USC EE503 – Probability for Electrical and Computer Engineers – Fall 2013

SAMPLE Midterm Exam

NAME:	....
STUDENT ID:	....

1. Consider a communication link where packets of information bits are sent through a data link that operates at the rate of R bit/s. At the output, a packet is either received correctly or it is completely lost (erased), with probability p. The erasure events are independent and identically distributed.

   Consider the following simple repetition protocol. A packet is repeated until it is correctly received. When a packet is correctly received, the transmitter removes it from its transmission queue and moves on to transmit the next packet in the queue buffer. It is assumed that an infinite sequence of packets are in the buffer, such that the queue is never empty.

   a) Compute the expected number of transmissions in order to correctly receive a packet.

   b) Compute the system rate, expressed as the average number of correctly received bits per second.

   Write your ANSWER in the space below (full derivation should be attached on extra pages):

2. We throw balls into 10 bins, one at a time and at random, and we stop when a ball is placed in an already occupied bin.
   Compute the probability that we stop at the 5-th ball.

   Write your ANSWER in the space below (full derivation should be attached on extra pages):

3. 52 cards are partitioned into 4 groups of size 13 cards each and each group is assigned to a player. Suppose that cards are distributed at random. Compute the probability that each player has an ace (notice: in a card deck there are 4 aces).

   Write your ANSWER in the space below (full derivation should be attached on extra pages):

4. Let X, Y be independent and identically distributed exponential RVs with parameter $\lambda = 1$.

   a) Find the joint pdf of $U = X + Y$ and $V = X/(X + Y)$;

   b) Show that V is uniformly distributed on $[0, 1]$.

   Write your ANSWER in the space below (full derivation should be attached on extra pages):

5. Consider the system defined by $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{Z}$, where $\mathbf{Y}, \mathbf{Z} \in \mathbb{R}^n$, $\mathbf{X} \in \mathbb{R}^k$, $\mathbf{A} \in \mathbb{R}^{n \times k}$. The matrix $\mathbf{A}$ is deterministic (constant), $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$, $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_x)$ and $\mathbf{Z}$ and $\mathbf{X}$ are mutually independent.

   a) Find an expression for the conditional mean $\mathbb{E}[\mathbf{X}|\mathbf{Y}]$ in terms of $\mathbf{A}, \mathbf{\Sigma}_x$ and σ^2.

   b) Find an expression for the corresponding conditional covariance matrix $\mathrm{cov}(\mathbf{X}|\mathbf{Y})$.

   Write your ANSWER in the space below (full derivation should be attached on extra pages):

6. Consider a set $X_1, \ldots, X_n$ of i.i.d. Gaussian random variables with mean μ and variance σ^2, and define the sample mean and the sample variance, given by

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

respectively.

a) Show that the above "estimators" of mean and variance are "unbiased", which means that their expected value is equal to the true mean and variance. This means that you have to prove the identities:

$$\mathbb{E}[\overline{X}] = \mu, \quad \text{and} \quad \mathbb{E}[S] = \sigma^2$$

b) Show that $\overline{X}$ is Gaussian, with mean μ and variance σ^2/n.

c) (more difficult ...) Show that $(n-1)S/\sigma^2$ is central Chi-squared with $n-1$ degrees of freedom and that $\overline{X}$ and S are statistically independent. *Hint for point c):* *1) define the centralized and normalized variables* $Y_i = (X_i - \mu)/\sigma$ *with sample mean* $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ *and show that* $\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = (n-1)S/\sigma^2$*; 2) Realize that the random vector* $\mathbf{Y} = (Y_1, \ldots, Y_n)^\mathsf{T}$ *has spherical symmetry, that is, for any orthogonal matrix* $\mathbf{R}$ *(what is an orthogonal matrix ??), you have that* $\mathbf{Z} = \mathbf{R}\mathbf{Y}$ *and* $\mathbf{Y}$ *have the same joint pdf, and in addition* $\|\mathbf{Z}\|^2 = \|\mathbf{R}\mathbf{Y}\|^2 = \mathbf{Y}^\mathsf{T}\mathbf{R}^\mathsf{T}\mathbf{R}\mathbf{Y} = \|\mathbf{Y}\|^2$*, i.e.,* $\mathbf{Z}$ *and* $\mathbf{Y}$ *have the same squared length; 3) choose* $\mathbf{R}$ *such that the first row is given by* $(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n})$ *and the other rows are any arbitrary orthogonal vectors. It follows that the first component of* $\mathbf{Z} = \mathbf{R}\mathbf{Y}$ *is* $Z_i = \frac{1}{n}\sum_{i=1}^{n} Y_i = \sqrt{n}\overline{Y}$*; 4) Consider the sum* $\sum_{i=2}^{n} Z_i^2 = \sum_{i=1}^{n} Z_i^2 - Z_1^2$ *and see how this is related to* $(n-1)S/\sigma^2$ *.... at this point the conclusion should be obvious!*