2016.

一. 1. $\frac{1}{3} C_5^2 \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^3$

2. $P(A|B)=1$, $0<P(A)<1$, $0<P(B)<1$, $\Rightarrow P(AB)=P(B)$

$$P(\bar{B}|\bar{A})=\frac{P(\bar{B}\cdot\bar{A})}{P(\bar{A})}=\frac{P(\overline{A\cup B})}{1-P(A)}=\frac{1-P(A)-P(B)+P(AB)}{1-P(A)}=1$$

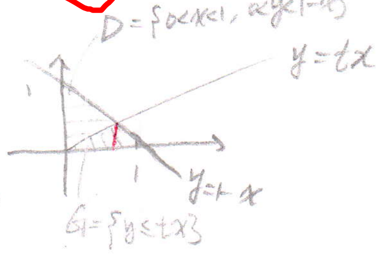
3. 由几何分布无记忆性. $X\sim G(p)$

$$P(X>6|X>3)=P(X>3)$$

又由例2.2.8 $P(X>3)=q^3$, 则 $P(X>6|X>3)=q^3=(1-p)^3$

二. $X\sim P(6)$, $Y|X=n \sim B(n,\frac{1}{3})$,

类似于例3.2.6的解题过程，可得 $Y\sim P(6\times\frac{1}{3})$ $P(2)$



三. (1) $f_X(x)=\int_{-\infty}^{+\infty} f(x,y)dy=\int_0^{1-x} 2dy=2(1-x)$, $0<x<1$,

$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)dx=\int_0^{1-y} 2dx=2(1-y)$, $0<y<1$,

(2) $Z=\frac{Y}{X}$: $(0,+\infty)$

当 $t<0$, $F_Z(t)=0$;

当 $t\geq 0$, $F_Z(t)=P(Z\leq t)=P(\frac{Y}{X}\leq t)=\iint_{G\cap D} f(x,y)dydx=\int_0^{\frac{1}{t+1}}\int_0^{tx} 2dydx+\int_{\frac{1}{t+1}}^{1}\int_0^{1-x} 2dydx$

$$=\int_0^{\frac{1}{t+1}} 2tx\,dx+\int_{\frac{1}{t+1}}^{1} 2(1-x)dx=\frac{t}{1+t}$$

$$f_Z(t)=\frac{dF_Z(t)}{dt}=\begin{cases}0 & 其他\\ \frac{1}{(1+t)^2}, & t\geq 0\end{cases}$$

四. 设球心为 (x_0,y_0,z_0)

$$(X-x_0)^2+(Y-y_0)^2+(Z-z_0)^2=1$$

$$E(X-x_0)^2+E(Y-y_0)^2+E(Z-z_0)^2=1$$

由对称性, $E(X-x_0)^2=\frac{1}{3}$, 又 $X\sim U(x_0-1,x_0+1)$, $E(X)=x_0$,

则 $D(X)=E(X-x_0)^2=\frac{1}{3}$

五. $\ln X \sim N(0, 4)$

设 $Y=\ln X$, 则 $X^3=(e^Y)^3=e^{3Y}$

$$E(X^3)=E(e^{3Y})=\int_{-\infty}^{+\infty} e^{3y}\frac{1}{\sqrt{2\pi}\,2}e^{-\frac{y^2}{8}}dy \underset{(u=\frac{y}{2})}{=} e^{18}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}(u-6)^2}du=e^{18}$$

六. $L(\theta)=\prod_{i=1}^{n}\frac{1}{2\theta}\exp[-\frac{|x_i|}{\theta}]$,

(1) $\ln L(\theta)=\sum_{i=1}^{n}[-\ln 2\theta-\frac{|x_i|}{\theta}]$, $\frac{d\ln L(\theta)}{d\theta}=\sum_{i=1}^{n}[-\frac{1}{\theta}-|x_i|(-1)\frac{1}{\theta^2}]=0$,

$\hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}|x_i|$, 则 $\frac{1}{n}\sum_{i=1}^{n}|X_i|$ 为 θ 的最大似然估计量.

(2) $E[\frac{1}{n}\sum_{i=1}^{n}|X_i|]=E[|X|]=\int_{-\infty}^{+\infty}|x|\frac{1}{2\theta}\exp[-\frac{|x|}{\theta}]dx=2\int_0^{\infty}x\frac{1}{2\theta}e^{-\frac{x}{\theta}}dx$

$$=\int_0^{\infty}\frac{x}{\theta}e^{-\frac{x}{\theta}}dx \underset{\substack{u=\frac{x}{\theta}\\(\theta>0)}}{=} \theta\cdot\int_0^{\infty}ue^{-u}du=\theta\cdot\Gamma(2)=\theta$$

$\frac{1}{n}\sum_{i=1}^{n}|X_i|$ 为 θ 的无偏估计量.

七 (1). 设 $E(X)=\mu$, $D(X)=\sigma^2$, 由 $X\sim N(\mu,\sigma^2)$ 及单正态总体抽样分布定理有

$$\frac{\bar{X}-\mu}{S/\sqrt{n}}\sim t(n-1)$$

$$P(-t_{\alpha/2}(n-1)\le\frac{\bar{X}-\mu}{S/\sqrt{n}}\le t_{\alpha/2}(n-1))=1-\alpha$$

$$P(\bar{X}-\frac{S}{\sqrt{n}}t_{\alpha/2}(n-1)\le\mu\le\bar{X}+\frac{S}{\sqrt{n}}t_{\alpha/2}(n-1))=1-\alpha$$

$$[\bar{X}-\frac{S}{\sqrt{n}}t_{\alpha/2}(n-1),\ \bar{X}+\frac{S}{\sqrt{n}}t_{\alpha/2}(n-1)]=[50.41-\frac{2.12}{4}2.1315,\ 50.41+\frac{2.12}{4}2.1315]$$
$$=[49.28,\ 51.54]$$

(2) $H_0:\sigma=2$ $H_1:\sigma\ne2$ 即 $H_0:\sigma^2=4$ $H_1:\sigma^2\ne4$.

$$\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$$

当 H_0 为真时, $\frac{(n-1)S^2}{4}\sim\chi^2(n-1)$

$$P(\frac{(n-1)S^2}{4}>\chi^2_{\alpha/2}(n-1) 或 \frac{(n-1)S^2}{4}<\chi^2_{1-\alpha/2}(n-1))=\alpha$$

拒绝域为 $(0,\ \chi^2_{1-\alpha/2}(n-1))\cup(\chi^2_{\alpha/2}(n-1),+\infty)=(0,\ 6.262)\cup(27.488,+\infty)$

$\frac{(n-1)S^2}{4}=\frac{15\times2.12^2}{4}=16.854<6.262$, 落入拒绝域,

在显著性水平 0.05 下, 拒绝 H_0

2014

一、$P(A_i比A_j先发生) = \sum_{k=1}^{n} P(A_i比A_j先发生|A_k)P(A_k)$

$= P(A_i) + \sum_{k\neq i,j} P(A_i比A_j先发生|A_k)P(A_k)$

$= P(A_i) + \sum_{k\neq i,j} P(A_i比A_j先发生)P(A_k)$

$= p_i + P(A_i比A_j先发生)(1-p_i-p_j)$

(类似于例1.6.5)

$P(A_i比A_j先发生) = \dfrac{p_i}{p_i+p_j}$

二、例2.2.9

三、(1) $\int_{-\infty}^{+\infty} f(x)dx = \int_{-1}^{0} a\,dx + \int_{0}^{2} b\,dx = a+2b=1$

$E(X) = \int_{-\infty}^{+\infty} x f(x)dx = \int_{-1}^{0} ax\,dx + \int_{0}^{2} bx\,dx = a\frac{x^2}{2}\Big|_{-1}^{0} + b\frac{1}{2}x^2\Big|_0^2 = -\frac{a}{2}+2b=1$

$\begin{cases} a+2b=1 \\ -\frac{a}{2}+2b=1 \end{cases}$ $b=\frac{1}{2}$, $a=0$

$f(x)=\begin{cases} \frac{1}{2}, & 0\le x<2 \\ 0, & 其他 \end{cases}$

(2) $Y=X^2$: $[0,4)$

当$y<0$, $F_Y(y)=0$; 当$y\geq 4$, $F_Y(y)=1$;

当$0\le y<4$, $F_Y(y)=P(Y\le y)=P(X^2\le y)=P(-\sqrt{y}\le X\le\sqrt{y})=\int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx$, ⊛

$0\le\sqrt{y}<2$, $-2\le-\sqrt{y}<0$, 则 ⊛ $=\int_0^{\sqrt{y}} \frac{1}{2}dx = \frac{\sqrt{y}}{2}$

$f_Y(y)=\begin{cases} 0, & 其他 \\ \frac{1}{4\sqrt{y}}, & 0<y<4 \end{cases}$

四

(1) $X \sim G(p)$

$$P(X=n, Y=m) = p\, q^{n-1} \cdot p\, C_{m-n-1}^{1}\, p\, q^{m-n-2} = (m-n-1)\, p^3 q^{m-3}$$

$m \geq n+2$

(2) $X \sim G(p)$, $P(X=n) = q^{n-1} p$, $n=1,2,\cdots$

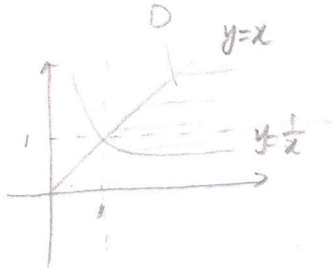
$$P(Y=m) = \sum_{n=1}^{m-2} P(Y=m, X=n) = \sum_{n=1}^{m-2} (m-n-1)\, p^3 q^{m-3} = p^3 q^{m-3} \frac{(m-2)(1+m-2)}{2}$$

$$= p^3 q^{m-3} \frac{(m-2)(m-1)}{2}$$

$m = 3, 4, \cdots$

五.

(1)



$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_{\frac{1}{x}}^{x} \frac{1}{2x^2 y}\,dy = \frac{1}{2x^2} \ln y \Big|_{\frac{1}{x}}^{x}$$

$$= \frac{1}{2x^2}\left(\ln x - \ln \frac{1}{x}\right) = \frac{\ln x}{x^2}, \quad x > 1.$$

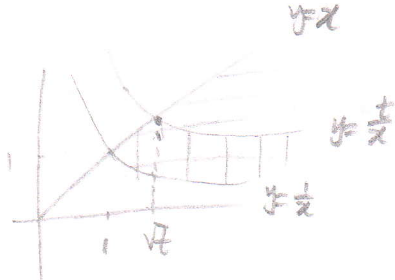
$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \begin{cases} \int_{\frac{1}{y}}^{+\infty} \frac{1}{2x^2 y}\,dx, & 0<y<1 \\ \int_{y}^{+\infty} \frac{1}{2x^2 y}\,dx, & 1 \leq y < \infty, \end{cases}$$

$$= \begin{cases} \frac{1}{2y}\left(-\frac{1}{x}\right)\Big|_{\frac{1}{y}}^{+\infty} = \frac{1}{2y} y = \frac{1}{2}. & 0<y<1 \\ \frac{1}{2y}\left(-\frac{1}{x}\right)\Big|_{y}^{+\infty} = \frac{1}{2y^2}, & 1 \leq y < \infty. \end{cases}$$

(2) $Z = XY$: $(1, +\infty)$

当 $t<1$, $F_Z(t) = 0$;

当 $t \geq 1$, $F_Z(t) = P(XY \leq t) = \iint_{G_t D} f(x,y)\,dy\,dx = \int_1^{\sqrt{t}} \int_{\frac{1}{x}}^{x} f(x,y)\,dy\,dx + \int_{\sqrt{t}}^{\infty} \int_{\frac{1}{x}}^{\frac{t}{x}} f(x,y)\,dy\,dx$



$$= \int_1^{\sqrt{t}} \int_{\frac{1}{x}}^{x} \frac{1}{2x^2 y}\,dy\,dx + \int_{\sqrt{t}}^{\infty} \int_{\frac{1}{x}}^{\frac{t}{x}} \frac{1}{2x^2 y}\,dy\,dx$$

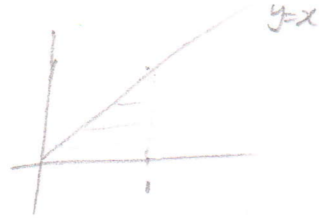
$G_t = \{yx \leq t\}$

$$= 1 - \frac{1}{\sqrt{t}} - \frac{\ln\sqrt{t}}{\sqrt{t}} + \frac{\ln t}{2\sqrt{t}} = 1 - \frac{1}{\sqrt{t}}$$

$$f_Z(t) = \begin{cases} \frac{1}{2} t^{-\frac{3}{2}}, & t \geq 1 \\ 0, & t < 1 \end{cases}$$

六. $E(X^2)=\int_0^1\int_0^x x^2\cdot 12y^2\,dy\,dx$

$=\int_0^1 4x^2\, y^3\Big|_0^x dx=\int_0^1 4x^5\,dx=\frac{4}{6}$



七. $E(X)=\frac{-\theta+\theta}{2}=0$ $D(X)=\frac{4\theta^2}{12}=\frac{\theta^2}{3}$

$E(X^2)=D(X)+(E(X))^2=\frac{\theta^2}{3}$

令 $E(X^2)=A_2$, $\frac{\theta^2}{3}=A_2$ $\hat{\theta}=\sqrt{3A_2}=\sqrt{3\frac{1}{n}\sum_{i=1}^{n}X_i^2}$

$E(\hat{\theta}^2)=E(\frac{3}{n}\sum_{i=1}^{n}X_i^2)=3E(X^2)=3\cdot\frac{\theta^2}{3}=\theta^2$

八. (1) $\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$

$P(\chi^2_{1-\alpha/2}(n-1)\le\frac{(n-1)S^2}{\sigma^2}\le\chi^2_{\alpha/2}(n-1))=1-\alpha$, $P(\frac{(n-1)S^2}{\chi^2_{\alpha/2}(n-1)}\le\sigma^2\le\frac{(n-1)S^2}{\chi^2_{1-\alpha/2}(n-1)})=1-\alpha$,

$[\frac{(n-1)S^2}{\chi^2_{\alpha/2}(n-1)},\frac{(n-1)S^2}{\chi^2_{1-\alpha/2}(n-1)}]$, $[\frac{24\times2.5^2}{39.364},\frac{24\times2.5^2}{12.401}]=[3.81,12.10]$

(2) H_0: $\mu=80$ H_1: $\mu\ne80$

$\frac{\bar{X}-\mu}{S/\sqrt{n}}\sim t(n-1)$

当H_0为真时, $\frac{\bar{X}-80}{S/\sqrt{n}}\sim t(n-1)$.

$P(|\frac{\bar{X}-80}{S/\sqrt{n}}|>t_{\alpha/2}(n-1))=\alpha$

$(-\infty,-t_{\alpha/2}(n-1))\cup(t_{\alpha/2}(n-1),+\infty)$ 为拒绝域. $(-\infty,-2.0639)\cup(2.0639,+\infty)$ 为拒绝域

$\frac{\bar{x}-80}{S/\sqrt{n}}=\frac{78.25-80}{2.5/5}=-3.5$ 落入拒绝域.

在显著性水平0.05下. 拒绝H_0