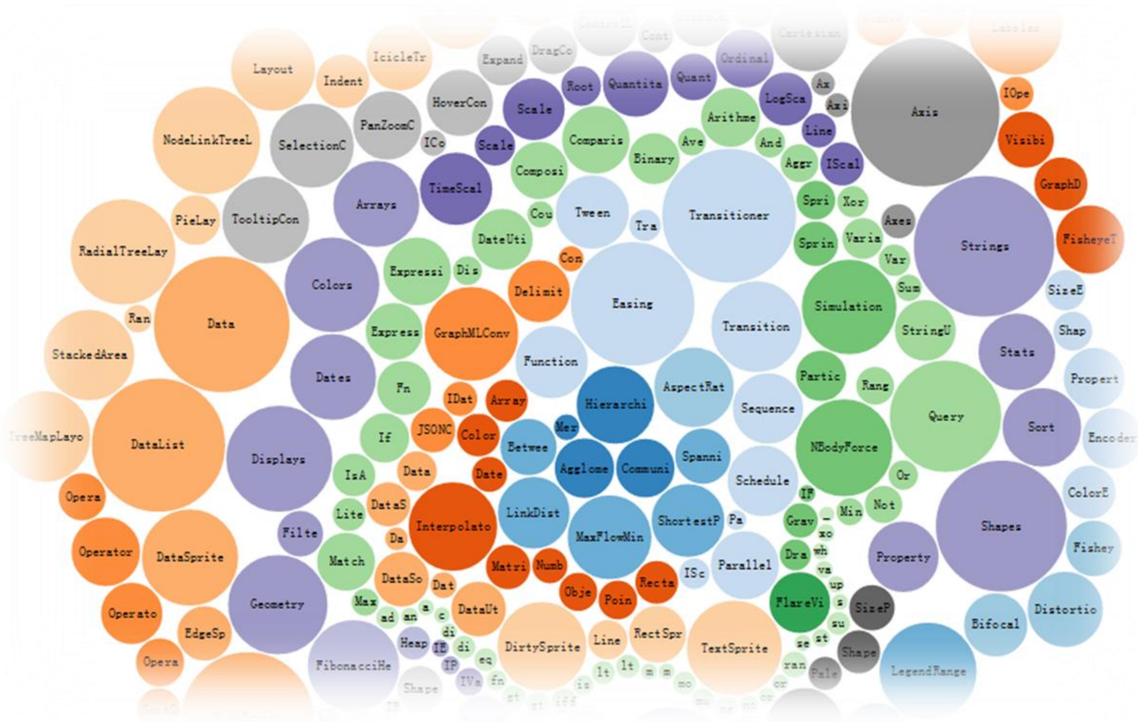

Graph Algorithms

A Brief Introduction

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目录

GRAPH AND ITS APPLICATIONS

Definitions and Applications

Konigsberg

- Once upon a time there was a city called Konigsberg in Prussia
- The capital of East Prussia until 1945
- Centre of learning for centuries, being home to Goldbach, Hilbert, Kant …

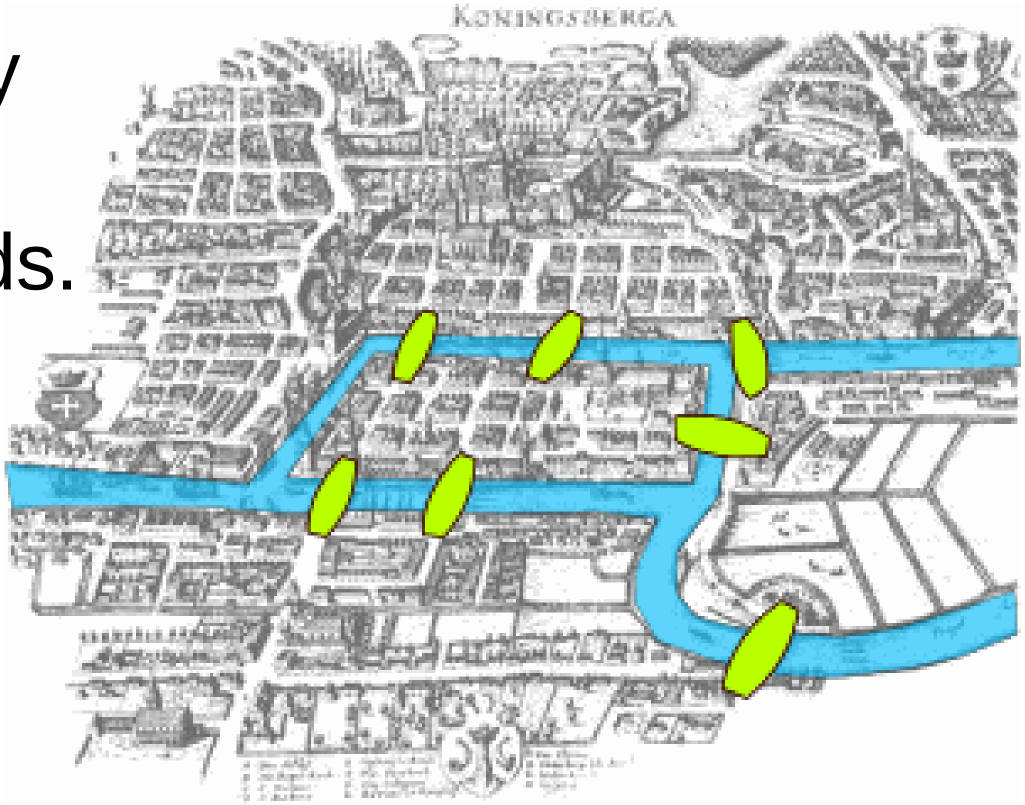
Position of Konigsberg

Seven Bridges

- Pregel river is passing through Konigsberg

- It separated the city into two mainland area and two islands.

- There are seven bridges connecting each area.

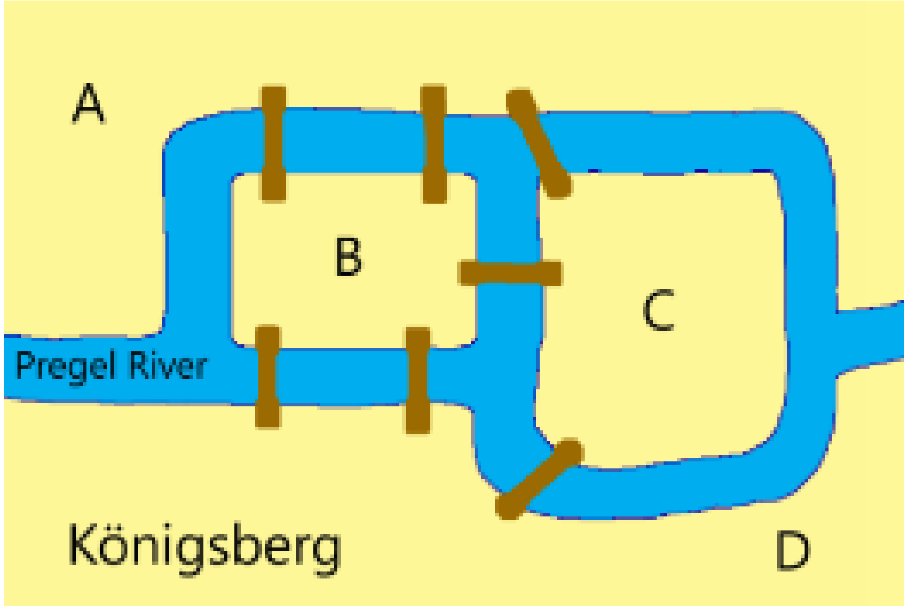


Seven Bridge Problem

- **A Tour Question:**

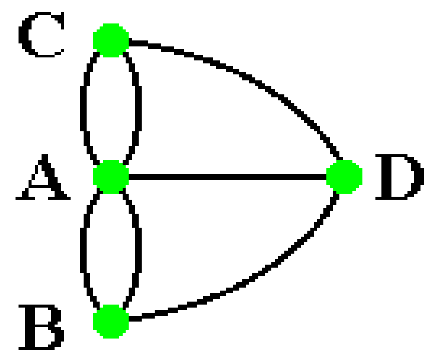
  Can we wander around the city, crossing each bridge once and only once?

Is there a solution?

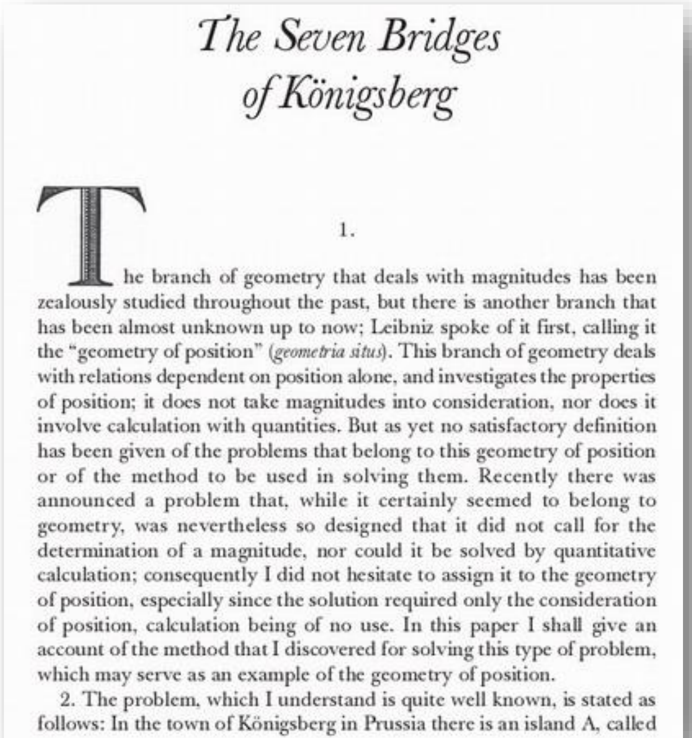


Euler's Solution

- Leonhard Euler Solved this problem in 1736



- Published the paper "The Seven Bridges of Konigsbery"



The Seven Bridges of Königsberg

1.

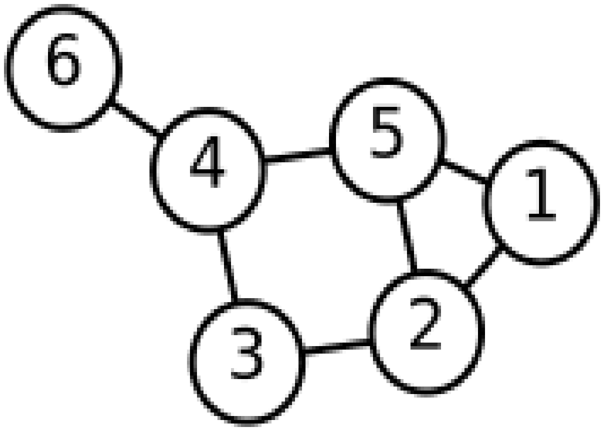
The branch of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibniz spoke of it first, calling it the "geometry of position" (*geometria situs*). This branch of geometry deals with relations dependent on position alone, and investigates the properties of position; it does not take magnitudes into consideration, nor does it involve calculation with quantities. But as yet no satisfactory definition has been given of the problems that belong to this geometry of position or of the method to be used in solving them. Recently there was announced a problem that, while it certainly seemed to belong to geometry, was nevertheless so designed that it did not call for the determination of a magnitude, nor could it be solved by quantitative calculation; consequently I did not hesitate to assign it to the geometry of position, especially since the solution required only the consideration of position, calculation being of no use. In this paper I shall give an account of the method that I discovered for solving this type of problem, which may serve as an example of the geometry of position.

2. The problem, which I understand is quite well known, is stated as follows: In the town of Königsberg in Prussia there is an island A, called

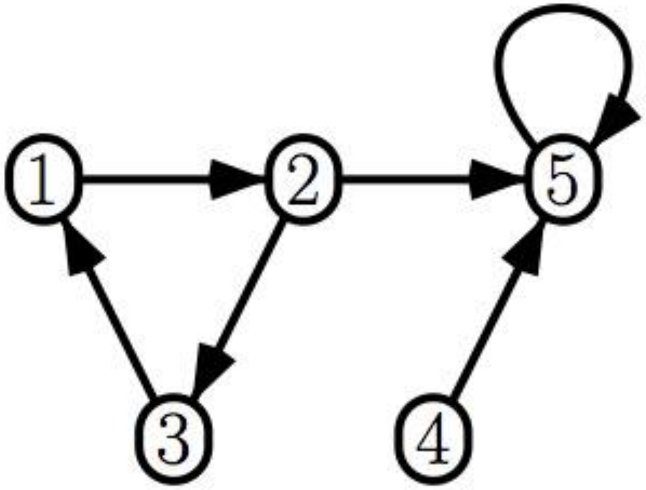
- The first negative solution The beginning of Graph Theory

Representing a Graph

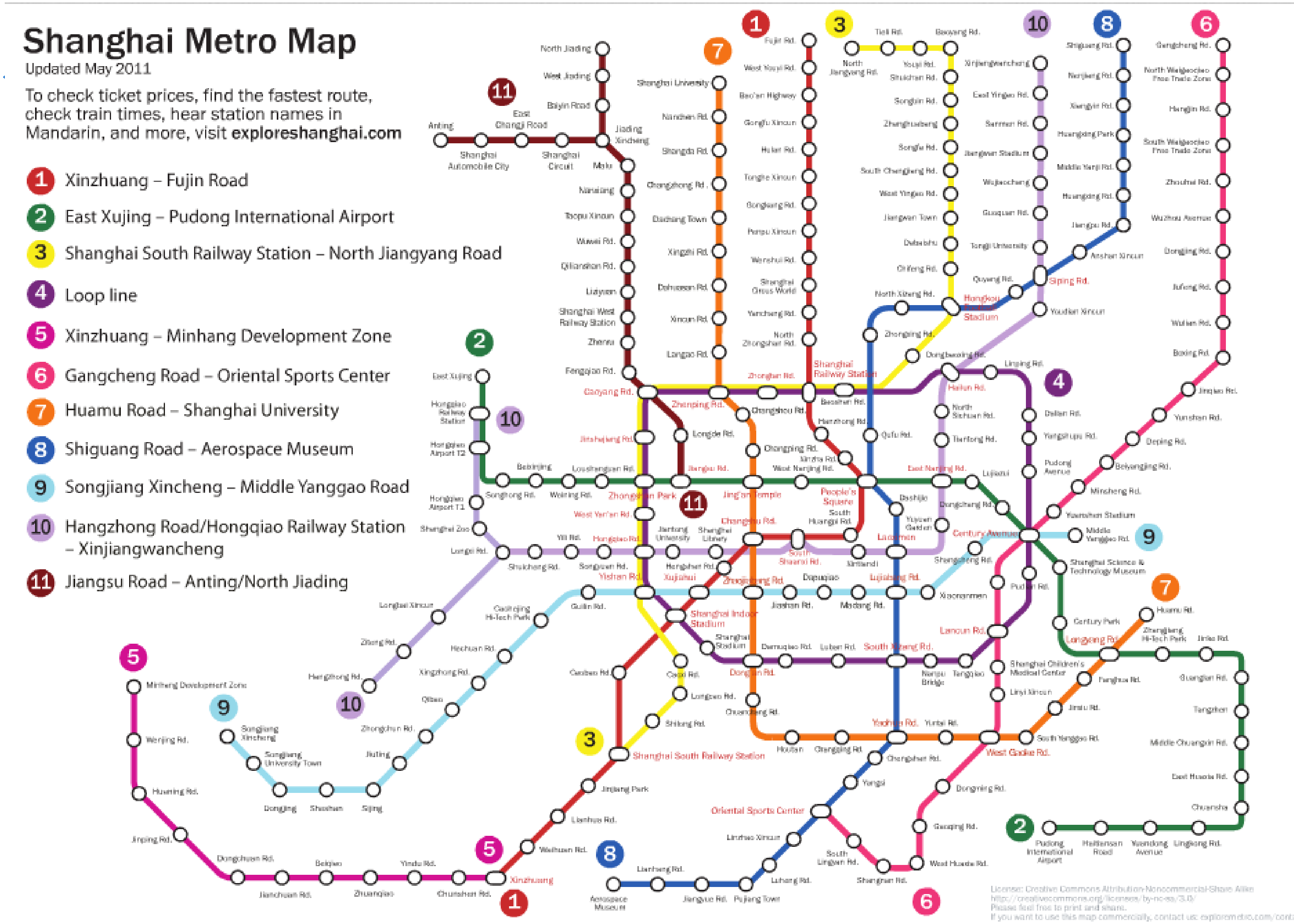
- Undirected Graph:
  - G=(V, E)
  - V: vertex
  - E: edges



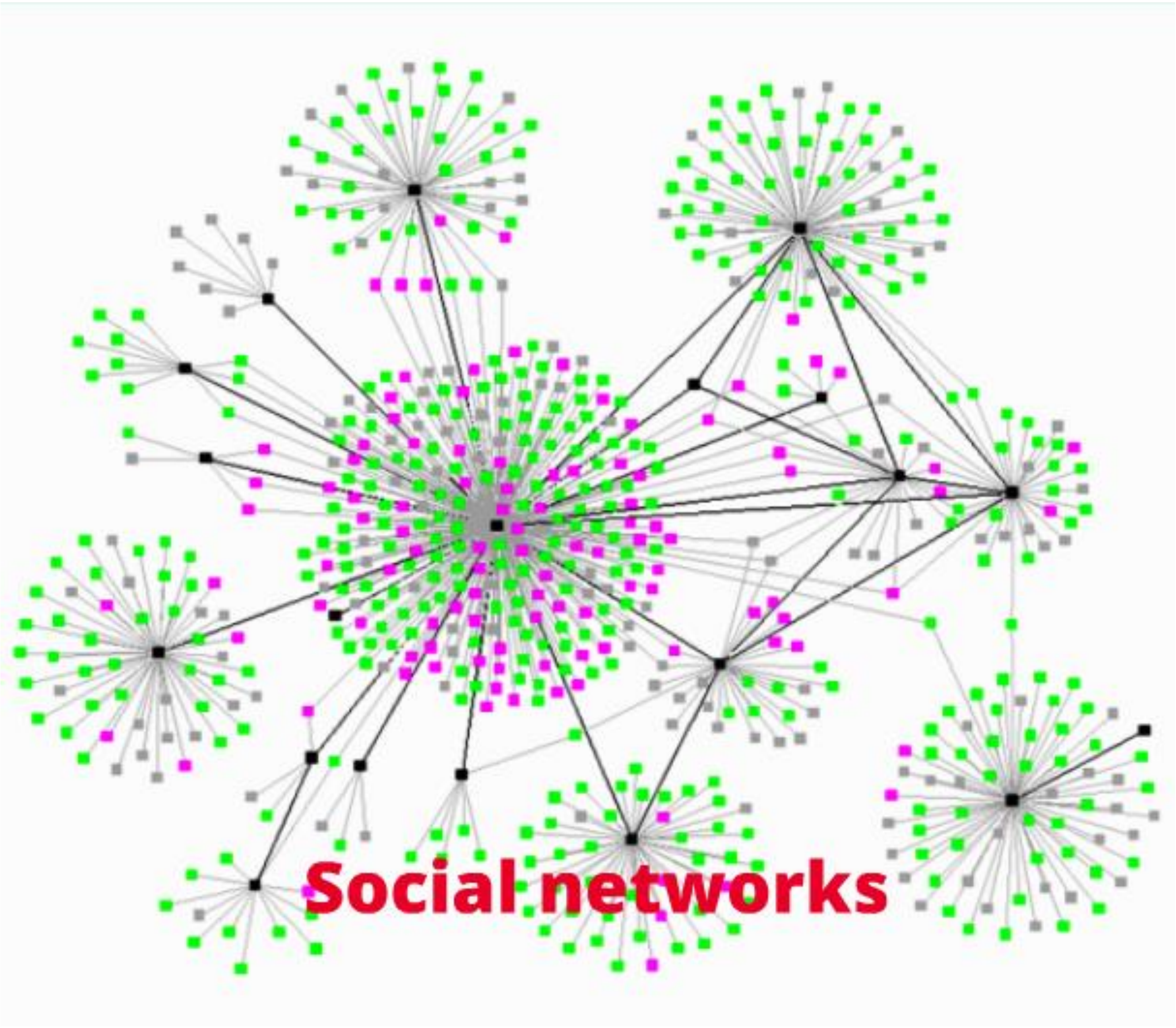
- Directed Graph:
  - G=(V, A)
  - V: vertex
  - A: arcs



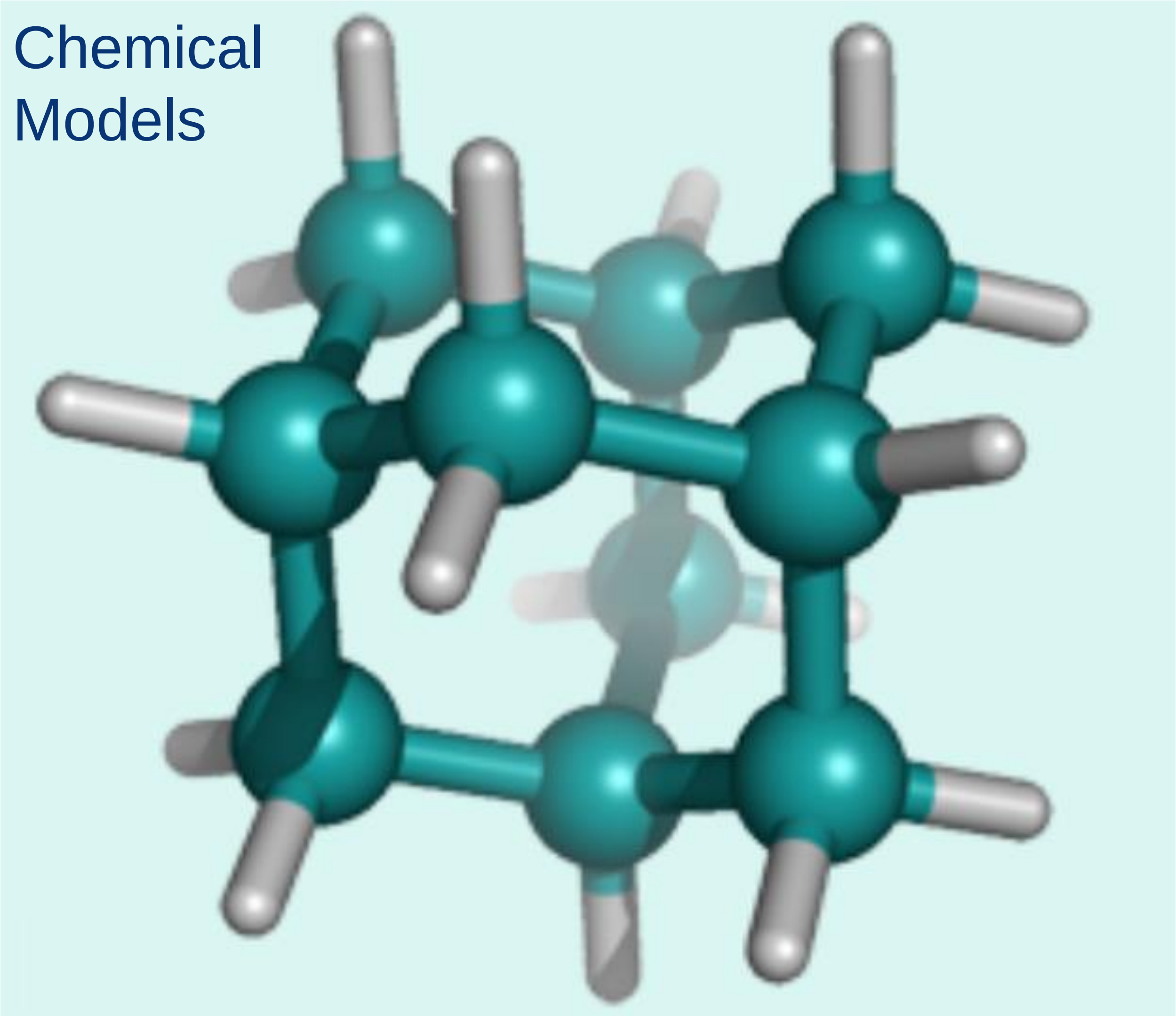
More Examples



More Examples (2)

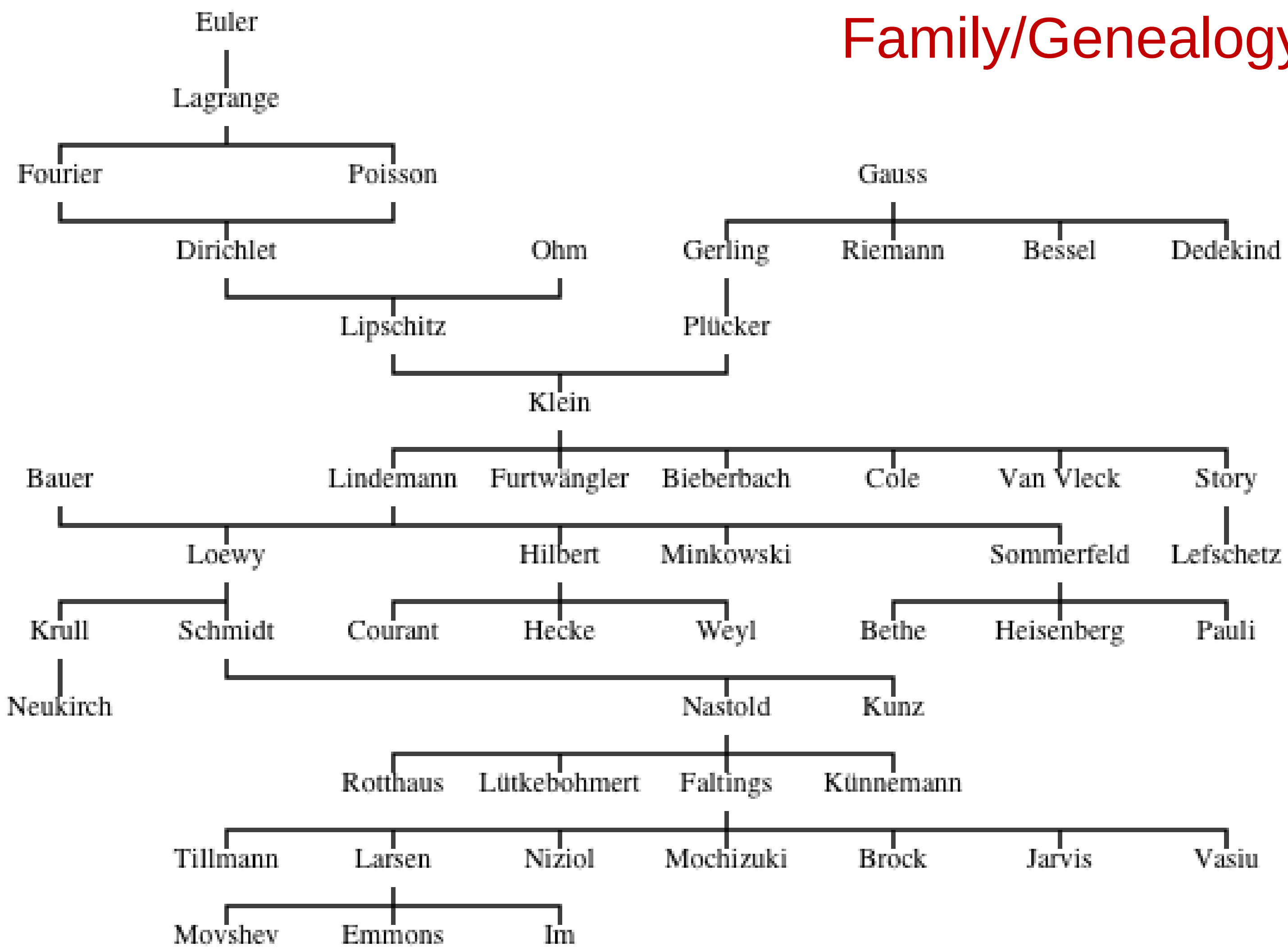


More Examples (3)

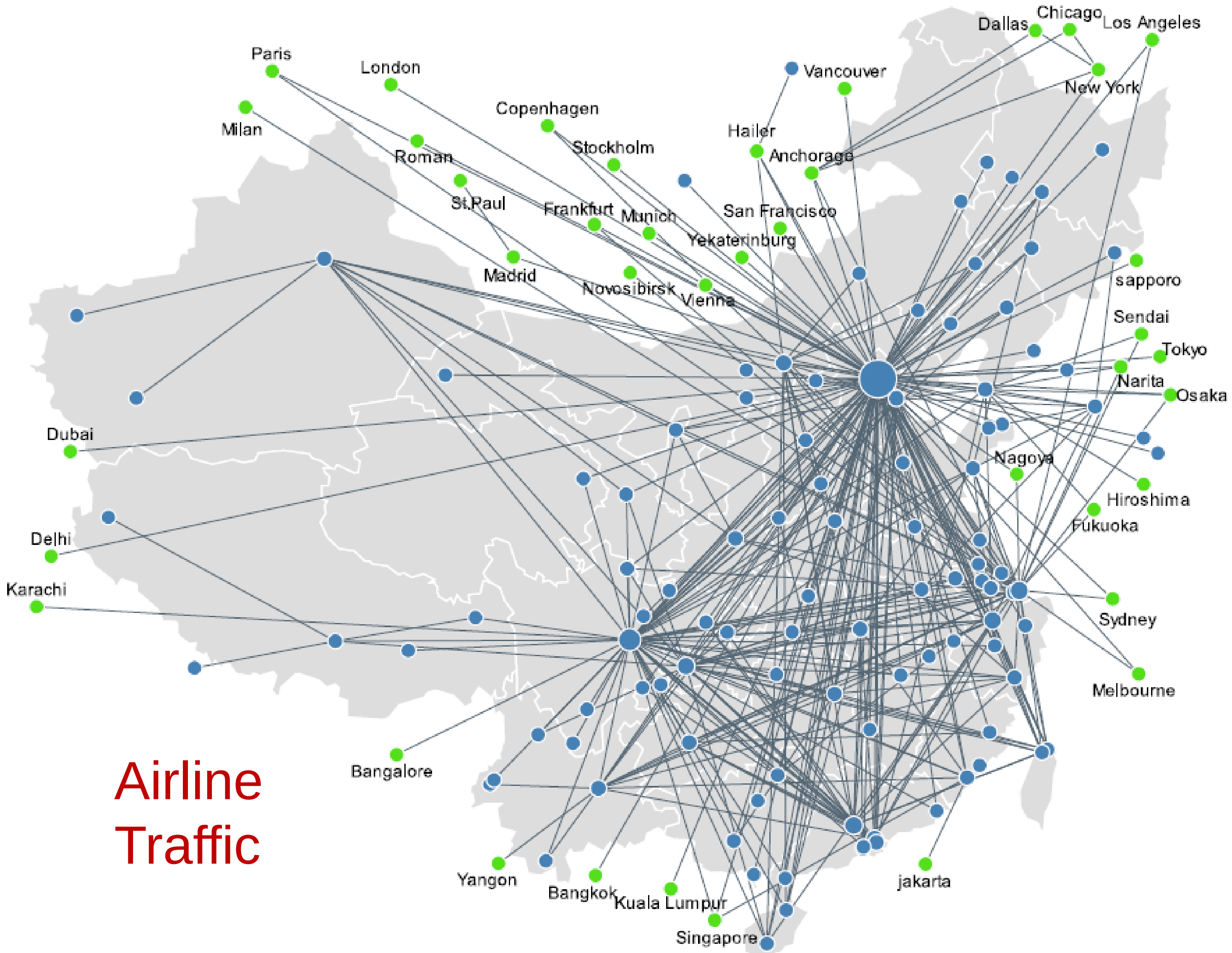


More Examples (4)

Family/Genealogy Tree



More Examples (5)



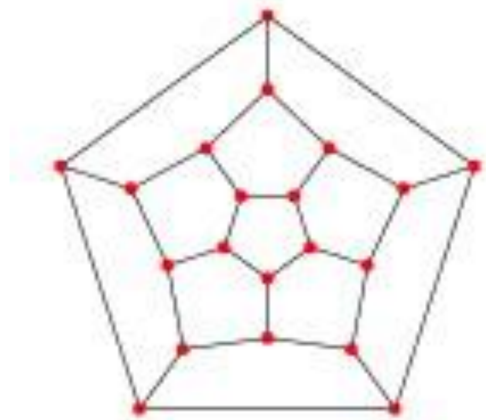
Airline Traffic

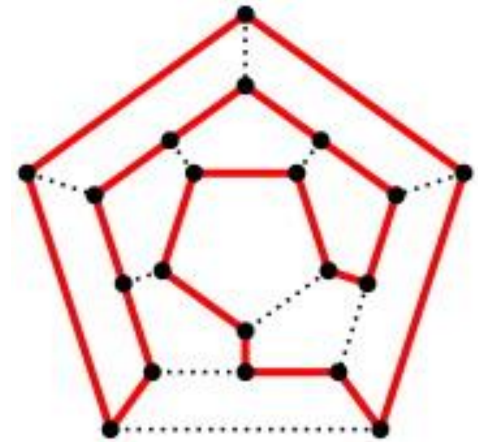
Icosian Game

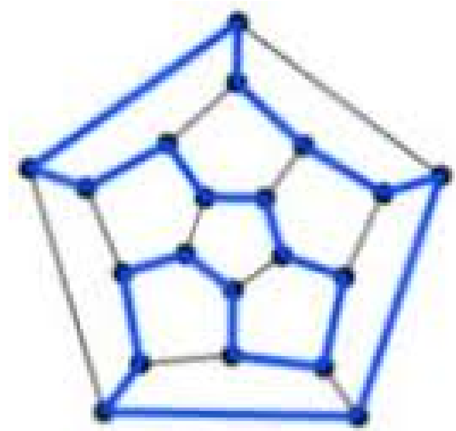
- In 1859, Sir William Rowan Hamilton developed the Icosian Game.
- Traverse the edges of an dodecahedron, i.e., a path such that every vertex is visited a single time, no edge is visited twice, and the ending point is the same as the starting point.
- Also refer as Hamiltonian Game.

Icosian Game

- Examples







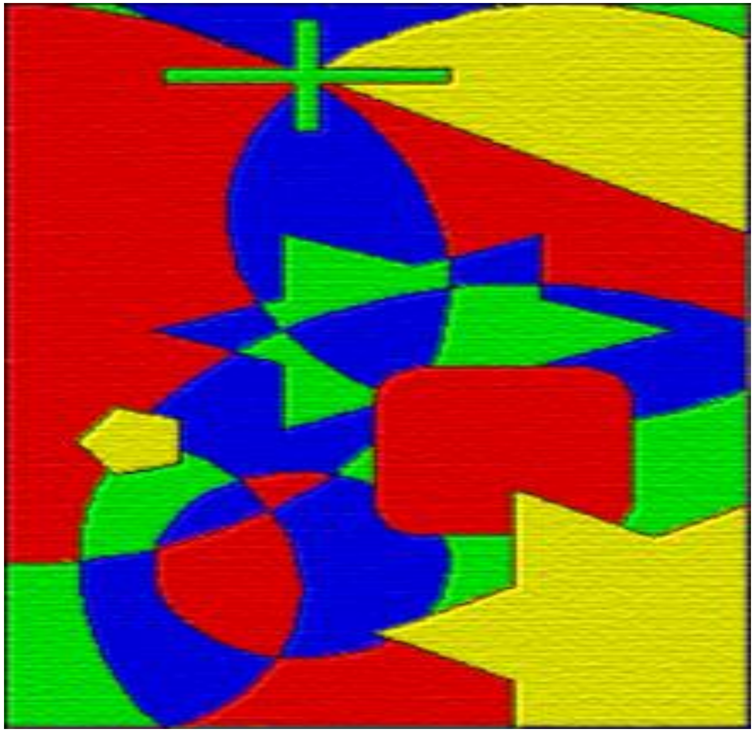
- 3D-Demo
  - http://mathworld.wolfram.com/IcosianGame.html

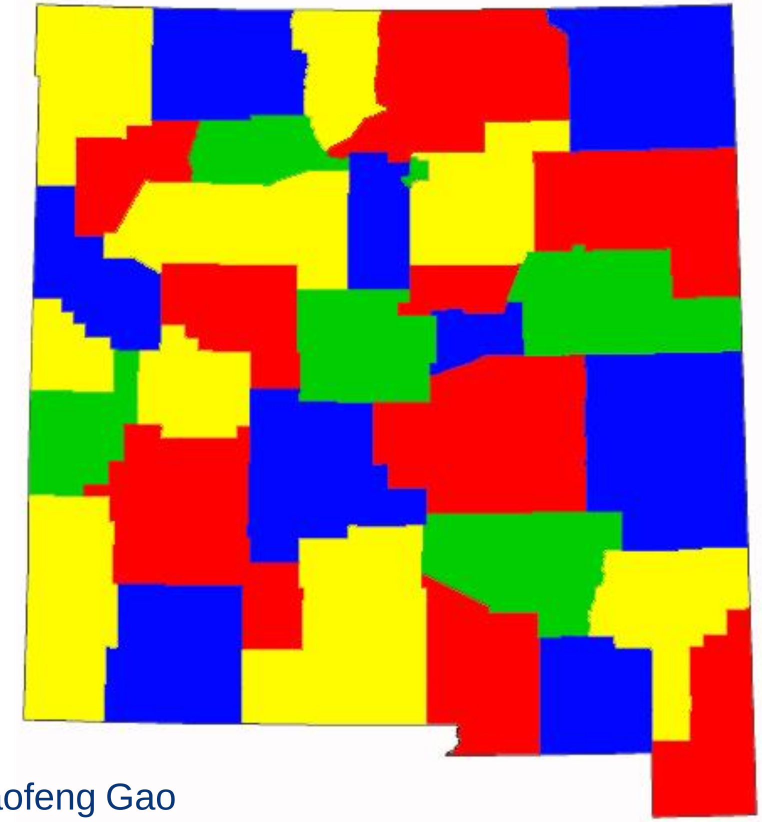
Four Color Theorem

- The four color theorem was stated, but not proved, in 1853 by Francis Guthrie.

Four Color Theorem

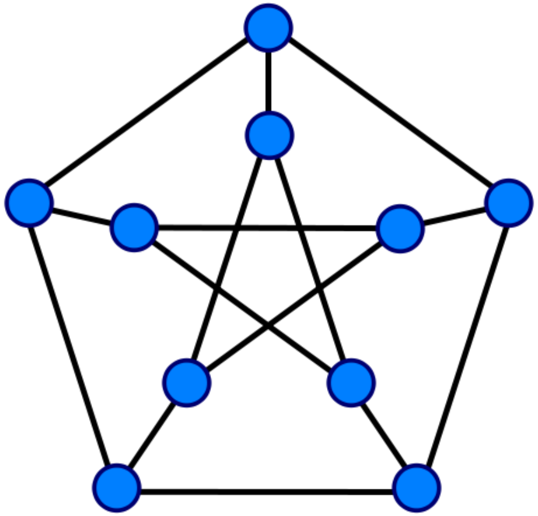
- The theorem asserts that four colors are enough to color any geographical map in such a way that no neighboring two countries are of the same color.



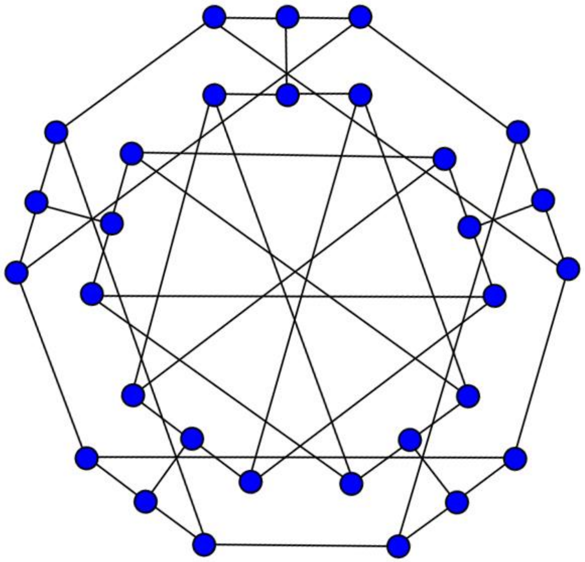


Snark Graph

- Each point has degree 3
- Strong connectivity
  (breaking any three edges will not violating the connectivity of the graph)

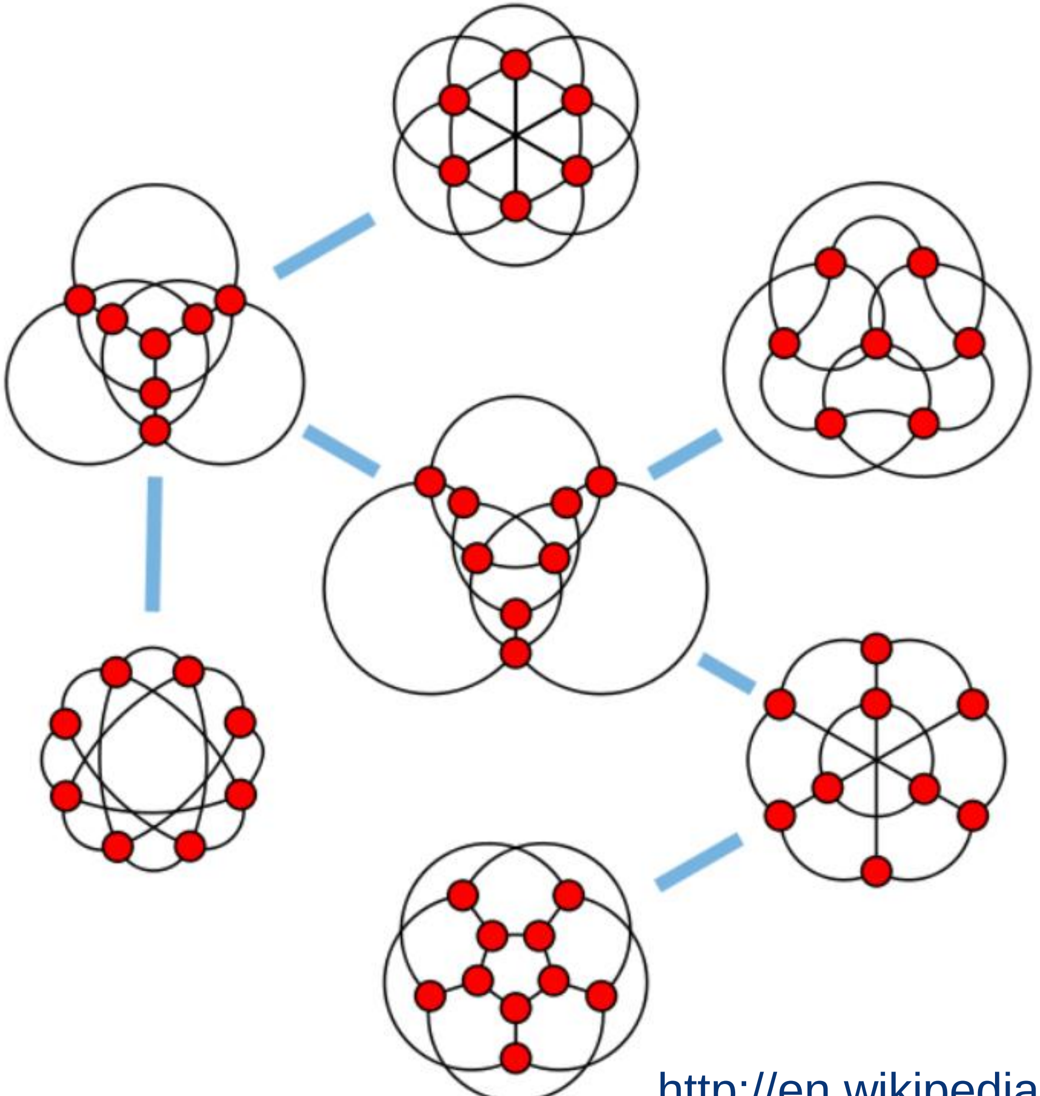


Single Star Snark
(Petersen Graph)



Double Star Snark

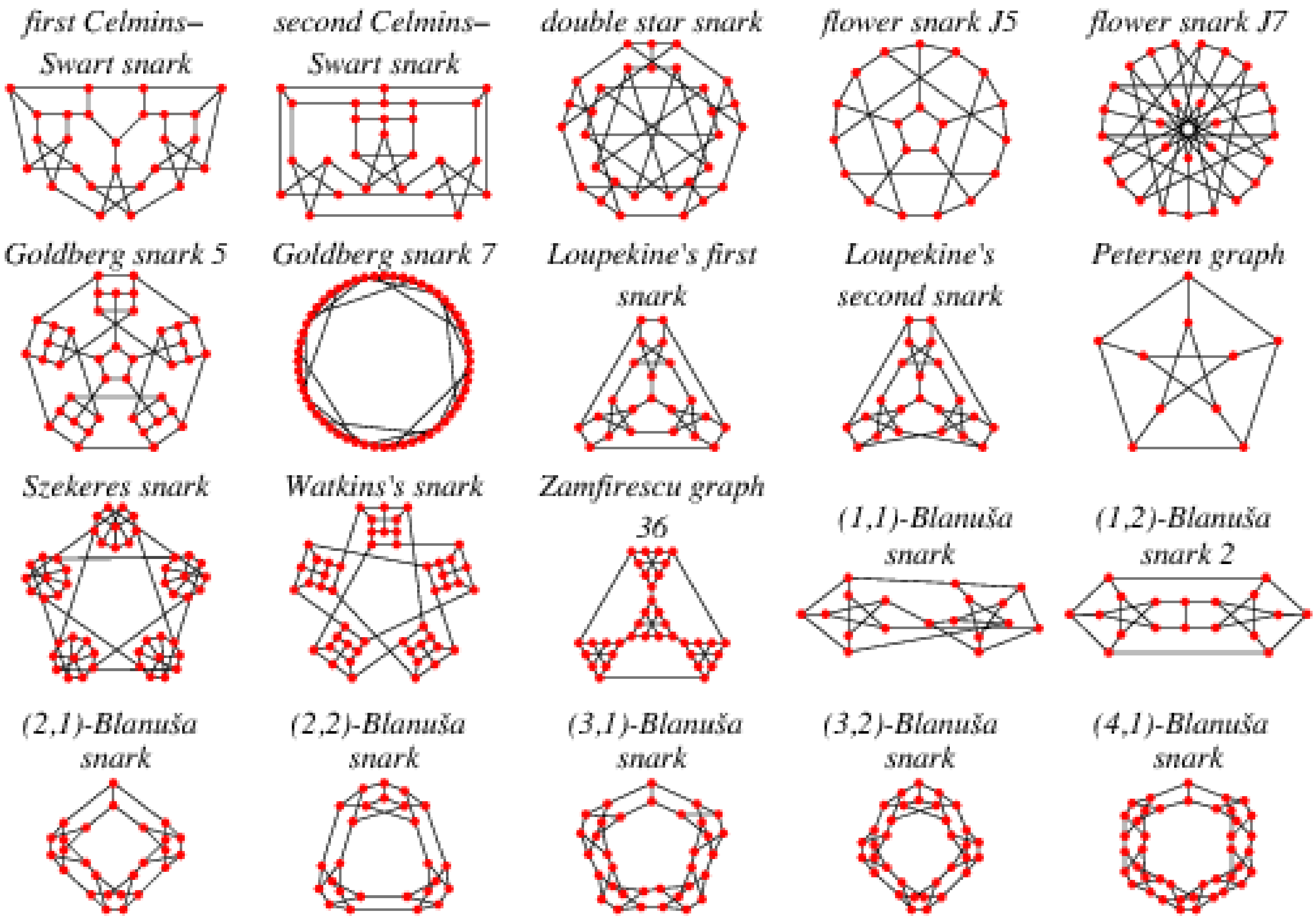
Petersen Family



Y-Δ

http://en.wikipedia.org/wiki/Petersen_graph

Snark Family

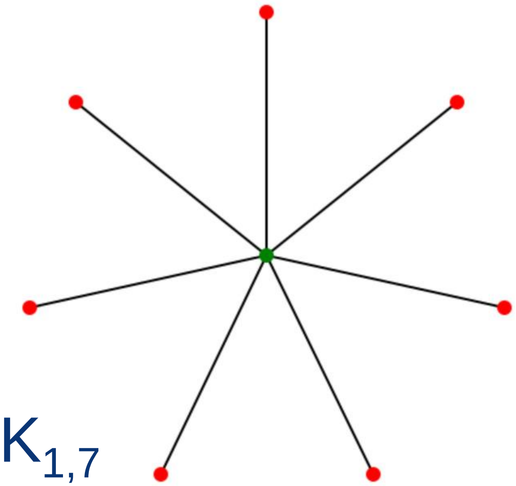


http://en.wikipedia.org/wiki/Snark_(graph_theory)

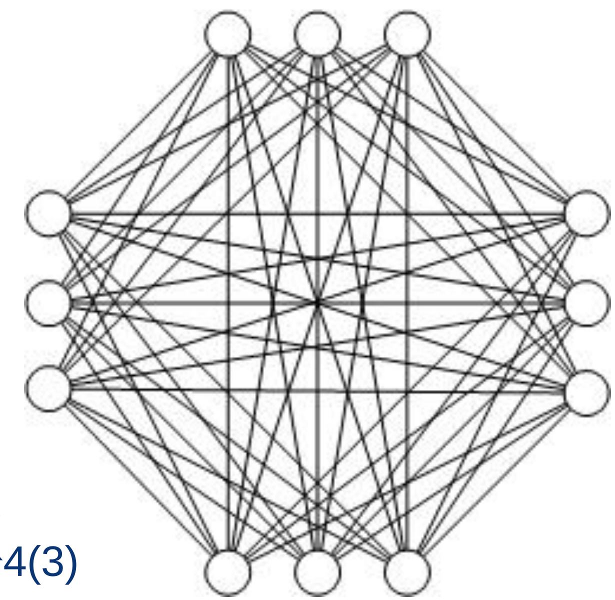
Well-Known Results

- **Complete Graph K_n**
- **Bipartite Graph $K_{m,n}$**
- **Star $K_{1,n}$**
- **r-Partite Graph $K_{r(m)}$**
- **Subgraph $H \subseteq G$**
  - Spanning/Induced Subgraph
- **Handshaking Theorem**

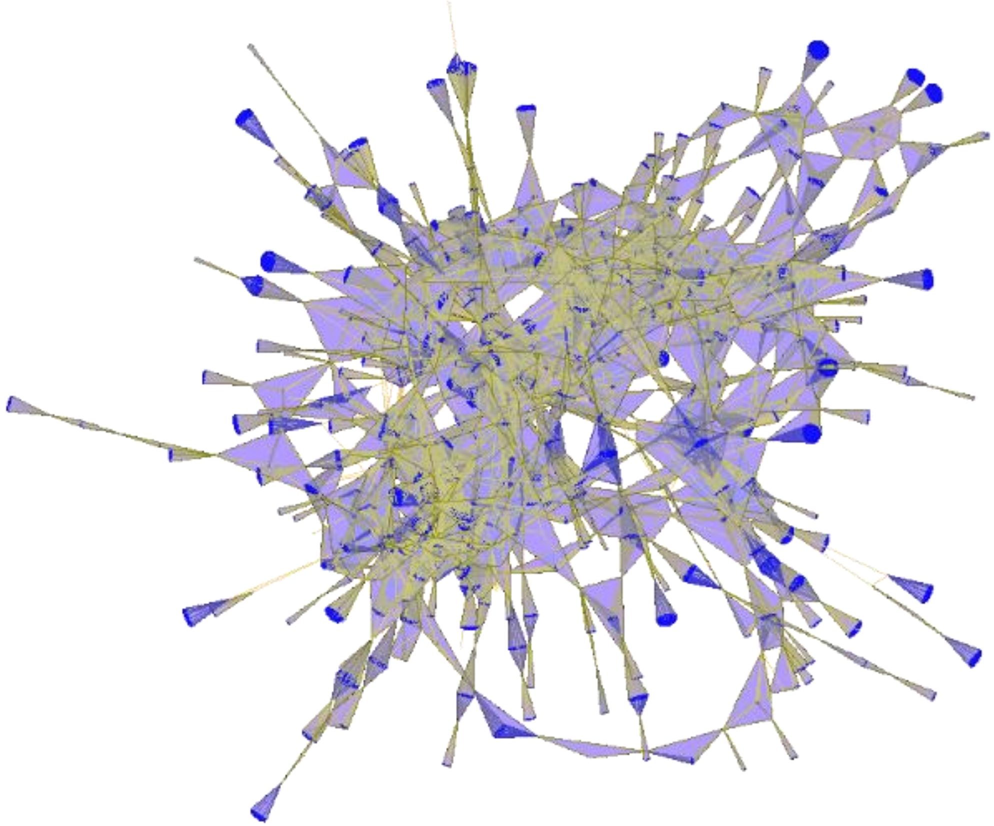
$$\sum_{v \in V} d(v) = 2|E|$$



$K_{1,7}$



$K_{4(3)}$



INTRODUCTION TO GRAPH ALGORITHMS

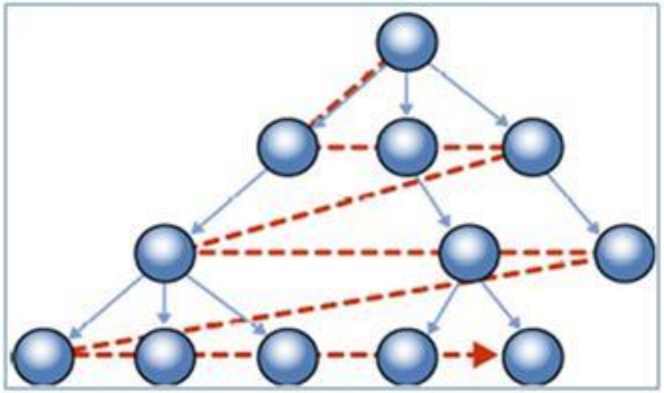
Three Categories

Algorithms on Graphs

- **Elementary Graph Algorithms**
  - Breadth-First Search
  - Depth-First Search
  - Minimum Spanning Tree
- **Shortest Path Problem**
  - Single-Source Shortest Path
  - All-Pairs Shortest Path
- **Maximum Flow**
  - Max-Flow vs Min-Cut
  - Applications

Breadth-First Search

- **Basic Strategy**
  - visit and inspect a node of a graph
  - gain access to visit the nodes that neighbor the currently visited node

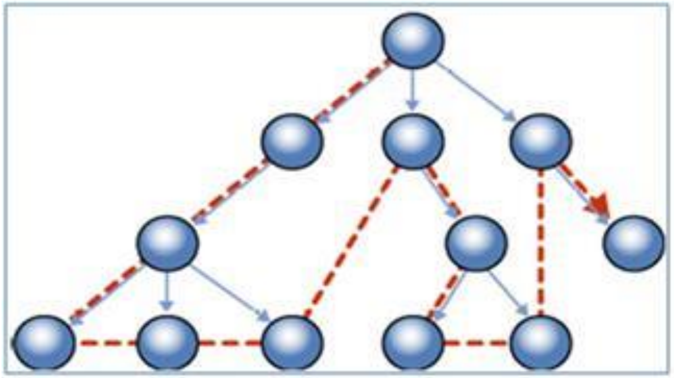


- **Applications and Theories**
  - Find nodes within one connected component
  - Find shortest path between two nodes u and v
  - Ford-Fulkerson method for computing the maximum flow in a flow network

Depth-First Search

❖ **Basic Strategy**

- Starts at arbitrary root
- Explores as far as possible along each branch before backtracking



❖ **Applications and Theories**

- Finding (strong) connected components
- Topological sorting
- Solving mazes puzzles with only one solution
- parenthesis theorem

Minimum Spanning Tree

Classical Algorithms

- Prim: maintain an optimal subtree
- Kruskal: maintain min-weight acyclic edge set
- Reverse-Delete: circle-deletion
- Borüvka Algorithm

Fundamental Results

- All greedy Approach with exchange property
- Correctness proof: cycle/cut property
- Efficiency: time complexity → heap

Single-Source Shortest Path

- **Dijkstra's Algorithm**
  - Greedy Approach
  - Graph with positive weights

- **Bellman-Ford Algorithm**
  - Dynamic Programming
  - Graph with negative weights (without negative-cycle)

All-Pair Shortest Path

- **Basic Dynamic Programming**
  - Matrix multiplication
  - Time Complexity: $\Theta(n^3 \lg n)$.
- **Floyd-Warshall algorithm**
  - Also dynamic programming, but faster
  - Time Complexity: $\Theta(n^3)$
- **Johnson's algorithm**
  - For sparse graph
  - Time Complexity: $O(VE + V^2 \lg V)$.

Maximum Flow Problem

Maximum Flow and Minimum Cut

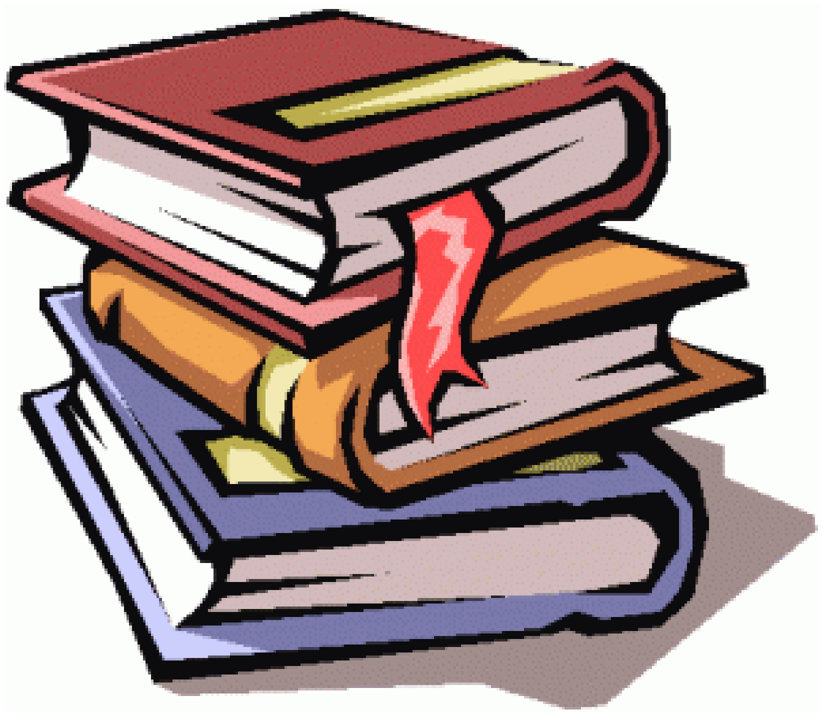
- Max-Flow Min-Cut Theorem: The value of the max flow is equal to the value of the min cut

Ford-Fulkerson Algorithm

- Find s-t flow of maximum value

Augmenting Path Algorithm

- Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths



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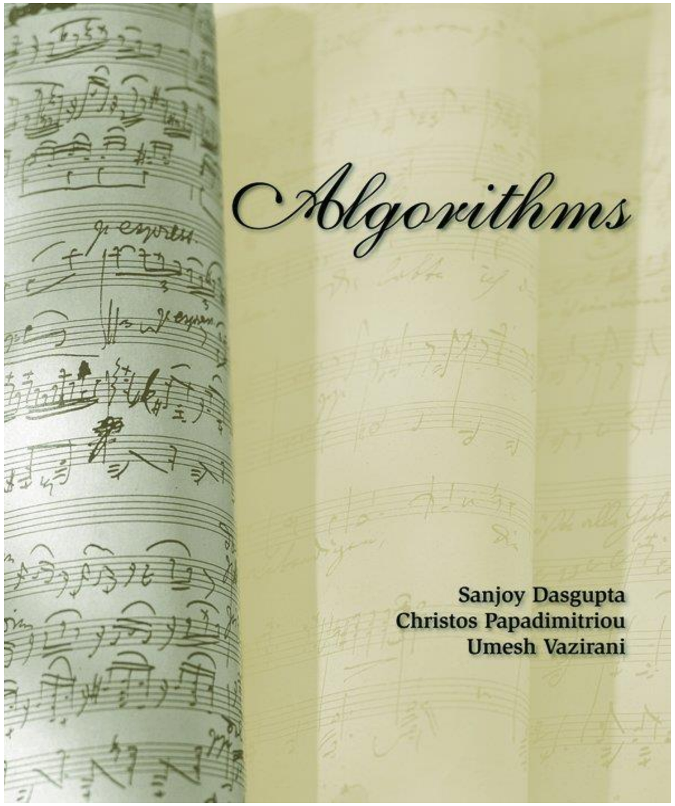

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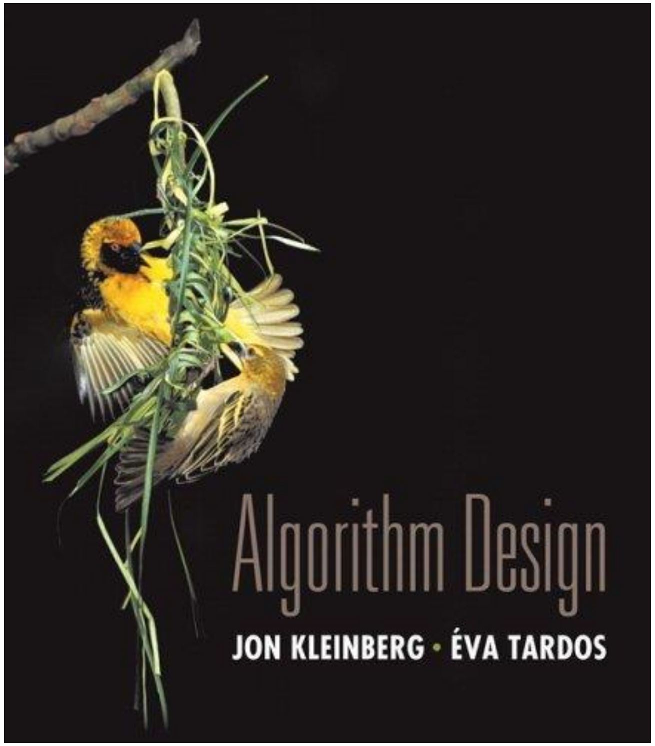

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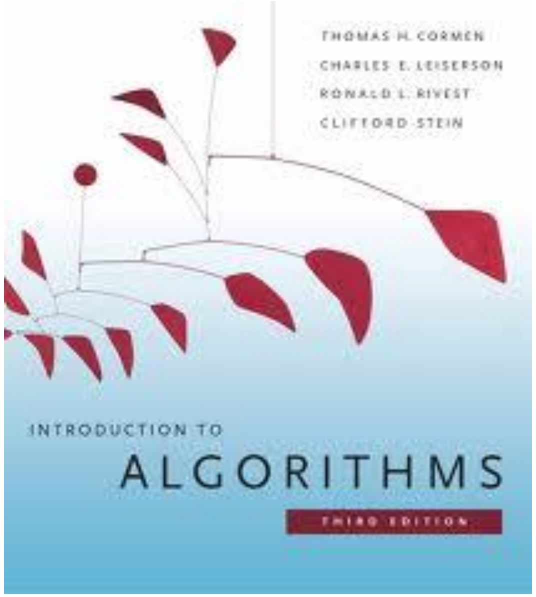
Edition: Third Edition



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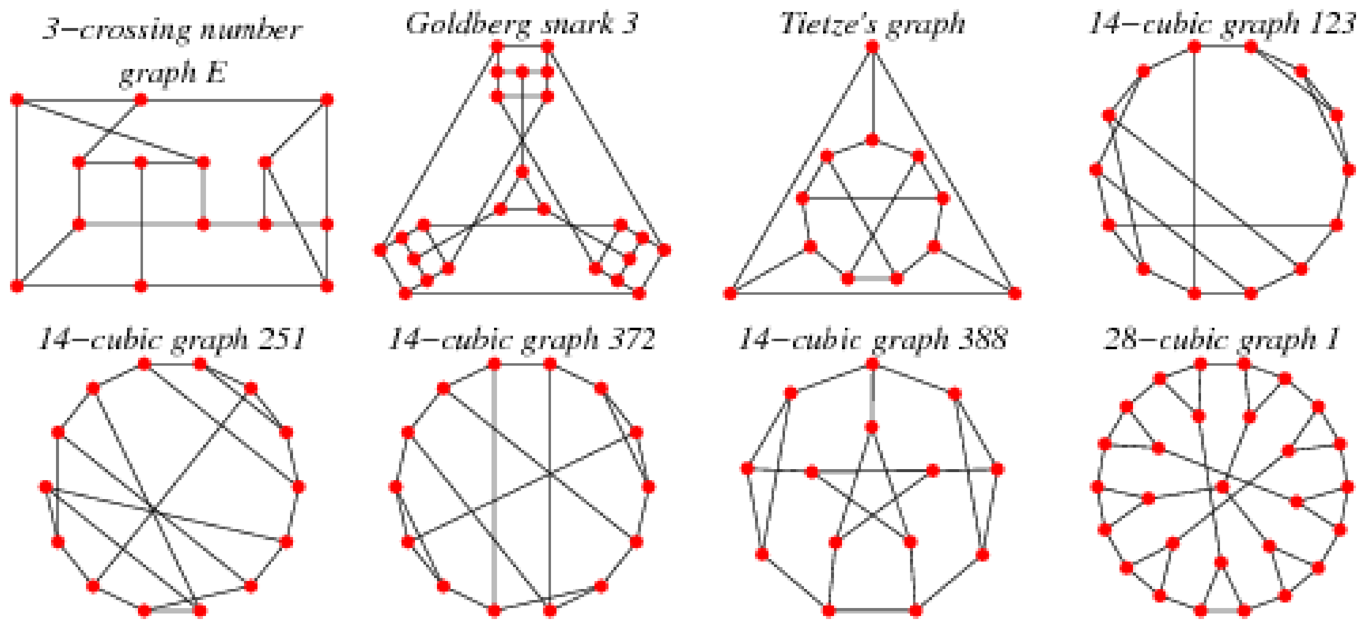

The End !

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