1. Assume that $X : \Omega \to \mathbb{R}$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Directly show that if $f : \mathbb{R} \to \mathbb{R}$ is an affine function of the form $f(x) = ax + b,\ a, b \in \mathbb{R}^{\geq 0}$), then $Y = f(X)$ is also a random vaiable.[1]

   Solution:

   It is obvious that $Y : \Omega \to \mathbb{R}$. We only need to show that $\{\omega | Y(\omega) \leq c\} \in \mathcal{F}, \forall c \in \mathbb{R}$ (i.e., it is an event):

   这句话啥意思

   If $a = 0$, $Y(\omega) = b$, so $\{\omega | Y(\omega) \leq c\}$ is either $\emptyset$ or Ω, which are both events.

   If $a > 0$,

   $$\{\omega | Y(\omega) \leq c\} = \{\omega | aX(\omega) + b \leq c\} = \{\omega | aX(\omega) \leq c - b\} = \{\omega | X(\omega) \leq (c - b)/a\}$$

   $\forall c \in \mathbb{R}, (c - b)/a \equiv d \in \mathbb{R}$, so $\{\omega | Y(\omega) \leq c\} = \{\omega | X(\omega) \leq d\}$, which is an event, because X is a random variable.

2. Assume that $\Omega = [0, 1/2]$ and $\mathbb{P}(B) = \int_B 2d\omega$ for any Borel subset of Ω. Calculate $\mathbb{P}(B|A)$ when $B = [0, 1/8]$ and $A = [1/16, 1/3)$. Are A and B independent?

   Solution:

   $$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

   But $A \cap B = [1/16, 1/3] \cap [0, 1/8] = [1/16, 1/8]$. Therefore, $\mathbb{P}(A \cap B) = \int_{1/16}^{1/8} 2d\omega = 1/8$. On the other hand, $\mathbb{P}(A) = \int_{1/16}^{1/3} 2d\omega = 13/24$ and $\mathbb{P}(B) = \int_0^{1/8} 2d\omega = 1/4$. Because $\mathbb{P}(A \cap B) \neq \mathbb{P}(A)\mathbb{P}(B)$, they are not independent.

3. Amy and Baichuan have $2k + 1$ coins, each with probability of a head equal to $1/2$. Baichuan tosses $k + 1$ coins, while Amy tosses the remaining k coins. Show that the probability that after all the coins have been tossed, Baichuan will have gotten more heads than Amy is $1/2$.

   Solution:

   定理
   conditional 计算题
   证明题

   Let B be the event that Baichuan tossed more heads. Let D be the event that after each has tossed k of their coins, Baichuan has more heads than Amy, let E be the event that under the same conditions, Amy has more heads than Baichuan, and let F be the event that they have the same number of heads. Since the coins are fair, we have $\mathbb{P}(D) = \mathbb{P}(E)$, and also $\mathbb{P}(F) = 1 - \mathbb{P}(D) - \mathbb{P}(E)$. Furthermore, we see that

   $$\mathbb{P}(B|D) = 1$$
   $$\mathbb{P}(B|E) = 0$$
   $$\mathbb{P}(B|F) = \frac{1}{2}$$

[1] Important Note: Posting the sample exam and its solutions to online forums or sharing it with other students is strictly prohibited. Instances will be reported to USC officials as academic dishonesty for disciplinary action.

Now we have, using the theorem of total probability,

$$
\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(D)\mathbb{P}(B|D) + \mathbb{P}(E)\cancel{\mathbb{P}(B|E)} + \mathbb{P}(F)\underbrace{\mathbb{P}(B|F)}_{\frac{1}{2}} \\
&= \mathbb{P}(D) + \frac{1}{2}\mathbb{P}(F) = \mathbb{P}(D) + \frac{1}{2}(1 - \mathbb{P}(D) - \mathbb{P}(E)) \\
&= \frac{1}{2} + \frac{1}{2}(\mathbb{P}(D) - \mathbb{P}(E)) = \frac{1}{2}
\end{aligned}
$$

as required. What is happening here is that Amy's probability of more heads than Baichuan is less than 1/2, so Baichuan has an advantage. However, the probability of equal number of heads is positive, and when added to Amy's probability of more heads, it gives 1/2.

4. A fair six-sided die is rolled repeatedly and independently. Let A_n be the event of rolling n sixes in $6n$ rolls, and let B_n be the event of rolling n or more sixes in $6n$ rolls. Do $\mathbb{P}(A_n)$ and $\mathbb{P}(B_n)$ change with n?

   Solution:

$$
\mathbb{P}(A_n) = \binom{6n}{n} (1/6)^n (5/6)^{(6n-n)}
$$

$$
\mathbb{P}(B_n) = \sum_{k=n}^{6n} \binom{6n}{k} (1/6)^k (5/6)^{(6n-k)}
$$

   Both $\mathbb{P}(A_n)$ and $\mathbb{P}(B_n)$ change with n.

5. A bizzare coin is tossed consecutively. The coin is designed so that the probability of seeing a head decreases: If $A_i = \{\text{Seeing a } H \text{ in the } i^{th} \text{ throw}\}$, then $\mathbb{P}(A_i) = 1/i^2$. What is the probability of seeing a head infinitely often, if the coin toss experiment is repeated ad-infinitum?

   Solution:

   Since $\sum_i^\infty 1/i^\alpha < \infty, \forall \alpha > 1$, $\sum_{i=1}^\infty \mathbb{P}(A_i) < \infty$. According to the first Borel-Cantelli Lemma, $\mathbb{P}(A_{\text{i.o.}}) = 0$.

6. 98% ~~percent~~ of all patients with a certain type of cancer in a hospital survive. However, 10% of all of those patients are given chemotherapy, and when chemotherapy is performed, the patient survives 90% of the time. If the randomly chosen patient with that type of cancer does not receive chemotherapy, what is the probability that the patient survives?[2]

   Solution:

[2] There are more questions on this sample test than a common midterm exam, so that students can practice more.

Let S be the event that a patient survives and let C be the event of receiving chemotherapy. Then:

$$\begin{aligned}\mathbb{P}(S|C^c) &= \frac{\mathbb{P}(S \cap C^c)}{\mathbb{P}(C^c)} = \frac{\mathbb{P}(S) - \mathbb{P}(S \cap C)}{\mathbb{P}(C^c)} = \frac{\mathbb{P}(S) - \mathbb{P}(S|C)\mathbb{P}(C)}{1 - \mathbb{P}(C)} = \frac{0.98 - 0.9 \times 0.1}{1 - 0.1} \\ &= \frac{89}{90}\end{aligned}$$