Solution!

EE503 Midterm Exam 1
Fall 2017

Instructor: Prof. Salman Avestimehr

October 5, 2017

- Do not open this exam until you are instructed to do so.
- You have two hours to complete this exam.
- The exam is closed-book and closed-note.
- You are allowed to use 1 page crib sheet (letter size – one side only).
- No calculators are permitted.
- No collaboration is permitted.
- For true or false questions, you need not show your work. For all other problems, you must explain all your reasonings to get full mark.

Name	
Problem 1 (12 pts)	
Problem 2 (12 pts)	
Problem 3 (18 pts)	
Problem 4 (16 pts)	
Problem 5 (12 pts)	
Problem 6 (14 pts)	
Problem 7 (16 pts)	
Total (100 pts)	

1. (12 points) **True or False – short questions.** *Circle* true or false as appropriate, or provide a short answer.

True False (a) (1 point) Suppose X is a $N(1,1)$ random variable. Then X is continuous and $f_X(1) > f_X(x)$ for all $x \neq 1$.

True False (b) (1 point) Suppose events A and B are independent. Then A^c and B^c are also independent. $P(A^c \cap B^c) = P(\overline{A \cup B}) = 1 - P(A \cup B) = (1 - P(A))(1 - P(B))$

True False (c) (1 point) A CDF may have infinitely many discontinuities.

True False (d) (1 point) If A, B, and C are events such that $A \cap B = \emptyset$ and $P(C) > 0$, then

$$P(A \cup B|C) = P(A|C) + P(B|C).$$

True False (e) (1 point) For any events A, B, and C we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C).$$

True False (f) (1 point) The function

$$f_X(x) = \begin{cases} x^2 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

is a valid PDF.

True False (g) (1 point) If X is Binomial(n,p) then $Y = n - X$ is Binomial$(n, 1-p)$.

True False (h) (1 point) Suppose X is a random variable with mean 0 and variance 1, and $Y = 3X + 1$. Then Y is a random variable with mean 1 and variance 10.

Short Question (i) (1 point) A random experiment has sample space $\Omega = \{a,b,c,d\}$. Suppose that $P(\{b,c\}) = P(\{b,d\}) = 5/12$, $P(\{a,b\}) = 2/3$, and $P(\{a,c\}) = 7/12$. Determine $P(\{a\})$.

$P(a) = 1 - P(\{b\}) - P(\{c\}) - P(\{d\}) = \frac{5}{12}$, $\left.\begin{array}{l} P(\{b\}) - P(\{c\}) = \frac{1}{12} \\ P(\{b\}) + P(\{c\}) = \frac{5}{12} \end{array}\right\} \Rightarrow \left.\begin{array}{l} P(\{b\}) = \frac{1}{4} \\ P(\{c\}) = \frac{1}{6} \end{array}\right\}$

$P(\{d\}) = \frac{5}{12} - P(\{b\}) =$

Short Question (j) (1 point) What is a random variable?

a function whose domain is a sample space and whose range is some set of real numbers is called RV.

Short Question (k) (2 points) Suppose that the number of telephone calls made in a day is an exponential random variable with mean 1500. Find a bound for the probability that more than 2017 calls are made in a day using Chebyshev's Inequality.

$P(X > 2017) = P(X - 1500 > 517) = P(|X - E(X)| > 517) \leq \frac{Var(X)}{517^2}$

$= \frac{1500^2}{517^2}$

2. (12 points) Three types of packets arrive at a router port. Twenty percent of the packets are "expedited forwarding (EF)" (e.g., live video streaming), 30 percent are "assured forwarding (AF)" (e.g., email and html packets), and 50 percent are "best effort (BE)" (e.g., youtube video). The arrival of the packets are independent of each other.

(a) (4 points) Find the probability that k out of N packets are EF.

any minor mistake -1
any major mistake -2 or -3

k from EF
N-k from non-EF

$$\text{Prob} = \binom{N}{k} (0.2)^k (0.8)^{N-k}$$

(b) (4 points) Suppose that packets arrive one at a time. Find the probability that k other packets are received before an EF packet arrives.

k other packets should not be EF.

$$\text{Prob} = (1-0.2)^k (0.2)$$

(c) (4 points) Find the probability that out of 20 packets, 8 are EF packets, 5 are AF packets, and 7 are BE packets.

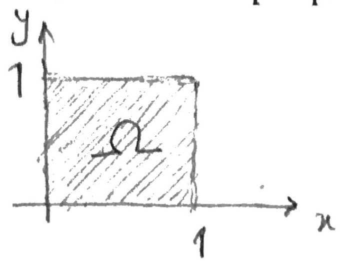
$$\text{Prob} = \frac{20!}{8!\,5!\,7!} (0.2)^8 (0.3)^5 (0.5)^7$$

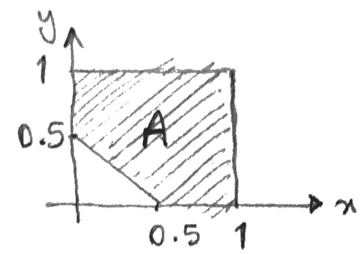
3. (18 points) Consider a probabilistic model with sample space $\Omega = [0,1] \times [0,1]$ (i.e., the unit square in $\mathbb{R}^2$), such that the probability of any event $A \subseteq \Omega$ is equal to the area of A. Consider the events

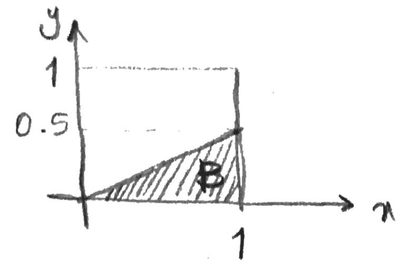
$$A = \{(x,y) \in \Omega | x + y \geq 0.5\}.$$

$$B = \{(x,y) \in \Omega | x \geq 2y\}.$$

(a) (4 points) Draw the sample space, and events A and B.







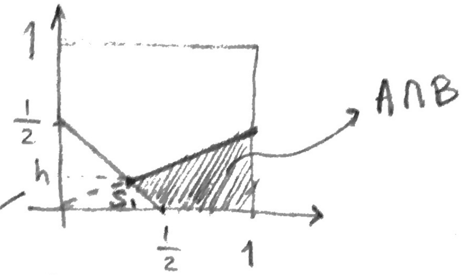
each mistake -1

(b) (4 points) Compute $P(A)$ and $P(B)$.

(2 points) $P(A)$ = area of region A in part a $= 1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{7}{8}$

(2 points) $P(B)$ = area of region B in part b $= \frac{1}{2}(1)\left(\frac{1}{2}\right) = \frac{1}{4}$

(c) (4 points) Compute $P(A \cap B)$ and $P(A \cup B^c)$. (Each 2 points)

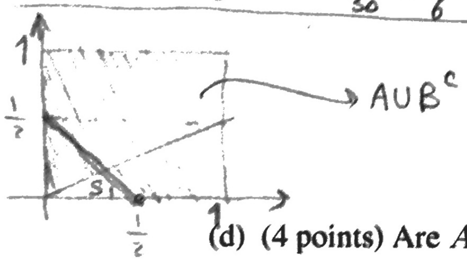


$h = y$, $x + y = 0.5$, $x = 2y$ $\Rightarrow h = \frac{5}{30} = \frac{1}{6}$

$\Rightarrow P(A \cap B) =$ area of shaded region

$=$ area (triangle, $\frac{1}{2}$, 1) $-$ area (S_1)

$= \frac{1}{4} - \frac{1}{2}\left(\frac{1}{2}\right)h = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}$



$P(A \cup B^c) = 1 - P(S_1) = 1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{6} = \frac{23}{24}$

(d) (4 points) Are A and B independent? Explain.

we should check $P(A \cap B) \stackrel{?}{=} P(A)P(B)$ to see if they are independent. Since

minor mistake -1
major mistake -3

$P(A \cap B) = \frac{5}{24} \neq P(A)P(B) = \frac{7}{8} \times \frac{1}{4}$,

A and B are NOT independant.

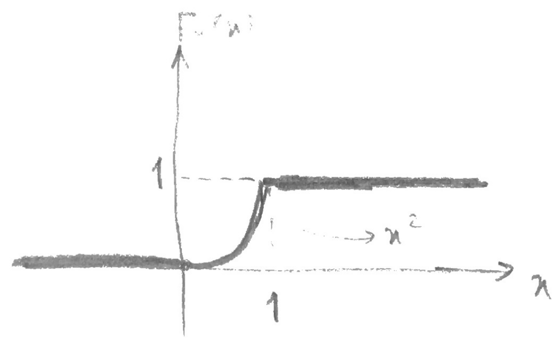
(e) (2 points) Compute $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{5}{24}}{\frac{1}{4}} = \frac{5}{6}$$

4. (16 points) Mehrdad tosses a dart and it lands on a point uniformly distributed within the unit circle centered at the origin.

(a) (4 points) Let the random variable X represent the distance of the point to the origin. Compute and plot the CDF of X.

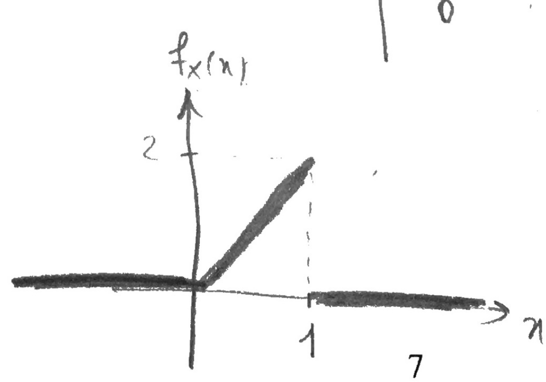
$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$



(b) (4 points) Is X continuous? If so, compute and plot its PDF. Is X discrete? If so, compute and plot its PMF.

X is continuous.

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

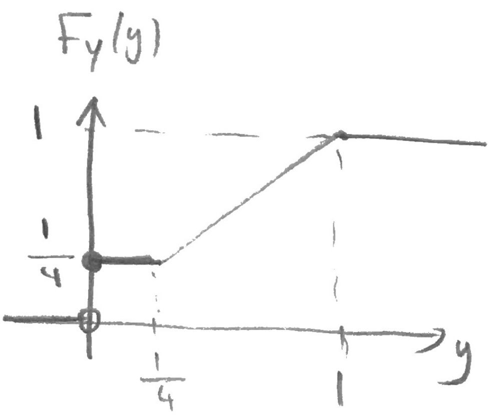


(c) (4 points) Suppose that Mehrdad receives a prize Y given by

$$Y = \begin{cases} 0 & \text{if } X \leq \frac{1}{2} \\ X^2 & \text{otherwise} \end{cases}$$

Compute and plot the CDF of Y.

$$P_Y(y) = P(Y \leq y) = \begin{cases} 0 & y<0 \\ P(X \leq \frac{1}{2}) & y=0 \\ P(X^2 \leq y) & \frac{1}{4} \leq y < 1 \\ 1 & 1 \leq y \end{cases}$$



$$= \begin{cases} 0 & y<0 \\ \frac{1}{4} & y=0 \\ P(X \leq \sqrt{y}) & \frac{1}{4} \leq y < 1 \\ 1 & 1 \leq y \end{cases} = \begin{cases} 0 & y<0 \\ \frac{1}{4} & 0 \leq y < \frac{1}{4} \\ y & \frac{1}{4} < y \leq 1 \\ 1 & y > 1 \end{cases}$$

$$= \begin{cases} 0 & y < \frac{1}{4} \\ y & \frac{1}{4} \leq y < 1 \\ 1 & 1 \leq y \end{cases}$$

(d) (4 points) Is Y continuous? If so, compute and plot its PDF. Is Y discrete? If so, compute and plot its PMF.

Since $P(Y=0) = \frac{1}{4} \neq 0 \Rightarrow$ not continuous

Since $F_Y(y)$ is not step-wise flat, Y is not discrete eit

5. (12 points) Suppose that the random variable X has the following CDF

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{2} + \frac{1}{3} & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

(a) (3 points) Is X discrete? Explain.

it is NOT discrete. $F_X(x)$ is NOT step-wise flat

($F_X(x) = \frac{x}{2} + \frac{1}{3}$ for $0 \le x < 1$)

(b) (3 points) Is X continuous? Explain.

it is NOT continuous cause probability of

$P(X=0) = \frac{1}{3}$ which is non-zero (in continuous RV,

the probability of the case RV takes specific value is zero)

(c) (3 points) Compute $P(0 \le X \le 1)$.

$$P(0 \le X \le 1) = P(X=0) + P(0 < X \le 1)$$
$$= \frac{1}{3} + F_X(1) - F_X(0) = 1$$

(d) (3 points) Compute $P(0 < X < 1)$.

$$P(0 < X < 1) = P(X < 1) - P(X \le 0)$$
$$= \left(\frac{1}{2} + \frac{1}{3}\right) - P(X=0)$$
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{3} = \frac{1}{2}$$

6. (14 points) A random variable X has PDF $f_X(.)$ where

$$f_X(x) = \begin{cases} cx^2(1-x), & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) (2 points) Find c.

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \Rightarrow \int_0^1 cx^2(1-x)\,dx = c\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 1$$

$$\Rightarrow c = 12$$

(b) (4 points) Find $P(\frac{1}{3} \leq X \leq \frac{2}{3})$.

$$P\left(\frac{1}{3} \leq X \leq \frac{2}{3}\right) = \int_{\frac{1}{3}}^{\frac{2}{3}} 12(x^2)(1-x)\,dx = 12\left[\frac{1}{3}\left(\frac{2}{3}\right)^3 - \frac{1}{4}\left(\frac{2}{3}\right)^4\right]$$

$$-12\left[\frac{1}{3}\left(\frac{1}{3}\right)^3 - \frac{1}{4}\left(\frac{1}{3}\right)^4\right] = 12 \times \frac{13}{4 \times 81} = \frac{39}{81}$$

$$\frac{13}{27}$$

(c) (4 points) Find the CDF $F_X(.)$ of X;

$$F_X(x) = \int_{y=-\infty}^{x} f_X(y)\,dy$$

$$= \begin{cases} 0 & x \le 0 \\ 4x^3 - 3x^4 & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$$

(d) (4 points) Calculate $E[X]$ and $var[X]$.

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^1 12x^3(1-x)\,dx = 3x^4 - \frac{12}{5}x^5 \Big|_0^1$$

$$= \frac{3}{5}$$

for variance, we first compute $E(x^2)$

$$E(x^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_0^1 12x^4(1-x)\,dx = \frac{12}{5}x^5 - 2x^6 \Big|_0^1 = \frac{2}{5}$$

$$Var(X) = E(X^2) - E^2(X) = \frac{2}{5} - \frac{9}{25} = \frac{1}{25}$$

7. (16 points) When a customer enters a branch of the USC Trust Company, the customer is either helped immediately or must wait a deterministic amount of time τ in order to be helped. The customer must wait with probability p. Once a teller starts helping the customer, the time until the transaction is finished is exponentially distributed with parameter λ. Let T denote the total time between when the customer enters the branch and the transaction is completed.

(a) (4 points) Compute and plot the CDF of T.

$$T = \underbrace{\omega}_{\text{waiting time}} + \underbrace{S}_{\text{time for transact (servic}}$$

$$F_T(t) = P(T \le t) = P(T \le t \mid wait)P(wait) + P(T \le t \mid no\ wait)P(no\ wa$$

$$= P(S \le t-\tau)\, p + P(S \le t)\,(1-p)$$

$$= \underbrace{\left[1-e^{-\lambda(t-\tau)}\right]}_{for\ t \ge \tau} p + \underbrace{\left[1-e^{-\lambda t}\right]}_{for\ t \ge 0}(1-p)$$

$$= \left(1-e^{-\lambda(t-\tau)}\right) p\, u(t-\tau) + \left(1-e^{-\lambda t}\right)(1-p)\, u(t$$

(b) (4 points) Is T continuous? If so, compute its PDF. Is T discrete? If so, compute its PMF.

T is continuous.

$$f_T(t) = \frac{d F_T(t)}{dt}$$

$$= \lambda p e^{-\lambda(t-\tau)} u(t-\tau) + \lambda(1-p) e^{-\lambda t} u(t)$$

(c) (4 points) Compute the chance that $T \geq 2\tau$.

$$P(T \geq 2\tau) = 1 - P(T < 2\tau)$$
$$= 1 - F_T(2\tau)$$
$$= 1 - p\left[1 - e^{-\lambda\tau}\right] - (1-p)\left[1 - e^{-\lambda 2\tau}\right]$$

(d) (4 points) Suppose the customer is still in the store after 2τ time units. What is the chance that the customer had to wait before being helped?

A = customer is still in the store after 2τ

B = customer had to wait before being helped

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)} = \frac{p e^{-\lambda\tau}}{1 - F_T(2\tau)}$$

$$= \frac{p e^{-\lambda\tau}}{1 - p(1 - e^{-\lambda\tau}) - (1-p)(1 - e^{-2\lambda\tau})}$$