PS 9

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9-1 | 7.4 CF cauchy RV. X, Y indep

ⓐ $X \sim$ cauchy $(\alpha = 1)$ $Y \sim$ cauchy $(\alpha = 4)$

$$\Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$$

$$\Phi_Z(\omega) = e^{-1|\omega|} \cdot e^{-4|\omega|}$$

$$\Phi_Z(\omega) = e^{-5|\omega|}$$

ⓑ pdf of Z:

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-5|\omega|} e^{-j\omega z} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{0} e^{5\omega} e^{-j\omega z} d\omega + \int_{0}^{\infty} e^{-5\omega} e^{-j\omega z} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{0} e^{\omega(5-jz)} d\omega + \int_{0}^{\infty} e^{\omega(-5-jz)} d\omega \right]$$

$$= \frac{1}{2\pi} \left(\left[\frac{e^{\omega(5-jz)}}{5-jz} \right]_{\omega=-\infty}^{\omega=0} + \left[\frac{e^{-\omega(+5+jz)}}{-5-jz} \right]_{\omega=0}^{\omega=\infty} \right)$$

$$= \frac{1}{2\pi} \left[\left[\frac{1}{5-jz} - 0 \right] + \left[0 - \frac{1}{-5-jz} \right] \right]$$

$$= \frac{1}{2\pi} \left[\frac{1}{5-jz} + \frac{1}{5+jz} \right]$$

$$= \frac{1}{2\pi} \left(\frac{5+j\omega}{25+z^2} + \frac{5-jz}{25+z^2} \right)$$

$$f_Z(z) = \frac{10}{2\pi(25+z^2)} = \frac{5}{\pi(25+z^2)}$$

9-2 | 7.5

$S_k = X_1 + X_2 + \dots X_k$, X_i's are indep RV's chi squared RVs w/ n_i degrees of freedom. Show S_k is chi square RV w/ $n = n_1 + \dots n_k$ degrees of freedom

$$\Phi_s(\omega) = \Phi_{X_1}(\omega) \cdot \Phi_{X_2}(\omega) \cdot \dots \Phi_{X_k}(\omega)$$

$$= \left(\frac{1}{1-2j\omega}\right)^{n_1/2} \cdot \left(\frac{1}{1-2j\omega}\right)^{n_2/2} \cdot \dots \left(\frac{1}{1-2j\omega}\right)^{n_k/2}$$

$$\Phi_s(\omega) = \left(\frac{1}{1-2j\omega}\right)^{(n_1+n_2+\dots n_k)/2} \longrightarrow$$ has form of a chi square RV w/ $\sum_{i=1}^{k} n_i$ degrees of freedom

9-3 | 7.7 $X \sim exp(2)$ $Y \sim exp(10)$ $Z = X + Y$, indep

(a) $\Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$

$$\Phi_Z(\omega) = \left(\frac{2}{2-j\omega}\right)\left(\frac{10}{10-j\omega}\right)$$

(b) $$\Phi_Z(\omega) = \frac{20}{20 - 12j\omega + \omega^2} = \frac{x}{2-j\omega} + \frac{y}{10-j\omega}$$

$$(10-j\omega)x + (2-j\omega)y = 20$$

$$x = \frac{2-j\omega}{20} \qquad y = \frac{10-j\omega}{20}$$

$$= \frac{10-2}{\cancel{4}\cdot 10, 2}\left(\frac{\cancel{2}}{2-j\omega}\right) - \frac{10-2}{\cancel{200}20}\left(\frac{1}{10-j\omega}\right)$$

$$= \frac{8}{20}\left[\frac{1}{2-j\omega} - \frac{1}{10-j\omega}\right]$$

$$f_Z(z) = \frac{8}{20}\left[e^{-2z} - e^{-10z}\right], \quad z > 0$$

9-4 X, Y indep $X \sim N(\mu_1, \sigma_1^2)$ $Y \sim N(\mu_2, \sigma_2^2)$

$Z = X + Y$

$\Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$

$\Phi_Z(\omega) = \exp\left[j\mu_1\omega - \frac{1}{2}\sigma_1^2\omega^2\right] \exp\left[j\omega\mu_2 - \frac{1}{2}\sigma_2^2\omega^2\right]$

$\Phi_Z(\omega) = \exp\left[j\omega(\mu_1+\mu_2) - \frac{1}{2}\omega^2(\sigma_1^2+\sigma_2^2)\right]$

So $\mu_Z = \mu_1 + \mu_2$ $\sigma_Z^2 = \sigma_1^2 + \sigma_2^2$

$f_Z(z) = \dfrac{\exp\left(\dfrac{-(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}$ $-\infty < z < \infty$

9-5

ⓐ $\Phi_X(\omega) = \dfrac{\lambda}{\lambda - j\omega}$

$E[X^3] = ?$ $\dfrac{d^3}{d\omega^3}\left(\dfrac{\lambda}{\lambda - j\omega}\right) = \dfrac{-6j\lambda}{(\lambda - j\omega)^4} \div -j$ j^3

$E[X^3] = \dfrac{6\lambda}{(\lambda - j\omega)^4}\Bigg|_{\omega=0} = \dfrac{6\lambda}{\lambda^4} = \dfrac{6}{\lambda^3}$

ⓑ $\Phi_X(\omega) = \frac{1}{2}(1 + e^{2j\omega})$

$\dfrac{d}{d\omega}\left(\dfrac{1+e^{2j\omega}}{2}\right) = \dfrac{2j}{2}e^{2j\omega} = je^{2j\omega}$

$E[X] = e^{2j\omega}\Big|_{\omega=0}$

$G_X(z) = \frac{1}{2}(1 + z^2)$

$E[X] = 1$

$P_X(1) = \frac{1}{1!}\left(\frac{2z}{2}\right)\Big|_{z=0} = 0$

9-5 c) $\Phi_x(\omega) = \frac{1}{2}(1 + e^{2j\omega})$

$E[X^2] = ?$ $\qquad \frac{d}{d\omega}\Phi_x(\omega) = je^{2j\omega}$

$E[X] = 1$ from $\qquad \frac{d^2}{d\omega^2}\Phi_x(\omega) = 2\cancel{j^2}\, e^{2j\omega}$

9-5 (b) $\qquad E[X^2] = 2e^{2j\omega}\Big|_{\omega=0} = 2$

so $\sigma_x^2 = E[X^2] - E[X]^2$

$\sigma_x^2 = 2 - 1^2 = 1$

9-6) $P(X=k) = A(0.8)^k$ for all $k = 0, 1, 2 \ldots$

(a) $P(X=0) = G_X(z)\Big|_{z=0} = A$

$$G_X(z) = \sum_{k=0}^{\infty} A(0.8)^k z^k$$

$$G_X(z) = \frac{A}{1-0.8z} \qquad A = 0.2$$

Because $P(X=k) = p(1-p)^k \Rightarrow \sum_{k=0}^{\infty} p(1-p)^k$

$$G_X(z) = \frac{0.2}{1-0.8z}, \quad \Phi_X(\omega) = \frac{0.2}{1-0.8e^{j\omega}}$$

(b) $$E[X] = \frac{dG(z)}{dz}\Big|_{z=1} = \frac{0.16}{(1-0.8z)^2}\Big|_{z=1} = \frac{0.16}{0.2^2} = \frac{0.16}{0.04}$$

$E[X] = 4$ ✓

(c) $P(X>1) = 1 - P(X=1) - P(X=0)$

$= 1 - [A + 0.8A]$

$P(X>1) = 1 - [0.2 + 0.16] = 0.64$

9-6 d)

$$\Phi_Y(\omega) = \left(e^{j\omega} + 2e^{2j\omega} + e^{3j\omega}\right)(0.25)(e^{-j\omega})$$
$$= \left[\frac{1}{4} + \frac{1}{2}e^{j\omega} + \frac{1}{4}e^{2j\omega}\right]$$

$$j \cdot E[Y] = \left.\frac{d\Phi_Y(\omega)}{d\omega}\right|_{\omega=0}$$
$$= \left.\frac{j}{2}\cancel{e^{j\omega}}^{1} + \frac{j}{2}\cancel{e^{2j\omega}}^{1}\right|_{\omega=0}$$
$$jE[Y] = j \longrightarrow E[Y] = 1$$

e)

$$G_Y(z) = \frac{1}{4} + \frac{z}{2} + \frac{z^2}{4}$$
$$= \frac{1}{2}\left[\frac{1}{2} + z + \frac{z^2}{2}\right]$$

$$P_Y(k) = \left.\frac{1}{k!}\frac{d^k}{dz^k}G_Y(z)\right|_{z=0}$$

$$P_Y(1) = \left.\frac{1}{2} + \frac{z}{2}\right|_{z=0} = \frac{1}{2}$$

$$P_Y(0) = \left.\frac{1}{4} + \frac{z}{2} + \frac{z^2}{4}\right|_{z=0} = \frac{1}{4}$$

So $P(Y>1) = 1 - P_Y(1) - P_Y(0) = \frac{1}{4}$

f) $Z = X+Y$

$$\Phi_Z(\omega) = \Phi_X(\omega)\cdot\Phi_Y(\omega)$$
$$= \left(\frac{0.2}{1-0.8e^{j\omega}}\right)\left(\frac{1}{4} + \frac{1}{2}e^{j\omega} + \frac{1}{4}e^{2j\omega}\right)$$

$$\left.\frac{d\Phi_Z(\omega)}{dz}\right|_{\omega=0} = 5j \longrightarrow E[Z] = 5$$

g) $P(Z=1) = ?$ $\quad G_Z(z) = \left(\frac{0.2}{1-0.8z}\right)\left(\frac{1}{4} + \frac{z}{2} + \frac{z^2}{4}\right)$

$$\left.\frac{dG_Z(z)}{dz}\right|_{z=0} = 0.22$$

So $P(Z=1) = \frac{1}{1}(0.22) = 0.22$

9-7) Lifetime $X_i \sim N(3, 0.25)$

$Y = \sum_{i=1}^{100} X_i$ $\quad E[Y] = 300 \quad \sigma_Y^2 = 25$

$\sigma_Y = \sqrt{25} = 5$

find $P(E[Y] - 4 < Y < E[Y] + 4)$

$= P(296 < Y < 304)$

$= \Phi\left(\frac{304 - 300}{5\sqrt{n}}\right) - \Phi\left(\frac{296 - 300}{5\sqrt{n}}\right)$

$= \Phi\left(\frac{4}{50}\right) - \Phi\left(-\frac{4}{50}\right)$ ✓