Finance - Independent Project

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1 Analytical Perspective

Financial markets are a canonical example of stochastic processes; their analysis should not be grounded solely in outcome space observations or first-principle reasoning applied directly to realized price paths. Nor should it rely on a fixed sample space, even though such a perspective is mathematically convenient. Financial time series are inherently non-cyclical: they evolve forward in time, do not repeat exact configurations, and continuously generate new structural conditions. From this perspective, a more natural object of study is not individual outcomes, but the evolution of the sample space itself. Specifically, this motivates analyzing how the geometry of the sample space changes over time, how dominant directions reorganize, and how many effective degrees of freedom are present at each stage.

Note: The framework and diagnostics were developed independently and are not adapted from an existing published model.

2 Mathematical Framework

I. State-Space Construction

Definitions. Let $\{P_t\}_{t=0}^{T}$ denote the asset price process and define returns by

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \qquad t = 1, \ldots, T.$$

Fix an embedding dimension $L \in N$. For each $t \geq L$, define the time-delay embedded state

$$x_t = \left(r_{t-L+1}, r_{t-L+2}, \ldots, r_t\right)^\top \in R^L.$$

Fix a rolling window length $H \in N$. At time t, define the rolling mean

$$\mu_t = \frac{1}{H} \sum_{i=1}^{H} x_{t-i},$$

and the centered embedded vectors

$$\tilde{x}_{t-i} = x_{t-i} - \mu_t, \qquad i = 1, \dots, H.$$

Define the rolling empirical covariance matrix

$$\Sigma_t = \frac{1}{H} \sum_{i=1}^{H} \tilde{x}_{t-i} \tilde{x}_{t-i}^\top \in R^{L \times L}.$$

Explanation and relevance: This section constructs a local state space for the return process using time-delay embedding and summarizes its second-order geometric structure through a rolling covariance matrix. Each embedded state x_t is analyzed relative to a local neighborhood formed by the preceding vectors $\{x_{t-H}, \dots, x_{t-1}\}$, which captures the geometric directions and dispersion structure immediately prior to time t. Because consecutive embedded vectors share overlapping returns, this local state space evolves gradually over time, reflecting the incremental nature of financial dynamics. All subsequent structural diagnostics are derived from Σ_t, which serves as the fundamental object encoding the evolving geometry of the system at time t.

II. Subspace Rotation and Structural Instability

Definitions. Let $\lambda_{t,1} \geq \lambda_{t,2} \geq \cdots \geq \lambda_{t,L} \geq 0$ denote the eigenvalues of Σ_t with corresponding orthonormal eigenvectors $u_{t,1}, \dots, u_{t,L}$.
Fix a subspace dimension $k \leq L$. Define the dominant PCA subspace basis

$$Q_t = [\, u_{t,1}, u_{t,2}, \dots, u_{t,k} \,] \in R^{L \times k}, \qquad Q_t^\top Q_t = I_k.$$

For consecutive time steps, define

$$M_t = Q_{t-1}^\top Q_t \in R^{k \times k}.$$

Let $\sigma_{t,1}, \dots, \sigma_{t,k}$ denote the singular values of M_t.
Define the subspace rotation metric

$$R_t = 1 - \frac{1}{k} \sum_{i=1}^{k} \sigma_{t,i}.$$

Explanation and relevance: The columns of Q_t span the dominant variance subspace of the local covariance structure at time t. Rather than treating these directions as fixed factors, the project interprets them as geometric objects evolving over time. The singular values of $Q_{t-1}^\top Q_t$ encode the principal angles between consecutive subspaces, providing a coordinate-free measure of how the dominant structure reorients. The rotation metric R_t therefore quantifies structural instability as motion on the Grassmann manifold, independent of scale or basis choice.

III. Structural Complexity via Effective Dimensionality

Definitions. Define the normalized eigenvalues

$$p_{t,i} = \frac{\lambda_{t,i}}{\sum_{j=1}^{L} \lambda_{t,j}}, \qquad i = 1, \ldots, L.$$

Define the spectral entropy

$$H_t = -\sum_{i=1}^{L} p_{t,i} \log p_{t,i}.$$

Define the entropy-based effective dimensionality

$$D_{\text{eff}}(t) = \exp(H_t).$$

For visualization, define smoothed quantities over a window W:

$$\bar{R}_t = \frac{1}{W} \sum_{j=0}^{W-1} R_{t-j}, \qquad \bar{D}_{\text{eff}}(t) = \frac{1}{W} \sum_{j=0}^{W-1} D_{\text{eff}}(t-j).$$

Explanation and relevance: While subspace rotation captures how dominant directions change, effective dimensionality captures how many directions are active. The entropy-based formulation provides a smooth, scale-invariant notion of intrinsic dimensionality that reflects whether variance is concentrated in a small number of modes or distributed across many directions. Together, R_t and $D_{\text{eff}}(t)$ describe complementary aspects of structure: orientation stability and spectral complexity. Their joint evolution characterizes regime-level changes in the geometry of the embedded return dynamics.

3 Empirical Illustrations

We empirically illustrate the proposed methodology using two examples: (1) the S&P 500 Index and (2) the EUR/USD currency pair. Both examples are analyzed using the same inputs: $L = 60$ (embedding dimension), $H = 20$ (rolling window length), and $k = 3$ (dominant PCA subspace dimension), as well as the same date range, 2000–2025.

I. Interpretation of Structural Extremes

In the empirical illustrations that follow, figures show the asset price series with structural extremes overlaid for interpretability. Specifically, the top 2.5% of subspace rotation values and the bottom 2.5% of effective dimensionality values are marked directly on the price series, using the same thresholds across all assets.

Subspace rotation is examined in the upper tail because large values indicate rapid reorientation of the sample-space geometry, reflecting unusually fast changes in the dominant variance directions. Effective dimensionality is examined in the lower tail because reductions in dimensionality indicate that variance collapses into fewer dominant modes, corresponding to structurally compressed regimes.

These thresholds are used solely for visualization and interpretation. The methodology itself does not rely on thresholding, event labeling, or predictive assumptions.

II. Plots and Analysis

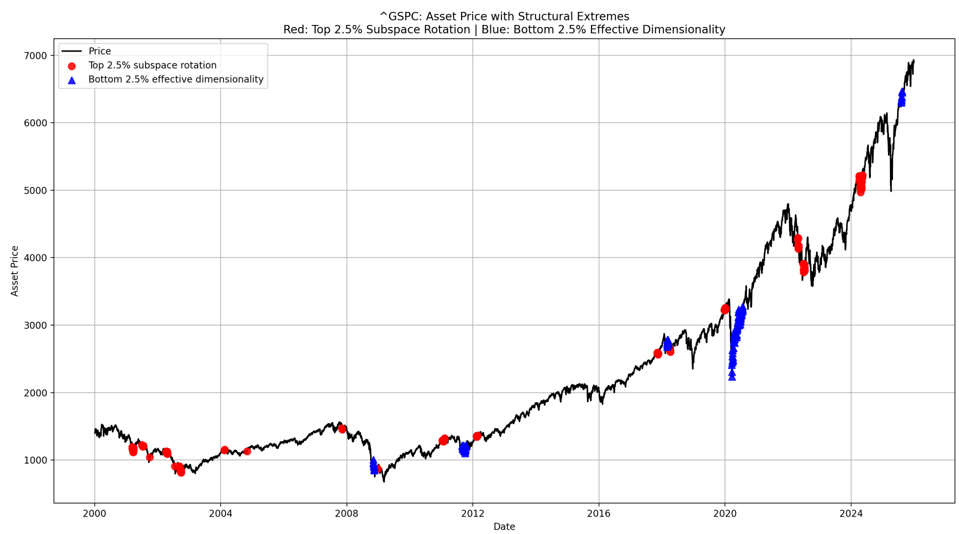

Figure 1: S&P 500 Index with structural extremes marked. Structural signals are shown for interpretation only and are not used for prediction.

The methodology defined above is applied to S&P 500 Index over the period 2000–2025, using the procedure defined in the Mathematical Framework. No asset-specific tuning or predictive assumptions are introduced; the analysis is intended as a structural illustration rather than a forecasting exercise.

In Figure 1, periods of elevated subspace rotation coincide with well-known episodes of structural reorganization in the market, such as the dot-com bubble in the early 2000s and the global financial crisis of 2008. During these periods, the dominant variance directions of the embedded return states reorient more rapidly than usual, indicating changes in the internal geometric structure of the system rather than changes in volatility alone.

Figure 1 also highlights periods of low effective dimensionality, most notably during the post-2008 recovery (around 2009) and the post-2020 period. In these phases, the reduction in effective dimensionality indicates that market variance is dominated by a smaller number of active directions, consistent with structurally compressed regimes in which fewer factors govern system behavior.

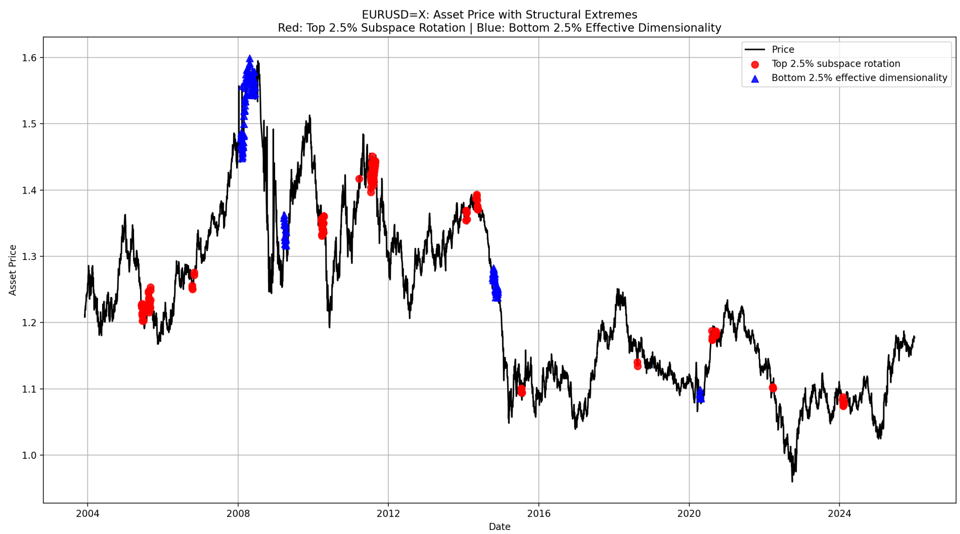

Figure 2: EUR/USD Currency Pair with structural extremes marked. Structural signals are shown for interpretation only and are not used for prediction.

The application of the methodology to the EUR/USD exchange rate over the period 2000–2025, using the same procedure defined for the SP 500 Index.

In Figure 2, periods of elevated subspace rotation coincide with well-known phases of structural reorganization in foreign exchange markets, particularly during the pre-2008 period and 2010–2012. During these periods, the dominant variance directions of the embedded return states reorient more rapidly than usual, reflecting changes in the underlying geometric structure of the exchange rate dynamics rather than changes in price direction alone.

Figure 2 also highlights periods of low effective dimensionality, most notably during the 2008–2009 crisis period and during subsequent strong directional regimes such as 2014–2015 and the post-2020 environment. In these phases, variance becomes concentrated in a smaller number of dominant modes, consistent with structurally compressed dynamics in which fewer macroeconomic factors govern system behavior.

4 Further Interests and Development of Methodology

Future development of the methodology will focus on refining the notion of subspace rotation as a measure of structural change. In the current formulation, subspace rotation is quantified using the singular values of $Q_{t-1}^{\top} Q_t$, which encode principal angles between consecutive dominant subspaces and provide a coordinate-free distance measure on the Grassmann manifold. While this approach is well-founded geometrically, it is not clear whether this metric is optimal for capturing all relevant aspects of structural instability in high-dimensional time series. Exploring alternative geometric distances or curvature-aware metrics on Grassmannian manifolds is a natural direction for further study.

A second direction for future work involves extending the framework beyond second-order structure. The current analysis is based on rolling covariance matrices, which capture variance geometry but ignore higher-order dependencies. Incorporating nonlinear or information-theoretic structures such as kernelized embeddings or higher-order moment tensors could allow the methodology to detect more subtle forms of structural change while preserving its geometric interpretation. Investigating how such extensions interact with effective dimensionality and subspace evolution is of particular interest.