RED LIGHT, GREEN LIGHT

MODELING TRAFFIC FLOW WITH QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

CAMERON GRAY

CLASSIFICATION OF FIRST-ORDER PDES

- Linear 1st Order PDEs are of the form $a(x, y)\, u_x + b(x, y)\, u_y = c(x, y)$
- Semilinear PDEs are of the form $a(x, y)\, u_x + b(x, y)\, u_y = c(x, y, u)$
- Quasilinear PDEs are of the form $a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u)$
- If $c(x, y, u) = 0$ then the equation is homogenous, else it is inhomogeneous
- Fully Nonlinear ODEs are of the form $F(x, y, u, u_x, u_y) = 0$
- We'll commonly be using a time coordinate t instead of y

THE METHOD OF CHARACTERISTICS

FOR LINEAR EQUATIONS (1)

- $a(x,y)\,u_x + b(x,y)\,u_y - c(x,y) = 0$ is equivalent to the vector equation $\langle a(x,y), b(x,y), c(x,y)\rangle \cdot \langle u_x, u_y, -1\rangle = 0$

- This means that the two vectors are perpendicular. $\langle u_x, u_y, -1\rangle$ is the normal vector to the surface defined by $S = \{(x, y, u(x,y))\}$, so $\langle a(x,y), b(x,y), c(x,y)\rangle$ is always tangent, and within, the surface.

- Let there be a curve $C \subset S$ where the curve is always tangent to $\langle a, b, c\rangle$, where the curve is parametrized by s: $C = \{(x(s), y(s), z(s))\}$ We can define s in such a way so that the tangent vector doesn't need to be normalized, giving the equation: $\frac{dC}{ds} = \langle a(x,y), b(x,y), c(x,y)\rangle$

THE METHOD OF CHARACTERISTICS

FOR LINEAR EQUATIONS (2)

- Separating vector components out, we the system of ODEs

$$\frac{dx}{ds} = a(x, y) \quad \frac{dy}{ds} = b(x, y) \quad \frac{dz}{ds} = c(x, y)$$

- The system of ODEs gives us a curve called a characteristic curve within the surface. This curve is not unique (conditions are needed), and so we can take the union of all these curves to get a set of surfaces (called integral surfaces) which are solutions to the original equation.

- In order to specify a unique surface which is a solution to the equation, conditions are needed.

EXAMPLE PROBLEM 1 (1)

$$u_t + v u_x = 0$$

$$a(x,t) = 1 \quad b(x,t) = v \quad c(x,t) = 0$$

$$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = v \quad \frac{dz}{ds} = 0$$

$$t(s) = s + c_1 \quad x(s) = vs + c_2 \quad z(s) = c_3$$

$$x - vt = c_4$$

$$z(x,t) = f(x - vt)$$

$$u(x,t) = f(x - vt)$$

INITIAL VALUE PROBLEMS

$$u(x,0) = \phi(x)$$

- There are many possible integral surfaces given, and in order to choose one, we force the integral surface we want to contain the curve $(\Gamma, \phi) = \{(x,0,\phi(x))\}$ so that the surface fits the initial conditions.

- We parametrize this curve by r, so that $\Gamma = \{(r,0)\}$

- We then create characteristic curves which satisfy the initial conditions $x(r,0) = r \quad t(r,0) = 0 \quad z(r,0) = \phi(r)$ and take the union of these characteristic curves to find the solution fitting the initial conditions.

- So the full set of equations to solve $a(x,t)\,u_t + b(x,t)\,u_x = c(x,t)$ is:

$$\frac{d}{ds}t(r,s) = a(x,t) \quad \frac{d}{ds}x(r,s) = b(x,t) \quad \frac{d}{ds}z(r,s) = c(x,t)$$
$$x(r,0) = r \quad t(r,0) = 0 \quad z(r,0) = \phi(r)$$

EXAMPLE PROBLEM 1 (2)

$$u_t + v u_x = 0 \quad u(x,0) = \phi(x) \quad \phi(x) = e^{-x^2}$$

$$t(r,s) = s + c_1(r) \quad x(r,s) = vs + c_2(r) \quad z(r,s) = c_3(r)$$

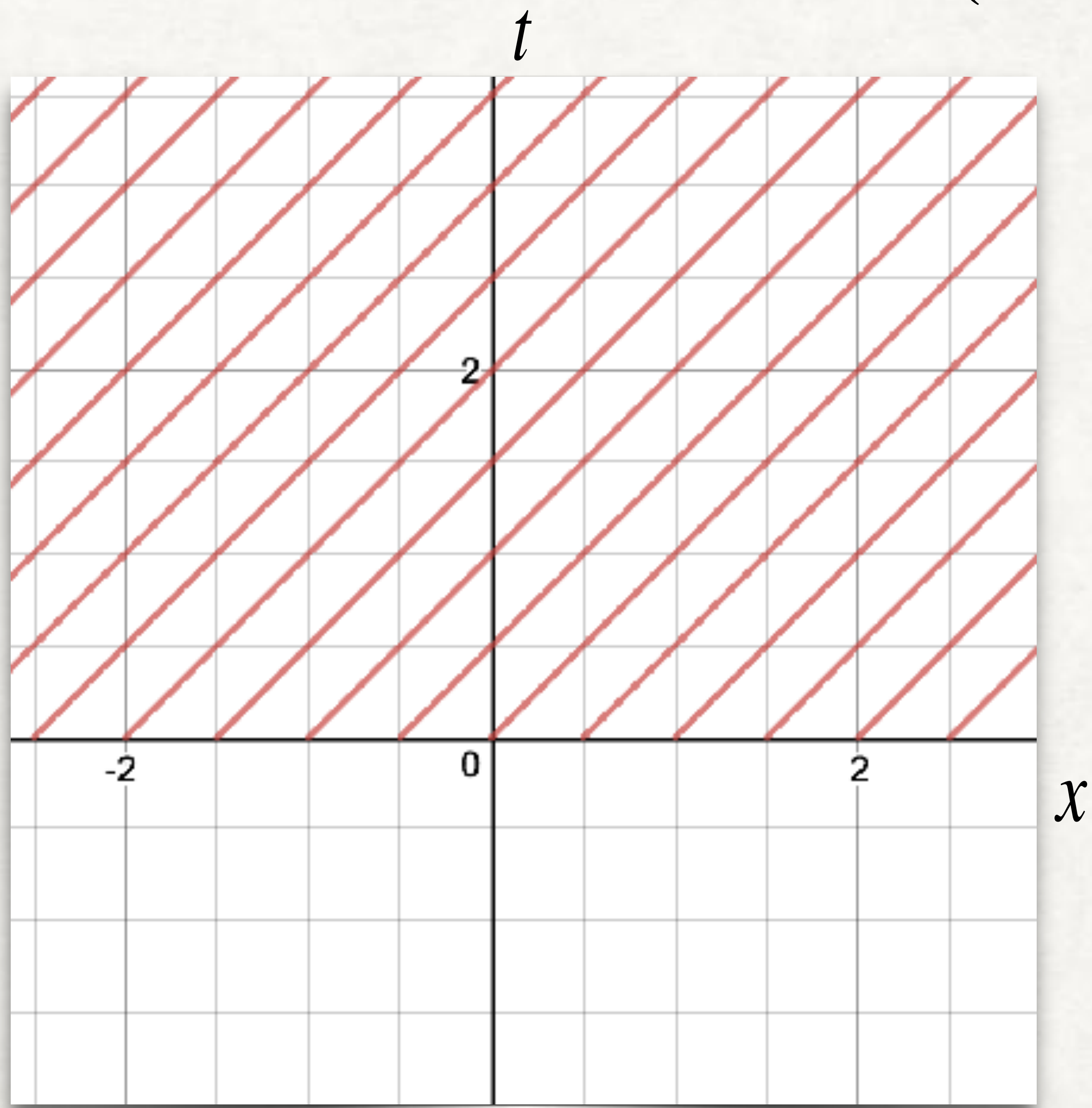
$$t(r,0) = 0 \quad x(r,0) = r \quad z(r,0) = \phi(r)$$

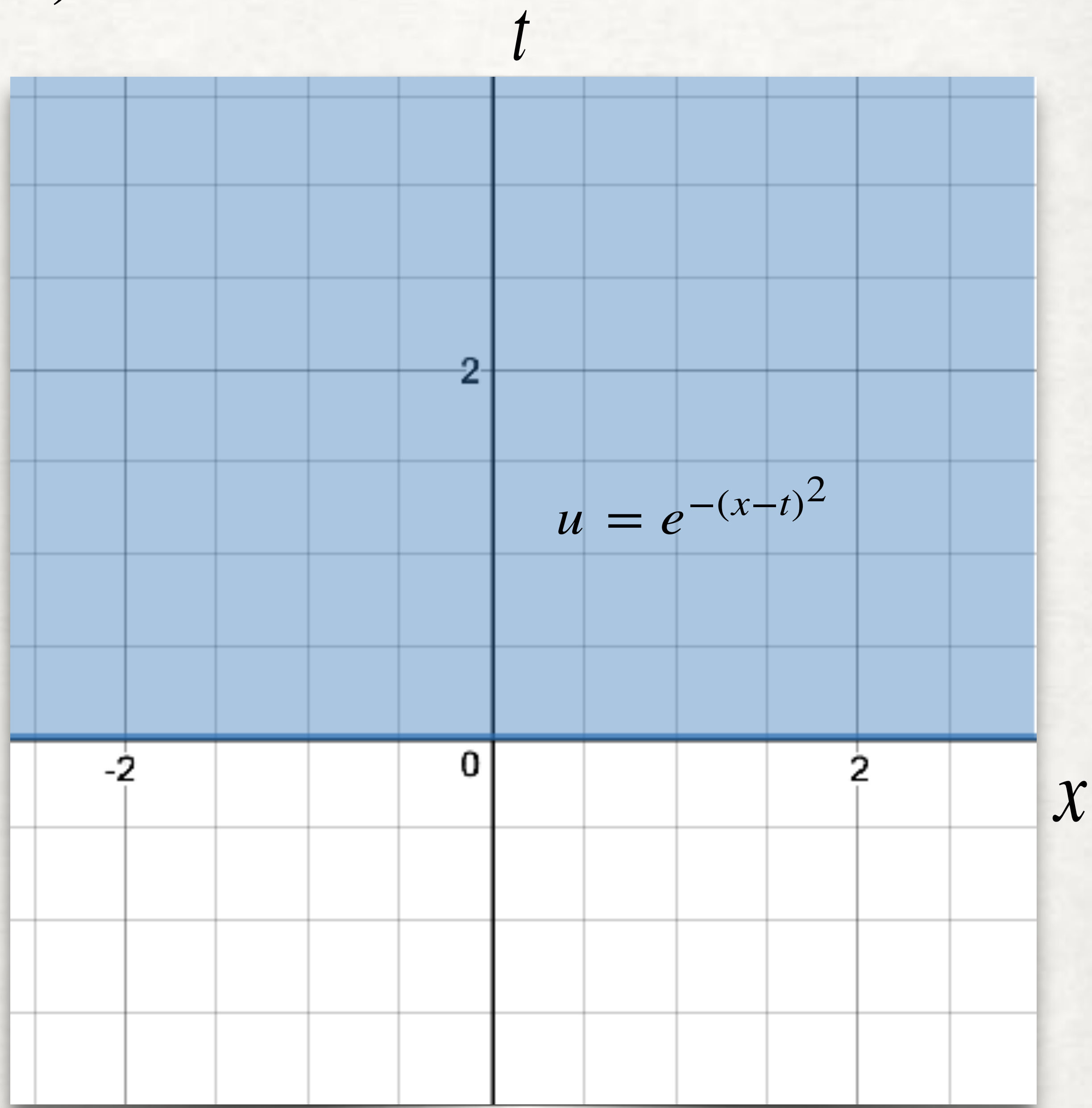
$$t(r,s) = s \quad x(r,s) = vs + r \quad z(r,s) = \phi(r)$$

$$x = vt + r \quad z(r,s) = e^{-r^2} \quad u(x,t) = e^{-(x-vt)^2}$$

EXAMPLE PROBLEM 1 (3)

$(v = 1.0)$

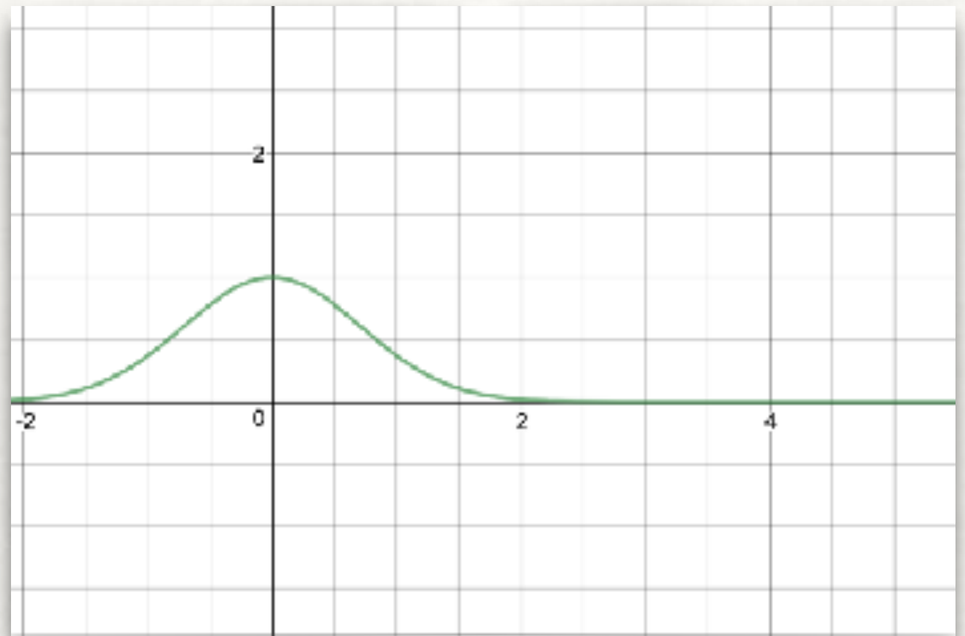




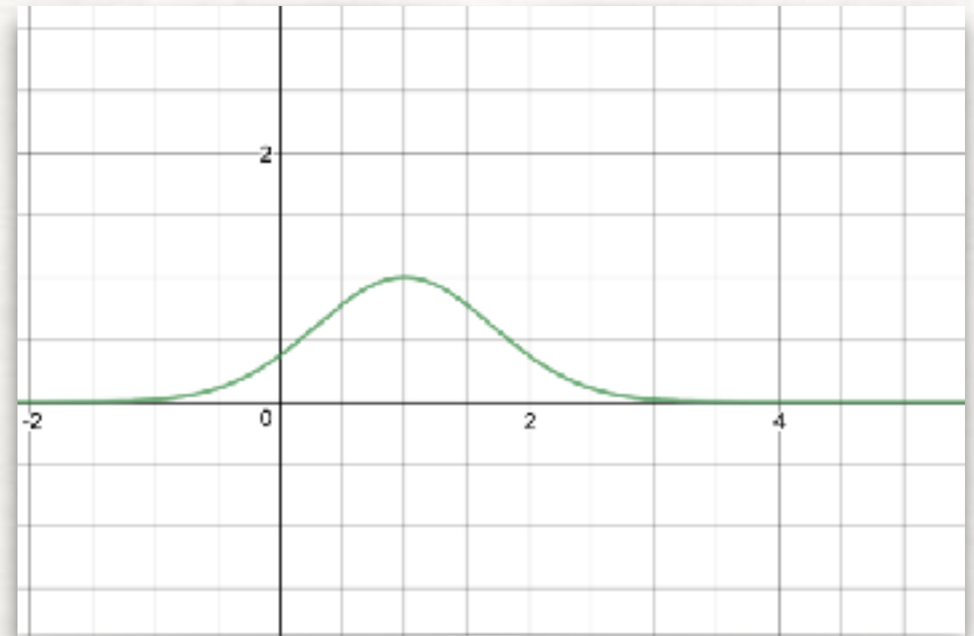
EXAMPLE PROBLEM 1 (4)

$(v = 1.0)$

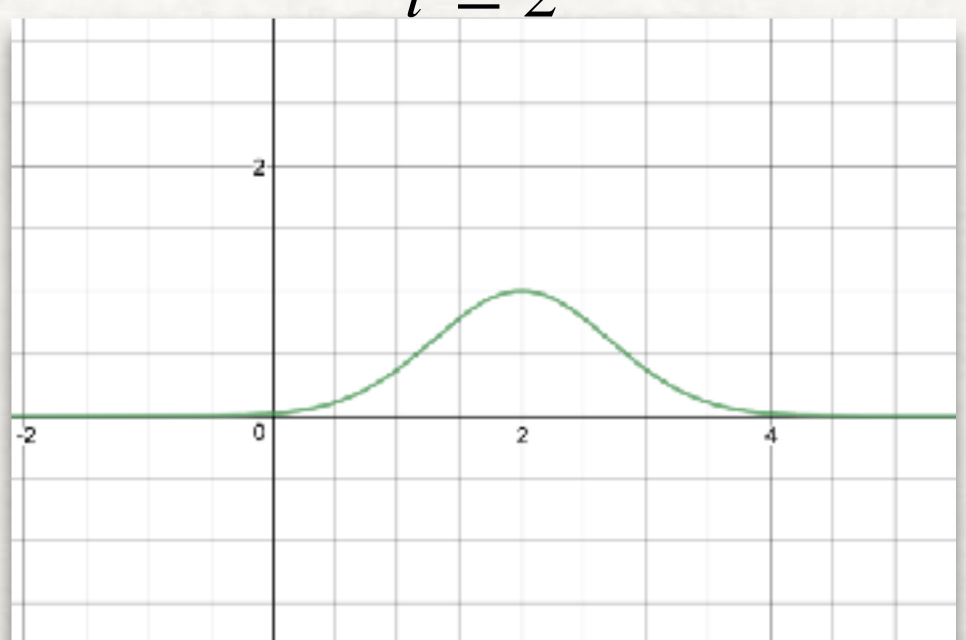
$t = 0$



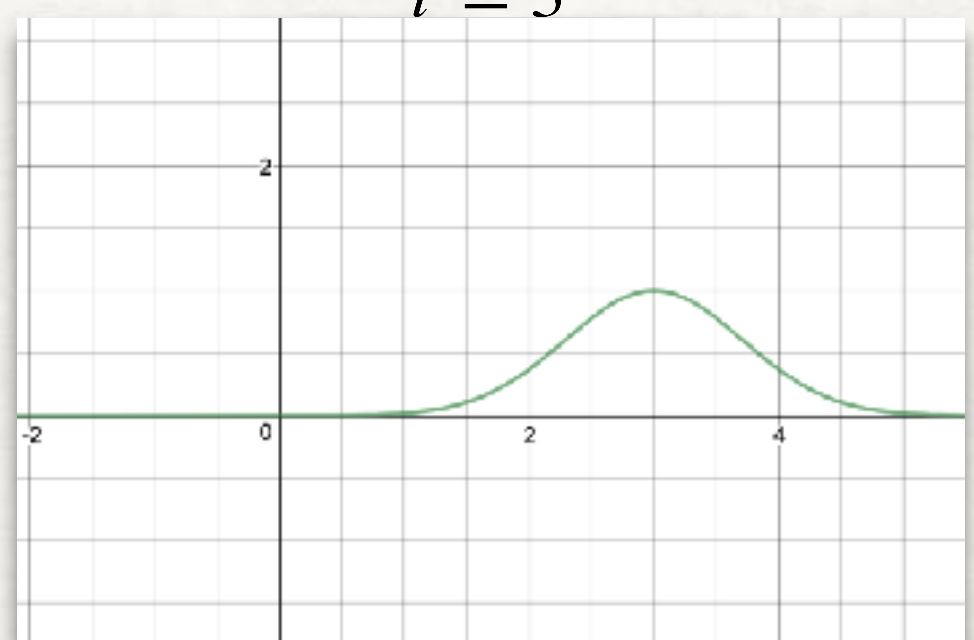
$t = 1$



$t = 2$

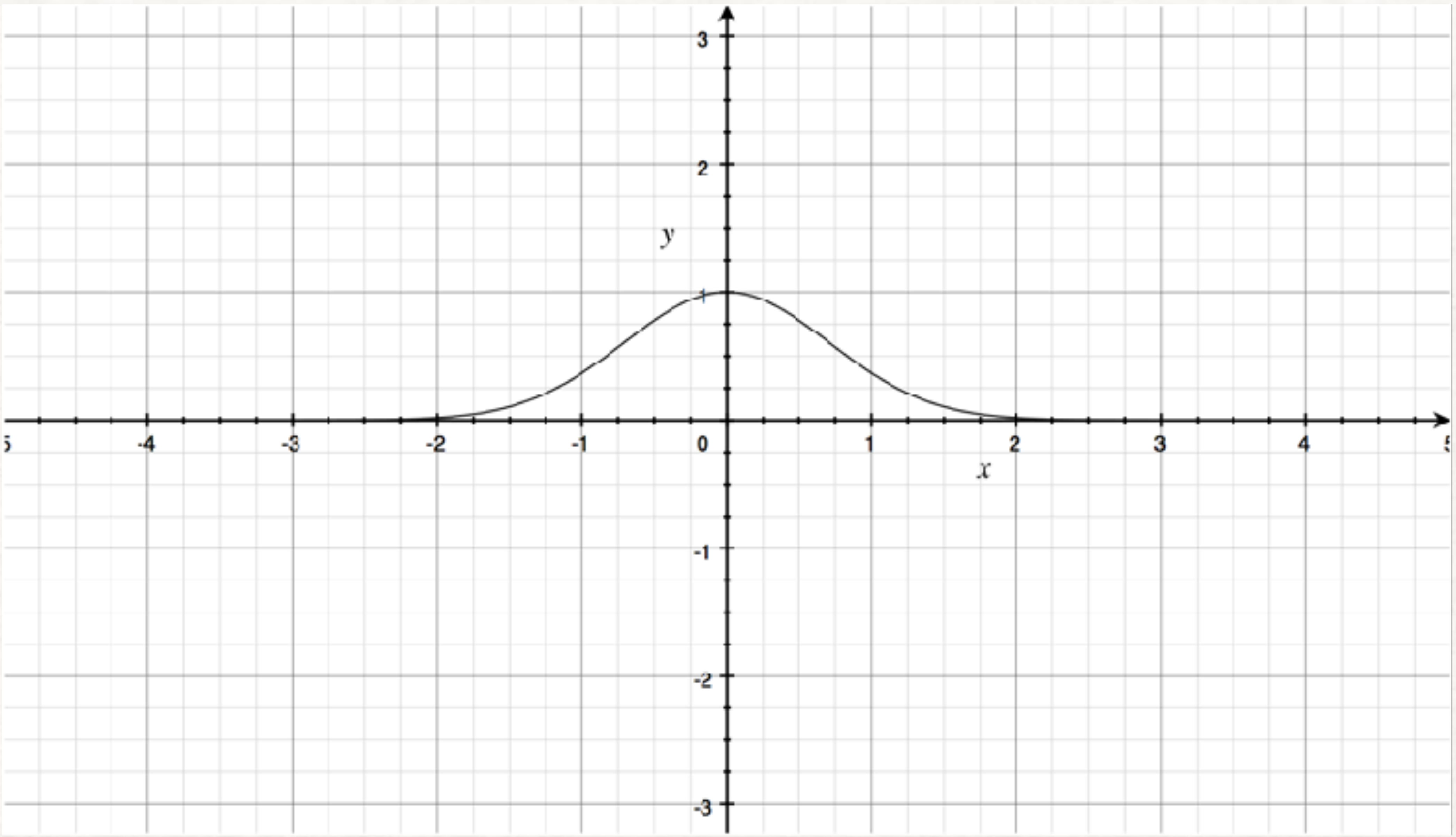


$t = 3$



EXAMPLE PROBLEM 1 (5)

$(v = 1.0)$



CAUCHY PROBLEMS

- Cauchy Problems have the general boundary condition $u|_\Gamma = \phi$, where the function $u(x,t)$ evaluated along the general curve through x-t space is equal to ϕ

- The curve Γ can be parametrized by r, so that $\Gamma = \{(\gamma_1(r), \gamma_2(r))\}$

- The curve must be nowhere tangent to the characteristics for a well-defined problem with a unique solution.

- For general linear Cauchy problems, we have the system of equations:

$$\frac{d}{ds}t(r,s) = a(x,t) \quad \frac{d}{ds}x(r,s) = b(x,t) \quad \frac{d}{ds}z(r,s) = c(x,t)$$

$$x(r,0) = \gamma_1(r) \quad t(r,0) = \gamma_2(r) \quad z(r,0) = \phi(r)$$

THE METHOD OF CHARACTERISTICS

FOR SEMILINEAR EQUATIONS

- For semilinear equations in the form $a(x,y)\,u_x + b(x,y)\,u_y = c(x,y,u)$ all that needs to be done is make c a function of x, y, and z instead of just x and y.

- Cauchy Problems are treated the same as in linear equations.

$$\frac{dx}{ds} = a(x,y) \quad \frac{dy}{ds} = b(x,y) \quad \frac{dz}{ds} = c(x,y,z)$$

EXAMPLE PROBLEM 2 (1)

$$u_t + v u_x = -u \quad u(x,0) = \phi(x) \quad \phi(x) = e^{-x^2}$$

$$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = v \quad \frac{dz}{ds} = -z$$

$$t(r,0) = 0 \quad x(r,0) = r \quad z(r,0) = \phi(r)$$

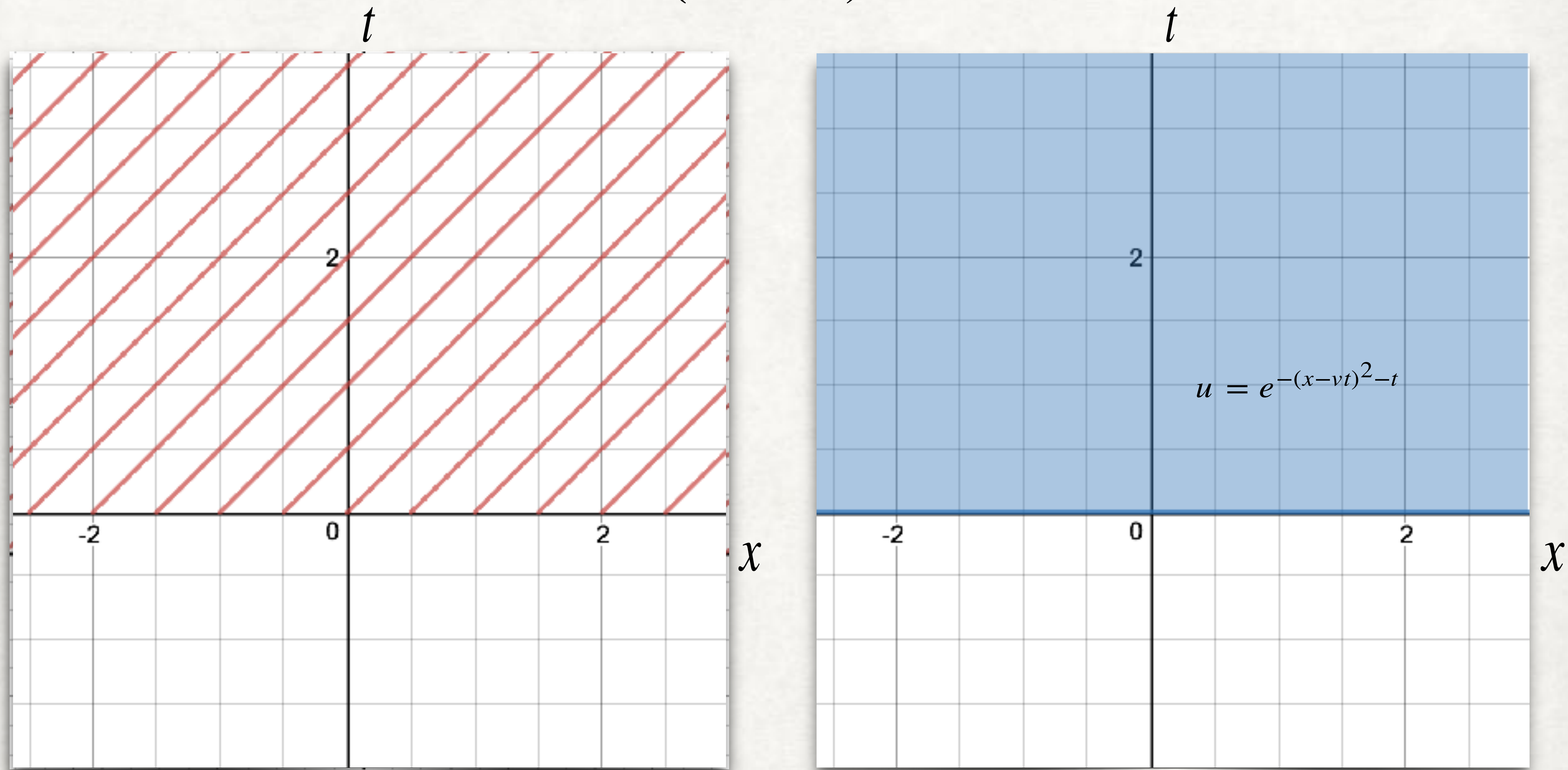
$$t = s + c_1(r) \quad x = vs + c_2(r) \quad z = c_3(r)\, e^{-s}$$

$$t = s \quad x = vs + r \quad z = \phi(r)\, e^{-s}$$

$$x = vt + r \quad u(x,t) = \phi(x - vt)\, e^{-t} \quad u(x,t) = e^{-(x-vt)^2 - t}$$

EXAMPLE PROBLEM 2 (2)

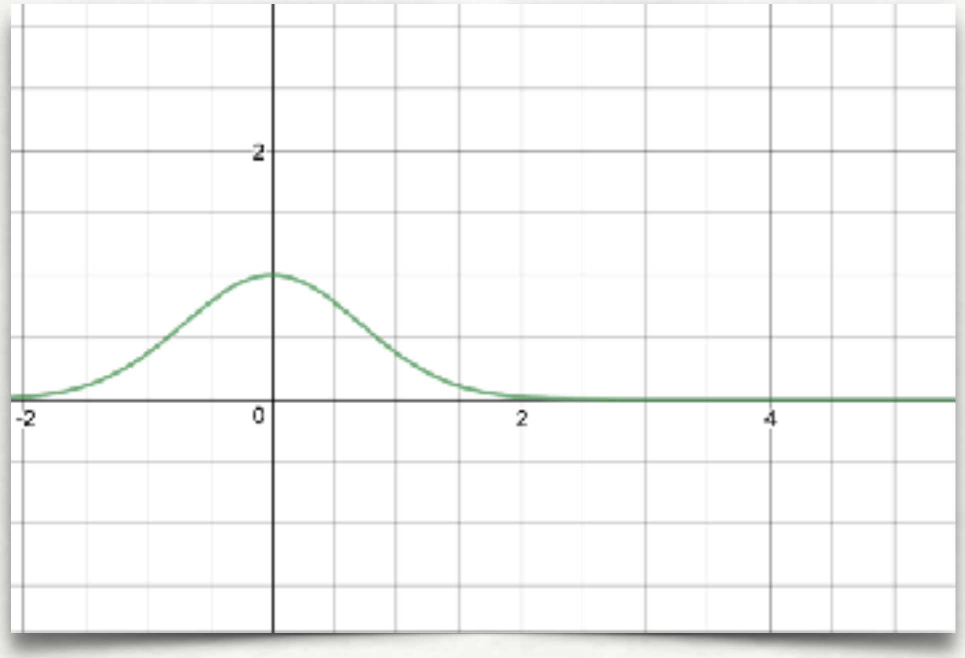
$(v = 1.0)$



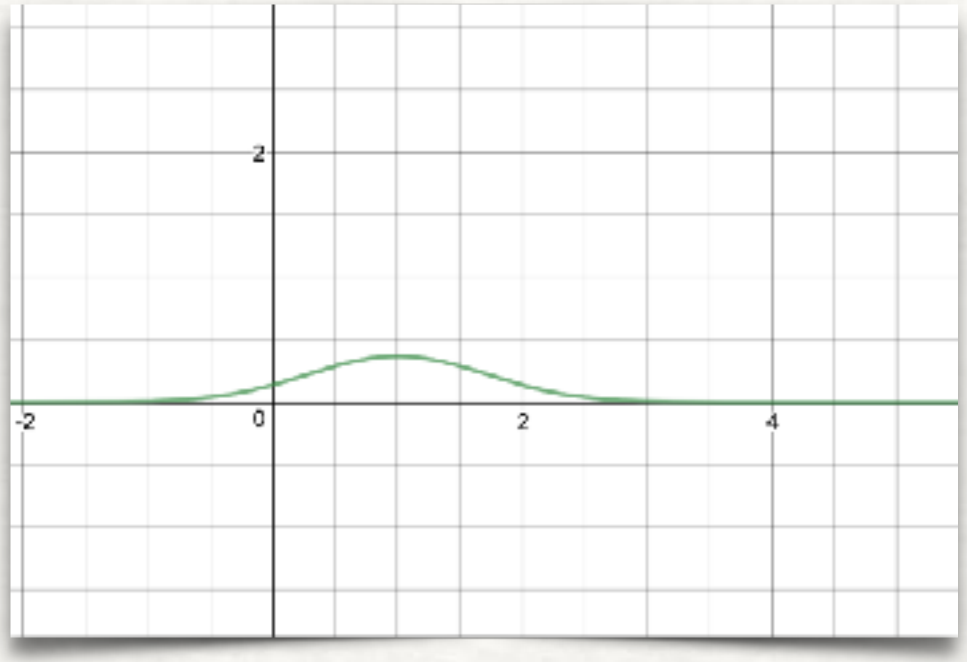
EXAMPLE PROBLEM 2 (3)

$(v = 1.0)$

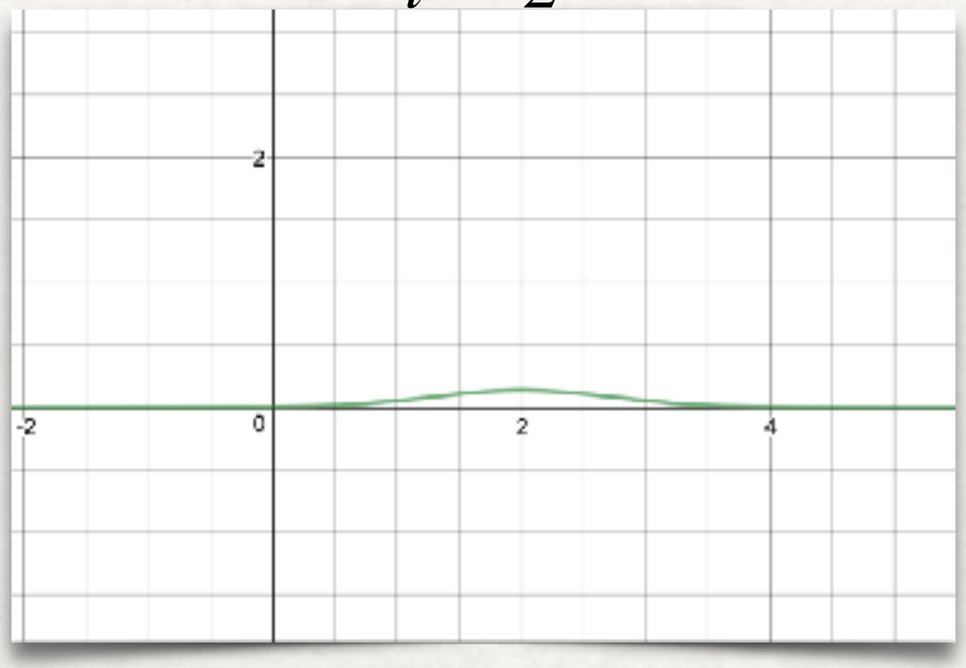
$t = 0$



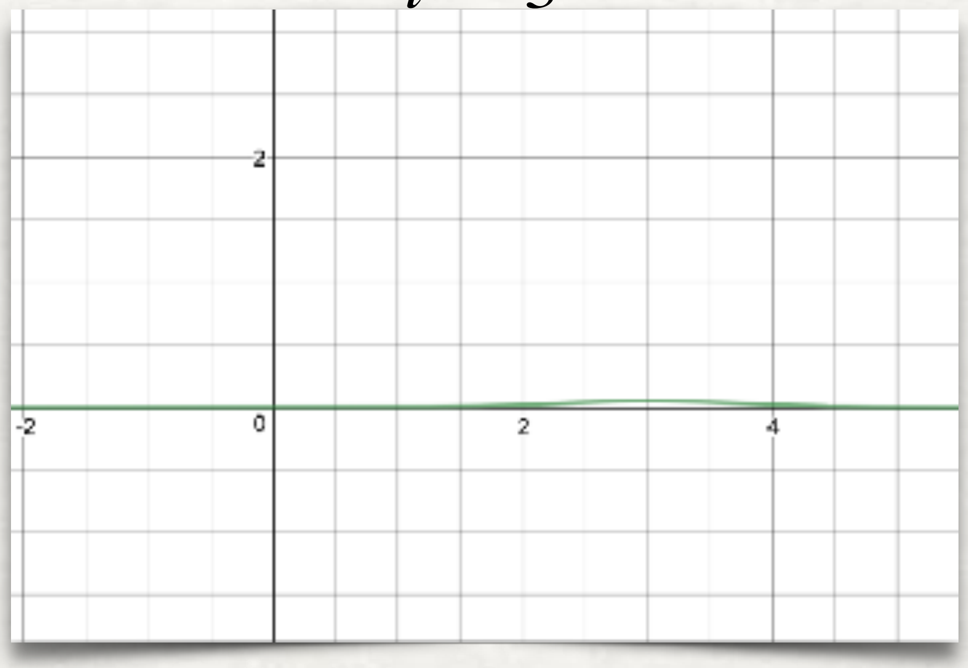
$t = 1$



$t = 2$

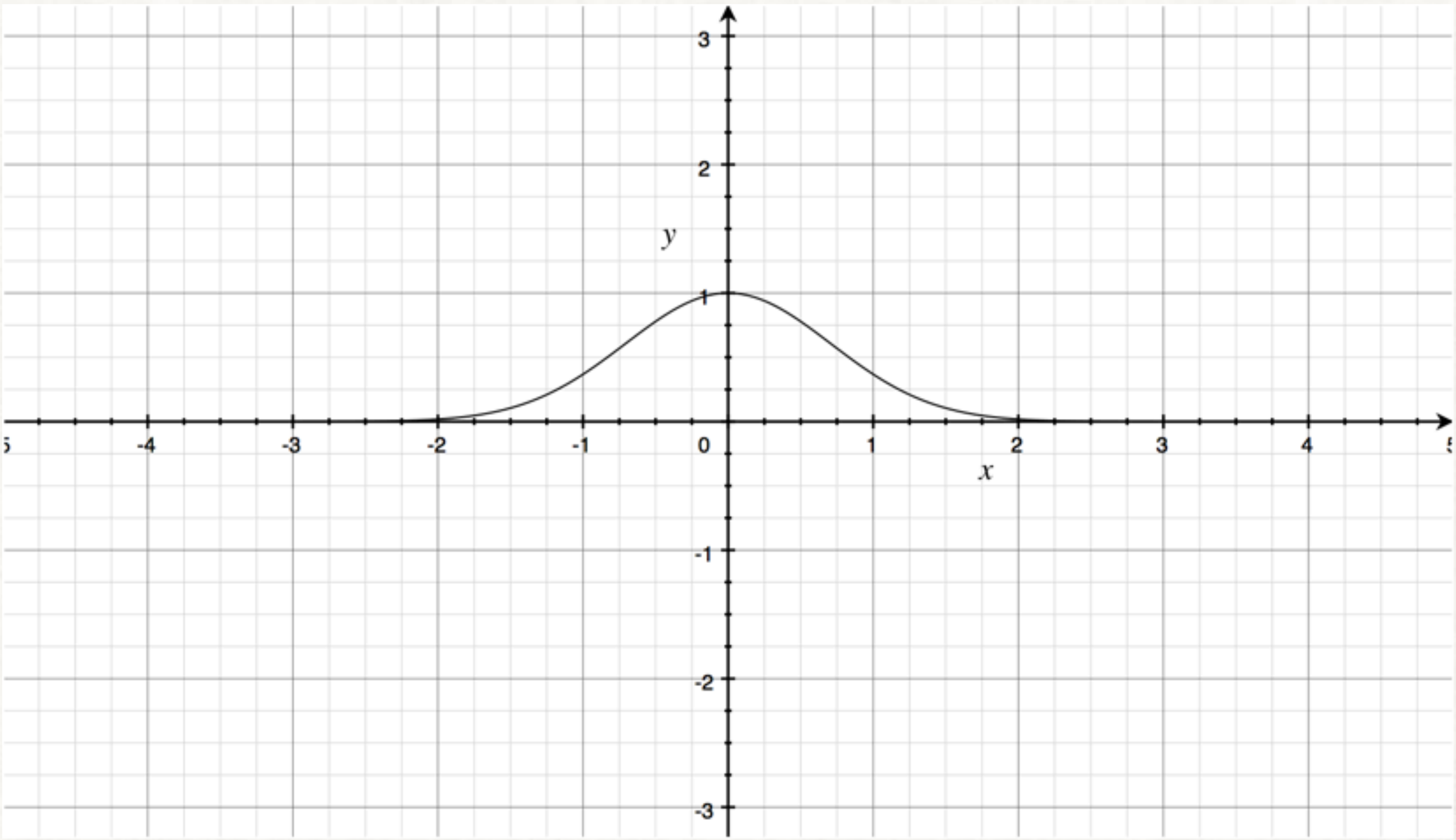


$t = 3$



EXAMPLE PROBLEM 2 (4)

$$(v = 1.0)$$



THE METHOD OF CHARACTERISTICS

FOR QUASILINEAR EQUATIONS

- The way of extending the Method of Characteristics to quasilinear equations is the same as extending it to semilinear equations. For a quasilinear Cauchy problem of the form:

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u) \quad u|_\Gamma = \phi$$

the system of equations becomes:

$$\frac{d}{ds}x(r, s) = a(x, y, z) \quad \frac{d}{ds}y(r, s) = b(x, y, z) \quad \frac{d}{ds}z(r, s) = c(x, y, z) \quad x(r,0) = \gamma_1(r) \quad y(r,0) = \gamma_2(r) \quad z(r,0) = \phi(r)$$

- We're only going to be looking at quasilinear equations which can be put in the form $u_t + [f(u)]_x = 0$, or $u_t + f'(u)\, u_x = 0$

- Burgers' Equation with initial values: $u_t + u\, u_x = 0 \quad u(x,0) = \phi(x)$

EXAMPLE PROBLEM 3 (1)

$$u_t + uu_x = 0 \quad u(x,0) = \phi(x) \quad \phi(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = z \quad \frac{dz}{ds} = 0$$

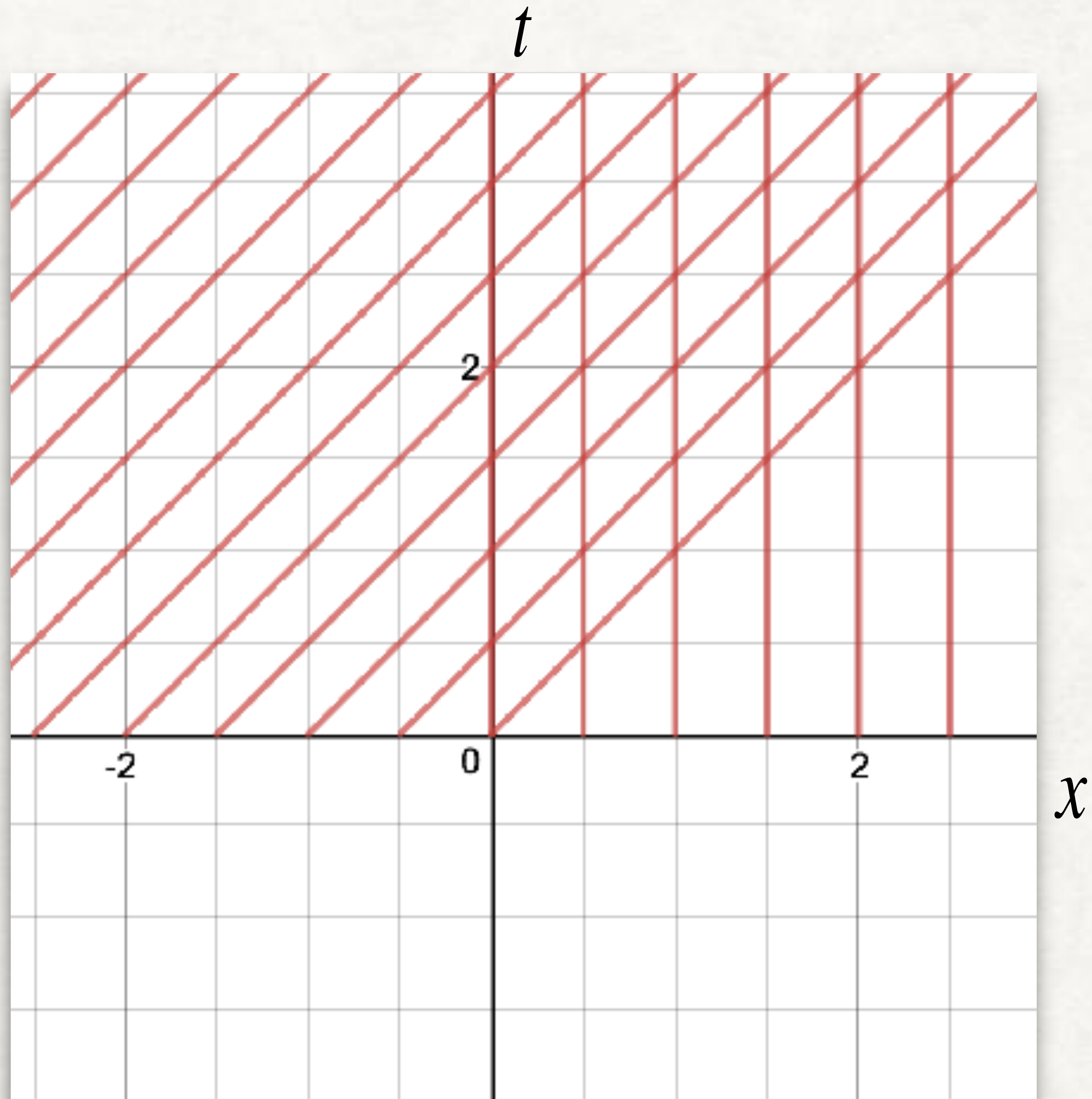
$$t(r,0) = 0 \quad x(r,0) = r \quad z(r,0) = \phi(r)$$

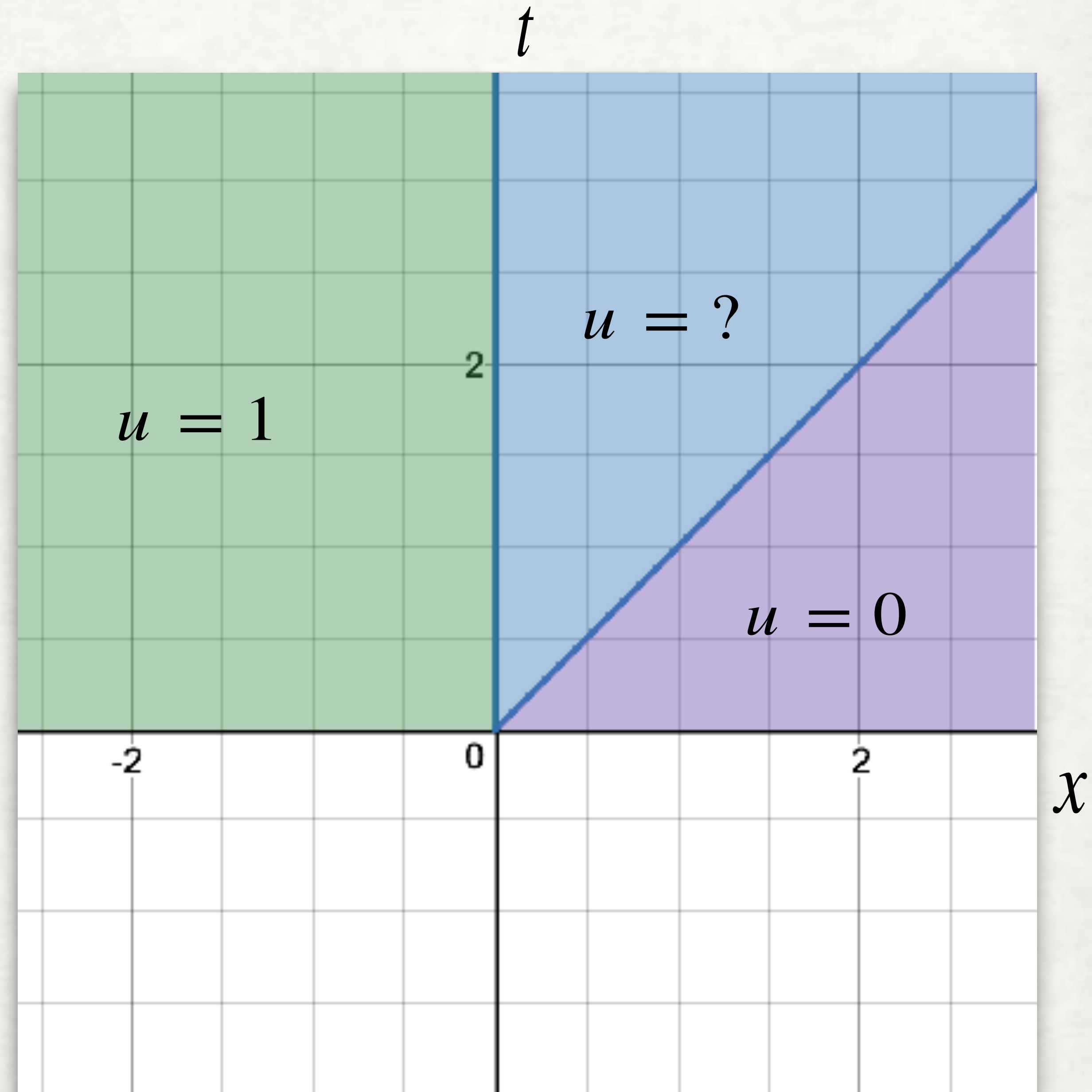
$$t = s + c_1(r) \quad x = zs + c_2(r) \quad z = c_3(r)$$

$$t = s \quad x = zs + r \quad z = \phi(r)$$

$$x = \begin{cases} t + r & \text{if } r < 0 \\ r & \text{if } r > 0 \end{cases} \qquad z = \begin{cases} 1 & \text{if } r < 0 \\ 0 & \text{if } r > 0 \end{cases}$$

EXAMPLE PROBLEM 3 (2)





RANKINE-HUGONIOT JUMP CONDITION

- The characteristics intersect, so we don't know what the value of u is in that area.
- To solve this problem, we draw a curve of discontinuity, called a shock wave, in that area, separating the solutions.
- For the equation $u_t + [f(u)]_x = 0$ the curve is given by $\zeta'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+}$ where $x = \zeta(t)$ is the curve, u^- is the value of u on the left side of the discontinuity, and u^+ is the value of u on the right side of the discontinuity.
- This is called the Rankine-Hugoniot jump condition.

EXAMPLE PROBLEM 3 (3)

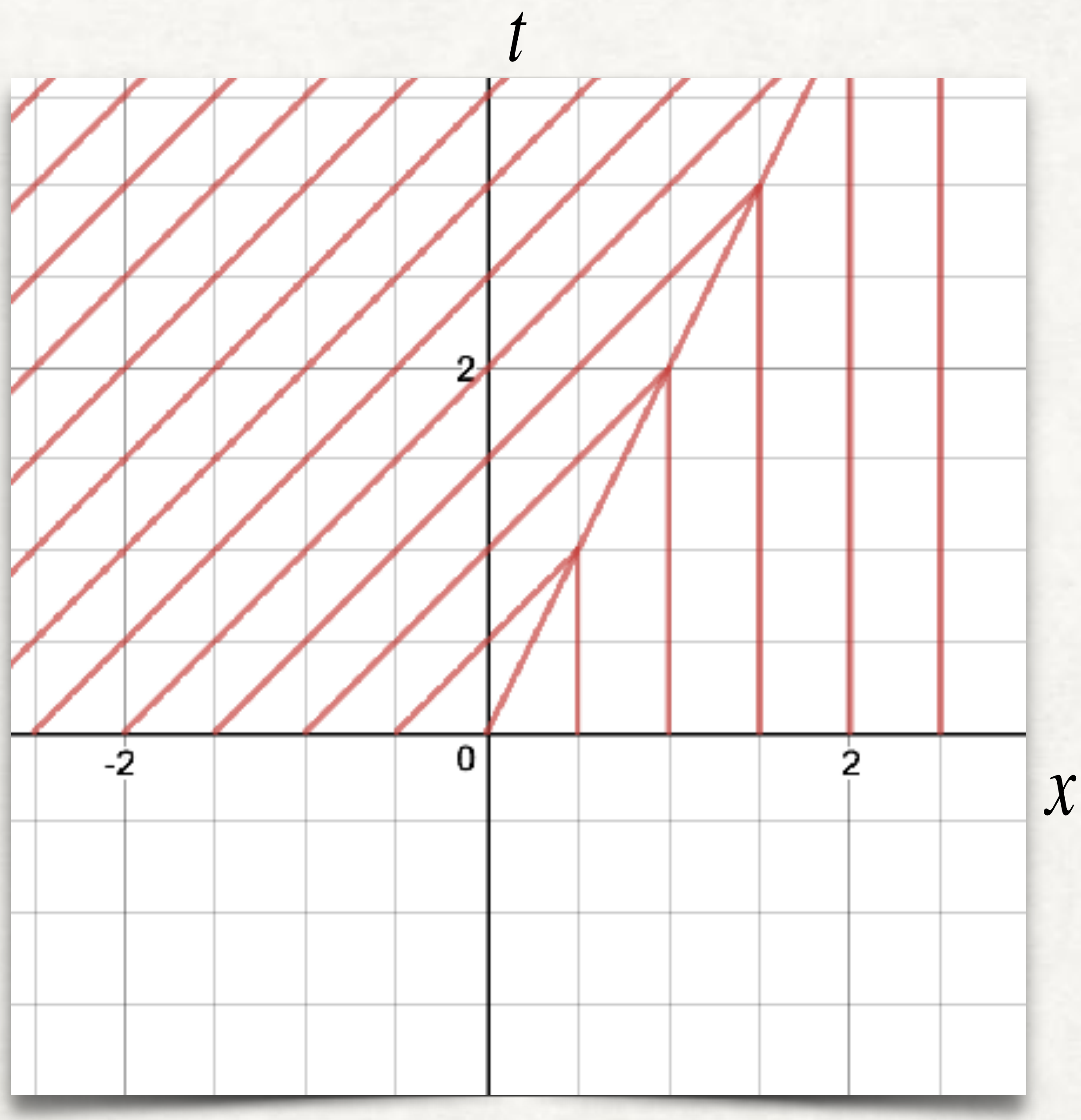
$$f(u) = \frac{1}{2}u^2$$

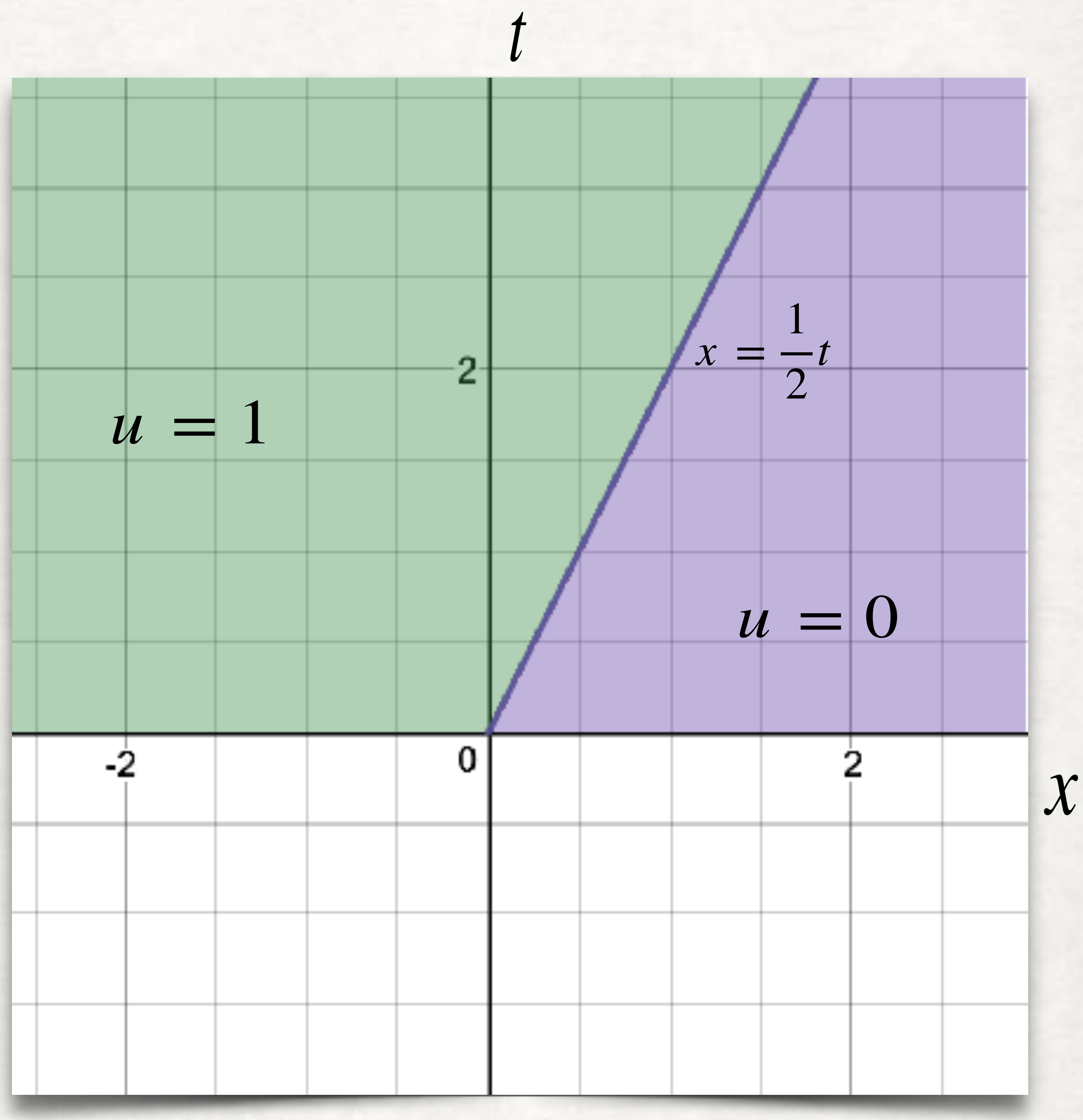
$$\zeta'(t) = \frac{\frac{1}{2}(1) - \frac{1}{2}(0)}{1 - 0} \qquad x(t) = \zeta(t) = \frac{1}{2}t + c_4$$

$$x(0) = 0$$

$$x = \frac{1}{2}t$$

EXAMPLE PROBLEM 3 (4)

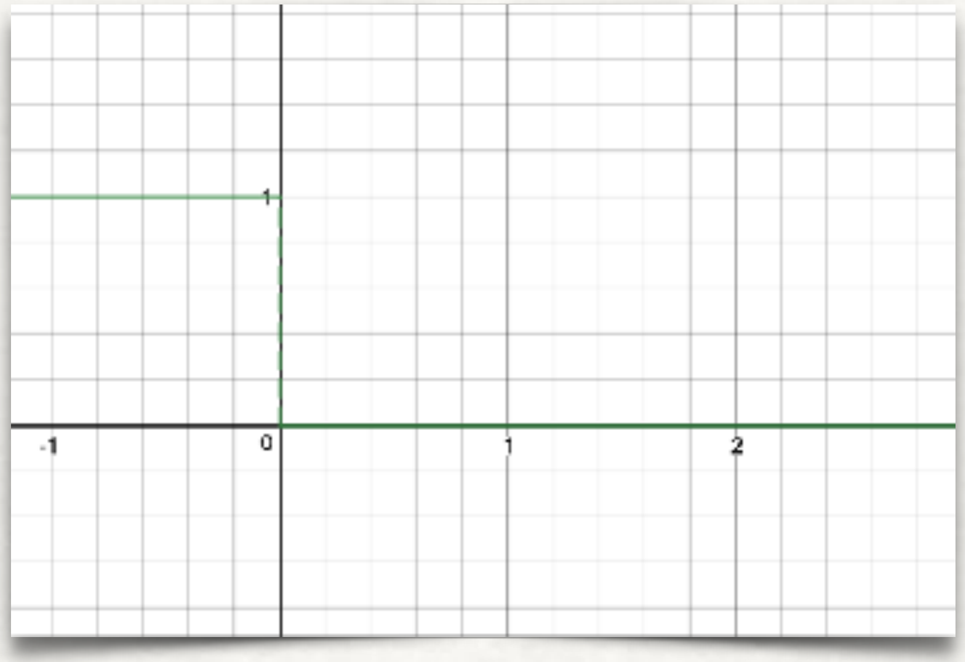




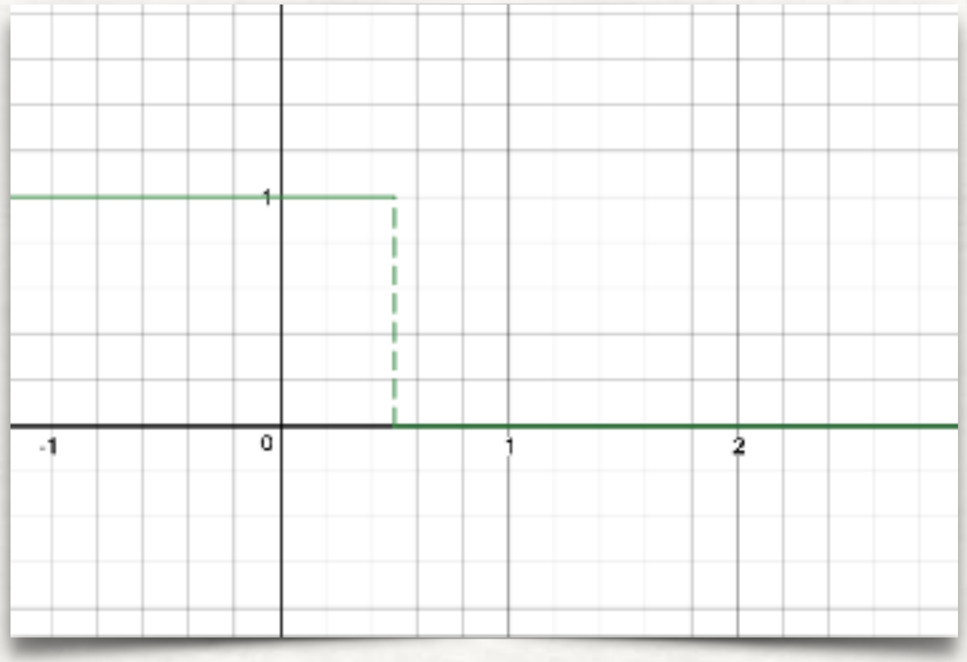
$$u(x,t) = \begin{cases} 1 & \text{if } x < \frac{1}{2}t \\ 0 & \text{if } x > \frac{1}{2}t \end{cases}$$

EXAMPLE PROBLEM 3 (5)

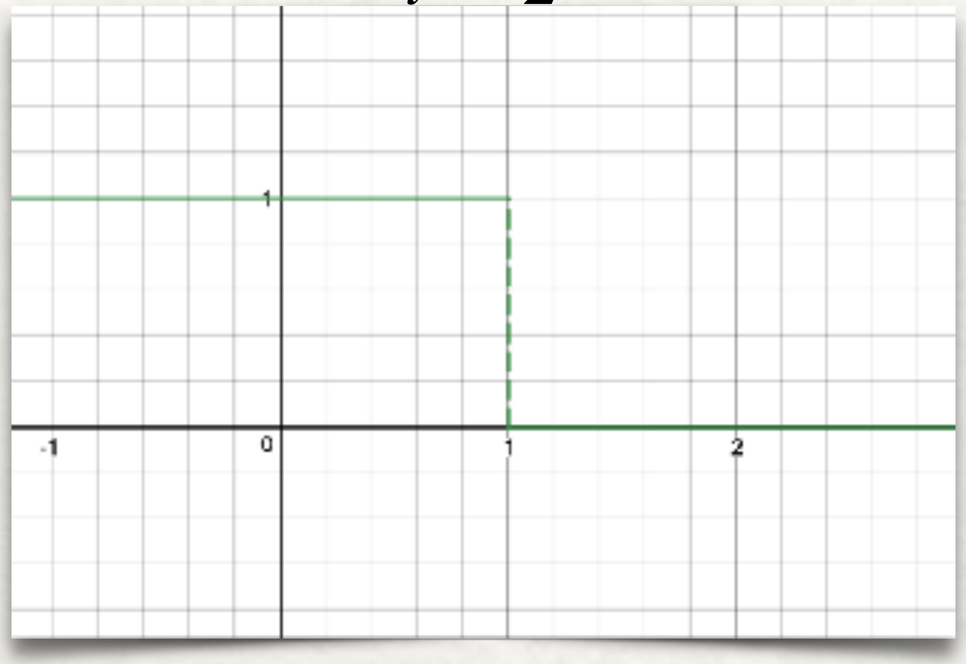
$t = 0$



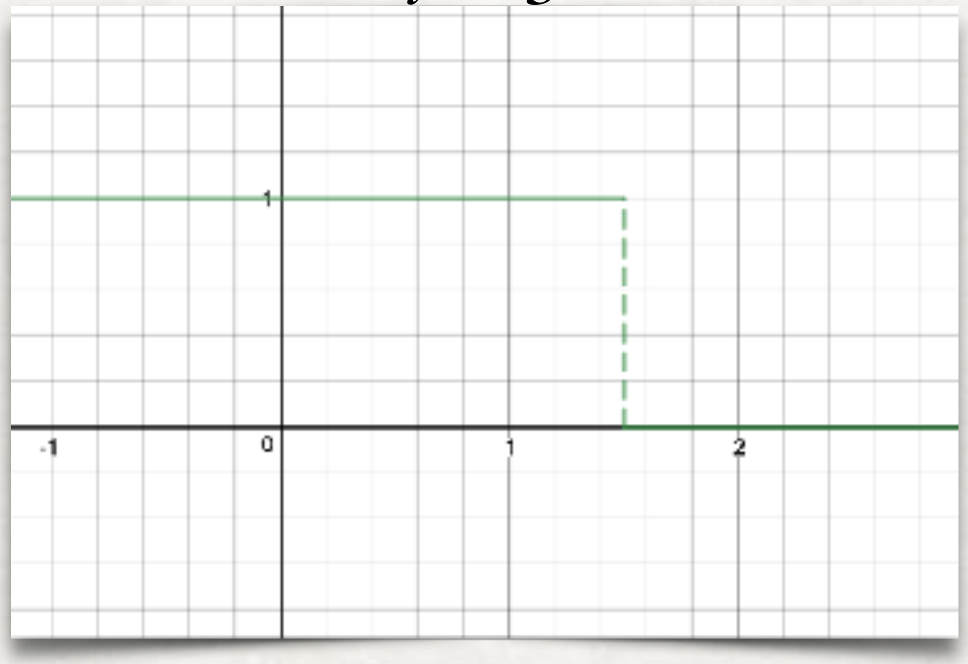
$t = 1$



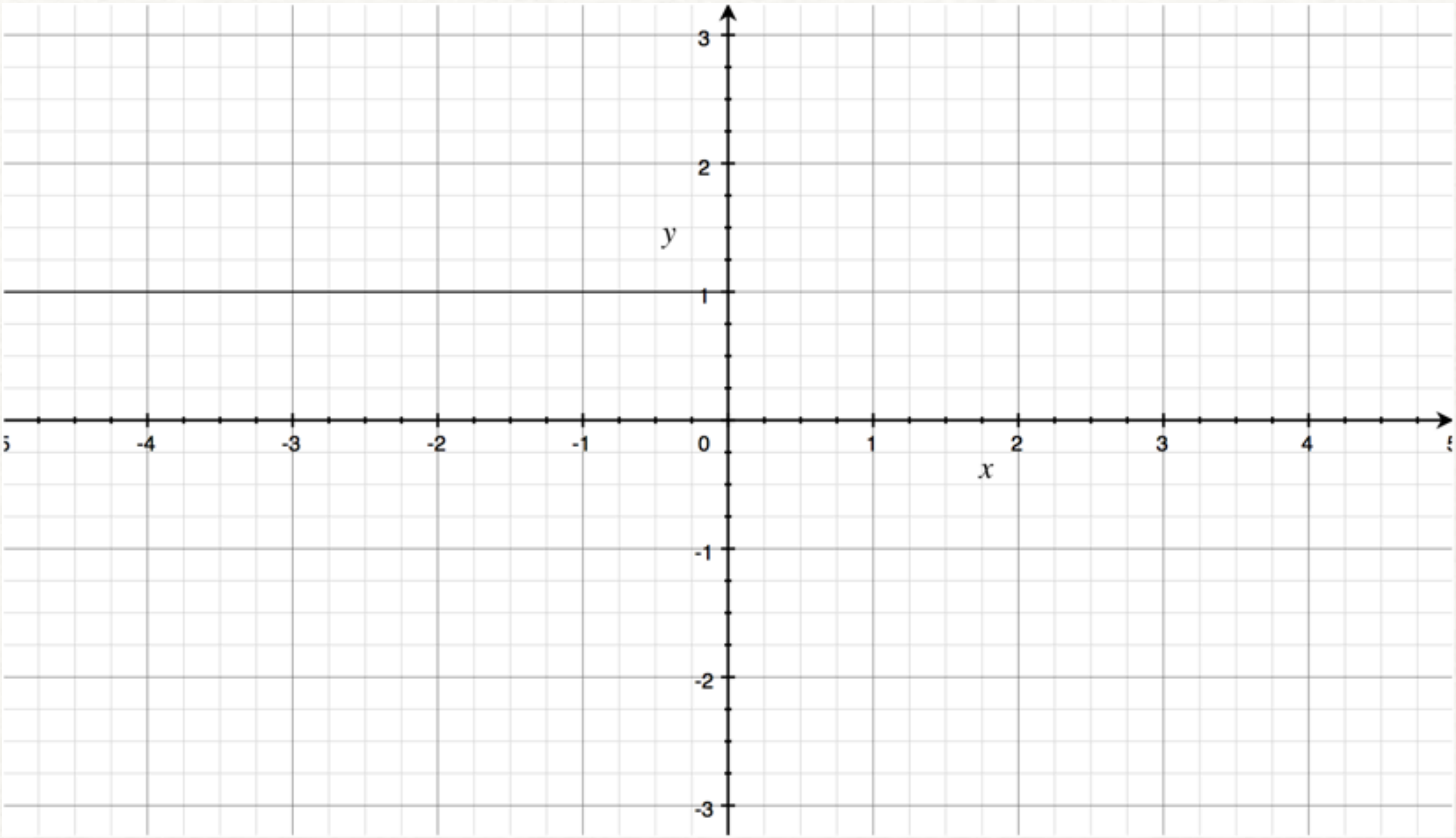
$t = 2$



$t = 3$



EXAMPLE PROBLEM 3 (6)



EXAMPLE PROBLEM 4 (1)

$$u_t + uu_x = 0 \quad u(x,0) = \phi(x) \quad \phi(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

$$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = z \quad \frac{dz}{ds} = 0$$

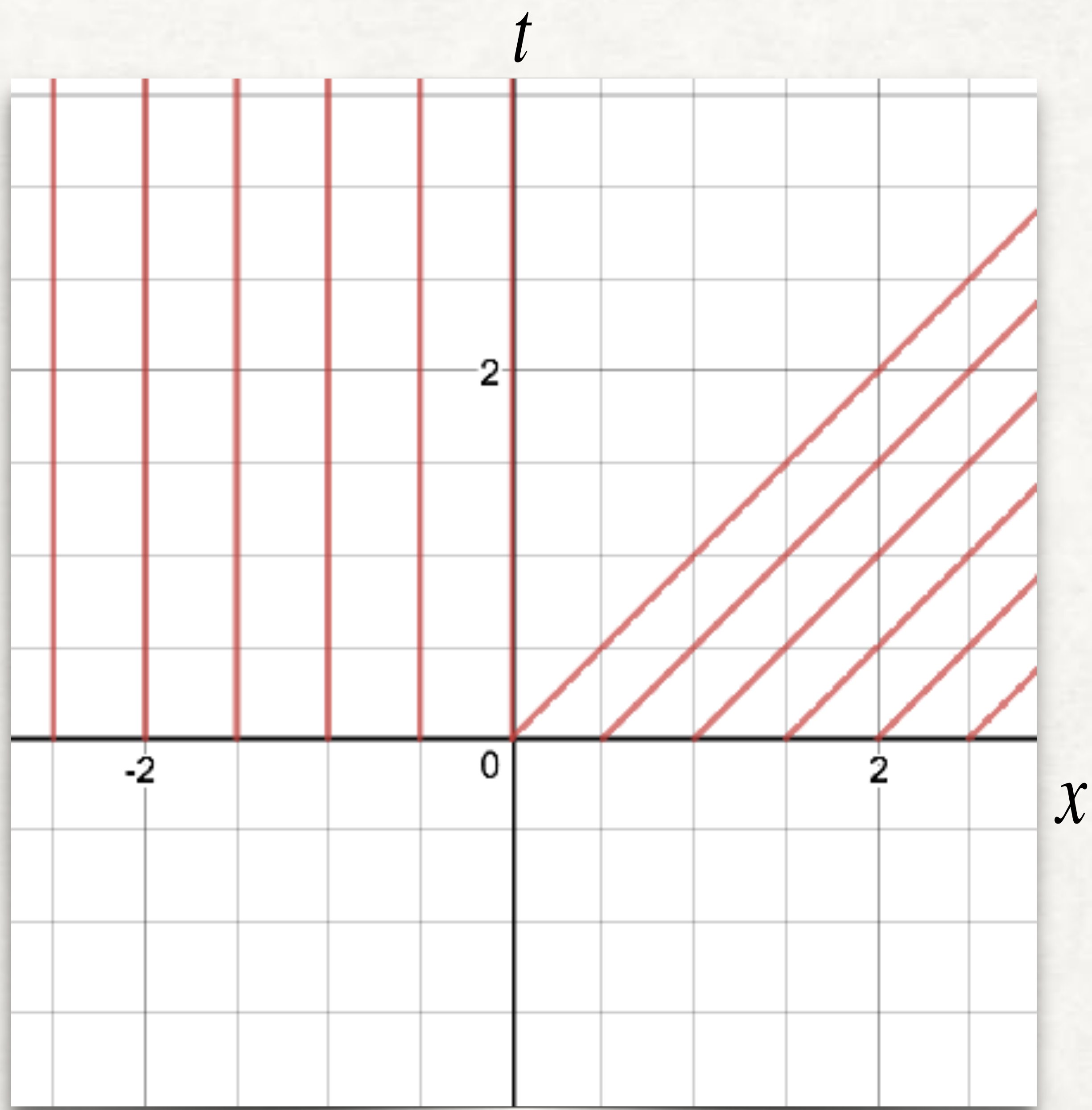
$$t(r,0) = 0 \quad x(r,0) = r \quad z(r,0) = \phi(r)$$

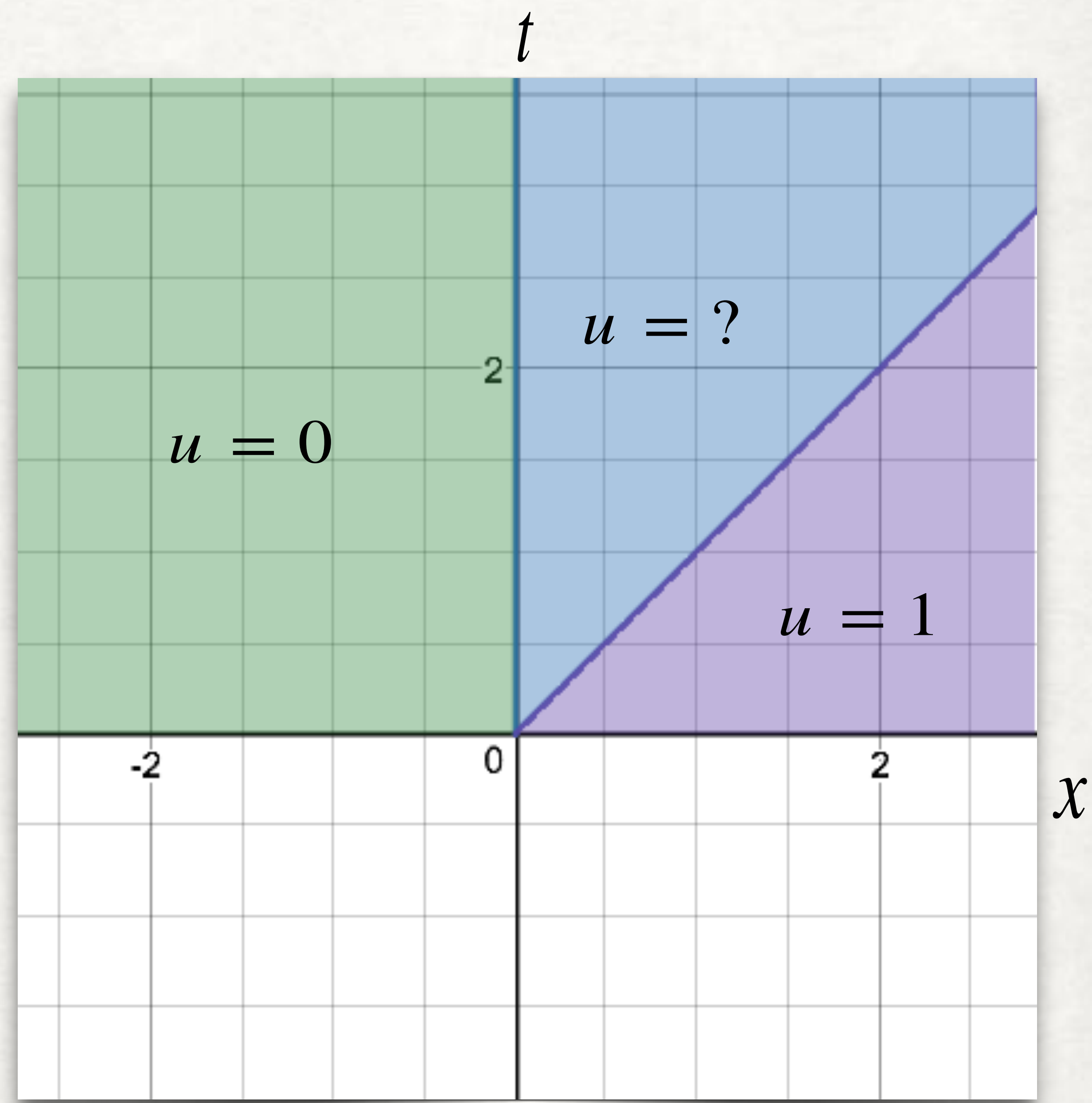
$$t = s + c_1(r) \quad x = zs + c_2(r) \quad z = c_3(r)$$

$$t = s \quad x = zs + r \quad z = \phi(r)$$

$$x = \begin{cases} r & \text{if } r < 0 \\ t + r & \text{if } r > 0 \end{cases} \qquad z = \begin{cases} 0 & \text{if } r < 0 \\ 1 & \text{if } r > 0 \end{cases}$$

EXAMPLE PROBLEM 4 (2)





EXPANSION FANS

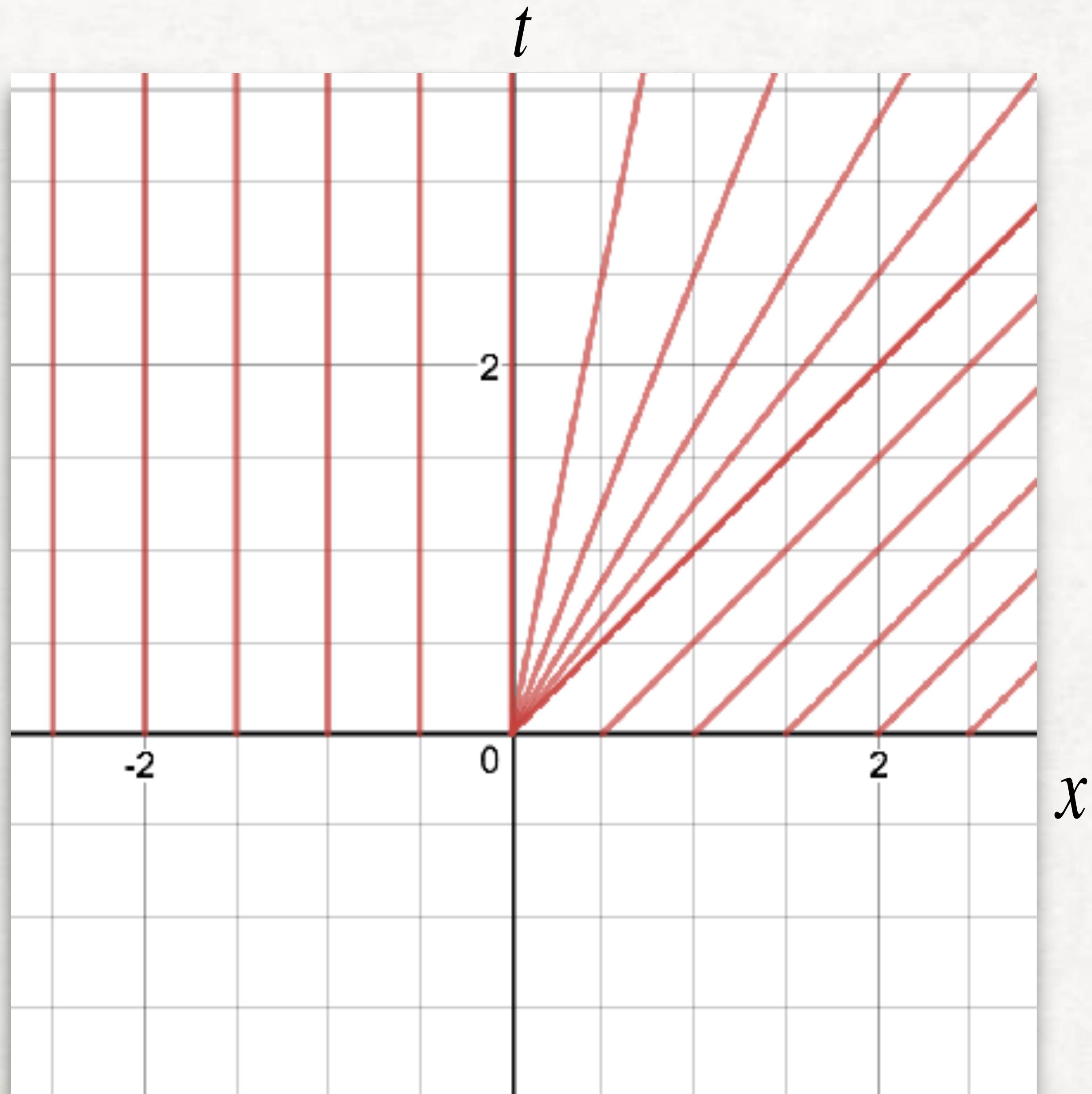
- There are no characteristics in an area, meaning we don't know what value u is there.

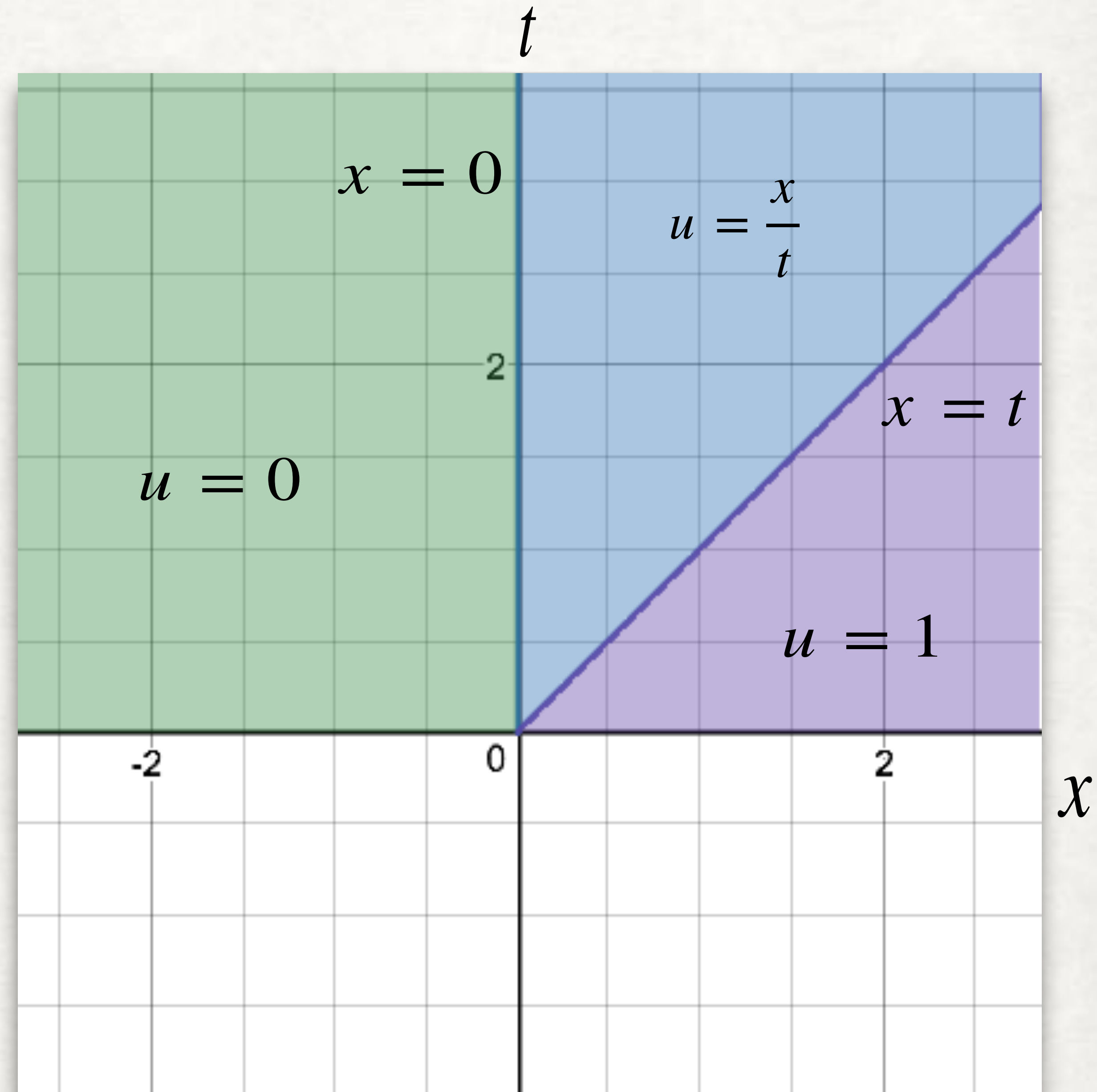
- In order to find a solution, we create an expansion fan, also called a rarefaction wave, in the missing area. Suppose that in the area, for the equation $u_t + f'(u)\, u_x = 0$, $u(x,t) = u\left(\frac{x}{t}\right)$

$$-\frac{x}{t^2} \cdot u'\left(\frac{x}{t}\right) + \frac{1}{t} \cdot f'(u) \cdot u'\left(\frac{x}{t}\right) = 0 \quad \frac{1}{t} \cdot u'\left(\frac{x}{t}\right)\left[f'(u) - \frac{x}{t}\right] = 0 \quad f'(u) = \frac{x}{t}$$

- So we can let $u\left(\frac{x}{t}\right) = (f')^{-1}\left(\frac{x}{t}\right)$ be the solution in the area if the boundaries are piecewise constant (the solution can be translated).

EXAMPLE PROBLEM 4 (3)

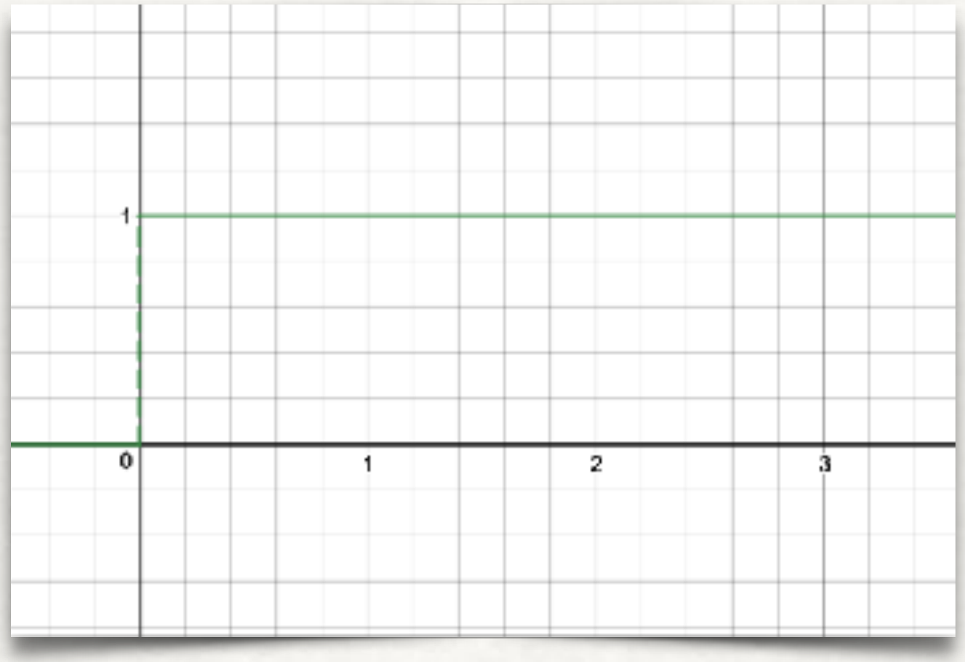




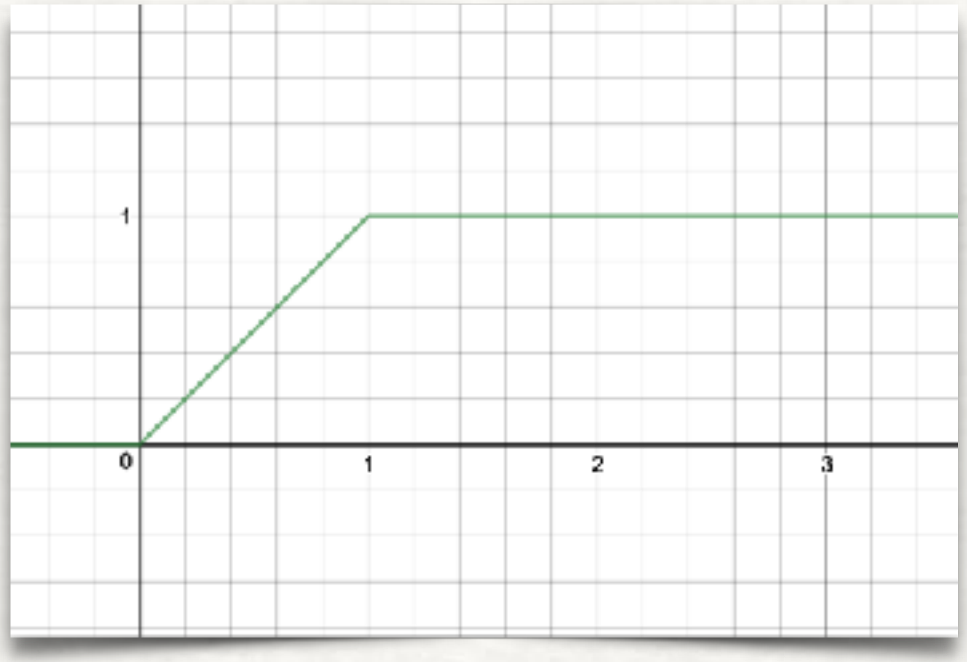
$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 < x < t \\ 1 & \text{if } x > t \end{cases}$$

EXAMPLE PROBLEM 4 (4)

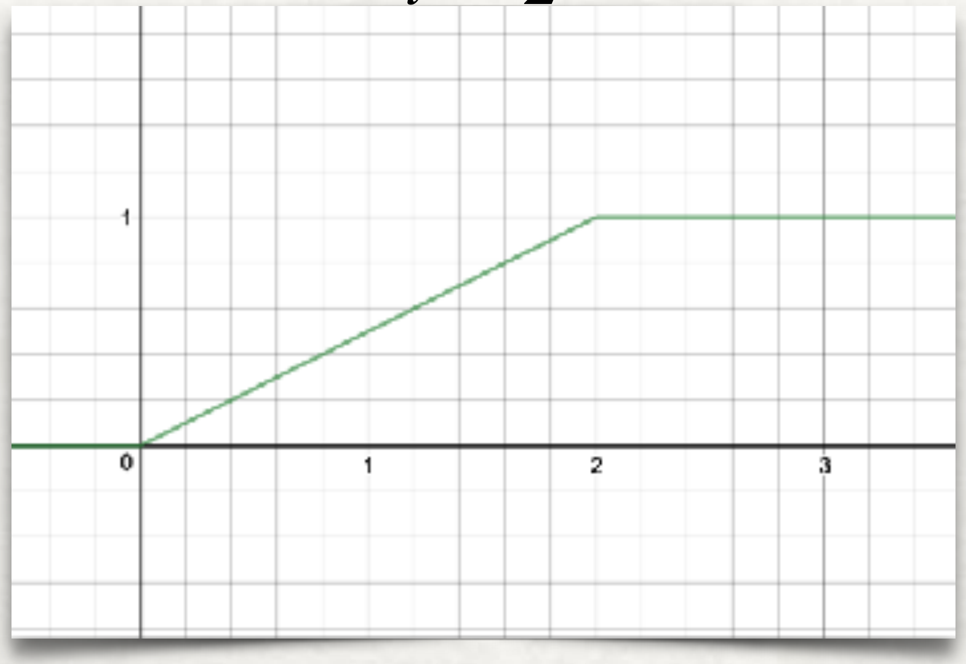
$t = 0$



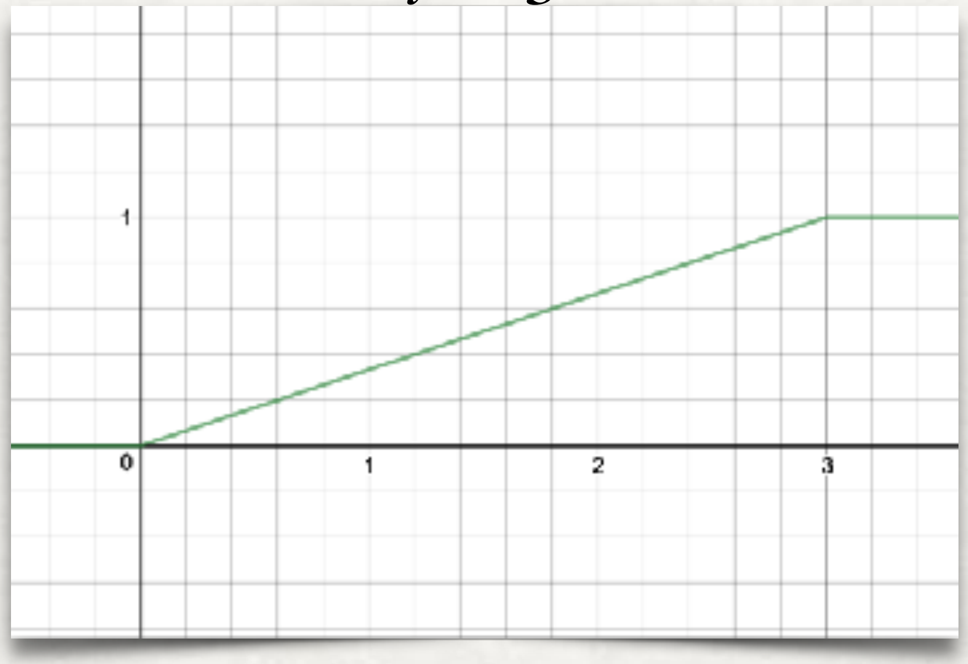
$t = 1$



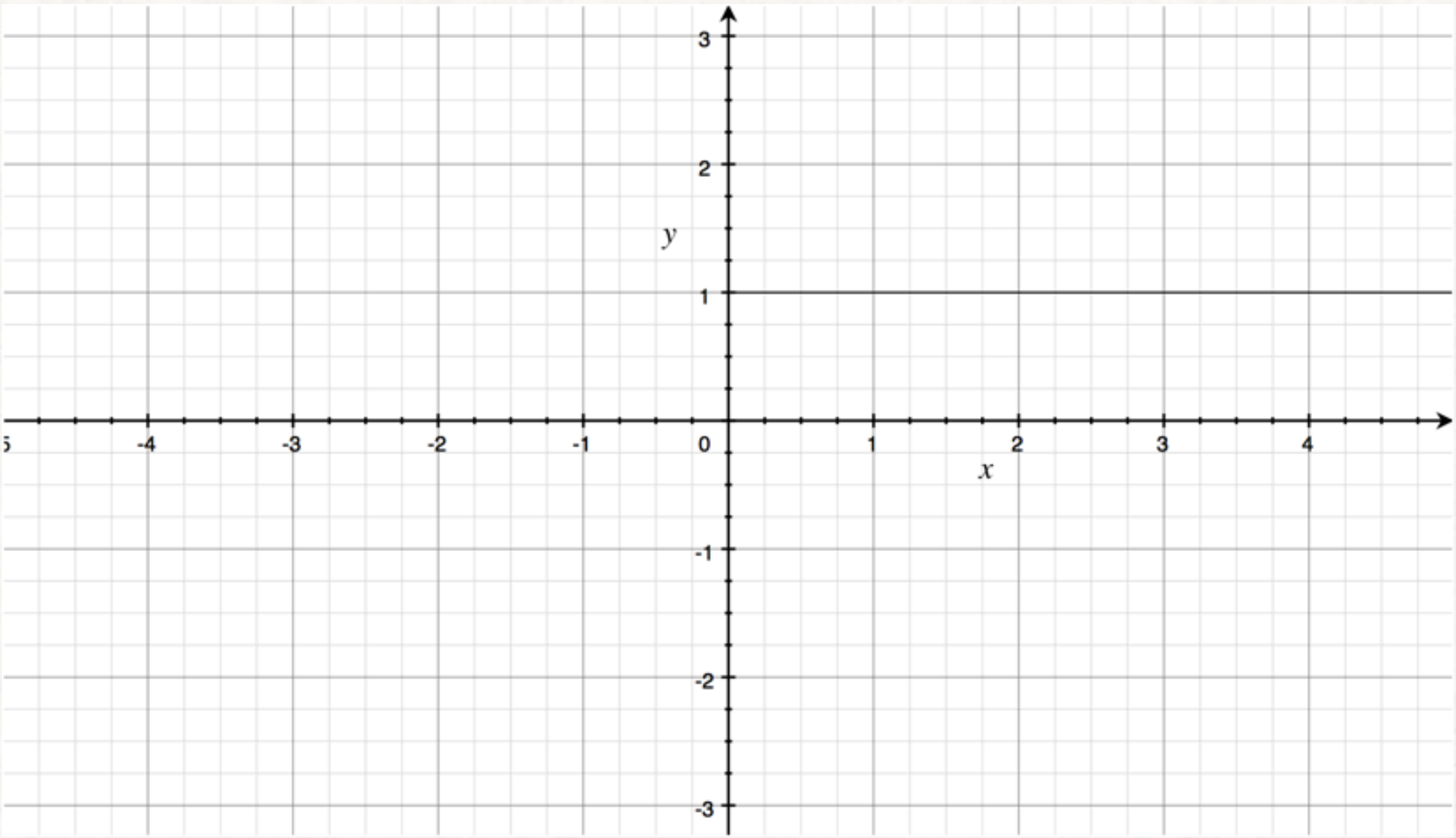
$t = 2$



$t = 3$



EXAMPLE PROBLEM 4 (5)



THE ENTROPY CONDITION

- Because the wave speed is $f'(u)$, we only want shock waves to occur if the left part of the wave passes the right, so we need the condition $f'(u^-) > \sigma > f'(u^+)$ for a shock wave, where $\sigma = \zeta'(t)$ is the speed of the discontinuity.

- We also need f to be uniformly convex, meaning that there exists a constant θ such that $f''(u) \geq \theta > 0$ for all u.

EXAMPLE PROBLEM 5 (1)

$$u_t + u u_x = 0 \quad u(x,0) = \phi(x) \quad \phi(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = z \quad \frac{dz}{ds} = 0$$

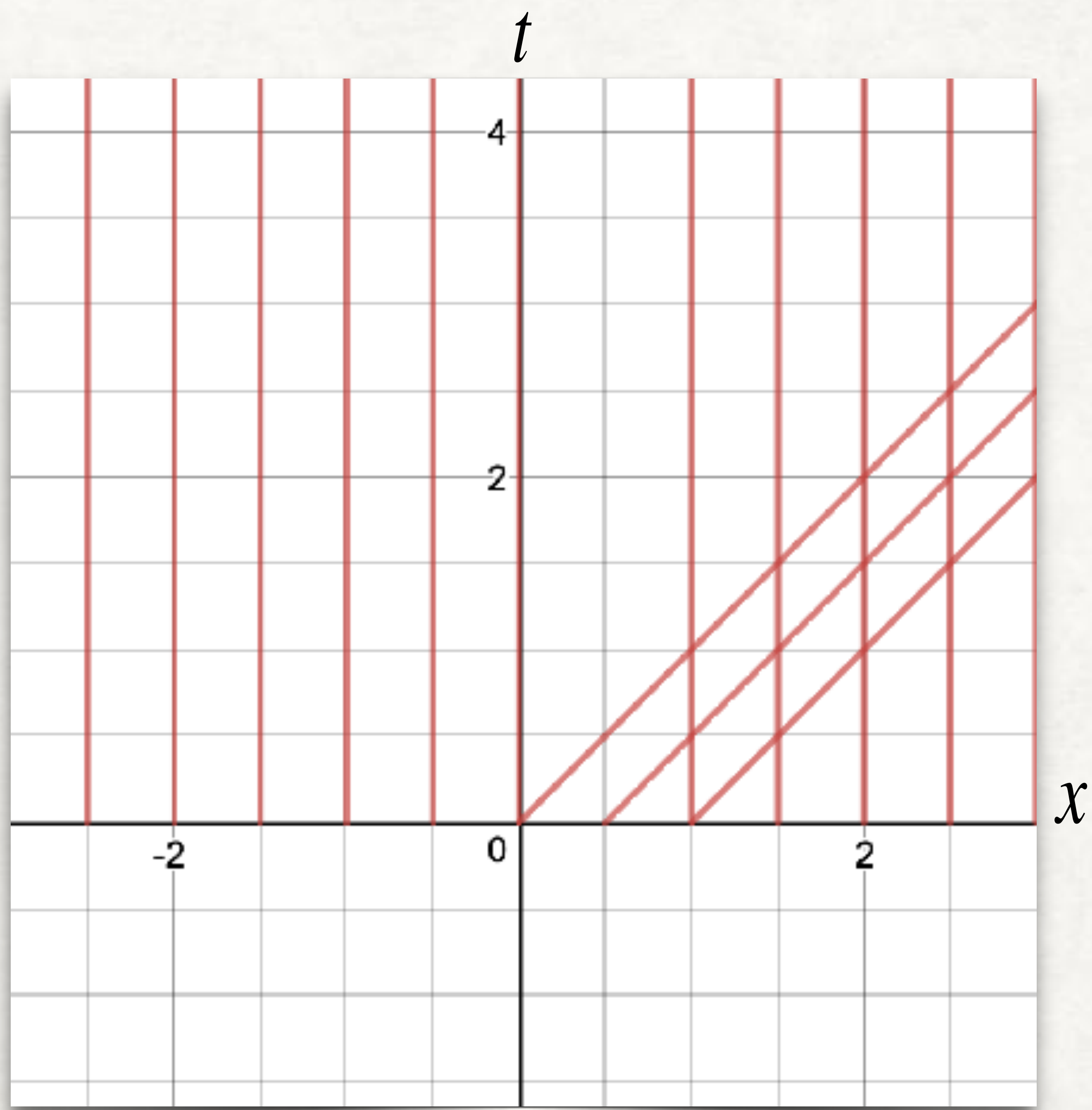
$$t(r,0) = 0 \quad x(r,0) = r \quad z(r,0) = \phi(r)$$

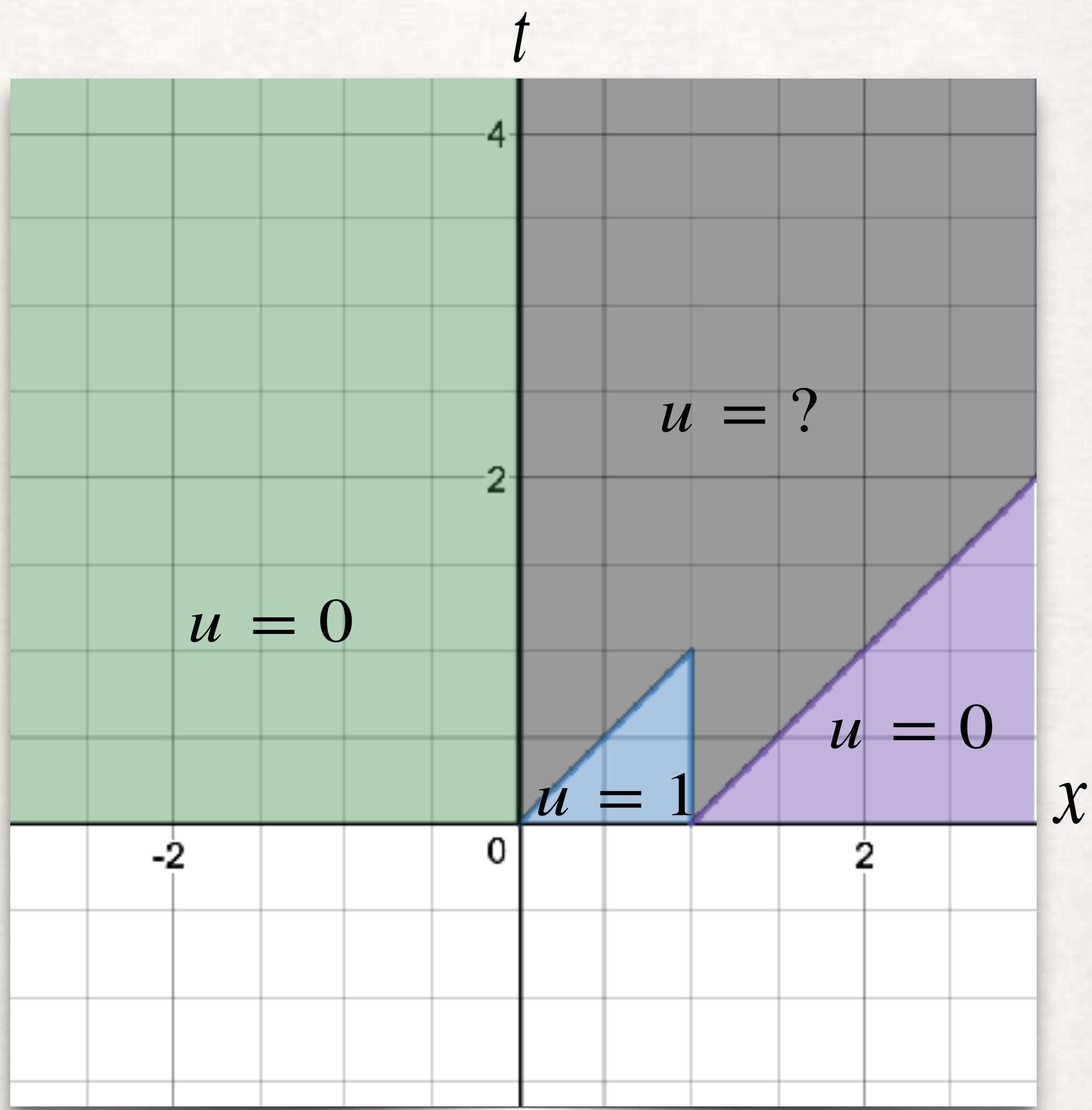
$$t = s + c_1(r) \quad x = zs + c_2(r) \quad z = c_3(r)$$

$$t = s \quad x = zs + r \quad z = \phi(r)$$

$$x = \begin{cases} r & \text{if } r < 0 \\ t + r & \text{if } 0 < r < 1 \\ r & \text{if } r > 1 \end{cases} \qquad z = \begin{cases} 0 & \text{if } r < 0 \\ 1 & \text{if } 0 < r < 1 \\ 0 & \text{if } r > 1 \end{cases}$$

EXAMPLE PROBLEM 5 (2)





From 0<t<2, we can create a shock wave where the r>1 and the 0<r<1 characteristics intersect, and a rarefaction wave between the r<0 and the 0<r<1 characteristics.

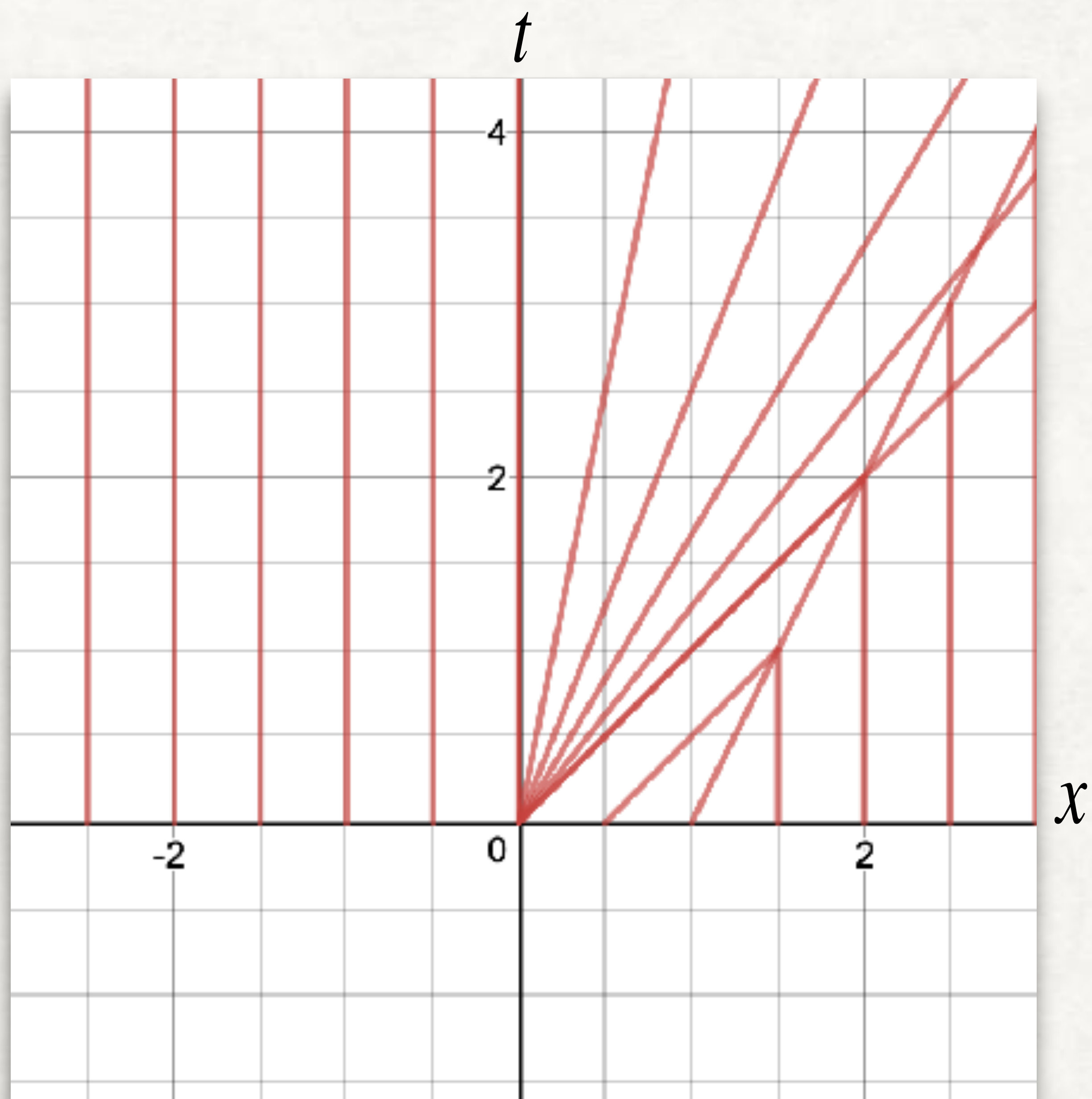
EXAMPLE PROBLEM 5 (3)

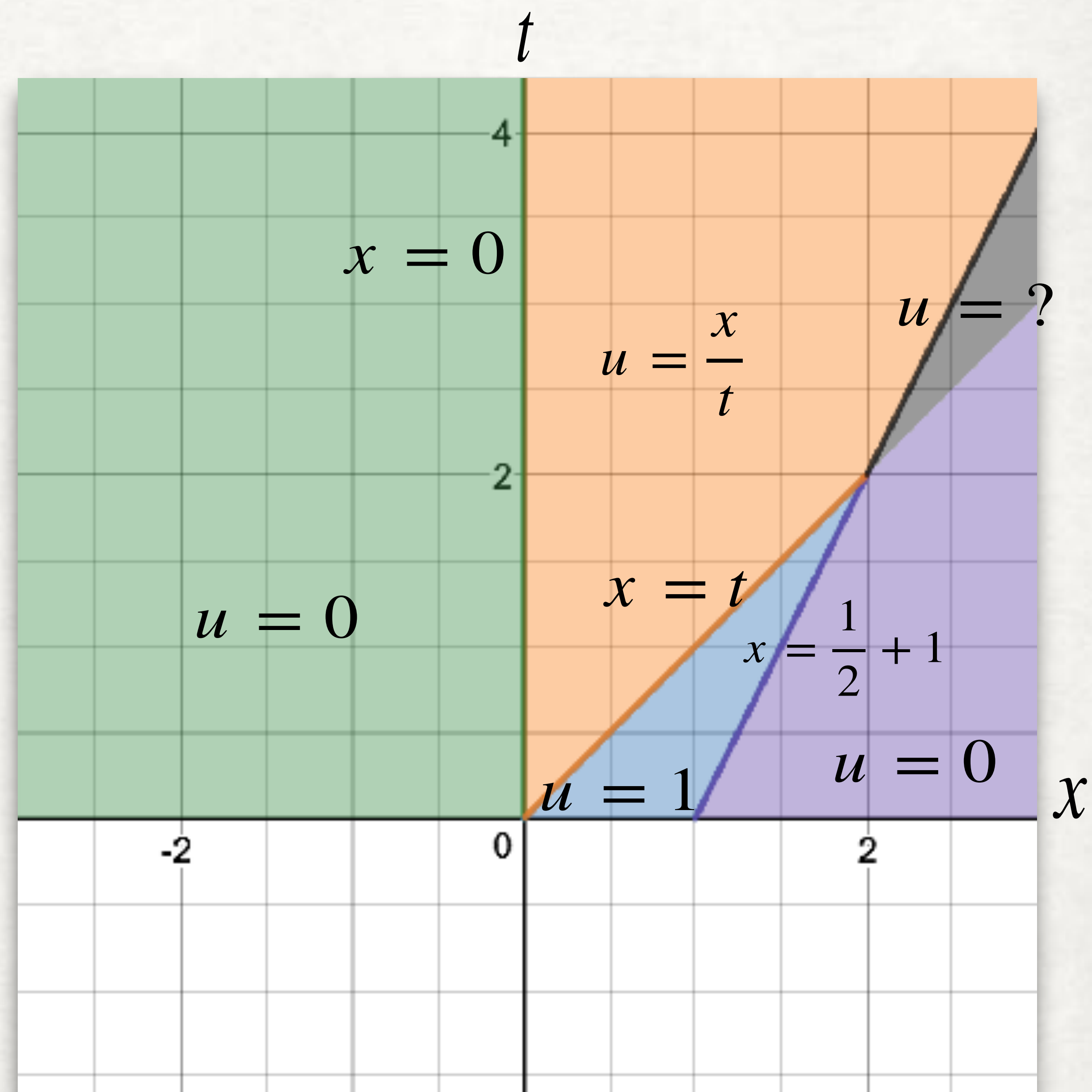
$$f(u) = \frac{1}{2}u^2 \quad f'(u) = u \quad (f')^{-1}(u) = u \quad (f')^{-1}\left(\frac{x}{t}\right) = \frac{x}{t}$$

$$\zeta'(t) = \frac{\frac{1}{2}(1) - \frac{1}{2}(0)}{1 - 0} \quad x(t) = \zeta(t) = \frac{1}{2}t + c_4$$

$$x(0) = 1 \quad x = \frac{1}{2}t + 1$$

EXAMPLE PROBLEM 5 (4)





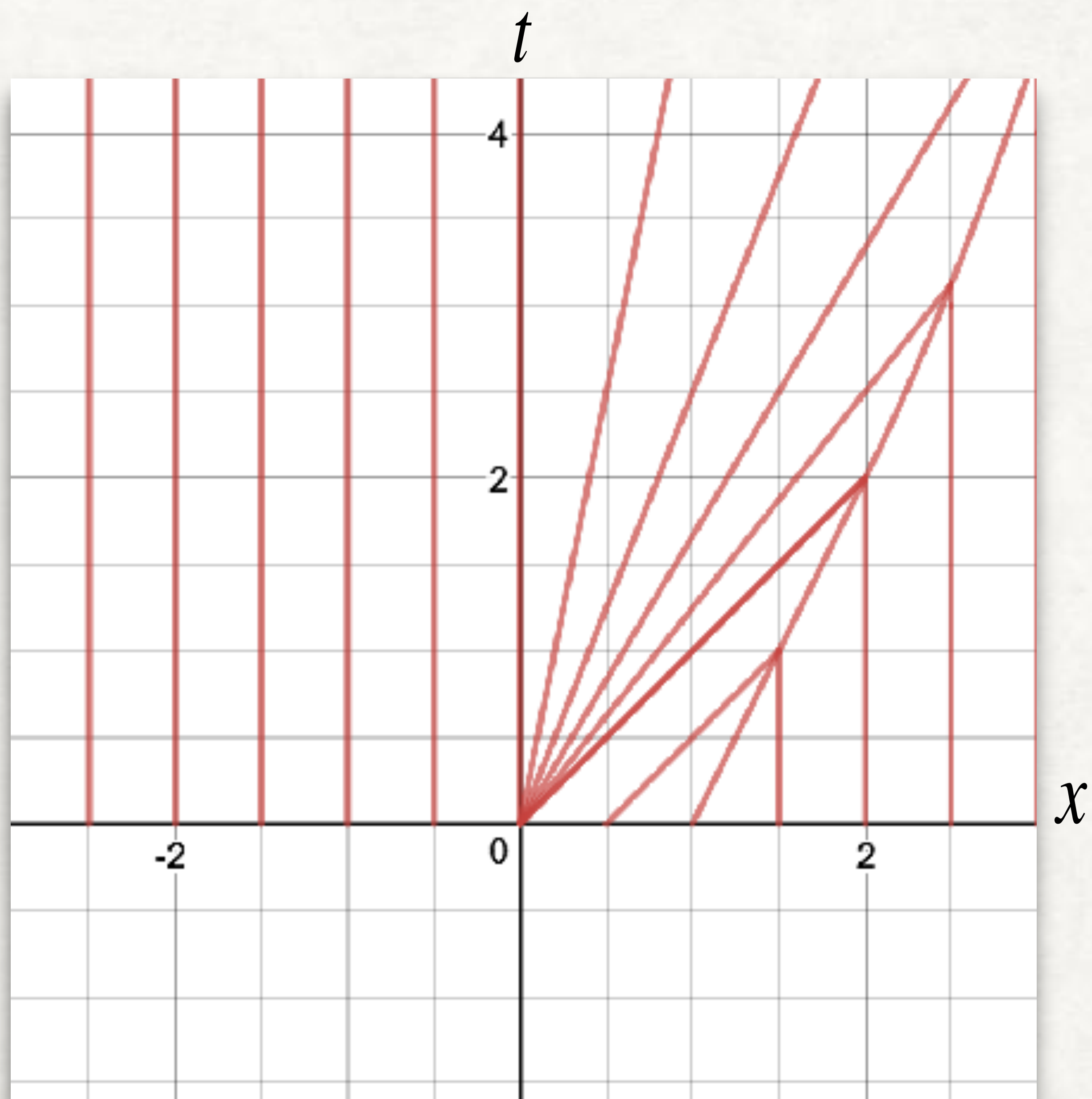
A new shock wave must be created to separate the rarefaction wave and the u=0 characteristics.

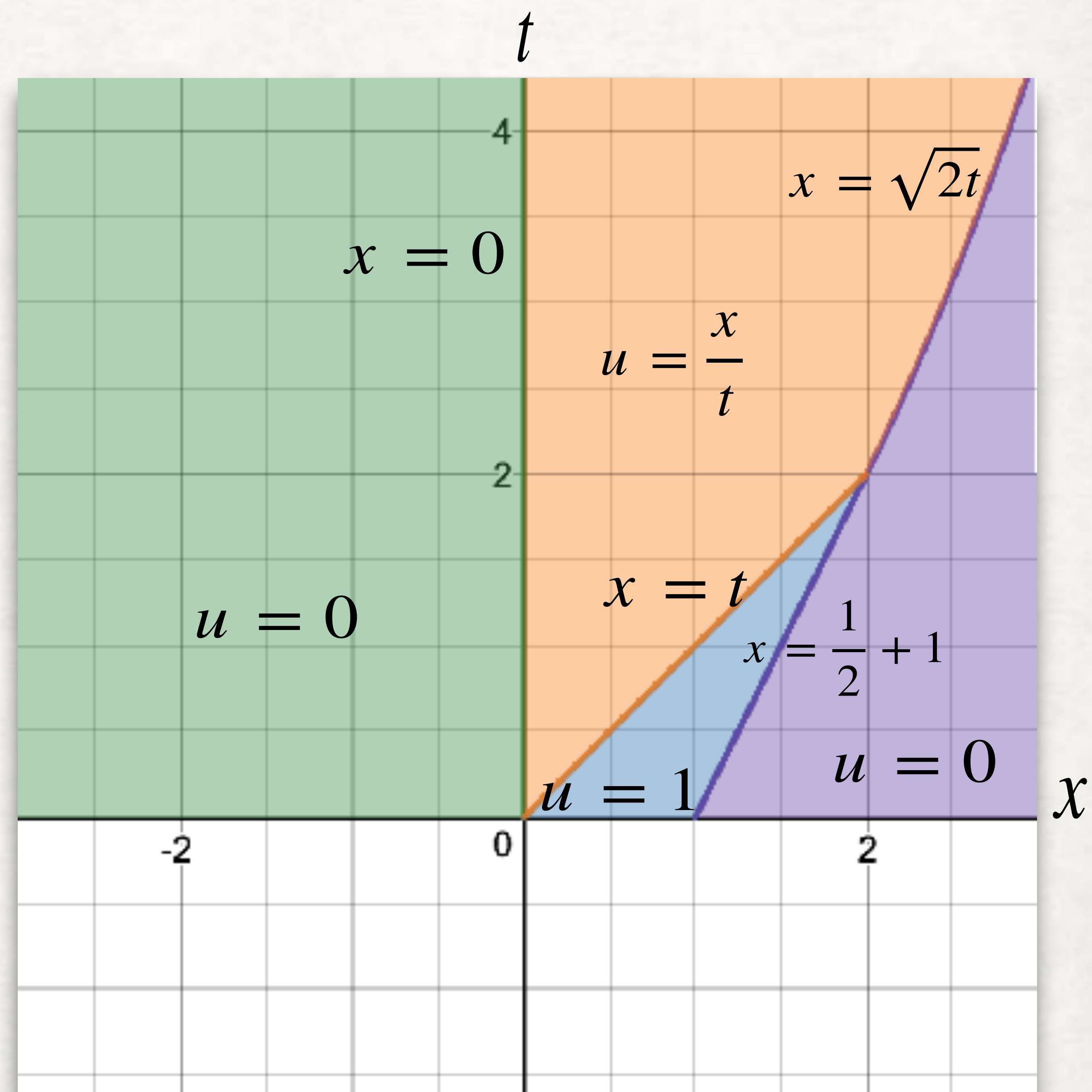
EXAMPLE PROBLEM 5 (5)

$$\zeta'(t) = \frac{\frac{1}{2}\left(\frac{x}{t}\right) - \frac{1}{2}(0)}{\frac{x}{t} - 0} \quad \frac{dx}{dt} = \frac{x}{2t} \quad x(t) = \zeta(t) = \sqrt{c_4 t}$$

$$x(2) = 2 \quad x = \sqrt{2t}$$

EXAMPLE PROBLEM 5 (6)

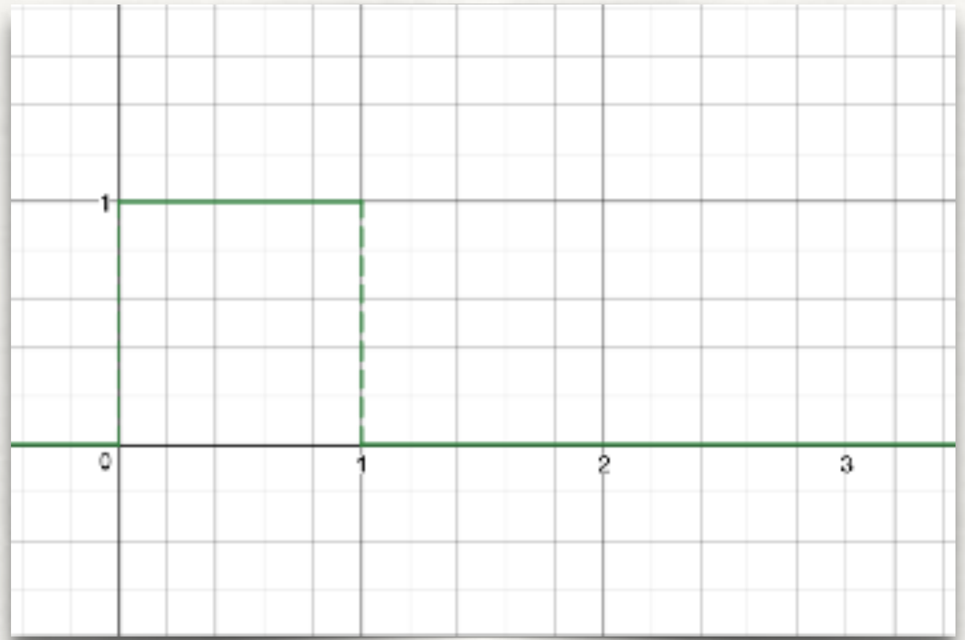




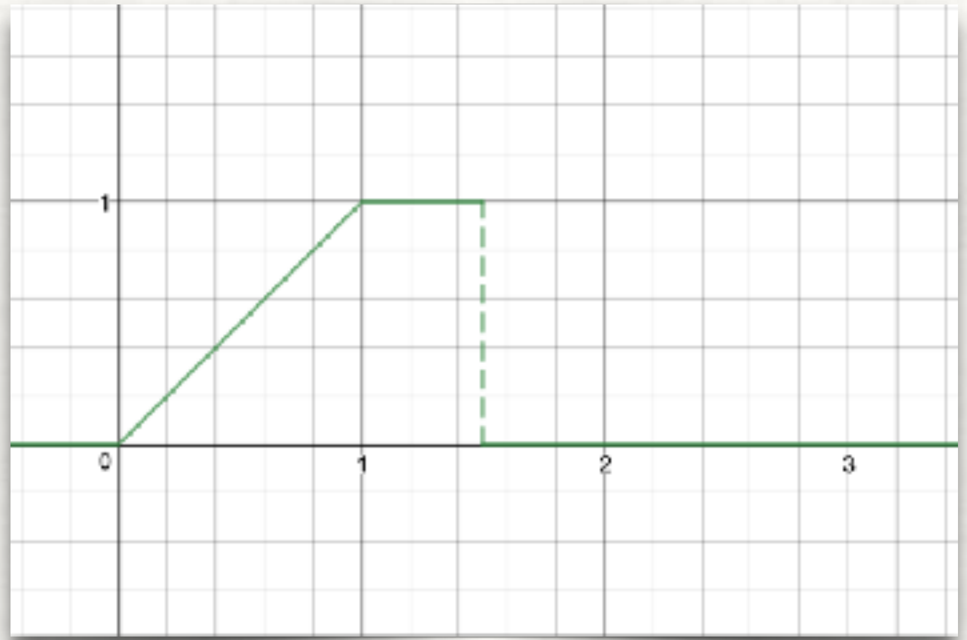
If $0 < t < 2$ then $u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 < x < t \\ 1 & \text{if } t < x < \frac{1}{2}t + 1 \\ 0 & \text{if } x > \frac{1}{2}t + 1 \end{cases}$ and if $t > 2$ then $u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 < x < \sqrt{2t} \\ 0 & \text{if } x > \sqrt{2t} \end{cases}$

EXAMPLE PROBLEM 5 (7)

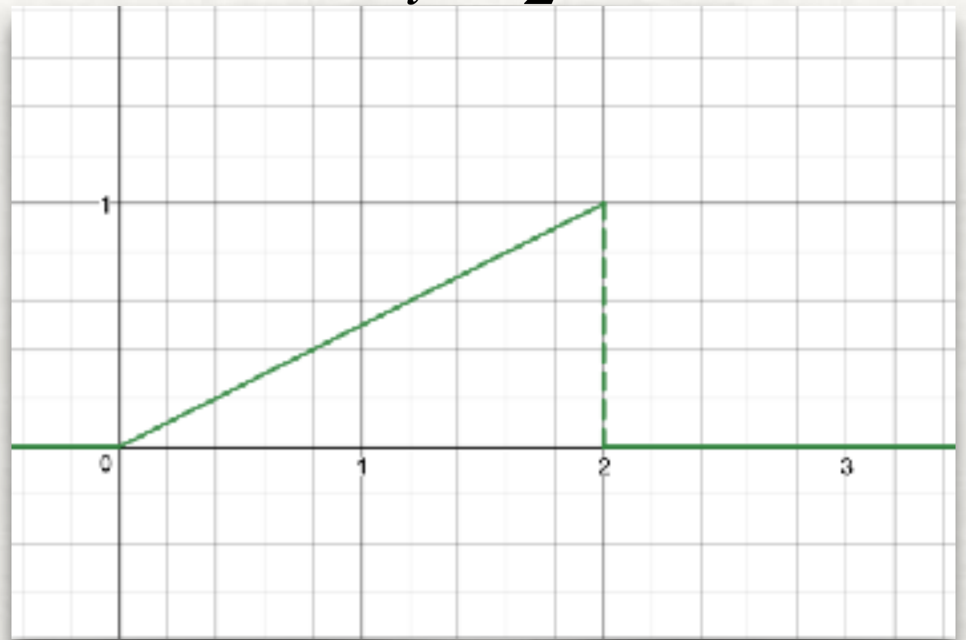
$t = 0$



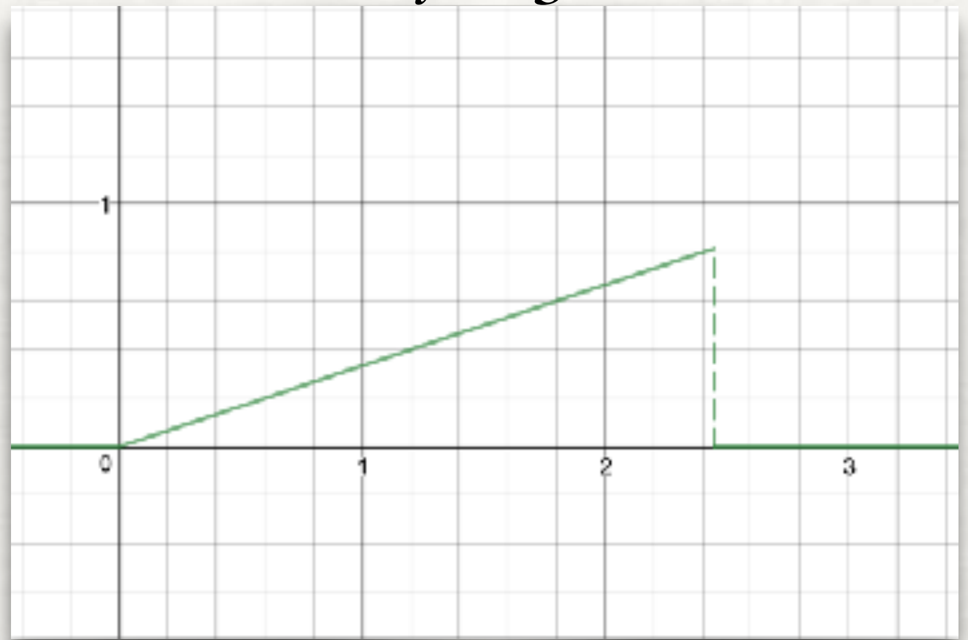
$t = 1$



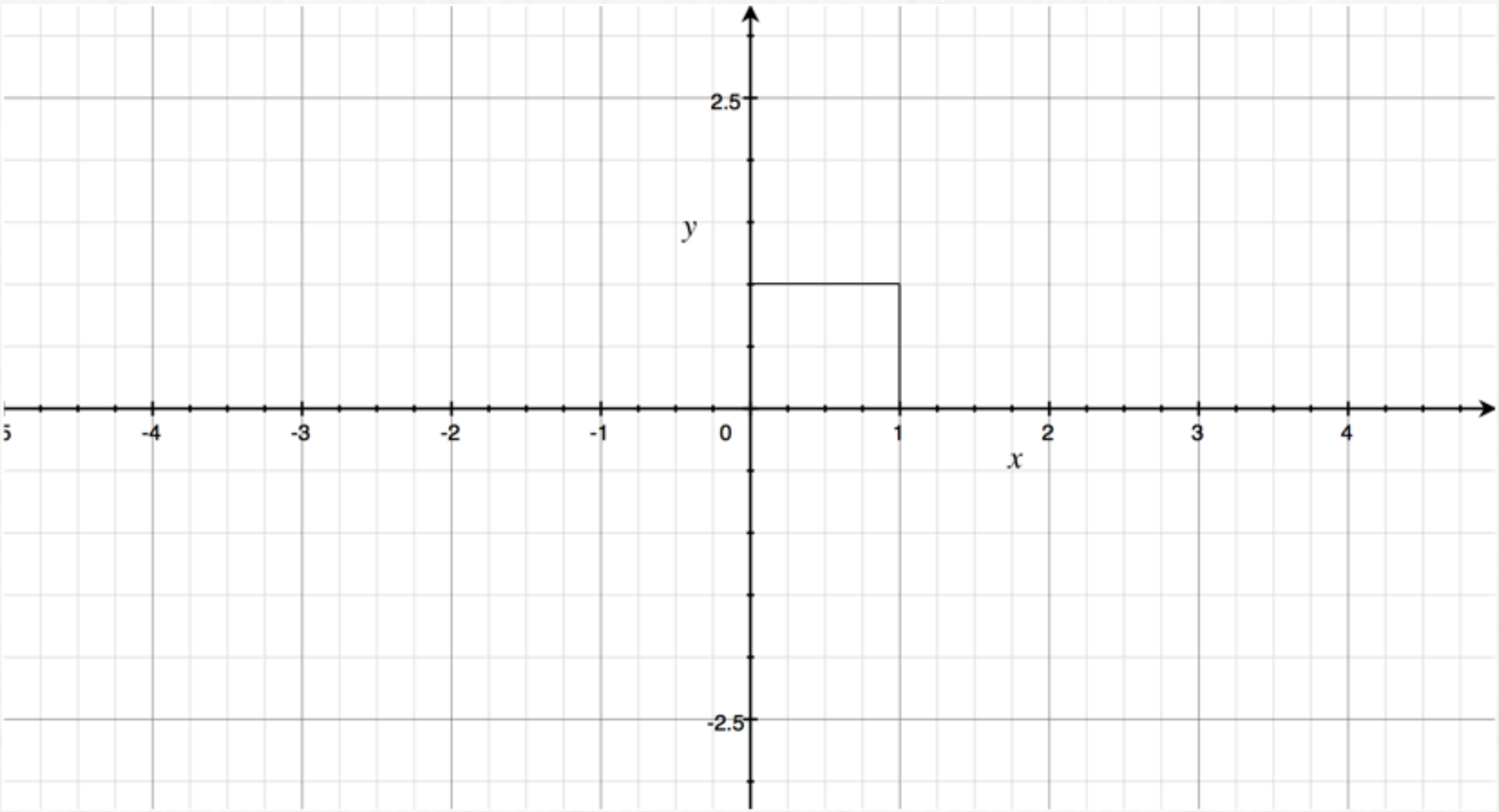
$t = 2$



$t = 3$



EXAMPLE PROBLEM 5 (8)



TRAFFIC MODELING (1)

- We'll let the car density at a point and time be given by $\rho(x,t)$ and the car velocity be given by $v(x,t)$
- The car flux will be given by $q(x,t) = \rho(x,t)\,v(x,t)$
- The continuity equation expresses "conservation of cars" and is given by $\frac{\partial}{\partial t}\rho(\vec{r},t) + \nabla \cdot q(\vec{r},t) = 0$. In one dimension, this is $\rho_t + q_x = 0$
- The velocity can usually be expressed as a function of the density, so we can write $v(\rho)$ and $q(\rho) = v(\rho)\,\rho$
- So, our traffic equation is $\rho_t + [v(\rho)\,\rho]_x = 0$ or $\rho_t + v'(\rho)\,\rho_x = 0$

TRAFFIC MODELING (2)

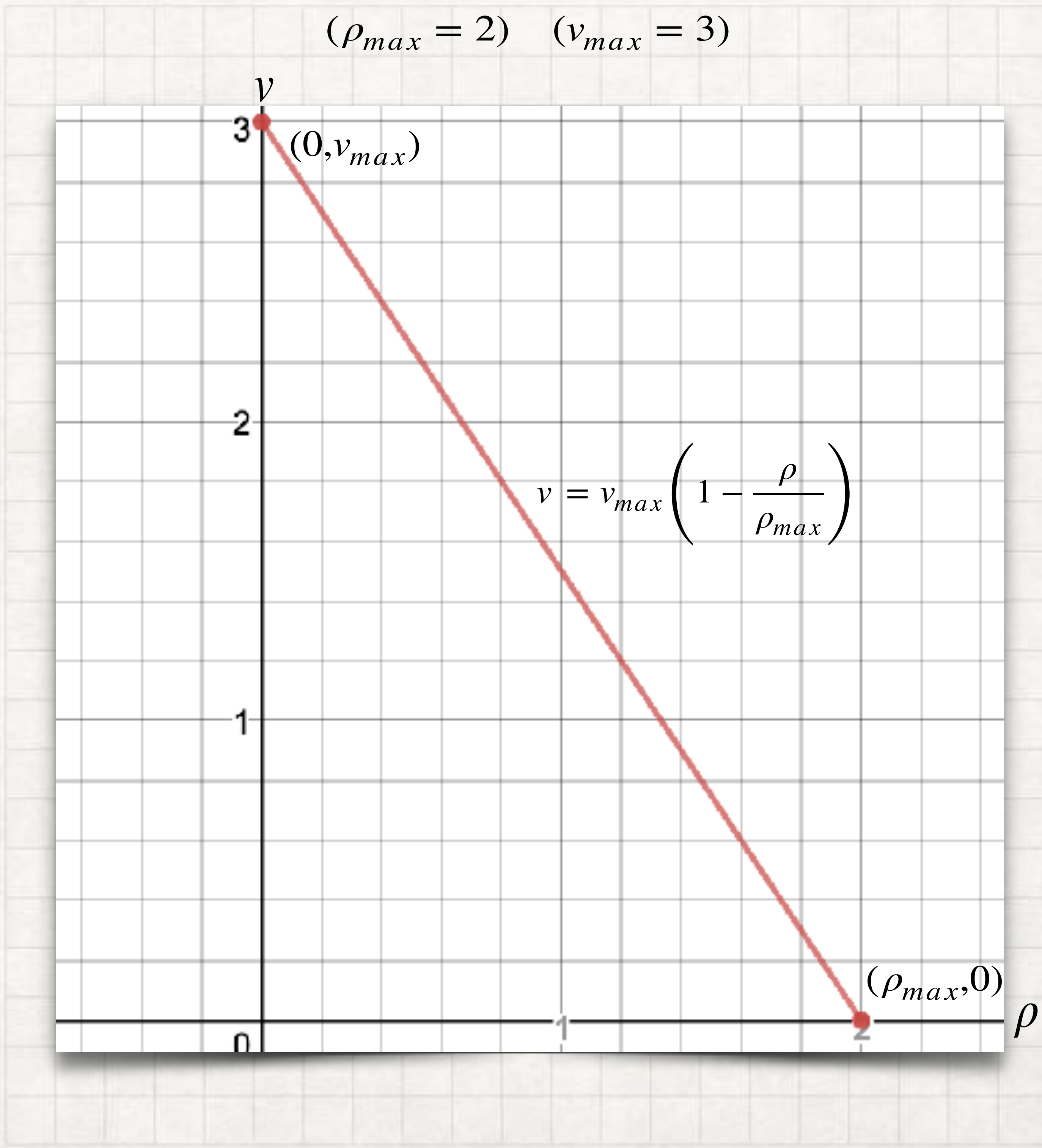
- A simple model for the velocity is $v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$ where v_{max} and ρ_{max} are the maximum velocity and density of the traffic.

- We'll use units where $v_{max} = \rho_{max} = 1$ for simplicity.

$$v(\rho) = 1 - \rho \qquad q(\rho) = \rho(1-\rho)$$

$$q'(\rho) = 1 - 2\rho$$

$$\rho_t + (1 - 2\rho)\,\rho_x = 0$$



EXAMPLE PROBLEM 6 (1)

A STOP LIGHT TURNING GREEN FROM RED

$$\rho_t + (1-2\rho)\rho_x = 0 \quad \rho(x,0) = \phi(x) \quad \phi(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = 1 - 2z \quad \frac{dz}{ds} = 0$$

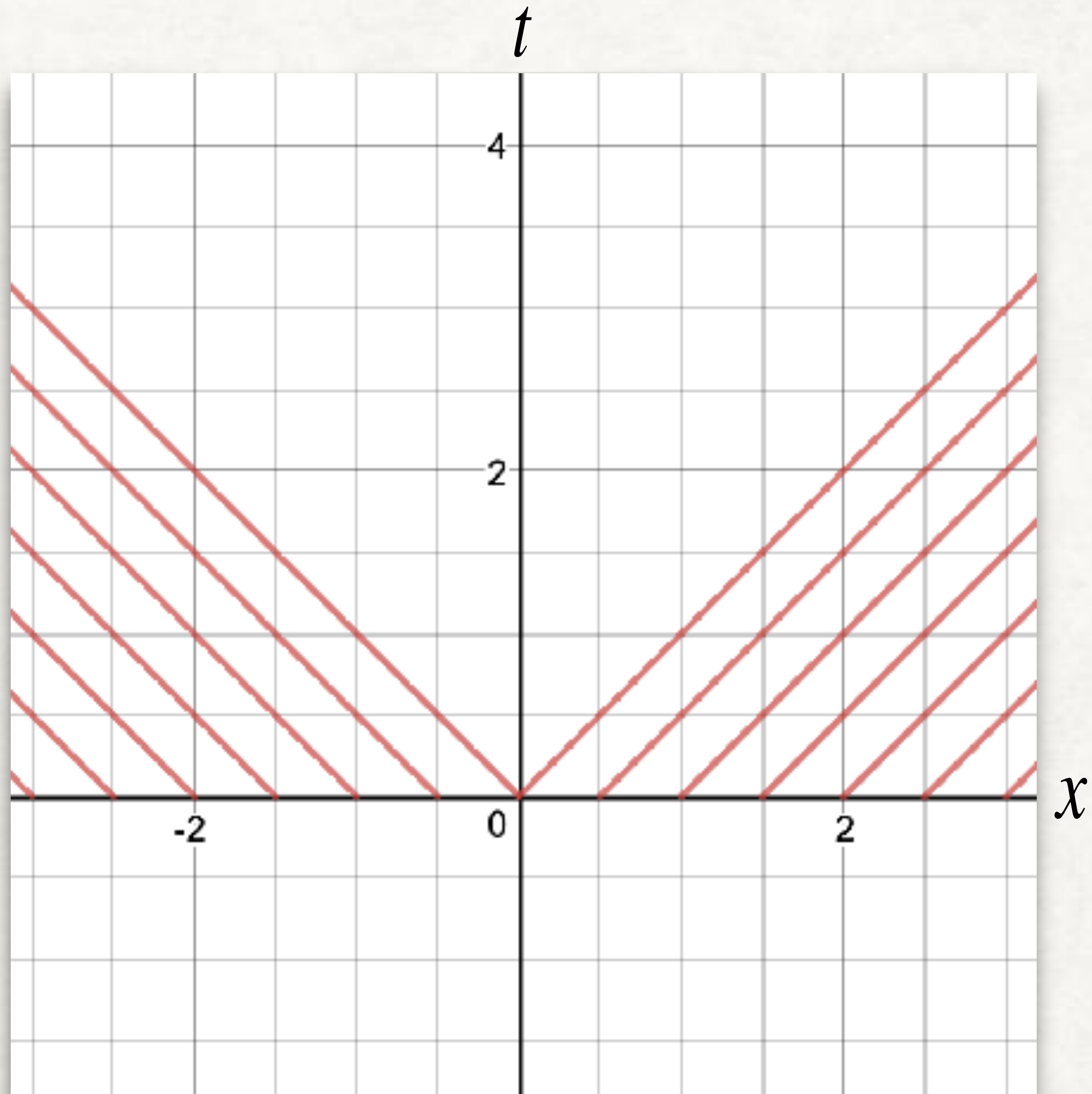
$$t(r,0) = 0 \quad x(r,0) = r \quad z(r,0) = \phi(r)$$

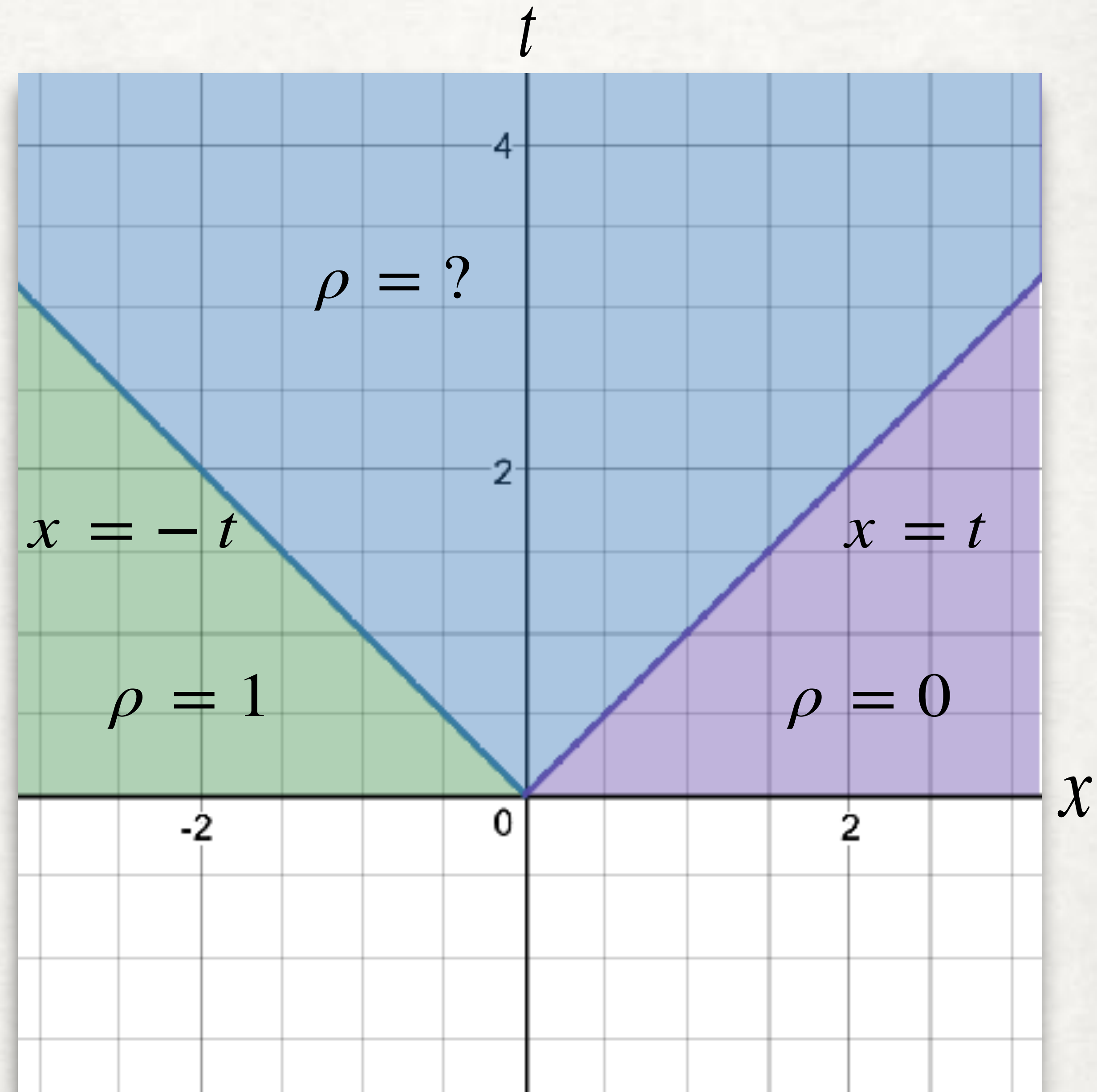
$$t = s + c_1(r) \quad x = (1-2z)s + c_2(r) \quad z = c_3(r)$$

$$t = s \quad x = (1-2\phi(r))s + r \quad z = \phi(r)$$

$$x = \begin{cases} -t + r & \text{if } r < 0 \\ t + r & \text{if } r > 0 \end{cases} \qquad z = \begin{cases} 1 & \text{if } r < 0 \\ 0 & \text{if } r > 0 \end{cases}$$

EXAMPLE PROBLEM 6 (2)





An expansion fan must be created to fill the area with no characteristics.

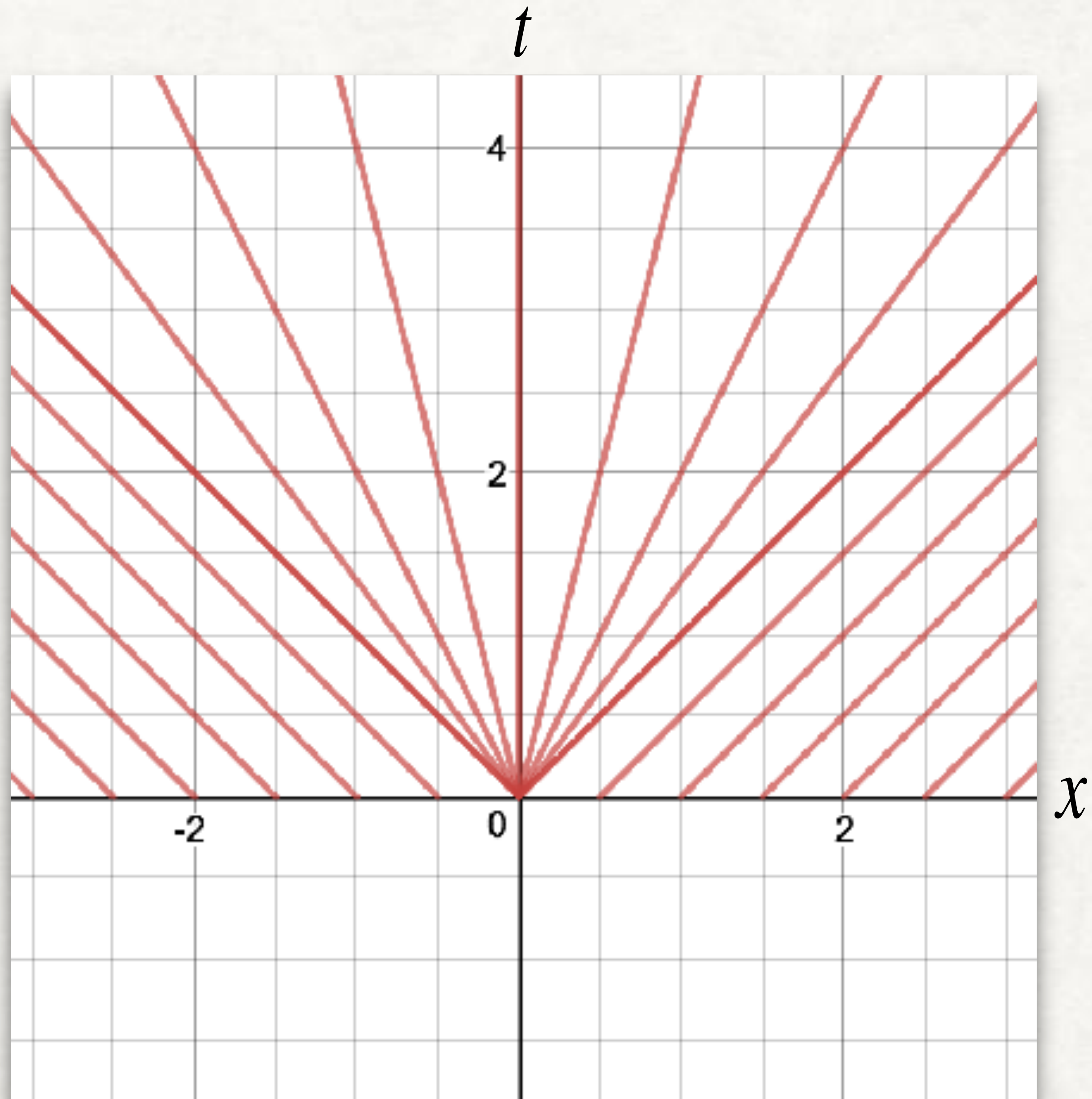
EXAMPLE PROBLEM 6 (3)

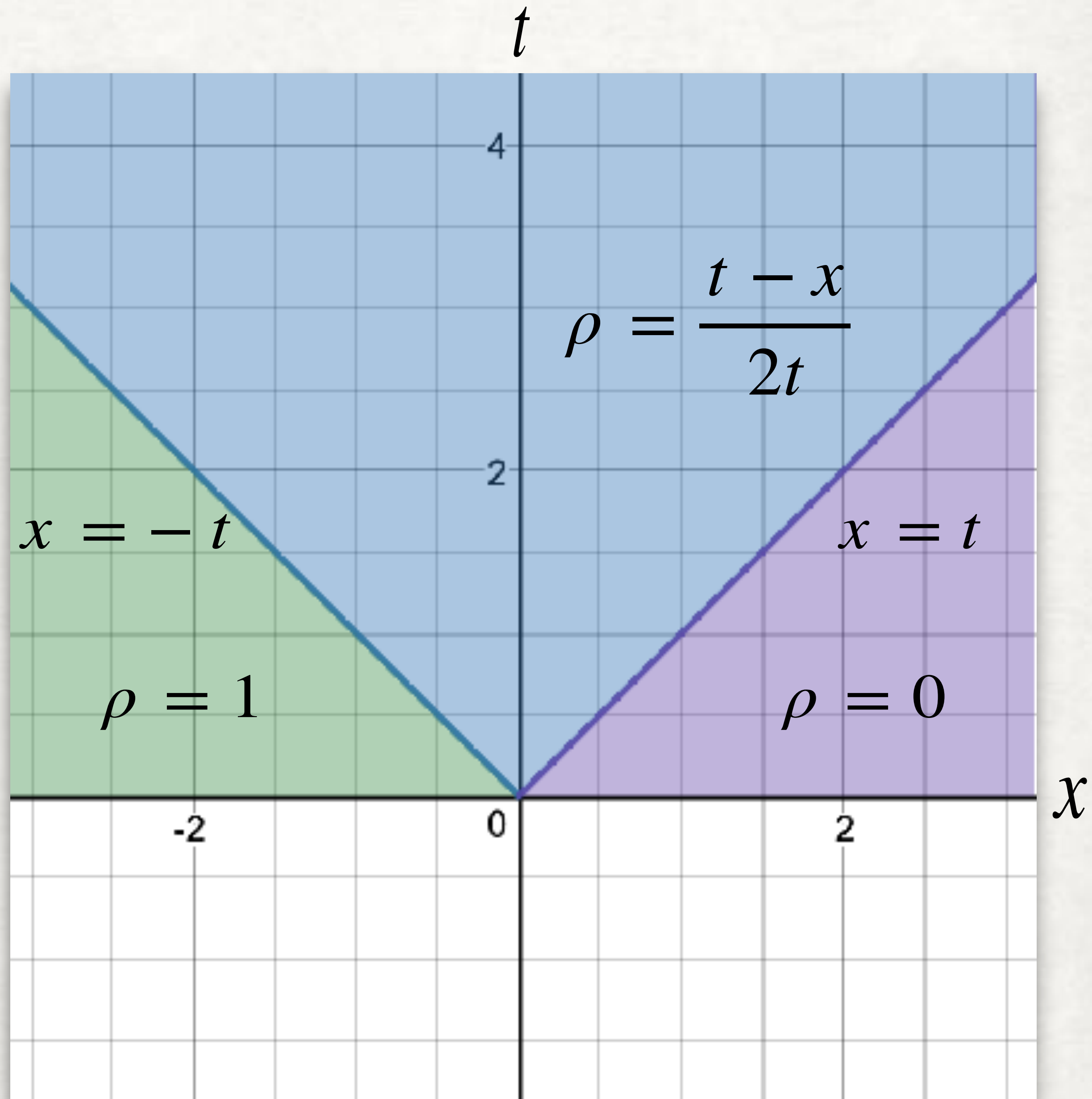
$$q'(\rho) = 1 - 2\rho$$

$$(q')^{-1}(\rho) = \frac{1-\rho}{2}$$

$$(q')^{-1}\left(\frac{x}{t}\right) = \frac{t-x}{2t}$$

EXAMPLE PROBLEM 6 (4)

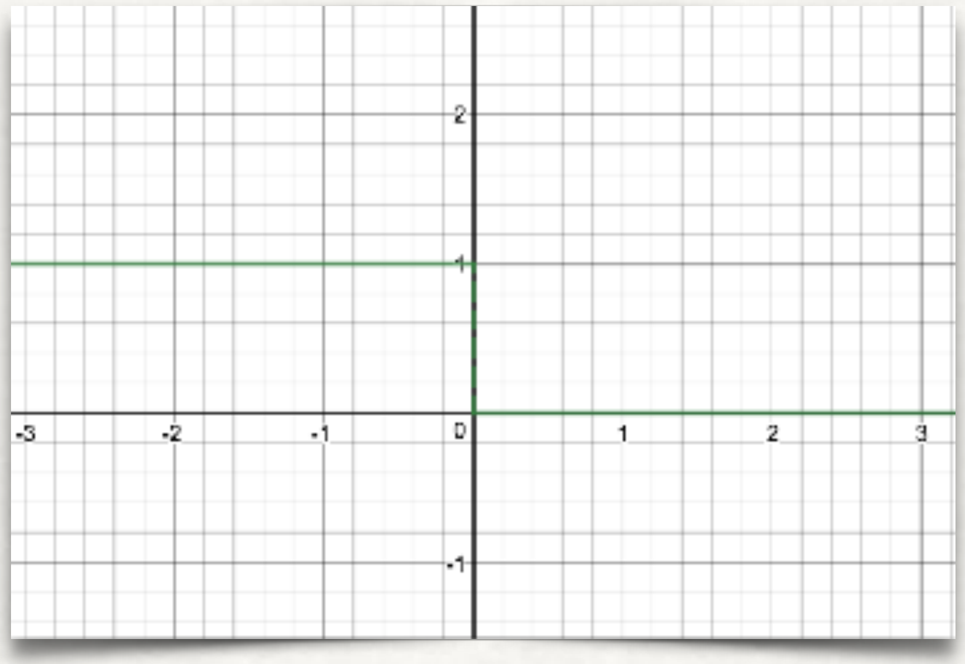




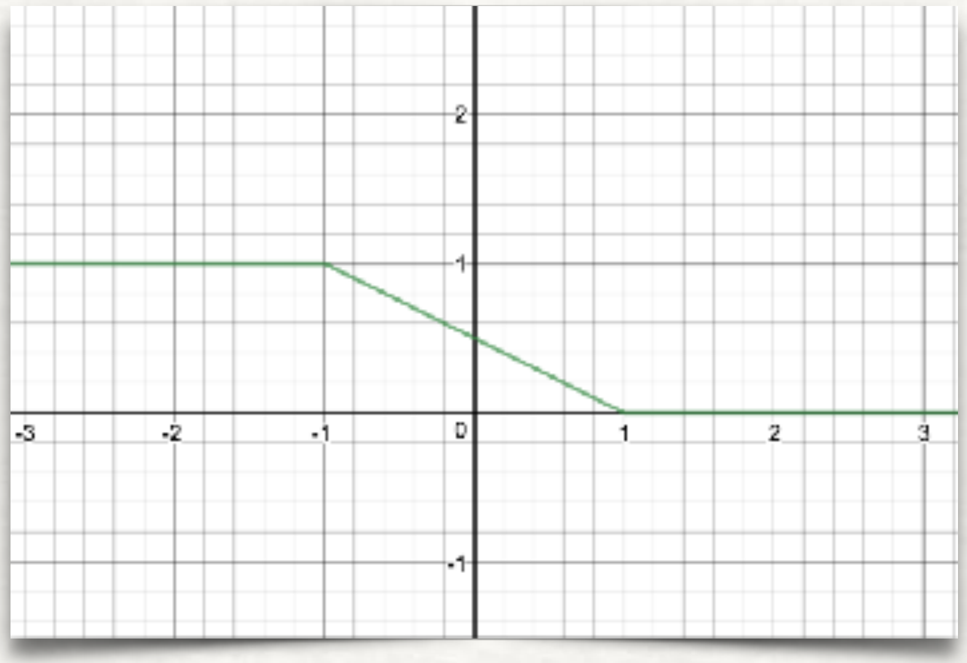
$$\rho(x,t) = \begin{cases} 1 & \text{if } x < -t \\ \dfrac{t-x}{2t} & \text{if } -t < x < t \\ 0 & \text{if } x > t \end{cases}$$

EXAMPLE PROBLEM 6 (5)

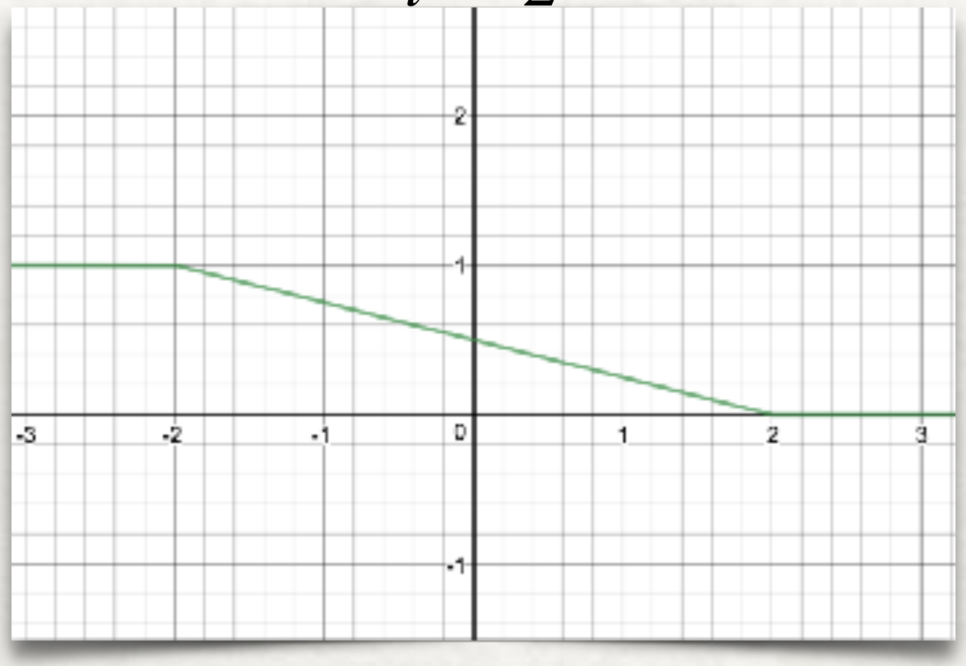
$t = 0$



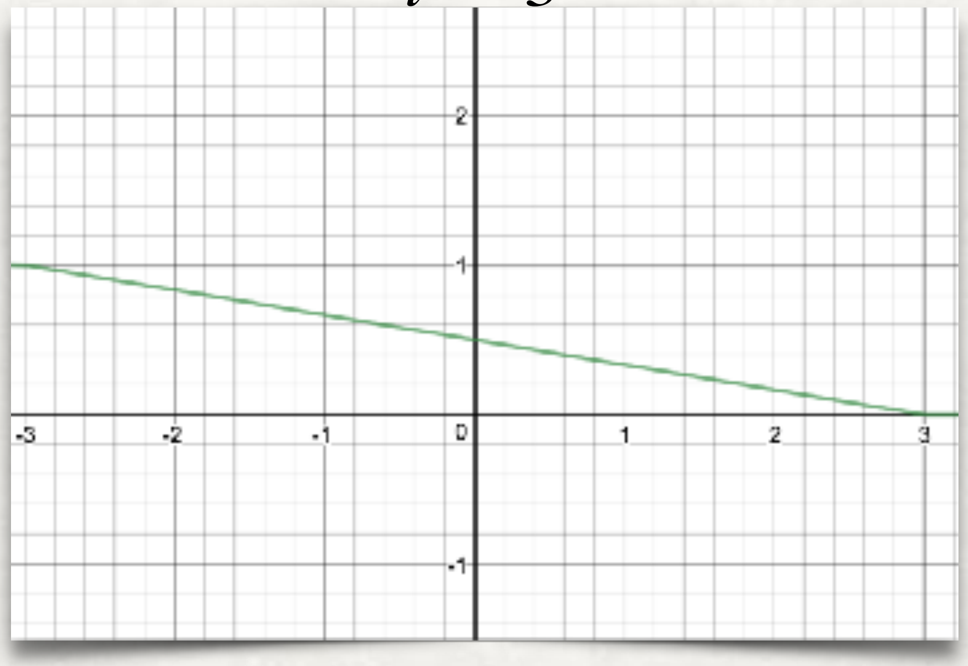
$t = 1$



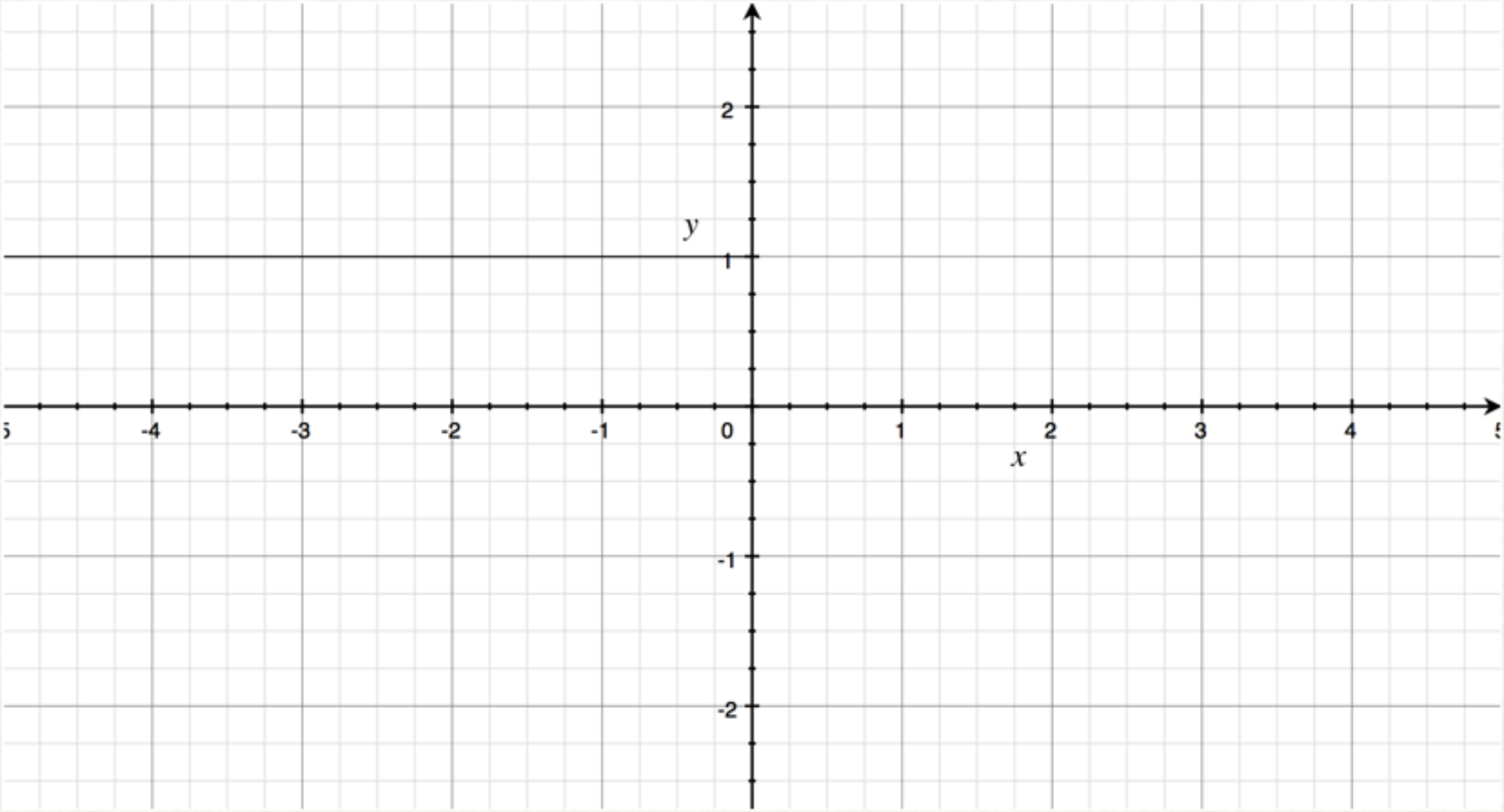
$t = 2$



$t = 3$



EXAMPLE PROBLEM 6 (6)



PATH OF A CAR (1)

- Let the position of the car be given by the function $x(t)$
- Let x_0 be the initial position of the car, and t_0 when the car starts moving. These are related by $t_0 = -x_0$
- The position of the car will be constant until $t = t_0$. Afterwards, the path will be given by the differential equation

$$\frac{dx}{dt} = 1 - \rho(x,t) = \frac{t+x}{2t}$$

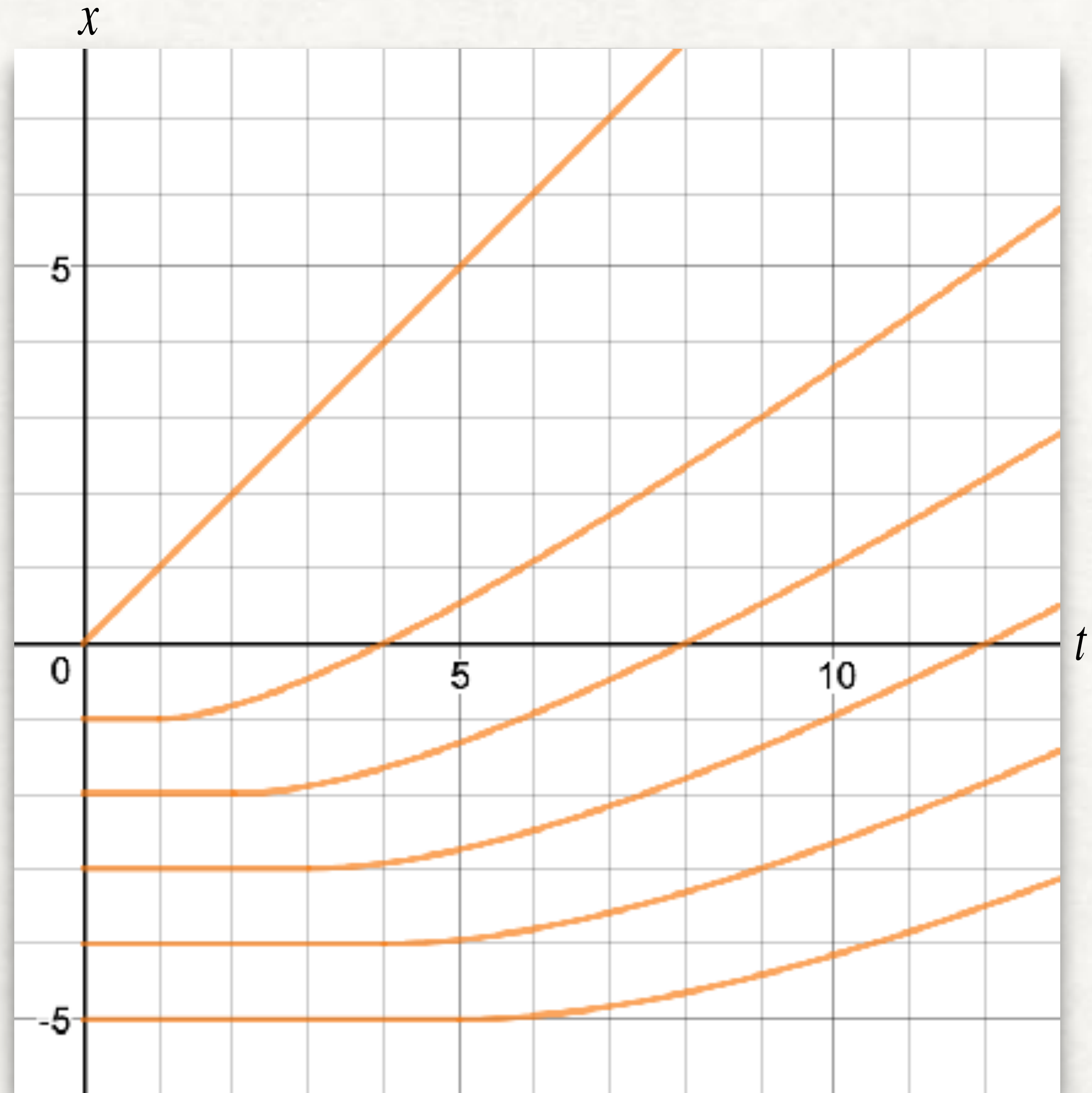
- So the path of the car is given by:

$$x = t + c_4\sqrt{t} \quad x(t_0) = x_0$$

$$x(t) = \begin{cases} x_0 & \text{if } 0 < t < t_0 \\ t - 2\sqrt{-x_0 t} & \text{if } t > t_0 \end{cases}$$

PATH OF A CAR (2)

PLOTTED AT $x_0 = -1, -2, -3, -4, -5$



$$x(t) = \begin{cases} x_0 & \text{if } 0 < t < t_0 \\ t - 2\sqrt{-x_0 t} & \text{if } t > t_0 \end{cases} \quad v(t) = \begin{cases} 0 & \text{if } 0 < t < t_0 \\ 1 - \sqrt{-\frac{x_0}{t}} & \text{if } t > t_0 \end{cases} \quad a(t) = \begin{cases} 0 & \text{if } 0 < t < t_0 \\ \sqrt{-\frac{x_0}{4t^3}} & \text{if } t > t_0 \end{cases}$$

OUTSIDE RESOURCES USED

- Stanford Math 220A Lecture Notes
  - http://web.stanford.edu/class/math220a/handouts/firstorder.pdf
  - http://web.stanford.edu/class/math220a/handouts/conservation.pdf
- Stephen Childress' "Notes on traffic flow"
  - http://www.math.nyu.edu/faculty/childres/traffic3.pdf