EE 503

Lecture #10

26 September 2019

2D PDF and CDFs
2 Random Variables

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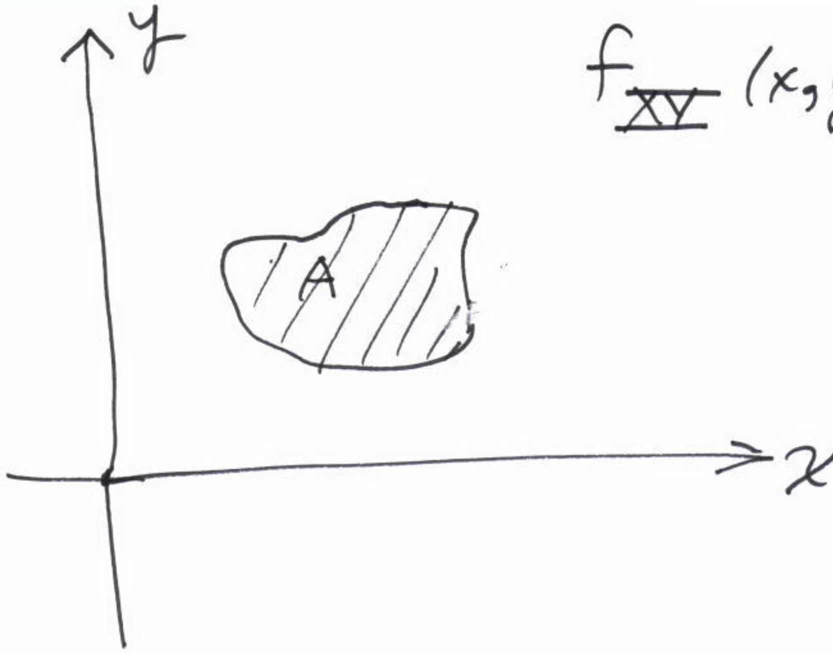
ps #5 posted

$$f_{\underline{XY}}(x,y) \geq 0$$



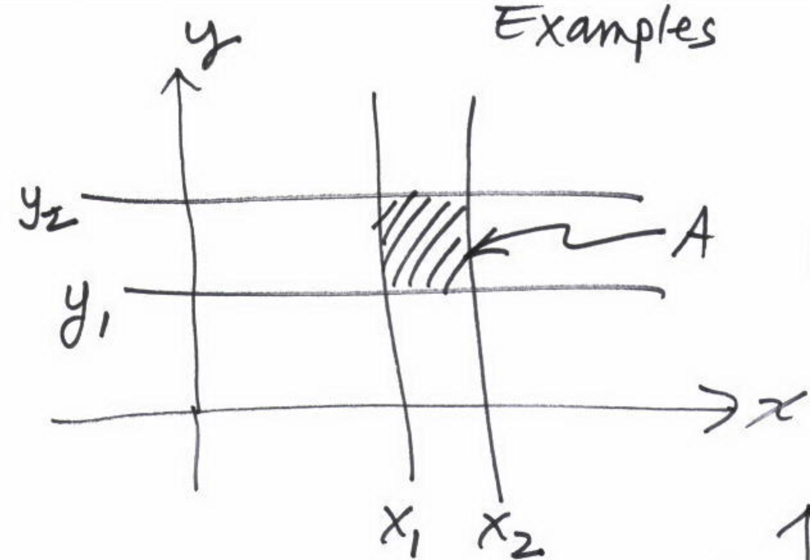
$$P((\underline{X},\underline{Y}) \in A) = P(A)$$

$$= \iint_A f_{\underline{XY}}(x,y)\,dx\,dy$$

= volume of the solid with a base A and surface $f_{\underline{XY}}(x,y)$

"base" = "footprint"

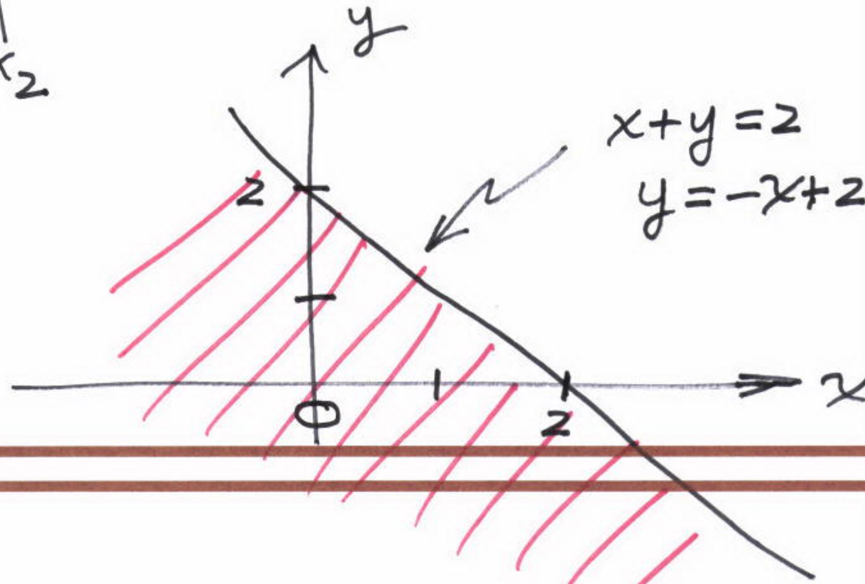
Examples

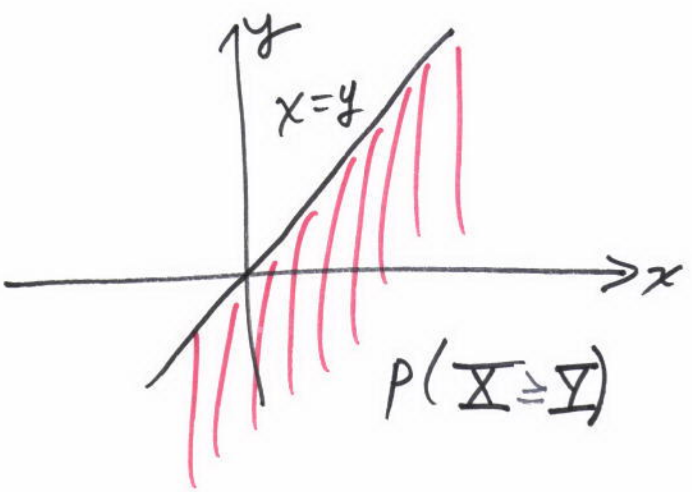


$P(A) =$ integral of $f_{XY}(x,y)$ over A

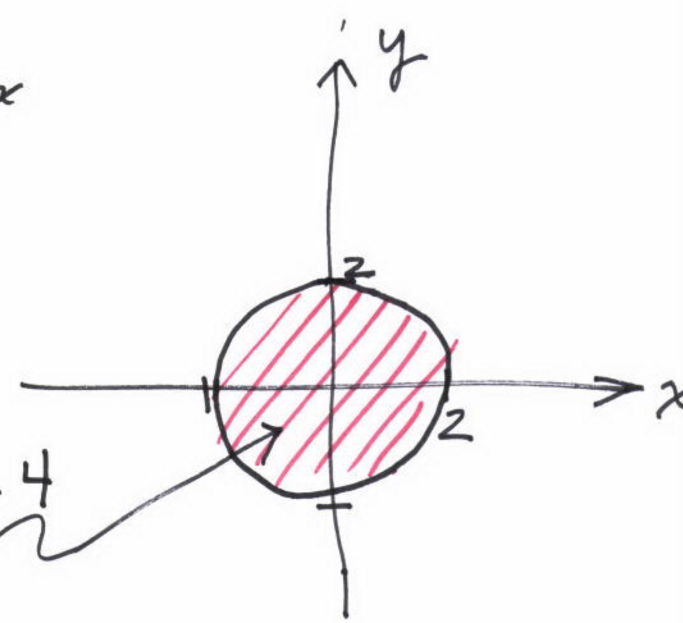
$P(X+Y \leq 2)$

$P(X+Y > 2)$





$P(X \geq Y)$



$X^2 + Y^2 \leq 4$

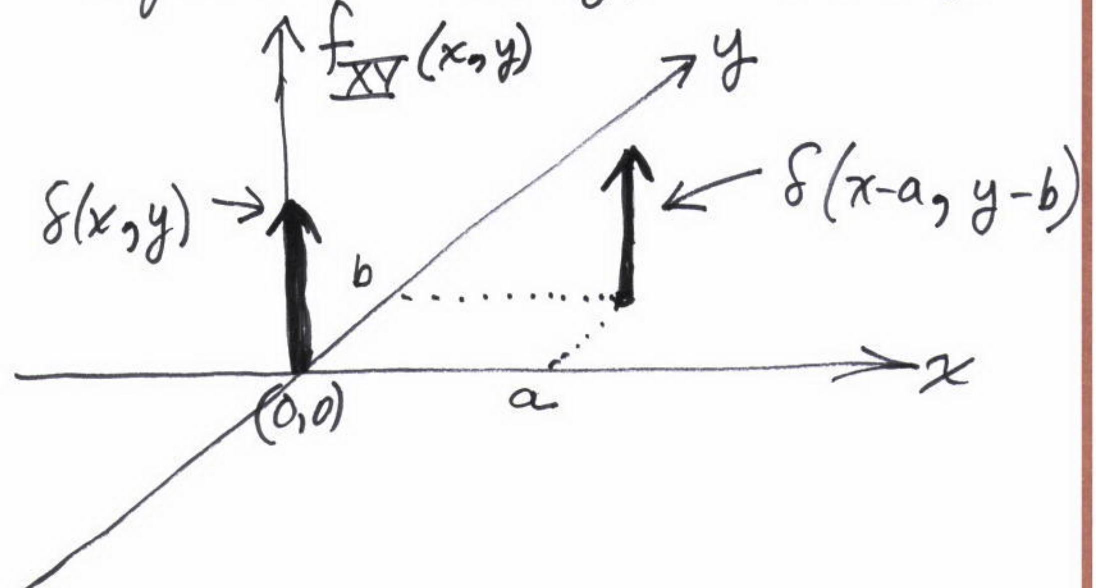
Review: 2D integration from calculus

Note: $f_{\underline{XY}}(x,y) \geq 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\underline{XY}}(x,y)\,dx\,dy = 1$$

for a discrete
2D PDF, the PDF consists of

2D δ functions $\delta(x,y) = \delta(x)\cdot\delta(y)$

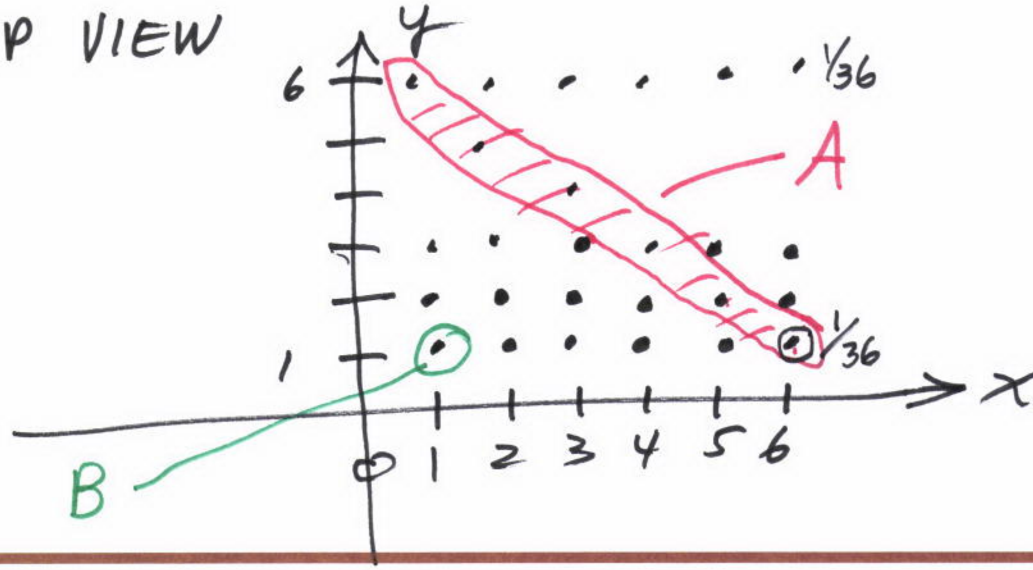


Example – Discrete 2D PDF

Roll a fair die 2 times

X = first roll Y = second roll

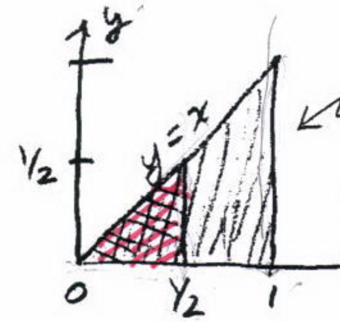
TOP VIEW



$$A = \{X + Y = 7\}$$

$$B = \{X + Y = 2\}$$

Example   2 RVs X, Y



defined in this footprint

$$f_{XY}(x,y) = \begin{cases} 8xy, & 0 \le y \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

a.) $P(X \le 1/2) = \int_{x=0}^{x=1/2}\int_{y=0}^{y=x} 8xy\,dy\,dx = \underline{\frac{1}{16}} = \int_y\int_x 8xy\,dx\,dy$

b.) find $f_Y(y)$

$f_Y(y) = 0, \quad y<0$

$f_Y(y) = \int_{x=y}^{x=1} 8xy\,dx = 4y(1-y^2) \quad 0 \le y \le 1$

$f_Y(y) = 0, \quad y>1$

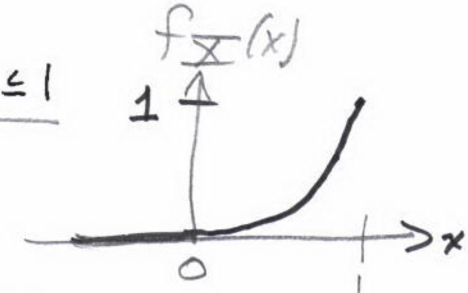
$f_Y(y) = \int f_{XY}(x,y)\,dx$

check is $\int_{-\infty}^{\infty} f_Y(y)\,dy \stackrel{?}{=} 1$

c.) $f_X(x) = 0, \quad \underline{x<0}$

$f_X(x) = \int_{y=0}^{y=x} 8xy\,dy = 4x^3, \quad \underline{0 \le x \le 1}$

$f_X(x) = \underline{0}, \quad x>1$



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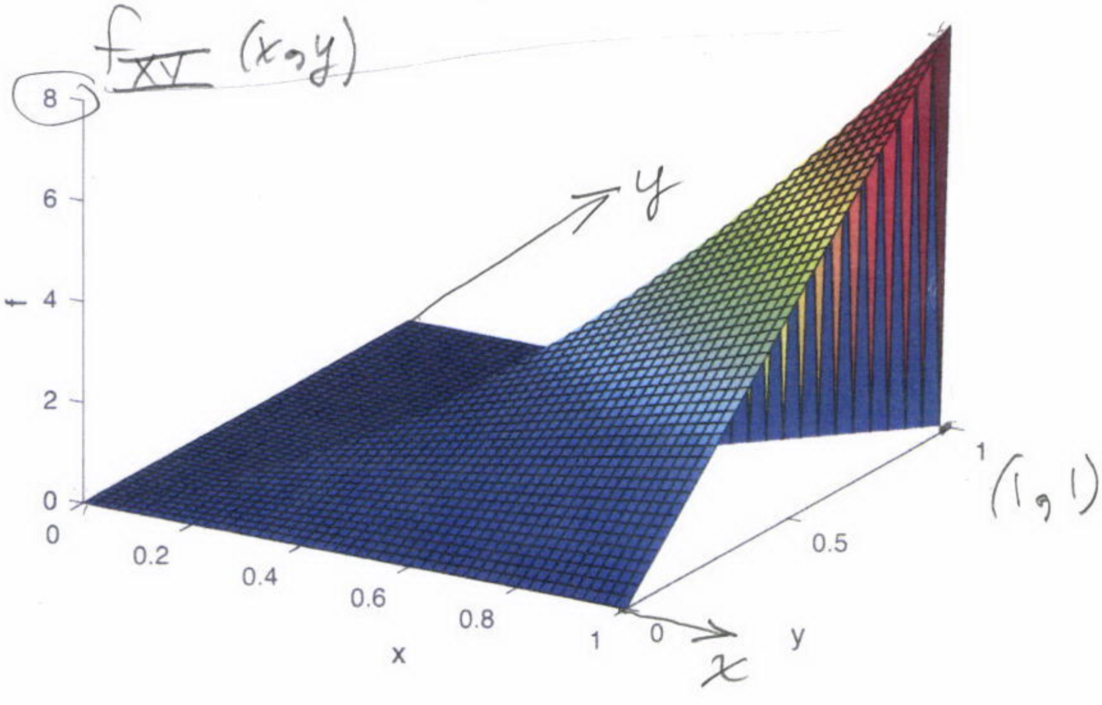
d.) independence? $f_X(x)\,f_Y(y) \ne f_{XY}(x,y)$

X, Y <u>not</u> independent

e.) $f_{X|Y}(x|y) = \dfrac{8xy}{4y(1-y^2)} = \begin{cases} \dfrac{2x}{(1-y^2)}, & 0 \le y \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$

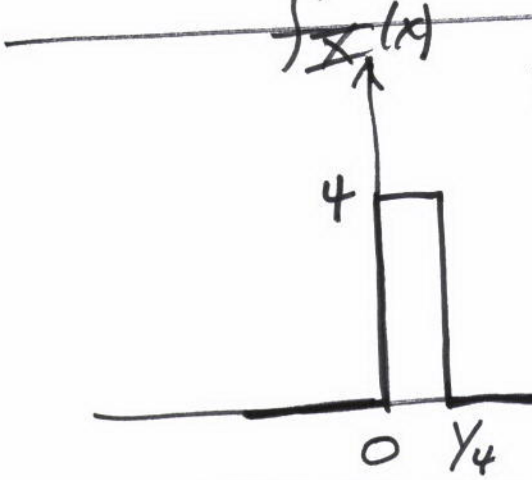
$= \dfrac{f_{XY}(x,y)}{f_Y(y)}$

$$f_{XY}(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0, & ew \end{cases}$$



what is $P(X=1, Y=1)$ for X, Y continuous RVs?

$$P(X=1, Y=1) = 0$$



$f_X(x) \sim X \sim \text{uniform}\ (0, 1/4)$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

Conditional PDFs – 2D LG3 4.4

look at

$$f_Y\big(y \mid \overbrace{x_1 \le X \le x_2}^{\text{"}M\text{"}}\big)$$

let $A = \{X = x\}$ and assume that $f_X(x) \neq 0$

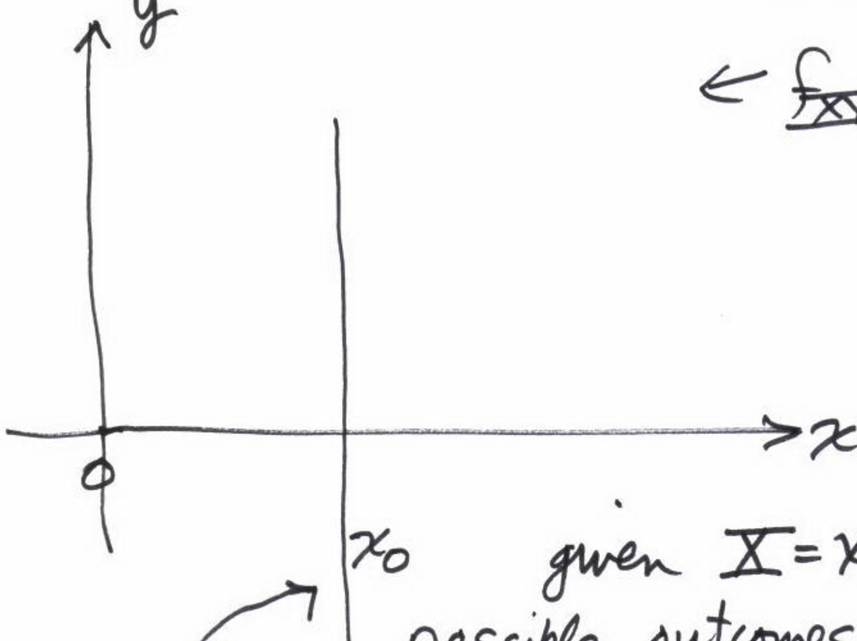
and assume X, Y are continuous RVs.

$$P(A) = P\{X = x\} = 0$$

Now, we do an experiment involving X, Y and the observed outcome is x_0

2D density

$\leftarrow f_{XY}(x,y)$ is a surface



given $X = x_0$, all possible outcomes must lie on this line

The conditional probability given this event must scale to 1 :

$$f_{Y|X}(y \mid X = x_0) = f_{Y|X}(y \mid x_0)$$

$$f_{Y|X}(y \mid x_0) = \frac{f_{XY}(y, x_0)}{f_X(x_0)}$$

$$\boxed{f_{Y|X}(y|x) = \frac{f_{XY}(y, x)}{f_X(x)}}$$

conditional density

$$\boxed{f_{XY}(x, y) = f_{Y|X}(y|x) \cdot f_X(x)} \quad (*)$$

$$\boxed{f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}}$$

here X is random

$Y = y$ is just a parameter in the conditional density

conditional DF

$$F_{X|Y}(x|y) = \int_{-\infty}^{\alpha = x} f_{X|Y}(\alpha|y)\, d\alpha$$

$$\boxed{F_{X|Y}(x|y) = \frac{\int_{-\infty}^{\alpha = x} f_{XY}(\alpha, y)\, d\alpha}{f_Y(y)}}$$

go back to (*)

$$f_{Y}(y) = \int_{x=-\infty}^{\infty} f_{XY}(x, y)\,dx$$

$$f_{Y}(y) = \int_{x=-\infty}^{\infty} f_{Y|X}(y|x)\, f_{X}(x)\,dx$$

Looks like "theorem of total probability"

Also

$$P(Y \in A \mid X = x) = P(Y \in A \mid X)$$

$$= \int_{y \in A} f_{Y}(y|x)\,dy$$

$$= \int_{y \in A} f_{Y}(y \mid X = x)\,dy$$

now

$$P(Y \in A) = \int_{-\infty}^{\infty} P(Y \in A \mid x)\, f_X(x)\, dx$$

= looks like "total probability

Conditional Probabilities Depending on Event B

1-D

$$f_Y(y \mid B) = \begin{cases} \dfrac{f_Y(y)}{P(B)}, & y \in B \\ 0, & y \notin B \end{cases}$$

assume $P(B) \neq 0$

2-D

$$f_{XY|B}(x, y \mid B) = \begin{cases} \dfrac{f_{XY}(x, y)}{P(B)}, & \text{if } (x, y) \in B \\ 0, & (x, y) \notin B \end{cases}$$

here B is some event defined on (X, Y)

Expectation - 2 RVs

$$E\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\,dx\,dy$$

$$E[g(x,y)\,|\,A] = \int_{-\infty}^{\infty}\int g(x,y)\, f_{XY}(x,y\,|\,A)\,dx\,dy$$

$$= \frac{\iint_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\,dx\,dy}{P(A)}$$

SPECIAL EXAMPLE

$A = \{X = x\}$

$$E(Y\,|\,X=x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{y \cdot f_{XY}(x,y)\,dy}{f_X(x)}$$ = a function of x

$$E(\underline{Y} \mid \underline{X}=x) = \underline{\text{a function of } x}$$

↑

Expectation and Sums Commute

$$E(\underline{X}+\underline{Y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f_{\underline{XY}}(x,y)\,dx\,dy$$

$$= \int\int_{-\infty}^{\infty} x \cdot f_{\underline{XY}}(x,y)\,dx\,dy + \int\int_{-\infty}^{\infty} y \cdot f_{\underline{XY}}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x \cdot f_{\underline{X}}(x)\,dx + \int_{-\infty}^{\infty} y \cdot f_{\underline{Y}}(y)\,dy$$

$$E(\underline{X}+\underline{Y}) = E(\underline{X}) + E(\underline{Y})$$

Moments

$$E(\underline{X}^k \underline{Y}^r) = \int\int_{-\infty}^{\infty} x^k y^r f_{\underline{XY}}(x,y)\,dx\,dy$$

$$= m_{kr}$$

Central Moment

$$E\left\{\left(\mathbf{X}-E(\mathbf{X})\right)^k\cdot\left(\mathbf{Y}-E(\mathbf{Y})\right)^r\right\}$$

$$=\int\int_{-\infty}^{\infty}\left(x-E(\mathbf{X})\right)^k\cdot\left(y-E(\mathbf{Y})\right)^r\cdot f_{\mathbf{XY}}(x,y)\,dx\,dy$$

$$=\mu_{kr}$$

Independence (Statistical Independence)

LG3 5.5

Two RVs $\mathbf{X},\mathbf{Y}$ are independent if and only if $\{\mathbf{X}\le x\}$ and $\{\mathbf{Y}\le y\}$ are <u>indep</u> for any x, y

or we need

$$P\{X \le x, Y \le y\} = P\{X \le x\} \cdot P\{Y \le y\}$$

$$\boxed{F_{XY}(x, y) = F_X(x) \cdot F_Y(y)}$$

also the density must factor ("separable" function)

$$\boxed{f_{XY}(x, y) = f_X(x) \cdot f_Y(y)}$$

if above is true,

X, Y are independent.

one way to do this is find $f_Y(y)$ and $f_X(x)$ from integration on $f_{XY}(x, y)$

Separable Expectations of Independent X, Y RVs

suppose $g(x, y) = g_1(x) \cdot g_2(y)$

$\Rightarrow$ separable function

$$E\{g_1(X)\cdot g_2(Y)\} = E\{g(X, Y)\}$$

$$\Rightarrow E[g_1(X)]\cdot E[g_2(Y)]$$

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example, if X, Y are indep.

and $g(x, y) = x\cdot y$

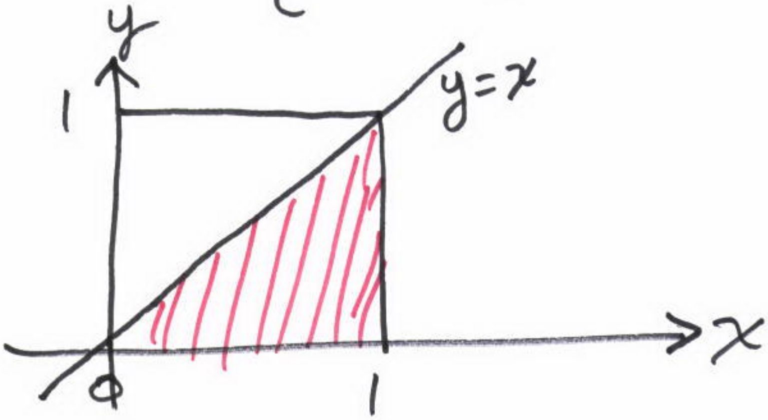
Then $E(\underline{XY}) = E(X)\cdot E(Y)$

<u>Examples</u> 2D RVs -

X, Y are indep. RVs with

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & ew \end{cases} \qquad f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & ew \end{cases}$$

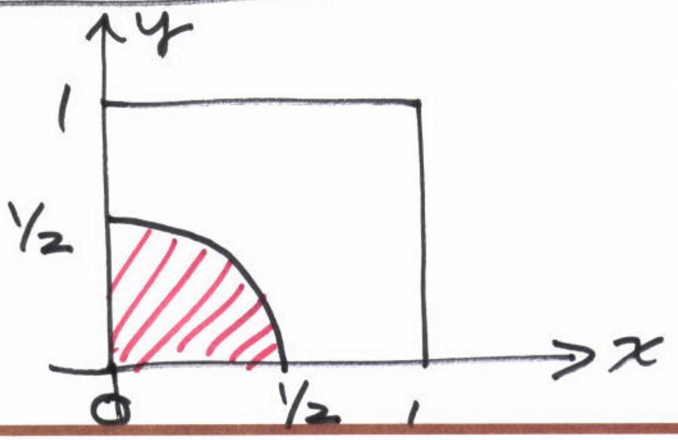
(1)



$P(X \ge Y)$

$$P(X \geq Y) = \int_{x=0}^{x=1} \int_{y=0}^{y=x} 4xy \, dy \, dx = \frac{1}{2}$$

(2) $P\{X^2 + Y^2 \leq 1/4\}$



(3) $P(\max(X, Y) \leq 1/2)$

$$\max(x, y) = \begin{cases} x, & x > y \\ y, & y > x \end{cases}$$