- **ray/plane intersection**

  - **ray definition in terms of its origin and a direction vector**

$\mathbf{R}_{\text{origin}} \equiv \mathbf{R}_0 \equiv [X_0\ Y_0\ Z_0]$
$\mathbf{R}_{\text{direction}} \equiv \mathbf{R}_d \equiv [X_d\ Y_d\ Z_d]$
where $X_d^2 + Y_d^2 + Z_d^2 = 1$ (i.e. normalized)

which defines a ray as

set of points on ray $\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{R}_d * t$, where $t > 0$.

  - **plane definition**

Plane $\equiv A * x + B * y + C * z + D = 0$
where $A^2 + B^2 + C^2 = 1$.

where the unit vector normal is

$\mathbf{P}_{\text{normal}} \equiv \mathbf{P}_n = [A\ B\ C]$

and where D is the distance from the origin to the plane

- **ray/plane intersection, cont.**

  - **distance from the ray's origin to the intersection with the plane P**

$$A * (X_0 + X_d * t) + B * (Y_0 + Y_d * t) + C * (Z_0 + Z_d * t) + D = 0$$

solving for t

$$t = \frac{-(A * X_0 + B * Y_0 + C * Z_0 + D)}{A * X_d + B * Y_d + C * Z_d}.$$

or in vector notation

$$t = \frac{-(\mathbf{P}_n \cdot \mathbf{R}_0 + D).}{\mathbf{P}_n \cdot \mathbf{R}_d}$$

- ray/plane intersection, cont.
  - achieving more efficiency
    - first calculate the dot product

$$v_d = \mathbf{P_n} \cdot \mathbf{R_d} = A * X_d + B * Y_d + C * Z_d.$$

    - if $v_d = 0$, the ray is parallel to the plane and no intersection occurs
    - if $v_d > 0$, the normal of the plane points away from the ray
      - the plane could be culled
    - otherwise, calculate the second dot product

$$v_0 = -(\mathbf{P_n} \cdot \mathbf{R_0} + D) = -(A * X_0 + B * Y_0 + C * Z_0 + D).$$

    - then calculate the ratio of the dot products

$$t = v_0 / v_d$$

      - if $v_d = 0$, $t = \infty$

- ray/plane intersection, cont.

  - if $t < 0$, the line defined by the ray intersects the plane behind the ray's origin (no actual intersection)

    - if v_0 and v_d have opposite signs, $t < 0$

  - otherwise, calculate the intersection point

$$r_i = [x_i \ y_i \ z_i] = [X_0 + X_d * t \quad Y_0 + Y_d * t \quad Z_0 + Z_d * t].$$

- **ray/plane intersection, cont.**

  - **example**

Given a plane [1 0 0 − 7] (which describes a plane where $x = 7$) and a ray with an origin of [2 3 4] and a direction of [0.577 0.577 0.577], find the intersection with a plane. Assume the plane is two-sided.

First calculate v_d

$$v_d = 1 * 0.577 + 0 * 0.577 + 0 * 0.577 = 0.577.$$

In this case, $v_d > 0$, so the plane points away from the ray. For this example the plane has two sides, so in this case there is no early termination. Calculate v_0:

$$v_0 = -(1 * 2 + 0 * 3 + 0 * 4 + (-7)) = 5.$$

Now calculate t:

$$t = 5/0.577 = 8.66.$$

Distance t is positive, so the point is not behind the ray. This value represents the distance from the ray's origin to the intersection point. The intersection point components are:

$$x_i = 2 + 0.577 * 8.66 = 7$$
$$y_i = 3 + 0.577 * 8.66 = 8$$
$$z_i = 4 + 0.577 * 8.66 = 9.$$

So $\mathbf{R}_i$ = [7 8 9]. To determine whether the plane's normal points in a direction towards the ray's origin, check if $v_d > 0$. It is, which means that the plane faces away from the ray. Simply negating the normal will give a normal which faces towards the ray, i.e. [− 1 0 0].

- **polygon intersection**
  - **point/polygon inside/outside testing**

    idea:

    - **shoot a ray from the point in an arbitrary direction in the plane of the polygon**
    - **count the number of line segments crossed**
    - **odd means in**
  - **definitions**
    - **the polygon is a set of N points**

$$\text{polygon} \equiv \text{set of } \mathbf{G}_n = [X_n \; Y_n \; Z_n], \text{ where } n = (0, 1, \ldots, N-1).$$

    - **the plane defined by these points is**

$$\text{plane} \equiv A * X + B * Y + C * Z + D = 0.$$

    - **the (not necessarily normalized) normal is**

$$\mathbf{P}_{\text{normal}} \equiv \mathbf{P}_{\text{n}} = [A \; B \; C]$$

- polygon intersection, cont.

  - algorithm

    - begin at the intersection point

$$\mathbf{R}_i \equiv [X_i \; Y_i \; Z_i]$$

    - project the polygon onto a two-dimensional plane

      - all points are specified by a pair (U,V)

      - the topology is unchanged

      - perhaps rotate about some axis until the normal becomes parallel to some other axis, say Z

        - the remaining two coordinates (X and Y in this case) generate the (U,V) coordinates in the plane

        - disadvantages

          - a rotation matrix must be generated for each polygon

          - a matrix multiplication must be performed for each coordinate

- algorithm, cont.

  . simpler projection

    - simply drop one of the x, y, z coordinates (preferably the one with the largest corresponding planar coefficient)

      . again, the topology is preserved

  . conduct the inside/outside test

    - translate the polygon so the intersection point is at the origin

    - subtract the intersection point's coordinates (U_i, V_i) from each vertex

    - shoot a ray along the U axis and count crossings

      . most edges are trivially crossed or trivially not crossed

      . edges which extend from diagonally opposite quadrants require the most calculation

    - if vertices lie right on the U axis, maybe displace them slightly

- more formal presentation of the algorithm

For the NV vertices $[X_n \; Y_n \; Z_n]$, where $n = 0$ to $NV - 1$, project these onto the dominant coordinate's plane, creating a list of vertices (U_n, V_n).

Translate the (U, V) polygon so that the intersection point is the origin. Call these points

$$(U_n', V_n').$$

Set the number of crossings NC to zero.

Set the sign holder SH as a function of V_0', the V' value of the first vertex of the first edge: (D2)

Set to -1 if V_0' is negative.
Set to $+1$ if V_0' is zero or positive.

For each edge of the polygon formed by points (U_a', V_a') and (U_b', V_b'), where $a = 0$ to $NV - 1$, $b = (a + 1) \bmod NV$:

Set the next sign holder NSH: (D3)

Set to -1 if V_b' is negative.
Set to $+1$ if V_b' is zero or positive.

If SH is not equal to NSH: (D4)

If U_a' is positive and U_b' is positive then (D5)
the line must cross $+U'$, so $NC = NC + 1$.

Else if either U_a' is positive or U_b' is positive then (D6)
the line might cross, so compute intersection on U' axis:

If $U_a' - V_a' * (U_b' - U_a')/(V_b' - V_a') > 0$ then (D7)
the line must cross $+U'$, so $NC = NC + 1$.

Set $SH = NSH$. (D8)

Next edge

If NC is odd, the point is inside the polygon, else it is outside. (D9)

example

Given a triangle:

$$\mathbf{G}_0 = [-3\ \ -3\ \ 7]$$
$$\mathbf{G}_1 = [\ \ 3\ \ -4\ \ 3]$$
$$\mathbf{G}_2 = [\ \ 4\ \ -5\ \ 4]$$

and an intersection point $\mathbf{R}_i = [-2\ \ -2\ \ 4]$, find if the point lies within the triangle. The plane equation is:

$$\mathbf{P} = [1\ \ 2\ \ 1\ \ -2]$$

The dominant coordinate in the plane equation is Y, so these coordinates are discarded leaving:

$$\mathbf{G}_{uv0} = [-3\ \ 7]$$
$$\mathbf{G}_{uv1} = [\ \ 3\ \ 3]$$
$$\mathbf{G}_{uv2} = [\ \ 4\ \ 4]$$
$$\mathbf{R}_{uvi} = [-2\ \ 4]$$

Translating the intersection point $[-2\ \ 4]$ to the coordinate system origin, the triangle is now:

$$\mathbf{G}'_{uv0} = [-1\ \ \ 3]$$
$$\mathbf{G}'_{uv1} = [\ \ 5\ \ -1]$$
$$\mathbf{G}'_{uv2} = [\ \ 6\ \ \ 0].$$

The first edge is defined by $(U'_a, V'_a) = (-1, 3)$, $(U'_b, V'_b) = (5, -1)$. The sign holder SH is $+1$ since V'_a is positive.

NSH is -1 since V'_b is negative. The first edge passes (D4), since SH and NSH don't match. Since U'_a is positive and U'_b is not, (D5) fails and (D6) passes, so the true intersection point must be calculated (D7):

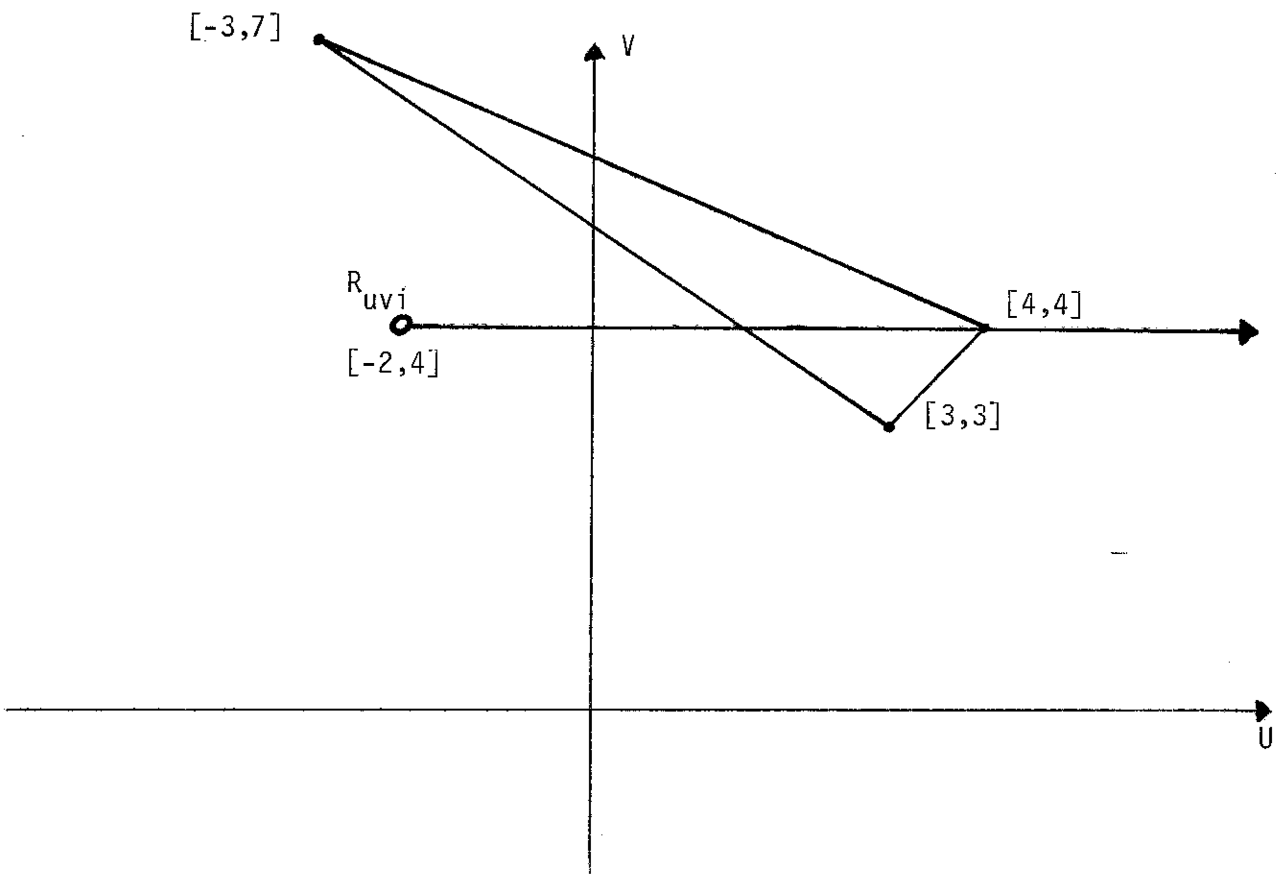
$$U'_a - V'_a * (U'_b - U'_a)/(V'_b - V'_a) \equiv -1 - 3 * (5 - (-1))/(-1 - 3) = 3.5.$$

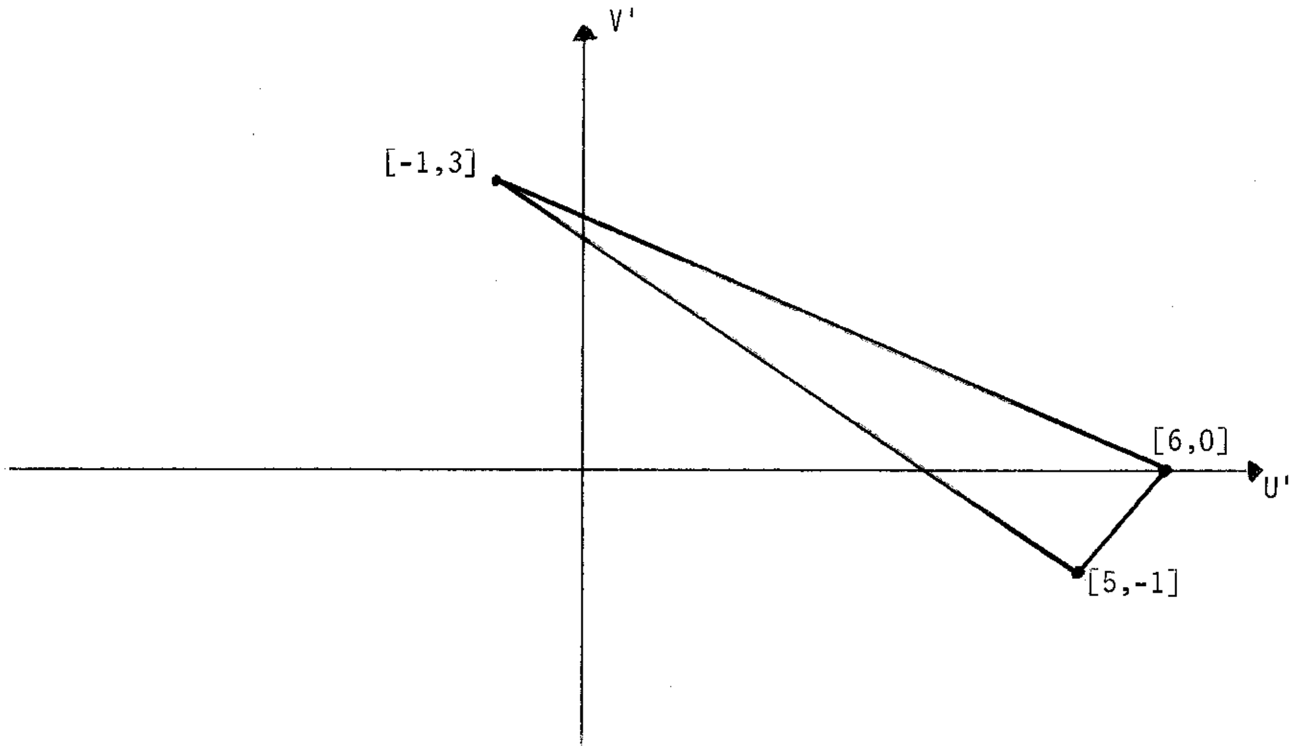
This means that the intersection point is on the $+U'$ axis at 3.5. By the (D7) test, this is considered to be a crossing, and so NC is incremented to 1. At the end of testing SH is changed to -1 by being set to NSH (D8).

The second edge is defined by $(U'_a, V'_a) = (5, -1)$, $(U'_b, V'_b) = (6,0)$. By (D3), NSH is $+1$, and since SH doesn't match NSH, (D4) is passed, so the line segment must intersect the U' axis. U'_a and U'_b are positive, so by (D5) a crossing takes place, and NC is incremented to 2. SH is set to $+1$ by NSH (D8).

The third edge is defined by $(U'_a, V'_a) = (6, 0)$, $(U'_b, V'_b) = (-1, 3)$. NSH is $+1$, and since SH matches NSH, no crossing takes place.

NC ends as 2, which is an even number, so the point is decided to be outside the polygon. Note how the vertex $(U', V') = (6, 0)$ lay on the $+U'$ axis, and how it was dealt with by considering the vertex to be consistently above the $+U'$ axis ray.

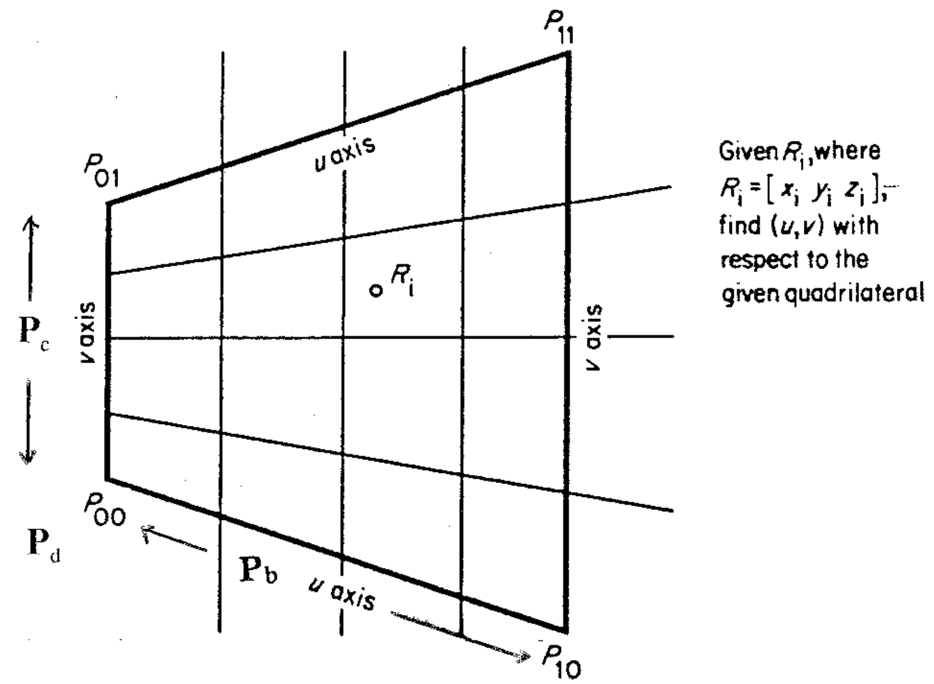

[-3,7]
V
R
uvi
[4,4]
[-2,4]
[3,3]
U


V'
[-1,3]
[6,0]
U'
[5,-1]

- convex quadrilateral inverse mapping

  - texture or color mapping may be desired

  - problem: convert the intersection point into a location within a convex quadrilateral expressed parametrically as (u,v) with respect to the four edges



  - calculate plane dependent factors

$$
\begin{aligned}
D_{u0} &= \mathbf{N}_c \cdot \mathbf{P}_d \\
D_{u1} &= \mathbf{N}_a \cdot \mathbf{P}_d + \mathbf{N}_c \cdot \mathbf{P}_b \\
D_{u2} &= \mathbf{N}_a \cdot \mathbf{P}_b \\
\mathbf{N}_a &= \mathbf{P}_a \otimes \mathbf{P}_n \\
\mathbf{N}_c &= \mathbf{P}_c \otimes \mathbf{P}_n
\end{aligned}
$$

where:

$\mathbf{P}_n$ **is the normal**

$$
\begin{aligned}
\mathbf{P}_a &= \mathbf{P}_{00} - \mathbf{P}_{10} + \mathbf{P}_{11} - \mathbf{P}_{01} \\
\mathbf{P}_b &= \mathbf{P}_{10} - \mathbf{P}_{00} \\
\mathbf{P}_c &= \mathbf{P}_{01} - \mathbf{P}_{00} \\
\mathbf{P}_d &= \mathbf{P}_{00}.
\end{aligned}
$$

- **convex quadrilateral inverse mapping, cont.**

  - **define a function for u describing the distance of the perpendicular plane (defined by that u and the quadrilateral's axes) from the coordinate system origin**

$$D(u) = (\mathbf{N}_c + \mathbf{N}_a * u) \cdot (\mathbf{P}_d + \mathbf{P}_b * u).$$

  - **given r_i, the distance of the perpendicular plane containing this point is**

$$D_r(u) = (\mathbf{N}_c + \mathbf{N}_a * u) \cdot \mathbf{R}_i.$$

  - **setting D(U) = to D_r(u), solving for u and simplifying**

$$A * u^2 + B * u + C = 0,$$

where

$$A = D_{u2}$$
$$B = D_{u1} - (\mathbf{R}_i \cdot \mathbf{N}_a)$$
$$C = D_{u0} - (\mathbf{R}_i \cdot \mathbf{N}_c).$$

which is a simple quadratic equation

  - **for efficiency, some factors should be computed and stored for each quadrilateral**

$$Q_{ux} = \mathbf{N}_a/(2 * D_{u2})$$
$$D_{ux} = -D_{u1}/(2 * D_{u2})$$
$$Q_{uy} = -\mathbf{N}_c/D_{u2}$$
$$D_{uy} = D_{u0}/D_{u2}.$$

- convex quadrilateral inverse mapping, cont.

  - solving for u (depending on whether or not the u axes are parallel)

    - if $D_{u2} = 0$

      - the u axes are parallel

$$u_p = -C/B = (\mathbf{N}_c \cdot \mathbf{R}_i - D_{u0})/(D_{u1} - \mathbf{N}_a \cdot \mathbf{R}_i).$$

    - if $D_{u2} \neq 0$

$$K_a = D_{ux} + (\mathbf{Q}_{ux} \cdot \mathbf{R}_i)$$
$$K_b = D_{uy} + (\mathbf{Q}_{uy} \cdot \mathbf{R}_i).$$

and

$$u_0 = K_a - \sqrt{(K_a^2 - K_b)}$$
$$u_1 = K_a + \sqrt{(K_a^2 - K_b)}.$$

of which at most one falls in the range 0...1

  - the value v is calculated similarly

$$D_{v0} = \mathbf{N}_b \cdot \mathbf{P}_d$$
$$D_{v1} = \mathbf{N}_a \cdot \mathbf{P}_d + \mathbf{N}_b \cdot \mathbf{P}_c$$
$$D_{v2} = \mathbf{N}_a \cdot \mathbf{P}_c$$
$$\mathbf{N}_a = \mathbf{P}_a \otimes \mathbf{P}_n$$
$$\mathbf{N}_b = \mathbf{P}_b \otimes \mathbf{P}_n$$

$$\mathbf{Q}_{vx} = \mathbf{N}_a/(2 * D_{v2})$$
$$D_{vx} = -D_{v1}/(2 * D_{v2})$$
$$\mathbf{Q}_{vy} = -\mathbf{N}_b/D_{v2}$$
$$D_{vy} = D_{v0}/D_{v2}.$$

. convex quadrilateral inverse mapping, cont.

- example

Given a quadrilateral:

$$
\begin{aligned}
\mathbf{P}_{00} &= [-5 \quad 1 \quad 2] \\
\mathbf{P}_{10} &= [-2 \ -3 \quad 6] \\
\mathbf{P}_{11} &= [\ \ 2 \ -1 \quad 4] \\
\mathbf{P}_{01} &= [\ \ 1 \quad 4 \ -1]
\end{aligned}
$$

and an intersection point $[-2 \ -1 \ 4]$, find the (u, v) inverse mapping. The plane equation is:

$$
\begin{aligned}
&B + C - 3 = 0, \text{ so} \\
&\mathbf{P}_n = [0 \ 1 \ 1].
\end{aligned}
$$

The factors can be calculated:

$$
\begin{aligned}
\mathbf{P}_a &= [-2 \ -1 \quad 1] \\
\mathbf{P}_b &= [\ \ 3 \ -4 \quad 4] \\
\mathbf{P}_c &= [\ \ 6 \quad 3 \ -3] \\
\mathbf{P}_d &= [-5 \quad 1 \quad 2]
\end{aligned}
$$

so:

$$
\begin{aligned}
\mathbf{N}_a &= [-2 \ -1 \ 1] \otimes [0 \ 1 \ 1] = [-2 \ 2 \ -2] \\
\mathbf{N}_c &= [6 \ 3 \ -3] \otimes [0 \ 1 \ 1] = [6 \ -6 \ 6] \\
D_{u0} &= [6 \ -6 \ 6] \cdot [-5 \ 1 \ 2] = -24 \\
D_{u1} &= [-2 \ 2 \ -2] \cdot [-5 \ 1 \ 2] + [6 \ -6 \ 6] \cdot [3 \ -4 \ 4] = 74 \\
D_{u2} &= [-2 \ 2 \ -2] \cdot [3 \ -4 \ 4] = -22.
\end{aligned}
$$

The other factors are:

$$
\begin{aligned}
\mathbf{Q}_{ux} &= [0.0455 \ \ -0.0455 \ 0.0455] \\
D_{ux} &= 1.68 \\
\mathbf{Q}_{uy} &= [0.272 \ \ -0.272 \ 0.272] \\
D_{uy} &= 1.09.
\end{aligned}
$$

Since $D_{v2} = 0$, the v axes must be parallel.

$$v_p = ([-8\ -3\ 3] \cdot [-2\ -1\ 4] - 43)/(-58 - [-2\ 2\ -2] \cdot [-2\ -1\ 4])$$
$$= 0.231.$$

The solution is then that the points lie at $(u, v) = (0.636,\ 0.231)$ within the quadrilateral.

Because $D_{u2} < 0$, the u axes are not parallel. This leads to

$$K_a = 1.68 + [0.0455\ \ -0.0455\ 0.0455] \cdot [-2\ -1\ 4] = 1.82$$
$$K_b = 1.09 + [0.272\ \ -0.272\ 0.272] \cdot [-2\ -1\ 4] = 1.91$$

so:

$$u_0 = 1.82 - \sqrt{(1.82 * 1.82 - 1.91)} = 0.636.$$

v is calculated by:

$$\mathbf{N}_a = [-2\ 2\ -2] \text{ (from before)}$$
$$\mathbf{N}_b = [3\ -4\ 4] \otimes [0\ 1\ 1] = [-8\ -3\ 3]$$
$$D_{v0} = [-8\ -3\ 3] \cdot [-5\ 1\ 2] = 43$$
$$D_{v1} = [-2\ 2\ -2] \cdot [-5\ 1\ 2] + [-8\ -3\ 3] \cdot [6\ 3\ -3] = -58$$
$$D_{v2} = [-2\ 2\ -2] \cdot [6\ 3\ -3] = 0.$$

- **triangle inverse mapping**
  - **one solution: pass the triangle to the previous algorithm, double the last vertex to give four vertices**

- ray/box intersection
    - the rectangular box is a commonly used form
    - useful for objects and for bounding volumes to speed intersection testing of complex objects
    - the slab method
        - a slab is a space between two parallel planes
        - the intersection of a set of slabs defines a bounding volume
        - the method relies on intersection of each pair of slabs by a ray
            - the ray hits the bounding volume only if the largest near value is not greater than the smallest far value
        - a simple bounding volume is aligned with the x, y and z axes

- **ray/box intersection, cont.**
  - **algorithm for determining if a box is hit**
    - **define the box by two coordinates**

$$\text{box's minimum extent} \equiv \mathbf{B}_l = [X_l \ Y_l \ Z_l]$$
$$\text{box's maximum extent} \equiv \mathbf{B}_h = [X_h \ Y_h \ Z_h].$$

    - **define a ray by its origin and a direction vector**

$$\mathbf{R}_{\text{origin}} \equiv \mathbf{R}_0 = [X_0 \ Y_0 \ Z_0]$$
$$\mathbf{R}_{\text{direction}} \equiv \mathbf{R}_d = [X_d \ Y_d \ Z_d]$$

which defines a ray as

$$\text{set of points on ray} \equiv \mathbf{R}(t) = \mathbf{R}_0 + \mathbf{R}_d * t$$

where t > 0

    - **set t_{near} = -infinity and t_{far} = infinity (i.e. large)**

- algorithm for determining if a box is hit, cont.

  . for each pair of planes PP associated with X, Y and Z (shown here for the set of x planes)

    - if the direction $X_d = 0$ then

      . the ray is parallel to the planes

      . if the origin X_o is not between the slabs ($X_0 < X_l$ or $X_o > X_h$), return false

    - else

      . calculate intersection distances of planes

      $t_1 = (X_l - X_o)/X_d$

      $t_2 = (X_h - X_o)/X_d$

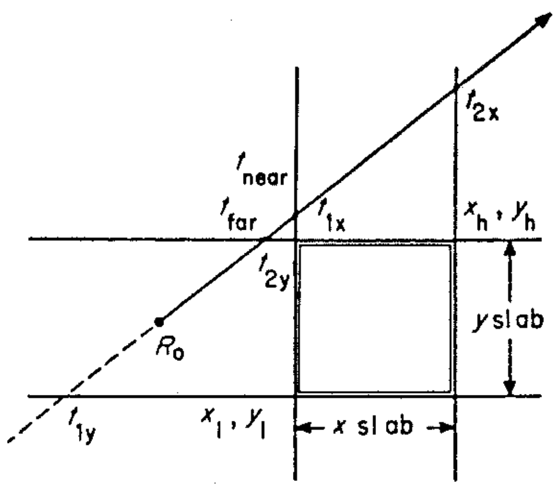
      . if $t_1 > t_2$, swap t_1 and t_2

      . if $t_1 > t_{near}$, set $t_{near} = t_1$

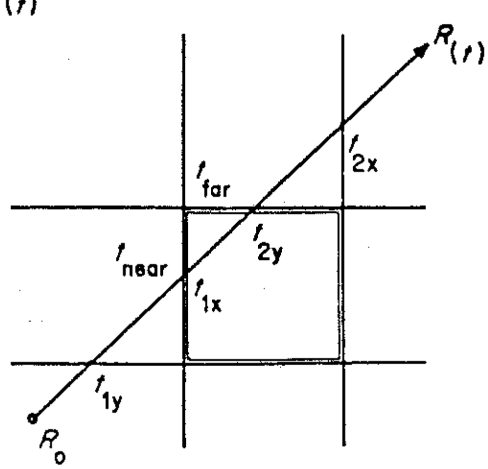
      if $t_2 < t_{far}$, set $t_{far} = t_2$

      . if $t_{near} > t_{far}$, box is missed; return false

      if $t_{far} < 0$, box is behind ray; return false



$t_{near} > t_{far}$, so ray misses box



$t_{near} < t_{far}$, and $t_{near} > 0$, so t_{near} is intersection distance

- algorithm for determining if a box is hit, cont.
  - if the box is hit
    - the intersection distance is equal to t_{near}
    - the ray's exit point is t_{far}

- **algorithm for determining if a box is hit, cont.**

  - **example**

Given a ray with origin [0 4 2] and direction [0.218 − 0.436 0.873] and a box with corners:

$$\mathbf{B}_l = [-1\ 2\ 1]$$
$$\mathbf{B}_h = [3\ 3\ 3]$$

find if the ray hits the box. The algorithm begins by looking at the X slab, defined by $X = -1$ and $X = 3$. The distances to these are:

$$t_{1x} = (-1 - 0)/0.218 = -4.59$$
$$t_{2x} = (3 - 0)/0.218 = 13.8$$

and so set $t_{near} = -4.59$ and $t_{far} = 13.8$. Neither $t_{near} > t_{far}$ (impossible for the first slabs test) nor $t_{far} < 0$, so the Y slab is examined:

$$t_{1y} = (2 - 4)/-0.436 = 4.59$$
$$t_{2y} = (3 - 4)/-0.436 = 2.29.$$

Since $t_{1y} > t_{2y}$, swap these values. Update $t_{near} = 2.29$ and $t_{far} = 4.59$. Again, neither test was failed, so check the Z slab:

$$t_{1z} = (1 - 2)/0.873 = -1.15$$
$$t_{2z} = (3 - 2)/0.873 = 1.15.$$

t_{near} is not updated and so is still 2.29, and $t_{far} = 1.15$. $t_{near} > t_{far}$ at this stage, so the ray must miss the box.

- **ray/quadric intersection and mapping**
  - **quadrics include cylinders, cones, ellipsoids, paraboloids, hyperboloids, etc. (spheres and planes are special subclasses)**
  - **for efficiency reasons, these simple objects often have their own intersection routines**
  - **the discussion here covers generalized intersection of these objects**
    - **a parametric ray formulation and an implicit surface equation are used to solve the intersection problem**

- **ray/quadric intersection**

  - **use the ray equation**

$$\mathbf{R}_{\text{origin}} \equiv \mathbf{R}_0 \equiv [X_0 \;\; Y_0 \;\; Z_0]$$
$$\mathbf{R}_{\text{direction}} \equiv \mathbf{R}_d \equiv [X_d \;\; Y_d \;\; Z_d]$$
where $X_d^2 + Y_d^2 + Z_d^2 = 1$ (i.e. normalized)

which defines a ray as

set of points on line $\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{R}_d * t$, where $t > 0$.

  - **the quadric surface equation is**

$$[X\ Y\ Z\ 1] * \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} * \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = 0$$

- **ray/quadric intersection, cont.**

  - **the matrix is labelled Q and is useful for performing transformations and other operations on the quadric**

    - **this equation is equivalent to**

      F(X,Y,Z) = 0, where

$$\begin{aligned} F(X, Y, Z) \equiv A^* X^2 + 2^* B^* X^* Y + 2^* C^* X^* Z + 2^* D^* X + \\ E^* Y^2 + 2^* F^* Y^* Z + 2^* G^* Y + \\ H^* Z^2 + 2^* I^* Z + J \end{aligned}$$

  - **substituting and solving for t yields coefficients for the quadratic formula**

$$\begin{aligned} A_q = A^* X_d^2 + 2^* B^* X_d{}^* Y_d + 2^* C^* X_d{}^* Z_d + \\ E^* Y_d^2 + 2^* F^* Y_d{}^* Z_d + \\ H^* Z_d^2 \end{aligned}$$

$$\begin{aligned} B_q = 2^*(A^* X_0{}^* X_d + B^*(X_0{}^* Y_d + X_d{}^* Y_0) + C^*(X_0{}^* Z_d + X_d{}^* Z_0) + \\ D^* X_d + E^* Y_0{}^* Y_d + F^*(Y_0{}^* Z_d + Y_d{}^* Z_0) + G^* Y_d + \\ H^* Z_0{}^* Z_d + I^* Z_d) \end{aligned}$$

$$\begin{aligned} C_q = A^* X_0^2 + 2^* B^* X_0{}^* Y_0 + 2^* C^* X_0{}^* Z_0 + 2^* D^* X_0 + \\ E^* Y_0^2 + 2^* F^* Y_0{}^* Z_0 + 2^* G^* Y_0 + \\ H^* Z_0^2 + 2^* I^* Z_0 + J. \end{aligned}$$

- **ray/quadric intersection, cont.**

  - **if $A_q \neq 0$, then if the quantity under the radical sign is less than 0; no intersection exists**

    - **otherwise calculate the roots**

      - **the smallest positive value of t is used to calculate the intersection point**

  - **if $A_q = 0$, $t = -C_q/B_q$**

  - **the intersection point is calculated as shown previously**

  - **the normal of a quadric surface is formed by taking partial derivatives of F with respect to X, Y and Z**

$$\mathbf{r}_n \equiv [x_n\ y_n\ z_n] = [dF/dX \quad dF/dY \quad dF/dZ]$$

$$x_n = 2^*(A^* x_i + B^* y_i + C^* z_i + D)$$
$$y_n = 2^*(B^* x_i + E^* y_i + F^* z_i + G)$$
$$z_n = 2^*(C^* x_i + F^* y_i + H^* z_i + I).$$

ray/quadric intersection, cont.

- example

Given a ray with an origin at [4 5 −3] and a direction vector of [0.577 0.577 −0.577], find the intersection point with an ellipsoid at [6 9 −2] with the axes lengths $X_a = 12$, $Y_a = 24$, $Z_a = 8$.

From basic analytic geometry, the ellipsoid's equation is:

$$\frac{(X-6)^2}{12^2} + \frac{(Y-9)^2}{24^2} + \frac{(Z-(-2))^2}{8^2} = 1$$

Simplifying, the quadric function is then:

$$F(X, Y, Z) \equiv 4^*X^2 - 48^*X + Y^2 - 18^*Y + 9^*Z^2 + 36^*Z - 315 = 0.$$

The equivalent matrix is formed by finding equivalences to the parameters A through J, and is:

$$[X\ Y\ Z\ 1] * \begin{bmatrix} 4 & 0 & 0 & -24 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 9 & 18 \\ -24 & -9 & 18 & 315 \end{bmatrix} * \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = 0$$

The coefficients for t are:

$$\begin{aligned} A_q = {} & 4^*0.577^*0.577 + 2^*0^*0.577^*0.577 + 2^*0^*0.577^*(-0.577) + \\ & 1^*0.577^*0.577 + 2^*0^*0.577^*(-0.577) + \\ & 9^*(-0.577)^*(-0.577) \\ = {} & 4.67 \end{aligned}$$

$$\begin{aligned} B_q = {} & 2^*(4^*4^*0.577 + 0^*(4^*0.577 + 0.577^*5) + 0^*(4^*(-0.577) + \\ & 0.577^*(-3)) - 24^*0.577 + \\ & 1^*5^*0.577 + 0^*(5^*(-0.577) + 0.577^*(-3)) - 9^*0.577 + \\ & 9^*(-3)^*(-0.577) + 18^*(-0.577)) \\ = {} & -3.46 \end{aligned}$$

$$\begin{aligned} C_q = {} & 4^*4^*4 + 2^*0^*4^*5 + 2^*0^*4^*(-3) + 2^*(-24)^*4 + \\ & 1^*5^*5 + 2^*0^*5^*(-3) + 2^*(-9)^*5 + \\ & 9^*(-3)^*(-3) + 2^*18^*(-3) - 315 \\ = {} & -535. \end{aligned}$$

The expression $B_q^2 - 4^* A_q{}^* C_q$ is positive, so an intersection point exists. The distance t is then either t_0 or t_1. First check t_0

$$t_0 = (-(-3.46) - \frac{\sqrt{((-3.46)^2 - 4^*4.67^*(-535))}}{2^*4.67}$$

$$= -10.3.$$

t_0 is negative (behind the ray), so check t_1:

$$t_1 = (-(-3.46) + \frac{\sqrt{((-3.46)^2 - 4^*4.67^*(-535))}}{2^*4.67}$$

$$= 11.1.$$

t_0 is positive, so this is the intersection point distance t. Note that the origin is inside the ellipsoid because only t_1 is positive. The intersection point is then

$$\begin{aligned} \mathbf{r}_i &= [4 + 0.577^*11.1 \quad 5 + 0.577^*11.1 \quad -3 + (-0.577)^*11.1] \\ &= [10.4 \quad 11.4 \quad -9.4]. \end{aligned}$$

Calculate the normal at the surface

$$\begin{aligned} x_n &= 4^*10.4 + 0^*11.4 + 0^*(-9.4) - 24 = 17.6 \\ y_n &= 0^*10.4 + 1^*11.4 + 0^*(-9.4) - 9 = 2.4 \\ z_n &= 0^*10.4 + 0^*11.4 + 9^*(-9.4) + 18 = -66.6. \end{aligned}$$

Normalizing, we get:

$$\mathbf{r}_n = [0.255 \quad 0.0348 \quad -0.966].$$

This is a vector whose dot product with $\mathbf{R}_d$:

$$[0.255\ 0.0348\ \ -0.966] \cdot [0.577\ 0.577\ \ -0.577]$$

is 0.725, which means that the surface normal faces in the direction of the ray. This means that the direction of the normal should be reversed so as to point toward the ray's origin.

- **standard inverse mappings**
  - **inverse mapping for a circle**
    - **mostly a problem of converting from Cartesian to polar coordinates**
    - **define a circle in the XY plane with its center at the origin and a radius C_r**

      $$X_c^2 + Y_c^2 = C_r^2$$

    - **an intersection point on the XY plane is given**

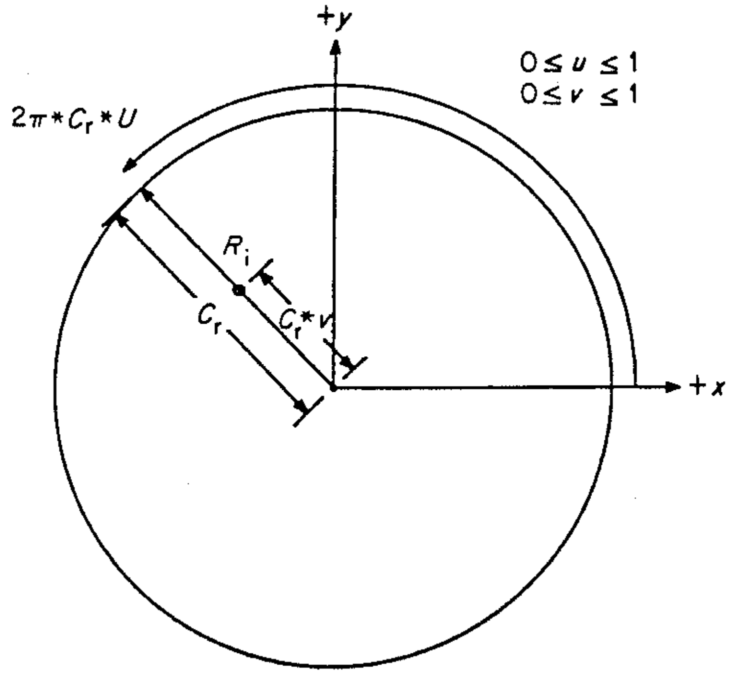
      $$\mathbf{R}_i = [X_i \; Y_i \; Z_i]$$

    - **the (u,v) coordinates**
      - **u and v range from 0 to 1**
      - **u starts at the +X axis and moves toward the +Y axis**
      - **v starts the origin and moves toward the edge of the circle**

$$v = \sqrt{((X_i^2 + Y_i^2)/C_r^2)}$$

$$u' = \frac{\arccos\left(X_i/\sqrt{(X_i^2 + Y_i^2)}\right)}{2 * \pi}$$

if $Y_i < 0$ then set $u = 1 - u'$, else set $u = u'$.


+y
0≤ u ≤ 1
0≤ v ≤ 1
2π*Cr*U
Ri
Cr
Cr*v
+x