Synchronization of Coupled Oscillators

Millennium Bridge from Kuramoto Model

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1. Introduction

Synchronization is a fundamental collective phenomenon in nonlinear dynamic systems, in which a population of interacting oscillators spontaneously develops the coherent behavior. Despite differences in their intrinsic frequencies, weak coupling can induce a transition form incoherence to partial phase locking. This transition is one of the simplest and most striking examples of self-organization in large dynamical systems.

The Kuramoto model provides a minimal mean-field framework for studying this effect, meaning that it makes each oscillator interact with the mean field generated with the whole population rather than with every other oscillator individually. It describes a population of phase oscillators with distributed natural frequencies and all-to-all sinusoidal coupling. As the coupling strength increases, the system undergoes a continuous phase transition characterized by the emergence of a nonzero order parameter. The critical coupling strength separating incoherence from synchronization can be predicted analytically and verified numerically.

In this project, we investigate the onset of synchronization through numerical integration of the Kuramoto model and compare the results with theoretical predictions for the critical coupling. We then apply the same synchronization mechanism to the Millennium Bridge model, where the interaction between pedestrians and structural motion produces a feedback instability. Through controlled numerical experiments, we demonstrate how collective phase locking leads to oscillations and validate the theoretical thresholds derived from the model.

2. Numerical integration of the Kuramoto Model

The Kuramoto Model

The Kuramoto model describes a population of N weakly coupled phase oscillators with distributed natural frequencies. In the all-to-all symmetric coupling case, the dynamics are given by

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \qquad i = 1, \ldots, N, \tag{1}$$

where $\theta_i \in [0, 2\pi)$ is the phase of the i-th oscillator, ω_i is its natural frequency, and $K > 0$ is the coupling strength.

The natural frequencies are assumed to be drawn from a unimodal and symmetric distribution $g(\omega)$, as discussed in [1]. In the mean-field formulation, the interaction term can be expressed in terms of the complex order parameter

$$r(t)e^{i\Psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}, \tag{2}$$

where $r(t) \in [0, 1]$ measures the degree of synchronization and $\Psi(t)$ denotes the average phase of the population.

Using this definition, the system can be rewritten as

$$\dot{\theta}_i = \omega_i + Kr \sin(\Psi - \theta_i), \tag{3}$$

which shows that each oscillator interacts with the mean field generated by the entire population. As shown in [1], the model exhibits a continuous phase transition: for coupling strengths below a critical value K_c, the incoherent state $r = 0$ is stable, whereas for $K > K_c$, a partially synchronized state with $r > 0$ emerges.

Subcritical Regime ($K < K_c$)

For coupling strength K=1, which lies below the theoretical critical value Kc, the order parameter r(t) exhibits persistent fluctuations without converging to a stable nonzero value. Although an initial transient increase is observed due to the random initial condition, this growth does not stabilize. Instead, the system settles into a fluctuating regime where r(t) oscillates around small values.

This behavior reflects the stability of the incoherent state predicted in Strogatz [2]. In this regime, the dispersion of natural frequencies dominates over the coupling term. While weak interactions may temporarily bring small groups of oscillators closer in phase, these clusters are unstable and dissolve over time. As a result, no macroscopic phase coherence emerges.

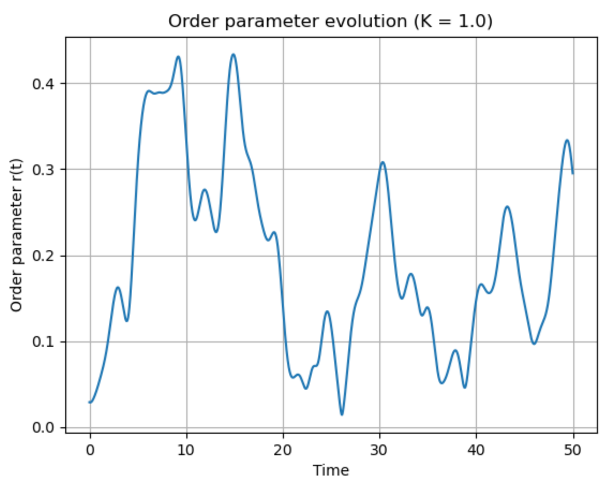

Figure 1: Order parameter below the critical coupling.

Subcritical Regime($K < K_c$)

When the coupling strength exceeds the critical threshold, the qualitative dynamics change dramatically. For K=3 and K=7, the order parameter grows rapidly from its initial small value and converges toward a stable plateau close to unity. This indicates the emergence of strong macroscopic synchronization.

The rapid increase of r(t) reflects the instability of the incoherent state in the supercritical regime. As soon as a small amount of coherence develops, the mean field strengthens, which in turn enhances the attractive coupling between oscillators. This positive feedback mechanism drives further phase alignment. The system therefore evolves toward a partially synchronized state in which a large fraction of oscillators rotate with a common collective frequency.

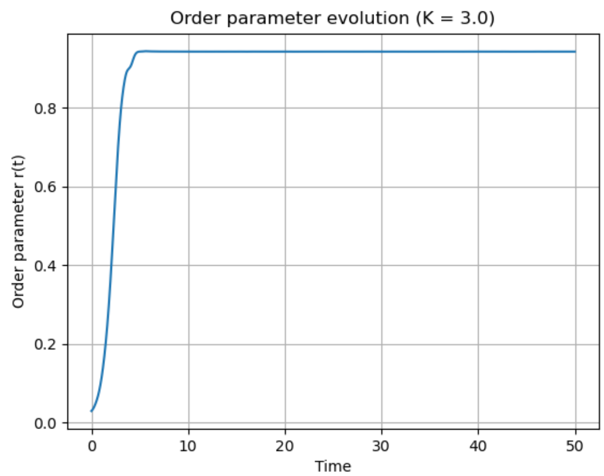

Figure 2: Order parameter above the critical coupling.

Side by side comparison

The side-by-side comparison between K=1 and K=7 clearly demonstrates the existence of a synchronization threshold. Both simulations begin from comparable random initial conditions, yet their long-term behavior differs qualitatively.

For K=1, the system remains incoherent, with only small finite-size fluctuations in r(t). For K=7, strong coherence develops and persists.

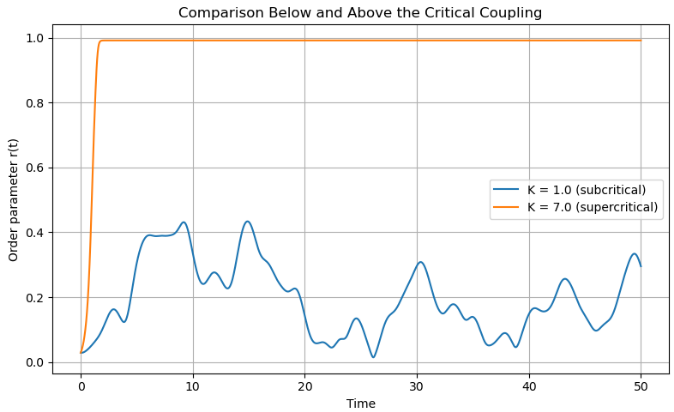

Figure 3: Comparison of the order parameter below and above the critical coupling.

This confirms the presence of a continuous phase transition, as described in [2]. The transition is not merely quantitative (i.e., stronger coupling producing slightly more alignment), but structural: the stability properties of the incoherent state change when the coupling crosses Kc. The order parameter evolves smoothly from near zero to a finite value, consistent with a supercritical bifurcation.

Oscillator Representation on the Circle

Subcritical Regime ($K < K_c$)

In the subcritical case, the oscillator distribution remains approximately uniform around the unit circle at all times. Although small transient deformations of the distribution can be observed, no persistent angular concentration develops.

Geometrically, the vectors corresponding to individual oscillators point in many different directions. When averaged, they nearly cancel out, resulting in a centroid close to the origin and a small value of r. This visual structure reflects statistical incoherence: the oscillators rotate at their natural frequencies without developing stable phase relationships.

The absence of clustering confirms that the incoherent state is dynamically stable in this regime, as predicted in [2]. Weak coupling slightly perturbs the phase distribution but is insufficient to overcome the dispersion in natural frequencies.

Supercritical Regime ($K < K_c$)

In contrast, the supercritical case exhibits a clear and progressive angular collapse of the oscillator distribution. Initially, the phases are nearly uniformly distributed. As time evolves, a visible cluster begins to form. Eventually, the majority of oscillators concentrate within a narrow angular sector.

In the rotating reference frame , the synchronized cluster becomes stationary. This is important: synchronization does not mean that oscillators stop moving, but rather that they rotate collectively with a common frequency. Removing the global rotation reveals true phase locking.

The geometric collapse of the distribution corresponds directly to the growth of the order parameter. As more oscillators align, the centroid moves away from the origin and r increases. In the strongly supercritical regime, the cluster becomes sharply localized, indicating near-complete phase coherence.

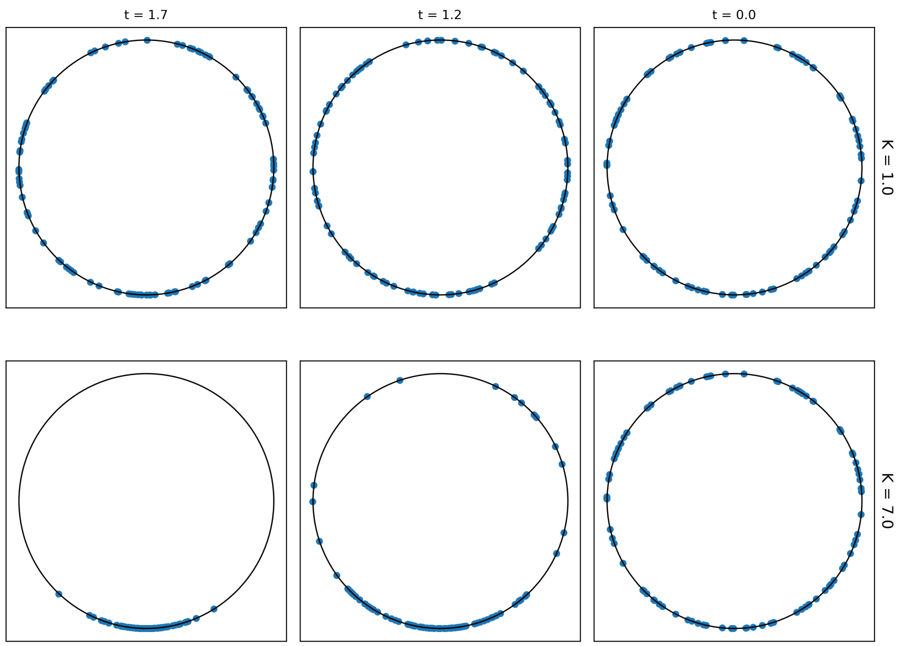

Figure 4: Comparison of oscillators with different K.

The circle snapshots reveal the microscopic process underlying the macroscopic transition. Synchronization emerges gradually rather than abruptly. Oscillators with natural frequencies close to the mean are entrained first. As the mean field strengthens, it pulls additional oscillators into the synchronized cluster.

3. Prediction and Numerical Verification of the Critical Coupling

To determine the critical coupling strength, the steady-state value of the order parameter r_∞ as a function of the coupling parameter K was computed. For each value of K, the system was integrated until it reached its saturation regime, and the order parameter was averaged over the final portion of the simulation to reduce fluctuations. The resulting values are displayed in the bifurcation diagram.

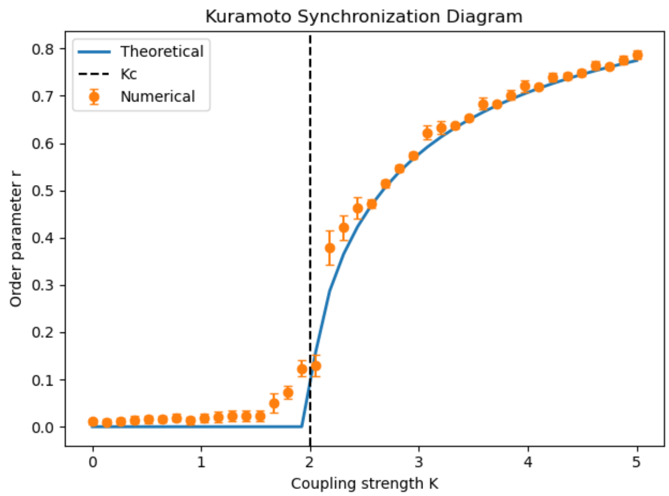

Figure 5: Bifurcation Diagram

In this part of the analysis, the natural frequencies were sampled from a Cauchy (Lorentzian) distribution,

$$g(\omega) = \frac{1}{\pi(1+\omega^2)}. \tag{4}$$

For this distribution, the self-consistency equation derived in [1] can be solved explicitly. The critical coupling is given by

$$K_c = \frac{2}{\pi g(0)}. \tag{5}$$

Since $g(0) = \frac{1}{\pi}$ for the Cauchy distribution, we obtain

$$K_c = 2. \tag{6}$$

Moreover, for $K > K_c$, the order parameter satisfies the explicit formula

$$r_\infty(K) = \sqrt{1 - \frac{K_c}{K}}, \tag{7}$$

while $r_\infty = 0$ for $K \leq K_c$.

The numerical bifurcation diagram confirms this prediction. For $K < 2$, the steady-state order parameter remains close to zero, indicating incoherence. As K crosses the theoretical threshold $K_c = 2$, a nonzero value of r_∞ emerges and increases smoothly with K. The dashed vertical line in the figure marks the theoretical critical coupling, and the numerical onset of synchronization occurs in close agreement with this value.

The agreement between the theoretical curve and the numerical data is particularly strong in this case because the Cauchy distribution allows an exact analytical solution of the self-consistency equation. The error bars represent finite-size fluctuations in the numerical simulations. Below the threshold, the small nonzero values arise purely from finite-size effects, while above K_c the fluctuations decrease as the synchronized cluster becomes more robust.

These results provide quantitative validation of the theoretical prediction and confirm that the synchronization transition in the Kuramoto model is continuous and accurately described by the mean-field analysis.

4. Millenium Bridge

Model Description

To model the Millennium Bridge phenomenon, we consider the coupled system introduced in [2]. The bridge is represented as a damped harmonic oscillator,

$$M\ddot{X}(t) + B\dot{X}(t) + KX(t) = \sum_{i=1}^{N} G \sin \theta_i(t), \tag{8}$$

where $X(t)$ denotes the lateral displacement of the bridge, M is the modal mass, B the damping coefficient, and K the stiffness. Each pedestrian contributes a periodic lateral force of amplitude G.

The pedestrians are modeled as phase oscillators whose stepping phases satisfy

$$\dot{\theta}_i = \Omega_i + CA(t) \sin\big(\Psi(t) - \theta_i + \alpha\big), \tag{9}$$

where $A(t)$ and $\Psi(t)$ are the instantaneous bridge amplitude and phase, respectively. The coupling parameter C measures the sensitivity of pedestrians to bridge motion.

The degree of synchronization is quantified by the order parameter

$$R(t)e^{i\Phi(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}. \tag{10}$$

As in the Kuramoto model, $R(t)$ measures crowd coherence.

According to [2], the critical crowd size is given by

$$N_c = \frac{2B\Omega}{\pi G C P(\Omega)}, \tag{11}$$

where $P(\Omega)$ is the pedestrian frequency distribution evaluated at the bridge frequency.

Crowd Ramp Experiment

To test this prediction, a crowd ramp experiment was performed numerically. The simulation begins with a small number of pedestrians and increases the crowd size in discrete increments at fixed time intervals. The critical time t_c is defined as the first time at which $N(t) \geq N_c$.

The stacked plots show the evolution of:

- the crowd size $N(t)$,
- the bridge amplitude $A(t)$,
- the synchronization level $R(t)$.

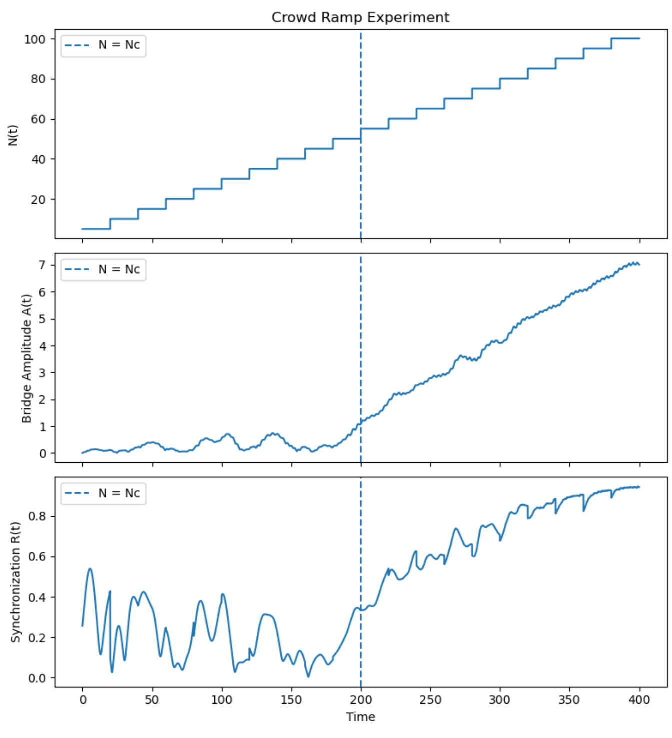

Figure 6: Crowd ramp

Before the critical crowd size is reached ($N < N_c$), the bridge amplitude remains small and fluctuates near zero. Although small oscillations occur due to random pedestrian forcing, damping prevents sustained growth. The synchronization measure $R(t)$ remains moderate and irregular, indicating incoherent pedestrian phases.

As the crowd size approaches the predicted threshold, a qualitative change occurs. Once $N(t)$ exceeds N_c, both the bridge amplitude and the synchronization measure begin to increase systematically. The growth of $R(t)$ indicates that pedestrians are becoming phase-aligned. This increased coherence enhances the collective lateral forcing, which in turn amplifies bridge motion.

The simulation therefore reproduces the positive feedback mechanism described in [2]: bridge motion promotes pedestrian synchronization, and synchronized pedestrians reinforce bridge motion. The onset of large-amplitude oscillations occurs close to the theoretically predicted value of N_c, confirming the validity of the analytical threshold.

Geometric Interpretation

The snapshots of the pedestrian phases provide a microscopic view of this transition. At early times, the pedestrians' phases are scattered and no coherent pattern is visible. As the crowd size increases beyond the threshold, the phase distribution becomes increasingly aligned. This angular concentration corresponds to the growth of the order parameter $R(t)$.

The simultaneous increase of $A(t)$ and $R(t)$ demonstrates that synchronization and structural wobbling emerge together. The instability is therefore not purely mechanical, but arises from the interaction between human gait dynamics and structural motion.

Overall, the numerical experiment confirms that the Millennium Bridge instability can be interpreted as a synchronization transition in a coupled oscillator–structure system.

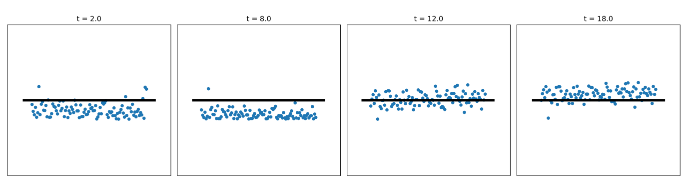

Figure 7: Snapshots of pedestrian phases and bridge displacement at increasing crowd sizes. Phase alignment becomes visible as the crowd exceeds the critical threshold.

5. Conclusion

In this project, we investigated the emergence of synchronization in systems of coupled oscillators through numerical simulations of the Kuramoto model and its application to the Millennium Bridge instability.

For the Kuramoto model, the numerical results confirmed the theoretical prediction of a critical coupling strength separating incoherent and synchronized regimes. The bifurcation diagram showed excellent agreement with the analytical solution in the case of the Cauchy distribution, including the correct prediction of the critical coupling $K_c = 2$ and the continuous onset of the order parameter. Both time-series analysis and geometric phase representations illustrated the microscopic mechanism of phase locking and its macroscopic manifestation in the order parameter.

In the Millennium Bridge model, the numerical crowd ramp experiment reproduced the predicted instability threshold. The onset of large-amplitude bridge oscillations occurred near the theoretical critical crowd size, and the growth of structural motion coincided with the emergence of pedestrian synchronization. This confirmed that the bridge wobbling can be interpreted as a synchronization transition in a coupled oscillator–structure system.

AI Usage and GitHub

To complete this assignment we made use of AI assistance for coding purposes and for language editing of our text.

The code was used to implement the models and create the visualizations found throughout the project can be found at the following GitHub link: https://github.com/candelaberzal/synchronization-of-coupled-oscillators

References

[1] S. H. Strogatz, *From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators*, Physica D 143 (2000), 1–20.

[2] S. H. Strogatz, D. M. Abrams, A. McRobie, B. Eckhardt, and E. Ott, *Crowd synchrony on the Millennium Bridge*, Nature 438 (2005), 43–44.