- Many experiments can be viewed as consisting of a sequence of independent subexperiments. In this chapter we presented the binomial, the multinomial, and the geometric probability laws as models that arise in this context.
- A Markov chain consists of a sequence of subexperiments in which the outcome of a subexperiment determines which subexperiment is performed next. The probability of a sequence of outcomes in a Markov chain is given by the product of the probability of the first outcome and the probabilities of all subsequent transitions.
- Computer simulation models use recursive equations to generate sequences of pseudo-random numbers.

CHECKLIST OF IMPORTANT TERMS

Axioms of Probability
Bayes' rule
Bernoulli trial
Binomial coefficient
Binomial theorem
Certain event
Conditional probability
Continuous sample space
Discrete sample space
Elementary event
Event
Event class
Independent events
Independent experiments
Initial probability assignment
Markov chain
Mutually exclusive events
Null event
Outcome
Partition
Probability law
Sample space
Set operations
Theorem on total probability
Tree diagram

ANNOTATED REFERENCES

There are dozens of introductory books on probability and statistics. The books listed here are some of my favorites. They start from the very beginning, they draw on intuition, they point out where mysterious complications lie below the surface, and they are fun to read! Reference [9] presents an introduction ot Octave and [10] gives an excellent introduction to computer simulation methods of random systems. Reference [11] is an online tutorial for Octave.

1. Y. A. Rozanov, *Probability Theory: A Concise Course*, Dover Publications, New York, 1969.
2. P. L. Meyer, *Introductory Probability and Statistical Applications*, Addison-Wesley, Reading, Mass., 1970.
3. K. L. Chung, *Elementary Probability Theory*, Springer-Verlag, New York, 1974.
4. Robert B. Ash, *Basic Probability Theory*, Wiley, New York, 1970.
5. L. Breiman, *Probability and Stochastic Processes*, Houghton Mifflin, Boston, 1969.
6. Terrence L. Fine, *Probability and Probabilistic Reasoning for Electrical Engineering*, Prentice Hall, Upper Saddle River, N.J., 2006.

7. W. Feller, *An Introduction to Probability Theory and Its Applications*, 3d ed., Wiley, New York, 1968.
8. A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis*, Dover Publications, New York, 1970.
9. P. J. G. Long, "Introduction to Octave," University of Cambridge, September 2005, available online.
10. A. M. Law and W. D. Kelton, *Simulation Modeling and Analysis*, McGraw-Hill, New York, 2000.

PROBLEMS

Section 2.1: Specifying Random Experiments

2.1. The (loose) minute hand in a clock is spun hard and the hour at which the hand comes to rest is noted.

(a) What is the sample space?

(b) Find the sets corresponding to the events: A = "hand is in first 4 hours"; B = "hand is between 2nd and 8th hours inclusive"; and D = "hand is in an odd hour."

(c) Find the events: $A \cap B \cap D$, $A^c \cap B$, $A \cup (B \cap D^c)$, $(A \cup B) \cap D^c$.

2.2. A die is tossed twice and the number of dots facing up in each toss is counted and noted in the order of occurrence.

(a) Find the sample space.

(b) Find the set A corresponding to the event "number of dots in first toss is not less than number of dots in second toss."

(c) Find the set B corresponding to the event "number of dots in first toss is 6."

(d) Does A imply B or does B imply A?

(e) Find $A \cap B^c$ and describe this event in words.

(f) Let C correspond to the event "number of dots in dice differs by 2." Find $A \cap C$.

2.3. Two dice are tossed and the magnitude of the difference in the number of dots facing up in the two dice is noted.

(a) Find the sample space.

(b) Find the set A corresponding to the event "magnitude of difference is 3."

(c) Express each of the elementary events in this experiment as the union of elementary events from Problem 2.2.

2.4. A binary communication system transmits a signal X that is either a $+2$ voltage signal or a -2 voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal.

(a) Find the sample space.

(b) Find the set of outcomes corresponding to the event "transmitted signal was definitely $+2$."

(c) Describe in words the event corresponding to the outcome $Y = 0$.

2.5. A desk drawer contains six pens, four of which are dry.

(a) The pens are selected at random one by one until a good pen is found. The sequence of test results is noted. What is the sample space?

(b) Suppose that only the number, and not the sequence, of pens tested in part a is noted. Specify the sample space.

(c) Suppose that the pens are selected one by one and tested until both good pens have been identified, and the sequence of test results is noted. What is the sample space?

(d) Specify the sample space in part c if only the number of pens tested is noted.

2.6. Three friends (Al, Bob, and Chris) put their names in a hat and each draws a name from the hat. (Assume Al picks first, then Bob, then Chris.)

(a) Find the sample space.

(b) Find the sets A, B, and C that correspond to the events "Al draws his name," "Bob draws his name," and "Chris draws his name."

(c) Find the set corresponding to the event, "no one draws his own name."

(d) Find the set corresponding to the event, "everyone draws his own name."

(e) Find the set corresponding to the event, "one or more draws his own name."

2.7. Let M be the number of message transmissions in Experiment E_6.

(a) What is the set A corresponding to the event "M is even"?

(b) What is the set B corresponding to the event "M is a multiple of 3"?

(c) What is the set C corresponding to the event "6 or fewer transmissions are required"?

(d) Find the sets $A \cap B$, $A - B$, $A \cap B \cap C$ and describe the corresponding events in words.

2.8. A number U is selected at random from the unit interval. Let the events A and B be: A = "U differs from 1/2 by more than 1/4" and B = "$1 - U$ is less than 1/2." Find the events $A \cap B$, $A^c \cap B$, $A \cup B$.

2.9. The sample space of an experiment is the real line. Let the events A and B correspond to the following subsets of the real line: $A = (-\infty, r]$ and $B = (-\infty, s]$, where $r \leq s$. Find an expression for the event $C = (r, s]$ in terms of A and B. Show that $B = A \cup C$ and $A \cap C = \varnothing$.

2.10. Use Venn diagrams to verify the set identities given in Eqs. (2.2) and (2.3). You will need to use different colors or different shadings to denote the various regions clearly.

2.11. Show that:

(a) If event A implies B, and B implies C, then A implies C.

(b) If event A implies B, then B^c implies A^c.

2.12. Show that if $A \cup B = A$ and $A \cap B = A$ then $A = B$.

2.13. Let A and B be events. Find an expression for the event "exactly one of the events A and B occurs." Draw a Venn diagram for this event.

2.14. Let A, B, and C be events. Find expressions for the following events:

(a) Exactly one of the three events occurs.

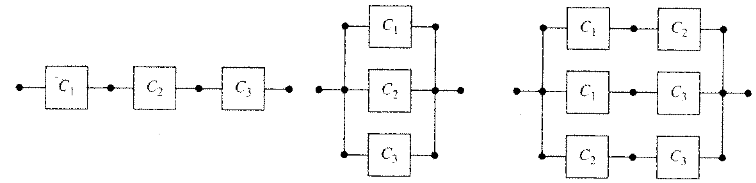
(b) Exactly two of the events occur.

(c) One or more of the events occur.

(d) Two or more of the events occur.

(e) None of the events occur.

2.15. Figure P2.1 shows three systems of three components, C_1, C_2, and C_3. Figure P2.1(a) is a "series" system in which the system is functioning only if all three components are functioning. Figure 2.1(b) is a "parallel" system in which the system is functioning as long as at least one of the three components is functioning. Figure 2.1(c) is a "two-out-of-three" system in which the system is functioning as long as at least two components are functioning. Let A_k be the event "component k is functioning." For each of the three system configurations, express the event "system is functioning" in terms of the events A_k.



(a) Series system (b) Parallel system (c) Two-out-of-three system

FIGURE P2.1

2.16. A system has two key subsystems. The system is "up" if both of its subsystems are functioning. Triple redundant systems are configured to provide high reliability. The overall system is operational as long as one of three systems is "up." Let A_{jk} correspond to the event "unit k in system j is functioning," for $j = 1, 2, 3$ and $k = 1, 2$.

(a) Write an expression for the event "overall system is up."

(b) Explain why the above problem is equivalent to the problem of having a connection in the network of switches shown in Fig. P2.2.

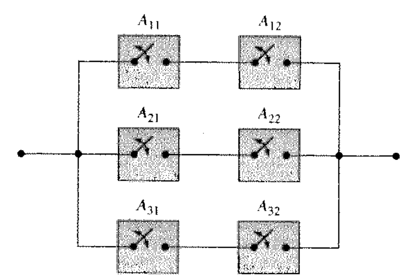


FIGURE P2.2

2.17. In a specified 6-AM-to-6-AM 24-hour period, a student wakes up at time t_1 and goes to sleep at some later time t_2.

(a) Find the sample space and sketch it on the x-y plane if the outcome of this experiment consists of the pair (t_1, t_2).

(b) Specify the set A and sketch the region on the plane corresponding to the event "student is asleep at noon."

(c) Specify the set B and sketch the region on the plane corresponding to the event "student sleeps through breakfast (7–9 AM)."

(d) Sketch the region corresponding to $A \cap B$ and describe the corresponding event in words.

2.18. A road crosses a railroad track at the top of a steep hill. The train cannot stop for oncoming cars and cars cannot see the train until it is too late. Suppose a train begins crossing the road at time t_1 and that the car begins crossing the track at time t_2, where $0 < t_1 < T$ and $0 < t_2 < T$.

(a) Find the sample space of this experiment.

(b) Suppose that it takes the train d_1 seconds to cross the road and it takes the car d_2 seconds to cross the track. Find the set that corresponds to a collision taking place.

(c) Find the set that corresponds to a collision is missed by 1 second or less.

2.19. A random experiment has sample space $S = \{-1, 0, +1\}$.

(a) Find all the subsets of S.

(b) The outcome of a random experiment consists of pairs of outcomes from S where the elements of the pair cannot be equal. Find the sample space S' of this experiment. How many subsets does S' have?

2.20. **(a)** A coin is tossed twice and the sequence of heads and tails is noted. Let S be the sample space of this experiment. Find all subsets of S.

(b) A coin is tossed twice and the number of heads is noted. Let S? be the sample space of this experiment. Find all subsets of S'.

(c) Consider parts a and b if the coin is tossed 10 times. How many subsets do S and S' have? How many bits are needed to assign a binary number to each possible subset?

Section 2.2: The Axioms of Probability

2.21. A die is tossed and the number of dots facing up is noted.

(a) Find the probability of the elementary events under the assumption that all faces of the die are equally likely to be facing up after a toss.

(b) Find the probability of the events: $A = \{\text{more than 3 dots}\}$; $B = \{\text{odd number of dots}\}$.

(c) Find the probability of $A \cup B$, $A \cap B$, A^c.

2.22. In Problem 2.2, a die is tossed twice and the number of dots facing up in each toss is counted and noted in the order of occurrence.

(a) Find the probabilities of the elementary events.

(b) Find the probabilities of events A, B, C, $A \cap B^c$, and $A \cap C$ defined in Problem 2.2.

2.23. A random experiment has sample space $S = \{a, b, c, d\}$. Suppose that $P[\{c, d\}] = 3/8$, $P[\{b, c\}] = 6/8$, and $P[\{d\}] = 1/8$, $P[\{c, d\}] = 3/8$. Use the axioms of probability to find the probabilities of the elementary events.

2.24. Find the probabilities of the following events in terms of $P[A]$, $P[B]$, and $P[A \cap B]$:

(a) A occurs and B does not occur; B occurs and A does not occur.

(b) Exactly one of A or B occurs.

(c) Neither A nor B occur.

2.25. Let the events A and B have $P[A] = x$, $P[B] = y$, and $P[A \cup B] = z$. Use Venn diagrams to find $P[A \cap B]$, $P[A^c \cap B^c]$, $P[A^c \cup B^c]$, $P[A \cap B^c]$, $P[A^c \cup B]$.

2.26. Show that

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C].$$

2.27. Use the argument from Problem 2.26 to prove Corollary 6 by induction.

2.28. A hexadecimal character consists of a group of three bits. Let A_i be the event "ith bit in a character is a 1."

(a) Find the probabilities for the following events: A_1, $A_1 \cap A_3$, $A_1 \cap A_2 \cap A_3$ and $A_1 \cup A_2 \cup A_3$. Assume that the values of bits are determined by tosses of a fair coin.

(b) Repeat part a if the coin is biased.

2.29. Let M be the number of message transmissions in Problem 2.7. Find the probabilities of the events A, B, C, C^c, $A \cap B$, $A - B$, $A \cap B \cap C$. Assume the probability of successful transmission is 1/2.

2.30. Use Corollary 7 to prove the following:

(a) $P[A \cup B \cup C] \leq P[A] + P[B] + P[C]$.

(b) $P\left[\bigcup_{k=1}^{n} A_k\right] \leq \sum_{k=1}^{n} P[A_k]$.

(c) $P\left[\bigcap_{k=1}^{n} A_k\right] \geq 1 - \sum_{k=1}^{n} P[A_k^c]$.

The second expression is called the **union bound**.

2.31. Let p be the probability that a single character appears incorrectly in this book. Use the union bound for the probability of there being any errors in a page with n characters.

2.32. A die is tossed and the number of dots facing up is noted.

(a) Find the probability of the elementary events if faces with an even number of dots are twice as likely to come up as faces with an odd number.

(b) Repeat parts b and c of Problem 2.21.

2.33. Consider Problem 2.1 where the minute hand in a clock is spun. Suppose that we now note the *minute* at which the hand comes to rest.

(a) Suppose that the minute hand is very loose so the hand is equally likely to come to rest anywhere in the clock. What are the probabilities of the elementary events?

(b) Now suppose that the minute hand is somewhat sticky and so the hand is 1/2 as likely to land in the second minute than in the first, 1/3 as likely to land in the third minute as in the first, and so on. What are the probabilities of the elementary events?

(c) Now suppose that the minute hand is very sticky and so the hand is 1/2 as likely to land in the second minute than in the first, 1/2 as likely to land in the third minute as in the second, and so on. What are the probabilities of the elementary events?

(d) Compare the probabilities that the hand lands in the last minute in parts a, b, and c.

2.34. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$.

(a) Find the probabilities of A, B, $A \cap B$, and $A \cap C$.

(b) Find the probabilities of $A \cup B$, $A \cup C$, and $A \cup B \cup C$, first, by directly evaluating the sets and then their probabilities, and second, by using the appropriate axioms or corollaries.

2.35. A number x is selected at random in the interval $[-1, 2]$. Numbers from the subinterval $[0, 2]$ occur half as frequently as those from $[-1, 0)$.

(a) Find the probability assignment for an interval completely within $[-1, 0)$; completely within $[0, 2]$; and partly in each of the above intervals.

(b) Repeat Problem 2.34 with this probability assignment.

2.36. The lifetime of a device behaves according to the probability law $P[(t, \infty)] = 1/t$ for $t > 1$. Let A be the event "lifetime is greater than 4," and B the event "lifetime is greater than 8."

(a) Find the probability of $A \cap B$, and $A \cup B$.

(b) Find the probability of the event "lifetime is greater than 6 but less than or equal to 12."

2.37. Consider an experiment for which the sample space is the real line. A probability law assigns probabilities to subsets of the form $(-\infty, r]$.

(a) Show that we must have $P[(-\infty, r]] \leq P[(-\infty, s]]$ when $r < s$.

(b) Find an expression for $P[(r, s]]$ in terms of $P[(-\infty, r]]$ and $P[(-\infty, s]]$

(c) Find an expression for $P[(s, \infty)]$.

2.38. Two numbers (x, y) are selected at random from the interval $[0, 1]$.

(a) Find the probability that the pair of numbers are inside the unit circle.

(b) Find the probability that $y > 2x$.

*Section 2.3: Computing Probabilities Using Counting Methods

2.39. The combination to a lock is given by three numbers from the set $\{0, 1, \ldots, 59\}$. Find the number of combinations possible.

2.40. How many seven-digit telephone numbers are possible if the first number is not allowed to be 0 or 1?

2.41. A pair of dice is tossed, a coin is flipped twice, and a card is selected at random from a deck of 52 distinct cards. Find the number of possible outcomes.

2.42. A lock has two buttons: a "0" button and a "1" button. To open a door you need to push the buttons according to a preset 8-bit sequence. How many sequences are there? Suppose you press an arbitrary 8-bit sequence; what is the probability that the door opens? If the first try does not succeed in opening the door, you try another number; what is the probability of success?

2.43. A Web site requires that users create a password with the following specifications:

- Length of 8 to 10 characters
- Includes at least one special character $\{!, @, \#, \$, \%, \wedge, \&, *, (,), +, =, \{, \}, |, <, >, \backslash, \_, \sim, [,], /, ?\}$
- No spaces
- May contain numbers (0–9), lower and upper case letters (a–z, A–Z)
- Is case-sensitive.

How many passwords are there? How long would it take to try all passwords if a password can be tested in 1 microsecond?

2.44. A multiple choice test has 10 questions with 3 choices each. How many ways are there to answer the test? What is the probability that two papers have the same answers?

2.45. A student has five different t-shirts and three pairs of jeans ("brand new," "broken in," and "perfect").

(a) How many days can the student dress without repeating the combination of jeans and t-shirt?

(b) How many days can the student dress without repeating the combination of jeans and t-shirt and without wearing the same t-shirt on two consecutive days?

2.46. Ordering a "deluxe" pizza means you have four choices from 15 available toppings. How many combinations are possible if toppings can be repeated? If they cannot be repeated? Assume that the order in which the toppings are selected does not matter.

2.47. A lecture room has 60 seats. In how many ways can 45 students occupy the seats in the room?

2.48. List all possible permutations of two distinct objects; three distinct objects; four distinct objects. Verify that the number is $n!$.

2.49. A toddler pulls three volumes of an encyclopedia from a bookshelf and, after being scolded, places them back in random order. What is the probability that the books are in the correct order?

2.50. Five balls are placed at random in five buckets. What is the probability that each bucket has a ball?

2.51. List all possible combinations of two objects from two distinct objects; three distinct objects; four distinct objects. Verify that the number is given by the binomial coefficient.

2.52. A dinner party is attended by four men and four women. How many unique ways can the eight people sit around the table? How many unique ways can the people sit around the table with men and women alternating seats?

2.53. A hot dog vendor provides onions, relish, mustard, ketchup, Dijon ketchup, and hot peppers for your hot dog. How many variations of hot dogs are possible using one condiment? Two condiments? None, some, or all of the condiments?

2.54. A lot of 100 items contains k defective items. M items are chosen at random and tested.

(a) What is the probability that m are found defective? This is called the *hypergeometric distribution*.

(b) A lot is accepted if 1 or fewer of the M items are defective. What is the probability that the lot is accepted?

2.55. A park has N raccoons of which eight were previously captured and tagged. Suppose that 20 raccoons are captured. Find the probability that four of these are found to be tagged. Denote this probability, which depends on N, by $p(N)$. Find the value of N that maximizes this probability. *Hint:* Compare the ratio $p(N)/p(N-1)$ to unity.

2.56. A lot of 50 items has 40 good items and 10 bad items.

(a) Suppose we test five samples from the lot, with replacement. Let X be the number of defective items in the sample. Find $P[X = k]$.

(b) Suppose we test five samples from the lot, without replacement. Let Y be the number of defective items in the sample. Find $P[Y = k]$.

2.57. How many distinct permutations are there of four red balls, two white balls, and three black balls?

2.58. A hockey team has 6 forwards, 4 defensemen, and 2 goalies. At any time, 3 forwards, 2 defensemen, and 1 goalie can be on the ice. How many combinations of players can a coach put on the ice?

2.59. Find the probability that in a class of 28 students exactly four were born in each of the seven days of the week.

2.60. Show that

$$\binom{n}{k} = \binom{n}{n-k}$$

2.61. In this problem we derive the multinomial coefficient. Suppose we partition a set of n distinct objects into J subsets $B_1, B_2, \ldots, B_J$ of size $k_1, \ldots, k_J$, respectively, where $k_i \geq 0$, and $k_1 + k_2 + \ldots + k_J = n$.

(a) Let N_i denote the number of possible outcomes when the ith subset is selected. Show that

$$N_1 = \binom{n}{k_1}, N_2 = \binom{n-k_1}{k_2}, \ldots, N_{J-1} = \binom{n-k_1-\cdots-k_{J-2}}{k_{J-1}}.$$

(b) Show that the number of partitions is then:

$$N_1 N_2 \ldots N_{J-1} = \frac{n!}{k_1!\, k_2! \ldots k_J!}.$$

Section 2.4: Conditional Probability

2.62. A die is tossed twice and the number of dots facing up is counted and noted in the order of occurrence. Let A be the event "number of dots in first toss is not less than number of dots in second toss," and let B be the event "number of dots in first toss is 6." Find $P[A|B]$ and $P[B|A]$.

2.63. Use conditional probabilities and tree diagrams to find the probabilities for the elementary events in the random experiments defined in parts a to d of Problem 2.5.

2.64. In Problem 2.6 (name in hat), find $P[B \cap C|A]$ and $P[C|A \cap B]$.

2.65. In Problem 2.29 (message transmissions), find $P[B|A]$ and $P[A|B]$.

2.66. In Problem 2.8 (unit interval), find $P[B|A]$ and $P[A|B]$.

2.67. In Problem 2.36 (device lifetime), find $P[B|A]$ and $P[A|B]$.

2.68. In Problem 2.33, let $A = \{\text{hand rests in last 10 minutes}\}$ and $B = \{\text{hand rests in last 5 minutes}\}$. Find $P[B|A]$ for parts a, b, and c.

2.69. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$. Find $P[A|B]$, $P[B|C]$, $P[A|C^c]$, $P[B|C^c]$.

2.70. In Problem 2.36, let A be the event "lifetime is greater than t," and B the event "lifetime is greater than $2t$." Find $P[B|A]$. Does the answer depend on t? Comment.

2.71. Find the probability that two or more students in a class of 20 students have the same birthday. *Hint:* Use Corollary 1. How big should the class be so that the probability that two or more students have the same birthday is 1/2?

2.72. A cryptographic hash takes a message as input and produces a fixed-length string as output, called the digital fingerprint. A brute force attack involves computing the hash for a large number of messages until a pair of distinct messages with the same hash is found. Find the number of attempts required so that the probability of obtaining a match is 1/2. How many attempts are required to find a matching pair if the digital fingerprint is 64 bits long? 128 bits long?

2.73. (a) Find $P[A|B]$ if $A \cap B = \varnothing$; if $A \subset B$; if $A \supset B$.
(b) Show that if $P[A|B] > P[A]$, then $P[B|A] > P[B]$.

2.74. Show that $P[A|B]$ satisfies the axioms of probability.
(i) $0 \le P[A|B] \le 1$
(ii) $P[S|B] = 1$
(iii) If $A \cap C = \varnothing$, then $P[A \cup C|B] = P[A|B] + P[C|B]$.

2.75. Show that $P[A \cap B \cap C] = P[A|B \cap C]P[B|C]P[C]$.

2.76. In each lot of 100 items, two items are tested, and the lot is rejected if either of the tested items is found defective.
(a) Find the probability that a lot with k defective items is accepted.
(b) Suppose that when the production process malfunctions, 50 out of 100 items are defective. In order to identify when the process is malfunctioning, how many items should be tested so that the probability that one or more items are found defective is at least 99%?

2.77. A nonsymmetric binary communications channel is shown in Fig. P2.3. Assume the input is "0" with probability p and "1" with probability $1 - p$.
(a) Find the probability that the output is 0.
(b) Find the probability that the input was 0 given that the output is 1. Find the probability that the input is 1 given that the output is 1. Which input is more probable?

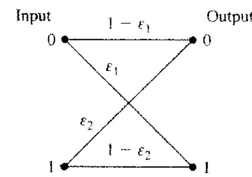


FIGURE P2.3

2.78. The transmitter in Problem 2.4 is equally likely to send $X = +2$ as $X = -2$. The malicious channel counts the number of heads in two tosses of a fair coin to decide by how much to reduce the magnitude of the input to produce the output Y.
(a) Use a tree diagram to find the set of possible input-output pairs.
(b) Find the probabilities of the input-output pairs.
(c) Find the probabilities of the output values.
(d) Find the probability that the input was $X = +2$ given that $Y = k$.

2.79. One of two coins is selected at random and tossed three times. The first coin comes up heads with probability p_1 and the second coin with probability $p_2 = 2/3 > p_1 = 1/3$.
(a) What is the probability that the number of heads is k?
(b) Find the probability that coin 1 was tossed given that k heads were observed, for $k = 0, 1, 2, 3$.
(c) In part b, which coin is more probable when k heads have been observed?
(d) Generalize the solution in part b to the case where the selected coin is tossed m times. In particular, find a threshold value T such that when $k > T$ heads are observed, coin 1 is more probable, and when $k < T$ are observed, coin 2 is more probable.
(e) Suppose that $p_2 = 1$ (that is, coin 2 is two-headed) and $0 < p_1 < 1$. What is the probability that we do not determine with certainty whether the coin is 1 or 2?

2.80. A computer manufacturer uses chips from three sources. Chips from sources A, B, and C are defective with probabilities .005, .001, and .010, respectively. If a randomly selected chip is found to be defective, find the probability that the manufacturer was A; that the manufacturer was C. Assume that the proportions of chips from A, B, and C are 0.5, 0.1, and 0.4, respectively.

2.81. A ternary communication system is shown in Fig. P2.4. Suppose that input symbols 0, 1, and 2 occur with probability 1/3 respectively.
(a) Find the probabilities of the output symbols.
(b) Suppose that a 1 is observed at the output. What is the probability that the input was 0? 1? 2?

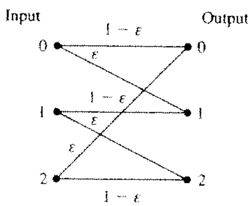


FIGURE P2.4

Section 2.5: Independence of Events

2.82. Let $S = \{1, 2, 3, 4\}$ and $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$. Assume the outcomes are equiprobable. Are A, B, and C independent events?

2.83. Let U be selected at random from the unit interval. Let $A = \{0 < U < 1/2\}$, $B = \{1/4 < U < 3/4\}$, and $C = \{1/2 < U < 1\}$. Are any of these events independent?

2.84. Alice and Mary practice free throws at the basketball court after school. Alice makes free throws with probability p_a and Mary makes them with probability p_m. Find the probability of the following outcomes when Alice and Mary each take one shot: Alice scores a basket; Either Alice or Mary scores a basket; both score; both miss.

2.85. Show that if A and B are independent events, then the pairs A and B^c, A^c and B, and A^c and B^c are also independent.

2.86. Show that events A and B are independent if $P[A|B] = P[A|B^c]$.

2.87. Let A, B, and C be events with probabilities $P[A]$, $P[B]$, and $P[C]$.

(a) Find $P[A \cup B]$ if A and B are independent.

(b) Find $P[A \cup B]$ if A and B are mutually exclusive.

(c) Find $P[A \cup B \cup C]$ if A, B, and C are independent.

(d) Find $P[A \cup B \cup C]$ if A, B, and C are pairwise mutually exclusive.

2.88. An experiment consists of picking one of two urns at random and then selecting a ball from the urn and noting its color (black or white). Let A be the event "urn 1 is selected" and B the event "a black ball is observed." Under what conditions are A and B independent?

2.89. Find the probabilities in Problem 2.14 assuming that events A, B, and C are independent.

2.90. Find the probabilities that the three types of systems are "up" in Problem 2.15. Assume that all units in the system fail independently and that a type k unit fails with probability p_k.

2.91. Find the probabilities that the system is "up" in Problem 2.16. Assume that all units in the system fail independently and that a type k unit fails with probability p_k.

2.92. A random experiment is repeated a large number of times and the occurrence of events A and B is noted. How would you test whether events A and B are independent?

2.93. Consider a very long sequence of hexadecimal characters. How would you test whether the relative frequencies of the four bits in the hex characters are consistent with independent tosses of coin?

2.94. Compute the probability of the system in Example 2.35 being "up" when a second controller is added to the system.

2.95. In the binary communication system in Example 2.26, find the value of ε for which the input of the channel is independent of the output of the channel. Can such a channel be used to transmit information?

2.96. In the ternary communication system in Problem 2.81, is there a choice of ε for which the input of the channel is independent of the output of the channel?

Section 2.6: Sequential Experiments

2.97. A block of 100 bits is transmitted over a binary communication channel with probability of bit error $p = 10^{-2}$.

(a) If the block has 1 or fewer errors then the receiver accepts the block. Find the probability that the block is accepted.

(b) If the block has more than 1 error, then the block is retransmitted. Find the probability that M retransmissions are required.

2.98. A fraction p of items from a certain production line is defective.

(a) What is the probability that there is more than one defective item in a batch of n items?

(b) During normal production $p = 10^{-3}$ but when production malfunctions $p = 10^{-1}$. Find the size of a batch that should be tested so that if any items are found defective we are 99% sure that there is a production malfunction.

2.99. A student needs eight chips of a certain type to build a circuit. It is known that 5% of these chips are defective. How many chips should he buy for there to be a greater than 90% probability of having enough chips for the circuit?

2.100. Each of n terminals broadcasts a message in a given time slot with probability p.

(a) Find the probability that exactly one terminal transmits so the message is received by all terminals without collision.

(b) Find the value of p that maximizes the probability of successful transmission in part a.

(c) Find the asymptotic value of the probability of successful transmission as n becomes large.

2.101. A system contains eight chips. The lifetime of each chip has a Weibull probability law: with parameters λ and $k = 2$: $P[(t, \infty)] = e^{-(\lambda t)^k}$ for $t \geq 0$. Find the probability that at least two chips are functioning after $2/\lambda$ seconds.

2.102. A machine makes errors in a certain operation with probability p. There are two types of errors. The fraction of errors that are type 1 is α, and type 2 is $1 - \alpha$.

(a) What is the probability of k errors in n operations?

(b) What is the probability of k_1 type 1 errors in n operations?

(c) What is the probability of k_2 type 2 errors in n operations?

(d) What is the joint probability of k_1 and k_2 type 1 and 2 errors, respectively, in n operations?

2.103. Three types of packets arrive at a router port. Ten percent of the packets are "expedited forwarding (EF)," 30 percent are "assured forwarding (AF)," and 60 percent are "best effort (BE)."

(a) Find the probability that k of N packets are not expedited forwarding.

(b) Suppose that packets arrive one at a time. Find the probability that k packets are received before an expedited forwarding packet arrives.

(c) Find the probability that out of 20 packets, 4 are EF packets, 6 are AF packets, and 10 are BE.

2.104. A run-length coder segments a binary information sequence into strings that consist of either a "run" of k "zeros" punctuated by a "one", for $k = 0, \ldots, m - 1$, or a string of m "zeros." The $m = 3$ case is:

String	Run-length k
1	0
01	1
001	2
000	3

Suppose that the information is produced by a sequence of Bernoulli trials with $P[\text{"one"}] = P[\text{success}] = p$.

(a) Find the probability of run-length k in the $m = 3$ case.

(b) Find the probability of run-length k for general m.

2.105. The amount of time cars are parked in a parking lot follows a geometric probability law with $p = 1/2$. The charge for parking in the lot is \$1 for each half-hour or less.

(a) Find the probability that a car pays k dollars.

(b) Suppose that there is a maximum charge of \$6. Find the probability that a car pays k dollars.

2.106. A biased coin is tossed repeatedly until heads has come up three times. Find the probability that k tosses are required. *Hint:* Show that $\{\text{"}k \text{ tosses are required"}\} = A \cap B$, where $A = \{\text{"}k\text{th toss is heads"}\}$ and $B = \{\text{"2 heads occurs in } k - 1 \text{ tosses"}\}$.

2.107. An urn initially contains two black balls and two white balls. The following experiment is repeated indefinitely: A ball is drawn from the urn; if the color of the ball is the same as the majority of balls remaining in the urn, then the ball is put back in the urn. Otherwise the ball is left out.

(a) Draw the trellis diagram for this experiment and label the branches by the transition probabilities.

(b) Find the probabilities for all sequences of outcomes of length 2 and length 3.

(c) Find the probability that the urn contains no black balls after three draws; no white balls after three draws.

(d) Find the probability that the urn contains two black balls after n trials; two white balls after n trials.

2.108. In Example 2.45, let $p_0(n)$ and $p_1(n)$ be the probabilities that urn 0 or urn 1 is used in the nth subexperiment.

(a) Find $p_0(1)$ and $p_1(1)$.

(b) Express $p_0(n + 1)$ and $p_1(n + 1)$ in terms of $p_0(n)$ and $p_1(n)$.

(c) Evaluate $p_0(n)$ and $p_1(n)$ for $n = 2, 3, 4$.

(d) Find the solution to the recursion in part b with the initial conditions given in part a.

(e) What are the urn probabilities as n approaches infinity?

*Section 2.7: Synthesizing Randomness: Number Generators

2.109. An urn experiment is to be used to simulate a random experiment with sample space $S = \{1, 2, 3, 4, 5\}$ and probabilities $p_1 = 1/3$, $p_2 = 1/5$, $p_3 = 1/4$, $p_4 = 1/7$, and $p_5 = 1 - (p_1 + p_2 + p_3 + p_4)$. How many balls should the urn contain? Generalize the result to show that an urn experiment can be used to simulate any random experiment with finite sample space and with probabilities given by rational numbers.

2.110. Suppose we are interested in using tosses of a fair coin to simulate a random experiment in which there are six equally likely outcomes, where $S = \{0, 1, 2, 3, 4, 5\}$. The following version of the "rejection method" is proposed:

1. Toss a fair coin three times and obtain a binary number by identifying heads with zero and tails with one.
2. If the outcome of the coin tosses in step 1 is the binary representation for a number in S, output the number. Otherwise, return to step 1.

(a) Find the probability that a number is produced in step 2.

(b) Show that the numbers that are produced in step 2 are equiprobable.

(c) Generalize the above algorithm to show how coin tossing can be used to simulate any random urn experiment.

2.111. Use the `rand` function in Octave to generate 1000 pairs of numbers in the unit square. Plot an x-y scattergram to confirm that the resulting points are uniformly distributed in the unit square.

2.112. Apply the rejection method introduced above to generate points that are uniformly distributed in the $x > y$ portion of the unit square. Use the `rand` function to generate a pair of numbers in the unit square. If $x > y$, accept the number. If not, select another pair. Plot an x-y scattergram for the pair of accepted numbers and confirm that the resulting points are uniformly distributed in the $x > y$ region of the unit square.

2.113. The *sample mean-squared value* of the numerical outcomes $X(1), X(2), \ldots X(n)$ of a series of n repetitions of an experiment is defined by

$$\langle X^2 \rangle_n = \frac{1}{n}\sum_{j=1}^{n} X^2(j).$$

(a) What would you expect this expression to converge to as the number of repetitions n becomes very large?

(b) Find a recursion formula for $\langle X^2 \rangle_n$ similar to the one found in Problem 1.9.

2.114. The *sample variance* is defined as the mean-squared value of the variation of the samples about the sample mean

$$\langle V^2 \rangle_n = \frac{1}{n}\sum_{j=1}^{n} \{X(j) - \langle X \rangle_n\}^2.$$

Note that the $\langle X \rangle_n$ also depends on the sample values. (It is customary to replace the n in the denominator with $n - 1$ for technical reasons that will be discussed in Chapter 8. For now we will use the above definition.)

(a) Show that the sample variance satisfies the following expression:

$$\langle V^2 \rangle_n = \langle X^2 \rangle_n - \langle X \rangle_n^2.$$

(b) Show that the sample variance satisfies the following recursion formula:

$$\langle V^2 \rangle_n = \left(1 - \frac{1}{n}\right)\langle V^2 \rangle_{n-1} + \frac{1}{n}\left(1 - \frac{1}{n}\right)(X(n) - \langle X \rangle_{n-1})^2,$$

with $\langle V^2 \rangle_0 = 0$.

2.115. Suppose you have a program to generate a sequence of numbers U_n that is uniformly distributed in $[0, 1]$. Let $Y_n = \alpha U_n + \beta$.

(a) Find α and β so that Y_n is uniformly distributed in the interval $[a, b]$.

(b) Let $a = -5$ and $b = 15$. Use Octave to generate Y_n and to compute the sample mean and sample variance in 1000 repetitions. Compare the sample mean and sample variance to $(a + b)/2$ and $(b - a)^2/12$, respectively.

2.116. Use Octave to simulate 100 repetitions of the random experiment where a coin is tossed 16 times and the number of heads is counted.

(a) Confirm that your results are similar to those in Figure 2.18.

(b) Rerun the experiment with $p = 0.25$ and $p = 0.75$. Are the results as expected?

*Section 2.8: Fine Points: Event Classes

2.117. In Example 2.49, Homer maps the outcomes from Lisa's sample space $S_L = \{r, g, t\}$ into a smaller sample space $S_H = \{R, G\}: f(r) = R, f(g) = G$, and $f(t) = G$.
Define the inverse image events as follows:

$$f^{-1}(\{R\}) = A_1 = \{r\} \quad \text{and} \quad f^{-1}(\{G\}) = A_2 = \{g, t\}.$$

Let A and B be events in Homer's sample space.

(a) Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) Show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(c) Show that $f^{-1}(A^c) = f^{-1}(A)^c$.

(d) Show that the results in parts a, b, and c hold for a general mapping f from a sample space S to a set S'.

2.118. Let f be a mapping from a sample space S to a finite set $S' = \{y_1, y_2, \ldots, y_n\}$.

(a) Show that the set of inverse images $A_k = f^{-1}(\{y_k\})$ forms a partition of S.

(b) Show that any event B of S' can be related to a union of A_k's.

2.119. Let A be any subset of S. Show that the class of sets $\{\varnothing, A, A^c, S\}$ is a field.

*Section 2.9: Fine Points: Probabilities of Sequences of Events

2.120. Find the countable union of the following sequences of events:

(a) $A_n = [a + 1/n, b - 1/n]$.

(b) $B_n = (-n, b - 1/n]$.

(c) $C_n = [a + 1/n, b)$.

2.121. Find the countable intersection of the following sequences of events:

(a) $A_n = (a - 1/n, b + 1/n)$.

(b) $B_n = [a, b + 1/n)$.

(c) $C_n = (a - 1/n, b]$.

2.122. **(a)** Show that the Borel field can be generated from the complements and countable intersections and unions of open sets (a, b).

(b) Suggest other classes of sets that can generate the Borel field.

2.123. Find expressions for the probabilities of the events in Problem 2.120.

2.124. Find expressions for the probabilities of the events in Problem 2.121.

Problems Requiring Cumulative Knowledge

2.125. Compare the binomial probability law and the hypergeometric law introduced in Problem 2.54 as follows.

(a) Suppose a lot has 20 items of which five are defective. A batch of ten items is tested without replacement. Find the probability that k are found defective for $k = 0, \ldots, 10$. Compare this to the binomial probabilities with $n = 10$ and $p = 5/20 = .25$.

(b) Repeat but with a lot of 1000 items of which 250 are defective. A batch of ten items is tested without replacement. Find the probability that k are found defective for $k = 0, \ldots, 10$. Compare this to the binomial probabilities with $n = 10$ and $p = 5/20 = .25$.

2.126. Suppose that in Example 2.43, computer A sends each message to computer B simultaneously over two unreliable radio links. Computer B can detect when errors have occurred in either link. Let the probability of message transmission error in link 1 and link 2 be q_1 and q_2 respectively. Computer B requests retransmissions until it receives an error-free message on either link.

(a) Find the probability that more than k transmissions are required.

(b) Find the probability that in the last transmission, the message on link 2 is received free of errors.

2.127. In order for a circuit board to work, seven identical chips must be in working order. To improve reliability, an additional chip is included in the board, and the design allows it to replace any of the seven other chips when they fail.

(a) Find the probability p_b that the board is working in terms of the probability p that an individual chip is working.

(b) Suppose that n circuit boards are operated in parallel, and that we require a 99.9% probability that at least one board is working. How many boards are needed?

2.128. Consider a well-shuffled deck of cards consisting of 52 distinct cards, of which four are aces and four are kings.

(a) Find the probability of obtaining an ace in the first draw.

(b) Draw a card from the deck and look at it. What is the probability of obtaining an ace in the second draw? Does the answer change if you had not observed the first draw?

(c) Suppose we draw seven cards from the deck. What is the probability that the seven cards include three aces? What is the probability that the seven cards include two kings? What is the probability that the seven cards include three aces and/or two kings?

(d) Suppose that the entire deck of cards is distributed equally among four players. What is the probability that each player gets an ace?

4. A. M. Law and W. D. Kelton, *Simulation Modeling and Analysis*, McGraw-Hill, New York, 2000.
5. N. L. Johnson, A. W. Kemp, and S. Kotz, *Univariate Discrete Distributions*, Wiley, New York, 2005.
6. Y. A. Rozanov, *Probability Theory: A Concise Course*, Dover Publications, New York, 1969.

PROBLEMS

Section 3.1: The Notion of a Random Variable

3.1. Let X be the maximum of the number of heads obtained when Carlos and Michael each flip a fair coin twice.

(a) Describe the underlying space S of this random experiment and specify the probabilities of its elementary events.

(b) Show the mapping from S to S_X, the range of X.

(c) Find the probabilities for the various values of X.

3.2. A die is tossed and the random variable X is defined as the number of full pairs of dots in the face showing up.

(a) Describe the underlying space S of this random experiment and specify the probabilities of its elementary events.

(b) Show the mapping from S to S_X, the range of X.

(c) Find the probabilities for the various values of X.

(d) Repeat parts a, b, and c, if Y is the number of full or partial pairs of dots in the face showing up.

(e) Explain why $P[X = 0]$ and $P[Y = 0]$ are not equal.

3.3. The loose minute hand of a clock is spun hard. The coordinates (x, y) of the point where the tip of the hand comes to rest is noted. Z is defined as the sgn function of the product of x and y, where $\text{sgn}(t)$ is 1 if $t > 0$, 0 if $t = 0$, and -1 if $t < 0$.

(a) Describe the underlying space S of this random experiment and specify the probabilities of its events.

(b) Show the mapping from S to S_X, the range of X.

(c) Find the probabilities for the various values of X.

3.4. A data source generates hexadecimal characters. Let X be the integer value corresponding to a hex character. Suppose that the four binary digits in the character are independent and each is equally likely to be 0 or 1.

(a) Describe the underlying space S of this random experiment and specify the probabilities of its elementary events.

(b) Show the mapping from S to S_X, the range of X.

(c) Find the probabilities for the various values of X.

(d) Let Y be the integer value of a hex character but suppose that the most significant bit is three times as likely to be a "0" as a "1". Find the probabilities for the values of Y.

3.5. Two transmitters send messages through bursts of radio signals to an antenna. During each time slot each transmitter sends a message with probability 1/2. Simultaneous transmissions result in loss of the messages. Let X be the number of time slots until the first message gets through.

(a) Describe the underlying sample space S of this random experiment and specify the probabilities of its elementary events.

(b) Show the mapping from S to S_X, the range of X.

(c) Find the probabilities for the various values of X.

3.6. An information source produces binary triplets $\{000, 111, 010, 101, 001, 110, 100, 011\}$ with corresponding probabilities $\{1/4, 1/4, 1/8, 1/8, 1/16, 1/16, 1/16, 1/16\}$. A binary code assigns a codeword of length $-\log_2 p_k$ to triplet k. Let X be the length of the string assigned to the output of the information source.

(a) Show the mapping from S to S_X, the range of X.

(b) Find the probabilities for the various values of X.

3.7. An urn contains 9 \$1 bills and one \$50 bill. Let the random variable X be the total amount that results when two bills are drawn from the urn without replacement.

(a) Describe the underlying space S of this random experiment and specify the probabilities of its elementary events.

(b) Show the mapping from S to S_X, the range of X.

(c) Find the probabilities for the various values of X.

3.8. An urn contains 9 \$1 bills and one \$50 bill. Let the random variable X be the total amount that results when two bills are drawn from the urn *with* replacement.

(a) Describe the underlying space S of this random experiment and specify the probabilities of its elementary events.

(b) Show the mapping from S to S_X, the range of X.

(c) Find the probabilities for the various values of X.

3.9. A coin is tossed n times. Let the random variable Y be the difference between the number of heads and the number of tails in the n tosses of a coin. Assume $P[\text{heads}] = p$.

(a) Describe the sample space of S.

(b) Find the probability of the event $\{Y = 0\}$.

(c) Find the probabilities for the other values of Y.

3.10. An m-bit password is required to access a system. A hacker systematically works through all possible m-bit patterns. Let X be the number of patterns tested until the correct password is found.

(a) Describe the sample space of S.

(b) Show the mapping from S to S_X, the range of X.

(c) Find the probabilities for the various values of X.

Section 3.2: Discrete Random Variables and Probability Mass Function

3.11. Let X be the maximum of the coin tosses in Problem 3.1.

(a) Compare the pmf of X with the pmf of Y, the number of heads in two tosses of a fair coin. Explain the difference.

(b) Suppose that Carlos uses a coin with probability of heads $p = 3/4$. Find the pmf of X.

3.12. Consider an information source that produces binary pairs that we designate as $S_X = \{1, 2, 3, 4\}$. Find and plot the pmf in the following cases:

(a) $p_k = p_1/k$ for all k in S_X.

(b) $p_{k+1} = p_k/2$ for $k = 2, 3, 4$.

(c) $p_{k+1} = p_k/2^k$ for $k = 2, 3, 4$.

(d) Can the random variables in parts a, b, and c be extended to take on values in the set $\{1, 2, \ldots\}$? If yes, specify the pmf of the resulting random variables. If no, explain why not.

3.13. Let X be a random variable with pmf $p_k = c/k^2$ for $k = 1, 2, \ldots$.

(a) Estimate the value of c numerically. Note that the series converges.

(b) Find $P[X > 4]$.

(c) Find $P[6 \leq X \leq 8]$.

3.14. Compare $P[X \geq 8]$ and $P[Y \geq 8]$ for outputs of the data source in Problem 3.4.

3.15. In Problem 3.5 suppose that terminal 1 transmits with probability 1/2 in a given time slot, but terminal 2 transmits with probability p.

(a) Find the pmf for the number of transmissions X until a message gets through.

(b) Given a successful transmission, find the probability that terminal 2 transmitted.

3.16. (a) In Problem 3.7 what is the probability that the amount drawn from the urn is more than \$2? More than \$50?

(b) Repeat part a for Problem 3.8.

3.17. A modem transmits a +2 voltage signal into a channel. The channel adds to this signal a noise term that is drawn from the set $\{0, -1, -2, -3\}$ with respective probabilities $\{4/10, 3/10, 2/10, 1/10\}$.

(a) Find the pmf of the output Y of the channel.

(b) What is the probability that the output of the channel is equal to the input of the channel?

(c) What is the probability that the output of the channel is positive?

3.18. A computer reserves a path in a network for 10 minutes. To extend the reservation the computer must successfully send a "refresh" message before the expiry time. However, messages are lost with probability 1/2. Suppose that it takes 10 seconds to send a refresh request and receive an acknowledgment. When should the computer start sending refresh messages in order to have a 99% chance of successfully extending the reservation time?

3.19. A modem transmits over an error-prone channel, so it repeats every "0" or "1" bit transmission five times. We call each such group of five bits a "codeword." The channel changes an input bit to its complement with probability $p = 1/10$ and it does so independently of its treatment of other input bits. The modem receiver takes a majority vote of the five received bits to estimate the input signal. Find the probability that the receiver makes the wrong decision.

3.20. Two dice are tossed and we let X be the difference in the number of dots facing up.

(a) Find and plot the pmf of X.

(b) Find the probability that $|X| \leq k$ for all k.

Section 3.3: Expected Value and Moments of Discrete Random Variable

3.21. (a) In Problem 3.11, compare $E[Y]$ to $E[X]$ where X is the maximum of coin tosses.

(b) Compare VAR[X] and VAR[Y].

3.22. Find the expected value and variance of the output of the information sources in Problem 3.12, parts a, b, and c.

3.23. (a) Find $E[X]$ for the hex integers in Problem 3.4.

(b) Find VAR[X].

3.24. Find the mean codeword length in Problem 3.6. How can this average be interpreted in a very large number of encodings of binary triplets?

3.25. (a) Find the mean and variance of the amount drawn from the urn in Problem 3.7.

(b) Find the mean and variance of the amount drawn from the urn in Problem 3.8.

3.26. Find $E[Y]$ and VAR[Y] for the difference between the number of heads and tails in Problem 3.9. In a large number of repetitions of this random experiment, what is the meaning of $E[Y]$?

3.27. Find $E[X]$ and VAR[X] in Problem 3.13.

3.28. Find the expected value and variance of the modem signal in Problem 3.17.

3.29. Find the mean and variance of the time that it takes to renew the reservation in Problem 3.18.

3.30. The modem in Problem 3.19 transmits 1000 5-bit codewords. What is the average number of codewords in error? If the modem transmits 1000 bits individually without repetition, what is the average number of bits in error? Explain how error rate is traded off against transmission speed.

3.31. (a) Suppose a fair coin is tossed n times. Each coin toss costs d dollars and the reward in obtaining X heads is $aX^2 + bX$. Find the expected value of the net reward.

(b) Suppose that the reward in obtaining X heads is a^X, where $a > 0$. Find the expected value of the reward.

3.32. Let $g(X) = I_A$, where $A = \{X > 10\}$.

(a) Find $E[g(X)]$ for X as in Problem 3.12a with $S_X = \{1, 2, \ldots, 15\}$.

(b) Repeat part a for X as in Problem 3.12b with $S_X = \{1, 2, \ldots, 15\}$.

(c) Repeat part a for X as in Problem 3.12c with $S_X = \{1, 2, \ldots, 15\}$.

3.33. Let $g(X) = (X - 10)^+$ (see Example 3.19).

(a) Find $E[X]$ for X as in Problem 3.12a with $S_X = \{1, 2, \ldots, 15\}$.

(b) Repeat part a for X as in Problem 3.12b with $S_X = \{1, 2, \ldots, 15\}$.

(c) Repeat part a for X as in Problem 3.12c with $S_X = \{1, 2, \ldots, 15\}$.

3.34. Consider the St. Petersburg Paradox in Example 3.16. Suppose that the casino has a total of $M = 2^m$ dollars, and so it can only afford a finite number of coin tosses.

(a) How many tosses can the casino afford?

(b) Find the expected payoff to the player.

(c) How much should a player be willing to pay to play this game?

Section 3.4: Conditional Probability Mass Function

3.35. (a) In Problem 3.11a, find the conditional pmf of X, the maximum of coin tosses, given that $X > 0$.

(b) Find the conditional pmf of X given that Michael got one head in two tosses.

(c) Find the conditional pmf of X given that Michael got one head in the first toss.

(d) In Problem 3.11b, find the probability that Carlos got the maximum given that $X = 2$.

3.36. Find the conditional pmf for the quaternary information source in Problem 3.12, parts a, b, and c given that $X < 4$.

3.37. (a) Find the conditional pmf of the hex integer X in Problem 3.4 given that $X < 8$.

(b) Find the conditional pmf of X given that the first bit is 0.

(c) Find the conditional pmf of X given that the 4th bit is 0.

3.38. (a) Find the conditional pmf of X in Problem 3.5 given that no message gets through in time slot 1.

(b) Find the conditional pmf of X given that the first transmitter transmitted in time slot 1.

3.39. **(a)** Find the conditional expected value of X in Problem 3.5 given that no message gets through in the first time slot. Show that $E[X \mid X > 1] = E[X] + 1$.

(b) Find the conditional expected value of X in Problem 3.5 given that a message gets through in the first time slot.

(c) Find $E[X]$ by using the results of parts a and b.

(d) Find $E[X^2]$ and VAR$[X]$ using the approach in parts b and c.

3.40. Explain why Eq. (3.31b) can be used to find $E[X^2]$, but it cannot be used to directly find VAR$[X]$.

3.41. **(a)** Find the conditional pmf for X in Problem 3.7 given that the first draw produced k dollars.

(b) Find the conditional expected value corresponding to part a.

(c) Find $E[X]$ using the results from part b.

(d) Find $E[X^2]$ and VAR$[X]$ using the approach in parts b and c.

3.42. Find $E[Y]$ and VAR$[Y]$ for the difference between the number of heads and tails in n tosses in Problem 3.9. *Hint:* Condition on the number of heads.

3.43. **(a)** In Problem 3.10 find the conditional pmf of X given that the password has not been found after k tries.

(b) Find the conditional expected value of X given $X > k$.

(c) Find $E[X]$ from the results in part b.

Section 3.5: Important Discrete Random Variables

3.44. Indicate the value of the indicator function for the event A, $I_A(\zeta)$, for each ζ in the sample space S. Find the pmf and expected of I_A.

(a) $S = \{1, 2, 3, 4, 5\}$ and $A = \{\zeta > 3\}$.

(b) $S = [0, 1]$ and $A = \{0.3 < \zeta \leq 0.7\}$.

(c) $S = \{\zeta = (x, y): 0 < x < 1, 0 < y < 1\}$ and $A = \{\zeta = (x, y): 0.25 < x + y < 1.25\}$.

(d) $S = (-\infty, \infty)$ and $A = \{\zeta > a\}$.

3.45. Let A and B be events for a random experiment with sample space S. Show that the Bernoulli random variable satisfies the following properties:

(a) $I_S = 1$ and $I_\varnothing = 0$.

(b) $I_{A \cap B} = I_A I_B$ and $I_{A \cup B} = I_A + I_B - I_A I_B$.

(c) Find the expected value of the indicator functions in parts a and b.

3.46. Heat must be removed from a system according to how fast it is generated. Suppose the system has eight components each of which is active with probability 0.25, independently of the others. The design of the heat removal system requires finding the probabilities of the following events:

(a) None of the systems is active.

(b) Exactly one is active.

(c) More than four are active.

(d) More than two and fewer than six are active.

3.47. Eight numbers are selected at random from the unit interval.

(a) Find the probability that the first four numbers are less than 0.25 and the last four are greater than 0.25.

(b) Find the probability that four numbers are less than 0.25 and four are greater than 0.25.

(c) Find the probability that the first three numbers are less than 0.25, the next two are between 0.25 and 0.75, and the last three are greater than 0.75.

(d) Find the probability that three numbers are less than 0.25, two are between 0.25 and 0.75, and three are greater than 0.75.

(e) Find the probability that the first four numbers are less than 0.25 and the last four are greater than 0.75.

(f) Find the probability that four numbers are less than 0.25 and four are greater than 0.75.

3.48. **(a)** Plot the pmf of the binomial random variable with $n = 4$ and $n = 5$, and $p = 0.10$, $p = 0.5$, and $p = 0.90$.

(b) Use Octave to plot the pmf of the binomial random variable with $n = 100$ and $p = 0.10$, $p = 0.5$, and $p = 0.90$.

3.49. Let X be a binomial random variable that results from the performance of n Bernoulli trials with probability of success p.

(a) Suppose that $X = 1$. Find the probability that the single event occurred in the kth Bernoulli trial.

(b) Suppose that $X = 2$. Find the probability that the two events occurred in the jth and kth Bernoulli trials where $j < k$.

(c) In light of your answers to parts a and b in what sense are the successes distributed "completely at random" over the n Bernoulli trials?

3.50. Let X be the binomial random variable.

(a) Show that

$$\frac{p_X(k+1)}{p_X(k)} = \frac{n-k}{k+1}\frac{p}{1-p} \qquad \text{where} \qquad p_X(0) = (1-p)^n.$$

(b) Show that part a implies that: (1) $P[X = k]$ is maximum at $k_{\max} = [(n+1)p]$, where $[x]$ denotes the largest integer that is smaller than or equal to x; and (2) when $(n+1)p$ is an integer, then the maximum is achieved at $k_{\max}$ and $k_{\max} - 1$.

3.51. Consider the expression $(a + b + c)^n$.

(a) Use the binomial expansion for $(a + b)$ and c to obtain an expression for $(a + b + c)^n$.

(b) Now expand all terms of the form $(a + b)^k$ and obtain an expression that involves the multinomial coefficient for $M = 3$ mutually exclusive events, A_1, A_2, A_3.

(c) Let $p_1 = P[A_1]$, $p_2 = P[A_2]$, $p_3 = P[A_3]$. Use the result from part b to show that the multinomial probabilities add to one.

3.52. A sequence of characters is transmitted over a channel that introduces errors with probability $p = 0.01$.

(a) What is the pmf of N, the number of error-free characters between erroneous characters?

(b) What is $E[N]$?

(c) Suppose we want to be 99% sure that at least 1000 characters are received correctly before a bad one occurs. What is the appropriate value of p?

3.53. Let N be a geometric random variable with $S_N = \{1, 2, \dots\}$.

(a) Find $P[N = k \mid N \leq m]$.

(b) Find the probability that N is odd.

3.54. Let M be a geometric random variable. Show that M satisfies the memoryless property: $P[M \geq k + j | M \geq j + 1] = P[M \geq k]$ for all $j, k > 1$.

3.55. Let X be a discrete random variable that assumes only nonnegative integer values and that satisfies the memoryless property. Show that X must be a geometric random variable. *Hint:* Find an equation that must be satisfied by $g(m) = P[M \geq m]$.

3.56. An audio player uses a low-quality hard drive. The initial cost of building the player is \$50. The hard drive fails after each month of use with probability 1/12. The cost to repair the hard drive is \$20. If a 1-year warranty is offered, how much should the manufacturer charge so that the probability of losing money on a player is 1% or less? What is the average cost per player?

3.57. A Christmas fruitcake has Poisson-distributed independent numbers of sultana raisins, iridescent red cherry bits, and radioactive green cherry bits with respective averages 48, 24, and 12 bits per cake. Suppose you politely accept 1/12 of a slice of the cake.

(a) What is the probability that you get lucky and get no green bits in your slice?

(b) What is the probability that you get really lucky and get no green bits and two or fewer red bits in your slice?

(c) What is the probability that you get extremely lucky and get no green or red bits and more than five raisins in your slice?

3.58. The number of orders waiting to be processed is given by a Poisson random variable with parameter $\alpha = \lambda/n\mu$, where λ is the average number of orders that arrive in a day, μ is the number of orders that can be processed by an employee per day, and n is the number of employees. Let $\lambda = 5$ and $\mu = 1$. Find the number of employees required so the probability that more than four orders are waiting is less than 10%. What is the probability that there are no orders waiting?

3.59. The number of page requests that arrive at a Web server is a Poisson random variable with an average of 6000 requests per minute.

(a) Find the probability that there are no requests in a 100-ms period.

(b) Find the probability that there are between 5 and 10 requests in a 100-ms period.

3.60. Use Octave to plot the pmf of the Poisson random variable with $\alpha = 0.1, 0.75, 2, 20$.

3.61. Find the mean and variance of a Poisson random variable.

3.62. For the Poisson random variable, show that for $\alpha < 1$, $P[N = k]$ is maximum at $k = 0$; for $\alpha > 1$, $P[N = k]$ is maximum at $[\alpha]$; and if α is a positive integer, then $P[N = k]$ is maximum at $k = \alpha$, and at $k = \alpha - 1$. *Hint:* Use the approach of Problem 3.50.

3.63. Compare the Poisson approximation and the binomial probabilities for $k = 0, 1, 2, 3$ and $n = 10$, $p = 0.1$; $n = 20$ and $p = 0.05$; and $n = 100$ and $p = 0.01$.

3.64. At a given time, the number of households connected to the Internet is a Poisson random variable with mean 50. Suppose that the transmission bit rate available for the household is 20 Megabits per second.

(a) Find the probability of the distribution of the transmission bit rate per user.

(b) Find the transmission bit rate that is available to a user with probability 90% or higher.

(c) What is the probability that a user has a share of 1 Megabit per second or higher?

3.65. An LCD display has 1000×750 pixels. A display is accepted if it has 15 or fewer faulty pixels. The probability that a pixel is faulty coming out of the production line is 10^{-5}. Find the proportion of displays that are accepted.

3.66. A data center has 10,000 disk drives. Suppose that a disk drive fails in a given day with probability 10^{-3}.

(a) Find the probability that there are no failures in a given day.

(b) Find the probability that there are fewer than 10 failures in two days.

(c) Find the number of spare disk drives that should be available so that all failures in a day can be replaced with probability 99%.

3.67. A binary communication channel has a probability of bit error of 10^{-6}. Suppose that transmissions occur in blocks of 10,000 bits. Let N be the number of errors introduced by the channel in a transmission block.

(a) Find $P[N = 0]$, $P[N \leq 3]$.

(b) For what value of p will the probability of 1 or more errors in a block be 99%?

3.68. Find the mean and variance of the uniform discrete random variable that takes on values in the set $\{1, 2, \ldots, L\}$ with equal probability. You will need the following formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

3.69. A voltage X is uniformly distributed in the set $\{-3, \ldots, 3, 4\}$.

(a) Find the mean and variance of X.

(b) Find the mean and variance of $Y = -2X^2 + 3$.

(c) Find the mean and variance of $W = \cos(\pi X/8)$.

(d) Find the mean and variance of $Z = \cos^2(\pi X/8)$.

3.70. Ten news Web sites are ranked in terms of popularity, and the frequency of requests to these sites are known to follow a Zipf distribution.

(a) What is the probability that a request is for the top-ranked site?

(b) What is the probability that a request is for one of the bottom five sites?

3.71. A collection of 1000 words is known to have a Zipf distribution.

(a) What is the probability of the 10 top-ranked words?

(b) What is the probability of the 10 lowest-ranked words?

3.72. What is the shape of the log of the Zipf probability vs. the log of the rank?

3.73. Plot the mean and variance of the Zipf random variable for $L = 1$ to $L = 100$.

3.74. An online video store has 10,000 titles. In order to provide fast response, the store caches the most popular titles. How many titles should be in the cache so that with probability 99% an arriving video request will be in the cache?

3.75. **(a)** Income distribution is perfectly equal if every individual has the same income. What is the Lorenz curve in this case?

(b) In a perfectly unequal income distribution, one individual has all the income and all others have none. What is the Lorenz curve in this case?

3.76. Let X be a geometric random variable in the set $\{1, 2, \ldots\}$.

(a) Find the pmf of X.

(b) Find the Lorenz curve of X. Assume L is infinite.

(c) Plot the curve for $p = 0.1, 0.5, 0.9$.

3.77. Let X be a zeta random variable with parameter α.

(a) Find an expression for $P[X \leq k]$.

(b) Plot the pmf of X for $\alpha = 1.5, 2$, and 3.

(c) Plot $P[X \le k]$ for $\alpha = 1.5, 2$, and 3.

Section 3.6: Generation of Discrete Random Variables

3.78. Octave provides function calls to evaluate the pmf of important discrete random variables. For example, the function `Poisson_pdf(x, lambda)` computes the pmf at x for the Poisson random variable.

(a) Plot the Poisson pmf for $\lambda = 0.5, 5, 50$, as well as $P[X \le k]$ and $P[X > k]$.

(b) Plot the binomial pmf for $n = 48$ and $p = 0.10, 0.30, 0.50, 0.75$, as well as $P[X \le k]$ and $P[X > k]$.

(c) Compare the binomial probabilities with the Poisson approximation for $n = 100$, $p = 0.01$.

3.79. The `discrete_pdf` function in Octave makes it possible to specify an arbitrary pmf for a specified S_X.

(a) Plot the pmf for Zipf random variables with $L = 10, 100, 1000$, as well as $P[X \le k]$ and $P[X > k]$.

(b) Plot the pmf for the reward in the St. Petersburg Paradox for $m = 20$ in Problem 3.34, as well as $P[X \le k]$ and $P[X > k]$. (You will need to use a log scale for the values of k.)

3.80. Use Octave to plot the Lorenz curve for the Zipf random variables in Problem 3.79a.

3.81. Repeat Problem 3.80 for the binomial random variable with $n = 100$ and $p = 0.1, 0.5$, and 0.9.

3.82. **(a)** Use the `discrete_rnd` function in Octave to simulate the urn experiment discussed in Section 1.3. Compute the relative frequencies of the outcomes in 1000 draws from the urn.

(b) Use the `discrete_pdf` function in Octave to specify a pmf for a binomial random variable with $n = 5$ and $p = 0.2$. Use `discrete_rnd` to generate 100 samples and plot the relative frequencies.

(c) Use `binomial_rnd` to generate the 100 samples in part b.

3.83. Use the `discrete_rnd` function to generate 200 samples of the Zipf random variable in Problem 3.79a. Plot the sequence of outcomes as well as the overall relative frequencies.

3.84. Use the `discrete_rnd` function to generate 200 samples of the St. Petersburg Paradox random variable in Problem 3.79b. Plot the sequence of outcomes as well as the overall relative frequencies.

3.85. Use Octave to generate 200 pairs of numbers, (X_i, Y_i), in which the components are independent, and each component is uniform in the set $\{1, 2, \ldots, 9, 10\}$.

(a) Plot the relative frequencies of the X and Y outcomes.

(b) Plot the relative frequencies of the random variable $Z = X + Y$. Can you discern the pmf of Z?

(c) Plot the relative frequencies of $W = XY$. Can you discern the pmf of Z?

(d) Plot the relative frequencies of $V = X/Y$. Is the pmf discernable?

3.86. Use Octave function `binomial_rnd` to generate 200 pairs of numbers, (X_i, Y_i), in which the components are independent, and where X_i are binomial with parameter $n = 8$, $p = 0.5$ and Y_i are binomial with parameter $n = 4$, $p = 0.5$.

(a) Plot the relative frequencies of the X and Y outcomes.

(b) Plot the relative frequencies of the random variable $Z = X + Y$. Does this correspond to the pmf you would expect? Explain.

3.87. Use Octave function `Poisson_rnd` to generate 200 pairs of numbers, (X_i, Y_i), in which the components are independent, and where X_i are the number of arrivals to a system in one second and Y_i are the number of arrivals to the system in the next two seconds. Assume that the arrival rate is five customers per second.

(a) Plot the relative frequencies of the X and Y outcomes.

(b) Plot the relative frequencies of the random variable $Z = X + Y$. Does this correspond to the pmf you would expect? Explain.

Problems Requiring Cumulative Knowledge

3.88. The fraction of defective items in a production line is p. Each item is tested and defective items are identified correctly with probability a.

(a) Assume nondefective items always pass the test. What is the probability that k items are tested until a defective item is identified?

(b) Suppose that the identified defective items are removed. What proportion of the remaining items is defective?

(c) Now suppose that nondefective items are identified as defective with probability b. Repeat part b.

3.89. A data transmission system uses messages of duration T seconds. After each message transmission, the transmitter stops and waits T seconds for a reply from the receiver. The receiver immediately replies with a message indicating that a message was received correctly. The transmitter proceeds to send a new message if it receives a reply within T seconds; otherwise, it retransmits the previous message. Suppose that messages can be completely garbled while in transit and that this occurs with probability p. Find the maximum possible rate at which messages can be successfully transmitted from the transmitter to the receiver.

3.90. An inspector selects every nth item in a production line for a detailed inspection. Suppose that the time between item arrivals is an exponential random variable with mean 1 minute, and suppose that it takes 2 minutes to inspect an item. Find the smallest value of n such that with a probability of 90% or more, the inspection is completed before the arrival of the next item that requires inspection.

3.91. The number X of photons counted by a receiver in an optical communication system is a Poisson random variable with rate λ_1 when a signal is present and a Poisson random variable with rate $\lambda_0 < \lambda_1$ when a signal is absent. Suppose that a signal is present with probability p.

(a) Find $P[\text{signal present} \mid X = k]$ and $P[\text{signal absent} \mid X = k]$.

(b) The receiver uses the following decision rule:

If $P[\text{signal present} \mid X = k] > P[\text{signal absent} \mid X = k]$, decide signal present; otherwise, decide signal absent.

Show that this decision rule leads to the following threshold rule:

If $X > T$, decide signal present; otherwise, decide signal absent.

(c) What is the probability of error for the above decision rule?

3.92. A binary information source (e.g., a document scanner) generates very long strings of 0's followed by occasional 1's. Suppose that symbols are independent and that $p = P[\text{symbol} = 0]$ is very close to one. Consider the following scheme for encoding the run X of 0's between consecutive 1's:

1. If $X = n$, express n as a multiple of an integer $M = 2^m$ and a remainder r, that is, find k and r such that $n = kM + r$, where $0 \le r < M - 1$;
2. The binary codeword for n then consists of a prefix consisting of k 0's followed by a 1, and a suffix consisting of the m-bit representation of the remainder r. The decoder can deduce the value of n from this binary string.
   - **(a)** Find the probability that the prefix has k zeros, assuming that $p^M = 1/2$.
   - **(b)** Find the average codeword length when $p^M = 1/2$.
   - **(c)** Find the compression ratio, which is defined as the ratio of the average run length to the average codeword length when $p^M = 1/2$.

CHAPTER 4

One Random Variable

In Chapter 3 we introduced the notion of a random variable and we developed methods for calculating probabilities and averages for the case where the random variable is discrete. In this chapter we consider the general case where the random variable may be discrete, continuous, or of mixed type. We introduce the cumulative distribution function which is used in the formal definition of a random variable, and which can handle all three types of random variables. We also introduce the probability density function for continuous random variables. The probabilities of events involving a random variable can be expressed as integrals of its probability density function. The expected value of continuous random variables is also introduced and related to our intuitive notion of average. We develop a number of methods for calculating probabilities and averages that are the basic tools in the analysis and design of systems that involve randomness.

4.1 THE CUMULATIVE DISTRIBUTION FUNCTION

The probability mass function of a discrete random variable was defined in terms of events of the form $\{X = b\}$. The cumulative distribution function is an alternative approach which uses events of the form $\{X \le b\}$. The cumulative distribution function has the advantage that it is not limited to discrete random variables and applies to all types of random variables. We begin with a formal definition of a random variable.

> **Definition:** Consider a random experiment with sample space S and event class $\mathcal{F}$. A **random variable** X is a function from the sample space S to R with the property that the set $A_b = \{\zeta : X(\zeta) \le b\}$ is in $\mathcal{F}$ for every b in R.

The definition simply requires that every set A_b have a well defined probability in the underlying random experiment, and this is not a problem in the cases we will consider. Why does the definition use sets of the form $\{\zeta : X(\zeta) \le b\}$ and not $\{\zeta : X(\zeta) = x_b\}$? We will see that all events of interest in the real line can be expressed in terms of sets of the form $\{\zeta : X(\zeta) \le b\}$.

The **cumulative distribution function** (cdf) of a random variable X is defined as the probability of the event $\{X \le x\}$:

$$F_X(x) = P[X \le x] \qquad \text{for } -\infty < x < +\infty, \tag{4.1}$$

CHECKLIST OF IMPORTANT TERMS

Characteristic function
Chebyshev inequality
Chernoff bound
Conditional cdf, pdf
Continuous random variable
Cumulative distribution function
Differential entropy
Discrete random variable
Entropy
Equivalent event
Expected value of X
Failure rate function
Function of a random variable
Laplace transform of the pdf
Markov inequality
Maximum entropy method
Mean time to failure (MTTF)
Moment theorem
nth moment of X
Probability density function
Probability generating function
Probability mass function
Random variable
Random variable of mixed type
Rejection method
Reliability
Standard deviation of X
Transformation method
Variance of X

ANNOTATED REFERENCES

Reference [1] is the standard reference for electrical engineers for the material on random variables. Reference [2] is entirely devoted to continuous distributions. Reference [3] discusses some of the finer points regarding the concept of a random variable at a level accessible to students of this course. Reference [4] presents detailed discussions of the various methods for generating random numbers with specified distributions. Reference [5] also discusses the generation of random variables. Reference [9] is focused on signal processing. Reference [11] discusses entropy in the context of information theory.

1. A. Papoulis and S. Pillai, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, New York, 2002.
2. N. Johnson et al., *Continuous Univariate Distributions*, vol. 2, Wiley, New York, 1995.
3. K. L. Chung, *Elementary Probability Theory*, Springer-Verlag, New York, 1974.
4. A. M. Law and W. D. Kelton, *Simulation Modeling and Analysis*, McGraw-Hill, New York, 2000.
5. S. M. Ross, *Introduction to Probability Models*, Academic Press, New York, 2003.
6. H. Cramer, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, N.J., 1946.
7. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C., 1964. Downloadable: *www.math.sfu.ca/~cbm/aands/*.
8. R. C. Cheng, "The Generation of Gamma Variables with Nonintegral Shape Parameter," *Appl. Statist.*, 26: 71–75, 1977.
9. R. Gray and L.D. Davisson, *An Introduction to Statistical Signal Processing*, Cambridge Univ. Press, Cambridge, UK, 2005.

10. P. O. Börjesson and C. E. W. Sundberg, "Simple Approximations of the Error Function $Q(x)$ for Communications Applications," *IEEE Trans. on Communications*, March 1979, 639–643.
11. R. G. Gallager, *Information Theory and Reliable Communication*, Wiley, New York, 1968.

PROBLEMS

Section 4.1: The Cumulative Distribution Function

4.1. An information source produces binary pairs that we designate as $S_X = \{1, 2, 3, 4\}$ with the following pmf's:
   (i) $p_k = p_1/k$ for all k in S_X.
   (ii) $p_{k+1} = p_k/2$ for $k = 2, 3, 4$.
   (iii) $p_{k+1} = p_k/2^k$ for $k = 2, 3, 4$.
   (a) Plot the cdf of these three random variables.
   (b) Use the cdf to find the probability of the events: $\{X \leq 1\}$, $\{X < 2.5\}$, $\{0.5 < X \leq 2\}$, $\{1 < X < 4\}$.

4.2. A die is tossed. Let X be the number of full pairs of dots in the face showing up, and Y be the number of full or partial pairs of dots in the face showing up. Find and plot the cdf of X and Y.

4.3. The loose minute hand of a clock is spun hard. The coordinates (x, y) of the point where the tip of the hand comes to rest is noted. Z is defined as the sgn function of the product of x and y, where $\text{sgn}(t)$ is 1 if $t > 0$, 0 if $t = 0$, and -1 if $t < 0$.
   (a) Find and plot the cdf of the random variable X.
   (b) Does the cdf change if the clock hand has a propensity to stop at 3, 6, 9, and 12 o'clock?

4.4. An urn contains 8 \$1 bills and two \$5 bills. Let X be the total amount that results when two bills are drawn from the urn without replacement, and let Y be the total amount that results when two bills are drawn from the urn *with* replacement.
   (a) Plot and compare the cdf's of the random variables.
   (b) Use the cdf to compare the probabilities of the following events in the two problems: $\{X = \$2\}$, $\{X < \$7\}$, $\{X \geq 6\}$.

4.5. Let Y be the difference between the number of heads and the number of tails in the 3 tosses of a fair coin.
   (a) Plot the cdf of the random variable Y.
   (b) Express $P[|Y| < y]$ in terms of the cdf of Y.

4.6. A dart is equally likely to land at any point inside a circular target of radius 2. Let R be the distance of the landing point from the origin.
   (a) Find the sample space S and the sample space of R, S_R.
   (b) Show the mapping from S to S_R.
   (c) The "bull's eye" is the central disk in the target of radius 0.25. Find the event A in S_R corresponding to "dart hits the bull's eye." Find the equivalent event in S and $P[A]$.
   (d) Find and plot the cdf of R.

4.7. A point is selected at random inside a square defined by $\{(x, y): 0 \leq x \leq b, 0 \leq y \leq b\}$. Assume the point is equally likely to fall anywhere in the square. Let the random variable Z be given by the minimum of the two coordinates of the point where the dart lands.
   (a) Find the sample space S and the sample space of Z, S_Z.

(b) Show the mapping from S to S_Z.

(c) Find the region in the square corresponding to the event $\{Z \le z\}$.

(d) Find and plot the cdf of Z.

(e) Use the cdf to find: $P[Z > 0]$, $P[Z > b]$, $P[Z \le b/2]$, $P[Z > b/4]$.

4.8. Let ζ be a point selected at random from the unit interval. Consider the random variable $X = (1 - \zeta)^{-1/2}$.

(a) Sketch X as a function of ζ.

(b) Find and plot the cdf of X.

(c) Find the probability of the events $\{X > 1\}$, $\{5 < X < 7\}$, $\{X \le 20\}$.

4.9. The loose hand of a clock is spun hard and the outcome ζ is the angle in the range $[0, 2\pi)$ where the hand comes to rest. Consider the random variable $X(\zeta) = 2\sin(\zeta/4)$.

(a) Sketch X as a function of ζ.

(b) Find and plot the cdf of X.

(c) Find the probability of the events $\{X > 1\}$, $\{-1/2 < X < 1/2\}$, $\{X \le 1/\sqrt{2}\}$.

4.10. Repeat Problem 4.9 if 80% of the time the hand comes to rest anywhere in the circle, but 20% of the time the hand comes to rest at 3, 6, 9, or 12 o'clock.

4.11. The random variable X is uniformly distributed in the interval $[-1, 2]$.

(a) Find and plot the cdf of X.

(b) Use the cdf to find the probabilities of the following events: $\{X \le 0\}$, $\{|X - 0.5| < 1\}$, and $C = \{X > -0.5\}$.

4.12. The cdf of the random variable X is given by:

$$F_X(x) = \begin{cases} 0 & x < -1 \\ 0.5 & -1 \le x \le 0 \\ (1 + x)/2 & 0 \le x \le 1 \\ 1 & x \ge 1. \end{cases}$$

(a) Plot the cdf and identify the type of random variable.

(b) Find $P[X \le -1]$, $P[X = -1]$, $P[X < 0.5]$, $P[-0.5 < X < 0.5]$, $P[X > -1]$, $P[X \le 2]$, $P[X > 3]$.

4.13. A random variable X has cdf:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - \frac{1}{4}e^{-2x} & \text{for } x \ge 0. \end{cases}$$

(a) Plot the cdf and identify the type of random variable.

(b) Find $P[X \le 2]$, $P[X = 0]$, $P[X < 0]$, $P[2 < X < 6]$, $P[X > 10]$.

4.14. The random variable X has cdf shown in Fig. P4.1.

(a) What type of random variable is X?

(b) Find the following probabilities: $P[X < -1]$, $P[X \le -1]$, $P[-1 < X < -0.75]$, $P[-0.5 \le X < 0]$, $P[-0.5 \le X \le 0.5]$, $P[|X - 0.5| < 0.5]$.

4.15. For $\beta > 0$ and $\lambda > 0$, the Weibull random variable Y has cdf:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-(x/\lambda)^\beta} & \text{for } x \ge 0. \end{cases}$$

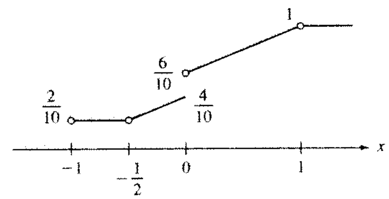


FIGURE P4.1

(a) Plot the cdf of Y for $\beta = 0.5$, 1, and 2.

(b) Find the probability $P[j\lambda < X < (j + 1)\lambda]$ and $P[X > j\lambda]$.

(c) Plot $\log P[X > x]$ vs. $\log x$.

4.16. The random variable X has cdf:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 0.5 + c\sin^2(\pi x/2) & 0 \le x \le 1 \\ 1 & x > 1. \end{cases}$$

(a) What values can c assume?

(b) Plot the cdf.

(c) Find $P[X > 0]$.

Section 4.2: The Probability Density Function

4.17. A random variable X has pdf:

$$f_X(x) = \begin{cases} c(1 - x^2) & -1 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find c and plot the pdf.

(b) Plot the cdf of X.

(c) Find $P[X = 0]$, $P[0 < X < 0.5]$, and $P[|X - 0.5| < 0.25]$.

4.18. A random variable X has pdf:

$$f_X(x) = \begin{cases} cx(1 - x^2) & 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find c and plot the pdf.

(b) Plot the cdf of X.

(c) Find $P[0 < X < 0.5]$, $P[X = 1]$, $P[.25 < X < 0.5]$.

4.19. (a) In Problem 4.6, find and plot the pdf of the random variable R, the distance from the dart to the center of the target.

(b) Use the pdf to find the probability that the dart is outside the bull's eye.

4.20. (a) Find and plot the pdf of the random variable Z in Problem 4.7.

(b) Use the pdf to find the probability that the minimum is greater than $b/3$.

4.21. **(a)** Find and plot the pdf in Problem 4.8.
(b) Use the pdf to find the probabilities of the events: $\{X > a\}$ and $\{X > 2a\}$.

4.22. **(a)** Find and plot the pdf in Problem 4.12.
(b) Use the pdf to find $P[-1 \le X < 0.25]$.

4.23. **(a)** Find and plot the pdf in Problem 4.13.
(b) Use the pdf to find $P[X = 0]$, $P[X > 8]$.

4.24. **(a)** Find and plot the pdf of the random variable in Problem 4.14.
(b) Use the pdf to calculate the probabilities in Problem 4.14b.

4.25. Find and plot the pdf of the Weibull random variable in Problem 4.15a.

4.26. Find the cdf of the Cauchy random variable which has pdf:

$$f_X(x) = \frac{\alpha/\pi}{x^2 + \alpha^2} \qquad -\infty < x < \infty.$$

4.27. A voltage X is uniformly distributed in the set $\{-3, -2, \dots, 3, 4\}$.
(a) Find the pdf and cdf of the random variable X.
(b) Find the pdf and cdf of the random variable $Y = -2X^2 + 3$.
(c) Find the pdf and cdf of the random variable $W = \cos(\pi X/8)$.
(d) Find the pdf and cdf of the random variable $Z = \cos^2(\pi X/8)$.

4.28. Find the pdf and cdf of the Zipf random variable in Problem 3.70.

4.29. Let C be an event for which $P[C] > 0$. Show that $F_X(x|C)$ satisfies the eight properties of a cdf.

4.30. **(a)** In Problem 4.13, find $F_X(x|C)$ where $C = \{X > 0\}$.
(b) Find $F_X(x|C)$ where $C = \{X = 0\}$.

4.31. **(a)** In Problem 4.10, find $F_X(x|B)$ where $B = \{$hand does not stop at 3, 6, 9, or 12 o'clock$\}$.
(b) Find $F_X(x|B^c)$.

4.32. In Problem 4.13, find $f_X(x|B)$ and $F_X(x|B)$ where $B = \{X > 0.25\}$.

4.33. Let X be the exponential random variable.
(a) Find and plot $F_X(x|X > t)$. How does $F_X(x|X > t)$ differ from $F_X(x)$?
(b) Find and plot $f_X(x|X > t)$.
(c) Show that $P[X > t + x|X > t] = P[X > x]$. Explain why this is called the memoryless property.

4.34. The Pareto random variable X has cdf:

$$F_X(x) = \begin{cases} 0 & x < x_m \\ 1 - \dfrac{x_m^\alpha}{x^\alpha} & x \ge x_m. \end{cases}$$

(a) Find and plot the pdf of X.
(b) Repeat Problem 4.33 parts a and b for the Pareto random variable.
(c) What happens to $P[X > t + x|X > t]$ as t becomes large? Interpret this result.

4.35. **(a)** Find and plot $F_X(x|a \le X \le b)$. Compare $F_X(x|a \le X \le b)$ to $F_X(x)$.
(b) Find and plot $f_X(x|a \le X \le b)$.

4.36. In Problem 4.6, find $F_R(r|R > 1)$ and $f_R(r|R > 1)$.

4.37. **(a)** In Problem 4.7, find $F_Z(z|b/4 \le Z \le b/2)$ and $f_Z(z|b/4 \le Z \le b/2)$.
(b) Find $F_Z(z|B)$ and $f_Z(z|B)$, where $B = \{x > b/2\}$.

4.38. A binary transmission system sends a "0" bit using a -1 voltage signal and a "1" bit by transmitting a $+1$. The received signal is corrupted by noise N that has a Laplacian distribution with parameter α. Assume that "0" bits and "1" bits are equiprobable.
(a) Find the pdf of the received signal $Y = X + N$, where X is the transmitted signal, given that a "0" was transmitted; that a "1" was transmitted.
(b) Suppose that the receiver decides a "0" was sent if $Y < 0$, and a "1" was sent if $Y \ge 0$. What is the probability that the receiver makes an error given that a $+1$ was transmitted? a -1 was transmitted?
(c) What is the overall probability of error?

Section 4.3: The Expected Value of X

4.39. Find the mean and variance of X in Problem 4.17.

4.40. Find the mean and variance of X in Problem 4.18.

4.41. Find the mean and variance of Y, the distance from the dart to the origin, in Problem 4.19.

4.42. Find the mean and variance of Z, the minimum of the coordinates in a square, in Problem 4.20.

4.43. Find the mean and variance of $X = (1 - \zeta)^{-1/2}$ in Problem 4.21. Find $E[X]$ using Eq. (4.28).

4.44. Find the mean and variance of X in Problems 4.12 and 4.22.

4.45. Find the mean and variance of X in Problems 4.13 and 4.23. Find $E[X]$ using Eq. (4.28).

4.46. Find the mean and variance of the Gaussian random variable by direct integration of Eqs. (4.27) and (4.34).

4.47. Prove Eqs. (4.28) and (4.29).

4.48. Find the variance of the exponential random variable.

4.49. **(a)** Show that the mean of the Weibull random variable in Problem 4.15 is $\Gamma(1 + 1/\beta)$ where $\Gamma(x)$ is the gamma function defined in Eq. (4.56).
(b) Find the second moment and the variance of the Weibull random variable.

4.50. Explain why the mean of the Cauchy random variable does not exist.

4.51. Show that $E[X]$ does not exist for the Pareto random variable with $\alpha = 1$ and $x_m = 1$.

4.52. Verify Eqs. (4.36), (4.37), and (4.38).

4.53. Let $Y = A\cos(\omega t) + c$ where A has mean m and variance σ^2 and ω and c are constants. Find the mean and variance of Y. Compare the results to those obtained in Example 4.15.

4.54. A limiter is shown in Fig. P4.2.

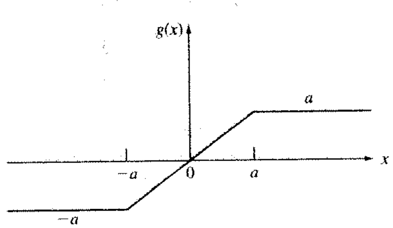


FIGURE P4.2

(a) Find an expression for the mean and variance of $Y = g(X)$ for an arbitrary continuous random variable X.

(b) Evaluate the mean and variance if X is a Laplacian random variable with $\lambda = a = 1$.

(c) Repeat part (b) if X is from Problem 4.17 with $a = 1/2$.

(d) Evaluate the mean and variance if $X = U^3$ where U is a uniform random variable in the unit interval, $[-1, 1]$ and $a = 1/2$.

4.55. A limiter with center-level clipping is shown in Fig. P4.3.

(a) Find an expression for the mean and variance of $Y = g(X)$ for an arbitrary continuous random variable X.

(b) Evaluate the mean and variance if X is Laplacian with $\lambda = a = 1$ and $b = 2$.

(c) Repeat part (b) if X is from Problem 4.22, $a = 1/2$, $b = 3/2$.

(d) Evaluate the mean and variance if $X = b\cos(2\pi U)$ where U is a uniform random variable in the unit interval $[-1, 1]$ and $a = 3/4$, $b = 1/2$.

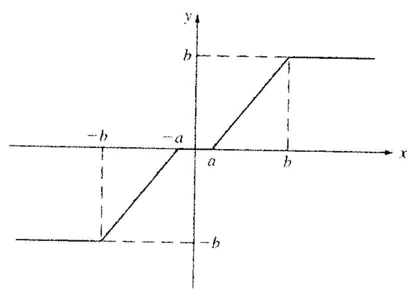


FIGURE P4.3

4.56. Let $Y = 3X + 2$.

(a) Find the mean and variance of Y in terms of the mean and variance of X.

(b) Evaluate the mean and variance of Y if X is Laplacian.

(c) Evaluate the mean and variance of Y if X is an arbitrary Gaussian random variable.

(d) Evaluate the mean and variance of Y if $X = b\cos(2\pi U)$ where U is a uniform random variable in the unit interval.

4.57. Find the nth moment of U, the uniform random variable in the unit interval. Repeat for X uniform in $[a, b]$.

4.58. Consider the quantizer in Example 4.20.

(a) Find the conditional pdf of X given that X is in the interval $(d, 2d)$.

(b) Find the conditional expected value and conditional variance of X given that X is in the interval $(d, 2d)$.

(c) Now suppose that when X falls in $(d, 2d)$, it is mapped onto the point c where $d < c < 2d$. Find an expression for the expected value of the mean square error: $E[(X - c)^2 | d < X < 2d]$.

(d) Find the value c that minimizes the above mean square error. Is c the midpoint of the interval? Explain why or why not by sketching possible conditional pdf shapes.

(e) Find an expression for the overall mean square error using the approach in parts c and d.

Section 4.4: Important Continuous Random Variables

4.59. Let X be a uniform random variable in the interval $[-2, 2]$. Find and plot $P[|X| > x]$.

4.60. In Example 4.20, let the input to the quantizer be a uniform random variable in the interval $[-4d, 4d]$. Show that $Z = X - Q(X)$ is uniformly distributed in $[-d/2, d/2]$.

4.61. Let X be an exponential random variable with parameter λ.

(a) For $d > 0$ and k a nonnegative integer, find $P[kd < X < (k + 1)d]$.

(b) Segment the positive real line into four equiprobable disjoint intervals.

4.62. The rth percentile, $\pi(r)$, of a random variable X is defined by $P[X \le \pi(r)] = r/100$.

(a) Find the 90%, 95%, and 99% percentiles of the exponential random variable with parameter λ.

(b) Repeat part a for the Gaussian random variable with parameters $m = 0$ and σ^2.

4.63. Let X be a Gaussian random variable with $m = 5$ and $\sigma^2 = 16$.

(a) Find $P[X > 4]$, $P[X \ge 7]$, $P[6.72 < X < 10.16]$, $P[2 < X < 7]$, $P[6 \le X \le 8]$.

(b) $P[X < a] = 0.8869$, find a.

(c) $P[X > b] = 0.11131$, find b.

(d) $P[13 < X \le c] = 0.0123$, find c.

4.64. Show that the Q-function for the Gaussian random variable satisfies $Q(-x) = 1 - Q(x)$.

4.65. Use Octave to generate Tables 4.2 and 4.3.

4.66. Let X be a Gaussian random variable with mean m and variance σ^2.

(a) Find $P[X \le m]$.

(b) Find $P[|X - m| < k\sigma]$, for $k = 1, 2, 3, 4, 5, 6$.

(c) Find the value of k for which $Q(k) = P[X > m + k\sigma] = 10^{-j}$ for $j = 1, 2, 3, 4, 5, 6$.

4.67. A binary transmission system transmits a signal X (-1 to send a "0" bit; $+1$ to send a "1" bit). The received signal is $Y = X + N$ where noise N has a zero-mean Gaussian distribution with variance σ^2. Assume that "0" bits are three times as likely as "1" bits.

(a) Find the conditional pdf of Y given the input value: $f_Y(y | X = +1)$ and $f_Y(y | X = -1)$.

(b) The receiver decides a "0" was transmitted if the observed value of y satisfies

$$f_Y(y | X = -1)P[X = -1] > f_Y(y | X = +1)P[X = +1]$$

and it decides a "1" was transmitted otherwise. Use the results from part a to show that this decision rule is equivalent to: If $y < T$ decide "0"; if $y \ge T$ decide "1".

(c) What is the probability that the receiver makes an error given that a $+1$ was transmitted? a -1 was transmitted? Assume $\sigma^2 = 1/16$.

(d) What is the overall probability of error?

4.68. Two chips are being considered for use in a certain system. The lifetime of chip 1 is modeled by a Gaussian random variable with mean 20,000 hours and standard deviation 5000 hours. (The probability of negative lifetime is negligible.) The lifetime of chip 2 is also a Gaussian random variable but with mean 22,000 hours and standard deviation 1000 hours. Which chip is preferred if the target lifetime of the system is 20,000 hours? 24,000 hours?

4.69. Passengers arrive at a taxi stand at an airport at a rate of one passenger per minute. The taxi driver will not leave until seven passengers arrive to fill his van. Suppose that passenger interarrival times are exponential random variables, and let X be the time to fill a van. Find the probability that more than 10 minutes elapse until the van is full.

4.70. **(a)** Show that the gamma random variable has mean:

$$E[X] = \alpha/\lambda.$$

(b) Show that the gamma random variable has second moment, and variance given by:

$$E[X^2] = \alpha(\alpha + 1)/\lambda^2 \text{ and } \mathrm{VAR}[X] = \alpha/\lambda^2.$$

(c) Use parts a and b to obtain the mean and variance of an m-Erlang random variable.

(d) Use parts a and b to obtain the mean and variance of a chi-square random variable.

4.71. The time X to complete a transaction in a system is a gamma random variable with mean 4 and variance 8. Use Octave to plot $P[X > x]$ as a function of x. Note: Octave uses $\beta = 1/2$.

4.72. **(a)** Plot the pdf of an m-Erlang random variable for $m = 1, 2, 3$ and $\lambda = 1$.

(b) Plot the chi-square pdf for $k = 1, 2, 3$.

4.73. A repair person keeps four widgets in stock. What is the probability that the widgets in stock will last 15 days if the repair person needs to replace widgets at an average rate of one widget every three days, where the time between widget failures is an exponential random variable?

4.74. **(a)** Find the cdf of the m-Erlang random variable by integration of the pdf. *Hint:* Use integration by parts.

(b) Show that the derivative of the cdf given by Eq. (4.58) gives the pdf of an m-Erlang random variable.

4.75. Plot the pdf of a beta random variable with: $a = b = 1/4, 1, 4, 8$; $a = 5, b = 1$; $a = 1, b = 3$; $a = 2, b = 5$.

Section 4.5: Functions of a Random Variable

4.76. Let X be a Gaussian random variable with mean 2 and variance 4. The reward in a system is given by $Y = (X)^+$. Find the pdf of Y.

4.77. The amplitude of a radio signal X is a Rayleigh random variable with pdf:

$$f_X(x) = \frac{x}{\alpha^2}e^{-x^2/2\alpha^2} \qquad x > 0, \quad \alpha > 0.$$

(a) Find the pdf of $Z = (X - r)^+$.

(b) Find the pdf of $Z = X^2$.

4.78. A wire has length X, an exponential random variable with mean 5π cm. The wire is cut to make rings of diameter 1 cm. Find the probability for the number of complete rings produced by each length of wire.

4.79. A signal that has amplitudes with a Gaussian pdf with zero mean and unit variance is applied to the quantizer in Example 4.27.

(a) Pick d so that the probability that X falls outside the range of the quantizer is 1%.

(b) Find the probability of the output levels of the quantizer.

4.80. The signal X is amplified and shifted as follows: $Y = 2X + 3$, where X is the random variable in Problem 4.12. Find the cdf and pdf of Y.

4.81. The net profit in a transaction is given by $Y = 2 - 4X$ where X is the random variable in Problem 4.13. Find the cdf and pdf of Y.

4.82. Find the cdf and pdf of the output of the limiter in Problem 4.54 parts b, c, and d.

4.83. Find the cdf and pdf of the output of the limiter with center-level clipping in Problem 4.55 parts b, c, and d.

4.84. Find the cdf and pdf of $Y = 3X + 2$ in Problem 4.56 parts b, c, and d.

4.85. The exam grades in a certain class have a Gaussian pdf with mean m and standard deviation σ. Find the constants a and b so that the random variable $y = aX + b$ has a Gaussian pdf with mean m' and standard deviation σ'.

4.86. Let $X = U^n$ where n is a positive integer and U is a uniform random variable in the unit interval. Find the cdf and pdf of X.

4.87. Repeat Problem 4.86 if U is uniform in the interval $[-1, 1]$.

4.88. Let $Y = |X|$ be the output of a full-wave rectifier with input voltage X.

(a) Find the cdf of Y by finding the equivalent event of $\{Y \le y\}$. Find the pdf of Y by differentiation of the cdf.

(b) Find the pdf of Y by finding the equivalent event of $\{y < Y \le y + dy\}$. Does the answer agree with part a?

(c) What is the pdf of Y if the $f_X(x)$ is an even function of x?

4.89. Find and plot the cdf of Y in Example 4.34.

4.90. A voltage X is a Gaussian random variable with mean 1 and variance 2. Find the pdf of the power dissipated by an R-ohm resistor $P = RX^2$.

4.91. Let $Y = e^X$.

(a) Find the cdf and pdf of Y in terms of the cdf and pdf of X.

(b) Find the pdf of Y when X is a Gaussian random variable. In this case Y is said to be a lognormal random variable. Plot the pdf and cdf of Y when X is zero-mean with variance 1/8; repeat with variance 8.

4.92. Let a radius be given by the random variable X in Problem 4.18.

(a) Find the pdf of the area covered by a disc with radius X.

(b) Find the pdf of the volume of a sphere with radius X.

(c) Find the pdf of the volume of a sphere in R^n:

$$Y = \begin{cases} (2\pi)^{(n-1)/2}\,X^n/(2 \times 4 \times \cdots \times n) & \text{for } n \text{ even} \\ 2(2\pi)^{(n-1)/2}\,X^n/(1 \times 3 \times \cdots \times n) & \text{for } n \text{ odd.} \end{cases}$$

4.93. In the quantizer in Example 4.20, let $Z = X - q(X)$. Find the pdf of Z if X is a Laplacian random variable with parameter $\alpha = d/2$.

4.94. Let $Y = \alpha \tan \pi X$, where X is uniformly distributed in the interval $(-1, 1)$.

(a) Show that Y is a Cauchy random variable.

(b) Find the pdf of $Y = 1/X$.

4.95. Let X be a Weibull random variable in Problem 4.15. Let $Y = (X/\lambda)^{\beta}$. Find the cdf and pdf of Y.

4.96. Find the pdf of $X = -\ln(1 - U)$, where U is a uniform random variable in $(0, 1)$.

Section 4.6: The Markov and Chebyshev Inequalities

4.97. Compare the Markov inequality and the exact probability for the event $\{X > c\}$ as a function of c for:
- **(a)** X is a uniform random variable in the interval $[0, b]$.
- **(b)** X is an exponential random variable with parameter λ.
- **(c)** X is a Pareto random variable with $\alpha > 1$.
- **(d)** X is a Rayleigh random variable.

4.98. Compare the Markov inequality and the exact probability for the event $\{X > c\}$ as a function of c for:
- **(a)** X is a uniform random variable in $\{1, 2, \ldots, L\}$.
- **(b)** X is a geometric random variable.
- **(c)** X is a Zipf random variable with $L = 10$; $L = 100$.
- **(d)** X is a binomial random variable with $n = 10, p = 0.5$; $n = 50, p = 0.5$.

4.99. Compare the Chebyshev inequality and the exact probability for the event $\{|X - m| > c\}$ as a function of c for:
- **(a)** X is a uniform random variable in the interval $[-b, b]$.
- **(b)** X is a Laplacian random variable with parameter α.
- **(c)** X is a zero-mean Gaussian random variable.
- **(d)** X is a binomial random variable with $n = 10, p = 0.5$; $n = 50, p = 0.5$.

4.100. Let X be the number of successes in n Bernoulli trials where the probability of success is p. Let $Y = X/n$ be the average number of successes per trial. Apply the Chebyshev inequality to the event $\{|Y - p| > a\}$. What happens as $n \to \infty$?

4.101. Suppose that light bulbs have exponentially distributed lifetimes with unknown mean $E[X]$. Suppose we measure the lifetime of n light bulbs, and we estimate the mean $E[X]$ by the arithmetic average Y of the measurements. Apply the Chebyshev inequality to the event $\{|Y - E[X]| > a\}$. What happens as $n \to \infty$? *Hint:* Use the m-Erlang random variable.

Section 4.7: Transform Methods

4.102.
- **(a)** Find the characteristic function of the uniform random variable in $[-b, b]$.
- **(b)** Find the mean and variance of X by applying the moment theorem.

4.103.
- **(a)** Find the characteristic function of the Laplacian random variable.
- **(b)** Find the mean and variance of X by applying the moment theorem.

4.104. Let $\Phi_X(\omega)$ be the characteristic function of an exponential random variable. What random variable does $\Phi_X^n(\omega)$ correspond to?

4.105. Find the mean and variance of the Gaussian random variable by applying the moment theorem to the characteristic function given in Table 4.1.

4.106. Find the characteristic function of $Y = aX + b$ where X is a Gaussian random variable. *Hint:* Use Eq. (4.79).

4.107. Show that the characteristic function for the Cauchy random variable is $e^{-|\omega|}$.

4.108. Find the Chernoff bound for the exponential random variable with $\lambda = 1$. Compare the bound to the exact value for $P[X > 5]$.

4.109.
- **(a)** Find the probability generating function of the geometric random variable.
- **(b)** Find the mean and variance of the geometric random variable from its pgf.

4.110.
- **(a)** Find the pgf for the binomial random variable X with parameters n and p.
- **(b)** Find the mean and variance of X from the pgf.

4.111. Let $G_X(z)$ be the pgf for a binomial random variable with parameters n and p, and let $G_Y(z)$ be the pgf for a binomial random variable with parameters m and p. Consider the function $G_X(z)\,G_Y(z)$. Is this a valid pgf? If so, to what random variable does it correspond?

4.112. Let $G_N(z)$ be the pgf for a Poisson random variable with parameter α, and let $G_M(z)$ be the pgf for a Poisson random variable with parameters β. Consider the function $G_N(z)\,G_M(z)$. Is this a valid pgf? If so, to what random variable does it correspond?

4.113. Let N be a Poisson random variable with parameter $\alpha = 1$. Compare the Chernoff bound and the exact value for $P[X \geq 5]$.

4.114.
- **(a)** Find the pgf $G_U(z)$ for the discrete uniform random variable U.
- **(b)** Find the mean and variance from the pgf.
- **(c)** Consider $G_U(z)^2$. Does this function correspond to a pgf? If so, find the mean of the corresponding random variable.

4.115.
- **(a)** Find $P[X = r]$ for the negative binomial random variable from the pgf in Table 3.1.
- **(b)** Find the mean of X.

4.116. Derive Eq. (4.89).

4.117. Obtain the nth moment of a gamma random variable from the Laplace transform of its pdf.

4.118. Let X be the mixture of two exponential random variables (see Example 4.58). Find the Laplace transform of the pdf of X.

4.119. The Laplace transform of the pdf of a random variable X is given by:

$$X^*(s) = \frac{a}{s+a}\,\frac{b}{s+b}.$$

Find the pdf of X. *Hint:* Use a partial fraction expansion of $X^*(s)$.

4.120. Find a relationship between the Laplace transform of a gamma random variable pdf with parameters α and λ and the Laplace transform of a gamma random variable with parameters $\alpha - 1$ and λ. What does this imply if X is an m-Erlang random variable?

4.121.
- **(a)** Find the Chernoff bound for $P[X > t]$ for the gamma random variable.
- **(b)** Compare the bound to the exact value of $P[X \geq 9]$ for an $m = 3, \lambda = 1$ Erlang random variable.

Section 4.8: Basic Reliability Calculations

4.122. The lifetime T of a device has pdf

$$f_T(t) = \begin{cases} 1/10T_0 & 0 < t < T_0 \\ 0.9\lambda e^{-\lambda(t-T_0)} & t \geq T_0 \\ 0 & t < T_0. \end{cases}$$

(a) Find the reliability and MTTF of the device.
(b) Find the failure rate function.
(c) How many hours of operation can be considered to achieve 99% reliability?

4.123. The lifetime T of a device has pdf

$$f_T(t) = \begin{cases} 1/T_0 & a \leq t \leq a + T_0 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the reliability and MTTF of the device.
(b) Find the failure rate function.
(c) How many hours of operation can be considered to achieve 99% reliability?

4.124. The lifetime T of a device is a Rayleigh random variable.
(a) Find the reliability of the device.
(b) Find the failure rate function. Does $r(t)$ increase with time?
(c) Find the reliability of two devices that are in series.
(d) Find the reliability of two devices that are in parallel.

4.125. The lifetime T of a device is a Weibull random variable.
(a) Plot the failure rates for $\alpha = 1$ and $\beta = 0.5$; for $\alpha = 1$ and $\beta = 2$.
(b) Plot the reliability functions in part a.
(c) Plot the reliability of two devices that are in series.
(d) Plot the reliability of two devices that are in parallel.

4.126. A system starts with m devices, 1 active and $m - 1$ on standby. Each device has an exponential lifetime. When a device fails it is immediately replaced with another device (if one is still available).
(a) Find the reliability of the system.
(b) Find the failure rate function.

4.127. Find the failure rate function of the memory chips discussed in Example 2.28. Plot $\ln(r(t))$ versus αt.

4.128. A device comes from two sources. Devices from source 1 have mean m and exponentially distributed lifetimes. Devices from source 2 have mean m and Pareto-distributed lifetimes with $\alpha > 1$. Assume a fraction p is from source 1 and a fraction $1 - p$ from source 2.
(a) Find the reliability of an arbitrarily selected device.
(b) Find the failure rate function.

4.129. A device has the failure rate function:

$$r(t) = \begin{cases} 1 + 9(1 - t) & 0 \leq t < 1 \\ 1 & 1 \leq t < 10 \\ 1 + 10(t - 10) & t \geq 10. \end{cases}$$

Find the reliability function and the pdf of the device.

4.130. A system has three identical components and the system is functioning if two or more components are functioning.
(a) Find the reliability and MTTF of the system if the component lifetimes are exponential random variables with mean 1.
(b) Find the reliability of the system if one of the components has mean 2.

4.131. Repeat Problem 4.130 if the component lifetimes are Weibull distributed with $\beta = 3$.

4.132. A system consists of two processors and three peripheral units. The system is functioning as long as one processor and two peripherals are functioning.
(a) Find the system reliability and MTTF if the processor lifetimes are exponential random variables with mean 5 and the peripheral lifetimes are Rayleigh random variables with mean 10.
(b) Find the system reliability and MTTF if the processor lifetimes are exponential random variables with mean 10 and the peripheral lifetimes are exponential random variables with mean 5.

4.133. An operation is carried out by a subsystem consisting of three units that operate in a series configuration.
(a) The units have exponentially distributed lifetimes with mean 1. How many subsystems should be operated in parallel to achieve a reliability of 99% in T hours of operation?
(b) Repeat part a with Rayleigh-distributed lifetimes.
(c) Repeat part a with Weibull-distributed lifetimes with $\beta = 3$.

Section 4.9: Computer Methods for Generating Random Variables

4.134. Octave provides function calls to evaluate the pdf and cdf of important continuous random variables. For example, the functions `\normal_cdf(x, m, var)` and `normal_pdf(x, m, var)` compute the cdf and pdf, respectively, at `x` for a Gaussian random variable with mean `m` and variance `var`.
(a) Plot the conditional pdfs in Example 4.11 if $v = \pm 2$ and the noise is zero-mean and unit variance.
(b) Compare the cdf of the Gaussian random variable with the Chernoff bound obtained in Example 4.44.

4.135. Plot the pdf and cdf of the gamma random variable for the following cases.
(a) $\lambda = 1$ and $\alpha = 1, 2, 4$.
(b) $\lambda = 1/2$ and $\alpha = 1/2, 1, 3/2, 5/2$.

4.136. The random variable X has the triangular pdf shown in Fig. P4.4.

(a) Find the transformation needed to generate X.

(b) Use Octave to generate 100 samples of X. Compare the empirical pdf of the samples with the desired pdf.

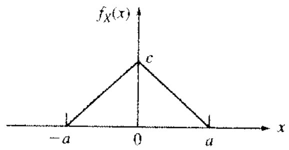


FIGURE P4.4

4.137. For each of the following random variables: Find the transformation needed to generate the random variable X; use Octave to generate 1000 samples of X; Plot the sequence of outcomes; compare the empirical pdf of the samples with the desired pdf.

(a) Laplacian random variable with $\alpha = 1$.

(b) Pareto random variable with $\alpha = 1.5, 2, 2.5$.

(c) Weibull random variable with $\beta = 0.5, 2, 3$ and $\lambda = 1$.

4.138. A random variable Y of mixed type has pdf

$$f_Y(x) = p\delta(x) + (1 - p)f_Y(x),$$

where X is a Laplacian random variable and p is a number between zero and one. Find the transformation required to generate Y.

4.139. Specify the transformation method needed to generate the geometric random variable with parameter $p = 1/2$. Find the average number of comparisons needed in the search to determine each outcome.

4.140. Specify the transformation method needed to generate the Poisson random variable with small parameter α. Compute the average number of comparisons needed in the search.

4.141. The following rejection method can be used to generate Gaussian random variables:

1. Generate U_1, a uniform random variable in the unit interval.
2. Let $X_1 = -\ln(U_1)$.
3. Generate U_2, a uniform random variable in the unit interval. If $U_2 \leq \exp\{-(X_1 - 1)^2/2\}$, accept X_1. Otherwise, reject X_1 and go to step 1.
4. Generate a random sign (+ or −) with equal probability. Output X equal to X_1 with the resulting sign.

(a) Show that if X_1 is accepted, then its pdf corresponds to the pdf of the absolute value of a Gaussian random variable with mean 0 and variance 1.

(b) Show that X is a Gaussian random variable with mean 0 and variance 1.

4.142. Cheng (1977) has shown that the function $Kf_Z(x)$ bounds the pdf of a gamma random variable with $\alpha > 1$, where

$$f_Z(x) = \frac{\lambda\alpha^\lambda x^{\lambda-1}}{(\alpha^\lambda + x^\lambda)^2} \quad \text{and} \quad K = (2\alpha - 1)^{1/2}.$$

Find the cdf of $f_Z(x)$ and the corresponding transformation needed to generate Z.

4.143. **(a)** Show that in the modified rejection method, the probability of accepting X_1 is $1/K$. *Hint:* Use conditional probability.

(b) Show that Z has the desired pdf.

4.144. Two methods for generating binomial random variables are: (1) Generate n Bernoulli random variables and add the outcomes; (2) Divide the unit interval according to binomial probabilities. Compare the methods under the following conditions:

(a) $p = 1/2, n = 5, 25, 50$;

(b) $p = 0.1, n = 5, 25, 50$.

(c) Use Octave to implement the two methods by generating 1000 binomially distributed samples.

4.145. Let the number of event occurrences in a time interval be a Poisson random variable. In Section 3.4, it was found that the time between events for a Poisson random variable is an exponentially distributed random variable.

(a) Explain how one can generate Poisson random variables from a sequence of exponentially distributed random variables.

(b) How does this method compare with the one presented in Problem 4.140?

(c) Use Octave to implement the two methods when $\alpha = 3, \alpha = 25$, and $\alpha = 100$.

4.146. Write a program to generate the gamma pdf with $\alpha > 1$ using the rejection method discussed in Problem 4.142. Use this method to generate m-Erlang random variables with $m = 2, 10$ and $\lambda = 1$ and compare the method to the straightforward generation of m exponential random variables as discussed in Example 4.57.

*Section 4.10: Entropy

4.147. Let X be the outcome of the toss of a fair die.

(a) Find the entropy of X.

(b) Suppose you are told that X is even. What is the reduction in entropy?

4.148. A biased coin is tossed three times.

(a) Find the entropy of the outcome if the sequence of heads and tails is noted.

(b) Find the entropy of the outcome if the number of heads is noted.

(c) Explain the difference between the entropies in parts a and b.

4.149. Let X be the number of tails until the first heads in a sequence of tosses of a biased coin.

(a) Find the entropy of X given that $X \geq k$.

(b) Find the entropy of X given that $X \leq k$.

4.150. One of two coins is selected at random: Coin A has $P[\text{heads}] = 1/10$ and coin B has $P[\text{heads}] = 9/10$.

(a) Suppose the coin is tossed once. Find the entropy of the outcome.

(b) Suppose the coin is tossed twice and the sequence of heads and tails is observed. Find the entropy of the outcome.

4.151. Suppose that the randomly selected coin in Problem 4.150 is tossed until the first occurrence of heads. Suppose that heads occurs in the kth toss. Find the entropy regarding the identity of the coin.

4.152. A communication channel accepts input I from the set $\{0, 1, 2, 3, 4, 5, 6\}$. The channel output is $X = I + N \bmod 7$, where N is equally likely to be +1 or −1.

(a) Find the entropy of I if all inputs are equiprobable.

(b) Find the entropy of I given that $X = 4$.

4.153. Let X be a discrete random variable with entropy H_X.
 (a) Find the entropy of $Y = 2X$.
 (b) Find the entropy of any invertible transformation of X.

4.154. Let (X, Y) be the pair of outcomes from two independent tosses of a die.
 (a) Find the entropy of X.
 (b) Find the entropy of the pair (X, Y).
 (c) Find the entropy in n independent tosses of a die. Explain why entropy is additive in this case.

4.155. Let X be the outcome of the toss of a die, and let Y be a randomly selected integer less than or equal to X.
 (a) Find the entropy of Y.
 (b) Find the entropy of the pair (X, Y) and denote it by $H(X, Y)$.
 (c) Find the entropy of Y given $X = k$ and denote it by $g(k) = H(Y \mid X = k)$. Find $E[g(X)] = E[H(Y \mid X)]$.
 (d) Show that $H(X, Y) = H_X + E[H(Y \mid X)]$. Explain the meaning of this equation.

4.156. Let X take on values from $\{1, 2, \ldots, K\}$. Suppose that $P[X = K] = p$, and let H_Y be the entropy of X given that X is not equal to K. Show that $H_X = -p \ln p - (1 - p) \ln(1 - p) + (1 - p)H_Y$.

4.157. Let X be a uniform random variable in Example 4.62. Find and plot the entropy of Q as a function of the variance of the error $X - Q(X)$. *Hint:* Express the variance of the error in terms of d and substitute into the expression for the entropy of Q.

4.158. A communication channel accepts as input either 000 or 111. The channel transmits each binary input correctly with probability $1 - p$ and erroneously with probability p. Find the entropy of the input given that the output is 000; given that the output is 010.

4.159. Let X be a uniform random variable in the interval $[-a, a]$. Suppose we are told that the X is positive. Use the approach in Example 4.62 to find the reduction in entropy. Show that this is equal to the difference of the differential entropy of X and the differential entropy of X given $\{X > 0\}$.

4.160. Let X be uniform in $[a, b]$, and let $Y = 2X$. Compare the differential entropies of X and Y. How does this result differ from the result in Problem 4.153?

4.161. Find the pmf for the random variable X for which the sequence of questions in Fig. 4.26(a) is optimum.

4.162. Let the random variable X have $S_X = \{1, 2, 3, 4, 5, 6\}$ and pmf (3/8, 3/8, 1/8, 1/16, 1/32, 1/32). Find the entropy of X. What is the best code you can find for X?

4.163. Seven cards are drawn from a deck of 52 distinct cards. How many bits are required to represent all possible outcomes?

4.164. Find the optimum encoding for the geometric random variable with $p = 1/2$.

4.165. An urn experiment has 10 equiprobable distinct outcomes. Find the performance of the best tree code for encoding (a) a single outcome of the experiment; (b) a sequence of n outcomes of the experiment.

4.166. A binary information source produces n outputs. Suppose we are told that there are k 1's in these n outputs.
 (a) What is the best code to indicate which pattern of k 1's and $n - k$ 0's occurred?
 (b) How many bits are required to specify the value of k using a code with a fixed number of bits?

4.167. The random variable X takes on values from the set $\{1, 2, 3, 4\}$. Find the maximum entropy pmf for X given that $E[X] = 2$.

4.168. The random variable X is nonnegative. Find the maximum entropy pdf for X given that $E[X] = 10$.

4.169. Find the maximum entropy pdf of X given that $E[X^2] = c$.

4.170. Suppose we are given two parameters of the random variable X, $E[g_1(X)] = c_1$ and $E[g_2(X)] = c_2$.
 (a) Show that the maximum entropy pdf for X has the form

$$f_X(x) = Ce^{-\lambda_1 g_1(x) - \lambda_2 g_2(x)}.$$

 (b) Find the entropy of X.

4.171. Find the maximum entropy pdf of X given that $E[X] = m$ and $\text{VAR}[X] = \sigma^2$.

Problems Requiring Cumulative Knowledge

4.172. Three types of customers arrive at a service station. The time required to service type 1 customers is an exponential random variable with mean 2. Type 2 customers have a Pareto distribution with $\alpha = 3$ and $x_m = 1$. Type 3 customers require a constant service time of 2 seconds. Suppose that the proportion of type 1, 2, and 3 customers is 1/2, 1/8, and 3/8, respectively. Find the probability that an arbitrary customer requires more than 15 seconds of service time. Compare the above probability to the bound provided by the Markov inequality.

4.173. The lifetime X of a light bulb is a random variable with

$$P[X > t] = 2/(2 + t) \text{ for } t > 0.$$

Suppose three new light bulbs are installed at time $t = 0$. At time $t = 1$ all three light bulbs are still working. Find the probability that at least one light bulb is still working at time $t = 9$.

4.174. The random variable X is uniformly distributed in the interval $[0, a]$. Suppose a is unknown, so we estimate a by the maximum value observed in n independent repetitions of the experiment; that is, we estimate a by $Y = \max\{X_1, X_2, \ldots, X_n\}$.
 (a) Find $P[Y \leq y]$.
 (b) Find the mean and variance of Y, and explain why Y is a good estimate for a when N is large.

4.175. The sample X of a signal is a Gaussian random variable with $m = 0$ and $\sigma^2 = 1$. Suppose that X is quantized by a nonuniform quantizer consisting of four intervals: $(-\infty, -a]$, $(-a, 0]$, $(0, a]$, and (a, ∞).
 (a) Find the value of a so that X is equally likely to fall in each of the four intervals.
 (b) Find the representation point $x_i = q(X)$ for X in $(0, a]$ that minimizes the mean-squared error, that is,

$$\int_0^a (x - x_1)^2 f_X(x)\, dx \text{ is minimized.}$$

 Hint: Differentiate the above expression with respect to x_i. Find the representation points for the other intervals.
 (c) Evaluate the mean-squared error of the quantizer $E[(X - q(X)^2]$.

4.176. The output Y of a binary communication system is a unit-variance Gaussian random with mean zero when the input is "0" and mean one when the input is "one". Assume the input is 1 with probability p.

(a) Find $P[\text{input is } 1 \mid y < Y < y + h]$ and $P[\text{input is } 0 \mid y < Y < y + h]$.

(b) The receiver uses the following decision rule:

If $P[\text{input is } 1 \mid y < Y < y + h] > P[\text{input is } 0 \mid y < Y < y + h]$, decide input was 1; otherwise, decide input was 0.

Show that this decision rule leads to the following threshold rule:

If $Y > T$, decide input was 1; otherwise, decide input was 0.

(c) What is the probability of error for the above decision rule?

CHAPTER 5

Pairs of Random Variables

Many random experiments involve several random variables. In some experiments a number of different quantities are measured. For example, the voltage signals at several points in a circuit at some specific time may be of interest. Other experiments involve the repeated measurement of a certain quantity such as the repeated measurement ("sampling") of the amplitude of an audio or video signal that varies with time. In Chapter 4 we developed techniques for calculating the probabilities of events involving a single random variable *in isolation*. In this chapter, we extend the concepts already introduced to two random variables:

- We use the joint pmf, cdf, and pdf to calculate the probabilities of events that involve the *joint* behavior of two random variables;
- We use expected value to define joint moments that summarize the behavior of two random variables;
- We determine when two random variables are independent, and we quantify their degree of "correlation" when they are not independent;
- We obtain conditional probabilities involving a pair of random variables.

In a sense we have already covered all the fundamental concepts of probability and random variables, and we are "simply" elaborating on the case of two or more random variables. Nevertheless, there are significant analytical techniques that need to be learned, e.g., double summations of pmf's and double integration of pdf's, so we first discuss the case of two random variables in detail because we can draw on our geometric intuition. Chapter 6 considers the general case of vector random variables. Throughout these two chapters you should be mindful of the forest (fundamental concepts) and the trees (specific techniques)!

5.1 TWO RANDOM VARIABLES

The notion of a random variable as a mapping is easily generalized to the case where two quantities are of interest. Consider a random experiment with sample space S and event class $\mathcal{F}$. We are interested in a function that assigns a pair of real numbers

2. L. Breiman, *Probability and Stochastic Processes*, Houghton Mifflin, Boston, 1969.
3. H. J. Larson and B. O. Shubert, *Probabilistic Models in Engineering Sciences*, vol. 1, Wiley, New York, 1979.
4. N. S. Jayant and P. Noll, *Digital Coding of Waveforms*, Prentice Hall, Englewood Cliffs, N.J., 1984.
5. N. Johnson et al., *Continuous Multivariate Distributions*, Wiley, New York, 2000.
6. H. Stark and J. W. Woods, *Probability, Random Processes, and Estimation Theory for Engineers*, Prentice Hall, Englewood Cliffs, N.J., 1986.
7. H. Anton, *Elementary Linear Algebra*, 9th ed., Wiley, New York, 2005.
8. C. H. Edwards, Jr., and D. E. Penney, *Calculus and Analytic Geometry*, 4th ed., Prentice Hall, Englewood Cliffs, N.J., 1994.

PROBLEMS

Section 5.1: Two Random Variables

5.1. Let X be the maximum and let Y be the minimum of the number of heads obtained when Carlos and Michael each flip a fair coin twice.
- **(a)** Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).
- **(b)** Find the probabilities for all values of (X, Y).
- **(c)** Find $P[X = Y]$.
- **(d)** Repeat parts b and c if Carlos uses a biased coin with $P[\text{heads}] = 3/4$.

5.2. Let X be the difference and let Y be the sum of the number of heads obtained when Carlos and Michael each flip a fair coin twice.
- **(a)** Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).
- **(b)** Find the probabilities for all values of (X, Y).
- **(c)** Find $P[X + Y = 1]$, $P[X + Y = 2]$.

5.3. The input X to a communication channel is "-1" or "1", with respective probabilities 1/4 and 3/4. The output of the channel Y is equal to: the corresponding input X with probability $1 - p - p_e$; $-X$ with probability p; 0 with probability p_e.
- **(a)** Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).
- **(b)** Find the probabilities for all values of (X, Y).
- **(c)** Find $P[X \neq Y]$, $P[Y = 0]$.

5.4. **(a)** Specify the range of the pair (N_1, N_2) in Example 5.2.
- **(b)** Specify and sketch the event "more revenue comes from type 1 requests than type 2 requests."

5.5. **(a)** Specify the range of the pair (Q, R) in Example 5.3.
- **(b)** Specify and sketch the event "last packet is more than half full."

5.6. Let the pair of random variables H and W be the height and weight in Example 5.1. The body mass index is a measure of body fat and is defined by $\text{BMI} = W/H^2$ where W is in kilograms and H is in meters. Determine and sketch on the plane the following events: $A = \{\text{"obese," BMI} \geq 30\}$; $B = \{\text{"overweight," } 25 \leq \text{BMI} < 30\}$; $C = \{\text{"normal," } 18.5 \leq \text{BMI} < 25\}$; and $D = \{\text{"underweight," BMI} < 18.5\}$.

5.7. Let (X, Y) be the two-dimensional noise signal in Example 5.4. Specify and sketch the events:
- **(a)** "Maximum noise magnitude is greater than 5."
- **(b)** "The noise power $X^2 + Y^2$ is greater than 4."
- **(c)** "The noise power $X^2 + Y^2$ is greater than 4 and less than 9."

5.8. For the pair of random variables (X, Y) sketch the region of the plane corresponding to the following events. Identify which events are of product form.
- **(a)** $\{X + Y > 3\}$.
- **(b)** $\{e^X > Ye^3\}$.
- **(c)** $\{\min(X, Y) > 0\} \cup \{\max\{X, Y\} < 0\}$.
- **(d)** $\{|X - Y| \geq 1\}$.
- **(e)** $\{|X/Y| > 2\}$.
- **(f)** $\{X/Y < 2\}$.
- **(g)** $\{X^3 > Y\}$.
- **(h)** $\{XY < 0\}$.
- **(i)** $\{\max(|X|, Y) < 3\}$.

Section 5.2: Pairs of Discrete Random Variables

5.9. **(a)** Find and sketch $p_{X,Y}(x, y)$ in Problem 5.1 when using a fair coin.
- **(b)** Find $p_X(x)$ and $p_Y(y)$.
- **(c)** Repeat parts a and b if Carlos uses a biased coin with $P[\text{heads}] = 3/4$.

5.10. **(a)** Find and sketch $p_{X,Y}(x, y)$ in Problem 5.2 when using a fair coin.
- **(b)** Find $p_X(x)$ and $p_Y(y)$.
- **(c)** Repeat parts a and b if Carlos uses a biased coin with $P[\text{heads}] = 3/4$.

5.11. **(a)** Find the marginal pmf's for the pairs of random variables with the indicated joint pmf.

(i)				(ii)				(iii)			
X/Y	−1	0	1	X/Y	−1	0	1	X/Y	−1	0	1
−1	1/6	1/6	0	−1	1/9	1/9	1/9	−1	1/3	0	0
0	0	0	1/3	0	1/9	1/9	1/9	0	0	1/3	0
1	1/6	1/6	0	1	1/9	1/9	1/9	1	0	0	1/3

- **(b)** Find the probability of the events $A = \{X > 0\}$, $B = \{X \geq Y\}$, and $C = \{X = -Y\}$ for the above joint pmfs.

5.12. A modem transmits a two-dimensional signal (X, Y) given by:

$$X = r\cos(2\pi\Theta/8) \quad \text{and} \quad Y = r\sin(2\pi\Theta/8)$$

where Θ is a discrete uniform random variable in the set $\{0, 1, 2, \ldots, 7\}$.
- **(a)** Show the mapping from S to S_{XY}, the range of the pair (X, Y).
- **(b)** Find the joint pmf of X and Y.
- **(c)** Find the marginal pmf of X and of Y.
- **(d)** Find the probability of the following events: $A = \{X = 0\}$, $B = \{Y \leq r/\sqrt{2}\}$, $C = \{X \geq r/\sqrt{2}, Y \geq r/\sqrt{2}\}$, $D = \{X < -r/\sqrt{2}\}$.

5.13. Let N_1 be the number of Web page requests arriving at a server in a 100-ms period and let N_2 be the number of Web page requests arriving at a server in the next 100-ms period. Assume that in a 1-ms interval either zero or one page request takes place with respective probabilities $1 - p = 0.95$ and $p = 0.05$, and that the requests in different 1-ms intervals are independent of each other.

(a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

(b) Find the joint pmf of X and Y.

(c) Find the marginal pmf for X and for Y.

(d) Find the probability of the events $A = \{X \geq Y\}$, $B = \{X = Y = 0\}$, $C = \{X > 5, Y > 3\}$.

(e) Find the probability of the event $D = \{X + Y = 10\}$.

5.14. Let N_1 be the number of Web page requests arriving at a server in the period $(0, 100)$ ms and let N_2 be the *total* combined number of Web page requests arriving at a server in the period $(0, 200)$ ms. Assume arrivals occur as in Problem 5.13.

(a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

(b) Find the joint pmf of N_1 and N_2.

(c) Find the marginal pmf for N_1 and N_2.

(d) Find the probability of the events $A = \{N_1 < N_2\}$, $B = \{N_2 = 0\}$, $C = \{N_1 > 5, N_2 > 3\}$, $D = \{|N_2 - 2N_1| < 2\}$.

5.15. At even time instants, a robot moves either $+\Delta$ cm or $-\Delta$ cm in the x-direction according to the outcome of a coin flip; at odd time instants, a robot moves similarly according to another coin flip in the y-direction. Assuming that the robot begins at the origin, let X and Y be the coordinates of the location of the robot after $2n$ time instants.

(a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

(b) Find the marginal pmf of the coordinates X and Y.

(c) Find the probability that the robot is within distance $\sqrt{2}$ of the origin after $2n$ time instants.

Section 5.3: The Joint cdf of x and y

5.16. (a) Sketch the joint cdf for the pair (X, Y) in Problem 5.1 and verify that the properties of the joint cdf are satisfied. You may find it helpful to first divide the plane into regions where the cdf is constant.

(b) Find the marginal cdf of X and of Y.

5.17. A point (X, Y) is selected at random inside a triangle defined by $\{(x, y): 0 \leq y \leq x \leq 1\}$. Assume the point is equally likely to fall anywhere in the triangle.

(a) Find the joint cdf of X and Y.

(b) Find the marginal cdf of X and of Y.

(c) Find the probabilities of the following events in terms of the joint cdf: $A = \{X \leq 1/2, Y \leq 3/4\}$; $B = \{1/4 < X \leq 3/4, 1/4 < Y \leq 3/4\}$.

5.18. A dart is equally likely to land at any point (X_1, X_2) inside a circular target of unit radius. Let R and Θ be the radius and angle of the point (X_1, X_2).

(a) Find the joint cdf of R and Θ.

(b) Find the marginal cdf of R and Θ.

(c) Use the joint cdf to find the probability that the point is in the first quadrant of the real plane and that the radius is greater than 0.5.

5.19. Find an expression for the probability of the events in Problem 5.8 parts c, h, and i in terms of the joint cdf of X and Y.

5.20. The pair (X, Y) has joint cdf given by:

$$F_{X,Y}(x, y) = \begin{cases} (1 - 1/x^2)(1 - 1/y^2) & \text{for } x > 1, y > 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Sketch the joint cdf.

(b) Find the marginal cdf of X and of Y.

(c) Find the probability of the following events: $\{X < 3, Y \leq 5\}$, $\{X > 4, Y > 3\}$.

5.21. Is the following a valid cdf? Why?

$$F_{X,Y}(x, y) = \begin{cases} (1 - 1/x^2y^2) & \text{for } x > 1, y > 1 \\ 0 & \text{elsewhere.} \end{cases}$$

5.22. Let $F_X(x)$ and $F_Y(y)$ be valid one-dimensional cdf's. Show that $F_{X,Y}(x, y) = F_X(x)F_Y(y)$ satisfies the properties of a two-dimensional cdf.

5.23. The number of users logged onto a system N and the time T until the next user logs off have joint probability given by:

$$P[N = n, X \leq t] = (1 - \rho)\rho^{n-1}(1 - e^{-n\lambda t}) \qquad \text{for } n = 1, 2, \ldots \quad t > 0.$$

(a) Sketch the above joint probability.

(b) Find the marginal pmf of N.

(c) Find the marginal cdf of X.

(d) Find $P[N \leq 3, X > 3/\lambda]$.

5.24. A factory has n machines of a certain type. Let p be the probability that a machine is working on any given day, and let N be the total number of machines working on a certain day. The time T required to manufacture an item is an exponentially distributed random variable with rate $k\alpha$ if k machines are working. Find and $P[T \leq t]$. Find $P[T \leq t]$ as $t \to \infty$ and explain the result.

Section 5.4: The Joint pdf of Two Continuous Random Variables

5.25. The amplitudes of two signals X and Y have joint pdf:

$$f_{X,Y}(x, y) = e^{-x/2}ye^{-y^2} \qquad \text{for } x > 0, y > 0.$$

(a) Find the joint cdf.

(b) Find $P[X^{1/2} > Y]$.

(c) Find the marginal pdfs.

5.26. Let X and Y have joint pdf:

$$f_{X,Y}(x, y) = k(x + y) \qquad \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1.$$

(a) Find k.

(b) Find the joint cdf of (X, Y).

(c) Find the marginal pdf of X and of Y.

(d) Find $P[X < Y]$, $P[Y < X^2]$, $P[X + Y > 0.5]$.

5.27. Let X and Y have joint pdf:

$$f_{X,Y}(x, y) = kx(1 - x)y \qquad \text{for } 0 < x < 1, 0 < y < 1.$$

(a) Find k.
(b) Find the joint cdf of (X, Y).
(c) Find the marginal pdf of X and of Y.
(d) Find $P[Y < X^{1/2}]$, $P[X < Y]$.

5.28. The random vector (X, Y) is uniformly distributed (i.e., $f(x, y) = k$) in the regions shown in Fig. P5.1 and zero elsewhere.

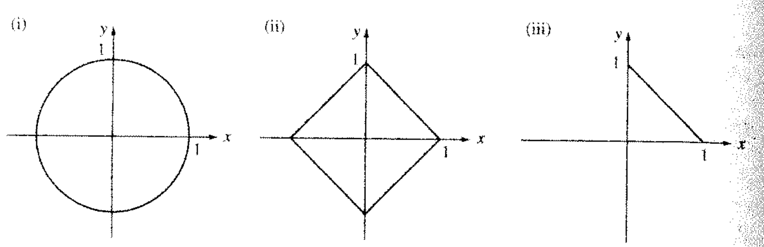


FIGURE P5.1

(a) Find the value of k in each case.
(b) Find the marginal pdf for X and for Y in each case.
(c) Find $P[X > 0, Y > 0]$.

5.29. **(a)** Find the joint cdf for the vector random variable introduced in Example 5.16.
(b) Use the result of part a to find the marginal cdf of X and of Y.

5.30. Let X and Y have the joint pdf:

$$f_{X,Y}(x, y) = ye^{-y(1+x)} \qquad \text{for } x > 0, y > 0.$$

Find the marginal pdf of X and of Y.

5.31. Let X and Y be the pair of random variables in Problem 5.17.
(a) Find the joint pdf of X and Y.
(b) Find the marginal pdf of X and of Y.
(c) Find $P[Y < X^2]$.

5.32. Let R and Θ be the pair of random variables in Problem 5.18.
(a) Find the joint pdf of R and Θ.
(b) Find the marginal pdf of R and of Θ.

5.33. Let (X, Y) be the jointly Gaussian random variables discussed in Example 5.18. Find $P[X^2 + Y^2 > r^2]$ when $\rho = 0$. *Hint:* Use polar coordinates to compute the integral.

5.34. The general form of the joint pdf for two jointly Gaussian random variables is given by Eq. (5.61a). Show that X and Y have marginal pdfs that correspond to Gaussian random variables with means m_1 and m_2 and variances σ_1^2 and σ_2^2 respectively.

5.35. The input X to a communication channel is +1 or −1 with probability p and $1 - p$, respectively. The received signal Y is the sum of X and noise N which has a Gaussian distribution with zero mean and variance $\sigma^2 = 0.25$.
(a) Find the joint probability $P[X = j, Y \le y]$.
(b) Find the marginal pmf of X and the marginal pdf of Y.
(c) Suppose we are given that $Y > 0$. Which is more likely, $X = 1$ or $X = -1$?

5.36. A modem sends a two-dimensional signal $\mathbf{X}$ from the set $\{(1,1), (1,-1), (-1,1), (-1,-1)\}$. The channel adds a noise signal (N_1, N_2), so the received signal is $\mathbf{Y} = \mathbf{X} + \mathbf{N} = (X_1 + N_1, X_2 + N_2)$. Assume that (N_1, N_2) have the jointly Gaussian pdf in Example 5.18 with $\rho = 0$. Let the distance between $\mathbf{X}$ and $\mathbf{Y}$ be $d(\mathbf{X}, \mathbf{Y}) = \{(X_1 - Y_1)^2 + (X_2 - Y_2)^2\}^{1/2}$.
(a) Suppose that $\mathbf{X} = (1, 1)$. Find and sketch region for the event $\{\mathbf{Y}$ is closer to $(1, 1)$ than to the other possible values of $\mathbf{X}\}$. Evaluate the probability of this event.
(b) Suppose that $\mathbf{X} = (1, 1)$. Find and sketch region for the event $\{\mathbf{Y}$ is closer to $(1, -1)$ than to the other possible values of $\mathbf{X}\}$. Evaluate the probability of this event.
(c) Suppose that $\mathbf{X} = (1, 1)$. Find and sketch region for the event $\{d(\mathbf{X}, \mathbf{Y}) > 1\}$. Evaluate the probability of this event. Explain why this probability is an upper bound on the probability that $\mathbf{Y}$ is closer to a signal other than $\mathbf{X} = (1, 1)$.

Section 5.5: Independence of Two Random Variables

5.37. Let X be the number of full pairs and let Y be the remainder of the number of dots observed in a toss of a fair die. Are X and Y independent random variables?

5.38. Let X and Y be the coordinates of the robot in Problem 5.15 after $2n$ time instants. Determine whether X and Y are independent random variables.

5.39. Let X and Y be the coordinates of the two-dimensional modem signal (X, Y) in Problem 5.12.
(a) Determine if X and Y are independent random variables.
(b) Repeat part a if even values of Θ are twice as likely as odd values.

5.40. Determine which of the joint pmfs in Problem 5.11 correspond to independent pairs of random variables.

5.41. Michael takes the 7:30 bus every morning. The arrival time of the bus at the stop is uniformly distributed in the interval [7:27, 7:37]. Michael's arrival time at the stop is also uniformly distributed in the interval [7:25, 7:40]. Assume that Michael's and the bus's arrival times are independent random variables.
(a) What is the probability that Michael arrives more than 5 minutes before the bus?
(b) What is the probability that Michael misses the bus?

5.42. Are R and Θ independent in Problem 5.18?

5.43. Are X and Y independent in Problem 5.20?

5.44. Are the signal amplitudes X and Y independent in Problem 5.25?

5.45. Are X and Y independent in Problem 5.26?

5.46. Are X and Y independent in Problem 5.27?

5.47. Let X and Y be independent random variables. Find an expression for the probability of the following events in terms of $F_X(x)$ and $F_Y(y)$.

(a) $\{a < X \leq b\} \cap \{Y > d\}$.

(b) $\{a < X \leq b\} \cap \{c \leq Y < d\}$.

(c) $\{|X| < a\} \cap \{c \leq Y \leq d\}$.

5.48. Let X and Y be independent random variables that are uniformly distributed in $[-1, 1]$. Find the probability of the following events:

(a) $P[X^2 < 1/2, |Y| < 1/2]$.

(b) $P[4X < 1, Y < 0]$.

(c) $P[XY < 1/2]$.

(d) $P[\max(X, Y) < 1/3]$.

5.49. Let X and Y be random variables that take on values from the set $\{-1, 0, 1\}$.

(a) Find a joint pmf for which X and Y are independent.

(b) Are X^2 and Y^2 independent random variables for the pmf in part a?

(c) Find a joint pmf for which X and Y are not independent, but for which X^2 and Y^2 are independent.

5.50. Let X and Y be the jointly Gaussian random variables introduced in Problem 5.34.

(a) Show that X and Y are independent random variables if and only if $\rho = 0$.

(b) Suppose $\rho = 0$, find $P[XY < 0]$.

5.51. Two fair dice are tossed repeatedly until a pair occurs. Let K be the number of tosses required and let X be the number showing up in the pair. Find the joint pmf of K and X and determine whether K and X are independent.

5.52. The number of devices L produced in a day is geometric distributed with probability of success p. Let N be the number of working devices and let M be the number of defective devices produced in a day.

(a) Are N and M independent random variables?

(b) Find the joint pmf of N and M.

(c) Find the marginal pmfs of N and M. (See hint in Problem 5.87b.)

(d) Are L and M independent random variables?

5.53. Let N_1 be the number of Web page requests arriving at a server in a 100-ms period and let N_2 be the number of Web page requests arriving at a server in the next 100-ms period. Use the result of Problem 5.13 parts a and b to develop a model where N_1 and N_2 are independent Poisson random variables.

5.54. **(a)** Show that Eq. (5.22) implies Eq. (5.21).

(b) Show that Eq. (5.21) implies Eq. (5.22).

5.55. Verify that Eqs. (5.22) and (5.23) can be obtained from each other.

Section 5.6: Joint Moments and Expected Values of a Function of Two Random Variables

5.56. **(a)** Find $E[(X + Y)^2]$.

(b) Find the variance of $X + Y$.

(c) Under what condition is the variance of the sum equal to the sum of the individual variances?

5.57. Find $E[|X - Y|]$ if X and Y are independent exponential random variables with parameters $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively.

5.58. Find $E[X^2e^Y]$ where X and Y are independent random variables, X is a zero-mean, unit-variance Gaussian random variable, and Y is a uniform random variable in the interval $[0, 3]$.

5.59. For the discrete random variables X and Y in Problem 5.1, find the correlation and covariance, and indicate whether the random variables are independent, orthogonal, or uncorrelated.

5.60. For the discrete random variables X and Y in Problem 5.2, find the correlation and covariance, and indicate whether the random variables are independent, orthogonal, or uncorrelated.

5.61. For the three pairs of discrete random variables in Problem 5.11, find the correlation and covariance of X and Y, and indicate whether the random variables are independent, orthogonal, or uncorrelated.

5.62. Let N_1 and N_2 be the number of Web page requests in Problem 5.13. Find the correlation and covariance of N_1 and N_2, and indicate whether the random variables are independent, orthogonal, or uncorrelated.

5.63. Repeat Problem 5.62 for N_1 and N_2, the number of Web page requests in Problem 5.14.

5.64. Let N and T be the number of users logged on and the time till the next logoff in Problem 5.23. Find the correlation and covariance of N and T, and indicate whether the random variables are independent, orthogonal, or uncorrelated.

5.65. Find the correlation and covariance of X and Y in Problem 5.26. Determine whether X and Y are independent, orthogonal, or uncorrelated.

5.66. Repeat Problem 5.65 for X and Y in Problem 5.27.

5.67. For the three pairs of continuous random variables X and Y in Problem 5.28, find the correlation and covariance, and indicate whether the random variables are independent, orthogonal, or uncorrelated.

5.68. Find the correlation coefficient between X and $Y = aX + b$. Does the answer depend on the sign of a?

5.69. Propose a method for estimating the covariance of two random variables.

5.70. **(a)** Complete the calculations for the correlation coefficient in Example 5.28.

(b) Repeat the calculations if X and Y have the pdf:

$$f_{X,Y}(x, y) = e^{-(x+|y|)} \qquad \text{for } x > 0, -x < y < x.$$

5.71. The output of a channel $Y = X + N$, where the input X and the noise N are independent, zero-mean random variables.

(a) Find the correlation coefficient between the input X and the output Y.

(b) Suppose we estimate the input X by a linear function $g(Y) = aY$. Find the value of a that minimizes the mean squared error $E[(X - aY)^2]$.

(c) Express the resulting mean-square error in terms of σ_X/σ_N.

5.72. In Example 5.27 let $X = \cos \Theta/4$ and $Y = \sin \Theta/4$. Are X and Y uncorrelated?

5.73. **(a)** Show that $\text{COV}(X, E[Y|X]) = \text{COV}(X, Y)$.

(b) Show that $E[Y|X = x] = E[Y]$, for all x, implies that X and Y are uncorrelated.

5.74. Use the fact that $E[(tX + Y)^2] \geq 0$ for all t to prove the Cauchy-Schwarz inequality:

$$(E[XY])^2 \leq E[X^2]E[Y^2].$$

Hint: Consider the discriminant of the quadratic equation in t that results from the above inequality.

Section 5.7: Conditional Probability and Conditional Expectation

5.75. **(a)** Find $p_Y(y \mid x)$ and $p_X(x \mid y)$ in Problem 5.1 assuming fair coins are used.
(b) Find $p_Y(y \mid x)$ and $p_X(x \mid y)$ in Problem 5.1 assuming Carlos uses a coin with $p = 3/4$.
(c) What is the effect on $p_X(x \mid y)$ of Carlos using a biased coin?
(d) Find $E[Y \mid X = x]$ and $E[X \mid Y = y]$ in part a; then find $E[X]$ and $E[Y]$.
(e) Find $E[Y \mid X = x]$ and $E[X \mid Y = y]$ in part b; then find $E[X]$ and $E[Y]$.

5.76. **(a)** Find $p_X(x \mid y)$ for the communication channel in Problem 5.3.
(b) For each value of y, find the value of x that maximizes $p_X(x \mid y)$. State any assumptions about p and p_e.
(c) Find the probability of error if a receiver uses the decision rule from part b.

5.77. **(a)** In Problem 5.11(i), which conditional pmf given X provides the most information about Y: $p_Y(y \mid -1)$, $p_Y(y \mid 0)$, or $p_Y(y \mid +1)$? Explain why.
(b) Compare the conditional pmfs in Problems 5.11(ii) and (iii) and explain which of these two cases is "more random."
(c) Find $E[Y \mid X = x]$ and $E[X \mid Y = y]$ in Problems 5.11(i), (ii), (iii); then find $E[X]$ and $E[Y]$.
(d) Find $E[Y^2 \mid X = x]$ and $E[X^2 \mid Y = y]$ in Problems 5.11(i), (ii), (iii); then find VAR$[X]$ and VAR$[Y]$.

5.78. **(a)** Find the conditional pmf of N_1 given N_2 in Problem 5.14.
(b) Find $P[N_1 = k \mid N_2 = 2k]$ for $k = 5, 10, 20$. *Hint:* Use Stirling's fromula.
(c) Find $E[N_1 \mid N_2 = k]$, then find $E[N_1]$.

5.79. In Example 5.30, let Y be the number of defects inside the region R and let Z be the number of defects outside the region.
(a) Find the pmf of Z given Y.
(b) Find the joint pmf of Y and Z.
(c) Are Y and Z independent random variables? Is the result intuitive?

5.80. **(a)** Find $f_Y(y \mid x)$ in Problem 5.26.
(b) Find $P[Y > X \mid x]$.
(c) Find $P[Y > X]$ using part b.
(d) Find $E[Y \mid X = x]$.

5.81. **(a)** Find $f_Y(y \mid x)$ in Problem 5.28(i).
(b) Find $E[Y \mid X = x]$ and $E[Y]$.
(c) Repeat parts a and b of Problem 5.28(ii).
(d) Repeat parts a and b of Problem 5.28(iii).

5.82. **(a)** Find $f_Y(y \mid x)$ in Example 5.27.
(b) Find $E[Y \mid X = x]$.
(c) Find $E[Y]$.
(d) Find $E[XY \mid X = x]$.
(e) Find $E[XY]$.

5.83. Find $f_Y(y \mid x)$ and $f_X(x \mid y)$ for the jointly Gaussian pdf in Problem 5.34.

5.84. **(a)** Find $f_X(t \mid N = n)$ in Problem 5.23.
(b) Find $E[X^t \mid N = n]$.
(c) Find the value of n that maximizes $P[N = n \mid t < X < t + dt]$.

5.85. **(a)** Find $p_Y(y \mid x)$ and $p_X(x \mid y)$ in Problem 5.12.
(b) Find $E[Y \mid X = x]$.
(c) Find $E[XY \mid X = x]$ and $E[XY]$.

5.86. A customer enters a store and is equally likely to be served by one of three clerks. The time taken by clerk 1 is a constant random variable with mean two minutes; the time for clerk 2 is exponentially distributed with mean two minutes; and the time for clerk 3 is Pareto distributed with mean two minutes and $\alpha = 2.5$.
(a) Find the pdf of T, the time taken to service a customer.
(b) Find $E[T]$ and VAR$[T]$.

5.87. A message requires N time units to be transmitted, where N is a geometric random variable with pmf $p_i = (1 - a)a^{i-1}$, $i = 1, 2, \ldots$. A single new message arrives during a time unit with probability p, and no messages arrive with probability $1 - p$. Let K be the number of new messages that arrive during the transmission of a single message.
(a) Find $E[K]$ and VAR$[K]$ using conditional expectation.
(b) Find the pmf of K. *Hint:* $(1 - \beta)^{-(k+1)} = \sum_{n=k}^{\infty} \binom{n}{k} \beta^{n-k}$.
(c) Find the conditional pmf of N given $K = k$.
(d) Find the value of n that maximizes $P[N = n \mid X = k]$.

5.88. The number of defects in a VLSI chip is a Poisson random variable with rate r. However, r is itself a gamma random variable with parameters α and λ.
(a) Use conditional expectation to find $E[N]$ and VAR$[N]$.
(b) Find the pmf for N, the number of defects.

5.89. **(a)** In Problem 5.35, find the conditional pmf of the input X of the communication channel given that the output is in the interval $y < Y \le y + dy$.
(b) Find the value of X that is more probable given $y < Y \le y + dy$.
(c) Find an expression for the probability of error if we use the result of part b to decide what the input to the channel was.

Section 5.8: Functions of Two Random Variables

5.90. Two toys are started at the same time each with a different battery. The first battery has a lifetime that is exponentially distributed with mean 100 minutes; the second battery has a Rayleigh-distributed lifetime with mean 100 minutes.
(a) Find the cdf to the time T until the battery in a toy first runs out.
(b) Suppose that both toys are still operating after 100 minutes. Find the cdf of the time T_2 that subsequently elapses until the battery in a toy first runs out.
(c) In part b, find the cdf of the total time that elapses until a battery first fails.

5.91. **(a)** Find the cdf of the time that elapses until both batteries run out in Problem 5.90a.
(b) Find the cdf of the remaining time until both batteries run out in Problem 5.90b.

5.92. Let K and N be independent random variables with nonnegative integer values.
(a) Find an expression for the pmf of $M = K + N$.
(b) Find the pmf of M if K and N are binomial random variables with parameters (k, p) and (n, p).
(c) Find the pmf of M if K and N are Poisson random variables with parameters α_1 and α_2, respectively.

5.93. The number X of goals the Bulldogs score against the Flames has a geometric distribution with mean 2; the number of goals Y that the Flames score against the Bulldogs is also geometrically distributed but with mean 4.

(a) Find the pmf of the $Z = X - Y$. Assume X and Y are independent.

(b) What is the probability that the Bulldogs beat the Flames? Tie the Flames?

(c) Find $E[Z]$.

5.94. Passengers arrive at an airport taxi stand every minute according to a Bernoulli random variable. A taxi will not leave until it has two passengers.

(a) Find the pmf until the time T when the taxi has two passengers.

(b) Find the pmf for the time that the first customer waits.

5.95. Let X and Y be independent random variables that are uniformly distributed in the interval $[0, 1]$. Find the pdf of $Z = XY$.

5.96. Let X_1, X_2, and X_3 be independent and uniformly distributed in $[-1, 1]$.

(a) Find the cdf and pdf of $Y = X_1 + X_2$.

(b) Find the cdf of $Z = Y + X_3$.

5.97. Let X and Y be independent random variables with gamma distributions and parameters (α_1, λ) and (α_2, λ), respectively. Show that $Z = X + Y$ is gamma-distributed with parameters $(\alpha_1 + \alpha_2, \lambda)$. *Hint:* See Eq. (4.59).

5.98. Signals X and Y are independent. X is exponentially distributed with mean 1 and Y is exponentially distributed with mean 1.

(a) Find the cdf of $Z = |X - Y|$.

(b) Use the result of part a to find $E[Z]$.

5.99. The random variables X and Y have the joint pdf

$$f_{X,Y}(x, y) = e^{-(x+y)} \qquad \text{for } 0 < y < x < 1.$$

Find the pdf of $Z = X + Y$.

5.100. Let X and Y be independent Rayleigh random variables with parameters $\alpha = \beta = 1$. Find the pdf of $Z = X/Y$.

5.101. Let X and Y be independent Gaussian random variables that are zero mean and unit variance. Show that $Z = X/Y$ is a Cauchy random variable.

5.102. Find the joint cdf of $W = \min(X, Y)$ and $Z = \max(X, Y)$ if X and Y are independent and X is uniformly distributed in $[0, 1]$ and Y is uniformly distributed in $[0, 1]$.

5.103. Find the joint cdf of $W = \min(X, Y)$ and $Z = \max(X, Y)$ if X and Y are independent exponential random variables with the same mean.

5.104. Find the joint cdf of $W = \min(X, Y)$ and $Z = \max(X, Y)$ if X and Y are the independent Pareto random variables with the same distribution.

5.105. Let $W = X + Y$ and $Z = X - Y$.

(a) Find an expression for the joint pdf of W and Z.

(b) Find $f_{W,Z}(z, w)$ if X and Y are independent exponential random variables with parameter $\lambda = 1$.

(c) Find $f_{W,Z}(z, w)$ if X and Y are independent Pareto random variables with the same distribution.

5.106. The pair (X, Y) is uniformly distributed in a ring centered about the origin and inner and outer radii $r_1 < r_2$. Let R and Θ be the radius and angle corresponding to (X, Y). Find the joint pdf of R and Θ.

5.107. Let X and Y be independent, zero-mean, unit-variance Gaussian random variables. Let $V = aX + bY$ and $W = cX + eY$.

(a) Find the joint pdf of V and W, assuming the transformation matrix A is invertible.

(b) Suppose A is not invertible. What is the joint pdf of V and W?

5.108. Let X and Y be independent Gaussian random variables that are zero mean and unit variance. Let $W = X^2 + Y^2$ and let $\Theta = \tan^{-1}(Y/X)$. Find the joint pdf of W and Θ.

5.109. Let X and Y be the random variables introduced in Example 5.4. Let $R = (X^2 + Y^2)^{1/2}$ and let $\Theta = \tan^{-1}(Y/X)$.

(a) Find the joint pdf of R and Θ.

(b) What is the joint pdf of X and Y?

Section 5.9: Pairs of Jointly Gaussian Variables

5.110. Let X and Y be jointly Gaussian random variables with pdf

$$f_{X,Y}(x, y) = \frac{\exp\{-2x^2 - y^2/2\}}{2\pi c} \qquad \text{for all } x, y.$$

Find VAR[X], VAR[Y], and COV(X, Y).

5.111. Let X and Y be jointly Gaussian random variables with pdf

$$f_{X,Y}(x, y) = \frac{\exp\left\{\frac{-1}{2}[x^2 + 4y^2 - 3xy + 3y - 2x + 1]\right\}}{2\pi} \qquad \text{for all } x, y.$$

Find $E[X]$, $E[Y]$, VAR[X], VAR[Y], and COV(X, Y).

5.112. Let X and Y be jointly Gaussian random variables with $E[Y] = 0$, $\sigma_1 = 1$, $\sigma_2 = 2$, and $E[X \mid Y] = Y/4 + 1$. Find the joint pdf of X and Y.

5.113. Let X and Y be zero-mean, independent Gaussian random variables with $\sigma^2 = 1$.

(a) Find the value of r for which the probability that (X, Y) falls inside a circle of radius r is 1/2.

(b) Find the conditional pdf of (X, Y) given that (X, Y) is not inside a ring with inner radius r_1 and outer radius r_2.

5.114. Use a plotting program (as provided by Octave or MATLAB) to show the pdf for jointly Gaussian zero-mean random variables with the following parameters:

(a) $\sigma_1 = 1, \sigma_2 = 1, \rho = 0$.

(b) $\sigma_1 = 1, \sigma_2 = 1, \rho = 0.8$.

(c) $\sigma_1 = 1, \sigma_2 = 1, \rho = -0.8$.

(d) $\sigma_1 = 1, \sigma_2 = 2, \rho = 0$.

(e) $\sigma_1 = 1, \sigma_2 = 2, \rho = 0.8$.

(f) $\sigma_1 = 1, \sigma_2 = 10, \rho = 0.8$.

5.115. Let X and Y be zero-mean, jointly Gaussian random variables with $\sigma_1 = 1, \sigma_2 = 2$, and correlation coefficient ρ.

(a) Plot the principal axes of the constant-pdf ellipse of (X, Y).

(b) Plot the conditional expectation of Y given $X = x$.

(c) Are the plots in parts a and b the same or different? Why?

5.116. Let X and Y be zero-mean, unit-variance jointly Gaussian random variables for which $\rho = 1$. Sketch the joint cdf of X and Y. Does a joint pdf exist?

5.117. Let $h(x, y)$ be a joint Gaussian pdf for zero-mean, unit-variance Gaussian random variables with correlation coefficient ρ_1. Let $g(x, y)$ be a joint Gaussian pdf for zero-mean, unit-variance Gaussian random variables with correlation coefficient $\rho_2 \neq \rho_1$. Suppose the random variables X and Y have joint pdf

$$f_{X,Y}(x, y) = \{h(x, y) + g(x, y)\}/2.$$

(a) Find the marginal pdf for X and for Y.

(b) Explain why X and Y are not jointly Gaussian random variables.

5.118. Use conditional expectation to show that for X and Y zero-mean, jointly Gaussian random variables, $E[X^2Y^2] = E[X^2]E[Y^2] + 2E[XY]^2$.

5.119. Let $\mathbf{X} = (X, Y)$ be the zero-mean jointly Gaussian random variables in Problem 5.110. Find a transformation A such that $\mathbf{Z} = A\mathbf{X}$ has components that are zero-mean, unit-variance Gaussian random variables.

5.120. In Example 5.47, suppose we estimate the value of the signal X from the noisy observation Y by:

$$\hat{X} = \frac{1}{1 + \sigma_N^2/\sigma_X^2} Y.$$

(a) Evaluate the mean square estimation error: $E[(X - \hat{X})^2]$.

(b) How does the estimation error in part a vary with signal-to-noise ratio σ_X/σ_N?

Section 5.10: Generating Independent Gaussian Random Variables

5.121. Find the inverse of the cdf of the Rayleigh random variable to derive the transformation method for generating Rayleigh random variables. Show that this method leads to the same algorithm that was presented in Section 5.10.

5.122. Reproduce the results presented in Example 5.49.

5.123. Consider the two-dimensional modem in Problem 5.36.

(a) Generate 10,000 discrete random variables uniformly distributed in the set $\{1, 2, 3, 4\}$. Assign each outcome in this set to one of the signals $\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$. The sequence of discrete random variables then produces a sequence of 10,000 signal points $\mathbf{X}$.

(b) Generate 10,000 noise pairs $\mathbf{N}$ of independent zero-mean, unit-variance jointly Gaussian random variables.

(c) Form the sequence of 10,000 received signals $\mathbf{Y} = (Y_1, Y_2) = \mathbf{X} + \mathbf{N}$.

(d) Plot the scattergram of received signal vectors. Is the plot what you expected?

(e) Estimate the transmitted signal by the quadrant that $\mathbf{Y}$ falls in: $\hat{X} = (\text{sgn}(Y_1), \text{sgn}(Y_2))$.

(f) Compare the estimates with the actually transmitted signals to estimate the probability of error.

5.124. Generate a sequence of 1000 pairs of independent zero-mean Gaussian random variables, where X has variance 2 and N has variance 1. Let $Y = X + N$ be the noisy signal from Example 5.47.

(a) Estimate X using the estimator in Problem 5.120, and calculate the sequence of estimation errors.

(b) What is the pdf of the estimation error?

(c) Compare the mean, variance, and relative frequencies of the estimation error with the result from part b.

5.125. Let $X_1, X_2, \ldots, X_{1000}$ be a sequence of zero-mean, unit-variance independent Gaussian random variables. Suppose that the sequence is "smoothed" as follows:

$$Y_n = (X_n + X_{N-1})/2 \text{ where } X_0 = 0.$$

(a) Find the pdf of (Y_n, Y_{n+1}).

(b) Generate the sequence of X_n and the corresponding sequence Y_n. Plot the scattergram of (Y_n, Y_{n+1}). Does it agree with the result from part a?

(c) Repeat parts a and b for $Z_n = (X_n - X_{N-1})/2$.

5.126. Let X and Y be independent, zero-mean, unit-variance Gaussian random variables. Find the linear transformation to generate jointly Gaussian random variables with means m_1, m_2, variances σ_1^2, σ_2^2, and correlation coefficient ρ. *Hint:* Use the conditional pdf in Eq. (5.64).

5.127. (a) Use the method developed in Problem 5.126 to generate 1000 pairs of jointly Gaussian random variables with $m_1 = 1, m_2 = -1$, variances $\sigma_1^2 = 1, \sigma_2^2 = 2$, and correlation coefficient $\rho = -1/2$.

(b) Plot a two-dimensional scattergram of the 1000 pairs and compare to equal-pdf contour lines for the theoretical pdf.

5.128. Let H and W be the height and weight of adult males. Studies have shown that H (in cm) and $V = \ln W$ (W in kg) are jointly Gaussian with parameters $m_H = 174$ cm, $m_V = 4.4$, $\sigma_H^2 = 42.36, \sigma_V^2 = 0.021$, and $\text{COV}(H, V) = 0.458$.

(a) Use the method in part a to generate 1000 pairs (H, V). Plot a scattergram to check the joint pdf.

(b) Convert the (H, V) pairs into (H, W) pairs.

(c) Calculate the body mass index for each outcome, and estimate the proportion of the population that is underweight, normal, overweight, or obese. (See Problem 5.6.)

Problems Requiring Cumulative Knowledge

5.129. The random variables X and Y have joint pdf:

$$f_{X,Y}(x, y) = c \sin(x + y) \qquad 0 \le x \le \pi/2, 0 \le y \le \pi/2.$$

(a) Find the value of the constant c.

(b) Find the joint cdf of X and Y.

(c) Find the marginal pdf's of X and of Y.

(d) Find the mean, variance, and covariance of X and Y.

5.130. An inspector selects an item for inspection according to the outcome of a coin flip: The item is inspected if the outcome is heads. Suppose that the time between item arrivals is an exponential random variable with mean one. Assume the time to inspect an item is a constant value t.

(a) Find the pmf for the number of item arrivals between consecutive inspections.

(b) Find the pdf for the time X between item inspections. *Hint:* Use conditional expectation.

(c) Find the value of p, so that with a probability of 90% an inspection is completed before the next item is selected for inspection.

5.131. The lifetime X of a device is an exponential random variable with mean $= 1/R$. Suppose that due to irregularities in the production process, the parameter R is random and has a gamma distribution.

(a) Find the joint pdf of X and R.

(b) Find the pdf of X.

(c) Find the mean and variance of X.

5.132. Let X and Y be samples of a random signal at two time instants. Suppose that X and Y are independent zero-mean Gaussian random variables with the same variance. When signal "0" is present the variance is σ_0^2, and when signal "1" is present the variance is $\sigma_1^2 > \sigma_0^2$. Suppose signals 0 and 1 occur with probabilities p and $1 - p$, respectively. Let $R^2 = X^2 + Y^2$ be the total energy of the two observations.

(a) Find the pdf of R^2 when signal 0 is present; when signal 1 is present. Find the pdf of R^2.

(b) Suppose we use the following "signal detection" rule: If $R^2 > T$, then we decide signal 1 is present; otherwise, we decide signal 0 is present. Find an expression for the probability of error in terms of T.

(c) Find the value of T that minimizes the probability of error.

5.133. Let $U_0, U_1, \ldots$ be a sequence of independent zero-mean, unit-variance Gaussian random variables. A "low-pass filter" takes the sequence U_i and produces the output sequence $X_n = (U_n + U_{n-1})/2$, and a "high-pass filter" produces the output sequence $Y_n = (U_n - U_{n-1})/2$.

(a) Find the joint pdf of X_n and X_{n-1}; of X_n and X_{n+m}, $m > 1$.

(b) Repeat part a for Y_n.

(c) Find the joint pdf of X_n and Y_m.

CHAPTER 6

Vector Random Variables

In the previous chapter we presented methods for dealing with two random variables. In this chapter we extend these methods to the case of n random variables in the following ways:

- By representing n random variables as a vector, we obtain a compact notation for the joint pmf, cdf, and pdf as well as marginal and conditional distributions.
- We present a general method for finding the pdf of transformations of vector random variables.
- Summary information of the distribution of a vector random variable is provided by an expected value vector and a covariance matrix.
- We use linear transformations and characteristic functions to find alternative representations of random vectors and their probabilities.
- We develop optimum estimators for estimating the value of a random variable based on observations of other random variables.
- We show how jointly Gaussian random vectors have a compact and easy-to-work-with pdf and characteristic function.

6.1 VECTOR RANDOM VARIABLES

The notion of a random variable is easily generalized to the case where several quantities are of interest. A **vector random variable X** is a function that assigns a vector of real numbers to each outcome ζ in S, the sample space of the random experiment. We use uppercase boldface notation for vector random variables. By convention $\mathbf{X}$ is a column vector (n rows by 1 column), so the vector random variable with components $X_1, X_2, \ldots, X_n$ corresponds to

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = [X_1, X_2, \ldots, X_n]^{\mathrm{T}}.$$

PROBLEMS

Section 6.1: Vector Random Variables

6.1. The point $\mathbf{X} = (X, Y, Z)$ is uniformly distributed inside a sphere of radius 1 about the origin. Find the probability of the following events:
(a) $\mathbf{X}$ is inside a sphere of radius r, $r > 0$.
(b) $\mathbf{X}$ is inside a cube of length $2/\sqrt{3}$ centered about the origin.
(c) All components of $\mathbf{X}$ are positive.
(d) Z is negative.

6.2. A random sinusoid signal is given by $X(t) = A \sin(t)$ where A is a uniform random variable in the interval $[0, 1]$. Let $\mathbf{X} = (X(t_1), X(t_2), X(t_3))$ be samples of the signal taken at times t_1, t_2, and t_3.
(a) Find the joint cdf of $\mathbf{X}$ in terms of the cdf of A if $t_1 = 0, t_2 = \pi/2$, and $t_3 = \pi$. Are $X(t_1), X(t_2), X(t_3)$ independent random variables?
(b) Find the joint cdf of $\mathbf{X}$ for $t_1, t_2 = t_1 + \pi/2$, and $t_3 = t_1 + \pi$. Let $t_1 = \pi/6$.

6.3. Let the random variables X, Y, and Z be independent random variables. Find the following probabilities in terms of $F_X(x)$, $F_Y(y)$, and $F_Z(z)$.
(a) $P[|X| < 5, Y < 4, Z^3 > 8]$.
(b) $P[X = 5, Y < 0, Z > 1]$.
(c) $P[\min(X, Y, Z) < 2]$.
(d) $P[\max(X, Y, Z) > 6]$.

6.4. A radio transmitter sends a signal $s > 0$ to a receiver using three paths. The signals that arrive at the receiver along each path are:

$$X_1 = s + N_1, X_2 = s + N_2, \text{and } X_3 = s + N_3,$$

where N_1, N_2, and N_3 are independent Gaussian random variables with zero mean and unit variance.
(a) Find the joint pdf of $\mathbf{X} = (X_1, X_2, X_3)$. Are X_1, X_2, and X_3 independent random variables?
(b) Find the probability that the minimum of all three signals is positive.
(c) Find the probability that a majority of the signals are positive.

6.5. An urn contains one black ball and two white balls. Three balls are drawn from the urn. Let $I_k = 1$ if the outcome of the kth draw is the black ball and let $I_k = 0$ otherwise. Define the following three random variables:

$$X = I_1 + I_2 + I_3,$$
$$Y = \min\{I_1, I_2, I_3\},$$
$$Z = \max\{I_1, I_2, I_3\}.$$

(a) Specify the range of values of the triplet (X, Y, Z) if each ball is put back into the urn after each draw; find the joint pmf for (X, Y, Z).
(b) In part a, are X, Y, and Z independent? Are X and Y independent?
(c) Repeat part a if each ball is not put back into the urn after each draw.

6.6. Consider the packet switch in Example 6.1. Suppose that each input has one packet with probability p and no packets with probability $1 - p$. Packets are equally likely to be destined to each of the outputs. Let X_1, X_2 and X_3 be the number of packet arrivals destined for output 1, 2, and 3, respectively.
(a) Find the joint pmf of X_1, X_2, and X_3 Hint: Imagine that every input has a *packet* go to a fictional port 4 with probality $1 - p$.
(b) Find the joint pmf of X_1 and X_2.
(c) Find the pmf of X_2.
(d) Are X_1, X_2, and X_3 independent random variables?
(e) Suppose that each output will accept at most one packet and discard all additional packets destined to it. Find the average number of packets discarded by the module in each T-second period.

6.7. Let X, Y, Z have joint pdf

$$f_{X,Y,Z}(x, y, z) = k(x + y + z) \quad \text{for} \quad 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1.$$

(a) Find k.
(b) Find $f_X(x \mid y, z)$ and $f_Z(z \mid x, y)$.
(c) Find $f_X(x), f_Y(y)$, and $f_Z(z)$.

6.8. A point $\mathbf{X} = (X, Y, Z)$ is selected at random inside the unit sphere.
(a) Find the marginal joint pdf of Y and Z.
(b) Find the marginal pdf of Y.
(c) Find the conditional joint pdf of X and Y given Z.
(d) Are X, Y, and Z independent random variables?
(e) Find the joint pdf of $\mathbf{X}$ given that the distance from $\mathbf{X}$ to the origin is greater than 1/2 and all the components of $\mathbf{X}$ are positive.

6.9. Show that $p_{X_1,X_2,X_3}(x_1, x_2, x_3) = p_{X_3}(x_3 \mid x_1, x_2)p_{X_2}(x_2 \mid x_1)p_{X_1}(x_1)$.

6.10. Let $X_1, X_2, \ldots, X_n$ be binary random variables taking on values 0 or 1 to denote whether a speaker is silent (0) or active (1). A silent speaker remains idle at the next time slot with probability 3/4, and an active speaker remains active with probability 1/2. Find the joint pmf for X_1, X_2, X_3, and the marginal pmf of X_3. Assume that the speaker begins in the silent state.

6.11. Show that $f_{X,Y,Z}(x, y, z) = f_Z(z \mid x, y)f_Y(y \mid x)f_X(x)$.

6.12. Let U_1, U_2, and U_3 be independent random variables and let $X = U_1, Y = U_1 + U_2$, and $Z = U_1 + U_2 + U_3$.
(a) Use the result in Problem 6.11 to find the joint pdf of X, Y, and Z.
(b) Let the U_i be independent uniform random variables in the interval $[0, 1]$. Find the marginal joint pdf of Y and Z. Find the marginal pdf of Z.
(c) Let the U_i be independent zero-mean, unit-variance Gaussian random variables. Find the marginal pdf of Y and Z. Find the marginal pdf of Z.

6.13. Let X_1, X_2, and X_3 be the multiplicative sequence in Example 6.7.
(a) Find, plot, and compare the marginal pdfs of X_1, X_2, and X_3.
(b) Find the conditional pdf of X_3 given $X_1 = x$.
(c) Find the conditional pdf of X_1 given $X_3 = z$.

6.14. Requests at an online music site are categorized as follows: Requests for most popular title with $p_1 = 1/2$; second most popular title with $p_2 = 1/4$; third most popular title with $p_3 = 1/8$; and other $p_4 = 1 - p_1 - p_2 - p_3 = 1/8$. Suppose there are a total number of requests in T seconds. Let X_k be the number of times category k occurs.

i) Find the joint pmf of (X_1, X_2, X_3).

i) Find the marginal pmf of (X_1, X_2). *Hint:* Use the binomial theorem.

) Find the marginal pmf of X_1.

l) Find the conditional joint pmf of (X_2, X_3) given $X_1 = m$, where $0 \le m \le n$.

he number N of requests at the online music site in Problem 6.14 is a Poisson random ıriable with mean α customers per second. Let X_k be the number of type k requests in seconds. Find the joint pmf of (X_1, X_2, X_3, X_4).

. random experiment has four possible outcomes. Suppose that the experiment is re- eated n independent times and let X_k be the number of times outcome k occurs. The ιint pmf of (X_1, X_2, X_3) is given by

$$p(k_1, k_2, k_3) = \frac{n!\,3!}{(n+3)!} = \binom{n+3}{3}^{-1} \quad \text{for} \quad 0 \le k_i \text{ and } k_1 + k_2 + k_3 \le n.$$

ı) Find the marginal pmf of (X_1, X_2).

ı) Find the marginal pmf of X_1.

:) Find the conditional joint pmf of (X_2, X_3) given $X_1 = m$, where $0 \le m \le n$.

he number of requests of types 1, 2, and 3, respectively, arriving at a service station in seconds are independent Poisson random variables with means $\lambda_1 t$, $\lambda_2 t$, and $\lambda_3 t$. Let $_1, N_2$, and N_3 be the number of requests that arrive during an exponentially distributed me T with mean αt.

ı) Find the joint pmf of N_1, N_2, and N_3.

ı) Find the marginal pmf of N_1.

:) Find the conditional pmf of N_1 and N_2, given N_3.

ı 6.2: Functions of Several Random Variables

' devices are installed at the same time. Let Y be the time until the first device fails.

ı) Find the pdf of Y if the lifetimes of the devices are independent and have the same Pareto distribution.

ı) Repeat part a if the device lifetimes have a Weibull distribution.

ı Problem 6.18 let $I_k(t)$ be the indicator function for the event "kth device is still work- ıg at time t." Let $N(t)$ be the number of devices still working at time t: $N(t) = I_1(t) +$ $_2(t) + \cdots + I_N(t)$. Find the pmf of $N(t)$ as well as its mean and variance.

. diversity receiver receives N independent versions of a signal. Each signal version has ı amplitude X_k that is Rayleigh distributed. The receiver selects that signal with the ırgest amplitude X_k^2. A signal is not useful if the squared amplitude falls below a thresh- ld γ. Find the probability that all N signals are below the threshold.

Iaykin) A receiver in a multiuser communication system accepts K binary signals from independent transmitters: $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_K)$, where Y_k is the received signal from ıe kth transmitter. In an ideal system the received vector is given by:

$$\mathbf{Y} = \mathbf{A}\mathbf{b} + \mathbf{N}$$

here $\mathbf{A} = [\alpha_k]$ is a diagonal matrix of positive channel gains, $\mathbf{b} = (b_1, b_2, \ldots, b_K)$ is ıe vector of bits from each of the transmitters where $b_k = \pm 1$, and $\mathbf{N}$ is a vector of **K**

independent zero-mean, unit-variance Gaussian random variables.

(a) Find the joint pdf of $\mathbf{Y}$.

(b) Suppose $\mathbf{b} = (1, 1, \ldots, 1)$, find the probability that all components of $\mathbf{Y}$ are positive.

6.22. **(a)** Find the joint pdf of $U = X_1$, $V = X_1 + X_2$, and $W = X_1 + X_2 + X_3$.

(b) Evaluate the joint pdf of (U, V, W) if the X_i are independent zero-mean, unit variance Gaussian random variables.

(c) Find the marginal pdf of V and of W.

6.23. **(a)** Find the joint pdf of the sample mean and variance of two random variables:

$$M = \frac{X_1 + X_2}{2} \qquad V = \frac{(X_1 - M)^2 + (X_2 - M)^2}{2}$$

in terms of the joint pdf of X_1 and X_2.

(b) Evaluate the joint pdf if X_1 and X_2 are independent Gaussian random variables with the same mean 1 and variance 1.

(c) Evaluate the joint pdf if X_1 and X_2 are independent exponential random variables with the same parameter 1.

6.24. **(a)** Use the auxiliary variable method to find the pdf of

$$Z = \frac{X}{X + Y}.$$

(b) Find the pdf of Z if X and Y are independent exponential random variables with the parameter 1.

(c) Repeat part b if X and Y are independent Pareto random variables with parameters $k = 2$ and $x_m = 1$.

6.25. Repeat Problem 6.24 parts a and b for $Z = X/Y$.

6.26. Let X and Y be zero-mean, unit-variance Gaussian random variables with correlation coefficient 1/2. Find the joint pdf of $U = X^2$ and $V = Y^4$.

6.27. Use auxilliary variables to find the pdf of $Z = X_1X_2X_3$ where the X_i are independent random variables that are uniformly distributed in $[0, 1]$.

6.28. Let X, Y, and Z be independent zero-mean, unit-variance Gaussian random variables.

(a) Find the pdf of $R = (X^2 + Y^2 + Z^2)^{1/2}$.

(b) Find the pdf of $R^2 = X^2 + Y^2 + Z^2$.

6.29. Let X_1, X_2, X_3, X_4 be processed as follows:

$$Y_1 = X_1, Y_2 = X_1 + X_2, Y_3 = X_2 + X_3, Y_4 = X_3 + X_4.$$

(a) Find an expression for the joint pdf of $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)$ in terms of the joint pdf of $\mathbf{X} = (X_1, X_2, X_3, X_4)$.

(b) Find the joint pdf of $\mathbf{Y}$ if X_1, X_2, X_3, X_4 are independent zero-mean, unit-variance Gaussian random variables.

Section 6.3: Expected Values of Vector Random Variables

6.30. Find $E[M]$, $E[V]$, and $E[MV]$ in Problem 6.23c.

6.31. Compute $E[Z]$ in Problem 6.27 in two ways:

(a) by integrating over $f_Z(z)$;

(b) by integrating over the joint pdf of (X_1, X_2, X_3).

6.32. Find the mean vector and covariance matrix for three multipath signals $\mathbf{X} = (X_1, X_2, X_3)$ in Problem 6.4.

6.33. Find the mean vector and covariance matrix for the samples of the sinusoidal signals $\mathbf{X} = (X(t_1), X(t_2), X(t_3))$ in Problem 6.2.

6.34. **(a)** Find the mean vector and covariance matrix for (X, Y, Z) in Problem 6.5a.

(b) Repeat part a for Problem 6.5c.

6.35. Find the mean vector and covariance matrix for (X, Y, Z) in Problem 6.7.

6.36. Find the mean vector and covariance matrix for the point (X, Y, Z) inside the unit sphere in Problem 6.8.

6.37. **(a)** Use the results of Problem 6.6c to find the mean vector for the packet arrivals X_1, X_2, and X_3 in Example 6.5.

(b) Use the results of Problem 6.6b to find the covariance matrix.

(c) Explain why X_1, X_2, and X_3 are correlated.

6.38. Find the mean vector and covariance matrix for the joint number of packet arrivals in a random time N_1, N_2, and N_3 in Problem 6.17. *Hint:* Use conditional expectation.

6.39. **(a)** Find the mean vector and covariance matrix (U, V, W) in terms of (X_1, X_2, X_3) in Problem 6.22b.

(b) Find the cross-covariance matrix between (U, V, W) and (X_1, X_2, X_3).

6.40. **(a)** Find the mean vector and covariance matrix of $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)$ in terms of those of $\mathbf{X} = (X_1, X_2, X_3, X_4)$ in Problem 6.29.

(b) Find the cross-covariance matrix between $\mathbf{Y}$ and $\mathbf{X}$.

(c) Evaluate the mean vector, covariance, and cross-covariance matrices if X_1, X_2, X_3, X_4 are independent random variables.

(d) Generalize the results in part c to $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_{n-1}, Y_n)$.

6.41. Let $\mathbf{X} = (X_1, X_2, X_3, X_4)$ consist of equal mean, independent, unit-variance random variables. Find the mean vector, covariance, and cross-covariance matrices of $\mathbf{Y} = \mathbf{AX}$:

(a) $$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 \\ 0 & 1 & 1/2 & 1/4 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

6.42. Let $W = aX + bY + c$, where X and Y are random variables.

(a) Find the characteristic function of W in terms of the joint characteristic function of X and Y.

(b) Find the characteristic function of W if X and Y are the random variables discussed in Example 6.19. Find the pdf of W.

6.43. **(a)** Find the joint characteristic function of the jointly Gaussian random variables X and Y introduced in Example 5.45. *Hint:* Consider X and Y as a transformation of the independent Gaussian random variables V and W.

(b) Find $E[X^2Y]$.

(c) Find the joint characteristic function of $X' = X + a$ and $Y' = Y + b$.

6.44. Let $X = aU + bV$ and $y = cU + dV$, where $|ad - bc| \neq 0$.

(a) Find the joint characteristic function of X and Y in terms of the joint characteristic function of U and V.

(b) Find an expression for $E[XY]$ in terms of joint moments of U and V.

6.45. Let X and Y be nonnegative, integer-valued random variables. The joint probability generating function is defined by

$$G_{X,Y}(z_1, z_2) = E[z_1^X z_2^Y] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} z_1^j z_2^k P[X = j, Y = k].$$

(a) Find the joint pgf for two independent Poisson random variables with parameters α_1 and α_2.

(b) Find the joint pgf for two independent binomial random variables with parameters (n, p) and (m, p).

6.46. Suppose that X and Y have joint pgf

$$G_{X,Y}(z_1, z_2) = e^{\alpha_1(z_1-1)+\alpha_2(z_2-1)+\beta(z_1z_2-1)}.$$

(a) Use the marginal pgf's to show that X and Y are Poisson random variables.

(b) Find the pgf of $Z = X + Y$. Is Z a Poisson random variable?

6.47. Let X and Y be trinomial random variables with joint pmf

$$P[X = j, Y = k] = \frac{n!\, p_1^j p_2^k (1 - p_1 - p_2)^{n-j-k}}{j!\, k!(n - j - k)!} \quad \text{for} \quad 0 \le j, k \text{ and } j + k \le n.$$

(a) Find the joint pgf of X and Y.

(b) Find the correlation and covariance of X and Y.

6.48. Find the mean vector and covariance matrix for (X, Y) in Problem 6.46.

6.49. Find the mean vector and covariance matrix for (X, Y) in Problem 6.47.

6.50. Let $\mathbf{X} = (X_1, X_2)$ have covariance matrix:

$$\mathbf{K_X} = \begin{bmatrix} 1 & 1/4 \\ 1/4 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{K_X}$.

(b) Find the orthogonal matrix $\mathbf{P}$ that diagonalizes $\mathbf{K_X}$. Verify that $\mathbf{P}$ is orthogonal and that $\mathbf{P}^T\mathbf{K_X}\mathbf{P} = \Lambda$.

(c) Express $\mathbf{X}$ in terms of the eigenvectors of $\mathbf{K_X}$ using the Karhunen-Loeve expansion.

6.51. Repeat Problem 6.50 for $\mathbf{X} = (X_1, X_2, X_3)$ with covariance matrix:

$$\mathbf{K_X} = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}.$$

square matrix $\mathbf{A}$ is said to be nonnegative definite if for any vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)^T$: $\boldsymbol{a}^T\mathbf{A}\,\boldsymbol{a} \geq 0$. Show that the covariance matrix is nonnegative definite. *Hint*: Use e fact that $E[(\boldsymbol{a}^T(\mathbf{X} - \mathbf{m_X}))^2] \geq 0$.

is positive definite if for any nonzero vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)^T$: $\boldsymbol{a}^T\mathbf{A}\,\boldsymbol{a} > 0$.

) Show that if all the eigenvalues are positive, then $\mathbf{K_X}$ is positive definite. *Hint*: Let $\mathbf{b} = \mathbf{P}^T\boldsymbol{a}$.

) Show that if $\mathbf{K_X}$ is positive definite, then all the eigenvalues are positive. *Hint*: Let $\boldsymbol{a}$ be an eigenvector of $\mathbf{K_X}$.

6.4: Jointly Gaussian Random Vectors

et $\mathbf{X} = (X_1, X_2)$ be the jointly Gaussian random variables with mean vector and covariance atrix given by:

$$\mathbf{m_X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{K_X} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}.$$

) Find the pdf of $\mathbf{X}$ in matrix notation.
) Find the pdf of $\mathbf{X}$ using the quadratic expression in the exponent.
) Find the marginal pdfs of X_1 and X_2.
d) Find a transformation A such that the vector $\mathbf{Y} = A\mathbf{X}$ consists of independent Gaussian random variables.
e) Find the joint pdf of $\mathbf{Y}$.

et $\mathbf{X} = (X_1, X_2, X_3)$ be the jointly Gaussian random variables with mean vector and ovariance matrix given by:

$$\mathbf{m_X} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{K_X} = \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 3/2 \end{bmatrix}.$$

a) Find the pdf of $\mathbf{X}$ in matrix notation.
b) Find the pdf of $\mathbf{X}$ using the quadratic expression in the exponent.
c) Find the marginal pdfs of X_1, X_2, and X_3.
d) Find a transformation A such that the vector $\mathbf{Y} = A\mathbf{X}$ consists of independent Gaussian random variables.
e) Find the joint pdf of $\mathbf{Y}$.

Let U_1, U_2, and U_3 be independent zero-mean, unit-variance Gaussian random variables and let $X = U_1$, $Y = U_1 + U_2$, and $Z = U_1 + U_2 + U_3$.
(a) Find the covariance matrix of (X, Y, Z).
(b) Find the joint pdf of (X, Y, Z).
(c) Find the conditional pdf of Y and Z given X.
(d) Find the conditional pdf of Z given X and Y.

Let X_1, X_2, X_3, X_4 be independent zero-mean, unit-variance Gaussian random variables that are processed as follows:

$$Y_1 = X_1 + X_2, Y_2 = X_2 + X_3, Y_3 = X_3 + X_4.$$

(a) Find the covariance matrix of $\mathbf{Y} = (Y_1, Y_2, Y_3)$.
(b) Find the joint pdf of $\mathbf{Y}$.
(c) Find the joint pdf of Y_1 and Y_2; Y_1 and Y_3.
(d) Find a transformation A such that the vector $\mathbf{Z} = A\mathbf{Y}$ consists of independent Gaussian random variables.

6.58. A more realistic model of the receiver in the multiuser communication system in Problem 6.21 has the K received signals $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_K)$ given by:

$$\mathbf{Y} = \mathbf{ARb} + \mathbf{N}$$

where $\mathbf{A} = [\alpha_k]$ is a diagonal matrix of positive channel gains, $\mathbf{R}$ is a symmetric matrix that accounts for the interference between users, and $\mathbf{b} = (b_1, b_2, \ldots, b_K)$ is the vector of bits from each of the transmitters. $\mathbf{N}$ is the vector of K independent zero-mean, unit-variance Gaussian noise random variables.
(a) Find the joint pdf of $\mathbf{Y}$.
(b) Suppose that in order to recover $\mathbf{b}$, the receiver computes $\mathbf{Z} = (\mathbf{AR})^{-1}\mathbf{Y}$. Find the joint pdf of $\mathbf{Z}$.

6.59. (a) Let $\mathbf{K}_3$ be the covariance matrix in Problem 6.55. Find the corresponding $\mathbf{Q}_2$ and $\mathbf{Q}_3$ in Example 6.23.
(b) Find the conditional pdf of X_3 given X_1 and X_2.

6.60. In Example 6.23, show that:

$$\tfrac{1}{2}(\mathbf{x}_n - \mathbf{m}_n)^T\mathbf{Q}_n(\mathbf{x}_n - \mathbf{m}_n) - \tfrac{1}{2}(\mathbf{x}_{n-1} - \mathbf{m}_{n-1})^T\mathbf{Q}_{n-1}(\mathbf{x}_{n-1} - \mathbf{m}_{n-1})$$
$$= Q_{nn}\{(x_n - m_n) + B\}^2 - Q_{nn}B^2$$
$$\text{where } B = \frac{1}{Q_{nn}}\sum_{j=1}^{n-1} Q_{jk}(x_j - m_j) \quad \text{and} \quad |\mathbf{K}_n|/|\mathbf{K}_{n-1}| = Q_{nn}.$$

6.61. Find the pdf of the sum of Gaussian random variables in the following cases:
(a) $Z = X_1 + X_2 + X_3$ in Problem 6.55.
(b) $Z = X + Y + Z$ in Problem 6.56.
(c) $Z = Y_1 + Y_2 + Y_3$ in Problem 6.57.

6.62. Find the joint characteristic function of the jointly Gaussian random vector $\mathbf{X}$ in Problem 6.54.

6.63. Suppose that a jointly Gaussian random vector $\mathbf{X}$ has zero mean vector and the covariance matrix given in Problem 6.51.
(a) Find the joint characteristic function.
(b) Can you obtain an expression for the joint pdf? Explain your answer.

6.64. Let X and Y be jointly Gaussian random variables. Derive the joint characteristic function for X and Y using conditional expectation.

6.65. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be jointly Gaussian random variables. Derive the characteristic function for $\mathbf{X}$ by carrying out the integral in Eq. (6.32). *Hint*: You will need to complete the square as follows:

$$(\mathbf{x} - j\mathbf{K}\omega)^T\mathbf{K}^{-1}(\mathbf{x} - j\mathbf{K}\omega) = \mathbf{x}^T\mathbf{K}^{-1}\mathbf{x} - 2j\mathbf{x}^T\omega + j^2\omega^T\mathbf{K}\omega.$$

6.66. Find $E[X^2Y^2]$ for jointly Gaussian random variables from the characteristic function.

6.67. Let $\mathbf{X} = (X_1, X_2, X_3, X_4)$ be zero-mean jointly Gaussian random variables. Show that $E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$.

Section 6.5: Mean Square Estimation

6.68. Let X and Y be discrete random variables with three possible joint pmf's:

(i)				(ii)				(iii)			
X/Y	-1	0	1	X/Y	-1	0	1	X/Y	-1	0	1
-1	1/6	1/6	0	-1	1/9	1/9	1/9	-1	1/3	0	0
0	0	0	1/3	0	1/9	1/9	1/9	0	0	1/3	0
1	1/6	1/6	0	1	1/9	1/9	1/9	1	0	0	1/3

(a) Find the minimum mean square error linear estimator for Y given X.
(b) Find the minimum mean square error estimator for Y given X.
(c) Find the MAP and ML estimators for Y given X.
(d) Compare the mean square error of the estimators in parts a, b, and c.

6.69. Repeat Problem 6.68 for the continuous random variables X and Y in Problem 5.26.

6.70. Find the ML estimator for the signal s in Problem 6.4.

6.71. Let N_1 be the number of Web page requests arriving at a server in the period (0, 100) ms and let N_2 be the *total* combined number of Web page requests arriving at a server in the period (0, 200) ms. Assume page requests occur every 1-ms interval according to independent Bernoulli trials with probability of success p.
(a) Find the minimum linear mean square estimator for N_2 given N_1 and the associated mean square error.
(b) Find the minimum mean square error estimator for N_2 given N_1 and the associated mean square error.
(c) Find the maximum a posteriori estimator for N_2 given N_1.
(d) Repeat parts a, b, and c for the estimation of N_1 given N_2.

6.72. Let $Y = X + N$ where X and N are independent Gaussian random variables with different variances and N is zero mean.
(a) Plot the correlation coefficient between the "observed signal" Y and the "desired signal" X as a function of the signal-to-noise ratio σ_X/σ_N.
(b) Find the minimum mean square error estimator for X given Y.
(c) Find the MAP and ML estimators for X given Y.
(d) Compare the mean square error of the estimators in parts a, b and c.

6.73. Let X, Y, Z be the random variables in Problem 6.7.
(a) Find the minimum mean square error linear estimator for Y given X and Z.
(b) Find the minimum mean square error estimator for Y given X and Z.
(c) Find the MAP and ML estimators for Y given X and Z.
(d) Compare the mean square error of the estimators in parts b and c.

6.74. (a) Repeat Problem 6.73 for the estimator of X_2, given X_1 and X_3 in Problem 6.13.
(b) Repeat Problem 6.73 for the estimator of X_3 given X_1 and X_2.

6.75. Consider the ideal multiuser communication system in Problem 6.21. Assume the transmitted bits b_k are independent and equally likely to be +1 or −1.
(a) Find the ML and MAP estimators for **b** given the observation **Y**.
(b) Find the minimum mean square linear estimator for **b** given the observation **Y**. How can this estimator be used in deciding what were the transmitted bits?

6.76. Repeat Problem 6.75 for the multiuser system in Problem 6.58.

6.77. A second-order predictor for samples of an image predicts the sample E as a linear function of sample D to its left and sample B in the previous line, as shown below:

line j	...	A	B	C ...
line $j + 1$	...	D	E	...

Estimate for $E = aD + bB$.

(a) Find a and b if all samples have variance σ^2 and if the correlation coefficient between D and E is ρ, between B and E is ρ, and between D and B is ρ^2.
(b) Find the mean square error of the predictor found in part a, and determine the reduction in the variance of the signal in going from the input to the output of the predictor.

6.78. Show that the mean square error of the two-tap linear predictor is given by Eq. (6.64).

6.79. In "hexagonal sampling" of an image, the samples in consecutive lines are offset relative to each other as shown below:

line j	...		A	B
line $j + 1$	...	C	D	

The covariance between two samples a and b is given by $\rho^{d(a,b)}$ where $d(a, b)$ is the Euclidean distance between the points. In the above samples, the distance between A and B, A and C, A and D, C and D, and B and D is 1. Suppose we wish to use a two-tap linear predictor to predict the sample D. Which two samples from the set $\{A, B, C\}$ should we use in the predictor? What is the resulting mean square error?

*Section 6.6: Generating Correlated Vector Random Variables

6.80. Find a linear transformation that diagonalizes **K**.

(a) $\mathbf{K} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$.

(b) $\mathbf{K} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

6.81. Generate and plot the scattergram of 1000 pairs of random variables **Y** with the covariance matrices in Problem 6.80 if:
(a) X_1 and X_2 are independent random variables that are each uniform in the unit interval;
(b) X_1 and X_2 are independent zero-mean, unit-variance Gaussian random variables.

6.82. Let $\mathbf{X} = (X_1, X_2, X_3)$ be the jointly Gaussian random variables in Problem 6.55.
(a) Find a linear transformation that diagonalizes the covariance matrix.
(b) Generate 1000 triplets of $\mathbf{Y} = \mathbf{AX}$ and plot the scattergrams for Y_1 and Y_2, Y_1 and Y_3, and Y_2 and Y_3. Confirm that the scattergrams are what is expected.

6.83. Let **X** be a jointly Gaussian random vector with mean $\mathbf{m_X}$ and covariance matrix $\mathbf{K_X}$ and let **A** be a matrix that diagonalizes $\mathbf{K_X}$. What is the joint pdf of $\mathbf{A}^{-1}(\mathbf{X} - \mathbf{m_X})$?

6.84. Let $X_1, X_2, \ldots, X_n$ be independent zero-mean, unit-variance Gaussian random variables. Let $Y_k = (X_k + X_{k-1})/2$, that is, Y_k is the moving average of pairs of values of X. Assume $X_{-1} = 0 = X_{n+1}$.
(a) Find the covariance matrix of the Y_k's.
(b) Use Octave to generate a sequence of 1000 samples $Y_1, \ldots, Y_n$. How would you check whether the Y_k's have the correct covariances?

6.85. Repeat Problem 6.84 with $Y_k = X_k - X_{k-1}$.

6.86. Let **U** be an orthogonal matrix. Show that if **A** diagonalizes the covariance matrix **K**, then $\mathbf{B} = \mathbf{UA}$ also diagonalizes **K**.

6.87. The transformation in Problem 6.56 is said to be "causal" because each output depends only on "past" inputs.
(a) Find the covariance matrix of X, Y, Z in Problem 6.56.
(b) Find a noncausal transformation that diagonalizes the covariance matrix in part a.

6.88. (a) Find a causal transformation that diagonalizes the covariance matrix in Problem 6.54.
(b) Repeat for the covariance matrix in Problem 6.55.

ms Requiring Cumulative Knowledge

et $U_0, U_1, \ldots$ be a sequence of independent zero-mean, unit-variance Gaussian ran-om variables. A "low-pass filter" takes the sequence U_i and produces the output se-uence $X_n = (U_n + U_{n-1})/2$, and a "high-pass filter" produces the output sequence $_n = (U_n - U_{n-1})/2$.

a) Find the joint pdf of X_{n+1}, X_n, and X_{n-1}; of X_n, X_{n+m}, and X_{n+2m}, $m > 1$.

b) Repeat part a for Y_n.

c) Find the joint pdf of X_n, X_m, Y_n, and Y_m.

d) Find the corresponding joint characteristic functions in parts a, b, and c.

et $X_1, X_2, \ldots, X_n$ be the samples of a speech waveform in Example 6.31. Suppose we ant to interpolate for the value of a sample in terms of the previous and the next sam-les, that is, we wish to find the best linear estimate for X_2 in terms of X_1 and X_3.

a) Find the coefficients of the best linear estimator (interpolator).

b) Find the mean square error of the best linear interpolator and compare it to the mean square error of the two-tap predictor in Example 6.31.

c) Suppose that the samples are jointly Gaussian. Find the pdf of the interpolation error.

et $X_1, X_2, \ldots, X_n$ be samples from some signal. Suppose that the samples are jointly aussian random variables with covariance

$$\text{COV}(X_i, X_j) = \begin{cases} \sigma^2 & \text{for } i = j \\ \rho\sigma^2 & \text{for } |i - j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

uppose we take blocks of two consecutive samples to form a vector $\mathbf{X}$, which is then lin-arly transformed to form $\mathbf{Y} = \mathbf{AX}$.

a) Find the matrix $\mathbf{A}$ so that the components of $\mathbf{Y}$ are independent random variables.

b) Let $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ be two consecutive blocks and let $\mathbf{Y}_i$ and $\mathbf{Y}_{i+1}$ be the corresponding transformed variables. Are the components of $\mathbf{Y}_i$ and $\mathbf{Y}_{i+1}$ independent?

multiplexer combines N digital television signals into a common communications line. V signal n generates X_n bits every 33 milliseconds, where X_n is a Gaussian random vari-ble with mean m and variance σ^2. Suppose that the multiplexer accepts a maximum otal of T bits from the combined sources every 33 ms, and that any bits in excess of T are iscarded. Assume that the N signals are independent.

a) Find the probability that bits are discarded in a given 33-ms period, if we let $T = m_a + t\sigma$, where m_a is the mean total bits generated by the combined sources, and σ is the standard deviation of the total number of bits produced by the combined sources.

b) Find the average number of bits discarded per period.

c) Find the long-term fraction of bits lost by the multiplexer.

d) Find the average number of bits per source allocated in part a, and find the average number of bits lost per source. What happens as N becomes large?

e) Suppose we require that t be adjusted with N so that the fraction of bits lost per source is kept constant. Find an equation whose solution yields the desired value of t.

f) Do the above results change if the signals have pairwise covariance ρ?

onsider the estimation of T given N_1 and arrivals in Problem 6.17.

a) Find the ML and MAP estimators for T.

b) Find the linear mean square estimator for T.

c) Repeat parts a and b if N_1 and N_2 are given.

CHAPTER 7

Sums of Random Variables and Long-Term Averages

Many problems involve the counting of the number of occurrences of events, the measurement of cumulative effects, or the computation of arithmetic averages in a series of measurements. Usually these problems can be reduced to the problem of finding, exactly or approximately, the distribution of a random variable that consists of the sum of n independent, identically distributed random variables. In this chapter, we investigate sums of random variables and their properties as n becomes large.

In Section 7.1, we show how the characteristic function is used to compute the pdf of the sum of independent random variables. In Section 7.2, we discuss the sample mean estimator for the expected value of a random variable and the relative frequency estimator for the probability of an event. We introduce measures for assessing the goodness of these estimators. We then discuss the laws of large numbers, which are theorems that state that the sample mean and relative frequency estimators converge to the corresponding expected values and probabilities as the number of samples is increased. These theoretical results demonstrate the remarkable consistency between probability theory and observed behavior, and they reinforce the relative frequency interpretation of probability.

In Section 7.3, we present the central limit theorem, which states that, under very general conditions, the cdf of a sum of random variables approaches that of a Gaussian random variable even though the cdf of the individual random variables may be far from Gaussian. This result enables us to approximate the pdf of sums of random variables by the pdf of a Gaussian random variable. The result also explains why the Gaussian random variable appears in so many diverse applications.

In Section 7.4 we consider sequences of random variables and their convergence properties. In Section 7.5 we discuss random experiments in which events occur at random times. In these experiments we are interested in the average rate at which events occur as well as the rate at which quantities associated with the events grow. Finally, Section 7.6 introduces computer methods based on the discrete Fourier transform that prove very useful in the numerical calculation of pmf's and pdf's from their transforms.

3. S. M. Ross, *Introduction to Probability Models*, Academic Press, New York, 2003.
4. J. A. Cadzow, *Foundations of Digital Signal Processing and Data Analysis*, Macmillan, New York, 1987.
5. P. L. Meyer, *Introductory Probability and Statistical Applications*, 2nd ed., Addison-Wesley, Reading, Mass., 1970.
6. J. W. Cooley, P. Lewis, and P. D. Welch, "The Fast Fourier Transform and Its Applications," *IEEE Transactions on Education*, vol. 12, pp. 27–34, March 1969.
7. H. J. Larson and B. O. Shubert, *Probability Models in Engineering Sciences*, vol. 1, Wiley, New York, 1979.
8. H. Stark and J. W. Woods, *Probability and Random Processes with Applications to Signal Processing*, 3d ed., Prentice Hall, Upper Saddle River, N.J., 2002.

PROBLEMS

Section 7.1: Sums of Random Variables

7.1. Let $Z = X + Y + Z$, where X, Y, and Z are zero-mean, unit-variance random variables with $\text{COV}(X, Y) = 1/2$, and $\text{COV}(Y, Z) = -1/4$ and $\text{COV}(X, Z) = 1/2$.
- **(a)** Find the mean and variance of Z.
- **(b)** Repeat part a assuming X, Y, and Z are uncorrelated random variables.

7.2. Let $X_1, \ldots, X_n$ be random variables with the same mean and with covariance function:

$$\text{COV}(X_i, X_j) = \begin{cases} \sigma^2 & \text{if } i = j \\ \rho\sigma^2 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\rho| < 1$. Find the mean and variance of $S_n = X_1 + \cdots + X_n$.

7.3. Let $X_1, \ldots, X_n$ be random variables with the same mean and with covariance function

$$\text{COV}(X_i, X_j) = \sigma^2\rho^{|i-j|},$$

where $|\rho| < 1$. Find the mean and variance of $S_n = X_1 + \cdots + X_n$.

7.4. Let X and Y be independent Cauchy random variables with parameters 1 and 4, respectively. Let $Z = X + Y$.
- **(a)** Find the characteristic function of Z.
- **(b)** Find the pdf of Z from the characteristic function found in part a.

7.5. Let $S_k = X_1 + \cdots + X_k$, where the X_i's are independent random variables, with X_i a chi-square random variable with n_i degrees of freedom. Show that S_k is a chi-square random variable with $n = n_1 + \cdots + n_k$ degrees of freedom.

7.6. Let $S_n = X_1^2 + \cdots + X_n^2$, where the X_i's are iid zero-mean, unit-variance Gaussian random variables.
- **(a)** Show that S_n is a chi-square random variable with n degrees of freedom. *Hint:* See Example 4.34.
- **(b)** Use the methods of Section 4.5 to find the pdf of

$$T_n = \sqrt{X_1^2 + \cdots + X_n^2}.$$

- **(c)** Show that T_2 is a Rayleigh random variable.
- **(d)** Find the pdf for T_3. The random variable T_3 is used to model the speed of molecules in a gas. T_3 is said to have the Maxwell distribution.

7.7. Let X and Y be independent exponential random variables with parameters 2 and 10, respectively. Let $Z = X + Y$.
- **(a)** Find the characteristic function of Z.
- **(b)** Find the pdf of Z from the characteristic function found in part a.

7.8. Let $Z = 3X - 7Y$, where X and Y are independent random variables.
- **(a)** Find the characteristic function of Z.
- **(b)** Find the mean and variance of Z by taking derivatives of the characteristic function found in part a.

7.9. Let M_n be the sample mean of n iid random variables X_j. Find the characteristic function of M_n in terms of the characteristic function of the X_j's.

7.10. The number X_j of raffle winners in classroom j is a binomial random variable with parameter n_j and p. Suppose that the school has K classrooms. Find the pmf of the total number of raffle winners in the school, assuming the X_j's are independent random variables.

7.11. The number of packet arrivals X_i at port i in a router is a Poisson random variable with mean α_i. Given that the router has k ports, find the pmf for the total number of packet arrivals at the router. Assume that the X_i's are independent random variables.

7.12. Let $X_1, X_2, \ldots$ be a sequence of independent integer-valued random variables, let N be an integer-valued random variable independent of the X_j, and let

$$S = \sum_{k=1}^{N} X_k.$$

- **(a)** Find the mean and variance of S.
- **(b)** Show that

$$G_S(z) = E(z^S) = G_N(G_X(z)),$$

where $G_X(z)$ is the generating function of each of the X_k's.

7.13. Let the number of smashed-up cars arriving at a body shop in a week be a Poisson random variable with mean L. Each job repair costs X_j dollars, the X_j's are iid random variables that are equally likely to be \$500 or \$1000.
- **(a)** Find the mean and variance of the total revenue R arriving in a week.
- **(b)** Find the $G_R(z) = E[z^R]$.

7.14. Let the number of widgets tested in an assembly line in 1 hour be a binomial random variable with parameters $n = 600$ and p. Suppose that the probability that a widget is faulty is a. Let S be the number of widgets that are found faulty in a 1-hour period.
- **(a)** Find the mean and variance of S.
- **(b)** Find $G_S(z) = E[z^S]$.

Section 7.2: The Sample Mean and the Laws of Large Numbers

7.15. Suppose that the number of particle emissions by a radioactive mass in t seconds is a Poisson random variable with mean λt. Use the Chebyshev inequality to obtain a bound for the probability that $|N(t)/t - \lambda|$ exceeds ε.

7.16. Suppose that 20% of voters are in favor of certain legislation. A large number n of voters are polled and a relative frequency estimate $f_A(n)$ for the above proportion is obtained.

Use Eq. (7.20) to determine how many voters should be polled in order that the probability is at least .95 that $f_A(n)$ differs from 0.20 by less than 0.02.

7.17. A fair die is tossed 20 times. Use Eq. (7.20) to bound the probability that the total number of dots is between 60 and 80.

7.18. Let X_i be a sequence of independent zero-mean, unit-variance Gaussian random variables. Compare the bound given by Eq. (7.20) with the exact value obtained from the Q function for $n = 16$ and $n = 81$.

7.19. Does the weak law of large numbers hold for the sample mean if the X_i's have the covariance functions given in Problem 7.2? Assume the X_i have the same mean.

7.20. Repeat Problem 7.19 if the X_i's have the covariance functions given in Problem 7.3.

7.21. (The **sample variance**) Let $X_1, \ldots, X_n$ be an iid sequence of random variables for which the mean and variance are unknown. The sample variance is defined as follows:

$$V_n^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - M_n)^2,$$

where M_n is the sample mean.

(a) Show that

$$\sum_{j=1}^{n}(X_j - \mu)^2 = \sum_{j=1}^{n}(X_j - M_n)^2 + n(M_n - \mu)^2.$$

(b) Use the result in part a to show that

$$E\left[k\sum_{j=1}^{n}(X_j - M_n)^2\right] = k(n-1)\sigma^2.$$

(c) Use part b to show that $E[V_n^2] = \sigma^2$. Thus V_n^2 is an unbiased estimator for the variance.

(d) Find the expected value of the sample variance if $n - 1$ is replaced by n. Note that this is a biased estimator for the variance.

Section 7.3: The Central Limit Theorem

7.22. **(a)** A fair coin is tossed 100 times. Estimate the probability that the number of heads is between 40 and 60. Estimate the probability that the number is between 50 and 55.

(b) Repeat part a for $n = 1000$ and the intervals [400, 600] and [500, 550].

7.23. Repeat Problem 7.16 using the central limit theorem.

7.24. Use the central limit theorem to estimate the probability in Problem 7.17.

7.25. The lifetime of a cheap light bulb is an exponential random variable with mean 36 hours. Suppose that 16 light bulbs are tested and their lifetimes measured. Use the central limit theorem to estimate the probability that the sum of the lifetimes is less than 600 hours.

7.26. A student uses pens whose lifetime is an exponential random variable with mean 1 week. Use the central limit theorem to determine the minimum number of pens he should buy at the beginning of a 15-week semester, so that with probability .99 he does not run out of pens during the semester.

7.27. Let S be the sum of 80 iid Poisson random variables with mean 0.25. Compare the exact value of $P[S = k]$ to an approximation given by the central limit theorem as in Eq. (7.30).

7.28. The number of messages arriving at a multiplexer is a Poisson random variable with mean 15 messages/second. Use the central limit theorem to estimate the probability that more than 950 messages arrive in one minute.

7.29. A binary transmission channel introduces bit errors with probability .15. Estimate the probability that there are 20 or fewer errors in 100 bit transmissions.

7.30. The sum of a list of 64 real numbers is to be computed. Suppose that numbers are rounded off to the nearest integer so that each number has an error that is uniformly distributed in the interval $(-0.5, 0.5)$. Use the central limit theorem to estimate the probability that the total error in the sum of the 64 numbers exceeds 4.

7.31. **(a)** A fair coin is tossed 100 times. Use the Chernoff bound to estimate the probability that the number of heads is greater than 90. Compare to an estimate using the central limit theorem.

(b) Repeat part a for $n = 1000$ and the probability that the number of heads is greater than 650.

7.32. A binary transmission channel introduces bit errors with probability .01. Use the Chernoff bound to estimate the probability that there are more than 3 errors in 100 bit transmissions. Compare to an estimate using the central limit theorem.

7.33. **(a)** When you play the rock/paper/scissors game against your sister you lose with probability 3/5. Use the Chernoff bound to estimate the probability that you win more than half of 20 games played.

(b) Repeat for 100 games.

(c) Use trial and error to find the number of games n that need to be played so that the probability that your sister wins more than 1/2 the games is 90%.

7.34. Show that the Chernoff bound for X, a Poisson random variable with mean α, is $P[X \geq a] \leq e^{-a\ln(a/\alpha)+a-\alpha}$ for $a > \alpha$. *Hint:* Use $E[e^{sX}] = e^{\alpha(e^s-1)}$.

7.35. Redo Problem 7.26 using the Chernoff bound.

7.36. Show that the Chernoff bound for X, a Gaussian random variable with mean μ and variance σ^2, is $P[X \geq a] \leq e^{-(a-\mu)^2/2\sigma^2}$, $a > \mu$. *Hint:* Use $E[e^{sX}] = e^{s\mu+s^2\sigma^2/2}$.

7.37. Compare the Chernoff bound for the Gaussian random variable with the estimates provided by Eq. (4.54).

7.38. **(a)** Find the Chernoff bound for the exponential random variable with rate λ.

(b) Compare the exact probability of $P[X \geq k/\lambda]$ with the Chernoff bound.

7.39. **(a)** Generalize the approach in Problem 7.38 to find the Chernoff bound for a gamma random variable with parameters λ and α.

(b) Use the result of part a to obtain the Chernoff bound for a chi-square random variable with k degrees of freedom.

*Section7.4: Convergence of Sequences of Random Variables

7.40. Let $U_n(\zeta)$, $W_n(\zeta)$, $Y_n(\zeta)$, and $Z_n(\zeta)$ be the sequences of random variables defined in Example 7.18.

(a) Plot the sequence of functions of ζ associated with each sequence of random variables.

(b) For $\zeta = 1/4$, plot the associated sample sequence.

7.41. Let ζ be selected at random from the interval $S = [0, 1]$, and let the probability that ζ is in a subinterval of S be given by the length of the subinterval. Define the following sequences of random variables for $n \geq 1$:

$$X_n(\zeta) = \zeta^n,\ Y_n(\zeta) = \cos^2 2\pi\zeta,\ Z_n(\zeta) = \cos^n 2\pi\zeta.$$

Do the sequences converge, and if so, in what sense and to what limiting random variable?

7.42. Let $b_i, i \geq 1$, be a sequence of iid, equiprobable Bernoulli random variables, and let ζ be the number between [0, 1] determined by the binary expansion

$$\zeta = \sum_{i=1}^{\infty} b_i 2^{-i}.$$

(a) Explain why ζ is uniformly distributed in [0, 1].

(b) How would you use this definition of ζ to generate the sample sequences that occur in the urn problem of Example 7.20?

7.43. Let X_n be a sequence of iid, equiprobable Bernoulli random variables, and let

$$Y_n = 2^n X_1 X_2 \ldots X_n.$$

(a) Plot a sample sequence. Does this sequence converge almost surely, and if so, to what limit?

(b) Does this sequence converge in the mean square sense?

7.44. Let X_n be a sequence of iid random variables with mean m and variance $\sigma^2 < \infty$. Let M_n be the associated sequence of arithmetic averages,

$$M_n = \frac{1}{n}\sum_{i=0}^{n} X_i.$$

Show that M_n converges to m in the mean square sense.

7.45. Let X_n and Y_n be two (possibly dependent) sequences of random variables that converge in the mean square sense to X and Y, respectively. Does the sequence $X_n + Y_n$ converge in the mean square sense, and if so, to what limit?

7.46. Let U_n be a sequence of iid zero-mean, unit-variance Gaussian random variables. A "low-pass filter" takes the sequence U_n and produces the sequence

$$X_n = \frac{1}{2}(U_n + U_{n-1}).$$

(a) Does this sequence converge in the mean square sense?

(b) Does it converge in distribution?

7.47. Does the sequence of random variables introduced in Example 7.20 converge in the mean square sense?

7.48. Customers arrive at an automated teller machine at discrete instants of time, $n = 1, 2, \ldots$. The number of customer arrivals in a time instant is a Bernoulli random variable with parameter p, and the sequence of arrivals is iid. Assume the machine services a customer in less than one time unit. Let X_n be the total number of customers served by the machine up to time n. Suppose that the machine fails at time N, where N is a geometric random variable with mean 100, so that the customer count remains at X_N thereafter.

(a) Sketch a sample sequence for X_n.

(b) Do the sample sequences converge almost surely, and if so, to what limit?

(c) Do the sample sequences converge in the mean square sense?

7.49. Show that the sequence $Y_n(\zeta)$ defined in Example 7.18 converges in distribution.

7.50. Let X_n be a sequence of Laplacian random variables with parameter $\alpha = n$. Does this sequence converge in distribution?

*Section 7.5: Long-Term Arrival Rates and Associated Averages

7.51. The customer arrival times at a bus depot are iid exponential random variables with mean 1 minute. Suppose that buses leave as soon as 30 seats are full. At what rate do buses leave the depot?

7.52. A faulty clock ticks forward every minute with probability $p = 0.1$ and it does not tick forward with probability $1 - p$. What is the rate at which this clock moves forward?

7.53. **(a)** Show that $\{N(t) \geq n\}$ and $\{S_n \leq t\}$ are equivalent events.

(b) Use part a to find $P[N(t) \leq n]$ when the X_i are iid exponential random variables with mean $1/\alpha$.

7.54. Explain why the following are not equivalent events:

(a) $\{N(t) \leq n\}$ and $\{S_n \geq t\}$.

(b) $\{N(t) > n\}$ and $\{S_n < t\}$.

7.55. A communication channel alternates between periods when it is error free and periods during which it introduces errors. Assuming that these periods are independent random variables of means $m_1 = 100$ hours and $m_2 = 1$ minute, respectively, find the long-term proportion of time during which the channel is error free.

7.56. A worker works at a rate r_1 when the boss is around and at a rate r_2 when the boss is not present. Suppose that the sequence of durations of the time periods when the boss is present and absent are independent random variables with means m_1 and m_2, respectively. Find the long-term average rate at which the worker works.

7.57. A computer (repairman) continuously cycles through three tasks (machines). Suppose that each time the computer services task i, it spends time X_i doing so.

(a) What is the long-term rate at which the computer cycles through the three tasks?

(b) What is the long-term proportion of time spent by the computer servicing task i?

(c) Repeat parts a and b if a random time W is required for the computer (repairman) to switch (walk) from one task (machine) to another.

7.58. Customers arrive at a phone booth and use the phone for a random time Y, with mean 3 minutes, if the phone is free. If the phone is not free, the customers leave immediately. Suppose that the time between customer arrivals is an exponential random variable with mean 10 minutes.

(a) Find the long-term rate at which customers use the phone.

(b) Find the long-term proportion of customers that leave without using the phone.

7.59. The lifetime of a certain system component is an exponential random variable with mean $T = 2$ months. Suppose that the component is replaced when it fails or when it reaches the age of $3T$ months.

(a) Find the long-term rate at which components are replaced.

(b) Find the long-term rate at which working components are replaced.

7.60. A data compression encoder segments a stream of information bits into patterns as shown below. Each pattern is then encoded into the codeword shown below.

Pattern	Codeword	Probability
1	100	.1
01	101	.09
001	110	.081
0001	111	.0729
0000	0	.6521

(a) If the information source produces a bit every millisecond, find the rate at which codewords are produced.

(b) Find the long-term ratio of encoded bits to information bits.

7.61. In Example 7.29 evaluate the proportion of time that the residual lifetime $r(t)$ exceeds c seconds for the following cases:

(a) X_j iid uniform random variables in the interval $[0, 2]$.

(b) X_j iid exponential random variables with mean 1.

(c) X_j iid Rayleigh random variables with mean 1.

(d) Calculate and compare the mean residual time in each of the above three cases.

7.62. Let the age $a(t)$ of a cycle be defined as the time that has elapsed from the last arrival up to an arbitrary time instant t. Show that the long-term proportion of time that $a(t)$ exceeds c seconds is given by Eq. (7.48).

7.63. Suppose that the cost in each cycle grows at a rate proportional to the age $a(t)$ of the cycle, that is,

$$C_j = \int_0^{X_j} a(t')\,dt'.$$

(a) Show that $C_j = X_j^2/2$.

(b) Show that the long-term rate at which the cost grows is $E[X^2]/2E[X]$.

(c) Show that the result in part b is also the long-term time average of $a(t)$, that is,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a(t')\,dt' = \frac{E[X^2]}{2E[X]}.$$

(d) Explain why the average residual life is also given by the above expression.

7.64. Calculate the mean age and mean residual life in Problem 7.63 in the following cases:

(a) X_j iid uniform random variables in the interval $[0, 2]$.

(b) X_j iid exponential random variables with mean 1.

(c) X_j iid Rayleigh random variables with mean 1.

7.65. (The Regenerative Method) Suppose that a queueing system has the property that when a customer arrives and finds an empty system, the future behavior of the system is completely independent of the past. Define a cycle to consist of the time period between two consecutive customer arrivals to an empty system. Let N_j be the number of customers served during the jth cycle and let T_j be the total delay of all customers served during the jth cycle.

(a) Use Theorem 2 to show that the average customer delay is given by $E[T]/E[N]$, that is,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} D_k = \frac{E[T]}{E[N]},$$

where D_k is the delay of the kth customer.

(b) How would you use this result to estimate the average delay in a computer simulation of a queueing system?

*Section 7.6: Calculating Distributions Using the Discrete Fourier Transform

7.66. Let the discrete random variable X be uniformly distributed in the set $\{0, 1, 2\}$.

(a) Find the $N = 3$ DFT for X.

(b) Use the inverse DFT to recover $P[X = 1]$.

7.67. Let $S = X + Y$, where X and Y are iid random variables uniformly distributed in the set $\{0, 1, 2\}$.

(a) Find the $N = 5$ DFT for S.

(b) Use the inverse DFT to find $P[S = 2]$.

7.68. Let X be a binomial random variable with parameter $n = 8$ and $p = 1/2$.

(a) Use the FFT to obtain the pmf of X from $\Phi_X(\omega)$.

(b) Use the FFT to obtain the pmf of $Z = X + Y$ where X and Y are iid binomial random variables with $n = 8$ and $p = 1/2$.

7.69. Let X_i be a discrete random variable that is uniformly distributed in the set $\{0, 1, \ldots, 9\}$. Use the FFT to find the pmf of $S_n = X_1 + \cdots + X_n$ for $n = 5$ and $n = 10$. Plot your results and compare them to Fig. 7.16.

7.70. Let X be the geometric random variable with parameter $p = 1/2$. Use the FFT to evaluate Eq. (7.55) to compute p'_k for $N = 8$ and $N = 16$. Compare the results to those given by Eq. (7.57).

7.71. Let X be a Poisson random variable with mean $L = 5$.

(a) Use the FFT to obtain the pmf from $\Phi_X(\omega)$. Find the value of N for which the error in Eq. (7.55) is less than 1%.

(b) Let $S = X_1 + X_2 + \cdots + X_5$, where the X_i are iid Poisson random variables with mean $L = 5$. Use the FFT to compute the pmf of S from $\Phi_X(\omega)$.

7.72. The probability generating function for the number N of customers in a certain queueing system (the so-called M/D/1 system discussed in Chapter 12) is

$$G_N(z) = \frac{(1-\rho)(1-z)}{1 - ze^{\rho(1-z)}},$$

where $0 \le \rho \le 1$. Use the FFT to obtain the pmf of N for $\rho = 1/2$.

7.73. Use the FFT to obtain approximately the pdf of a Laplacian random variable from its characteristic function. Use the same parameters as in Example 7.33 and compare your results to those shown in Fig. 7.17.

7.74. Use the FFT to obtain approximately the pdf of $Z = X + Y$, where X and Y are independent Laplacian random variables with parameters $\alpha = 1$ and $\alpha = 2$, respectively.

7.75. Use the FFT to obtain approximately the pdf of a zero-mean, unit-variance Gaussian random variable from its characteristic function. Experiment with the values of N and ω_0 and compare the results given by the FFT with the exact values.

7.76. Figures 7.2 through 7.4 for the cdf of the sum of iid Bernoulli, uniform, and exponential random variables were obtained using the FFT. Reproduce the results shown in these figures.

Problems Requiring Cumulative Knowledge

7.77. The number X of type 1 defects in a system is a binomial random variable with parameters n and p, and the number Y of type 2 defects is binomial with parameters m and r.

(a) Find the probability generating function for the total number of defects in the system.

(b) Find an expression for the probability that the total number of defects is k.

(c) Let $n = 32$, $p = 1/10$, and $m = 16$, $r = 1/8$. Use the FFT to evaluate the pmf for the total number of defects in the system.

7.78. Let U_n be a sequence of iid zero-mean, unit-variance Gaussian random variables. A "low-pass filter" takes the sequence U_n and produces the sequence

$$X_n = \frac{1}{2}U_n + \left(\frac{1}{2}\right)^2 U_{n-1} + \cdots + \left(\frac{1}{2}\right)^n U_1.$$

(a) Find the mean and variance of X_n.

(b) Find the characteristic function of X_n. What happens as n approaches infinity?

(c) Does this sequence of random variables converge? In what sense?

7.79. Let S_n be the sum of a sequence of X_i's that are jointly Gaussian random variables with mean μ and with the covariance function given in Problem 7.2.

(a) Find the characteristic function of S_n.

(b) Find the mean and variance of $S_n - S_m$.

(c) Find the joint characteristic function of S_n and S_m. *Hint:* Assuming $n > m$, condition on the value of S_m.

(d) Does S_n converge in the mean square sense?

7.80. Repeat Problem 7.79 with the sequence of X_i's given as jointly Gaussian random variables with mean and covariance functions given in Problem 7.3.

7.81. Let Z_n be the sequence of random variables defined in the formulation of the central limit theorem, Eq. (7.26a). Does Z_n converge in the mean square sense?

7.82. Let X_n be the sequence of independent, identically distributed outputs of an information source. At time n, the source produces symbols according to the following probabilities:

Symbol	Probability	Codeword
A	1/2	0
B	1/4	10
C	1/8	110
D	1/16	1110
E	1/16	1111

(a) The self-information of the output at time n is defined by the random variable $Y_n = -\log_2 P[X_n]$. Thus, for example, if the output is C, the self-information is $-\log_2 1/8 = 3$. Find the mean and variance of Y_n. Note that the expected value of the self-information is equal to the entropy of X (cf. Section 4.10).

(b) Consider the sequence of arithmetic averages of the self-information:

$$S_n = \frac{1}{n}\sum_{k=1}^{n} Y_k.$$

Do the weak law and strong law of large numbers apply to S_n?

(c) Now suppose that the outputs of the information source are encoded using the variable-length binary codewords indicated above. Note that the length of the codewords corresponds to the self-information of the corresponding symbol. Interpret the result of part b in terms of the rate at which bits are produced when the above code is applied to the information source outputs.

CHAPTER 8

Statistics

Probability theory allows us to model situations that exhibit randomness in terms of random experiments involving sample spaces, events, and probability distributions. The axioms of probability allow us to develop an extensive set of tools for calculating probabilities and averages for a wide array of random experiments. The field of statistics plays the key role of bridging probability models to the real world. In applying probability models to real situations, we must perform experiments and gather data to answer questions such as:

- What are the values of parameters, e.g., mean and variance, of a random variable of interest?
- Are the observed data consistent with an assumed distribution?
- Are the observed data consistent with a given parameter value of a random variable?

Statistics is concerned with the gathering and analysis of data and with the drawing of conclusions or inferences from the data. The methods from statistics provide us with the means to answer the above questions.

In this chapter we first consider the estimation of parameters of random variables. We develop methods for obtaining point estimates as well as confidence intervals for parameters of interest. We then consider hypothesis testing and develop methods that allow us to accept or reject statements about a random variable based on observed data. We will apply these methods to determine the goodness of fit of distributions to observed data.

The Gaussian random variable plays a crucial role in statistics. We note that the Gaussian random variable is referred to as the **normal random variable** in the statistics literature.

8.1 SAMPLES AND SAMPLING DISTRIBUTIONS

The origin of the term "statistics" is in the gathering of data about the *population* in a state or locality in order to draw conclusions about properties of the population, e.g., potential tax revenue or size of pool of potential army recruits. Typically the

CHECKLIST OF IMPORTANT TERMS

Acceptance region
Alternative hypothesis
Bayes decision rule
Bayes estimator
Chi-square goodness-of-fit test
Composite hypothesis
Confidence interval
Confidence level
Consistent estimator
Cramer-Rao inequality
Critical region
Critical value
Decision rule
Efficiency
False alarm probability
Fisher information
Invariance property
Likelihood function
Likelihood ratio function
Log likelihood function
Maximum likelihood method
Maximum likelihood test
Mean square estimation error
Method of batch means
Neyman-Pearson test
Normal random variable
Null hypothesis
Point estimator
Population
Power
Probability of detection
Random sample
Rejection region
Sampling distribution
Score function
Significance level
Significance test
Simple hypothesis
Statistic
Strongly consistent estimator
Type I error
Type II error
Unbiased estimator

ANNOTATED REFERENCES

Bulmer [1] is a classic introductory textbook on statistics. Ross [2] and Wackerly [3] provide excellent and up-to-date introductions to statistics. Bickel [4] provides a more advanced treatment. Cramer [5] is a classic text that provides careful development of many traditional statistical methods. Van Trees [6] has influenced the application of statistical methods in modern communications. [10] provides a very useful online resource for learning probability and statistics.

1. M.G. Bulmer, *Principles of Statistics*, Dover Publications, New York, 1979.
2. S. M. Ross, *Introduction to Probability and Statistics for Engineers and Scientists*, Elsevier Academic Press, Burlington, Mass., 2004.
3. D. M. Wackerly, W. Mendenhall, and R. L. Scheaffer, *Mathematical Statistics with Applications*, Duxbury, Pacific Grove, Calif., 2002.
4. P. J. Bickel and K. A. Doksum, *Mathematical Statistics*, Prentice Hall, Upper Saddle River, N.J., 2007.
5. H. Cramer, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, N.J., 1999.
6. H. L. Van Trees, *Detection, Estimation, and Modulation Theory*, John Wiley & Sons, New York, 1968.
7. A. O. Allen, *Probability, Statistics, and Queueing Theory*, Academic Press, New York, 1978.
8. S. Kokoska and D. Zwillinger, *Standard Probability and Statistics Tables and Formulae*, Chapman & Hall, Boca Raton, Fl., 2000.

9. M. Hardy, "An Illuminating Counter-example," *American Mathematical Monthly*, March 2003, pp.234–238.
10. *Virtual Laboratory in Probability and Statistics*, www.math.uah.edu/stat.
11. C. H. Edwards, Jr., and D. E. Penney, *Calculus and Analytic Geometry*, 4th ed., Prentice Hall, Englewood Cliffs, N.J., 1984.

PROBLEMS

Note: Statistics involves working with data. For this reason the problems in this section incorporate exercises that involve the generation of random samples of random variables using the methods introduced in Chapters 3, 4, 5, and 6. These exercises can be skipped without loss of continuity.

Section 8.1: Samples and Sampling Distributions

8.1. Let X be a Gaussian random variable with mean 10 and variance 4. A sample of size 9 is obtained and the sample mean, minimum, and maximum of the sample are calculated.
- **(a)** Find the probability that the sample mean is less than 9.
- **(b)** Find the probability that the minimum is greater than 8.
- **(c)** Find the probability that the maximum is less than 12.
- **(d)** Find n so the sample mean is within 1 of the true mean with probability 0.95.
- **(e)** Generate 100 random samples of size 9. Compare the probabilities obtained in parts a, b, and c to the observed relative frequencies.

8.2. The lifetime of a device is an exponential random variable with mean 50 months. A sample of size 25 is obtained and the sample mean, maximum, and minimum of the sample are calculated.
- **(a)** Estimate the probability that the sample mean differs from the true mean by more than 1 month.
- **(b)** Find the probability that the longest-lived sample is greater than 100 months.
- **(c)** Find the probability that the shortest-lived sample is less than 25 months.
- **(d)** Find n so the sample mean is within 5 months of the true mean with probability 0.9.
- **(e)** Generate 100 random samples of size 25. Compare the probabilities obtained in parts a, b, and c to the observed relative frequencies.

8.3. Let the signal X be a uniform random variable in the interval $[-3, 3]$, and suppose that a sample of size 50 is obtained.
- **(a)** Estimate the probability that the sample mean is outside the interval $[-0.5, 0.5]$.
- **(b)** Estimate the probability that the maximum of the sample is less than 2.5.
- **(c)** Estimate the probability that the sample mean of the squares of the samples is greater than 3.
- **(d)** Generate 100 random samples of size 50. Compare the probabilities obtained in parts a, b, and c to the observed relative frequencies.

8.4. Let X be a Poisson random variable with mean $\alpha = 2$, and suppose that a sample of size 16 is obtained.
- **(a)** Estimate the probability that the sample mean is greater than 2.5.
- **(b)** Estimate the probability that the sample mean differs from the true mean by more than 0.5.

(c) Find n so the sample mean differs from the true mean by more than 0.5 with probability 0.95.

(d) Generate 100 random samples of size 16. Compare the probabilities obtained in parts a and b to the observed relative frequencies.

8.5. The interarrival time of queries at a call center are exponential random variables with mean interarrival time 1/4. Suppose that a sample of size 9 is obtained.

(a) The estimator $\hat{\lambda}_1 = 1/\overline{X}_9$ is used to estimate the arrival rate. Find the probability that the estimator differs from the true arrival rate by more than 1.

(b) Suppose the estimator $\hat{\lambda}_2 = 1/9 \min(X_1, \ldots, X_9)$ is used to estimate the arrival rate. Find the probability that the estimator differs from the true arrival rate by more than 1.

(c) Generate 100 random samples of size 9. Compare the probabilities obtained in parts a and b to the observed relative frequencies.

8.6. Let the sample $X_1, X_2, \ldots, X_n$ consist of iid versions of the random variable X. The method of moments involves estimating the moments of X as follows:

$$\hat{m}_k = \frac{1}{n}\sum_{j=1}^{n} X_j^k.$$

(a) Suppose that X is a uniform random variable in the interval $[0, \theta]$. Use $\hat{m}_1$ to find an estimator for θ.

(b) Find the mean and variance of the estimator in part a.

8.7. Let X be a gamma random variable with parameters α and $\beta = 1/\lambda$.

(a) Use the first two moment estimators $\hat{m}_1$ and $\hat{m}_2$ of X (defined in Problem 8.6) to estimate the parameters α and β.

(b) Describe the behavior of the estimators in part a as n becomes large.

8.8. Let $\mathbf{X} = (X, Y)$ be a pair of random variables with known means, μ_1 and μ_2. Consider the following estimator for the covariance of X and Y:

$$\hat{C}_{X,Y} = \frac{1}{n}\sum_{j=1}^{n}(X_j - \mu_1)(Y_j - \mu_2).$$

(a) Find the expected value and variance of this estimator.

(b) Explain the behavior of the estimator as n becomes large.

8.9. Let $\mathbf{X} = (X, Y)$ be a pair of random variables with unknown means and covariances. Consider the following estimator for the covariance of X and Y:

$$\hat{K}_{X,Y} = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \overline{X}_n)(Y_j - \overline{Y}_n).$$

(a) Find the expected value of this estimator.

(b) Explain why the estimator approaches the estimator in Problem 8.8 for n large. *Hint:* See Eq. (8.15).

8.10. Let the sample $X_1, X_2, \ldots, X_n$ consist of iid versions of the random variable X. Consider the maximum and minimum statistics for the sample:

$$W = \min(X_1, \ldots, X_n) \quad \text{and} \quad Z = \max(X_1, \ldots, X_n).$$

(a) Show that the pdf of Z is $f_Z(x) = n[F_X(x)]^{n-1} f_X(x)$.

(b) Show that the pdf of W is $f_W(x) = n[1 - F_X(x)]^{n-1} f_X(x)$.

Section 8.2: Parameter Estimation

8.11. Show that the mean square estimation error satisfies $E[(\hat{\Theta} - \theta)^2] = \text{VAR}[\hat{\Theta}] + B(\hat{\Theta})^2$.

8.12. Let the sample X_1, X_2, X_3, X_4 consist of iid versions of a Poisson random variable X with mean $\alpha = 4$. Find the mean and variance of the following estimators for α and determine whether they are biased or unbiased.

(a) $\hat{\alpha}_1 = (X_1 + X_2)/2$.

(b) $\hat{\alpha}_2 = (X_3 + X_4)/2$.

(c) $\hat{\alpha}_3 = (X_1 + 2X_2)/3$.

(d) $\hat{\alpha}_4 = (X_1 + X_2 + X_3 + X_4)/4$.

8.13. **(a)** Let $\hat{\Theta}_1$ and $\hat{\Theta}_2$ be unbiased estimators for the parameter θ. Show that the estimator $\hat{\Theta} = p\hat{\Theta}_1 + (1 - p)\hat{\Theta}_2$ is also an unbiased estimator for θ, where $0 \le p \le 1$.

(b) Find the value of p in part a that minimizes the mean square error.

(c) Find the value of p that minimizes the mean square error if $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are the estimators in Problems 8.12a and 8.12b.

(d) Repeat part c for the estimators in Problems 8.12a and 8.12d.

(e) Let $\hat{\Theta}_1$ and $\hat{\Theta}_2$ be unbiased estimators for the first and second moments of X. Find an estimator for the variance of X. Is it biased?

8.14. The output of a communication system is $Y = \theta + N$, where θ is an input signal and N is a noise signal that is uniformly distributed in the interval $[0, 2]$. Suppose the signal is transmitted n times and that the noise terms are iid random variables.

(a) Show that the sample mean of the outputs is a biased estimator for θ.

(b) Find the mean square error of the estimator.

8.15. The number of requests at a Web server is a Poisson random variable X with mean $\alpha = 2$ requests per minute. Suppose that n 1-minute intervals are observed and that the number N_0 of intervals with zero arrivals is counted. The probability of zero arrivals is then estimated by $\hat{p}_0 = N_0/n$. To estimate the arrival rate α, $\hat{p}$ is set equal to the probability of zero arrivals in one minute:

$$\hat{p}_0 = N_0/n = P[X = 0] = \frac{\alpha^0}{0!}e^{-\alpha} = e^{-\alpha}.$$

(a) Solve the above equation for $\hat{\alpha}$ to obtain an estimator for the arrival rate.

(b) Show that $\hat{\alpha}$ is biased.

(c) Find the mean square error of $\hat{\alpha}$.

(d) Is $\hat{\alpha}$ a consistent estimator?

8.16. Generate 100 samples size 20 of the Poisson random variables in Problem 8.15.

(a) Estimate the arrival rate α using the sample mean estimator and the estimator from Problem 8.15.

(b) Compare the bias and mean square error of the two estimators.

8.17. To estimate the variance of a Bernoulli random variable X, we perform n iid trials and count the number of successes k and obtain the estimate $\hat{p} = k/n$. We then estimate the variance of X by

$$\hat{\sigma}^2 = \hat{p}(1 - \hat{p}) = \frac{k}{n}\left(1 - \frac{k}{n}\right).$$

(a) Show that $\hat{\sigma}^2$ is a biased estimator for the variance of X.

(b) Is $\hat{\sigma}^2$ a consistent estimator for the variance of X?

(c) Find a constant c, so that $c\hat{\sigma}^2$ is an unbiased estimator for the variance of X.

(d) Find the mean square errors of the estimators in parts b and c.

8.18. Let $X_1, X_2, \ldots, X_n$ be a random sample of a uniform random variable that is uniformly distributed in the interval $[0, \theta]$. Consider the following estimator for θ:

$$\hat{\Theta} = \max\{X_1, X_2, \ldots, X_n\}.$$

(a) Find the pdf of $\hat{\Theta}$ using the results of Problem 8.10.

(b) Show that $\hat{\Theta}$ is a biased estimator.

(c) Find the variance of $\hat{\Theta}$ and determine whether it is a consistent estimator.

(d) Find a constant c so that $c\hat{\Theta}$ is an unbiased estimator.

(e) Generate a random sample of 20 uniform random variables with $\theta = 5$. Compare the values provided by the two estimators in 100 separate trials.

(f) Generate 1000 samples of the uniform random variable, updating the estimator value every 50 samples. Can you discern the bias of the estimator?

8.19. Let $X_1, X_2, \ldots, X_n$ be a random sample of a Pareto random variable:

$$f_X(x) = k\frac{\theta^k}{x^{k+1}} \quad \text{for } \theta \le x$$

with $k = 2.5$. Consider the estimator for θ:

$$\hat{\Theta} = \min\{X_1, X_2, \ldots X_n\}.$$

(a) Show that $\hat{\Theta}$ is a biased estimator and find the bias.

(b) Find the mean squared error of $\hat{\Theta}$.

(c) Determine whether $\hat{\Theta}$ is a consistent estimator.

(d) Use Octave to generate 1000 samples of the Pareto random variable. Update the estimator value every 50 samples. Can you discern the bias of the estimator?

(e) Repeat part d with $k = 1.5$. What changes?

8.20. Generate 100 samples of sizes 5, 10, 20 of exponential random variables with mean 1. Compare the histograms of the estimates given by the biased and unbiased estimators for the sample variance.

8.21. Find the variance of the sample variance estimator in Example 8.8. *Hint:* Assume $m = 0$.

8.22. Generate 100 samples of size 20 of pairs of zero-mean, unit-variance jointly Gaussian random variables with correlation coefficient $\rho = 0.50$. Compare the histograms of the estimates given by the estimators for the sample covariance in Problems 8.8 and 8.9.

8.23. Repeat the scenario in Problem 8.22 for the following estimator for the correlation coefficient between two random variables X and Y:

$$\hat{\rho}_{X,Y} = \frac{\sum_{j=1}^{n}(X_j - \bar{X}_n)(Y_j - \bar{Y}_n)}{\sqrt{\sum_{j=1}^{n}(X_j - \bar{X}_n)^2 \sum_{j=1}^{n}(Y_j - \bar{Y}_n)^2}}.$$

Section 8.3: Maximum Likelihood Estimation

8.24. Let X be an exponential random variable with mean $1/\lambda$.

(a) Find the maximum likelihood estimator $\hat{\Theta}_{ML}$ for $\theta = 1/\lambda$.

(b) Find the maximum likelihood estimator $\hat{\Theta}_{ML}$ for $\theta = \lambda$.

(c) Find the pdfs of the estimators in part a.

(d) Is the estimator in part a unbiased and consistent?

(e) Repeat 20 trials of the following experiment: Generate a sample of 16 observations of the exponential random variable with $\lambda = 1/2$ and find the values given by the estimators in parts a and b. Show a histogram of the values produced by the estimators.

8.25. Let $X = \theta + N$ be the output of a noisy channel where the input is the parameter θ and N is a zero-mean, unit-variance Gaussian random variable. Suppose that the output is measured n times to obtain the random sample $X_i = \theta + N_i$ for $i = 1, \ldots, n$.

(a) Find the maximum likelihood estimator $\hat{\Theta}_{ML}$ for θ.

(b) Find the pdf of $\hat{\Theta}_{ML}$.

(c) Determine whether $\hat{\Theta}_{ML}$ is unbiased and consistent.

8.26. Show that the maximum likelihood estimator for a uniform random variable that is distributed in the interval $[0, \theta]$ is $\hat{\Theta} = \max\{X_1, X_2, \ldots, X_n\}$. *Hint:* You will need to show that the maximum occurs at an endpoint of the interval of parameter values.

8.27. Let X be a Pareto random variable with parameters α and x_m.

(a) Find the maximum likelihood estimator for α assuming x_m is known.

(b) Show that the maximum likelihood estimators for α and x_m are:

$$\hat{\alpha}_{ML} = n\left[\sum_{j=1}^{n} \log\left(\frac{X_j}{\hat{x}_{m,ML}}\right)\right]^{-1} \quad \text{and} \quad \hat{x}_{m,ML} = \min(X_1, X_2, \ldots, X_n).$$

(c) Discuss the behavior of the estimators in parts a and b as n becomes large and determine whether they are consistent.

(d) Repeat five trials of the following experiment: Generate a sample of 100 observations of the Pareto random variable with $\alpha = 2.5$ and $x_m = 1$ and obtain the values given by the estimators in part b. Repeat for $\alpha = 1.5$ and $x_m = 1$, and $\alpha = 0.5$ and $x_m = 1$.

8.28. (a) Show that the maximum likelihood estimator for the parameter $\theta = \alpha^2$ of the Rayleigh random variable is

$$\hat{\alpha}^2_{ML} = \frac{1}{2n}\sum_{j=1}^{n} X_j^2.$$

(b) Is the estimator is unbiased?

(c) Repeat 50 trials of the following experiment: Generate a sample of 16 observations of the Rayleigh random variable with $\alpha = 2$ and find the values given by the estimator in part a. Show a histogram of the values produced by the estimator.

8.29. (a) Show that the maximum likelihood estimator for $\theta = a$ of the beta random variable with $b = 1$ is

$$\hat{a}_{ML} = \left[\frac{1}{n}\sum_{j=1}^{n} \log X_j\right]^{-1}.$$

(b) Generate a sample of 100 observations of the beta random variable with $b = 1$ and $a = 0.5$ to obtain the estimate for a. Repeat for $a = 1$, $a = 2$, and $a = 3$.

8.30. Let X be a Weibull random variable with parameters α and β (see Eq. 4.102).

(a) Assuming that β is known, show that the maximum likelihood estimator for $\theta = \alpha$ is:

$$\hat{\alpha}_{ML} = \left[\frac{1}{n}\sum_{j=1}^{n} X_j^{\beta}\right]^{-1}.$$

5. S. M. Ross, *Stochastic Processes*, Wiley, New York, 1983.
6. D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes*, Chapman and Hall, London, 1972.
7. R. G. Gallager, *Discrete Stochastic Processes*, Kluwer Academic Press, Boston, 1996.
8. J. Kohlas, *Stochastic Methods of Operations Research*, Cambridge University Press, London, 1982.
9. H. Anton, *Elementary Linear Algebra*, Wiley, New York, 1981.
10. A. M. Langville and C. D. Meyer, *Google's PageRank and Beyond*, Princeton University Press, Princeton, NJ, 2006.

PROBLEMS

Section 11.1: Markov Processes

11.1. Let M_n denote the sequence of sample means from an iid random process X_n:

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

(a) Is M_n a Markov process?

(b) If the answer to part a is yes, find the following state transition pdf:

$$f_{M_n}(x \mid M_{n-1} = y).$$

11.2. An urn initially contains five black balls and five white balls. The following experiment is repeated indefinitely: A ball is drawn from the urn; if the ball is white, it is put back in the urn, otherwise it is left out. Let X_n be the number of black balls remaining in the urn after n draws from the urn.

(a) Is X_n a Markov process? If so, find the appropriate transition probabilities and the corresponding trellis diagram.

(b) Do the transition probabilities depend on n?

(c) Repeat part a if the urn initially has K black balls and K white balls.

11.3. An urn initially contains two black balls and two white balls. The following experiment is repeated indefinitely: A ball is drawn from the urn; with probability a, the color of the ball is changed to the other color and is then put back in the urn, otherwise it is put back without change. Let X_n be the number of black balls in the urn after n draws from the urn.

(a) Is X_n a Markov process? If so, find the appropriate transition probabilities.

(b) Do the transition probabilities depend on n?

(c) Repeat part a if $a = 1$. What changes?

(d) Repeat parts a and c if the urn contains K black balls and K white balls.

11.4. Michael and Marisa initially have four pens each. Out of the total of eight pens, half are good and half are dry. The following experiment is repeated indefinitely: Michael and Marisa exchange a randomly selected pen from their set. Let X_n be the number of good pens in Marisa's set after n draws.

(a) Is X_n a Markov process? If so, find the appropriate transition probabilities.

(b) Do the transition probabilities depend on n?

(c) Repeat part a if Michael and Marisa initially have a total of K good pens and K dry pens.

11.5. Does a Markov process have independent increments? *Hint:* Use the process in Problem 11.2 to support your answer.

11.6. Let X_n be the Bernoulli iid process, and let Y_n be given by

$$Y_n = X_n + X_{n-1}.$$

It was shown in Example 11.2 that Y_n is not a Markov process. Consider the vector process defined by $\mathbf{Z}_n = (X_n, X_{n-1})$.

(a) Show that $\mathbf{Z}_n$ is a Markov process.

(b) Find the state transition diagram for $\mathbf{Z}_n$.

11.7. (a) Show that the following autoregressive process is a Markov process:

$$Y_n = rY_{n-1} + X_n \qquad Y_0 = 0,$$

where X_n is an iid process.

(b) Find the transition pdf if X_n is an iid Gaussian sequence.

11.8. The amount of water in an aquifer at year end is a random variable X_n. The amount of water drawn from the aquifer in a year is a random variable D_n and the amount restored by rainfall is W_n.

(a) Find a set of equations to describe the total amount of water X_n in the aquifer over time.

(b) Under what conditions is X_n a Markov process?

Section 11.2: Discrete-Time Markov Chains

11.9. Let X_n be an iid integer-valued random process. Show that X_n is a Markov process and give its one-step transition probability matrix.

11.10. An information source generates iid bits for X_n for which $P[0] = a = 1 - P[1]$.

(a) Suppose that X_n is transmitted over a binary symmetric channel with error probability ε. Find the probabilities of the outputs of the channel.

(b) Suppose that X_n is transmitted over K consecutive identical and independent binary symmetric channels. Does the sequence of channel outputs form a Markov chain?

(c) Find the K-step transition probabilities that relate the input bits from the source to the outputs of the Kth channel.

(d) What are the probabilities of the outputs of the Kth channel as $K \to \infty$?

11.11. Each time unit a data multiplexer receives a packet with probability a, and/or transmits a packet from its buffer with probability b. Assume that the multiplexer can hold at most N packets. Let X_n be the number of packets in the multiplexer at time n.

(a) Show that the system can be modeled by a Markov chain.

(b) Find the transition probability matrix P.

(c) Find the stationary pmf.

11.12. Let X_n be the Markov chain defined for the urn experiment in Problem 11.2.

(a) Find the one-step transition probability matrix P for X_n.

(b) Find the two-step transition probability matrix P^2 by matrix multiplication. Check your answer by computing $p_{54}(2)$ and comparing it to the corresponding entry in P^2.

(c) What happens to X_n as n approaches infinity? Use your answer to guess the limit of P^n as $n \to \infty$.