PS7 due 10/24

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7-1 textbook problem 5.82

$f_{xy} = kx(1-x)y$ where $k = 12$ from prev HW

ⓐ find $f_Y(y|x) = \begin{cases} \frac{f_Y(x)}{P_r(x)} & 0<x<1 \\ 0 & ew \end{cases}$

From PS6: $f_X(x) = 6x(1-x)$ $\quad 0<x<1$ $\qquad f_Y(y) = 2y$ $\quad 0<y<1$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} \qquad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$$

$$\boxed{f_{Y|X}(y|X) = \frac{\cancel{12}^{2} \cancel{x(1-x)} y}{\cancel{6} \cancel{x(1-x)}} = 2y \qquad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}}$$

ⓑ $E[Y|X=x] = \int_0^1 y \, f_{Y|X}(y|x)\,dy$

$$= \int_0^1 2y^2\,dy = \frac{2y^3}{3}\Big]_0^1 = \frac{2}{3}$$

ⓒ $E[Y] = \int_0^1 y \, f_Y(y)\,dy$

$$= \int_0^1 2y^2\,dy = \frac{2}{3}$$

ⓓ done on back side

ⓔ $E[XY] = = \int_0^1 \int_0^1 xy\,12x(1-x)y\,dx\,dy$

$$= \int_0^1 \int_0^1 12x^2y^2 - 12x^3y^2\,dx\,dy$$

$$= \int_0^1 \left[\frac{12x^2y^3}{3} - \frac{12x^3y^3}{3}\right]_{y=0}^{y=1} dx$$

$$= \int_0^1 4x^2 - 4x^3\,dx$$

$$= \frac{4x^3}{3} - \frac{4x^4}{4}\Big]_0^1 = \frac{4}{3} - 1 = \frac{1}{3} = E[XY]$$

(d) $E[XY \mid X = x] = \int_0^1 xy \, 2y \, dy$

$$E[XY \mid X = x] = x \cdot \left[\frac{2y^3}{3}\right]_0^1 = \frac{2x}{3}$$

$\boxed{7-2}$ 4.76

reward $Y = (X)^+ \quad X \sim N(\mu = 2, \sigma^2 = 4)$

$$(X^+) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$$

$$f_Y(y) = f_X(x \mid X \geq 0) = \begin{cases} \dfrac{f_X(x)}{Pr(X \geq 0)} , & x \geq 0 \\ 0 , & ew \end{cases}$$

$$= \begin{cases} \dfrac{\exp\left(\frac{-(x-2)^2}{8}\right)}{2\sqrt{2\pi}\left(1 - \Phi\left(\frac{-2}{4}\right)\right)} & x \geq 0 \\ 0 & ew \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{\exp\left(\frac{-(x-2)^2}{8}\right)}{2\sqrt{2\pi}\left(1 - \Phi\left(-\frac{1}{2}\right)\right)} , & X \geq 0 \\ 0 & ew \end{cases}$$

$\boxed{7\text{-}3}$ $E[Y] = aE[X] + b = a\mu + b = \mu'$

$var(Y) = a^2 Var(X) = a^2\sigma^2 = \sigma'^2$

$a = \frac{\sigma'}{\sigma} \longrightarrow \frac{\sigma'}{\sigma}\mu + b = \mu'$

$b = \mu' - \frac{\sigma'}{\sigma}\mu$

$\boxed{7\text{-}4}$ $f_X(x) = \begin{cases} k(2-x) & 0 \le x \le 2 \\ 0 , & ew \end{cases}$

(a)

$\int_0^2 k(2-x)\,dx = 1$ $\qquad F_X(x) = \int_0^x \frac{1}{2}(2-\alpha)\,d\alpha$

$\left[2x - \frac{x^2}{2}\right]_0^2 = \frac{1}{k}$ $\qquad = \frac{1}{2}\left[2\alpha - \frac{\alpha^2}{2}\right]_0^2$

$4 - 2 = \frac{1}{k} \Rightarrow k = \frac{1}{2}$ $\qquad F_X(x) = \frac{1}{2}\left[2x - \frac{x^2}{2}\right]$

(b) $Y = X^2 = g(x)$ $\qquad \left.\frac{dg(x)}{dx}\right]_{x=y^{1/2}} = 2x\Big]_{x=y^{1/2}}$

$y^{1/2} = x = g'(y)$ $\qquad = 2y^{1/2}$

$f_x(y^{1/2}) = \frac{1}{2}(2-\sqrt{y})$

$f_Y(y) = \begin{cases} \dfrac{\frac{1}{2}(2-\sqrt{y})}{2y^{1/2}} , & 0 \le y \le 4 \\ 0 , & ew \end{cases}$

$E[X^2] = \frac{2}{3} \rightarrow$ integral done on wolfram Alpha for verification.

$E[Y] = \int_0^4 \frac{1}{4}(2\sqrt{y} - y)\,dy$

$\frac{1}{4}\left[\frac{\cancel{4}2y^{3/2}}{3/\cancel{2}} - \frac{y^2}{2}\right]_0^4 = \frac{4^{3/2}}{3} - \frac{16}{8} = \frac{2}{3}$

7-5)

$$f_X(x) = \frac{ce^{-cx}}{(1+e^{-cx})^2} \qquad -\infty < x < \infty$$

$$g(x) = Y = \frac{1}{1+e^{-cx}}$$

$$y^{-1} = 1 + e^{-cx}$$

$$y^{-1} - 1 = e^{-cx}$$

$$\ln\left(\tfrac{1}{y} - 1\right) = -cx$$

$$\frac{-\ln\left(\tfrac{1}{y}-1\right)}{c} = X = g^{-1}(y)$$

$= -\infty$ @ $y < 0$

$0 < y < 1$ increasing

$= \infty$ @ $y > 1$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{\left.\frac{dg(x)}{dx}\right]_{x=g^{-1}(y)}} \qquad 0 < y < 1$$

$$\frac{dg(x)}{dx} = (1+e^{-cx})^{-1} = \left.\frac{ce^{-cx}}{(1+e^{-cx})^2}\right|_{x=g^{-1}(y)}$$

$$= \frac{c\exp\left[+\frac{\cancel{c}\ln\left(\frac{1}{y}-1\right)}{\cancel{c}}\right]}{\left(1+\exp\left[+\frac{\cancel{c}\ln\left(\frac{1}{y}-1\right)}{\cancel{c}}\right]\right)}$$

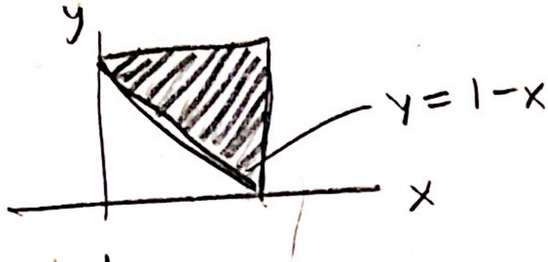
$$= \frac{c\left(\frac{1}{y}-1\right)}{1+\left(\frac{1}{y}-1\right)} = cy\left(\tfrac{1}{y}-1\right) = c(1-y)$$

$$f_X(g^{-1}(y)) = \frac{c\exp\left[-\frac{c\ln\left(\frac{1}{y}-1\right)}{c}\right]}{\left(1+e^{-c\left(\ln\frac{1}{y}-1\right)/c}\right)} = c(1-y)$$

$$\text{so } f_Y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{ew} \end{cases}$$

interesting b/c it maps f_X back to uniform density function

7-6 $f_{XY} = \begin{cases} 2 , & 0<x<1,\ 0<y<1,\ x+y \geq 1 \\ 0 , & ew \end{cases}$



(a) $E(Z) = E(XY) = \int_0^1 \int_{y=1-x}^1 2xy\,dy\,dx$

$$\int_0^1 \left[y^2\right]_{1-x}^1 x\,dx$$

$$1-(1-x)^2\ x\,dx$$

$$1-(1+x^2-2x)\ x\,dx$$

$$\int_0^1 2x^2 - x^3\,dx$$

$$\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{2}{3} - \frac{1}{4}$$

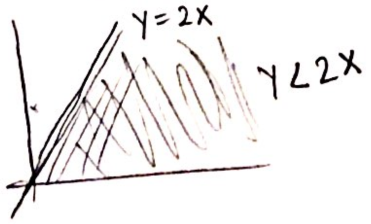
$$E[XY] = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$$

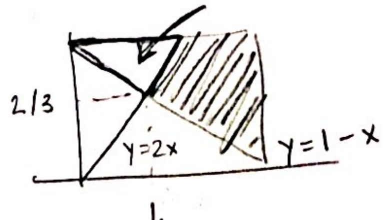
(b) $P(A) = P(X > 0.75)$

$$f_X(x) = \int_{1-x}^1 f_{xy}\,dy = \int_{1-x}^1 2\,dy = 2y\Big]_{1-x}^1 = 2x$$

$$\int_{0.75}^1 2x\,dx = x^2\Big]_{0.75}^1 = 1 - 0.75^2 = 0.4375$$

(c)





$2x = 1-x$
$3x = 1$
$x = 1/3$

Because f_{xy} doesn't vary w/ X or Y we can find the area $\overset{1/2}{\nabla}\,]\,1/3$ $\quad \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{12}$.

$$\iint_A f_{xy}\,dx\,dy = 2\cdot\frac{1}{12} = \frac{1}{6} = P(Y > 2X)$$

so $\boxed{P(2X > Y) = \frac{5}{6}}$

(d) done in (b) $f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & ew \end{cases}$

(e) $f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{2}{2x} = x^{-1}$

(f) $f_Y(y) = \int_{1-y}^{1} 2\,dx = 2y$

so $f_{XY} \neq f_X \cdot f_Y \longrightarrow$ NOT indep

(g) $E[Y \mid X = x] = \int_{1-x}^{1} y \cdot x^{-1}\,dy = \frac{1}{x}\left[\frac{y^2}{2}\right]_{1-x}^{1} = \frac{1}{2x}\left[1 - (1-x)^2\right]$

$= \frac{1}{2x}\left[\cancel{1} - (\cancel{1} + x^2 - 2x)\right] = \frac{1}{2x}\left[2x - x^2\right] = \frac{1}{2}\left[1 - x\right]$

(h) $f_{X|A}(x|A) = \begin{cases} \dfrac{f_X(x)}{P(A)}, & x \in A \\ 0, & ew \end{cases}$

$f_{X|A}(x|A) = \begin{cases} \dfrac{2x}{0.4375}, & x > 0.75 \\ 0, & ew \end{cases}$

(i) $f_{Y|A}(y|A) = \begin{cases} \dfrac{f_{Y|X}(y|x)}{P(A)}, & x > 0.75 \\ 0, & ew \end{cases}$

$f_{Y|A}(y|A) = \begin{cases} \dfrac{x^{-1}}{0.4375}, & x > 0.75 \\ 0, & ew \end{cases}$

$\boxed{7-7}$ 5.101 → cauchy

$X \sim N(0,1) \quad Y \sim N(0,1)$

show $Z = X/Y \sim$ cauchy RV

$$f_{XY} = \frac{e^{-\left(\frac{x^2}{2} + \frac{y^2}{2}\right)}}{2\pi} \qquad z = \frac{x}{y} \rightarrow zy = x$$

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f(zy, y)\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |y| e^{-\left(\frac{y^2}{2} + \frac{z^2y^2}{2}\right)} dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |y| e^{-y^2\left(\frac{1}{2} + \frac{z^2}{2}\right)} dy$$

$$= \frac{1}{\pi} \int_{0}^{\infty} y e^{-y^2\left(\frac{1}{2} + \frac{z^2}{2}\right)} dy$$

$$= \frac{1}{2\pi\left(\frac{1}{2} + \frac{z^2}{2}\right)} \left[0 - (-1)\right] =$$

$$f_Z(z) = \frac{1}{2\pi\left(\frac{1}{2} + \frac{z^2}{2}\right)} = \frac{1}{\pi(1+z^2)}$$

cauchy w/ $\alpha = 1$ ✓

7-8 $\quad X \sim N(\mu_1, \sigma_1^2) \quad Y \sim N(\mu_2, \sigma_2^2)$

From Example 7.3 pg 362 in textbook:

S_n = sum of independent Guassian RVs

w/ $\mu_1, \mu_2 \ldots \mu_n$ & $\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2$

then

$$\text{mean}(S_n) = \sum_{i=1}^{n} \mu_i \qquad \text{var}(S_n) = \sum_{i=1}^{n} \sigma_i^2$$

$$\mu_z = \mu_1 + \mu_2 \qquad \sigma_z^2 = \sigma_1^2 + \sigma_2^2$$

so

$$f_z(z) = \frac{\exp\left[\frac{-(z-\mu_z)^2}{2\sigma_z^2}\right]}{\sqrt{2\pi\sigma_z^2}}$$

$$f_z(z) = \frac{\exp\left[\frac{-(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}\right]}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2}}$$