Charlie André

PS11

11-1

$P(\text{good}) = 0.995$

ⓐ $P(\text{2 bad in 1000 chips}) = \binom{1000}{2}(0.995)^{998}(0.005)^2$

$= 0.0839 \quad = 8.39\%$ chance

ⓑ $P(r=2)$ Demoivre Laplace

$P(r=2, n=1000) \approx \frac{1}{\sqrt{2\pi npq}} \exp\left[\frac{-(r-np)^2}{2npq}\right]$

$P(r=2, n=1000)$ $\exp\left[\cancel{\frac{-(2-995)^2}{2(1000)(0.995)(0.005)}}\right]$

$= \frac{1}{\sqrt{2\pi(1000)(0.995)(0.005)}}$ $\cancel{\exp[-99100]}$ e^0

$= \frac{1}{5.59096}$ → NOT very close!

$= 0.1788 = 17.88\%$

$\cancel{= 0.1788} \quad \cancel{\int_{y=1.5}^{2.5} \exp\left[\frac{-(y-995)^2}{2000(0.995)(0.005)}\right] dy} \to e^0$

7.17 fair die tossed 20 times. eqn 20 to bound prob that tot # of dots $60 \to 80$

$\mu = 3.5 \qquad 20 \times 3.5 = 70$ ✓

$P[|M_n - \mu| < \epsilon] \geq 1 - \frac{\sigma^2}{n\epsilon^2}$

$\sigma_x^2 = \frac{1}{6}[1^2 + 2^2 + 3^2 \ldots] - 3.5^2$

$P[|M_n - 3.5| < \frac{1}{2}] \geq 1 - \frac{\sigma^2}{5}$

$\sigma_x^2 = 2.91666$

$P[|M_{20} - 3.5| < \frac{1}{2}] \geq 0.416667$

7.24 CLT

20 die tosses

$Pr(|\text{sum of dots} - 70| < 10) = ?$

use continuity correction near mean

$Pr(60 < S_n < 80) = ?$

$= \Phi\left(\frac{80 + 1/2 - \mu_{SN}}{\sigma_{SN}\sqrt{n}}\right) - \Phi\left(\frac{60 - 1/2 - \mu_{SN}}{\sigma_{SN}\sqrt{n}}\right)$

$\mu_{SN} = 70$ σ_{SN}^2 (from prev) $= 2.9166$

$= \Phi\left(\frac{9.5}{\sqrt{20}\,(2.917)}\right) - \Phi\left(\frac{-10.5}{\sqrt{20}\,(2.917)}\right)$

$= 0.8086$

11-4 text 8.1 (a) → (d)

(a) $X \sim N(\mu = 10, var = 4)$ $N = 9$

$\overline{X}_n \sim N(\mu = 10, var = \frac{4}{9})$

$Pr(\overline{X_N} < 9)$

$= \Phi\left(\frac{9-10}{\sqrt{4/9}}\right) = \Phi\left(\frac{-1}{2/3}\right) = 0.0668$

(b) $Pr(\min > 8) = Pr(\text{All } X_i\text{'s} > 8)$

$= Pr(X_i > 8)^n$

$= \left[1 - \Phi\left(\frac{8-10}{2}\right)\right]^9$

$= 0.2112$

(c) $Pr(\max < 12) = Pr(\text{All } X_i\text{'s} < 12) = Pr(X_i < 12)^9$

$= \left(\Phi\left(\frac{12-10}{2}\right)\right)^9 = 0.2112$

8.1d) n such that $P(|\overline{X}_n - \mu| < 1)$

$$P(|\overline{X}_n - 10| < 1) \geq 1 - \frac{\sigma_x^2}{n\,1^2} = 0.95$$

$$0.95 = 1 - \frac{\sigma_x^2}{n} \rightarrow \frac{\sigma_x^2}{n} = 0.05$$

$$\frac{\sigma_x^2}{0.05} = 20\,\sigma_x^2 = n$$

$$\sigma_x^2 = \frac{4}{9} \longrightarrow n = 8.88 \quad \text{so}$$

$n \geq 9$

n must be 9 or greater

11-5 $E[X] = \mu$, unknown σ_x^2

$$\hat{V} = \frac{1}{n}\sum_{j=1}^{n}(x_j - \mu)^2$$

$$= \frac{1}{n}\sum_{j=1}^{N}(x_j - \mu + \mu - \overline{X}_n)^2$$

$$= \frac{1}{n}\sum_{j=1}^{n}\left[(X_j-\mu)^2 + 2(X_j-\mu)(\mu-\overline{X}_n) + (\mu-\overline{X}_n)^2\right]$$

$$= \frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)^2 + \frac{2}{n}(\mu-\overline{X}_n)\sum_{j=1}^{n}(X_j-\mu) + \frac{1}{n}\sum_{j=1}^{n}(\mu-\overline{X}_n)^2$$

$$= \frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)^2 + 2(\mu-\overline{X}_n)(\overline{X}_n-\mu) + (\mu-\overline{X}_n)^2$$

$$= \frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)^2 - (\overline{X}_n-\mu)^2$$

$$\text{so } E[\hat{V}] = ? = E\left[\frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)^2 - (\overline{X}_n-\mu)^2\right]$$

$$= \frac{1}{n}\,n\,\sigma_{x_i}^2 - E\left[(\overline{X}_n-\mu)^2\right]$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2 \neq \sigma^2 \rightarrow \text{Biased estimator}$$

11-6 8.12 $X_1\ X_2\ X_3\ X_4$ IID, poisson, $\alpha = 4$

$\sigma_x^2 = \alpha$

(a) $\hat{\alpha}_1 = (X_1 + X_2)/2$ Expect $E[X] = \alpha$

$E[\hat{\alpha}_1] = E[\frac{1}{2}(X_1 + X_2)] = \frac{1}{2}(E[X_1] + E[X_2])$

$= \frac{1}{2}[4+4] = 4$ ✓ → unbiased.

$4 = E[\alpha]$

$var(\hat{\alpha}_1) = \frac{var(X_1 + X_2)}{4} = \frac{2}{4} var(X) = \frac{1}{2} var(X) = \frac{\alpha}{2}$

(b) $\hat{\alpha}_2 = \frac{X_3 + X_4}{2}$

$E[\hat{\alpha}_2] = \frac{1}{2}(E[X_3] + E[X_4]) = \frac{1}{2} \cdot 2\alpha = \alpha$ → unbiased

$var(\hat{\alpha}_2) = \frac{var(X_3 + X_4)}{4} = \frac{var(X)}{2} = \frac{\alpha}{2}$

(c) $\hat{\alpha}_3 = \frac{X_1 + 2X_2}{3}$

$E[\hat{\alpha}_3] = \frac{1}{3}(E[X_1] + 2E[X_2]) = \frac{1}{3}(\alpha + 2\alpha) = \alpha$ → unbiased

$var(\hat{\alpha}_3) = \frac{1}{9} var(X_1 + 2X_2) = \frac{1}{9} 5 var(X) = \frac{5}{9} var(X)$

$= \frac{5}{9}\alpha$

(d) $\hat{\alpha}_4 = \frac{1}{4}[X_1 + X_2 + X_3 + X_4]$

$E[\hat{\alpha}_4] = \frac{1}{4}[4E[X]] = E[X] = \alpha$ → unbiased

$var(\hat{\alpha}_4] = \frac{1}{16}[4 var(X)] = \frac{4\alpha}{16} = \frac{\alpha}{4}$

11-7 | 8.28 a, b

ⓐ show that MLE for $\theta = \alpha^2$ of Rayleigh RV =

$$\hat{\alpha}^2_{ML} = \frac{1}{2n}\sum_{j=1}^{n} x_j^2 \qquad f_X(x) = \frac{x}{\theta}\exp\left[\frac{-x^2}{2\theta}\right]$$

$$l(\theta) = f_X(x_1 \ldots x_n \mid \theta) = \frac{x_1}{\theta}\exp\left[\frac{-x_1^2}{2\theta}\right] \cdot \ldots \frac{x_n}{\theta}\exp\left[\frac{-x_n^2}{2\theta}\right]$$

$$\ln(l(\theta)) = L(\theta) = \sum_{i=1}^{n} \ln\frac{x_i}{\theta} - \frac{x_i^2}{2\theta} \qquad \frac{d}{d\theta}\left(\ln\frac{x_i}{\theta}\right) = \frac{\left(\frac{-x}{\theta^2}\right)}{\left(\frac{x}{\theta}\right)} = \frac{-1}{\theta}$$

$$\frac{d}{d\theta}L(\theta) = \sum_{i=1}^{n}\frac{x_i^2}{2\theta^2} - \frac{1}{\theta} = 0 \qquad \frac{d}{d\theta}\left(\frac{-x_i^2}{2\theta}\right) = \frac{+x_i^2}{2\theta^2}$$

$$-\frac{n}{\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2 = 0$$

$$\frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2 = \frac{n}{\theta}$$

$$\frac{1}{2}\sum_{i=1}^{n} x_i^2 = n\theta$$

$$\frac{1}{2n}\sum_{i=1}^{n} x_i^2 = \theta \quad ✓$$

ⓑ unbiased?

$$E[\hat{\alpha}^2_{ML}] = E\left[\frac{1}{2n}\sum_{i=1}^{n} x_i^2\right] = \frac{1}{2n}\sum_{i=1}^{n} E[x_i^2]$$

$$\sigma_X^2 = E[X^2] - E[X]^2$$

$$2\alpha^2 - \frac{\pi}{2}\alpha^2 = E[X^2] - \alpha^2\frac{\pi}{2}$$

$$E[X^2] = 2\alpha^2$$

$$= \frac{1}{2n}\left[n \cdot 2\alpha^2\right]$$

$$= \alpha^2 \longrightarrow \text{unbiased } ✓$$