FRACTALS

Introduction

- fractal geometry dates from 1975
    - systematic fractal geometry was developed at the IBM T.J. Watson Research Center
    - Benoit B. Mandelbrot
- Mandelbrot's fractal geometry provides
    - a description of many seemingly complex forms found in nature
    - a mathematical model for these forms
- statistical self-similarity
    - an essential quality of fractals in nature
    - a characteristic which may be quantified by a fractal dimension

Two perceptions

- **Galileo (1623)**

  Philosophy is written in this grand book - I mean universe - which stands continuously open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.

- **Mandelbrot (1982)**

  clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line

Uses

- within the past decade, fractal geometry and its concepts have become central tools in
  - physics
  - chemistry
  - biology
  - geology
  - meteorology
  - materials science
  - military science
- fractals have become of interest to graphic designers and filmmakers
  - exciting shapes
  - artificial but realistic worlds

Differences between traditional Euclidean shapes and fractals

- **Euclidean shapes**
  - more than 2000 years old
  - based on characteristic size or scale
    - radius of a sphere
    - size of a cube
  - to suit man made objects
    - machine shops are Euclidean factories
  - described by formula
- **fractals**
  - about 20 years old
  - no specific size or scaling
    - self-similar
    - independent of scale
  - appropriate for natural shapes
  - described algorithmically (recursively)

The von Koch snowflake curve

- exact self-similarity (or scaling)

    - each portion, when magnified, can reproduce exactly a larger portion

- the limiting curve crams infinite length into a finite area of the plane without intersecting itself

- the algorithm is concise, simple to describe and easy to compute

    - yet no algebraic formula specifies the points of the curve

Dimension

- a line segment can be divided into N identical parts, each of which is 1/Nth the size of the original

- a two-dimensional object, such as a square area in the plane, can be divided into N self-similar objects each of which is $1/(N^{1/2})$ the scale of the original

- a three-dimensional object, such as a solid cube, can be divided into N self-similar little cubes each of which is $1/(N^{1/3})$ the scale of the original

- a d-dimensional object can be divided into N self-similar copies of itself, each of which is $1/(N^{1/d})$ the scale of the original

- fractal or similarity dimension is given by

$$D = \log(N)/\log(1/r)$$

  - r is the scaling factor which produces the copy from the original ($1/(N^{1/d})$ in the previous example)

    - any segment of the von Koch curve is composed of 4 subsegments, each of which is scaled down by a factor of 1/3

      - its fractal dimension $D = \log(4)/\log(3)$, or about 1.26

      - it fills more space than a line ($D = 1$), but less than a plane ($D = 2$)

      - its dimension measures its wiggliness, or how much of space it fills

      - it's still a curve, since removing a single point cuts it in two pieces

Statistical self-similarity

- **a large scale view may be insufficient to predict exact details of a magnified view**

- **in nature, objects can look statistically similar while being different in detail at different length scales**

Mandelbrot landscapes

- actual landscapes can be statistically self-similar only over a finite (but often quite large) range of distances
    - the size of the planet or the force of gravity may limit the height of mountains, for example
    - the smallest scales may be limited by the basic grain size or by the atomic nature of particles
- nevertheless, Mandelbrot's geometry is by far the best approximation
- most algorithms add random irregularities at smaller and smaller scales
- the higher the D, the rougher the surface
    - 2.15 is appropriate for much of the earth
    - fractal geometry specifies only the relative height variations of the landscape at different length scales
- a flat-bottomed basin can be approximated mathematically by
    - taking height variations
    - scaling them by a power law
        - a power greater than 1 flattens lower elevations emphasizing peaks
        - a power less than 1 flattens peaks and increases steepness at lower elevations
        - fractal dimension is not affected

Fractally distributed craters

- craters are circular with height profiles similar to the effect of dropping marbles in mud

- realism is achieved by proper distribution of sizes

  - for the moon, distribution is fractal or power-law, with many more small craters than large ones

  - the number of craters having an area $a > A$ varies as $1/A$

Fractal planet

- Brownian motion on a sphere

- an earth-like surface of a fractal planet is a generalization of Brownian motion or a random walk on a sphere

    - the surface becomes the result of many independent surface displacements or faults

    - each fault encircles the sphere in a random direction and divides it into hemispheres which are displaced relative to each other in height

    - the surface of the sphere is mapped onto a rectangular array similar to a flat projection map of the earth

    - 10,000 faults produce a plausible planet with D = 2.5

    - the data is mapped back onto the sphere

    - color can be based on height

    - once the fractal dimension is in an appropriate range, the mathematical models rarely fail to evoke natural images

    - nature favors shapes with a fractal dimension about 0.2 to 0.3 greater than the Euclidean dimension

        - coastlines: 1.2

        - landscapes: 2.2

        - clouds: 3.3

        - clusters of clusters of galaxies: 3.2

Fractal flakes and clouds

- by adding irregularities, fractals with topological dimension 3 can be raised to fractal dimension between $3 < D < 4$

  - useful for representing temperature or water vapor density as a function of 3-dimensional position

- such constructions are best viewed with their zerosets

  - for a temperature distribution $T(x,y,z)$ with fractal dimension $3 < D < 4$, the zerosets are all points where $T(x,y,z) - T_0 = 0$

    - lines on a topographic map are zerosets

  - the zerosets have fractal dimension $D - 1$

- displaying the zerosets

  - all points for which $T(x,y,z) > T_0$ are opaque

  - all other points are transparent

  - realistic looking clouds can be produced by allowing local light scattering and opacity to vary continuously as $T(x,y,z)$

Scaling randomness in time: $1/f^{beta}$-noises

- unpredictable changes of any quantity V varying with time are known as noise

- spectral density, $S_V(f)$, gives an estimate of the mean square fluctuations at frequency f and, consequently, of variations over a time scale of order 1/f

- 1/f-noise is any fluctuating quantity V(t) with $S_V(f)$ varying as $1/f^{beta}$ over many decades with $0.5 < beta < 1.5$

  - the origin of 1/f-noise remains a mystery after more than 60 years of investigation

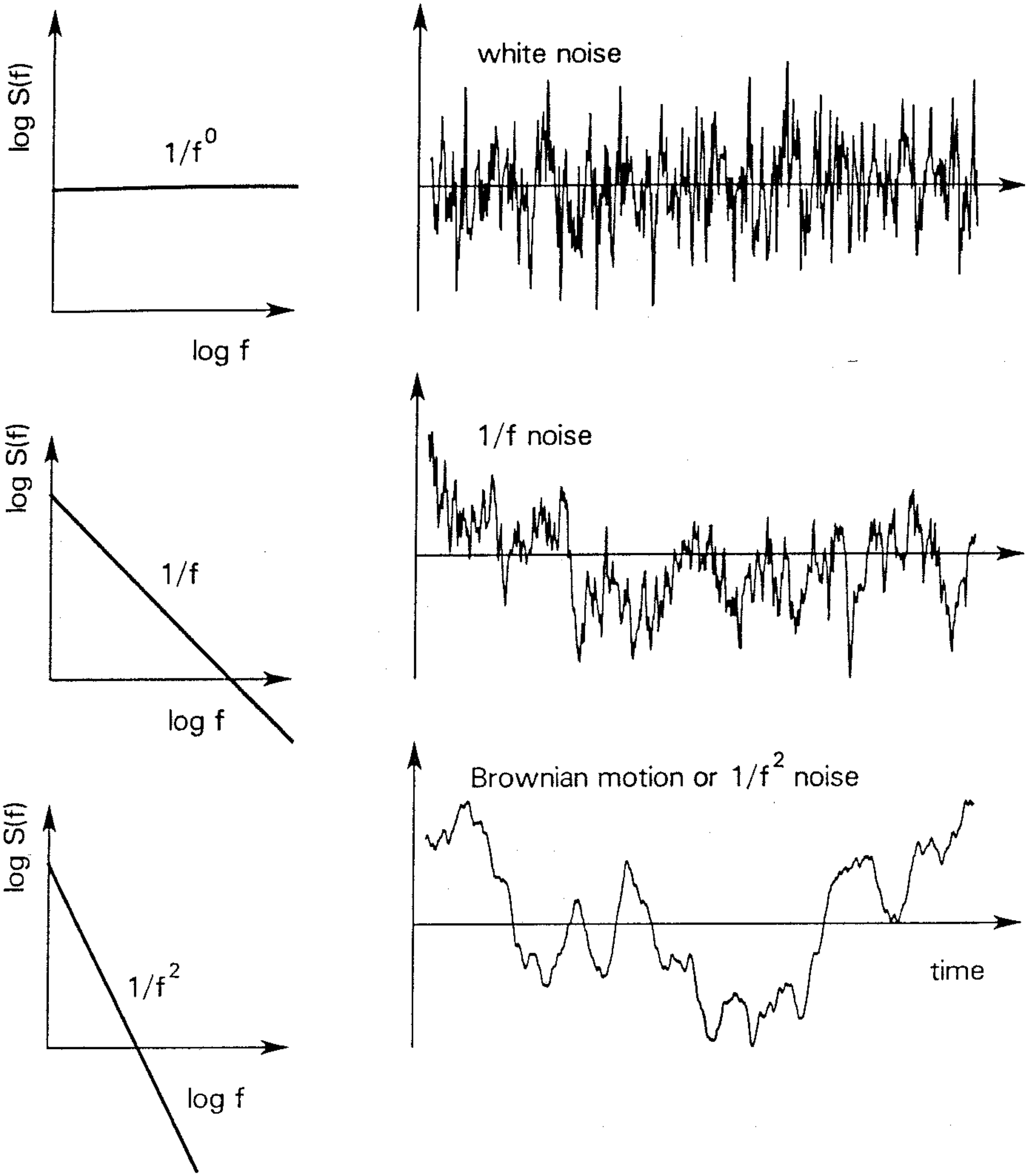
  - it is the most common noise found in nature

    - in all electronic components from simple carbon resistors to all semiconducting devices

    - in all time measuring devices from the ancient hourglass to the most accurate atomic clocks

    - in ocean flows to changes in yearly flood levels of the Nile as recorded by the ancient Egyptians

    - other examples ranging from music to the flow of automobiles on an expressway

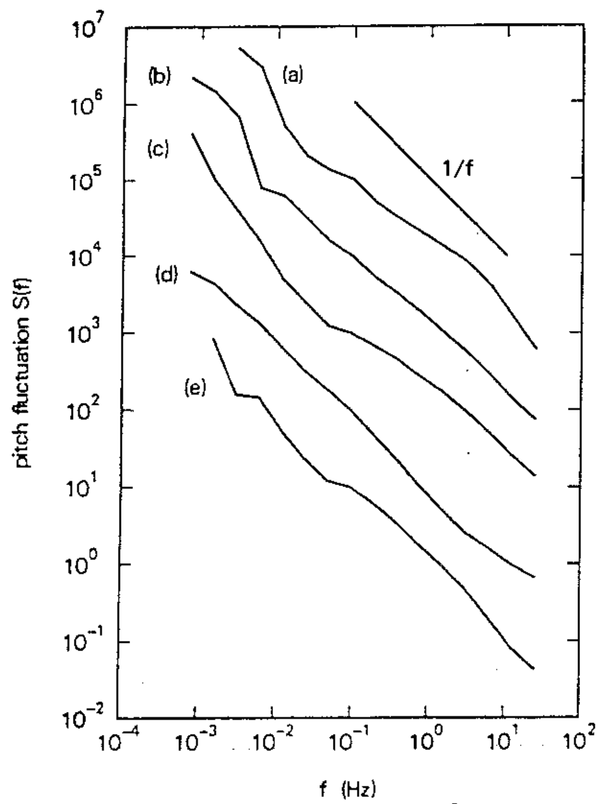

log S(f)
$1/f^0$
log f
white noise
log S(f)
1/f
log f
1/f noise
log S(f)
$1/f^2$
log f
Brownian motion or $1/f^2$ noise
time

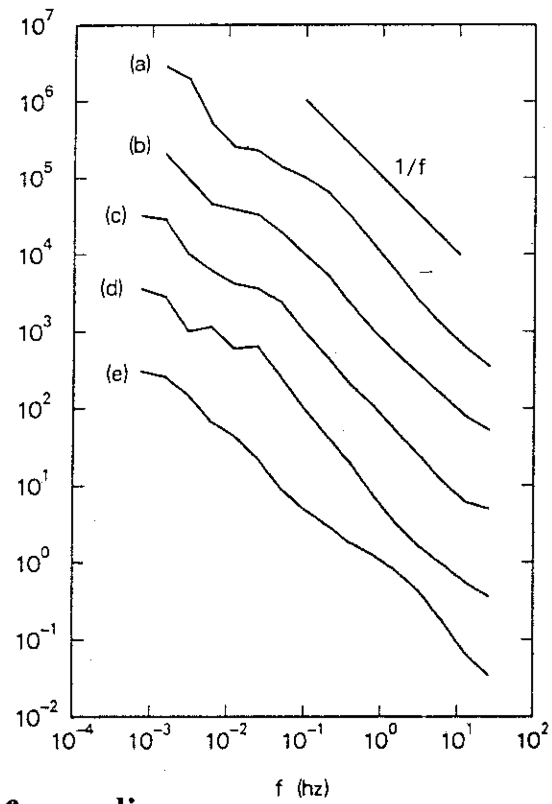
Mathematical Models: fractional Brownian motion (fBm)

- a most useful mathematical model for random fractals found in nature
    - mountainous terrain and clouds
- an extension of the central concept of Brownian motion from physics and mathematics
- most natural computer graphics fractal simulations are based on an extension of fBm to higher dimensions

Fractal music

- almost all musical melodies mimic 1/f-noise
- lines connecting notes in a score look like a plot of 1/f-noise





frequency of sampling

Pitch fluctuations from different musical cultures
(a) the Ba-Benzele Pygmies
(b) traditional music of Japan
(c) classical ragas of India
(d) folk songs of old Russia
(e) American blues

Pitch fluctuations in western music
(a) Medieval music up to 1300
(b) Beethoven, 3rd Symphony
(c) Debussey, piano works
(d) R. Strauss, ein Heldenlebe
(e) the Beatles, Sgt. Pepper

- the higher the frequency of observation, the less the variation
- music apparently imitates the way our world changes in time
- music generated from 1/f-noise is closer to real music than music generated with "white" or "brown" noise
  - results are recognizably musical

Fractional Brownian motion, $V_H(t)$

- a single valued function of one variable
- it resembles a mountainous horizon or the fluctuations of an economic variable
- scaling is characterized by a parameter H, $0 < H < 1$
  - small H produces the roughest traces
  - large H produces the smoothest traces

$$\Delta V \propto \Delta t^H$$

- in the usual Brownian motion or random walk, the sum of independent increments or steps leads to a variation that scales as the square root of the number of steps
  - $H = 0.5$ corresponds to a trace of Brownian motion

Self-affinity

- self-similar shapes repeat statistically or exactly under a magnification

- however, fBm traces repeat statistically only when the t and V directions are magnified by different amounts

    - if t is magnified by r to become rt, V must be magnified by r^H to become $r^H V$

    - for a random walk (H = 0.5), four times as many steps must be taken to go twice as far

- this nonuniform scaling is known as self-affinity

    - shapes are statistically invariant under transformations that scale different coordinates by different amounts

Zerosets

- like traditional Euclidean shapes, fractals typically reduce their dimension by one under intersection with a plane

    - the intersection of a fractal curve in the plane (with fractal dimension $1 < D < 2$) with a straight line is a set of points of dimension D-1

    - if the direction of the intersecting line is chosen to eliminate one of the coordinates, the self-affine curve reduces to a self-similar set of points

- the zeroset of fBm is the intersection of the trace of $V_H(t)$ with the t axis (the set of all points such that $V_H(t) = 0$)

    - a disconnected set of points with topological dimension 0 and fractal dimension $D_0 = 1-H$ between 0 and 1

    - the fractal dimension $D = D_0 + 1$

$$= 2 - H$$

Self-affinity in higher dimensions: Mandelbrot landscapes and clouds

- **Mandelbrot proposed modelling of the Earth's irregular surface as a generalization of traces of fBm**

- **t is replaced by x and y in the plane to give $V_H(x,y)$ as the surface altitude at position x,y**

- **the altitude variations of a hiker following any straight line path at constant speed in the xy-plane is a fractional Brownian motion**

- **if the hiker travels a distance delta-r in the xy-plane, the typical altitude variation delta-V is given by**

$$\Delta V \propto \Delta r^H$$

- **the fractal dimension D must be greater than the topological dimension 2 of the surface**

D = 3 - H for a fractal landscape $V_H(x,y)$

- **the intersection of a vertical plane with the surface $V_H(x,y)$ is a self-affine fBm trace characterized by H**
  - **it has a fractal dimension one less than the surface itself**

Self-affinity in higher dimensions: Mandelbrot landscapes and clouds, cont.

- the zeroset of $V_H(x,y)$, is its intersection with a horizontal plane
  - it has a fractal dimension $D_0 = 2 - H$
  - the intersection produces a family of (possibly disconnected) curves
    - they form the (self-similar) coastlines of the $V_H(x,y)$ landscape
    - these are statistically invariant under changes of magnification
- the generalization of fBm can continue to still higher dimensions to produce, for example, a self-affine fractal temperature or density distribution $V_H(x,y,z)$
  - $D = 4 - H$ for a fractal cloud $V_H(x,y,z)$
  - the zeroset $V_H(x,y,z) = \text{constant}$ gives a self-similar fractal with $D_0 = 3 - H$

Algorithms for Random Fractals

- mathematical expertise is not needed to generate a fractal

- the complexity of a fractal, measured by the length of the shortest computer program that generates it, is very low

- two major varieties of fractals

  - deterministic fractals

    - composed of several scaled down and rotated copies of themselves

      - the von Koch snowflake curve

      - the Julia sets

    - a mapping rule is repeated over and over in a usually recursive scheme

    - the structure is contained in any (large or small) region

  - random fractals

    - these allow for simulation of natural phenomena

Several categories of algorithms

- an approximation of a random fractal at some resolution is input
  - the algorithm produces an improved approximation
    - outputs can become new inputs until the desired resolution is achieved
    - example: midpoint displacement methods
- only one approximation of a random fractal is computed for the final resolution
  - example: Fourier filtering
- the approximation is obtained via an iteration
  - example: the random cut method

Two methods to graphically display fractal surfaces

- flat, two-dimensional top view and parallel projection for a three-dimensional view
- three-dimensional perspective projection, using an extended floating horizon method

One-dimensional Brownian motion

- the simplest random fractal
  - also at the heart of all the generalizations presented here
- definitions
  - X(t) is a one-dimensional random process whose values are random variables $X(t_1)$, $X(t_2)$, etc.
  - the increment $X(t_2) - X(t_1)$ has a Gaussian distribution
  - the mean square increments have a variance proportional to the time differences

$$E\left[|X(t_2) - X(t_1)|^2\right] \propto |t_2 - t_1|$$

  - the increments of X are statistically self-similar
    - two random functions X(t) and $[1/(r)^{0.5})]X(rt)$ are statistically indistinguishable
    - the second is a properly rescaled version of the first
    - example: if X(16t) is divided by 4, it produces Brownian motion which looks like X(t)

One-dimensional Brownian motion, cont.

- **integrating white noise**
  - **the integral of uncorrelated white Gaussian noise W has increments with Gaussian distributions**

$$X(t) = \int_{-\infty}^{t} W(s)\,ds$$

  - **the mean square increments have a variance proportional to the time differences**
  - **the random variables W(t) are uncorrelated and have the same normal distribution N(0,1)**
  - **the graph of a sample Brownian motion X(t) has fractal dimension 1.5**
    - **the intersection of the graph with a horizontal line has dimension 0.5**
  - **algorithm for white noise**

```
ALGORITHM WhiteNoiseBM  (X, N, seed)
Title        Brownian motion by integration of white Gaussian noise

Arguments    X[]          array of reals of size N
             N            size of array X
             seed         seed value for random number generator
Variables    i            integer

BEGIN
    X[0] := 0
    InitGauss (seed)
    FOR i := 1 TO N-1 DO
        X[i] := X[i-1] + Gauss () / (N-1)
    END FOR
END
```

One-dimensional Brownian motion, cont.

- algorithm to initialize random number generators

ALGORITHM	**InitGauss** (seed)	
Title	Initialization of random number generators	
Arguments	seed	seed value for random number generator
Globals	Arand	rand() returns values between 0 and Arand, system dependent
	Nrand	number of samples of rand() to be taken in Gauss()
	GaussAdd	real parameter for the linear transformation in Gauss()
	GaussFac	real parameter for the linear transformation in Gauss()
Functions	srand()	initialization of system random numbers

```
BEGIN
    Nrand := 4
    Arand := power (2, 31) - 1
    GaussAdd := sqrt (3 * Nrand)
    GaussFac := 2 * GaussAdd / (Nrand * Arand)
    srand (seed)
END
```

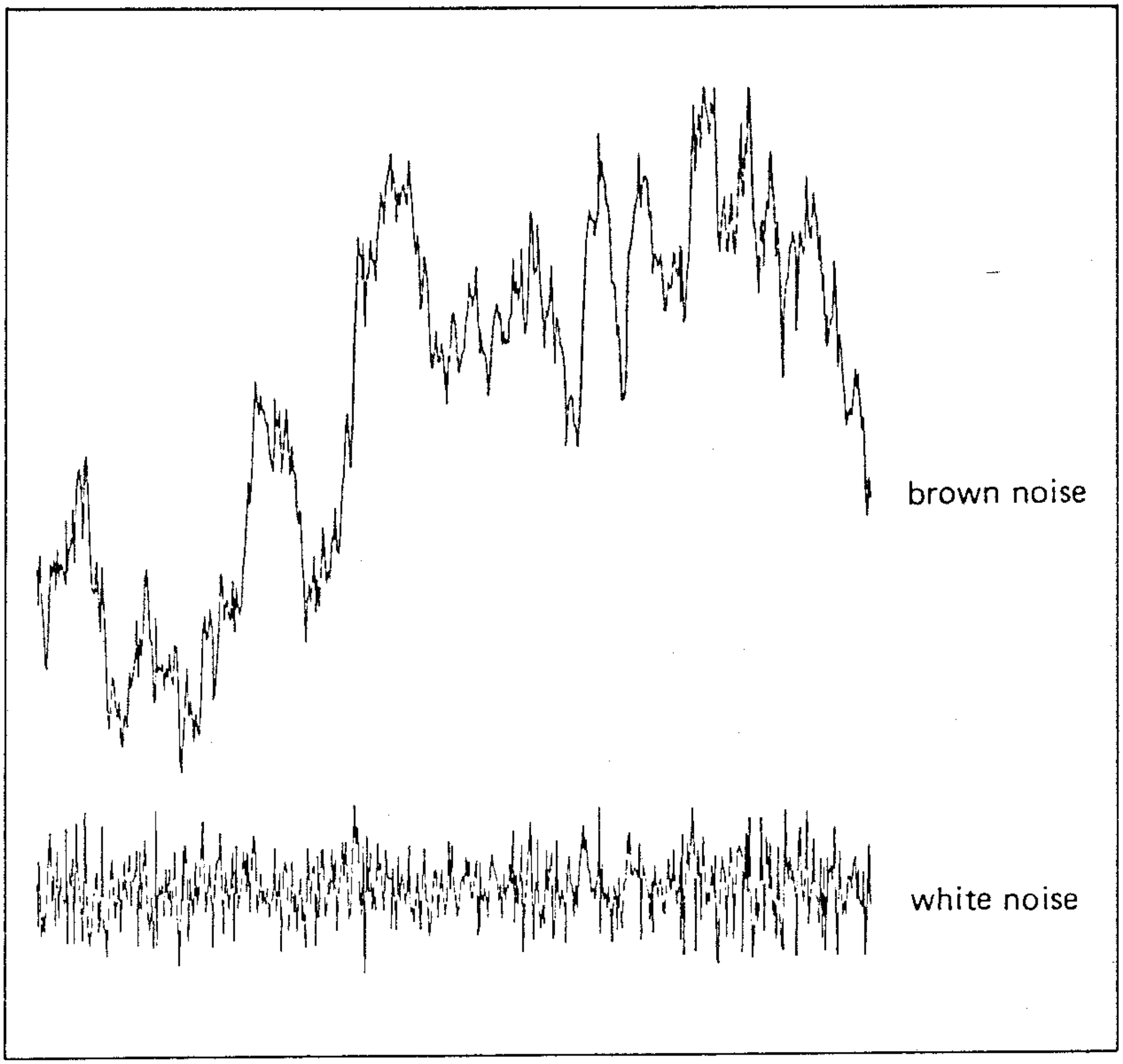
- algorithm to return the Gaussian random number

ALGORITHM	**Gauss** ()	
Title	Function returning Gaussian random number	
Globals	Nrand	number of samples of rand() to be taken in Gauss()
	GaussAdd	real parameter for the linear transformation in Gauss()
	GaussFac	real parameter for the linear transformation in Gauss()
Locals	sum	real
	i	integer
Functions	rand()	system function for random numbers

```
BEGIN
    sum := 0
    FOR i := 1 TO Nrand DO
        sum := sum + rand ()
    END FOR
    RETURN (GaussFac * sum - GaussAdd)
END
```

One-dimensional Brownian motion, cont.

- **Brownian motion in one dimension**



One-dimensional Brownian motion, cont.

- Generating Gaussian random numbers

  - Gauss() returns a sample of a random variable with normal distribution (mean 0 and variance 1)

  - a routine rand(), assumed to be available, returns random numbers uniformly distributed over some interval [0,A]

    - typically, A is $2^{31} - 1$ or $2^{15} - 1$

  - srand(seed), also assumed to exist, introduces a seed value for rand()

  - taking certain linearly scaled averages of the values returned by rand() approximates a Gaussian random variable as follows

    - a random variable Y is standardized by subtracting its expected value and dividing by its standard deviation

      $Z = (Y - E(Y))/(s.d.\ Y)$

    - if Z_n is the standardized sum of any n identically distributed random variables, then the probability distribution of Z_n tends to the normal distribution as n approaches infinity

One-dimensional Brownian motion, cont.

- if Y_i is the ith value returned by rand() i := 1 to n, then

$$E(Y_i) = \frac{1}{2}A, \text{ var } Y_i = \frac{1}{12}A^2$$

and

$$E\left(\sum_{i=1}^{n} Y_i\right) = \frac{n}{2}A, \text{ var}\left(\sum_{i=1}^{n} Y_i\right) = \frac{n}{12}A^2$$

and

$$Z_n = \frac{\sum_{i=1}^{n} Y_i - \frac{n}{2}A}{\sqrt{\frac{n}{12}}A} = \frac{1}{A}\sqrt{\frac{12}{n}}\sum_{i=1}^{n} Y_i - \sqrt{3n}$$

which is the approximate Gaussian variable

- n = 3 or 4 yields satisfactory results

One-dimensional Brownian motion, cont.

- random midpoint displacement

  - random midpoint displacement produces Brownian motion

  - the process is computed for times t between 0 and 1

    - start by setting X(0) = 0 and by selecting X(1) as a sample of a Gaussian random variable with mean 0 and variance sigma^2 = var(X(1) - X(0))

    - then

$$\operatorname{var}(X(t_2) - X(t_1)) = |t_2 - t_1|\sigma^2$$

for $0 \leq t_1 \leq t_2 \leq 1$

    - set X(1/2) to be the average of X(0) and X(1) plus some Gaussian random offset D_1 with mean 0 and variance delta_1^2

    - then

$$X(\frac{1}{2}) - X(0) = \frac{1}{2}(X(1) - X(0)) + D_1$$

    - both X(1/2) - X(0) and X(1) - X(1/2) have mean value 0

    - also

$$\operatorname{var}(X(\frac{1}{2}) - X(0)) = \frac{1}{4}\operatorname{var}(X(1) - X(0)) + \Delta_1^2 = \frac{1}{2}\sigma^2$$

    - therefore

$$\Delta_1^2 = \frac{1}{4}\sigma^2$$

- continuing

$$X(\frac{1}{4}) - X(0) = \frac{1}{2}(X(0) + X(\frac{1}{2})) + D_2$$

- and both X(1/2) - X(1/4) and X(1/4) - X(0) are Gaussian with mean 0

- for

$$\mathrm{var}(X(\frac{1}{4}) - X(0)) = \frac{1}{4}\,\mathrm{var}(X(\frac{1}{2}) - X(0)) + \Delta_2^2 = \frac{1}{4}\sigma^2$$

to hold

$$\Delta_2^2 = \frac{1}{8}\sigma^2$$

- in general

$$\Delta_n^2 = \frac{1}{2^{n+1}}\sigma^2$$

as the variance of the displacement of D_n

- corresponding to time differences delta t = 2^{-n}, a random element of variance $2^{-(n+1)}$sigma^2 is added, which is proportional to delta t