PRICING AUTOCALLABLES UNDER LOCAL-STOCHASTIC VOLATILITY

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Motivation

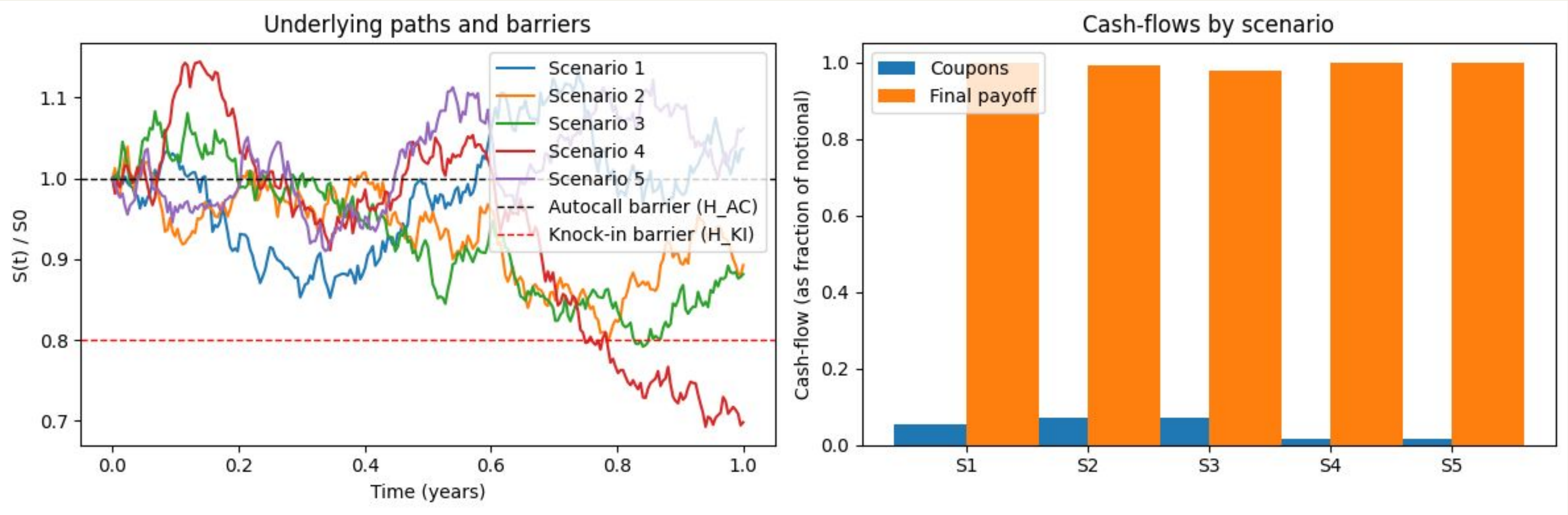
- Autocallable notes = large share of equity structured products
- Strong path-dependence: barriers + coupons
- Market practice: pricing mostly under **Local Volatility (LV)**
- Question: *How different are prices under LV vs Local–Stochastic Volatility (LSV)?*
- Impact on **model risk**, coupon levels and risk management

What is an autocallable (ABRC)?

- Notional N, *maturity* $\boldsymbol{T^E}$, *observation dates* $\boldsymbol{T_i}$
- *Autocall barrier* H^{AC}: $\boldsymbol{early\ redemption\ N + coupon}$
- *Knock-in barrier* H: $\boldsymbol{activation\ of\ short\ put}$

Final payoff:

- *no knock-in* $\rightarrow \boldsymbol{N + coupons}$
- knock-in & $S_T < K \ \rightarrow$ $\boldsymbol{loss\ proportional\ to\ (K - S_T)}$



Numerical example: ABRC

Notional: \$1,000

Underlying: Stock XYZ, initial price = \$100

Maturity: 12 months

Observation dates: T_1=3m, T_2=6m, T_3=9m, T_4=12m

Autocall barrier HAC: 100% → \$100

Knock-in barrier H: 70% → \$70

Coupon: 10% annual → paid only if autocall occurs

Final payoff:

-- If no knock-in → \$1,000 + coupon

-- If knock-in → receive shares (10 shares) + no coupon

Scenario 1: Autocall at T_2 (6 months)

Price at T_1 (3m): \$95 < \$100 → no autocall

Price at T_2 (6m): \$105 ≥ \$100 → autocall triggered!

You get:

- \$1,000 (notional)
- \$50 (half-year coupon)

→ **Total:** \$1,050

→ Exit early — no risk of knock-in

Scenario 2: Knock-in at T_3, no autocall

Price at T_1 (3m): \$90 → no autocall

Price at T_2 (6m): \$85 → no autocall

Price at T_3 (9m): \$68 ≤ \$70 → knock-in activated!

Price at T_4 (12m): \$90 still < \$100 → no autocall

You get:

- 10 shares of XYZ → 10 × \$90 = \$900
- No coupon (since no autocall)

→ **Total:** \$900 → Loss of \$100

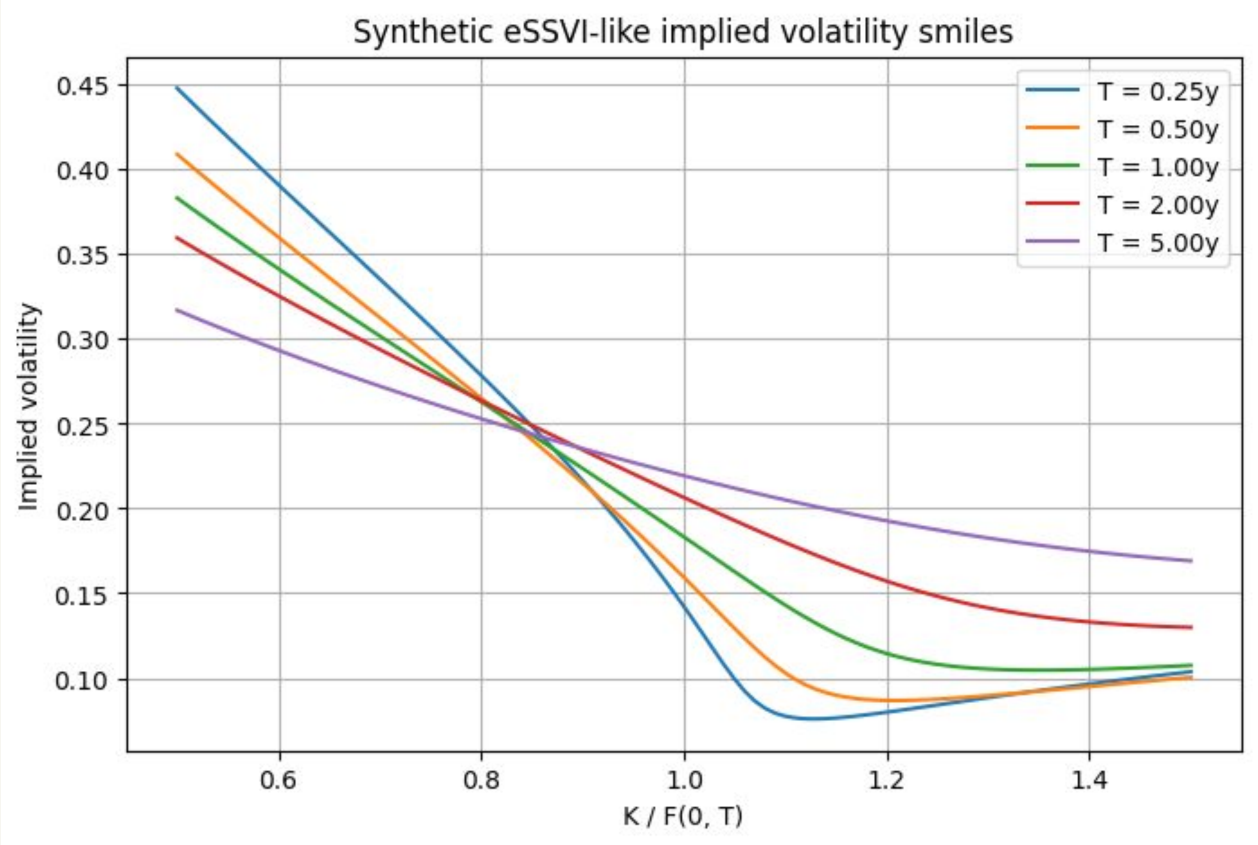
Local Volatility

- Dynamics:

$$dS_t = (r - q)S_t\,dt + \sigma_{LV}(t, S_t)S_t\,dW_t$$

Calibrated from market IV surface (Dupire)

- **Pros:**
  - *Exact fit to European option prices*
  - *PDE/tree friendly, static replication*
- **Cons:**
  - *Deterministic vol: no vol-of-vol*
  - *Unrealistic smile dynamics / forward smiles*



Heston-Stohastic volatility

- Dynamics:

$$dS_t = (r - q)S_t\,dt + \sqrt{V_t}S_t\,dW_t^S$$

$$dV_t = \kappa(\theta - V_t)\,dt + \eta\sqrt{V_t}\,dW_t^V$$

$$\mathrm{corr}\left(dW^S, dW^V\right) = \rho$$

- Parameters: $\kappa, \theta, \eta, \rho, V_0$
- **Pros:** random volatility, leverage effect, smile dynamics
- **Cons:** difficult exact calibration to full IV surface

Local-Stohastic volatility

- Hybrid model:

$$\boldsymbol{dS_t = (r-q)S_t\,dt + L(t,S_t)\sqrt{V_t}S_t\,dW_t^S}$$

- $\boldsymbol{V_t} - Heston - type\ variance$
- $\boldsymbol{L(t,S_t)} - leverage\ function\ correcting\ LV$
- Goal:
  - *Match European prices (as LV)*
  - *Keep realistic smile dynamics (as SV)*
- Two-step calibration:
  1. *IV surface → LV(Dupire / eSSVI)*
  2. *Heston +* $L(t,S_t)$ *from conditional expectations*

Gyöngy theorem & leverage function

- Gyöngy (1986): LV can reproduce one-dimensional marginals of SV model
- In LSV:

$$L(t,S_t)^2 = \frac{\sigma_{LV}(t,S)^2}{E^Q[V_t|S_t = S]}$$

- where by Dupire's formula:

$$\sigma_{LV}^2(t,S) = 2\left.\frac{\frac{\partial C}{\partial T} + \big(r(T) - q(T)\big)K\frac{\partial C}{\partial K} + q(T)C}{K\frac{2\partial^2 C}{\partial K^2}}\right|_{T=t,K=S}$$

- Implies:
  - *LSV and LV have same S_T distributions*
  - *→ same prices for all Europeans*

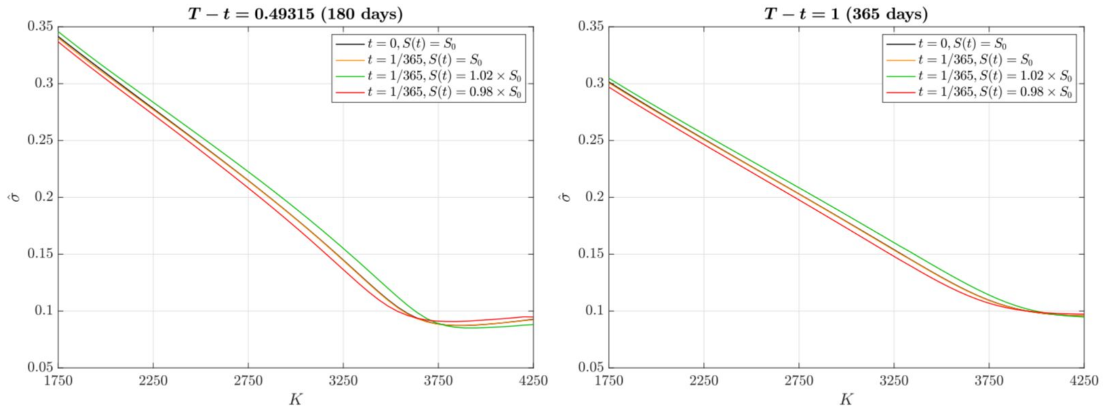


FIGURE 3. $T - t = 180$ and 365 days IV smile dynamics in the Heston LSV model calibrated to SPX options as of January 23, 2020. In black the current IV smile. In orange, green, and red the forward IV smiles resulting from different levels at time $t = 1/365$.

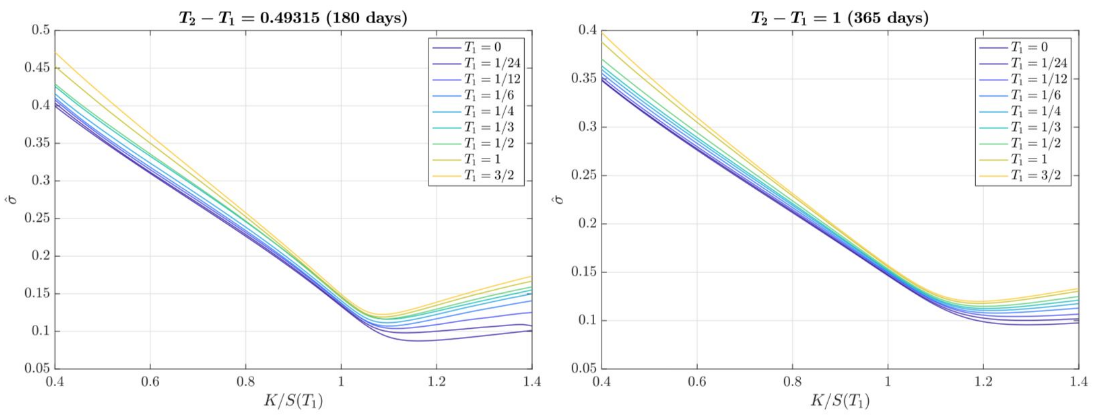


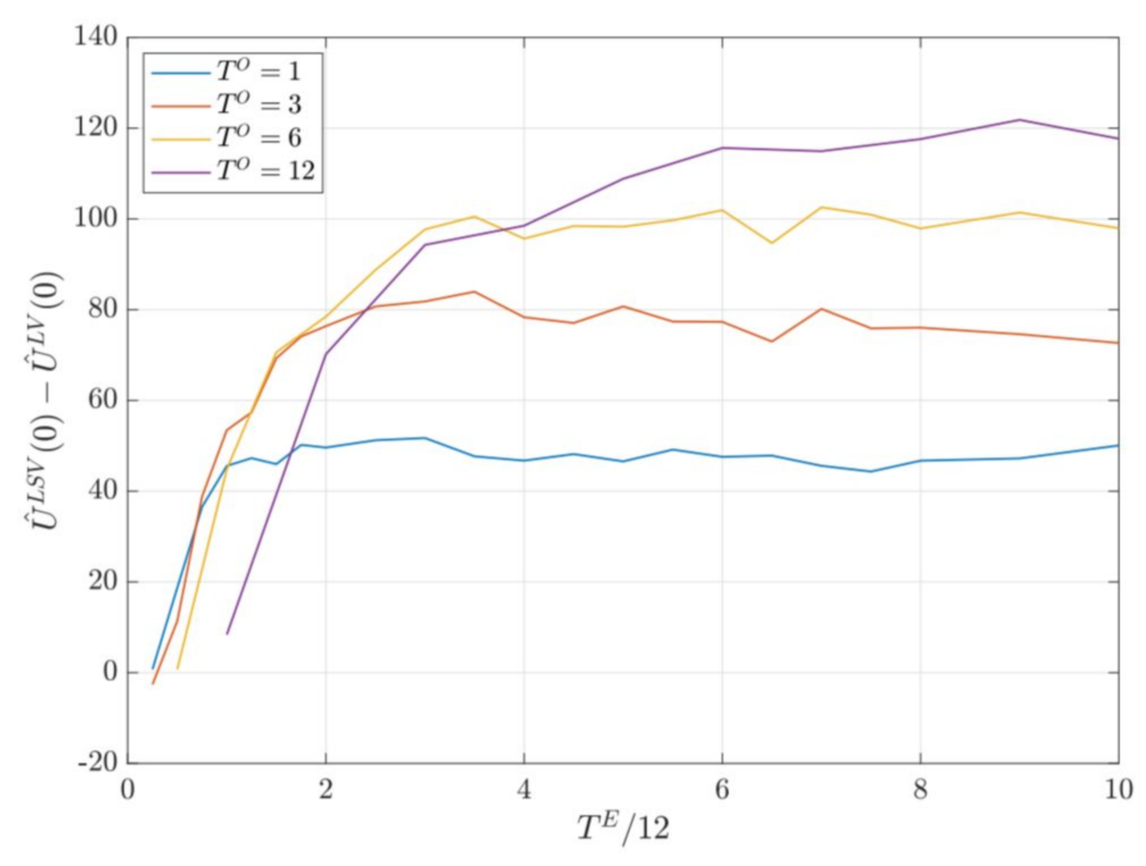
FIGURE 4. 180 and 365 days spot- and forward-starting implied volatility smiles generated by the Heston local-stochastic volatility model calibrated to SPX options as of January 23, 2020. T_1 denotes the forward start date and $T_2 - T_1$ the residual maturity.

Empirical LV vs LSV

- Data: SPX options (IvyDB), eSSVI IV surface
- LV vs Heston LSV, calibrated to same IV surface
- LV misprices:
- term structure of **forward-start smiles**
- ATMF skew dynamics
- LSV reproduces both current and forward IV smiles better

Pricing results (LV vs LSV)

- **Product:** single-asset autocallable BRC on SPX
- Compare prices $U^{LSV}(0)\ vs\ U^{LV}(0)$
- **Main findings:**
  - $U^{LSV} > U^{LV}$ *almost always*
  - *Differences up to **200-250 bps** of notional*
  - *Larger gaps for:*
    - Long maturities $\boldsymbol{T^E}$
    - Space observations $\boldsymbol{T^O}$
    - High coupons & coupons barriers
    - Strong vol-of-vol $\boldsymbol{\eta}$, negative $\boldsymbol{\rho}$



My replication: setup

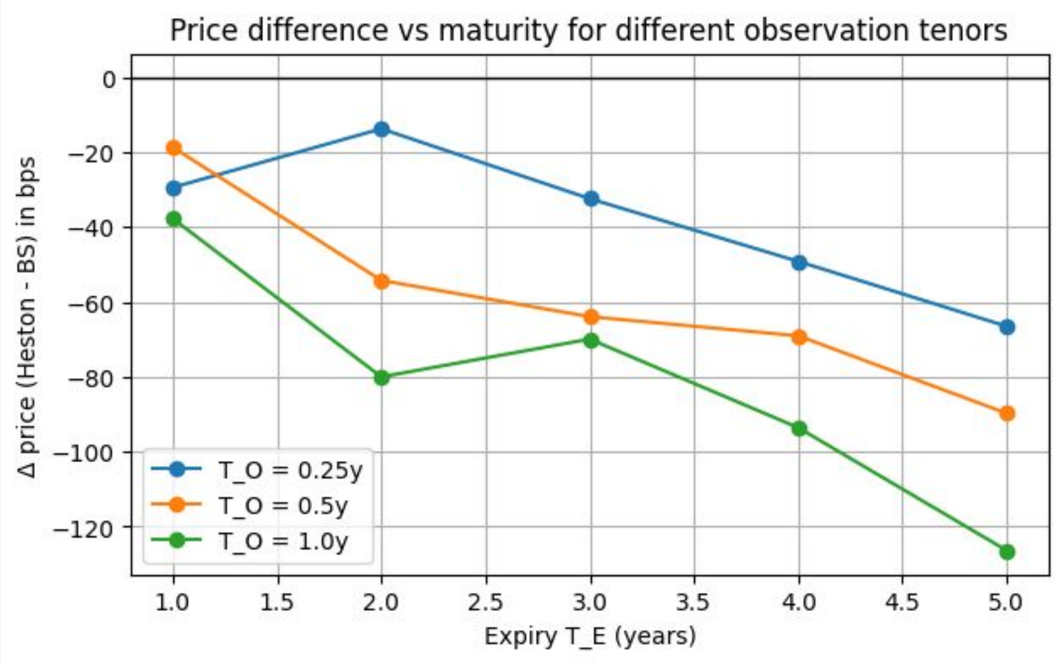
- LV proxy: **Black–Scholes** with constant vol $\sigma = 20\%$
- LSV-lite proxy: **Heston** with $(\boldsymbol{\kappa}; \boldsymbol{\theta}; \boldsymbol{\eta}; \boldsymbol{\rho}; \boldsymbol{V_0}) = (\mathbf{2.0}; \mathbf{0.04}; \mathbf{0.6}; -\mathbf{0.7}; \mathbf{0.04})$
- Synthetic eSSVI-like IV surface
- **Benchmark ABRC:**
  - $T^E = 1 - 5y;\ T^O = 0.25 - 1y$
  - $K = 90\%; H = 80\%;\ H^{AC} = 100\%$
  - $Coupons\ Y = 0 - 12\%, coupon\ barrier\ H^Y$

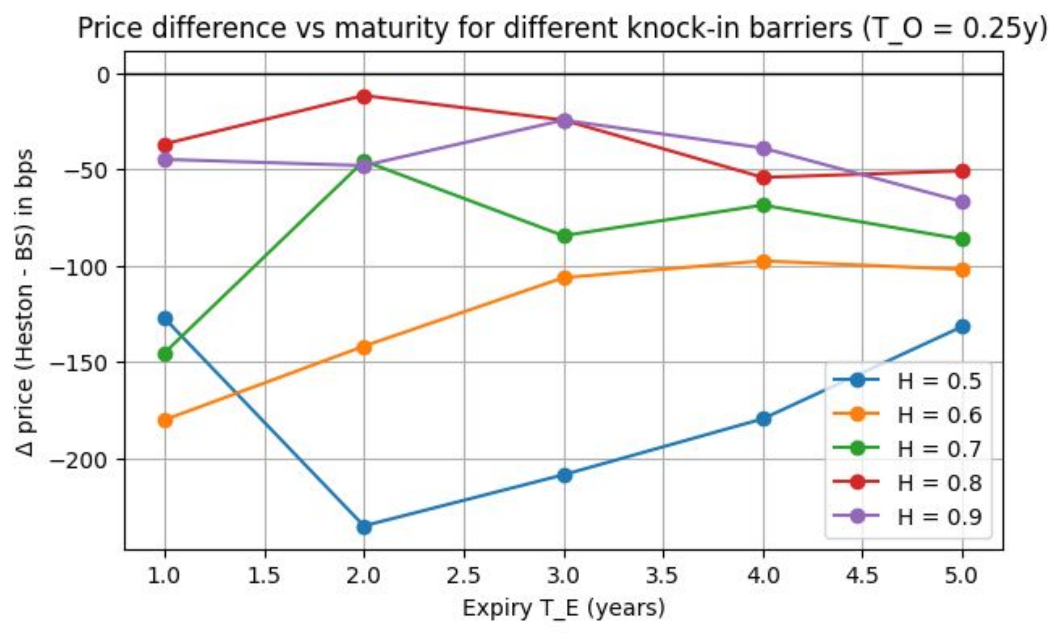
Replication: structure sensitivity I

- Price difference:

$$\Delta = U^{Heston} - U^{BS}$$

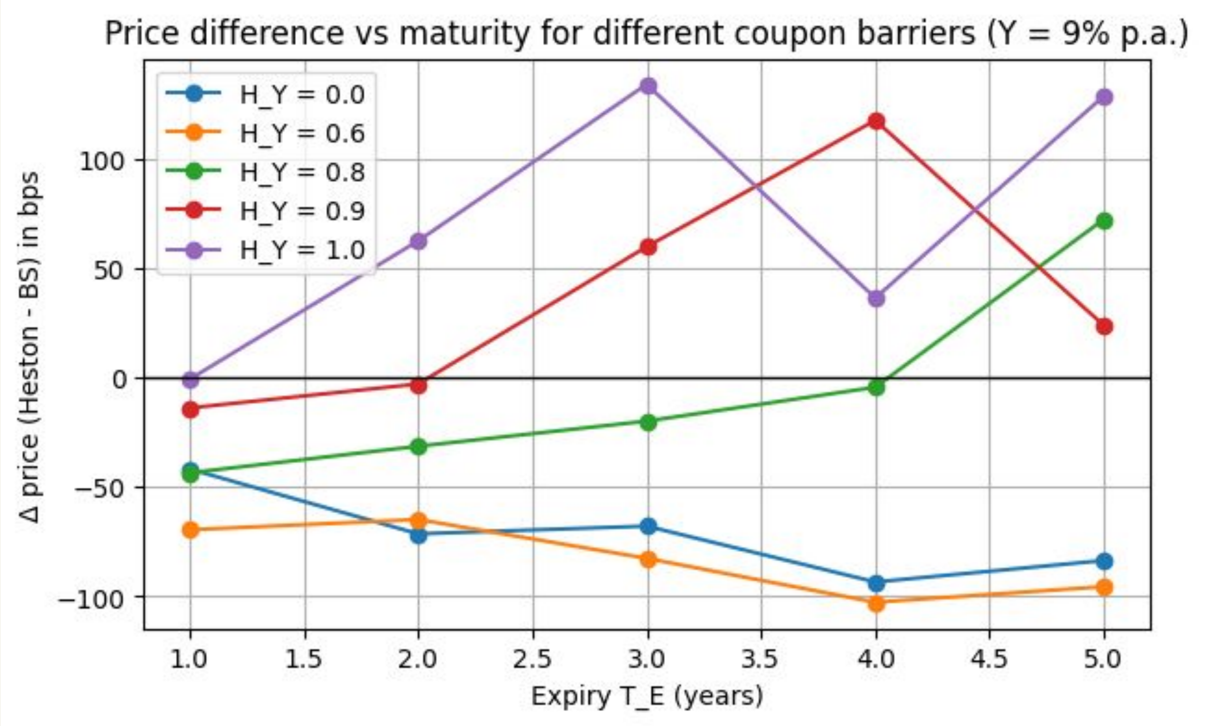
- Maturity & observation tenor:
  - $|\Delta|$ $increases\ with\ maturity\ T^E$
  - $|\Delta|$ $larger\ for\ rarer\ observations (0.5 - 1y\ vs\ 0.25y)$
- Knock-in barrier H:
  - *deep barriers (50–60%) → large |Δ| (up to ~200 bps)*
  - *higher barriers (80–90%) → smaller |Δ|*





Replication: structure sensitivity II

- **Coupon level $\boldsymbol{Y}$:**
  - $Y = 0\%: \Delta > 0$ ($Heston\ price\ higher$)
  - *Higher coupons (6-12%)* $\rightarrow \Delta < 0, |\Delta|$ *up to ~150-200 bps*
- **Coupon barrier $\boldsymbol{H^Y}$:**
  - $H^Y = 0: moderate\ |\Delta|\ (-40 \ldots -90\ bps)$
  - *High* H^Y $(90 - 100\%) \rightarrow$ *sign flip,* $\Delta > 0, up\ to \sim 100+$ *bps*
- **Interpretation:** coupons and coupon barriers strongly amplify model risk



Limitations of my replication

- BS vs Heston instead of full LV vs Heston–LSV
- No joint calibration to real IV surface (synthetic eSSVI only)
- No real market data (IvyDB SPX)
- Simplified product:
  - *no autocall coupon*
  - *simplified memory / single underlying*
- **Result:** sign of Δ may differ from article, focus on **qualitative** patterns

Takeways & Conclusion

- Autocallables are highly model–sensitive (barriers + coupons + path dependence)
- LV calibrated to vanillas may significantly misprice ABRCs
- LSV-type models (or at least SV) better capture **vol-of-vol** and **forward smile risk**
- Structural features that drive model risk:
  - *maturity & observation frequency*
  - *knock-in & coupon barriers*
  - *coupon level*
- My replication confirms strong dependence of prices on model choice and structure, even in a simplified BS vs Heston setting

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