PS4 Due Thursday 09/26/19 Charlie André

$\boxed{4\text{-}1}$ text problem 3.11

ⓐ Borrowing work done last week for X:

$Pr(X=0) = \frac{1}{16}$

$Pr(X=1) = \frac{8}{16}$

$Pr(X=2) = \frac{7}{16}$

For Y: $Pr(Y=0) = Pr(T_1=0) \cdot Pr(T_2=0)$

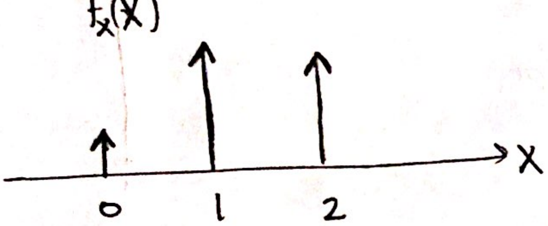
$Pr(Y=0) = \frac{1}{4}$

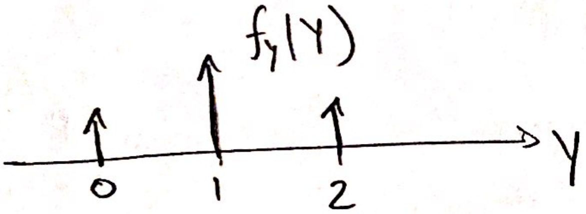
$Pr(Y=1) = Pr(T_1=T, T_2=H) + Pr(T_1=H, T_2=T) =$

$Pr(Y=1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$Pr(Y=2) = Pr(T_1=H) \cdot Pr(T_2=H)$

$Pr(Y=2) = \frac{1}{4}$





By enforcing X = max of 2 attempts at Y, we shift the pdf towards 2, increasing likelihood of getting higher value for X than for Y

ⓑ Carlos uses a coin with prob of heads = $p = \frac{3}{4}$

Find $f_X(X)$ C = # of heads from Carlos M = # Heads, Michael

$\cancel{Pr(X=0) = Pr(C=0) \cdot Pr(M=0) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{64}}$

$\cancel{Pr(X=1) = Pr(C=0, M=1) + Pr(C=1, M=0) + Pr(C=1, M=1)}$

$\cancel{= \left(\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2}\right) + \left(\frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}\right) \cdot 2 + \left(\frac{3}{4} \cdot \frac{1}{4}\right.}$

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1) fundamental events of # of heads from Carlos :

c \ m			
	0	1	2
	1	1	2
	2	2	2

$Pr(C=0) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$Pr(C=1) = 2 \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{8} = \frac{6}{16}$

$Pr(C=2) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$ ✓

$Pr(X=0) = Pr(C=0, M=0) = \frac{1}{16} \cdot \frac{1}{4} = \frac{1}{64}$

$Pr(X=1) = Pr(C=0, M=1) + Pr(C=1, M=0) + Pr(C=1, M=1)$

$Pr(X=1) = \frac{1}{16} \cdot \frac{1}{2} + \frac{6}{16} \cdot \frac{1}{4} + \frac{6}{16} \cdot \frac{1}{2}$

$Pr(X=1) = \frac{2}{64} + \frac{6}{64} + \frac{12}{64} = \frac{10}{32} = \frac{20}{64}$

$Pr(X=2) = Pr(C=0, M=2) + Pr(C=1, M=2) + Pr(C=2, M=2)$
$+ Pr(C=2, M=1) + Pr(C=2, M=0)$

$= \left(\frac{1}{16} \cdot \frac{1}{4}\right) + \left(\frac{6}{16} \cdot \frac{1}{4}\right) + \left(\frac{9}{16} \cdot \frac{1}{4}\right) + \left(\frac{9}{16} \cdot \frac{1}{2}\right) + \left(\frac{9}{16} \cdot \frac{1}{4}\right)$

$Pr(X=2) = \frac{1}{64} + \frac{6}{64} + \frac{9}{64} + \frac{18}{64} + \frac{9}{64} = \frac{43}{64}$

3.21

ⓐ $E[Y] = Pr(Y=0)\cdot 0 + Pr(Y=1)\cdot 1 + Pr(Y=2)\cdot 2$

$$E[Y] = \frac{1}{4}\cdot 0 + 1\cdot\frac{1}{2} + \frac{1}{4}\cdot 2 = 1$$

$$E[X] = \frac{1}{16}\cdot 0 + \frac{1}{2}\cdot 1 + \frac{7}{16}\cdot 2 = \frac{11}{8}$$

$E[X] > E[Y] \longrightarrow$ expected as

$\max(A_1, A_2)$ should usually be bigger than either A_1 or A_2 when $pdf(A_1) = pdf(A_2)$

ⓑ $var(X) = ?$

$$var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \frac{1}{16}\cdot 0^2 + \frac{1}{2}\cdot 1^2 + \frac{7}{16}\cdot 2^2 = \frac{9}{4} = \frac{144}{64}$$

$$(E[X])^2 = \left(\frac{11}{8}\right)^2 = \frac{121}{64}$$

$$var(Y) = E[Y^2] - (E[Y])^2$$

$$E[Y^2] = \frac{1}{4}\cdot 0^2 + \frac{1}{2}1^2 + \frac{1}{4}\cdot 2^2 = \frac{3}{2}$$

$$(E[Y])^2 = 1^2 = 1$$

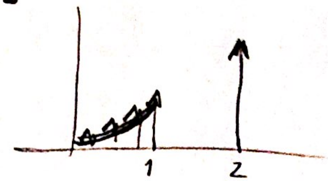
$$var(X) = E[X^2] - (E[X])^2 = \frac{144}{64} - \frac{121}{64} = \frac{23}{64}$$

$var(Y) = \frac{3}{2} - 1 = \frac{1}{2}$ $\quad var(X) < var(Y)$,

makes sense because $f_Y(y)$ is more evenly spread on $[0, 2]$ and $f_X(x)$ is more concentrated towards higher end of range.

4-3) $f_X(x) = x^2\left[u(x) - u(x-1)\right] + a\,\delta(x-2)$

(a) $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = 1$



$\displaystyle\int_0^1 x^2\,dx + a\cdot 1 = 1$

$\displaystyle\frac{x^3}{3}\Big]_0^1 + a = 1 \longrightarrow \boxed{a = \frac{2}{3}}$

$\displaystyle E[x] = \int_0^1 x\cdot x^2\,dx + \int_{-\infty}^{\infty} x\cdot\frac{2}{3}\,\delta(x-2)\,dx$

$\displaystyle E[x] = \frac{x^4}{4}\Big]_0^1 + 2\cdot\frac{2}{3} = \frac{1}{4} + \frac{4}{3} = \frac{3}{12} + \frac{15}{12}$

$\displaystyle E[x] = \frac{18}{12}$

$\sigma_x^2 = E[x^2] - (E[x])^2$

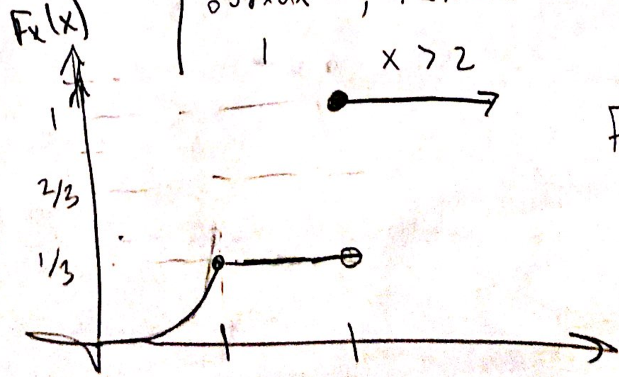
$\displaystyle E[x^2] = \int_0^1 x^2\cdot x^2\,dx + \int_{-\infty}^{\infty} x^2\,\frac{2}{3}\,\delta(x-2)\,dx$

$\displaystyle \frac{x^5}{5}\Big]_0^1 + 4 = 4\frac{1}{5} = \frac{21}{5}$

$\displaystyle \sigma_x^2 = \frac{21}{5} - \left(\frac{18}{12}\right)^2 = 1.95$

(b) $F_X(x) = \begin{cases} \int f_X\,dx, & 0 < x < 1 \\ \int_0^1 f_X\,dx, & 1 < x < 2 \\ 1 & x > 2 \end{cases}$

$\displaystyle \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3}$

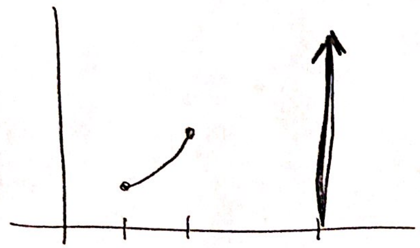


$F_X(x) = \begin{cases} \frac{x^3}{3} & 0 < x < 1 \\ \frac{1}{3} & 1 < x < 2 \\ 1 & x > 2 \end{cases}$

4-3 c W = event $X \geq 0.5$

find $f_X(X|W)$ & $E(X|W)$

$f_X(x|w)$



$$P_r(X = x \mid W) = \frac{P_r(X = x, X \geq 0.5)}{P_r(X \geq 0.5)}$$

$$P_r(X \geq 0.5) = \int_{0.5}^{1} x^2\,dx + \frac{2}{3}$$

$$= 1 - \int_{0}^{0.5} x^2\,dx$$

$$P_r(X \geq 0.5) = 1 - \frac{0.5^3}{3} = 0.95833$$

$$f_X(x|w) = \begin{cases} 0 & x < 0.5 \\ \frac{x^2}{0.9583} & 0.5 < x < 1 \\ \frac{(\frac{2}{3})}{0.9583} & x = 2 \end{cases}$$

$$F_X(x|w) = \begin{cases} 0 & x < 0.5 \\ \frac{x^3}{3(0.9583)} & 0.5 < x < 1 \\ 0.3478 & 1 < x < 2 \\ 1 & x > 2 \end{cases}$$

this jump $\approx \frac{(\frac{2}{3})}{0.9583}$

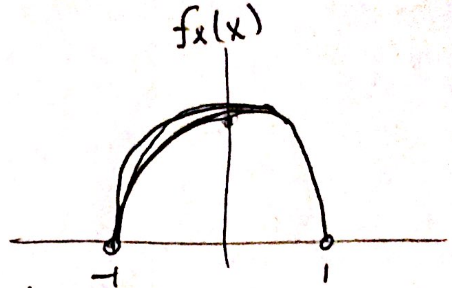
$0.6957 \approx 1 - 0.3478$

0.6522

Close enough

4-4) 4.17

(a) $f_X(x) = \begin{cases} c(1-x^2) & -1 \le x \le 1 \\ 0, & \text{ow} \end{cases}$



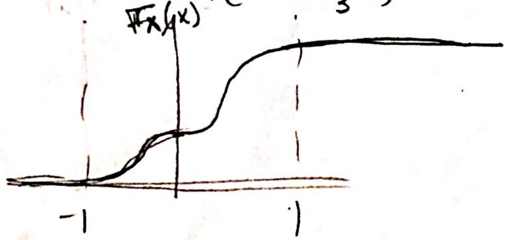
$c \int_{-1}^{1} 1 - x^2 \, dx \Rightarrow \left(x - \frac{x^3}{3} \right) \Big]_{-1}^{1} = \frac{1}{c}$

$\left[1 - \frac{1}{3}\right] - \left[-1 + \frac{1}{3}\right] = \frac{1}{c}$

$\frac{4}{3} = \frac{1}{c} \Rightarrow c = \frac{3}{4}$

(b) $F_X(x) = \int f_X(x)\,dx$ on $-1 < x < 1$

$F_X(x) = \frac{3}{4}\left(x - \frac{x^3}{3}\right)$



$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{3}{4}\left(x - \frac{x^3}{3}\right) & -1 < x < 1 \\ 1 & x > 1 \end{cases}$

(c) $P(X = 0) = f_X(0) = \frac{3}{4}$

$P(0 < x < 0.5) = \int_0^{0.5} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_0^{0.5}$

$P(0 < x < 0.5) = \frac{3}{4}\left[\frac{1}{2} - \frac{1}{24}\right] = \frac{3}{8} - \frac{1}{32} = \frac{11}{32}$

$P(|x - 0.5| < 0.25) = ?$ $\quad x - 0.5 < 0.25 \;\cap\; x - 0.5 > -0.25$

$= P(x < 0.75) + P(x > 0.25) - P(-\frac{3}{4} < x < \frac{3}{4})$

$\cancel{\int_{-1}^{0.75} \frac{3}{4}(1-x^2)dx + \int_{0.75}^{1} \frac{3}{4}(1-x^2)dx - \int_{-0.75}^{0.75} \frac{3}{4}(1-x^2)dx}$

$\cancel{=}$

continued on next pg.

4-4 cont

$$\cancel{= F(0.75) - F(-1) + F(1) - F(-0.75) - F(0.75) + F(-\tfrac{3}{4})}$$

$$\cancel{= F(1) - F(-1) = 1}$$

$$= P\left(\tfrac{1}{4} < X < \tfrac{3}{4}\right) = F\left(\tfrac{3}{4}\right) - F\left(\tfrac{1}{4}\right)$$

$$= \frac{3}{4}\left[\left(\frac{3}{4} - \frac{(3/4)^3}{3}\right) - \left(\frac{1}{4} - \frac{(1/4)^3}{3}\right)\right]$$

$$\frac{3}{4}\left[\frac{1}{2} - 0.4063\right] = 0.0703$$

$\boxed{4-5}$ 4.39 μ_x and σ_x^2 from 4.17

$$\mu_x = \int_{-1}^{1} \frac{3}{4} x \cdot (1-x^2)\,dx$$

$$= \int_{-1}^{1} \frac{3}{4}(x - x^3)\,dx = \frac{3}{4}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{1}$$

$$\mu_x = \frac{3}{4}\left[\left(\frac{1}{2} - \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{4}\right)\right] = 0$$

$$\sigma_x^2 = E\{x^2\} - \mu_x^2$$

$$E[x^2] = \int_{-1}^{1} \frac{3}{4}(x^2 - x^4)\,dx = \frac{3}{4}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1}$$

$$= \frac{3}{4}\left[\left(\frac{1}{3} - \frac{1}{5}\right) - \left(-\frac{1}{3} + \frac{1}{5}\right)\right]$$

$$= \frac{3}{4}\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{6}{12} - \frac{6}{20} = \frac{2}{10}$$

so $\sigma_x^2 = \left(\frac{1}{2} - \frac{3}{10}\right) - 0 = \frac{1}{2} - \frac{3}{10} = \frac{2}{10}$

4-6 $X \sim U(0,1)$

ⓐ $Pr(|X-\mu_x| > \sigma_x)$

$\mu_x = \frac{1}{2}$ $\quad E[x^2] = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3}$

so $\sigma_x^2 = E[x^2] - (E[x])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

so $\sigma_x = \frac{1}{\sqrt{12}}$

$X - \frac{1}{2} > \frac{1}{\sqrt{12}}$ $\qquad X - \frac{1}{2} < \frac{-1}{\sqrt{12}}$

$X > \frac{+1}{2} + \frac{1}{\sqrt{12}}$ $\qquad X < \frac{1}{2} - \frac{1}{\sqrt{12}}$

$= Pr\left(\frac{1}{2} - \frac{1}{\sqrt{12}} < x < \frac{1}{2} + \frac{1}{\sqrt{12}}\right)$

$$F_x(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

so $Pr(x - \sigma_x < x < x + \sigma_x) =$

$(x + \sigma_x) - (x - \sigma_x) = 2\sigma_x = \frac{2}{\sqrt{12}}$

ⓑ within $4\sigma_x$

$Pr(x - 4\sigma_x < x < x + 4\sigma_x) =$

$(x + 4\sigma_x) - (x - 4\sigma_x) = 8\sigma_x = \frac{8}{\sqrt{12}}$

4-7) $P(H) = p$ flipped till n^{th} head appears

$X = \#$ of flips required

$$P(X=k) = \binom{k-1}{n-1} p^n (1-p)^{k-n}$$

need $(n-1)H$ for $X = k-1$ then one more head on n^{th} flip

$$P(n-1 \text{ Heads on } k-1 \text{ flips}) = \binom{k-1}{n-1} p^{n-1} (1-p)^{(k-1)-(n-1)}$$

$$= \binom{k-1}{n-1} p^{n-1} (1-p)^{k-n}$$

$$P(\text{next flip} = \text{Heads}) = p$$

so $P(X=k) = P(n-1 \text{ Heads on } k-1 \text{ flips}, \text{ next flip heads})$

$$P(X=k) = \binom{k-1}{n-1} p^n (1-p)^{k-n}$$

QED