CURVED SURFACES
(Section 10-2 in *Computer Graphics*)

- Introduction
- Parametric Equations
- Bézier Curves
- B-spline Curves
- Bézier Surfaces
- B-spline Surfaces

Introduction to Curved Surfaces

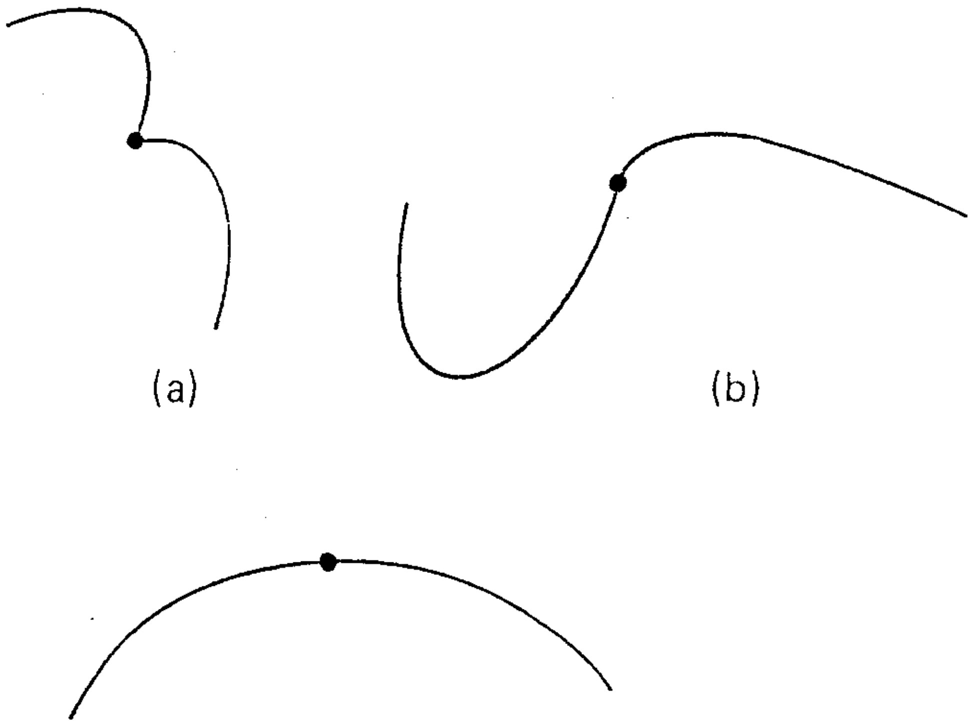
- two surface generation methods
  - mathematical functions define the surfaces
    - representation in analytic form
      - y = f(x)
      - z = g(x)
      - changes in slope may mean changing the independent variable
      - awkward for multivalued functions
    - see figure 10-6 on page 194
  - a set of user-specified data points
    - see figure 10-7 on page 194

Parametric Equations

- any point on a curve can be represented by
  $P(u) = (x(u), y(u), z(u))$
  $0 \leq u \leq 1$, usually
- example: a circle in the xy plane of radius r centered at the origin
  $x(u) = r\cos(2\pi u)$
  $y(u) = r\sin(2\pi u)$
  $z(u) = 0$
- approximations to other curves can be represented by polynomials
- sometimes, different polynomials are used for different portions of the curve

Parametric Equations, continued

- continuity between sections of the curve becomes important
  - zero-order continuity means the curves meet (a)
  - first-order continuity means the tangent lines of the adjoining sections match at the joint (b)
  - second-order continuity means the curvatures of the adjoining sections match at the joint (c)



(a) (b)

(c)

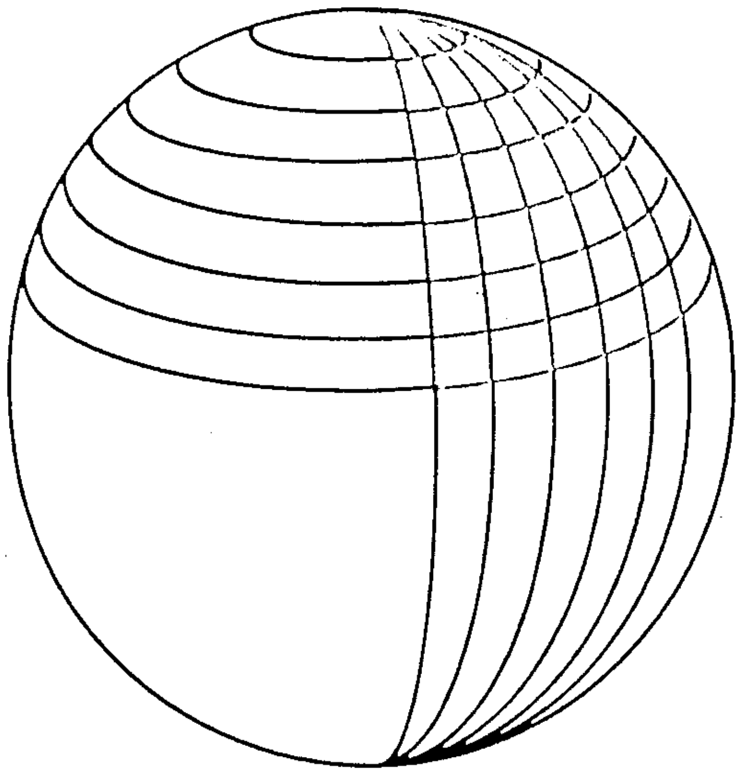
parametric equations for surfaces

- $P(u, v) = (x(u, v), y(u, v), z(u, v))$
- $0 \leq u, v \leq 1$, usually
- example: a sphere of radius r centered at the origin

  $x(u, v) = r\sin(\pi u)\cos(2\pi v)$

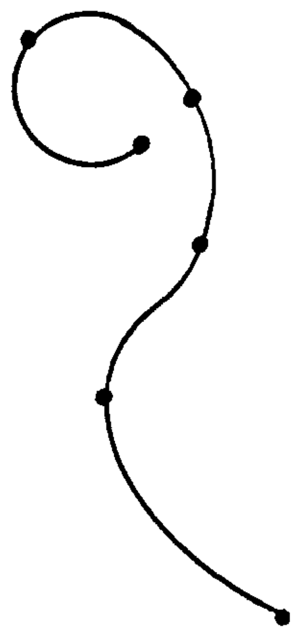
  $y(u, v) = r\sin(\pi u)\sin(2\pi v)$

  $z(u, v) = r\cos(\pi u)$

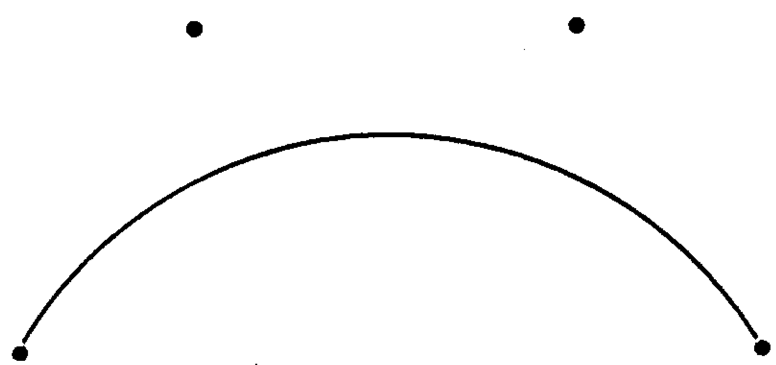


setting up parametric polynomial equations

- control points to indicate the shape of the curve
- sometimes the control points are interpolated



- sometimes the control points are approximated



Bézier Curves

- developed for Renault automobile bodies
- the Bézier coordinate function is

$$P(u) = \sum_{k=0}^{n} p_k \, B_{k,n}(u)$$

where

- $p_k = (x_k, y_k, z_k)$, k = 0 to n, are the n+1 control points
- each $B_{k,n}$ is a polynomial function called a blending function

$$B_{k,n}(u) = C(n,k)u^k (1 - u)^{n-k}$$

- the C(n,k) represent the binomial coefficient

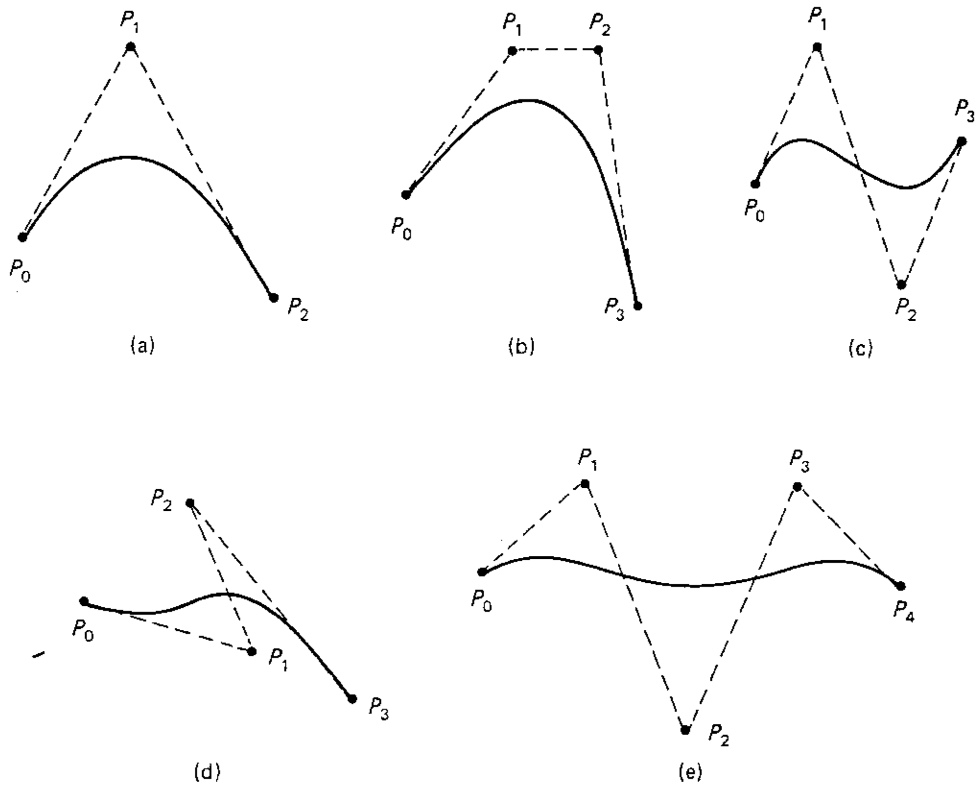
$$C(n,k) = \frac{n!}{k!\,(n - k)!}$$

- individual coordinates are represented by

$$x(u) = \sum_{k=0}^{n} x_k B_{k,n}(u)$$

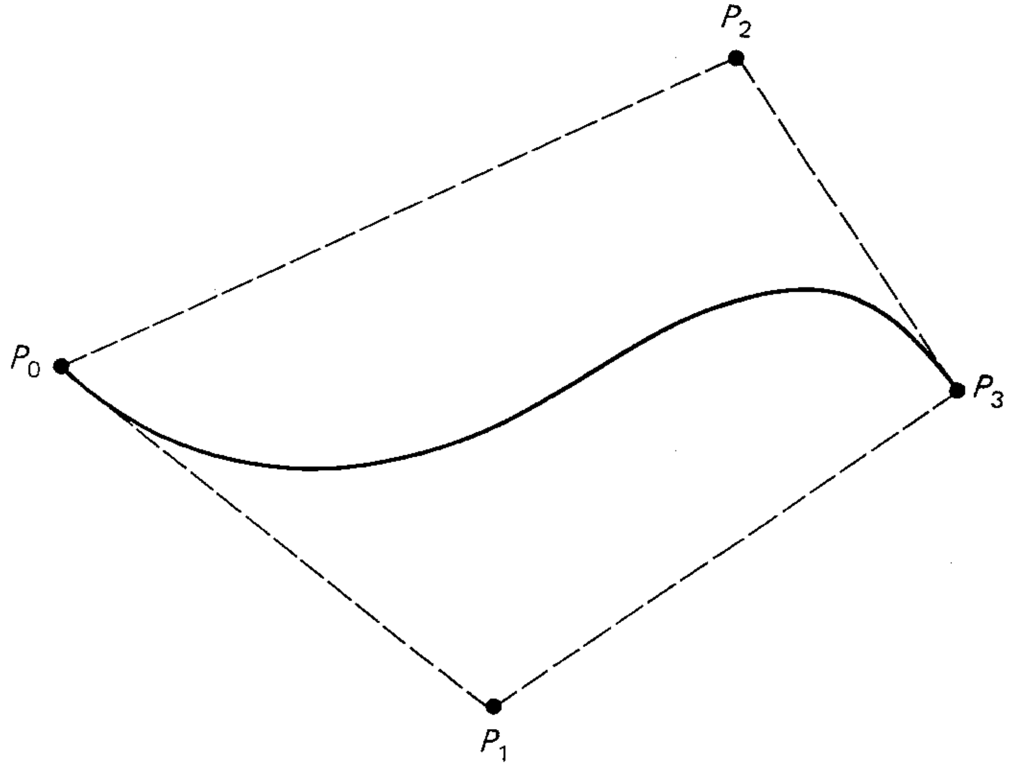
$$y(u) = \sum_{k=0}^{n} y_k B_{k,n}(u)$$

$$z(u) = \sum_{k=0}^{n} z_k B_{k,n}(u)$$


P_1
P_0
P_2
(a)
P_1
P_2
P_0
P_3
(b)
P_1
P_3
P_0
P_2
(c)
P_2
P_0
P_1
P_3
(d)
P_1
P_3
P_0
P_4
P_2
(e)

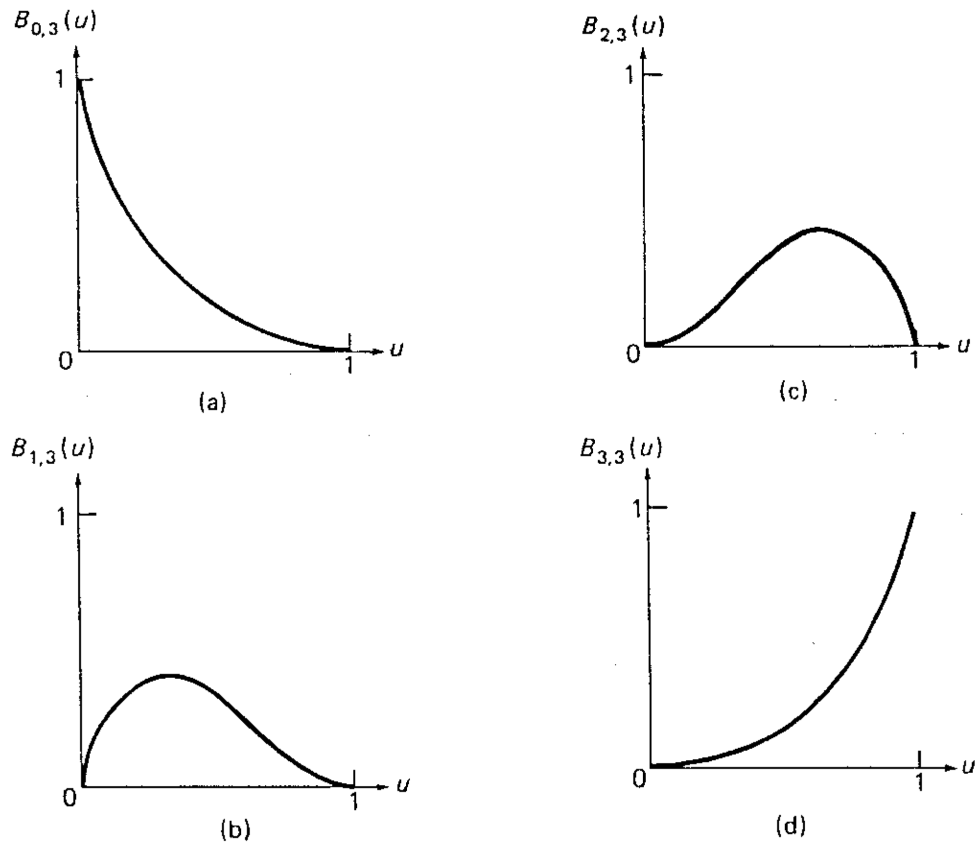
Bézier Curves, continued

- the curve lies within a convex hull



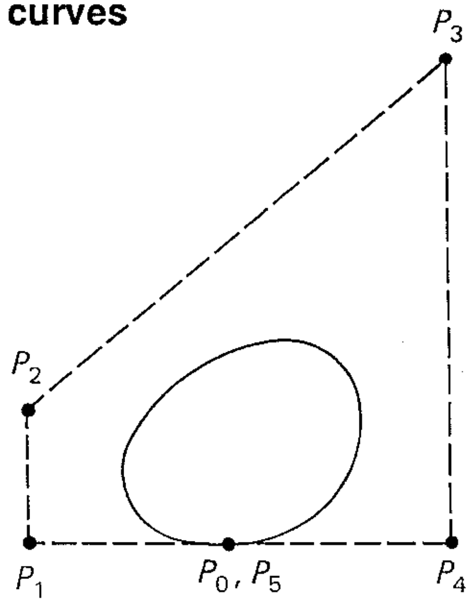
Bézier Curves, continued

- the blending functions

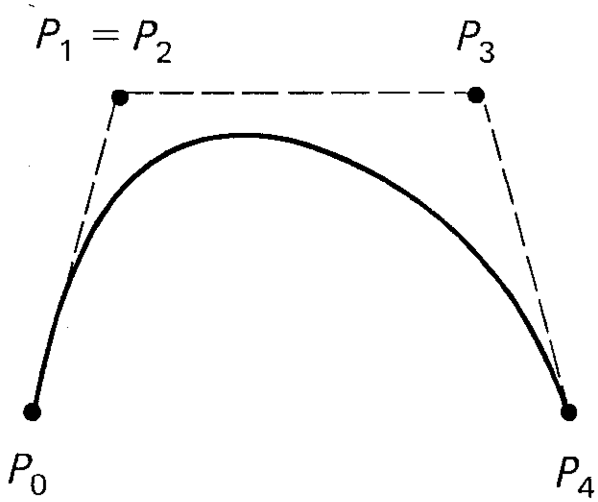


Bézier Curves, continued

- closed curves

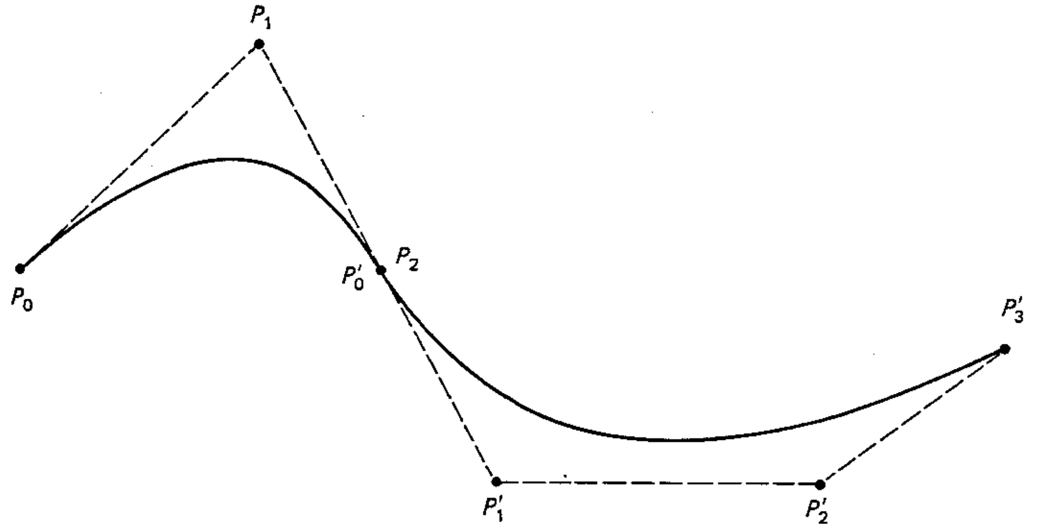


- multiple copositional control points



Bézier Curves, continued

- the degree of the polynomial is one less than the number of control points
- curves can be pieced together
  - avoids high order polynomials
  - provides local control
  - continuity at the joint can be controlled



Bézier Curve Example

GIVEN: four control points at (0,0), (1,2), (4,2) and (5,0)

FIND: the Bézier curve

$B_{0,3} = C(3,0)u^{0}(1-u)^{3-0} = [3!/(0!3!)](1-u)^3 = (1-u)^3$
$B_{1,3} = 3u(1-u)^2$
$B_{2,3} = 3u^2(1-u)$
$B_{3,3} = u^3$

$$P(u) = \sum_{i=0}^{3} p_i B_i(u)$$

$$P(u) = p_o B_{o,n}(u) + p_1 B_{1,n}(u) + p_2 B_{2,n}(u) + p_3 B_{3,n}(u)$$

$$(x,y) = (0,0)(1-u)^3 + (1,2)[3u(1-u)^2] + (4,2)[3u^2(1-u)] + (5,0)u^3$$

or

$$x = 0(1-u)^3 + 3u(1-u)^2 + 12u^2(1-u) + 5u^3$$
$$= -4u^3 + 6u^2 + 3u$$

$$y = 0(1-u)^3 + 6u(1-u)^2 + 6u^2(1-u) + 0u^3$$
$$= -6u^2 + 6u$$

B-spline Curves

- the B-spline coordinate function is

$$P(u) = \sum_{k=0}^{n} p_k N_{k,t}(u)$$

where

- p_k, k = 0 to n, are the n+1 control points
- each $N_{k,t}$ is a blending function, defined recursively

$$N_{k,1} = \begin{cases} 1 & \text{if } u_k \leqslant u < u_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

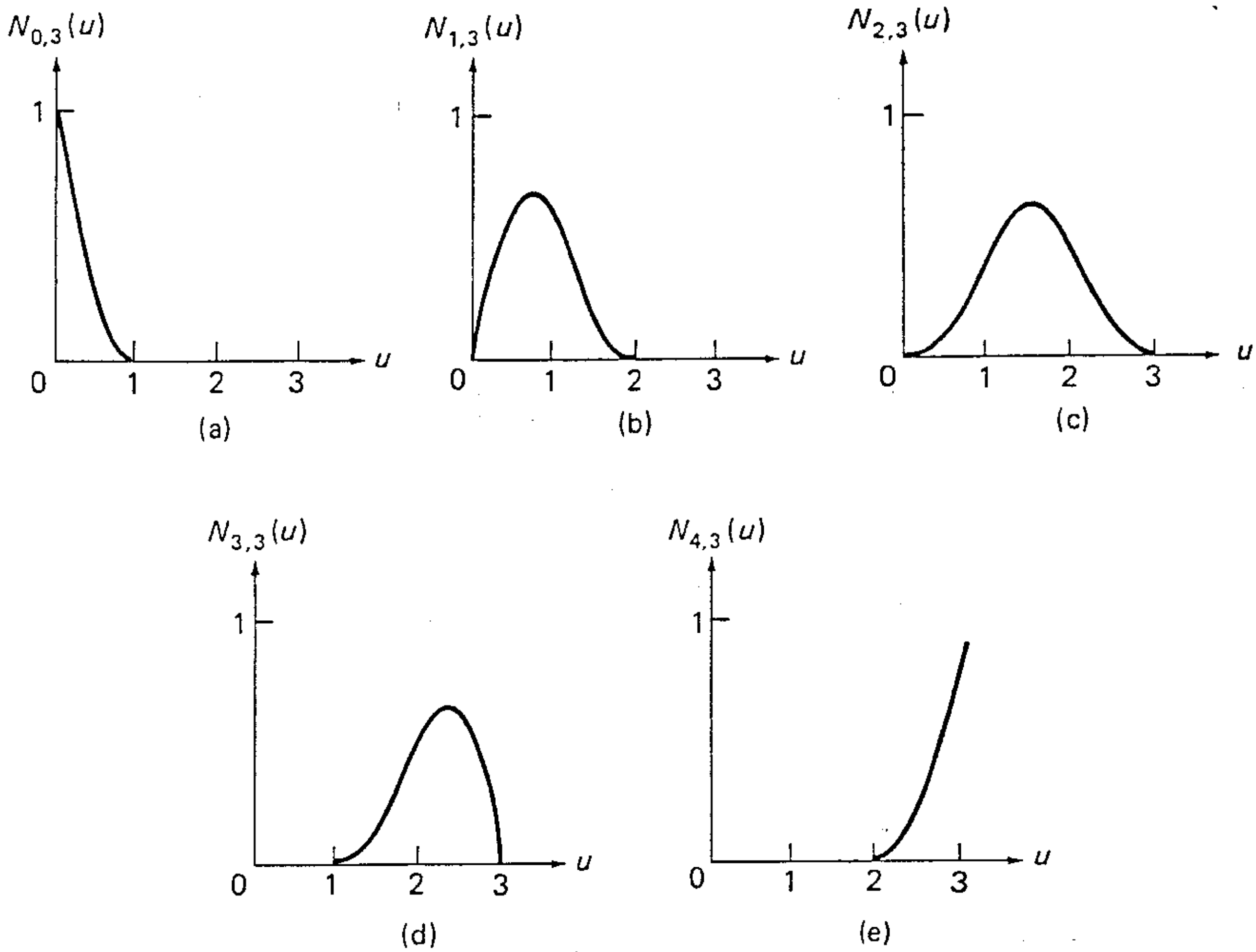
$$N_{k,t}(u) = \frac{u - u_k}{u_{k+t-1} - u_k} N_{k,t-1}(u) + \frac{u_{k+t} - u}{u_{k+t} - u_{k+1}} N_{k+1,\,t-1}(u)$$

- the defining positions or breakpoints u are defined by

$$u_j = \begin{cases} 0 & \text{if } j < t \\ j - t + 1 & \text{if } t \leqslant j \leqslant n \\ n - t + 2 & \text{if } j > n \end{cases}$$

B-spline Curves, continued

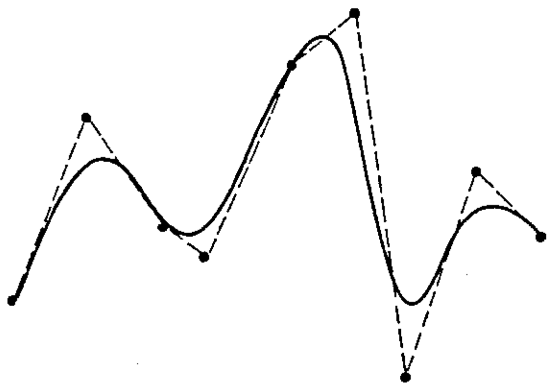
the blending functions using 5 control points



(a) (b) (c) (d) (e)

B-spline Curves, continued

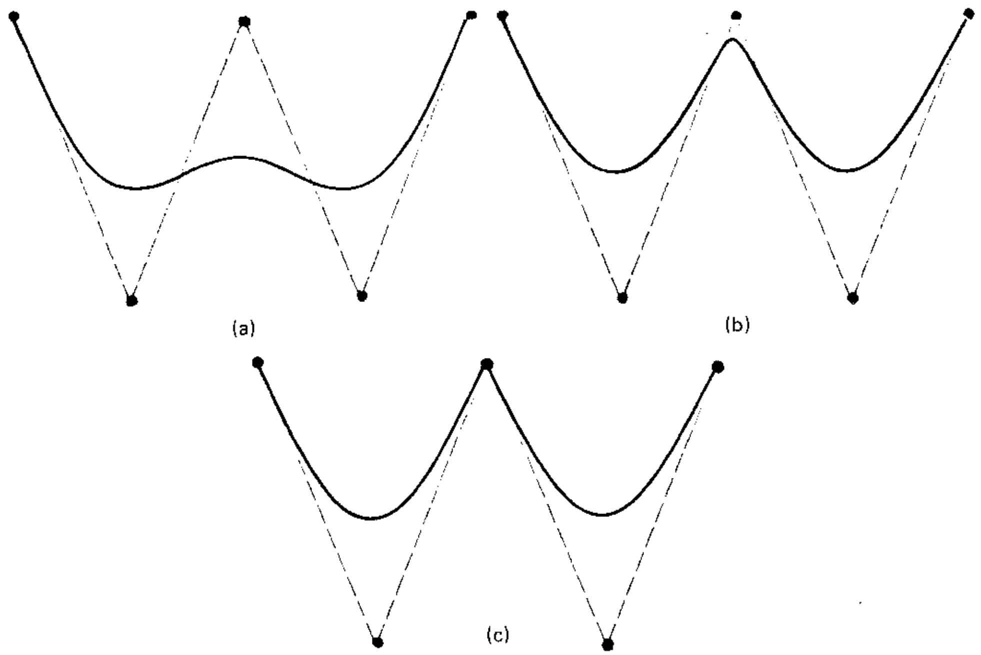
- local control by repositioning the third control point



(a) (b)

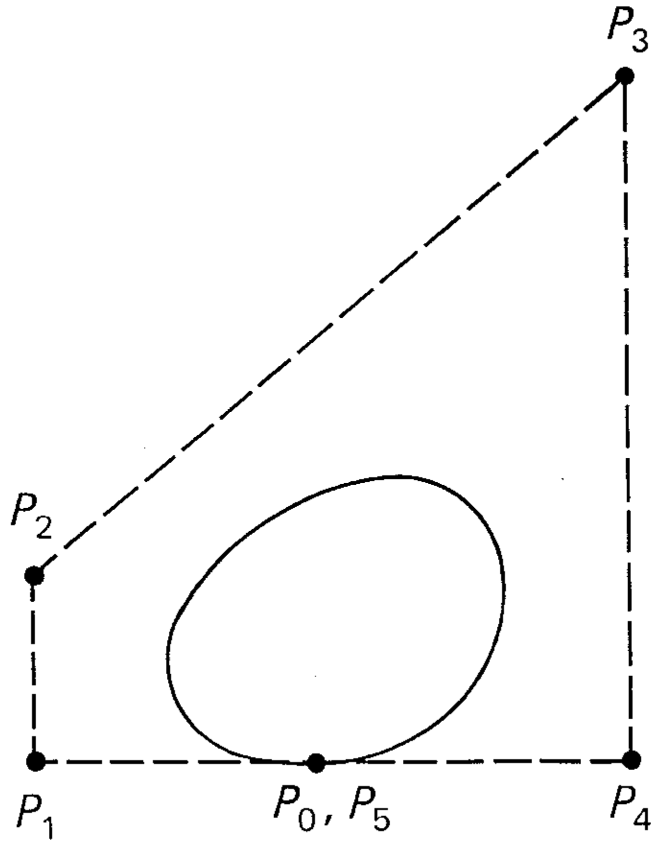
B-spline Curves, continued

- an increase in the number of control points does not increase the degree of the curve
- no need to piece sections together
- multiple coincident control points
  - one, two and three control points at the center position



B-spline Curves, continued

- closed curves



- convex hulls

other curves

CURRENT SYNOPSIS OF PARAMETRIC CURVE FORMS

PARAMETRIC CURVE REPRESENTATIONS	Basis (Blending) Functions	Continuity *	Points On/Off Curve	Convex Hull, Variation Diminishing	Global/ Local Control	Additional Control Parameters
B-Spline (Cubic)	B-Spline	C^2	Off	Yes	Local	None
Rational	B-Spline	C^2	Off	Yes	Local	Weights
Beta-Spline (Cubic)	B-Spline	G^2	Off	Yes	Local	β1-Bias β2-Tension
Beta2-Spline (Cubic)	B-Spline	G^2	Off	Yes	Local	β2-Tension
Bézier (Any Order)	Bernstein	C^∞	Off	Yes	Global	None
Rational	Bernstein	C^∞	Off	Yes	Global	Weights
Cardinal Spline (Cubic)	Hermite	C^2	On	No	Global	None
Cincy Parabola (Quadratic)	Standard	C^0	On	No	Local	None
Hermite (Cubic)	Hermite	C^1	On	No	Local	Endpoint Tangents
Overhauser (Cubic)	Parabolic Blending	C^1	On	No	Local	None
Q-Spline (Quintic)	Standard	C^2	On	No	Local	None

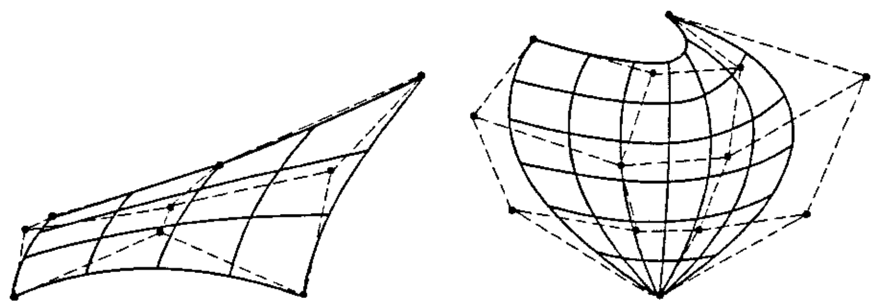
*** C - parametric continuity**
G - geometric continuity

Bézier Surfaces

- two sets of Bézier curves can represent surfaces specified by control points
- the Bézier coordinate function

$$P(u,v) = \sum_{j=0}^{m} \sum_{k=0}^{n} p_{j,k} B_{j,m}(u)\, B_{k,n}(v)$$

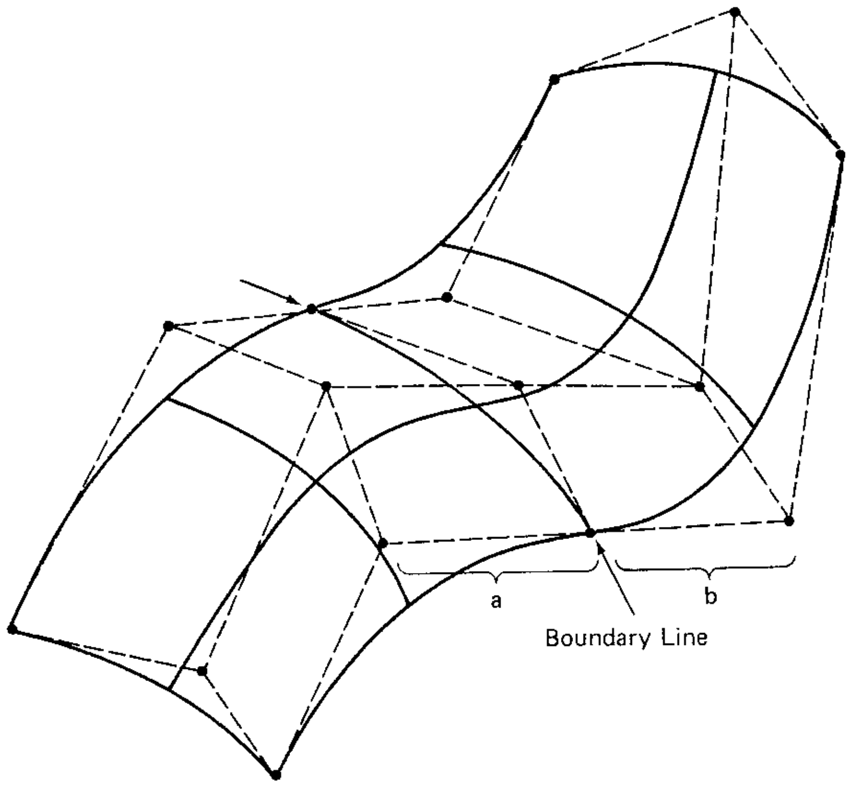
where
- $p_{j,k}$ represent the (m + 1)-by-(n + 1) control
  - points



(a) (b)

Bézier Surfaces, continued

- transition from one section to another



- first order continuity
  - ratio of a to b is constant for each line of control points across the boundary line

B-spline Surfaces

- the B-spline coordinate function

$$P(u, v) = \sum_{j=0}^{m} \sum_{k=0}^{n} p_{j,k} N_{j,s}(u)\, N_{k,t}(v)$$

where

- $p_{j,k}$ represent the (m + 1)-by-(n + 1) control points

CURVED SURFACES

- Parametric Equations
- Bézier curves
- B-spline curves
- Bézier surfaces
- B-spline surfaces