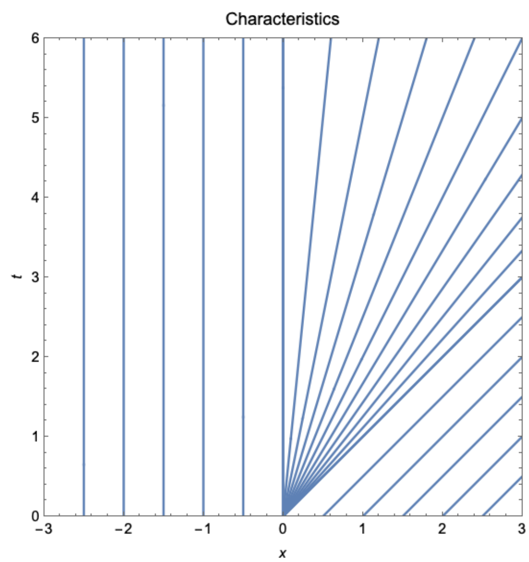

Figure 8: Adding in a rarefaction wave

weak solutions to choose from. To choose one, we must impose a condition on the set of solutions. One possible condition is that characteristics don't originate from curves of discontinuity. This is called the entropy condition and means that shocks don't generate information via characteristics; they can only absorb it [3]. More quantitatively, suppose that we have a characteristic intersecting a curve of discontinuity of speed σ. If it is from the left, then the speed of the characteristic must be greater than the characteristic, or else it would be "catching up" to it. If it is intersecting from the right, it must be slower. As the speed of a characteristic is $q'(\rho)$, we thus get the condition:

$$q'(\rho_l) > \sigma > q'(\rho_r). \tag{33}$$

For our example, any discontinuity in the unknown region would require

$$\rho_l > \sigma > \rho_r. \tag{34}$$

As even creating multiple discontinuities would require at least one on either side, we would need some discontinuity with $0 > \sigma > 1$, so any solution with discontinuities does not satisfy the entropy condition.

Though it is beyond the scope of this paper, one can prove that (after putting the entropy condition in a more precise form) if $q(\rho)$ is convex, then there is a unique weak solution that satisfies the entropy condition. See chapter 3.4.3 of [3] for more details.

Theorem 5.1. *Suppose that for the continuity equation, we have a rarefaction wave originating from (x_1, t_1) when $t > t_1$. Additionally, suppose that q' is invertible. Within the rarefaction wave, the solution is given by*

$$\rho(x,t) = G\left(\frac{x - x_1}{t - t_1}\right) \tag{35}$$

where $G = (q')^{-1}$. If the boundaries of the rarefaction wave are also characteristics extending from (x_1, t_1), it will be continuous.

Proof. Consider any point (x, t) within the rarefaction wave. The slope of a straight line between (x, t) and (x_1, t_1) is given by $\frac{x-x_1}{t-t_1}$. As this is one of the characteristics in the rarefaction wave by assumption (the characteristics of the continuity equation are straight lines), the slope must be equal to

$$\frac{x - x_1}{t - t_1} = q'(\rho(x_1, t_1)). \tag{36}$$

However, since ρ is constant along characteristics, we must have $\rho(x,t) = \rho(x_1,t_1)$. Thus, since q' is invertible, we have

$$\rho(x,t) = G\left(\frac{x-x_1}{t-t_1}\right). \tag{37}$$

By our assumptions on q, G is continuous, and thus the solution is continuous within the rarefaction wave, so long as $t > t_1$. If the boundaries outside of the wave are also characteristics extending from (x_1,t_1), they will be constant slope lines as well, and as ρ is determined by the slope of the characteristics, they must be equal to the boundaries inside the wave. □

We can see that this solution satisfies the PDE when $t > t_1$, assuming G is differentiable (see [6]). We have:

$$\begin{aligned}\rho_t + q'(\rho)\rho_x &= -G'\left(\frac{x-x_1}{t-t_1}\right)\frac{x-x_1}{(t-t_1)^2} + \frac{x-x_1}{t-t_1}G'\left(\frac{x-x_1}{t-t_1}\right)\frac{1}{t-t_1} \\ &= 0.\end{aligned}$$

6 Traffic Flow

In order to model traffic flow, we must decide on a phase velocity $v(\rho)$ for an individual car. Two possible assumptions to make are that there is a maximum velocity of cars, v_0, and a maximum density of cars ρ_0. At zero density, $\rho = 0$, the cars will attain their maximum speed, and at the maximum density, $\rho = \rho_0$, we will have a traffic jam where the speed of the cars is zero. Rescaling our units, we can choose to set $\rho_0 = v_0 = 1$, giving us the conditions

$$v(0) = 1 \tag{38}$$

$$v(1) = 0. \tag{39}$$

We can also assume that $v(\rho)$ is non-increasing: increasing the density of cars will never cause the individual cars to get faster. A simple model that is used in [1] is a linear model, where we take

$$v(\rho) = 1 - \rho \tag{40}$$

$$q(\rho) = \rho(1 - \rho) \tag{41}$$

$$q'(\rho) = 1 - 2\rho \tag{42}$$

shown in figure 9.

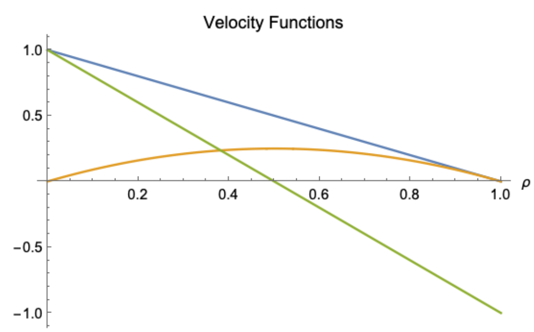

Figure 9: Phase velocity, flux, and group velocity

We note something interesting here: while $v(\rho)$, the velocity of an individual car, is always positive, $q'(\rho)$, the velocity that information travels as, can be negative. This means that we can have characteristics travelling both to the left and the right. This shouldn't be too surprising: an individual car passes from characteristic to characteristic and by our model will always go forward, but the density and thus the speed of the traffic itself goes backward. We can expect that high-density traffic will propagate backwards: traffic jams travel behind you. A more complicated theoretical model can also be found in [1], though we will be using the linear model in the following situations.

6.1 Red Light to Green Light

For our first example, consider a stop light turning green from red. At $t = 0$, we will have no cars to the right of the light, $\rho = 0$, while to the left of the light all of the cars will be fully backed up, with $\rho = 1$. This gives us the initial value problem:

$$\rho_t + q'(\rho)\rho_x = 0 \tag{43}$$

$$\rho(x, 0) = g(x) \tag{44}$$

$$q(\rho) = \rho(1 - \rho) \tag{45}$$

with the initial condition

$$g(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0. \end{cases} \tag{46}$$

From our previous example, our characteristics, parameterized by $x_0 \in \mathbb{R}$, are

$$x = (1 - 2g(x_0))t + x_0 \tag{47}$$

and are straight lines of constant ρ (figure 10).

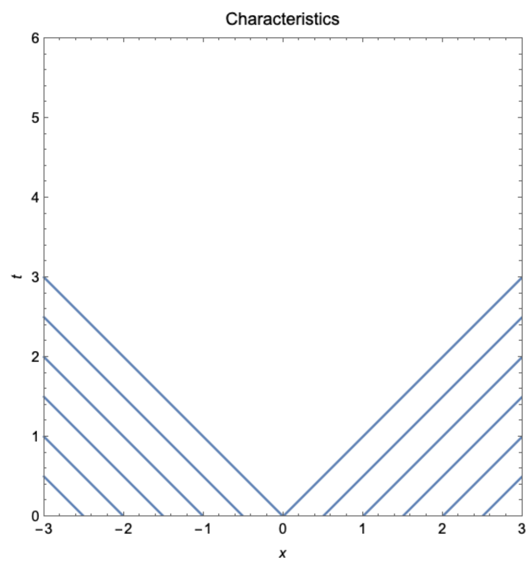

Figure 10: Characteristics of red to green light

We see that there are no characteristics in the center region, which suggests that we need a rarefaction wave. From theorem 5.1, we can take

$$\rho(x,t) = G\left(\frac{x}{t}\right) = \frac{1}{2}\left(1 - \frac{x}{t}\right) \tag{48}$$

within the rarefaction wave (figure 11). We can check that solution is continuous: on the right boundary, $x = t$, we see $\rho = 0$, and on the left boundary, $x = -t$, we see $\rho = 1$. As there are no curves of discontinuity, the solution satisfies the entropy condition, and we have

$$\rho(x,t) = \begin{cases} 1 & x \le -t \\ \frac{1}{2}\left(1 - \frac{x}{t}\right) & -t < x < t \\ 0 & x \ge t \end{cases} \tag{49}$$

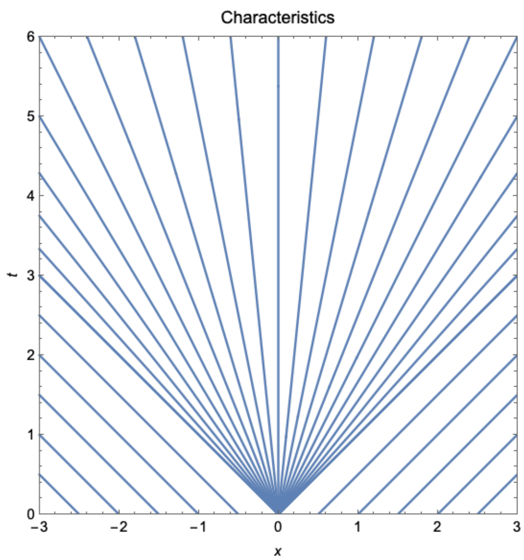

Figure 11: Characteristics including a rarefaction wave

A plot of the solution at various times is shown in figure 12.

6.2 Green Light to Red Light

Suppose that we have a constant density of cars ρ_1, traveling at a constant velocity before a green light turns red at $x = 0$ and $t = 0$. Similarly to [1], we can model this by requiring that for all $t > 0$, $\rho(0,t) = 1$ when approached

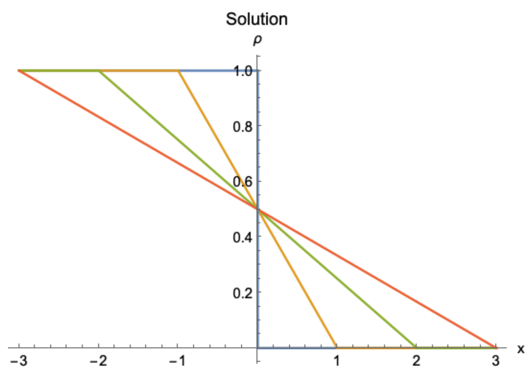

Figure 12: Red light to green light

from the left, and $\rho(0,t) = 0$ when approached from the right. This means that cars driving towards it are stopped, and there are no cars driving away from it. This changes our problem from an initial value problem, so we don't require that the solution holds at $x = 0$. While the initial value of ρ_1 means that we have characteristics of slope $1 - 2\rho_1$ everywhere else, our stop light means that for all time, we will have characteristics of slope -1 extending from the t-axis to the left, and of slope $+1$ to the right. Our characteristics are shown in figure 13.

We can see that there must be a shock extending to the left of the stop light, stopping the cars to the left. From the Rankine-Hugoniot condition, the speed of this shock will be

$$\sigma_L = \frac{q(\rho_1) - q(1)}{\rho_1 - 1} = -\rho_1 \tag{50}$$

Additionally, we will have a second shock goes extending to the right of the

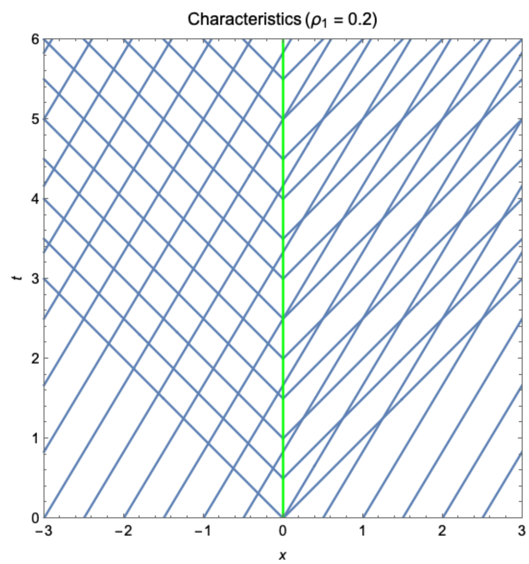

Figure 13: Characteristics of a green to red light before adding shocks

stop light, of speed

$$\sigma_R = \frac{q(0) - q(\rho_1)}{0 - \rho_1} = 1 - \rho_1 \tag{51}$$

Note that this is, in fact, the speed of a car at density ρ_1, which fits our expectation that the density behind the final car that gets through the light will be zero. These are shown in figure 14

The entropy condition holds at both shocks: for the first, we have

$$q'(\rho_1) = 1 - 2\rho_1 > -\rho_1 > q'(1) = -1 \tag{52}$$

and for the second

$$q'(0) = 1 > 1 - \rho_1 > q'(\rho_1) = 1 - 2\rho_1. \tag{53}$$

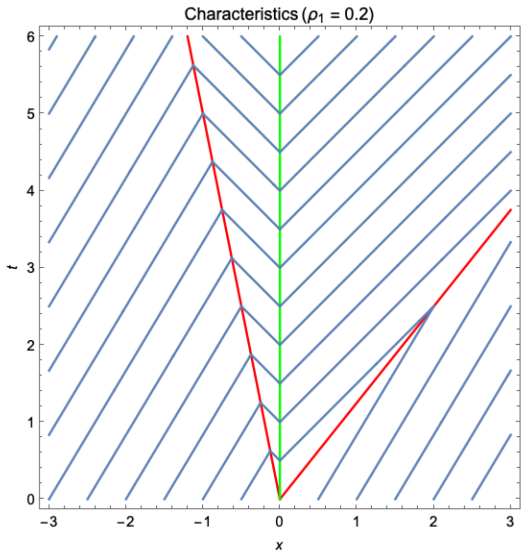

Figure 14: Characteristics of a green to red light after adding shocks

Our solution is thus

$$\rho(x,t) = \begin{cases} \rho_1 & x \leq -\rho_1 t \\ 1 & -\rho_1 t < x \leq 0 \\ 0 & 0 < x < (1-\rho_1)t \\ \rho_1 & x \geq (1-\rho_1)t. \end{cases} \quad (54)$$

6.3 Green Light to Red Light to Green Light

Now, we take the two scenarios together. At $t = t_1$, the red light from the previous scenario turns green again. The region surrounding our stop light will stop emitting characteristics and will become a rarefaction wave centered around $(0, t_1)$ as in the first scenario, with $\rho(x,t) = \frac{1}{2}\left(1 - \frac{x}{t-t_1}\right)$. This will change our two shocks when the rarefaction catches up with them. It will reach the left shock at

$$
\begin{aligned}
t_L &= t_1 + \rho_1 t_L \\
t_L &= \frac{t_1}{1-\rho_1}
\end{aligned}
$$

and the right shock at

$$
\begin{aligned}
t_R &= t_1 + (1-\rho_1) t_R \\
t_R &= \frac{t_1}{\rho_1}.
\end{aligned}
$$

For the left shock, the slope will now be dependant on the position. The differential equation for the speed becomes

$$
\begin{aligned}
\sigma_L = \frac{dx}{dt} &= \frac{q(\rho_1) - q(\rho(x,t))}{\rho_1 - \rho(x,t)} = \frac{\rho_1(1-\rho_1) - \frac{1}{2}\left(1 - \frac{x}{t-t_1}\right)\left(1 - \frac{1}{2}\left(1 - \frac{x}{t-t_1}\right)\right)}{\rho_1 - \frac{1}{2}\left(1 - \frac{x}{t-t_1}\right)} \\
&= \frac{1}{2}\left(1 - 2\rho_1 + \frac{x}{t-t_1}\right)
\end{aligned}
$$

which is an ordinary differential equation. With the initial condition $x_L(t_L) = -\rho_1 t_L$ it has a solution of

$$
x_L(t) = (1-2\rho_1)(t-t_1) - 2\sqrt{\rho_1(1-\rho_1)t_1}\sqrt{t-t_1}. \tag{55}
$$

Similarly, the right shock has

$$
\begin{aligned}
\sigma_R = \frac{dx}{dt} &= \frac{q(\rho(x,t)) - q(\rho_1)}{\rho(x,t) - \rho_1} = \frac{\rho_1(1-\rho_1) - \frac{1}{2}\left(1 - \frac{x}{t-t_1}\right)\left(1 - \frac{1}{2}\left(1 - \frac{x}{t-t_1}\right)\right)}{\rho_1 - \frac{1}{2}\left(1 - \frac{x}{t-t_1}\right)} \\
&= \frac{1}{2}\left(1 - 2\rho_1 + \frac{x}{t-t_1}\right)
\end{aligned}
$$

and with $x_R(t_R) = (1-\rho_1)t_R$ has

$$x_R(t) = (1-2\rho_1)(t-t_1) + 2\sqrt{\rho_1(1-\rho_1)t_1}\sqrt{t-t_1}. \tag{56}$$

These new shocks will never intersect. The left one is shown in figure 15.

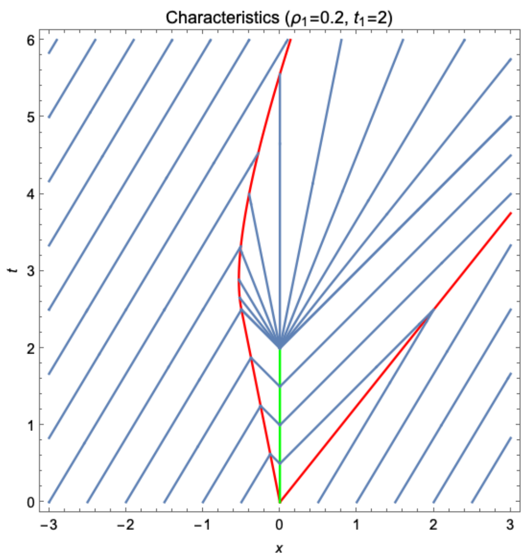

Figure 15: Shocks for a green to red to green light.

Further discussion of this example is found in [1]. By using this and the preceding examples, we could potentially create more complicated models, such as a series of traffic lights. This could allow us to answer questions such as how long two lights should be spread apart in time to allow the maximum number of cars through.

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