EE 503 PS 5 Due 10/03

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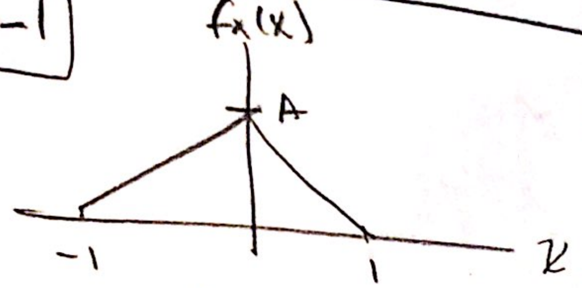
5-1



(a) $A = ?$ by geometry $\int f_x(x)\,dx = 1$

$1 \cdot A \cdot 2 \cdot \frac{1}{2} = A = \underline{1}$

(b) $M = \{x \leq 0.5\}$

$$f_x(x) = \begin{cases} x+1, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & ew \end{cases}$$

$$P(M) = \int_{-\infty}^{0.5} f_x(x)$$

$$= \int_{-1}^{0} x+1\,dx + \int_{0}^{0.5} 1-x\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{0.5}$$

$$= \frac{1}{2} + \left[\frac{1}{2} - \frac{1}{8}\right] = \frac{7}{8}$$

(c) $E[x] = ?$

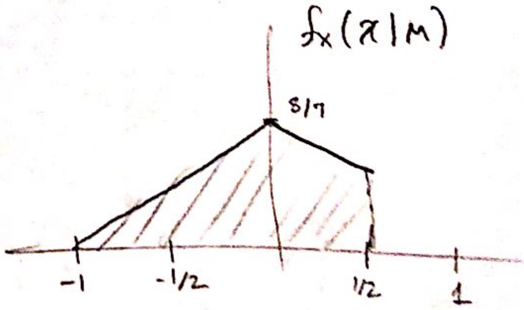
$$= \int_{-1}^{0} x^2+x\,dx + \int_{0}^{1} x-x^2\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{0}^{1}$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 0$$

(d) $f_{x|M}(x|M)$

$$= \begin{cases} \dfrac{x+1}{7/8}, & -1 < x < 0 \\ \dfrac{1-x}{7/8}, & 0 < x < 0.5 \\ 0, & \text{otherwise} \end{cases}$$



(e) $$E[x|M] = \int_{-1}^{0} \frac{x^2+x}{7/8}\,dx + \int_{0}^{1/2} \frac{x-x^2}{7/8}\,dx$$

$$\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{0}^{0.5} \Rightarrow \frac{1}{8} - \frac{1/8}{3}$$

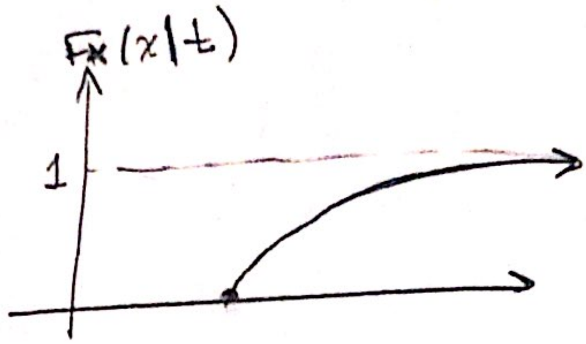
$$= \frac{8}{7}\left[\frac{1}{3} - \frac{1}{2}\right] + \frac{8}{7}\left[\frac{1}{8} - \frac{1/8}{3}\right]$$

$$= \frac{8}{7}\left[\frac{-1}{6}\right] + \left(\frac{1}{7} - \frac{1}{21}\right]$$

$$E[x|M] = -0.19047 + 0.09523 = -0.09524$$

5-2 | Text #4.33 $X \sim \exp$ <u>memoryless property</u>

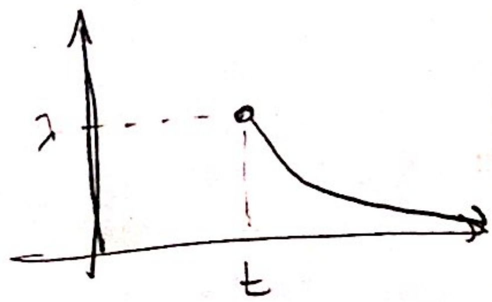
ⓐ $f_X(x \mid X>t) = ?$ $\begin{cases} 0, & x<t \\ \dfrac{f_X(x)}{1-F_X(t)}, & x>t \end{cases}$



$$F_X(x \mid X>t) = \begin{cases} 0, & x<t \\ F_X(x-t) & x>t \end{cases}$$

- $F_X(x \mid X>t)$ is the same curve, shifted by t

ⓑ $f_X(x|t) = \begin{cases} 0 & x<t \\ \lambda \exp(-\lambda(x-t)) & x>t \end{cases}$



ⓒ show $P[X> t+x \mid X>t] = P(X>x)$

$$\frac{1-F_X(t+x)}{1-F_X(t)} = 1 - F_X(x) \longrightarrow \text{def } z(x) = 1-F_X(x)$$

prove <u>inverse</u> $P[X<t+x \mid X<t] = P(X<x)$

$$\frac{F_X(t+x)}{F_X(t)} = F_X(x)$$

memoryless b/c it does not change shape depending on a shift or where it starts

$$\frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$$

$$e^{-\lambda(t+\cdot)} = e^{-\lambda x - \lambda t} \checkmark$$

QED.

5-3

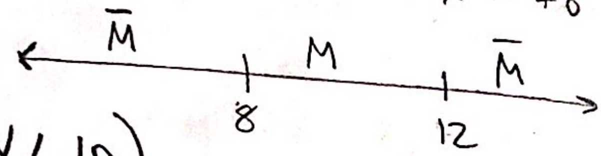
$X \sim N(10, 1)$ $\quad M = \{(X-10)^2 < 4\}$

$$f_{X|M}(x|M) = ? = \begin{cases} \dfrac{f_X(x)}{P(M)} & X \in M \\ 0, & ew \end{cases}$$

$P(M) = ?$ $\quad M = \{(X-10)^2 < 4$

$X - 10 = 2$
$x = 12$

$x - 10 = -2$
$x = +8$



$P(M) = P(8 \le X \le 10)$

$\sigma = \sigma^2 = 1$

$$f_{X|M}(x|M) = \begin{cases} 0 & x < 8 \\ \dfrac{f_X(x)}{F_X(10) - F_X(8)} & 8 < x < 10 \\ 0 & x > 10 \end{cases}$$

same shape as a standard Normal, w/ shifted mean

So
$$f_{X|M}(x|M) = \begin{cases} 0, & x < 8 \\ \dfrac{\exp\left(\dfrac{-(x-10)^2}{2}\right)}{\sqrt{2\pi} \cdot [\phi(2) - \phi(-2)]}, & 8 < x < 10 \\ 0, & x > 10 \end{cases}$$

$F_X(12) = \phi\left(\frac{12-M}{\sigma}\right) = \phi(2)$

$F_X(8) = \phi\left(\frac{8-10}{\sigma}\right) = \phi(-2)$

5-4 # of veh waiting $= X$ $f_X(x) = A(1-\frac{x}{5})[u(x) - u(x-5)]$

(a) $A = ?$ $A \cdot \int_0^5 1 - \frac{x}{5} dx = 1$

$$x - \frac{x^2}{10} \Big]_0^5 = \frac{1}{A}$$

$$\frac{50}{10} - \frac{25}{10} = \frac{25}{10} \rightarrow A = \frac{10}{25}$$

(b) CDF for X

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{10}{25}(x - \frac{x^2}{10}) & 0 < x < 5 \\ 1 & x > 5 \end{cases}$$

Integral done above, didn't need to redo.

(c) $Pr(X = 2) = 0 \rightarrow$ by result of being a cont RV

(d) $Pr(X < 3) = F_X(3) = \frac{10}{25}(3 - \frac{3^2}{10}) = \frac{10}{25}(\frac{21}{10})$

$$F_X(3) = \frac{21}{25}$$

(e) $E[x] = \frac{10}{25}\int_0^5 x - \frac{x^2}{5} dx = \frac{10}{25}\left[\frac{x^2}{2} - \frac{x^3}{15}\right]_0^5 = \frac{10}{25}\left[\frac{25}{2} - \frac{125}{15}\right] = 1.666$

$E[x^2] = \frac{10}{25}\int_0^5 x^2 - \frac{x^3}{5} dx = \left[\frac{x^3}{3} - \frac{x^4}{20}\right]_0^5 = \frac{10}{25}\left[\frac{125}{3} - \frac{625}{20}\right] = 4.1666$

$\sigma_X^2 = E[x^2] - (E[x])^2$

$\sigma_X^2 = 4.166 - 1.666^2 = 1.3911$

5-4 (f) $W = \{1 < x < 3\}$

$Pr(x < 2 \mid W) = ?$

$$f_x(x \mid W) = \begin{cases} \frac{f_x(x)}{Pr(W)} & 1 < x < 3 \\ 0, & ew \end{cases}$$

$$Pr(W) = \int_1^3 \left(\frac{10}{25}\right)\left(1 - \frac{x}{5}\right) dx$$

$$= \frac{10}{25}\left[x - \frac{x^2}{10}\right]_1^3 = \frac{10}{25}\left[\left(3 - \frac{9}{10}\right) - \left(1 - \frac{1}{10}\right)\right]$$

$$Pr(W) = \frac{10}{25}\left[2 - \frac{8}{10}\right] = 0.48$$

$$Pr(x<2 \mid W) = \int_1^2 \left(\frac{10}{25}\right)\left(\frac{1}{0.16}\right)\left(1 - \frac{x}{5}\right) dx$$

$$\cancel{(2.5)\left[x - \frac{x^2}{10}\right]_1^2 = 2.5\left[\left(2 - \frac{4}{10}\right) - \left(1 - \frac{1}{10}\right)\right]}$$

$$\cancel{= 2.5\left[\frac{16}{10} - \frac{9}{10}\right] = 1.75}$$

$$= \frac{1}{0.46} \cdot \int_0^2 f_x(x)\, dx = \frac{1}{0.48}\int_1^2 \left(\frac{10}{25}\right)\left(1 - \frac{x}{5}\right) dx$$

$$= 0.8333\left[x - \frac{x^2}{10}\right]_1^2 = 0.833\left[\left(2 - \frac{4}{10}\right) - \left(1 - \frac{1}{10}\right)\right]$$

$$= 0.5833$$

(g) $$E[X \mid W] = \left(\frac{10}{25}\right)\left(\frac{1}{0.48}\right) \int_1^3 x - \frac{x^2}{5}\, dx$$

$$= 0.833\left[\frac{x^2}{2} - \frac{x^3}{15}\right]_1^3$$

$$= 0.833\left[\left(\frac{9}{2} - \frac{27}{15}\right) - \left(\frac{1}{2} - \frac{1}{15}\right)\right]$$

$$E[X \mid W] = 1.888$$

5-5 | 3.59   requests / min ~ X

$E[X] = \alpha = 6000 \Rightarrow 100$ request / second

time b/t events $Y \sim$ exp w/ mean $\frac{1}{\alpha} = \frac{1}{6000}$

(a) Pr(no requests in 100 ms) = Pr(time b/t requests = 100 ms)

exp RV $E[Y] = \frac{1}{\lambda} \rightarrow E[Y] = \frac{1}{6000} \Rightarrow \lambda = 6000$

$Pr(Y > 100\text{ ms}) = 1 - Pr(Y < 100\text{ ms})$

$= 1 - F_Y(100\text{ ms})$

$F_Y(100\text{ ms}) = F_Y(0.1)$ 6000 req / min

Pr(No requests for $\frac{1}{600}$ of a minute)

$= Pr\left(Y > \frac{1}{600}\right) = 1 - F_Y\left(\frac{1}{600}\right)$

$= 1 - \left(1 - e^{-\lambda \frac{1}{600}}\right)$

$= \exp\left(\frac{-1}{600 \cdot 6000}\right) =$

5-5 | 3.59 $X \sim \alpha = 6000$ req/min

$Y \sim \alpha = 100$ req/second

(a) $Z \sim \lambda = \frac{1}{100}$ second b/t request avg

$Pr\left(Z > \frac{10}{100} \text{ seconds b/t request}\right) = 1 - F_Z(0.1)$

$= 1 - \left[1 - e^{-(100)(0.1)}\right] = e^{-10} = 4.54 \times 10^{-5}$

(b) Pr(5-10 reqs in 100 ms) $Y \sim \alpha = 10$ req/0.1 sec

$Z \sim \lambda = \frac{1}{10}$ sec/req

$F(10) - F(5) = ?$

$$\sum_{k=5}^{10} \frac{\alpha^k}{k!} e^{-\alpha} = \boxed{\sum_{k=5}^{10} \frac{10^k}{k!} e^{-10}}$$

5-6 $P(X > x) = \dfrac{2}{x+2}$ for $x > 0$ lifetime x

3 bulbs indep start @ $x = 0$

ⓐ $\Pr(\text{one bulb works}) = \Pr(2 \text{ break}, 1 \text{ works})$

$$= \binom{3}{1}\left(\frac{2}{x+2}\right)\left(1 - \frac{2}{x+2}\right)^2$$

ⓑ all are working after $x = 1$ → cond prob

$$\Pr(\text{All working at } x=1) = \left(\frac{2}{3}\right)^3 = 0.2963$$

$\Pr(\text{one working at } x=9 \mid \text{All work at } x=1) =$

$$\frac{\Pr(\text{one works @ } x=9)}{\Pr(\text{All work @ } x=1)} = \frac{\binom{3}{1}\left(\frac{2}{x+2}\right)\left(1-\frac{2}{x+2}\right)^2}{0.2963}$$

$$= \frac{3\left(\frac{2}{11}\right)\left(1 - \frac{2}{11}\right)^2}{0.2963}$$

$= 1.2323$ → can't be right

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rework → $P(X > x \mid X > 1) = \dfrac{\left(\frac{2}{x+2}\right)}{\left(\frac{2}{3}\right)} = 1.5\left(\dfrac{2}{x+2}\right)$

So: $\Pr(\text{one works @ } x=9) = \binom{3}{1}\left(\dfrac{3}{11}\right)\left(1 - \dfrac{3}{11}\right)^2$

$$\boxed{= 0.433}$$