Electromagnetism (PHYS 201)

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Lectures delivered at the University of Liverpool, 2021-22

December 15, 2021

Module coordinator

Module organizer, lecture delivery and workshops:

Professor Constantinos (Costas) Andreopoulos
Chair of Experimental Particle Physics

http://costas.andreopoulos.eu

Contact information:

- Office: Oliver Lodge 316
- E-Mail: constantinos.andreopoulos@cern.ch
- Tel: 01517-943201 (Liverpool), 01235-445091 (Rutherford Appleton Lab)

Office hours

My **regular office hours** for face-to-face contact would be on:

- Thursday, 10:15 - 13:00, or
- by appointment.

If you prefer the flexibility (and safety) of online meetings, you are welcome to book an appointment for **1-on-1 Zoom meeting**:

- Request an appointment visiting this URL:
  `doodle.com/mm/costasandreopoulos/book-a-time`
- At the agreed appointment time, connect to this Zoom channel:
  `liverpool-ac-uk.zoom.us/j/93479179469?pwd=elBqaWM0anJhUmxEa29vTXpQWGNkdz09`

Step-by-step instructions can be found in the PHYS201 Canvas page.

Other module staff

Staff contributing to face-to-face workshops:

- Dr. Marco Roda, Marco.Roda @nospam liverpool.ac.uk
- Prof. Christos Touramanis,touraman @nospam liverpool.ac.uk
- Prof. Joost Vossebeld, vossebel @nospam liverpool.ac.uk

Demonstrators (with workshop script marking responsibilities):

- Mr. Levon Abelian
- Ms. Eloisa Arena
- Mr. Thomas Beesley
- Mr. Alessandro Biondini
- Ms. Holly Tann

Blended delivery

The following might change in response to new COVID restrictions.
Please watch the PHYS201 Canvas module for updates.

PHYS201:

- **is taught in 12 weeks**
- **is organised for delivery in 12 x 2-hr lectures**
  - lectures are given online on Zoom
  - single 2-hr lecture per week (Tuesdays at 10:00 am)
  - lectures start in week 1
- **provides 11 x 1-hour workshops ("feedback sessions")**
  - workshops allow face-to-face interaction in *small* groups
  - held at various times and places (see your timetable)
  - for each group, a single 1-hr workshop per week
  - workshops start in week 2
    - workshops offset by $+1$ week wrt the corresponding lecture
    - plenty of time for new concepts to settle, before attacking relevant problems

Please find a detailed PHYS201 weekly schedule on Canvas.

More on workshops

Out of the 11 workshops, the last one will be used for a revision.

On the first 10 workshops (starting on week 2):

- Problem sets published at the start of each week (9 am on Monday)
- You have 1 week to work on your problems, and you should submit your solutions by 10 am on the following Monday.

Each problem set will contain:

- A small number of problems for **summative assessment**
  - Your mark on these problems contributes towards your final mark
- A larger number of problems for **formative assessment**
  - Your mark on these problems **does not** contribute on your final mark.
  - Try as many as you can - the more you do, the more feedback you receive.
  - If pressed for time, you can choose problems on topics you stuggle the most.

Model solutions for the entire problem set will be provided.

- Solutions will be published at 10 am on Thursday, **3 days after the nominal submission deadline**.
- Summative coursework can not be accepted after that point.

Learning resources

- Very **detailed lectures notes**, including several worked problems
  - As a baseline, can be your only reading (not recommended)
  - Full set already uploaded on Canvas ✓
- **Recorded lectures from 2020-21** (material is unchanged)
  - Full set already uploaded on Canvas ✓
  - **Optional**, but if you want to work at faster pace than scheduled you can!
- **Recordings of 2021-22 lectures**
  - Will be uploaded to Canvas shortly after the corresponding lecture delivery
- **Extensive bibliography** and links to Liverpool library
  - Already uploaded on Canvas ✓
- **Additional reading and (external) bite-sized videos** (more worked examples, experiments and demonstrations, relevant research etc)
  - All links already available on Canvas ✓
  - Clearly **optional**, for those of you who want to go at greater depth
- **Detailed model solutions to workshop problems**
  - Will be uploaded to Canvas soon after your submission (allowing for late ones)

Feedback policy and opportunities

Receiving feedback requires your active participation. In red, I highlight what you need to do!

The would be numerous opportunities to receive feedback:

- Verbal feedback in face-to-face workshops and online lectures
  - Please try and speak up! In online Zoom sessions, raise your hand to speak (or just interrupt me), or use the Zoom chat.
- Written feedback on your workshop scripts
  - Provided by demonstrators using model solutions, within $\sim$1 week from submission.
  - I will be working with demonstrators to ensure the quality of their feedback.
  - You can't receive feedback if you don't submit. Don't miss the deadlines!
  - The more of the formative assessment you complete, the more feedback you get.
- Detailed model solutions and a marking scheme for each workshop
  - The ultimate workshop feedback!
  - Study the published solutions, contrast with your own, and reflect.
- Feedback during online sessions
  - Common issues / misconceptions seen in workshops, will be addressed in lectures
- More personalised feedback on 1-on-1 meetings (face to face, or on Zoom)
  - On demand! Please come to my office or request an appointment.
- Canvas discussion boards, e-mail.
  - Participate!

Assessment

- **2-hour open book examination on January 2022** (Weight: 70%)
- **10 summative assessments** (a small and clearly designated subset of problems) **in the context of weekly workshops** (Weight: 30%)
  - Each of the 10 summative assessments contributes equally.
  - Workshop scripts will be marked by postgraduate demonstrators.
  - To ensure uniformity:
    - A detailed marking scheme will be available to each demonstrator.
    - I will work closely with demonstrators to resolve marking issues.
    - I will sample and remark a random 10% of all scripts.
  - Problems sets will be published at 9 am on Monday, in weeks 2-11, with a deadline a week later (10 am on next Monday).

Late penalties, deadline extensions, penalties for copying

Late penalties should be applied as specified in the Code of Practice on Assessment (see Section 6.2 of the main document). The Code of Practice on Assessment does not provide for students to request extensions to coursework deadlines, unless such extensions are allowed under a student's Learning Support Plan. Students submitting coursework late because of unforeseen medical or other extenuating circumstances may instead apply for exemption from late penalties.
Such exemption cannot be granted by individual Module Co-ordinators, but should be decided by Year Co-ordinators, who will work within agreed guidelines to ensure consistency.
Suspected cases of academic misconduct will be handled as described in Appendix L, in the Code of Practice on Assessment.
`www.liverpool.ac.uk/media/livacuk/tqsd/code-of-practice-on-assessment/code_of_practice_on_assessment.pdf`

What is examinable?

- **Everything that is in the PHYS201 slides <u>is examinable</u>**
  - Unless I explicitly state otherwise
- **Everything I discuss during the lectures <u>is examinable</u>** (whether it is in the PHYS201 slides or not)
  - Unless I explicitly state otherwise
- Most of what I will discuss is in the slides, but not everything!
  - Do not skip the lectures!

Aims of the course

- To introduce the fundamental concepts and principles of electrostatics, magnetostatics, electromagnetism and Maxwell's equations, and electromagnetic waves.
- To introduce differential vector analysis in the context of electromagnetism.
- To introduce circuit principles and analysis (EMF, Ohm's law, Kirchhoff's rules, RC and RLC circuits)
- To introduce the formulation of Maxwell's equations in the presence of dielectric and magnetic materials.
- To develop the ability of students to apply Maxwell's equations to simple problems involving dielectric and magnetic materials.
- To develop the concepts of field theories in Physics using electromagnetism as an example.
- To introduce light as an electromagnetic wave.

Source: PHYS 201 module page in ORBIT.

Syllabus

- Electric charge, Coulomb's law, Charge density
- Electric field, Principle of Superposition
- Electric flux, Gauss' law (integral form)
- Mutual potential energy of point charges, electric potential
- Calculating the field from the potential (gradient)
- Circulation, charges on conductors
- Gauss' law in differential form (divergence)
- Circulation law in differential form (curl)
- Poisson's and Laplace's laws and solutions
- Electric dipole
- Electrostatics and conductors, method of images
- Gauss' and Stokes' theorems
- EMF, potential difference, electric current, current density
- Resistance, Ohm's law
- Circuits, Kirkhhoff's rules

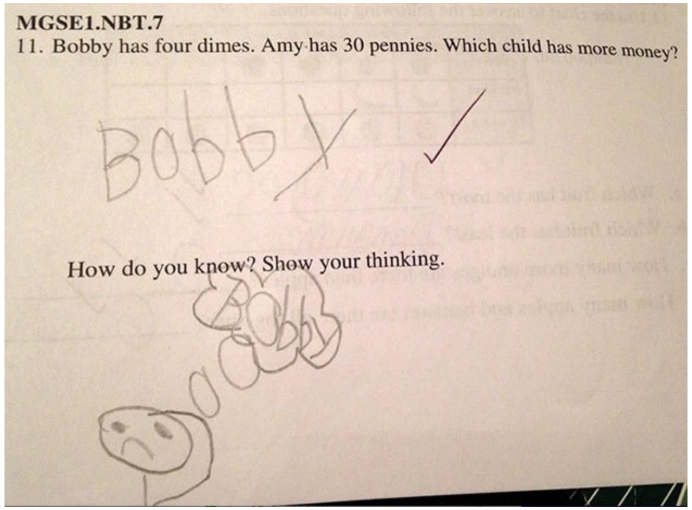
Syllabus cont'd

- Capacitance, calculation of capacitance for simple cases, RC circuits
- Dielectrics, polarization, electric displacement field
- Capacitance in the presence of dielectrics, force on a dielectric
- Magnetism, magnetic field, Biot-Savart law
- Lorentz force, force between currents
- Charged particle motion in magnetic field, velocity filter
- Magnetic dipole field, Ampere's law in integral and differential forms
- Maxwell's equations in vacuum for steady conditions
- Vector potential
- Magnetic materials, magnetization, magnetic field strength
- Maxwell's equations in the presence of materials for steady conditions
- Motion of conductors inside magnetic fields, Faraday's and Lenz's laws
- Time-varying fields, Maxwell's equations for the most general case
- Derivation of electromagnetic waves from Maxwell's equations, speed of light
- LCR circuits

Source: PHYS 201 module page in ORBIT.

Our approach will be calculus based

- We will explore several **interesting physics concepts**.
- Our approach will be calculus-based.
- The electromagnetic phenomena are described by a set of **very beautiful mathematical equations (Maxwell equations)**.



Mathematical dexterity necessary to

- sharpen your instinct
- help you understand the connections between different physics concepts
- tackle practical problems

Dust off your maths textbooks! Its importance can not be overstated.

Recommended textbooks

Will follow mainly:

- D.J. Griffiths, 'Introduction to Electrodynamics', Prentice Hall, 1999
  Paper copies available in the library.

You may also find useful:

- The Feynman Lectures in Physics
  Available online: *http://www.feynmanlectures.caltech.edu*
- L.S. Grant and W.R. Phillips, 'Electromagnetism', Wiley, 2013
  An ebook version available in the library.
- W.J. Duffin, 'Electricity and Magnetism', McGraw-Hill, 1990
  Paper copies available in the library.
- Daniel Fleisch, 'A Student's Guide to Maxwell's Equations', Cambridge, 2008
  Paper copies and an ebook version available in the library.
- Daniel Fleisch, 'A Student's Guide to Vectors and Tensors', Cambridge, 2011
  Paper copies and an ebook version available in the library.
- George Arfken, 'Mathematical methods for physicists', Oxford, 2012
  An ebook version available in the library.

Slides / Handouts

There are 4 types of slides: Blue, Red, Orange, and Green.

The electrostatic potential energy is stored in the field

We can now prove a statement I made earlier: That **the electrostatic potential energy is stored in the electric field**.

The potential energy stored in a system of N charges can be written as:

$$U = \frac{1}{2}\sum_{i,j=1;i\neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0|\vec{r}_{ij}|} = \frac{1}{2}\sum_i q_i \sum_{j=1;j\neq i}^{N} \frac{q_j}{4\pi\epsilon_0|\vec{r}_{ij}|} = \frac{1}{2}\sum_i^N q_i V(\vec{r}_i)$$

where

$$V(\vec{r}_i) = \sum_{j=1;j\neq i}^{N} \frac{q_j}{4\pi\epsilon_0|\vec{r}_{ij}|}$$

is the potential at $\vec{r}_i$ due to all charges other than q_i.

The above result can be adapted for the continuous case too, using the (by now) familiar substitutions:

$$U = \frac{1}{2}\sum_i^N q_i V(\vec{r}_i) \rightarrow \frac{1}{2}\int_V \rho(\vec{r})V(\vec{r})d\tau$$

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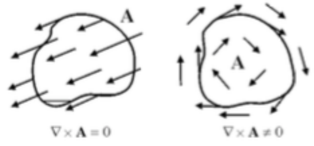
Reminder: Curl

We skipped that from our brief reminder in lecture 1. Time to remember what it is.

The curl ($\vec{\nabla} \times \vec{A}$) of a vector field $\vec{A} = \left(A_x, A_y, A_z\right)$ is defined as:

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$



The curl of a vector field, evaluated at a specific space point, tells us how much the field curls about that point.

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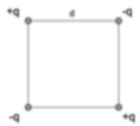
Worked example: 4 charges at corners of a square

Question



Four charges each of magnitude q are located at the four corners of a square of side d such that like charges occupy the corners across the diagonals.
Calculate the work done in assembling these charges.

Work done is $W = U_{12} + U_{23} + U_{34} + U_{41} + U_{13} + U_{24}$ where $U_{ij} = \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$

Numbering charges clockwisely from top left one, so that charges 1 and 3 are positive and 2 and 4 negative:

$$W = \frac{q^2}{4\pi\epsilon_0}\left\{-\frac{1}{a} - \frac{1}{a} - \frac{1}{a} - \frac{1}{a} + \frac{1}{\sqrt{2}a} + \frac{1}{\sqrt{2}a}\right\} = -\frac{q^2}{4\pi\epsilon_0 a}\left(4 - \sqrt{2}\right)$$

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PHYS201 scientific programming task for Lecture 1

Write a Python program that:

- **Can read an arbitrary distribution of N discrete charges (in 2-D),**
  e.g. by accepting as input a text file with N rows, where the i^{th} row contains the coordinates x_i, y_i in m, and the charge q_i in C.
- **Allows you to visualise the input charge distribution,**
  e.g using appropriately-positioned circles whose color or size represents amount of charge.
- **Allows you to visualise the electric field lines** in the viscinity of the charge distrubution.

Test you program, reproducing some of the simpler field maps shown before.

Document your program and submit it to the PHYS201 GitHub repository!

Upload images of the most interesting field map you are able to generate!

You can use any external Python module that helps you do the job!

- `matplotlib.pyplot` seems like a suitable tool allowing you to draw vector fields. Google it!

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"Blue slides"

The main body of each PHYS201 lecture.

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where

$$V(\vec{r_i}) = \sum_{j=1;j\neq i}^{N} \frac{q_j}{4\pi\epsilon_0 |\vec{r_{ij}}|}$$

is the potential at $\vec{r_i}$ due to all charges other than q_i.

The above result can be adapted for the continuous case too, using the (by now) familiar substitutions:

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Please check Canvas regularly for corrections and updates.

"Red slides"

Things you should already know / **reminders**.

This material is here for easy reference! Typically, I will skip most of these slides during the lecture.

Please study the reminders in advance of each lecture.

Reminder: Curl

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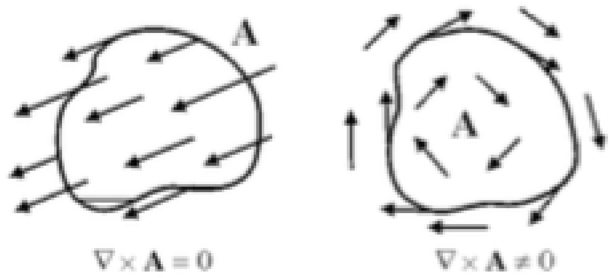
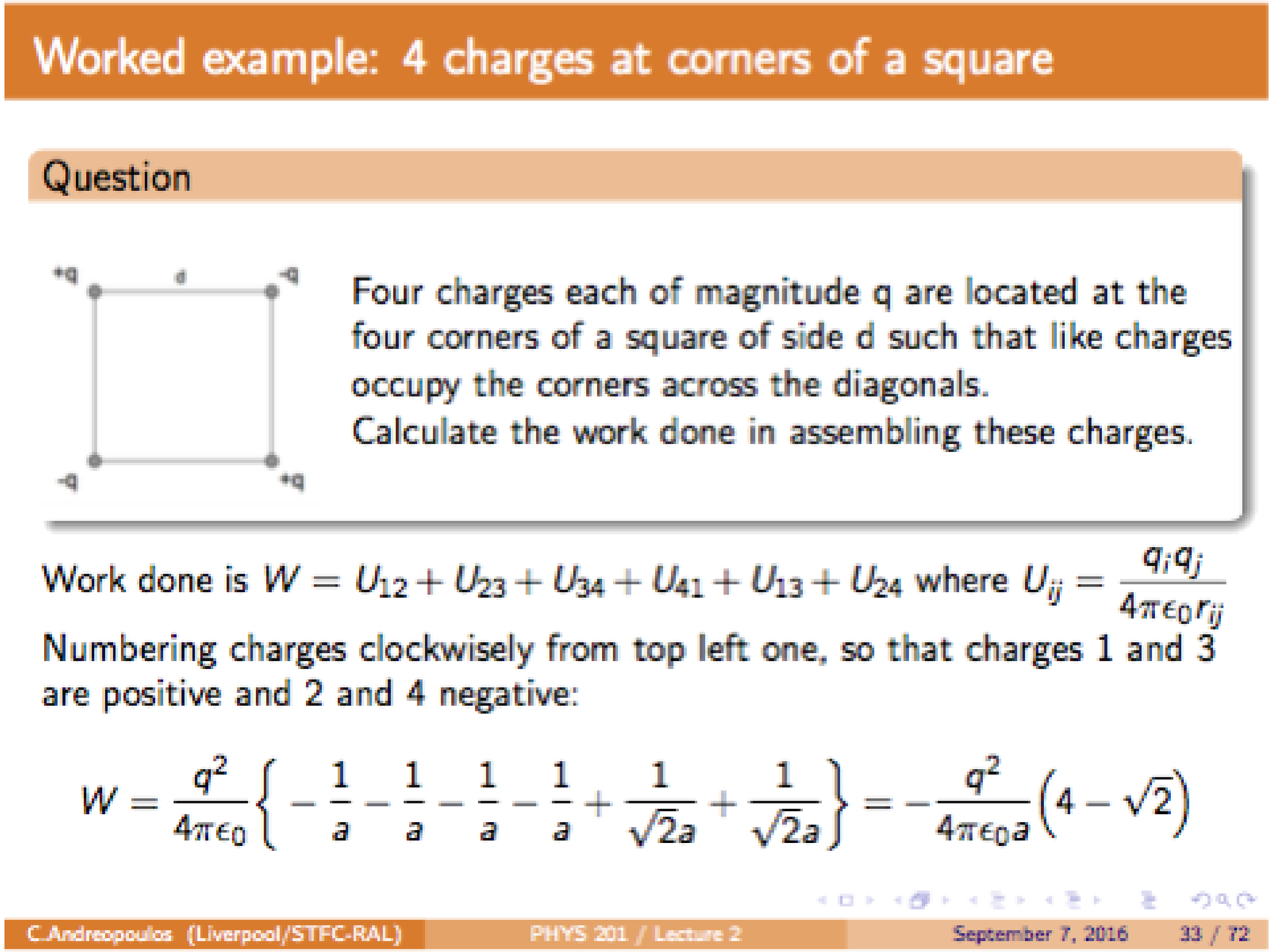


The curl of a vector field, evaluated at a specific space point, tells us how much the field curls about that point.

Please check Canvas regularly for corrections and updates.

Questions and worked examples.

Will study as many as we have time for. More may be added in the digital version to address specific class needs.



Worked example: 4 charges at corners of a square

Question

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$$W = \frac{q^2}{4\pi\epsilon_0}\left\{-\frac{1}{a} - \frac{1}{a} - \frac{1}{a} - \frac{1}{a} + \frac{1}{\sqrt{2}a} + \frac{1}{\sqrt{2}a}\right\} = -\frac{q^2}{4\pi\epsilon_0 a}\left(4 - \sqrt{2}\right)$$

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Please check Canvas regularly for corrections and updates.

"Green slides"

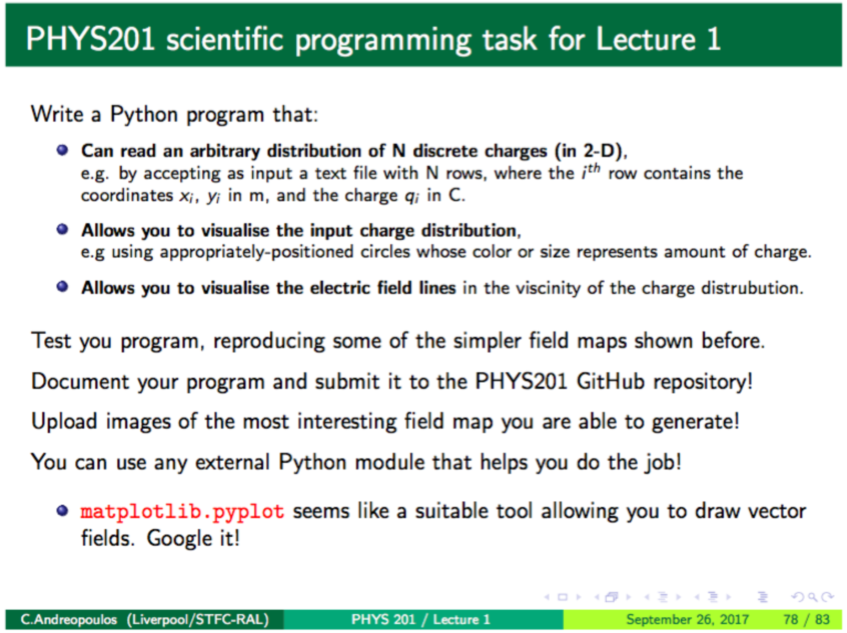
Optional tasks (1/lecture) for which the analytical solution is too complex

Solve numerically, using C++/ROOT, Python or any other language (*)

Will be adding tasks, along with helpful hints, throughout the semester.

No marks awarded (optional tasks) but:

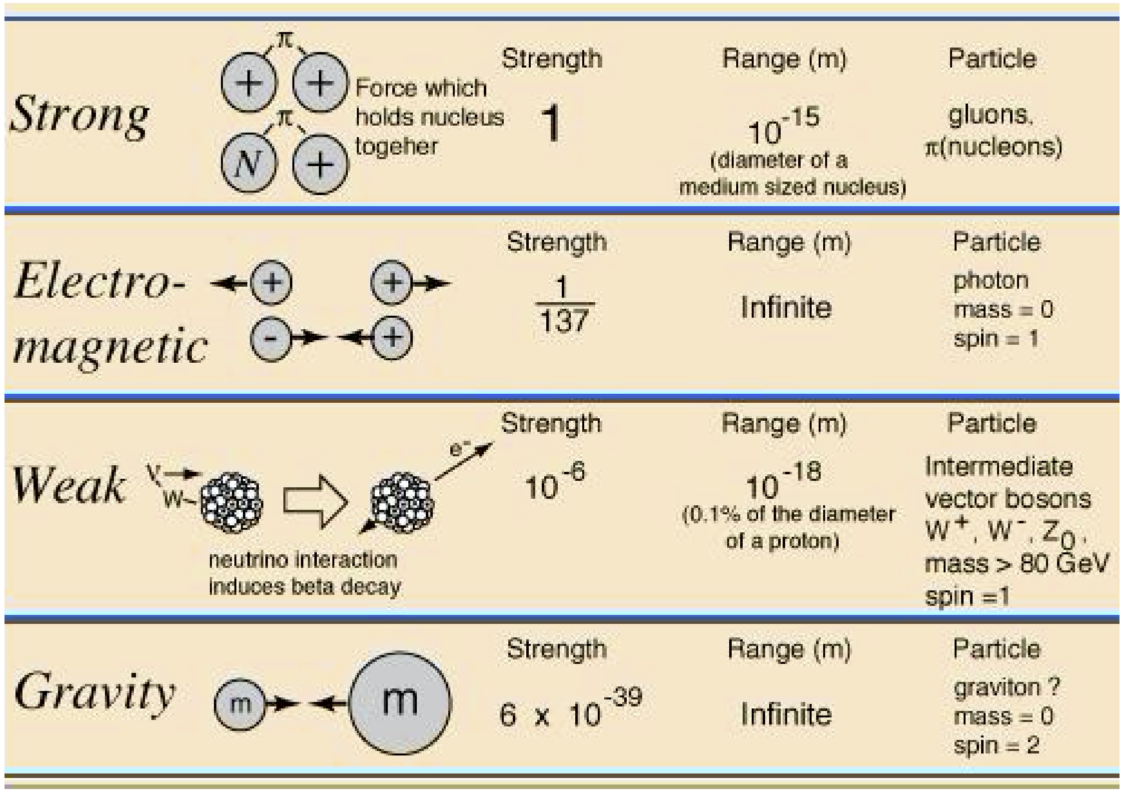
- Improve your understanding of concepts!
- Gain experience in scientific computing!



Why is Electromagnetism one of your core modules?

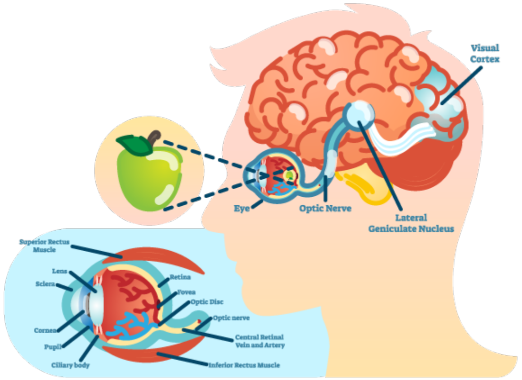
Electromagnetism is one of the fundamental forces

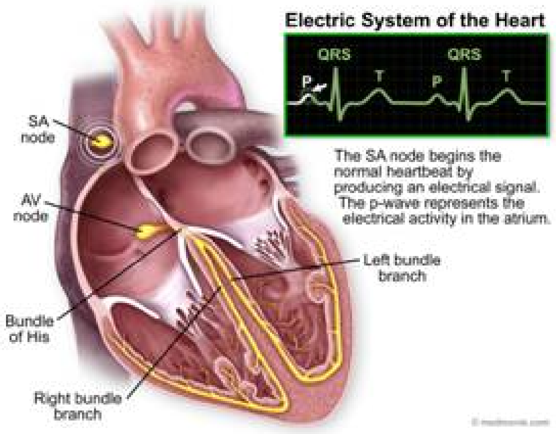
Main goal of physics: Study of the **fundamental constituents of matter and of the forces between them**.



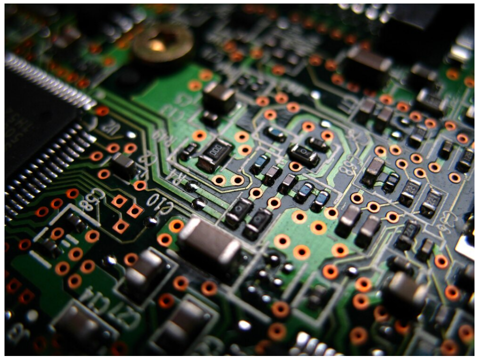
- Strong nuclear: keeps the atomic nucleus together
- Electromagnetism: the subject of this course.
- Weak nuclear: responsible for nuclear β decays
- Gravity: Pins you down on the Earth and rotates you around the Sun

Electromagnetism



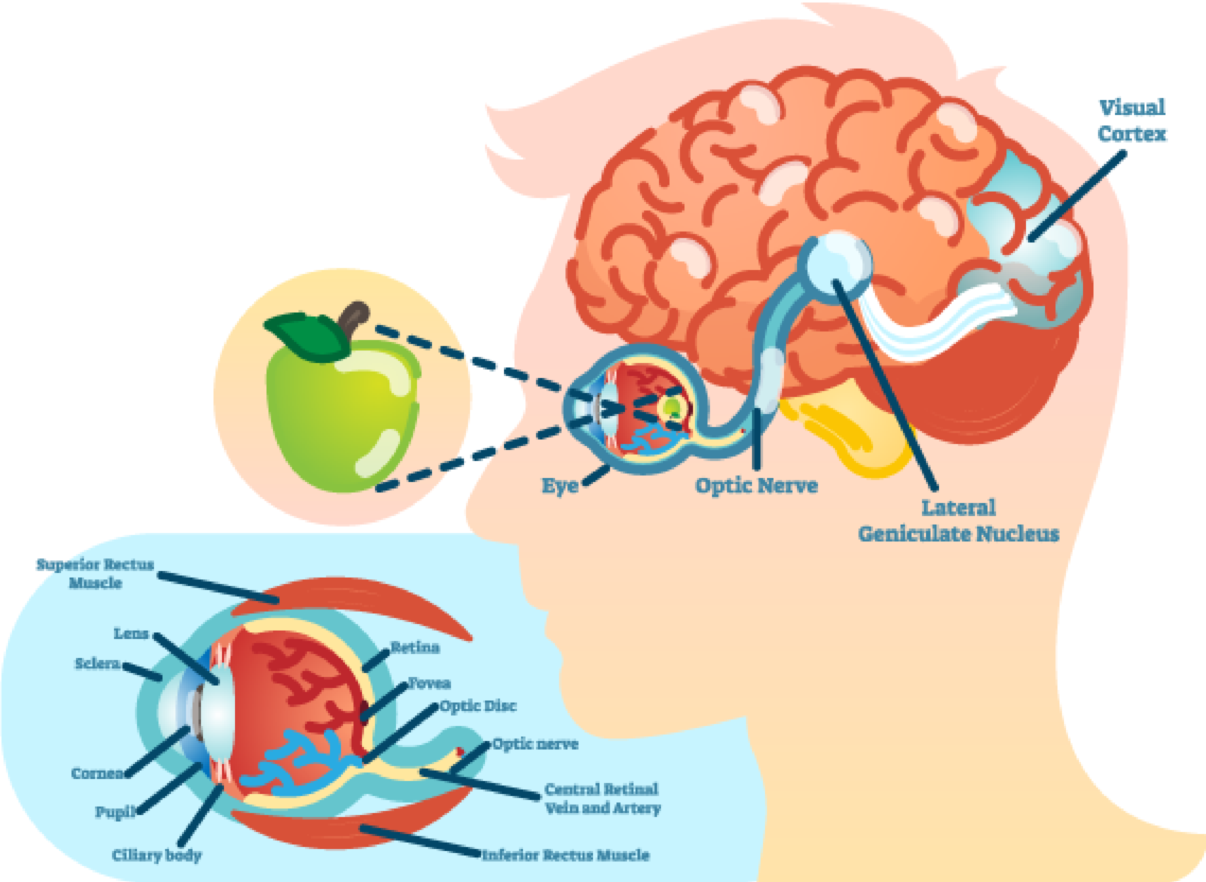


Electromagnetism **underpins life, shapes our environment and our perception of it, and is at the heart of technological innovations.**

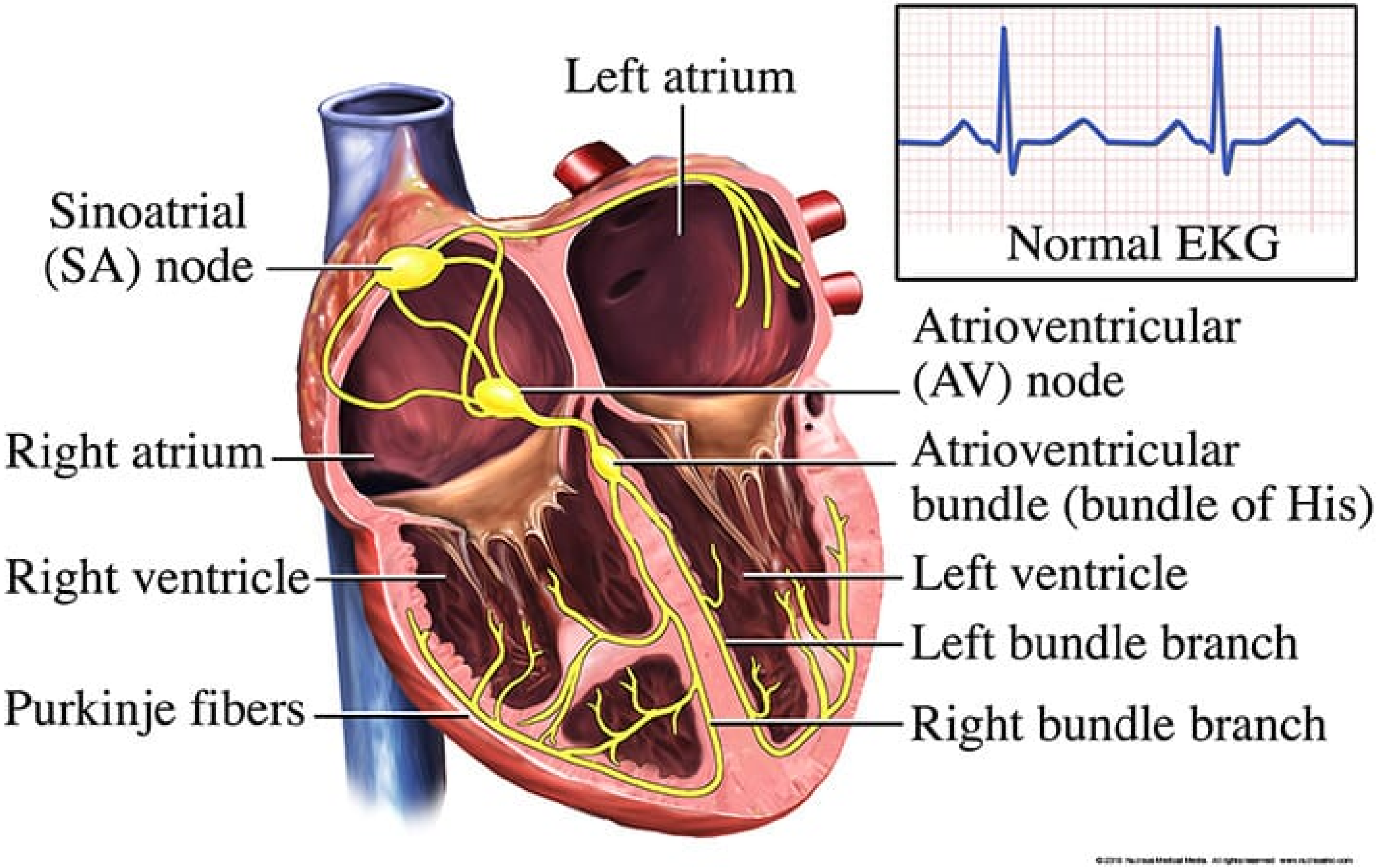


Electromagnetism underpins life / chemistry

You see me because **light (an electromagnetic wave)** was scattered off me and interacted with photo-sensitive tissue in the retina of your eyes. This tissue generates **electric signals** that travel to your brain via the optic nerve. In the visual cortex, through the electrical activity of many layers of neurons, your brain processes the input image.



The contraction of the muscles in your heart caused by **electrical impulses** from the sinoatrial node.

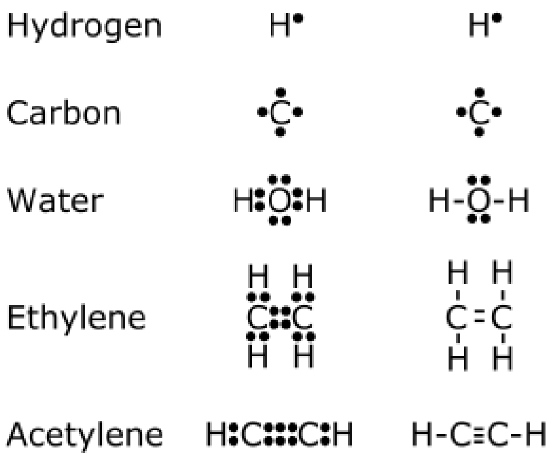


Shapes our environment and our perception of it

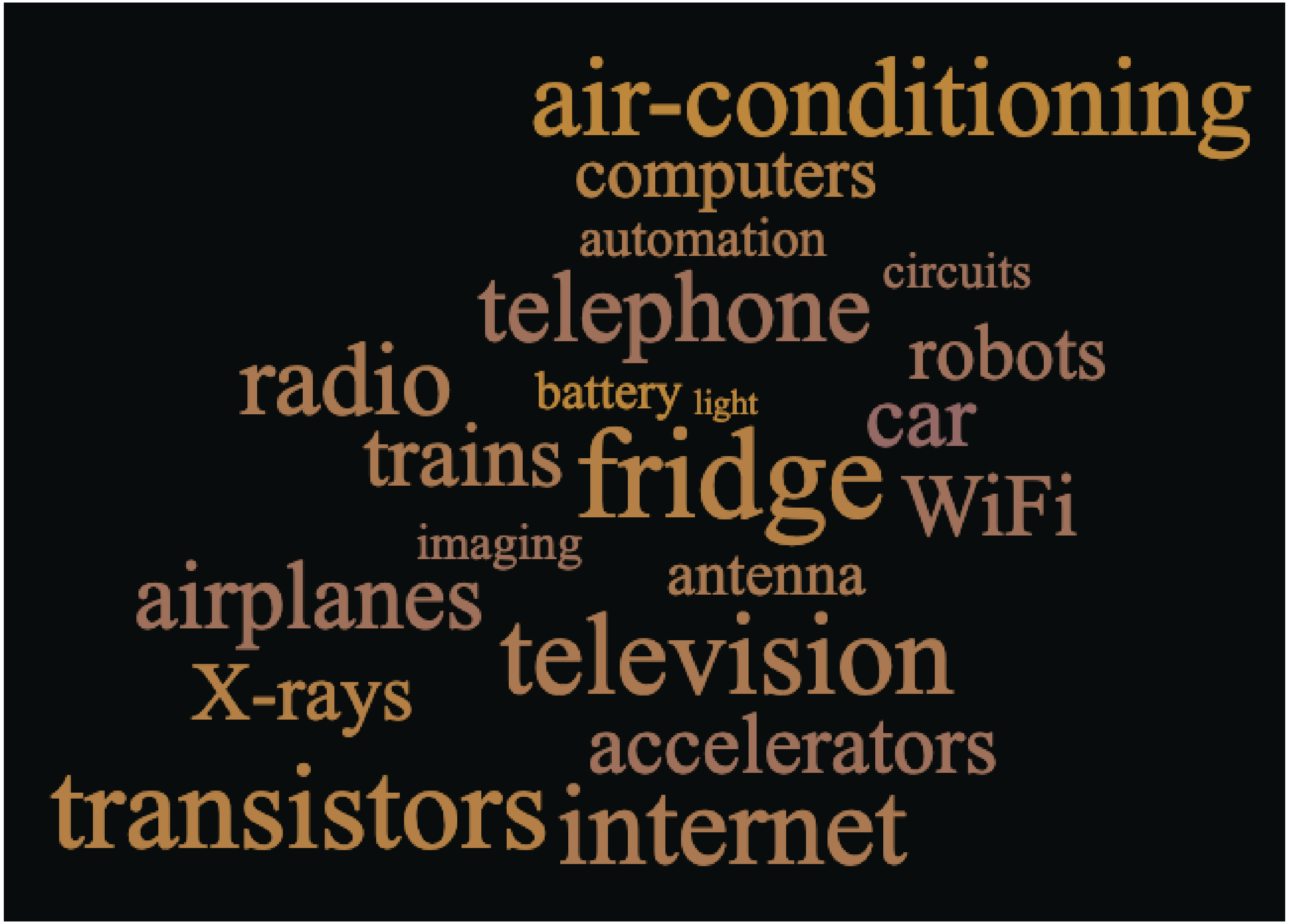
The feel of touching your desk, is the the **electromagnetic repulsion between the electron clouds** in your hand and in your desk.

Vision/colour.

Chemical bonds arise from electric attraction between oppositely charged ions, or through the sharing of electric charges (electrons).

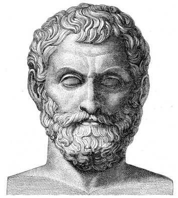


Is at the heart of technological innovations



Electric and magnetic forces known from ancient times

Thales (∼625-545 BC), a Greek Philosopher from Miletus (Asia Minor) has recorded observations of the properties of **amber** and **lodestones**.



- **Amber**: a yellow-orange-brown fossilised tree resin
  - Amber is known as *'electron'* in Greek, the name given to first charged particle discovered and the origin of the term *electricity*.
- When rubbed with fur, amber could attract small bodies like small pieces of straw.

- **Lodestone**: naturally magnetised piece of the mineral magnetite
  - Lodestones were first found in the Greek colony of *Magnesia*, in what is now Asia minor, hence the term *magnetism*.
- Lodestone are attracted to iron and other lodestones.

No further advance till the early modern period

Electric and magnetic forces had an impact on early thinking.

- Electricity and magnetism perplexed Thales: How can an (inanimate) object attract other objects?
- This led him to believe that amber and lodestone may be alive?
  - And then, perhaps, everything is living.

There were some early applications

- Magnetic compass (Chinese, 12th century AD; others)

But no further advance in understanding till the early modern period.

- William Gilbert (1540-1603)

Evolution of electromagnetic theory and applications

- The subject as we know it was **developed in less than a century**
  - $\sim$1785: Coulomb publishes his law
  - $\sim$1864: Maxwell publishes his famous theory
    - unity of electric and magnetic phenomena and understanding of light
- Several applications followed the development of the theory
  - 1880: first wired-up house
  - 1891: electric fan
  - 1901: vacuum cleaner
  - 1909: washing machine and iron
  - 1918: refrigerator and dishwasher
  - ...

Try to imagine your life without electricity!

Electromagnetism in modern physics

It is a wonderful example of a **covariant theory**

- The **Special theory of relativity** had its origins in Classical Electrodynamics

Classical Electrodynamics coupled with Quantum Mechanics gives rise to **Quantum Electrodynamics (QED)**

- Experimentally tested to 1 in 10^{11} parts
    - 1 in 10^{11}: Like predicting the distance from New York to Los Angeles to about half the thickness of usual printer paper...
- **One of the most successful theories** ever built, and the most precisely tested one in the history of science.
- **A prototype for other Quantum Field Theories**

PHYS 201 / Lecture 1
Electric charge; Coulomb's law; Superposition principle; Electric field

Professor Costas Andreopoulos[1,2], *FHEA*

[1]University of Liverpool, Department of Physics

[2]U.K. Research & Innovation (UKRI), Science & Technology Facilities Council, Rutherford Appleton Laboratory, Particle Physics Department

Lectures delivered at the University of Liverpool, 2021-22

December 15, 2021

Plan for Lecture 1

- **Electric charge**
  - The source of electric phenomena
- **Coulomb's law**
  - Describes the force between two point charges
- **Superposition principle**
  - Allows the calculation of the force on a charge from an array of other charges
- **Continuous distributions of charge**
  - Make the leap from discrete to continuous charge distributions described by a charge density
  - Reformulate Coulomb's law for continuous charge distributions
- **Electric field**
  - This new concept will provide a more fundamental way to think about electric forces in terms of a field that permeates space. We will think of ways to visualise it.

Electric charge

The electric charge is an **intrinsic property of matter**

- It comes in **two varieties**
  - There are positive electric charges and negative electric charges
  - The electric charge is an algebraic quantity
    - A positive charge and an equal negative charge amounts to no charge.
- It is **quantised**
  - Free charges are multiples of the electron charge
- It is **conserved**
  - The charge is conserved globally
    - i.e. the total charge of the universe is fixed for all time
  - But a stronger statement can also be made: It is conserved locally
    - i.e. charge can not disappear from here and re-appear in another corner of the world, which would still conserve the charge globally

Quiz

Question

What is wrong with the following radioactive decay?

$$^{238}_{92}U \rightarrow \, ^{234}_{90}Th + \, ^{4}_{3}Li$$

Answer

Count the numbers of protons on each side!

Question

Electrons and positrons are produced by the nuclear transformations of protons and neutrons known as β decay. If a neutron transforms into a proton, is an electron or a positron produced?

Answer

$n \rightarrow p + e^- + \bar{\nu}_e$

Electric charge

In SI, the electric charge unit is measured in **Coulomb (C)**.

- The absolute value of the charge of an electron is about 1.6×10^{-19} C.
- The amount of charge that travels through an alkaline AA battery till it is fully discharged is about 5000 C.
- Electrostatic shocks are caused by charges of few μC (10^{-6} C)

In SI, the Coulomb is a *derived* unit.

One Coulomb is

- the charge carried by a current of one Ampere flowing for one second, or
- the charge on a capacitor of one Farad held to a potential of one Volt.

Of course, we don't know about Amperes, Farads and Volts just yet.
For now, just remember that the electric charge unit is the Coulomb (C).

Point charges

Later in this lecture we will talk about *extended* charge distributions.

However, initially (and throughout the course) we will also be talking about *point* charges.

- A point charge is... well, exactly that. An amount of charge concentrated in a single geometrical point.
- Obviously, this is an approximation.
- However, this is a good approximation when the distances involved in a problem under investigation are far larger than the actual "size" of the charges.

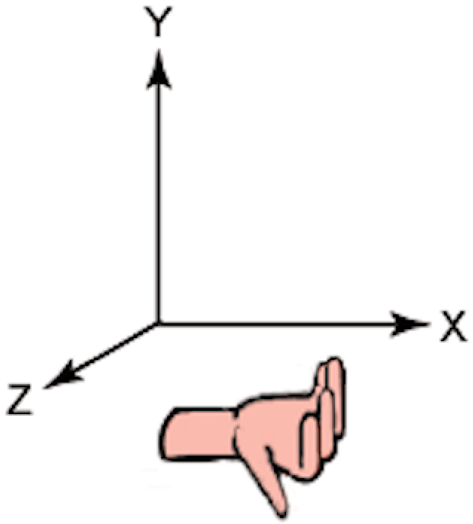
Reminder: Scalars and Vectors

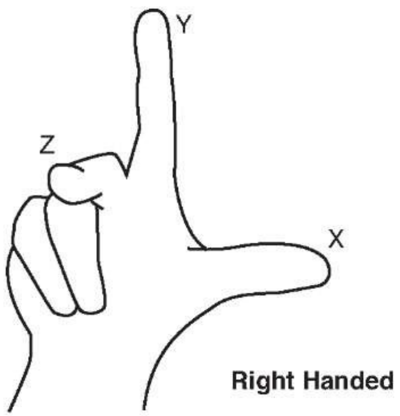
- **Scalars**
  - They are quantities that **have magnitude but no direction**
- **Vectors**
  - They are quantities that **have both magnitude and direction**
  - Note that vectors have magnitude and direction **but no location**
    - a displaced vector has the same magnitude and direction

Can you give a few examples of scalar and vector quantities?

Reminder: Vectors

Let me draw a *right-handed* coordinate system in a 3-D *Euclidian* space (below).





Explaining "Euclidian":

- The term refers to the *mathematical structure* of the space: i.e. the relationships describing the distance and angles between a set of points, the symmetries (rotations, reflections) of the space etc.
- From Greek mathematician Euclid of Alexandria, c. 300 BC.
- The term does not specify the dimensionality (indeed, a Euclidian space can be 1-D, 2-D, 3-D, ...)
- Not all spaces are Euclidian: Non-Euclidian spaces (e.g. curved spaces) also exist and used in modern physics.
- **The physical space (as the canvas for usual empirical phenomena) is Euclidian**.

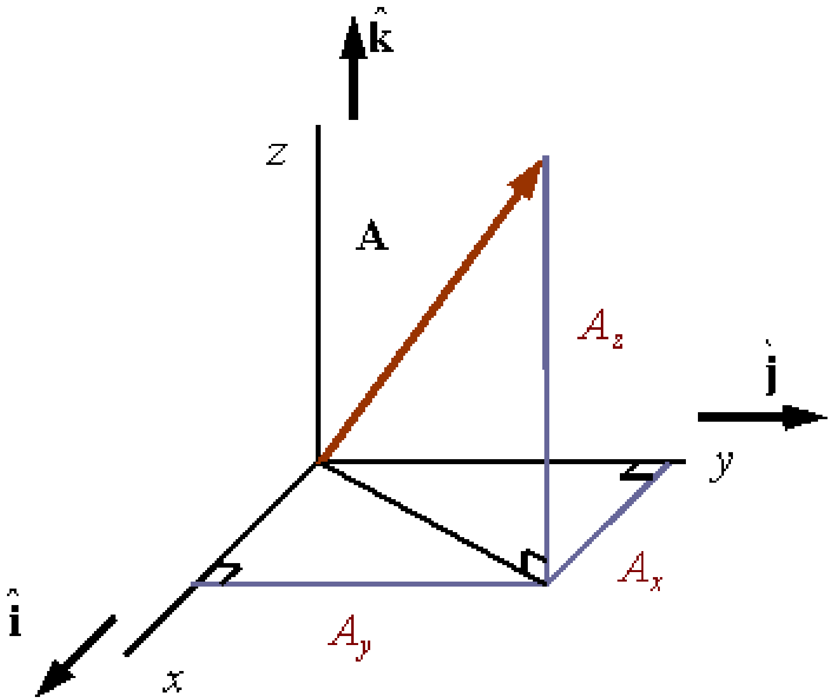
The coordinate system is called **right**-handed for a reason! See on left and **take notice**. If you are right-handed, can't be bothered to leave the pencil from your right hand and, instead, you use your other hand to determine the directions (as I have seen several students do) then you are in trouble! (Unless you have two right hands.)

Reminder: Vectors

Now, let's draw a **vector** $\vec{A}$ in our coordinate system.

Since vectors have magnitude and direction but no location, let's shift it, for convenience, so that it starts at the origin of our coordinate system.

Let A_x, A_y, A_z be the components of $\vec{A}$ along the x,y and z axis respectively.



We usually write $\vec{A}$ in terms of its components as:

$$\vec{A} = \left(A_x, A_y, A_z\right)$$

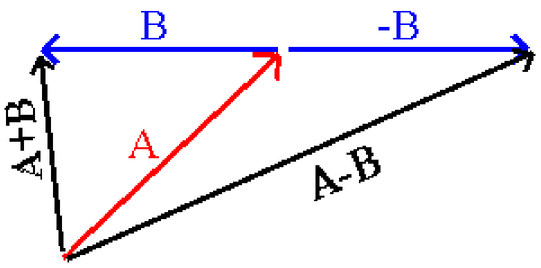
or

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

where $\hat{i}$, $\hat{j}$, $\hat{k}$ (also usually denoted as $\hat{x}$, $\hat{y}$, $\hat{z}$) are unit vectors along the x, y and z axis respectively: $\hat{i}$=(1,0,0), $\hat{j}$=(0,1,0), $\hat{k}$=(0,0,1).

We can also write $\vec{A} = |\vec{A}| \cdot \hat{A}$ where $|\vec{A}|$ is the magnitude of $\vec{A}$ ($|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$) and $\hat{A}$ is a unit vector ($|\hat{A}| = 1$) along $\vec{A}$.

Reminder: Vector addition and subtraction



Let $\vec{A} = \left(A_x, A_y, A_z\right)$ and $\vec{B} = \left(B_x, B_y, B_z\right)$.

Then

$$\vec{A} + \vec{B} = \left(A_x, A_y, A_z\right) + \left(B_x, B_y, B_z\right) = \left(A_x + B_x, A_y + B_y, A_z + B_z\right)$$

and

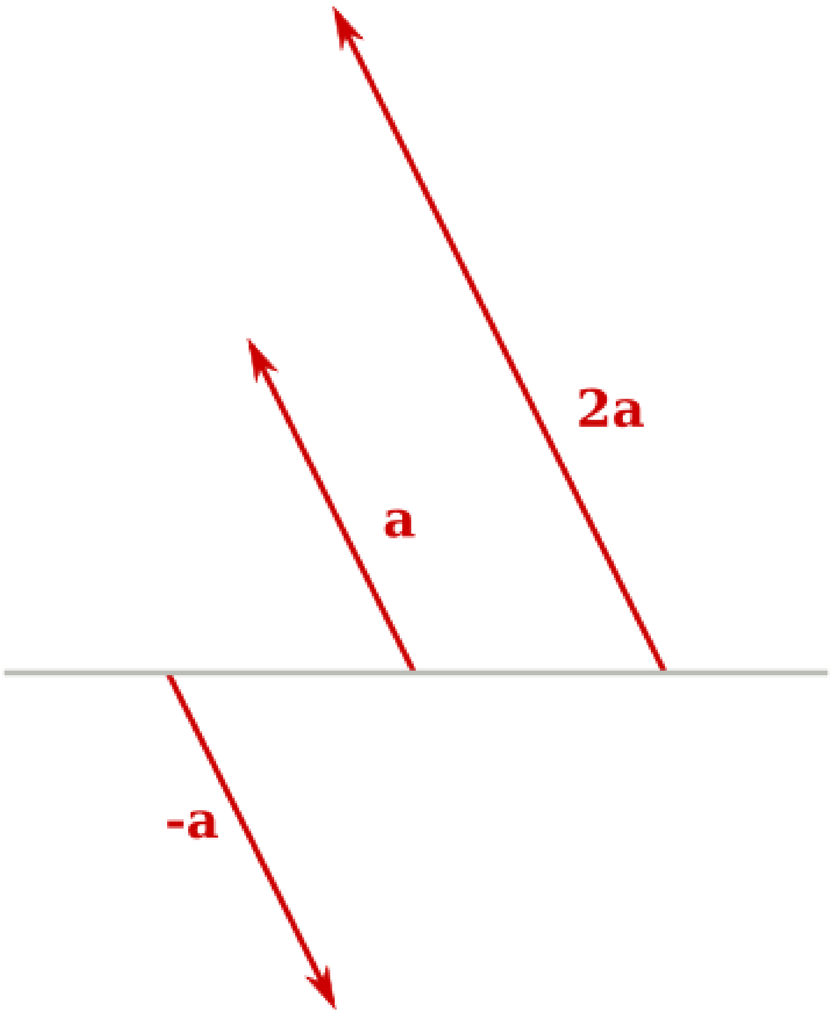
$$\vec{A} - \vec{B} = \left(A_x, A_y, A_z\right) - \left(B_x, B_y, B_z\right) = \left(A_x - B_x, A_y - B_y, A_z - B_z\right)$$

Reminder: Vector multiplication

We have 3 kinds of multiplication involving vectors:

- Product $\lambda\vec{A}$ of a vector $\vec{A}$ with a scalar λ
- **Dot product** $\vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$
- **Cross product** $\vec{A} \times \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$

Reminder: Product $\lambda\vec{A}$ of a vector $\vec{A}$ with a scalar λ



Let $\vec{A} = \Big(A_x, A_y, A_z\Big)$.

Then

$$\lambda\vec{A} = \lambda\Big(A_x, A_y, A_z\Big) = \Big(\lambda A_x, \lambda A_y, \lambda A_z\Big)$$

Reminder: Dot product $\vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$

Dot product $\vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$ is defined as

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

i.e. as the sum of the products of the x, y and z components of $\vec{A}$ and $\vec{B}$.

It can also be written as

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|cos\theta$$

where $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ (similarly for $|\vec{B}|$) and θ is the angle between $\vec{A}$ and $\vec{B}$. So, the dot product $\vec{A} \cdot \vec{B}$ is product of the magnitude of $\vec{A}$ with the magnitude of the projection of $\vec{B}$ along $\vec{A}$ (or vice versa).

Please note that the **dot product** $\vec{A} \cdot \vec{B}$ **is scalar**.

Reminder: Dot product $\vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$

As we have seen, the dot product of two vectors $\vec{A}$, $\vec{B}$ can be written as

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|cos\theta$$

One can easily see that:

- If $\vec{A}$ and $\vec{B}$ are perpendicular ($\theta = \pi/2$) then $\vec{A} \cdot \vec{B} = 0$.
- If $\vec{A}$ and $\vec{B}$ are collinear ($\theta = 0$) then $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|$.
- The dot product of a vector $\vec{A}$ with itself is $\vec{A} \cdot \vec{A} = |\vec{A}|^2$.

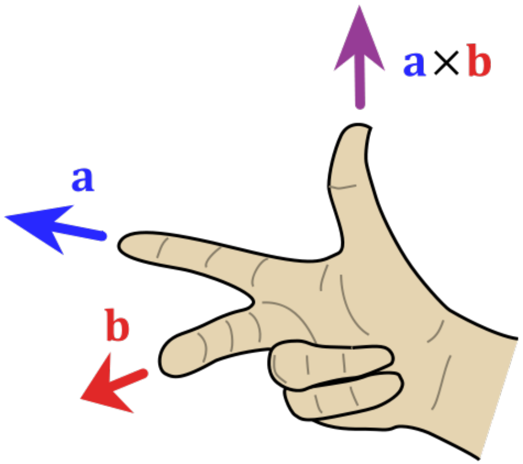
Note that the dot product is

- distributive [$\vec{A} \cdot \left(\vec{B} + \vec{C}\right) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$], and
- commutative [$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$].

Reminder: Cross product $\vec{A} \times \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$

The cross product $\vec{A} \times \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$ is defined as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\hat{i} - (A_x B_z - A_z B_x)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$



It can also be written as

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \, sin\theta \; \hat{n}$$

where $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ (similarly for $|\vec{B}|$), θ is the angle between $\vec{A}$ and $\vec{B}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ as shown on the left.

Note that the **cross product** $\vec{A} \times \vec{B}$ **is a vector** (actually, *pseudo*-vector).

Reminder: Cross product $\vec{A} \times \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$

As we have seen, the cross product of two vectors $\vec{A}$, $\vec{B}$ can be written as

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \; sin\theta \; \hat{n}$$

One can easily see that:

- If $\vec{A}$ and $\vec{B}$ are perpendicular ($\theta = \pi/2$) then $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|$.
- If $\vec{A}$ and $\vec{B}$ are collinear ($\theta = 0$) then $\vec{A} \times \vec{B} = 0$.
- The cross product of a vector $\vec{A}$ with itself is $\vec{A} \times \vec{A} = 0$.

Note that the cross product is

- distributive [$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$],
- not commutative [$\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$].

Reminder: Useful identities / Triple products

Scalar triple product $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$:

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$$

Notice how the order is preserved (cyclically) and that result is scalar.

Vector triple product $\vec{A} \times \left(\vec{B} \times \vec{C}\right)$:

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{A} \cdot \vec{C}\right) - \vec{C} \cdot \left(\vec{A} \cdot \vec{B}\right)$$

This the so-called **BAC-CAB rule**. Notice that result is a vector.

Higher vector products can be reduced by repeated applications of above.

Quiz

Recall that:

- The product $\lambda\vec{A}$ of a vector $\vec{A}$ with a scalar λ is a vector
- The **dot product** $\vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$ is a scalar
- The **cross product** $\vec{A} \times \vec{B}$ of two vectors $\vec{A}$, $\vec{B}$ is a vector

You should be able to answer the following:

Question

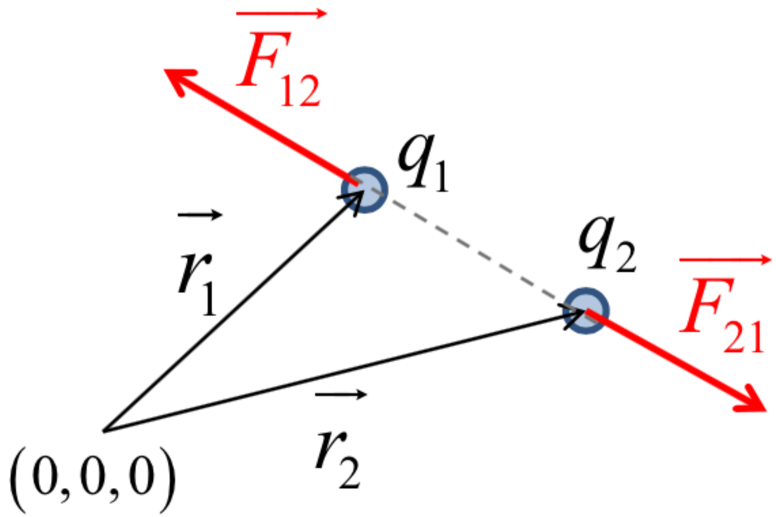
What is the product $(\vec{A} \cdot \vec{B}) \times \vec{C}$ of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$?
Is it a scalar or a vector?

Answer

The product is ill-formed and it means nothing. It is the cross product of a scalar with a vector!

Coulomb's law

Coulomb's law describes the **electrical force between two charges**.



(drawn for like charges)

The force $\vec{F}_{12}$ exerted on test charge 1 by charge 2 (notice convention) is:

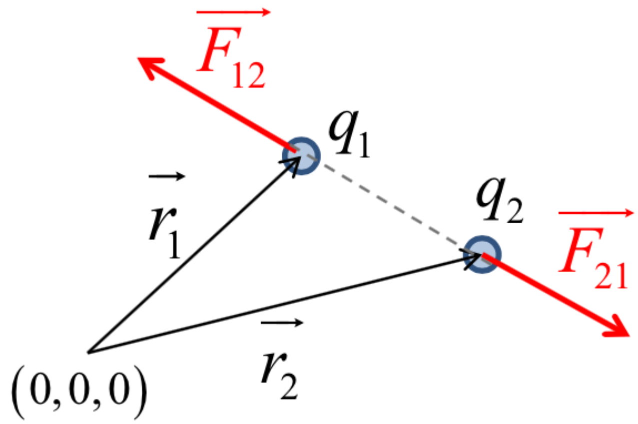
$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} \hat{r}_{12}$$

where:

- ϵ_0 is the permittivity of free space ($\epsilon_0 = 8.854 \times 10^{-12}\ C^2/Nm^2$), and
- $\hat{r}_{12}$ is a unit vector ($|\hat{r}_{12}|=1$) in the direction of $\vec{r}_1 - \vec{r}_2$ (i.e. pointing from q_2 to q_1).

Coulomb's law

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}\hat{r}_{12}$$



(drawn for like charges)

Observations: The force

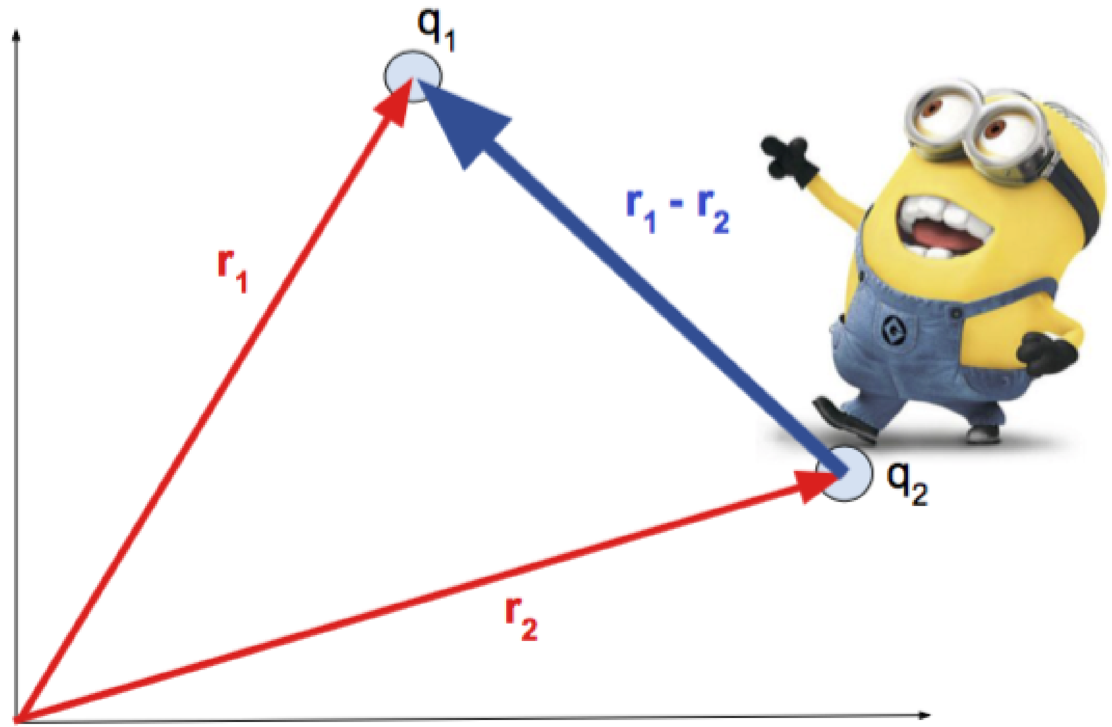
- is proportional to the magnitude of each charge
- is inversely proportional to the square of the distance between charges
- has infinite range
- has a direction along the line connecting the two charges
- can be attractive or repulsive, depending on the charge signs

Coulomb's law: Please notice the direction of $\vec{r}_1$ - $\vec{r}_2$

$\vec{r}_1$ - $\vec{r}_2$ is the vector pointing from q_2 to q_1, not the other way around (A common mistake!)

$\hat{r}_{12}$ is a unit vector along $\vec{r}_1$ - $\vec{r}_2$:

$$\hat{r}_{12} = \widehat{r_1 - r_2}$$



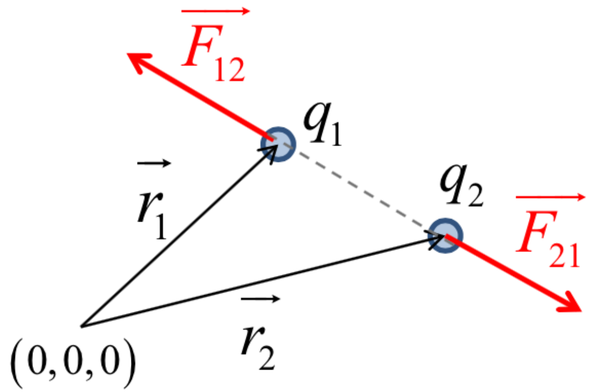
$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}\hat{r}_{12}$$

- $\vec{F}_{12}$ points along $\hat{r}_{12}$ if $q_1 \cdot q_2 > 0$
- $\vec{F}_{12}$ has opposite direction from $\hat{r}_{12}$ if $q_1 \cdot q_2 < 0$

Coulomb's law: Notice the difference between $\vec{F}_{12}$ and $\vec{F}_{21}$

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}\hat{r}_{12}$$

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{|\vec{r}_2 - \vec{r}_1|^2}\hat{r}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}(-\hat{r}_{12})$$



(drawn for like charges)

$$\vec{F}_{21} = -\vec{F}_{12} \quad \begin{cases} \vec{F}_{12} \text{ is the force exerted on test charge 1 by charge 2} \\ \\ \vec{F}_{21} \text{ is the force exerted on test charge 2 by charge 1} \end{cases}$$

Take notice of the convention and the relative minus sign between $\vec{F}_{12}$ and $\vec{F}_{21}$.
A common mistake is to calculate $\vec{F}_{12}$ when $\vec{F}_{21}$ is asked (or vice versa).

Coulomb's law: $\hat{r}/|\vec{r}|^2$ vs $\vec{r}/|\vec{r}|^3$

We wrote the force exerted on test charge 1 by a test charge 2 ($\vec{F}_{12}$) as:

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} (r_1 \hat{-} r_2)$$

Sometimes, we will also be writing it as:

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2)$$

The two expressions are **completely equivalent**.

A surprising large number of students doesn't see that, and several students mix the two equations (e.g. using a unit vector $\hat{r}_{12}$ but then a cube power of $|\vec{r}_1 - \vec{r}_2|$). If you too are confused, please study the next slide carefully.

Coulomb's law: $\hat{r}/|\vec{r}\,|^2$ vs $\vec{r}/|\vec{r}\,|^3$

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}\hat{r}_{12} \quad \text{is the same as} \quad \vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3}\vec{r}_{12}$$

Notice that:

$$\frac{\vec{r}}{r^3} = \frac{r \cdot \hat{r}}{r^3} = \frac{\hat{r}}{r^2}$$

where:

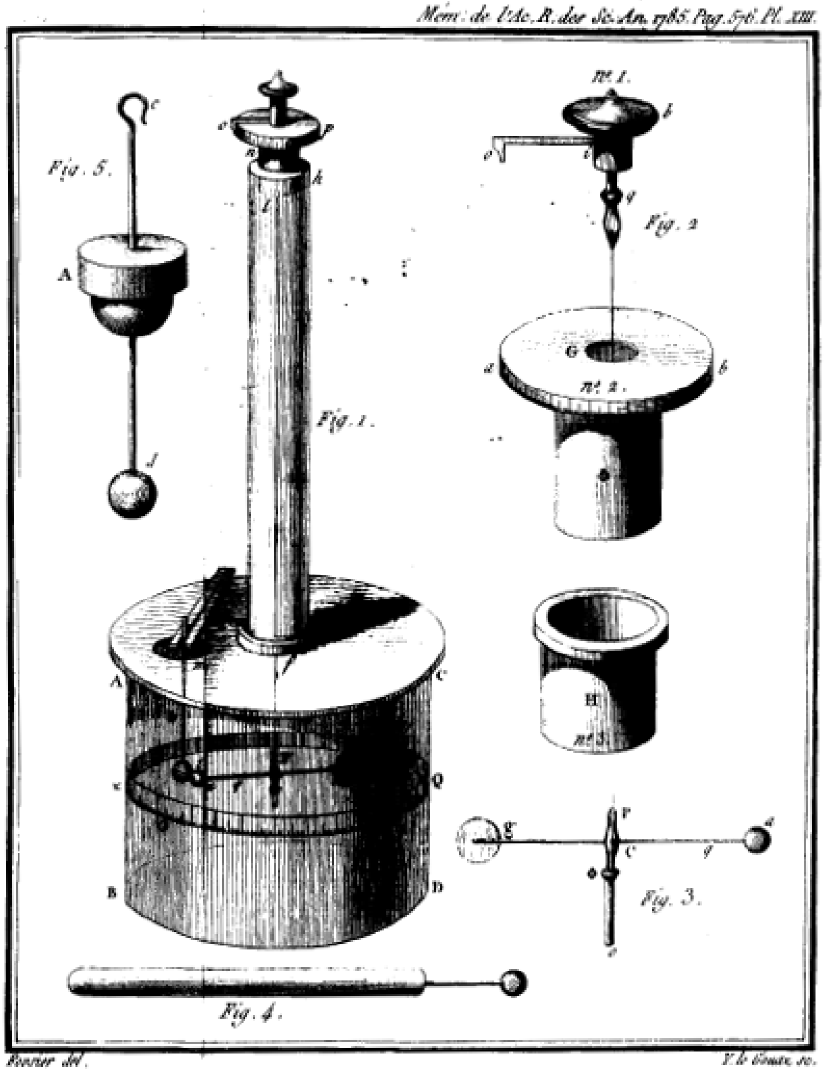
- $\vec{r}$ is a vector (a quantity that is characterised both by a magnitude and a direction),
- r is the magnitude of $\vec{r}$, and
- $\hat{r}$ is a unit vector along $\vec{r}$ (has the same direction as $\vec{r}$, but its magnitude is 1).

If you still don't see the above equivalence, please come and talk to me.

Coulomb's law

Charles-Augustin de Coulomb (1736-1806), French physicist.

His law was included in his 1785 publication Premier Mémoire sur l'Électricité et le Magnétisme.



Coulomb deduced his law using a **torsion balance** experiment:

An insulating rod, with a metal-coated (charged) ball attached on one end, is suspended by a silk fiber which acts as a weak torsion spring. A second charged ball brought near to the first one twists the silk fiber by a certain angle. The electrical force can be deduced from the force it took to twist the fiber as much as it did.

Coulomb's law

Henry Cavendish (1731-1810), British physicist.

- In the early 1770's, **Henry Cavendish**, an English physicist, had already discovered the dependence of the electrical force upon the distance and charge. He never published it!
  - Cavendish measured that the force was inversely proportional to $n = 2.00 \pm 0.02$
  - His experiment (using 2 concentric conducting rings) was a direct application of Gauss' law (published a century later)!
- And others before Cavendish (Daniel Bernoulli, Alessandro Volta, Joseph Priestley, John Robinson) either suspected the inverse square law or performed measurements.
- Later, **Maxwell**, refining the Cavendish technique measured $n = 2.0 \pm 0.00005$ [Maxwell, Vol 1, p. 80]
- In 1936 **Plimpton and Lawton** found that Coulomb's exponent differs from two by less than one part in a billion!

Coulomb's law / Quiz

The electrical force like the gravitational force, but 'a **billion** - **billion** - **billion** - **billion** times stronger' [Feynman Lectures].

Question [Feynman Lectures]

Within our body we have both positive charges (protons) and negative charges (electrons) in equal amounts. The attractive and repulsive forces "cancel out" so we do not attract or repel each other electrically. If our bodies had just 1% more protons than electrons, the repelling force at a distance of an arm's length would be enough to lift a weight equal to that of

- yourself?
- everybody in this class?
- everything in this city?
- the moon?
- more??

Answer

Actually, the repelling force at a distance of an arm's length would be enough to lift a weight equal to that of the entire Earth!

Note on e/m units

- There is arbitrariness in the choice of the fundamental units.
  - Traditionally, we take the mass, length and time as basic units (but this is not necessary and other choices are also in use).

- Take Coulomb's law:

$$|\vec{F}_{12}| \propto \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}$$

  - the constant of proportionality is automatically defined if the magnitude and dimensions of the unit of charge has been defined in advance, or
  - one can choose the constant of proportionality arbitrarily and then use Coulomb's law to define the unit of charge.

- For e/m quantities there is no compelling tradition.
  - Many systems of units: **electrostatic** (esu), **electromagnetic** (emu), **Gaussian**, **Heaviside - Lorentz**, **SI / rationalised**, ...

Note on e/m units

- What's more, various e/m equations look somewhat different in different systems.

$$|\vec{F}_{12}| = \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} \;\; (\textit{Gaussian}), \quad |\vec{F}_{12}| = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} \;\; (\textit{SI}), \quad |\vec{F}_{12}| = \frac{1}{4\pi} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} \;\; (\textit{HL})$$

- Different popular textbooks use different systems of units: A source of confusion!
- If you are interested in the issue of systems of units for e/m quantities, glance though the Appendix of J.D.Jackson's Classical Electrodynamics (ISBN-13: 978-0471309321).
- The Gaussian and SI/rationalised are the most commonly used system of units.
- The slides, as well as in all recommended textbooks, the SI/rationalised system of units is used.

Worked example

Question

Find the force on charge q_1, 20 μC, due to charge q_2, -300 μC, where q_1 is at position $\vec{r}_1 = $ (0, 1, 2) m and q_2 is at position $\vec{r}_2 = $ (2, 0, 0).

As we have seen, the force on charge q_1 due to charge q_2 is:

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2)$$

The position $\vec{r}_1 - \vec{r}_2$ of q_1 relative to q_2 is:

$$\vec{r}_1 - \vec{r}_2 = (0, 1, 2)\ m - (2, 0, 0)\ m = (-2, 1, 2)\ m$$

The magnitude of $\vec{r}_1 - \vec{r}_2$ is:

$$|\vec{r}_1 - \vec{r}_2| = \sqrt{(-2)^2 + 1^2 + 2^2}\ m = \sqrt{9}\ m = 3\ m$$

Worked example

Therefore:

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2) \Rightarrow$$

$$\vec{F}_{12} = (9.0 \times 10^9 \ \frac{N \cdot m^2}{C^2}) \cdot \frac{(20 \times 10^{-6} \ C)(-300 \times 10^{-6} \ C)}{(3 \ m)^3} \cdot (-2, 1, 2) \ m \Rightarrow$$

$$\vec{F}_{12} = (9.0 \times 10^9 \ \frac{N \cdot m^2}{C^2}) \cdot \frac{-6000 \times 10^{-12} \ C^2}{27 \ m^3} \cdot (-2, 1, 2) \ m \Rightarrow$$

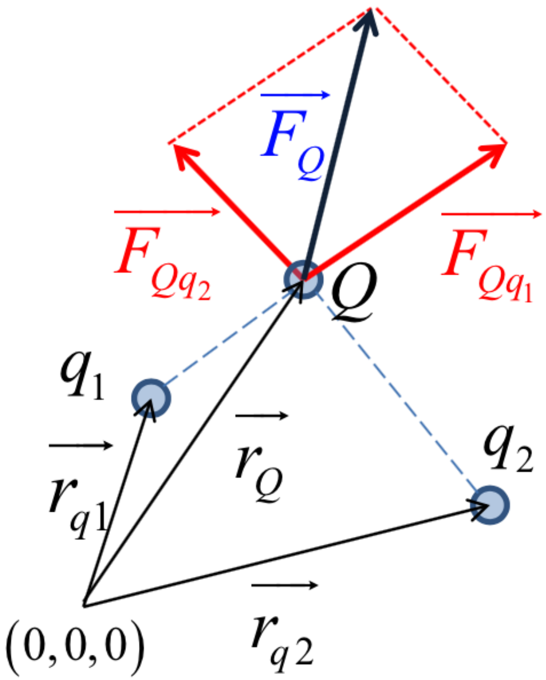
$$\vec{F}_{12} = -\frac{(9.0 \times 10^9) \cdot (6 \times 10^{-9})}{27} \cdot (-2, 1, 2) \ N \Rightarrow$$

$$\vec{F}_{12} = -\frac{54}{27} \cdot (-2, 1, 2) \ N = -2 \cdot (-2, 1, 2) \ N \Rightarrow$$

$$\vec{F}_{12} = (4, -2, -4) \ N$$

Superposition principle

The forces exerted by several charges on a test charge superimpose each other undisturbed: The total force is the vector sum of forces.



(drawn for like charges)

The total force $\vec{F}_Q$ exerted on test charge Q by test charges q_1, q_2 is:

$$\vec{F}_Q = \sum_{i=1}^{2} F_{Qq_i} \Rightarrow$$

$$\vec{F}_Q = \sum_{i=1}^{2} \frac{1}{4\pi\epsilon_0} \frac{Qq_i}{|\vec{r}_Q - \vec{r}_{q_i}|^2} \hat{r}_{Qq_i} \Rightarrow$$

$$\vec{F}_Q = \sum_{i=1}^{2} \frac{1}{4\pi\epsilon_0} \frac{Qq_i}{|\vec{r}_Q - \vec{r}_{q_i}|^3} (\vec{r}_Q - \vec{r}_{q_i})$$

Superposition principle

Superposition is not a logical necessity, but an experimental fact.

Here is how things could have been different:

- Example 1: If the electromagnetic force were proportional to the square of the total source charge [Griffiths].

  Then, obviously

$$Q^2(q_1+q_2)^2 \neq Q^2 q_1^2 + Q^2 q_2^2$$

Superposition is not a logical necessity, but an experimental fact.

Here is how things could have been different:

- Example 2: If there were many-body forces [Greiner].

  The Coulomb force between charges is merely a **2-body force**: This means that the force between two charges is undisturbed from the presence of other bodies and many-body forces do not occur.

  A many-body force is one where the force between bodies 1 and 2 would also depend on the positions and charges of other bodies. Below is an example of a hypothetical 3-body force (r_s is the centre of gravity between q_1, q_2):

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{r_1} - \vec{r_2}}{|(\vec{r_1} - \vec{r_2})\left(1 + \frac{q_3^2}{q_1 q_2} \cdot \frac{|\vec{r_1}-\vec{r_2}|}{|\vec{r_s}-\vec{r_3}|}\right)|^3}$$

Coulomb force due to an array of charges q_1, q_2, ..., q_n

So, starting from the basic Coulomb force between two charges and using the superposition principle, we can compute the total force exerted on a charge Q, by an array of charges q_1, q_2, ..., q_n.

The total force on Q is the vector sum of all forces due to each of q_1, q_2, ..., q_n individually:

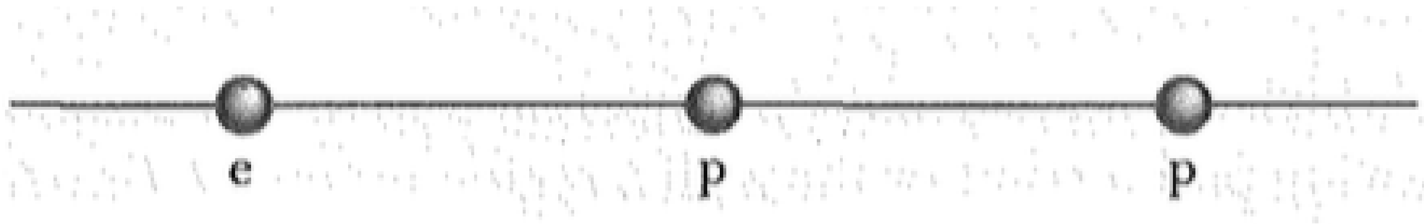
$$\vec{F}_Q = \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{Q q_i}{|\vec{r}_Q - \vec{r}_{q_i}|^3} (\vec{r}_Q - \vec{r}_{q_i})$$

Quiz

Recall Coulomb's force and the superposition principle.

Question

The figure shows two protons (symbol p) and one electron (symbol e) on an axis.

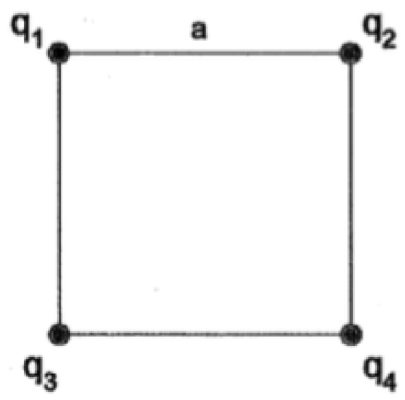


What is the direction of

1. the electrostatic force on the central proton due to the electron,
2. the electrostatic force on the central proton due to the other proton, and
3. the net electrostatic force on the central proton?

Worked example

Question



Four charges are located at the four corners of a square of side a. The charges are $q1 = q4 = Q$ and $q2 = q3 = q$. What is the ratio Q/q if the net electrostatic force on particle 1 is zero?

The Coulomb force exerted on charge q_i due to a charge q_j at distance $\vec{r}$ is given by:

$$\vec{F}_{ij} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_i q_j}{|\vec{r}|^3} \vec{r}$$

Take the x-axis to be parallel with the line connecting the charges q_1 and q_2, and pointing from q_1 to q_2. We will calculate the x-component of the force exerted on q_1 due to each of q_2, q_3 and q_4.

Worked example

$$F_{12;x} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{a^2} = \frac{1}{4\pi\epsilon_0} \cdot \frac{Qq}{a^2}$$

$$F_{13;x} = 0$$

$$F_{14;x} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_4}{(\sqrt{2}\ a)^2} \cdot cos\left(\frac{\pi}{4}\right) = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q^2}{2a^2} \cdot \frac{1}{\sqrt{2}}$$

Thefore, the x-component of the total force experienced by q_1 is:

$$F_{1;x} = F_{12;x} + F_{13;x} + F_{14;x} = \frac{1}{4\pi\epsilon_0} \cdot \left\{ \frac{Q^2}{2\sqrt{2}a^2} + \frac{qQ}{a^2} \right\}$$

Setting $F_{1;x} = 0$, we have:

$$\frac{Q^2}{2\sqrt{2}a^2} + \frac{qQ}{a^2} = 0 \Rightarrow \frac{Q}{2\sqrt{2}} + q = 0 \Rightarrow \frac{Q}{q} = -2\sqrt{2} \Rightarrow \frac{Q}{q} = -2.83$$

Continuous charge distributions

To describe complex distributions of charges, we use the concept of **charge density**. It describes the amount of charge per unit volume, area or length.

- If charge Q is distributed continuously over a volume, the (volume) charge density $\rho(\vec{r})$ is given by:

$$\rho(\vec{r}) = \frac{dQ(\vec{r})}{d\tau} \Rightarrow Q = \int_{volume} d\tau \; \rho(\vec{r})$$

- If charge Q is distributed continuously over a surface, use surface charge density $(\sigma(\vec{r}))$

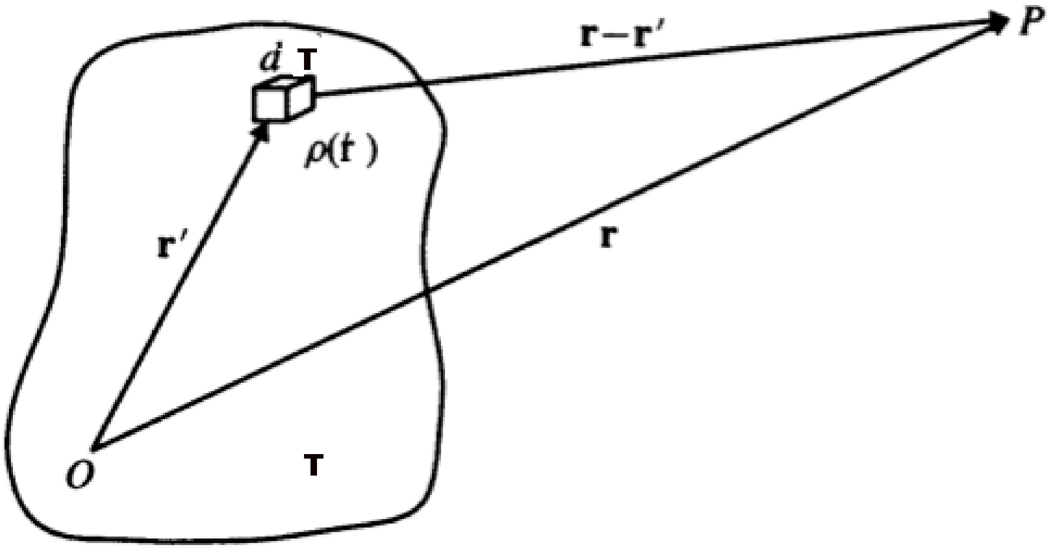
$$\sigma(\vec{r}) = \frac{dQ(\vec{r})}{dS} \Rightarrow Q = \int_{area} dS \; \sigma(\vec{r})$$

- Charge distributed (continuously) along a line: use linear charge density $(\lambda(\vec{r}))$

$$\lambda(\vec{r}) = \frac{dQ(\vec{r})}{dl} \Rightarrow Q = \int_{line} dl \; \lambda(\vec{r})$$

Coulomb force due to a continuous charge distribution

Here, we are considering charge distributed in a volume (similarly for charge distributed on a surface or along a line).



We will start from the discrete case, and do the following substitutions:

- $\vec{r}_{q_i} \rightarrow \vec{r}'$
- $\vec{r}_Q \rightarrow \vec{r}$
- $q_i \rightarrow dq(\vec{r}') = d\tau' \rho(\vec{r}')$
- $\sum \rightarrow \int$

We no longer enumerate positions with the charge ID, and replace the sum / discrete charges with integral / charge density. Therefore:

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{|\vec{r}_Q - \vec{r}_{q_i}|^3} (\vec{r}_Q - \vec{r}_{q_i}) \xrightarrow{\sum \rightarrow \int} \vec{F}_Q = \frac{Q}{4\pi\epsilon_0} \int_\tau d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')$$

Oddities of Coulomb's force

Coulomb's force, as we introduced it, has some oddities:

- **Action at a distance**
  - In early theories, all physical interaction was reduced to *collision*.
  - But what is the medium that carries Coulomb's force?
- **Instantaneous**
  - But nothing, including the influence of one charge on another, can travel faster than the speed of light.

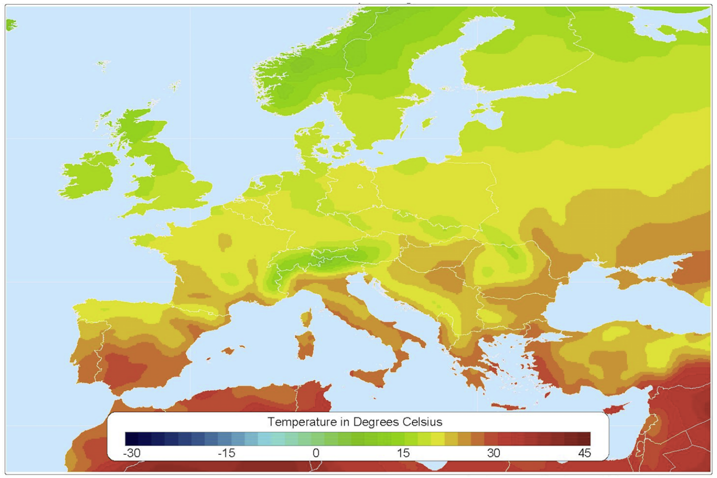
Reminder: Fields

A field is a physical quantity that permeates all space!

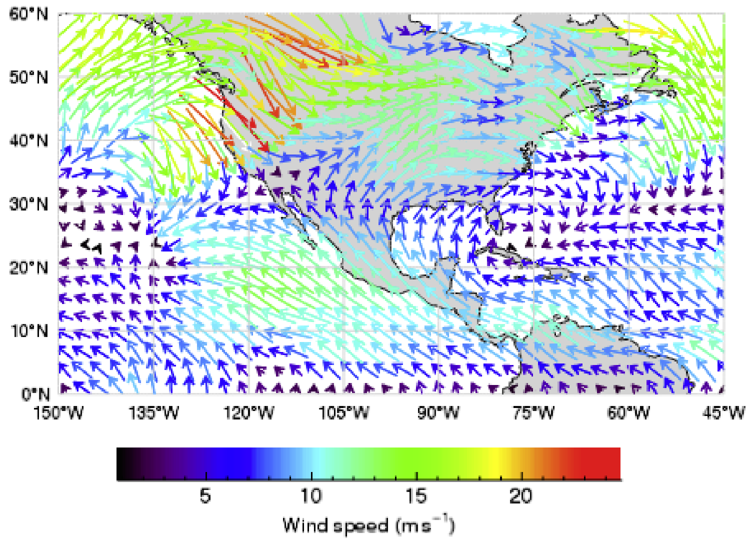
If that physical quantity

- is just a number (a scalar) we have a **scalar field**
- has magnitute and direction (is a vector) we have a **vector field**

Temperature map (a scalar field)



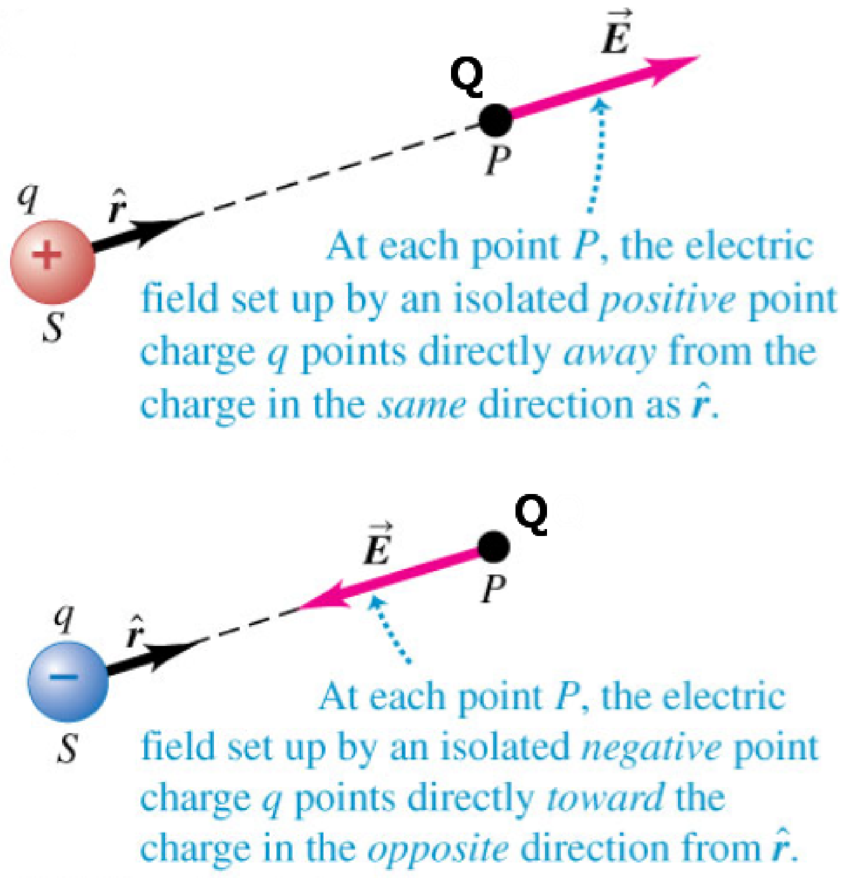
Wind map (a vector field)



In electromagnetism, we will see several scalar and vector fields.

Electric field

It is useful to introduce a new concept: The **electric field** $\vec{E}$



Defined as force exerted on test charge Q, placed in position r, per unit charge.

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q(\vec{r})}{Q}$$

$\vec{E}$ is a field - it **permeates all space**. A charge creates an electric field around it. Other charges feel an electrical force because of the existence of this field.

Here we consider stationary charges (field sources): $\vec{E}$ doesn't depend on t.

$\vec{E}$ has units of NC^{-1} $(= Vm^{-1})$.

Electric field

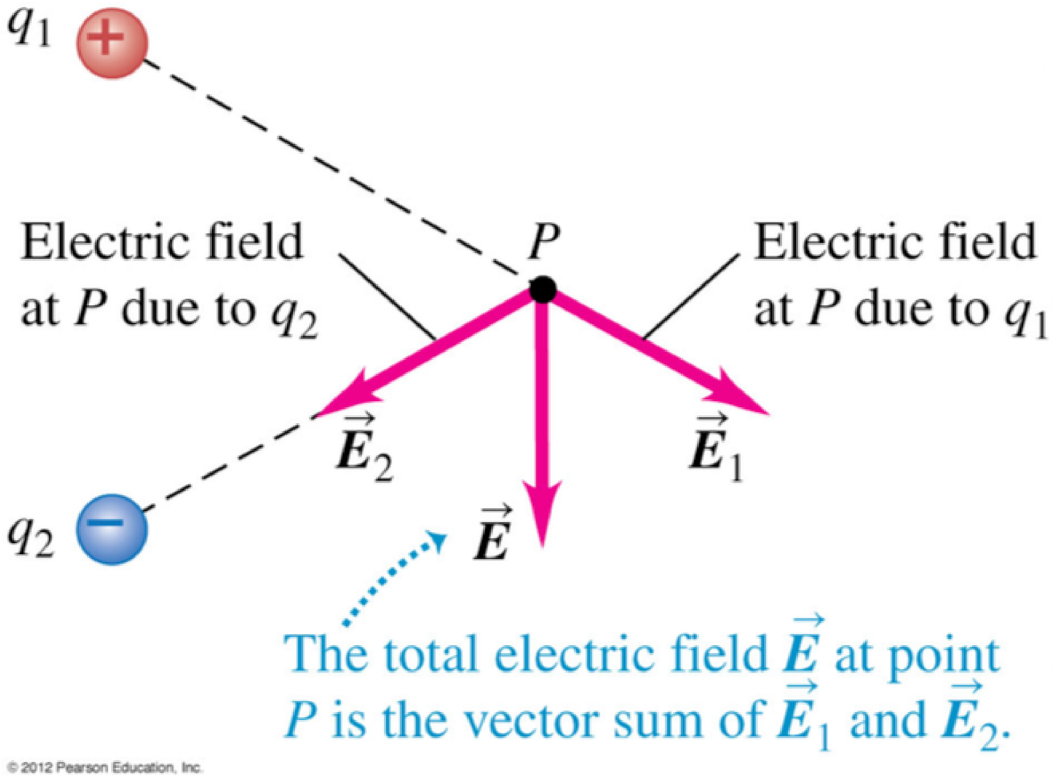
The electric field was defined as the force exerted on a test charge, placed in position $\vec{r}$, per unit charge.

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q(\vec{r})}{Q}$$

Be careful with this definition!

- The electric field times the charge of a body does not necessarily give us the force exerted on that body.
- If that charge is too large, it may disrupt the array of charges creating the electric field.
- The electric field (magnitude and direction) is the **constant limiting value** of the force/charge ratio **as the charge becomes smaller and smaller**.

Electric field due to an array of charges



What about the electric **field created by an array of charges**?

The **superposition principle** applies.

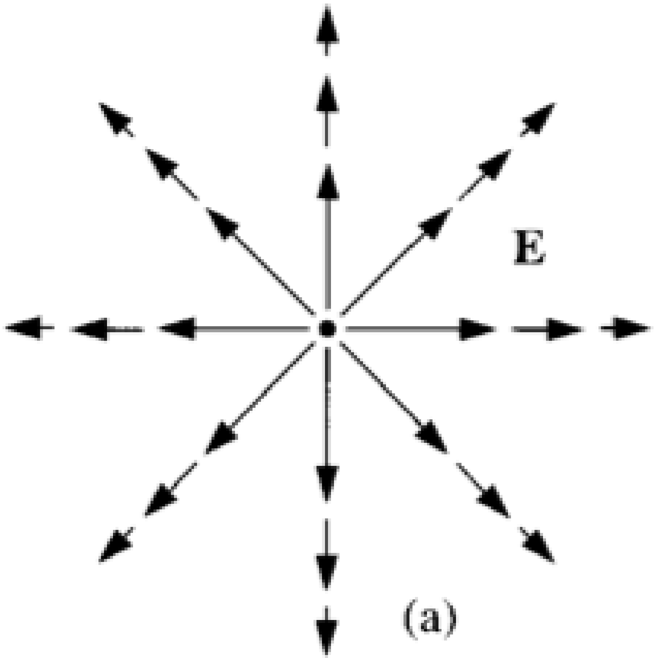
The electric field at position P (where test charge Q is inserted) is given by:

$$\vec{E} = \frac{\vec{F}_Q}{Q} = \frac{\sum_{i=1}^{n} \vec{F}_Q q_i}{Q} = \sum_{i=1}^{n} \vec{E}_i$$

Visualising the electric field

It was easy to visualise the electric force (a vector). But how can we also visualise and represent the electric field (a vector in every point in space)?

Take a simple case: A positive point charge q at the origin.



The electric field $\vec{E}(\vec{r})$ at a point $\vec{r}$ is given by:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

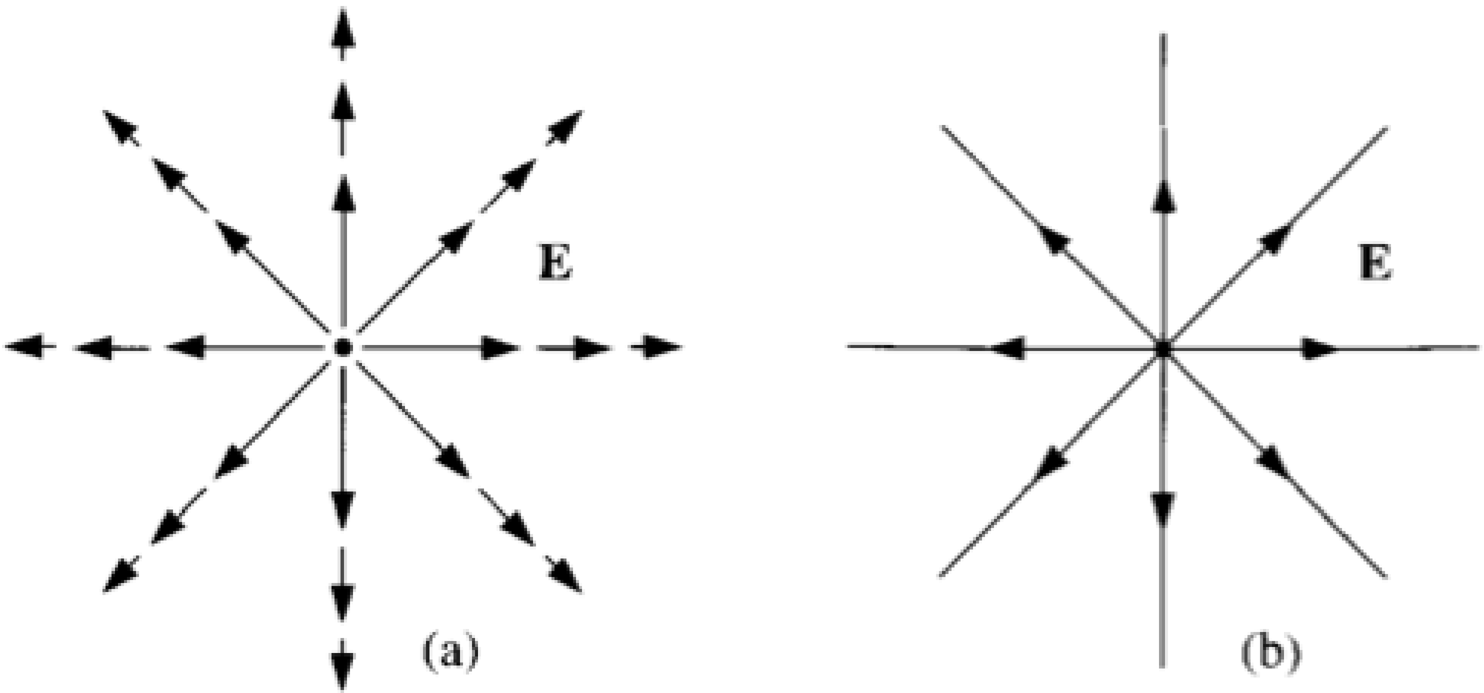
To visualise the field, we can draw field vectors at various points.

The field gets smaller away from the charge, so the vectors get shorter.

Visualising the electric field

It is impractical to visualise the field as shown before.
There is a better way: Connect the arrows! This forms *field lines*.

- Using field lines we can still visualise the field direction.
- How about the strength (previously represented by the vector size)?
  - The field strength is now indicated by the **density** of the field lines.

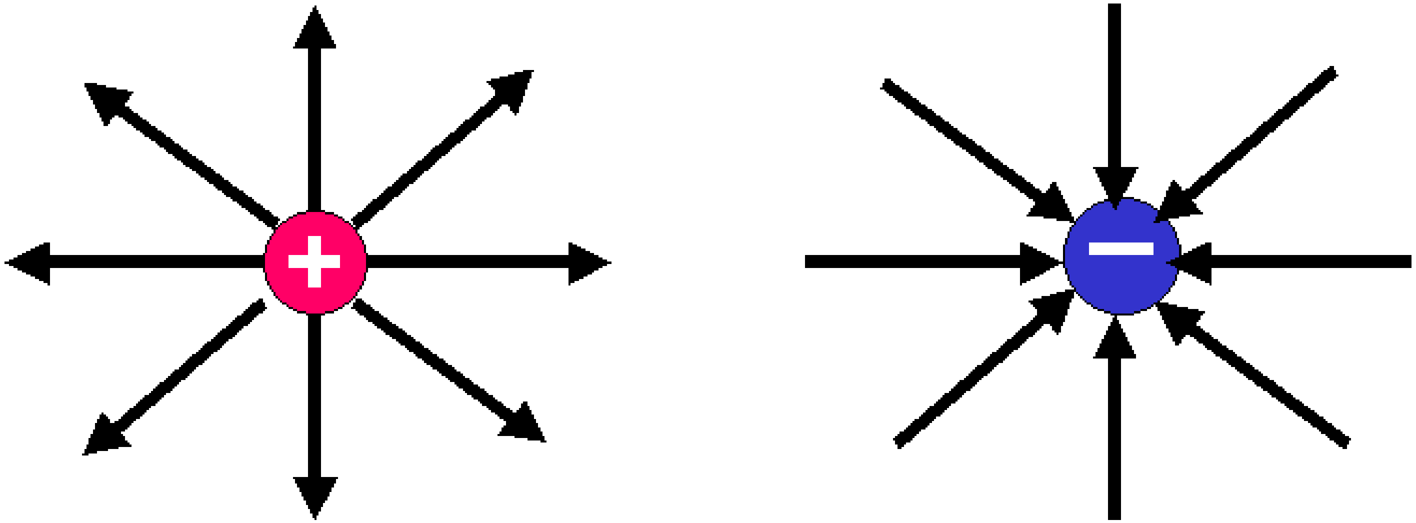


Visualising the electric field

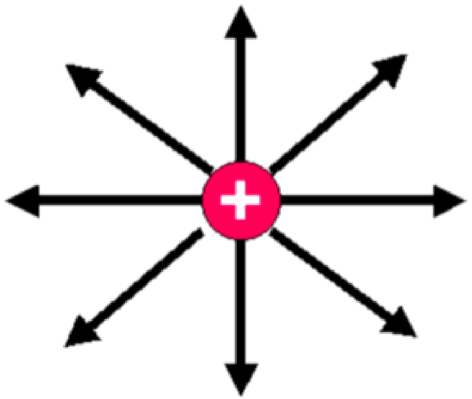
The electric field $\vec{E}(\vec{r})$ at a point $\vec{r}$, for the simple configuration we are studying, is given by:

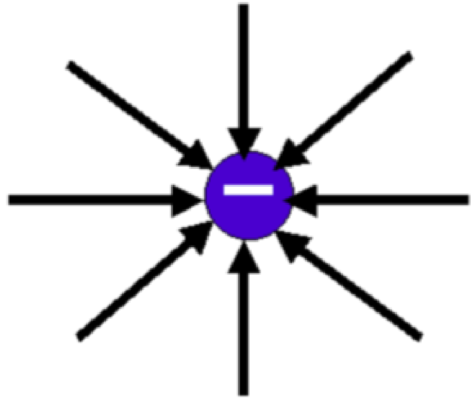
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

Obviously, the direction of the field (and thus the *flow* of field lines) depends on the sign of charge q:

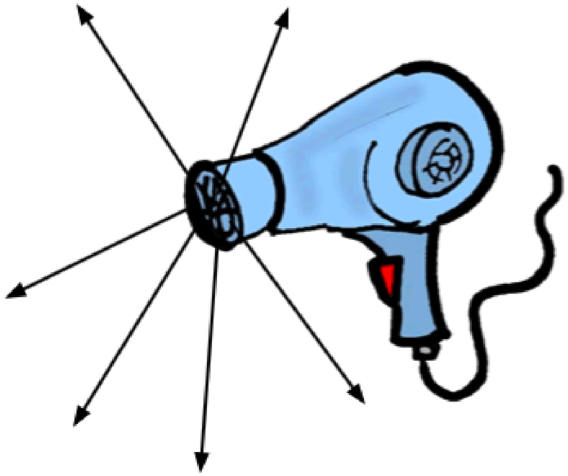


Think of the electric field as the velocity field of air. Then the field lines represent the *flow* of air.



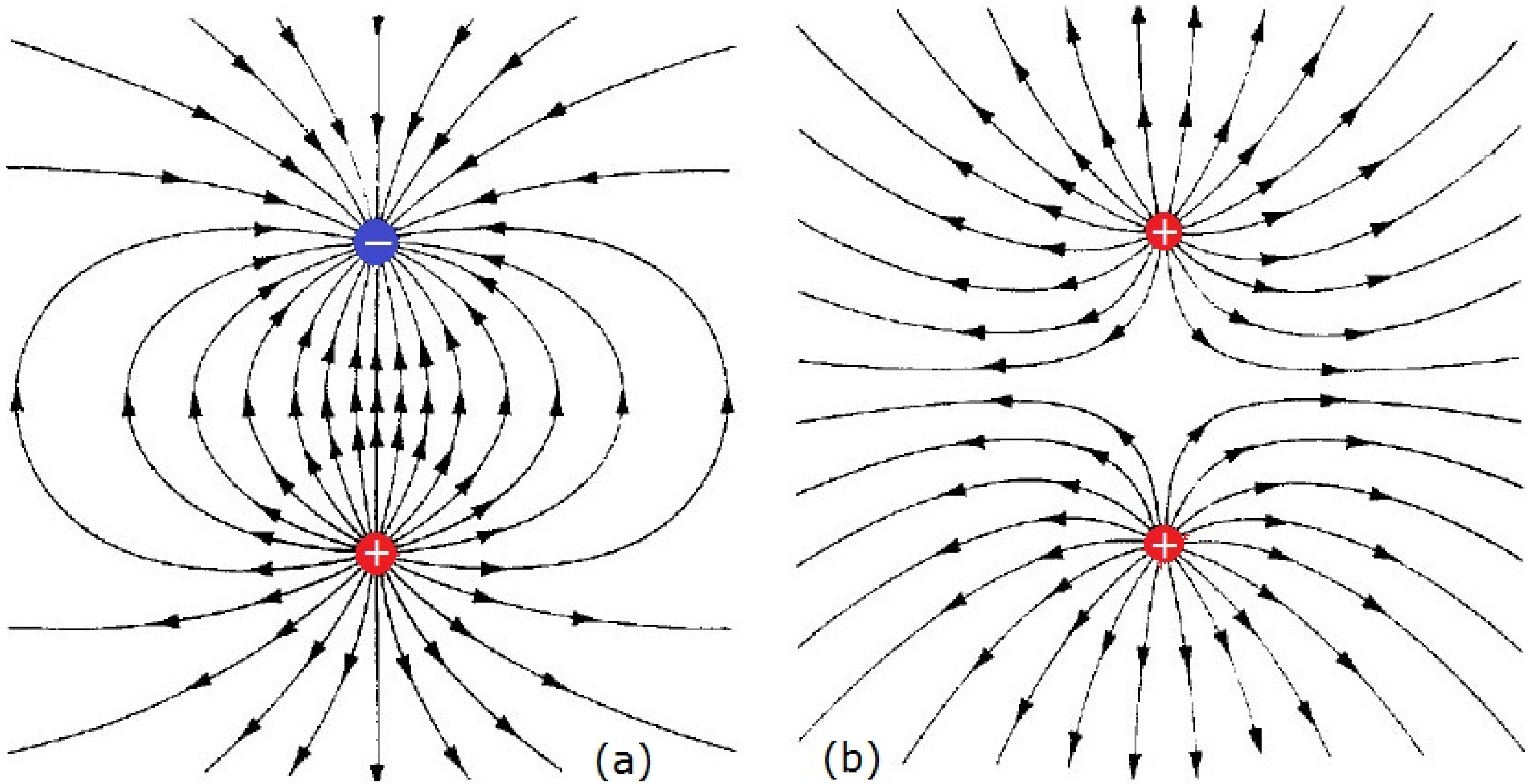


Positive charges behave like *hair dryers*: They blow out air.



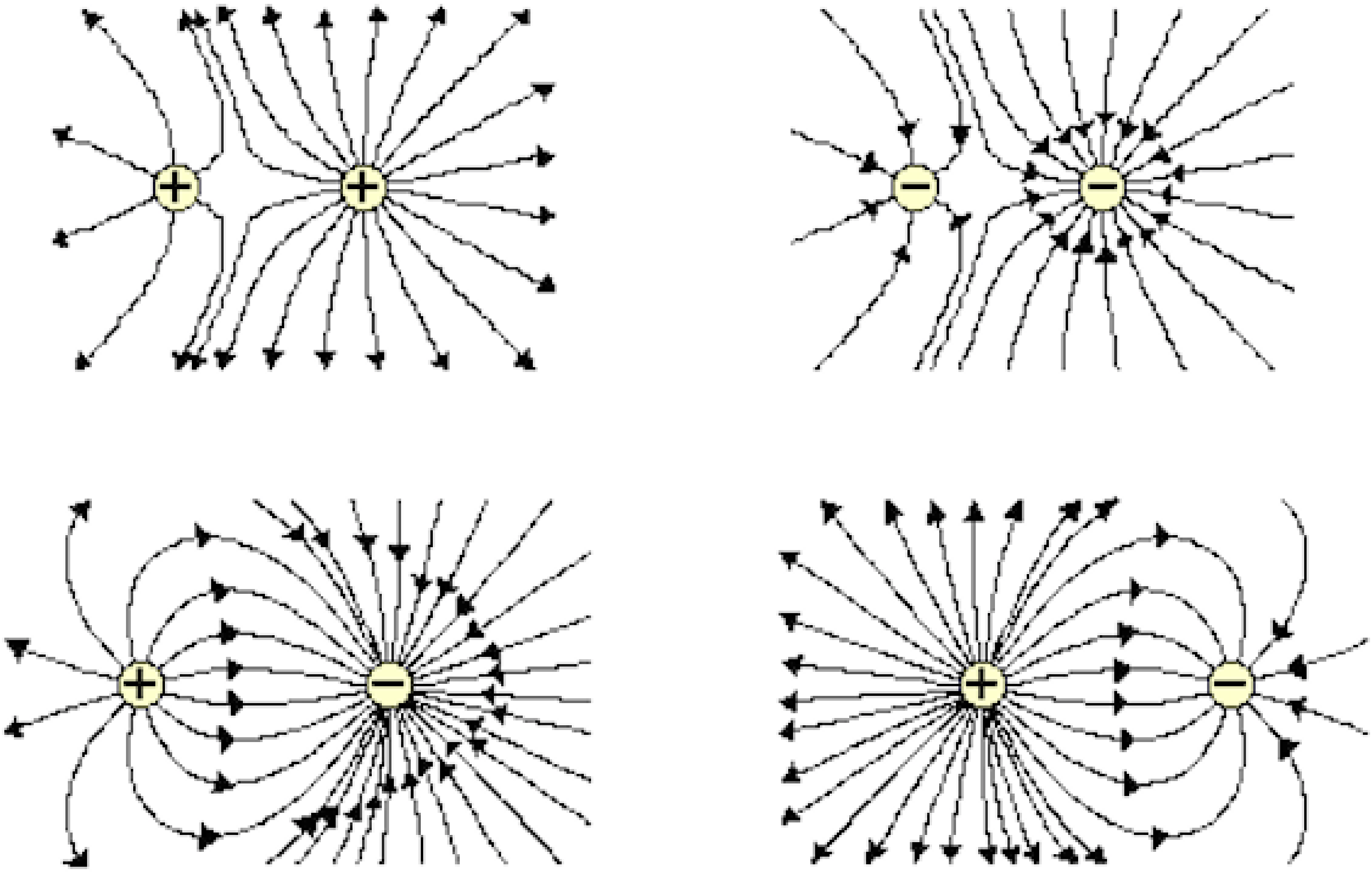
Negative charges behave like *vacuum cleaners*: They suck in air.

Electric field lines: 2 equal charges



Field lines start from positive charges and end up in negative charges.
The force is tangential to the field lines.

Electric field lines: 2 unequal charges

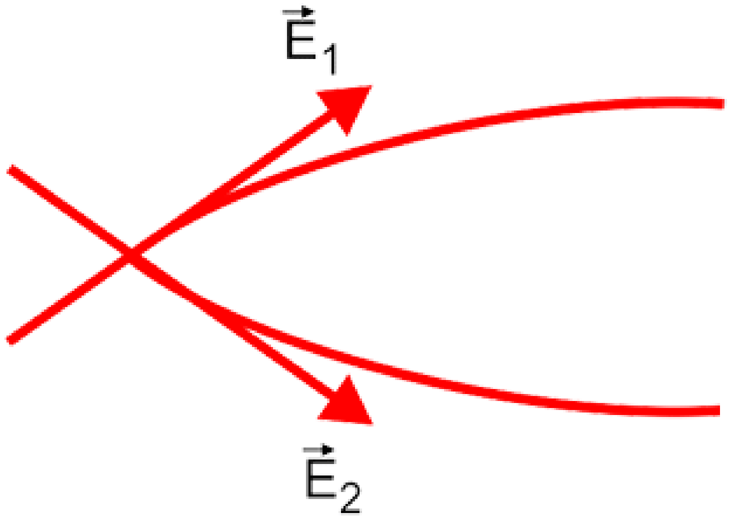


Question: Do you understand which of the two charges in each pair is the largest?

Quiz

Question

Can electric field lines cross?



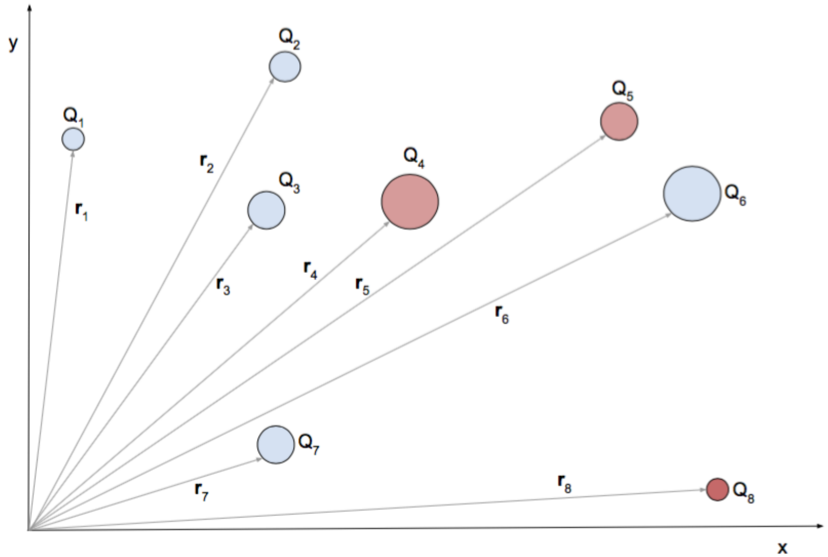
Answer

Of course not! That would mean that there are points that have two different electric field directions!

Quiz

Question

Now that you understand the basics, can you draw the electric field map for the following configuration (light red: negative charges, light blue: positive charges, amount of charge proportional to the circle area)?



Answer

That's what computers are for!

Electric field due to a continuous charge distribution

As we saw earlier, the force due to a continuous distribution of charge (distributed in a volume τ) is:

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0} \int_\tau d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')$$

Therefore, the electric field in position $\vec{r}$ is:

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q}{Q} = \frac{1}{4\pi\epsilon_0} \int_\tau d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')$$

The above, is one of the most general results we obtained. In principle, it's all we need to know! It gives us the electric field $\vec{E}$, and thus the electric force on any charge, due to an arbitrary distribution of charge.

However, the integral involved is very difficult to evaluate. Much of what follows is about how to compute $\vec{E}$ by *avoiding* that complex integration.

Lecture 1 - Main points to remember

- **Electric charge**
  - The source of electric phenomena
  - An intrinsic property of matter
  - Comes in two varieties (positive and negative)
  - An algebraic quantity (+q + (-q) = 0)
  - It is quantised
  - It is conserved (globally and locally)
  - SI unit: Coulomb (C) [= 1 A · 1 s]

- **Coulomb's law**
  - Describes the force between two point charges
  - The force $\vec{F}_{12}$ exerted on test charge 1 by charge 2 is:

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}\hat{r}_{12} \quad or \quad \vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2)$$

Lecture 1 - Main points to remember (cont'd)

- **Superposition principle**
  - Allows the calculation of the total force on a charge Q from an array of other charges q_1, q_2, ..., q_n

$$\vec{F}_Q = \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{Q q_i}{|\vec{r}_Q - \vec{r}_{q_i}|^3} (\vec{r}_Q - \vec{r}_{q_i})$$

  - Total force is the vector sum of forces.
  - Not a logical necessity: An experimental fact!

- **Continuous distributions of charge**
  - Made the leap from discrete to continuous charge distributions described by a charge density
  - Reformulated Coulomb's law for continuous charge distributions

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0} \int_\tau d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')$$

Lecture 1 - Main points to remember (cont'd)

- **Electric field**
  - A more fundamental way to think about electric forces in terms of a field that permeates space.
  - Defined the electric field $\vec{E}$ as the force exerted on a test charge Q, placed in position $\vec{r}$, per unit charge.

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q(\vec{r})}{Q}$$

- **Visualizing the electric field - field lines**
  - Field direction indicated by the direction of field lines
  - Field strength indicated by the density of field lines
  - Force tangential to field lines
  - Field lines start from positive charges and end on negative ones: Positive charges are "sources", negative charges are "sinks"
  - Field lines can not cross

At the next lecture (Lecture 2)

- **Electric flux**
  - The "flow" of an electric field through a surface.
- **Gauss' law**
  - Relates the flux through a closed surface with the net charge contained in it.
  - Our first Maxwell equation! Will study both its 'differential' and 'integral' forms.

Optional reading for Lecture 1

Worked example: Electric field and force of 3 point charges

Question

Three charges are arranged as following: $Q_1 = 1$ nC at (-1,-1,0) cm, $Q_2 = 1$ nC at (1,-1,0) cm and $Q_3 =$ -2nC at (0,2,0) cm. Calculate:

1. The electric field (vector) at the point (0,0,0).
2. The force acting on charge Q_3.

The electric field $\vec{E}$ at $\vec{r}$, due to Q_1, Q_2, and Q_3, is given by the superposition of the fields produced by each point charge:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{2} \frac{Q_i}{|\vec{r}-\vec{r_i}|^3} \Big(\vec{r}-\vec{r_i}\Big)$$

Therefore, at $\vec{r}=\vec{0}$, it is given by:

$$\vec{E}(\vec{0}) = \frac{1}{4\pi\epsilon_0} \Big\{ \frac{Q_1}{|\vec{r_1}|^3}\Big(-\vec{r_1}\Big) + \frac{Q_2}{|\vec{r_2}|^3}\Big(-\vec{r_2}\Big) + \frac{Q_3}{|\vec{r_3}|^3}\Big(-\vec{r_3}\Big) \Big\}$$

Worked example: Electric field and force of 3 point charges

Substituting the given quantities into the previous equation, we get:

$$\vec{E}(\vec{0}) = (9.0 \times 10^{9}\ \frac{N \cdot m^2}{C^2})\Big\{\frac{(1.0 \times 10^{-9}\ C)}{\sqrt{2}^3 \times 10^{-6}\ m^3}\Big(1,1,0\Big) \times 10^{-2} m +$$

$$+\frac{(1.0 \times 10^{-9}\ C)}{\sqrt{2}^3 \times 10^{-6}\ m^3}\Big(-1,1,0\Big) \times 10^{-2} m + \frac{(-2.0 \times 10^{-9}\ C)}{2^3 \times 10^{-6}\ m^3}\Big(0,-2,0\Big) \times 10^{-2} m\Big\} \Rightarrow$$

$$\vec{E}(\vec{0}) = 9.0 \times 10^{9}\Big\{\frac{2.0 \times 10^{-11}}{\sqrt{2}^3 \times 10^{-6}} + \frac{4.0 \times 10^{-11}}{2^3 \times 10^{-6}}\Big\}\Big(0,1,0\Big)\ \frac{N}{C} \Rightarrow$$

$$\vec{E}(\vec{0}) = 9.0 \times 10^{9}\Big\{0.707 \times 10^{-5} + 0.500 \times 10^{-5}\Big\}\Big(0,1,0\Big)\ \frac{N}{C} \Rightarrow$$

$$\vec{E}(\vec{0}) = 1.0863 \times 10^{5}\Big(0,1,0\Big)\ \frac{N}{C}$$

Worked example: Electric field and force of 3 point charges

The force $\vec{F}_3$ acting on Q_3 is the vector sum of $\vec{F}_{31}$ and $\vec{F}_{32}$.

$$\vec{F}_{31} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_3}{|\vec{r}_3 - \vec{r}_1|^3}\Big(\vec{r}_3 - \vec{r}_1\Big) \Rightarrow$$

$$\vec{F}_{31} = (9.0\times 10^9 \; \frac{N\cdot m^2}{C^2})\frac{(1.0\times 10^{-9}\; C)(-2.0\times 10^{-9}\; C)}{\sqrt{10.0}^3 \times 10^{-6}\; m^3}\Big(1,3,0\Big)\times 10^{-2}\; m$$

$$\vec{F}_{32} = \frac{1}{4\pi\epsilon_0}\frac{Q_2 Q_3}{|\vec{r}_3 - \vec{r}_2|^3}\Big(\vec{r}_3 - \vec{r}_2\Big) \Rightarrow$$

$$\vec{F}_{32} = (9.0\times 10^9 \; \frac{N\cdot m^2}{C^2})\frac{(1.0\times 10^{-9}\; C)(-2.0\times 10^{-9}\; C)}{\sqrt{10.0}^3 \times 10^{-6}\; m^3}\Big(-1,3,0\Big)\times 10^{-2}\; m$$

$$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32} = (9.0\times 10^9 \; \frac{N\cdot m^2}{C^2})\frac{(-2.0\times 10^{-18}\; C^2)}{\sqrt{10.0}^3 \times 10^{-6}\; m^3}\Big(0,6,0\Big)\times 10^{-2}\; m \Rightarrow$$

$$\vec{F}_3 = -34.1\Big(0,1,0\Big)\; \mu N$$

Worked example: Equilibrium position of charge

Question

In the figure below, particle 1 of charge $+1.0\ \mu$C and particle 2 of charge -3.0 μC are held at separation $L = 10.0$ cm on an x axis. If particle 3 of unknown charge q_3 is to be located such that the net electrostatic force on it from particles 1 and 2 is zero, what must be the x and y coordinates of particle 3?

There is no equilibrium position for q_3 between the two fixed charges, because it is being pulled by one and pushed by the other (since q_1 and q_2 have different signs). In this region, the two forces on q3 ($\vec{F}_{31}$ and $\vec{F}_{32}$) have components that cannot cancel.

Worked example: Equilibrium position of charge

For the same reason, there are no equilibrium positions off-axis (with the axis defined as that which passes through the two fixed charges), and therefore y=0.

On the semi-infinite region of the axis that is nearest q_2 and furthest from q_1 an equilibrium position for q_3 cannot be found because $|q_1| < |q_2|$ and the magnitude of force exerted by q_2 is everywhere (in that region) stronger than that exerted by q_1 on q_3.

Thus, we must look in the semi-infinite region of the axis which is nearest q_1 and furthest from q_2 (x<0).

Worked example: Equilibrium position of charge

In that region, the net force on q_3 has magnitude which is given by:

$$F_3 = \left| \frac{1}{4\pi\epsilon_0} \frac{|q_1 q_3|}{|x|^2} - \frac{1}{4\pi\epsilon_0} \frac{|q_2 q_3|}{(L+|x|)^2} \right|$$

Setting F_3 equal to zero, the above expression yields:

$$\frac{|q_1|}{|x|^2} - \frac{|q_2|}{(L+|x|)^2} = 0 \Rightarrow \left(\frac{L+|x|}{|x|}\right)^2 = \frac{|q_2|}{|q_1|} \Rightarrow \frac{L}{|x|} + 1 = \sqrt{\frac{|q_2|}{|q_1|}} \Rightarrow$$

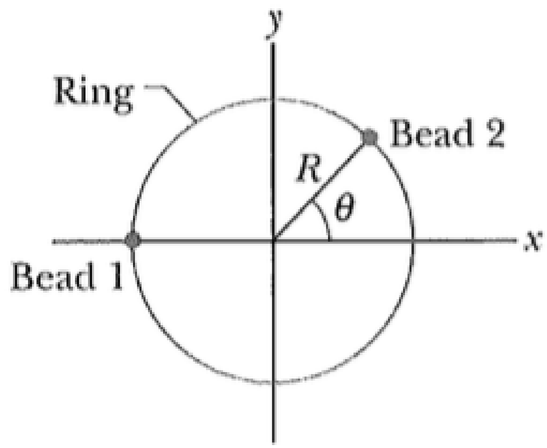
$$|x| = \frac{L}{\sqrt{\frac{|q_2|}{|q_1|}} - 1} \Rightarrow$$

$$|x| = \frac{10\ cm}{\sqrt{\frac{|-3\ \mu C|}{|+1\ \mu C|}} - 1} = \frac{10\ cm}{\sqrt{3} - 1} \approx 13.66\ cm \Rightarrow x = -13.66\ cm$$

Worked example: Field of two charged beads on ring

Question

The figure below shows a plastic ring of radius $R = 50$ cm. Two small charged beads are on the ring: Bead 1 of charge $+2\ \mu$C is fixed in place at the left side; bead 2 of charge $+6\ \mu$C can be moved along the ring. The two beads produce a net electric field of magnitude E at the centre of the ring. At what (a) positive and (b) negative value of angle θ should bead 2 be positioned so that $E = 2 \times 10^5$ N/C?



Worked example: Field of two charged beads on ring

The net electric field components along x and y are

$$E_x = \frac{q_1}{4\pi\epsilon_0 R^2} - \frac{q_2 cos\theta}{4\pi\epsilon_0 R^2} \quad and \quad E_y = -\frac{q_2 sin\theta}{4\pi\epsilon_0 R^2}$$

The magnitude of the net electric field is:

$$E = \sqrt{E_x^2 + E_y^2} = \sqrt{(\frac{q_1}{4\pi\epsilon_0 R^2} - \frac{q_2 cos\theta}{4\pi\epsilon_0 R^2})^2 + (-\frac{q_2 sin\theta}{4\pi\epsilon_0 R^2})^2}$$

$$= \frac{1}{4\pi\epsilon_0 R^2}\sqrt{(q_1 - q_2 cos\theta)^2 + (q_2 sin\theta)^2}$$

$$= \frac{1}{4\pi\epsilon_0 R^2}\sqrt{q_1^2 + q_2^2 cos^2\theta + q_2^2 sin^2\theta - 2q_1 q_2 cos\theta}$$

$$= \frac{1}{4\pi\epsilon_0 R^2}\sqrt{q_1^2 + q_2^2 - 2q_1 q_2 cos\theta}$$

Worked example: Field of two charged beads on ring

Therefore:

$$(E4\pi\epsilon_0 R^2)^2 = q_1^2 + q_2^2 - 2q_1 q_2 cos\theta \Rightarrow cos\theta = \frac{q_1^2 + q_2^2 - E^2(4\pi\epsilon_0)^2 R^4}{2q_1 q_2}$$

Substituting $E = 2 \times 10^5$ N/C $q_1 = 2 \times 10^{-6}$ C, $q_2 = 6 \times 10^{-6}$ C, and $R = 0.5$ m, in the equation above, we have:

$$cos\theta = \frac{(2 \times 10^{-6}\ C)^2 + (6 \times 10^{-6}\ C)^2 - (\frac{2\times10^5\ N/C}{8.99\times10^9\ N{\cdot}m^2/C^2})^2(0.5\ m)^4}{2(2 \times 10^{-6}\ C)(6 \times 10^{-6}\ C)} \Rightarrow$$

$$cos\theta \approx \frac{3}{8} = 0.375 \Rightarrow \theta \approx \pm 68^o$$

PHYS201 scientific programming task for Lecture 1

Write a program that:

- **Can read an arbitrary distribution of N discrete charges (in 2-D)**, e.g. by accepting as input a text file with N rows, where the i^{th} row contains the coordinates x_i, y_i in m, and the charge q_i in C.
- **Allows you to visualise the input charge distribution**, e.g using appropriately-positioned circles whose color or size represents amount of charge.
- **Allows you to visualise the electric field lines** in the vicinity of the charge distrubution.

Test your program, reproducing some of the simpler field maps shown before.

PHYS 201 / Lecture 2
Electric flux; Gauss' law

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Lectures delivered at the University of Liverpool, 2021-22

December 15, 2021

- **Electric charge**
  - The source of electric phenomena
  - An intrinsic property of matter
  - Comes in two varieties (positive and negative)
  - An algebraic quantity (+q + (-q) = 0)
  - It is quantised
  - It is conserved (globally and locally)
  - SI unit: Coulomb (C) [= 1 A · 1 s]

- **Coulomb's law**
  - Describes the force between two point charges
  - The force $\vec{F}_{12}$ exerted on test charge 1 by charge 2 is:

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}\hat{r}_{12} \quad or \quad \vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2)$$

- **Superposition principle**
  - Allows the calculation of the total force on a charge Q from an array of other charges q_1, q_2, ..., q_n

$$\vec{F}_Q = \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{Qq_i}{|\vec{r}_Q - \vec{r}_{q_i}|^3} (\vec{r}_Q - \vec{r}_{q_i})$$

  - Total force is the vector sum of forces.
  - Not a logical necessity: An experimental fact!

- **Continuous distributions of charge**
  - Made the leap from discrete to continuous charge distributions described by a charge density
  - Reformulated Coulomb's law for continuous charge distributions

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0} \int_\tau d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')$$

Lecture 1 - Revision (cont'd)

- **Electric field**
  - A more fundamental way to think about electric forces in terms of a field that permeates space.
  - Defined the electric field $\vec{E}$ as the force exerted on a test charge Q, placed in position $\vec{r}$, per unit charge.

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q(\vec{r})}{Q}$$

- **Visualizing the electric field - field lines**
  - Field direction indicated by the direction of field lines
  - Field strength indicated by the density of field lines
  - Force tangential to field lines
  - Field lines start from positive charges and end on negative ones: Positive charges are "sources", negative charges are "sinks"
  - Field lines can not cross

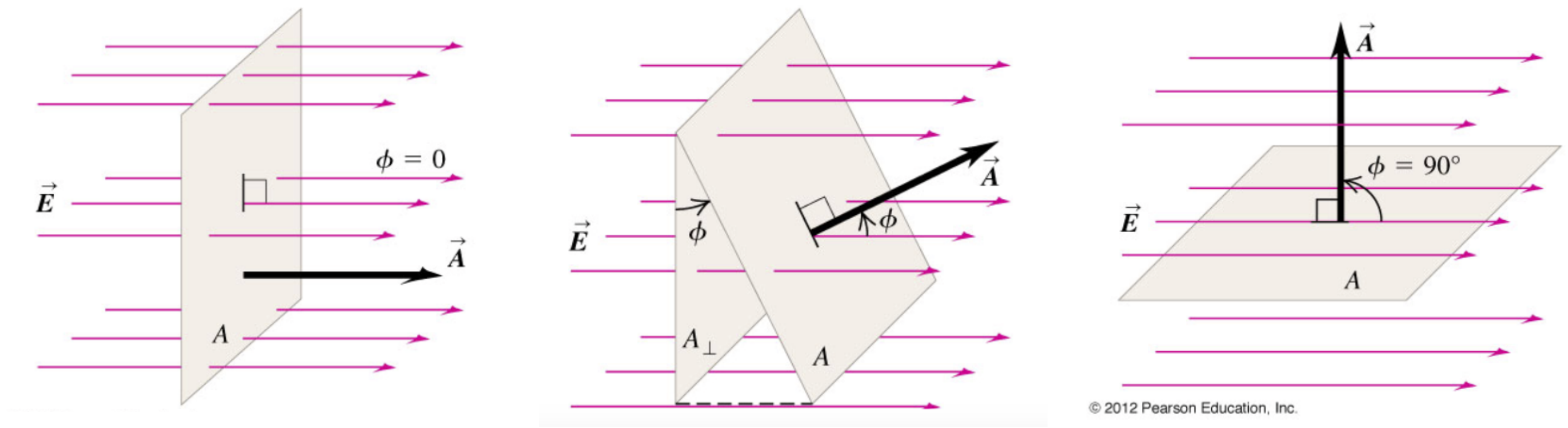
Plan for Lecture 2

- **Electric flux**
  - The "flow" of an electric field (i.e. the number of electric field lines) through a surface.
- **Gauss' law**
  - Relates the flux through a closed surface with the net charge contained in it.
  - Our first Maxwell equation! Will study both its 'differential' and 'integral' forms.

Electric flux

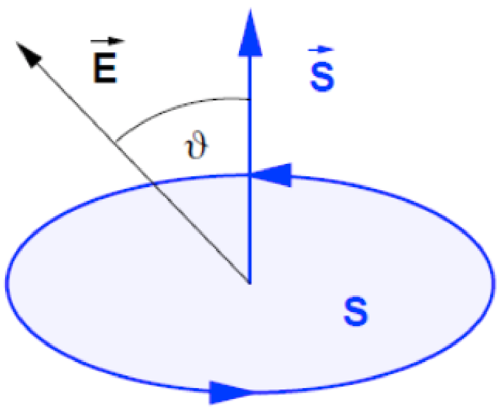
The flux Φ_E of the electric field $\vec{E}$ through a surface S is the *number of field lines* flowing through S.

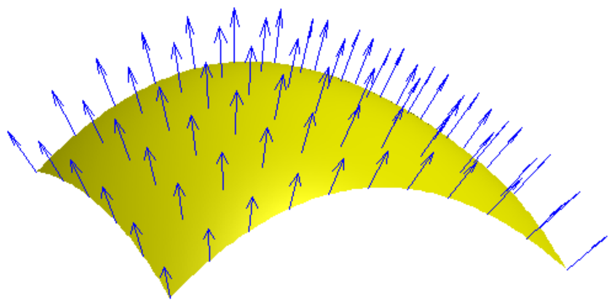
- The number of field lines, and thus Φ_E is proportional to $|\vec{E}|$ and, obviously, proportional to S.
- The flux decreases if the surface is not perpendicular to the field, and becomes zero when the surface is parallel to the field.



We can write the electric flux as $\Phi_E = |\vec{E}|\ S\ cos\theta$.

Electric flux





Note that if the surface is closed, it is customary to define the direction of each element dS to point outwards.

It is useful to express a surface S as a vector: The surface vector $\vec{S}$ is perpendicular to the surface, and has a size which is proportional to its area. The vector can point to either side of S.
Then the electric flux Φ_E can be expressed as the dot product of $\vec{E}$ and $\vec{S}$:

$$\Phi_E = \vec{E} \cdot \vec{S} = |\vec{E}| \; |\vec{S}| \; cos\theta$$

Of course, S may not be flat, or $\vec{E}$ might change along S. But we can always divide S in infinitesimally small areas dS that are approximately flat and within which $\vec{E}$ remains constant.
Then the flux $d\Phi_E$ through each element dS is given by $d\Phi_E = \vec{E} \cdot d\vec{S}$ and the total flux Φ_E by the integral:

$$\Phi_E = \int_S \vec{E} \cdot d\vec{S}$$

Worked example: Calculation of electric flux (Easy)

Question

A surface has the area vector $\vec{A} = (2\hat{x}+3\hat{y})$ m^2. What is the flux of a uniform electric field through the area if the field is

- $\vec{E} = 4\hat{x}$ N/C, and
- $\vec{E} = 4\hat{z}$ N/C?

The flux of a uniform electric field $\vec{E}$ through a planar surface $\vec{A}$ is given by

$$\Phi = \vec{E} \cdot \vec{A}$$

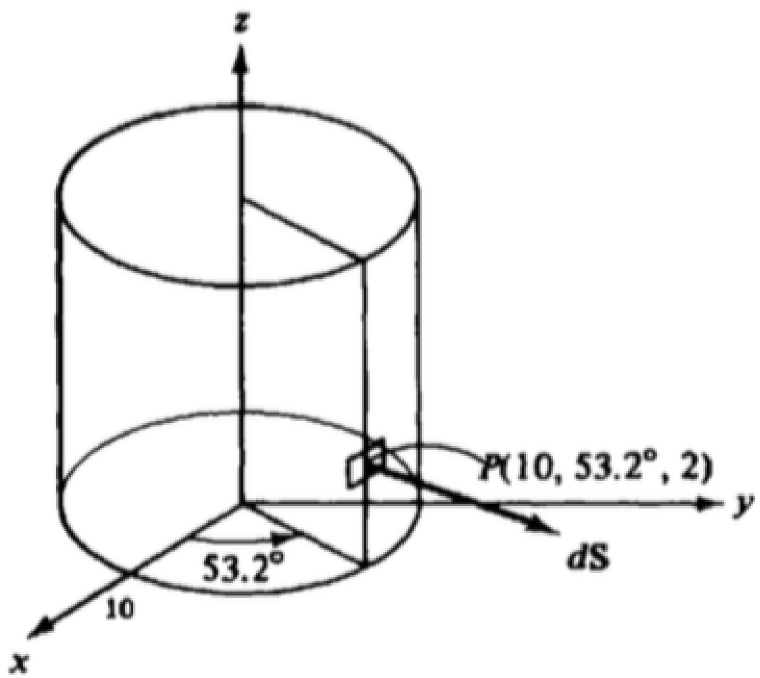
Therefore, for the 2 cases:

- $\Phi = \Big\{4\hat{x} \cdot (2\hat{x} + 3\hat{y})\Big\} \frac{N \cdot m^2}{C} = \Big\{8(\hat{x} \cdot \hat{x}) + 12(\hat{x} \cdot \hat{y})\Big\} \frac{N \cdot m^2}{C} = 8 \frac{N \cdot m^2}{C}$
- $\Phi = \Big\{4\hat{z} \cdot (2\hat{x} + 3\hat{y})\Big\} \frac{N \cdot m^2}{C} = \Big\{8(\hat{z} \cdot \hat{x}) + 12(\hat{z} \cdot \hat{y})\Big\} \frac{N \cdot m^2}{C} = 0 \frac{N \cdot m^2}{C}$

Question

An electric field $\vec{E}$ is given by:

$$\vec{E} = \{2x\,\frac{N}{m\cdot C}\}\hat{x} + \{2\,\frac{N}{C} - 2y\,\frac{N}{m\cdot C}\}\hat{y} + \{4z\,\frac{N}{m\cdot C}\}\hat{z}$$



Determine the electric flux crossing a 1 mm $\times$ 1 mm area on the surface of the cylindrical shell at:

$$r = 10\ m,\ \ z = 2\ m,\ \phi = 53.2^o$$

Worked example: Calculation of electric flux

At the point P:

$$x = rcos\phi = (10\ m)cos53.2^o = (10\ m)0.6 = 6\ m$$

$$y = rsin\phi = (10\ m)sin53.2^o = (10\ m)0.8 = 8\ m$$

The electric field $\vec{E}$ varies very little over the small given area. Taking it to be constant is a good approximation.
Therefore, I will just substitute x = 6 m, y = 8 m, and z = 2 m in the given expression for $\vec{E}$:

$$\vec{E} = \{2 \cdot (6\ m)\ \frac{N}{m \cdot C}\}\hat{x} + \{2\ \frac{N}{C} - 2 \cdot (8\ m)\frac{N}{m \cdot C}\}\hat{y} + \{4 \cdot (2\ m)\ \frac{N}{m \cdot C}\}\hat{z} \Rightarrow$$

$$\vec{E} = \{12\hat{x} - 14\hat{y} + 8\hat{z}\}\frac{N}{C}$$

Worked example: Calculation of electric flux

Now, I need to express the area $d\vec{S}$ in vector form.

Because the area is very small ($|d\vec{S}| = 1$ mm^2) in comparison to the size of the cylinder (r = 10 m), we can consider the area $d\vec{S}$ to be *planar*.

Also, as one can easily see, the vector $d\vec{S}$ lies on the xy plane.

Therefore:

$$d\vec{S} = dS_x\hat{x} + dS_y\hat{y}$$

where:

$$dS_x = |d\vec{S}|cos\phi = (10^{-6}\ m^2)\ cos53.2^o = 0.6 \times 10^{-6}\ m^2$$

$$dS_y = |d\vec{S}|sin\phi = (10^{-6}\ m^2)\ sin53.2^o = 0.8 \times 10^{-6}\ m^2$$

Putting everything together:

$$d\vec{S} = \left\{0.6\hat{x} + 0.8\hat{y}\right\} \times 10^{-6}\ m^2$$

Worked example: Calculation of electric flux

We're now ready to calculate the flux $d\Phi$ of the electric field $\vec{E}$ through the small patch $d\vec{S}$:

$$d\Phi = \vec{E} \cdot d\vec{S} \Rightarrow$$

$$d\Phi = \left\{12\hat{x} - 14\hat{y} + 8\hat{z}\right\}\frac{N}{C} \cdot \left\{0.6\hat{x} + 0.8\hat{y}\right\} \times 10^{-6}\ m^2 \Rightarrow$$

$$d\Phi = \left\{12 \cdot 0.6 - 14 \cdot 0.8 + 8 \cdot 0\right\} \times 10^{-6}\frac{N \cdot m^2}{C} = \left\{7.2 - 11.2 + 0\right\} \times 10^{-6}\frac{N \cdot m^2}{C} \Rightarrow$$

$$d\Phi = -4 \times 10^{-6}\frac{N \cdot m^2}{C}$$

Q: What is the significance of the minus sign?

Electric flux through a closed surface

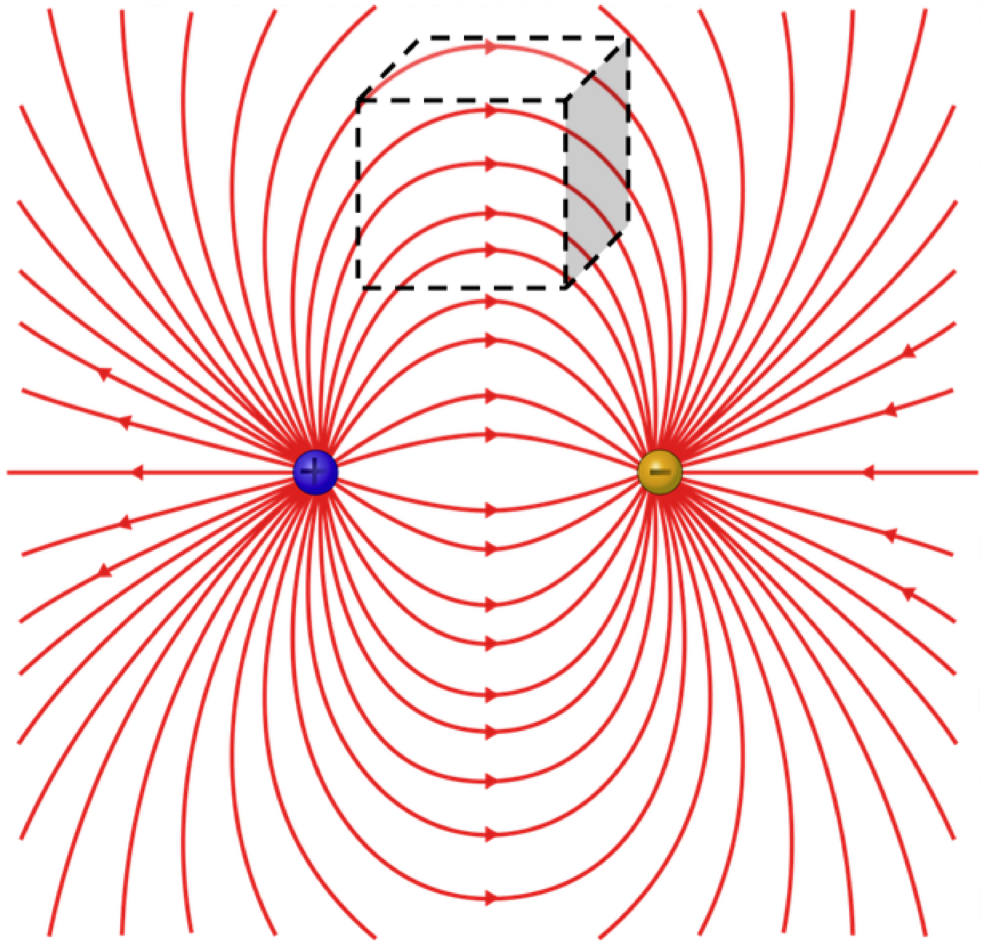
What is the electric flux through the closed surface shown?

No charge is contained in the closed surface.

Since flux lines start from positive charges and end in negative charges, every line that enters the closed surface eventually exits.

Field lines that enter and field lines that exit contribute with different signs in $\oint \vec{E} d\vec{S}$ (recall that the surface vector points outwards).

The flux is 0.

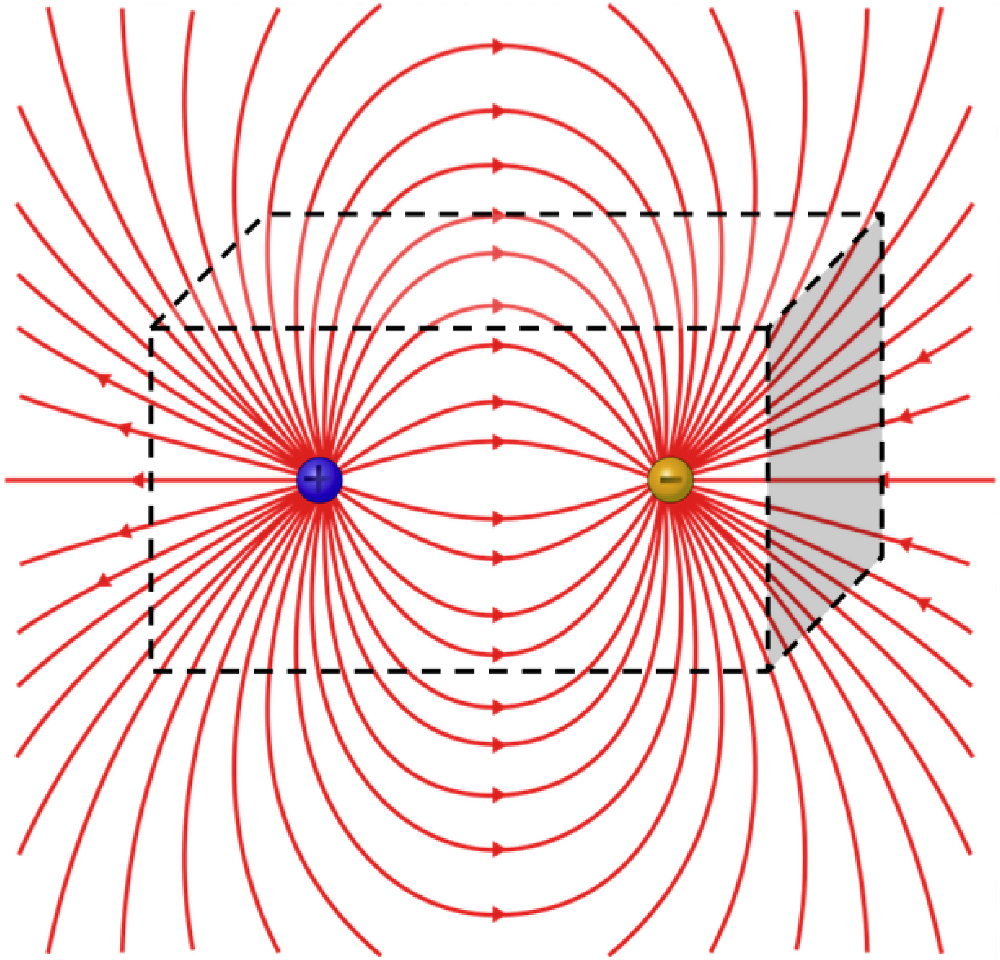


Electric flux through a closed surface

What is the electric flux through the closed surface shown?

Both a positive and an equal negative charge are contained within the closed surface. Certain lines originating from the positive charge can reach the negative charge without exiting the closed surface. Lines that do exit though the closed surface, re-enter to terminate on the negative charge.

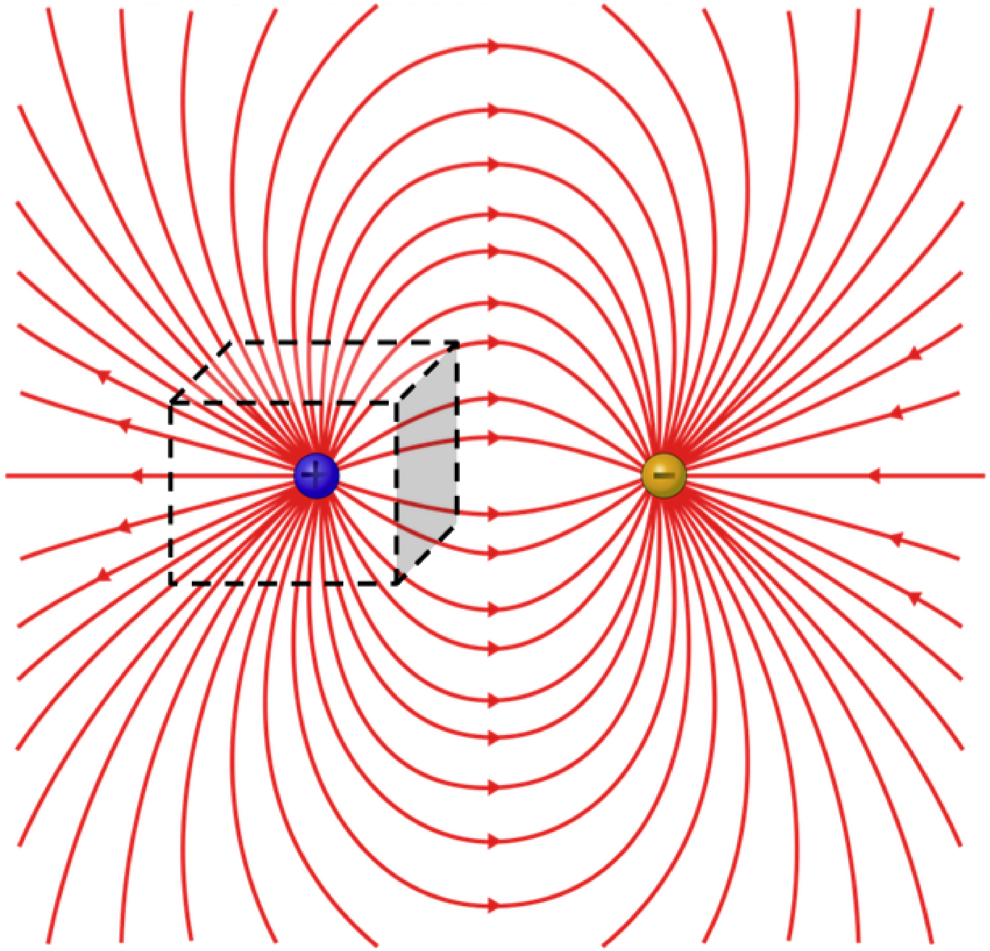
The flux is 0 again.



Electric flux through a closed surface

What is the electric flux through the closed surface shown?

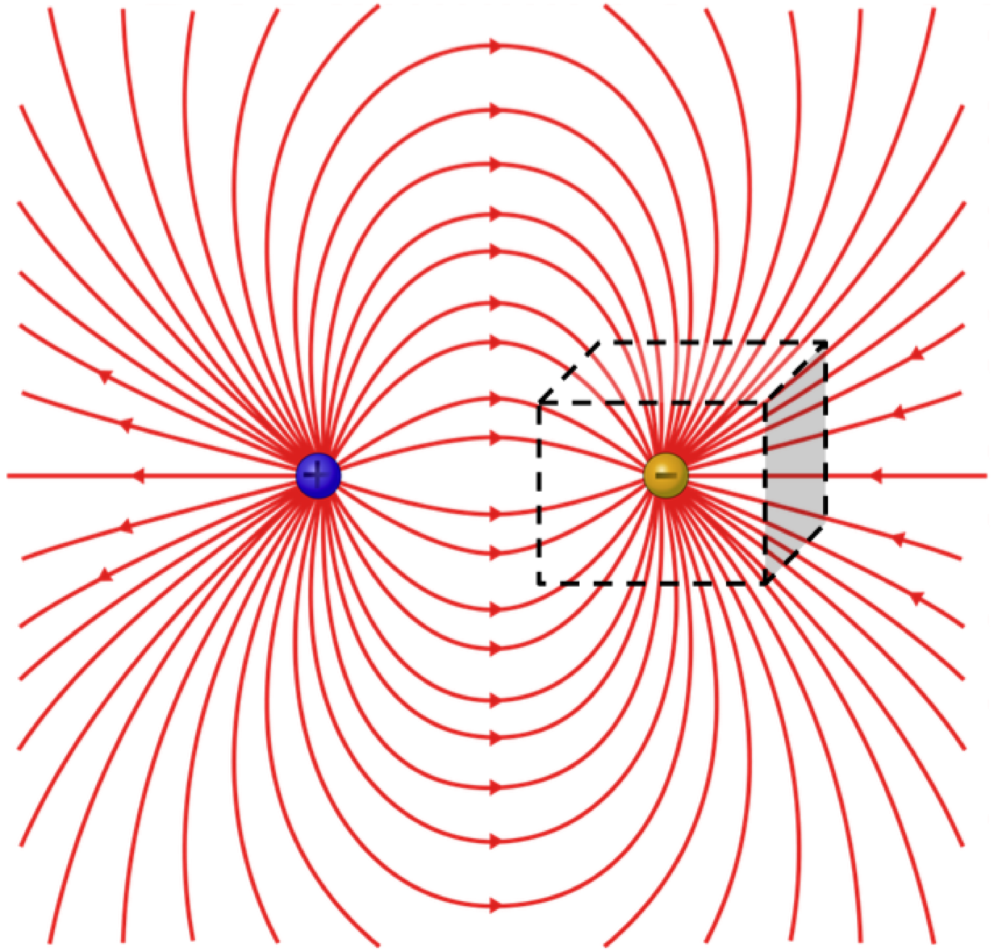
All field lines are exiting. The flux is non-zero and positive.



Electric flux through a closed surface

What is the electric flux through the closed surface shown?

Now all field lines are entering. The flux is non-zero. It has the same value as in the previous page, but now it is negative.



In the previous few slides, the *essence* of **Gauss' law** was illustrated.

Carl Friedrich Gauss
(1777-1855)
German mathematician

The electric flux $\Phi_E = \oint \vec{E} d\vec{S}$ through a closed surface is related to the net charge Q_{enc} enclosed within the surface.

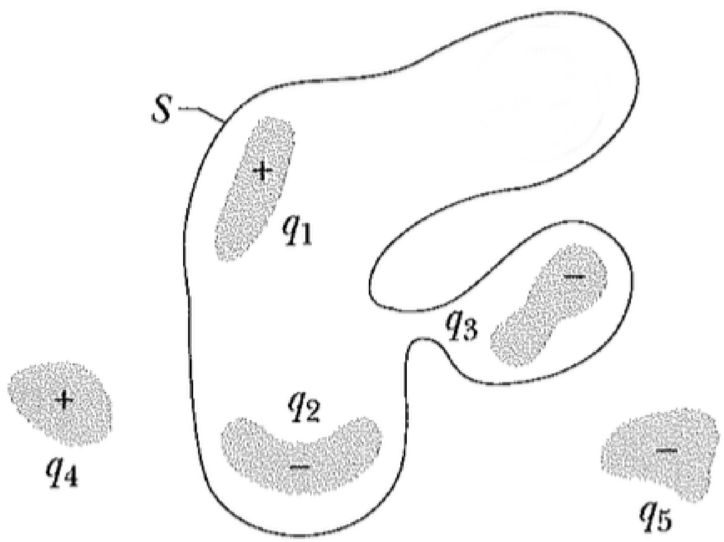
As it turns out, this relation is simply:

$$\Phi_E = \frac{Q_{enc}}{\epsilon_0}$$

It doesn't matter how elaborate the shape of the surface or the enclosed charge distribution (and the resulting field) may be:
The above simple relation **always holds**!

Worked example: Finding Φ from Q_{enc} (Easy)

Question



The figure on the left shows 5 charged lumps of a material. The cross-section of the surface S is indicated.
What is the *net electric flux* through the surface if:

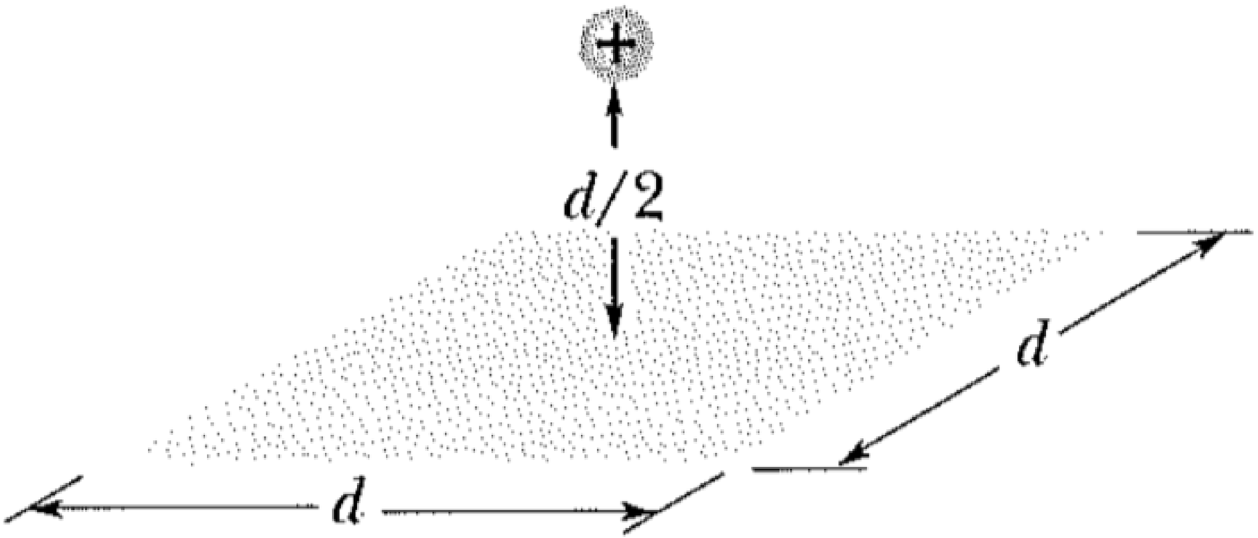
- $q_1 = q_4 = 3.1$ nC,
- $q_2 = q_5 =$ -5.9 nC, and
- $q_3 =$ -3.1 nC?

$$\Phi = \frac{Q_{enc}}{\epsilon_0} = \frac{q_1 + q_2 + q_3}{\epsilon_0} = \frac{(3.1 - 5.9 - 3.1) \times 10^{-9}\ C}{8.85 \times 10^{-12}\ C^2/(N \cdot m^2)} = -0.67 \times 10^3 \frac{N \cdot m^2}{C}$$

Worked example: Finding Φ from Q_{enc}

Question

A proton is a distance $d/2$ directly above the center of a square of side d. What is the magnitude of the electric flux through the square?



Worked example: Finding Φ from Q_{enc}

Consider a closed Gaussian surface in the shape of a cube of edge length d, with the proton (of charge q) positioned in the centre of the cube. Gauss's law tells us that the flux Φ through that surface is given by:

$$\Phi = \frac{q}{\epsilon_0}$$

Because of the symmetry of the configuration, the same electric flux Φ_{face} passes through each of the 6 faces of the cube, so:

$$\Phi = 6\Phi_{face}$$

Therefore:

$$\Phi_{face} = \frac{q}{6\epsilon_0} \Rightarrow$$

$$\Phi_{face} = \frac{1.6 \times 10^{-19}\ C}{6(8.85 \times 10^{-12}\ \frac{C^2}{N \cdot m^2})} = 3 \times 10^{-9}\ \frac{N \cdot m^2}{C}$$

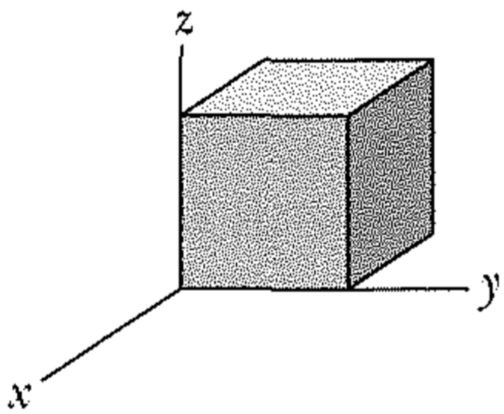
Worked example: Finding Q_{enc} from Φ

Question

The figure below shows a closed surface in the shape of a cube of edge length 2 m. It lies in a region where the non-uniform electric field $\vec{E}$ is given by the following:

$$\vec{E} = (3x + 4)\hat{x} + 6\hat{y} + 7\hat{z}.$$

In the above expression, $|\vec{E}|$ has units of N/C if x is given in m. What is the net charge contained by the cube?



Worked example: Finding Q_{enc} from Φ

It is easy to see that the constant terms of $\vec{E}$ make no contribution to the total flux, therefore we can consider only the non-constant component.

Let $\vec{E}_{nc}$ be that non-constant component of the field, which is given by:

$$\vec{E}_{nc} = (3x\frac{N \cdot m}{C})\hat{x}$$

Let $\vec{S}_{x+}$ ($\vec{S}_{x-}$) be the area vector of the front (back) face of the cube lying on the yz plane. That vector points towards positive (negative) x.

For a given x, $\vec{E}_{nc}$ is constant along the yz plane. Since the field is uniform and the surfaces are planar, the flux through the 2 cube faces lying on yz planes can be written as:

$$\Phi_x = \vec{S}_{x+} \cdot \vec{E}_{nc}(x = 0\ m) + \vec{S}_{x-} \cdot \vec{E}_{nc}(x = -2\ m)$$

Worked example: Finding Q_{enc} from Φ

Because the magnitude of $\vec{E}_{nc}$ is proportional to x and the first term is evaluated at x=0, the first term is zero. What remains is:

$$\Phi_x = \vec{S}_{x-} \cdot \vec{E}_{nc}(x = -2\ m) = (4\ m^2)(-\hat{x}) \cdot (-6\ N/C)\hat{x} = 24\ N \cdot m^2/C$$

Since $\vec{E}_{nc}$ points along $\hat{x}$, the flux Φ_y (Φ_z) through the cube faces lying on xz (xy) planes is zero. Therefore, the total flux through all 6 faces of the cube is:

$$\Phi = 24\ N \cdot m^2/C$$

Finally, from Gauss's law:

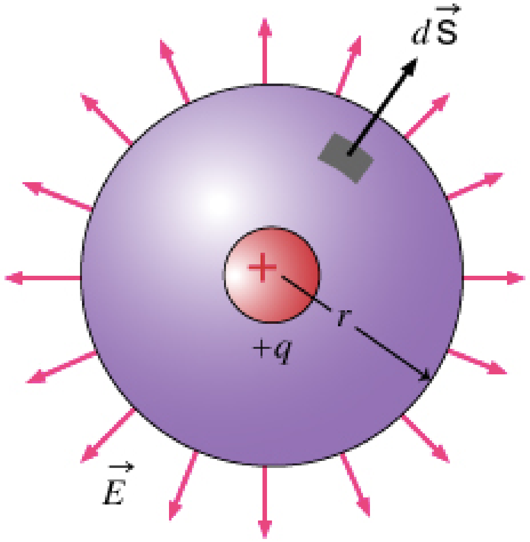
$$\Phi = \frac{q_{enc}}{\epsilon_0} \Rightarrow q_{enc} = \Phi\epsilon_0 \Rightarrow$$

$$q_{enc} = (24\ N \cdot m^2/C)(8.85 \times 10^{-12}\ \frac{C^2}{N \cdot m^2}) \Rightarrow$$

$$q_{enc} = 2.124 \times 10^{-10}\ C$$

Gauss' law - Derivation for a simple case

We will derive Gauss' law for a very simple case: A single charge Q in the centre of a sphere of radius R.



$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$d\vec{S} = dS\hat{r}$$

$$\Phi_E = \oint_{sphere} \vec{E} \cdot d\vec{S} \Rightarrow$$

$$\Phi_E = \oint_{sphere} \left(\frac{Q}{4\pi\epsilon_0 R^2}\hat{r}\right) \cdot (dS\hat{r}) \xRightarrow{\hat{r}\cdot\hat{r}=1}$$

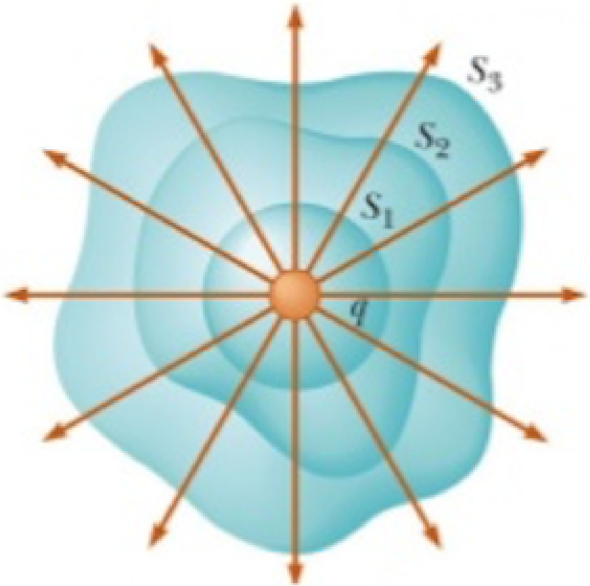
$$\Phi_E = \frac{Q}{4\pi\epsilon_0 R^2} \oint_{sphere} dS \Rightarrow$$

$$\Phi_E = \frac{Q}{4\pi\epsilon_0 R^2} 4\pi R^2 \Rightarrow$$

$$\mathbf{\Phi_E = \frac{Q}{\epsilon_0}}$$

Gauss' law - Generalisation for any surface

Although the law was derived for a very simple case, it is a **general result**.

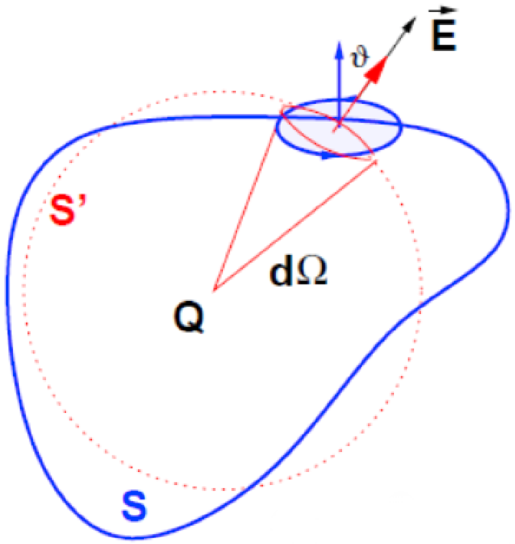


The same result would have been obtained

- if the charge was not in the centre but in any other position within the sphere, or
- if I had any other closed surface shape instead of a sphere.

Only one thing matters for the calculation of Φ_E: The **net charge within the surface**.

Gauss' law - Generalisation for any surface



Consider element dS on a general surface S and the corresponding element dS' on a sphere S', both covering the same solid angle $d\Omega$

By definition, on the spherical surface S':

$$d\Omega = \frac{dS'}{r^2} \Rightarrow dS' = r^2 d\Omega$$

By construction:

$$dS' = dScos\theta \Rightarrow dS = \frac{dS'}{cos\theta} \Rightarrow dS = \frac{r^2 d\Omega}{cos\theta}$$

So, the flux Φ_E through the general surface S is

$$\Phi_E = \int_S \vec{E} \cdot d\vec{S} = \int_S EdScos\theta = \int_\Omega E \frac{r^2 d\Omega}{\cancel{cos\theta}} \cancel{cos\theta} =$$

$$\int_\Omega Er^2 d\Omega = \int_{S'} EdS' = \int_{S'} \vec{E} d\vec{S}' = \Phi_E^{sphere} = \frac{Q}{\epsilon_0}$$

Gauss' law - Generalization for any charge distribution

Our result for a *single charge* enclosed by an arbitrary closed surface S is:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

What is the **form of Gauss' law if the surface S encloses an array of charges** (or a continuous charge distribution)? The result can be obtained via a straightforward application of the superposition principle.

$$\left.\begin{array}{l} \oint_S \vec{E_1} \cdot d\vec{S} = \frac{Q_1}{\epsilon_0} \\ \\ \oint_S \vec{E_2} \cdot d\vec{S} = \frac{Q_2}{\epsilon_0} \\ \\ \dots \\ \\ \oint_S \vec{E_n} \cdot d\vec{S} = \frac{Q_n}{\epsilon_0} \end{array}\right\} \oint_S \left(\sum_{i=1}^{n} \vec{E_i}\right) \cdot d\vec{S} = \frac{\sum_{i=1}^{n} Q_i}{\epsilon_0} \Rightarrow \oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

Integral form of Gauss' law

Our result for any array of charges (or continuous charge distribution), with net charge Q_{enc}, enclosed by an arbitrary closed surface S is:

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

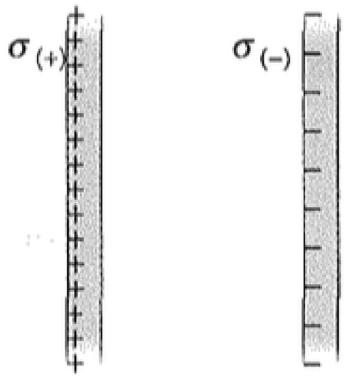
This is known as the **integral form of Gauss' law**.

This form is useful in cases where the problem at hand has a symmetry. The symmetry can be exploited to simplify the calculation of the integral.

Worked example (Planar symmetry)

Question

The figure on the left shows two large, parallel, non-conducting sheets, each with a fixed uniform charge on one side.



The magnitudes of the surface charge densities are $\sigma_{(+)} = 6.8\ \mu\text{C}/\text{m}^2$ for the positively charged sheet, and $\sigma_{(-)} =$ - $4.3\ \mu\text{C}/\text{m}^2$ for the negatively charged sheet. Find the electric field $\vec{E}$ (a) to the left of the sheets, (b) between the sheets, and (c) to the right of the sheets.

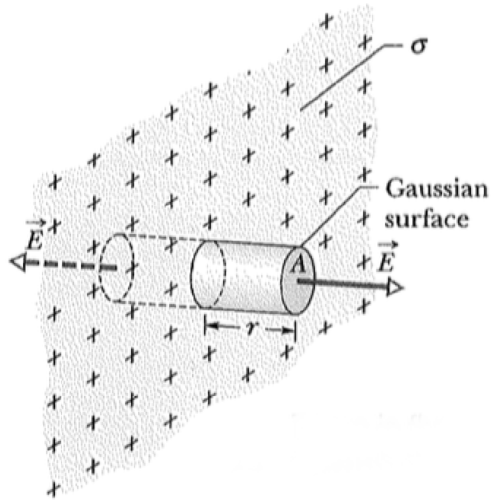
Note: We will revisit this example when we study the **parallel plate capacitor** in Lecture 4.

Worked example (Planar symmetry)

Consider an infinite positively charged plane with surface charge density σ and the (appropriately chosen) cylindrical *Gaussian* surface shown below.

The symmetry of the problem indicates that $\vec{E}$ is **perpendicular to the sheet**: There is no flux through the curved surface but only through the 2 circular end caps.

Additionally, because the plane is positively charged, the field $\vec{E}$ is directed outwards.



The surface charge enclosed by the Gaussian surface is $Q_{enc} = \sigma A$.

Therefore, applying Gauss' law, we get:

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow EA + EA = \frac{\sigma A}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$

The electric field stays the same for any point, irrespectively of its distance from the plane.

Worked example (Planar symmetry)

Repeating the exercise for a negatively charged plane yield of the same magnitude. with the difference that it would point inwards.

For the case of the 2 parallel plates, the electric field $\vec{E}$ anywhere in space is given by the superposition of 2 fields:

- The field $\vec{E}_{(+)}$, due to the positively charged plane, with direction away from that plane and magnitude:

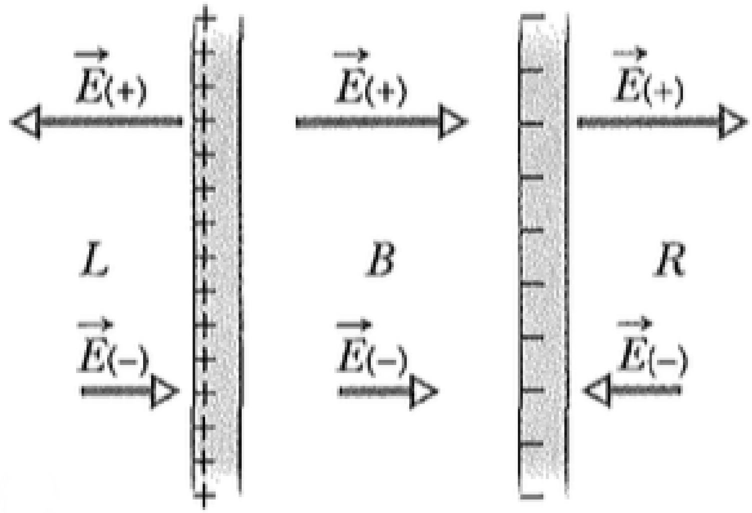
$$E_{(+)} = \frac{\sigma_{(+)}}{2\epsilon_0} = \frac{6.8 \times 10^{-6}\ C/m^2}{2(8.85 \times 10^{-12}\ C/(N \cdot m^2))} = 3.84 \times 10^5 N/C$$

- The field $\vec{E}_{(-)}$, due to the positively charged plane, with direction away from that plane and magnitude:

$$E_{(-)} = \frac{\sigma_{(-)}}{2\epsilon_0} = \frac{4.3 \times 10^{-6}\ C/m^2}{2(8.85 \times 10^{-12}\ C/(N \cdot m^2))} = 2.43 \times 10^5 N/C$$

Worked example (Planar symmetry)

We can now work out the electric field on the left of the planes, between the planes, and on the right of the planes.



- On the **left** of the planes:
  $E_L = E_{(+)} - E_{(-)} = 1.4 \times 10^5 \; N/C$ directed to the left.
- **Between** the planes:
  $E_B = E_{(+)} + E_{(-)} = 6.3 \times 10^5 \; N/C$ directed to the right.
- On the **right** of the planes:
  $E_R = E_{(+)} - E_{(-)} = 1.4 \times 10^5 \; N/C$ directed to the right.

Worked example (Cylindrical symmetry)

Question

A long, straight, thin plastic rod is uniformly charged with $+10$ nC/m. Calculate the magnitude of the electric field at 10 cm away from the rod. What is the direction of the electric field at any point away from the rod?

Consider a cylindrical surface with the rod running along its axis.

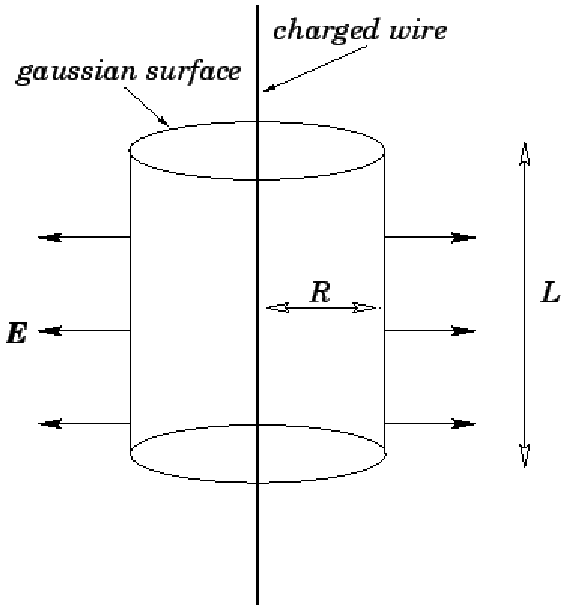
Gauss' law in integral form is:

$$\oint \vec{E}(r) \cdot d\vec{S} = \frac{Q_{encl}}{\epsilon_0}$$

Assuming the rod is infinitely long (i.e. ignoring fringe effects) the electric field is everywhere perpendicular to the rod and points away from it. Therefore, all the flux is going out of the sides of the cylinder.

Worked example (Cylindrical symmetry)

For a length L and linear charge density λ:



$$\oint \Big(E(r)\hat{r}\Big) \cdot \Big(dS\hat{r}\Big) = \frac{\lambda L}{\epsilon_0} \Rightarrow$$

$$E(r) 2\pi r L = \frac{\lambda L}{\epsilon_0} \Rightarrow$$

$$E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$$

At the given distance away from the rod (10 cm), the magnitude of the electric field is:

$$E = 2 \cdot (9.0 \times 10^9 \; \frac{N \cdot m^2}{C^2}) \frac{10 \times 10^{-9} \; C/m}{0.1 \; m} \Rightarrow E = 1.8 \; kV/m$$

Worked example (Spherical symmetry)

Question

A solid non-conductive sphere with radius R is uniformly charged and carries total charge Q. Calculate the electric field E at radius r from its centre for the following two cases:

1. Inside the sphere ($r < R$).
2. On the surface or outside the sphere ($r \geq R$).

How is the result for case 2 above related to that for a point charge Q at the origin?

The charge density is:

$$\rho = \frac{Q}{\tau} = \frac{Q}{4\pi R^3/3} = \frac{3Q}{4\pi R^3}$$

Worked example (Spherical symmetry)

Exploiting the spherical symmetry of the problem, the flux of the electric field at any distance r is:

$$\Phi(r) = 4\pi r^2 E(r)$$

From Gauss' law:

$$\Phi(r) = \frac{Q_{encl}(r)}{\epsilon_0}$$

Therefore:

$$4\pi r^2 E(r) = \frac{Q_{encl}(r)}{\epsilon_0}$$

$$E(r) = \frac{Q_{encl}(r)}{4\pi\epsilon_0 r^2}$$

Worked example (Spherical symmetry)

For $r < R$:

$$Q_{encl}(r) = \rho\tau(r) = \frac{3Q}{4\pi R^3} \cdot \frac{4\pi r^3}{3} = Q\frac{r^3}{R^3}$$

$$E(r) = \frac{Q\frac{r^3}{R^3}}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0 R^3} r$$

For $r \geq R$:

$$Q_{encl}(r) = Q$$

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$$

The result for the case of $r \geq R$ is identical to that for point charge Q at the origin.

Integral and differential forms of Gauss' law

Our result for any array of charges (or continuous charge distribution), with net charge Q_{enc}, enclosed by an arbitrary closed surface S is:

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

This is known as the **integral form of Gauss' law**.

This form is useful in cases where the problem at hand has a symmetry. The symmetry can be exploited to simplify the calculation of the integral. Will will see several examples of the above in the workshops.

It is very useful to obtain Gauss' law in its **differential form**.
This is a more practical form for the analytical or numerical solution of problems that do not have a symmetry that we can exploit.

Before we can proceed, we need to continue brushing up our calculus skills.

Reminder: Gradient, Divergence, Curl and all that

- The $\vec{\nabla}$ (*nabla*) operator

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

- The nabla operator is a kind of a vector
  - It is a *vector operator* that acts upon the quantity on the right
  - It appears as a vector as long as we do not make a distinction between "acting upon" and "multiplying"

- Just as we have 3 kinds of vector multiplications (with scalar, dot product, cross product), we have 3 ways the nabla operator can act
  - $\vec{\nabla}$ (a scalar function) $\rightarrow$ **gradient**
  - $\vec{\nabla} \cdot$ (a vector function) $\rightarrow$ **divergence**
  - $\vec{\nabla} \times$ (a vector function) $\rightarrow$ **curl**

Reminder: Gradient, Divergence, Curl and all that

<u>Gradient</u>: This is a **generalization to many dimensions** of the concept of the derivative of a 1-dimensional function.

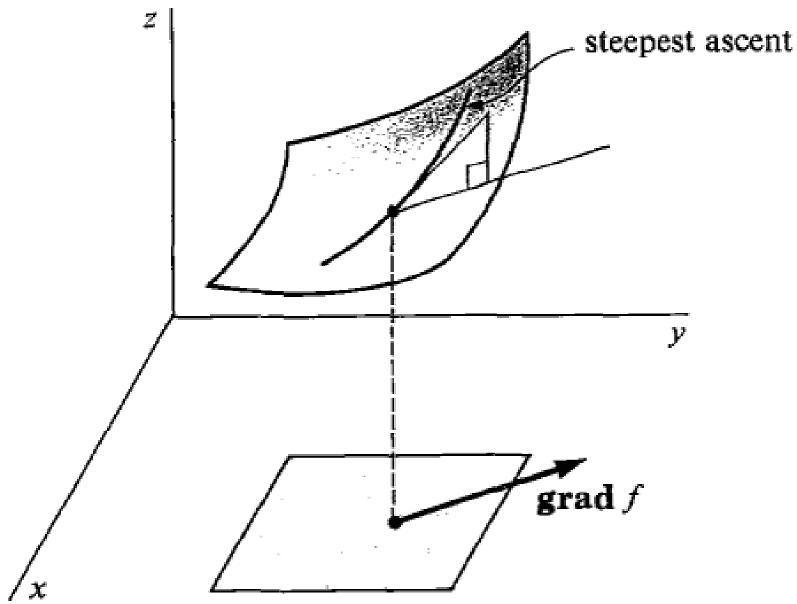
Let $f(x, y)$ be a scalar field in a 2-dimensional space. Its gradient is:

$$\vec{\nabla} f(x, y) = \left(\frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y} \right)$$

Like the usual derivative, $\vec{\nabla} f(x, y)$ tells us how fast $f(x, y)$ changes as we move in space.

Notice that $\vec{\nabla} f(x, y)$ is a vector. It tells us, not only how fast $f(x, y)$ changes, but also the **direction of the steepest ascent**.

Similarly for 3 or more dimensions.



Reminder: Gradient, Divergence, Curl and all that

In the previous slide, we used a **partial derivative**. It is worth stopping to make sure you all understand what this is.

Consider a function f of several variables x,y,z,... Notice that

$$\frac{\partial f(x,y,z,...)}{\partial x} \quad \textbf{is not the same as} \quad \frac{df(x,y,z,...)}{dx}$$

A partial derivative, with respect to x, of a function of several variables, e.g. f(x,y,z,...), is the derivative with respect to x **while all other variables are held constant**.

On the other hand, the total derivative takes into account, not only the change of the function as x changes, but also **adds in all the changes due to the indirect dependencies**:

$$\frac{df(x,y,z,...)}{dx} = \frac{\partial f(x,y,z,...)}{\partial x} + \frac{\partial f(x,y,z,...)}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial f(x,y,z,...)}{\partial z} \cdot \frac{\partial z}{\partial x} + ...$$

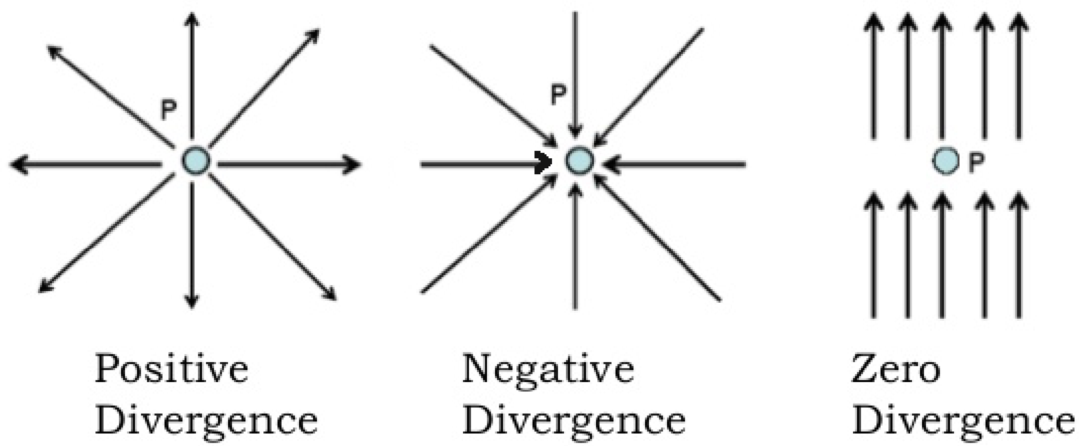
Reminder: Gradient, Divergence, Curl and all that

Divergence: Let $\vec{F}(x,y) = \Big(F_x(x,y),\ F_y(x,y)\Big)$ be a vector field in a 2-dimensional space. Its divergence is given by:

$$\vec{\nabla}\cdot\vec{F}(x,y) = \frac{\partial F_x(x,y)}{\partial x} + \frac{\partial F_y(x,y)}{\partial y}$$

Similarly for 3 or more dimensions.

Notice that the divergence of a vector field is a scalar.
Its value for a particular point expresses the magnitude of the vector field's source (or sink) at that point.

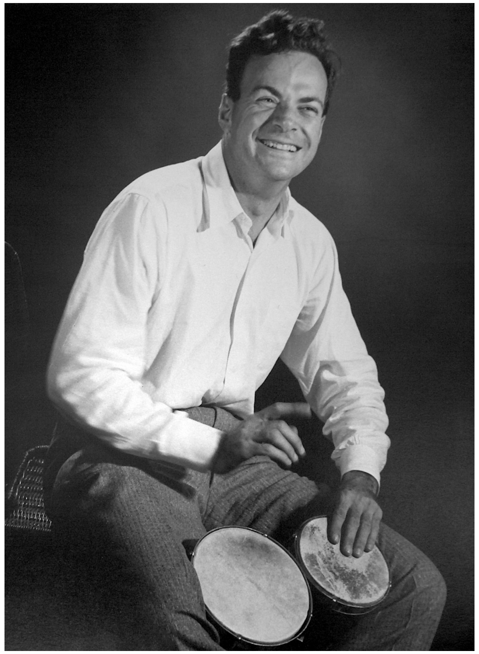


Curl: This won't be introduced till later so let's forget it for now.

Reminder: Gradient, Divergence, Curl and all that

Confused?

Here is a piece of advice that comes from Feynman.



- If you are solving a particular problem where $\vec{\nabla}\vec{E}$ appears, **do not just stare at it** if you are not quite sure what it means or if it confuses you.
- This is a nice compact notation that is supposed to make our lives easier, not more difficult.
- But it takes a while to get used to compact notations and be sure that we truly understand what they mean
- If you find that $\vec{\nabla}\vec{E}$ confuses you, write it out as

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

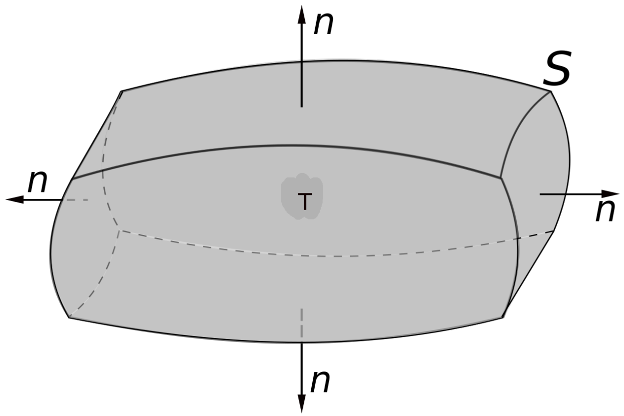
- There is **no shame at writing the components and not using the fancy compact notation**

Marking your scripts, I may excuse you for being confused by $\vec{\nabla} \cdot \vec{E}$.

But not for ignoring Feynman's advice!

Reminder: Gauss' theorem (Divergence theorem)

Gauss' theorem: **Not to be confused with Gauss' law** we saw earlier.



- Assume a volume τ whose boundary is the closed surface S
- Assume a vector field $\vec{F}$ which is
  - defined anywhere in τ
  - continuous / differentiable anywhere in τ

The integral of the flux of the field $\vec{F}$ over the closed surface S is equal to the integral of the divergence of $\vec{F}$ in the volume τ

$$\oint_S \vec{F} \cdot d\vec{S} = \int_\tau \vec{\nabla} \cdot \vec{F} d\tau$$

Deriving the differential form of Gauss' law

Expressing Q_{enc} in terms of the density ρ, the integral form of Gauss' law is:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} Q_{enc} = \frac{1}{\epsilon_0} \int \rho d\tau$$

Applying Gauss' theorem to the electric field:

$$\oint_S \vec{E} \cdot d\vec{S} = \int_\tau \vec{\nabla} \cdot \vec{E} d\tau$$

Therefore:

$$\int_\tau \vec{\nabla} \cdot \vec{E} d\tau = \frac{1}{\epsilon_0} \int \rho d\tau \Rightarrow \int_\tau \left(\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\epsilon_0} \right) d\tau = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

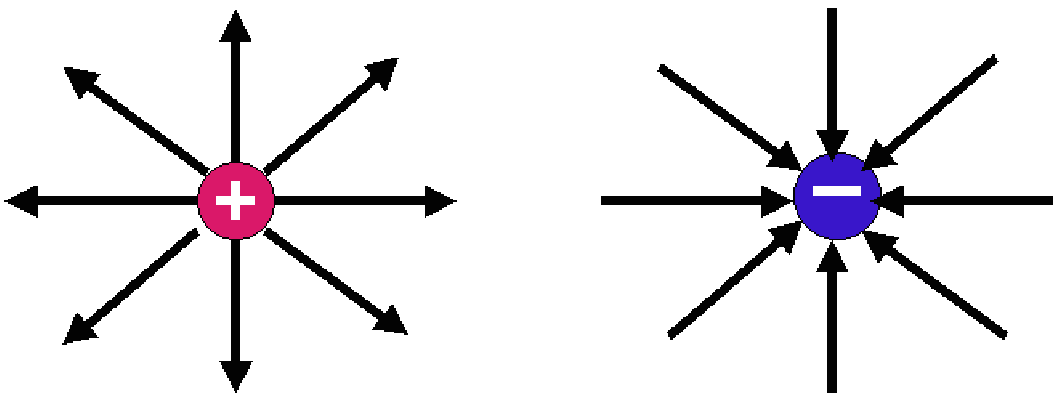
The above is the **differential form of Gauss' law**.
It tells us that the divergence of the electric field at each point is proportional to the local charge density.

The differential form of Gauss' law

Differential form of Gauss' law: The divergence of the electric field at each point is proportional to the local charge density.

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

Recall the geometrical interpretation of the divergence of a vector field.



Gauss' law tells us, and it is easier to see that in its differential form, that the **electric charge is the source of the electric field**.

Lecture 2 - Main points to remember

- **Electric flux**
  - The electric flux Φ_E is the number of field lines of the electric field $\vec{E}$ flowing through a surface S

$$\Phi_E = \int_S \vec{E} \cdot d\vec{S}$$

- **Gauss' law**
  - Our first Maxwell equation!
  - In integral form (useful if a symmetry can be exploited to simplify the integral evaluation): Relates the flux through a closed surface with the net charge contained in it

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

  - In differential form: Relates the divergence of the electric field with the local charge density

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

At the next lecture (Lecture 3)

- **Potential energy of a charge in an electrostatic field**
    - we will generalise for discrete and continuous charge distributions
- **Electric potential**
- **Circuital law for electrostatics** in differential and integral forms
- Continue brushing up our relevant calculus skills

Optional reading for Lecture 2

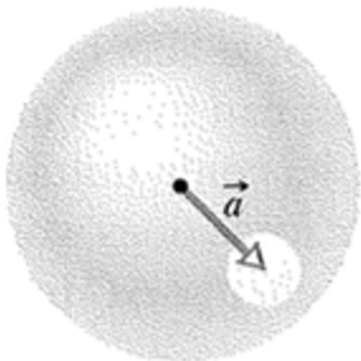
Worked example: Exploiting the superposition principle

Question

A nonconductive solid sphere has a uniform volume charge density ρ.

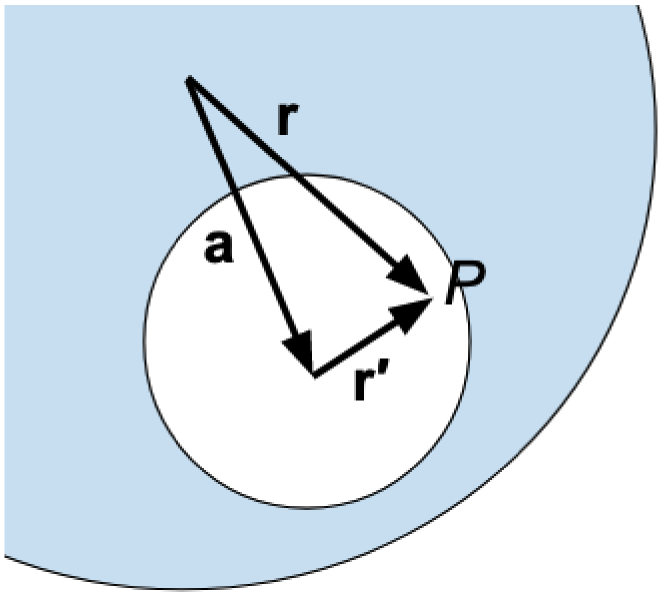
Let $\vec{r}$ be the vector from the centre of the sphere to a general point P within the sphere. As you should be able to easily confirm, the electric field $\vec{E}$ at a point $\vec{r}$ within the sphere is given by $\vec{E}(\vec{r}) = \rho\vec{r}/(3\epsilon_0)$.

If a spherical cavity is hollowed out of the sphere, as shown below, using superposition concepts, show that the electric field at all points within the cavity is uniform and equal to $\vec{E}(\vec{r}) = \rho\vec{a}/(3\epsilon_0)$ where $\vec{a}$ is the position vector from the centre of the sphere to the centre of the cavity.



Worked example: Exploiting the superposition principle

The charge distribution in case of a uniformly charged solid sphere with a cavity is equivalent to that of a whole uniformly charged solid sphere of charge density ρ plus a smaller uniformly charged solid sphere of charge density -ρ that fills the void.



The field produced by each uniformly charged sphere is given.
By superposition, the field at a point P within the cavity (at distance $\vec{r}$ from the centre of the larger sphere) is given by

$$\vec{E}(\vec{r}) = \frac{\rho \vec{r}}{3\epsilon_0} + \frac{(-\rho)\vec{r}'}{3\epsilon_0} = \frac{\rho \vec{r}}{3\epsilon_0} + \frac{(-\rho)(\vec{r} - \vec{a})}{3\epsilon_0} \Rightarrow$$

$$\vec{E}(\vec{r}) = \frac{\rho \vec{a}}{3\epsilon_0}$$

Worked example: Spherical shell with non-uniform density

Question

A spherical shell of inner radius a and outer radius b has volume charge density $\rho = kr^2$ $(a \leq r \leq b)$, where k is a constant and r is the radial distance from the centre of the shell.
Find the magnitude of the electric field $\vec{E}$ produced by this charge distribution at radial distances i) $r < a$, ii) $a \leq r < b$, and iii) $b \leq r$.

Due to the obvious spherical symmetry, the electric field $\vec{E}$ is a radial vector, and it is is going to be a function of only the radial distance r.

We have 3 distinct regions: i) $r < a$, ii) $a \leq r < b$, and iii) $b \leq r$.

For the calculation of $\vec{E}$ we will be applying Gauss's theorem:

$$\int_S \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_{\tau(S)} \rho d\tau$$

Worked example: Spherical shell with non-uniform density

For $r < a$, for *any possible* closed surface S that stays within that region of no charge, Gauss's theorem gives

$$\int_S \vec{E} \cdot d\vec{S} = 0$$

Since this happens for all surfaces, it can not be the result of an accidental cancellation and it has to be that the integrand itself is 0. Therefore:

$$E(r < a) = 0 \tag{1}$$

For $a \leq r < b$, exploiting the spherical symmetry, we will be applying Gauss's theorem for a concentric spherical surface $S(r)$ of radius r.

Because both $\vec{E}$ and the surface vector $d\vec{S}$ are collinear, and the norm of $\vec{E}$ is constant, everywhere on $S(r)$, the flux calculation is simplified:

$$\int_{S(r)} \vec{E} \cdot d\vec{S} = \int_{S(r)} EdS = E \int_{S(r)} dS = E4\pi r^2$$

Worked example: Spherical shell with non-uniform density

The charge Q_{enc}, in the volume $\tau(S)$, enclosed by a spherical surface $S(r)$ of radius r is given by:

$$Q_{enc} = \int_{\tau(S)} \rho d\tau = \int_a^r \left(ku^2\right) 4\pi u^2 du = 4\pi k \int_a^r u^4 du = \frac{4\pi k}{5} u^5 \Big|_a^r \Rightarrow$$

$$Q_{enc} = \frac{4\pi k}{5}\left(r^5 - a^5\right)$$

Therefore, Gauss's law can be expressed as:

$$E\cancel{4\pi} r^2 = \frac{\cancel{4\pi} k}{5\epsilon_0}\left(r^5 - a^5\right)$$

This yields:

$$E(a \le r < b) = \frac{k}{5\epsilon_0}\left(r^3 - \frac{a^5}{r^2}\right)$$

Worked example: Spherical shell with non-uniform density

For $b \le r$, one can apply a similar analysis. In this case, Q_{enc} is no longer a function of r since all spherical surfaces $S(r)$ include all charge. From the previous expression for Q_{enc}, setting r equal to b, we have:

$$Q_{enc} = \frac{4\pi k}{5}\left(b^5 - a^5\right)$$

Gauss's law can be expressed as:

$$E\cancel{4\pi}r^2 = \frac{\cancel{4\pi}k}{5\epsilon_0}\left(b^5 - a^5\right)$$

This yields:

$$E(b \le r) = \frac{k}{5\epsilon_0}\left(\frac{b^5 - a^5}{r^2}\right)$$

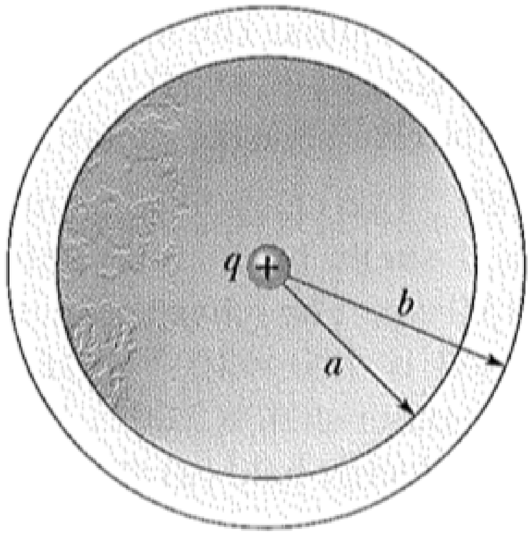
Summarizing, we found that:

$$E = \begin{cases} 0 & \text{for } r < a \\ \frac{k}{5\epsilon_0}\left(r^3 - \frac{a^5}{r^2}\right) & \text{for } a \le r < b \\ \frac{k}{5\epsilon_0}\left(\frac{b^5 - a^5}{r^2}\right) & \text{for } b \le r \end{cases}$$

Worked example: Spherical shell and point charge

Question

The figure below shows a nonconducting spherical shell of inner radius a and outer radius b. The shell has (within its thickness) a positive volume charge density $\rho(r) = A/r$, where A is a constant and r is the distance from the center of the shell. In addition, a small ball of charge q is located at that center. What value should A have if the electric field in the shell ($a < r < b$) is to be uniform?



Worked example: Spherical shell and point charge

Gauss's law in integral form is:

$$\Phi = \frac{Q_{enc}}{\epsilon_0}$$

The electric flux Φ over a closed surface S is defined as:

$$\Phi = \oint_S \vec{E} \cdot d\vec{S}$$

Consider a spherical Gaussian surface with radius r, where $a < r < b$. Exploiting the spherical symmetry of the problem, we can write:

$$\Phi = \oint_S EdS = E \oint_S dS = E 4\pi r^2$$

Worked example: Spherical shell and point charge

The charge in the volume $\tau(S)$, enclosed by the spherical Gaussian surface S with radius r, is the charge q at the origin and the fraction of the charge of the shell that is distributed at radial distances between a and r. Therefore:

$$Q_{enc} = q + \int_{\tau(S)} \rho d\tau \Rightarrow$$

$$Q_{enc} = q + \int_a^r \frac{A}{r'} 4\pi r'^2 dr' \Rightarrow Q_{enc} = q + 4\pi A \int_a^r r' dr' \Rightarrow$$

$$Q_{enc} = q + 4\pi A \frac{r'^2}{2}\Big|_a^r \Rightarrow Q_{enc} = q + 2\pi A(r^2 - a^2)$$

Combining the above exressions for Φ and Q_{enc}, we have:

$$E 4\pi r^2 = \frac{1}{\epsilon_0}\Big(q + 2\pi A(r^2 - a^2)\Big)$$

Worked example: Spherical shell and point charge

Solving for E, we find:

$$E = \frac{q + 2\pi A(r^2 - a^2)}{4\pi\epsilon_0 r^2} \Rightarrow E = \frac{1}{4\pi\epsilon_0}\Big(\frac{q}{r^2} + 2\pi A - \frac{2\pi A a^2}{r^2}\Big)$$

The requirement that E is uniform in the shell, implies that:

$$\frac{q}{r^2} - \frac{2\pi A a^2}{r^2} = 0$$

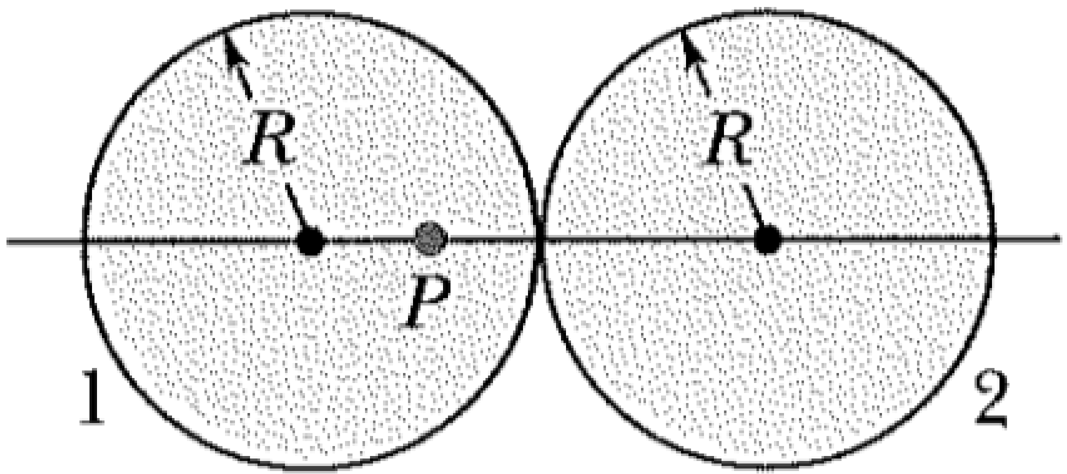
Therefore:

$$A = \frac{q}{2\pi a^2}$$

Worked example: Two uniformly charged spheres

Question

The figure below shows, in cross section, two solid spheres with uniformly distributed charge throughout their volumes. Each has radius R. Point P lies on a line connecting the centres of the spheres, at radial distance R/2.00 from the centre of sphere 1. If the net electric field at point P is zero, what is the ratio q_2/q_1 of the total charges?



Worked example: Two uniformly charged spheres

The electric field inside and outside a uniformly charged, nonconductive, solid sphere of radius R was calculated earlier in this Lecture.

The results are summarised below:

For $r < R$:

$$E(r) = \frac{Q\frac{r^3}{R^3}}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0 R^3} r$$

For $r \geq R$:

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$$

The electric field $\vec{E}$ at point P is the superposition of 2 fields: the field $\vec{E}_1$ because of the presence of sphere 1 and the field $\vec{E}_2$ because of the presense of sphere 2:

$$\vec{E}(P) = \vec{E}_1(P) + \vec{E}_2(P)$$

Worked example: Two uniformly charged spheres

Applying our previous result for the electric field:

$$\vec{E}_1(P) = \left\{\frac{Q}{4\pi\epsilon_0 R^3} r\right\}\hat{x} \xrightarrow{Q=q_1,\ r=R/2} \vec{E}_1(P) = \left\{\frac{q_1/2}{4\pi\epsilon_0 R^2}\right\}\hat{x}$$

and:

$$\vec{E}_2(P) = \left\{\frac{Q}{4\pi\epsilon_0 r^2}\right\}(-\hat{x}) \xrightarrow{Q=q_2,\ r=R+R/2=3R/2} \vec{E}_2(P) = -\left\{\frac{4q_2/9}{4\pi\epsilon_0 R^2}\right\}\hat{x}$$

Therefore, the total field is:

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0 R^2}\left(\frac{q_1}{2} - \frac{4q_2}{9}\right)\hat{x}$$

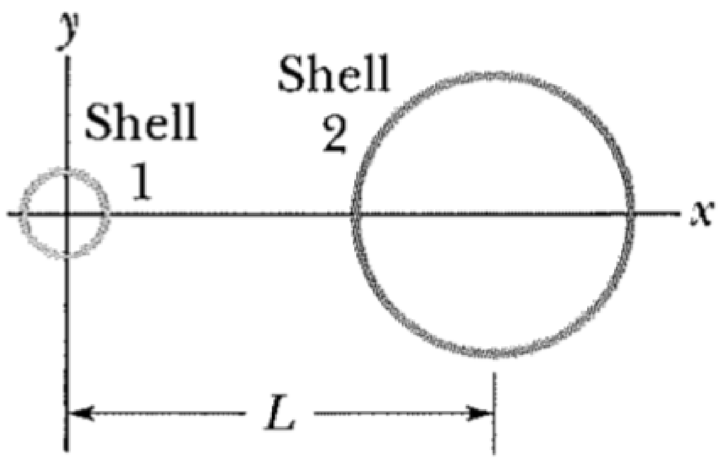
If E(P)=0, then:

$$\frac{q_1}{2} - \frac{4q_2}{9} = 0 \Rightarrow \frac{q_2}{q_1} = \frac{9}{8} \Rightarrow \frac{q_2}{q_1} = 1.125$$

Worked example: Two spherical shells

Question

The figure on the left shows two nonconducting spherical shells fixed in place on an x axis. Shell 1 has uniform surface charge density $+4.0\ \mu C/m^2$ on its outer surface and radius 0.50 cm, and shell 2 has uniform surface charge density -2.0 $\mu C/m^2$ on its outer surface and radius 2.0 cm. The shell centres are separated by a distance $L = 6.0$ cm. Other than at $x = \infty$, where on the x axis is the net electric field equal to zero?



Worked example: Two spherical shells

Consider Gauss's law in its integral form:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

Exploiting the symmetry of a system with charge Q distributed uniformly on a spherical, nonconducting shell or radius R centred at the origin of the coordinate system, we can easily calculate the electric field:

- For $r < R$, the enclosed charge is 0 and from the symmetry of the problem we deduce that $\vec{E}(r) = 0$.
- For $r \geq R$, all charge Q is enclosed, and the electric field can be written (as if we had a point charge at the origin) as $\vec{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$.

The electric field $\vec{E}$ produced by the two spherical shells 1 and 2 is given superposition principle:

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

where $\vec{E}_1$ ($\vec{E}_2$) is the field of sphere 1 (2).

Worked example: Two spherical shells

The field produced by each shell is pointing radially in (as in the case of the negatively charged shell 2) or out (as in the case of the positively charged shell 1). Therefore, for all points on the x axis, $\vec{E}_1$ and $\vec{E}_2$ point along $\pm\hat{x}$.

Within shell 1, $\vec{E}_1 = 0$, and within shell 2, $\vec{E}_2 = 0$. Therefore, within the shells, the net electric field $\vec{E}_1 + \vec{E}_2$ cannot be zero there.

For all x values between the shells ($R_1 < x < L - R_2$), the fields $\vec{E}_1$ and $\vec{E}_2$ point in the same direction and, therefore, the net electric field $\vec{E}_1 + \vec{E}_2$ cannot be zero there.

The charge contained by shells 1 and 2 can be calculated as follows:

$$Q_1 = 4\pi R_1^2 \sigma_1 = 4\pi (5 \times 10^{-3}\ m)^2 (4 \times 10^{-6}\ C/m^2) = 4\pi \times 10^{-10}\ C$$

$$|Q_2| = 4\pi R_2^2 |\sigma_2| = 4\pi (2 \times 10^{-2}\ m)^2 (2 \times 10^{-6}\ C/m^2) = 8 \times 4\pi \times 10^{-10}\ C$$

Worked example: Two spherical shells

Since $|Q_2| > Q_1$, $|\vec{E}_2| > |\vec{E}_1|$ for all values of $x > L + R_2$ (closer to shell 2 than to shell 1). Therefore, the net field $\vec{E}_1 + \vec{E}_2$ cannot be zero there.

Following from the above, the only range of x values where the net field $\vec{E}_1 + \vec{E}_2$ can be zero, is $x < -R_1$. In that range, $\vec{E}_1$ and $\vec{E}_2$ have opposite directions, and magnitudes given by:

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{|x|^2} \xrightarrow{4\pi R_1^2 \sigma_1} E_1 = \frac{R_1^2\sigma_1}{\epsilon_0}\frac{1}{|x|^2}$$

$$E_2 = \frac{1}{4\pi\epsilon_0}\frac{|Q_2|}{(L+|x|)^2} \xrightarrow{|Q_2| = 4\pi R_2^2 |\sigma_2|} E_2 = \frac{R_2^2|\sigma_2|}{\epsilon_0}\frac{1}{(L+|x|)^2}$$

The requirement that the net electric field is zero, implies that:

$$E_1 = E_2$$

Worked example: Two spherical shells

Therefore:

$$\frac{R_1^2 \sigma_1}{\epsilon_0} \frac{1}{|x|^2} = \frac{R_2^2 |\sigma_2|}{\epsilon_0} \frac{1}{(L + |x|)^2} \Rightarrow$$

$$\left(\frac{L + |x|}{|x|}\right)^2 = \frac{R_2^2 |\sigma_2|}{R_1^2 \sigma_1} \Rightarrow \frac{L}{|x|} + 1 = \frac{R_2}{R_1} \sqrt{\frac{|\sigma_2|}{\sigma_1}} \Rightarrow$$

$$|x| = \frac{L}{\frac{R_2}{R_1} \sqrt{\frac{|\sigma_2|}{\sigma_1}} - 1} \Rightarrow$$

$$|x| = \frac{6 \ cm}{\frac{2.0}{0.5} \sqrt{\frac{2}{4}} - 1} = \frac{6 \ cm}{\frac{4}{\sqrt{2}} - 1} \approx 3.28 \ cm \Rightarrow x \approx -3.28 \ cm$$

Worked example: Electric field of two charged sheets

Question

Two uniform infinite sheets of electric charge densities $+\sigma$ and $-\sigma$ intersect at a right angle. Find the magnitude and direction of the electric field everywhere and sketch the electric field lines.

The magnitude of the electric field caused by a single, infinite sheet of uniform charge density σ is given by:

$$E = \frac{\sigma}{2\epsilon_0}$$

The direction of the field is perpendicular to the sheet and it is pointing towards (away from) the sheet, if it is negatively (positively) charged.
The field in the problem of two uniform infinite sheets can be obtained by invoking the superposition principle and superimposing undisturbed

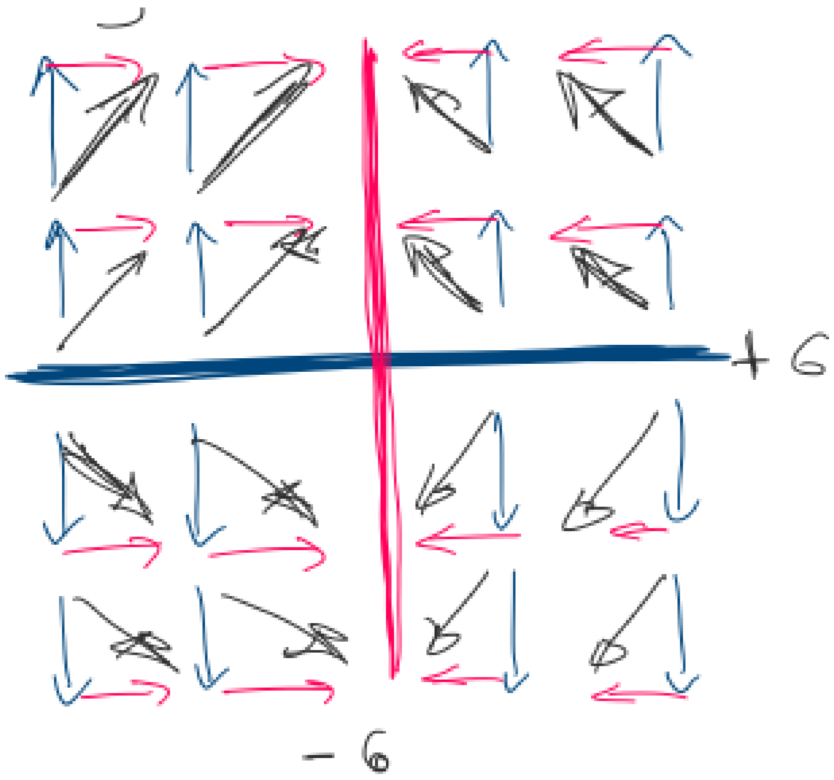
- the field $\vec{E_{(+)}}$ of an infinite sheet of uniform charge density $+\sigma$, and
- the field $\vec{E_{(-)}}$ of an infinite sheet of uniform charge density $-\sigma$.

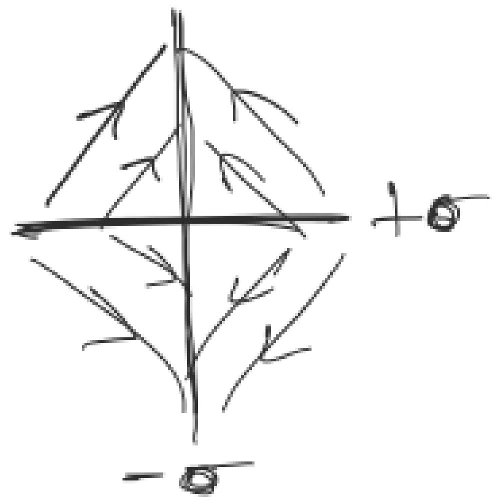
Worked example: Electric field of two charged sheets

The magnitude of the combined field ($\vec{E}$) is:

$$E = \sqrt{E_{(+)}^2 + E_{(-)}^2} = \sqrt{\left(\frac{+\sigma}{2\epsilon_0}\right)^2 + \left(\frac{-\sigma}{2\epsilon_0}\right)^2} \Rightarrow E = \frac{\sqrt{2}\sigma}{2\epsilon_0}$$

The field lines are sketched below:





PHYS201 scientific programming task for Lecture 2

If you did the previous task, you already have a program to calculate the electric field (in 2-D) for an arbitrary distribution of discrete charges.

Generalize your previous program:

- Move from a 2-D to a **3-D calculation**.
- Add an option to **specify a continuous distribution of charge** (i.e. work with a user-defined charge density function $\rho(\vec{r})$)

What you will be doing, is to perform the following numerical integration:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_\tau d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')$$

Can you test Gauss' law numerically?

PHYS201 scientific programming task for Lecture 2

As an example, use the following charge density in spherical coordinates:

$$\rho = \begin{cases} \frac{\rho_0}{(r/r_0)^2} e^{-r/r_0} cos^2\phi, & \text{if } r < 5r_0 \\ \\ 0, \text{otherwise} \end{cases}$$

where $\rho_0 = 0.16$ C/m^3 and $r_0 = 10$ cm.

Calculate numerically the amount of charge Q enclosed in a sphere of radius r, as a function or r:

$$Q(r) = \int_0^r \int_{4\pi} d\tau \rho(\vec{r'})$$

Confirm that your distribution plateaus to a value of Q_{tot} for r $> 5r_0$, as the sphere encloses all regions of non-zero charge density. What is the value of Q_{tot}?

Calculate the electric flux through the surface of a sphere with radius r $= 5r_0$ and confirm that:

$$\epsilon_0 \oint \vec{E} \cdot d\vec{S} = Q_{tot}$$

PHYS 201 / Lecture 3
Electrical potential energy and potential, Circuital law, Poisson and Laplace equations, Boundary conditions, Uniqueness theorem

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Lectures delivered at the University of Liverpool, 2021-22

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Lecture 2 - Revision

- **Electric flux**
  - The electric flux Φ_E is the number of field lines of the electric field $\vec{E}$ flowing through a surface S

$$\Phi_E = \int_S \vec{E} \cdot d\vec{S}$$

- **Gauss' law**
  - Our first Maxwell equation!
  - In integral form (useful if a symmetry can be exploited to simplify the integral evaluation): Relates the flux through a closed surface with the net charge contained in it

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

  - In differential form: Relates the divergence of the electric field with the local charge density

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Plan for Lecture 3

- How much **work** do I need to do **to bring charges close together?**

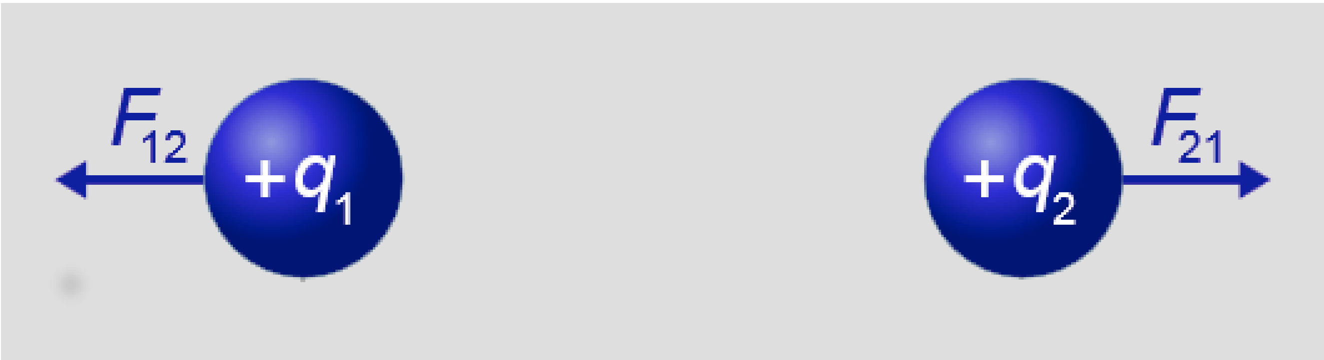
  We will calculate the work done to assemble
  - discrete systems of 2, 3 and N charges, and
  - a continuous distribution of charge characterised by density $\rho(\vec{r})$

- Will see that the work done is **path-independent**.
- Will explain that the work becomes the **electric potential energy** of the system of charges and that it is stored in the electric field.
- Will introduce the concept of **electric potential**.
- Will study the **circuital law** in differential and integral forms.
- Will introduce the **Poisson** and **Laplace** equations.
- In parallel, will continue revising some relevant maths:
  - The Laplace operator, the "curl" of a vector field, boundary problems and the *uniqueness theorem*

Assembling a system of 2 same-sign charges

Consider two same-sign charges separated by an infinite distance.

It is easy to understand that **in order to bring the two charges closer together I need to do work.**

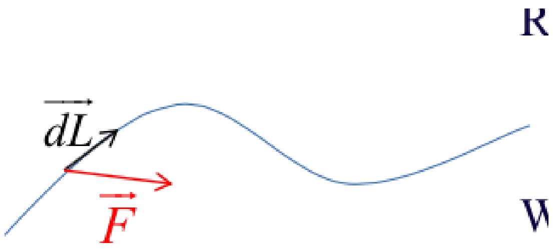


Why?

Because **they repel each other.** It is like pushing in a spring!

Reminder: Work

A force is said to **do work** (denoted with W) if, when it is acting on a body, there is a **displacement of the point of application in the direction of the force**.



The work dW done by a force $\vec{F}$ displacing the point of application by $d\vec{\ell}$, is given by the dot product:

$$dW = \vec{F} \cdot d\vec{\ell}$$

Notice that work is a **scalar**. Its SI unit is a **Joule (J)**: It is the work done by a force of 1 N over a distance of 1 m.

Note that **a force perpendicular to the direction of motion does no work**.

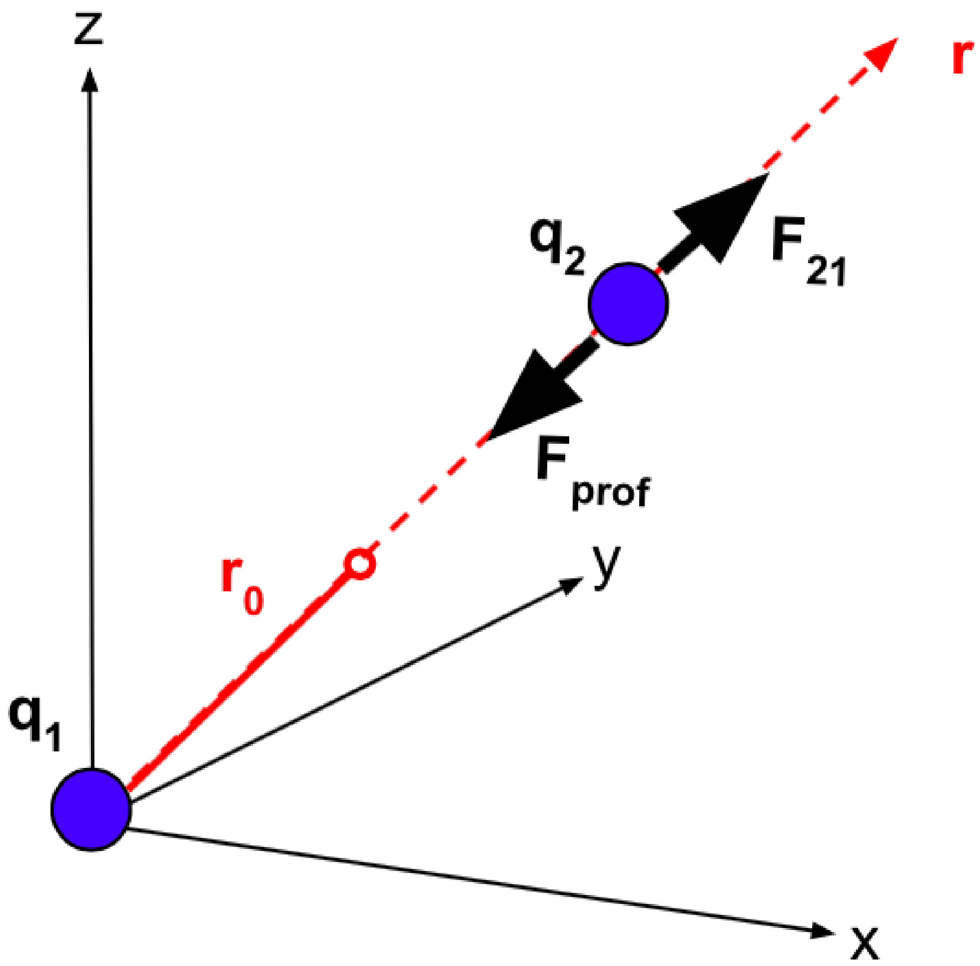
The **work can be positive or negative**. By convention, we take work to be negative if it opposes the motion, i.e. $\theta > 90^o$.

The total work along a trajectory is given by integrating (summing up) the work done for each infinitesimal displacement $d\vec{\ell}$: $W = \int dW = \int \vec{F} \cdot d\vec{\ell}$.

Work is very closely related to energy.

Assembling a system of 2 same-sign charges

We will answer the following question: **How much work is needed to bring 2 same-sign charges q_1 and q_2 from infinity to a distance r_0?**



<u>Note</u>:

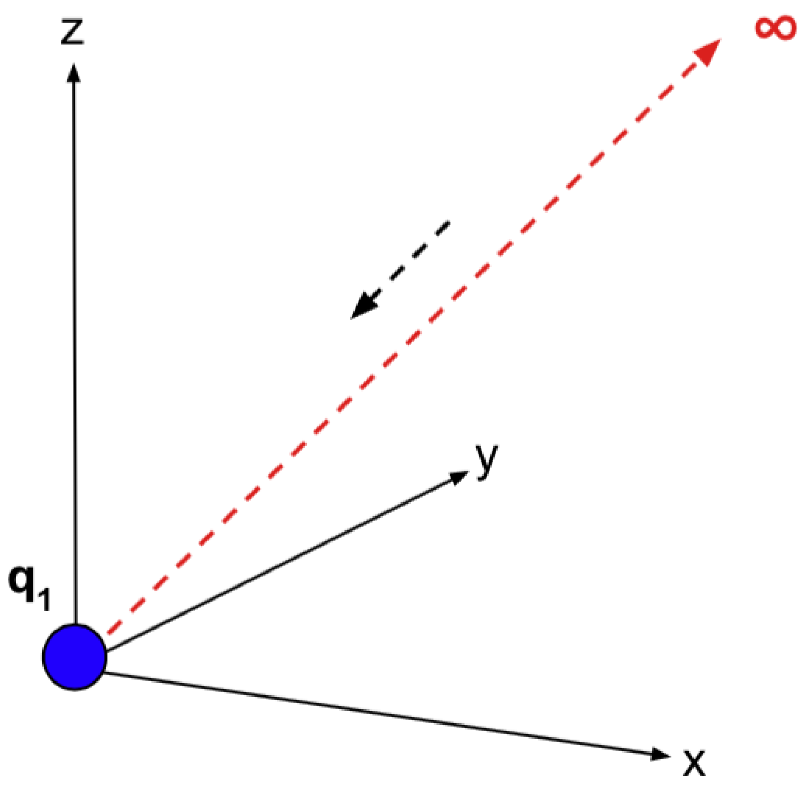
The two like charges repel each other.

I will calculate **the work done by <u>me</u> (F_{prof}) against the action of the field (F_{21})**, not the work done by the field force (F_{21}).

The difference between the two is a sign.

Work done assembling a system of 2 same-sign charges

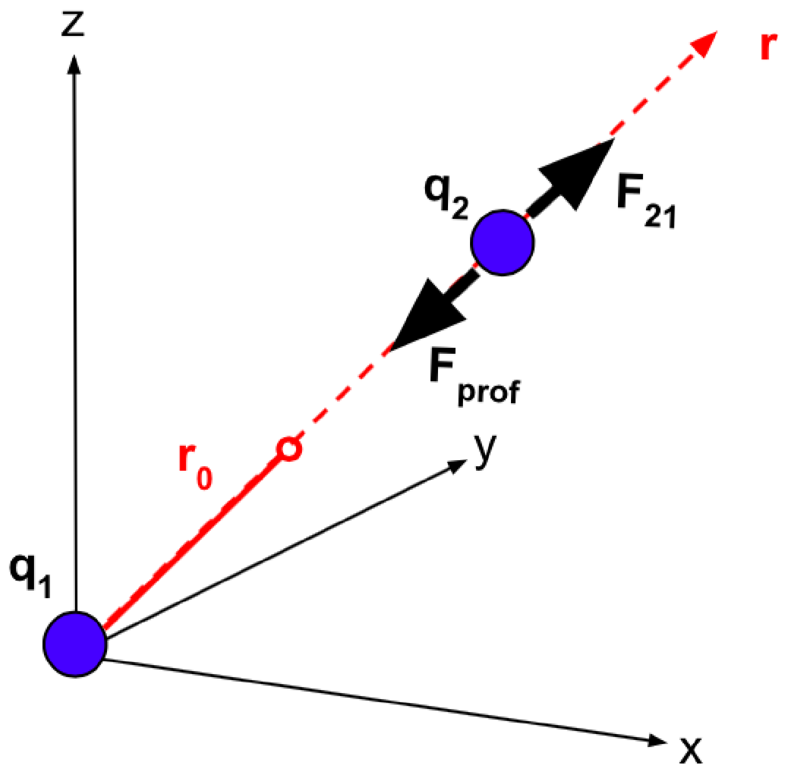
Placing charge q_1:



- Initially all charges are at *"infinity"*
  - i.e. far enough so that the force between them is so small that it can be neglected
- I reach out all the way to infinity and grab charge q_1
- I bring the charge q_1 at the origin of my coordinate system
  - There is no opposing force (other charges are still infinitely away)
- That was an "easy" task
  - I did **no work!**

Work done assembling a system of 2 same-sign charges

Placing charge q_2:



- I reach out all the way to infinity again and grab charge q_2 (same-sign as q_1).
- I begin to bring the charge q_2 closer to q_1, intending to pin it at $\vec{r_0}$.
- There is an **opposing force** due to q_1
- The force $\vec{F}_{21}$ exerted on charge q_2 due to charge q_1 is:

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

- I need to overcome this force so I apply a force $\vec{F}_{prof}$ which is:

$$\vec{F}_{prof} = -\vec{F}_{21}$$

Work done assembling a system of 2 same-sign charges

The **work done by me** ($\vec{F}_{prof}$) as I try **to overcome the repulsive force** ($\vec{F}_{21}$) between q_1 and q_2 and move q_2 by an infinitesimally small distance $d\vec{\ell}$ is:

$$dW_{prof} = \vec{F}_{prof} \cdot d\vec{\ell} = -\vec{F}_{21} \cdot d\vec{\ell} = -\Big(\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}\Big) \cdot d\vec{\ell}$$

The force is a radial vector. If I only move q_2 colinearly with the force ($d\vec{\ell} = d\vec{r}$), the above dot product simplifies to:

$$dW_{prof} = -\frac{q_1 q_2}{4\pi\epsilon_0}\frac{dr}{r^2}$$

The total work done to bring the q_2 from infinity to the position $\vec{r}_0$ is found by integrating dW_{prof} from infinity (∞) to r_0:

$$W_{prof} = \int_{\infty}^{r_0} dW_{prof} = -\frac{q_1 q_2}{4\pi\epsilon_0}\int_{\infty}^{r_0}\frac{dr}{r^2}$$

Work done assembling a system of 2 same-sign charges

I am sure that you can all calculate this integral:

$$W_{prof} = -\frac{q_1 q_2}{4\pi\epsilon_0} \int_{\infty}^{r_0} \frac{dr}{r^2}$$

The *anti-derivative* of $1/r^2$ is $-1/r$, so the integral becomes:

$$W_{prof} = -\frac{q_1 q_2}{4\pi\epsilon_0} \Big(-\frac{1}{r} \Big) \Big|_{\infty}^{r_0} = \frac{q_1 q_2}{4\pi\epsilon_0} \Big(\frac{1}{r} \Big) \Big|_{\infty}^{r_0} = \frac{q_1 q_2}{4\pi\epsilon_0} \Big(\frac{1}{r_0} - lim_{x\to\infty} \frac{1}{x} \Big) \Rightarrow$$

$$W_{prof} = \frac{q_1 q_2}{4\pi\epsilon_0 r_0}$$

Mini reminder: Anti-derivatives

The *anti-derivative* of a function f is a differentiable function F whose derivative is equal to f (i.e. $\mathrm{F}' = \mathrm{f}$). Confirm that $(\text{-}1/\mathrm{r})' = 1/\mathrm{r}^2$.

Work done assembling a system of 2 ~~same-sign~~ charges

We showed that the total work I need to do to assemble a system of 2 same-sign charges q_1 and q_2 at distance r_0 is given by:

$$W_{prof} = \frac{q_1 q_2}{4\pi\epsilon_0 r_0}$$

The same result would have been obtained if, instead of considering 2 same-sign charges, I had used one positive and one negative charge.

Notice that W_{prof} involves the product $q_1 q_2$:

- **For same sign charges** $q_1 q_2 > 0$:
  To assemble that system I need to do **positive work** (see convention discussed earlier). I need to spend energy because the force between the two charges is repulsive.
- **For charges with different sign** $q_1 q_2 < 0$:
  To assemble that system I need to do **negative work** as the force is attractive.

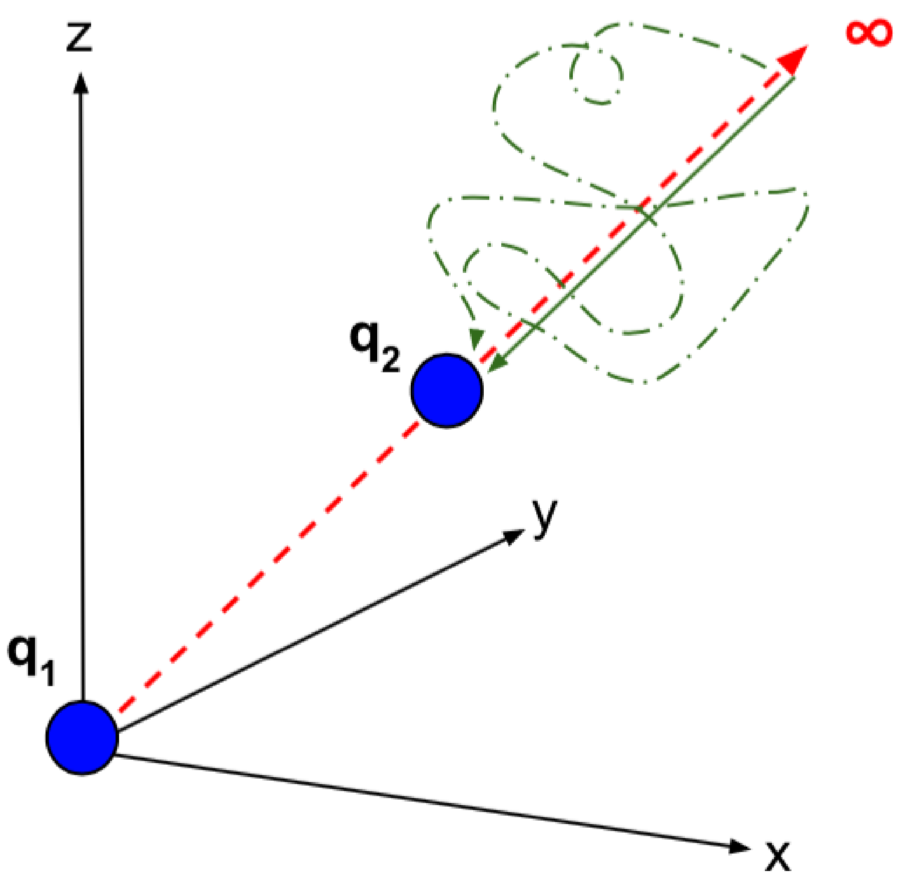
Electrostatic potential energy of a system of 2 charges

The work I did (W_{prof}) became the **electrostatic potential energy of the system** of two charges (usually denoted with U):

$$W_{prof} = U = \frac{q_1 q_2}{4\pi\epsilon_0 r_0}$$

- The electrostatic potential energy is the energy of the system of (q_1, q_2), **not the energy of each of** q_1 **and** q_2 **individually**.
- As we shall see later in this course, this electrostatic potential energy is **stored in the electric field** produced by the two charges.
- The potential energy (like the work done to assemble the system) can be positive or negative.

Path-independence of work



We showed that the total work W_{prof} that I need to do to assemble a system of 2 same-sign charges q_1 and q_2 at distance r_0, and hence the electrostatic potential energy U stored in the system, is given by:

$$W_{prof} = U = \frac{q_1 q_2}{4\pi\epsilon_0 r_0}$$

I obtained that result by moving q_2 along a *straight* line that was collinear with the electric force.

There is an obvious question:
Would a **longer curved trajectory give me a different result**?

Path-independence of work

The answer to the previous question is no.

- The work done by the electric force is path-independent.
- We say then that **the electric force is conservative**.

It is not difficult to understand that physically:

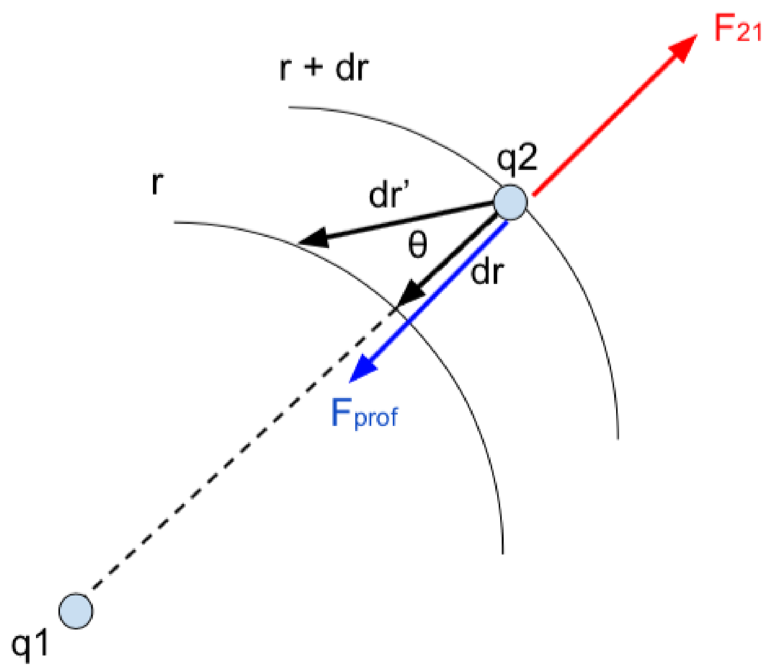
- Work is converted to potential energy stored in the electric field.
- But identical charge configurations (given charges placed at given positions) produce the exact same field.
- Recall the general result for $\vec{E}$:

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q}{Q} = \frac{1}{4\pi\epsilon_0} \int_\tau d\tau' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

  What matters is the distribution of charge, not how it got there.
- So the work done should be independent of the path.

Path-independence of work

We can show the path-independence mathematically. It is sufficient to show this for charge moving **between two infinitesimally separated spherical shells**.



The work done by moving q_2 along $d\vec{r}$ is:

$$dW = \vec{F}_{prof} \cdot d\vec{r} = |\vec{F}_{prof}||d\vec{r}|$$

whereas the work done by moving it along $d\vec{r}'$ is:

$$dW' = \vec{F}_{prof} \cdot d\vec{r}' = |\vec{F}_{prof}||d\vec{r}'|cos\theta$$

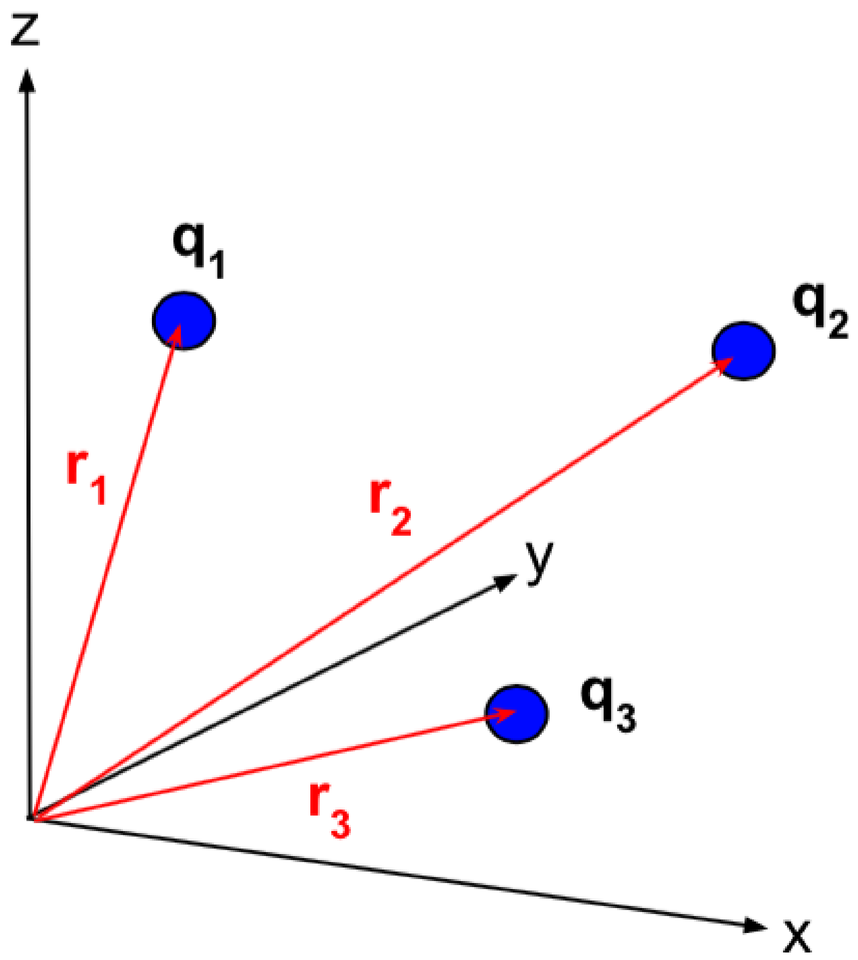
But $|d\vec{r}| = |d\vec{r}'|cos\theta$, therefore:

$$dW = dW'$$

Hint: Since the result is path-independent, choose a path that simplifies the calculations (usually, by exploiting the symmetries of the problem)

Generalisation for 3 charges

How about a slightly more complex system that has 3 charges q_1, q_2, q_3?



The question now becomes:

How much work is needed to assemble (or, how much potential energy is stored in) **a system of 3 charges?**

Generalisation for 3 charges

- To assemble a system of 3 charges, I need to add a charge q_3 to a system of 2 charges q_1 and q_2.
- We already know how much work is needed to bring the first 2 charges together or, equivalently, what is the electrostatic potential energy stored in this system.
- If q_1 is brought at position $\vec{r}_1$ and q_2 at position $\vec{r}_2$, the energy stored in the system is:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_{12}|}$$

where $|\vec{r}_{12}| = |\vec{r}_1 - \vec{r}_2|$ is the distance between q_1 and q_2.

(Here, I changed the notation slightly ($r_0 \rightarrow |\vec{r}_{12}|$) to help me generalise the result obtained previously for 2 charges.)

Generalisation for 3 charges

So, the problem is reduced to finding out **how much work is needed to add the charge q_3 to the system of q_1 and q_2.**

The superposition principle applies (the total force on a charge due to an array of other charges is the vector sum of the individual forces):

$$\vec{F} = \sum_i \vec{F}_i$$

So, the total field force $\vec{F}_3$ exerted on q_3 is:

$$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32}$$

where $\vec{F}_{31}$ ($\vec{F}_{32}$) is the force exerted on q_3 due to q_1 (q_2).

Generalisation for 3 charges

Therefore, the work that I need to do against the action of the field (i.e. against $\vec{F}_3$) is:

$$W_{prof} = \int \vec{F}_{prof} \cdot d\vec{\ell} = -\int \vec{F}_3 \cdot d\vec{\ell} = -\int \vec{F}_{31} \cdot d\vec{\ell} - \int \vec{F}_{32} \cdot d\vec{\ell} \Rightarrow$$

$$W_{prof} = W_{prof;1} + W_{prof;2}$$

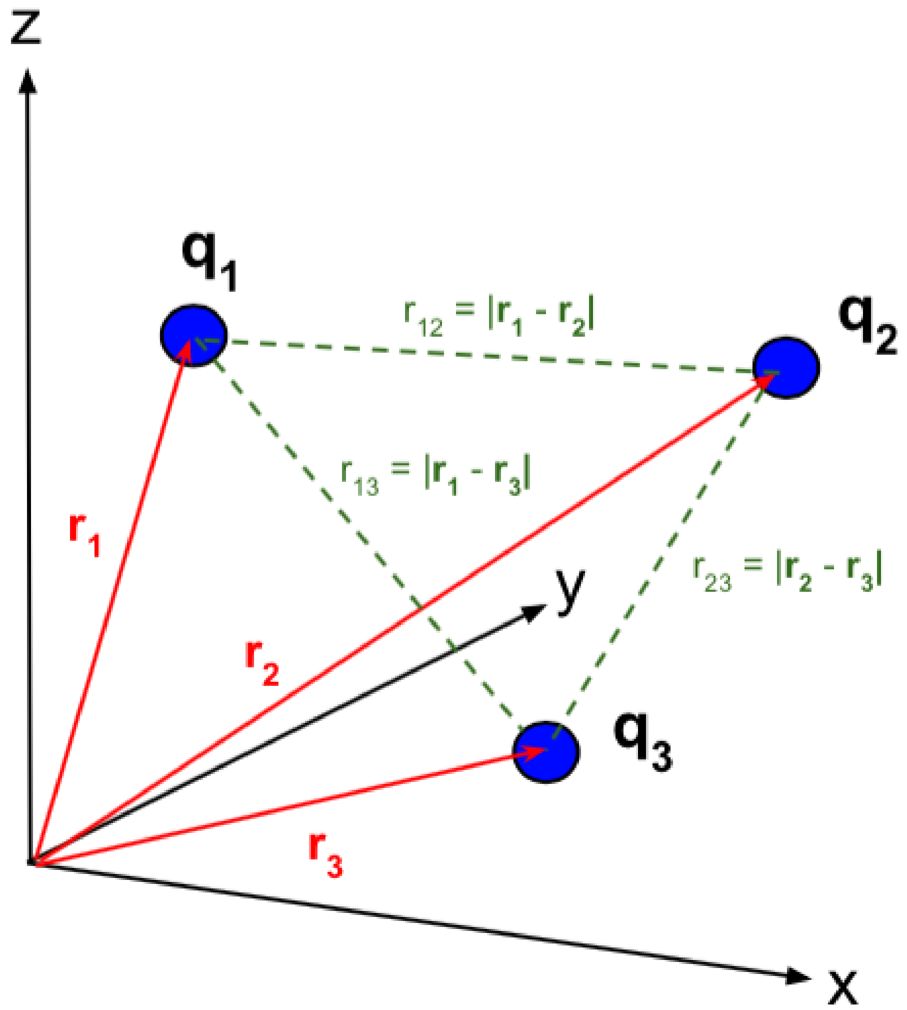
What this tells us is that, in order to bring q_3 at position $\vec{r}_3$, after q_1 was brought at position $\vec{r}_1$ and q_2 at $\vec{r}_2$, I need to do work which is the sum of:

- the work I would need to do if only q_1 was in place, and
- the work I would need to do if only q_2 was in place.

My 3-charge problem is reduced to a number of 2-charge problems (for which I know the solutions).

Generalisation for 3 charges

So the previous observation allows us to calculate the work (or total potential energy) **by considering all possible sub-systems of 2 charges**.



How many distinct sub-systems of 2 charges are there in the system of 3 charges?

The obvious answer is 3:

- (q_1, q_2) separated by distance $|\vec{r}_{12}|$,
- (q_1, q_3) separated by distance $|\vec{r}_{13}|$, and
- (q_2, q_3) separated by distance $|\vec{r}_{23}|$.

Generalisation for 3 charges

Therefore, the work I need to do to assemble (and, consequently the total potential energy stored in) the system of 3 charges is:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_{12}|} + \frac{q_1 q_3}{4\pi\epsilon_0 |\vec{r}_{13}|} + \frac{q_2 q_3}{4\pi\epsilon_0 |\vec{r}_{23}|}$$

Here, I made no assumption about the charge signs:

- They can be anything, in any combination (+++, ++-, +-+, -++, ...)
- For some of the pairs above I may need to do positive work, while for others I may need to do negative work.
- The total work I need to do is the algebraic sum.
- The total work may have either sign, depending on which terms dominate.

What to remember:
My 3-charge problem was reduced to a number of 2-charge problems.
Each pair contributes a $\frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$ term to the total potential energy.

Generalisation for N charges

We can generalise our result for a **system of N charges** q_1, q_2, q_3,..., q_N.

There is a clear recipe for calculating the total potential energy:

- Find all distinct pairs of charges.
- Every distinct pair will contribute with a $\frac{q_i q_j}{4\pi\epsilon_0|\vec{r_{ij}}|}$ term.

Therefore the total potential energy can be written as:

$$U = \sum_{\text{all pairs}} \frac{q_i q_j}{4\pi\epsilon_0|\vec{r_{ij}}|}$$

The question now becomes **how to enumerate all possible pairs**.

Generalisation for N charges

How many pairs are there in the system of N charges?

The number of groups of k items we can select from a collection of n items ($k <= n$), if the order of the selection does not matter, is given by the **Binomial Coefficient**:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)...(n-k+1)}{k(k-1)...1}$$

For example:

- for N = 4 charges, the numbers of pairs is $\binom{4}{2} = 6$
- for N = 10 charges, the numbers of pairs is $\binom{10}{2} = 45$

We need a way to take all pairs into account systematically **so that we do not neglect or double-count terms**.

Generalisation for N charges

Let's think about **a systematic procedure for forming pairs**:

- We **start with charge** q_1 and we pair it with all charges.

  The pairs are $\cancel{(q_1, q_1)}$, (q_1, q_2), (q_1, q_3),..., (q_1, q_N).

  - We do not include "self-energy" terms (i.e. we don't consider the terms obtained by pairing each charge with "itself") so we neglect (q_1, q_1).

- Let's **move to charge** q_2 and pair it with all charges.

  The new pairs are $\cancel{(q_2, q_1)}$,$\cancel{(q_2, q_2)}$,(q_2, q_3),...,(q_2, q_N).

  - As before, we don't include the self-energy term (q_2, q_2).
  - But we also neglect (q_2, q_1). This is the same as (q_1, q_2) that was already taken into account.

You probably start to see the pattern here.

Generalisation for N charges

To take into account all distinct pairs, we sum over the charges q_i and, for each q_i, we sum over the charges q_j but with j starting from i+1 (not from 1):

$$U = \sum_{i=1} \sum_{j=i+1} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

We need to be **careful towards the end of the double sum**:

- If i=N-1 (the last but one charge), then there is only one pair left to be formed with j=N: (q_{N-1}, q_N).
- So my double sum **ends when i=N-1 and j=N**.

Therefore, the expression for the total potential energy of a system of N charges becomes:

$$U = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

Generalisation for N charges

The table below shows all $q_i q_j$ terms we included (✓). We **can re-write the sum in a more symmetric way**.

		j							
		q_1	q_2	q_3	q_4	q_5	…	q_{N-1}	q_N
i	q_1	-	✓	✓	✓	✓	…	✓	✓
	q_2	-	-	✓	✓	✓	…	✓	✓
	q_3	-	-	-	✓	✓	…	✓	✓
	q_4	-	-	-	-	✓	…	✓	✓
	q_5	-	-	-	-	-	…	✓	✓
	…	…	…	…	…	…	…	…	…
	…	…	…	…	…	…	…	…	…
	q_{N-1}	-	-	-	-	-	…	-	✓
	q_N	-	-	-	-	-	…	-	-

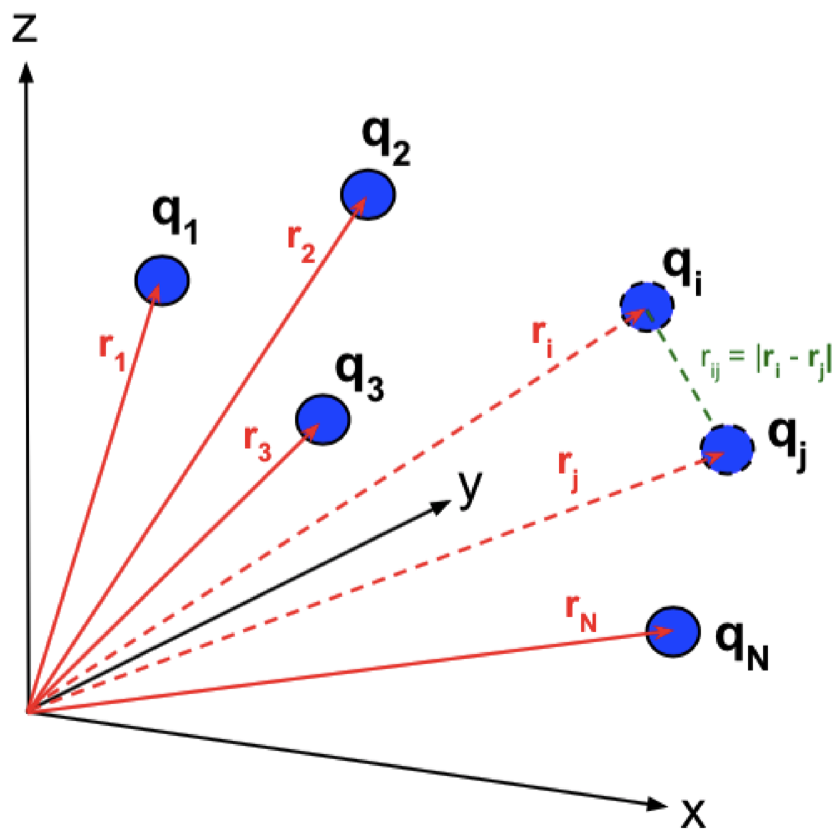
- What happens if I just sum up over all charges, without the $j > i$ condition, excluding only the self-terms?
- Obviously, I **count each pair twice**: E.g, I consider both (q_1, q_2) and (q_2, q_1).

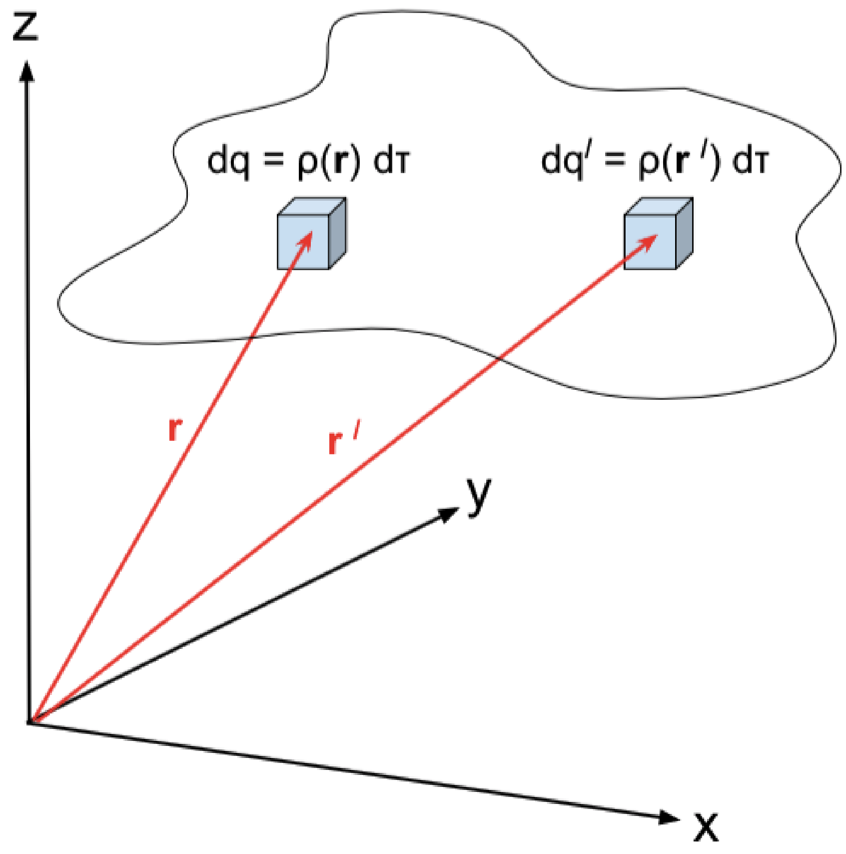
So I can just rewrite the sum as:

$$U = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|} \rightarrow \frac{1}{2} \sum_{i,j=1; i \neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

Generalisation for continuous charge distributions

We will now make the **leap from discrete to continuous distributions of charge** characterised by a volume charge density $\rho(\vec{r})$.





Generalisation for continuous charge distributions

Our starting point, is the result obtained for N charges:

$$U = \frac{1}{2} \sum_{i,j=1;i\neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r_{ij}}|}$$

This result can be readily adapted for the continuous case by making the following substitutions:

- $q_i \rightarrow dq(\vec{r}) = \rho(\vec{r}) d\tau$
- $q_j \rightarrow dq(\vec{r'}) = \rho(\vec{r'}) d\tau'$
- $|\vec{r_{ij}}| \rightarrow |\vec{r} - \vec{r'}|$
- $\sum \rightarrow \int$

The electrostatic potential energy for a continuous charge distribution characterised by density ρ can be written as:

$$U = \frac{1}{2} \int_{vol} d\tau \int_{vol} d\tau' \frac{\rho(\vec{r})\rho(\vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|}$$

Summary of basic results

Potential energy stored in a:

- **system of 2 charges**:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{12}|}$$

- **system of 3 charges**:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{12}|} + \frac{q_1 q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{13}|} + \frac{q_2 q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{23}|}$$

- **system of N charges**:

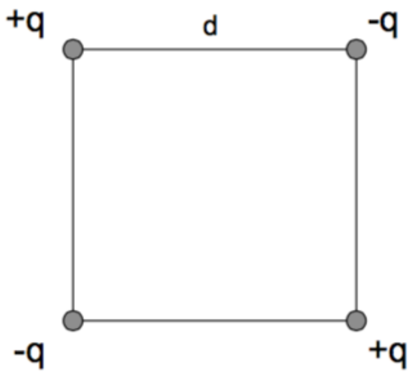
$$U = \frac{1}{2} \sum_{i,j=1;i\neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

- **continuous charge distribution** (with density ρ over a volume τ):

$$U = \frac{1}{2} \int_{vol} d\tau \int_{vol} d\tau' \frac{\rho(\vec{r})\rho(\vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|}$$

Worked example: Potential of 4 charges on square

Question



Four charges each of magnitude q are located at the four corners of a square of side d such that like charges occupy the corners across the diagonals.
Calculate the work done in assembling these charges.

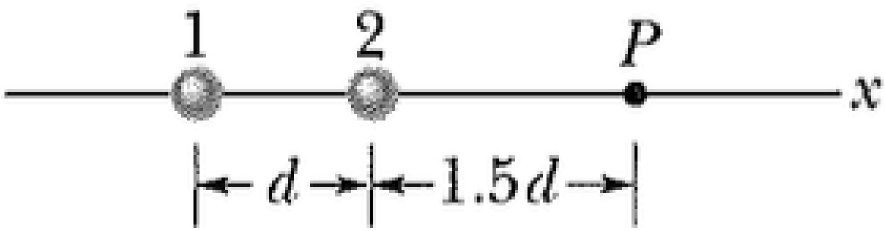
Work done is $W = U_{12} + U_{23} + U_{34} + U_{41} + U_{13} + U_{24}$ where $U_{ij} = \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$
Numbering charges clock-wisely from top left one, so that charges 1 and 3 are positive and 2 and 4 negative:

$$W = \frac{1}{4\pi\epsilon_0}\left\{ -\frac{q^2}{d} - \frac{q^2}{d} - \frac{q^2}{d} - \frac{q^2}{d} + \frac{q^2}{\sqrt{2}d} + \frac{q^2}{\sqrt{2}d} \right\} = -\frac{q^2}{4\pi\epsilon_0 d}\left(4 - \sqrt{2}\right)$$

Worked example: Potential of 3 charges on a line

Question

Two particles of charges q_1 and q_2 are fixed to an x axis, as shown below. If a third particle, of charge $+6.0\ \mu$C, is brought from an infinite distance to point P, the three-particle system has the same electric potential energy as the original two-particle system. What is the charge ratio q_2/q_1?



Worked example: Potential of 3 charges in line

By adding a particle with charge q at point P, at a distance r_1 from q_1 and at distance r_2 from q_2, the change in the electrostatic potential energy of the system is given by:

$$\Delta U = \frac{q_1 q}{4\pi\epsilon_0 r_1} + \frac{q_2 q}{4\pi\epsilon_0 r_2}$$

Since $\Delta U = 0$, we have:

$$\frac{q_1 q}{4\pi\epsilon_0 r_1} + \frac{q_2 q}{4\pi\epsilon_0 r_2} = 0 \Rightarrow$$

$$\frac{q_1}{r_1} + \frac{q_2}{r_2} = 0 \xrightarrow{r_1=2.5d,\ r_2=1.5d} \frac{q_1}{2.5} + \frac{q_2}{1.5} = 0 \Rightarrow$$

$$\frac{q_2}{q_1} = -\frac{1.5}{2.5} = -\frac{3}{5} = -0.6$$

Force and potential energy

Earlier, we saw that

$$U = \int \vec{F} \cdot d\vec{\ell}$$

So **given a force** (on all points along our trajectory) **we can calculate** the work done and thus **the electrostatic potential energy**.

But what if we wanted to solve the opposite problem?
Suppose that, somehow, we knew the potential energy for adding a charge anywhere on space and we wanted to calculate the force on that charge?

This would require *inverting* the above equation.

As you have also seen in mechanics, we can express the force in terms of the gradient of U:

$$\vec{F} = -\vec{\nabla} U$$

Worked example: Finding $\vec{F}$ from U

Question

Starting from the Coulomb force between two charges Q and q, we estimated the following potential energy U for the system of two charges:

$$U = \frac{Qq}{4\pi\epsilon_0 |\vec{r}|}$$

where $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. Can you derive the Coulomb force between charges Q and q starting from the above potential energy U?

The force is given by:

$$\vec{F} = -\vec{\nabla} U$$

The gradient of U is:

$$\vec{\nabla} U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right) = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}$$

Worked example: Finding $\vec{F}$ from U

Substituting U from we have:

$$\vec{\nabla} U = \frac{Qq}{4\pi\epsilon_0} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{1}{|\vec{r}|}$$

The partial derivative $\frac{\partial}{\partial x} \frac{1}{|\vec{r}|}$ is calculated as follows:

$$\frac{\partial}{\partial x} \frac{1}{|\vec{r}|} = \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{\partial}{\partial x} \left(x^2 + y^2 + z^2 \right)^{-\frac{1}{2}} =$$

$$= -\frac{1}{2} \left(x^2 + y^2 + z^2 \right)^{-\frac{3}{2}} (2x) = \frac{-x}{\left(\sqrt{x^2 + y^2 + z^2} \right)^3} = \frac{-x}{|\vec{r}|^3}$$

Similarly:

$$\frac{\partial}{\partial y} \frac{1}{|\vec{r}|} = \frac{-y}{|\vec{r}|^3} \quad \textit{and} \quad \frac{\partial}{\partial z} \frac{1}{|\vec{r}|} = \frac{-z}{|\vec{r}|^3}$$

Worked example: Finding $\vec{F}$ from U

Therefore:

$$\vec{\nabla} U = \frac{Qq}{4\pi\epsilon_0}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{|\vec{r}|} = \frac{Qq}{4\pi\epsilon_0}\left(\frac{-x}{|\vec{r}|^3}, \frac{-y}{|\vec{r}|^3}, \frac{-z}{|\vec{r}|^3}\right) = -\frac{Qq}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$

Since $\vec{F} = -\vec{\nabla} U$ the force $\vec{F}$ is given by:

$$\vec{F} = \frac{Qq}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$

This is our well-known Coulomb force between charges Q and q.

Note: Since the potential energy has only radial dependence, the above calculation could be simplified by using the gradient in spherical coordinates:

$$\vec{\nabla} U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} + \frac{1}{r sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$$

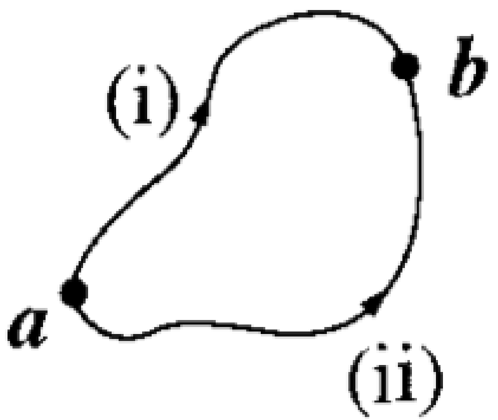
Try this on your own.

Circuital law for Electrostatics

As we saw earlier, the work $W = \int \vec{F} \cdot d\vec{\ell}$ done to move a charge from point a to b is independent of the path followed between a and b.

Since $\vec{E} = \vec{F}/Q$, the quantity $\int \vec{E} \cdot d\vec{\ell}$ is also path independent.

It is not difficult to see that, for a closed path, $\oint \vec{E} \cdot d\vec{\ell} = 0$.



$$\int_{a \; along \; i}^{b} \vec{E} \cdot d\vec{\ell} = \int_{a \; along \; ii}^{b} \vec{E} \cdot d\vec{\ell} \Rightarrow$$

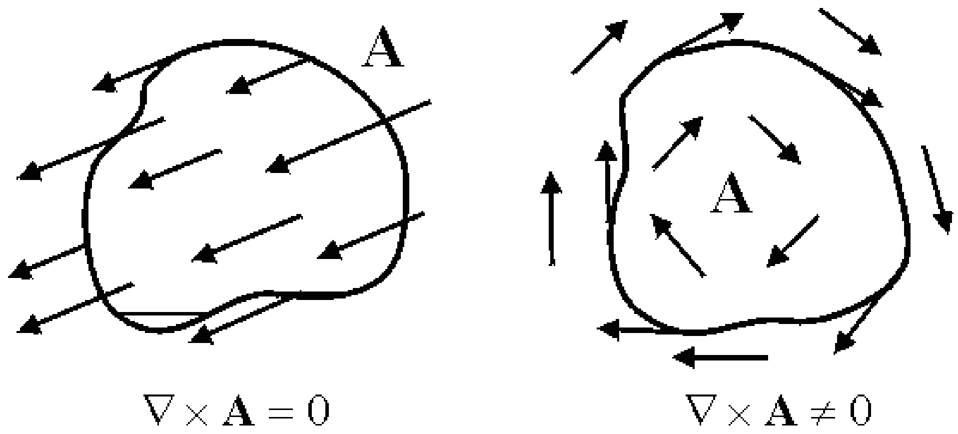
$$\int_{a \; along \; i}^{b} \vec{E} \cdot d\vec{\ell} = -\int_{b \; along \; ii}^{a} \vec{E} \cdot d\vec{\ell} \Rightarrow$$

$$\oint \vec{E} \cdot d\vec{\ell} = 0$$

Reminder: Curl

The curl ($\vec{\nabla} \times \vec{A}$) of a vector field $\vec{A} = \Big(A_x, A_y, A_z\Big)$ is defined as:

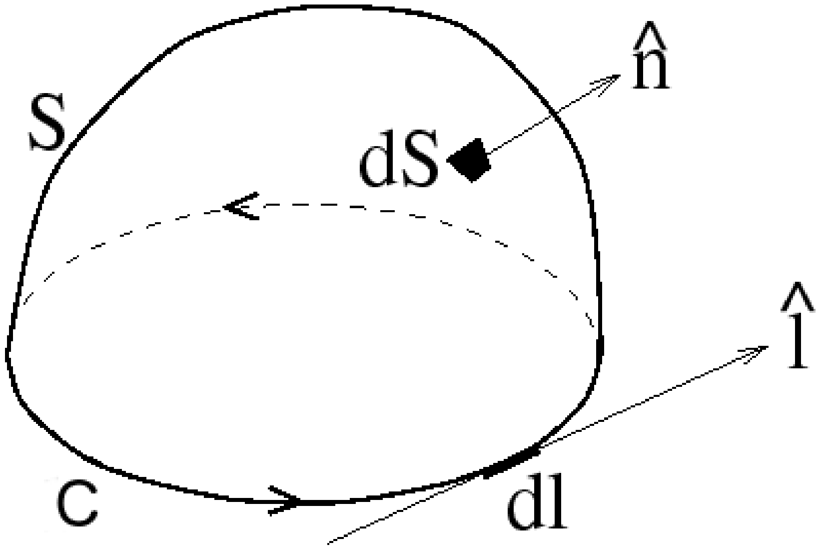
$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \Big(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\Big)\hat{x} - \Big(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\Big)\hat{y} + \Big(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\Big)\hat{z}$$



The curl of a vector field, evaluated at a specific space point, tells us how much the field curls about that point.

Reminder: Stokes' theorem

Stokes' theorem **relates the circulation of a vector field $\vec{F}$ around a closed line C with the flux of the curl of the vector field $\vec{F}$ through the open surface S** defined by the closed line C.



$$\oint_C \vec{F} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

Circuital law for Electrostatics

Applying the Stoke's theorem for the electric field $\vec{E}$ we have

$$\oint \vec{E} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S}$$

But we know that for any closed path

$$\oint \vec{E} \cdot d\vec{\ell} = 0$$

Therefore

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = 0$$

If the above is true for any surface S, then

$$\vec{\nabla} \times \vec{E} = 0$$

$\vec{E}$ is a *special* vector field: One that has **no rotation** for all points in space.

Our first two Maxwell equations for Electrostatics

	Integral form	*Differential form*
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = Q_{enclosed}/\epsilon_0$	$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$

Electric potential

Now I will introduce the new concept of the **electric potential (V)**.

- **Not be confused with electric potential energy (U)**.
  The choice of names is unfortunate.

We define the electric potential V at a position $\vec{r}$ to be the **amount of work required to bring a charge Q at position $\vec{r}$, divided by the charge Q**

$$V = \frac{W}{Q}$$

In SI, the potential has units of **Volts** (V).
It is a derived unit: A Volt is a Joule per Coulomb (J/C).

Electric potential

The potential V is **scalar field**:

- Like $\vec{E}$, it permeates all space.
- Unlike $\vec{E}$, it associates a scalar (not a vector) with every space point.

The **electric field $\vec{E}$ and the potential V are related** as follows:

$$\vec{E} = -\vec{\nabla} V$$

The fact that we can express $\vec{E}$, a vector field, as the gradient fo a scalar field is due to a *special property* of $\vec{E}$. We shall see that special property expressed in the *circuital law* in just a while.

Electric potential of a point charge

Assume that charge q is at the origin and charge Q is brought at position $\vec{r}$. As we saw earlier, the work W done to bring Q at $\vec{r}$ is:

$$W = \frac{qQ}{4\pi\epsilon_0 |\vec{r}|}$$

Therefore, the potential at position $\vec{r}$ due to charge q is:

$$V = \frac{W}{Q} = \frac{q}{4\pi\epsilon_0 |\vec{r}|}$$

We can generalise the above, relaxing the requirement that the point charge q is at the origin. If, instead, q is placed at position $\vec{r}\,'$, then the electric potential at $\vec{r}$ can be written as:

$$V = \frac{q}{4\pi\epsilon_0 |\vec{r} - \vec{r}\,'|}$$

Potential of group of charges / continuous distribution

The potential at position $\vec{r}$ due to a point charge q placed at $\vec{r}\,'$ is:

$$V = \frac{q}{4\pi\epsilon_0 |\vec{r} - \vec{r}\,'|}$$

Note that the **potential obeys the superposition principle**.

Therefore, the potential at position $\vec{r}$ due to charges q_1, q_2,..., q_N placed at positions $\vec{r}_1$, $\vec{r}_2$,..., $\vec{r}_N$ is given by:

$$V = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|}$$

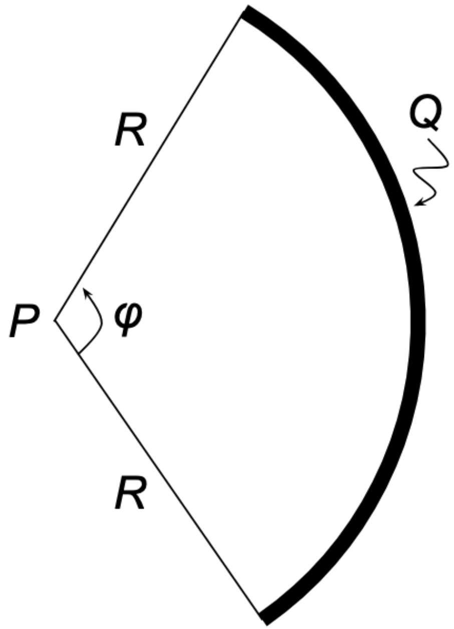
Similarly, the potential at position $\vec{r}$ due to a continuous charge distribution ρ is given by:

$$V = \frac{1}{4\pi\epsilon_0} \int_{\tau'} \frac{\rho(\vec{r}\,')d\tau'}{|\vec{r} - \vec{r}\,'|}$$

Worked example: Electric potential of circular arc of charge

Question

A plastic rod having a uniformly distributed charge Q has been bent into a circular arc of radius R and central angle ϕ. With $V=0$ at infinity, what is the electric potential at P, the centre of curvature of the rod?



Placing, for convenience but without loss of generality, the origin of the coordinate system at P, the potential at $\vec{r} = \vec{0}$ can be written as:

$$V = \frac{1}{4\pi\epsilon_0} \int_{rod} \frac{dq(\vec{r}\,')}{|\vec{r}\,'|}$$

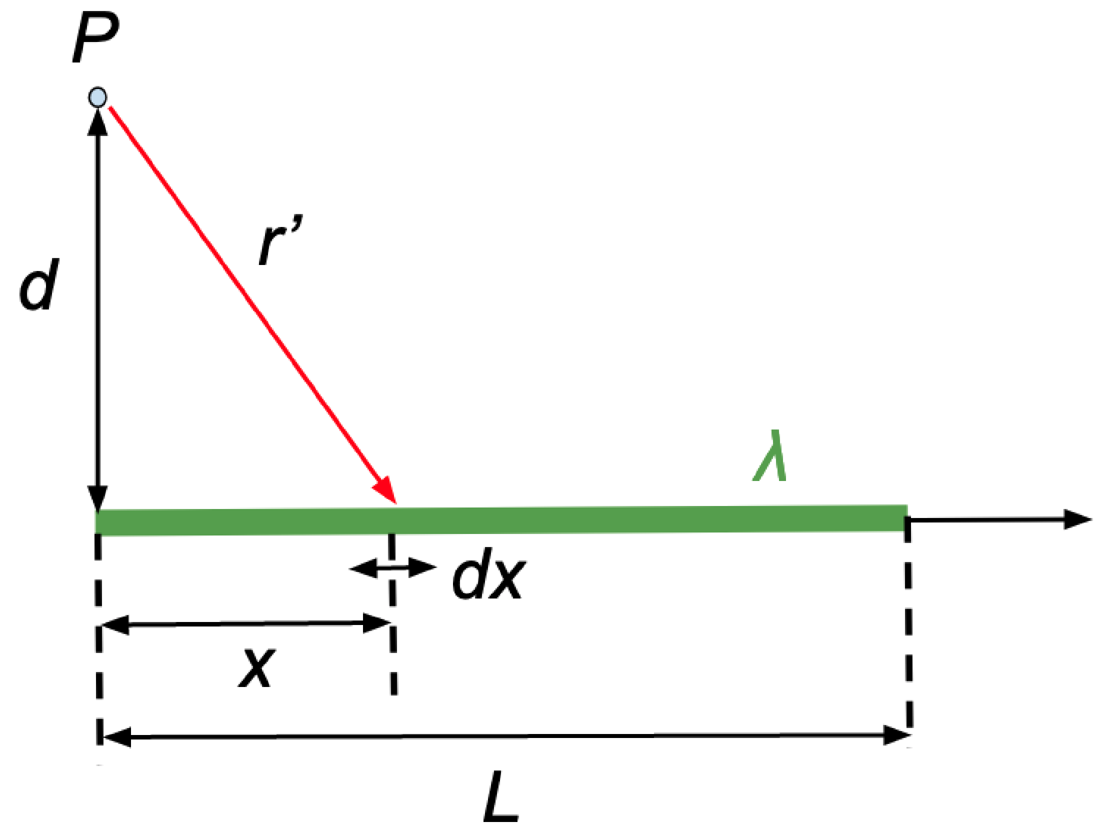
But $|\vec{r}\,'| = R$ is a constant. Therefore:

$$V = \frac{1}{4\pi\epsilon_0 R} \int_{rod} dq(\vec{r}\,') = \frac{Q}{4\pi\epsilon_0 R}$$

Worked example: Electric potential of non-conducting rod

Question

Thin non-conducting rod of length L has a positive charge of uniform linear density λ. Determine the electric potential V due to the rod at point P, at perpendicular distance d from the left end of the rod.



Worked example: Electric potential of non-conducting rod

Placing, for convenience but without loss of generality, the origin of the coordinate system at P, the potential at $\vec{r} = \vec{0}$ can be written as:

$$V = \frac{1}{4\pi\epsilon_0} \int_{rod} \frac{dq(\vec{r}\,')}{r'}$$

The amount of charge dq at position $\vec{r}\,'$ is given by $dq = \lambda dx$, therefore the potential becomes:

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{dx}{r'}$$

To carry out the integration, we need to relate r' and x. From the previous schematic, we find:

$$x^2 + d^2 = r^2 \Rightarrow r = \sqrt{x^2 + d^2}$$

Worked example: Electric potential of non-conducting rod

Therefore the potential can be written as:

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{dx}{\sqrt{x^2 + d^2}} = \frac{\lambda}{4\pi\epsilon_0} ln\Big(x + \sqrt{x^2 + d^2}\Big)\Big|_0^L$$

Evaluating the integral, we have:

$$V = \frac{\lambda}{4\pi\epsilon_0} ln\Big(x + \sqrt{x^2 + d^2}\Big)\Big|_0^L \Rightarrow$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \Big\{ ln\Big(L + \sqrt{L^2 + d^2}\Big) - ln\Big(d\Big) \Big\} \Rightarrow$$

$$V = \frac{\lambda}{4\pi\epsilon_0} ln\Big(\frac{L + \sqrt{L^2 + d^2}}{d}\Big)$$

A p.d.e. (*) for the electric potential

Let's consider what is involved in calculating $\vec{E}$: Since it is a vector field we need to calculate all three components (E_x,E_y and E_z).

We have already seen that (Gauss's law; 1^{st} Maxwell equation):

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

On its own, the divergence of a vector field is **not enough to uniquely determine** that vector field.
However, in this lecture, we also saw another differential equation for $\vec{E}$ (Circuital law; 2^{nd} Maxwell equation):

$$\vec{\nabla} \times \vec{E} = 0$$

So we have **two coupled differential equations to compute** $\vec{E}$:
The components of $\vec{E}$ are **interdependent**. It all sounds too complicated!

(*) *p.d.e.: partial differential equation.*

A p.d.e. for the electric potential

Can we simplify things?

- Let's start by exploiting the fact that:

$$\vec{\nabla} \times \vec{E} = 0$$

- We know (see your basic calculus textbooks) that **the curl of the gradient of any scalar function λ is always 0**:

$$\vec{\nabla} \times \left(\vec{\nabla} \lambda \right) = 0$$

- So, $\vec{E}$ is the gradient of a scalar function λ ($\vec{E} = \vec{\nabla} \lambda$).
- We already know which is that scalar function. It is the electric potential V introduced earlier ($\lambda = -V$):

$$\vec{E} = -\vec{\nabla} V$$

The minus sign is conventional.

A p.d.e. for the electric potential: The Poisson equation

So we can **reduce the problem of calculating** $\vec{E}$ (a vector for which we need to calculate all 3 components E_x, E_y and E_z) **to the simpler problem of calculating a single scalar function V**.

Starting from Gauss's law, and substituting $\vec{E} = -\vec{\nabla} V$ we have:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \vec{\nabla} \cdot \Big(-\vec{\nabla} V \Big) = \frac{\rho}{\epsilon_0} \Rightarrow$$

$$\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$$

The above is known as the **Poisson equation**.

($\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is known as the *Laplace operator* - see later).

A p.d.e. for the electric potential: The Laplace equation

We are often interested in **finding the electric field away from charges** (in regions where the charge density ρ is 0).

Then, Poisson's equation:

$$\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$$

becomes:

$$\vec{\nabla}^2 V = 0$$

The above is known as the **Laplace equation**.

Reminder: The Laplace operator

The Laplace operator is a 2^{nd} order differential operator defined as follows:

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

So the Laplacian of a scalar function f is:

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Laplacian **is the divergence of the gradient** of the scalar function f:

$$\vec{\nabla}^2 f = \vec{\nabla} \cdot \left(\vec{\nabla} f \right)$$

It represents a quantity which is important in several physical processes: The Laplacian $\vec{\nabla}^2 f(\vec{r})$ of a scalar function f at a point $\vec{r}$ tells you **how much $f(\vec{r})$ differs from its average over a small volume around $\vec{r}$.**

Worked example: Finding ρ from V

Question

A potential V is given by:

$$V(\vec{r}) = V(r) = V_0 e^{-(r/a)^2}$$

where $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.
Find out the charge density ρ that is responsible for it.

Starting from Poisson's equation we can solve for ρ:

$$\vec{\nabla}^2 V(\vec{r}) = -\rho(\vec{r})/\epsilon_0 \Rightarrow \rho(\vec{r}) = -\epsilon_0 \vec{\nabla}^2 V(\vec{r})$$

The Laplacian of V is:

$$\vec{\nabla}^2 V = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big) V = V_0 \Big(\frac{\partial^2 e^{-(r/a)^2}}{\partial x^2} + \frac{\partial^2 e^{-(r/a)^2}}{\partial y^2} + \frac{\partial^2 e^{-(r/a)^2}}{\partial z^2}\Big)$$

Worked example: Finding ρ from V

Using the chain rule

$$\frac{\partial e^{-(r/a)^2}}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r}e^{-(r/a)^2}$$

where

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}2x = \frac{x}{r}$$

and

$$\frac{\partial}{\partial r}e^{-(r/a)^2} = e^{-(r/a)^2}\frac{\partial}{\partial r}\Big(-\frac{r^2}{a^2}\Big) = e^{-(r/a)^2}\Big(\frac{-2r}{a^2}\Big)$$

Therefore

$$\frac{\partial e^{-(r/a)^2}}{\partial x} = -\frac{2x}{a^2}e^{-(r/a)^2}$$

Worked example: Finding ρ from V

Continuing, in the same manner, to compute the The second partial derivative wrt to x of $e^{-(r/a)^2}$ is:

$$\frac{\partial^2}{\partial x^2} e^{-(r/a)^2} = \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial x} e^{-(r/a)^2}\Big)$$

We already calculated (see previous page) that:

$$\frac{\partial}{\partial x} e^{-(r/a)^2} = -\frac{2x}{a^2} e^{-(r/a)^2}$$

Therefore:

$$\frac{\partial^2}{\partial x^2} e^{-(r/a)^2} = \frac{\partial}{\partial x}\Big(-\frac{2x}{a^2} e^{-(r/a)^2}\Big) = \Big(-\frac{2}{a^2}\frac{\partial}{\partial x} x\Big) e^{-(r/a)^2} - \frac{2x}{a^2}\Big(\frac{\partial}{\partial x} e^{-(r/a)^2}\Big)$$

$$= -\frac{2}{a^2} e^{-(r/a)^2} - \frac{2x}{a^2}\Big(-\frac{2x}{a^2} e^{-(r/a)^2}\Big) = \Big(\frac{4x^2}{a^4} - \frac{2}{a^2}\Big) e^{-(r/a)^2}$$

Worked example: Finding ρ from V

Similarly:

$$\frac{\partial^2}{\partial y^2} e^{-(r/a)^2} = \left(\frac{4y^2}{a^4} - \frac{2}{a^2}\right) e^{-(r/a)^2} \quad and \quad \frac{\partial^2}{\partial z^2} e^{-(r/a)^2} = \left(\frac{4z^2}{a^4} - \frac{2}{a^2}\right) e^{-(r/a)^2}$$

Therefore:

$$\vec{\nabla}^2 V = V_0 \vec{\nabla}^2 e^{-(r/a)^2} = V_0 \left(4\frac{x^2+y^2+z^2}{a^4} - \frac{6}{a^2}\right) e^{-(r/a)^2} \Rightarrow$$

$$\vec{\nabla}^2 V = V_0 \left(\frac{4r^2}{a^4} - \frac{6}{a^2}\right) e^{-(r/a)^2}$$

So, finally, the charge density is given by:

$$\rho(r) = -\epsilon_0 \vec{\nabla}^2 V(r) = -\epsilon_0 V_0 \left(\frac{4r^2}{a^4} - \frac{6}{a^2}\right) e^{-(r/a)^2}$$

Since the problem has spherical symmetry ($V(\vec{r})$ is a function of $r = |\vec{r}|$ alone) it would have been easier to use the Laplacian operator in spherical coordinates (Try this on your own): $\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin^2\phi}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 sin\phi}\frac{\partial}{\partial \phi}\left(sin\phi \frac{\partial}{\partial \phi}\right)$.

Solving Poisson's equation

Previously, we used Poisson's equation to solve a simple problem: We found the charge density $\rho(\vec{r})$ responsible for a particular potential $V(\vec{r})$.

In practical applications we are usually interested in the *inverse* problem: We start from a known charge density and want to calculate the potential.

Unfortunately, the Poisson equation is *the wrong way around*.

$$\vec{\nabla}^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

Whereas one simply needs to differentiate the potential to get the density, one needs to invert the Poisson equation to compute the potential:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} d^3r'$$

Solving Poisson's equation

Inverting the Poisson equation gives us:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} d^3r'$$

In general, it is not easy to calculate this integral analytically.

- Moreover, in some practical applications, we may not know the density ρ everywhere in space. We may know only V at some boundaries of the region of interest.

It is usually preferable to look at a problem in its differential form:

The Poisson equation $\vec{\nabla}^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$ **together with appropriate boundary conditions is equivalent to the above integral**.

Notice that without the appropriate boundary conditions, the Poisson equation by itself does not determine V.

Boundary conditions

Take the Poisson equation in the absence of charges (Laplace equation). For now, let's just consider the 1-dimensional case:

$$\frac{d^2V(x)}{dx^2} = 0$$

One can easily solve the above and obtain:

$$\frac{d^2V(x)}{dx^2} = 0 \Rightarrow \frac{dV(x)}{dx} = c_1 \Rightarrow V(x) = c_1 x + c_2$$

The above is not a solution, but a **class of solutions**.

The exact solution is known if I can determine the constants c_1 and c_2. For example, I can determine c_1, c_2 if I know:

- the value of V(x) at 2 points, or
- or, the derivative and the value of V(x) at 1 point.

Boundary conditions

How about the Laplace equation at higher dimensions?

As we have seen, in the usual 3 dimensions we have:

$$\vec{\nabla}^2 V(\vec{r}) = \frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = 0$$

The main question is: **What are the appropriate boundary conditions now that there are partial derivatives involved?**

This is a much harder problem:

- Not a finite number of constants in the solution!
- Not easy to say whether a set of boundary conditions is acceptable.

A uniqueness theorem is a proof that a given set of boundary conditions is sufficient.

- It guarantees that a set of boundary conditions yields a unique solution.

The simpler set of boundary conditions is the so-called **Dirichlet** boundary condition: The value of V is specified everywhere on the (closed) boundary surface S of the volume where we are interested to know V.

- The statement that the above boundary conditions yield a unique solution is usually referred to as the **first uniquness theorem**.

Notice a ramification of the uniquness theorem: It doesn't matter how arrive at a solution (by mathematical genius or luck). If you have a solution that satisfies the given boundary conditions then that is *the only* solution.

- **To bring together a collection of charges I need to do work** (for example, In case of two like-sign charges I need to exert a force against the action of the field)

$$W = \int \vec{F} \cdot d\vec{\ell}$$

- The work done can be positive or negative.
- The work done is **path-independent**
  - I do the same work regardless of the path followed to bring the charges in their positions.
  - We say that the electric force is **conservative**.
- The work done is converted to **electric potential energy**
- We calculated the potential energy for systems of 2, 3 and N charges as well as continuous distributions of charge.

Lecture 3 - Main points to remember (cont'd)

For a

- system of 2 charges:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{12}|}$$

- system of 3 charges:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{12}|} + \frac{q_1 q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{13}|} + \frac{q_2 q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{23}|}$$

- system of N charges:

$$U = \frac{1}{2} \sum_{i,j=1; i \neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

- continuous charge distribution (with density ρ over a volume τ):

$$U = \frac{1}{2} \int_{\tau} d\tau \int_{\tau'} d\tau' \frac{\rho(\vec{r})\rho(\vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|}$$

Lecture 3 - Main points to remember (cont'd)

- The electric **potential energy is stored in the electric field**:

$$U = \frac{\epsilon_0}{2} \int_{all\ space} |\vec{E}(\vec{r})|^2 d\tau$$

- **Relationship between force and potential energy**:

$$U = \int \vec{F} \cdot d\vec{\ell}$$

$$\vec{F} = -\vec{\nabla} U$$

- **Electric potential (V)**: A scalar field
  - It is the amount of work required to bring a charge Q at position $\vec{r}$, divided by the charge Q.
  - SI units: Volts (V)
    - Derived unit: One Joule per Coulomb

Lecture 3 - Main points to remember (cont'd)

- We also studied the **circuital law for Electrostatics**
  - Our second set of Maxwell's equations.

- Maxwell's equation we know so far:

	Integral form	*Differential form*
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = Q_{enclosed}/\epsilon_0$	$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$

- Because (in Electrostatics) the electric field has no rotation it can be expressed as the gradient of a scalar function (the electric potential):

$$\vec{E} = -\vec{\nabla} V$$

Lecture 3 - Main points to remember (cont'd)

- Need both the divergence and the curl of $\vec{E}$ (both Gauss' and circuital laws) to determine all three components of $\vec{E}$.
  - Gauss' and circuital laws provide a coupled system of 1^{st} order p.d.e's.
- I can combine the Gauss' and circuital laws into a single 2^{nd} order p.d.e for the electric potential V: **Poisson equation**

$$\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$$

- Away from sources (ρ=0) Poisson's equation becomes $\vec{\nabla}^2 V = 0$ which is known as the **Laplace equation**.
- Using the Poisson (or Laplace) equations one can determine V (and, thus, $\vec{E}$) only using the appropriate **boundary conditions**.
- **A uniqueness theorem** is a proof that a given set of boundary conditions is sufficient.

At the next lecture (Lecture 4)

What happens when materials are placed in an electric field?

We will study the two main types of materials with regards to their electrical properties:

- Conductors
- Dielectrics (or insulators)

and we will discuss some useful concepts:

- Capacitance
- The electric dipole
- Polarization

Optional reading for Lecture 3

The electrostatic potential energy is stored in the field

We can now prove a statement I made earlier: That **the electrostatic potential energy is stored in the electric field**.

The potential energy stored in a system of N charges can be written as:

$$U = \frac{1}{2}\sum_{i,j=1;i\neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|} = \frac{1}{2}\sum_{i}^{N} q_i \sum_{j=1;j\neq i}^{N} \frac{q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|} = \frac{1}{2}\sum_{i}^{N} q_i V(\vec{r}_i)$$

where

$$V(\vec{r}_i) = \sum_{j=1;j\neq i}^{N} \frac{q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

is the potential at $\vec{r}_i$ due to all charges other than q_i.

The above result can be adapted for the continuous case too, using the (by now) familiar substitutions:

$$U = \frac{1}{2}\sum_{i}^{N} q_i V(\vec{r}_i) \rightarrow \frac{1}{2}\int_V \rho(\vec{r}) V(\vec{r}) d\tau$$

The electrostatic potential energy is stored in the field

The expression we have for the potential energy U of a continuous charge distribution described by charge density ρ is:

$$U = \frac{1}{2} \int_V \rho(\vec{r}) V(\vec{r}) d\tau$$

We will show that U is related to a volume integral of $|\vec{E}|^2$.

Using Gauss's law, the charge density ρ can be written as:

$$\vec{\nabla} \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} \Rightarrow \rho(\vec{r}) = \epsilon_0 \vec{\nabla} \vec{E}(\vec{r})$$

Substituting ρ into the expression for U, we have:

$$U = \frac{\epsilon_0}{2} \int_V \left(\vec{\nabla} \vec{E}(\vec{r}) \right) V(\vec{r}) d\tau$$

The electrostatic potential energy is stored in the field

I would like to express U in terms of $\vec{E}$ only.

Since $\vec{\nabla} V = -\vec{E}$, I will try to move the $\vec{\nabla}$ in the previous expression so that it operates on V instead on $\vec{E}$.

The trick is to operate with $\vec{\nabla}$ on the product $\vec{E}V$:

$$\vec{\nabla}\Big(\vec{E}(\vec{r})V(\vec{r})\Big) = \Big(\vec{\nabla}\vec{E}(\vec{r})\Big)V(\vec{r}) + \vec{E}(\vec{r})\Big(\vec{\nabla}V(\vec{r})\Big) \Rightarrow$$

$$\Big(\vec{\nabla}\vec{E}(\vec{r})\Big)V(\vec{r}) = \vec{\nabla}\Big(\vec{E}(\vec{r}))V(\vec{r})\Big) - \vec{E}(\vec{r})\Big(\vec{\nabla}V(\vec{r})\Big)$$

Substituting the above in the last equation of the previous page we have:

$$U = \frac{\epsilon_0}{2}\int_V \vec{\nabla}\Big(\vec{E}(\vec{r}))V(\vec{r})\Big)d\tau - \frac{\epsilon_0}{2}\int_V \vec{E}(\vec{r})\Big(\vec{\nabla}V(\vec{r})\Big)d\tau$$

The electrostatic potential energy is stored in the field

Using Gauss' theorem, the 1^{st} term of the previous expression for U becomes:

$$\int_V \vec{\nabla}\Big(\vec{E}(\vec{r}))V(\vec{r})\Big)d\tau = \oint_S \Big(\vec{E}(\vec{r}))V(\vec{r})\Big)d\vec{S}$$

Using $\vec{\nabla}V = -\vec{E}$, the 2^{nd} term of the previous expression for U becomes:

$$\int_V \vec{E}(\vec{r})\Big(\vec{\nabla}V(\vec{r})\Big)d\tau = -\int_V \vec{E}(\vec{r})\vec{E}(\vec{r})d\tau = -\int_V |\vec{E}(\vec{r})|^2 d\tau$$

The equation for the electric potential U can be rewritten as:

$$U = \frac{\epsilon_0}{2}\oint_S \Big(\vec{E}(\vec{r}))V(\vec{r})\Big)d\vec{S} + \frac{\epsilon_0}{2}\int_V |\vec{E}(\vec{r})|^2 d\tau$$

As r $\rightarrow \infty$, the surface term $\oint_S \Big(...\Big)d\vec{S} \rightarrow 0$. Therefore:

$$U = \frac{\epsilon_0}{2}\int_{all\ space} |\vec{E}(\vec{r})|^2 d\tau$$

The Dirichlet boundary condition yields unique solutions

This can be understood as follows:

- Imagine a volume with a charge density ρ which is known at all points.
- Also assume that the potential is known everywhere on the boundaries (i.e. on the surface surrounding the volume)
- Assume that there are two distinct solutions, $V_1(\vec{r})$ and $V_2(\vec{r})$.
  - $\vec{\nabla}^2 V_1(\vec{r}) = -\rho(\vec{r})/\epsilon_0$
  - $\vec{\nabla}^2 V_2(\vec{r}) = -\rho(\vec{r})/\epsilon_0$
- Now, consider the function $V_3(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r})$
  - At the boundary, $V_3(\vec{r}) = 0$ (since both V_1, V_2 satisfy the same boundary condition)
  - Also, $V_3(\vec{r})$ satisfies the equation $\vec{\nabla}^2 V_3 = 0$ and, as such, changes monotonically inside the volume and has no minima/maxima.
- V_3 ranges between 0 and ...0, so V_3 is 0 everywhere in the given volume.
- This contradicts the assumption that two distinct solutions V_1, V_2 can exist. $V_1 = V_2$ everywhere in the volume, so a unique solution exists.

$$\vec{\nabla}^2 V_3 = \vec{\nabla}^2 (V_1 - V_2) = \vec{\nabla}^2 V_1 - \vec{\nabla}^2 V_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0$$

Worked example: Hypothetical form of electric force

Question

Suppose that instead of the Coulomb force law, one found experimentally that the force between two charges q_1 and q_2 was

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1 - \sqrt{C|\vec{r}_1 - \vec{r}_2|}}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2)$$

where C is a constant.

- Find an expression for the electric field $\vec{E}$ surrounding a point charge q at the origin of the coordinate system.
- Choose an arbitrary closed path around a point charge q and calculate the line integral $\oint \vec{E} \cdot d\vec{\ell}$. Compare with the Coulomb result.
- Find $\oint \vec{E} \cdot d\vec{S}$ over a spherical surface of radius R with a point charge q at its centre. Compare with the Coulomb result.
- Calculate $\vec{\nabla} \cdot \vec{E}$ and compare with the Coulomb result.

Worked example: Hypothetical form of electric force

Assuming, without loss of generality, that the point charge q is at the origin of the coordinate system and that a positive test charge Q is brought at distance $\vec{r}$ from q, the electric field at $\vec{r}$ is given by:

$$\vec{E}(\vec{r}) = \frac{\vec{F}_{Qq}}{Q}$$

where $\vec{F}_{Qq}$ is the force exerted on Q because of q.
Using the given expression of the electric force, and using $r = |\vec{r}|$, $\vec{F}_{Qq}$ can be written as:

$$\vec{F}_{Qq} = \frac{Qq}{4\pi\epsilon_0} \frac{1-\sqrt{Cr}}{r^3} \vec{r} \xRightarrow{\vec{r}=r\hat{r}} \vec{F}_{Qq} = \frac{Qq}{4\pi\epsilon_0} \frac{1-\sqrt{Cr}}{r^2} \hat{r}$$

Therefore, the electric field at point $\vec{r}$ can be written as:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1-\sqrt{Cr}}{r^2} \hat{r}$$

Worked example: Hypothetical form of electric force

The path integral of the electric field along a closed path L can be written as:

$$\oint_L \vec{E} \cdot d\vec{\ell} = \frac{q}{4\pi\epsilon_0} \oint_L \frac{1-\sqrt{Cr}}{r^2} \hat{r} \cdot d\vec{\ell}$$

Using:

$$\hat{r} \cdot d\vec{\ell} = d\ell cos\theta = dr$$

we find:

$$\oint_L \vec{E} \cdot d\vec{\ell} = \frac{q}{4\pi\epsilon_0} \oint_L \frac{1-\sqrt{Cr}}{r^2} dr = \frac{q}{4\pi\epsilon_0} \oint_L \left(r^{-2} - \sqrt{C} r^{-3/2} \right) dr$$

$$= \frac{q}{4\pi\epsilon_0} \left(-\frac{1}{r'} + 2\sqrt{C} \frac{1}{\sqrt{r'}} \right) \Bigg|_r^r = 0$$

This is the same as the corresponding Coulomb result.

Worked example: Hypothetical form of electric force

The flux of the electric field through the closed surface S or a sphere with radius R can be written as:

$$\oint_{S(R)} \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\epsilon_0} \oint_{S(R)} \frac{1-\sqrt{Cr}}{r^2} \hat{r} \cdot d\vec{S}$$

Since the integration surface is a sphere centred at the origin of the coordinate system, $d\vec{S}$ is a radial vector:

$$d\vec{S} = dS\hat{r}$$

and, therefore, the surface integral can be written as:

$$\oint_{S(R)} \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\epsilon_0} \oint_{S(R)} \frac{1-\sqrt{Cr}}{r^2} dS$$

Over the integration surface, at $r = R$ the integrand has a constant value of: $\frac{q}{4\pi\epsilon_0} \frac{1-\sqrt{CR}}{R^2}$.

Worked example: Hypothetical form of electric force

Therefore, the expression for electric flux through S becomes:

$$\oint_{S(R)} \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\epsilon_0} \frac{1-\sqrt{CR}}{R^2} \oint_{S(R)} dS = \frac{q}{4\pi\epsilon_0} \frac{1-\sqrt{CR}}{\cancel{R^2}} 4\pi \cancel{R^2} \Rightarrow$$

$$\oint_{S(R)} \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0}\left(1-\sqrt{CR}\right)$$

This differs by corresponding Coulomb result: Therefore, the expression for electric flux through S becomes:

$$\oint_{S(R)} \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0}$$

by a term equal to $-\sqrt{CR}\frac{q}{\epsilon_0}$.

Worked example: Hypothetical form of electric force

The given electric field is a radial one and, therefore, it is easier to compute its divergence by expressing the divergence operator in spherical coordinates:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \Big(\frac{q}{4\pi\epsilon_0} \frac{1-\sqrt{Cr}}{r^2} \Big) \right)$$

Therefore:

$$\vec{\nabla} \cdot \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \Big(1 - \sqrt{Cr} \Big) = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \Big(-\frac{1}{2}\sqrt{\frac{C}{r}} \Big) = -\frac{q\sqrt{C}}{8\pi\epsilon_0 r^{5/2}}$$

This differs from the Coulomb result:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

which, for a point charge q at the origin, can be written as

$$\vec{\nabla} \cdot \vec{E} = \frac{q}{\epsilon_0} \delta(r)$$

PHYS201 scientific programming task for Lecture 3

We will attempt to **solve numerically the Laplace equation in 2-D**, for some given boundary conditions, and determine the potential V!

The Laplace equation in 2-D takes the form

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$

We will solve this equation for all x, y in the square area defined by:

$$0 < x < L \quad \text{and} \quad 0 < y < L$$

Our boundary conditions are:

$$V(x,0) = V_0, \quad V(x,L) = 0, \quad V(0,y) = 0, \quad V(L,y) = 0$$

Take L = 1 m and $V_0 = 1$ V.

PHYS201 scientific programming task for Lecture 3

Hint: Solve the Laplace equation numerically, using the *finite difference method*. Consider the Taylor expansions of a function f(x) around x:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$$

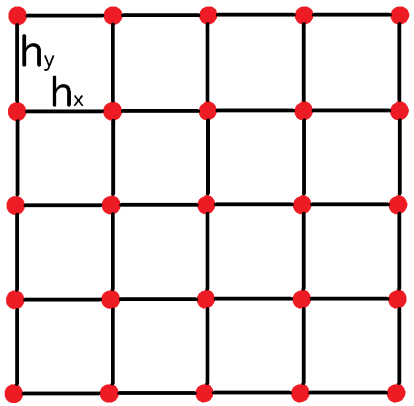
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - O(h^3)$$

where h is a small distance.

Adding the two equations, we obtain the *first central difference approximation* for the second derivative of f(x):

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4) \Rightarrow$$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$



Now, consider a uniform mesh (grid), as shown on the left, where the spacing between neighbouring points along x is h_x and the spacing between neighbouring points along y is h_y.

Using the *first central difference approximation*, the Laplace equation for any point on the 2-D grid can be written as:

$$\frac{V(x+h_x,y) - 2V(x,y) + V(x-h_x,y)}{h_x^2} + \frac{V(x,y+h_y) - 2V(x,y) + V(x,y-h_y)}{h_y^2} = 0$$

If the grid is uniform ($h_x = h_y = h$), then the above equation becomes:

$$V(x+h,y) + V(x-h,y) + V(x,y+h) + V(x,y-h) - 4V(x,y) = 0$$

You have a set of such equations, one for each grid point, which you need to **solve simultaneously** in order to determine V(x,y) for each grid point.

PHYS 201 / Lecture 4
Conductors; Capacitance; Dielectrics; Dipoles; Polarization

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December 15, 2021

Lecture 3 - Revision

- **To bring together a collection of charges I need to do work** (for example, In case of two like-sign charges I need to exert a force against the action of the field)

$$W = \int \vec{F} \cdot d\vec{\ell}$$

- The work done can be positive or negative.
- The work done is **path-independent**
  - I do the same work regardless of the path followed to bring the charges in their positions.
  - We say that the electric force is **conservative**.
- The work done is converted to **electric potential energy**
- We calculated the potential energy for systems of 2, 3 and N charges as well as continuous distributions of charge.

Lecture 3 - Revision (cont'd)

For a

- system of 2 charges:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{12}|}$$

- system of 3 charges:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{12}|} + \frac{q_1 q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{13}|} + \frac{q_2 q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{23}|}$$

- system of N charges:

$$U = \frac{1}{2} \sum_{i,j=1;i\neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

- continuous charge distribution (with density ρ over a volume τ):

$$U = \frac{1}{2} \int_{\tau} d\tau \int_{\tau'} d\tau' \frac{\rho(\vec{r})\rho(\vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|}$$

Lecture 3 - Revision (cont'd)

- The electric **potential energy is stored in the electric field**:

$$U = \frac{\epsilon_0}{2} \int_{all\ space} |\vec{E}(\vec{r})|^2 d\tau$$

- **Relationship between force and potential energy**:

$$U = \int \vec{F} \cdot d\vec{\ell}$$

$$\vec{F} = -\vec{\nabla} U$$

- **Electric potential (V)**: A scalar field
  - It is the amount of work required to bring a charge Q at position $\vec{r}$, divided by the charge Q.
  - SI units: Volts (V)
    - Derived unit: One Joule per Coulomb

Lecture 3 - Revision (cont'd)

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  - Our second set of Maxwell's equations.
- Maxwell's equation we know so far:

	Integral form	*Differential form*
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Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$

- Because (in Electrostatics) the electric field has no rotation it can be expressed as the gradient of a scalar function (the electric potential):

$$\vec{E} = -\vec{\nabla} V$$

Lecture 3 - Revision (cont'd)

- Need both the divergence and the curl of $\vec{E}$ (both Gauss' and circuital laws) to determine all three components of $\vec{E}$.
  - Gauss' and circuital laws provide a coupled system of 1^{st} order p.d.e's.
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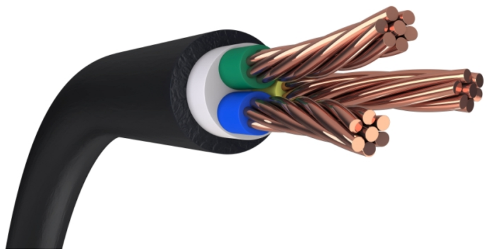
- Away from sources (ρ=0) Poisson's equation becomes $\vec{\nabla}^2 V = 0$ which is known as the **Laplace equation**.
- Using the Poisson (or Laplace) equations one can determine V (and, thus, $\vec{E}$) only using the appropriate **boundary conditions**.
- **A uniqueness theorem** is a proof that a given set of boundary conditions is sufficient.

Plan for Lecture 4

- So far we studied electrostatics in the *vacuum*.
- In this lecture we will discuss **what happens when materials are placed within an electric field**.
- With regards to their electrical properties, there are 2 main types of materials
    - materials that conduct electricity: **conductors**
    - materials that do not conduct electricity: **insulators (dielectrics)**
- We will start by discussing what happens when we place a conductor in an electrostatic field
- We will discuss the concept of capacitance, and study in more detail a simple capacitor: Parallel plate capacitor.
- Then, we also see what happens when we place a dielectric inside an electrostatic field and we will discuss electrical dipoles and the concept of polarisation.

Conductors

A **conductor** is an object or type of material which **contains electric charges that are relatively free to move** through that object or material.



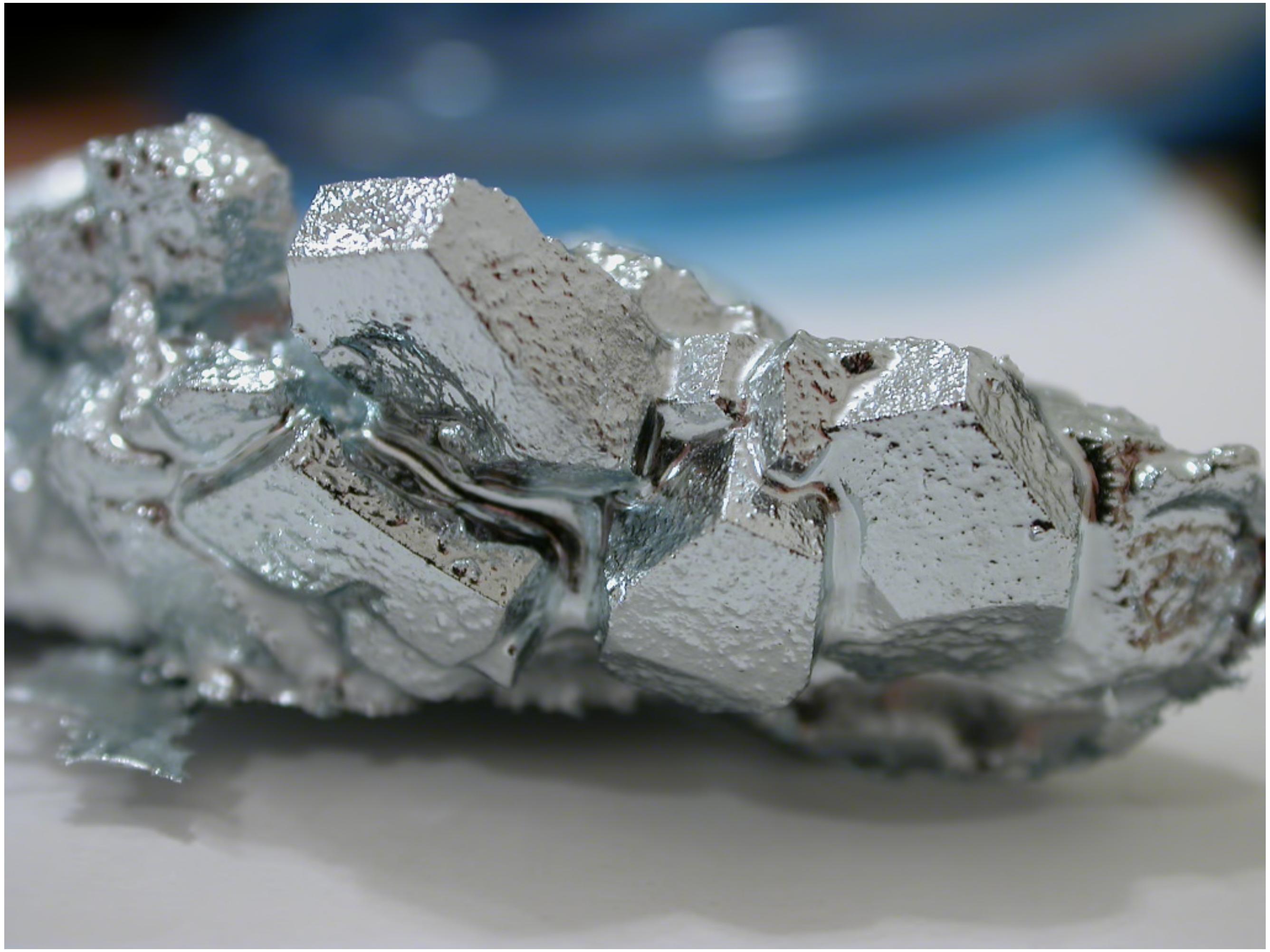
Free electric charges:

- In most case, these are **electrons** (e.g. in metals).
- But in some cases the positive charges may also be able to move (e.g. in electrolytes such as salt water).

A **perfect conductor** has an **unlimited supply of free charges**.

Supply of free charges

What provides that *unlimited supply* of free charges?
We will examine the most typical conducting substance: **Metals**.

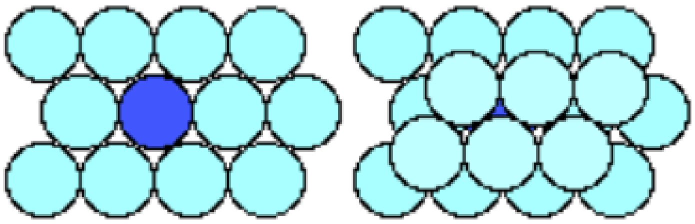


Crystal structure of metals

In a metallic substance, the **atoms are usually closed packed and neatly arranged in symmetrical structures we call crystals**.

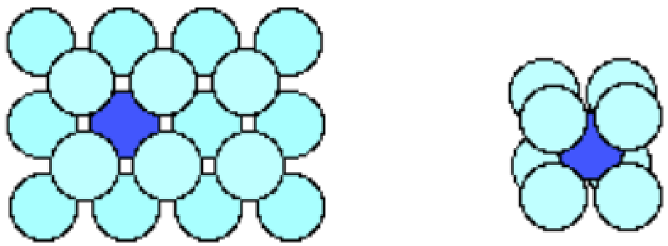
- Think how grocers pack oranges: Nature does something very similar when it organizes atoms in crystals.
- Atoms in crystals typically have 12 or 8 touching neighbours.

12 neighbours



6 neighbours on each plane + 3 on each of the two neighbouring planes.

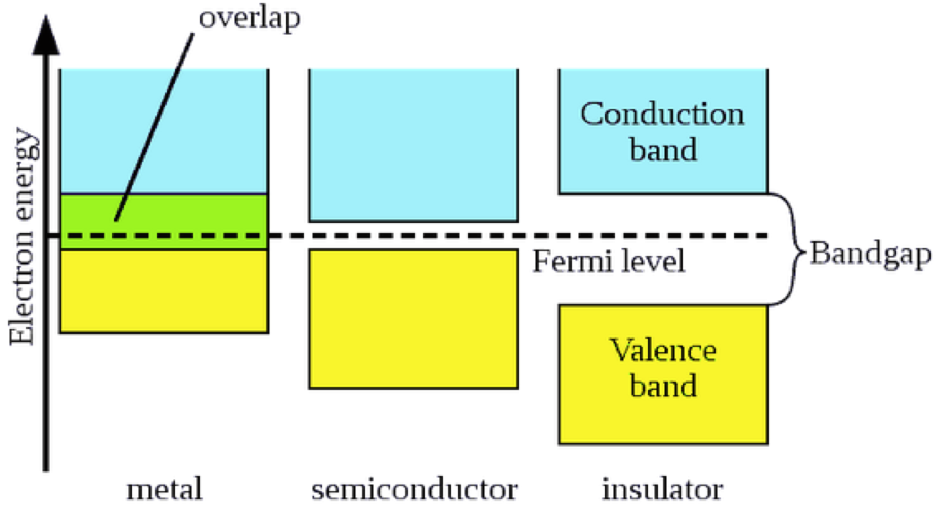
8 neighbours (less efficient packing)



No touching neighbour on same plane, but 4 from each of the planes below and above.

Supply of free charges

Atoms lose one or more electrons from their outer shells and these electrons are free to move. In the case of metals the positive charges are packed together and can not move.



- Energy levels quantized.
- Closely spaced levels: Energy "bands".
- Nature seeks to minimize the total energy but not all electrons in the lowest state (Pauli exclusion principle).
- Levels up to the Fermi level are filled.
- Metals have many energy levels close to the Fermi level: Electrons can jump between states.

A **perfect conductor** has an **unlimited supply of free charges**.

- There are no perfect conductors, but many substances come close!
- For example, in copper, the free charge density is 1.8×10^{10} C/m^3.

Placing a conductor within an electric field

What happens if we place a conductor (e.g. a chunk of metal) **within an external electric field?**

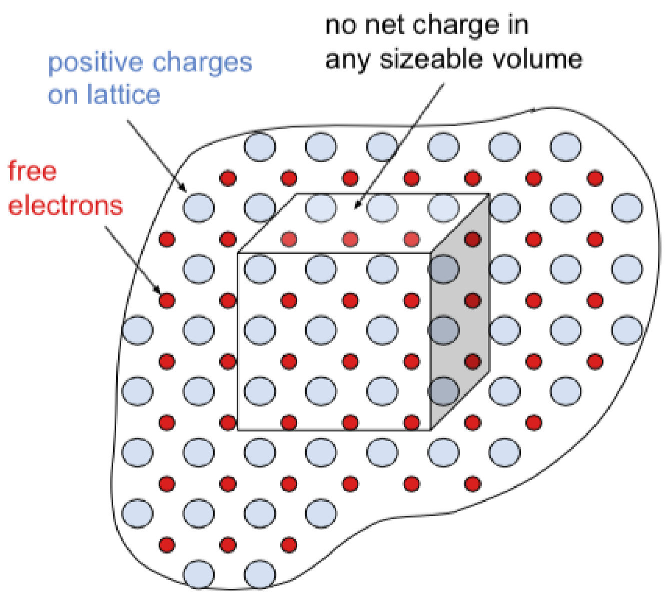
- The electric field vanishes everywhere inside a conductor.
- The potential in constant inside a conductor.
- Charge accumulates in the surface.
- The electric field on the surface of a conductor has no tangential component.

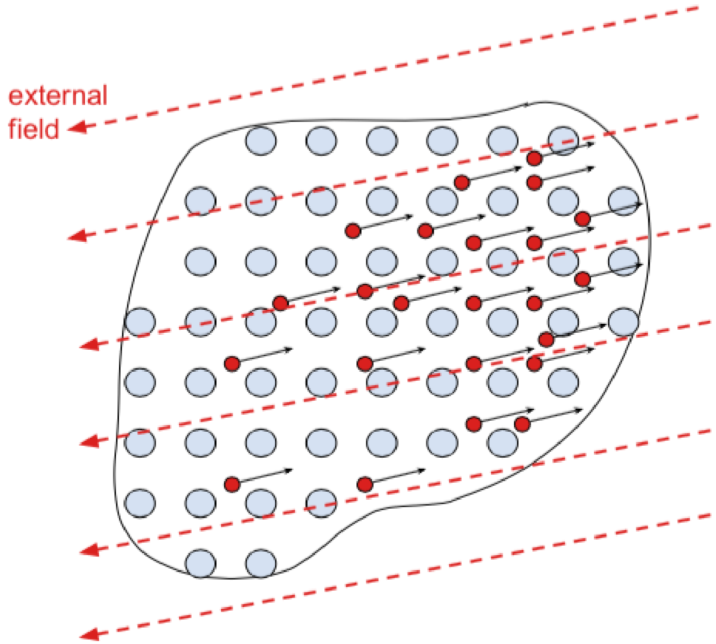
How do these effects come about?

The effect of the external electric field

Assume that our conductor is uncharged, so it is neutral (*). Of course, it is made up from positive and negative charges. Within an external field:

- The free negative charges will move opposite to the field.
- The positive charges feel a force but are pinned and can not move.



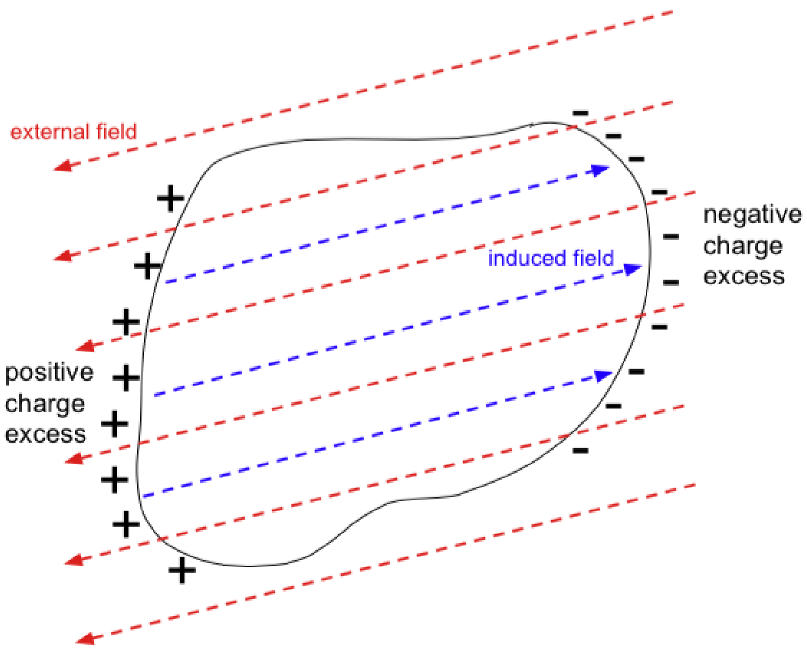


(*) It is neutral on average: i.e. in *macroscopic* volumes that contain few $\times \sim$100 atoms. Obviously much smaller volumes can have net charge: For example, a volume that includes only an atomic nucleus will be positively charged.

The electric field inside a conductor

The motion of charges creates a **macroscopic accumulation of charge**:

- An excess of negative charge on the side opposite to the direction of $\vec{E}$.
- A deficit of negative charge (i.e. an excess of positive charge) on the side pointed to by $\vec{E}$.



- **The induced charges create an electric field of their own**.
- The electric field within the conductor is the vector sum of the the external and induced fields.
- The induced electric field **opposes the external field**.

Eventually **the electric field vanishes everywhere inside the conductor.**

The electric potential inside a conductor

Recall that the electric field $\vec{E}$ and electric potential V are related as follows:

$$\vec{E} = -\vec{\nabla} V = -\Big(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\Big)$$

The electric field vanishes everywhere inside a conductor.

Therefore, **the potential in constant inside a conductor.**

We say that the conductor is an **equipotential**.

Induced charge inside a conductor

As we have seen, charges are free to move. Where to?

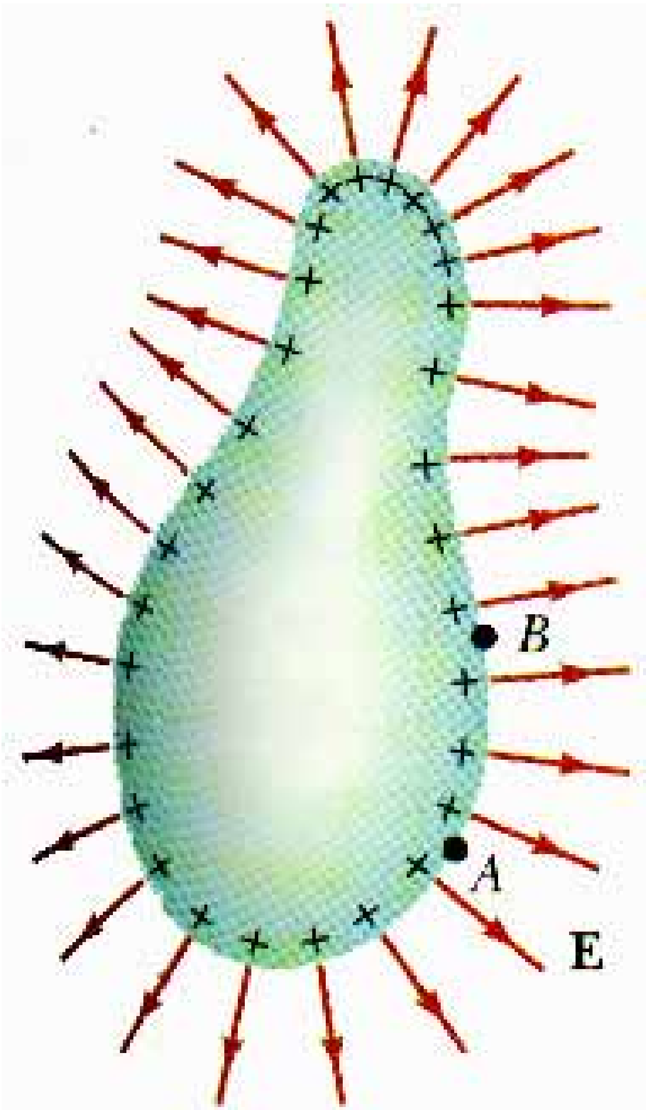
The surface is the only place where induced charges could be!

- It follows from Gauss's law:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

- If $|\vec{E}| = 0$ inside a conductor, then $\rho(r)$=0 too.
- The induced charges can not live in the volume of the conductor, so they have to be on its surface.

Field on the surface of the conductor



For the same reasons that the electric field is cancelled off within the conductor, **it is perpendicular to the surface just outside the conductor**.

If there was a tangential component on the surface, charge would move so as to cancel off that component!

Field on the surface of the conductor

To be convinced, consider a charge Q moving on the surface of the conductor:

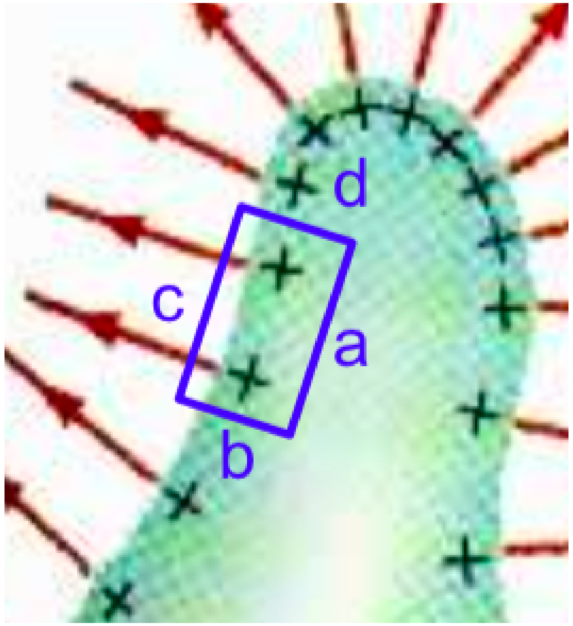
- The electric field exerts a force upon Q.
- The surface of the conductor is at the same potential V.
- Therefore, as long as Q stays on the surface, the electric force acting on Q does no work.

Recall that $W = \int \vec{F} \cdot d\vec{\ell}$.

If the electric force does no work, it is because **it has no tangential component on the surface of the conductor**, i.e. it is always perpendicular to the surface.

Field on the surface of the conductor

Another easy way to see that the electric field just outside the surface of a conductor has no tangential component is by a straightforward application of the circuital law.



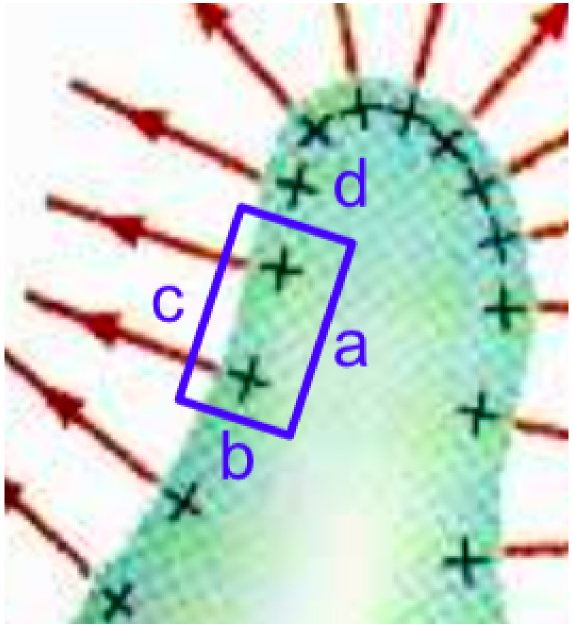
We know that the circulation of the electric field $\vec{E}$ along a closed trajectory is 0:

$$\oint \vec{E} \cdot d\vec{\ell} = 0.$$

Applying the circuital law for the closed trajectory shown on the left, we have:

$$\oint \vec{E} \cdot d\vec{\ell} = \int_a \vec{E} \cdot d\vec{\ell} + \int_b \vec{E} \cdot d\vec{\ell} + \int_c \vec{E} \cdot d\vec{\ell} + \int_d \vec{E} \cdot d\vec{\ell} = 0.$$

Field on the surface of the conductor



- $\vec{E} = 0$ in the conductor so the segment 'a' does not contribute to the integral ($\int_a \vec{E} \cdot d\vec{\ell} = 0$).
- The segments exiting the conductor ('b' and 'd') can be made infinitesimally small, so they do not contribute to the integral either ($\int_b \vec{E} \cdot d\vec{\ell} = \int_d \vec{E} \cdot d\vec{\ell} = 0$).
- Therefore $\oint \vec{E} \cdot d\vec{\ell} = 0$ implies that $\int_c \vec{E} \cdot d\vec{\ell} = 0$
- The segment 'c' can be anywhere just outside the surface of the conductor and, since the segments 'b' and 'd' are infinitesimally small, 'c' is parallel to the surface of the conductor.
- Therefore, the only way that $\int_c \vec{E} \cdot d\vec{\ell}$ to be always 0 is for $\vec{E}$ to be always perpendicular to the surface of the conductor.

Distribution of charge over the surface of a conductor

In general, the induced charges are **not uniformly distributed**.

Consider two conducting spheres with different radii.

The sphere with radius r_1 has charge Q_1, while the sphere with radius r_2 has charge Q_2. The spheres are brought into electrical contact and then separated.

What is the charge density after separation?

After the separation the two spheres have charge Q'_1, Q'_2 and they are at the same potential V:

$$V = \frac{Q'_1}{4\pi\epsilon_0 r_1} = \frac{Q'_2}{4\pi\epsilon_0 r_2}$$

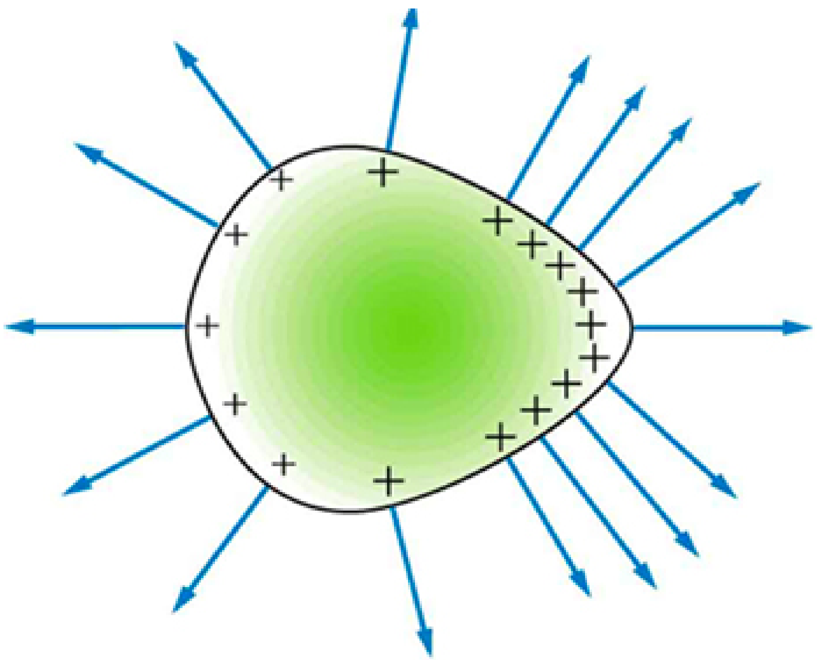
Distribution of charge over the surface of a conductor

Therefore the surface charge densities are:

$$\sigma_1' = \frac{Q_1'}{4\pi r_1^2} = \frac{4\pi\epsilon_0 r_1 V}{4\pi r_1^2} = \frac{\epsilon_0 V}{r_1} \quad \textit{and} \quad \sigma_2' = \frac{\epsilon_0 V}{r_2}$$

Thus $\sigma' \propto \frac{1}{r}$.

This is a more general conclusion.



The surface charge density is smaller in the areas where curvature is smaller.

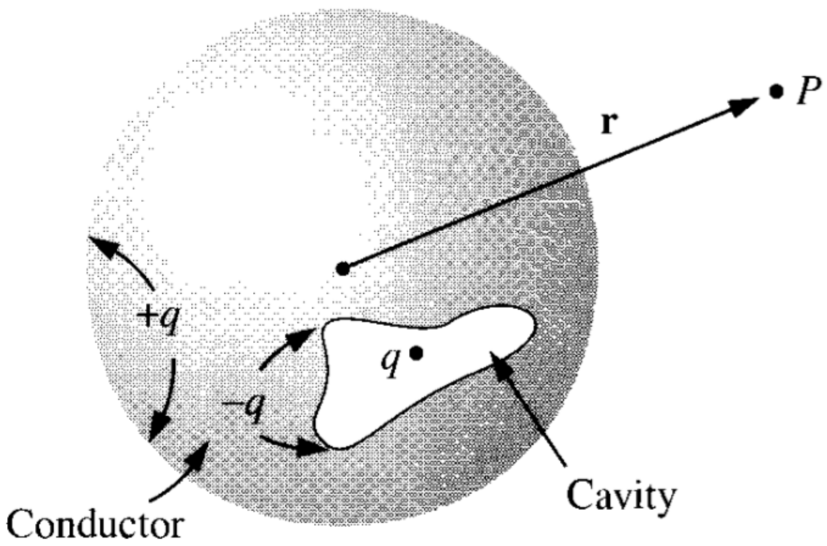
Summary: Placing a conductor within an electric field

- The electric field vanishes everywhere inside a conductor.
- The potential in constant inside a conductor.
- Charge accumulates in the surface.
- The electric field on the surface of a conductor has no tangential component.

Quiz

Question

An uncharged spherical conductor entered at the origin has a cavity with an arbitrary shape carved out of it. Somewhere within the cavity is a charge q. What is the field outside the sphere?



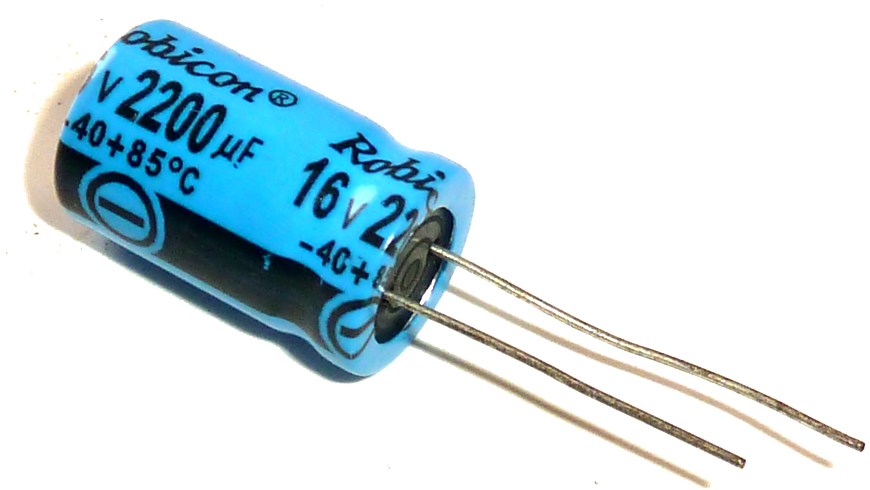
Will discuss in the lecture. For a detailed written answer see Griffiths.

Capacitance

Capacitance denotes the ability of a body to store electric charge.

A system that has capacitance is called a **capacitor**.

As we shall see, capacitors store energy (in their electric field).



Capacitance of an isolated conductor

At first, let's consider an **isolated conductor** with **net charge** Q in it.

- If Q is positive that means that we took away electrons or,
- if Q is negative that means that we gave it an excess of electrons.

The charge Q distributes itself in the conductor.

- That charge distribution is expressed with the charge density ρ.
- We don't know what that charge density is: The charge will not be uniformly distributed, unless the conductor is a uniform sphere.

The one thing we do know for sure is that the integral of ρ over the volume of the conductor is Q

$$Q = \int_{\tau} \rho(\vec{r}) d\tau$$

Capacitance of isolated conductor

As we saw earlier, the electric field is 0 within the conductor, therefore the conductor is an **equipotential**. The potential is:

$$V = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\rho(\vec{r})}{r} d\tau$$

Let's tweak the amount of charge by a factor k (e.g. let's double it (k=2)):

$$Q \rightarrow Q' = kQ \Rightarrow \rho(\vec{r}) \rightarrow \rho'(\vec{r}) = k\rho(\vec{r})$$

The potential changes by the same amount:

$$V \rightarrow V' = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\rho'(\vec{r})}{r} d\tau = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{k\rho(\vec{r})}{r} d\tau = k \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\rho(\vec{r})}{r} d\tau = kV$$

Therefore, **the ratio Q/V remains constant**.
Capacitance is the constant of proportionality (Q/V).

Capacitance

Capacitance is the constant ratio Q/V.

- In SI, capacitance has units of Coulomb/Volt (= Farad, F)
  - So, a 1 F capacitor can store charge of 1 C at 1 V.

The Farad is a very large unit of capacitance.

- The Earth has a capacitance of ...0.0007 F!
- In common electrical circuits, capacitors of the order of pF (10^{-12}F) - μF (10^{-6}F) are used.

Consider a common AAA battery:

- produces a nominal voltage of 1.5 V, and
- has a lifetime of $\sim$1 A*h = 1 C/s * 3600 s = 3600 C

You would need a $\sim$2000 F capacitor to store the same amount of charge in the same potential difference! **Impractical to use capacitors "as batteries"**.
But an advantage of a capacitor is that it can discharge very very quickly, whereas a battery takes a long time.

Capacitance of a system of two conductors

Consider an **isolated system of 2 conductors**.

If I move charge from one to another, the two conductors become oppositely charged.

- the one that loses e- becomes positively charged (+Q) and is held at potential V_+, while
- the one that gains e- becomes negatively charged (-Q) and is held at potential V_-

I can define the **capacitance of the system of two conductors** as the ratio of charge Q stored in each of the conductors over the potential difference between the two conductors:

$$C = \frac{Q}{V_+ - V_-}$$

Capacitance

Capacitance is really always defined for a system of two conductors.

$$C = \frac{Q}{V_+ - V_-}$$

What about the capacitance of the *single* conductor we saw earlier?
One can always assume that there is a second conductor at infinity held at zero potential.

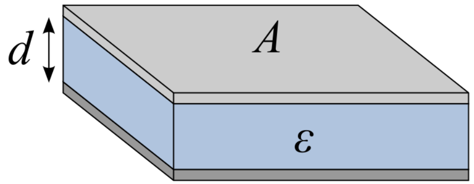
The capacitance is an **intrinsically positive quantity**.

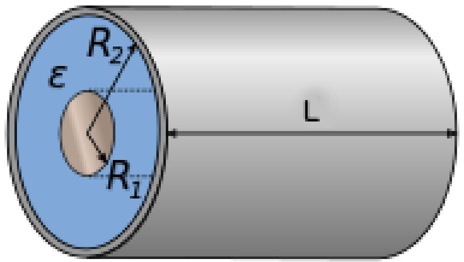
- Q is the charge of the positive conductor.
- $V_+ - V_-$ is the positive potential difference between the positively and negatively charged conductors

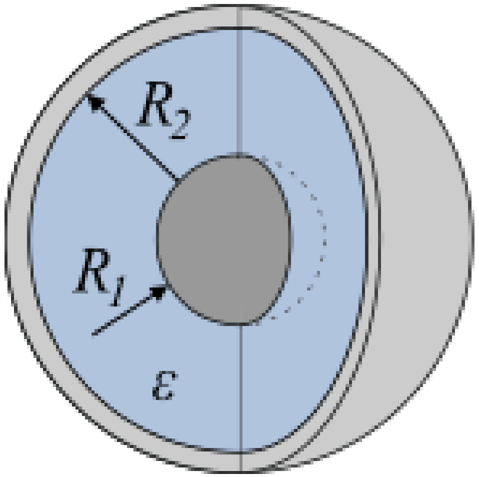
It is **easy to calculate the capacitance for simple geometries**.

For example:

- for a system of parallel plates, or
- systems with cylindrical or spherical symmetry







In this lecture we will study the parallel plate capacitor in some detail.

But, first, we will discuss how we can go about calculating the capacitance in all cases.

Calculating the capacitance

For system that exhibits some spatial symmetry, Gauss' law in integral form provides an easy way to calculate the electric field $\vec{E}$ and express it in terms of the charge Q stored in one of the conductors (by using an appropriate Gaussian surface):

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

If $\vec{E}$ is known, one can calculate the potential difference V between the two conductors as:

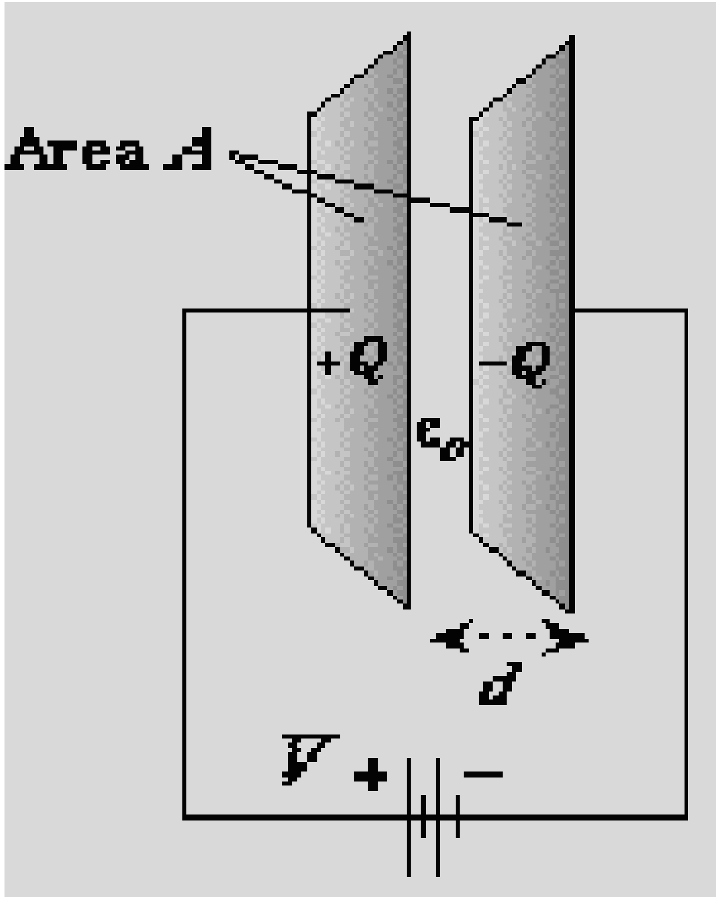
$$V := \Delta V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{\ell}$$

Knowing the charge Q in each conductor and the potential difference V between the two conductors, we calculate the capacitance C:

$$C = \frac{Q}{V}$$

A simple system: Parallel plate capacitor

We will study a simple system: The **parallel plate capacitor**.
It consists of two planar conductors with infinitesimally small thickness.

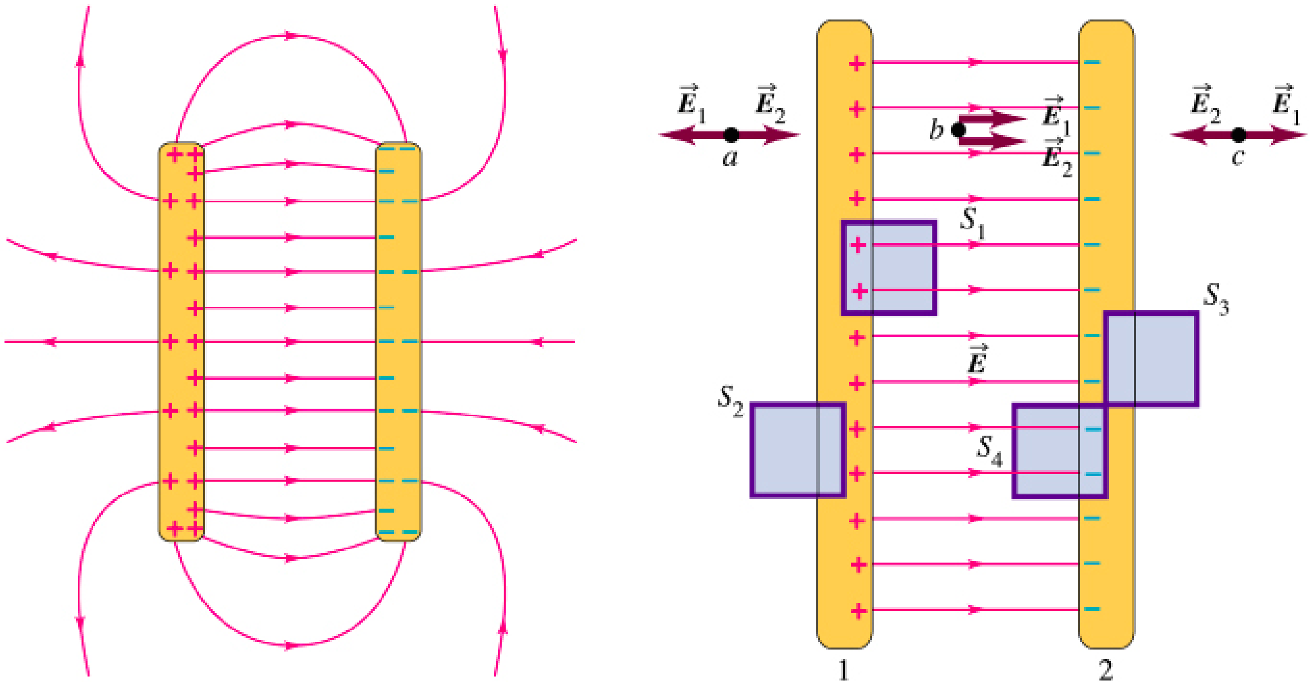


Assume that:

- Each plate has an area A.
- The two plates are at a distance d apart.
- The plates have a surface charge density σ and they are oppositely charged:
  - one plate has a charge $Q = \sigma A$, while
  - the other has charge $-Q = -\sigma A$
- The +Q plate is at x=0, while the -Q plate is at x=d.
- For now, let's consider that the two plates are separated by vacuum.

We **consider ideal systems** (*): They produce symmetric field with no fringe effects (see Fig. (a)) at the edges.

The field produced by the parallel plate capacitor is shown in Fig. (b).



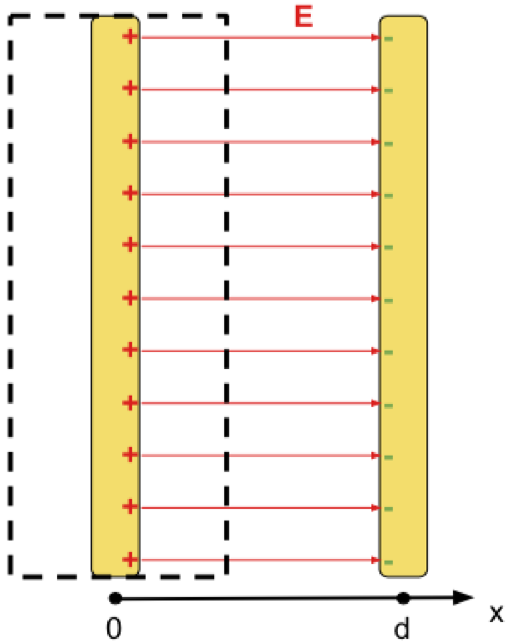
(a) (b)

(*) Our main emphasis is on the concepts.

Capacitance of parallel plate capacitor

Consider the Gaussian surface shown below (dashed lines) around the plate with positive charge $Q = \sigma A$.

Gauss's law gives us the electric field $\vec{E}$ between the two plates.



$$\oint_A \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow \oint_A (E\hat{x}) \cdot (dS\hat{x}) = \frac{\sigma A}{\epsilon_0} \Rightarrow$$

$$E(\hat{x} \cdot \hat{x}) \oint_A dS = \frac{\sigma A}{\epsilon_0} \xRightarrow{\hat{x}\cdot\hat{x}=1} EA = \frac{\sigma A}{\epsilon_0} \Rightarrow$$

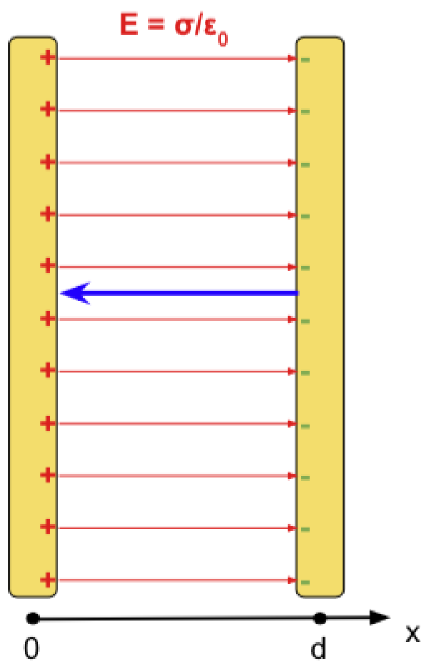
$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{x}$$

Capacitance of parallel plate capacitor

The potential difference V between the two plates is given by the following path integral of the electric field (notice the sign conventions):

$$V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{\ell}$$

Since V is path-independent, I will choose a path (from the negative to the positive plate) that simplies the calculation (shown in blue below).

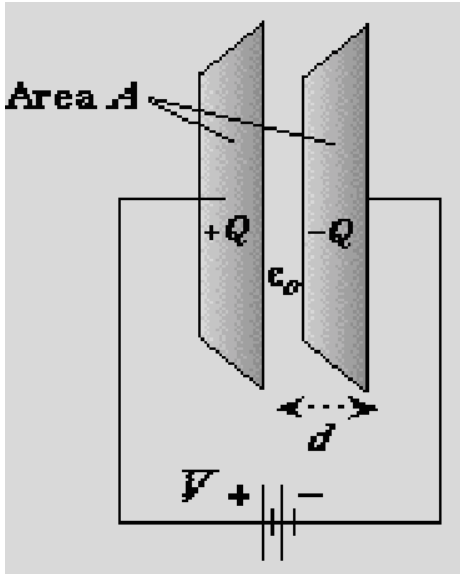


$$V = -\int_d^0 (E\hat{x}) \cdot \left(dx\hat{x}\right) \Rightarrow$$

$$V = -E(\hat{x} \cdot \hat{x}) \int_d^0 dx \Rightarrow V = Ed \xRightarrow{E=\frac{\sigma}{\epsilon_0}}$$

$$V = \frac{\sigma d}{\epsilon_0}$$

Capacitance of parallel plate capacitor



Capacitance (example):
A parallel plate capacitor with plates of area A = 100 mm^2 separated by a distance d = 10 mm has a capacitance of 0.0885 pF.
If the potential difference between the two plates is 1 V this capacitor can hold charge of 0.0885 pC.

Using Gauss' law we calculated the field between the two plates:

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{x}$$

This allowed us to calculate the potential difference between the two plates:

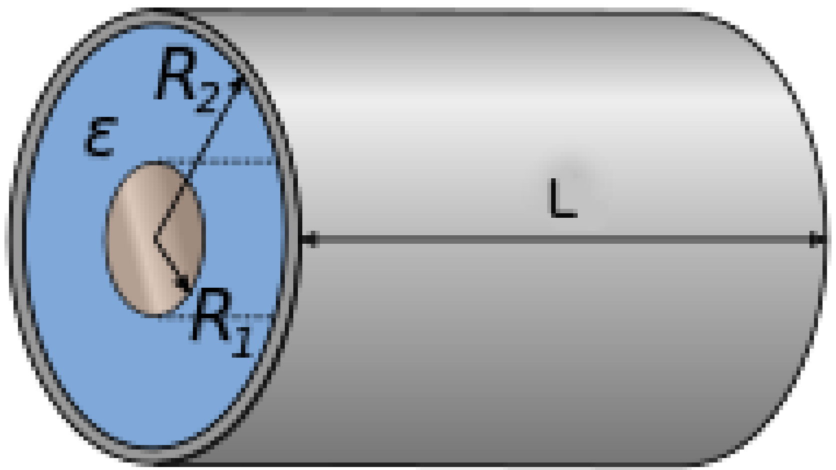
$$V = \frac{\sigma d}{\epsilon_0}$$

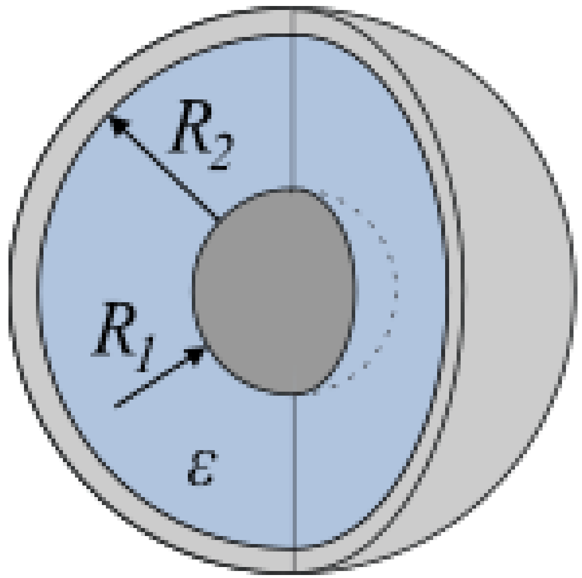
Therefore, the capacitance is:

$$C = \frac{Q}{\Delta V} = \frac{\sigma A}{\sigma d/\epsilon_0} \Rightarrow C = \epsilon_0 \frac{A}{d}$$

Notice that it depends only on the geometrical characteristics of the capacitor (and ϵ_0).

Repeat the capacitance calculation for systems with cylindrical and spherical symmetry, and confirm the answers given below.





$$C = \frac{2\pi\epsilon L}{ln(R_2/R_1)}$$

$$C = \frac{4\pi\epsilon}{\frac{1}{R_1} - \frac{1}{R_2}}$$

Energy stored in a capacitor

The electric field between the two plates of the parallel plate capacitor is

$$\mathbf{E} = \frac{\sigma}{\epsilon_\mathbf{0}}$$

We know there is (electric potential) **energy stored in the electric field**.

It is not difficult to realize that it **required work to charge the two plates of the capacitor**.

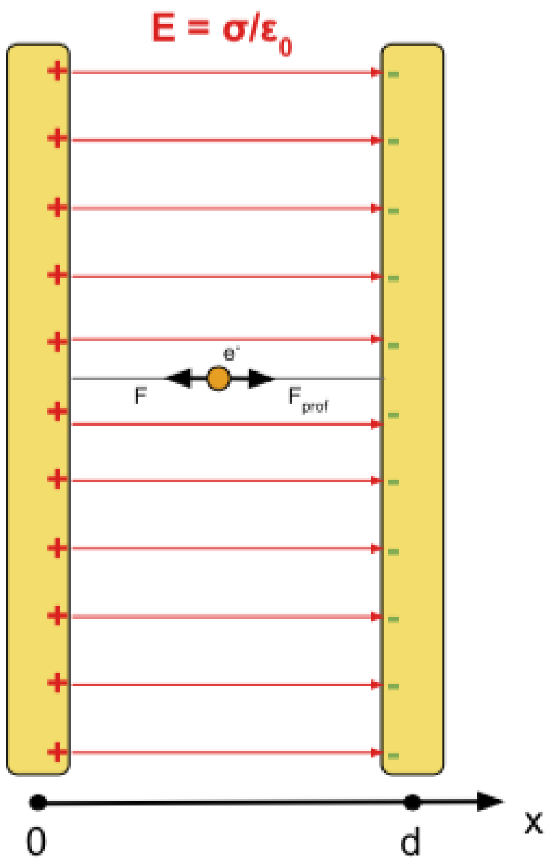
- This work became the energy now stored in the electric field.

I will **calculate the energy stored in the capacitor**.

- This exercise is not very different from the one we did in a previous lecture, when we calculated the work done to assemble systems of 2, 3 and N charges, as well as continuous distributions of charge.

Energy stored in a capacitor

To charge one plate with positive charge and the other with negative charge, I need to move electrons from one to another.



Consider I am already some way though that process and that I need to move yet another electron:

- There is already some charge on both plates and an electric field is formed between them.
- It becomes difficult to move an electron away from the positive (+) and towards the negative (-) plate: The electric field exerts a force ($\vec{F}$) on the opposite direction.
- I need to exert an equal opposite force ($\vec{F}_{prof}$) to overcome the action of the field.

Energy stored in a capacitor

Assume that I want to move an infinitesimal amount of negative charge -$|\delta q|$ from the positive (+) plate at x=0 to the negative (-) plate at x=d.

The work I do (against the action of the field) is:

$$\delta W_{prof} = \int_{+}^{-} \vec{F}_{prof} \cdot d\vec{\ell} = -\int_{+}^{-} \vec{F} \cdot d\vec{\ell}$$

where $\vec{F}$ is the field force and $\vec{F}_{prof}$ is the force that I apply ($\vec{F}_{prof} = -\vec{F}$).

The force $\vec{F}$ exerted on the negative charge -$|\delta q|$ is given by:

$$\vec{F} = -|\delta q|\vec{E}$$

Note: $\vec{F}$ and $\vec{E}$ have different directions because the charge is negative.

Energy stored in a capacitor

Therefore, the work I do can be written as:

$$\delta W_{prof} = |\delta q| \int_{+}^{-} \vec{E} \cdot d\vec{\ell} = |\delta q| \Big(- \int_{-}^{+} \vec{E} \cdot d\vec{\ell} \Big) = |\delta q| \Big(V_{+} - V_{-} \Big) = |\delta q| V$$

where V is the difference between the potential of the positive (V_{+}) and the negative (V_{-}) plate.
The work done move charge Q from one plate to another, is calculate by integrating over $|\delta q|$:

$$W_{prof} = \int \delta W_{prof} = \int_{0}^{Q} |\delta q| V$$

From the definition of capacitance:

$$C = \frac{|q|}{V_{+} - V_{-}} = \frac{|q|}{V} \Rightarrow V = \frac{1}{C} |q|$$

Energy stored in a capacitor

The work done is:

$$W_{prof} = \int_0^Q |\delta q| V \xrightarrow{V=\frac{1}{C}|q|} W_{prof} = \int_0^Q \frac{1}{C}|q||\delta q| \Rightarrow \mathbf{W_{prof} = \frac{Q^2}{2C}}$$

Since $C = Q/V$, we can also write the above as:

$$W_{prof} = \frac{Q^2}{2C} = \frac{\Big(CV\Big)^2}{2C} \Rightarrow \mathbf{W_{prof} = \frac{1}{2}CV^2}$$

or, equivalently, as:

$$W_{prof} = \frac{Q^2}{2C} = \frac{Q^2}{2\frac{Q}{V}} \Rightarrow \mathbf{W_{prof} = \frac{1}{2}QV}$$

Again, I note that the work I did (W_{prof}) becomes the **electric potential energy** (U) of the parallel plate capacitor.

Energy stored in a capacitor

In the previous lecture, we said that the energy stored in the electric field $\vec{E}$ is the integral, over all space, of $|\vec{E}|^2$.

$$U = \frac{\epsilon_0}{2} \int_{all\ space} |\vec{E}(\vec{r})|^2 d\tau$$

Can we show that for the parallel plate capacitor?

Recall that, on the previous slide we calculated that the potential energy U stored in the parallel plate capacitor (using W_{prof} and U interchangeably):

$$U = \frac{1}{2} C V^2$$

And, earlier in the lecture, we calculated the capacitance C of the parallel plate capacitor, as well as the potential difference V between its two plates:

$$C = \epsilon_0 \frac{A}{d} \qquad V = \frac{\sigma d}{\epsilon_0}$$

Energy stored in a capacitor

Therefore:

$$U = \frac{1}{2} C V^2 \xrightarrow{C=\epsilon_0 \frac{A}{d} \quad V=\frac{\sigma d}{\epsilon_0}} U = \frac{1}{2} \left(\epsilon_0 \frac{A}{d}\right) \left(\frac{\sigma d}{\epsilon_0}\right)^2 \Rightarrow U = \frac{\epsilon_0}{2} \Big(Ad\Big) \left(\frac{\sigma}{\epsilon_0}\right)^2$$

Notice that:

- $A \cdot d$ is the volume between the two conducting plates
  - This is the only volume where $|\vec{E}| \neq 0$
- σ / ϵ_0 is the electric field between the plates

Therefore, the above expression for U can be written as

$$U = \frac{\epsilon_0}{2} \Big(Volume\Big) |\vec{E}|^2$$

which is what we sought to show.

Worked example

Question

The charge centre of a thundercloud, 3 km above ground, carries a charge of 20 C. Assume the charge centre to be a circle of radius 1 km and model the cloud/ground system as a parallel plate capacitor.
Find:

1. The capacitance of the system.
2. The potential difference between the charge centre and the ground.
3. The average electric field strength between the cloud and the ground.
4. The electrical energy stored in the system.

Worked example

$$C = \frac{\epsilon_0 A}{d} = \frac{(8.854 \times 10^{-12}\ \frac{C^2}{N \cdot m^2}) \cdot \left(\pi \cdot (1.0 \times 10^3\ m)^2\right)}{3 \times 10^3\ m} = 9.3 \times 10^{-9}\ F$$

$$V = \frac{Q}{C} = \frac{20\ C}{9.3 \times 10^{-9}\ F} = 2.2 \times 10^9\ V$$

$$E = \frac{V}{d} = \frac{2.2 \times 10^9\ V}{3 \times 10^3\ m} = 7.3 \times 10^5\ V/m$$

$$U = \frac{1}{2}QV = \frac{1}{2}(20\ C)(2.2 \times 10^9\ V) = 2.2 \times 10^{10}\ J$$

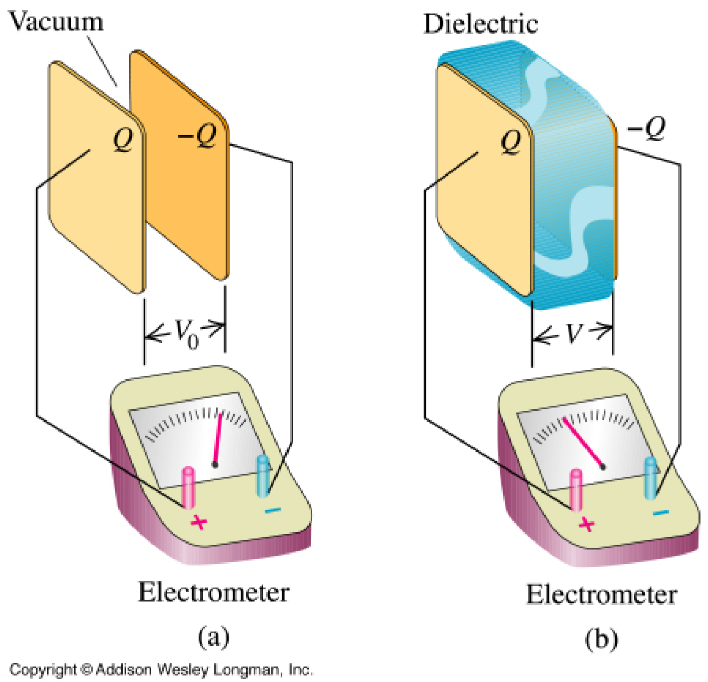
Dielectrics

Another class of materials exists that, unlike conducting materials, **does not conduct electricity** (they do not contain free-moving charges).

They are called **insulators** or **dielectrics**.

Are they *"blind"* to the existence of an external electric field?

Will see that **this is not the case.**



Even from the early days in the development of electromagnetism, it was noticed (by Faraday) that **if you insert a dielectric material within a parallel plate capacitor its capacitance increases substantially!**

This means that by inserting a dielectric between the plates one **can store the same amount of charge at a much smaller potential difference**.

Dielectrics

Recall that the potential difference V between the two plates is given by

$$V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{\ell}$$

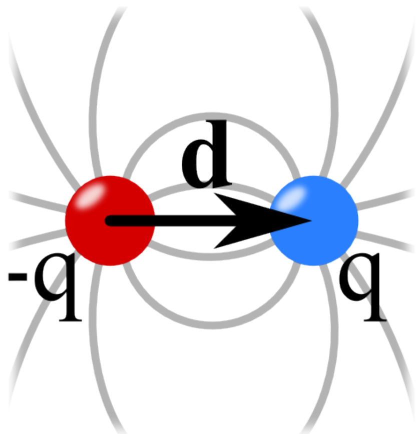
So, by inserting dielectric material within a parallel plate capacitor, the electric field between the two plates (i.e. within the dielectric) is now much smaller!

How is this possible?

- Unlike conductors, the dielectric has no free charges whose redistribution within the volume of the material creates an opposing field.
- What is the physics mechanism that is responsible for the reduction of the strength of the electric field?

Before we answer the above question, let's examine the **electric dipole**.

Electric dipoles and electric dipole moment



Let's consider two point charges +q and -q separated by a *small distance* d: This is an **electric dipole**.

How small is a "*small distance*"? 1 mile? 1 m? 1 mm?

- Small compared with the distances where we are interested to know the field.

Dipoles are a very common configuration in nature.

An electric dipole its described by its **electric dipole moment** $\vec{p} = q\vec{d}$

- The dipole moment is a vector quantity.
- Its direction is from the negative to the positive charge.

The potential field $V(\vec{r})$ due to an **electric dipole** is

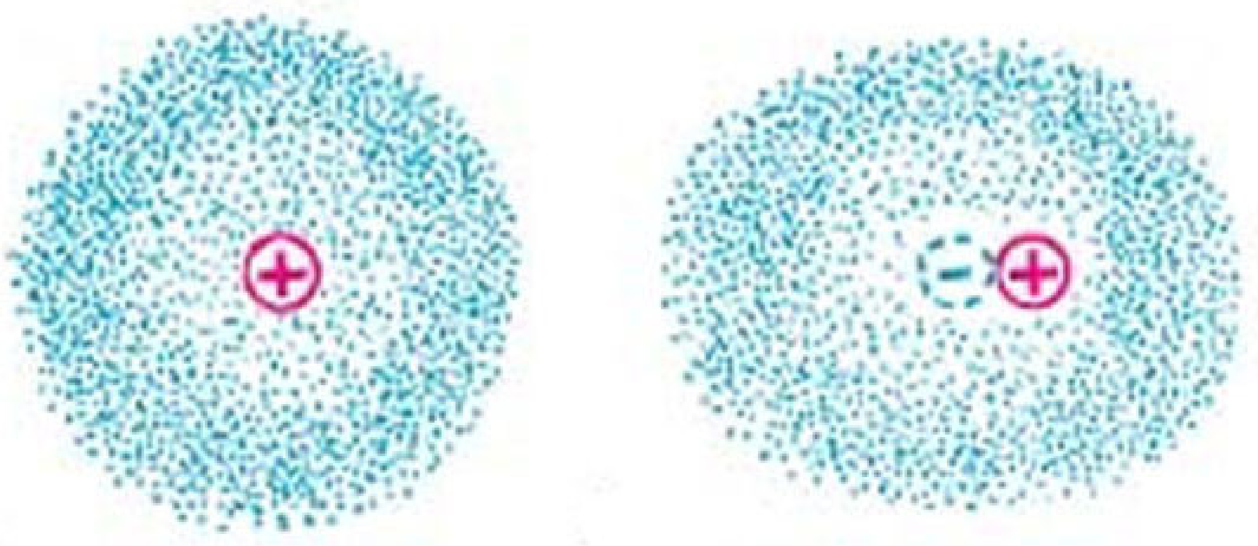
$$V \approx \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

Examples of electric dipoles in nature (1)

Consider an unpolarised atom within an external electric field.

- Initially the centres of gravity of the negative and positive charge distributions coincide.
- But, in the presence of an external electric field, the electron cloud and the nucleus are *pulled apart*.

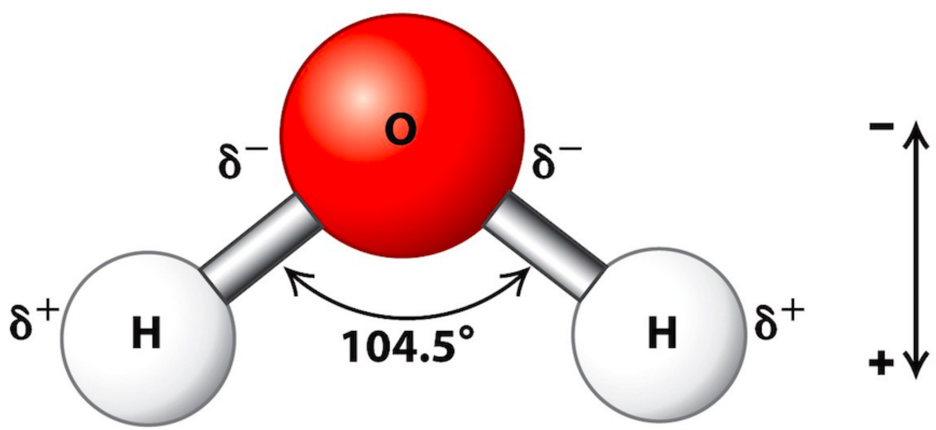
In the presence of a field, the atom **becomes an electric dipole**.



Examples of electric dipoles in nature (2)

Complex molecules can be polarised even in the absence of an external electric field.

Perhaps the most common example is the water molecule.

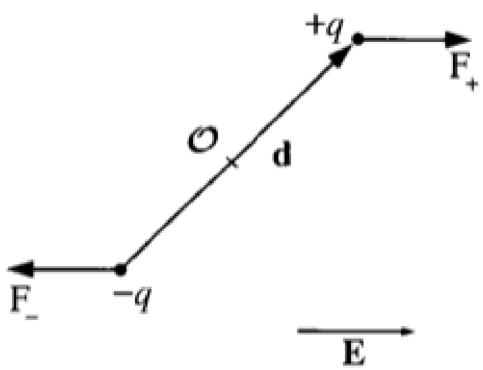


- The 2 H atoms form an angle of 105^o with the O atom.
- There is a net negative charge around the O atom.
  - Electrons spend more time around the O atom than around the H atoms.
- Consequently, there is net positive charge around the two H atoms.

Although this is more complex configuration than a simple dipole, when viewed from a distance it is very similar to a dipole.

Polar molecules in the presence of an electric field

In substances with polar molecules, the **electric dipole moments of the polar molecules are randomly oriented**.



An external electric field produces a **torque $\vec{T}$ on an electric dipole** $\vec{p}$:

$$\vec{T} = \vec{p} \times \vec{E}$$

Therefore, in the presence of an external electric field, **polar molecules will tend to get aligned along the direction of the field**.

Macroscopic electric polarisation of dielectrics

The key observations:

- An electric field can induce an electric dipole moment in previously unpolarised atoms. The dipole moments are aligned with the electric field.
- An electric field exerts a torque on polar molecules (molecules with permanent electric dipole moment) aligning the previously randomised dipole moment vectors along the direction of the field.

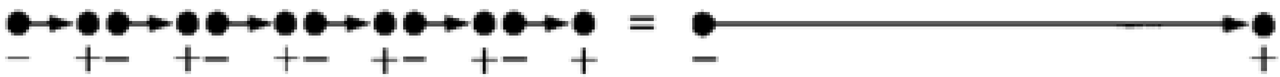
The dielectric becomes **macroscopically polarised**.

The **polarisation** $\vec{P}$ is defined as the **amount of electric dipole moment per unit volume**.

- The polarised material creates a field of its own.
- This polarisation field opposes the external field that created the polarisation in the first place.
- Unlike in conductors, the field cancelation is not complete.

Polarization charges

Let's consider what would happen if the microscopic dipoles were all perfectly aligned along a direction.

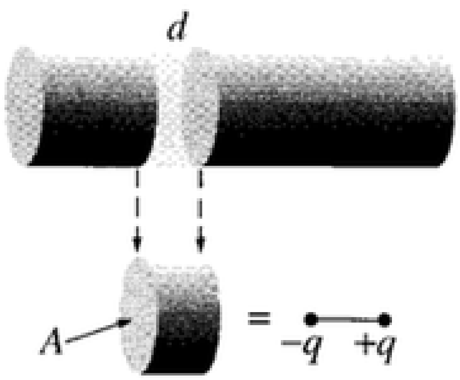


The positive charge at the "head" of a dipole cancels the negative charge at the "tail" of the neighbouring dipole. At the end, the only unbalanced charges appear on the surface.

A polarised material has a surface charge density due to unbalanced polarisation charges.

Surface density of polarization charge

What is the surface charge density due to polarisation charges?



Consider a cylindrical dielectric, with cross-section A, whose axis runs parallel to an external electric field. Take a slide of length d: polarisation charge ±q is accumulated on the cross-sectional areas.

Let P be the polarisation. The total electric dipole moment of the slice is:

$$P\Big(A \cdot d\Big) = q \cdot d \Rightarrow q = A \cdot P$$

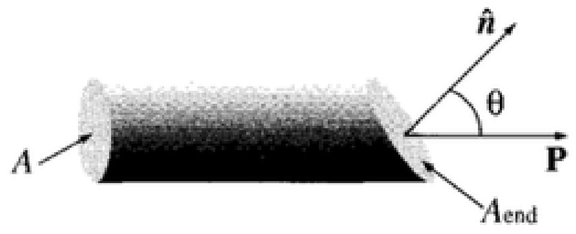
If the surface charge density is σ_P (charge per area) then:

$$q = \sigma_P \cdot A$$

Therefore, the charge density is:

$$\sigma_P = P$$

Surface density of polarization charge



If the end cap was at an angle θ with respect to the polarisation vector, it would still contain the same amount of charge Q, but the area of the endcap would be larger ($A/cos\theta$)

In that case, the charge density would have been

$$\sigma_P = Pcos\theta$$

So the surface density of polarisation charges can be expressed as the dot product of the polarisation vector $\vec{P}$ with a unit vector $\hat{n}$ normal to the surface

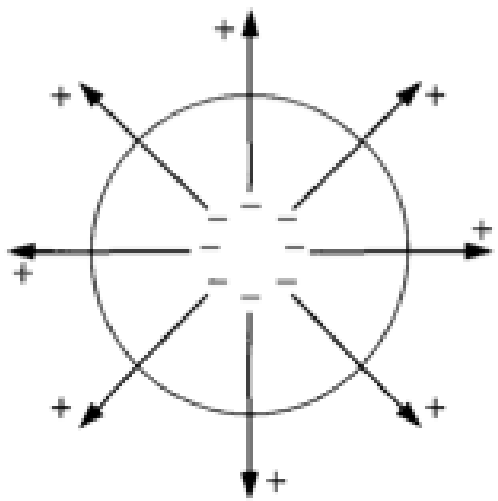
$$\sigma_P = \vec{P} \cdot \hat{n}$$

Volume density of polarization charge

Are the polarisation charges always on the surface?

The answer to this question depends on the alignment of the microscopic dipole moments. Consider a case where the polarisation field diverges:

- At a point of positive divergence, I have accumulation of negative volume (not surface) polarisation charge.
- Similarly, negative divergence causes the accumulation of positive polarisation charge.



The volume density of polarisation charges can be written as

$$\rho_P = -\vec{\nabla} \cdot \vec{P}$$

So charge can also accumulate in the volume of an insulator, as long as the polarisation field diverges.

Electric field produced by a polarized dielectric

An external electric field $\vec{E}_0$ **induces both surface and volume charges** (polarization charges) to a dielectric.

The dielectric becomes the **source of a new electric field** $\vec{E}_P$.

$\vec{E}_P$ is the field produced by a surface charge density $\sigma_P = \vec{P} \cdot \hat{n}$ and a volume charge density $\rho_P = -\vec{\nabla} \cdot \vec{P}$:

$$V_P(\vec{r}) = \frac{1}{4\pi\epsilon_0} \oint \frac{1}{r} \sigma_P(\vec{r'}) dS' + \frac{1}{4\pi\epsilon_0} \oint \frac{1}{r} \rho_P(\vec{r'}) d\tau'$$

$$\vec{E}_P(\vec{r}) = -\vec{\nabla} V_P(\vec{r})$$

The total field in the presence of the dielectric is:

$$\vec{E}(\vec{r}) = \vec{E}_0(\vec{r}) + \vec{E}_P(\vec{r})$$

As we mentioned already, $\vec{E}_P$ opposes $\vec{E}_0$.

Electric field produced by a polarized dielectric

In order to calculate the total electric field $\vec{E}$, we need to calculate the field due to induced charges $\vec{E}_P$ and add it to the external field $\vec{E}_0$.
To calculate $\vec{E}_P$, we need the surface and volume charge densities:

$$\sigma_P = \vec{P} \cdot \hat{n} \quad \textit{and} \quad \rho_P = -\vec{\nabla} \cdot \vec{P}$$

Both σ_P and ρ_P depend on the polarisation field $\vec{P}$. How is it calculated?

For many materials (called **linear dielectrics**) the polarisation is proportional to the total electric field $\vec{E}$ (assuming that $\vec{E}$ is not strong enough)

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

The factor χ_e **is the electric susceptibility** and depends on the material.
The factor ϵ_0 is there mainly so that χ_e becomes dimensionless.

Gauss's law in dielectrics

We will **generalise Gauss's law in the presence of dielectrics** (where besides free charges we also have induced polarisation charges).

Recall Gauss' law (in differential form) in vacuum:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

where ρ is the total charge density and, now, includes both free (ρ_f) and polarisation charges (ρ_P):

$$\vec{\nabla}\vec{E} = \frac{\rho_f + \rho_P}{\epsilon_0}$$

The density of polarisation charges can be expressed in terms of the polarisation vector, therefore:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f - \vec{\nabla} \cdot \vec{P}}{\epsilon_0}$$

Gauss's law in dielectrics

Collecting the divergences together:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f - \vec{\nabla} \cdot \vec{P}}{\epsilon_0} \Rightarrow \epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_f - \vec{\nabla} \cdot \vec{P} \Rightarrow \epsilon_0 \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P} = \rho_f \Rightarrow$$

$$\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \vec{P} \right) = \rho_f$$

We define the **electric displacement** $\vec{D}$ as:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

In SI, the electric displacement $\vec{D}$ has **units of** C/m^2.

Gauss's law in dielectrics

The electric displacement vector $\vec{D}$ satisfies the differential equation:

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

Notice that ρ_f is the density of **free charges**.

This is **Gauss' law**, in differential form, in the presence of dielectrics.

We have generalised Gauss's law, taking into account **both free and polarisation charges**.

The equivalent integral form can be obtained using Gauss's theorem:

$$\oint \vec{D} \cdot d\vec{S} = Q_f$$

Notice that Q_f is the **free charge**.

Gauss's law in dielectrics

As we have seen, we can then write the polarization $\vec{P}$ in terms of $\vec{E}$:

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

Now, let's focus on dielectric materials which are:

- **Linear**: χ_e independent of the magnitude of $\vec{E}$
- **Isotropic**: χ_e independent of the direction of $\vec{E}$.
- **Homogeneous**: χ_e independent of the position.

Therefore, the electric displacement becomes:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \chi_e \epsilon_0 \vec{E} = \Big(1 + \chi_e\Big) \epsilon_0 \vec{E} = \epsilon_r \epsilon_0 \vec{E}$$

where $\epsilon_r = 1 + \chi_e$ is the **relative permittivity** of the dielectric material (sometimes also called **dielectric constant**).

Gauss's law in dielectrics

You probably think that $\vec{D}$ is much like $\vec{E}$: They both obey Gauss's law.

Notice that this is not quite true.

In the previous lecture, we stressed that the divergence of a vector field is not sufficient to uniquely determine all components of the field. One also needs the curl of the field.

We used that freedom provided by $\vec{\nabla} \times \vec{E} = 0$ to express $\vec{E}$ as the gradient of a scalar field (the potential V).

What is the rotation of $\vec{D}$?

$$\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \left(\epsilon_0 \vec{E} + \vec{P}\right) = \epsilon_0 \vec{\nabla} \times \vec{E} + \vec{\nabla} \times \vec{P} \Rightarrow \vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$$

So, in principle, the curl of $\vec{D}$ is not always 0.

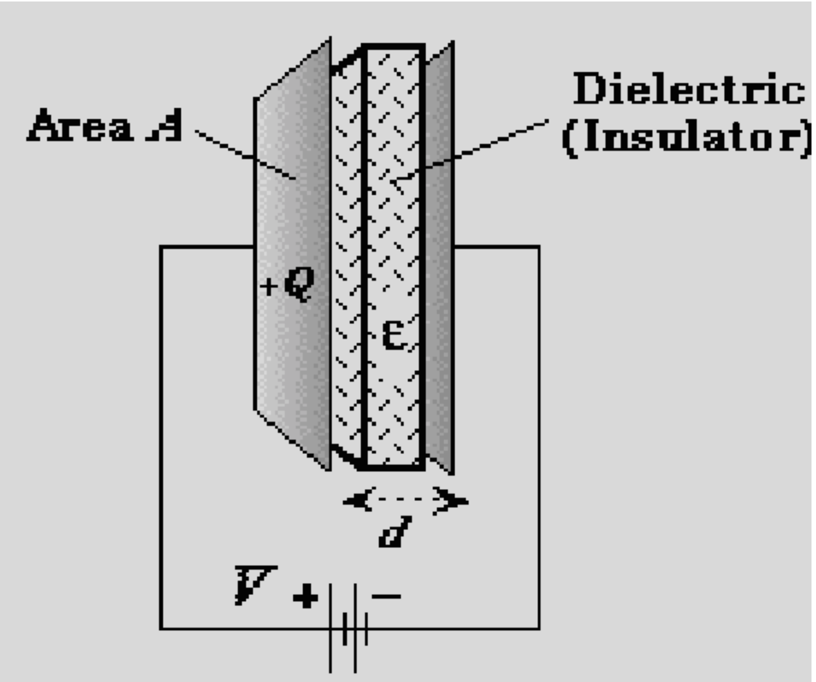
- There is no scalar function (potential) whose gradient is $\vec{D}$.

Notice that for linear materials:

$$\vec{\nabla} \times \vec{P} = \vec{\nabla} \times \left(\chi_e \epsilon_0 \vec{E}\right) = \epsilon_r \epsilon_0 \vec{\nabla} \times \vec{E} = 0$$

Parallel plate capacitor *with dielectric*

Let's consider again the parallel plate capacitor we studied before.



This time, we will examine what happens to its capacity if we **insert a dielectric (insulator) within its two oppositely charged plates**.

We will follow the exact same procedure we followed earlier in this lecture, but now we will **use Gauss's law in the presence of dielectrics**.

$$\oint \vec{D} \cdot d\vec{S} = Q_f$$

Parallel plate capacitor *with dielectric*

I will use the same Gaussian surface as before and apply Gauss's law in integral form. Both $\vec{D}$ and $d\vec{S}$ point along $\hat{x}$, and there is flux only through the right side of the Gaussian surface.

$$\oint \vec{D} \cdot d\vec{S} = Q_f \Rightarrow DA = Q_f \Rightarrow \vec{D} = \frac{Q_f}{A}\hat{x}$$

As we have seen the electric displacement $\vec{D}$ is:

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E}$$

Therefore the total electric field $\vec{E}$ is:

$$\vec{E} = \frac{Q_f}{\epsilon_r \epsilon_0 A}\hat{x}$$

Parallel plate capacitor *with dielectric*

The potential difference between the two charged plates is

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{\ell} = -\int_{d}^{0} \vec{E}\cdot d\vec{\ell} = -\int_{d}^{0} \Big(\frac{Q_f}{\epsilon_r \epsilon_0 A}\hat{x}\Big)\cdot dl\hat{x} \Rightarrow = -\frac{Q_f}{\epsilon_r \epsilon_0 A}\int_{d}^{0} dl \Rightarrow$$

$$V = \frac{Q_f d}{\epsilon_r \epsilon_0 A}$$

Therefore, the capacitance is

$$C = \frac{Q_f}{\Delta V} = \frac{Q_f}{\frac{Q_f d}{\epsilon_r \epsilon_0 A}} \Rightarrow \mathbf{C} = \frac{\epsilon_\mathbf{r} \epsilon_\mathbf{0} \mathbf{A}}{\mathbf{d}} \Rightarrow \mathbf{C} = \epsilon_\mathbf{r} \mathbf{C_0}$$

In the presence of a dielectric, the capacitance is multiplied by the relative permittivity ϵ_r.

Worked example

Question

A parallel plate capacitor with a given charge Q_0 has an electric field of $3.2 \times 10^5 \ V/m$ in vacuum and $2.5 \times 10^5 \ V/m$ when filled with a dielectric. Find:

1. The relative permittivity of the dielectric.
2. the charge density on the surface of the dielectric.

$$C_{diel} = \epsilon_r C_0 \xrightarrow{C=Q/V} \frac{\cancel{Q_0}}{V_{diel}} = \epsilon_r \frac{\cancel{Q_0}}{V_0} \Rightarrow \epsilon_r = \frac{V_0}{V_{diel}} \xrightarrow{V=-\int \vec{E}d\vec{\ell}}$$

$$\epsilon_r = \frac{E_0 \cancel{d}}{E_{diel} \cancel{d}} = \frac{3.2 \times 10^5 \ V/m}{2.5 \times 10^5 \ V/m} \Rightarrow \epsilon_r = 1.28$$

$$\chi_e = \epsilon_r - 1 = 1.28 - 1 \Rightarrow \chi_e = 0.28$$

$$\sigma_P = P = \chi_e \epsilon_0 E = 0.28 \cdot (8.854 \times 10^{-12}) \cdot (2.5 \times 10^5) \ C/m^2 = 6.2 \times 10^{-7} C/m^2$$

Worked example

Question

A parallel plate capacitor filled with air (for which vacuum is a good approximation in our problem) has a capacitance of 12.5 μF and is connected to a power supply which keeps it at constant potential difference of 24 V. A piece of material with a relative permittivity of 3.75 is placed between the plates completely filling the space between them. How much energy is stored in the capacitor before and after the dielectric is inserted?

In air (treated as vacuum): $U_0 = \frac{1}{2} C_0 V^2 = 3.6\ mJ$

In dielectric: $U = \frac{1}{2} C V^2 = \frac{1}{2} \epsilon_r C_0 V^2 = 13.5\ mJ$

Therefore: $\Delta U = U - U_0 = 9.9\ mJ$

The energy stored in the capacitor is increased.

Lecture 4 - Main points to remember

- With regards to electrical properties, there are 2 types of materials
  - materials that conduct electricity: **conductors**
  - materials that do not conduct electricity: **insulators (dielectrics)**
- A conductor is an object or type of material which **contains electric charges that are relatively free to move**
  - A **perfect conductor** has an **unlimited supply of free charges**.
  - There are no perfect conductors, but many substances come close!
    - e.g. the free charge density in copper is 1.8×10^{10} C$/m^3$.
- If we place a **conductor within an external electric field**:
  - The electric field vanishes everywhere inside a conductor.
  - The potential is constant inside a conductor.
  - Charge accumulates in the surface.
  - The field on the surface of a conductor has no tangential component.

Lecture 4 - Main points to remember (cont'd)

- Capacitance (C) denotes the ability of a body to store electric charge. For a system of two conductors, one with charge +Q held at potential V_+ and one with charge -Q held at potential V_-, the capacitance is a positive quantity defined as:

$$C = \frac{Q}{V_+ - V_-}$$

  Its SI unit is the **Farad (F)** defined as one Coulomb per Volt.

- Calculating the capacitance for simple systems:
  - Use Gauss' law to calculate the electric field $\vec{E}$ in terms of the charge Q stored in one of the conductors: $\oint_S \vec{E} \cdot d\vec{S} = Q/\epsilon_0$
  - Once $\vec{E}$ is known, calculate the potential difference V between the two conductors as: $V := \Delta V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{\ell}$
  - From the known charge Q in the positive conductor and the potential difference V between the conductors, calculate $C = Q/V$

Lecture 4 - Main points to remember (cont'd)

- We studied a simple system: The **parallel plate capacitor**
- The electric field between the plates was found to be:

$$E = \frac{\sigma}{\epsilon_0}$$

- The parallel plate capacitor has capacitance:

$$C = \epsilon_0 \frac{A}{d}$$

  It depends only on the geometrical characteristics of the capacitor.
- Energy stored in the parallel plate capacitor:

$$U = \frac{Q^2}{2C} \quad or \quad U = \frac{1}{2} C V^2 \quad or \quad U = \frac{1}{2} Q V$$

  and confirmed that:

$$U = \frac{\epsilon_0}{2} \int_{all\ space} |\vec{E}(\vec{r})|^2 d\tau$$

Lecture 4 - Main points to remember (cont'd)

- **Electric dipole**: Point charges +q and -q at a *small distance* d.
- An electric dipole its described by its **electric dipole moment** $\vec{p} = q\vec{d}$
  - A vector pointing from the negative to the positive charge
- An electric dipole creates a potential field V given by

$$V \approx \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

- An electric field $\vec{E}$ exerts to an electric dipole with moment $\vec{p}$ a torque $\vec{T} = \vec{p} \times \vec{E}$
- Electric fields induce dipole moments in the direction of the field (or align towards the direction of the field polar molecules with permanent dipole moments) and generate macroscopic polarisation.
- The **polarisation** $\vec{P}$ of a material is defined as the **amount of electric dipole moment per unit volume**.

Lecture 4 - Main points to remember (cont'd)

- The polarisation induces surface and volume polarisation charges. The corresponding densities are $\sigma_P = \vec{P} \cdot \hat{n}$ and $\rho_P = -\vec{\nabla} \cdot \vec{P}$
- In the presence of dielectrics, Gauss's law need to be generalised to include both free charges we also have induced polarisation charges:

$$\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \vec{P}\right) = \rho_f \quad \textit{and} \quad \oint \left(\epsilon_0 \vec{E} + \vec{P}\right) \cdot d\vec{S} = Q_f$$

  where ρ_f is the free charge density and Q_f the amount of free charge.
- The vector $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ is the electric displacement vector
  - In SI, the electric displacement unit is C/m^2.
- For **linear dielectrics**, $\vec{P} = \chi_e \epsilon_0 \vec{E}$ and, therefore, $\vec{D} = \epsilon_r \epsilon_0 \vec{E}$, where χ_e is the electric susceptibility of the material and $\epsilon_r = 1 + \chi_e$ is the relative permittivity (dielectric constant).
- A dielectric with relative permittivity ϵ_r inserted between the plates of a parallel plate capacitor, increases its capacitance by a factor of ϵ_r.

Maxwell's equation we know so far

In vacuum (static case):

	Integral form	*Differential form*
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = Q_{enclosed}/\epsilon_0$	$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$

In the presence of materials (static case):

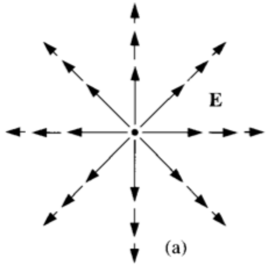
	Integral form	*Differential form*
Gauss's law	$\oint \vec{D} d\vec{S} = Q_{enclosed;free}$	$\vec{\nabla} \cdot \vec{D} = \rho_{free}$
Circuital law	$\oint \vec{E} d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$

At the next lecture (Lecture 5)

- Electric current.
- Magnetic phenomena.
- Electric charge inside a magnetic field.
- Cyclotron motion.
- Magnetic force on an electric current.
- The Biot-Savart law.
- Magnetic field around a straight wire.

Optional reading for Lecture 4

Electric dipole field



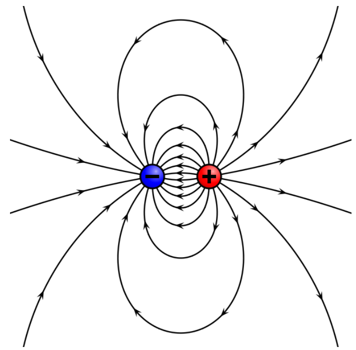
As we know, the potential field $V(\vec{r})$ due to an **electric monopole** (i.e. a point charge q) has an $1/r$ dependence:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Consequently, its electric field $\vec{E}(\vec{r})$ ($\vec{E} = -\vec{\nabla}V$) has an $1/r^2$ dependence.

We will see that the potential field $V(\vec{r})$ due to an **electric dipole** is

$$\mathbf{V} \approx \frac{\mathbf{1}}{\mathbf{4}\pi\epsilon_{\mathbf{0}}}\frac{\tilde{\mathbf{p}}\hat{\mathbf{r}}}{\mathbf{r^2}}$$

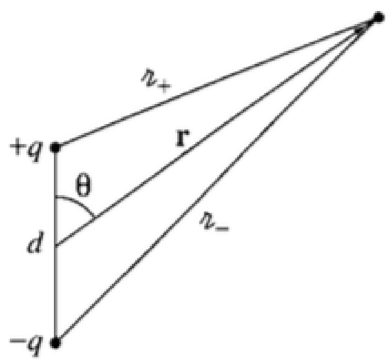


Therefore, it falls off as $1/r^2$, faster than the monopole potential. Consequently, the dipole electric field $\vec{E}(\vec{r})$ has an $1/r^3$ dependence.

Calculating the electric dipole field

As we know, the potential field due to a single charge q is given by:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$



Now, I have 2 charges: a positive and a negative one. The superposition principle applies. The potential at a distance r from the centre of the dipole is:

$$V = \frac{1}{4\pi\epsilon_0}\Big(\frac{q}{r_+} - \frac{q}{r_-}\Big)$$

I need to express r_+ and r_- in terms of r. From the law of cosines (generalisation of the Pythagorean theorem):

$$r_+^2 = r^2 + (d/2)^2 - r \cdot d \cdot cos\theta = r^2\Big(1 - \frac{d}{r}cos\theta + \frac{d^2}{4r^2}\Big) \xRightarrow{r>>d}$$

$$r_+^2 \approx r^2\Big(1 - \frac{d}{r}cos\theta\Big)$$

Calculating the electric dipole field

For r_-, the expressions are similar but involves $cos(\pi - \theta) = -cos\theta$ and, therefore, there is an extra minus sign.

$$r_-^2 \approx r^2 \left(1 + \frac{d}{r} cos\theta\right)$$

For convenience let me rename the small term involving d/r as ϵ:

$$\frac{d}{r} cos\theta = \epsilon$$

We have that

$$r_+^2 \approx r^2(1-\epsilon) \Rightarrow r_+ \approx r(1-\epsilon)^{1/2} \Rightarrow \frac{1}{r_+} \approx \frac{1}{r}(1-\epsilon)^{-1/2} \Rightarrow \frac{1}{r_+} \approx \frac{1}{r}(1+\frac{1}{2}\epsilon)$$

and, similarly

$$\frac{1}{r_-} \approx \frac{1}{r}(1 - \frac{1}{2}\epsilon)$$

Calculating the electric dipole field

Therefore

$$\frac{1}{r_+} - \frac{1}{r_-} \approx \left(\frac{1}{r}(1 + \frac{1}{2}\epsilon)\right) - \left(\frac{1}{r}(1 - \frac{1}{2}\epsilon)\right) = \frac{1}{r}\epsilon \xrightarrow{\epsilon = \frac{d}{r}cos\theta}$$

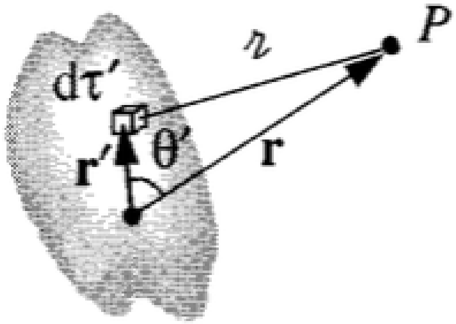
$$\frac{1}{r_+} - \frac{1}{r_-} \approx \frac{d}{r^2}cos\theta$$

The electric dipole potential is given by

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) \Rightarrow V \approx \frac{1}{4\pi\epsilon_0}\frac{qdcos\theta}{r^2} \Rightarrow$$

$$V \approx \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

So the potential of a dipole falls off as $1/r^2$ ($1/r$ for a monopole).
Consequently, the electric field of a dipole falls off as $1/r^3$.



For an **arbitrary charge distribution** characterised by a density ρ the potential can be expanded to:

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_{vol} (r')^n P_n(cos\theta')\rho(\vec{r'})d\tau'$$

where θ' is the angle between $\vec{r}$ and $\vec{r'}$. Notice that there is no r dependence in the integral.

This is called the **multipole expansion**.

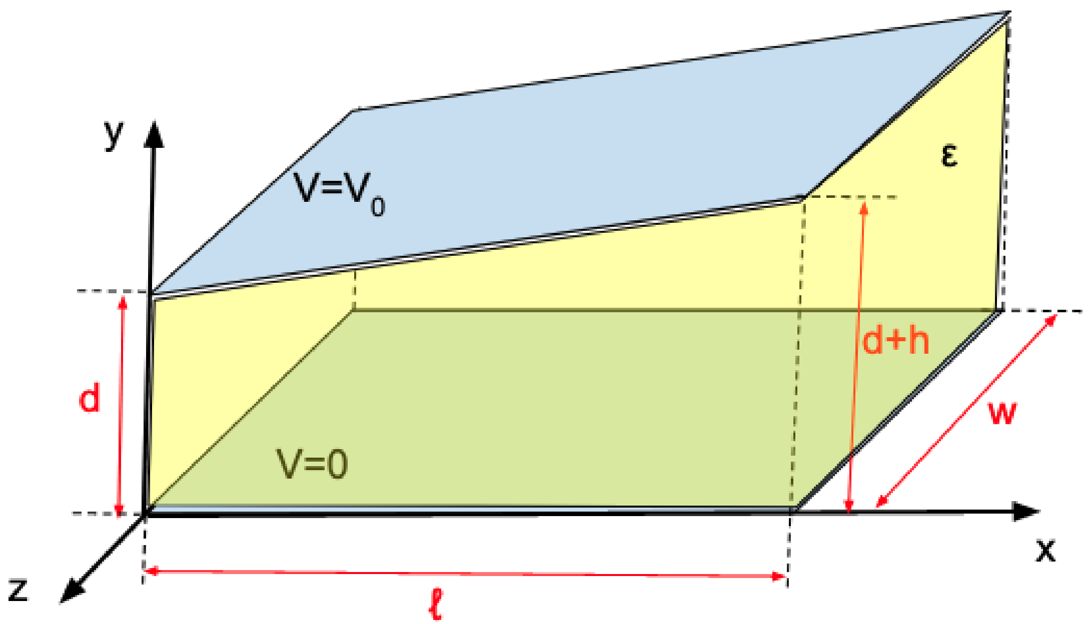
- The first $(1/r)$ term is the **monopole** term
- The second $(1/r^2)$ term is the **dipole** term
- $1/r^3$ term: **quadruple** term
- $1/r^4$ term: **octopole** term
- ...

Worked example: The not-so-parallel plate capacitor

Question

In the figure below, you are given the not-so-parallel plate capacitor.

- Neglecting edge effects, when a voltage difference V_0 is placed across the two conductors, find the potetial everywhere between the plates.
- When the wedge is filled with a medium of dielectric constant ϵ, find the capacitance of the system in terms of the constants given.

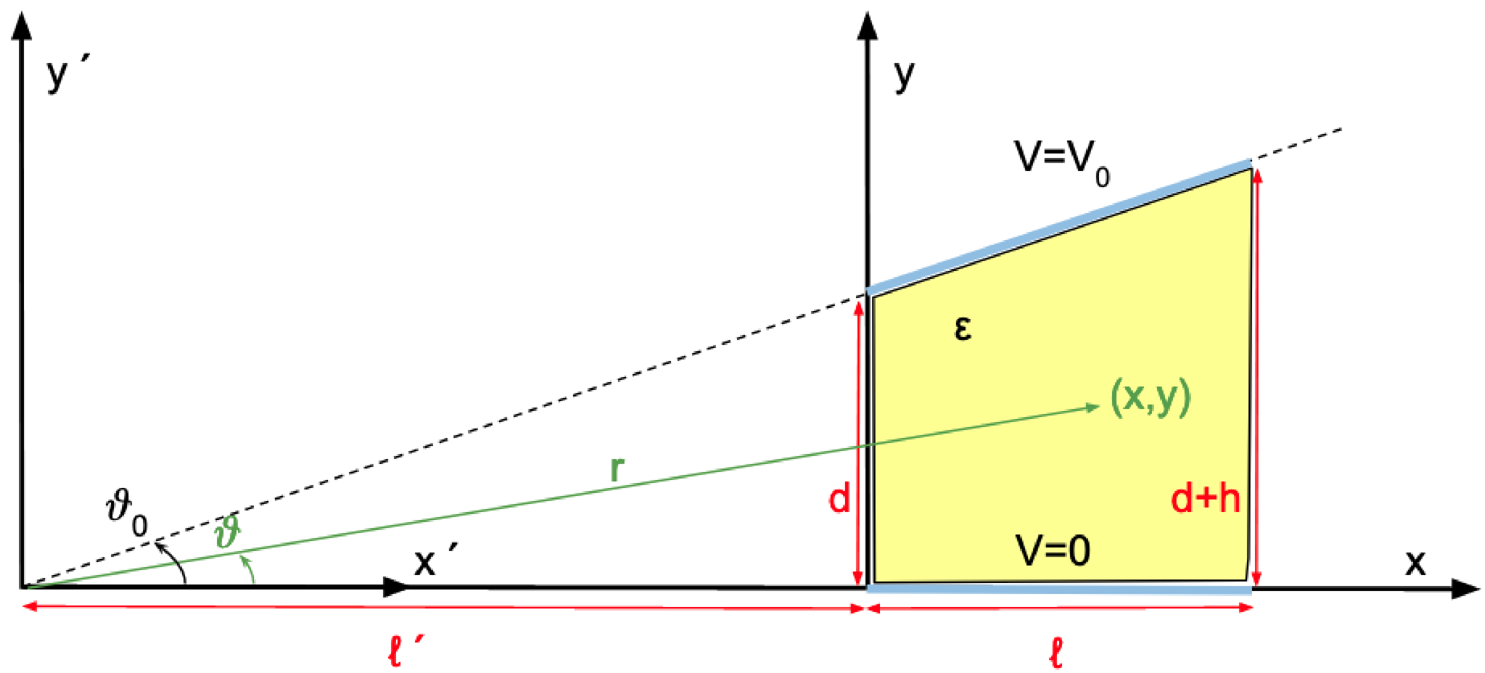


Worked example: The not-so-parallel plate capacitor

Neglecting edge effects, the problem is a two-dimensional one.

Symmetry dictates that the electric field is parallel to the xy plane and it is *independent* of z.

A convenient coordinate system for our analysis is $O'(x', y', z')$ which is shifted from $O(x, y, z)$ by a distance ℓ' along the x axis, so imaginary extrapolations of the capacitor plates intersect at $x' = 0$, as shown below.



Worked example: The not-so-parallel plate capacitor

The quantities ℓ', θ_0, θ and r introduced above (see schematic) can be easily related to the given quantities ℓ, d, h and the coordinates x, y of a point within the capacitor in the original coordinate system (O).

$$tan\theta_0 = \frac{d+h}{\ell+\ell'} = \frac{d}{\ell'} \Rightarrow \ell' = \ell\frac{d}{h}$$

$$tan\theta_0 = \frac{d}{\ell'} = \frac{\cancel{d}}{\ell\frac{\cancel{d}}{h}} \Rightarrow \theta_0 = arctan\Big(\frac{h}{\ell}\Big)$$

$$tan\theta = \frac{y}{x+\ell'} = \frac{y}{x+\ell\frac{d}{h}} \Rightarrow \theta = arctan\Big(\frac{y}{x+\ell\frac{d}{h}}\Big)$$

$$cos\theta = \frac{x+\ell'}{r} = \frac{x+\ell\frac{d}{h}}{r} \xRightarrow{cos\theta\approx 1} r = x+\ell\frac{d}{h}$$

Worked example: The not-so-parallel plate capacitor

The potetial between the plates can be found by solving Poisson's equation

$$\vec{\nabla}^2 V = \frac{\rho}{\epsilon_0} \xrightarrow{\rho=0} \vec{\nabla}^2 V = 0$$

In polar coordinates, this is written as

$$\frac{1}{r^2}\frac{d^2 V(\theta)}{d\theta} = 0 \Rightarrow \frac{d^2 V(\theta)}{d\theta} = 0$$

The equation has the following solution

$$V(\theta) = A + B\theta$$

The costants A, B can be derived from the boundary conditions

$$V(\theta = 0) = 0 \Rightarrow A = 0$$

$$V(\theta = \theta_0) = V_0 \xrightarrow{A=0} B\theta_0 = V_0 \Rightarrow B = \frac{V_0}{\theta_0}$$

Worked example: The not-so-parallel plate capacitor

Therefore, the solution $V(\theta)$ is given by

$$V(\theta) = \frac{V_0}{\theta_0}\theta$$

Using the expression for θ_0 that was derived previously, $V(\theta)$ can be written as

$$V(\theta) = \frac{V_0}{arctan\left(\frac{h}{\ell}\right)}\theta$$

In terms of coordinates x, y in the original coordinate system (O), the potential can be expressed as

$$V(x,y) = \frac{V_0}{arctan\left(\frac{h}{\ell}\right)} \arctan\left(\frac{y}{x + \ell\frac{d}{h}}\right)$$

Worked example: The not-so-parallel plate capacitor

The capacitance of the system will be calculated from

$$C = \frac{|Q|}{|V_0|}$$

where the unknown charge Q can be estimated from

$$Q = \oint \vec{D} \cdot d\vec{S}$$

The field $\vec{D}$ is related to electric field $\vec{E}$, and therefore the potential V, as

$$\vec{D} = \epsilon \vec{E} \xRightarrow{\vec{E} = -\vec{\nabla} V} \vec{D} = -\epsilon \vec{\nabla} V$$

Therefore

$$\vec{D} = -\epsilon \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} \xRightarrow{V = \frac{V_0}{\theta_0} \theta} \vec{D} = -\frac{1}{r} \frac{\epsilon V_0}{\theta_0} \hat{\theta}$$

Worked example: The not-so-parallel plate capacitor

Using the above expression for $\vec{D}$, Q is calculated as

$$Q = \oint \vec{D} \cdot d\vec{S} = \oint DdS \xRightarrow{dS=wdx,\ D=-\frac{1}{r}\frac{\epsilon V_0}{\theta_0}} Q = -\frac{w\epsilon V_0}{\theta_0} \int_0^{\ell} \frac{1}{r} dx \xRightarrow{r=x+\ell\frac{d}{h}}$$

$$Q = -\frac{w\epsilon V_0}{\theta_0} \int_0^{\ell} \frac{1}{x+\ell\frac{d}{h}} dx = -\frac{w\epsilon V_0}{\theta_0} ln(x+\ell\frac{d}{h})\Big|_0^{\ell} \Rightarrow$$

$$Q = -\frac{w\epsilon V_0}{\theta_0} ln\frac{d+h}{d} \xRightarrow{\theta_0=arctan\left(\frac{h}{\ell}\right)} Q = -\frac{w\epsilon V_0}{arctan\left(\frac{h}{\ell}\right)} ln\left(\frac{d+h}{d}\right)$$

Therefore

$$C = \frac{|Q|}{|V_0|} = \frac{w\epsilon}{arctan\left(\frac{h}{\ell}\right)} ln\left(\frac{d+h}{d}\right)$$

Worked example: Pulling a dielectric out of a capacitor

Question

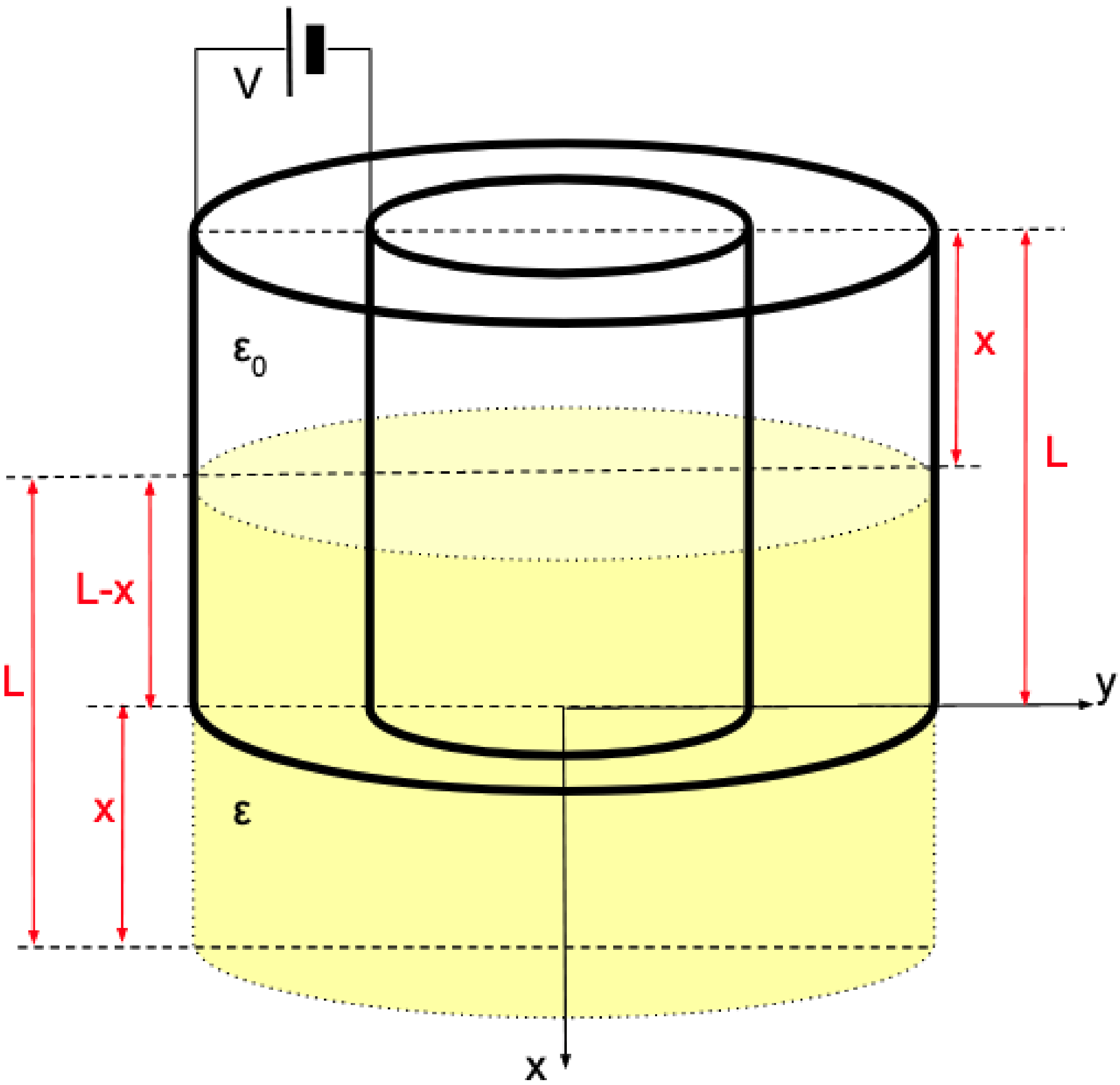
A cylindrical capacitor of length L consists of an inner metallic wire of radius a, and a thin outer metallic shell of radius b. The space in between is filled with a dielectric with permittivity ϵ.
As we have seen in the workshops, if edge effects are ignored, the capacitance C of the cylindrical capacitor is given by

$$C = \frac{2\pi\epsilon L}{ln(b/a)}.$$

Suppose that the dielectric is pulled partly out of the capacitor, and that the capacitor is connected to a battery of electromotive force V.

1. Find the force necessary to hold the dielectric in this position.
2. In which direction must the force be applied?

Worked example: Pulling a dielectric out of a capacitor



Worked example: Pulling a dielectric out of a capacitor

If the dielectric is pulled out of the cylindrical capacitor by a length x, then length $L - x$ remains inside the capacitor.
The part of the capacitor with length x that does not have a dielectric has capacitance C_1 given by

$$C_1 = \frac{2\pi\epsilon_0 x}{ln(b/a)}$$

whereas the part of the capacitor with length $L - x$ that does have a dielectric has a capacitance C_2 given by

$$C_2 = \frac{2\pi\epsilon(L - x)}{ln(b/a)}.$$

Those two capacitors have a common potential difference across the inner and outer conductors (connected parallel) and, therefore, the combined capacitance C is

$$C = C_1 + C2 = \frac{2\pi\epsilon_0 x}{ln(b/a)} + \frac{2\pi\epsilon(L - x)}{ln(b/a)} = \frac{2\pi\epsilon}{ln(b/a)}\Big(L + (\frac{\epsilon_0}{\epsilon} - 1)x\Big)$$

Worked example: Pulling a dielectric out of a capacitor

The work provided by the battery as it charges the capacitor becomes energy stored in the capacitor and mechanical work of the force F that moves the dielectric with respect to the capacitor. We can write

$$dW_{battery} = dU_{capacitor} + dW_{mechanical} \Rightarrow$$

$$VdQ = d\left(\frac{1}{2}CV^2\right) + Fdx$$

Since V is constant, the above can be written as

$$VdQ = \frac{1}{2}V^2 dC + Fdx$$

Using the definition of capacitance, $C = \frac{Q}{V}$, we can write

$$dC = \frac{dQ}{V} \Rightarrow V^2 dC = VdQ$$

From all the above, we obtain

$$V^2 dC = \frac{1}{2}V^2 dC + Fdx \Rightarrow \frac{1}{2}V^2 dC = Fdx$$

Worked example: Pulling a dielectric out of a capacitor

As the dielectric exits the capacitor, x increases and, therefore, $dx > 0$. When x increases we expect the capacitance C to decrease. Given that $\epsilon_0/\epsilon < 1$, this can be easily deduced from the expression for C,

$$C = \frac{2\pi\epsilon}{ln(b/a)}\Big(L + (\frac{\epsilon_0}{\epsilon} - 1)x\Big).$$

Therefore, for $dx > 0$, $dC < 0$. Considering this and observing the expression

$$\frac{1}{2}V^2 dC = Fdx,$$

we see that the term Fdx needs to be negative. Therefore, for $dx > 0$ (direction of exiting the capacitor), the direction of the force on the dielectric is opposite, and the dielectric is pulled into the capacitor.
If we wanted to hold the dielectric in a fixed position, we would need to apply an opposite force $\vec{F'}(= -\vec{F})$, pointing *away* from the capacitor.

Worked example: Pulling a dielectric out of a capacitor

The magnitude of my force $F' = F$

$$\frac{1}{2}V^2 dC = Fdx \Rightarrow F = \frac{1}{2}V^2 \frac{dC}{dx}$$

Differentiating the expression for C we derived earlier, we find

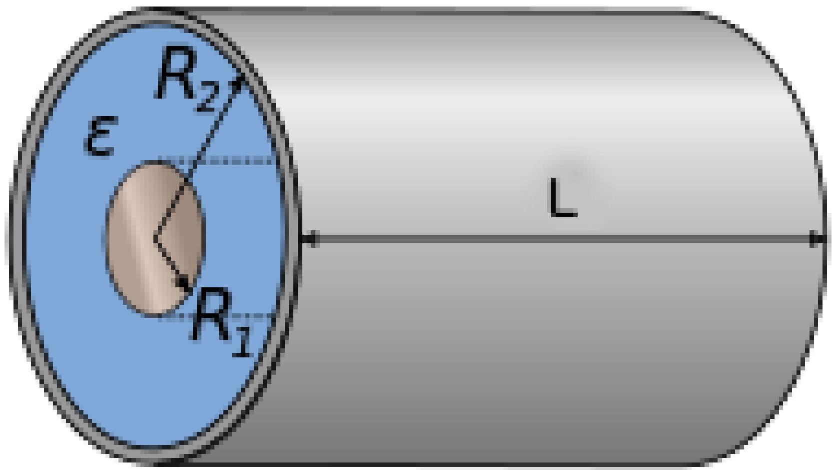
$$\frac{dC}{dx} = \frac{2\pi\epsilon}{ln(b/a)}\Big(\frac{\epsilon_0}{\epsilon} - 1\Big)$$

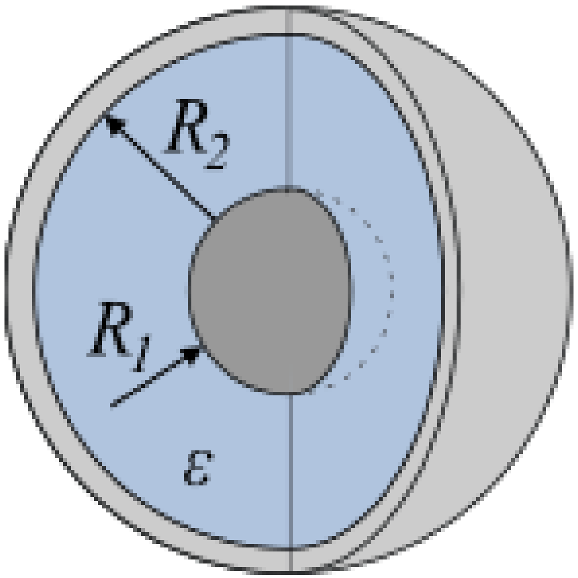
Thefore, the magnitude of the force is given by

$$F = \frac{1}{2}V^2 \frac{2\pi\epsilon}{ln(b/a)}\Big(\frac{\epsilon_0}{\epsilon} - 1\Big) \Rightarrow F = \frac{\pi\epsilon V^2}{ln(b/a)}\Big(\frac{\epsilon_0}{\epsilon} - 1\Big)$$

Question

Calculate the capacitance of a cylindrical and a spherical capacitor.





$$C = \frac{2\pi\epsilon L}{ln(R_2/R_1)}$$

$$C = \frac{4\pi\epsilon}{\frac{1}{R_1} - \frac{1}{R_2}}$$

Worked example: Cylindrical and spherical capacitors

As a Gaussian surface, for the calculation for a cylindrical capacitor, we choose a cylinder of length L and radius ρ ($R_1 \leq \rho \leq R_2$), closed by end caps. It is coaxial with the cylinders of radii R_1 and R_2 and encloses the central cylinder and thus also the charge Q on that cylinder. Let's assume, without loss of generality, that the inner conductor is positively charged and the outer conductor is negatively charged.
From Gauss' law:

$$\Phi_E = Q/\epsilon_0 \Rightarrow \oint_S \vec{E} \cdot d\vec{S} = Q/\epsilon_0$$

The electric field $\vec{E}$ is collinear with the normal to the Gaussian surface and, due to cylindrical symmetry, it has the same magnitude everywhere on that surface. There is no electric flux through the end caps.
The overall flux through the closed Gaussian surface is:

$$\oint_S \vec{E} \cdot d\vec{S} = \oint_S E \cdot dS = E \oint_S dS = ES = E(2\pi\rho L)$$

Worked example: Cylindrical and spherical capacitors

Therefore:

$$E(2\pi\rho L) = Q/\epsilon_0 \Rightarrow E = \frac{Q}{2\pi\epsilon_0 L\rho}$$

In vector form the electric field is written as:

$$\vec{E} = \frac{Q}{2\pi\epsilon_0 L\rho} \cdot \hat{\rho}$$

where $\hat{\rho}$ is the axial radial unit vector of the cylindrical coordinate system used (it is perpendicular to the axis of the two cylindrical conductors and it points outwards).
The potential difference V between the positively and negatively charged conductors is:

$$V := V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{\ell}$$

This integral is path-independent. The integral is simplified if $\vec{E}$ and $d\vec{\ell}$ are colinear. Since the electric field points along $\hat{\rho}$, we chose $d\vec{\ell} = d\rho \cdot \hat{\rho}$.

Worked example: Cylindrical and spherical capacitors

Therefore:

$$V = -\frac{Q}{2\pi\epsilon_0 L}\int_{R_2}^{R_1}\frac{d\rho}{\rho}$$

$$= -\frac{Q}{2\pi\epsilon_0 L} ln(\rho)\Big|_{R_2}^{R_1} = -\frac{Q}{2\pi\epsilon_0 L}(lnR_1 - lnR_2) = \frac{Q}{2\pi\epsilon_0 L}(lnR_2 - lnR_1) \Rightarrow$$

$$V = \frac{Q}{2\pi\epsilon_0 L} ln(R_2/R_1)$$

Therefore, the capacitance of a cylindrical capacitor is given by:

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\epsilon_0 L} ln(R_2/R_1)} = 2\pi\epsilon_0 \frac{L}{ln(R_2/R_1)}$$

Worked example: Cylindrical and spherical capacitors

As a Gaussian surface, for the calculation for a spherical capacitor, we choose a sphere of radius r ($R_1 \leq r \leq R_2$) which is concentric with the spherical shells of radii R_1 and R_2. Let's assume, without loss of generality, that the inner conductor is positively charged and the outer conductor is negatively charged.
From Gauss' law:

$$\Phi_E = Q/\epsilon_0 \Rightarrow \oint_S \vec{E} \cdot d\vec{S} = Q/\epsilon_0$$

The electric field $\vec{E}$ is collinear with the normal to the chosen Gaussian surface and, due to spherical symmetry, it has the same magnitude everywhere on that surface:

$$\oint_S \vec{E} \cdot d\vec{S} = \oint_S E \cdot dS = E \oint_S dS = ES = E(4\pi r^2)$$

Worked example: Cylindrical and spherical capacitors

Therefore:

$$E(4\pi r^2) = Q/\epsilon_0 \Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}$$

In vector form the electric field is written as:

$$E = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

where $\hat{r}$ is the radial unit vector of the spherical coordinate system used.
The potential difference V between the positively and negatively charged conductors is:

$$V := V_+ - V_- = -\int_-^+ \vec{E} d\vec{\ell}$$

The above path-independent integral is simplified if $\vec{E}$ and $d\vec{\ell}$ are collinear ($d\vec{\ell} = dr \cdot \hat{r}$).

Worked example: Cylindrical and spherical capacitors

Therefore:

$$V = -\frac{Q}{4\pi}\int_{R_2}^{R_1}\frac{dr}{r^2}$$

$$= -\frac{Q}{4\pi\epsilon_0}(-\frac{1}{r})|_{R_2}^{R_1} = -\frac{Q}{4\pi\epsilon_0}(-\frac{1}{R_1}+\frac{1}{R_2}) = \frac{Q}{4\pi\epsilon_0}(\frac{1}{R_1}-\frac{1}{R_2}) \Rightarrow \frac{Q}{4\pi\epsilon_0}\frac{R_2-R_1}{R_1 R_2}$$

$$V = \frac{Q}{4\pi\epsilon_0}\frac{R_2-R_1}{R_1 R_2}$$

Therefore, the capacitance of a spherical capacitor is given by:

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\frac{R_2-R_1}{R_1 R_2}} = 4\pi\epsilon_0\frac{R_1 R_2}{R_2-R_1}$$

Worked example: Capacitor with inhomogeneous dielectric

Question

The volume between two concentric conducting spherical surfaces of radii a and b ($a < b$), is filled with an inhomogeneous dielectric with permittivity

$$\epsilon = \frac{\epsilon_0}{1 + Kr}$$

where K is a constant and r is the radial coordinate. The displacement field $\vec{D}$ is related to the electric field $\vec{E}$ by the usual formula $\vec{D} = \epsilon \vec{E}$.
A charge Q is placed on the inner surface, while the outer one is grounded.
Find:

- The magnitude and direction of the electric displacement in the region $a < r < b$.
- The capacitance of the device.
- The volume polarization charge density in $a < r < b$.
- The surface polarization charge density at $r = a$ and $r = b$.

Worked example: Capacitor with inhomogeneous dielectric

The integral form of Gauss's law for the electric displacement field $\vec{D}$ is

$$\oint \vec{D} \cdot d\vec{S} = Q_{free}$$

Due to spherical symmetry of the problem, if we choose as integration surface the surface of a sphere of radius r (a ¡ r ¡ b), the vectors $\vec{D}$ and $d\vec{S}$ are both radial, and $|\vec{D}| = D$ is constant over the integration surface. Therefore:

$$\oint DdS = Q_{free} \Rightarrow D \oint dS = Q_{free} \Rightarrow D4\pi r^2 = Q_{free} \Rightarrow$$

$$D(r) = \frac{Q_{free}}{4\pi r^2}$$

The radial vector $\vec{D}$ can be written in vector form as:

$$\vec{D}(\vec{r}) = \frac{Q_{free}}{4\pi r^2}\hat{r}$$

Worked example: Capacitor with inhomogeneous dielectric

The displacement and electric field vectors are related by:

$$\vec{D}(\vec{r}) = \epsilon \vec{E}(\vec{r})$$

Therefore:

$$\vec{E}(\vec{r}) = \frac{Q_{free}}{4\pi\epsilon r^2}\hat{r}$$

The expression given for the permittivity of the inhomogeneous dielectric is

$$\epsilon = \frac{\epsilon_0}{1 + Kr}$$

Substituting the above expression for ϵ into the earlier expression for $\vec{E}$, we have:

$$\vec{E}(\vec{r}) = \frac{Q_{free}}{4\pi\epsilon_0}\frac{1 + Kr}{r^2}\hat{r}$$

Worked example: Capacitor with inhomogeneous dielectric

The potential difference V between the two concentric spherical surfaces of radii a and b is given by:

$$V = \int_a^b \vec{E}(\vec{r}) \cdot d\vec{\ell}$$

Choosing a radial integration path:

$$d\vec{\ell} = d\vec{r}$$

and carrying out the integration from a to b, we find:

$$V = \frac{Q_{free}}{4\pi\epsilon_0} \int_a^b \frac{1+Kr}{r^2} \hat{r} \cdot d\vec{r} = \frac{Q_{free}}{4\pi\epsilon_0} \int_a^b \frac{1+Kr}{r^2} dr = \frac{Q_{free}}{4\pi\epsilon_0} \Big(\int_a^b \frac{dr}{r^2} + K \int_a^b \frac{dr}{r} \Big)$$

$$= \frac{Q_{free}}{4\pi\epsilon_0} \Big(-\frac{1}{r} + Kln(r) \Big) \Big|_a^b = \frac{Q_{free}}{4\pi\epsilon_0} \Big(-\frac{1}{b} + Kln(b) + \frac{1}{a} - Kln(a) \Big) \Rightarrow$$

Worked example: Capacitor with inhomogeneous dielectric

$$V = \frac{Q_{free}}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b} + Kln(\frac{b}{a})\right)$$

The capacitance of the device is given by:

$$C = \frac{Q_{free}}{V}$$

Substituting V, we find:

$$C = \frac{\cancel{Q_{free}}}{\frac{\cancel{Q_{free}}}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b} + Kln(\frac{b}{a})\right)} = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b} + Kln(\frac{b}{a})} \Rightarrow$$

$$C = \frac{4\pi\epsilon_0 ab}{b - a + Kabln(\frac{b}{a})}$$

Worked example: Capacitor with inhomogeneous dielectric

The polarization vector $\vec{P}$ is given by:

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

The factor $\chi_e \epsilon_0$ can be expressed in terms of the given ϵ as follows:

$$1 + \chi_e = \epsilon_r = \frac{\epsilon}{\epsilon_0} \Rightarrow (1+\chi_e)\epsilon_0 = \epsilon \Rightarrow \chi_e \epsilon_0 = \epsilon - \epsilon_0$$

Using the above, the earlier expression for $\vec{P}$ can be written as:

$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}$$

Substituting the given expression of ϵ, we have:

$$\vec{P} = \left(\frac{\cancel{\epsilon_0}}{1+Kr} - \cancel{\epsilon_0}\right) \frac{Q_{free}}{4\pi\cancel{\epsilon_0}} \frac{1+Kr}{r^2}\hat{r} = \frac{Q_{free}}{4\pi}\left(\frac{1}{1+Kr} - 1\right)\frac{1+Kr}{r^2}\hat{r}$$

$$= \frac{Q_{free}}{4\pi} \frac{\cancel{1} - \cancel{1} - K\cancel{r}}{\cancel{1+Kr}} \frac{\cancel{1+Kr}}{r^{\cancel{2}}}\hat{r} \Rightarrow \vec{P} = -\frac{Q_{free} K}{4\pi r}\hat{r}$$

Worked example: Capacitor with inhomogeneous dielectric

The volume polarization charge density ρ_P is given by:

$$\rho_P = -\vec{\nabla} \cdot \vec{P}$$

Substituting in the above equation the expression for $\vec{P}$, we have:

$$\rho_P = -\vec{\nabla} \cdot \Big(- \frac{Q_{free} K}{4\pi r} \hat{r} \Big)$$

As $\vec{P}$ is radial vector, it is easier to evaluate its divergence expressing the divergence operator in spherical coordinates. Doing so, we find the volume polarization charge density ρ_P to be

$$\rho_P = -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \Big(- \frac{Q_{free} K}{4\pi r} \Big) \right) \Rightarrow$$

$$\rho_P = \frac{Q_{free} K}{4\pi r^2}$$

Worked example: Capacitor with inhomogeneous dielectric

The surface polarization charge densities σ_P ar $r = a, b$ are given by the projection of $\vec{P}$ along the corresponding surface vectors.
Using the expression of $\vec{P}$ we found earlier, we get:

$$\sigma_P(r = a) = \vec{P} \cdot (-\hat{r}) = \frac{Q_{free} K}{4\pi a}$$

and

$$\sigma_P(r = b) = \vec{P} \cdot (+\hat{r}) = -\frac{Q_{free} K}{4\pi b}$$

PHYS 201 / Lecture 5
Electric current / link with magnetic phenomena; Magnetic field, Lorentz force; Cyclotron motion; Biot-Savart law and applications

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Lectures delivered at the University of Liverpool, 2021-22

December 15, 2021

Lecture 4 - Revision

- With regards to electrical properties, there are 2 types of materials
  - materials that conduct electricity: **conductors**
  - materials that do not conduct electricity: **insulators (dielectrics)**
- A conductor is an object or type of material which **contains electric charges that are relatively free to move**
  - A **perfect conductor** has an **unlimited supply of free charges**.
  - There are no perfect conductors, but many substances come close!
    - e.g. the free charge density in copper is 1.8×10^{10} C$/m^3$.
- If we place a **conductor within an external electric field**:
  - The electric field vanishes everywhere inside a conductor.
  - The potential is constant inside a conductor.
  - Charge accumulates in the surface.
  - The field on the surface of a conductor has no tangential component.

- Capacitance (C) denotes the ability of a body to store electric charge. For a system of two conductors, one with charge +Q held at potential V_+ and one with charge -Q held at potential V_-, the capacitance is a positive quantity defined as:

$$C = \frac{Q}{V_+ - V_-}$$

  Its SI unit is the **Farad (F)** defined as one Coulomb per Volt.
- Calculating the capacitance for simple systems:
  - Use Gauss' law to calculate the electric field $\vec{E}$ in terms of the charge Q stored in one of the conductors: $\oint_S \vec{E} \cdot d\vec{S} = Q/\epsilon_0$
  - Once $\vec{E}$ is known, calculate the potential difference V between the two conductors as: $V := \Delta V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{\ell}$
  - From the known charge Q in the positive conductor and the potential difference V between the conductors, calculate $C = Q/V$

Lecture 4 - Revision (cont'd)

- We studied a simple system: The **parallel plate capacitor**
- The electric field between the plates was found to be:

$$E = \frac{\sigma}{\epsilon_0}$$

- The parallel plate capacitor has capacitance:

$$C = \epsilon_0 \frac{A}{d}$$

  It depends only on the geometrical characteristics of the capacitor.
- Energy stored in the parallel plate capacitor:

$$U = \frac{Q^2}{2C} \quad or \quad U = \frac{1}{2}CV^2 \quad or \quad U = \frac{1}{2}QV$$

  and confirmed that:

$$U = \frac{\epsilon_0}{2} \int_{all\ space} |\vec{E}(\vec{r})|^2 d\tau$$

Lecture 4 - Revision (cont'd)

- **Electric dipole**: Point charges +q and -q at a *small distance* d.
- An electric dipole its described by its **electric dipole moment** $\vec{p} = q\vec{d}$
  - A vector pointing from the negative to the positive charge
- An electric dipole creates a potential field V given by

$$V \approx \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

- An electric field $\vec{E}$ exerts to an electric dipole with moment $\vec{p}$ a torque $\vec{T} = \vec{p} \times \vec{E}$
- Electric fields induce dipole moments in the direction of the field (or align towards the direction of the field polar molecules with permanent dipole moments) and generate macroscopic polarisation.
- The **polarisation** $\vec{P}$ of a material is defined as the **amount of electric dipole moment per unit volume**.

- The polarisation induces surface and volume polarisation charges. The corresponding densities are $\sigma_P = \vec{P} \cdot \hat{n}$ and $\rho_P = -\vec{\nabla} \cdot \vec{P}$
- In the presence of dielectrics, Gauss's law need to be generalised to include both free charges we also have induced polarisation charges:

$$\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \vec{P}\right) = \rho_f \quad \textit{and} \quad \oint \left(\epsilon_0 \vec{E} + \vec{P}\right) \cdot d\vec{S} = Q_f$$

  where ρ_f is the free charge density and Q_f the amount of free charge.
- The vector $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ is the electric displacement vector
  - In SI, the electric displacement unit is C/m^2.
- For **linear dielectrics**, $\vec{P} = \chi_e \epsilon_0 \vec{E}$ and, therefore, $\vec{D} = \epsilon_r \epsilon_0 \vec{E}$, where χ_e is the electric susceptibility of the material and $\epsilon_r = 1 + \chi_e$ is the relative permittivity (dielectric constant).
- A dielectric with relative permittivity ϵ_r inserted between the plates of a parallel plate capacitor, increases its capacitance by a factor of ϵ_r.

Maxwell's equation we know so far

In vacuum (static case):

	Integral form	*Differential form*
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = Q_{enclosed}/\epsilon_0$	$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$

In the presence of materials (static case):

	Integral form	*Differential form*
Gauss's law	$\oint \vec{D} d\vec{S} = Q_{enclosed;free}$	$\vec{\nabla} \cdot \vec{D} = \rho_{free}$
Circuital law	$\oint \vec{E} d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$

Plan for Lecture 5

- Electric current
  - The *microscopic* view
- Magnetic phenomena
  - Connection with the electric current
- The magnetic field
- Magnetic force on a charge
  - Cyclotron motion and cyclotrons
- Magnetic force on a current
- Magnetic forces do no work
- Calculating magnetic fields: The Biot-Savart law
  - Magnetic field around a straight wire

Electric current

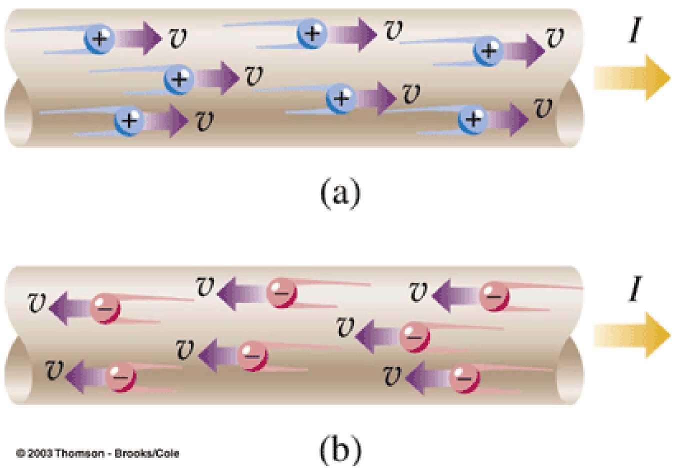
An **electric current is a flow of electric charge.**

It is represented by the amount of charge passing through per unit time.

$$I = \frac{dQ}{dt}$$

In SI, the unit of the electric current is the **Ampere (A)**. One Ampere is a charge change of 1 C over a period of 1 s.

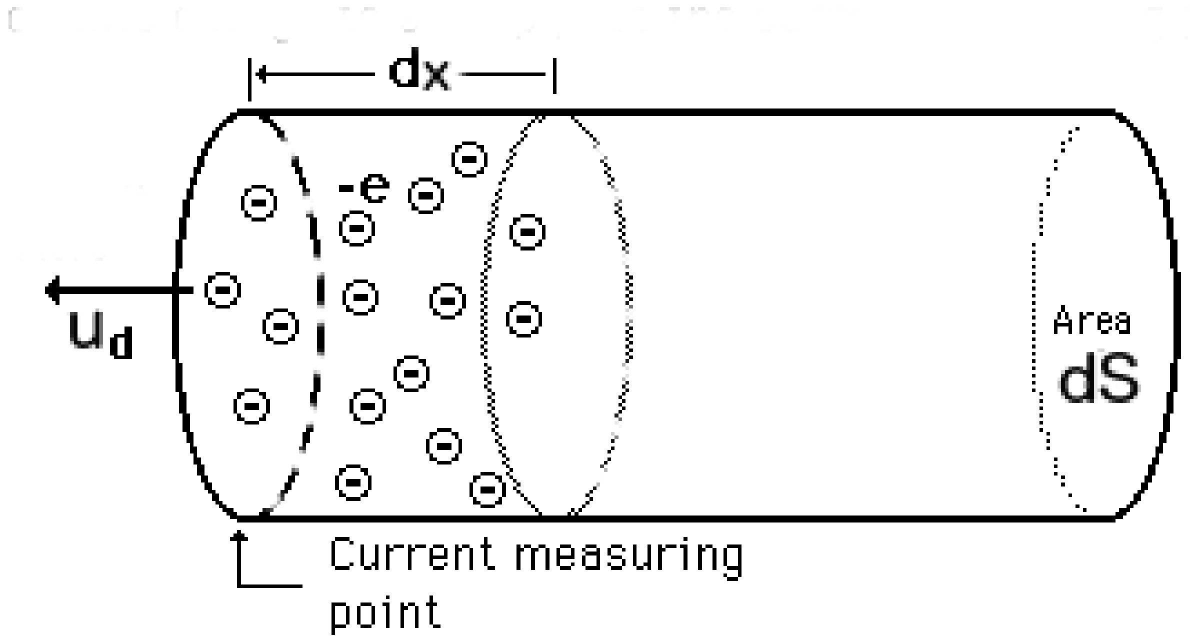
- Actually, this is how we define the Coulomb! We will see the actual definition of the Ampere once we have studied the magnetic force between two parallel conductors.



Note that, *by convention*, the current direction is **the direction of flow of positive charges**.

Electric current: The *microscopic* view

Consider a “tube” within a conducting material:



Let

- n be the carrier density,
- q be the charge of each carrier, and
- $\vec{u}_d$ be the average carrier velocity

The amount of charge passing through a cross-section of area dS is:

$$dQ = nqd\vec{x} \cdot d\vec{S} = nq\Big(\vec{u}_d dt\Big) \cdot d\vec{S}$$

Electric current: The *microscopic* view

Therefore, the current I through the area dS is given by:

$$I = \frac{dQ}{dt} = \frac{nq\left(\vec{u}_d dt\right) d\vec{S}}{dt} = nq\vec{u}_d \cdot d\vec{S}$$

The total amount of current flowing through the entire surface S is given by integrating the above result over S:

$$I = \int_S nq\vec{u}_d \cdot d\vec{S}$$

We can define a **current density** $\vec{j}$ as follows:

$$I = \int_S \vec{j} \cdot d\vec{S}$$

The current density is the **current per unit area of cross-section**. You can easily see from the above that:

$$\vec{j} = nq\vec{u}_d$$

Electric current: The *microscopic* view

In general

$$\vec{j} = \sigma \vec{E}$$

where σ is the **conductivity** of the material. It is an intrinsic property of a material and a **measure of its ability to conduct an electric current.**

- for a perfect insulator: σ=0, whereas for a perfect conductor: $\sigma=\infty$

In SI, the unit of conductivity is $1/(\Omega \cdot m)$ $(= S/m)$.

The inverse of conductivity is called, **resistivity** (ρ):

$$\rho = \frac{1}{\sigma}$$

Typical values:

Material	ρ $(\Omega \cdot m)$ at 20^{o}C	σ $(\Omega^{-1} \cdot m^{-1})$ at 20^{o}C
Graphene	1.00×10^{-8}	1.00×10^{8}
Copper	1.68×10^{-8}	5.96×10^{7}
Glass	10^{11}-10^{15}	10^{-15}-10^{-11}

Electric current: The *microscopic* view

Assume that a current I is **flowing out of a volume** τ through its surrounding closed surface S. The current flowing through the surface S is given by the negative (*) rate of change of the charge contained in the volume τ:

$$I = -\frac{dQ}{dt}$$

where Q is the volume integral of the charge density ρ:

$$Q = \int_\tau \rho d\tau$$

Therefore

$$I = -\frac{d}{dt}\int_\tau \rho d\tau = -\int_\tau \frac{d\rho}{dt} d\tau$$

$$I = \oint_S \vec{j} \cdot d\vec{S} \xrightarrow{\oint_S \vec{F}\cdot d\vec{S} = \int_\tau \vec{\nabla}\cdot\vec{F} d\tau} I = \int_\tau \vec{\nabla} \cdot \vec{j} d\tau$$

(*) Current flows out, so the amount of charge in **decreases**.

Electric current: The *microscopic* view

Equating the right-hand sides of the previous two equations, we have:

$$\int_\tau \vec{\nabla} \cdot \vec{j} d\tau = - \int_\tau \frac{d\rho}{dt} d\tau \Rightarrow \int_\tau \left(\vec{\nabla} \cdot \vec{j} + \frac{d\rho}{dt} \right) d\tau = 0$$

$$\vec{\nabla} \cdot \vec{j} + \frac{d\rho}{dt} = 0$$

The divergence of the current density is zero except for points where the net charge is introduced to or removed from the system.

This result is known as the **continuity equation**.

Charge conservation leads to current conservation.

Worked example

Question

Near Earth, the density of protons in the solar wind (a stream of particles from the Sun) is 8.70 cm^{-3}, and their speed is 470 km/s.

- Find the current density of these protons.
- If the Earth magnetic field did not deflect the protons, what total current would Earth receive?

The radius of the Earth is 6.37 $\times$ 10^6 m.

The magnitude of the current density vector $\vec{j}$ is:

$$|\vec{j}| = nq|\vec{u}| \Rightarrow$$

$$|\vec{j}| = \left(\frac{8.70}{10^{-6}\ m^3}\right)\left(1.60 \times 10^{-19}\ C\right)\left(470 \times 10^3\ m/s\right) = 6.54 \times 10^{-7} A/m^2$$

Worked example

Although the total surface area of Earth is $4\pi R^2_{Earth}$ (that of a sphere), the area to be used in a computation of how many protons in an approximately unidirectional beam (the solar wind) will be captured by Earth is its projected area. In other words, for the beam, the encounter is with a "target" of circular area πR^2_{Earth} .

The rate of charge transport implied by the influx of protons is:

$$I = \left(\pi R^2_{Earth}\right) |\vec{j}| \Rightarrow$$

$$I = \pi \left(6.37 \times 10^6 \ m\right)^2 \left(6.54 \times 10^{-7} \ A/m^2\right) = 8.34 \times 10^7 A$$

Magnetic phenomena

As we have seen, magnetic phenomena (as well as electric ones), were known from the antiquity.

Lodestones (naturally magnetised pieces of the mineral *magnetite*, first found in Magnesia, Asia Minor) were known to be **attracted to iron and other lodestones**.

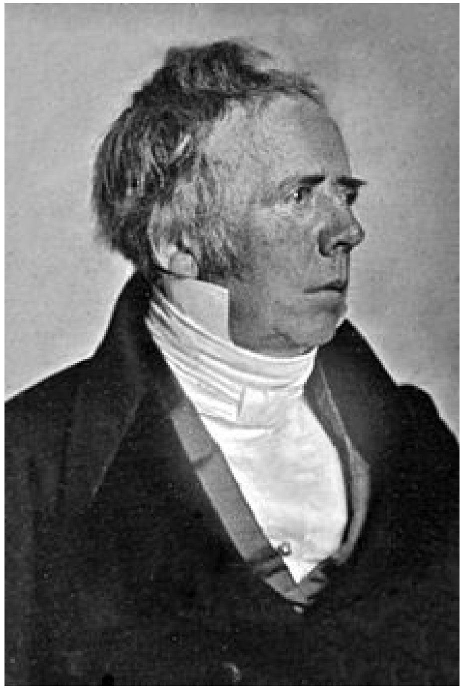
Early application: The **magnetic compass** (12th century AD).

Magnetic phenomena, were seemingly unrelated to electric ones.

Magnetic phenomena and electric current

A **compass is deflected in the presence of a current** (but not in presence of stationary charges): **A current generates a magnetic field!**

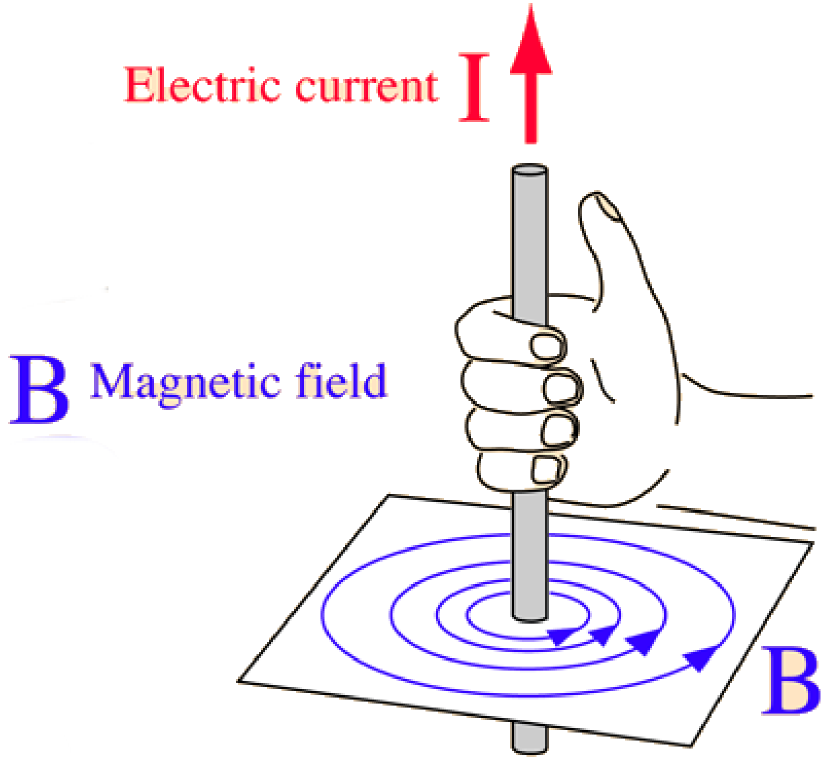
First observed by Oersted in the early 1800's.



Hans Christian Oersted
(1777-1851)
Danish physicist

Magnetic phenomena and electric current

A current generates a magnetic field!

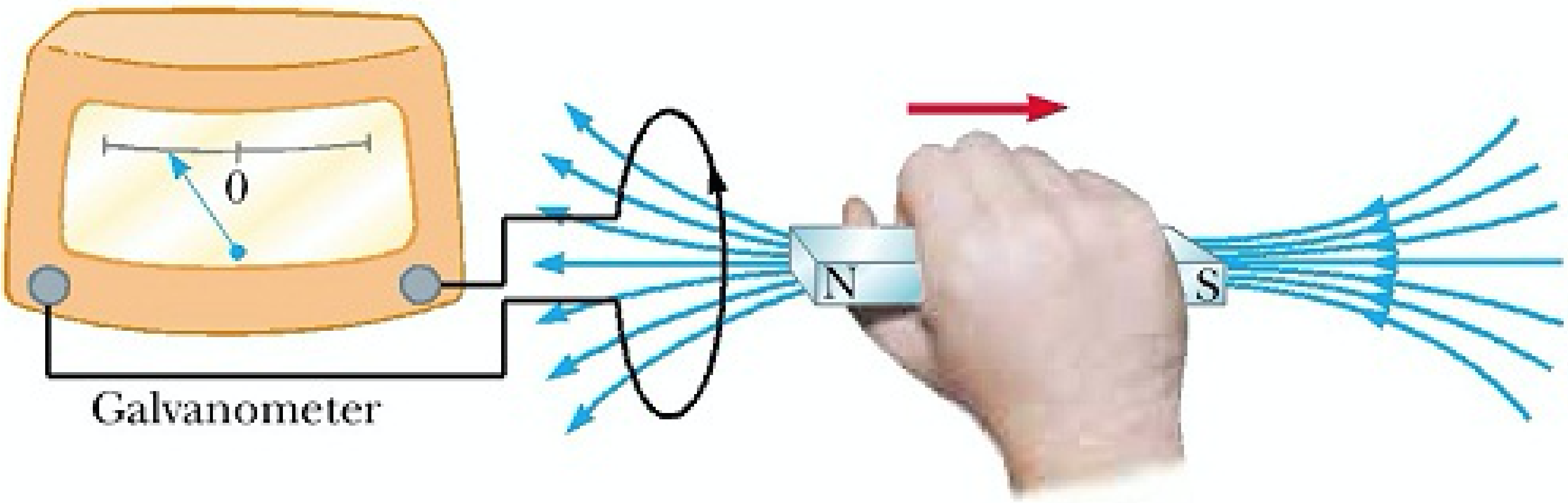


Interestingly, the field doesn't point towards to (or away from) the wire:
It circles around it. *(We will calculate this field later in this lecture.)*

Magnetic phenomena and electric current

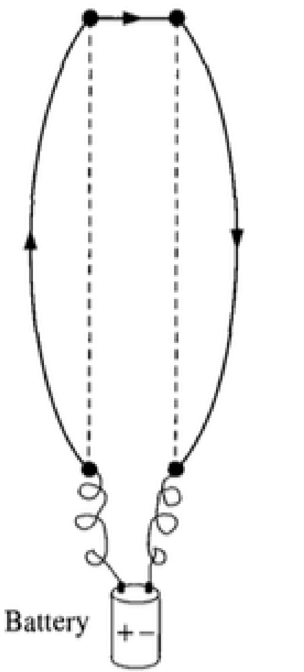
Not only a compass would be deflected in the presence of a current, but the opposite seems to be happening as well:

The motion of a magnet near a conducting loop induces a current in the loop (but there is no current if the magnet is stationary)!

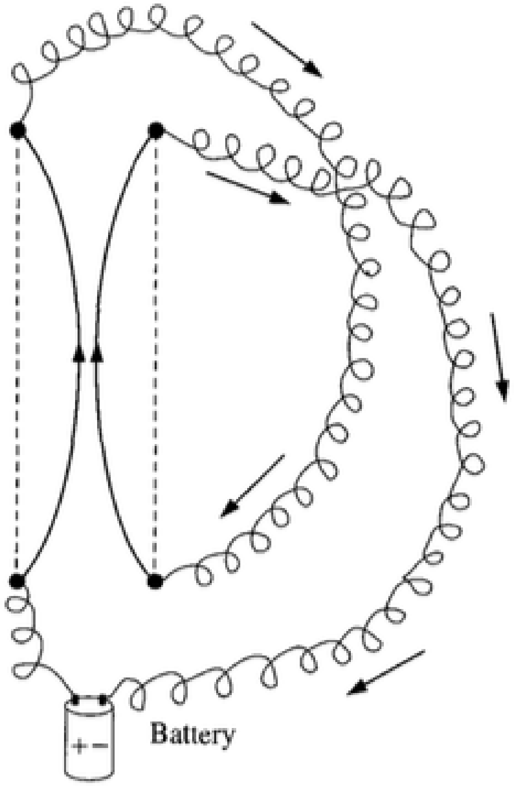


Magnetic phenomena and electric current

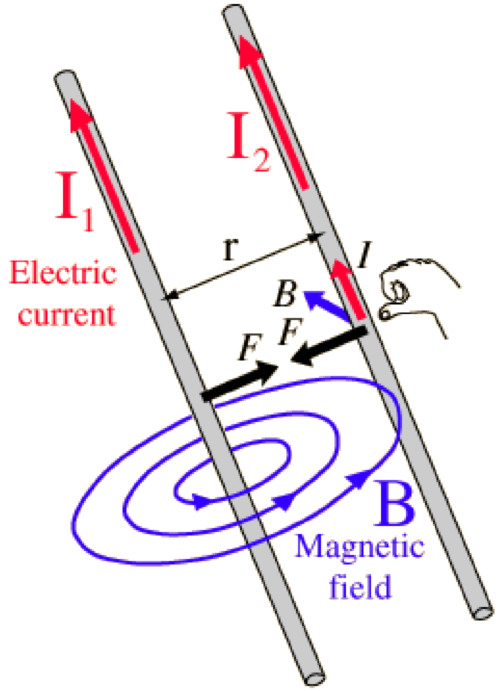
As result of the fact that a current generates a magnetic field, **a magnetic force is exerted between two wires!**



(a) Currents in opposite directions repel.



(b) Currents in same directions attract.



(We will calculate this force later in this lecture series.)

Magnetic phenomena and electric current

Let's reflect on the astonishing early observations:

- **Moving charges (electric currents) generate magnetic fields!**
- **Moving magnetic fields generate electric currents!**
- **There are magnetic forces between electric currents!**

Contrary to earlier beliefs, **magnetic and electric phenomena have a common origin: the electric charge!**

They are not different phenomena, but **different manifestations of a single interaction!**

The magnetic field

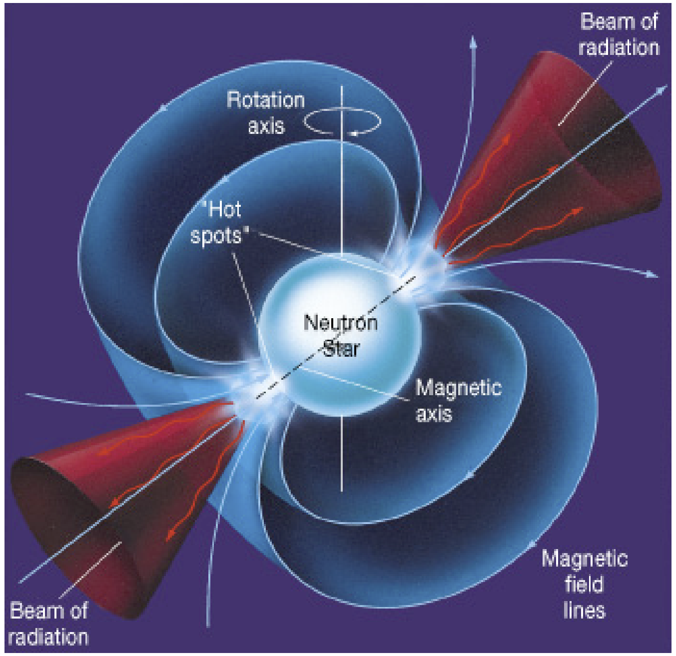
The **magnetic field is the magnetic effect of electric currents and magnetic materials**.

- It is a **vector field**: It permeates all space and associates a vector with each point.

In SI, the unit of the magnetic field is the **Tesla (T)**.

- Named in honour of Nikola Tesla (1856-1943), a Serbian-American physicist, engineer and inventor.
- The Tesla is a derived unit
  - A charge of 1 C with a velocity of 1 m/s perpendicular to a magnetic field of 1 T, experiences a magnetic force of 1 N ($T = N \cdot s \cdot C^{-1} \cdot m^{-1}$)
  - Other common (equivalent) definitions: $T = V \cdot s \cdot m^{-2} = N \cdot A \cdot m^{-1} = J \cdot A \cdot m^{-2} = Wb \cdot m^{-2}$
- Another commonly used unit is the CGS one: the **Gauss (G)**
  - 1 G $= 10^{-4}$ T
  - Earth's magnetic field is around 1 G.

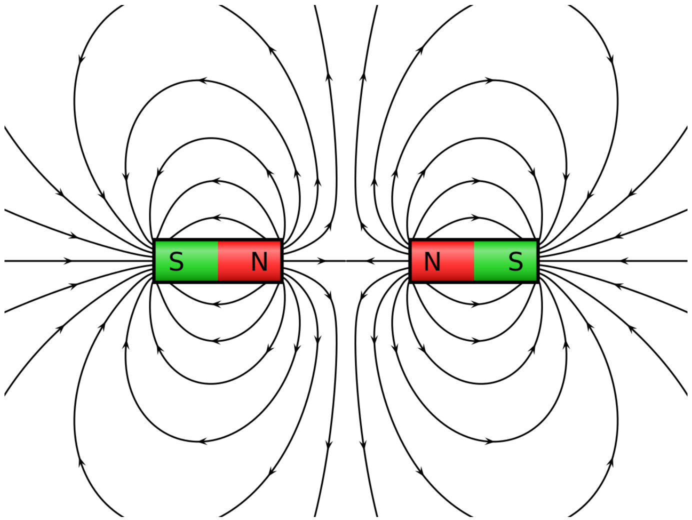
Typical magnetic fields



- **Human brain**:
  1 pT (10^{-12} T) / 10 nG = (10^{-8} G)
- **Earth's** magnetic field:
  Somewhat less than 10^{-4} T / 1 G
- **Refrigerator** magnet:
  5 mT ($5\ 10^{-3}$ T) / 50 G
- **LHC** dipole magnet:
  10 T / 10^{5} G
- Field that you need to **levitate a frog**:
  16 T / 1.6×10^{5} G
- **Neutron stars** (magnetar):
  1 MT (10^{6} T) - 100 GT (10^{11} T) /
  10 GG (10^{10} G) - 1 PG (10^{15} G)

"Magnetic charges"

There are several similarities between electrostatics and magnetostatics:



- We have two "types of magnetic charges" (we call them **North** and **South** *poles*)
- Same poles repel each other, while opposite poles attract each other.
- Magnetic field lines always start from a North pole and end in a South pole.

However, **there is one striking difference**: Whereas single positive or a single negative charges (electric monopoles) exist in isolation, **magnetic monopoles do not exist** (or they have not been found yet (*)).

(*) Or, maybe we have: Google "Blas Cabrera, Valentine's day magnetic monopole"

Magnetic force on an electric charge

What is the force that a magnetic field exerts on a moving charge?

It was observed experimentally that the force $\vec{F}$ exerted by a magnetic field $\vec{B}$, on a charge q moving with velocity $\vec{u}$ has a magnitude $|\vec{F}|$ that is proportional to all q, $|\vec{u}|$ and $|\vec{B}|$:

$$|\vec{F}| \propto q|\vec{u}||\vec{B}|$$

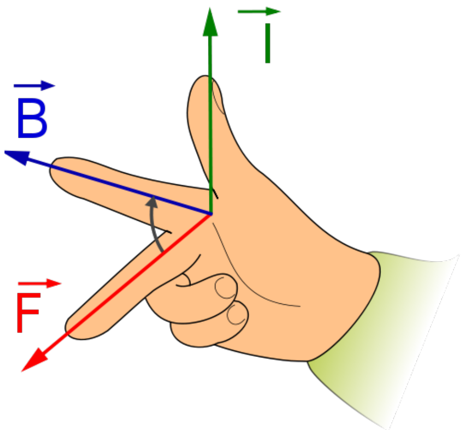
It was observed that:

- magnetic force vanishes if $\vec{u}$ and $\vec{B}$ are parallel
- force is increasing as the angle between $\vec{u}$ and $\vec{B}$ increases
- the maximum force is exerted when $\vec{u}$ and $\vec{B}$ are perpendicular to each other
- F is not on the same plane as $\vec{u}$ and $\vec{B}$, but perpendicular to the plane defined by $\vec{u}$ and $\vec{B}$

All the above can be summarised as:

$$\vec{F} = q\vec{u} \times \vec{B}$$

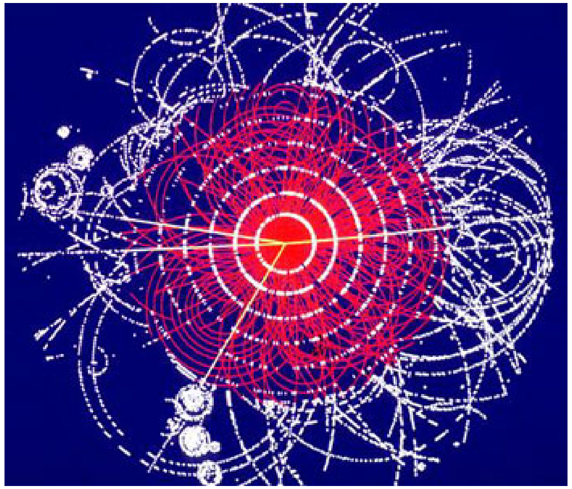
Magnetic force on an electric charge

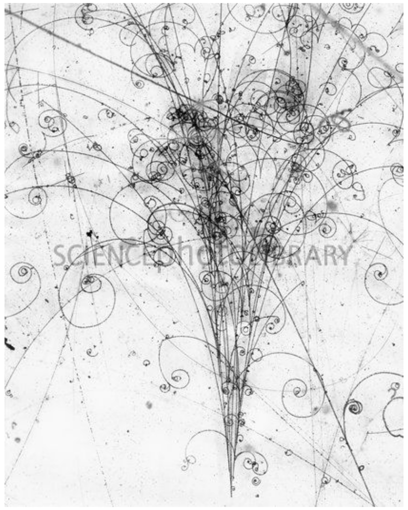


The magnetic force is given by:

$$\vec{F} = q\vec{u} \times \vec{B}$$

Use the **right-hand rule** to find the direction of $\vec{F}$.





The **magnetic force** exerted on a moving charge is **always perpendicular to its velocity** and acts as a **centripetal force**.

Charges within magnetic fields tend to **follow curved trajectories**.

Worked example

Question

A particle with charge q = -1.24 $\times 10^{-8}$ C is moving in a magnetic field $\vec{B} = (1.4\ T)\hat{x}$ with velocity $\vec{u} = (4.19 \times 10^4\ m/s)\hat{x} + (-3.85 \times 10^4\ m/s)\hat{y}$. Calculate in vector form the force exerted on the particle by the field.

The magnetic force $\vec{F}$ exerted on the charged particle is:

$$\vec{F} = q\vec{u} \times \vec{B} \Rightarrow$$

$$\vec{F} = (-1.24 \times 10^{-8} C)\Big\{(4.19 \times 10^4\ m/s)\hat{x} + (-3.85 \times 10^4\ m/s)\hat{y}\Big\} \times \Big\{(1.4\ T)\hat{x}\Big\}$$

Notice that $\hat{x} \times \hat{x} = 0$ and $\hat{y} \times \hat{x} = -\hat{z}$, and therefore:

$$\vec{F} = \Big(-1.24 \times 10^{-8} C\Big)\Big(-3.85 \times 10^4\ m/s\Big)\Big(1.4\ T\Big)\Big(-\hat{z}\Big) \Rightarrow$$

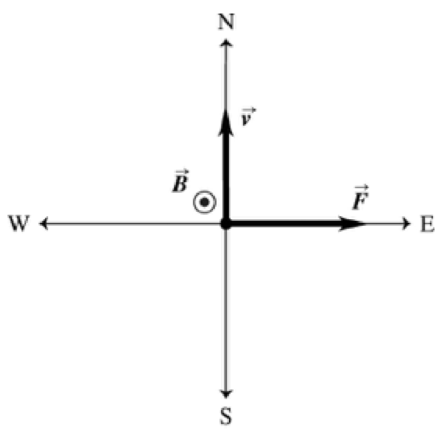
$$\vec{F} = \Big((-1.24)(-3.85)1.4 \times 10^{-4} N\Big)\Big(-\hat{z}\Big) \Rightarrow \vec{F} = -\Big(6.6836 \times 10^{-4} N\Big)\hat{z}$$

Worked example

Question

In a 1.25T magnetic field directed vertically upward, a particle with a charge of magnitude 8.5 μC which moves initially northward at 4.75 km/s is deflected towards the east.

1. What is the sign of the charge of the particle?
2. Find the magnetic force on the particle.



1. Curl your right-hand fingers from $\vec{u}$ to $\vec{B}$ through the smallest angle. This is the direction of $\vec{u} \times \vec{B}$, and it points east. $\vec{F}$ ($= q\ \vec{u} \times \vec{B}$) is in this direction too, so the charge is positive.
2. Let ϕ be the angle between $\vec{u}$ and $\vec{B}$. The force exerted on the particle is:

$$F = q{\cdot}u{\cdot}B{\cdot}sin\phi = \left(8.50{\times}10^{-6}\ C\right)\left(4.75{\times}10^{3}\ \frac{m}{s}\right)\left(1.25\ T\right)sin\frac{\pi}{2} = 0.0505\ N$$

Lorentz force

The total force felt by a charged body in the presence of both electric and magnetic fields is called the **Lorentz force**.

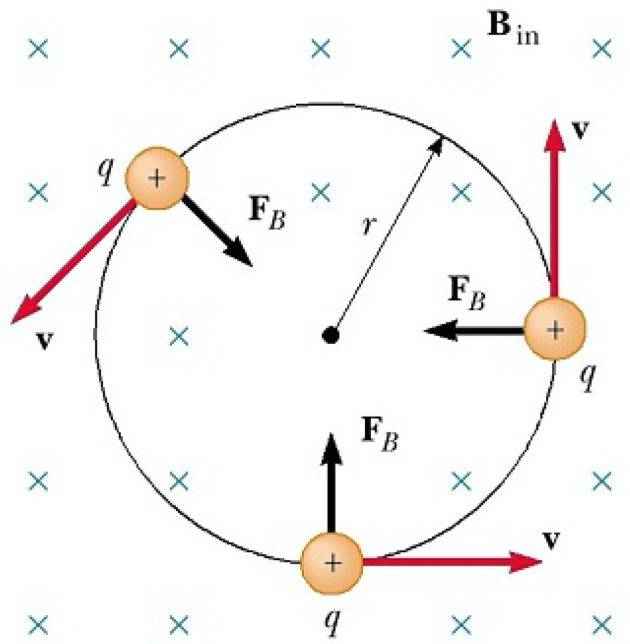
$$\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

It was first derived by Oliver Heaviside or James Maxwell. Hendrik Lorentz derived it a few years later.

The **magnetic force is much smaller than the electric force** unless the particle is moving at a velocity that is a significant fraction of the speed of light.

Cyclotron motion

Let's study the **simple trajectory** of a particle moving with a **constant velocity** u in a **constant magnetic field** B.



Assume that a positive charge enters the magnetic field B with a velocity u that is perpendicular to the magnetic field which points inwards (see on the left).

The direction of the magnetic force, found using the right-hand rule, is also shown.

The magnetic force provides **centripetal acceleration** and the charge will start moving counter-clockwise, along a circle (of radius r), with a velocity that is constant in magnitude.

Cyclotron motion

The magnetic force for a charge q with velocity u perpendicular to B is:

$$F = quB$$

The centripetal force (mass times centripetal acceleration) is:

$$F = m\frac{u^2}{r}$$

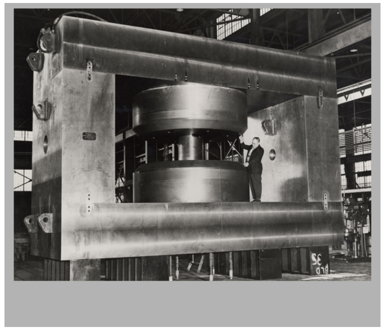
Therefore:

$$m\frac{u^2}{r} = quB \Rightarrow m\frac{u}{r} = qB \Rightarrow mu = qBr \Rightarrow \mathbf{p} = \mathbf{qBr}$$

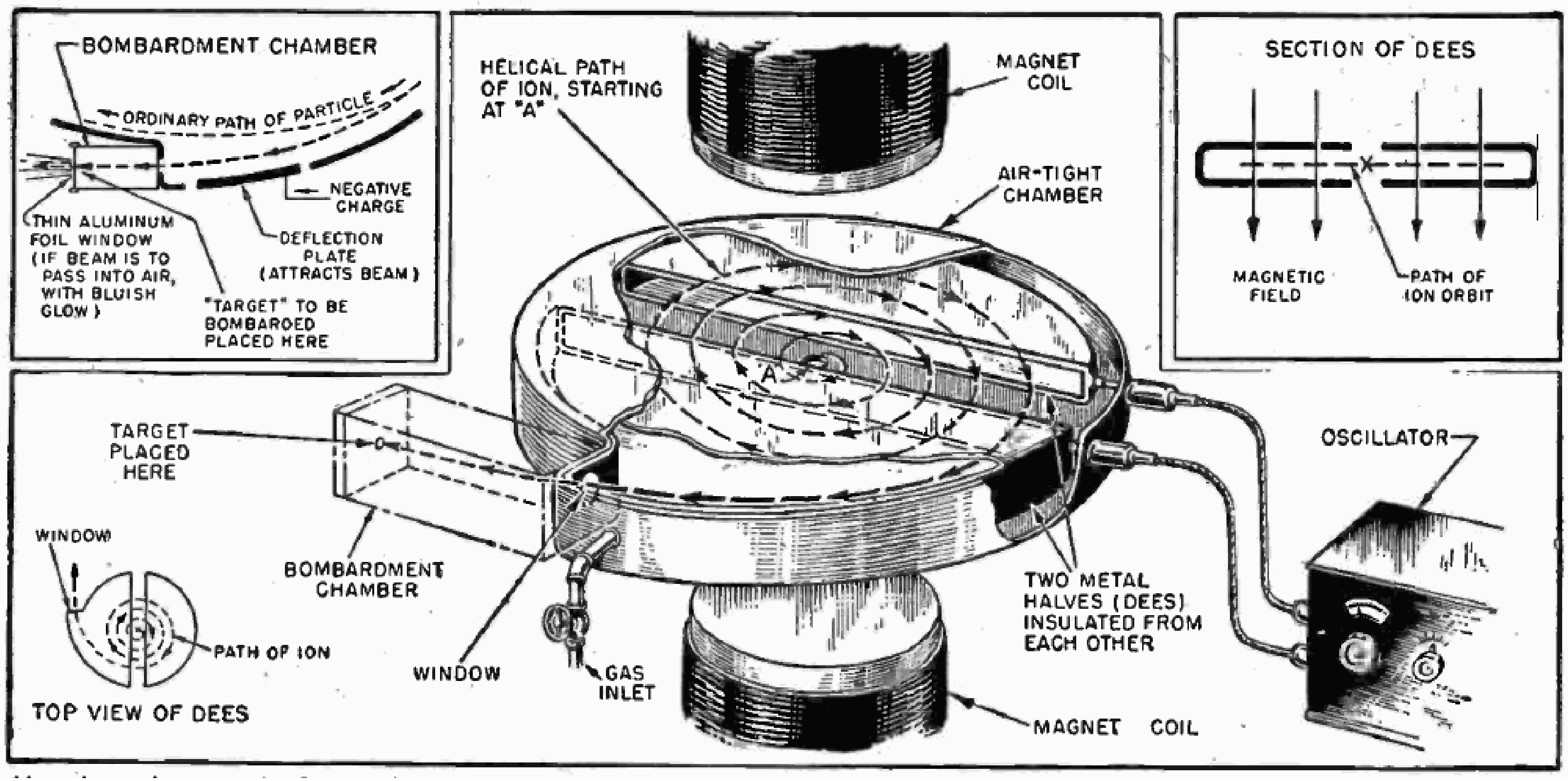
where p is the particle momentum. This is known as the **cyclotron formula** and describes the motion of a charged particle in a cyclotron.

The period T (time for one revolution) is independent of the particle velocity and it depends only on the particle type and the magnetic field:

$$T = \frac{2\pi r}{u} \xrightarrow{mu=qBr} T = \frac{2\pi m}{qB}$$



A cyclotron is a type of particle accelerator.
It was invented by Ernest Lawrence in 1932 and it was the most powerful type of accelerator till it was superseded by the synchrotron in the 1950's.



The Liverpool Cyclotron

James Chadwick

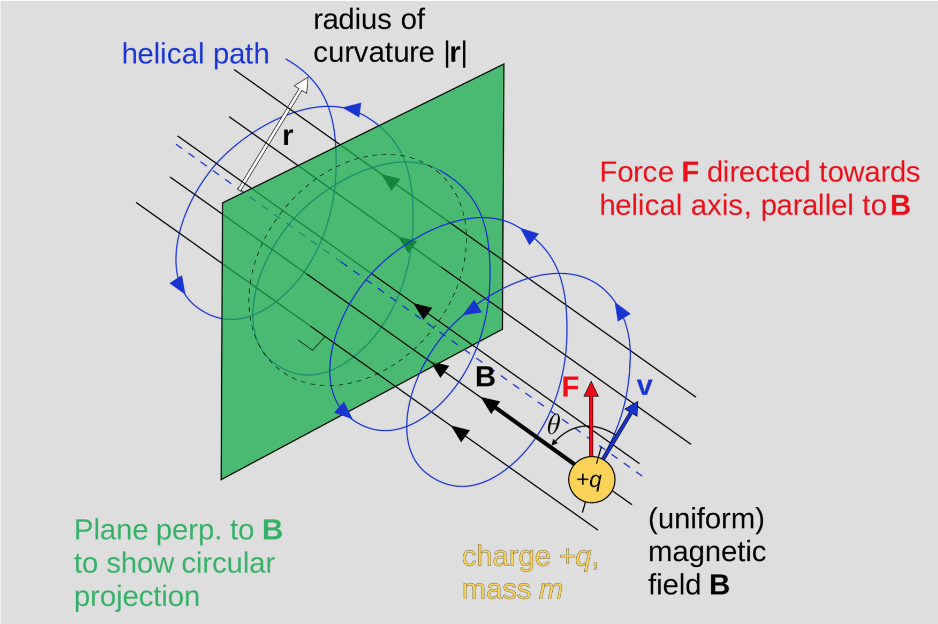
(1891-1974)

Liverpool Metropolitan

Cathedral

- When **Chadwick** (who discovered the neutron, Nobel Prize in Physics 1935) was the Head of the Physics Department at the University of Liverpool, (some time before the WW2) this was the home of a very large cyclotron.
  - You can see pieces of the D's in the VGM.
- And, in the 50's the University of Liverpool had a new synchrocyclotron, somewhere in the grounds of the Metropolitan Cathedral, that was the most powerful accelerator in Europe at the time.

If the charged particle was entering the magnetic field at an angle with the field lines then it would do **two motions**:

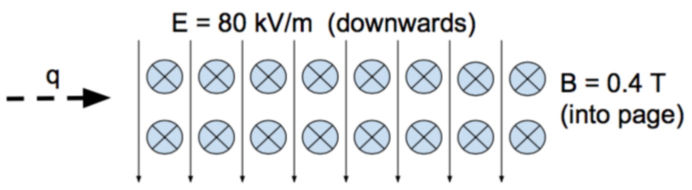


- On the plane perpendicular to the field, it would move in a **circular trajectory** with constant velocity $u_\perp$ (velocity component perpendicular to the field).
  Note: $F_\perp = q\vec{u}_\perp \times \vec{B} = qu_\perp B$
- It would also move along a **straight line** in the direction of the field with constant velocity $u_{||}$ (component parallel to the field).
  Note: $F_{||} = q\vec{u}_{||} \times \vec{B} = 0$

The combination of the two motions causes a **helical trajectory**.

Worked example

Question



A beam of particles with charge q enters a region where there is a uniform electric field $\vec{E}$, of magnitude 80 kV/m, directed downwards. In the same region, perpendicular to $\vec{E}$ and directed into the page is a magnetic field $|\vec{B}| = 0.4$T. As the particles enter that region their velocity is perpendicular to $\vec{E}$ and $\vec{B}$.

1. If the speed of the particles is properly chosen, the particles will not be deflected by these crossed electric and magnetic fields. What speed is selected in this case?
2. If the electric field is cut off and the same magnetic field is maintained, the charged particles move in the magnetic field in a circular path of radius 1.14 cm. Determine the ratio of the electric charge to the mass of the particles.

Worked example

The magnetic force is: $F_m = quB$, and the electric force is: $F_e = qE$

For no deflection:

$$F_m = F_e \Rightarrow quB = qE \Rightarrow u = \frac{E}{B} = \frac{80 \times 10^3 \ V/m}{0.4 \ T} = 2 \times 10^5 m/s$$

Once the electric field is cut off, the particle moves in a circular path with the magnetic force acting as the centripetal force.

$$\frac{mu^2}{r} = quB \Rightarrow \frac{q}{m} = \frac{u}{Br} = \frac{E}{B^2 r} \Rightarrow$$

$$\frac{q}{m} = \frac{80 \times 10^3 \ V/m}{\left(0.4 \ T\right)^2 \left(1.14 \times 10^{-2} m\right)} = 4.38 \times 10^7 C/kg$$

Worked example

Question

The pole pieces (the "dees") of a cyclotron are 50 cm in diameter, in a uniform magnetic field of 15,000 G.

1. Find approximate values for the kinetic energies up to which (a) protons and (b) α-particles (He^4 nuclei) could be accelerated.
2. What oscillator frequency would be required in each case?

The charge of the proton is $q_p = 1.6 \times 10^{-19} C$ and the mass of the proton is $m_p = 1.67 \times 10^{-27} kg$.

The cyclotron formula is:

$$mu = qBr$$

where m is the particle mass, q its charge, u its velocity, r the radius of curvature and B the magnetic field.

Worked example

Hence the velocity of the particle is:

$$u = \frac{qBr}{m}$$

The kinetic energy of the accelerated particle is:

$$K = \frac{1}{2}mu^2 = \frac{1}{2}m\left(\frac{qBr}{m}\right)^2 = \frac{q^2B^2r^2}{2m}$$

The kinetic energy K is maximal when the particle moves in a trajectory where the radius of curvature r has the largest possible value.

The maximum value of r is the radius (call it R) of the dees:

$$r = R = \frac{50\ cm}{2} = 25cm$$

Worked example

Thefore, the maximum kinetic energy for protons is given by:

$$K_{max;p} = \frac{q_p^2 B^2 R^2}{2m_p} = \frac{(1.6 \times 10^{-19} C)^2 (1.5 T)^2 (0.25 m)^2}{2(1.67 \times 10^{-27} kg)} \xRightarrow{T = kg/(Cs),\ J = kg \cdot m^2/s^2}$$

$$K_{max;p} = 1.08 \times 10^{-12} J$$

An α particle is about 4 times heavier than a proton ($m_\alpha = 4m_p$) and it has twice the charge ($q_\alpha = 2q_p$). Thefore, the maximum kinetic energy for α's is given by:

$$K_{max;\alpha} = \frac{q_\alpha^2 B^2 R^2}{2m_\alpha} = \frac{\left(2q_p\right)^2 B^2 R^2}{2\left(4m_p\right)} = \frac{\cancel{4} q_p^2 B^2 R^2}{2 \cdot \cancel{4} m_p} = \frac{q_p^2 B^2 R^2}{2m_p} \Rightarrow$$

$$K_{max;\alpha} = K_{max;p}$$

Worked example

The oscillator frequency f is given by:

$$u = \left(2\pi f\right) r \Rightarrow f = \frac{u}{2\pi r}$$

Substituting the previous expression for u ($u = qBr/m$), we have:

$$f = \frac{u}{2\pi r} = \frac{qBr}{2\pi rm} \Rightarrow f = \frac{qB}{2\pi m}$$

For protons:

$$f_p = \frac{q_p B}{2\pi m_p} = \frac{(1.6 \times 10^{-19} C)(1.5 T)}{2\pi(1.67 \times 10^{-27} kg)} = \frac{1.6 \cdot 1.5}{2\pi \cdot 1.67} \times 10^8 \frac{C \cdot T}{kg} \xRightarrow{T = kg/(Cs)}$$

$$f_p = 0.22 \times 10^8 \frac{C}{kg} \frac{kg}{Cs} = 0.22 \times 10^8 Hz = 22 MHz$$

For α's, the corresponding frequency is:

$$f_\alpha = \frac{q_\alpha B}{2\pi m_\alpha} = \frac{2 q_p B}{2\pi \cdot 4 m_p} = \frac{1}{2} \frac{q_p B}{2\pi m_p} = \frac{1}{2} f_p = \frac{1}{2} 22 MHz = 11 MHz$$

Magnetic force on current

As we have seen, the force on a single moving charge q is:

$$\vec{F} = q\vec{u} \times \vec{B}$$

How about the **force on a current** that consists of several moving charges?

$$q... \rightarrow \int_V \rho ... d\tau$$

$$\vec{F} = q\vec{u} \times \vec{B} \rightarrow \int_\tau \rho\vec{u} \times \vec{B} d\tau \xrightarrow{\vec{j}=\rho\vec{u}} \vec{F} = \int_\tau \vec{j} \times \vec{B} d\tau$$

For a current through a conducting wire

$$\vec{F} = I \int_L d\vec{\ell} \times \vec{B}$$

Magnetic forces do no work on electric charges.

Recall that a force $\vec{F}$ is said to do work dW if, when it is acting on a body, there is a displacement $d\vec{\ell}$ of the point of application in the direction of the force. The total work done is defined as:

$$W = \int dW = \int \vec{F} \cdot d\vec{\ell}$$

The magnetic force is $\vec{F} = q\vec{u} \times \vec{B}$ and therefore, it is always perpendicular to the velocity. Substituting the magnetic force, and expressing the displacement $d\vec{\ell}$ in terms of the velocity $\vec{u}$ ($d\vec{\ell} = \vec{u}dt$) we have:

$$W = \int \vec{F} \cdot d\vec{\ell} = q \int \left(\vec{u} \times \vec{B}\right) \cdot \vec{u}dt$$

That quantity is obviously 0! The cross product $\vec{u} \times \vec{B}$ is perpendicular to $\vec{u}$ and, thus, its dot product with $\vec{u}$ yields $W = 0$.

Generation of magnetic fields

We have talked extensively about the magnetic field $\vec{B}$, but **we do not know how to calculate it yet**...

In electrostatics we several examples of how to calculate the electric field $\vec{E}$ produced by a collection of charges. For example, the electric field $\vec{E}(\vec{r})$ at position $\vec{r}$, produced by a continuous distribution of charge, characterised by a charge density ρ was given by:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq(\vec{r'})}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')$$

This is just an appropriate generalisation of Coulomb's law.

As discussed, the above integral is difficult to evaluate and, in fact, we developed machinery to avoid it. But, *in principle*, this above integral tells us all that we need to know:

How to relate constant electric charges to constant electric fields.

The Biot-Savart law

It there a similar expression in magnetostatics?

Left: Jean Baptiste Biot (1774-1862)
Right: Felix Savart (1791-1841)

We seek an expression that **relates steady currents (*) to constant magnetic fields.**

Such a relationship was discovered in 1820 by Biot and Savart.

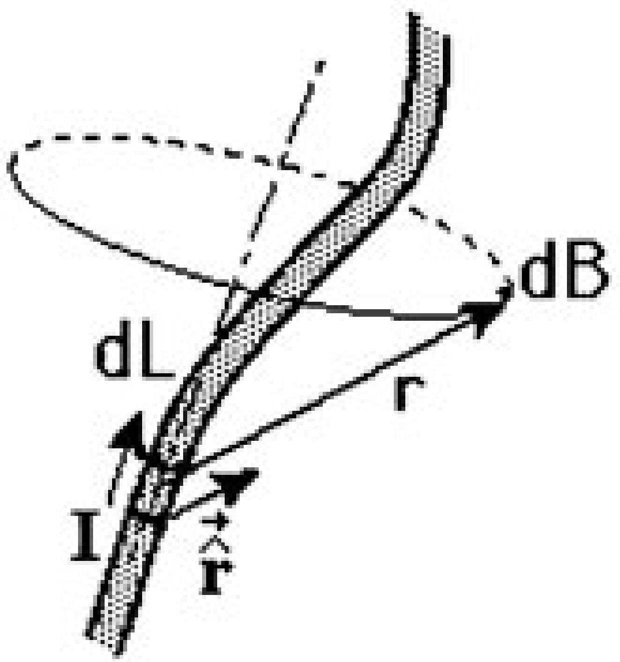
It is now known as the **Biot-Savart law** (pronounced: *bee-oh-suh-vahr*).

(*) Notice that:

- There is no *true* steady current. It is a suitable approximation when changes are slow.
- A moving point charge does not constitute a steady current!
- A steady current *does not pile up*. Therefore $\partial\rho/\partial t = 0$ everywhere in space and hence, recalling the continuity equation, $\vec{\nabla}\vec{j} = 0$.

The Biot-Savart law

Assume a current I flowing along a conducting wire as shown below.



The magnetic field $d\vec{B}$ because of the current flowing through the infinitesimal conducting wire element $d\vec{\ell}$ is given by:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

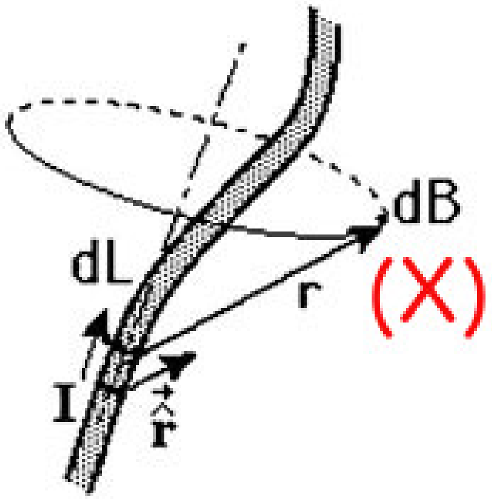
where:

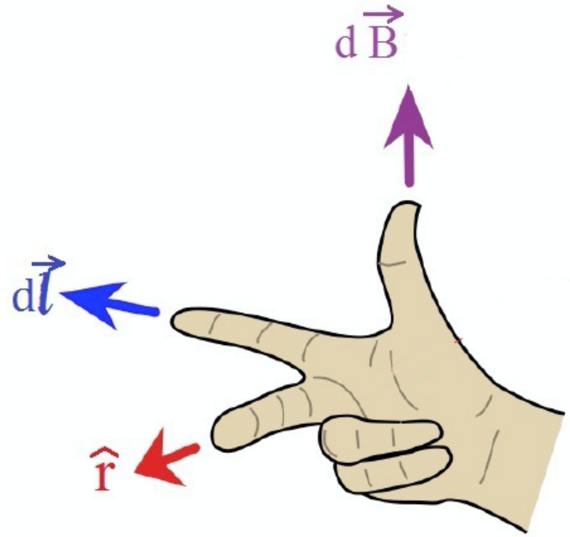
- $\vec{r}$ is the distance from $d\vec{\ell}$ to the point where we want to know the field, and
- μ_0 is a constant ($4\pi \times 10^{-7}\ N/A^2$) called the **permeability of free space**

Notice the $1/r^2$ **dependence** of the magnetic field (similar to the electric field).

The Biot-Savart law

Notice that the direction of $d\vec{B}$ is determined by the cross-product $d\vec{\ell} \times \vec{r}$.





The **superposition principle is valid for magnetostatics**.
The magnetic field $\vec{B}$ produced by the current flowing across the *entire* length L of the conducting wire is the vector sum of the fields due to each infinitesimal element $d\vec{\ell}$:

$$\vec{B} = \int_L d\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

The Biot-Savart law - Other forms

If we want to express the distance between the element $d\vec{\ell}$ and the point where we want to know the field (P) in terms of the distance $\vec{r}'(\vec{r})$ of $d\vec{\ell}(P)$ from the origin O, the previous expression becomes:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{\ell} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

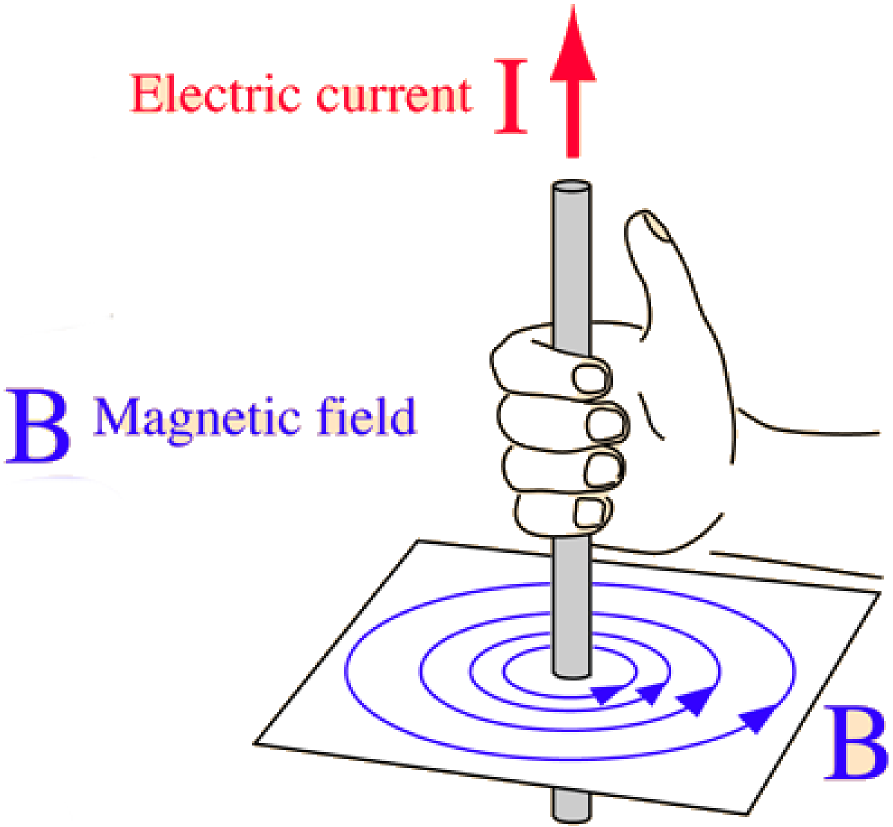
The most general form of the Biot-Savart law, giving the magnetic field $\vec{B}(\vec{r})$ due to an arbitrary volume current density $\vec{j}$ in volume τ is:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_\tau d\tau' \frac{\vec{j}(\vec{r}') \times (\vec{r} - \vec{r}')}{|(\vec{r} - \vec{r}')|^3}$$

For a surface current density $\vec{K}$, on surface S the above simplifies to:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S dS' \frac{\vec{K}(\vec{r}') \times (\vec{r} - \vec{r}')}{|(\vec{r} - \vec{r}')|^3}$$

Magnetic field around a straight wire



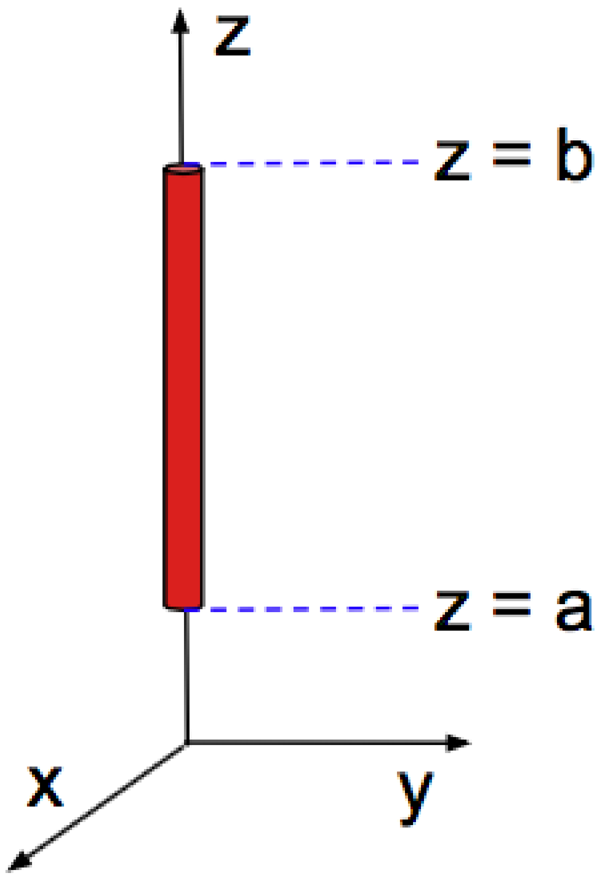
We will use the Biot-Savart law:

$$\vec{B} = \int_L \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

to calculate the magnetic field $\vec{B}$ generated by a *simple* (*) current configuration: **A single straight conducting wire**.

(*) unfortunatelly, even for a simple configuration the magnetic field is not all the simple to calculate analytically.

Magnetic field around a straight wire



Assume that:

- there is current I flowing along a wire in the positive z direction, and
- the conductor has a finite length extending from z=a to z=b
  - later, I will consider a wire of infinite length ($a \to -\infty$ and $b \to +\infty$)

For convenience, without loss of generality, assume that the wire has x=0, y=0.

I will try to **calculate the magnetic field at all points on the plane z=0**.

- since, eventually, I will consider a wire with infinite length, this choice is as good as any.

Magnetic field around a straight wire

The Biot-Savart law states that the magnetic field $\vec{B}$ at a point $\vec{r}$ is:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_a^b \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

The element $d\vec{\ell}$ is along the positive z axis, so I can write it as:

$$d\vec{\ell} = dz\hat{z} = \Big(0, 0, dz\Big)$$

The vector $\vec{r}$ points from the element $d\vec{\ell}$, which is somewhere on the z axis (between z=a and z=b) to a point on the plane z=0. So:

$$\vec{r} = \Big(x, y, 0\Big) - \Big(0, 0, z\Big) = \Big(x, y, -z\Big)$$

The magnitude of $\vec{r}$ is:

$$r = |\vec{r}| = \Big(x^2 + y^2 + (-z)^2\Big)^{1/2} = \Big(x^2 + y^2 + z^2\Big)^{1/2}$$

Magnetic field around a straight wire

We can now evaluate the $d\vec{\ell} \times \vec{r}$ cross product appearing in the Biot-Savart law:

$$d\vec{\ell} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & dz \\ x & y & -z \end{vmatrix} = \begin{vmatrix} 0 & dz \\ y & -z \end{vmatrix} \hat{x} - \begin{vmatrix} 0 & dz \\ x & -z \end{vmatrix} \hat{y} + \begin{vmatrix} 0 & 0 \\ x & y \end{vmatrix} \hat{z} =$$

$$= \Big(- ydz \Big) \hat{x} - \Big(- xdz \Big) \hat{y} + 0\hat{z} \Rightarrow d\vec{\ell} \times \vec{r} = \Big(- y, x, 0 \Big) dz$$

So, substituting everything back into the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_a^b \frac{d\vec{\ell} \times \vec{r}}{r^3} = \frac{\mu_0 I}{4\pi} \int_a^b \frac{\Big(- y, x, 0 \Big)}{(x^2 + y^2 + z^2)^{3/2}} dz$$

In principle we are done, but we now have to evaluate the integrals above.

Magnetic field around a straight wire

First, some observations. The equation:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_a^b \frac{\Big(-y, x, 0 \Big)}{(x^2+y^2+z^2)^{3/2}} dz$$

gives us all 3 components B_x, B_y and B_z of the $\vec{B}$ vector. We notice that the magnetic field has no z component. The field vectors lie on (x,y) plane with components:

$$B_x(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_a^b \frac{-y}{(x^2+y^2+z^2)^{3/2}} dz = -y \frac{\mu_0 I}{4\pi} \int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}}$$

$$B_y(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_a^b \frac{x}{(x^2+y^2+z^2)^{3/2}} dz = x \frac{\mu_0 I}{4\pi} \int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}}$$

where $\rho^2 = x^2 + y^2$ is a *constant* for the integration over dz.

Magnetic field around a straight wire

Using the result we obtained for the integral $\int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}}$ (see Optional reading), we can now calculate the components of $\vec{B}$

$$B_x(\vec{r}) = -y\frac{\mu_0 I}{4\pi}\int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}} = \frac{\mu_0 I}{4\pi}\cdot\frac{-y}{\rho^2}\Big[\frac{z}{(\rho^2+z^2)^{1/2}}\Big]\Big|_{z=a}^{z=b}$$

$$B_y(\vec{r}) = x\frac{\mu_0 I}{4\pi}\int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}} = \frac{\mu_0 I}{4\pi}\cdot\frac{x}{\rho^2}\Big[\frac{z}{(\rho^2+z^2)^{1/2}}\Big]\Big|_{z=a}^{z=b}$$

and

$$B_z(\vec{r}) = 0$$

Putting it all together, we have:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{\rho^2}\Big[\frac{b}{(\rho^2+b^2)^{1/2}} - \frac{a}{(\rho^2+a^2)^{1/2}}\Big]\Big(-y, x, 0\Big)$$

Magnetic field around a straight wire

We will now obtain an approximation of the following equation

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{\rho^2}\Big[\frac{b}{(\rho^2+b^2)^{1/2}} - \frac{a}{(\rho^2+a^2)^{1/2}}\Big]\Big(-y, x, 0\Big)$$

as $a \rightarrow -\infty$ and $b \rightarrow +\infty$. Setting a = -b:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{\rho^2}\frac{2b}{(\rho^2+b^2)^{1/2}}\Big(-y, x, 0\Big)$$

If we allow b to become infinite, then:

$$\lim_{b\rightarrow\infty}\frac{2b}{(\rho^2+b^2)^{1/2}} \overset{b>>\rho}{\approx} \lim_{x\rightarrow\infty}\frac{2b}{(b^2)^{1/2}} = \lim_{x\rightarrow\infty}\frac{2b}{b} = 2$$

and, therefore, the magnetic field $\vec{B}(\vec{r})$ becomes:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho^2}\Big(-y, x, 0\Big)$$

The previous result can be rewritten as:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\Big(-\frac{y}{\rho}, \frac{x}{\rho}, 0\Big)$$

Let's call:

$$\hat{\phi} = \Big(-\frac{y}{\rho}, \frac{x}{\rho}, 0\Big)$$

It can be easily seen that $\hat{\phi}$ is the azimuthal unit vector:

$$\hat{\phi}\cdot\hat{\phi} = 1 \quad \textit{and} \quad \hat{\rho}\cdot\hat{\phi} = 0$$

Our result can be summarised as:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$$

Lecture 5 - Main points to remember

- An **electric current is a flow of electric charge.** It is represented by the amount of charge passing though per unit time.

$$I = \frac{dQ}{dt}$$

  In SI, the unit of the electric current is the **Ampere (A)**.

- The current density $\vec{j}$ is the **current per unit area of cross-section**:

$$\vec{j} = nq\vec{u}_d$$

  where n is the charge carrier density and $\vec{u}_d$ their average velocity.

- In general:

$$\vec{j} = \sigma\vec{E}$$

  where σ is the **conductivity** of the material (SI unit: $1/(\Omega \cdot m)$). The inverse quantity $\rho = 1/\sigma$ is called **resistivity**.

Lecture 5 - Main points to remember (cont'd)

- Magnetic and electric phenomena have a common origin. Remember the empirical evidence:
  - Electric currents generate magnetic fields!
  - Moving magnetic fields generate electric currents!
  - There are magnetic forces between electric currents!
- The magnetic field (a vector field) is the magnetic effect of electric currents and magnetic materials (SI unit: **Tesla (T)**)
- The magnetic force on an electric charge q moving with velocity $\vec{u}$ in a magnetic field $\vec{B}$ is given by: $\vec{F} = q\vec{u} \times \vec{B}$
- Consequently, the magnetic force on a current is $\vec{F} = I \int_L d\vec{\ell} \times \vec{B}$
- Magnetic forces do no work on electric charges.
- In the presence of both a magnetic field $\vec{B}$ and an electric field $\vec{E}$, the total (so-called Lorentz) force on charge q is: $\vec{F} = q\Big(\vec{E} + \vec{u} \times \vec{B}\Big)$

Lecture 5 - Main points to remember (cont'd)

- Biot-Savart law (expresses $\vec{B}$ in terms of the current I):

$$\vec{B} = \int_L d\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

where the integral is over the elements $d\vec{\ell}$ along the conductor, and $\vec{r}$ is the distance from $d\vec{\ell}$ to the point where we want to know the field.

- Biot-Savart in action: Magnetic field around a wire with current I:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$$

where ρ is the distance from the wire and $\hat{\phi}$ the azimuthal unit vector.

At the next lecture (Lecture 6)

We will continue studying **magnetostatics**

- Magnetic force between two parallel conductors
- Magnetic dipole moments
- Principles of DC motors
- The curl and divergence of the magnetic field
- The vector potential

Optional reading for Lecture 5

Estimating $\int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}}$

We can prove that:

$$\int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}} = \frac{z}{\rho^2(z^2+\rho^2)^{1/2}}\Bigg|_a^b$$

One way to calculate this integral is to change variables ($z \rightarrow u$) and perform an integration over the variable u. A clever variable transformation will leave us with a much simpler integral to calculate.
Let's try the following variable transformation:

$$z \rightarrow u = tan^{-1}\Big(\frac{z}{\rho})\Big)$$

Therefore:

$$z = \rho tan(u) \quad and \quad dz = \rho\Big(tan(u)\Big)du \Rightarrow dz = \frac{\rho}{cos^2(u)}du$$

Estimating $\int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}}$

With that variable transformation, the integrand becomes:

$$\frac{1}{(\rho^2+z^2)^{3/2}} \rightarrow \frac{1}{(\rho^2 tan^2(u)+\rho^2)^{3/2}} = \frac{1}{\rho^3(tan^2(u)+1)^{3/2}} = \frac{1}{\rho^3(\frac{sin^2(u)}{cos^2(u)}+1)^{3/2}} =$$

$$= \frac{1}{\rho^3(\frac{sin^2(u)+cos^2(u)}{cos^2(u)})^{3/2}} = \frac{1}{\rho^3(\frac{1}{cos^2(u)})^{3/2}} = \frac{cos^3(u)}{\rho^3}$$

Therefore:

$$\int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}} = \int_{u(a)}^{u(b)} \Big(\frac{cos^3(u)}{\rho^3}\Big)\Big(\frac{\rho}{cos^2(u)}du\Big) = \frac{1}{\rho^2}\int_{u(a)}^{u(b)} cos(u)du =$$

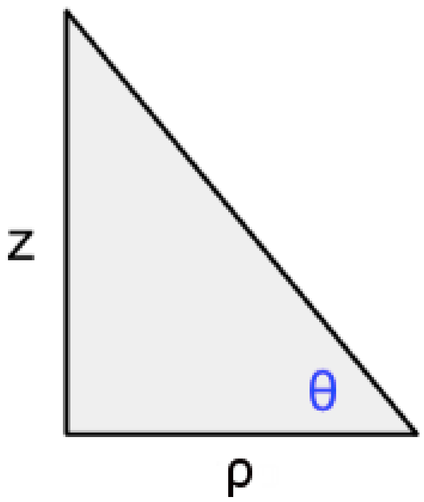
$$= \frac{1}{\rho^2} sin(u)\Big|_{u(a)}^{u(b)} = \frac{1}{\rho^2} sin\Big[tan^{-1}\Big((\frac{z}{\rho})\Big)\Big]\Big|_a^b$$

Estimating $\int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}}$

In order to evaluate the term

$$sin\left[tan^{-1}\left(\frac{z}{\rho}\right)\right]\Big|_a^b$$

appearing in the previous expression, consider the triangle below.



We have:

$$tan(\theta) = \frac{z}{\rho} \Rightarrow \theta = tan^{-1}\left(\frac{z}{\rho}\right) \Rightarrow sin(\theta) = sin\left[tan^{-1}\left(\frac{z}{\rho}\right)\right]$$

But:

$$sin(\theta) = \frac{z}{(z^2+\rho^2)^{1/2}}$$

Therefore:

$$sin\left[tan^{-1}\left(\frac{z}{\rho}\right)\right] = \frac{z}{(z^2+\rho^2)^{1/2}}$$

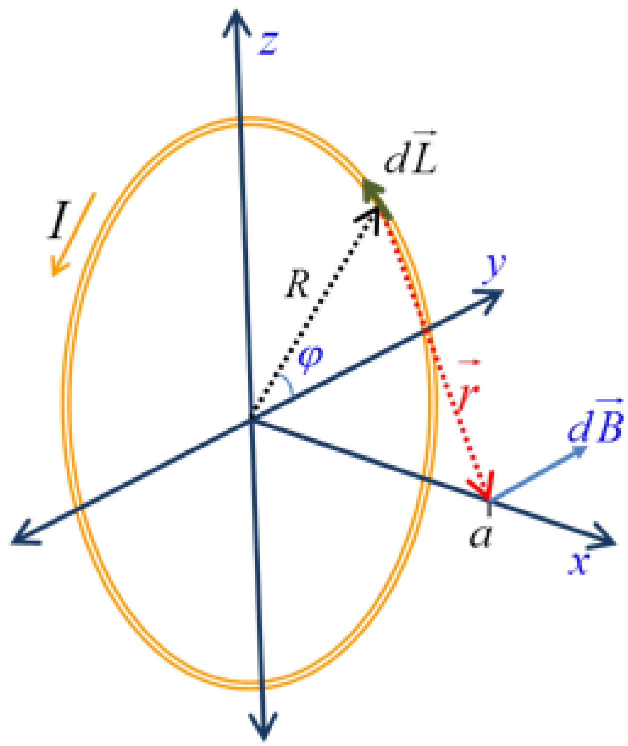
So, indeed, we showed that:

$$\int_a^b \frac{dz}{(\rho^2+z^2)^{3/2}} = \frac{z}{\rho^2(z^2+\rho^2)^{1/2}}\Bigg|_a^b$$

Worked example: Magnetic field of circular loop

Question

Calculate the magnetic field along the axis of a circular loop carrying a steady current I.



Worked example: Magnetic field of circular loop

To calculate the magnetic field $\vec{B}$ produced by the circular loop, we will use the Biot-Savart law:

$$\vec{B} = \int_L d\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

The loop axis is the x-axis. For a point at distance a on that axis, the location vector $\vec{r}_{axis}$ can be written as:

$$\vec{r}_{axis} = \Big(a, 0, 0\Big)$$

The circular loop is on the $y - z$ plane ($x = 0$). The location vector $\vec{r}_{loop}$ for any point on the loop is:

$$\vec{r}_{loop} = \Big(0, Rcos\phi, Rsin\phi\Big)$$

The vector $\vec{r}$ connecting points on the loop and the loop axis is:

$$\vec{r} = \vec{r}_{axis} - \vec{r}_{loop} = \Big(a, 0, 0\Big) - \Big(0, Rcos\phi, Rsin\phi\Big) = \Big(a, -Rcos\phi, -Rsin\phi\Big)$$

Worked example: Magnetic field of circular loop

Therefore, the magnitude of $\vec{r}$ is:

$$r = |\vec{r}| = \left(a^2 + R^2 cos^2\phi + R^2 sin^2\phi\right)^{1/2} = \left(a^2 + R^2\right)^{1/2}$$

The element $d\vec{\ell}$ that is tangential to the loop can be written as:

$$d\vec{\ell} = \left(0, -Rsin\phi, Rcos\phi\right) d\phi$$

The cross-product of $d\vec{\ell}$ and $\vec{r}$ is:

$$d\vec{\ell} \times \vec{r} = \left(0, -Rsin\phi, Rcos\phi\right) \times \left(a, -Rcos\phi, -Rsin\phi\right) d\phi \Rightarrow$$

$$d\vec{\ell} \times \vec{r} = \left(R^2, aRcos\phi, aRsin\phi\right) d\phi$$

Worked example: Magnetic field of circular loop

Substituting everything into the Biot-Savart equation, we have:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{\ell} \times \vec{r}}{r^3} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{\Big(R^2, aRcos\phi, aRsin\phi\Big)}{\Big(a^2 + R^2\Big)^{3/2}} d\phi \Rightarrow$$

$$\vec{B} = \frac{\mu_0 I}{4\pi\Big(a^2 + R^2\Big)^{3/2}} \int_0^{2\pi} \Big(R^2, aRcos\phi, aRsin\phi\Big) d\phi \Rightarrow$$

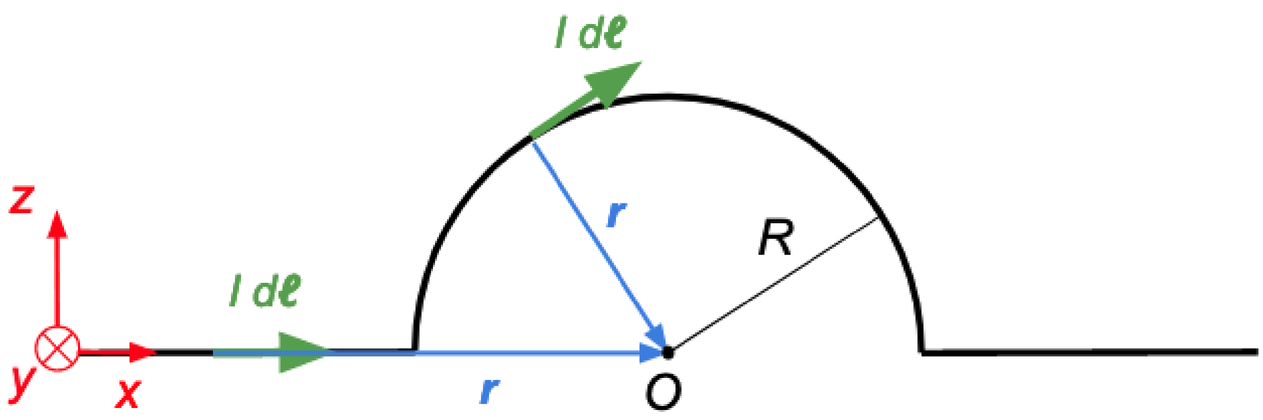
$$\vec{B} = \frac{\mu_0 I}{4\pi\Big(a^2 + R^2\Big)^{3/2}} \Big(R^2 \int_0^{2\pi} d\phi, aR \int_0^{2\pi} cos\phi d\phi, aR \int_0^{2\pi} sin\phi d\phi\Big) \Rightarrow$$

$$\vec{B} = \frac{\mu_0 I R^2}{2\Big(a^2 + R^2\Big)^{3/2}} \Big(1, 0, 0\Big)$$

Worked example: Field of straight wire semi-circular bent

Question

An infinitely long wire carries a current I. It is bent so as to have a semi-circular detour around the origin, with radius R. Calculate the magnetic field $\vec{B}$ at the origin.



The field $\vec{B}$ of a current element $Id\vec{\ell}$ is given by the Biot-Savart law:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

Worked example: Field of straight wire semi-circular bent

The straight parts of the wire do not contribute to the magnetic field at the origin O since, as it can be seen from the schematic above, for these parts of the wire, $d\vec{\ell}$ and $\vec{r}$ are parallel and, therefore:

$$d\vec{\ell} \times \vec{r} = 0$$

Only the semicircular bent contributes to the magnetic field at the origin. On that bent, $d\vec{\ell}$ and $\vec{r}$ are always perpendicular to each other and the law of Biot-Savart can be simplified as:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{r d\ell}{r^3} \hat{y} = \frac{\mu_0 I}{4\pi} \frac{d\ell}{r^2} \hat{y} \xrightarrow{r=R, d\ell=Rd\theta} d\vec{B} = \frac{\mu_0 I}{4\pi R} d\theta \hat{y}$$

Thefore, the magnetic field due to the full semi-circular bent is given by:

$$\vec{B} = \int_{semicircle} d\vec{B} = \frac{\mu_0 I}{4\pi R} \left(\int_0^{\pi} d\theta \right) \hat{y} \Rightarrow \vec{B} = \frac{\mu_0 I}{4R} \hat{y}$$

Worked example: Field of rotating cylindrical shell

Question

Consider a cylindrical shell of radius R and length L, carrying charge with surface charge density σ and rotating with angular velocity ω.
Find the mangnetic field $\vec{B}$ at point P, at distance d from the end of the shell, on the cylinder axis.

The most general form of the Biot-Savart law, giving the magnetic field $\vec{B}(\vec{r})$ due to an arbitrary volume current density $\vec{j}$ is:

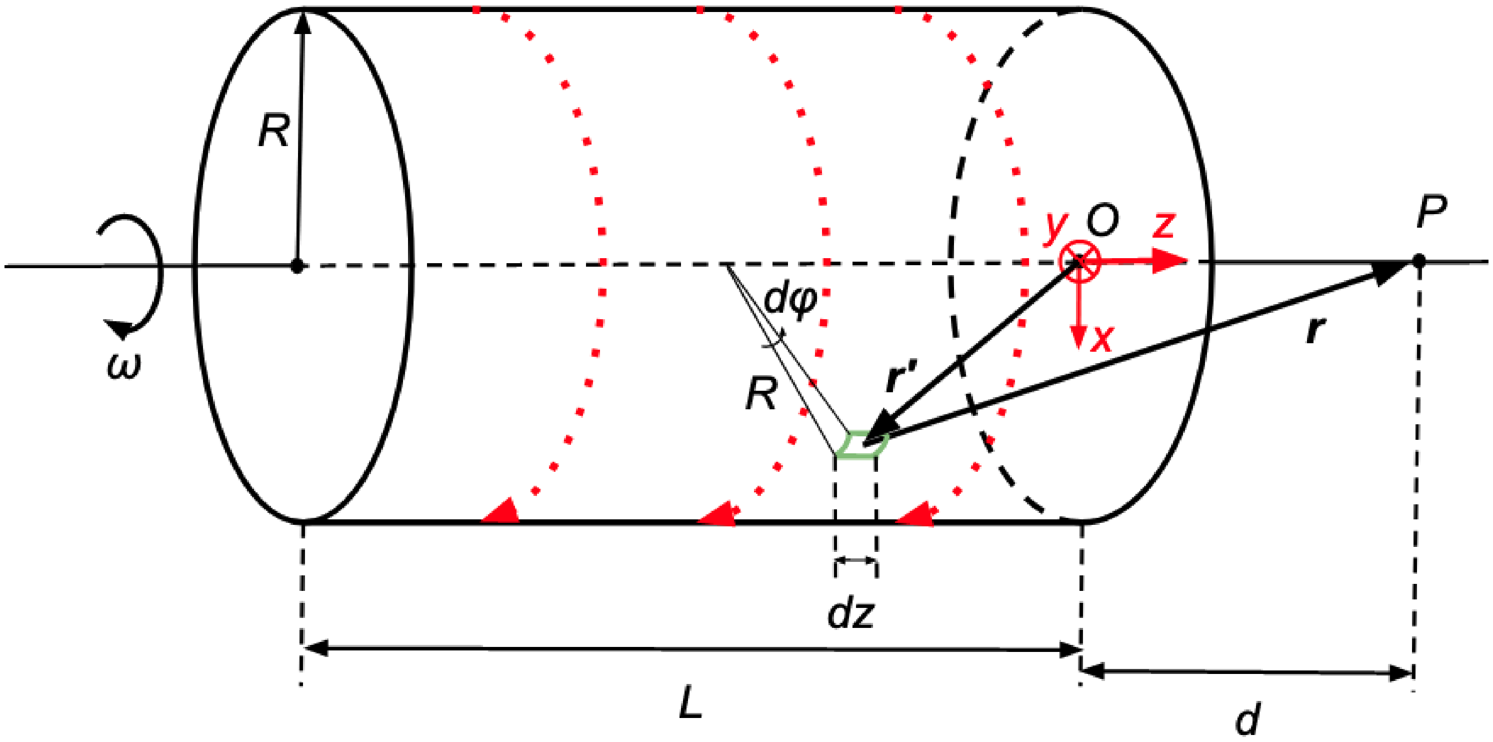
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_\tau d\tau' \frac{\vec{j}(\vec{r}') \times (\vec{r} - \vec{r}')}{|(\vec{r} - \vec{r}')|^3}$$

where the integration is over volume τ.
For a surface current density $\vec{K}$, on surface S the above simplifies to:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S dS' \frac{\vec{K}(\vec{r}') \times (\vec{r} - \vec{r}')}{|(\vec{r} - \vec{r}')|^3}$$

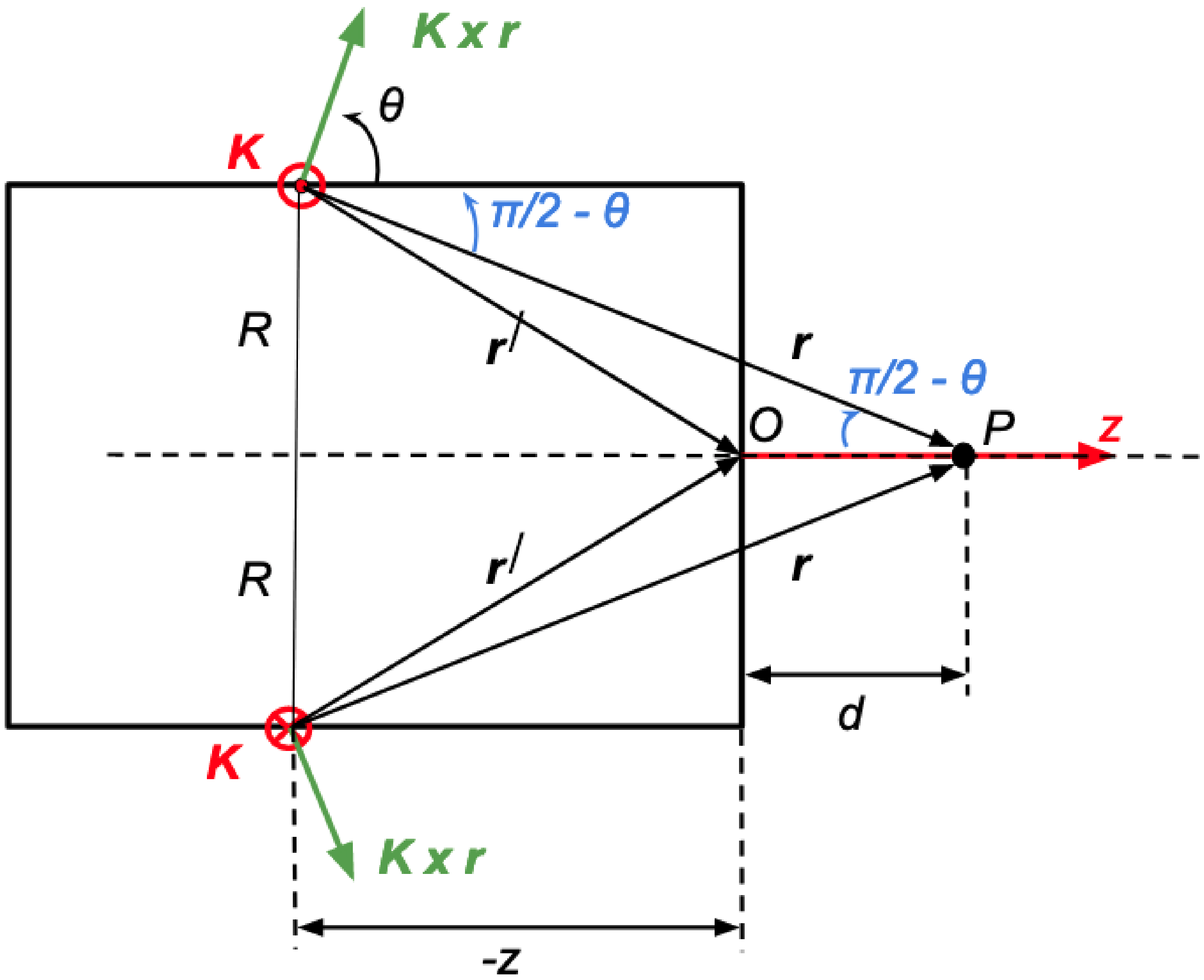
The previous expression, using the coordinate system, and the definitions of vectors $\vec{r}$ and $\vec{r}'$ shown in the figure below, yields the following expression for the magnetic field $\vec{B}$ at point P.

$$\vec{B} = \frac{\mu_0}{4\pi} \int_{cylinder} dS' \frac{\vec{K}(\vec{r}') \times \hat{r}}{r^2}$$



Worked example: Field of rotating cylindrical shell

It is useful to draw a projection of the previous schematic on the *xz* plane. Angles and distances which are going to be used later, are defined here.



Worked example: Field of rotating cylindrical shell

If $\vec{u}$ is the velocity of each point on the surface of the cylinder, the surface current density $\vec{K}$ is given by:

$$\vec{K} = \sigma \vec{u} = \sigma \omega R \hat{\phi}$$

where $\hat{\phi}$ is the azimuthal unit vector.
Since $\vec{K}(\vec{r}')$ and $\hat{r}$ are perpendicular, the magnitude of $\vec{K}(\vec{r}') \times \hat{r}$ is:

$$|\vec{K}(\vec{r}') \times \hat{r}| = |\vec{K}(\vec{r}')| \cancelto{1}{|\hat{r}|} \cancelto{1}{sin\frac{\pi}{2}} = |\vec{K}(\vec{r}')| = \sigma \omega R$$

Due to symmetry, only the z component of $\vec{K}(\vec{r}') \times \hat{r}$ will contribute to the magnetic field, as the other components cancel out. The z component is:

$$\vec{K}(\vec{r}') \times \hat{r}|_z = \sigma \omega R cos\theta$$

Worked example: Field of rotating cylindrical shell

From the schematic shown a projection of the cylinder on the xz plane, we can see that:

$$cos\theta = sin(\frac{\pi}{2} - \theta) = \frac{R}{r}$$

Therefore, the only non-vanishing contribution of $\vec{K}(\vec{r}') \times \hat{r}$ to the magnetic field at P can be written as:

$$\vec{K}(\vec{r}') \times \hat{r} = \frac{\sigma\omega R^2}{r}\hat{z}$$

Substituting the above into Biot-Savart's expression for the magnetic field at P, and using $dS' = Rd\phi dz$, we find:

$$\vec{B} = \hat{z}\frac{\mu_0\sigma\omega R^3}{4\pi}\int_{cylinder} d\phi dz \frac{1}{r^3} = \hat{z}\frac{\mu_0\sigma\omega R^3}{2}\int_{-L}^{0}\frac{dz}{r^3}$$

Worked example: Field of rotating cylindrical shell

To carry out the integration, we need to express r in terms of z. Looking the previous schematic, we see that:

$$r^2 = R^2 + (d+z)^2$$

Therefore, the magnetic field $\vec{B}$ at P is given by:

$$\vec{B} = \hat{z}\frac{\mu_0 \sigma \omega R^3}{2} \int_{-L}^{0} \frac{dz}{\left(R^2 + (d+z)^2\right)^{3/2}} \xrightarrow{\ell = d+z}$$

$$\vec{B} = \hat{z}\frac{\mu_0 \sigma \omega R^3}{2} \int_{d-L}^{d} \frac{d\ell}{\left(R^2 + \ell^2\right)^{3/2}}$$

Earlier in this lecture (field of infinite straight conductor), we proved that:

$$\int_a^b \frac{dz}{(\rho^2 + z^2)^{3/2}} = \frac{z}{\rho^2 (z^2 + \rho^2)^{1/2}} \Bigg|_a^b$$

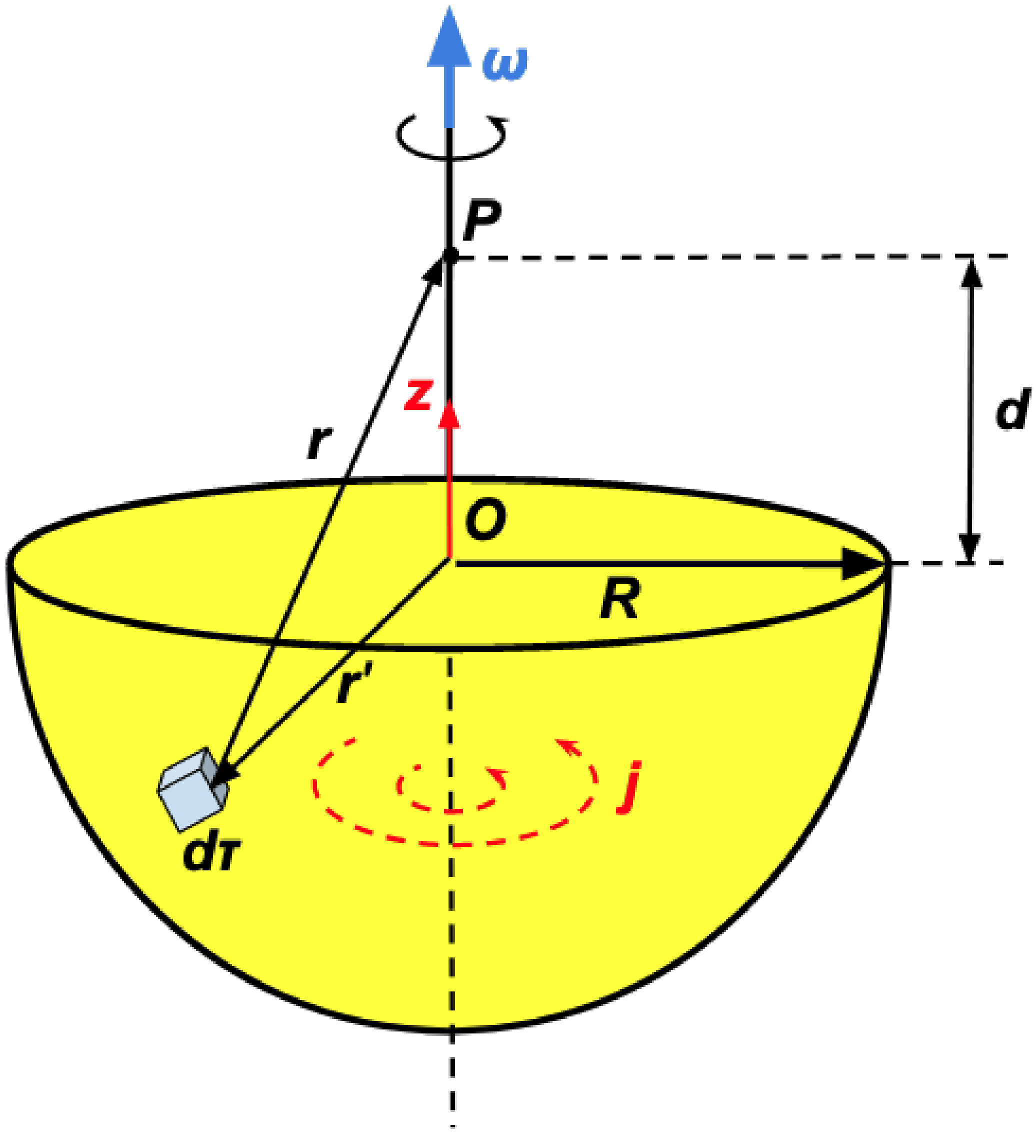
Using the above expression to help us evaluate the integral, we find for the magnetic field $\vec{B}$ at P:

$$\vec{B} = \hat{z}\frac{\mu_0 \sigma \omega R^3}{2} \frac{\ell}{R^2(\ell^2 + R^2)^{1/2}}\Bigg|_{d-L}^{L} \Rightarrow$$

$$\vec{B} = \hat{z}\frac{\mu_0 \sigma \omega R^3}{2} \Bigg\{ \frac{L}{R^2(L^2 + R^2)^{1/2}} - \frac{(d-L)}{R^2((d-L)^2 + R^2)^{1/2}} \Bigg\} \Rightarrow$$

$$\vec{B} = \hat{z}\frac{\mu_0 \sigma \omega R}{2} \Bigg\{ \frac{L}{\sqrt{L^2 + R^2}} - \frac{(d-L)}{\sqrt{(d-L)^2 + R^2}} \Bigg\}$$

Worked example: Field of rotating hemisphere



If you have calculated the magnetic field of a rotating cyclindrical shell, you can also try to calculate the field of hemisphere of radius R and volume charge density ρ, rotating with angular velocity ω, at distance d above its centre.

Worked example: Field of current element

Question

A current element $Id\vec{\ell}$ is located at the origin. The current is in the direction of the $+z$ axis. Is the x component of the the field at a point $P(x, y, z)$:

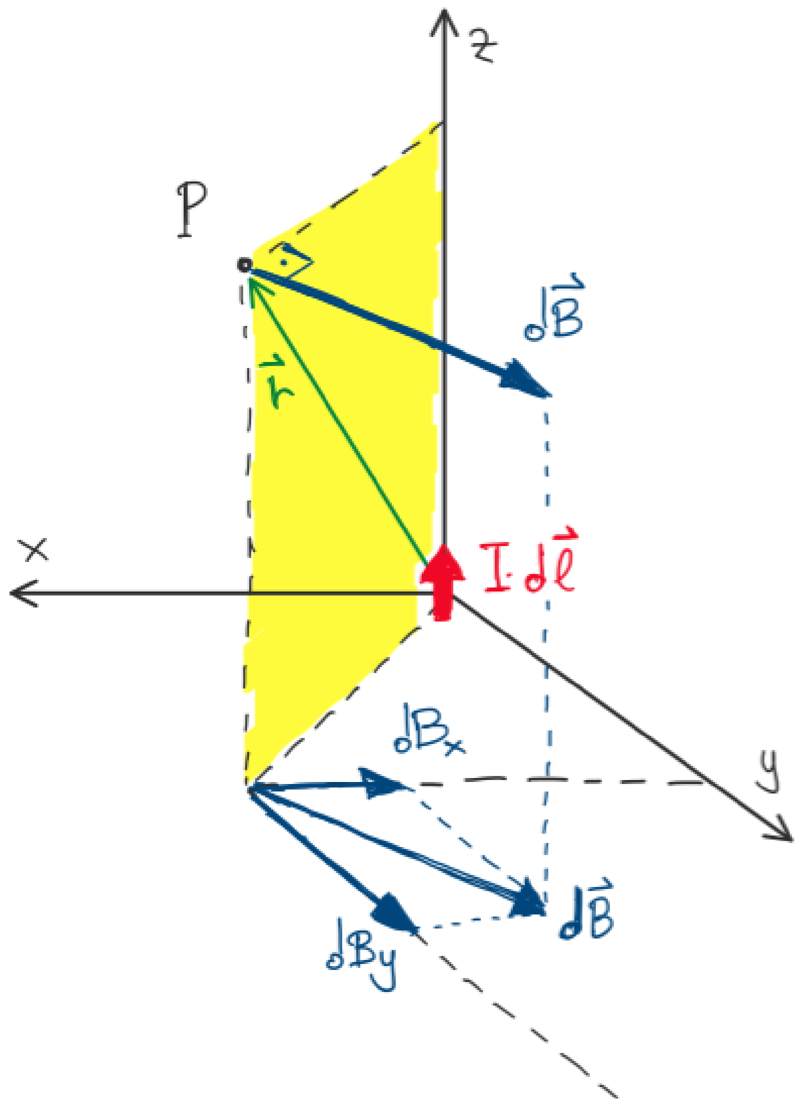
- 0,
- $\frac{\mu_0 I}{4\pi} \frac{y}{(x^2 + y^2 + z^2)^{3/2}} d\ell$, or
- $\frac{\mu_0 I}{4\pi} \frac{x}{(x^2 + y^2 + z^2)^{3/2}} d\ell$?

Justify your answer.

The magnetic field $\vec{B}$ of the current element $Id\vec{\ell}$ is given by the Biot-Savart law:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

Worked example: Field of current element



$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

where, for the given problem (see schematic on the left):

$$d\vec{\ell} = \Big(0, 0, d\ell\Big)$$

$$\vec{r} = \Big(x, y, z\Big)$$

$$r = |\vec{r}| = \Big(x^2 + y^2 + z^2\Big)^{1/2}$$

Worked example: Field of current element

The cross-product $d\vec{\ell} \times \vec{r}$ can be written as:

$$d\vec{\ell} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & d\ell \\ x & y & z \end{vmatrix} = \begin{vmatrix} 0 & d\ell \\ y & z \end{vmatrix} \hat{x} - \begin{vmatrix} 0 & d\ell \\ x & z \end{vmatrix} \hat{y} + \begin{vmatrix} 0 & 0 \\ x & y \end{vmatrix} \hat{z}$$

$$= \Big(-yd\ell \Big)\hat{x} - \Big(-xd\ell \Big)\hat{y} + 0\hat{y} = \Big(-y, x, 0 \Big) d\ell$$

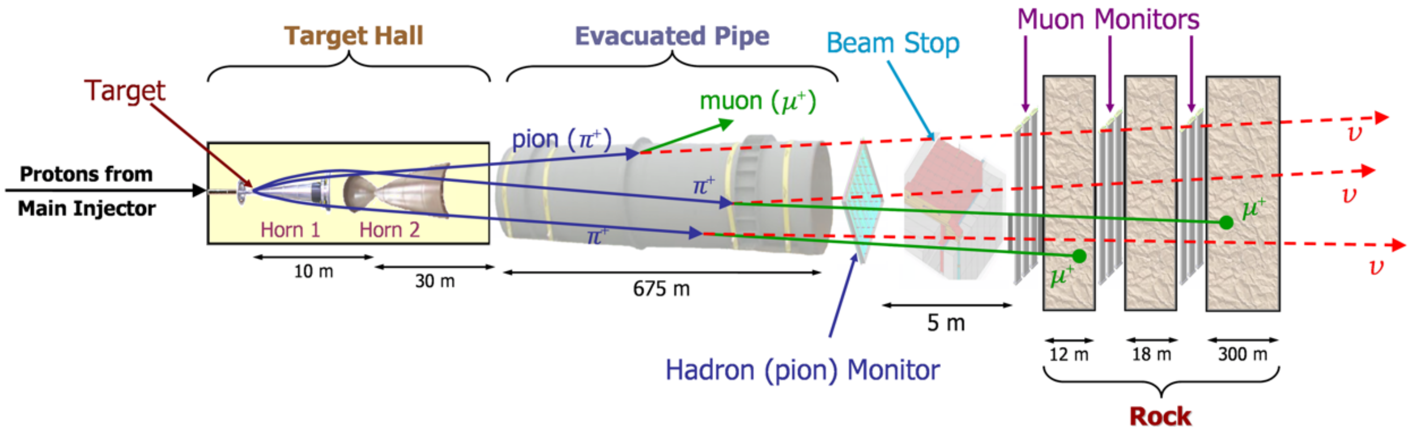
Therefore, the magnetic field at P, due to the element $Id\vec{\ell}$, is:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\Big(-y, x, 0 \Big) d\ell}{\Big(x^2 + y^2 + z^2 \Big)^{3/2}}$$

which has an x-component given by:

$$dB_x = -\frac{\mu_0 I}{4\pi} \frac{yd\ell}{\Big(x^2 + y^2 + z^2 \Big)^{3/2}}$$

Background information: **Making a (conventional) neutrino beam**

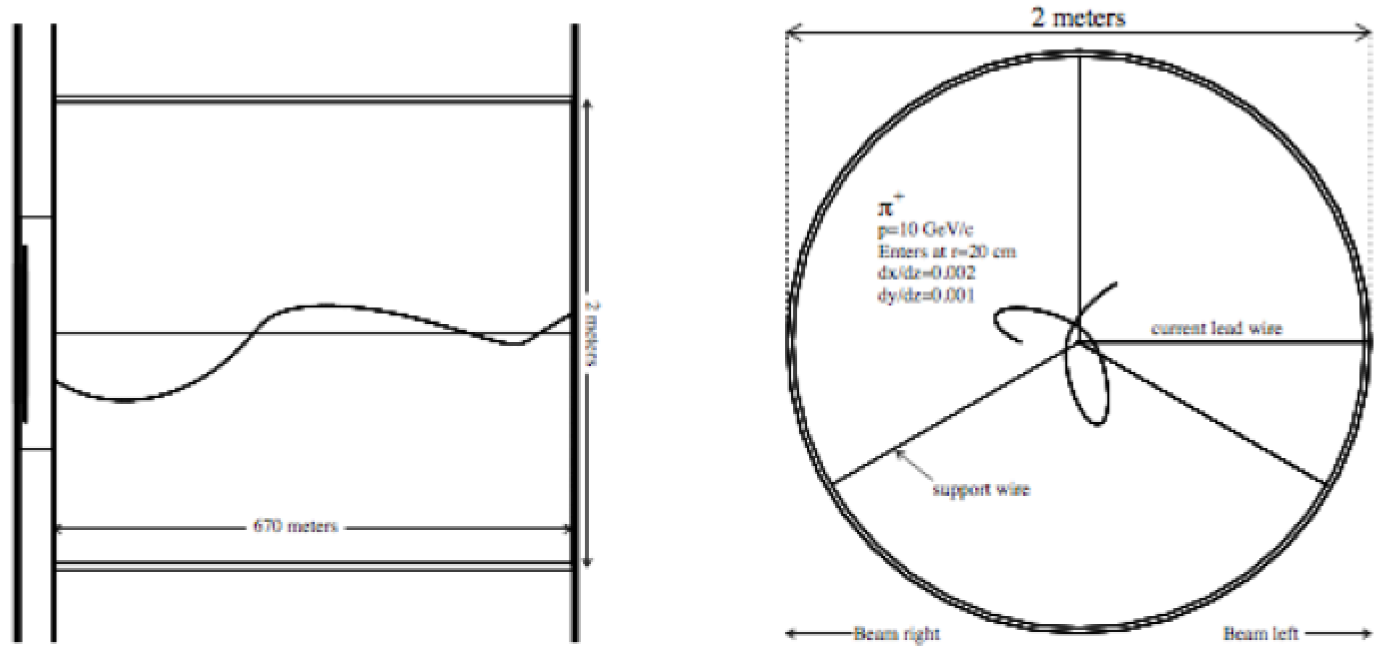


Where do our ν's come from?

$$\pi^+ \to \nu_\mu + \mu^+$$
$$\pi^- \to \bar{\nu}_\mu + \mu^-$$
$$\mu^+ \to \bar{\nu}_\mu + \nu_e + e^+$$
$$\mu^- \to \bar{\nu}_e + \nu_\mu + e^-$$
$$K^+ \to \nu_\mu + \mu^+$$
$$K^+ \to \nu_e + \pi^0 + e^+$$
$$K^+ \to \nu_\mu + \pi^0 + \mu^+$$
$$K^- \to \bar{\nu}_\mu + \mu^-$$
$$K^- \to \bar{\nu}_e + \pi^0 + e^-$$
$$K^- \to \bar{\nu}_\mu + \pi^0 + \mu^-$$
$$K^0_L \to \bar{\nu}_\mu + \pi^+ + \mu^-$$
$$K^0_L \to \nu_\mu + \pi^- + \mu^+$$
$$K^0_L \to \bar{\nu}_e + \pi^+ + e^-$$
$$K^0_L \to \nu_e + \pi^- + e^+$$

Background information: **The hadron hose**

This is an additional **focusing element**. It is a wire located in the decay pipe, downstream of the target and focussing horns. The wire is pulsed with current creating a toroidal magnetic field within the decay volume.
See J.Hylen et al., https://arxiv.org/pdf/hep-ex/0210051.pdf



Sample orbit for a 10 GeV π^+.

PHYS201 scientific programming task for Lecture 5

- Write a program to **visualise trajectories of charged particles** moving in the magnetic field produced by the hadron hose.
- In this lecture, we studied the two main elements for this calculation.
  - The magnetic field produced by a long wire.
  - The magnetic force on a moving electric charge.
- The Runge-Kutta method (look it up) can be used for solving the equation of motion numerically and calculating the trajectories.
- Assume that the decay pipe has a diameter of 2 m and a length of 700 m. Positively-charged pions, with a lifetime of 2.6×10^{-8} μs enter the decay pipe from the centre of its upstream face.
  The pions have energies which are uniformly distributed in the 1-5 GeV range, and their direction is isotropic in the $\theta = \pm 10^{o}$ range, where θ is the angle between the hose and the pion direction.
  Optimize the hose current, so as to maximize the number of pion decays within the decay volume and, hence, the neutrino flux.

PHYS 201 / Lecture 6
Force between conductors; Magnetic dipole moment; DC motors; Field of a toroidal coil; Curl and divergence of the magnetic field; Vector potential

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UNIVERSITY OF LIVERPOOL

Science & Technology Facilities Council
Rutherford Appleton Laboratory

Lecture 5 - Revision

- An **electric current is a flow of electric charge.** It is represented by the amount of charge passing though per unit time.

$$I = \frac{dQ}{dt}$$

  In SI, the unit of the electric current is the **Ampere (A)**.

- The current density $\vec{j}$ is the **current per unit area of cross-section**:

$$\vec{j} = nq\vec{u}_d$$

  where n is the charge carrier density and $\vec{u}_d$ their average velocity.

- In general:

$$\vec{j} = \sigma \vec{E}$$

  where σ is the **conductivity** of the material (SI unit: $1/(\Omega \cdot m)$). The inverse quantity $\rho = 1/\sigma$ is called **resistivity**.

Lecture 5 - Revision (cont'd)

- Magnetic and electric phenomena have a common origin. Remember the empirical evidence:
  - Electric currents generate magnetic fields!
  - Moving magnetic fields generate electric currents!
  - There are magnetic forces between electric currents!
- The magnetic field (a vector field) is the magnetic effect of electric currents and magnetic materials (SI unit: **Tesla (T)**)
- The magnetic force on an electric charge q moving with velocity $\vec{u}$ in a magnetic field $\vec{B}$ is given by: $\vec{F} = q\vec{u} \times \vec{B}$
- Consequently, the magnetic force on a current is $\vec{F} = I \int_L d\vec{\ell} \times \vec{B}$
- Magnetic forces do no work on electric charges.
- In the presence of both a magnetic field $\vec{B}$ and an electric field $\vec{E}$, the total (so-called Lorentz) force on charge q is: $\vec{F} = q\Big(\vec{E} + \vec{u} \times \vec{B}\Big)$

Lecture 5 - Revision (cont'd)

- Biot-Savart law (expresses $\vec{B}$ in terms of the current I):

$$\vec{B} = \int_L d\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

where the integral is over the elements $d\vec{\ell}$ along the conductor, and $\vec{r}$ is the distance from $d\vec{\ell}$ to the point where we want to know the field.

- Biot-Savart in action: Magnetic field around a wire with current I:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$$

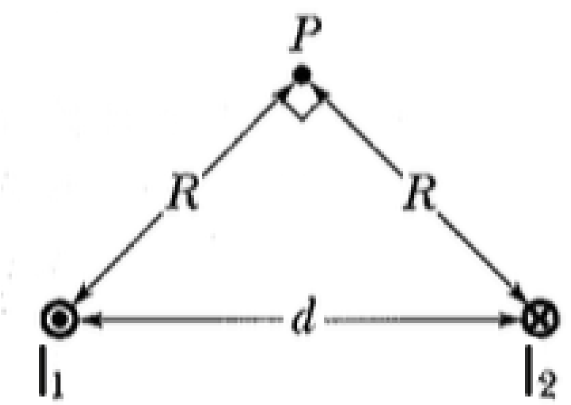
where ρ is the distance from the wire and $\hat{\phi}$ the azimuthal unit vector.

Plan for Lecture 6

- *Expand on concepts studied in the previous lecture and study worked examples.*
- Force between two parallel conductors
  - Definition of the Ampere
- Current loop
- Torque on a current loop
- Magnetic dipole moment
- The (DC) electric motor
- The curl and circulation of the magnetic field: Ampere's law
- The divergence and the flux of the magnetic field
- The vector potential

Worked example

Question



The figure on the left shows two long parallel wires carrying currents I_1 and I_2 in opposite directions. What are the magnitude and direction of the net magnetic field at point P?
Assume the following values: $I_1 = 15$ A, $I_2 = 32$ A, and d $= 5.3$ cm.

The net magnetic field $\vec{B}$ at point P is the vector sum of the magnetic fields due to the currents in the two wires.
The magnetic field $\vec{B}_k$ around a wire k with current I_k:

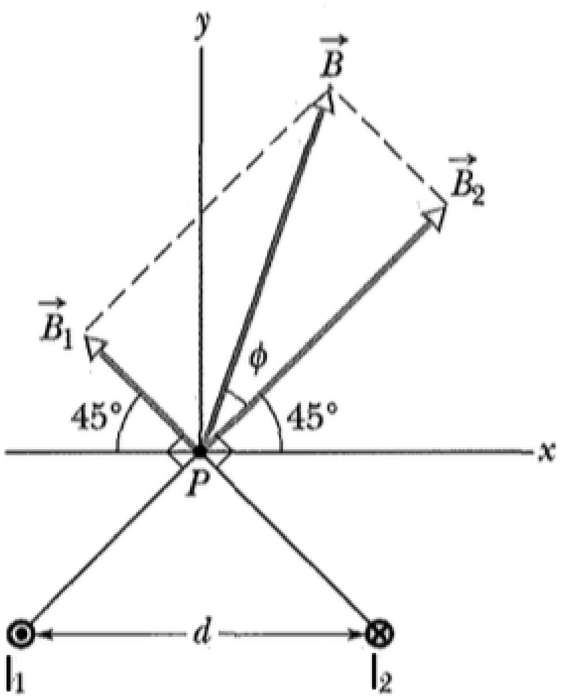
$$\vec{B}_k(\vec{r}) = \frac{\mu_0 I_k}{2\pi\rho}\hat{\phi}$$

where ρ is the distance from the wire and $\hat{\phi}$ the azimuthal unit vector.

Worked example

The point P is at a distance R from both currents I_1 and I_2. Those currents produce magnetic fields $\vec{B}_1$ and $\vec{B}_2$ with magnitudes:

$$B_1 = \frac{\mu_0 I_1}{2\pi R} \;,\quad B_2 = \frac{\mu_0 I_2}{2\pi R}$$



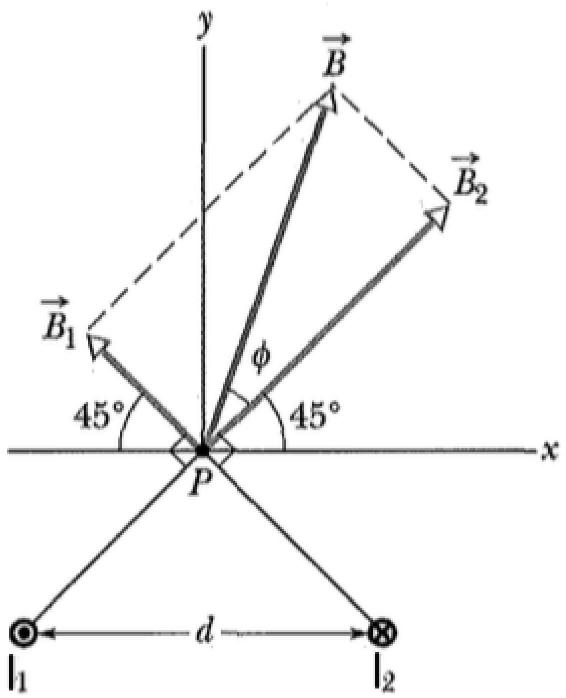
The fields $\vec{B}_1$ and $\vec{B}_2$ have the azimuthal directions shown on the left (notice the current direction and use the right-hand rule).
Note that the angles between sides R an d are 45^o.
The distance R is given in terms of the given distance d by:

$$R = dcos\frac{\pi}{4}$$

Therefore:

$$B_1 = \frac{\mu_0 I_1}{2\pi dcos\frac{\pi}{4}} \;,\quad B_2 = \frac{\mu_0 I_2}{2\pi dcos\frac{\pi}{4}}$$

Worked example



The net magnetic field $\vec{B}$ is given by:

$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

The magnitude of $\vec{B}$ is:

$$B = \sqrt{B_1^2 + B_2^2} = \frac{\mu_0}{2\pi d cos\frac{\pi}{4}}\sqrt{I_1^2 + I_2^2} \Rightarrow$$

$$B = \frac{(4\pi \times 10^{-7}\ T \cdot m/A)}{(2\pi)(5.3 \times 10^{-2}\ m)(cos\frac{\pi}{4})}\sqrt{(15\ A)^2 + (32\ A)^2} \Rightarrow$$

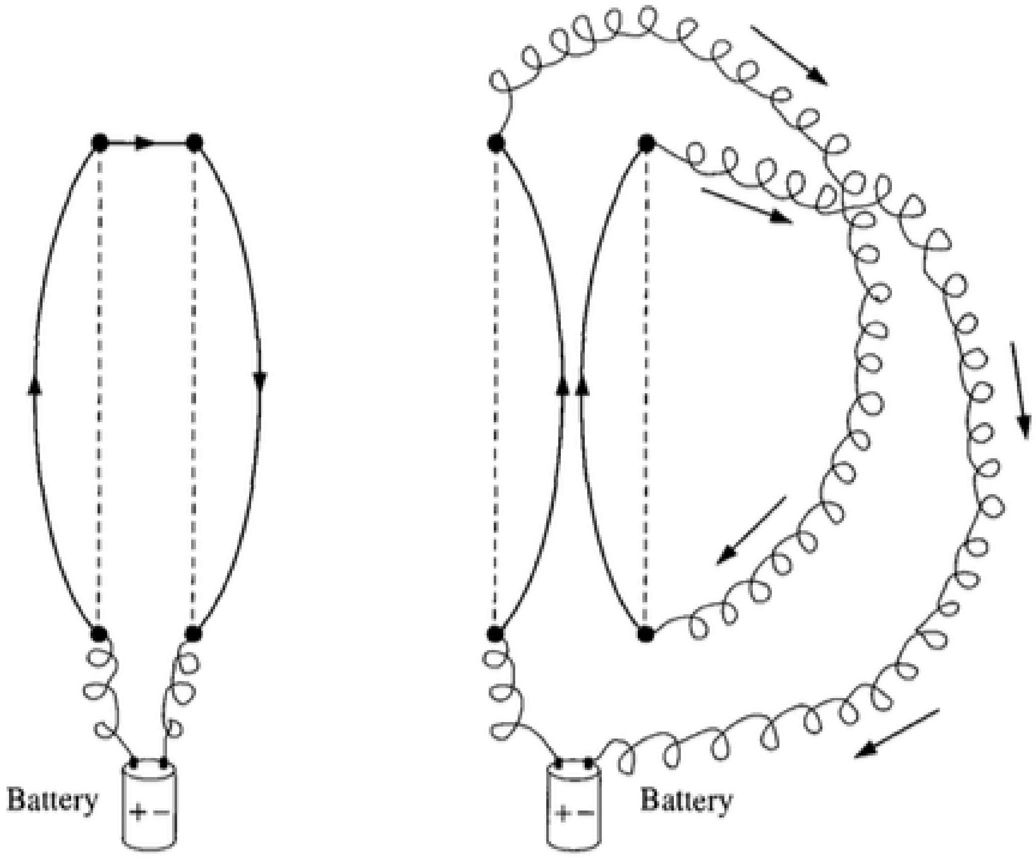
$$B \approx 190\mu T$$

The angle ϕ between the directions of $\vec{B}$ and $\vec{B}_2$ is given by:

$$\tan\phi = \frac{B_1}{B_2} \Rightarrow \tan\phi = \frac{I_1}{I_2} \Rightarrow \tan\phi = \frac{15\ A}{32\ A} \Rightarrow \tan\phi = \frac{15\ A}{32\ A} \Rightarrow \phi = 25^o$$

Therefore, the angle between $\vec{B}$ and the x-axis is: $\phi + 45^0 = 70^o$.

Force between two parallel conductors



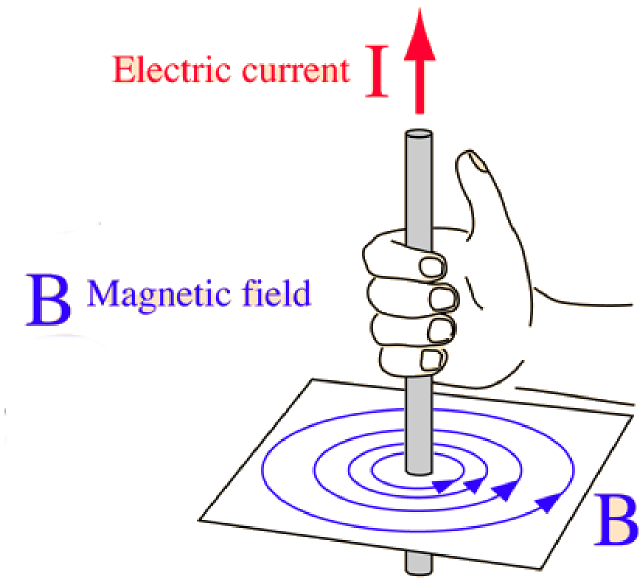
(a) Currents in opposite directions repel.

(b) Currents in same directions attract.

As we saw in the previous lecture, there is a magnetic force exerted between two wires.
This force is:

- attractive if both currents flow in the same direction, and
- repulsive if the two currents flow in opposite directions.

Force between two parallel conductors



We know the the magnetic field $\vec{B}$ produced by a constant current flowing across an infinite straight wire.

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\phi} = \frac{\mu_0 I}{2\pi\rho}\Big(-\frac{y}{\rho}, \frac{x}{\rho}, 0\Big)$$

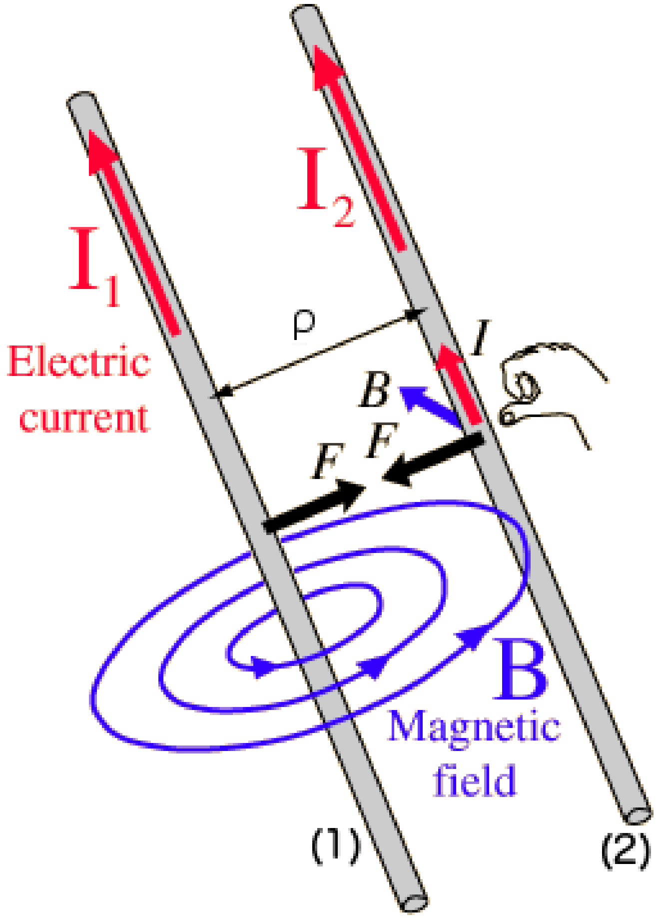
We also know the force exerted on a current within a magnetic field:

$$\vec{F} = I \int_L d\vec{\ell} \times \vec{B}$$

The two above ingredients allow us to calculate **the force between two parallel conductors.**

Force between two parallel conductors

Consider the 2 wires (1) and (2) shown below.



- We will calculate the **force $\vec{F}$ exerted on the current in wire (2)**.
- This force is **caused by the magnetic field $\vec{B}$** at the location of wire (2) **due to the current in wire (1)**. (*)

Convince yourselves, using the right-hand rule, that the magnetic force is:

- **attractive** if both currents flow in the **same direction**, and
- **repulsive** if the two currents flow in **opposite directions**.

(*) We will see that an **interchange of (1) and (2) leaves the result unchanged**: i.e. the same result is obtained if we consider the force on wire (1) due to the field produced by wire (2).

Force between two parallel conductors

The force on wire (2) due to the field produced by wire (1) is:

$$\vec{F}_{21} = I_2 \int d\vec{\ell}_2 \times \vec{B}_1$$

where an infinitesimal element $d\vec{\ell}_2$ on wire (2) can be written as (*):

$$d\vec{\ell}_2 = \Big(x, y, z + dz\Big) - \Big(x, y, z\Big) = \Big(0, 0, dz\Big)$$

and the magnetic field due to wire (1) is:

$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi\rho}\Big(-\frac{y}{\rho}, \frac{x}{\rho}, 0\Big)$$

Putting everything together, the force $\vec{F}_{21}$ can be written as:

$$\vec{F}_{21} = \frac{\mu_0 I_1 I_2}{2\pi\rho} \int \Big(0, 0, dz\Big) \times \Big(-\frac{y}{\rho}, \frac{x}{\rho}, 0\Big)$$

(*) We take both wires to be along the z axis. Wire (1) passes through (x,y)=(0,0), whereas wire (2) passes through an arbitrary point on (x,y) plane.

Force between two parallel conductors

The cross product appearing in the previous equation can be calculated as:

$$\Big(0, 0, dz\Big) \times \Big(-\frac{y}{\rho}, \frac{x}{\rho}, 0\Big) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & dz \\ -\frac{y}{\rho} & \frac{x}{\rho} & 0 \end{vmatrix} =$$

$$\hat{x}\begin{vmatrix} 0 & dz \\ \frac{x}{\rho} & 0 \end{vmatrix} - \hat{y}\begin{vmatrix} 0 & dz \\ -\frac{y}{\rho} & 0 \end{vmatrix} + \hat{z}\begin{vmatrix} 0 & 0 \\ -\frac{y}{\rho} & \frac{x}{\rho} \end{vmatrix} = \Big(-\frac{x}{\rho}, -\frac{y}{\rho}, 0\Big)dz$$

Therefore:

$$\vec{F}_{21} = \frac{\mu_0 I_1 I_2}{2\pi\rho}\int_L \Big(-\frac{x}{\rho}, -\frac{y}{\rho}, 0\Big)dz = -\frac{\mu_0 I_1 I_2}{2\pi\rho}\int_L \Big(\frac{x}{\rho}, \frac{y}{\rho}, 0\Big)dz$$

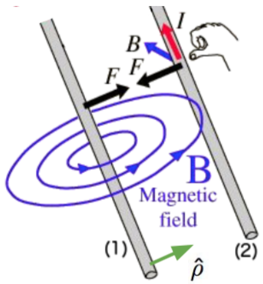
The vector $\Big(\frac{x}{\rho}, \frac{y}{\rho}, 0\Big)$ is a constant for the integration over the length L of the wire, hence:

$$\vec{F}_{21} = -\frac{\mu_0 I_1 I_2}{2\pi\rho}\Big(\frac{x}{\rho}, \frac{y}{\rho}, 0\Big)\int_L dz$$

Force between two parallel conductors

Performing the trivial integration over z, we have:

$$\vec{F}_{21} = -\frac{\mu_0 I_1 I_2 L}{2\pi\rho}\left(\frac{x}{\rho}, \frac{y}{\rho}, 0\right)$$



What is $\left(\frac{x}{\rho}, \frac{y}{\rho}, 0\right)$? It can be easily seen that it is the unit vector $\hat{\rho}$, pointing along the (shortest) distance between wires (1) and (2).

$$\hat{\rho} = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0\right) \; , \; \hat{\rho}\cdot\hat{\rho} = \left(\frac{x}{\rho}\right)^2 + \left(\frac{y}{\rho}\right)^2 = 1$$

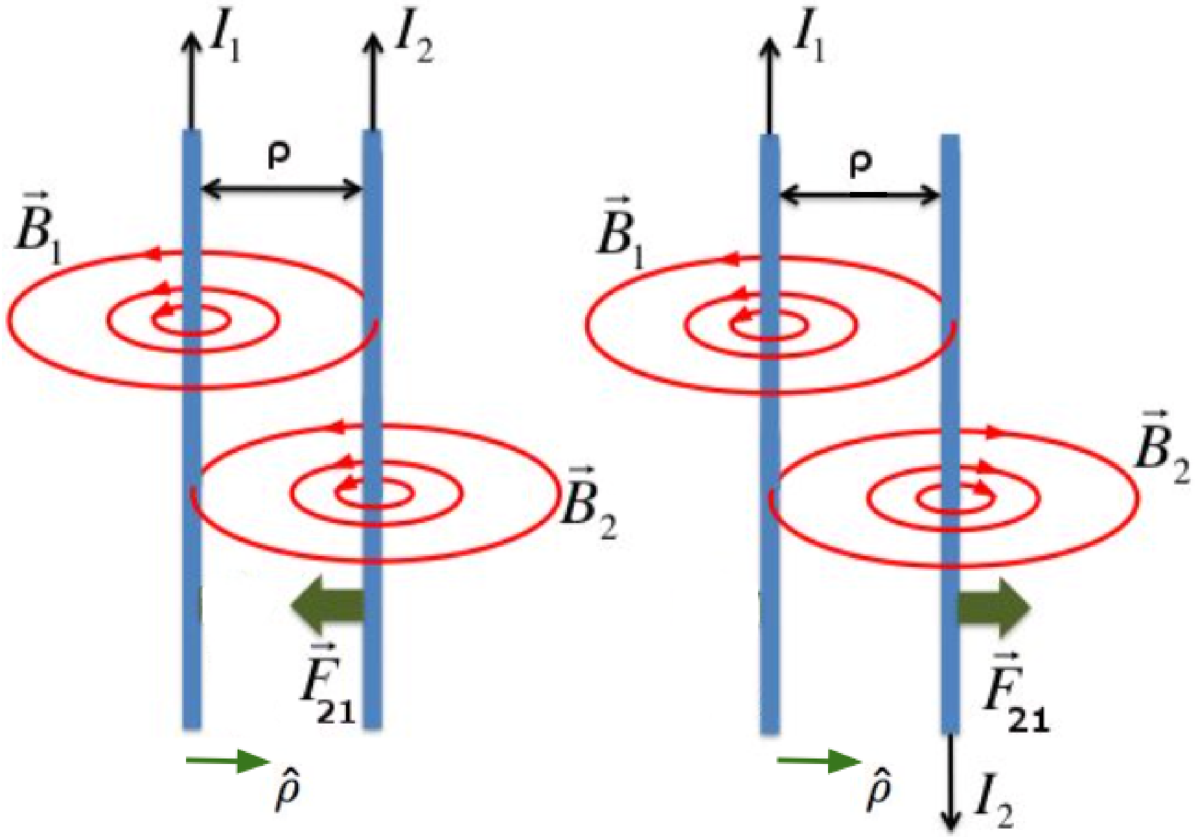
The force $\vec{F}_{21}$ on wire (2) due to the field of wire (1) is written as:

$$\vec{F}_{21} = -\frac{\mu_0 I_1 I_2 L}{2\pi\rho}\hat{\rho}$$

Force between two parallel conductors

The force $\vec{F}_{21}$ on wire (2) due to the field of wire (1):

$$\vec{F}_{21} = -\frac{\mu_0 I_1 I_2 L}{2\pi\rho}\hat{\rho}$$

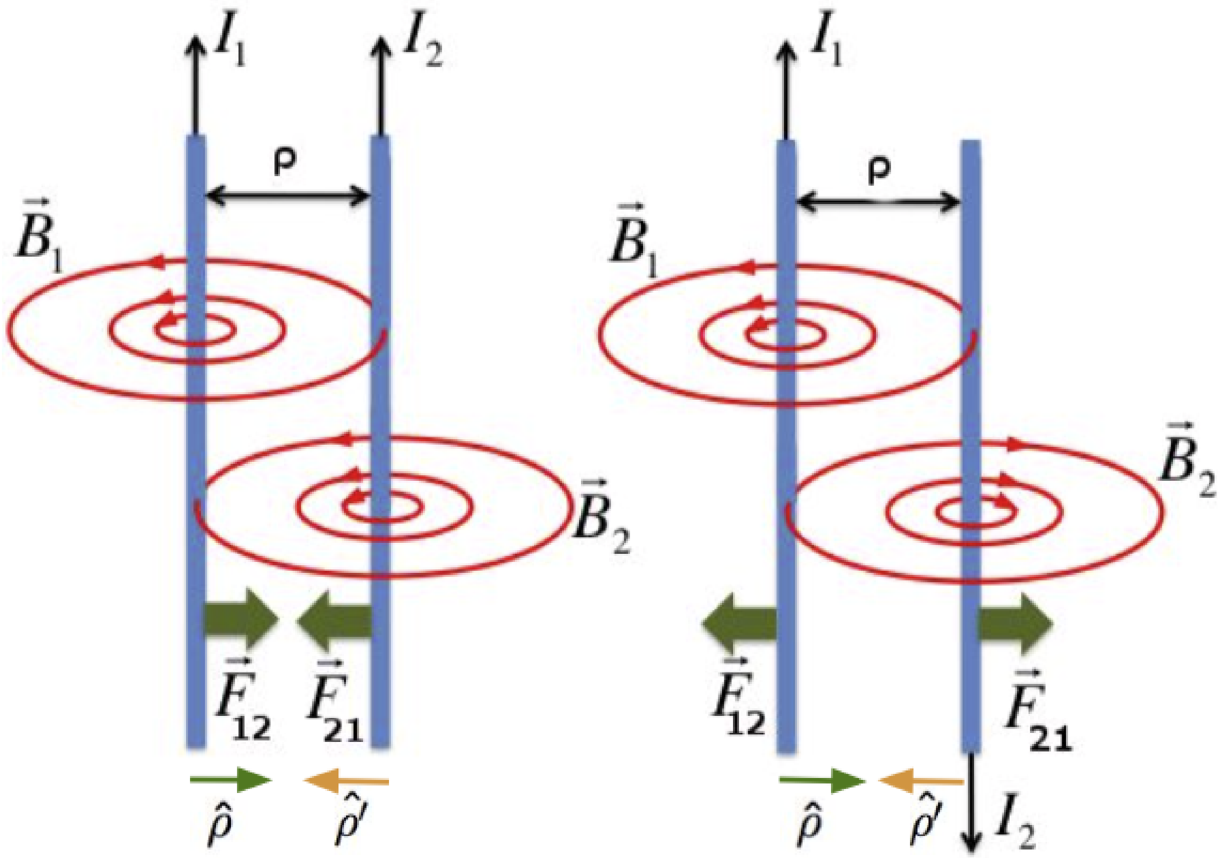


The presence of $-\hat{\rho}$ tells us that, if I_1 and I_2 have the same direction, the force $\vec{F}_{21}$ pulls the wire (2) towards the wire (1).

Force between two parallel conductors

Consequently, the force $\vec{F}_{12}$ on wire (1) due to the field of wire (2) is given by:

$$\vec{F}_{12} = -\frac{\mu_0 I_1 I_2 L}{2\pi\rho}\hat{\rho}' = \frac{\mu_0 I_1 I_2 L}{2\pi\rho}\hat{\rho}$$



The relative minus sign is because the corresponding unit vector $\hat{\rho}'$ starting from wire (2) and pointing towards wire (1) has the opposite direction wrt $\hat{\rho}$: $\hat{\rho}' = -\hat{\rho}$

Force between two parallel conductors

The force $\vec{F}_{21}$ on wire (2) due to the field of wire (1):

$$\vec{F}_{21} = -\frac{\mu_0 I_1 I_2 L}{2\pi\rho}\hat{\rho}$$

The force $\vec{F}_{12}$ on wire (1) due to the field of wire (2):

$$\vec{F}_{12} = -\frac{\mu_0 I_1 I_2 L}{2\pi\rho}\hat{\rho}' = \frac{\mu_0 I_1 I_2 L}{2\pi\rho}\hat{\rho}$$

Notice that in both cases, the magnitude of the force is the same, as expected:

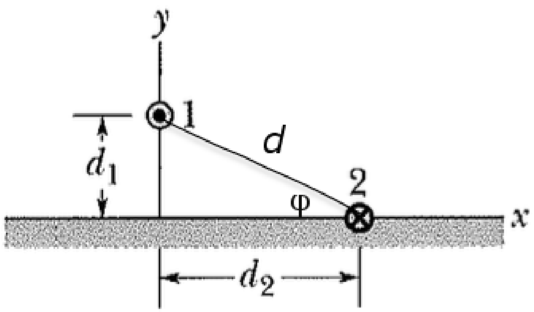
$$|\vec{F}_{21}| = |\vec{F}_{12}| = \frac{\mu_0 I_1 I_2 L}{2\pi\rho}$$

From now on we will drop the indices from the force and write it as:

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi\rho}$$

Worked example

Question



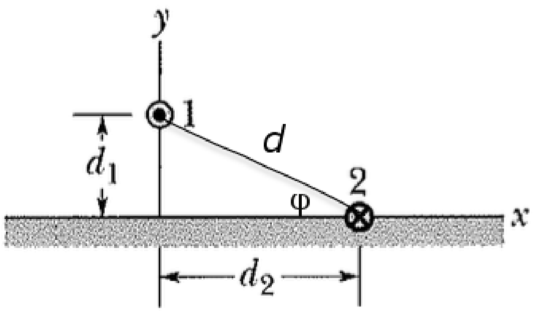
The figure on the left shows wire 1 in cross section; the wire is long and straight, carries a current of 4.00 mA out of the page, and is at distance $d_1 = 2.40$ cm from a surface. Wire 2, which is parallel to wire 1 and also long, is at horizontal distance $d_2 = 5.00$ cm from wire 1 and carries a current of 6.80 mA into the page.

What is the x-component of the magnetic force *per unit length* on wire 2 due to wire 1?

The magnitude of the force per unit length between the wires is given by:

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi d} \Rightarrow \frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi\sqrt{d_1^2 + d_2^2}}$$

Worked example



The x-component is found by multiplying the amplitude with:

$$cos\phi = \frac{d_2}{\sqrt{d_1^2 + d_2^2}}$$

Therefore, the x-component of the magnetic force *per unit length* on wire 2 due to wire 1 is:

$$\frac{F_x}{L} = \frac{\mu_0 I_1 I_2}{2\pi\sqrt{d_1^2 + d_2^2}} \cdot \frac{d_2}{\sqrt{d_1^2 + d_2^2}} \Rightarrow \frac{F_x}{L} = \frac{\mu_0 I_1 I_2 d_2}{2\pi(d_1^2 + d_2^2)} \Rightarrow$$

$$\frac{F_x}{L} = \frac{(4\pi \times 10^{-7}\ T \cdot m/A)(4.00 \times 10^{-3}\ A)(6.80 \times 10^{-3}\ A)(0.050\ m)}{2\pi\Big((0.024\ m)^2 + (0.050\ m)^2\Big)} \Rightarrow$$

$$\frac{F_x}{L} = 8.84 \times 10^{-11}\ N/m$$

The definition of an Ampere

The force between two conducting wires

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi\rho}$$

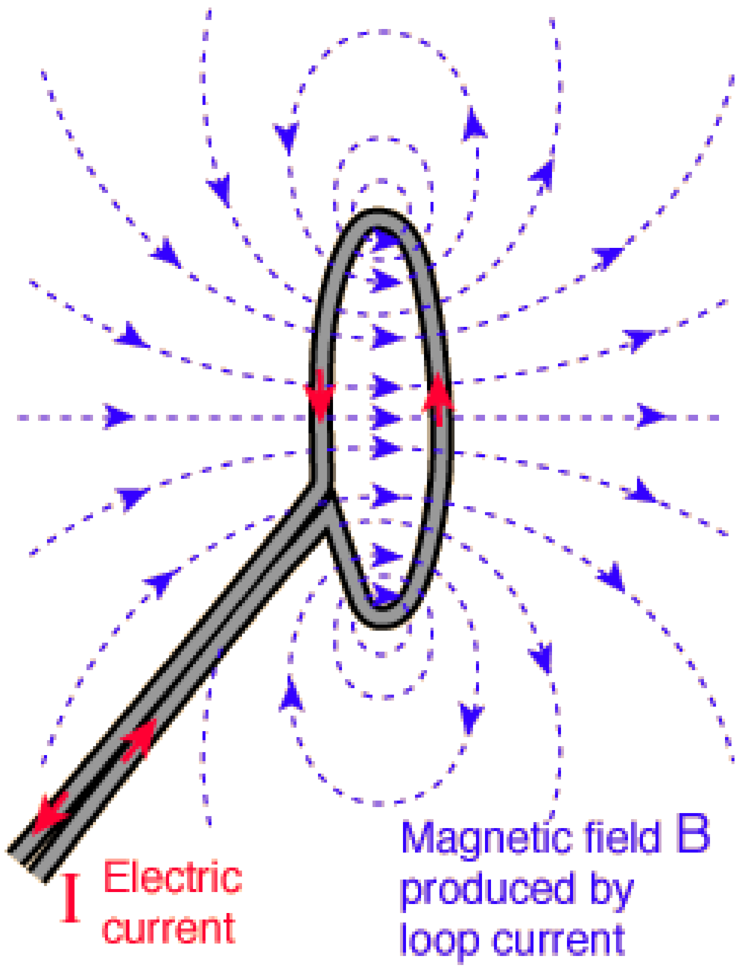
is used to **define the SI unit of current (Ampere)**

Assume:

- 2 straight parallel conductors, placed in vacuum
- The conductors are 1 m apart
- They have infinite length and negligible cross section.

One **Ampere is the amount of current that**, if maintained between those conductors **produces a force of 2** $\times$ 10^{-7} **N per metre of length.**

Current loop



Let's consider a **current loop** as the one shown on the left.

A current loop is a **magnetic dipole** which is the magnetic analogue of the electric dipole we saw at an earlier lecture. The analogies will become clearer later in this lecture series.

A current loop within a magnetic field

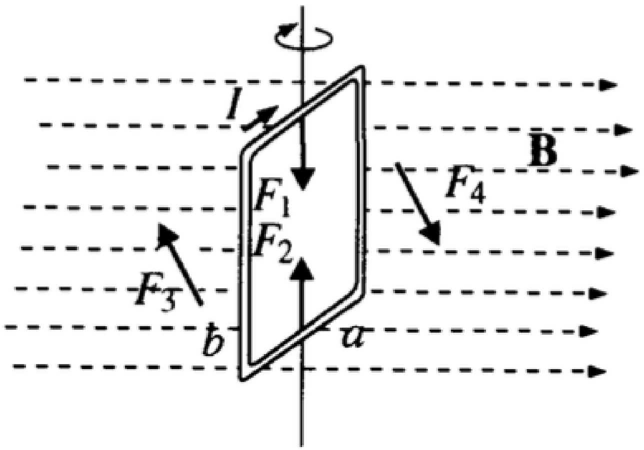
Let's study a **conducting loop within a magnetic field**.

For simplicity, we will examine:

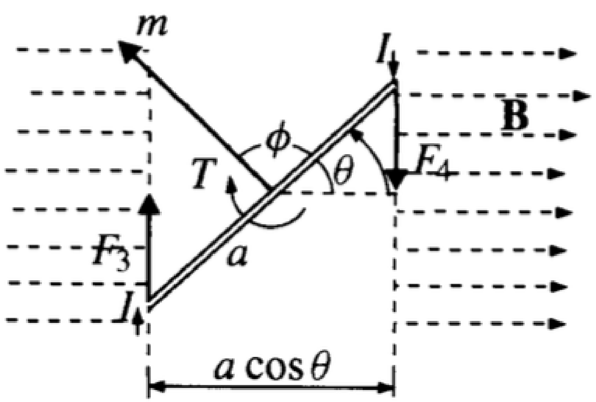
- a loop with a **rectangular shape**,
- circulated by a **constant current I**,
- placed within a **uniform magnetic field B** that is **perpendicular to the long sides**.

We will assume that the loop can **rotate freely around an axis** parallel to the long sides going through the centre of the short sides (see figure).

Force on the sides of a current loop



(a) perspective

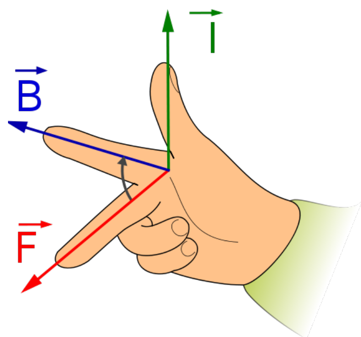


(b) top view

We will study the force exerted on each side of the loop. This force, as we have seen, is given by:

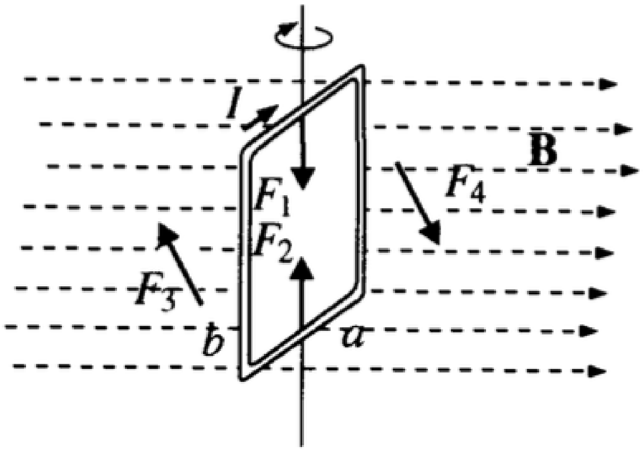
$$\vec{F} = I \int_L d\vec{\ell} \times \vec{B}$$

Its direction is given by the right-hand rule.

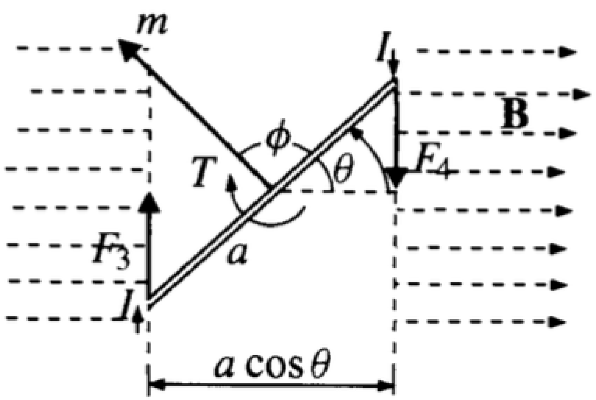


Let **a** be the length of the short sides, and **b** the length of the long sides. As the loop rotates, the angle between the short sides and the magnetic field $\vec{B}$ changes: Let's call this angle θ (see figure).

Force on the sides of a current loop



(a) perspective



(b) top view

$|\vec{B}|$ is constant over the loop, but, as it (the loop) rotates, the angle between the short sides (and, hence, their current) and $\vec{B}$ changes.

Therefore, the force on the short sides is:

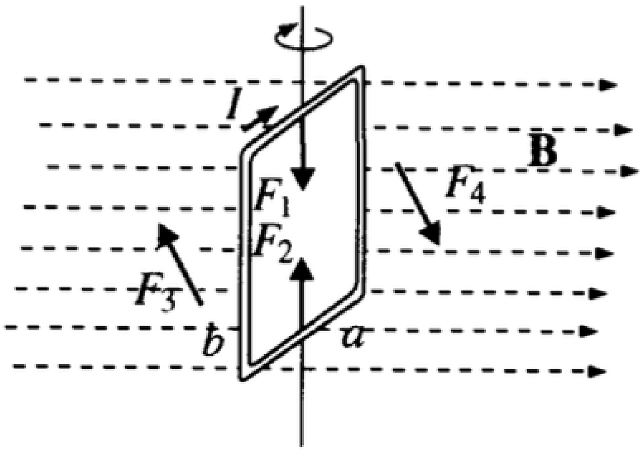
$$|\vec{F_1}| = |\vec{F_2}| = Ia|\vec{B}|sin\theta$$

where the angle θ was defined previously.
Obviously, as the loop rotates the force on the short sides varies between:

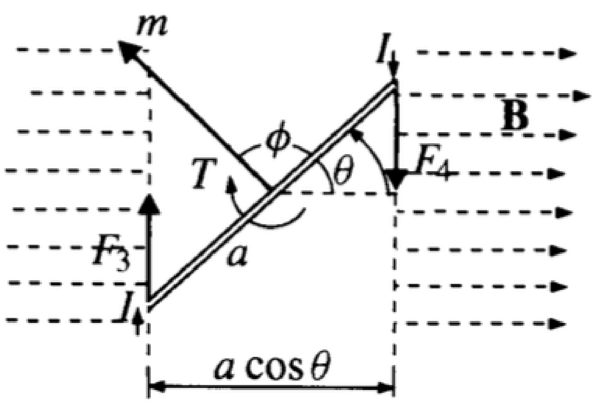
$$|\vec{F_1}| = |\vec{F_2}| = \begin{cases} 0, \text{for } \theta = 0, \text{ and} \\ Ia|\vec{B}|, \text{for } \theta = \pi/2 \end{cases}$$

On the other hand, the long sides are perpendicular to $\vec{B}$ and, hence, the force on them is constant and is given by:

$$|\vec{F_3}| = |\vec{F_4}| = Ib|\vec{B}|$$



(a) perspective



(b) top view

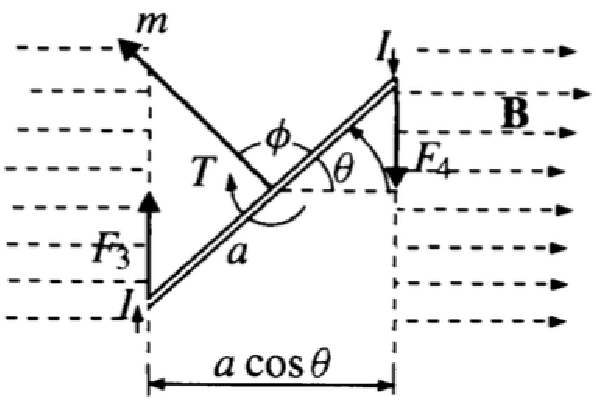
Using the right-hand rule, one can easily see that the forces on the short sides ($\vec{F}_1$ and $\vec{F}_2$) are either:

- both pointing inwards trying to *squeeze* the loop (for $0 < \theta < \pi$), or
- both pointing outwards trying to *expand* the loop (for $\pi < \theta < 2\pi$).

On either case, the forces on the short sides are parallel to the axis or rotation, hence they exert **no torque on the loop**.

On the other hand, the forces on the two long sides ($\vec{F}_3$ and $\vec{F}_4$) have constant but opposite directions. They are perpendicular to the axis of rotation and, therefore, **they exert a torque on the loop**.

Torque on a current loop



(b) top view

The torque $\vec{T}$ exerted by a force $\vec{F}$ is:

$$\vec{T} = \vec{r} \times \vec{F}$$

where $\vec{r}$ is the displacement vector, pointing from the axis of rotation to the point where the force is applied.

In our case (see figure above) $\vec{T}$ (which points into the page, $\hat{in}$) is:

$$\vec{T} = \left\{ \frac{a}{2} |F_3| sin\left(\frac{\pi}{2} + \theta\right) + \frac{a}{2} |F_4| sin\left(\frac{\pi}{2} + \theta\right) \right\} \hat{in} \xrightarrow{|F_3|=|F_4|=Ib|\vec{B}|}$$

$$\vec{T} = \cancel{2} \frac{a}{\cancel{2}} \left(Ib|\vec{B}|\right) sin\left(\frac{\pi}{2}+\theta\right) \hat{in} = I\left(a\ b\right)|\vec{B}| sin\left(\frac{\pi}{2}+\theta\right) \hat{in} \xrightarrow{|\vec{S}|=ab,\ \ \phi=\frac{\pi}{2}+\theta=\angle(\vec{S},\vec{B})}$$

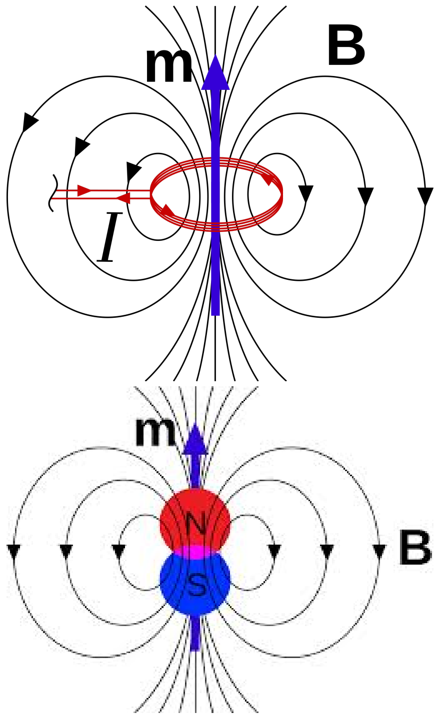
$$\vec{T} = \left(I\vec{S}\right) \times \vec{B}$$

Magnetic dipole moment

The **magnetic dipole moment** $\vec{m}$ of a current loop is defined as

$$\vec{m} = I\vec{S}$$

The magnetic moment of a magnet is a quantity that **determines the torque it will experience in an external magnetic field.**



The magnetic dipole moment is **a vector**. it has both a magnitude ($|\vec{m}|$ = I $|\vec{S}|$) and a direction:

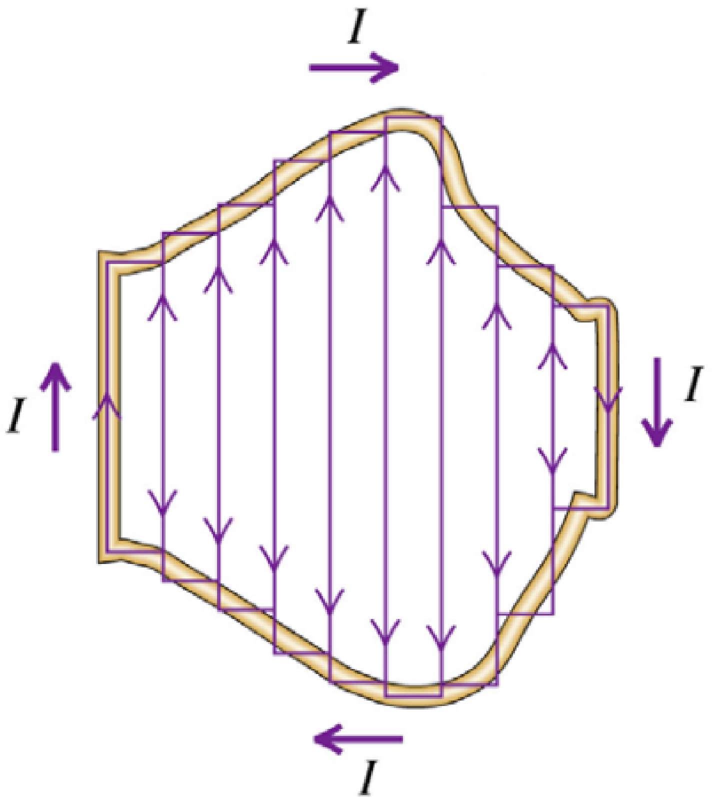
- Curl your right hand in the direction of the current in the loop: Your thumb points towards the magnetic dipole moment.
- You can also think that it points from the south to north pole of the magnet.

Magnetic dipole moment for any planar loop

Is the simple definition of the magnetic dipole moment $\vec{m}$

$$\vec{m} = I\vec{S}$$

a consequence of considering a simple loop with a rectangular shape?

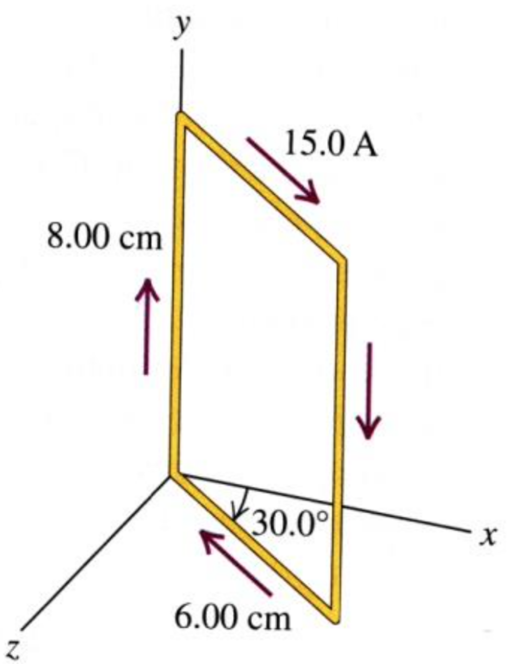


What if the current loop had an arbitrary shape?

Any planar current loop of any shape can be approximated by a set of rectangular loops.

Worked example

Question



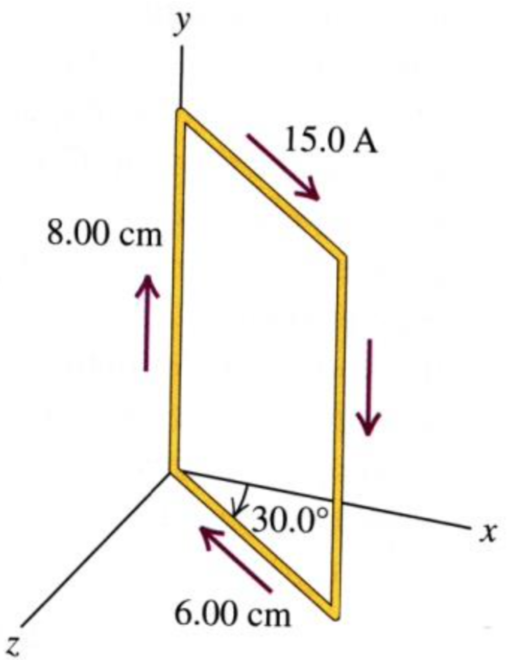
The rectangular loop in the figure is pivoted around the y-axis and carries a current of 15.0 A in the direction shown.

1. If the loop is in uniform magnetic field with magnitude 0.48 T in the +x direction, find the magnitude and direction of the torque required to hold the loop in the position shown.
2. What torque would be required if the loop were pivoted about an axis through its centre, parallel to the y-axis?

(1) The torque exerted by a field $\vec{B}$ on a magnetic dipole moment $\vec{m}$ is:

$$\vec{T} = \vec{m} \times \vec{B}$$

Worked example



$\vec{B}$ is along $\hat{x}$ and $\vec{m}$ is perpendicular to the loop and towards the direction of your thumb if your right-hand curls in the direction of the current.

In the position shown, $\vec{m}$ is on the xz plane pointing towards $\hat{n}$ = - $\cos\frac{\pi}{6}\ \hat{z} + \sin\frac{\pi}{6}\ \hat{x}$.

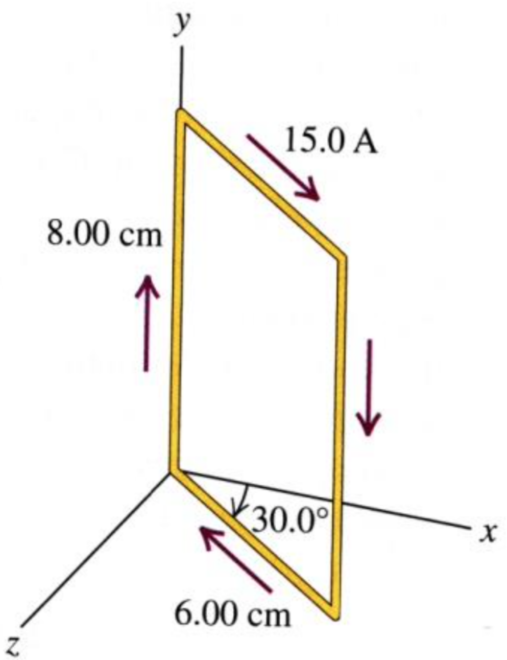
Therefore, the direction of the torque exerted by the magnetic field is towards the negative y axis $(-\hat{y})$:
To keep the loop in place, you must provide a torque in the opposite direction $(+\hat{y})$.

The magnitude of $\vec{T}$ is:

$$|\vec{T}| = |\vec{m}| \cdot |\vec{B}| \cdot sin\Big(\angle(\vec{m}, \vec{B})\Big) \Rightarrow |\vec{T}| = \Big(I \cdot S\Big) \cdot |\vec{B}| \cdot sin\Big(\angle(\vec{m}, \vec{B})\Big) \Rightarrow$$

$$|\vec{T}| = \Big(15.0\ A \cdot 0.08\ m \cdot 0.06\ m\Big) \cdot \Big(0.48\ T\Big) \cdot sin\Big(\frac{\pi}{2} - \frac{\pi}{6}\Big) \Rightarrow$$

Worked example



$$|\vec{T}| = \left(15.0 \cdot 0.08 \cdot 0.06 \cdot 0.48 \; sin\frac{\pi}{3}\right) \; N \cdot m \Rightarrow$$

$$|\vec{T}| \approx 0.03 \; N \cdot m$$

(2) If the loop was pivoted through its center, then there would be a torque on both sides of the loop parallel to the rotation axis. However, the lever arm is only half as large, so the total torque in each case is identical to the value found in part (1).

The electric motor

As we have seen:

- an electric currents induces a magnetic field, and
- a moving magnet (or an otherwise varying magnetic field) induces an electric current

And, as we also saw earlier today:

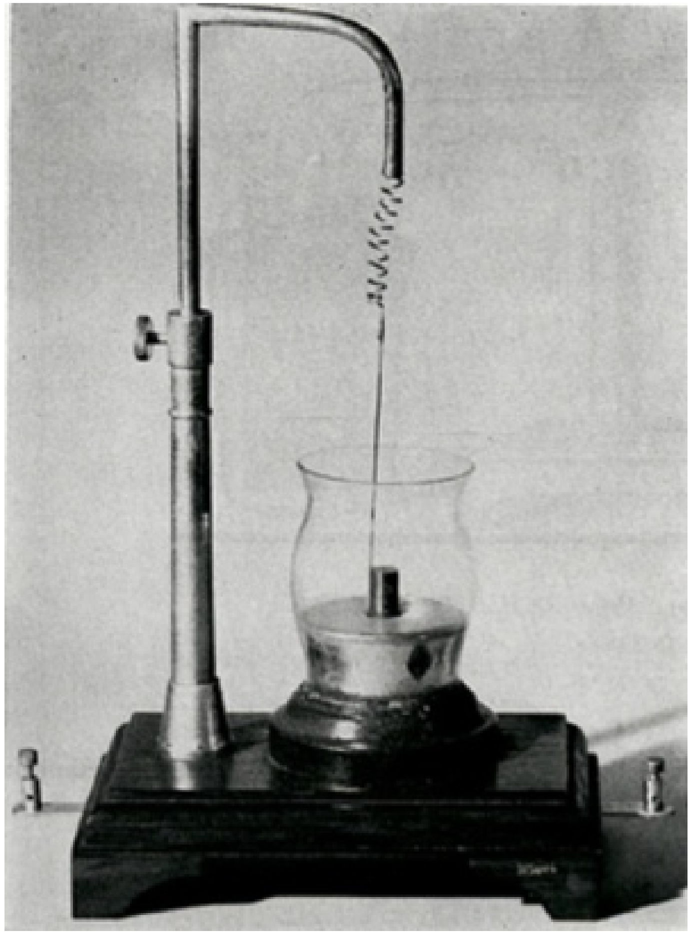
- A magnetic field exerts a torque on a current loop and spins it!

The exciting question, very early on, was: **Could we exploit these phenomena to build engines to do work?**

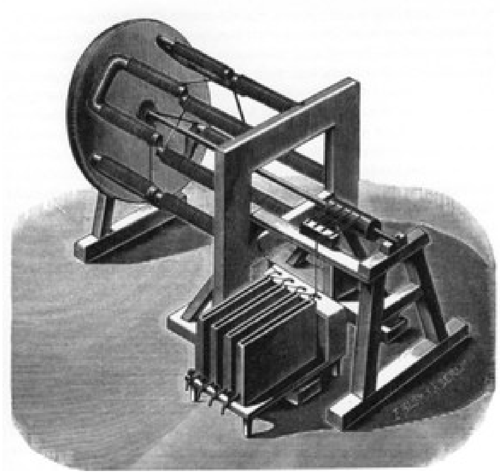
- The relevant empirical laws were already known by the early 1820's.
- Already at the end of that decade there were early versions of **electrical motors.**

Read *'The invention of the electric motor 1800-1854'* by Martin Doppelbauer, available at: https://www.eti.kit.edu/english/1376.php

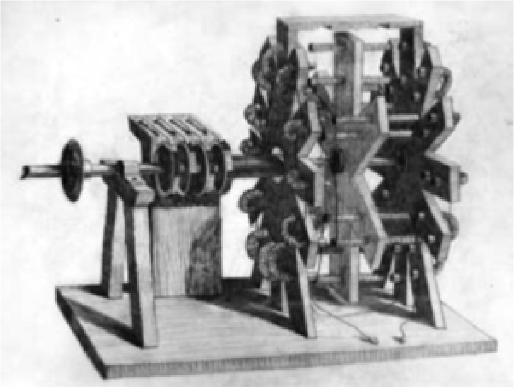
The electric motor



Michael Faraday, 1821: A vertically suspended wire moves in a circular orbit around a magnet.



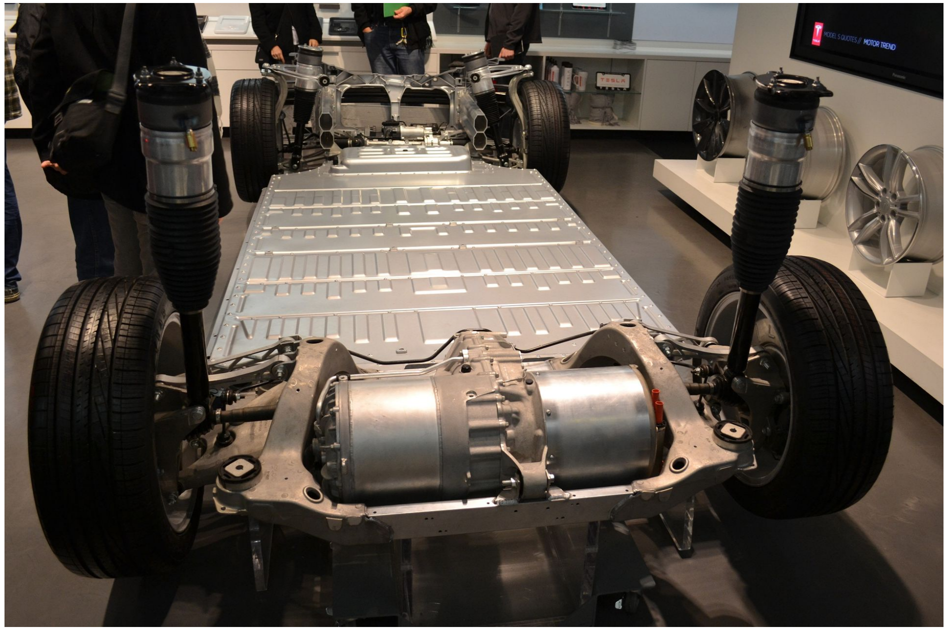
Moritz Jacobi, 1834: First (?) electric motor (15 W)



Moritz Jacobi, 1838: Improved electric motors (300 W, 1/4 hp) - Could propel a boat with a dozen passengers on a 7.5 km journey at a speed of 2.5 km/h.

The electric motor

An electric motor is a machine that converts electrical energy into mechanical energy.

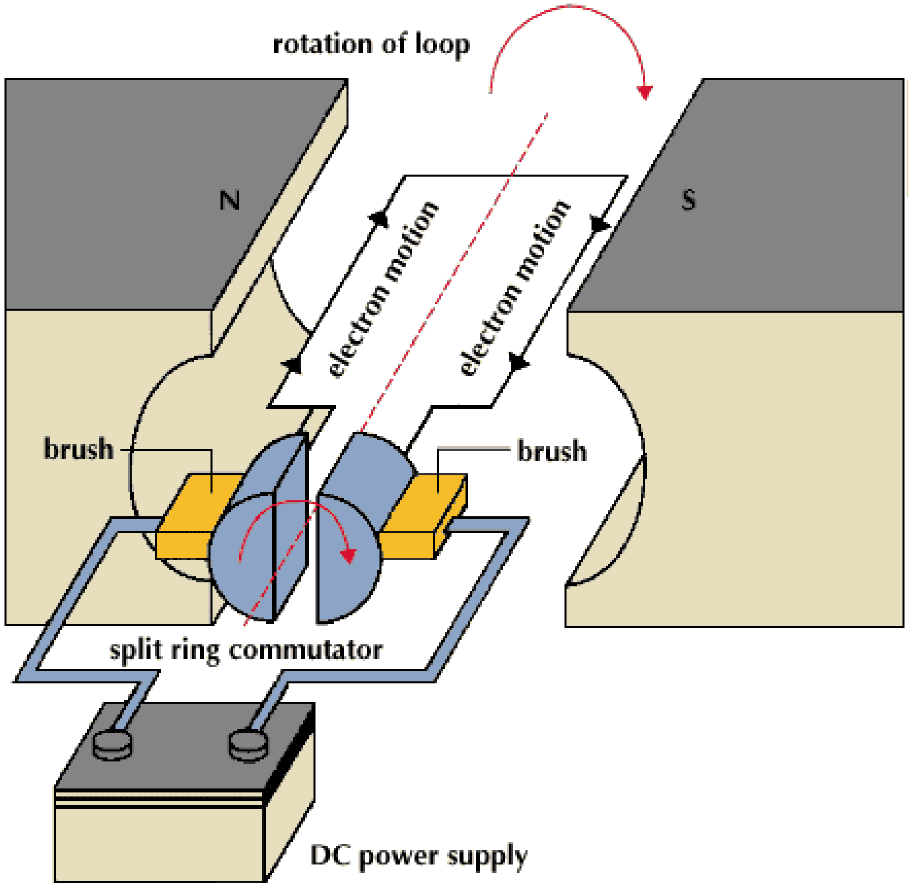


Electric motor on Tesla Model S

- It exploits the interaction between an electric motor's magnetic field and current loops to generate force within the motor.
- It can be powered by
  - direct current (DC) sources (e.g. batteries), or
  - alternating current (AC) sources (e.g. the power grid).

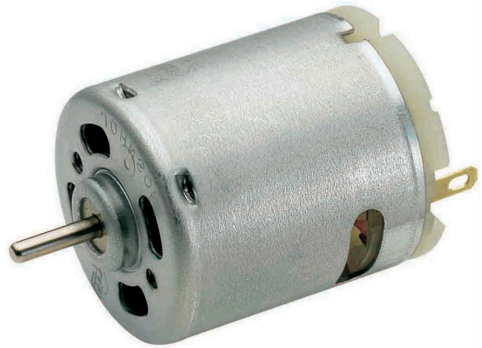
The DC electric motor

We will study a typical **DC electric motor**.



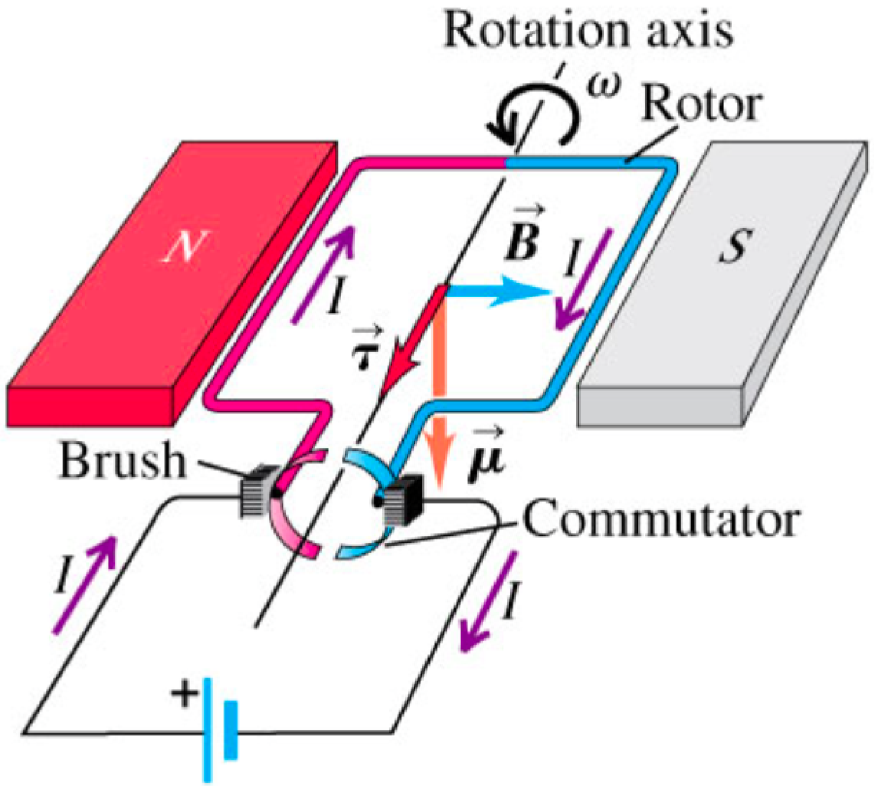
There are two main parts:

- The part which is stationary (called **stator**) and includes the magnet.
- The part which rotates (called **rotor**) - It turns a shaft to provide the mechanical power it generates (i.e. can be coupled to pulleys or gears and can do work).

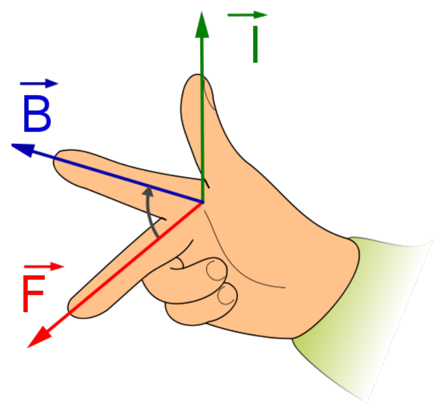


The DC electric motor

(a) Brushes are aligned with commutator segments.

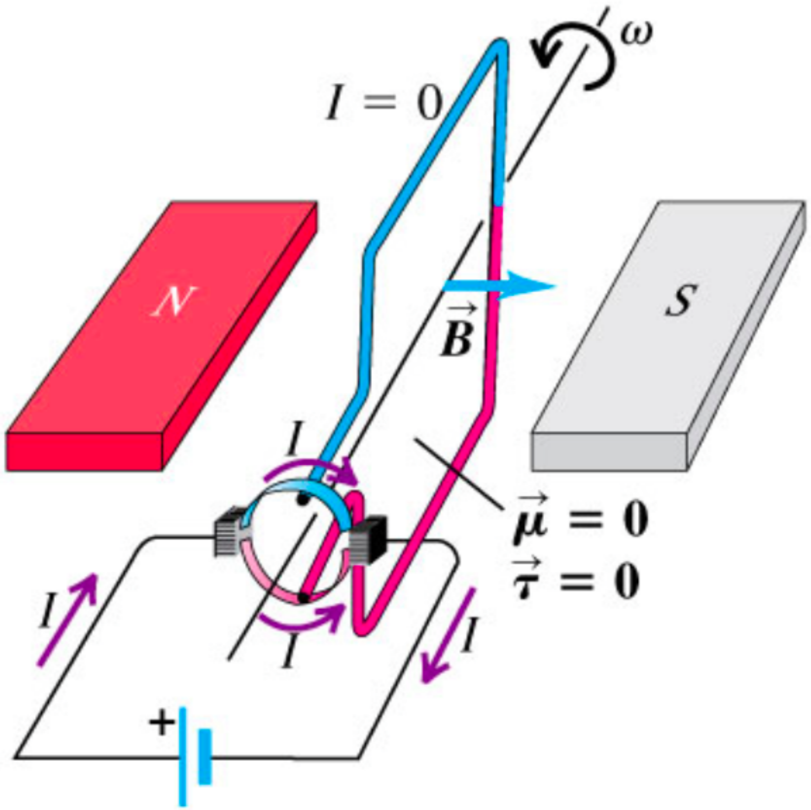


- Current flows into the red side of the rotor and out of the blue side.
- Magnetic forces:



  - on left-hand side: downwards
  - on right-hand side: upwards
- The magnetic torque causes the rotor to **spin counterclockwise**.

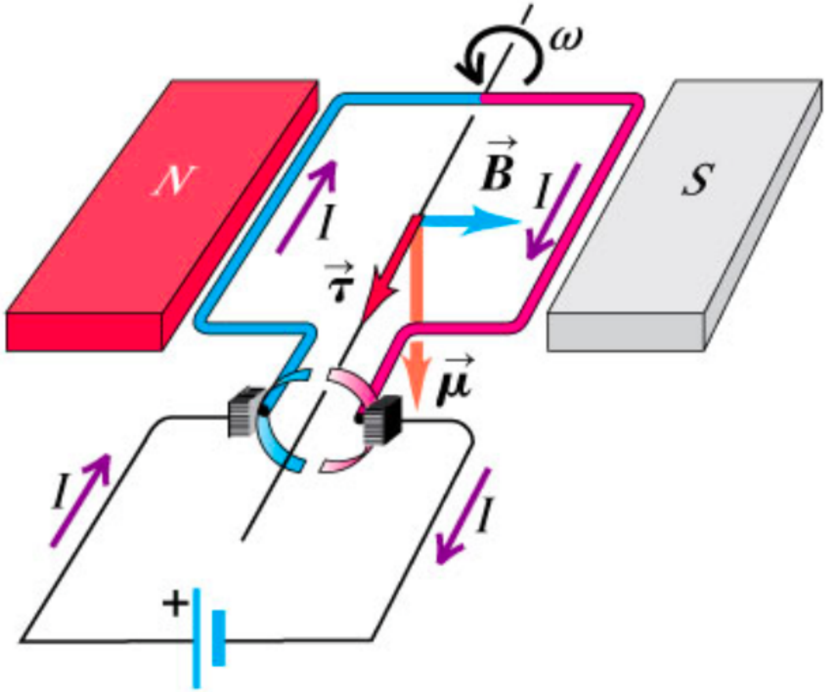
(b) The rotor has turned 90^o.



- Each brush is in contact with both commutator segments, so the current bypasses the rotor altogether.
- **No magnetic torque acts on the rotor**

The DC electric motor

(c) The rotor has turned 180^o.

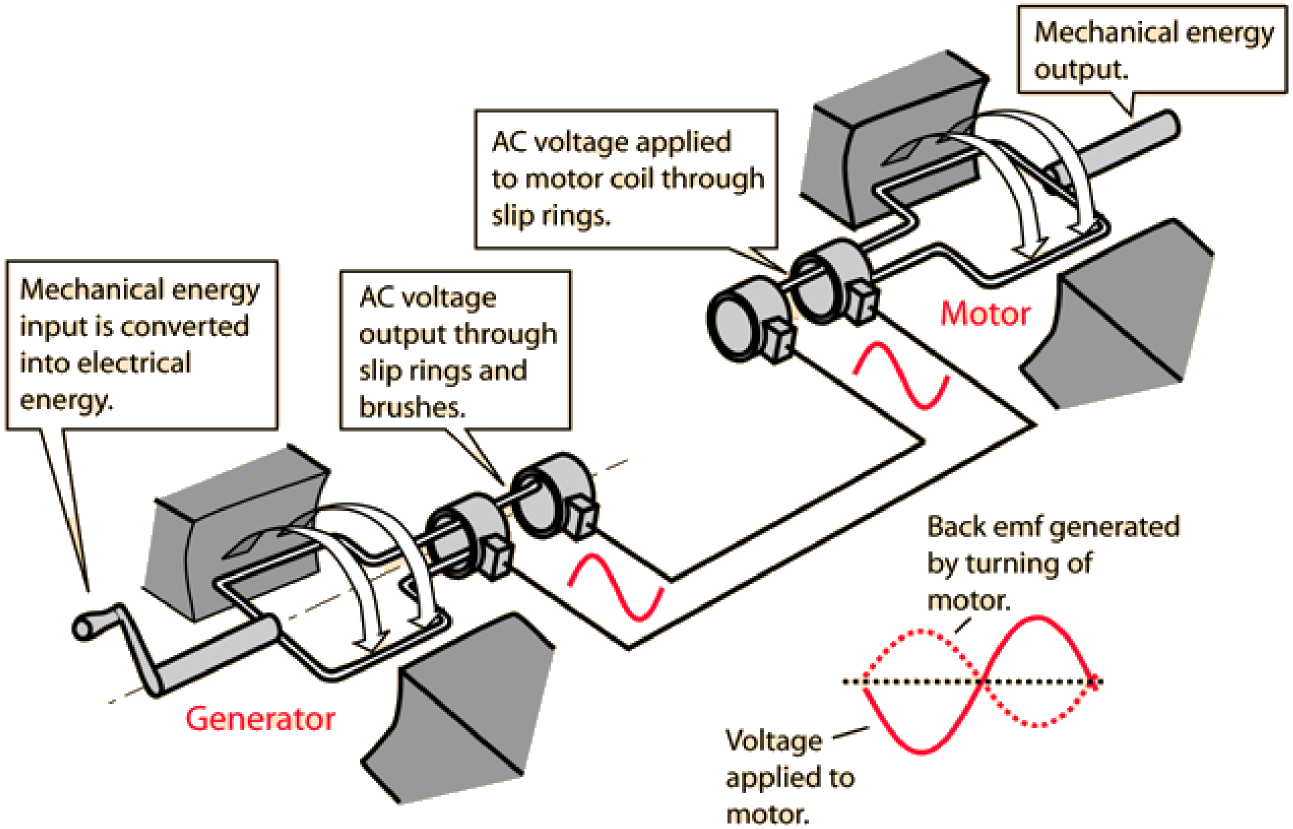


- The brushes are again aligned with commutator segments This time the current flows into the blue side of the rotor and out of the red side.
- Magnetic forces:
  - on left-hand side: downwards
  - on right-hand side: upwards
- Therefore the magnetic torque again causes the rotor to **spin counterclockwise**.

Question: What would happen if the current flow was unchanged (e.g. always into the red side of the rotor and out of the blue side as in (a))?

The DC electric motor and generators

Note that the machine shown on the slide **is either a motor, or a generator!** There is a **reciprocity between the two**.



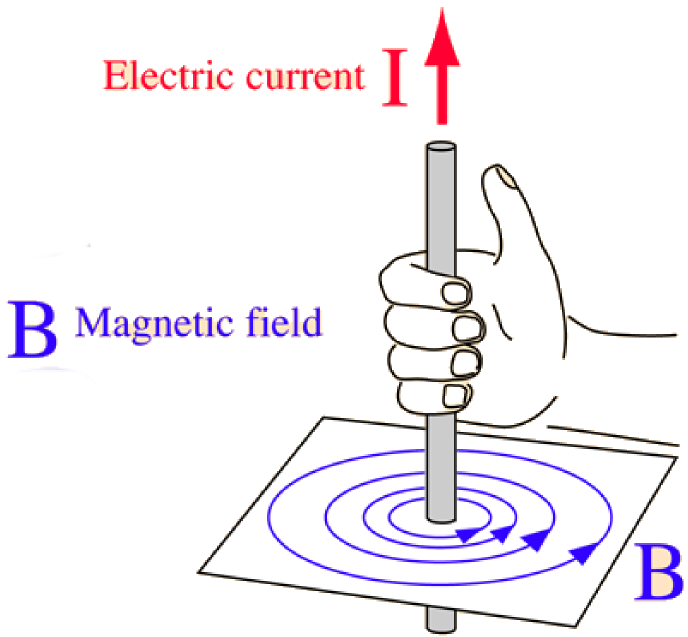
- Examine what happens if, instead of putting a current through the loop to make it rotate, you rotate the loop manually.
- As loop rotates into a magnetic field, so there is an EMF in the circuit and current flows!

The curl of the magnetic field

Recall the **circuital law** of electrostatics:

$$\vec{\nabla} \times \vec{E} = 0$$

It tells us that the **electric field has no rotation** anywhere in space. Hence, there are **no closed field lines** in an electrostatic field.

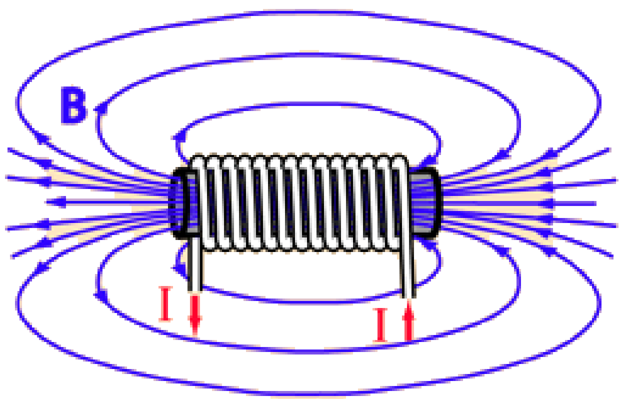


How about the magnetic field?

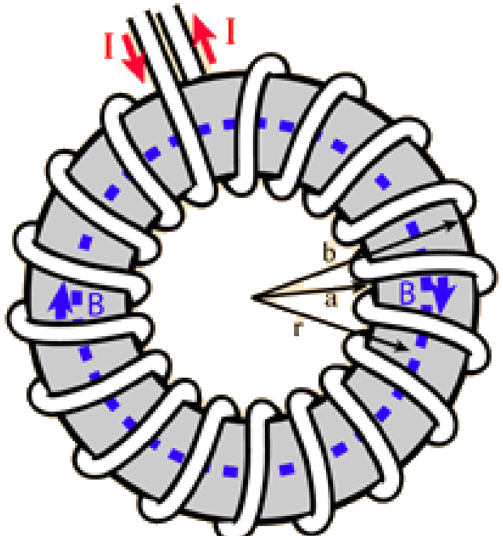
We have studied one example in detail (the magnetic field of an infinite straight wire) where we saw a very different behaviour:

The magnetic field lines **loop back on themselves** (the magnetic field lines are always closed)!

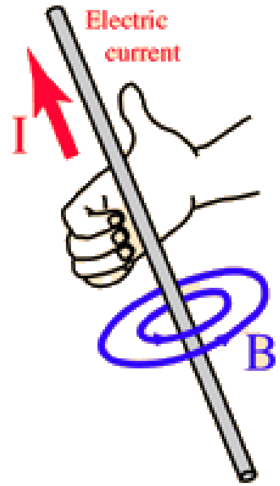
Magnetic field examples: The field lines **loop back on themselves**!



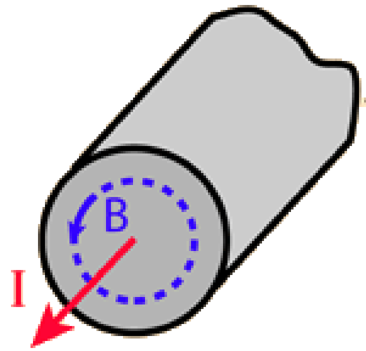
Magnetic field inside a long solenoic.



Magnetic field inside a toroidal coil.



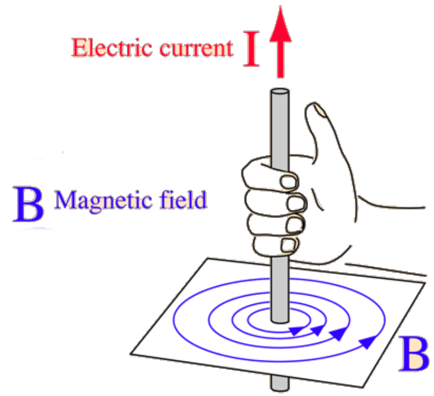
Magnetic field from a long straight wire.



Magnetic field inside a conductor.

The curl of the magnetic field

In the case of the infinitely-long straight conducting wire with current I, we can easily calculate the line integral $\oint \vec{B} d\vec{\ell}$ of the magnetic field $\vec{B}$ along a closed circular trajectory.



As we have seen, $\vec{B}$ is given by: $\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$ whereas an infinitesimal step $d\vec{\ell}$ on a circular trajectory or radius ρ is given by: $d\vec{\ell} = \rho \, d\phi \, \hat{\phi}$

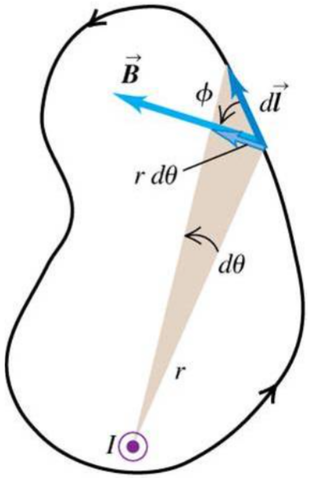
Therefore:

$$\oint \vec{B} \cdot d\vec{\ell} = \oint \left(\frac{\mu_0 I}{2\pi\rho}\hat{\phi}\right) \cdot \left(\rho d\phi \hat{\phi}\right) = \frac{\mu_0 I}{2\pi\cancel{\rho}}\cancel{\rho} \oint \left(\hat{\phi} \cdot \hat{\phi}\right) d\phi = \frac{\mu_0 I}{2\pi} \oint d\phi = \frac{\mu_0 I}{\cancel{2\pi}}\cancel{2\pi} \Rightarrow$$

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

The curl of the magnetic field

The previous result can be **generalised for any closed path** around an infinite conductor.



$$\oint \vec{B}\cdot d\vec{\ell} = \oint |\vec{B}||d\vec{\ell}|cos\phi \Rightarrow \oint \vec{B}\cdot d\vec{\ell} = \oint \left(\frac{\mu_0 I}{2\pi \cancel{r}}\right)\left(\cancel{r}\; d\theta\right) \Rightarrow$$

$$\oint \vec{B}\cdot d\vec{\ell} = \frac{\mu_0 I}{2\pi}\oint d\theta = \frac{\mu_0 I}{\cancel{2\pi}}\cancel{2\pi} \Rightarrow \oint \vec{B}\cdot d\vec{\ell} = \mu_0 I$$

It can also be **generalised for any number of conductors (currents)** passing through the loop. In general:

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{encl}$$

where:

$$I_{encl} = \int_S \vec{j}\cdot d\vec{S}$$

André Ampère, 1775-1836
French physicist and mathematician.

The general result we just saw is known as **Ampere's law**:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl} = \mu_0 \int_S \vec{j} \cdot d\vec{S}$$

The line integral of the magnetic field $\vec{B}$ along a closed path L is proportional to the current passing though any surface S defined by L.

We can draw some illuminating parallels with Gauss's law in electrostatics:

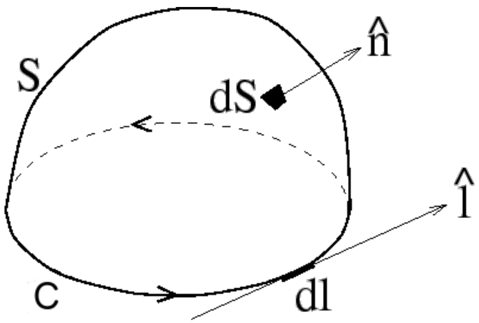
$$\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} Q_{encl} = \frac{1}{\epsilon_0} \int_\tau \rho d\tau$$

The surface integral of the electric field $\vec{E}$ through a closed surface S is proportional to the net charge contained in the volume τ defined by S.

Ampere's law (differential form)

Ampere's law in integral form is:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl} = \mu_0 \int_S \vec{j} \cdot d\vec{S}$$



Stoke's theorem relates the line integral of a vector field with the surface integral of the curl of the field:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{S}$$

Applying Stoke's theorem Ampere's law becomes:

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_0 \int_S \vec{j} \cdot d\vec{S} \Rightarrow \int_S \left(\vec{\nabla} \times \vec{B} - \mu_0 \vec{j}\right) \cdot d\vec{S} = 0$$

The above integral to be 0 for every surface S, hence:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

Application of Ampere's Law: Field of toroidal coil

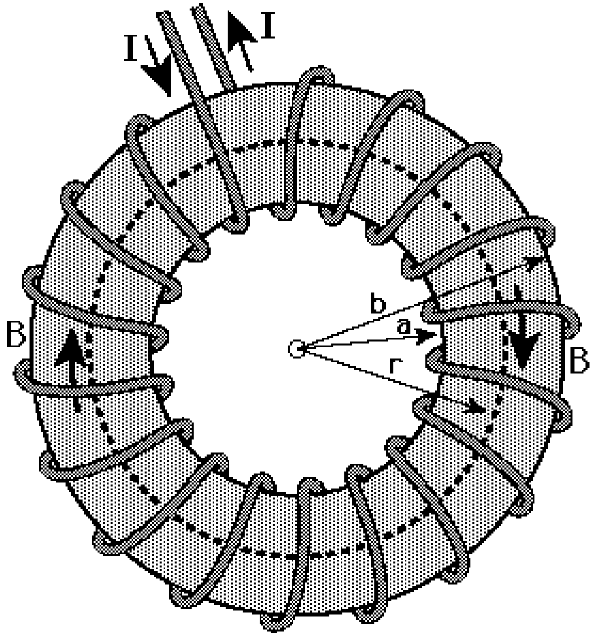
Consider a toroidal (*) coil with N equally spaced circular windings carying current I.

What is the **magnetic field inside and outside the torus?**

(*) A **torus** is the surface generated by rotating a circle in 3-dimensional space about an axis that is coplanar with the circle (i.e. a doughnut shape)

Application of Ampere's Law: Field of toroidal coil

Inside the torus, the field is directed tangentially, as shown below, and it depends on the radius r.



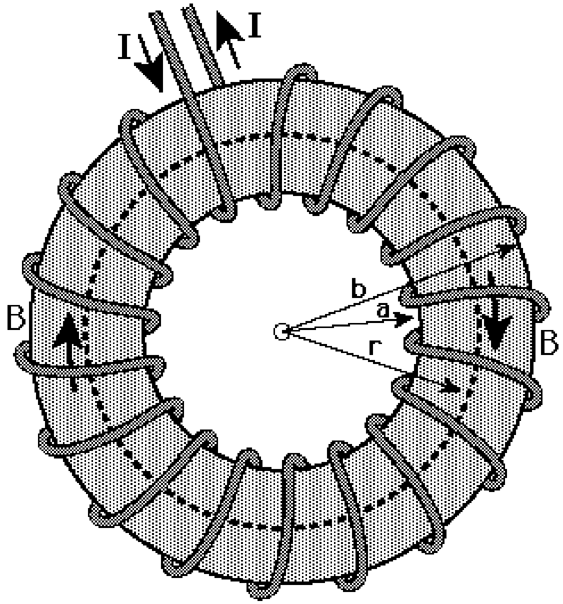
Consider the path shown with a dashed line on the schematic: The magnitude of the magnetic field $|\vec{B}|$ is constant along the chosen path, and $\vec{B}$ is collinear with $d\vec{\ell}$. Therefore:

$$\oint_L \vec{B} \cdot d\vec{\ell} = |\vec{B}| 2\pi r$$

Since there are N windings, each carrying current I, the total current enclosed by the chosen path is:

$$I_{encl} = NI$$

Application of Ampere's Law: Field of toroidal coil



Ampere's law in integral form is:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl}$$

So inside the torus:

$$|\vec{B}| 2\pi r = \mu_0 N I \Rightarrow |\vec{B}| = \frac{\mu_0 N I}{2\pi r}$$

Hence the results of the magnetic field $\vec{B}$ can be summarised as follows:

$$|\vec{B}| = \begin{cases} 0, \text{for } r < a, \\ \frac{\mu_0 n I}{2\pi r}, \text{for } a < r < b, \text{ and} \\ 0, \text{for } r > b \end{cases}$$

Calculus reminders

You should try to prove both of these at home

BAC-CAB rule:

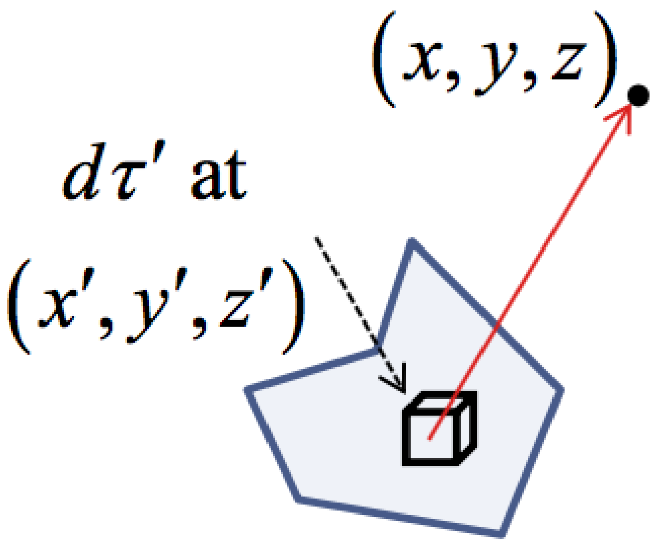
$$\vec{\nabla} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{A}\right) - \vec{A} \cdot \left(\vec{\nabla} \times \vec{B}\right)$$

The curl of $\vec{r}/|\vec{r}|^3$:

$$\vec{\nabla} \times \frac{\vec{r}}{|\vec{r}|^3} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} \end{vmatrix} = 0$$

- Recall that $\vec{r}/|\vec{r}|^3$ appears both in the Coulomb and Biot-Savart law.

Generalisation of the Biot-Savart law for a current density $\vec{j}$



At an earlier lecture, we studied the Biot-Savart law for a constant current I flowing over a linear path:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

where the line integral is over the elements $d\vec{\ell}$ along the conductor, and $\vec{r}$ is the distance from $d\vec{\ell}$ to the point where we want to know the field.

The Biot-Savart law can be trivially generalised for an arbitrary current density $\vec{j}$:

$$\vec{B} = \frac{\mu_0}{4\pi} \int_{\tau'} \frac{\vec{j}(\vec{r}') \times \vec{r}}{r^3} d\tau'$$

The divergence and flux of the magnetic field

Starting from the Biot-Savart law:

$$\vec{B} = \frac{\mu_0}{4\pi} \int_{\tau'} \frac{\vec{j}(\vec{r}') \times \vec{r}}{r^3} d\tau'$$

we will calculate the divergence of $\vec{B}$. The operator $\vec{\nabla}$ commutes with the integral and is applied to the integrant. The BAC-CAB rules gives:

$$\vec{\nabla} \cdot \left(\vec{j}(\vec{r}') \times \frac{\vec{r}}{r^3} \right) = \frac{\vec{r}}{r^3} \left(\vec{\nabla} \times \vec{j}(\vec{r}') \right) - \vec{j}(\vec{r}') \left(\vec{\nabla} \times \frac{\vec{r}}{r^3} \right)$$

The operator $\vec{\nabla}$ involves derivative of the unprimed variables whereas the density $\vec{j}$ is a function of primed variables only, hence: $\vec{\nabla} \times \vec{j}(\vec{r}') = 0$.

As we have also seen: $\vec{\nabla} \times \frac{\vec{r}}{r^3} = 0$.

Therefore:

$$\vec{\nabla} \cdot \vec{B} = 0$$

The divergence and flux of the magnetic field

The equation

$$\vec{\nabla} \cdot \vec{B} = 0$$

expresses the fact that **there are no magnetic monopoles**. Indeed, there is no point in space from where magnetic field lines *emanate from*, and they always loop back on themselves.

For the same reason, we expect that the magnetic flux through a closed surface is zero: The field lines that from a closed surface eventually re-enter as they loop back on themselves.
Gauss' theorem tells us that:

$$\oint_S \vec{B} \cdot d\vec{S} = \int_\tau \vec{\nabla} \cdot \vec{B} d\tau$$

But $\vec{\nabla} \cdot \vec{B} = 0$, hence:

$$\oint_S \vec{B} \cdot d\vec{S} = 0$$

The vector potential

Recall that, in electrostatics, the circuital law:

$$\vec{\nabla} \times \vec{E} = 0$$

allowed us to write the electric field $\vec{E}$ as the gradient of a scalar field (recall that $\vec{\nabla} \times \left(\vec{\nabla}\phi\right) = 0$ for *any* scalar function ϕ). That scalar field was the potential V:

$$\vec{E} = -\vec{\nabla} V$$

By substituting the above into Gauss's law, we found that the scalar potential V satisfies the so-called *Poisson* equation:

$$\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$$

This single 2^{nd} order p.d.e for the electric potential V, is equivalent with the following coupled system of two 1^{st} order p.d.e's: $\vec{\nabla} \times \vec{E} = 0$ and $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$. Using the appropriate boundary conditions we can solve the Poisson's equation to determine V and, thus, determine $\vec{E}$.

The vector potential

Similarly, the relation

$$\vec{\nabla} \cdot \vec{B} = 0$$

allows us to express the magnetic field $\vec{B}$ as the rotation of a vector field (recall that $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A} \right) = 0$ for *any* vector field $\vec{A}$):

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

We call $\vec{A}$ the **vector potential**.

Substituting the above definition into Ampere's law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \Rightarrow \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \mu_0 \vec{j} \Rightarrow$$

$$\vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{j}$$

The vector potential

One can add to V *any function* λ whose gradient is zero ($\vec{\nabla}\lambda = 0$):

$$V \rightarrow V' = V + \lambda$$

and the **physics would remain unchanged**!

Similarly, we can add to $\vec{A}$ *any function* $\vec{\Lambda}$ whose curl vanishes ($\vec{\nabla} \times \vec{\Lambda} = 0$):

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\Lambda}$$

and the **physics would remain unchanged**!

We can use this freedom to eliminate the divergence of $\vec{A}$ ($\vec{\nabla}\vec{A} = 0$). With this condition, $\vec{A}$ satisfies the following p.d.e.:

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{j}$$

This expression is very similar to our known *Poisson* equation. In fact, it is nothing more than a *vector of Poisson* equations.

Maxwell's equation we know so far

In vacuum (static case):

	Integral form	*Differential form*
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = Q_{enclosed}/\epsilon_0$	$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

Lecture 6 - Main points to remember

- At distance ρ, the force between two parallel straight conductors of length L carrying current I_1 and I_2, respectively, is:

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi\rho}$$

  The force is attractive if both currents flow in the same direction, and repulsive if the two currents flow in opposite directions.
- One Ampere is the amount of current that, if maintained between those conductors produces a force of 2×10^{-7} N per metre of length.
- The magnetic dipole moment $\vec{m}$ of a current loop is defined as

$$\vec{m} = I\vec{S}$$

- The magnetic moment $\vec{T}$ of a magnet is a quantity that determines the torque it will experience in an external magnetic field.

$$\vec{T} = \vec{m} \times \vec{B}$$

Lecture 6 - Main points to remember (cont'd)

- **Ampere's law** in integral form:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl} = \mu_0 \int_S \vec{j} \cdot d\vec{S}$$

The line integral of the magnetic field $\vec{B}$ along a closed path L is proportional to the current passing though any surface S defined by L.

- **Ampere's law** in differential form:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

The curl $\vec{B}$ is proportional to the local current density $\vec{j}$.

- The magnetic field inside a toroidal coil with n windings, each carrying current I, is:

$$|\vec{B}| = \frac{\mu_0 n I}{2\pi r}$$

where r is the distance from the centre of the coil.

Lecture 6 - Main points to remember (cont'd)

- The relation:

$$\vec{\nabla} \cdot \vec{B} = 0$$

allows us to express $\vec{B}$ as the curl of a vector field $\vec{A}$:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

We call $\vec{A}$ the **vector potential**.

- There is some freedom in determining $\vec{A}$:
  - Adding to it a function whose curl is zero leaves the physics unchanged.
  - We use this freedom to eliminate the divergence of $\vec{A}$ ($\vec{\nabla} \cdot \vec{A} = 0$).
- The vector potential $\vec{A}$ satisfies the following equation:

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{j}$$

(each component of $\vec{A}$ satisfies a Poisson equation).

At the next lecture (Lecture 7)

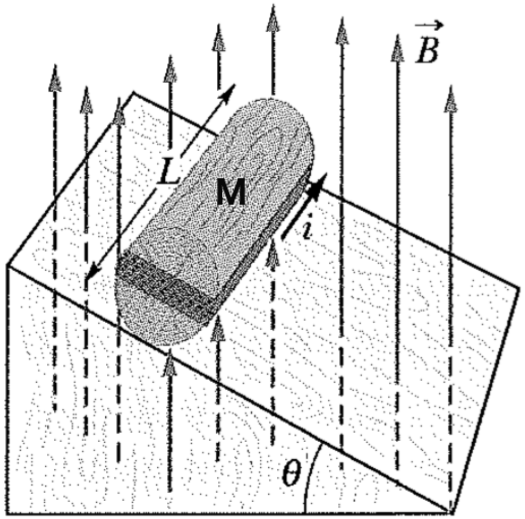
In the next lecture, we will study the magnetic properties of materials:

- diamagnetism
- paramagnetism
- ferromagnetism

Optional reading for Lecture 6

Worked example: Current loop on incline

Question



The figure on the left shows a wood cylinder of mass M = 0.250 kg and length L = 0.100 m, with N = 10.0 turns of wire wrapped around it longitudinally, so that the plane of the wire coil contains the long central axis of the cylinder. The cylinder is released on a plane inclined at an angle θ to the horizontal, with the plane of the coil parallel to the incline plane. If there is a vertical uniform magnetic field of magnitude 0.500 T, what is the least current i through the coil that keeps the cylinder from rolling down the plane?

Worked example: Current loop on incline

We take the x axis to be positive down the incline. Then the x component of Newton's second law for the center of mass yields:

$$Mgsin\theta - f = Ma$$

where f is the force of friction, acting up the incline at the point of contact a is the cylinder's acceleration, and $Mgsin\theta$ is the force of gravity in the direction of x.
Since the plane of the loop is parallel to the incline, the dipole moment $\vec{m}$ is normal to the incline and it has a maginutude of:

$$|\vec{m}| = I|\vec{S}| = NiL2r$$

where $I = Ni$ is the total current flowing in the loop (N turns of current i) and $|\vec{S}| = L2r$ is the area of the loop (L is the length of the cylinder and r is its radius).

Worked example: Current loop on incline

The magnetic field produces a torque with magnitude:

$$|\vec{T_B}| = |\vec{m} \times \vec{B}| = |\vec{m}||\vec{B}|sin\theta$$

The force of friction produces a torque with magnitude:

$$|\vec{T_f}| = fr$$

$\vec{T_B}$ produces an angular acceleration in the counterclockwise direction, and $\vec{T_f}$ produces an angular acceleration in the clockwise direction.

Newton's second law for rotation about the center of the cylinder, gives:

$$|\vec{m}||\vec{B}|sin\theta - fr = I\alpha$$

where I is the mass moment of inertia and α is the angular acceleration.

Worked example: Current loop on incline

Since we want the current that holds the cylinder in place, we set $a = 0$ and $\alpha = 0$.

Therefore:

$$Mgsin\theta - f = Ma \Rightarrow f = Mgsin\theta$$

and

$$|\vec{m}||\vec{B}|sin\theta - fr = 0 \Rightarrow$$

$$(NiL2r)|\vec{B}|sin\theta - (Mgsin\theta)r = 0 \Rightarrow$$

$$i = \frac{Mg}{2NLB} \Rightarrow$$

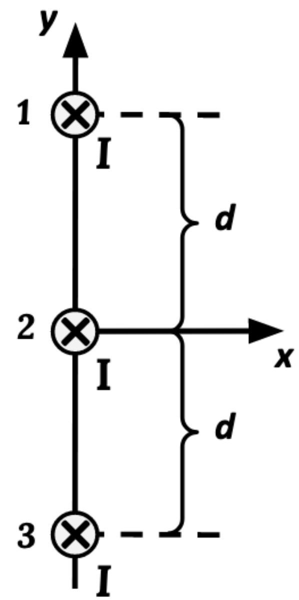
$$i = \frac{(0.250\ kg)(9.8\ m/s^2)}{2(10.0)(0.100\ m)(0.500\ T)} = 2.45\ A$$

Worked example: System of 3 wires

Question

Consider 3 straight, infinitely long, coplanar, equally spaced wires with zero radius, each carrying a current I in the same direction.

1. Find the positions where the magnetic field is zero.
2. Sketch the magnetic field lines.
3. If the middle wire is displaced a very small distance x ($x << d$) to the right (as shown in the diagram) while the other 2 wires are held fixed, show that it will excecute a simple harmonic oscillation. If the wire has linear mass density λ (mass per unit length) find the angular frequency of oscillation.



Worked example: System of 3 wires

The magnetic field $\vec{B}$, produced by each long straight wire carrying a current I, it is azimuthal, and it is given by

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

where r is the distance from the wire.

Since the three wires are coplanar, and they carry current in the same direction producing similar azimulthal magnetic fields, the points of zero magnetic field must be located between the wires, and be on the same plane as the wires. There must be two zero-field points on each plane perpendicular to the three wires. On each such plane, the zero-field points are between the wires and lie on the axis that connects the wires.

Let y be the distance between the middle wire and one of the zero-field points. For each such point, the field from the closest wire that lies on one side will cancel out the anti-parallel fields from the other two wires that lie on the opposite side.

Worked example: System of 3 wires

Therefore, we can write:

$$\frac{\mu_0 I}{2\pi(d-y)} = \frac{\mu_0 I}{2\pi y} + \frac{\mu_0 I}{2\pi(d+y)}$$

The above yields:

$$\frac{1}{d-y} = \frac{1}{y} + \frac{1}{d+y} \Rightarrow$$

$$\frac{1}{d-y} = \frac{d+2y}{y(d-y)} \Rightarrow$$

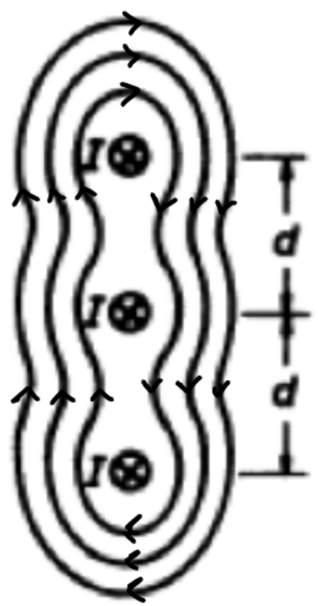
$$y(d+y) = (d-y)(d+2y) \Rightarrow$$

$$\cancel{yd} + y^2 = d^2 + \cancel{2yd} - \cancel{yd} - 2y^2 \Rightarrow$$

$$3y^2 = d^2 \Rightarrow$$

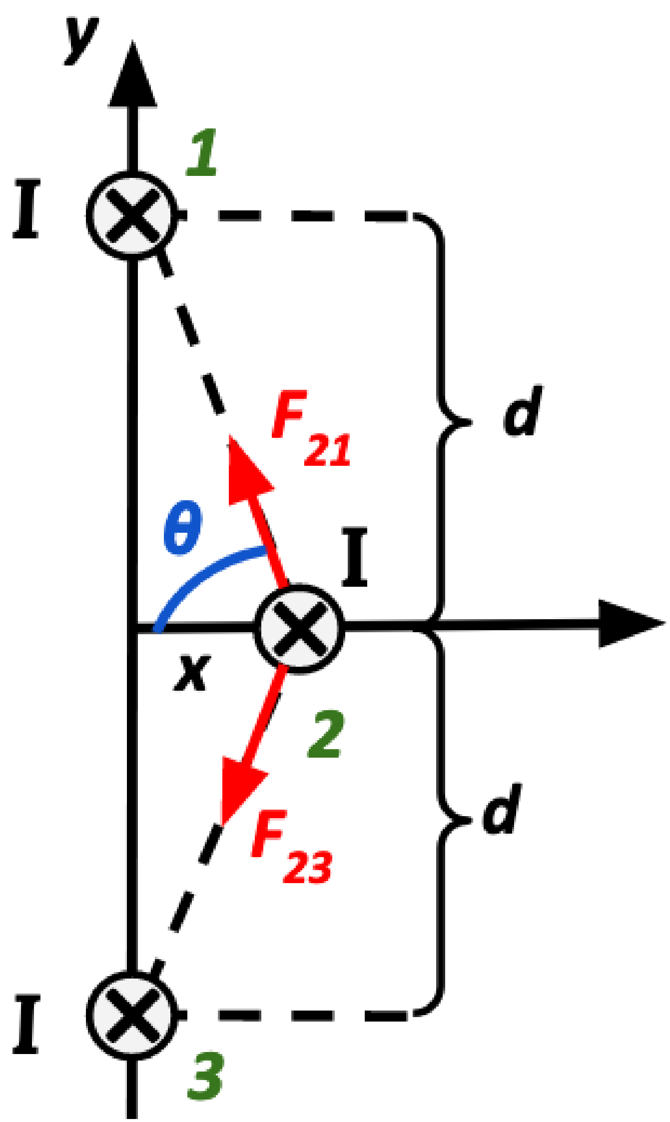
$$y = \pm\frac{d}{\sqrt{3}}$$

The magnetic field lines, are shown below:



Worked example: System of 3 wires

If the middle wire (2) is displaced by a small distance x in the direction of the positive x axis, the (attractive) forces $\vec{F}_{21}$ and $\vec{F}_{23}$ exerted on wire 2 because of wires 1 and 3, correspondingly, are shown in the figure on the right.



Worked example: System of 3 wires

These forces will have the same magnitude, which given by:

$$F_{21} = F_{23} = \frac{\mu_0 I^2}{2\pi\sqrt{d^2 + x^2}} L$$

where L is the length of the wires.
As it can be seen, the y components of $\vec{F}_{21}$ and $\vec{F}_{23}$ cancel out. The (negative) x components of both of these forces will contribute to the total force $\vec{F}_2$ exerted on wire 2, which is given by:

$$\vec{F}_2 = -\Big(F_{21} cos\theta + F_{23} cos\theta\Big)\hat{x}$$

Using the expressions we derived for F_{21} and F_{23}, the above becomes:

$$\vec{F}_2 = -\cancel{2}\frac{\mu_0 I^2}{\cancel{2}\pi\sqrt{d^2 + x^2}} L cos\theta \hat{x}$$

Worked example: System of 3 wires

Given that:

$$cos\theta = \frac{x}{\sqrt{d^2 + x^2}}$$

the previous expression for $\vec{F}_2$ can be written as:

$$\vec{F}_2 = -\frac{\mu_0 I^2}{\pi} \frac{x}{d^2 + x^2} L\hat{x} \xRightarrow{d^2+x^2 \approx d^2}$$

$$\vec{F}_2 = -\frac{\mu_0 I^2 L}{\pi d^2} x\hat{x}$$

As it can be seen, the force is proportional and opposite to the displacement, Hence the motion of the wire is a simple harmonic oscillation about the equilibrium position at x=0.

Using Newton's second law, we can write:

$$\vec{F}_2 = m\frac{d^2x}{dt^2}\hat{x} \Rightarrow$$

Worked example: System of 3 wires

$$-\frac{\mu_0 I^2 L}{\pi d^2} x\hat{x} = m\frac{d^2x}{dt^2}\hat{x} \xRightarrow{m=\lambda L}$$

$$-\frac{\mu_0 I^2 L}{\pi d^2} x = \lambda L \frac{d^2x}{dt^2} \Rightarrow$$

$$\frac{d^2x}{dt^2} + \frac{\mu_0 I^2}{\pi \lambda d^2} x$$

which has the form of the known equation of motion for a harmonic oscillator ($\ddot{x} + \omega^2 x = 0$). From this we can deduce that the angular frequency of oscillation, ω, is given by:

$$\omega = \sqrt{\frac{\mu_0 I^2}{\pi \lambda d^2}} = \frac{I}{d}\sqrt{\frac{\mu_0}{\pi \lambda}}$$

Worked example: Circular coil in uniform field

Question

A circular coil, 0.05 m in radius, with 30 turns of wire, lies on the horizontal plane. It carries a current of 5.0 A counterclockwise (as viewed from the top). The coil is in a uniform magnetic field of 1.2 T directed to the right. Find the magnetic moment and the torque on the coil in vector form.

The magnetic dipole moment $\vec{m}$ of a current loop is defined as:

$$\vec{m} = I\vec{S}$$

where I is the current flowing in the loop and $\vec{S}$ is its area.
The area of a circular coil with radius r = 0.05 m is

$$S = \pi r^2 = \pi (0.05\ m)^2 = 0.00785\ m^2$$

The current I_{turn} flowing through each turn of wire is 5.0 A and the coil has N = 30 turns. Hence, the total current is:

$$I = NI_{turn} = 30 \cdot 5\ A = 150\ A$$

Worked example: Circular coil in uniform field

Therefore, the magnitude of the magnetic moment $\vec{m}$ is:

$$|\vec{m}| = I \cdot S = \Big(150\ A\Big) \cdot \Big(0.00785\ m^2\Big) \approx 1.18\ A \cdot m^2$$

The vector $\vec{m}$ is perpendicular to the loop and points in the same direction as your thumb if your right-hand curls in the direction of the current. The given circular coil lies on the horizontal plane (which we assume to be the xy plane, with x pointing on the right) with I flowing counterclockwisely.

Therefore, the direction of the magnetic moment is along $\hat{z}$:

$$\vec{m} \approx \Big(1.18\ A \cdot m^2\Big)\hat{z}$$

The torque exerted by a magnetic field is:

$$\vec{T} = \vec{m} \times \vec{B} \Rightarrow$$

$$\vec{T} = \Big(1.18\ A \cdot m^2\Big)\hat{z} \times \Big(1.2\ T\Big)\hat{x} \approx \Big(1.42\ N \cdot m\Big)\Big(\hat{z} \times \hat{x}\Big) = \Big(1.42\ N \cdot m\Big)\hat{y}$$

Worked example: Magnetic field of cylindrical conductor

Question

A long non-magnetic cylindrical conductor with inner radius a and outer radius b carries a current I which flows along the direction of the axis of symmetry of the cylinder. Assume that the current density in the conductor is uniform.

The current I produces a magnetic field $\vec{B}$. Find the magnetic field $\vec{B}$ as a function of the radial co-ordinate r and express it in vector form for the following cases:

- inside the hollow space ($r < a$),
- within the cylindrical conductor ($a < r < b$), and
- outside the conductor ($r > a$).

Worked example: Magnetic field of cylindrical conductor

For the calculation of the magnetic field in this cylindrically-symmetric problem we will use Ampere's circuital law for a closed path L

$$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

where I_{enc} is the current that flows through the open surface $S(L)$ that has the closed path L as its boundary:

$$I_{enc} = \int_{S(L)} \vec{j} \cdot d\vec{S}$$

The uniform current density j, within the conductor, can be written as the ratio of the total current I over the area of a cross-section of the cylinder perpendicular to the direction of the current flow. If the axis of the cylinder is the z axis, and the current flows in the positive direction, we can write:

$$\vec{j} = \frac{I}{\pi(b^2 - a^2)}\hat{z} \qquad (\text{for } a < r < b)$$

Worked example: Magnetic field of cylindrical conductor

For *any* closed path L in the region $r < a$, $I_{enc} = 0$. Ampere's law yields:

$$\oint_L \vec{B} \cdot d\vec{\ell} = 0$$

Since this is true for any arbitrary closed path, it can not be the result of an accidental cancellation and it has to be the integrand itself that is zero. Therefore, for $r < a$:

$$B = 0$$

For $a < r < b$, in order to exploit the cylindrical symmetry of the problem in the application of Ampere's law, we choose a circular integration path L with radius r whose centre is on the symmetry axis of the cylinder.
The current I_{enc} that flows through the area S of a circle that has L as its boundary and it is perpendicular to $\vec{j}$, is given by:

$$I_{enc} = \int_{S(L)} \vec{j} \cdot d\vec{S} = jS = I \frac{r^2 - a^2}{b^2 - a^2}$$

Worked example: Magnetic field of cylindrical conductor

Substituting the above in Ampere's circuital law, and considering that, due to symmetry, $\vec{B}$ is an azimuthal vector and, along L, the vectors $\vec{B}$ and $d\vec{\ell}$ are co-linear, we obtain:

$$B 2\pi r = \mu_0 I \frac{r^2 - a^2}{b^2 - a^2} \Rightarrow B = \frac{\mu_0 I}{2\pi r} \frac{r^2 - a^2}{b^2 - a^2}$$

Therefore, for $a < r < b$:

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \frac{r^2 - a^2}{b^2 - a^2} \hat{\phi}$$

For $r > b$, we can use similar procedure and arguments as in the case of $a < r < b$, but with:

$$I_{enc} = I$$

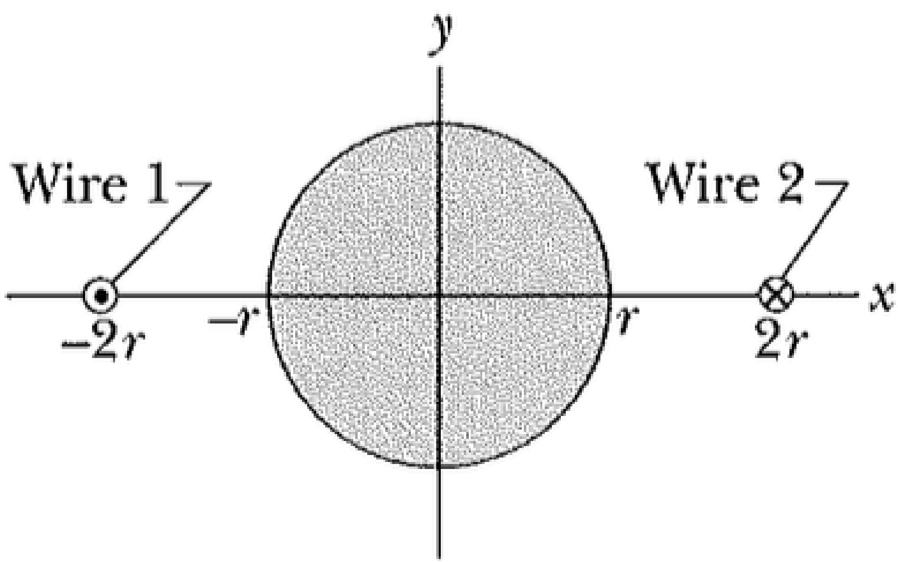
since our integration path includes the full cylindrical conductor.
Therefore, for $r > b$:

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

Worked example: Magnetic flux through half cylinder

Question

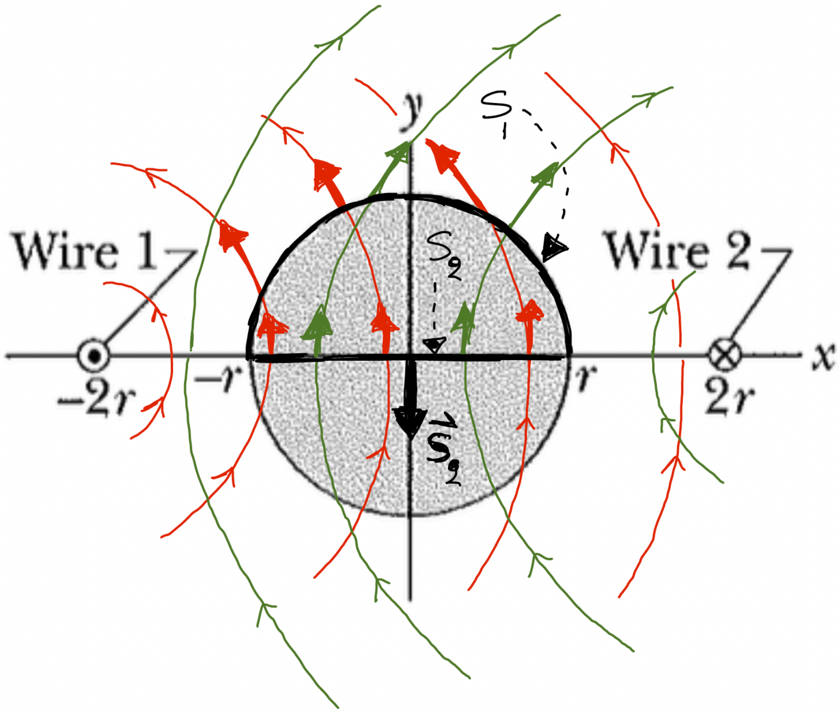
Two wires, parallel to a z axis and a distance $4r$ apart, carry equal currents i in opposite directions, as shown below. A circular cylinder of radius r and length L has its axis on the z axis, midway between the wires. Use Gauss' law for magnetism to derive an expression for the net outward magnetic flux through the half of the cylindrical surface above the x axis.



Worked example: Magnetic flux through half cylinder

Let S_1 be the half of the cylindrical surface (above the x axis) for which we are interested in calculating the magnetic flux, and S_2 be the portion of the xz plane that lies within the cylinder.

Each of the wires 1 and 2 produces an azimulthal magnetic field, as shown on the figure below. The total magnetic field has no z component.



Worked example: Magnetic flux through half cylinder

From Gauss' law for magnetism, the flux through the closed surface formed by S_1, S_2 and two end-cups on the xy plane (through which there is no flux, since the field of each wire is azimuthal) is zero. Therefore:

$$\Phi_{S_1} + \Phi_{S_2} = 0 \Rightarrow \Phi_{S_1} = -\Phi_{S_2}$$

So we can obtain the flux through the half-cylindrical surface S_1, by calculating the flux through the planar surface S_2. The latter is a much easier calculation.

The total magnetic flux through the surface S_2 is given by

$$\Phi_{S_2} = \int_{S_2} \vec{B} \cdot d\vec{S}$$

The total field $\vec{B}$ is the superposition of the fields $\vec{B}_1$ and $\vec{B}_2$ produced by the wires 1 and 2 respectively.

$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

Worked example: Magnetic flux through half cylinder

The field produced by a long straight wire with current I_k is given by

$$\vec{B}_k = \frac{\mu_0 I_k}{2\pi r_k} \hat{\phi}_k$$

where r_k is the distance from the wire and $\hat{\phi}_k$ is the azimuthal direction around that wire.

The current in wire 1 travels along $\hat{z}$ (out of the page) and therefore its magnetic field, as viewed from above, rotates anticlockwisely. On the surface S_2, the field $\vec{B}_1$ points along $\hat{y}$. Similarly, the current in wire 2 travels along -$\hat{z}$ (into the page) and therefore its magnetic field, as viewed from above, rotates clockwisely. On the surface S_2, the field $\vec{B}_2$ points along $\hat{y}$ too.

So, in summary, on the surface S_2:

$$\vec{B}_1 = \frac{\mu_0 i}{2\pi r_1}\hat{y} = \frac{\mu_0 i}{2\pi(2r+x)}\hat{y} \quad \text{and} \quad \vec{B}_2 = \frac{\mu_0 i}{2\pi r_2}\hat{y} = \frac{\mu_0 i}{2\pi(2r-x)}\hat{y}$$

Worked example: Magnetic flux through half cylinder

By convention, the unit vector for each element of a closed surface points outwards, so the surface vector of S_2 points towards -$\hat{y}$. Therefore, Φ_{S_2} is:

$$\Phi_{S_1} = -\Phi_{S_2} = -\int_{S_2} (\vec{B}_1 + \vec{B}_2) \cdot d\vec{S}$$

$$= \frac{\mu_0 iL}{2\pi}\{\int_{-r}^{+r} \frac{dx}{2r+x} + \int_{-r}^{+r} \frac{dx}{2r-x}\} = \frac{\mu_0 iL}{2\pi}\{\int_{-r}^{+r} \frac{d(2r+x)}{2r+x} - \int_{-r}^{+r} \frac{d(2r-x)}{2r-x}\}$$

$$= \frac{\mu_0 iL}{2\pi}\{ln(2r+x)|_{-r}^{+r} - ln(2r-x)|_{-r}^{+r}\}$$

$$= \frac{\mu_0 iL}{2\pi}\{(ln(3r) - ln(r)) - (ln(r) - ln(3r)\} = \frac{\mu_0 iL}{2\pi}(2ln(3r) - 2ln(r)) = \frac{\mu_0 iL}{2\pi} 2ln3 \Rightarrow$$

$$\Phi_{S_1} = \frac{\mu_0 i}{\pi} L ln3$$

PHYS 201 / Lecture 7
Magnetization; H-field; Ampere's law in materials; Diamagnetism, Paramagnetism and Ferromagnetism;

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Rutherford Appleton Laboratory

- At distance ρ, the force between two parallel straight conductors of length L carrying current I_1 and I_2, respectively, is:

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi\rho}$$

  The force is attractive if both currents flow in the same direction, and repulsive if the two currents flow in opposite directions.
- One Ampere is the amount of current that, if maintained between those conductors produces a force of 2×10^{-7} N per metre of length.
- The magnetic dipole moment $\vec{m}$ of a current loop is defined as

$$\vec{m} = I\vec{S}$$

- The magnetic moment $\vec{T}$ of a magnet is a quantity that determines the torque it will experience in an external magnetic field.

$$\vec{T} = \vec{m} \times \vec{B}$$

- **Ampere's law** in integral form:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl} = \mu_0 \int_S \vec{j} \cdot d\vec{S}$$

The line integral of the magnetic field $\vec{B}$ along a closed path L is proportional to the current passing though any surface S defined by L.

- **Ampere's law** in differential form:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

The curl $\vec{B}$ is proportional to the local current density $\vec{j}$.

- The magnetic field inside a toroidal coil with n windings, each carrying current I, is:

$$|\vec{B}| = \frac{\mu_0 n I}{2\pi r}$$

where r is the distance from the centre of the coil.

Lecture 6 - Revision (cont'd)

- The relation:

$$\vec{\nabla} \cdot \vec{B} = 0$$

allows us to express $\vec{B}$ as the curl of a vector field $\vec{A}$:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

We call $\vec{A}$ the **vector potential**.

- There is some freedom in determining $\vec{A}$:
  - Adding to it a function whose curl is zero leaves the physics unchanged.
  - We use this freedom to eliminate the divergence of $\vec{A}$ ($\vec{\nabla} \cdot \vec{A} = 0$).
- The vector potential $\vec{A}$ satisfies the following equation:

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{j}$$

(each component of $\vec{A}$ satisfies a Poisson equation).

Plan for Lecture 7

In this lecture:

- We will discuss the magnetic properties of materials (**diamagnetism**, **paramagnetism** and **ferromagnetism** (*)) and develop arguments to understand the physical origins.
- We will complete the study of **Maxwell's eqs. in materials (for static fields)**.

(*) There are other types of magnetism too: *antiferromagnetism*, *ferrimagnetism*, *superparamagnetism*, *metamagnetism*. We will neglect these.
(**) In this lecture, quite often, I will remind you of what I discussed a few weeks ago in the lecture on dielectrics and polarisation.

The physical origins of the magnetisation and polarisation effects are very different. But, surprisingly, **the maths are very very similar**.

Magnetic properties of materials

The material which has the **most striking and well known magnetic properties is iron (Fe).**

Similar properties are exhibited by

- **nickel (Ni)**,
- **cobalt (Co)**,
- **gadolinium (Gd)**, and
- **dysprosium (Dy)**.

We call these materials **ferromagnets**. Ferromagnetism is due to a quantum effect called *exchange coupling*.

Not only these materials can have a significant magnetisation when inside an external magnetic field, they also **retain their magnetisation in the absence of an external magnetic field.**

But **other substances get magnetised too.**

- It might seem odd but, **water** can be magnetised!
- **Wood** can be magnetised!
- **Frogs** can be magnetised!
- And, of course, **you** can be magnetised too!

But you can not attract wood with a horseshoe magnet. And you need an enormous magnetic field ($\sim$ 15 T) to levitate a frog.

The magnetic effects for these materials are **very very weak!**

- $\sim$million times weaker than the effects in ferromagnetic materials.

Moreover, water, wood and frogs do not remain magnetized once the external magnetic field is removed.

Magnetic properties of materials

The materials that exhibit weaker and non-permanent magnetic effects have a rather **odd behaviour** in the following sense:

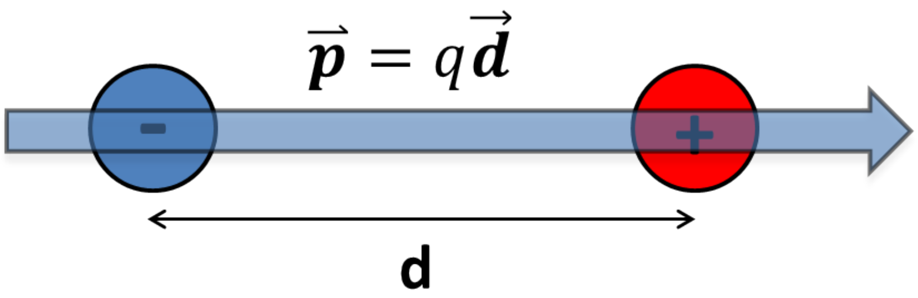
Recall that an **electric field polarises a dielectric in** (more or less) **the direction of the field**. However, in the presence of a magnetic field, some substances get magnetised in the direction of the field and some in the opposite direction!

- Substances that get **magnetised in the direction of the magnetic field**, are called **paramagnetic**.
- Substances that get **magnetised in the direction opposite to the magnetic field**, are called **diamagnetic**.

What physics underpins that difference in the magnetic behaviour?

We will develop (..wrong) classical arguments to understand these effects.

Reminder: Electric dipole moments and polarization

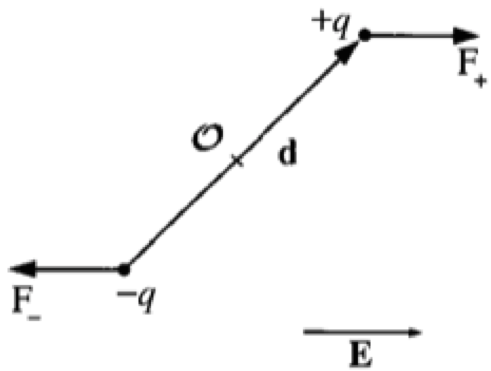


An electric dipole its described by its electric dipole moment $\vec{p}$ (a vector, pointing from the - to the + charge):

$$\vec{p} = q\vec{d}$$

An external electric field **induces a dipole moment** in the direction of the field, or **rotates polar molecules** towards the direction of the field.

The **torque** $\vec{T}$ is given by: $\vec{T} = \vec{p} \times \vec{E}$

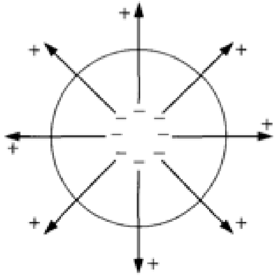


The net effect is that matter gets **polarised at a macroscopic level** (not just at the level of individual atoms or molecules).

We defined the **polarisation** $\vec{P}$ as the **amount of electric dipole moment per unit volume**.

Reminder: Electric dipole moments and polarization

In a polarised material, there is accumulation of **induced charge**:
Both **surface** and **volume** charge is induced.



We convinced ourselves that the density of the surface charge is $\sigma_P = \vec{P}\hat{n}$ where $\hat{n}$ is a unit vector normal to the surface, whereas the density of the volume charge is $\rho_P = -\vec{\nabla}\vec{P}$.

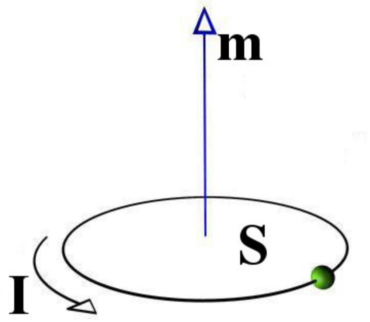
The charges induced by polarisation, **create their own electric field $\vec{E}_P$ that opposes the external electric field.**

$$\vec{E}_P(\vec{r}) = -\vec{\nabla} V_P(\vec{r})$$

$$V_P(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{|\vec{r}-\vec{r}'|}\sigma_P(\vec{r}')dS' + \frac{1}{4\pi\epsilon_0}\int \frac{1}{|\vec{r}-\vec{r}'|}\rho_P(\vec{r}')d\tau'$$

Magnetic dipole moment and magnetization

The situation in magnetism is very very similar (at least mathematically).



We defined the **magnetic dipole moment** $\vec{m}$ as:

$$\vec{m} = I\vec{S}$$

An external magnetic field $\vec{B}$ exerts a torque $\vec{T}$ on a magnetic dipole $\vec{m}$ which is given by:

$$\vec{T} = \vec{m} \times \vec{B}$$

This will tend to **align** the previously randomised **magnetic moments** and **create magnetisation at a macroscopic level**.

We define **magnetisation** $\vec{M}$ as the amount of **magnetic dipole moment per unit volume**.

Magnetization-induced currents

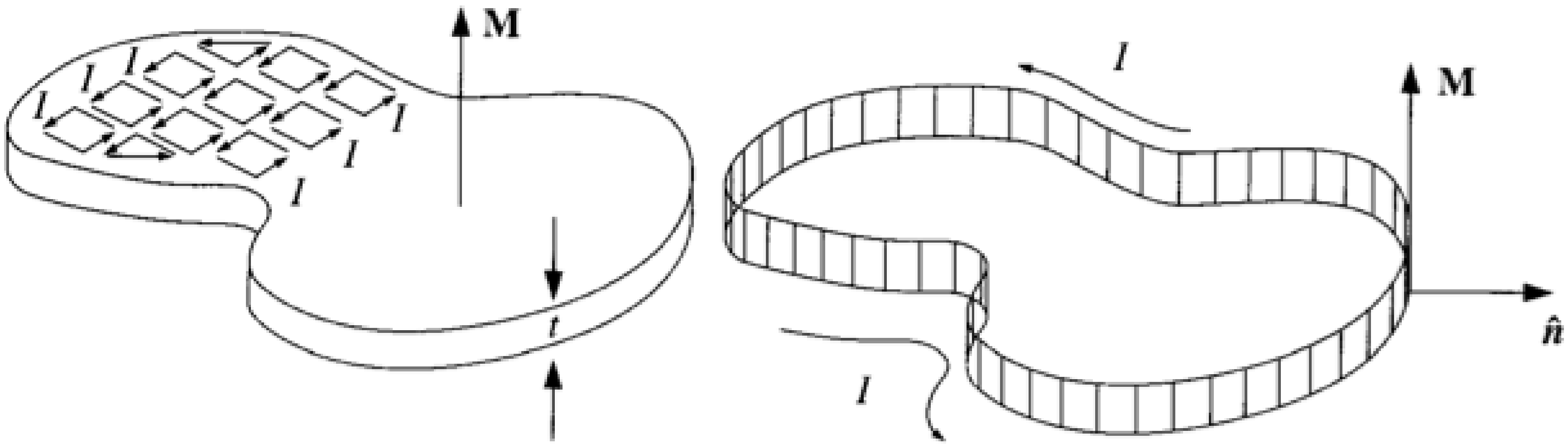
The **magnetisation induces surface and volume currents**.

We can easily be convinced, although we will not show it mathematically, that the **density of the surface current** is:

$$j_m^{surf} = \vec{M} \times \hat{n}$$

whereas the **density of the volume current** is:

$$j_m^{vol} = \vec{\nabla} \times \vec{M}$$



Correspondence between quantities

Electrostatics		Magnetostatics	
electric dipole moment	$\vec{p} = q\vec{d}$	$\vec{m} = I\vec{S}$	magnetic dipole moment
torque within $\vec{E}$ field	$\vec{T} = \vec{p} \times \vec{E}$	$\vec{T} = \vec{m} \times \vec{B}$	torque within a $\vec{B}$ field
polarization	$\vec{P} = \frac{(e.d.m)}{volume}$	$\vec{M} = \frac{(m.d.m)}{volume}$	magnetization
surface charge density	$\sigma_P = \vec{P} \cdot \hat{n}$	$j_m^{surf} = \vec{M} \times \hat{n}$	surface current density
volume charge density	$\rho_P = -\vec{\nabla} \cdot \vec{P}$	$j_m^{vol} = \vec{\nabla} \times \vec{M}$	volume current density

Recall **Ampere's law in vacuum**:

$$\oint_L \vec{B}\cdot d\vec{\ell} = \mu_0 \int_{S(L)} \vec{j}\cdot d\vec{S} \;\; (\textit{integral form}), \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{j} \;\; (\textit{differential form})$$

Ampere's law,

- in integral form, connects the line integral of the magnetic field $\vec{B}$ along a closed path L with the current I flowing through the surface S(L) defined by the closed path L, and,
- in differential form, connects the rotation of the magnetic field, at any point in space, with the local current density $\vec{j}$.

Our **objective is to re-formulate Ampere's law for materials** where, **in addition to free currents**, we have **magnetization-induced currents**.

Recall that we did something very similar with Gauss's law in electrostatics.

Reminder: Gauss' law in materials

We started from Gauss' law in vacuum:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

and wrote the total charge density ρ as the sum of the free (ρ_f) and polarisation (ρ_P) charge densities:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f + \rho_P}{\epsilon_0}$$

The polarisation charge density ρ_P is given, in terms of the polarisation field $\vec{P}$, by:

$$\rho_P = -\vec{\nabla} \cdot \vec{P}$$

and, therefore, Gauss' law becomes:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f - \vec{\nabla} \cdot \vec{P}}{\epsilon_0}$$

Reminder: Gauss' law in materials

We collected all the divergences together and we obtained:

$$\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \vec{P}\right) = \rho_f$$

This is the differential form of Gauss's law in materials.

We defined the electric displacement $\vec{D}$ as:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

and, therefore, Gauss's law in materials was written as:

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

In integral form, Gauss's law in materials becomes:

$$\oint_S \vec{D} \cdot d\vec{S} = \int_{\tau(S)} \rho_f d\tau = Q_f$$

Ampere's law in materials

Similarly, starting from Ampere's law in vacuum:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

we can write the total current density $\vec{j}$ as the vector sum of the free ($\vec{j}_f$) and magnetization ($\vec{j}_m$) current densities:

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big(\vec{j}_f + \vec{j}_m \Big)$$

Expressing the magnetization current density $\vec{j}_m$ in terms of the magnetization field $\vec{M}$:

$$\vec{j}_m = \vec{\nabla} \times \vec{M}$$

we can write Ampere's law as:

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big(\vec{j}_f + \vec{\nabla} \times \vec{M} \Big)$$

Ampere's law in materials

Dividing with μ_0 and collecting the curls together and get:

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_0} = \vec{j}_f + \vec{\nabla} \times \vec{M} \Rightarrow \vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{j}_f$$

We define the **magnetic field strength** or **magnetic field intensity** $\vec{H}$ as:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

we can write the differential form of Ampere's law in materials as:

$$\vec{\nabla} \times \vec{H} = \vec{j}_f$$

Using Stokes' theorem, as we have done several times in past lectures, we can go to the integral form of Ampere's law in materials which is:

$$\oint_L \vec{H} \cdot d\vec{\ell} = \int_{S(L)} \vec{j}_f \cdot d\vec{S} = I_f$$

The "*H field*"

In SI, the quantity $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ has **units of A/m**.

$\vec{H}$ plays a role analogous to $\vec{D}$ in electrostatics: It allows us to write Ampere's law in terms of the free currents alone.

We called $\vec{H}$ **magnetic field strength** or **magnetic field intensity**.

- Some textbooks also call $\vec{H}$ the ... **auxiliary field**.
- Other textbooks call $\vec{H}$ the **magnetic field**, and then call $\vec{B}$ something else (typically, magnetic induction). Confusion is inevitabe!
- Sometimes, $\vec{H}$ is called the **magnetising field** - I like this best.
- Griffiths makes a good suggestion: *"H has no sensible name. Just call it H."* (at least we all agree on the symbol).

Typically, it is easier to think in terms of $\vec{H}$ and $\vec{E}$: At the Lab we control free currents (hence $\vec{H}$) and the voltage of EMF sources (hence $\vec{E}$).
But let's not be confused: $\vec{B}$ **and** $\vec{E}$ **are the fundamental quantities**.

Reminder: Electric susceptibility

The Gauss' law in materials is: $\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \vec{P}\right) = \rho_f$

The polarisation vector $\vec{P}$ can be expressed in terms of $\vec{E}$:

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

where χ_e is the so-called **electric susceptibility** (dimensionless).

For **linear dielectrics** (and low intensity fields) χ_e is a constant that does not depend on $\vec{E}$. Therefore, Gauss' law can be written as:

$$\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \chi_e \epsilon_0 \vec{E}\right) = \rho_f \Rightarrow (1+\chi_e)\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_f \Rightarrow \epsilon_r \epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_f \Rightarrow \epsilon \vec{\nabla} \cdot \vec{E} = \rho_f$$

where the factor $\epsilon_r = 1 + \chi_e$ is the **relative permittivity** or **dielectric constant** (dimensionless) and $\epsilon = \epsilon_r \epsilon_0$ is the **permittivity** of the dielectric (SI unit: $A \cdot s \cdot V^{-1} \cdot m^{-1}$).

Magnetic susceptibility

Ampere's law in materials is: $\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \mu_0 \vec{j_f}$

If the analogy with electrostatics was exact, we would write $\vec{M}$ in terms of $\vec{B}$. However, this is where the analogy breaks. Instead we typically write:

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is the **magnetic susceptibility**. For **linear materials**, χ_m is a constant independent of the value of $\vec{H}$. Expressing $\vec{B}$ in terms of $\vec{H}$:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \vec{B} = \mu_0 \Big(\vec{H} + \vec{M} \Big) \xRightarrow{\vec{M}=\chi_m \vec{H}} \vec{B} = \Big(1 + \chi_m \Big) \mu_0 \vec{H} \Rightarrow$$

$$\vec{B} = \mu_r \mu_0 \vec{H} \Rightarrow \vec{B} = \mu \vec{H}$$

where $\mu_r = 1 + \chi_\mu$ is the **relative permeability** (dimensionless) and $\mu = \mu_r \mu_0$ is the **permeability** of the material (SI unit: $V \cdot s \cdot A^{-1} \cdot m^{-1}$).

Correspondence between quantities

Electrostatics		Magnetostatics	
electric dipole moment	$\vec{p} = q\vec{d}$	$\vec{m} = I\vec{S}$	magnetic dipole moment
torque within $\vec{E}$ field	$\vec{T} = \vec{p} \times \vec{E}$	$\vec{T} = \vec{m} \times \vec{B}$	torque within a $\vec{B}$ field
polarization	$\vec{P} = \frac{(e.d.m)}{volume}$	$\vec{M} = \frac{(m.d.m)}{volume}$	magnetization
surface charge density	$\sigma_P = \vec{P} \cdot \hat{n}$	$j_m^{surf} = \vec{M} \times \hat{n}$	surface current density
volume charge density	$\rho_P = -\vec{\nabla} \cdot \vec{P}$	$j_m^{vol} = \vec{\nabla} \times \vec{M}$	volume current density
electric displacement	$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$	$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$	magnetizing field
Gauss' law in materials	$\vec{\nabla} \cdot \vec{D} = \rho_f$	$\vec{\nabla} \times \vec{H} = \vec{j}_f$	Ampere's law in materials
	$\oint_S \vec{D} \cdot d\vec{S} = Q_f$	$\oint_L \vec{H} \cdot d\vec{\ell} = I_f$	

Maxwell's equations we know so far

Static case in vacuum		
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int \rho d\tau = \frac{Q}{\epsilon_0}$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot d\vec{S} = \mu_0 I$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

Static case within materials		
Gauss's law	$\oint \vec{D} \cdot d\vec{S} = \int \rho_f d\tau = Q_f$	$\vec{\nabla} \cdot \vec{D} = \rho_f$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{H} \cdot d\vec{\ell} = \int \vec{j_f} \cdot d\vec{S} = I_f$	$\vec{\nabla} \times \vec{H} = \vec{j_f}$

Diamagnetic materials

In **diamagnetic** substances, the magnetization is in a direction **opposite** to that of an externally applied magnetizing field.

- Diamagnetism is a **weak and non-permanent effect**.
- Diamagnetic materials are **repelled** by the applied magnetic field.

Reminder:

$$\vec{M} = \chi_m \vec{H}$$

$$\vec{B} = \mu \vec{H}$$

$$\mu = \mu_0 \Big(1 + \chi_m\Big)$$

Since $\vec{M}$ and $\vec{H}$ are anti-parallel: $\chi_m < 0$

Diamagnetism is a weak effect: $|\chi_m| << 1$
and, therefore: $\mu/\mu_0 < 1$

Typical values of χ_m for diamagnetic substances:

Substance	χ_m at $T = 0^o C$
H_2	$-2.3 \cdot 10^{-9}$
$H_2 O$	$-1.2 \cdot 10^{-5}$
N_2	$-0.7 \cdot 10^{-8}$
Ag	$-2.5 \cdot 10^{-5}$

Paramagnetic materials

In **paramagnetic** substances, the magnetization is in the same direction as that of an externally applied magnetizing field.

- Paramagnetism is a **weak and non-permanent effect**.
- Paramagnetic materials are **attracted** by the applied magnetic field.

Reminder:

$$\vec{M} = \chi_m \vec{H}$$

$$\vec{B} = \mu \vec{H}$$

$$\mu = \mu_0 \Big(1 + \chi_m \Big)$$

Since $\vec{M}$ and $\vec{H}$ are parallel: $\chi_m > 0$

Paramagnetism is a weak effect: $|\chi_m| << 1$
and, therefore: $\mu/\mu_0 > 1$

Typical values of χ_m for paramagnetic substances:

Substance	χ_m at $T = 20^o C$
O_2	$1.8 \cdot 10^{-8}$
Pt	$2.7 \cdot 10^{-5}$
Al	$2.1 \cdot 10^{-5}$

Ferromagnetic materials

For **ferromagnetic** materials, the **magnetisation** produced by an external field is **much much larger**.

- Previously quoted values of the magnetic susceptibility for paramagnetic and diamagnetic materials that were at most in the few $\times 10^{-5}$ range.
- For iron, nickel and Cobalt, the magnetic susceptibility is $\sim 10^{+6}$!

In ferromagnetic materials the **magnetic susceptibility is not a constant**

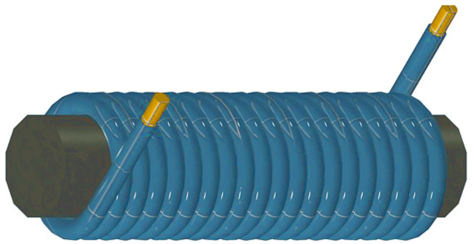
- i.e. ferromagnets are not linear materials.

As matter of fact, the **magnetic susceptibility does not have a single value for a given M and H!**

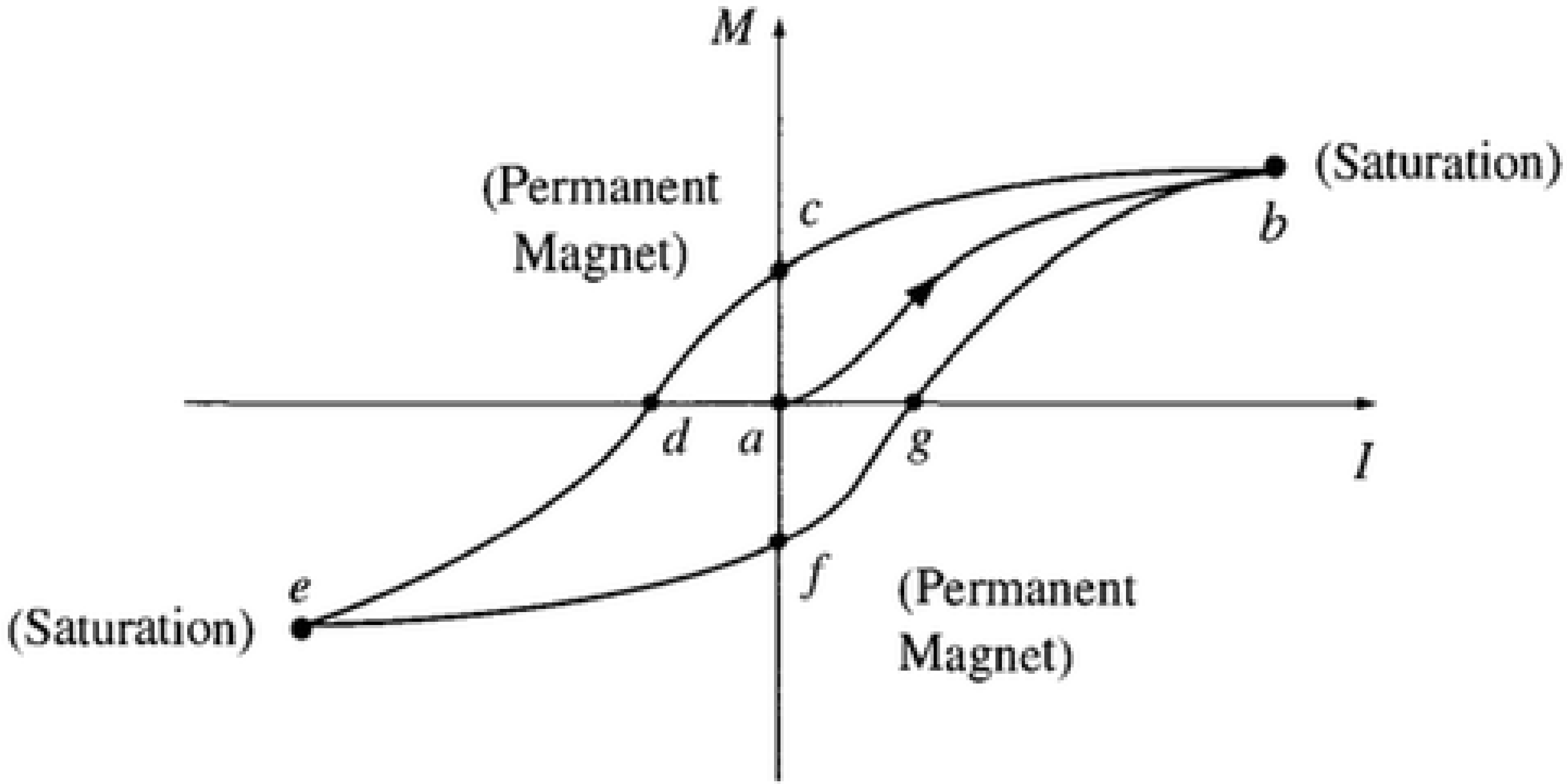
- i.e. the magnetisation depends on its magnetisation history!

Ferromagnetic materials **retain a magnetisation** even when the magnetising field is removed!

Hysteresis loop



Consider the simple electromagnet shown on the left. Current I running in the coil turns creates a magnetic field and magnetizes an iron core. Let's consider the magnetization M as function of the current intensity I.



Physical origins

What physics underpins the different magnetic behaviour of diamagnetic, paramagnetic and ferromagnetic materials?

I will attempt to give some **classical arguments** and descriptions aiming to develop a qualitative understanding.

But, keep in mind that, our classical explanations, although simple and intuitive, are **ultimately wrong**.

- They are **quantum-mechanical effects**. A detailed description of diamagnetism, paramagnetism and ferromagnetism is beyond the scope of this introductory course.
- In fact, our explanation is wrong even in the context of classical physics (See *'Classical physics gives neither diamagnetism nor paramagnetism', Section 34.6 in Feynman lectures, available online*).

But we will stick to these classical arguments for now.

Physical origin of diamagnetism

Consider an electron orbiting a nucleus anticlockwisely on the xy plane:
A **magnetic dipole moment** is associated with the orbital motion:

$$\vec{m} = -\frac{1}{2} e u R \hat{z}$$

If a magnetic field $\vec{B}$ is applied, the magnetic moment changes in a direction **opposite to the magnetic field** (See 'Optional reading'):

$$\Delta\vec{m} = -\frac{e^2 R^2}{4 m_e} \vec{B}$$

Electrons rotate around the nucleus in **random directions**: There is no net magnetization as their **magnetic dipole moments cancel out**.

In an external field, $\Delta\vec{m}$ is **antiparallel to that field for <u>all</u> electrons**: This creates **net magnetisation opposite to the external field**.

Diamagnetism is a **universal phenomenon**.

- It occurs in all substances, even paramagnetic ones.

Physical origin of paramagnetism

Just as the orbital angular momentum of the electron is responsible for a magnetic dipole moment, **so is its spin**!

- Nothing "spins". Spin is an *intrinsic* angular momentum of the electron.
- So, intrinsically, *every electron is a small magnetic dipole*.

In the atomic shells, electrons are arranged **in pairs of opposite spin**: The intrinsic magnetic dipole moments of the pair are canceled out.

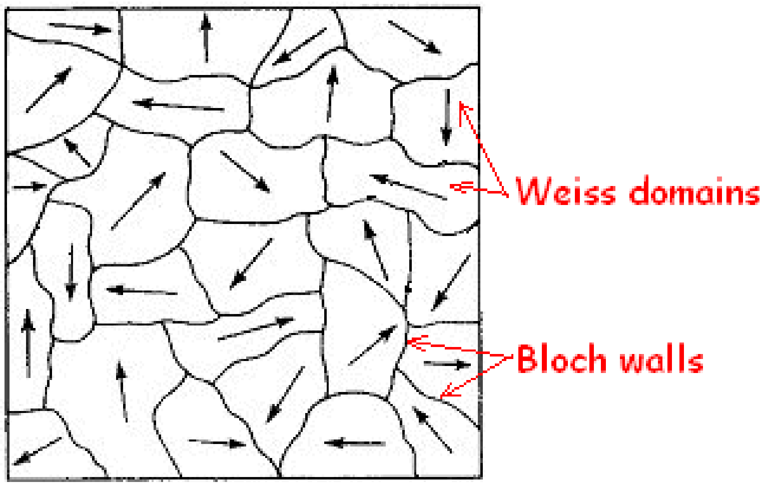
But in atoms with **odd number** of electrons, there is an **unmatched electron**: Its intrinsic magnetic dipole moment is not canceled out.

In the presence of a magnetic field, **the unmatched electron in each atom**, feels a torque ($T = \vec{m} \times \vec{B}$) and it is **aligned with field**: This creates a net macroscopic **magnetisation along the field**.

Realigning the spin of an electron is easier than rearranging the atomic orbits. Thus, even though diamagnetism is universal, paramagnetism is the dominant effect for some substances.

Physical origin of ferromagnetism

Ferromagnetism, like paramagnetism, involves the magnetic dipole moments due to the **spin of unpaired atomic electrons**.



Within regions (called *Weiss* domains), every magnetic dipole *prefers* to be aligned at the same direction as its neighbours. Therefore, each Weiss domain has a macroscopic magnetisation.

For a sizeable chunk of material, there is a large numbers of domains magnetized at random directions (*) so there is **no net magnetisation**.

(*) actually, there may be preferential direction within a crystal, but a sizeable chunk of material contains a large number or randomly oriented crystals.

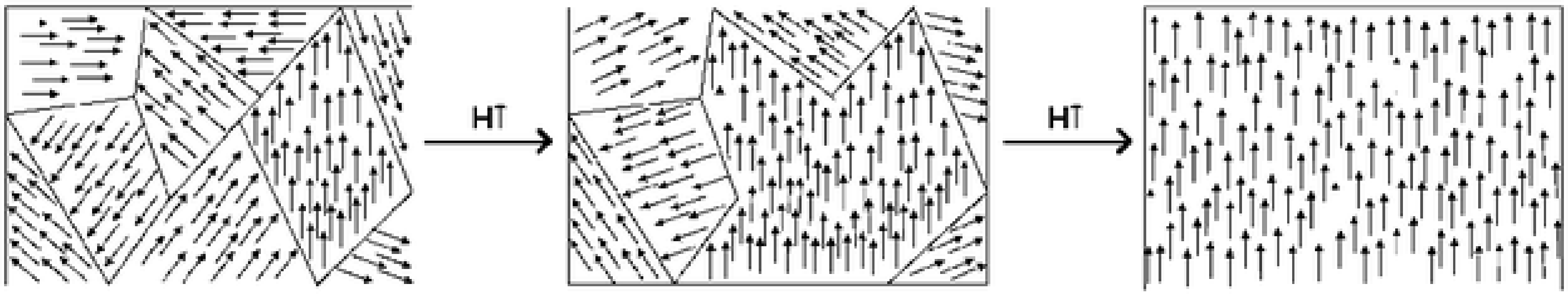
Physical origin of ferromagnetism

A magnetic field $\vec{B}$ will try to align each dipole with it ($\vec{T} = \vec{m} \times \vec{B}$), but dipoles prefer to stay aligned with their neighbours.

Now, consider dipoles neighbouring Weiss domains whose magnetization is already in the direction of $\vec{B}$. These dipole would feel:

- the influence of the torque due to $\vec{B}$, and
- the influence of the neighbouring dipoles already aligned with $\vec{B}$.

The end result is that **Weiss domains re-arrange their boundaries**. Domains with magnetization in the direction of $\vec{B}$ grow - others shrink. There is now **net magnetisation towards** $\vec{B}$.

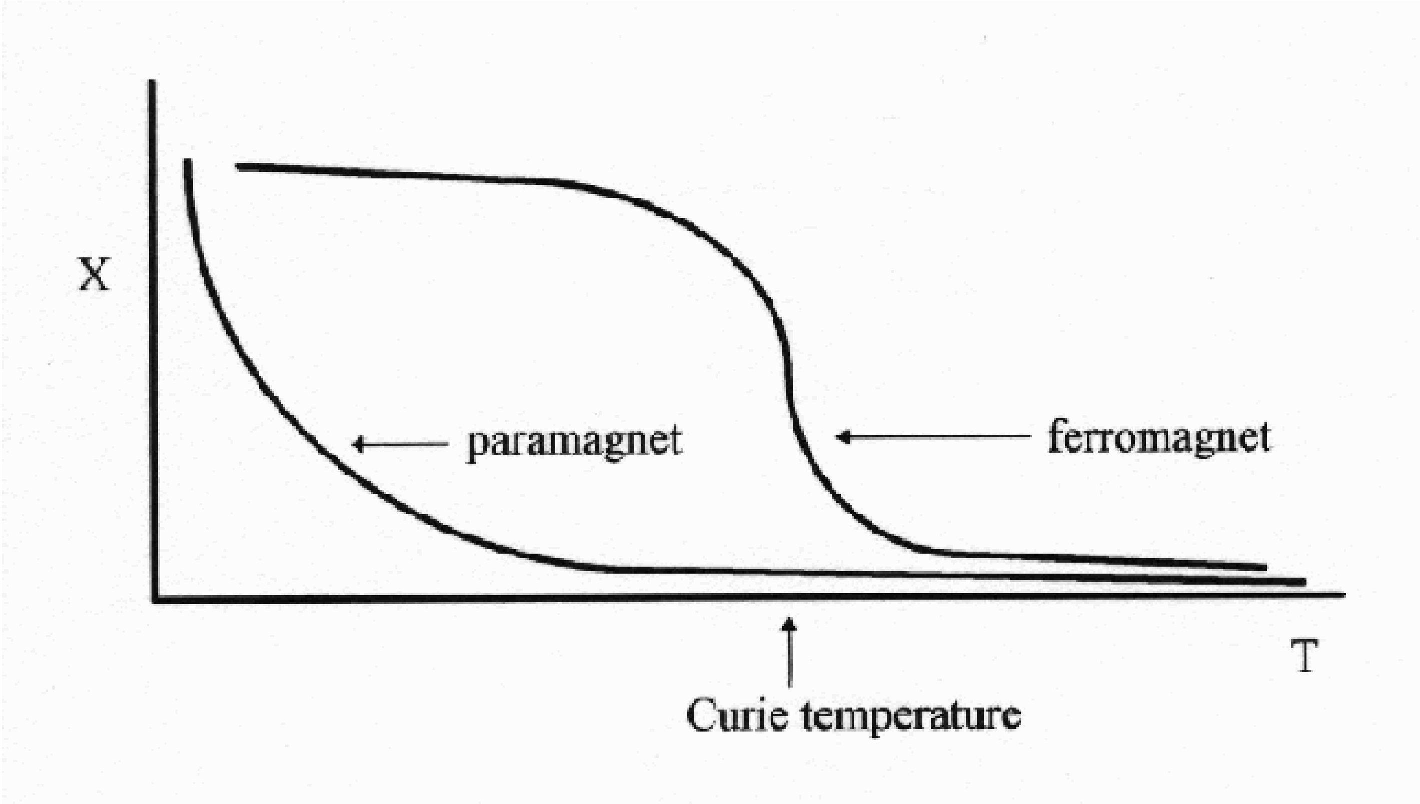


Physical origin of ferromagnetism

Random thermal motions tend to destroy the ordering / dipole alignment.

- Random thermal motion increases with temperature.

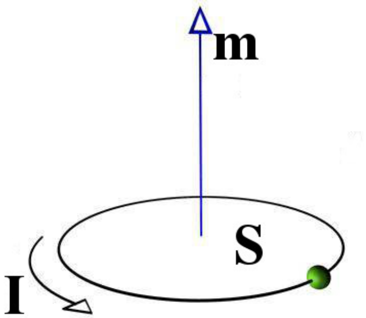
There is a given temperature called **Curie point** (for iron this is at 770^o C) where a ferromagnetic material undergoes a **phase transition** and becomes paramagnetic.



Lecture 7 revision

In the last lecture:

- We completed the study of **Maxwell's eqs. in materials (for static fields)** and emphasized the analogies between electrostatics and magnetostatics.
- We discussed the magnetic properties of materials (**diamagnetism**, **paramagnetism** and **ferromagnetism**) and developed arguments to understand the physical origins.
- We found out how to **extend Maxwell's eqs. in vacuum** in the case of **time-dependent fields**.



We defined the **magnetic dipole moment** $\vec{m}$ as:

$$\vec{m} = I\vec{S}$$

An external magnetic field $\vec{B}$ exerts a torque $\vec{T}$ on a magnetic dipole $\vec{m}$ which is given by:

$$\vec{T} = \vec{m} \times \vec{B}$$

This will tend to **align** the previously randomised **magnetic moments** and **create magnetisation at a macroscopic level**.

We define **magnetisation** $\vec{M}$ as the amount of **magnetic dipole moment per unit volume**.

Lecture 7 revision (Magnetization-induced currents)

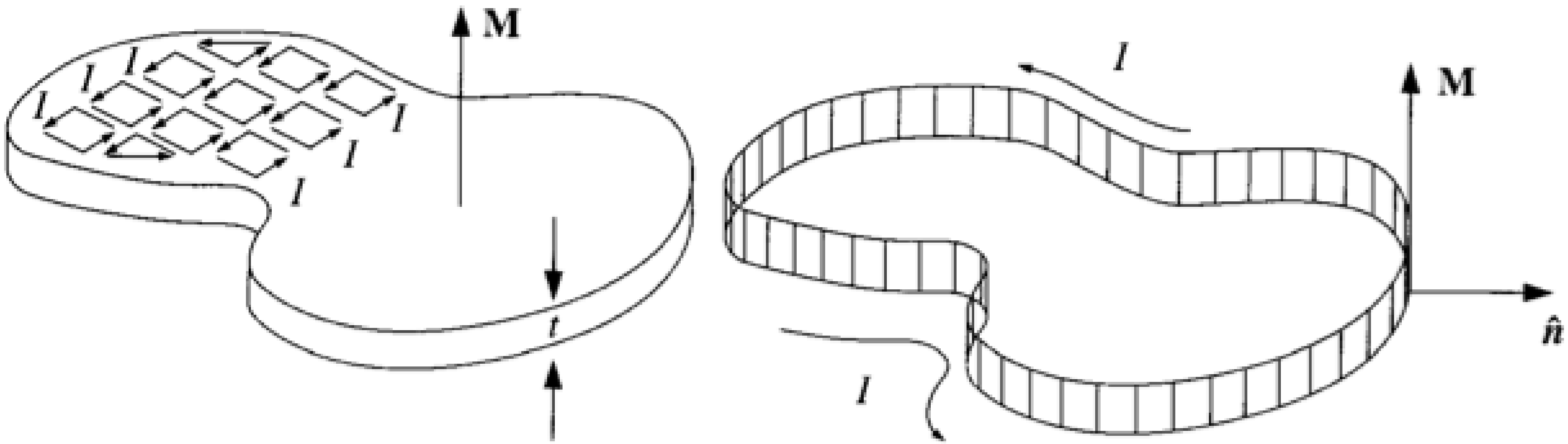
The **magnetisation induces surface and volume currents**.

We can easily be convinced, although we will not show it mathematically, that the **density of the surface current** is:

$$j_m^{surf} = \vec{M} \times \hat{n}$$

whereas the **density of the volume current** is:

$$j_m^{vol} = \vec{\nabla} \times \vec{M}$$



Lecture 7 revision (Ampere's law in materials)

We started from Ampere's law in vacuum:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

By writing the total current density $\vec{j}$ as the vector sum of the free ($\vec{j}_f$) and magnetization ($\vec{j}_m$) current densities, and expressing $\vec{j}_m$ in terms of the magnetization field $\vec{M}$ ($\vec{j}_m = \vec{\nabla} \times \vec{M}$), we finally wrote Ampere's law as:

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{j}_f$$

We defined the **magnetic field strength** or **magnetic field intensity** $\vec{H}$ as:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

In SI, the quantity $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ has **units of A/m**.

Lecture 7 revision (Linear materials)

Ampere's law in materials is: $\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \mu_0 \vec{j}_f$

If the analogy with electrostatics was exact, we would write $\vec{M}$ in terms of $\vec{B}$. However, this is where the analogy breaks. Instead we typically write:

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is the **magnetic susceptibility**. For **linear materials**, χ_m is a constant independent of the value of $\vec{H}$. Expressing $\vec{B}$ in terms of $\vec{H}$:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) \xrightarrow{\vec{M} = \chi_m \vec{H}} \vec{B} = \left(1 + \chi_m \right) \mu_0 \vec{H} \Rightarrow$$

$$\vec{B} = \mu_r \mu_0 \vec{H} \Rightarrow \vec{B} = \mu \vec{H}$$

where $\mu_r = 1 + \chi_\mu$ is the **relative permeability** (dimensionless) and $\mu = \mu_r \mu_0$ is the **permeability** of the material (SI unit: $V \cdot s \cdot A^{-1} \cdot m^{-1}$).

Lecture 7 revision (Correspondence between quantities)

Electrostatics		Magnetostatics	
electric dipole moment	$\vec{p} = q\vec{d}$	$\vec{m} = I\vec{S}$	magnetic dipole moment
torque within $\vec{E}$ field	$\vec{T} = \vec{p} \times \vec{E}$	$\vec{T} = \vec{m} \times \vec{B}$	torque within a $\vec{B}$ field
polarization	$\vec{P} = \frac{(e.d.m)}{volume}$	$\vec{M} = \frac{(m.d.m)}{volume}$	magnetization
surface charge density	$\sigma_P = \vec{P} \cdot \hat{n}$	$j_m^{surf} = \vec{M} \times \hat{n}$	surface current density
volume charge density	$\rho_P = -\vec{\nabla} \cdot \vec{P}$	$j_m^{vol} = \vec{\nabla} \times \vec{M}$	volume current density
electric displacement	$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$	$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$	magnetizing field
Gauss' law in materials	$\vec{\nabla} \cdot \vec{D} = \rho_f$	$\vec{\nabla} \times \vec{H} = \vec{j}_f$	Ampere's law in materials
	$\oint_S \vec{D} \cdot d\vec{S} = Q_f$	$\oint_L \vec{H} \cdot d\vec{\ell} = I_f$	

Lecture 7 revision (Magnetic properties of materials)

- We also discussed the **magnetic properties of materials** and developed (classical) arguments to understand the **physical origins**.
- The material which has the **most striking and well known magnetic properties is iron (Fe).**
  - Nickel (Ni), Cobalt (Co), Gadolinium (Gd) and Dysprosium (Dy) behave similarly. We call these materials **ferromagnets**.
  - Not only these materials can have a significant magnetisation when inside an external magnetic field, they also **retain their magnetisation in the absence of an external magnetic field.**
- But **other substances get magnetised too** (water, wood, frogs,...)
  - The magnetic effects for these materials are **very very weak!**
  - Moreover, water, wood, and frogs **do not remain magnetized** once the external magnetic field is removed.
  - In a presense of an external magnetic field these substances can be magnetised either in the direction of the field (**paramagnetic** substances), or opposite to it (**diamagnetic** substances).

Lecture 7 revision (Maxwell's eqs. for the static case)

Static case in vacuum		
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int \rho d\tau$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot d\vec{S}$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

Static case within materials		
Gauss's law	$\oint \vec{D} \cdot d\vec{S} = \int \rho_f d\tau$	$\vec{\nabla} \cdot \vec{D} = \rho_f$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{H} \cdot d\vec{\ell} = \int \vec{j_f} \cdot d\vec{S}$	$\vec{\nabla} \times \vec{H} = \vec{j_f}$

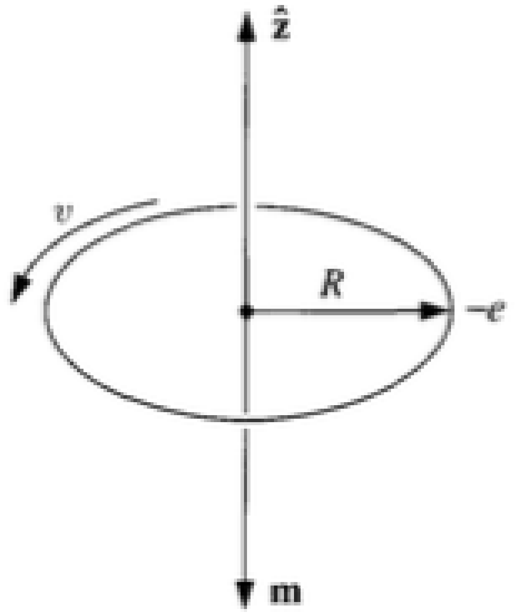
At the next lecture (Lecture 8)

- We will study how we need to **extend Maxwell's eqs. in vaccum** in the case of **time-dependent fields**.

Optional reading for Lecture 7

Physical origin of diamagnetism

Consider an electron orbiting a nucleus:



For time intervals much larger than the period of rotation T, we may think of the orbiting electron as a *steady* current I:

$$I = \frac{q}{T} = -\frac{eu}{2\pi R}$$

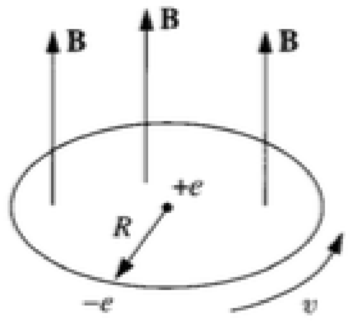
where u is the electron velocity and R is the radius of its orbit (taken to be circular).

The **magnetic dipole moment associated with that orbital motion** is:

$$\vec{m} = I\vec{S} = \left(-\frac{eu}{2\pi R}\right)\left(\pi R^2 \hat{z}\right) \Rightarrow \vec{m} = -\frac{1}{2}euR\hat{z}$$

Physical origin of diamagnetism

Within an external $\vec{B}$ field, there is a significant effect on the electron orbit.



In the absence of a $\vec{B}$ field, Coulomb's force (due to the nucleus) provides centripetal acceleration:

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e\frac{u^2}{R}$$

In the presence of a $\vec{B}$ field, there is an additional force $-e\Big(\vec{u}\times\vec{B}\Big)$.

Assuming (for simplicity) that $\vec{B}$ is perpendicular to the orbital plane, then:

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\bar{u}B = m_e\frac{\bar{u}^2}{R} \Rightarrow m_e\frac{u^2}{R} + e\bar{u}B = m_e\frac{\bar{u}^2}{R} \Rightarrow$$

$$e\bar{u}B = \frac{m_e}{R}\Big(\bar{u}^2 - u^2\Big) \Rightarrow e\bar{u}B = \frac{m_e}{R}\Big(\bar{u}+u\Big)\Big(\bar{u}-u\Big)$$

Physical origin of diamagnetism

Assuming that the change $\Delta u = \bar{u} - u$ is small, so that $\bar{u} + u \approx 2u \approx 2\bar{u}$, we can write the previous equation as:

$$e\bar{u}B = \frac{m_e}{R}\Big(\bar{u} + u\Big)\Big(\bar{u} - u\Big) \Rightarrow e\cancel{\bar{u}}B = \frac{m_e}{R}2\cancel{\bar{u}}\Delta u \Rightarrow$$

$$\Delta u = \frac{eRB}{2m_e}$$

In the presence of a $\vec{B}$ field, the electron velocity increases by Δu. This changes its magnetic dipole moment:

$$\vec{m} = -\frac{1}{2}euR\hat{z}$$

The change $\Delta\vec{m}$ is given by:

$$\Delta\vec{m} = -\frac{1}{2}e\Delta uR\hat{z} = -\frac{1}{2}e\left(\frac{eRB}{2m_e}\right)R\hat{z} \Rightarrow \Delta\vec{m} = -\frac{e^2R^2}{4m_e}\vec{B}$$

Worked example: Free current on circular wire

Question

An infinite straight wire with a circular cross-section of radius R is lying along the z axis and it has an internal $\vec{H}$ field given by

$$\vec{H} = \frac{J_0}{r}\left(\frac{1}{a^2} sin(ar) - \frac{r}{a} cos(ar)\right)\hat{\phi}$$

where r is the radial distance from the centre of the circular conductor, $\hat{\phi}$ is the azimuthal unit vector, J_0 is a constant current density, and $a = \pi/(2R)$. Find an expression for the total free current in the conductor, and give its direction.

The free current I can be computed from Ampere's law:

$$I = \oint_L \vec{H} \cdot d\vec{\ell}$$

Worked example: Free current on circular wire

Using the given $\vec{H}$ field, and using $a = \pi/2R$, as given, the previous expression for I becomes:

$$I = \oint_L \frac{J_0}{r}\Big(\frac{2R}{\pi^2} sin(\frac{\pi r}{2R}) - \frac{2Rr}{\pi} cos(\frac{\pi r}{2R})\Big)\hat{\phi} \cdot d\vec{\ell}$$

To calculate the total free current in the circular conductor, we choose a closed circular path with radius equal to the radius of the conductor. The element $d\vec{\ell}$ can be written as:

$$d\vec{\ell} = Rd\phi\ \hat{\phi}$$

Using the above expression for $d\vec{\ell}$, and substituting $r = R$, into the our last expression for I, we find:

$$I = \int_0^{2\pi} \frac{10^4}{R}\Big(\frac{4R^2}{\pi^2} sin(\frac{\pi R}{2R}) - \frac{2R^2}{\pi} cos(\frac{\pi R}{2R})\Big)\hat{\phi} \cdot \Big(Rd\phi\ \hat{\phi}\Big) \xRightarrow{\hat{\phi}\cdot\hat{\phi}=1}$$

Worked example: Free current on circular wire

$$I = \frac{J_0}{R}\Big(\frac{4R^2}{\pi^2}\cancelto{1}{sin(\frac{\pi}{2})} - \frac{2R^2}{\pi}\cancelto{0}{cos(\frac{\pi}{2})}\Big) R \int_0^{2\pi} d\phi \Rightarrow$$

$$I = \frac{J_0}{\cancel{R}}\frac{4R^2}{\pi^{\cancel{2}}}\cancel{R}2\cancel{\pi} \Rightarrow$$

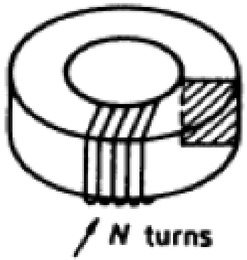
$$I = \frac{8J_0}{\pi}R^2$$

Since $\vec{H}$ is an azimuthal vector, the free current flows in the positive z axis.

Worked example: Toroid with square iron core

Question

A toroid having an iron core of square cross-section and permeability μ is wound with N closely spaced turns of wire carrying a current I. Find an expression for the magnitude of the magnetisation M everywhere inside the iron core.



The magnetisation M inside the iron core is:

$$M = \frac{B}{\mu_0} - H$$

According to Ampere's circuital law:

$$\oint_L \vec{H} \cdot d\vec{\ell} = I_{free} = NI$$

where I_{free} is the free current through an open area $S(L)$ whose boundary is the closed path L, and I is the current on each of the N windings.

Worked example: Toroid with square iron core

Considering the cylindrical symmetry of the problem, $\vec{H}$ is azimuthal. Therefore, for a closed path L which is concentric to the iron core, perpendicular to the symmetry axis of the toroid, and lies within the core:

$$\oint_L \vec{H} \cdot d\vec{\ell} = H 2\pi r$$

Therefore, Ampere's circuital law yields:

$$H 2\pi r = NI \Rightarrow H = \frac{NI}{2\pi r}$$

The magnetic field B can be computed from H as follows:

$$B = \mu H$$

Substituting the above into our initial expession for M, we find:

$$M = \frac{\mu H}{\mu_0} - H = \Big(\frac{\mu}{\mu_0} - 1\Big) H = \Big(\frac{\mu}{\mu_0} - 1\Big) \frac{NI}{2\pi r}$$

PHYS 201 / Lecture 8
Generalizing Maxwell's eqs for time-dependent fields: Faraday's law and Maxwell's correction to Ampere's law

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Science & Technology Facilities Council
Rutherford Appleton Laboratory

Lecture 7 revision

In the last lecture:

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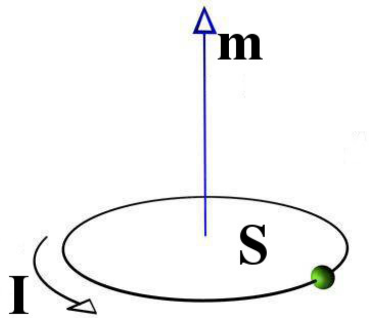
- We discussed the magnetic properties of materials (**diamagnetism**, **paramagnetism** and **ferromagnetism**) and developed arguments to understand the physical origins.
- We found out how to **extend Maxwell's eqs. in vacuum** in the case of **time-dependent fields**.



We defined the **magnetic dipole moment** $\vec{m}$ as:

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An external magnetic field $\vec{B}$ exerts a torque $\vec{T}$ on a magnetic dipole $\vec{m}$ which is given by:

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We define **magnetisation** $\vec{M}$ as the amount of **magnetic dipole moment per unit volume**.

Lecture 7 revision (Magnetization-induced currents)

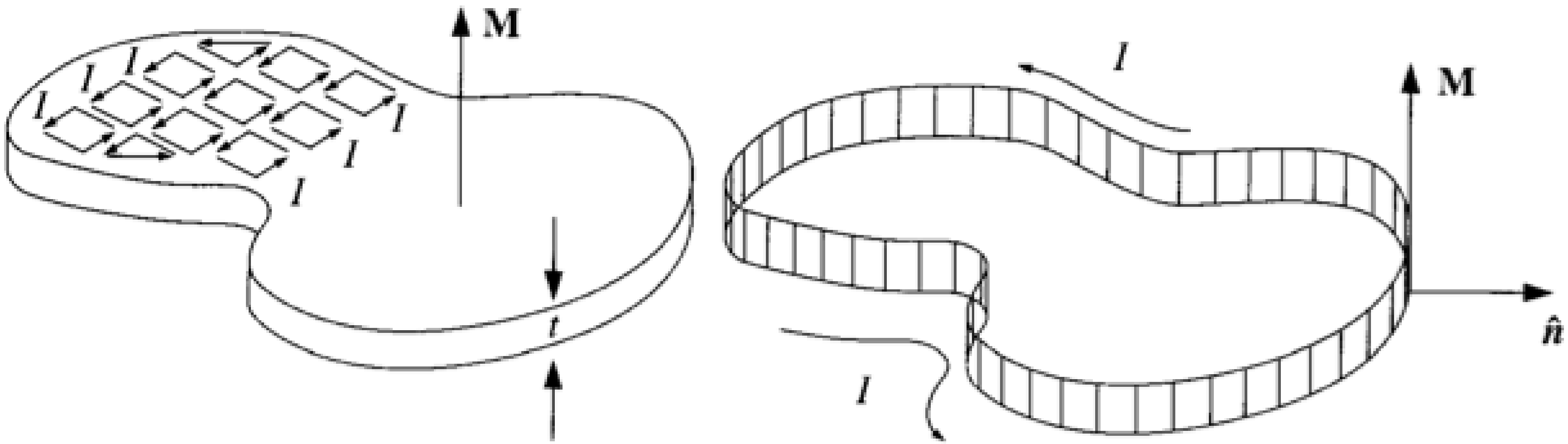
The **magnetisation induces surface and volume currents**.

We can easily be convinced, although we will not show it mathematically, that the **density of the surface current** is:

$$j_m^{surf} = \vec{M} \times \hat{n}$$

whereas the **density of the volume current** is:

$$j_m^{vol} = \vec{\nabla} \times \vec{M}$$



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By writing the total current density $\vec{j}$ as the vector sum of the free ($\vec{j}_f$) and magnetization ($\vec{j}_m$) current densities, and expressing $\vec{j}_m$ in terms of the magnetization field $\vec{M}$ ($\vec{j}_m = \vec{\nabla} \times \vec{M}$), we finally wrote Ampere's law as:

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We defined the **magnetic field strength** or **magnetic field intensity** $\vec{H}$ as:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

In SI, the quantity $\vec{H} = \dfrac{\vec{B}}{\mu_0} - \vec{M}$ has **units of A/m**.

Lecture 7 revision (Linear materials)

Ampere's law in materials is: $\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \mu_0 \vec{j_f}$

If the analogy with electrostatics was exact, we would write $\vec{M}$ in terms of $\vec{B}$. However, this is where the analogy breaks. Instead we typically write:

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is the **magnetic susceptibility**. For **linear materials**, χ_m is a constant independent of the value of $\vec{H}$. Expressing $\vec{B}$ in terms of $\vec{H}$:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) \xrightarrow{\vec{M} = \chi_m \vec{H}} \vec{B} = \left(1 + \chi_m \right) \mu_0 \vec{H} \Rightarrow$$

$$\vec{B} = \mu_r \mu_0 \vec{H} \Rightarrow \vec{B} = \mu \vec{H}$$

where $\mu_r = 1 + \chi_\mu$ is the **relative permeability** (dimensionless) and $\mu = \mu_r \mu_0$ is the **permeability** of the material (SI unit: $V \cdot s \cdot A^{-1} \cdot m^{-1}$).

Lecture 7 revision (Correspondence between quantities)

Electrostatics		Magnetostatics	
electric dipole moment	$\vec{p} = q\vec{d}$	$\vec{m} = I\vec{S}$	magnetic dipole moment
torque within $\vec{E}$ field	$\vec{T} = \vec{p} \times \vec{E}$	$\vec{T} = \vec{m} \times \vec{B}$	torque within a $\vec{B}$ field
polarization	$\vec{P} = \frac{(e.d.m)}{volume}$	$\vec{M} = \frac{(m.d.m)}{volume}$	magnetization
surface charge density	$\sigma_P = \vec{P} \cdot \hat{n}$	$j_m^{surf} = \vec{M} \times \hat{n}$	surface current density
volume charge density	$\rho_P = -\vec{\nabla} \cdot \vec{P}$	$j_m^{vol} = \vec{\nabla} \times \vec{M}$	volume current density
electric displacement	$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$	$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$	magnetizing field
Gauss' law in materials	$\vec{\nabla} \cdot \vec{D} = \rho_f$	$\vec{\nabla} \times \vec{H} = \vec{j_f}$	Ampere's law in materials
	$\oint_S \vec{D} \cdot d\vec{S} = Q_f$	$\oint_L \vec{H} \cdot d\vec{\ell} = I_f$	

Lecture 7 revision (Magnetic properties of materials)

- We also discussed the **magnetic properties of materials** and developed (classical) arguments to understand the **physical origins**.
- The material which has the **most striking and well known magnetic properties is iron (Fe).**
    - Nickel (Ni), Cobalt (Co), Gadolinium (Gd) and Dysprosium (Dy) behave similarly. We call these materials **ferromagnets**.
    - Not only these materials can have a significant magnetisation when inside an external magnetic field, they also **retain their magnetisation in the absence of an external magnetic field.**
- But **other substances get magnetised too** (water, wood, frogs,...)
    - The magnetic effects for these materials are **very very weak!**
    - Moreover, water, wood, and frogs **do not remain magnetized** once the external magnetic field is removed.
    - In a presense of an external magnetic field these substances can be magnetised either in the direction of the field (**paramagnetic** substances), or opposite to it (**diamagnetic** substances).

Lecture 7 revision (Maxwell's eqs. for the static case)

Static case in vacuum		
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int \rho d\tau$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot d\vec{S}$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

Static case within materials		
Gauss's law	$\oint \vec{D} \cdot d\vec{S} = \int \rho_f d\tau$	$\vec{\nabla} \cdot \vec{D} = \rho_f$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{H} \cdot d\vec{\ell} = \int \vec{j}_f \cdot d\vec{S}$	$\vec{\nabla} \times \vec{H} = \vec{j}_f$

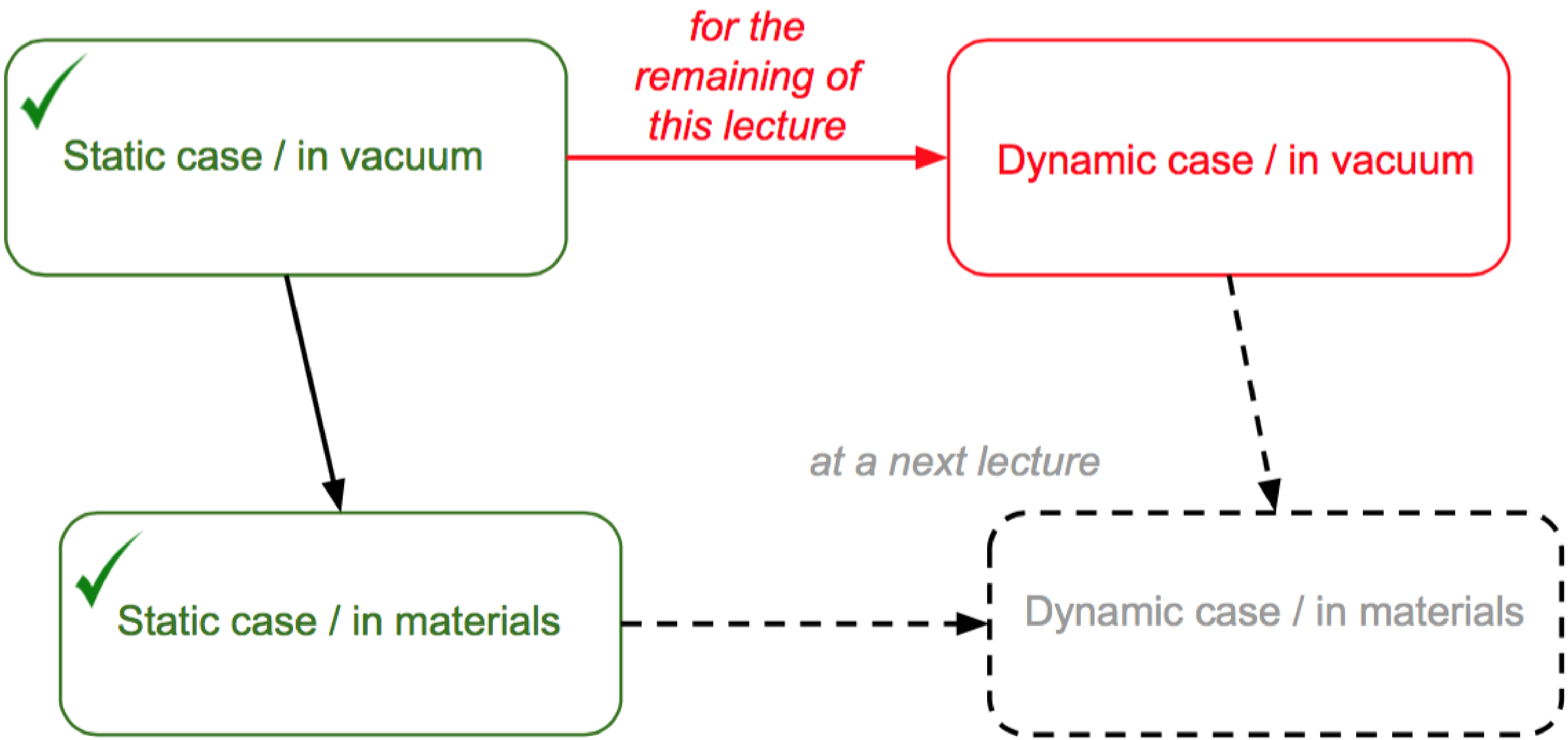
Plan for Lecture 8

In this lecture:

- We will study how we need to **extend Maxwell's eqs. in vaccum** in the case of **time-dependent fields**.

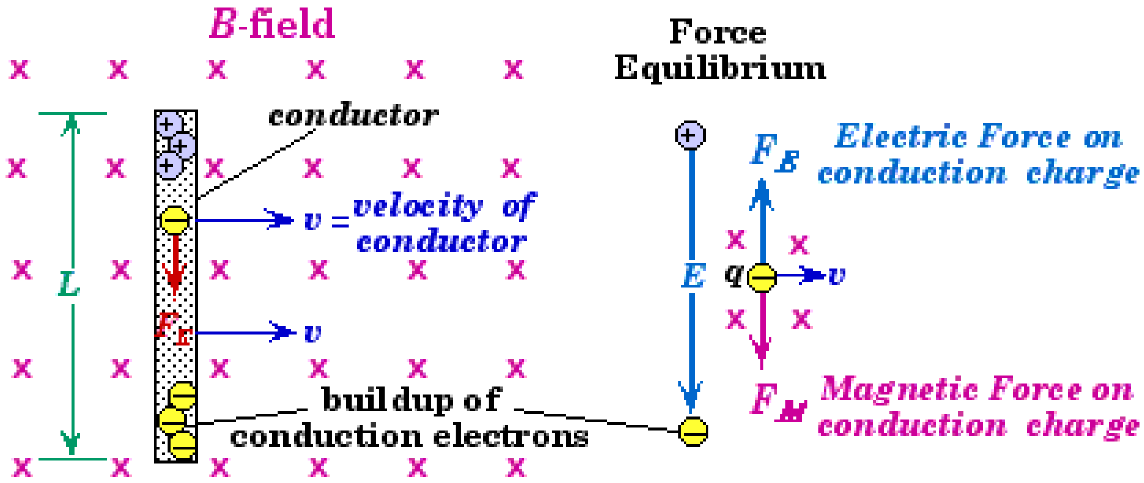
Time-dependent fields

Having completed the study of Maxwell's equations for the static case (both in vaccum and in materials), we will now turn our attention to the case where the electric and magnetic field can vary with time.



A conductor moving in a magnetic field

Consider a conductor with length L moves with velocity $\vec{u}$ inside a homogenous magnetic field $\vec{B}$, as shown below:

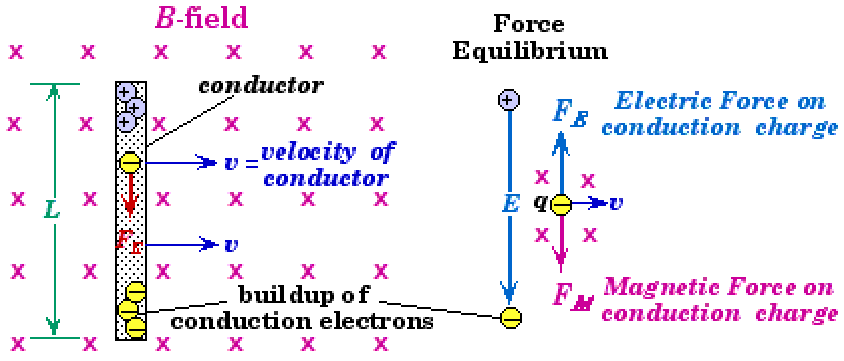


Each electron in the conductor feels a magnetic force $\vec{F}_M = q\vec{u} \times \vec{B}$.

That magnetic force **induces the build-up of charge** which **produces an electric field** $\vec{E}$: Each electron feels an electric force $\vec{F}_E = q\vec{E}$.

The resulting **electric force** $\vec{F}_E$ **opposes the magnetic force** $\vec{F}_M$.

A conductor moving in a magnetic field



At equilibrium, the electric and magnetic forces are equal in strength:

$$\vec{F}_E = \vec{F}_M \Rightarrow \cancel{q}\vec{E} = \cancel{q}\vec{u} \times \vec{B} \Rightarrow$$

$$\vec{E} = \vec{u} \times \vec{B}$$

An electrical potential difference develops between the ends of the moving conductor, which becomes a source of EMF:

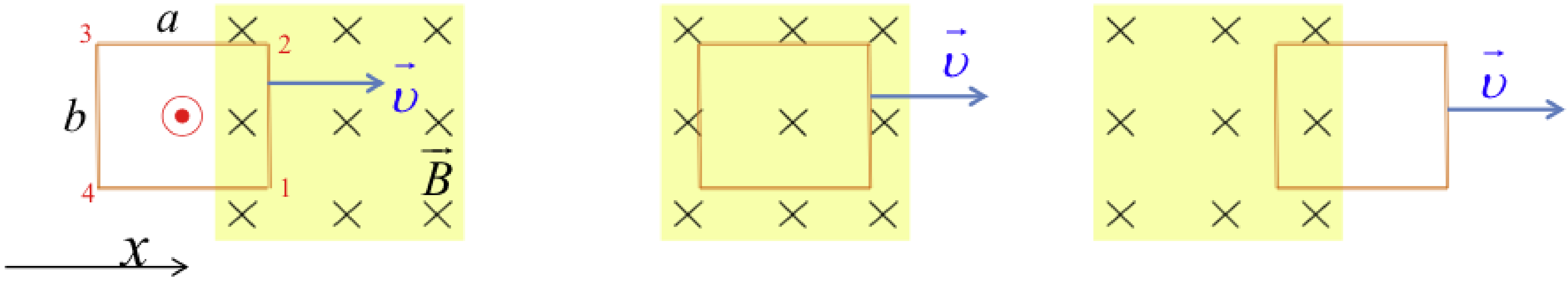
$$\mathcal{E} = \int_L \vec{E} \cdot d\vec{\ell} = \int_L \left(\vec{u} \times \vec{B}\right) \cdot d\vec{\ell}$$

For a closed circuit:

$$\mathcal{E} = \oint_L \vec{E} \cdot d\vec{\ell} = \oint_L \left(\vec{u} \times \vec{B}\right) \cdot d\vec{\ell}$$

A circuit moving through a magnetic field

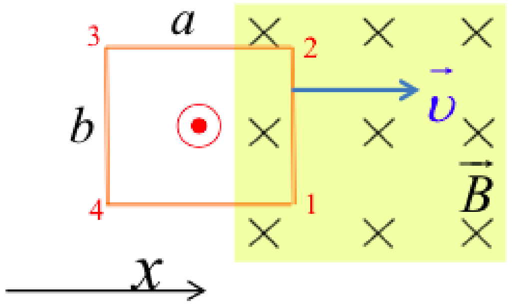
Now let's consider a simple rectangular circuit moving through a region that has a magnetic field $\vec{B}$:



Conventions:

- $\vec{B}$ is towards the page.
- Assuming current I flows anti-clockwise.
- Looping within the circuit: anti-clockwise
- Following the direction of the current I with our right-hand, the thumb points towards you - let's use that direction for the surface vector

A circuit moving through a magnetic field



Let's consider what happens as the circuit enters in the region of the magnetic field.

EMF:

- Side '12': $\mathcal{E} = \int \left(\vec{u} \times \vec{B}\right) \cdot d\vec{\ell} = uBb$
- Side '34' ($\vec{B} = 0$): $\mathcal{E} = 0$
- Side '23' and '14' ($\vec{u} \times \vec{B} \perp d\vec{\ell}$): $\mathcal{E} = 0$
- Summing-up the EMFs for all 4 sides:

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{\ell} = uBb$$

Change in flux:

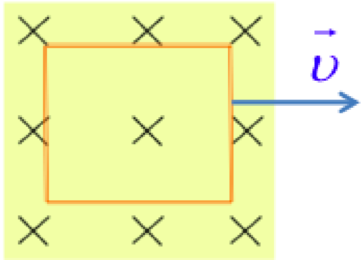
$$\frac{d\Phi_M}{dt} = \frac{d}{dt}\Big(\int \vec{B} \cdot d\vec{S}\Big) = \frac{d}{dt}\Big(\vec{B} \cdot \int d\vec{S}\Big) =$$

$$\frac{d}{dt}\Big(\vec{B} \cdot \vec{S}\Big) \overset{\angle(\vec{B},\vec{S})=\pi}{=\!=\!=\!=} \frac{d}{dt}\Big(-BS\Big) \overset{S=bx}{=\!=\!=}$$

$$\frac{d}{dt}\Big(-Bbx\Big) = -\frac{dx}{dt}Bb \Rightarrow$$

$$\frac{d\Phi_M}{dt} = -uBb$$

A circuit moving through a magnetic field



Let's consider what happens when the circuit is entirely within the region of the magnetic field.

EMF:

- Side '12': $\mathcal{E} = \int \left(\vec{u} \times \vec{B}\right) \cdot d\vec{\ell} = uBb$
- Side '34': $\mathcal{E} = \int \left(\vec{u} \times \vec{B}\right) \cdot d\vec{\ell} = -uBb$
  (since $\vec{u} \times \vec{B}$ is same as for side '12', but $d\vec{\ell}$ has opposite direction).
- Side '23' and '14' ($\vec{u} \times \vec{B} \perp d\vec{\ell}$): $\mathcal{E} = 0$
- Summing-up the EMFs for all 4 sides:

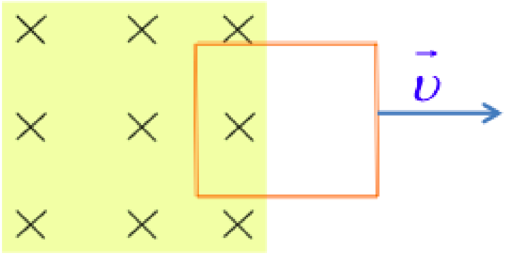
$$\mathcal{E} = \oint \vec{E} \cdot d\vec{\ell} = 0$$

Change in flux:

The circuit is entirely within the magnetic field so, as it moves, there are as many magnetic field line entering its surface as ones exiting. The rate of change of the magnetic flux is 0.

$$\frac{d\Phi_M}{dt} = 0$$

A circuit moving through a magnetic field



Finally, let's consider what happens as the circuit exits the region of the magnetic field.

EMF:

- Side '12' ($\vec{B} = 0$): $\mathcal{E} = 0$
- Side '34': $\mathcal{E} = \int \left(\vec{u} \times \vec{B}\right) \cdot d\vec{\ell} = -uBb$
- Side '23' and '14' ($\vec{u} \times \vec{B} \perp d\vec{\ell}$): $\mathcal{E} = 0$
- Summing-up the EMFs for all 4 sides:

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{\ell} = -uBb$$

Change in flux:

$$\frac{d\Phi_M}{dt} = \frac{d}{dt}\Big(\int \vec{B} \cdot d\vec{S}\Big) = \frac{d}{dt}\Big(\vec{B} \cdot \int d\vec{S}\Big) =$$

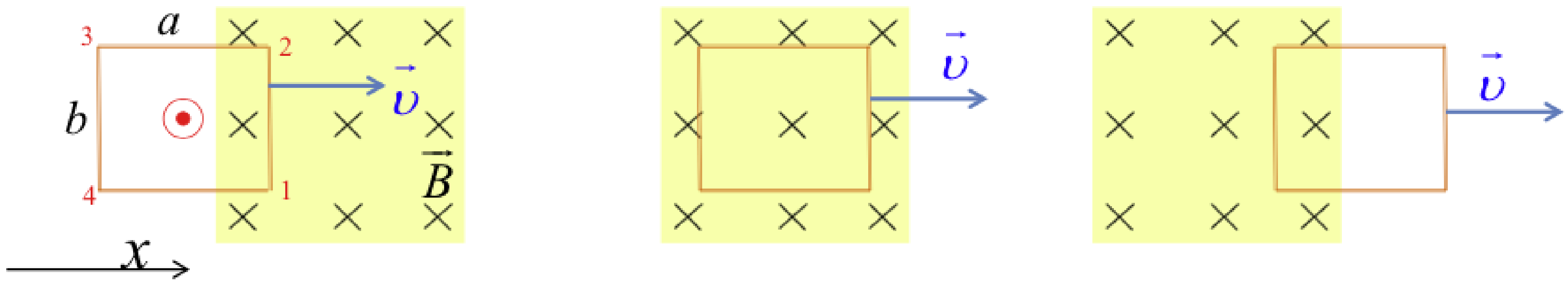
$$\frac{d}{dt}\Big(\vec{B} \cdot \vec{S}\Big) \overset{\angle(\vec{B},\vec{S})=\pi}{=\!=\!=\!=} \frac{d}{dt}\Big(- BS\Big) \overset{S=bx}{=\!=\!=}$$

$$\frac{d}{dt}\Big(- Bbx\Big) = -\frac{dx}{dt}Bb = -(-u)Bb \Rightarrow$$

$$\frac{d\Phi_M}{dt} = uBb$$

A circuit moving through a magnetic field

A rectangular circuit moving through a region with magnetic field $\vec{B}$:



In summary:

	$\oint_L \vec{E} \cdot d\vec{\ell}$	$\frac{d\Phi_M}{dt}$
left	uBb	-uBb
centre	0	0
right	-uBb	uBb

So, indeed, in all cases:

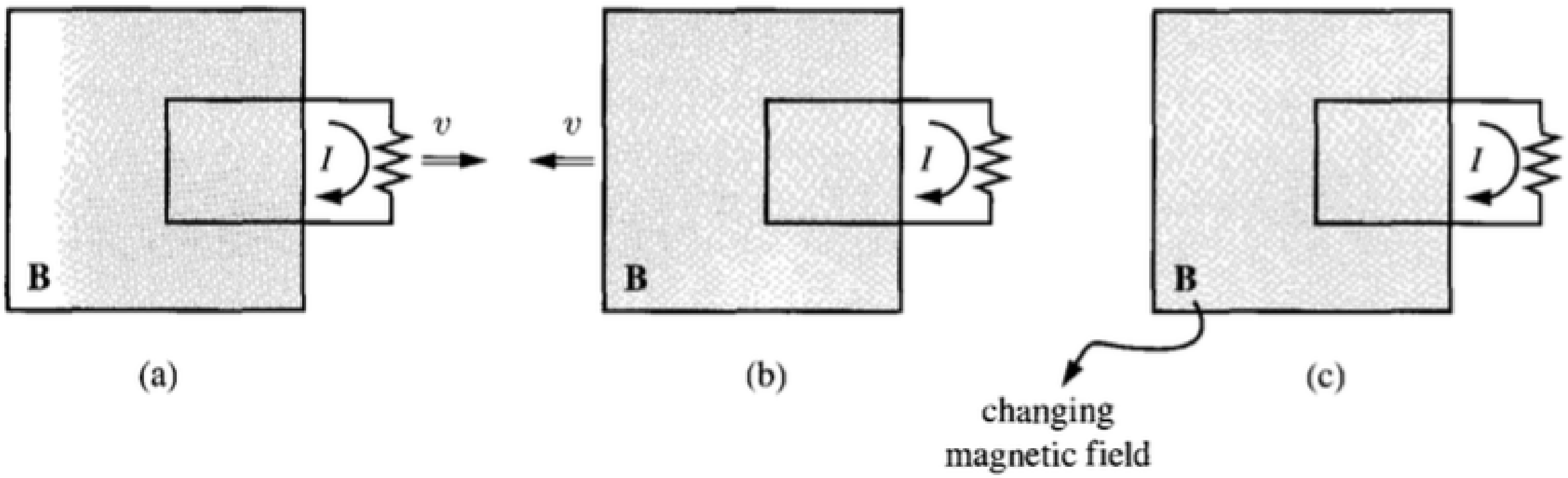
$$\mathcal{E} = \oint_L \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_M}{dt}$$

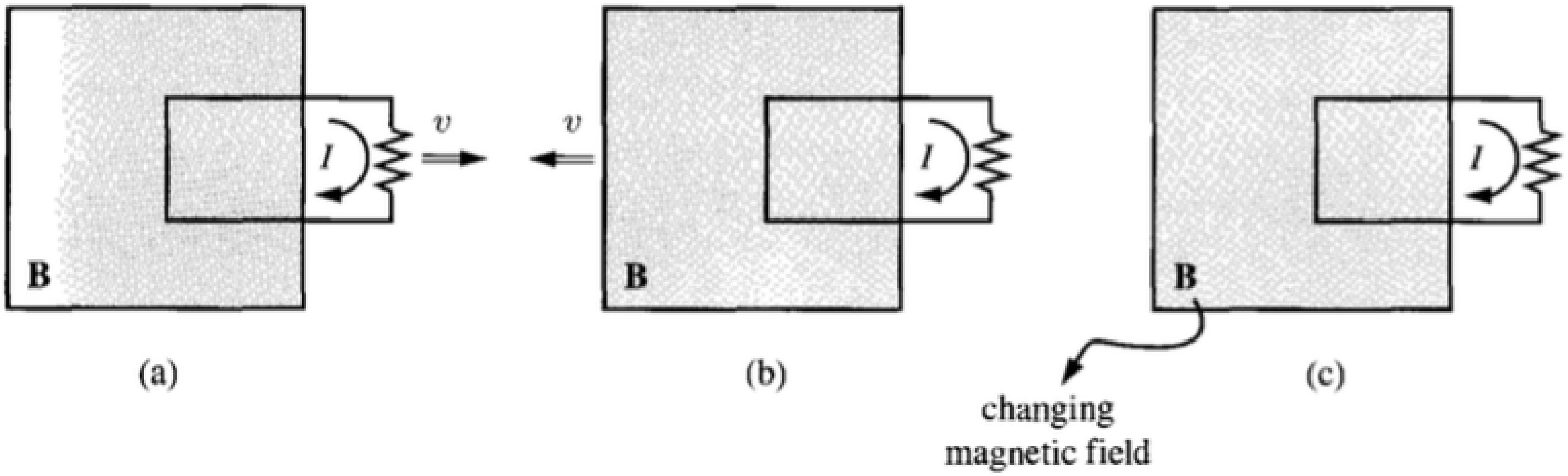
Faraday's observations

In 1831 Michael Faraday reported on a series of experiments.

A current flows in a wire loop when:

(a) the loop is pulled through a magnetic field,
(b) the loop is at rest but the magnet moves in the opposite direction, and
(c) both the loop and the magnet are at rest but the strength of the magnetic field is varied





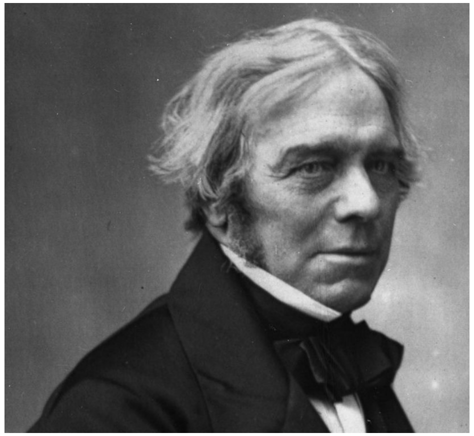
- In cases (a) and (b) it is the **magnetic field** that is **responsible for the EMF** (as in the example we studied earlier).
- But in case (c) both the loop and the magnet are stationary and the force felt by the electrons can not be magnetic. It is the **electric field** that is **responsible for the EMF**!

This led Faraday to realize that:

A time-varying magnetic field induces an electric field

Faraday's law / Lenz's law

In all cases the **motional EMF** is directly related to the **change of the magnetic flux Φ_M though the circuit**:



Michael Faraday (1791 - 1867).

$$\mathcal{E} = \oint_L \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_M}{dt}$$

This is the so-called **Faraday's law** (integral form).

Lenz' law (1845): The EMF induced by a changing flux has a polarity such that the current flowing gives rise to a flux which opposes the change of flux.

The minus sign is a consequence of the **conservation of energy** and of **Newton's 3rd law** of motion: Induction is a an "inertial reaction". The system develops a current which tries to maintain the flux constant.

Heinrich Lenz (1804 - 1865).

Differential form of Faraday's law

The integral form of Faraday's law is:

$$\oint_L \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

Using Stoke's theorem, we obtain its differential form:

$$\oint_L \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int_{S(L)} \vec{B} \cdot d\vec{S} \Rightarrow \int_{S(L)} \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{S} = -\int_{S(L)} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \Rightarrow$$

$$\int_{S(L)} \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{S} = 0 \Rightarrow \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Worked example

Question

A circular wire loop on the x-y plane has a radius $r_0 = 1$ cm at time t $= 0$. A homogeneous magnetic field of 3×10^{-3} T in the positive z direction permeates the loop.

1. Find the magnetic flux through the loop at t=0.
2. The radius of the loop increases with $r(t) = r_0 + u \cdot t$ with u $= 1$ cm/s. Find the magnetic flux through the loop at t $= 4$ s.
3. Derive an expression for the EMF in the loop as a function of time.

The flux through the loop is given by $\Phi = \int \vec{B} \cdot d\vec{S}$.

The loop is lying on the x-y plane, hence the surface vector $d\vec{S}$ that is normal to the loop surface is along the z axis. $\vec{B}$ is also along the z axis.

Therefore, the above dot product simplifies to $\Phi = \int B dS$.

Worked example

Since $\vec{B}$ is homogeneous:

$$\Phi = B \int dS = B\left(\pi r^2\right)$$

where r is the radius of the loop.

In our case, r and hence Φ are functions of time:

$$\Phi(t) = B\left(\pi r(t)^2\right)$$

At time t $=$ 0:

$$\Phi(t=0) = B\left(\pi r_0^2\right) = \left(3 \times 10^{-3}\ T\right)\left(3.14 \cdot 0.01^2\ m^2\right) = 9.4 \times 10^{-7}\ T \cdot m^2$$

Worked example

The radius of the loop increases as $r(t) = r_0 + u \cdot t$, with u = 1 cm/s.

At t = 4 s, the radius of the loop is:

$$r(t = 4\ s) = 0.01\ cm + \Big(0.01\ m/s\Big) \cdot \Big(4\ s\Big) = 0.05\ m$$

Therefore the flux through the loop is now:

$$\Phi(4\ s) = \Big(3 \times 10^{-3}\ T\Big)\Big(3.14 \cdot 0.05^2\ m^2\Big) = 2.35 \times 10^{-5}\ T \cdot m^2$$

Worked example

According to Faraday's law, the EMF $\mathcal{E} = \oint \vec{E} d\vec{\ell}$ developed in the wire loop is equal to the negative rate of change of the magnetic flux through the loop:

$$\mathcal{E} = -\frac{d\Phi(t)}{dt} = -\frac{d}{dt}\left\{B\left(\pi r(t)^2\right)\right\} = -\pi B\frac{d}{dt}\left\{r(t)^2\right\}$$

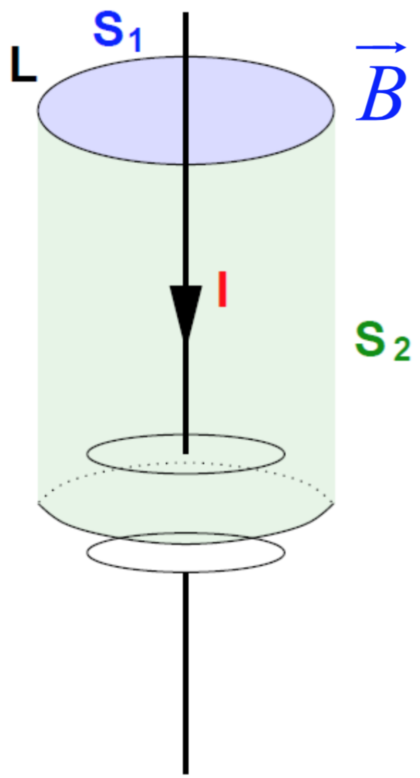
Recall that $r(t) = r_0 + u \cdot t$, therefore:

$$\mathcal{E} = -\pi B\frac{d}{dt}\left\{\left(r_0 + u \cdot t\right)^2\right\} = -\pi B\frac{d}{dt}\left\{r_0^2 + 2 \cdot r_0 \cdot u \cdot t + u^2 \cdot t^2\right\}$$

$$= -\pi B\left(2 \cdot r_0 \cdot u + 2 \cdot u^2 \cdot t\right) = -2\pi Bu\left(r_0 + u \cdot t\right) = -2\pi Bur(t)$$

A problem with Ampere's law

Consider a current I charging a parallel plate capacitor.



Apply Ampere's law for the flat (light blue) surface at the top:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

So **there is a magnetic field along L**.

Now, apply Ampere's law for the "tophat" (light green) surface made of the cylindrical part (no current flowing through it) and the circle between the two conducting plates (also no current):

$$\oint_L \vec{B} \cdot d\vec{\ell} = 0$$

So **there is NO magnetic field along L**.

We get **contradictory predictions** for the same path L! What is wrong?

A problem with Ampere's Law

We get **contradictory predictions** for the same path L! What is wrong?

When we studied Faraday's law, we saw that a change in the magnetic field $\vec{B}$ is responsible for creating an electric field $\vec{E}$:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

We should expect that the opposite may be true as well! A change in the electric field $\vec{E}$ may modify the magnetic field $\vec{B}$.

In the example studied previously, the electric field $\vec{E}$ change with time.

How should we extend Ampere's law to take the effects of a changing electric field into account?

Extending Ampere's law

Let's start from the differential form of Ampere's law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

We can calculate the divergence of both sides:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) = \mu_0 \vec{\nabla} \cdot \vec{j}$$

The left-hand side is 0 (the divergence of the curl of a vector field is always 0). So the right-hand side has to be 0 as well.

$$\vec{\nabla} \vec{j} = 0$$

But this is generally not true! Recall the continuity equation that expresses the local conservation of charge:

$$\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$$

Extending Ampere's law

Ampere's law, as we know it so far, is at odds with the fundamental principle of the local conservation of charge. Can we reconcile the two? The charge density is given by Gauss' law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$$

Taking the partial derivative of ρ with respect to time:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\left(\epsilon_0 \vec{\nabla} \cdot \vec{E}\right) = \vec{\nabla} \cdot \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Then, the continuity equation:

$$\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$$

becomes:

$$\vec{\nabla} \cdot \vec{j} = -\vec{\nabla} \cdot \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Extending Ampere's law

We have written the continuity equation as:

$$\vec{\nabla} \cdot \vec{j} = -\vec{\nabla} \cdot \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

which suggests that:

$$\vec{\nabla} \cdot \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0$$

This is interesting hint!

Maxwell realised that all it takes to fix Ampere's law is to do the following substitution:

$$\vec{j} \rightarrow \vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The term $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ is the density of the so-called called **displacement current**.

Extending Ampere's law

With this extension (adding the displacement current), Ampere's law is no longer at odds with the local conservation of charge, as expressed with the continuity equation. Indeed:

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Big) \Rightarrow$$

$$\vec{\nabla} \cdot \Big(\vec{\nabla} \times \vec{B} \Big) = \vec{\nabla} \cdot \Big(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \Big) \Rightarrow$$

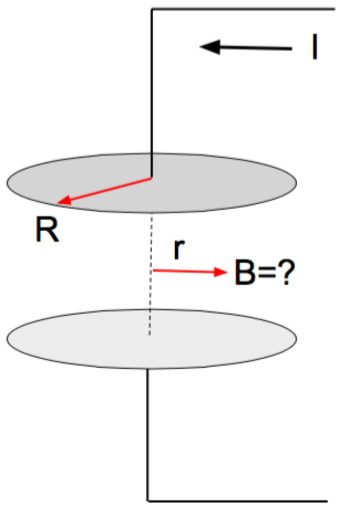
$$0 = \cancel{\mu_0} \vec{\nabla} \cdot \vec{j} + \epsilon_0 \cancel{\mu_0} \frac{\partial}{\partial t} \Big(\vec{\nabla} \cdot \vec{E} \Big) \Rightarrow$$

$$0 = \vec{\nabla} \cdot \vec{j} + \cancel{\epsilon_0} \frac{\partial}{\partial t} \Big(\frac{\rho}{\cancel{\epsilon_0}} \Big) \Rightarrow$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j}$$

Worked example

Question



A parallel-plate capacitor with circular plates of radius R is being charged as shown in the figure on the left. Derive an expression for the magnetic field at radius r (in the volume within the two plates) for the case $r \leq R$.

A magnetic field can be created either by a current or a changing electric field. This is expressed by Ampere's law:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 \int_S \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S}$$

Worked example

There is no current between the capacitor plates, but the electric flux there is changing. Thus, Ampere's law reduces to:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \int_S \left(\frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S}$$

We choose a circular Amperian loop with radius r $\leq$ R.

The magnetic field $\vec{B}$ at all points along the loop is tangent to the loop, as is the path element $d\vec{\ell}$. Thus $\vec{B}$ and $d\vec{\ell}$ are parallel or antiparallel at each point of the loop.

For simplicity, assume that we step along the loop in such a way so that $\vec{B}$ and $d\vec{\ell}$ are parallel. Therefore:

$$\oint_L \vec{B} \cdot d\vec{\ell} = \oint_L B \, d\ell$$

Worked example

Due to the circular symmetry of the plates, we can also assume that $\vec{B}$ has the same magnitude at every point around the loop. Thus, B can be taken outside the integral.

The integral that remains is $\oint_L d\ell$ which simply gives the circumference $2\pi r$ of the loop. Therefore, the integral becomes:

$$\oint_L \vec{B} \cdot d\vec{\ell} = B \left(2\pi r\right)$$

Substituting the above result into Ampere's law gives:

$$B \left(2\pi r\right) = \mu_0 \epsilon_0 \int_S \left(\frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{S} \Rightarrow B = \frac{\mu_0 \epsilon_0}{2\pi r} \int_S \left(\frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{S} \Rightarrow$$

$$B = \frac{\mu_0 \epsilon_0}{2\pi r} \frac{\partial}{\partial t} \int_S \vec{E} \cdot d\vec{S}$$

Worked example

We assume that the electric field $\vec{E}$ is uniform between the capacitor plates and directed perpendicular to the plates. The electric flux through the Amperian loop is simply EA, where A is the (constant) area encircled by the loop within the electric field. The previous equation becomes:

$$B = \frac{\mu_0 \epsilon_0}{2\pi r} \frac{d}{dt}\Big(EA\Big) = \frac{\mu_0 \epsilon_0}{2\pi r} A \frac{dE}{dt}$$

The area A that is encircled by the Amperial loop within the electric field is the full area πr^2 of the loop because the loop's radius r is less than (or equal to) the plate radius R. Substituting πr^2 for A, we have:

$$B = \frac{\mu_0 \epsilon_0}{2\pi r} \pi r^2 \frac{dE}{dt} \Rightarrow B = \frac{\mu_0 \epsilon_0 r}{2} \frac{dE}{dt}$$

Worked example

Ignoring edge effects, the electric field E is uniform within the plates of the parallel plate capacitor and vanishes outside the volume within the two plates. Using a cylindrical Gaussian surface whose bases are parallel to the plates and which encloses the upper plate, we have:

$$E \pi R^2 = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{1}{\pi R^2 \epsilon_0} Q$$

Therefore, the rate of change of E is given by:

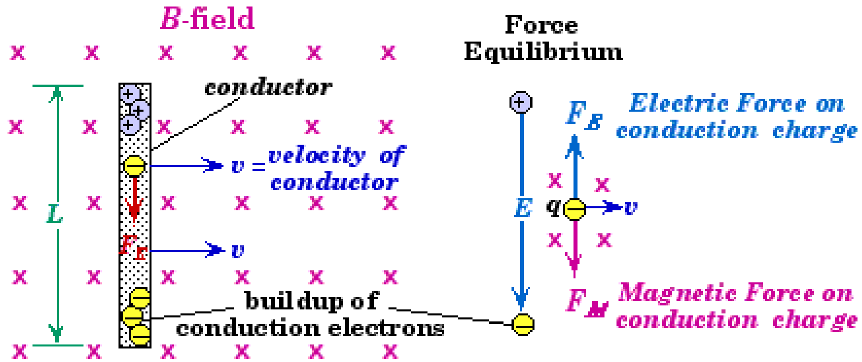
$$\frac{dE}{dt} = \frac{1}{\pi R^2 \epsilon_0} \frac{dQ}{dt} \xRightarrow{I = dQ/dt} \frac{dE}{dt} = \frac{1}{\pi R^2 \epsilon_0} I$$

Substituting the expression for dE/dt in the expression we obtained earlier for B, we have:

$$B = \frac{\mu_0 \epsilon_0 r}{2} \frac{1}{\pi R^2 \epsilon_0} I \Rightarrow B = \frac{\mu_0 I r}{2 \pi R^2}$$

This is the required expression for the magnetic field at radius r $\leq$ R.

We consider a conductor with length L moves with velocity $\vec{u}$ inside a homogenous magnetic field $\vec{B}$, as shown on the right.



Each electron in the conductor feels a magnetic force $\vec{F}_M = q\vec{u} \times \vec{B}$.

That magnetic force **induces the build-up of charge** which **produces an electric field** $\vec{E}$: Each electron feels an electric force $\vec{F}_E = q\vec{E}$.

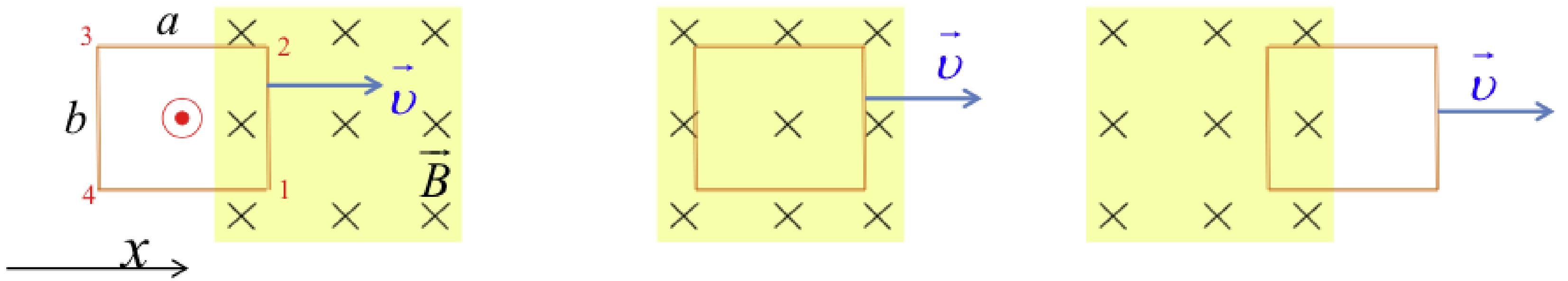
The resulting **electric force** $\vec{F}_E$ **opposes the magnetic force** $\vec{F}_M$.

An electrical potential difference develops between the ends of the moving conductor, which becomes a source of EMF:

$$\mathcal{E} = \int_L \vec{E} \cdot d\vec{\ell} = \int_L \left(\vec{u} \times \vec{B}\right) \cdot d\vec{\ell}$$

Lecture 8 revision (Circuit moving in a magnetic field)

A rectangular circuit moving through a region with magnetic field $\vec{B}$:



In summary:

	$\oint_L \vec{E} \cdot d\vec{\ell}$	$\frac{d\Phi_M}{dt}$
left	uBb	-uBb
centre	0	0
right	-uBb	uBb

So, indeed, in all cases:

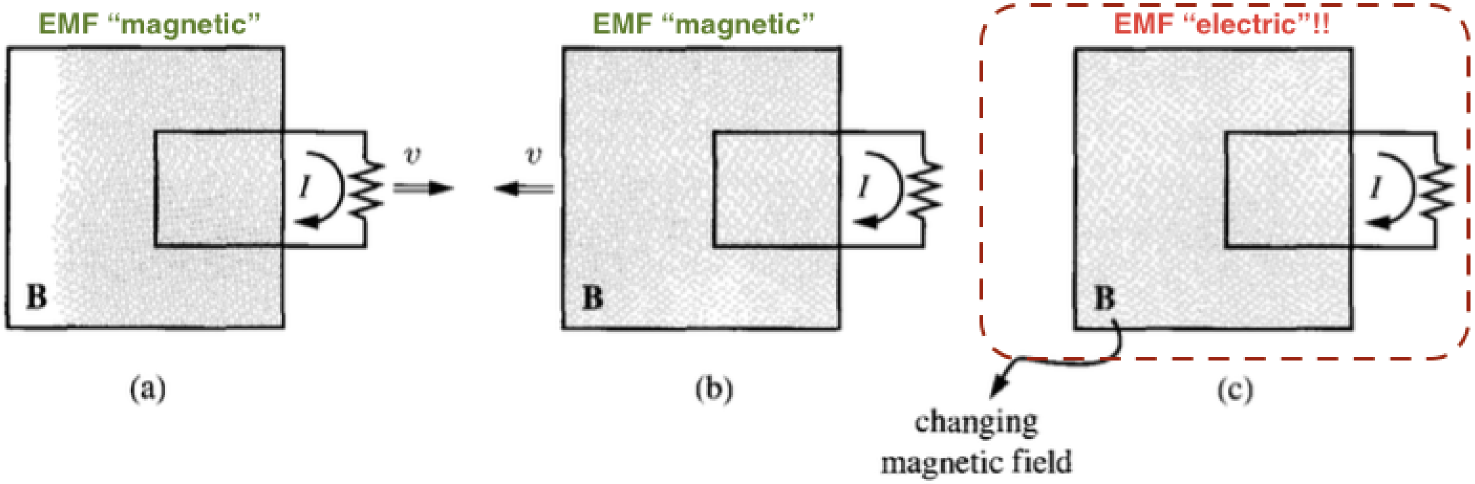
$$\mathcal{E} = \oint_L \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_M}{dt}$$

Lecture 8 revision (Faraday's observations)

In 1831 Michael Faraday reported on a series of experiments.

A current flows in a wire loop when:

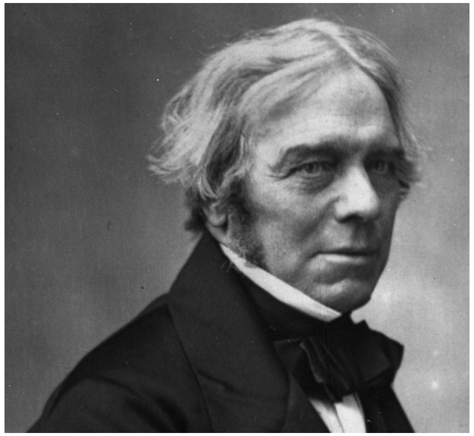
- (a) the loop is pulled through a magnetic field,
- (b) the loop is at rest but the magnet moves in the opposite direction, and
- (c) both the loop and the magnet are at rest but the strength of the magnetic field is varied



This led Faraday to realize that:

A time-varying magnetic field induces an electic field

In all cases the **motional EMF** is directly related to the **change of the magnetic flux Φ_M though the circuit**:



Michael Faraday (1791 - 1867).

$$\mathcal{E} = \oint_L \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_M}{dt}$$

This is the so-called **Faraday's law**.

Lenz' law (1845): The EMF induced by a changing flux has a polarity such that the current flowing gives rise to a flux which opposes the change of flux.

The minus sign is a consequence of the **conservation of energy** and of **Newton's 3rd law** of motion: Induction is a an "inertial reaction". The system develops a current which tries to maintain the flux constant.

Heinrich Lenz (1804 - 1865).

We also studied a case where Ampere's law led to paradoxical results.

We also saw that Ampere's law (as we knew it) was inconsistent with the continuity equation (which expresses the local conservation of charge).

The problem of course was that we took a law from magnetostatics and applied it in a different context (electrodynamics) where it is no longer valid.

Maxwell realised that all it takes to fix Ampere's law is to do the following substitution:

$$\vec{j} \rightarrow \vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The term $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ is the density of the so-called **displacement current**.

Lecture 8 revision (Maxwell's eqs for the dynamic case)

Compared with what we had seen in the study of electrostatics and magnetostatics, the study of time-dependent fields (electrodynamics) brought the following complication:

- **Electric fields are produced** not only by electric charges, but also **by changing magnetic fields!**

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- **Magnetic fields are produced** not only by electric currents, but also **by changing electric fields!**

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

The full list of Maxwell equations for the static and dynamic cases in vacuum is shown on the next slide. Notice that all 4 equations are coupled in the dynamic case.

Lecture 8 revision (Maxwell's eqs. for the dynamic case)

Static case (in vacuum)		
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int \rho d\tau$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot d\vec{S}$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

Generalization of above for the dynamic case (in vacuum)		
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int \rho d\tau$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Faraday's law	$\oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Gauss's law (magn)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{S}$	$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$

Lecture 8 - Main points to remember

Compared with what we had seen in the study of electrostatics and magnetostatics, the study of time-dependent fields (electrodynamics) brought the following complication:

- **Electric fields are produced** not only by electric charges, but also **by changing magnetic fields!**

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- **Magnetic fields are produced** not only by electric currents, but also **by changing electric fields!**

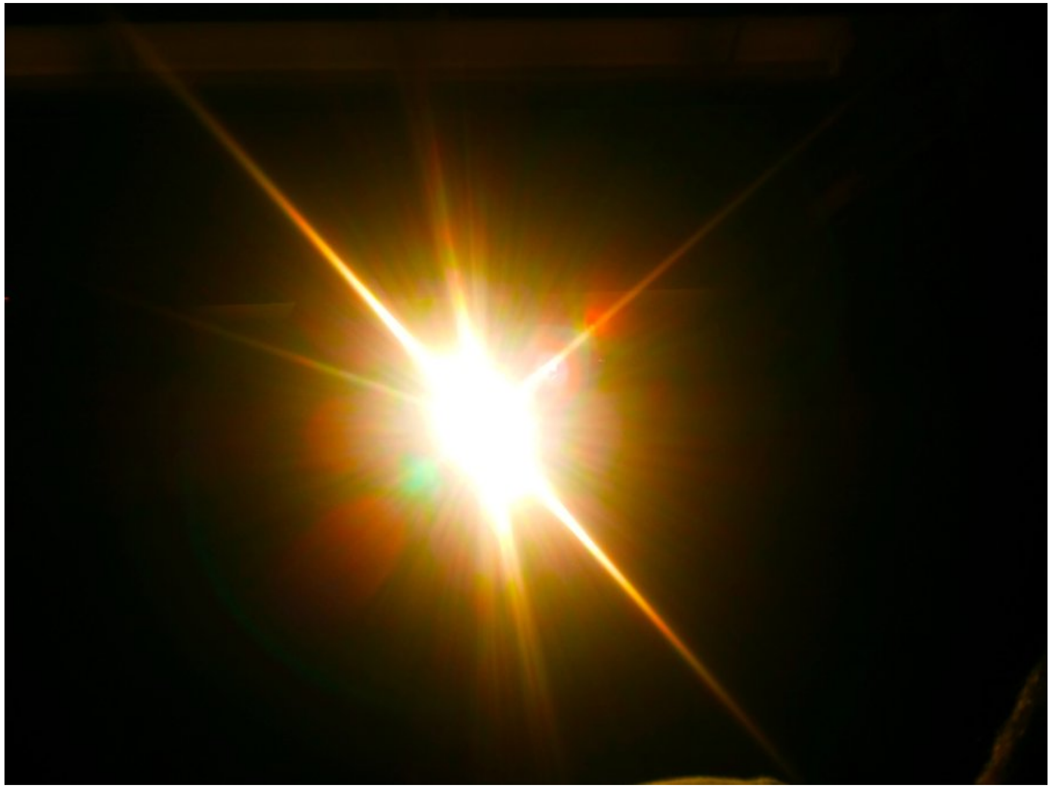
$$\vec{\nabla} \times \vec{B} = \mu_0 \Big(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Big)$$

The full list of Maxwell equations for the static and dynamic cases in vacuum is shown on the next slide. Notice that whereas the 2 equations involving $\vec{E}$ and the 2 equations involving $\vec{B}$ were decoupled in the static case, all 4 equations are coupled in the dynamic case.

At the next lecture (Lecture 9)

- We will study the solutions of the time-dependent Maxwell equations in vaccum away from sources ($\rho = 0$, $\vec{j} = \vec{0}$)

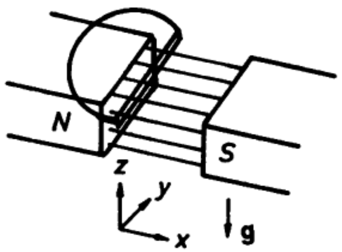
- **Light**!



Optional reading for Lecture 8

Worked example: Wire falling in magnetic field

Question



A long straight wire parallel to the y axis lies in a uniform magnetic field $\vec{B} = B\hat{x}$, as shown in the figure on the left. The mass per unit length and resistance per unit length of the wire are λ_m and λ_r respectively.

The wire may be considered to extend to the edges of the field, where the ends are connected to one another by a massless perfect conductor which lies outside the field. Fringing effects can be neglected.
The wire is allowed to fall under the influence of gravity ($\vec{g} = -g\hat{z}$).
Find:

- the electromotive force developed across the wire,
- the current flowing along the wire,
- the magnetic force acting on the wire, and
- the terminal velocity of the wire as it falls through the magnetic field.

Worked example: Wire falling in magnetic field

As the wire falls within the magnetic field with a velocity u, the EMF ε developed across the wire is given by:

$$\varepsilon = \int (\vec{u} \times \vec{B}) \cdot d\vec{\ell} = uB\ell$$

where ℓ is the length of the wire (which is fully within the magnetic field). The resistance R of the wire of length ℓ is:

$$R = \lambda_r \ell$$

Therefore, the current that will flow in the wire is given by:

$$I = \frac{\varepsilon}{R} = \frac{uB\cancel{\ell}}{\lambda_r \cancel{\ell}} = \frac{uB}{\lambda_r}$$

This current is flowing towards the $-\hat{y}$ direction.

Worked example: Wire falling in magnetic field

The force exerted on the current-carrying wire because of the magnetic field is:

$$\vec{F}_B = I \int d\vec{\ell} \times \vec{B}$$

Substituting I from the previous expression, and considering $d\vec{\ell}$ to be in the direction of the current, we have:

$$\vec{F}_B = \frac{uB}{\lambda_r} \int \Big(dy(-\hat{y})\Big) \times \Big(B\hat{x}\Big) = \frac{uB^2}{\lambda_r} \Big(\int dy\Big)\Big(-\hat{y}) \times \hat{x}\Big) \Rightarrow$$

$$\vec{F}_B = \frac{uB^2 \ell}{\lambda_r} \hat{z}$$

This force will oppose the downwards gravitational force $\vec{F}_g$.

Worked example: Wire falling in magnetic field

The wire carries a mass given by:

$$m = \lambda_m \ell$$

and, therefore, the force of gravity is:

$$\vec{F}_g = m\vec{g} = -\lambda_m \ell g \hat{z}$$

The wire will reach its terminal velocity u' when the two forces cancel each other out exactly. Therefore, u' is given by the condition:

$$\frac{u' B^2 \cancel{\ell}}{\lambda_r} = \lambda_m \cancel{\ell} g$$

Solving for u' we find:

$$u' = \frac{\lambda_r \lambda_m g}{B^2}$$

Worked example: Displacement current in capacitor

Question

At what rate must the potential difference between the plates of a parallel plate capacitor with a 2.0 μF capacitance be changed to produce a displacement current of 1.5 A?

Let the area plate be A and the plate separation be d. The displacement current I_d is:

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt} \xrightarrow{\Phi_E = EA} I_d = \epsilon_0 \frac{d}{dt}(EA) = \epsilon_0 A \frac{dE}{dt}$$

$$\xrightarrow{E=V/d} I_d = \epsilon_0 A \frac{d}{dt}\left(\frac{V}{d}\right) = \frac{\epsilon_0 A}{d}\frac{dV}{dt} \xrightarrow{C=\epsilon_0 A/d} I_d = C\frac{dV}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{I_d}{C} = \frac{15.0\ A}{2 \times 10^{-6}\ F} = 7.5 \times 10^5\ V/s$$

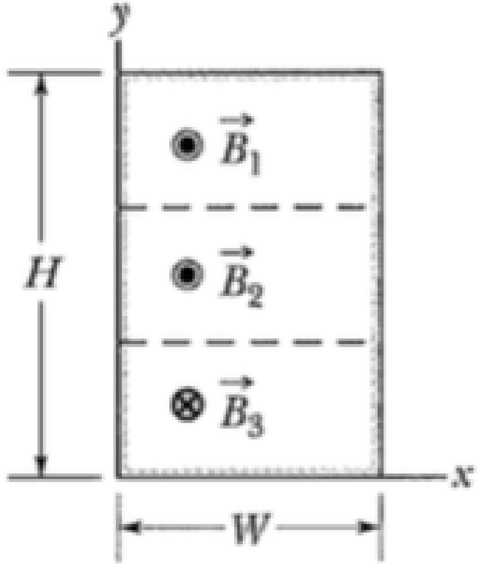
Worked example: Induced current in rectangular wire loop

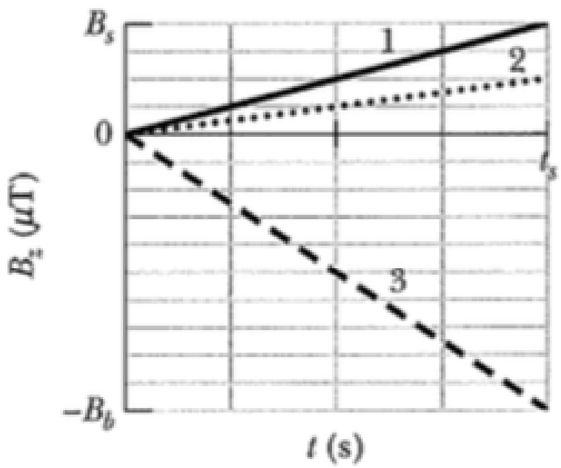
Question

The figure below shows a wire that forms a rectangle (W = 20 cm, H = 30 cm) and has a resistance of 5.0 mΩ. Its interior is split into three equal areas, with magnetic fields $\vec{B}_1$, $\vec{B}_2$ and $\vec{B}_3$. The fields are uniform within each region and directly out of or into the page as indicated.
The graph below gives the change in the z components (B_z) of the three fields with time t; the vertical axis scale is set by $B_s = 4.0\ \mu$T and $B_b = 2.5 B_s$ and the horizontal axis scale is set by $t_s = 2.0$ s.

What are the

- magnitude, and
- direction of the current induced in the wire?





Worked example: Induced current in rectangular wire loop

The induced emf is:

$$\varepsilon = -\sum_i \frac{d\Phi_{B;i}}{dt}$$

If the surface vector of the loop is collinear with $\vec{B}_1$ and $\vec{B}_2$, the flux due to $\vec{B}_1$ and $\vec{B}_2$ is positive, whereas the flux due to $\vec{B}_3$ is negative. Therefore:

$$\varepsilon = \frac{1}{3}HW\Big\{-\frac{dB_1}{dt} - \frac{dB_2}{dt} + \frac{dB_3}{dt}\Big\} =$$

$$\frac{(0.30\ m)(0.20\ m)}{3}\Big\{-\frac{4\times10^{-6}\ T}{2.0\ s} - \frac{2\times10^{-6}\ T}{2.0\ s} + \frac{10\times10^{-6}\ T}{2.0\ s}\Big\} = 4\times10^{-8}\ V$$

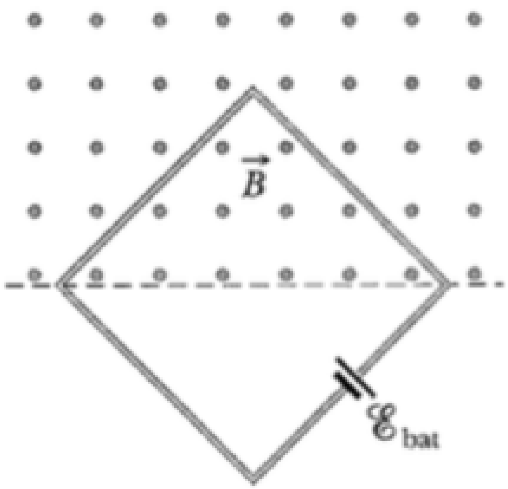
The plus sign means that the emf is dominated by changes in $\vec{B}_3$.
The current induced by ε is:

$$I = \frac{|\varepsilon|}{R} = \frac{4\times10^{-8}\ V}{5\times10^{-3}\ \Omega} \approx 8\ \mu A$$

By Lenz's law, the induced emf (and current) resist to these changes in the magnetic flux. Therefore, the direction of the current is counter-clockwise.

Worked example: Loop in uniform time-dependent $\vec{B}$

Question



A square wire loop with 2.00 m sides is perpendicular to a uniform magnetic field, with half the area of the loop in the field, as shown in the figure on the left. The loop contains an ideal battery with emf $\varepsilon_{bat} =$ 20 V. If the magnitude of the field varies with time according to $B = 0.0420$ - $0.870t$, with B in Teslas and t in seconds, what are the

- net emf in the circuit, and
- the direction of the (net) emf around the loop?

Let L be the length of a side of the square circuit. Then the magnetic flux through the circuit is:

$$\Phi_B = \frac{1}{2} L^2 B$$

Worked example: Loop in uniform time-dependent $\vec{B}$

The induced emf is:

$$\varepsilon_{induced} = -\frac{d\Phi_B}{dt} = -\frac{1}{2}L^2\frac{dB}{dt}$$

The rate of change of the given field is:

$$\frac{dB}{dt} = \frac{d}{dt}(0.0420 - 0.870t) = -0.870 \ T/s$$

Therefore, the induced emf is:

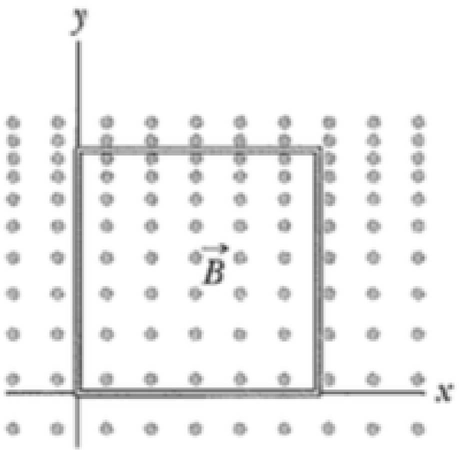
$$\varepsilon_{induced} = -\frac{1}{2}(2.0 \ m)^2(-0.870 \ T/s) = 1.74 \ V$$

The magnetic field is out of the page and decreasing so the induced emf is counterclockwise around the circuit, in the same direction as the emf of the battery. The total emf is:

$$\varepsilon_{total} = \varepsilon_{induced} + \varepsilon_{battery} = 1.74 \ V + 20 \ V = 21.74 \ V$$

Worked example: Loop in non-uniform time-dependent $\vec{B}$

Question



As seen in the figure on the left, a square loop of wire has sides of length 2.0 cm. A magnetic field is directed out of the page; its magnitude is given by $B = 4.0t^2y$, where B is in Teslas, t is in seconds, and y is in meters. At $t = 2.5$ s, what are the

- magnitude, and
- direction of the emf induced in the loop?

Consider a (thin) strip of area of height dy and width $\ell = 0.020$ m. The strip is located at position y $(0 < y < \ell)$. The magnetic field in that thin strip is uniform and, therefore, the magnetic flux through that strip is:

$$d\Phi_B = BdA = (4t^2y)(\ell dy)$$

Worked example: Loop in non-uniform time-dependent $\vec{B}$

The total flux through the square loop is:

$$\Phi_B = \int d\Phi_B = \int_0^{\ell} 4t^2 y \ell dy = 4t^2 \ell \int_0^{\ell} y dy = 2t^2 \ell^3$$

Thus, Faraday's law yields:

$$\varepsilon = -\frac{d\Phi_B}{dt} = -4t\ell^3$$

At $t = 2.5$ s, the magnitude of the emf is:

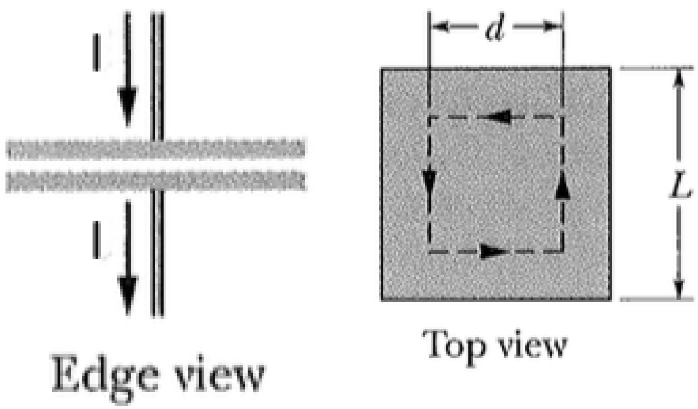
$$|\varepsilon| = 4(2.5\ s)(0.02\ m)^3 = 8.0 \times 10^{-5}\ V$$

The emf direction is clockwise, by Lenz's law.

Worked example: Charging a parallel-plate capacitor

Question

A parallel-plate capacitor has square plates of edge length $L = 1.0$ m. A current of 2.0 A charges the capacitor, producing a uniform electric field $\vec{E}$ between the plates, with $\vec{E}$ perpendicular to the plates.



- What is the displacement current I_d through the region between the plates?
- What is dE/dt in this region?
- What is the displacement current encircled by the square dashed path of edge length $d = 0.50$ m?
- What is $\oint \vec{B} \cdot d\vec{\ell}$ around this square dashed path?

Worked example: Charging a parallel-plate capacitor

As the current I charges the capacitor, the electric field between the plates of the capacitor is changing. This produces a displacement current I_d between the plates, which is given by:

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

Let A be the area of a plate, d the plate separation, and E the magnitude of the electric field between the plates. E is uniform, and it is given by:

$$E = \frac{V}{d}$$

where V is the potential difference across the plates.
The current into the positive plate of the capacitor is

$$I = \frac{dQ}{dt} = \frac{d}{dt}(CV) = C\frac{dV}{dt} = \frac{\epsilon_0 A}{d}\frac{d(Ed)}{dt} = \epsilon_0 A \frac{dE}{dt} = \epsilon_0 \frac{d\Phi_E}{dt} = I_d$$

At any time, the conduction current I in the wires equals the displacement current I_d in the gap between the plates, and thus $I_d = 2.0$ A.

Worked example: Charging a parallel-plate capacitor

The rate of change of the electic field is:

$$I_d = \epsilon_0 A \frac{dE}{dt} \Rightarrow \frac{dE}{dt} = \frac{I_d}{\epsilon_0 A} \Rightarrow$$

$$\frac{dE}{dt} = \frac{2.0\ A}{(8.85 \times 10^{-12}\ F/m)(1.0\ m^2)} = 2/3 \times 10^{11} \frac{V}{m \cdot s}$$

The displacement current I'_d through the indicated path is

$$I'_d = I_d \frac{d^2}{L^2} = (2.0\ A) \frac{(0.5\ m)^2}{(1.0\ m)^2} = (2.0\ A) \frac{1}{4} = 0.5\ A$$

From Ampere's law, the integral of the magnetic field around the indicated path is

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I'_d = (1.26 \times 10^{-6}\ T \cdot m/A)(0.5\ A) = 6.3 \times 10^{-7}\ T \cdot m$$

PHYS 201 / Lecture 9
Electromagnetic Waves

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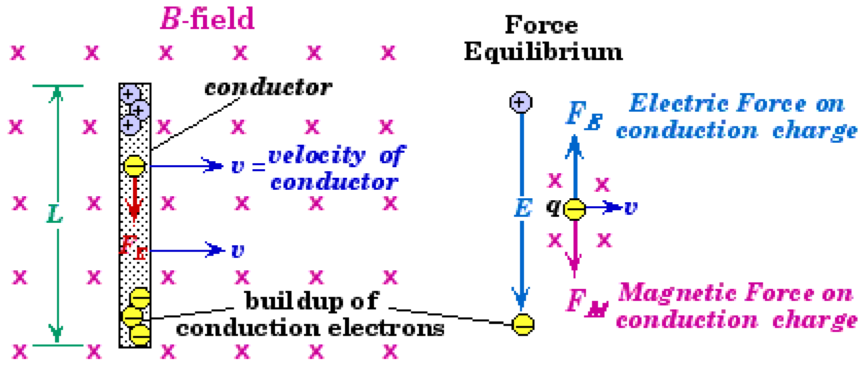
Lectures delivered at the University of Liverpool, 2021-22

December 15, 2021

Lecture 8 revision (Conductors moving in a magnetic field)

We consider a conductor with length L moves with velocity $\vec{u}$ inside a homogenous magnetic field $\vec{B}$, as shown on the right.



Each electron in the conductor feels a magnetic force $\vec{F}_M = q\vec{u} \times \vec{B}$.

That magnetic force **induces the build-up of charge** which **produces an electric field** $\vec{E}$: Each electron feels an electric force $\vec{F}_E = q\vec{E}$.

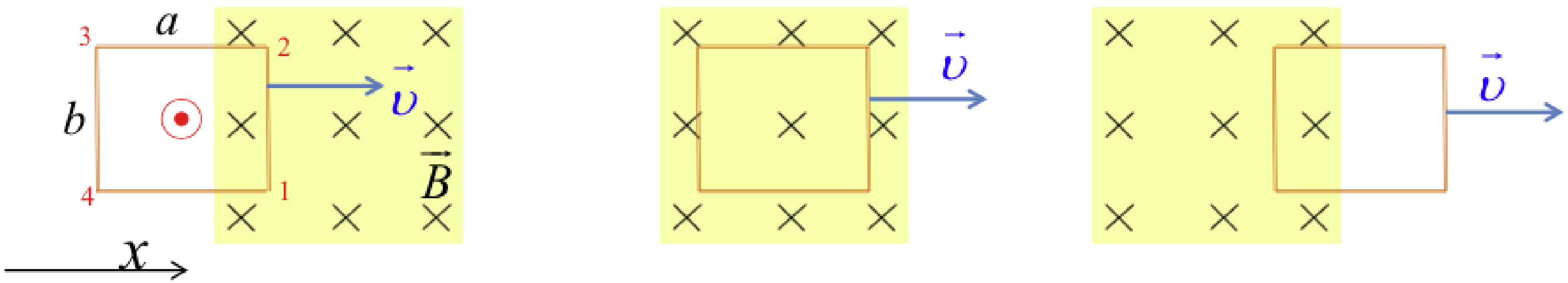
The resulting **electric force** $\vec{F}_E$ **opposes the magnetic force** $\vec{F}_M$.

An electrical potential difference develops between the ends of the moving conductor, which becomes a source of EMF:

$$\mathcal{E} = \int_L \vec{E} \cdot d\vec{\ell} = \int_L \left(\vec{u} \times \vec{B}\right) \cdot d\vec{\ell}$$

Lecture 8 revision (Circuit moving in a magnetic field)

A rectangular circuit moving through a region with magnetic field $\vec{B}$:



In summary:

	$\oint_L \vec{E} \cdot d\vec{\ell}$	$\frac{d\Phi_M}{dt}$
left	uBb	-uBb
centre	0	0
right	-uBb	uBb

So, indeed, in all cases:

$$\mathcal{E} = \oint_L \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_M}{dt}$$

Lecture 8 revision (Faraday's observations)

In 1831 Michael Faraday reported on a series of experiments.

A current flows in a wire loop when:

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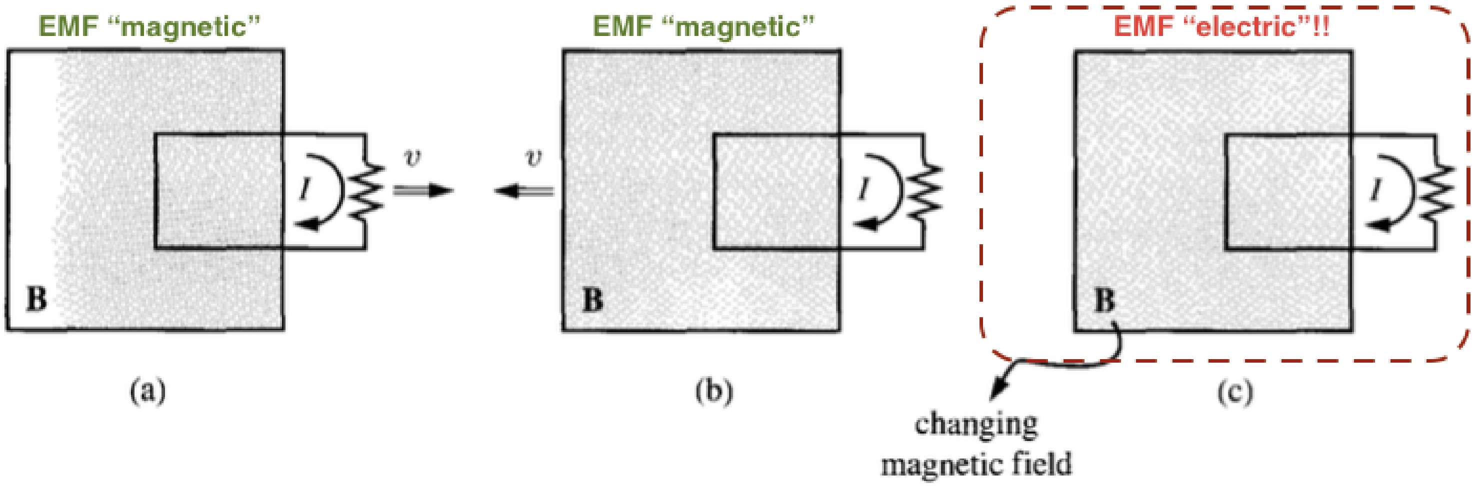
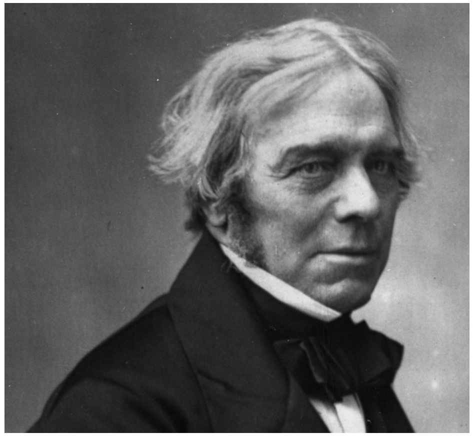
b) the loop is at rest but the magnet moves in the opposite direction, and
c) both the loop and the magnet are at rest but the strength of the magnetic field is varied



This led Faraday to realize that:

A time-varying magnetic field induces an electic field

In all cases the **motional EMF** is directly related to the **change of the magnetic flux Φ_M though the circuit**:



Michael Faraday (1791 - 1867).

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This is the so-called **Faraday's law**.

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Heinrich Lenz (1804 - 1865).

We also studied a case where Ampere's law led to paradoxical results.

We also saw that Ampere's law (as we knew it) was inconsistent with the continuity equation (which expresses the local conservation of charge).

The problem of course was that we took a law from magnetostatics and applied it in a different context (electrodynamics) where it is no longer valid.

Maxwell realised that all it takes to fix Ampere's law is to do the following substitution:

$$\vec{j} \rightarrow \vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The term $\epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ is the density of the so-called **displacement current**.

Lecture 8 revision (Maxwell's eqs for the dynamic case)

Compared with what we had seen in the study of electrostatics and magnetostatics, the study of time-dependent fields (electrodynamics) brought the following complication:

- **Electric fields are produced** not only by electric charges, but also **by changing magnetic fields!**

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- **Magnetic fields are produced** not only by electric currents, but also **by changing electric fields!**

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Big)$$

The full list of Maxwell equations for the static and dynamic cases in vacuum is shown on the next slide. Notice that all 4 equations are coupled in the dynamic case.

Lecture 8 revision (Maxwell's eqs. for the dynamic case)

Static case (in vacuum)		
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int \rho d\tau$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot d\vec{S}$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

Generalization of above for the dynamic case (in vacuum)		
Gauss's law	$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int \rho d\tau$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Faraday's law	$\oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Gauss's law (magn)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{S}$	$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$

Lecture 8 - Revision

Compared with what we had seen in the study of electrostatics and magnetostatics, the study of time-dependent fields (electrodynamics) brought the following complication:

- **Electric fields are produced** not only by electric charges, but also **by changing magnetic fields!**

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- **Magnetic fields are produced** not only by electric currents, but also **by changing electric fields!**

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Big)$$

The full list of Maxwell equations for the static and dynamic cases in vacuum is shown on the next slide. Notice that whereas the 2 equations involving $\vec{E}$ and the 2 equations involving $\vec{B}$ were decoupled in the static case, all 4 equations are coupled in the dynamic case.

Plan for Lecture 9

- Maxwell's equations in vacuum in absence of sources
- Reminder on waves
- The electric and magnetic fields satisfy the wave equation
- Electromagnetic waves and light
- Energy carried by electromagnetic waves
- The Poynting theorem

Maxwell's equations in vacuum in absence of sources

Let me start from the known Maxwell's equations in vacuum, and consider the case where there are:

- **no charges** ($\rho = 0$)
- and **no currents** ($\vec{j} = \vec{0}$).

$$\vec{\nabla} \cdot \vec{E} = \frac{\overset{0}{\cancel{\rho}}}{\cancel{\epsilon_0}} \Rightarrow \vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \overset{0}{\cancel{\mu_0 \vec{j}}} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

A change in one field feeds the other (even in absence of sources): Faraday's law tells us that a changing magnetic field generates an electric field, and Maxwell's correction to Ampere's law tells us that a changing electric field generates a magnetic field.

Can this interplay generate waves?

Reminder: Waves

A wave is a **disturbance that travels through space.**

As this disturbance travels it transfers energy.

If there is no absorption or dispersion, that disturbance propagates with a constant velocity and a fixed ‘shape’.

Reminder: Types of waves

There are **three main types of waves**:

- **Mechanical waves**
  - These are the waves you are most familiar with
    - For example, sound waves, sea waves, seismic waves.
  - Mechanical waves **need a medium** to propagate into. They propagate by deforming that medium.
    - When I speak, I disturb the air molecules in front of my mouth.
    - They collide with neighbouring molecules and bounce back.
    - This disturbance propagates, and it reaches and vibrates your eardrums.
- **Electromagnetic waves** $\leftarrow$ The subject of this lecture
  - Less familiar, probably, although you interact with them constantly.
  - They **do not require a medium to propagate.**
- **Matter waves** $\leftarrow$ Will study in future modules
  - You are probably very unfamiliar with these, but they take centre stage in quantum mechanics (*wave-particle duality*).
  - Waves associated with fundamental particles (such as the electron), composite particles (such as the proton), atoms or even molecules.

Reminder: The wave equation

A **wave equation** describes how the disturbance propagates in time.

Let $\phi(\vec{r}, t)$ be a function that describes that disturbance as a function of position in space and time. Then it satisfies the following equation:

$$\vec{\nabla}^2 \phi(\vec{r}, t) = \frac{1}{u^2} \frac{\partial^2 \phi(\vec{r}, t)}{\partial t^2}$$

where **u is the wave velocity**.

A mathematician would say that the wave equation is a **2nd order linear partial differential equation (p.d.e)**.

- This particular type of p.d.e. is called *hyperbolic*.

Reminder: Waves

Look carefully at the wave equation:

$$\vec{\nabla}^2 \phi(\vec{r}, t) = \frac{1}{u^2} \frac{\partial^2 \phi(\vec{r}, t)}{\partial t^2}$$

Here, we encounter $\vec{\nabla}^2$ again. This is the so-called *Laplace operator* that we have already seen at a previous lecture (Poisson equation).

The Laplacian $\vec{\nabla}^2 \phi$ is defined as **the divergence of the gradient** of the scalar function ϕ:

$$\vec{\nabla}^2 \phi(\vec{r}, t) = \vec{\nabla} \cdot \left(\vec{\nabla} \phi(\vec{r}, t) \right) = \frac{\partial^2 \phi(\vec{r}, t)}{\partial x^2} + \frac{\partial^2 \phi(\vec{r}, t)}{\partial y^2} + \frac{\partial^2 \phi(\vec{r}, t)}{\partial z^2}$$

Recall that it represents a quantity which is important in several physical processes: The Laplacian $\vec{\nabla}^2 \phi(\vec{r})$ of a scalar function ϕ at a point $\vec{r}$ tells you how much $\phi(\vec{r})$ differs from its average over a small volume around $\vec{r}$.

Reminder: Solutions of the wave equation

In one dimension, say x, the wave equation becomes:

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} = \frac{1}{u^2} \frac{\partial^2 \phi(x,t)}{\partial t^2}$$

where u is the wave velocity.

What are the solutions of the wave equation?

Any (any whatsoever) function ϕ of x and t, **where x and t appear only in the combination of x-ut**, is a solution.

$$\phi(x,t) = \psi(z) = \psi(x - ut)$$

For example, the following function is a solution of the wave equation:

$$\phi = sin^3\Big(x - ut\Big) - \frac{e^{(x-ut)}}{cos^{\frac{7}{2}}\Big(x - ut\Big)} + \Big(x - ut\Big)^{1.298}$$

We can easily see that using the chain rule.

Reminder: Solutions of the wave equation

Any function of x and t, where x and t appear only in the combination of x-ut, is a solution of the wave equation.

Let $\psi(z)$ be a function of x-ut (z = x-ut). The partial derivative of $\psi(z)$ with respect to time is:

$$\frac{\partial\psi(z)}{\partial t} = \frac{\partial z}{\partial t} \cdot \frac{\partial\psi(z)}{\partial z} = \left\{ \frac{\partial}{\partial t}\Big(x - ut\Big) \right\} \cdot \frac{\partial\psi(z)}{\partial z} = -u\frac{\partial\psi(z)}{\partial z}$$

Differentiating once again, we have:

$$\frac{\partial^2\psi(z)}{\partial t^2} = \frac{\partial}{\partial t}\Big(\frac{\partial\psi(z)}{\partial t}\Big) = \frac{\partial}{\partial t}\Big(- u\frac{\partial\psi(z)}{\partial z}\Big) = \frac{\partial z}{\partial t}\frac{\partial}{\partial z}\Big(- u\frac{\partial\psi(z)}{\partial z}\Big) =$$

$$= \left\{ \frac{\partial}{\partial t}\Big(x - ut\Big) \right\} \frac{\partial}{\partial z}\Big(- u\frac{\partial\psi(z)}{\partial z}\Big) = -u\frac{\partial}{\partial z}\Big(- u\frac{\partial\psi(z)}{\partial z}\Big) = u^2\frac{\partial^2\psi(z)}{\partial z^2} \Rightarrow$$

$$\frac{1}{u^2}\frac{\partial^2\psi(z)}{\partial t^2} = \frac{\partial^2\psi(z)}{\partial z^2} (*)$$

Reminder: Solutions of the wave equation

Similarly, the partial derivative of $\psi(z)$ with respect to x is:

$$\frac{\partial\psi(z)}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\partial\psi(z)}{\partial z} = \left\{ \frac{\partial}{\partial x}\Big(x - ut\Big) \right\} \cdot \frac{\partial\psi(z)}{\partial z} = \frac{\partial\psi(z)}{\partial z}$$

Differentiating once again, we have:

$$\frac{\partial^2\psi(z)}{\partial x^2} = \frac{\partial}{\partial x}\Big(\frac{\partial\psi(z)}{\partial x}\Big) = \frac{\partial}{\partial x}\Big(\frac{\partial\psi(z)}{\partial z}\Big) = \frac{\partial z}{\partial x}\frac{\partial}{\partial z}\Big(\frac{\partial\psi(z)}{\partial z}\Big) =$$

$$= \left\{ \frac{\partial}{\partial x}\Big(x - ut\Big) \right\} \frac{\partial}{\partial z}\Big(\frac{\partial\psi(z)}{\partial z}\Big) = \frac{\partial^2\psi(z)}{\partial z^2} \Rightarrow \frac{\partial^2\psi(z)}{\partial x^2} = \frac{\partial^2\psi(z)}{\partial z^2} (**)$$

The right-hand sides of Eqs. (*) and (**) are equal, and so are the left-hand sides:

$$\frac{\partial^2\psi(z)}{\partial x^2} = \frac{1}{u^2}\frac{\partial^2\psi(z)}{\partial t^2}$$

Therefore, $\psi(z) = \psi(x - ut)$ is a solution of the wave equation.

Reminder: Solutions of the wave equation

So, any function ϕ of x and t, **where x and t appear only in the combination of x-ut**, is a solution.

$$\phi(x, t) = \psi(x - ut)$$

Can you think of **another family of solutions?**

Any function ϕ of x and t, **where x and t appear only in the combination of x+ut**, is also a solution.

$$\phi(x, t) = \chi(x + ut)$$

The wave equation is **linear**: A superimposition of solutions is a solution.

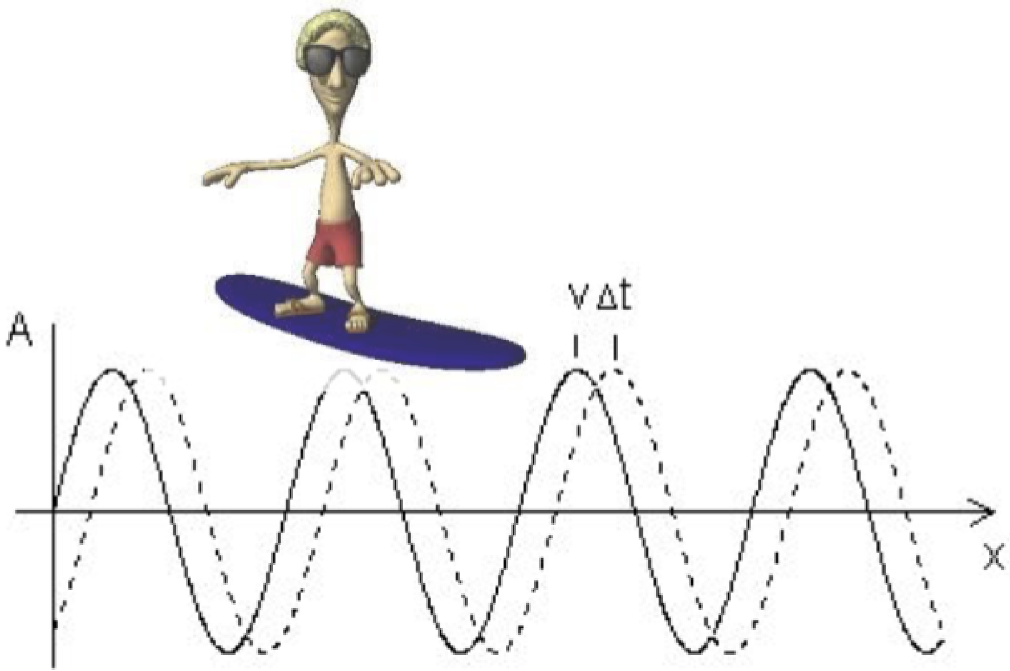
Therefore, most general solution to the wave equation can be written as:

$$\phi(x, t) = \psi(x - ut) + \chi(x + ut)$$

Reminder: Direction of wave propagation

What does this particular combination of variables (x-ut, x+ut) tell us about the direction of wave propagation?

- $\psi(x - ut)$ describes a wave moving in the positive direction of x.
- $\chi(x + ut)$ describes a wave moving in the negative direction of x.



Convince yourselves that the guy on the left should ride a $\psi(x - ut)$ wave, not a $\chi(x + ut)$ one.

The electric field satisfies a wave equation

Now that we have refreshed our memory on waves, let me return to my original question. Maxwell's equation in vacuum, in the absence of charges ($\rho = 0$) and currents ($\vec{j} = \vec{0}$) are shown below:

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

A change in one field feeds the other (even in absence of sources).

Can this interplay generate waves?

The electric field satisfies a wave equation

We will start from Faraday's law in vacuum:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

We will take the curl of both sides and use identity shown below. Then, we will use Gauss's and Ampere's law in vacuum and in absence of sources:

$$\vec{\nabla} \cdot \vec{E} = \cancelto{0}{\frac{\rho}{\epsilon_0}} = 0 \quad \textit{and} \quad \vec{\nabla} \times \vec{B} = \cancelto{0}{\mu_0 \vec{j}} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

to substitute the resulting $\vec{\nabla} \cdot \vec{E}$ and $\vec{\nabla} \times \vec{B}$.

Mini reminder from calculus

For any vector field $\vec{F}$: $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{F}\right) - \vec{\nabla}^2 \vec{F}$

The **curl of the curl** of a vector field is the **gradient of the divergence** of the field minus the **divergence of the gradient** (Laplacian) of the field.

The electric field satisfies a wave equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E} \right) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) \Rightarrow$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} \xRightarrow{\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F} \right) = \vec{\nabla} \left(\vec{\nabla} \cdot \vec{F} \right) - \vec{\nabla}^2 \vec{F}}$$

$$\vec{\nabla} \left(\vec{\nabla} \cdot \vec{E} \right) - \vec{\nabla}^2 \vec{E} = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \xRightarrow{\vec{\nabla} \cdot \vec{E} = \cancelto{0}{\frac{\rho}{\epsilon_0}} = 0 \;\; and \;\; \vec{\nabla} \times \vec{B} = \cancelto{0}{\mu_0 \vec{j}} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

$$-\vec{\nabla}^2 \vec{E} = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \Rightarrow \vec{\nabla}^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

The magnetic field satisfies a wave equation

The calculation for $\vec{B}$ proceeds along similar lines.

We will start from Ampere's law in vacuum and in absence of sources:

$$\vec{\nabla} \times \vec{B} = \cancelto{0}{\mu_0 \vec{j}} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

We will take the curl of both sides and use identity used before. Then, we will use the fact that there are no magnetic monopoles and Faraday's law: Gauss's and Ampere's law in vacuum and in absence of sources:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad and \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

to substitute the resulting $\vec{\nabla} \cdot \vec{B}$ and $\vec{\nabla} \times \vec{E}$.

The magnetic field satisfies a wave equation

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} \right) = \vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \Rightarrow$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} \right) = \mu_0 \epsilon_0 \vec{\nabla} \times \frac{\partial \vec{E}}{\partial t} \xRightarrow{\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F} \right) = \vec{\nabla} \left(\vec{\nabla} \cdot \vec{F} \right) - \vec{\nabla}^2 \vec{F}}$$

$$\vec{\nabla} \left(\vec{\nabla} \cdot \vec{B} \right) - \vec{\nabla}^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{E} \right) \xRightarrow{\vec{\nabla} \cdot \vec{B} = 0 \;\; and \;\; \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$$-\vec{\nabla}^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\frac{\partial \vec{B}}{\partial t} \right) \Rightarrow \vec{\nabla}^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

A most peculiar wave!

- The electric and magnetic fields satisfy a wave equation.
- Electric and magnetic field "disturbances" can propagate in space in absence of charges or currents.
- That disturbance **can not die off**!
  - A time-varying electric field generates a time-varying magnetic field.
  - A time-varying magnetic field generates a time-varying electric field.

  The change of one field feeds the other!
- An electric wave can't exist without a magnetic one (and vice versa) They are part of the same phenomenon: **electromagnetic waves**.
- Unlike mechanical waves, electromagnetic waves **don't need a medium to propagate into**!

The speed of electromagnetic waves in vacuum

$\vec{E}$ and $\vec{B}$ satisfy the following wave equations:

$$\vec{\nabla}^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Recall that earlier we wrote the wave equation as:

$$\vec{\nabla}^2 \phi(\vec{r}, t) = \frac{1}{u^2} \frac{\partial^2 \phi(\vec{r}, t)}{\partial t^2}$$

where u is the wave velocity.

Therefore the electric and magnetic waves propagate in vacuum with the same speed which, from above, can be identified with:

$$u = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

The speed of electromagnetic waves in vacuum

Can you confirm that $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$ has velocity units?

In SI:

- the permeability μ_0 is given in $\frac{N}{A^2}$, while
- the permitivity ϵ_0 is given in $\frac{C^2}{N \cdot m^2}$.

The product $\mu_0 \epsilon_0$ has units of:

$$\frac{\cancel{N}}{A^2} \cdot \frac{C^2}{\cancel{N} \cdot m^2} = \frac{1}{A^2} \cdot \frac{C^2}{m^2} = \frac{1}{\left(\frac{C}{s}\right)^2} \cdot \frac{C^2}{m^2} = \frac{s^2}{\cancel{C^2}} \cdot \frac{\cancel{C^2}}{m^2} = \frac{s^2}{m^2}$$

Indeed, $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$ has SI units of $\frac{m}{s}$.

The speed of electromagnetic waves in vacuum

What is the numerical value of $\dfrac{1}{\sqrt{\mu_0 \epsilon_0}}$?

$$u = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{\left(8.854 \cdot 10^{-12}\ \frac{N}{A}\right) \cdot \left(4\pi \cdot 10^{-7}\ \frac{C}{N \cdot m}\right)}} = \mathbf{299,792,458\ m/s}$$

Do you recognize this number?

Measurements of the speed of light before Maxwell's time

At the time that Maxwell was developing his theory, the speed of light was already known. In fact, it was known from the late 17th century.

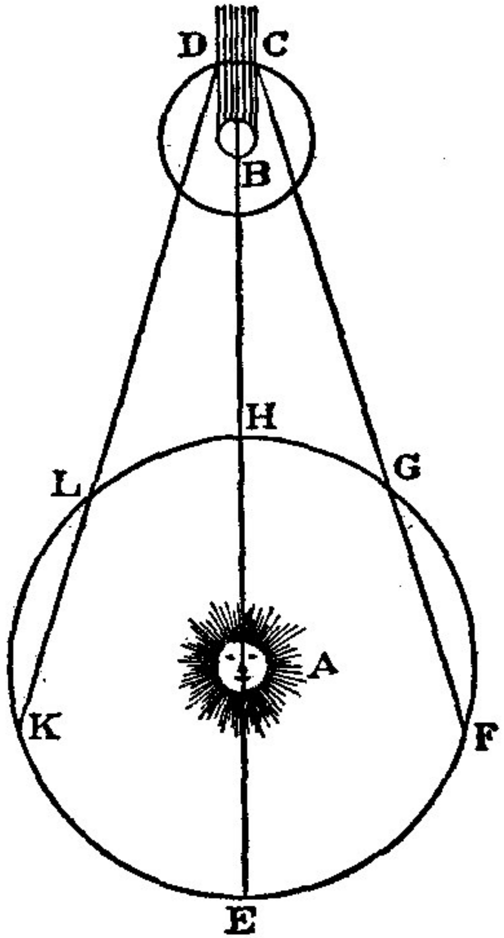


FIG. 70.

In 1670's, **Olaf Roemer** measured the speed of light by observing the **period of the moons of Jupiter.**

- The apparent period at Earth is different than true period due to the velocity of the Earth.
- The variation of the apparent period was

$$\Delta T = T \cdot \frac{2u}{c}$$

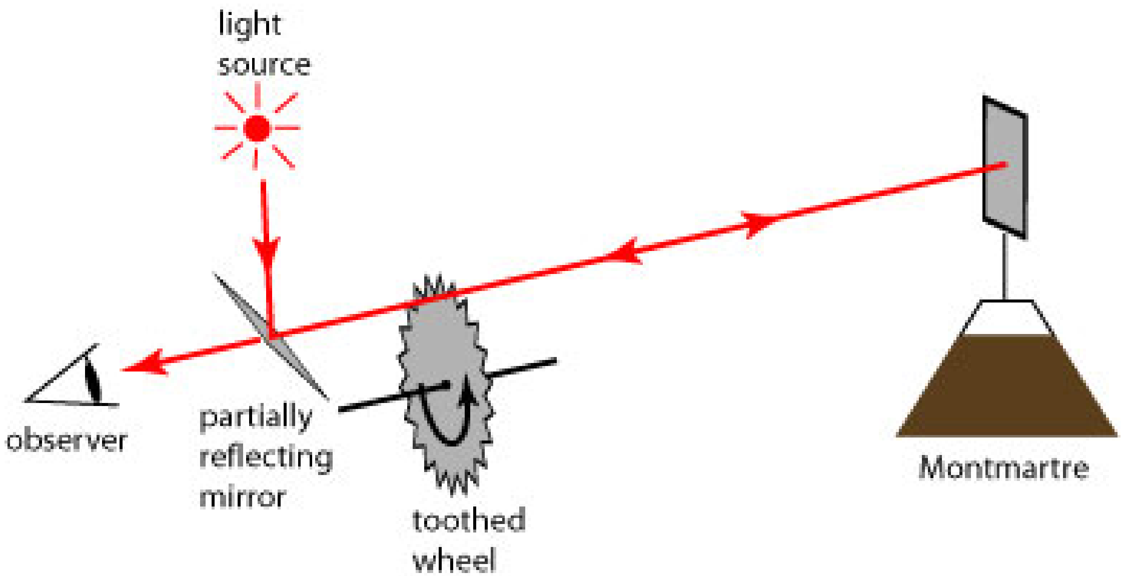
  where T the true period, u the velocity of the earth and c the speed of light.
- This allowed the speed of light to be determined to $\sim$ 30%.

Measurements of the speed of light before Maxwell's time

And in 1848-49, a few years before Maxwell published his famous treatise, **Hippolyte Fizeau** measured the speed of light to $\sim$5%.

Light passes between two teeth of the rotating gear and gets reflected from a mirror. If the reflected light again passes between two teeth of the rotating gear, the speed of light can be calculated from the gear-mirror distance and the speed of rotation.

Fizeau determined the speed of light between an intense light source and a mirror about 8 km distant. The light source was interrupted by a rotating cogwheel with 720 notches that could be rotated at a variable speed of up to hundreds of times a second.



Light is an electromagnetic wave!

Maxwell (with typical British understatement) said that the result was *"somewhat of a surprise"* !!

James Clerk Maxwell
(1831 - 1879)

"I made out the equations before I had any suspicion of the nearness between the two values of the velocity of propagation of magnetic effects and of light, so I think I have reasons to believe that the magnetic and luminiferous media are identical".

'We can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena. (Maxwell discussed his ideas in terms of a model in which the vacuum was like an elastic solid.)

Light is an electromagnetic wave!

Let's just think how amazing and unexpected that was!

What was ϵ_0 ?
It is the constant of proportionality in Coulomb's law that, connects charge densities and the resulting electric field $\vec{E}$:

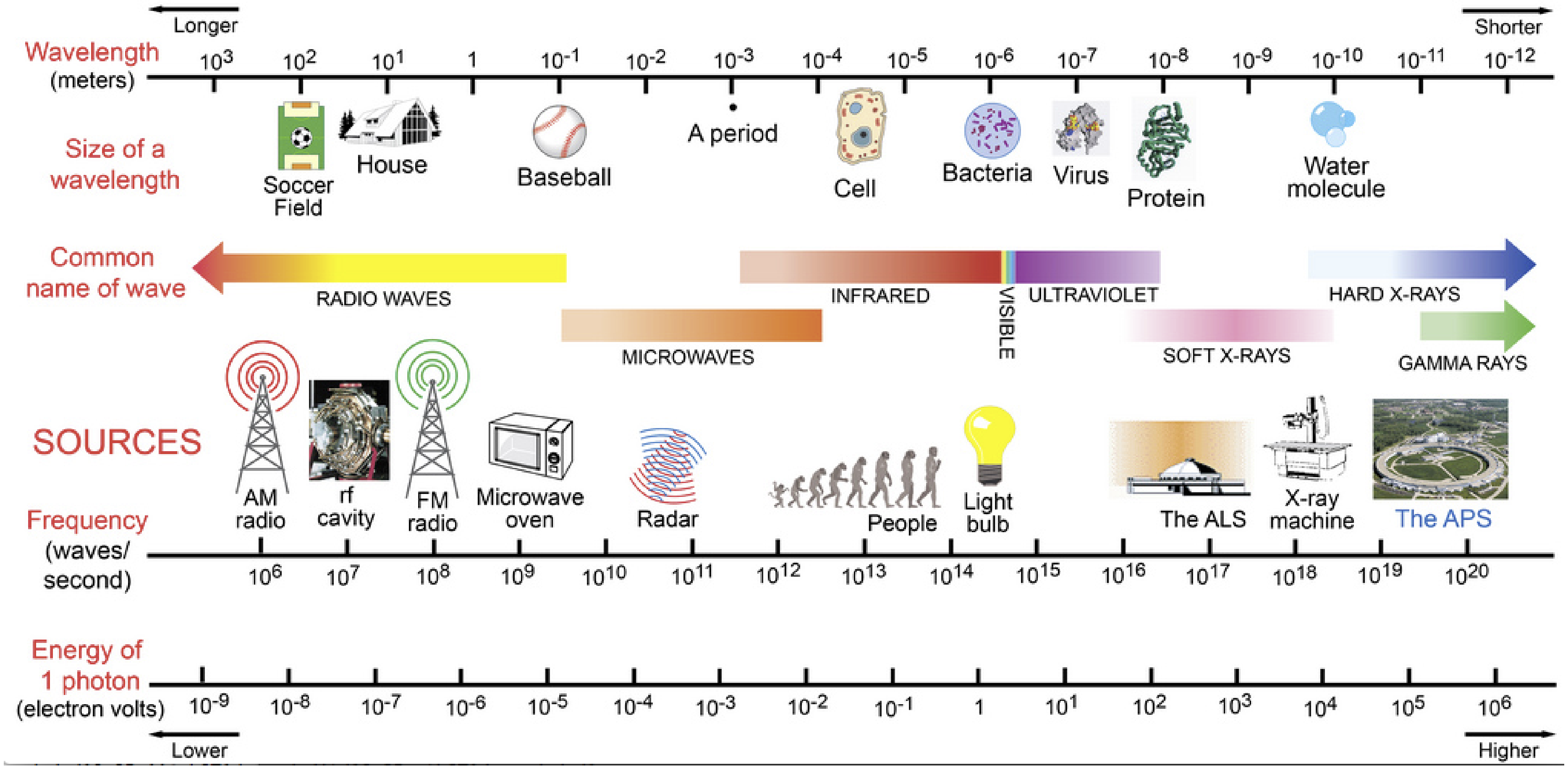
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_{\mathbf{0}}} \int d\tau' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

What was μ_0 ?
It is the constant of proportionality in the Biot-Savart law that connects currents and the resulting magnetic field $\vec{B}$:

$$\vec{B}(\vec{r}) = \frac{\mu_{\mathbf{0}}}{4\pi} \int d\tau' \vec{j}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

The fact that the speed of the speed of light is determined by these two (apparently unrelated to light) constants must have stunned Maxwell (or, in his works, it was *"somewhat of a surprise"*).

The electromagnetic spectrum



Worked example

Question

An electric field $\vec{E}$ is given by:

$$\vec{E} = E_0 \cdot sin\Big(k \cdot (x - ct)\Big) \cdot \hat{y}$$

where k is the propagation number.

a) Show that it fulfils the wave equation.
b) Find the corresponding magnetic field.
c) Comment on directions and amplitudes.

Worked example

First, we will calculate the Laplacian ($\vec{\nabla} \cdot \vec{E}$) of the given electric field $\vec{E}$:

$$\vec{E} = E_0 sin\Big(k(x - ct)\Big)\hat{y}$$

We have:

$$\vec{\nabla}^2\vec{E} = \frac{\partial^2\vec{E}}{\partial x^2} + \cancelto{0}{\frac{\partial^2\vec{E}}{\partial y^2}} + \cancelto{0}{\frac{\partial^2\vec{E}}{\partial z^2}} = \frac{\partial^2}{\partial x^2}\Bigg\{E_0 sin\Big(k(x - ct)\Big)\hat{y}\Bigg\} =$$

$$= E_0\Bigg\{\frac{\partial^2}{\partial x^2} sin\Big(k(x - ct)\Big)\Bigg\}\hat{y} = E_0\Bigg\{k\frac{\partial}{\partial x} cos\Big(k(x - ct)\Big)\Bigg\}\hat{y} =$$

$$= E_0\Bigg\{ - k^2 sin\Big(k(x - ct)\Big)\Bigg\}\hat{y} = -k^2\Bigg\{E_0 sin\Big(k(x - ct)\Big)\hat{y}\Bigg\} \Rightarrow$$

$$\vec{\nabla}^2\vec{E} = -k^2\vec{E}$$

Worked example

Then, we will calculate the second-order partial derivative with respect to time ($\partial^2/\partial t^2$) of the given electric field $\vec{E}$:

$$\vec{E} = E_0 sin\Big(k(x-ct)\Big)\hat{y}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{\partial^2}{\partial t^2}\Big\{E_0 sin\Big(k(x-ct)\Big)\hat{y}\Big\} =$$

$$= E_0\Big\{\frac{\partial^2}{\partial t^2} sin\Big(k(x-ct)\Big)\Big\}\hat{y} = E_0\Big\{-kc\frac{\partial}{\partial t} cos\Big(k(x-ct)\Big)\Big\}\hat{y} =$$

$$= E_0\Big\{-k^2c^2 sin\Big(k(x-ct)\Big)\Big\}\hat{y} = -k^2c^2\Big\{E_0 sin\Big(k(x-ct)\Big)\hat{y}\Big\} \Rightarrow$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -k^2c^2\vec{E}$$

Worked example

So, we found that:

$$\vec{\nabla}^2 \vec{E} = -k^2 \cdot \vec{E}$$

and:

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -k^2 \cdot c^2 \cdot \vec{E}$$

Hence:

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \cdot \vec{\nabla}^2 \vec{E}$$

The electric field $\vec{E} = E_0 \cdot sin\Big(k \cdot (x - c \cdot t)\Big) \cdot \hat{y}$ fullfils a wave equation with velocity c.

Worked example

To find the corresponding magnetic field $\vec{B}$, we will use Faraday's law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

First we will calculate $\vec{\nabla} \times \vec{E}$, and then integrate the result over time.

We have:

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{x} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_y & E_z \end{vmatrix} - \hat{y} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ E_x & E_z \end{vmatrix} + \hat{z} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ E_x & E_y \end{vmatrix} \Rightarrow$$

$$\vec{\nabla} \times \vec{E} = \hat{x} \cdot \Big(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\Big) - \hat{y} \cdot \Big(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\Big) + \hat{z} \cdot \Big(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\Big)$$

Worked example

Recall that $\vec{E}$ has only a y component ($E_x = E_z = 0$), which has only x dependence ($E_y = E_y(x)$ with no y or z dependence). Therefore:

$$\vec{\nabla} \times \vec{E} = \hat{x} \cdot \left(\cancelto{0}{\frac{\partial E_z}{\partial y}} - \cancelto{0}{\frac{\partial E_y}{\partial z}} \right) - \hat{y} \cdot \left(\cancelto{0}{\frac{\partial E_z}{\partial x}} - \cancelto{0}{\frac{\partial E_x}{\partial z}} \right) + \hat{z} \cdot \left(\frac{\partial E_y}{\partial x} - \cancelto{0}{\frac{\partial E_x}{\partial y}} \right) \Rightarrow$$

$$\vec{\nabla} \times \vec{E} = \hat{z} \cdot \frac{\partial E_y}{\partial x}$$

The first-order partial derivative of $E_y(x)$ with respect to x is:

$$\frac{\partial E_y}{\partial x} = \frac{\partial}{\partial x} \left\{ E_0 \cdot sin\Big(k \cdot (x - c \cdot t) \Big) \right\} = E_0 \cdot k \cdot cos\Big(k \cdot (x - c \cdot t) \Big)$$

Hence:

$$\vec{\nabla} \times \vec{E} = \left\{ E_0 \cdot k \cdot cos\Big(k \cdot (x - c \cdot t) \Big) \right\} \cdot \hat{z}$$

Worked example

Substituting the previous result in Faraday's law, we have:

$$-\frac{\partial \vec{B}}{\partial t} = \Big\{ E_0 \cdot k \cdot cos\Big(k \cdot (x - c \cdot t)\Big)\Big\} \cdot \hat{z}$$

Integrating over time:

$$\vec{B} = \Big\{ - E_0 \cdot k \cdot \int cos\Big(k \cdot (x - c \cdot t)\Big) dt \Big\} \cdot \hat{z} =$$

$$= \Big\{ - E_0 \cdot k \cdot \frac{sin\Big(k \cdot (x - c \cdot t)\Big)}{-k \cdot c} \Big\} \cdot \hat{z} + \vec{C} = \Big\{ \frac{E_0}{c} \cdot sin\Big(k \cdot (x - c \cdot t)\Big)\Big\} \cdot \hat{z} + \vec{C}$$

In the absence of currents there are no time invariant components of $\vec{B}$ ($\vec{C} = \vec{0}$) hence:

$$\vec{B} = \Big\{ \frac{E_0}{c} \cdot sin\Big(k \cdot (x - c \cdot t)\Big)\Big\} \cdot \hat{z}$$

Worked example

Our electric and magnetic fields are given by:

$$\vec{E} = \left\{ E_0 \cdot sin\Big(k \cdot (x - c \cdot t)\Big) \right\} \cdot \hat{y} \quad and \quad \vec{B} = \left\{ \frac{E_0}{c} \cdot sin\Big(k \cdot (x - c \cdot t)\Big) \right\} \cdot \hat{z}$$

Notice that the **amplitude of the magnetic field is smaller to that of the electric field by a factor of c**.

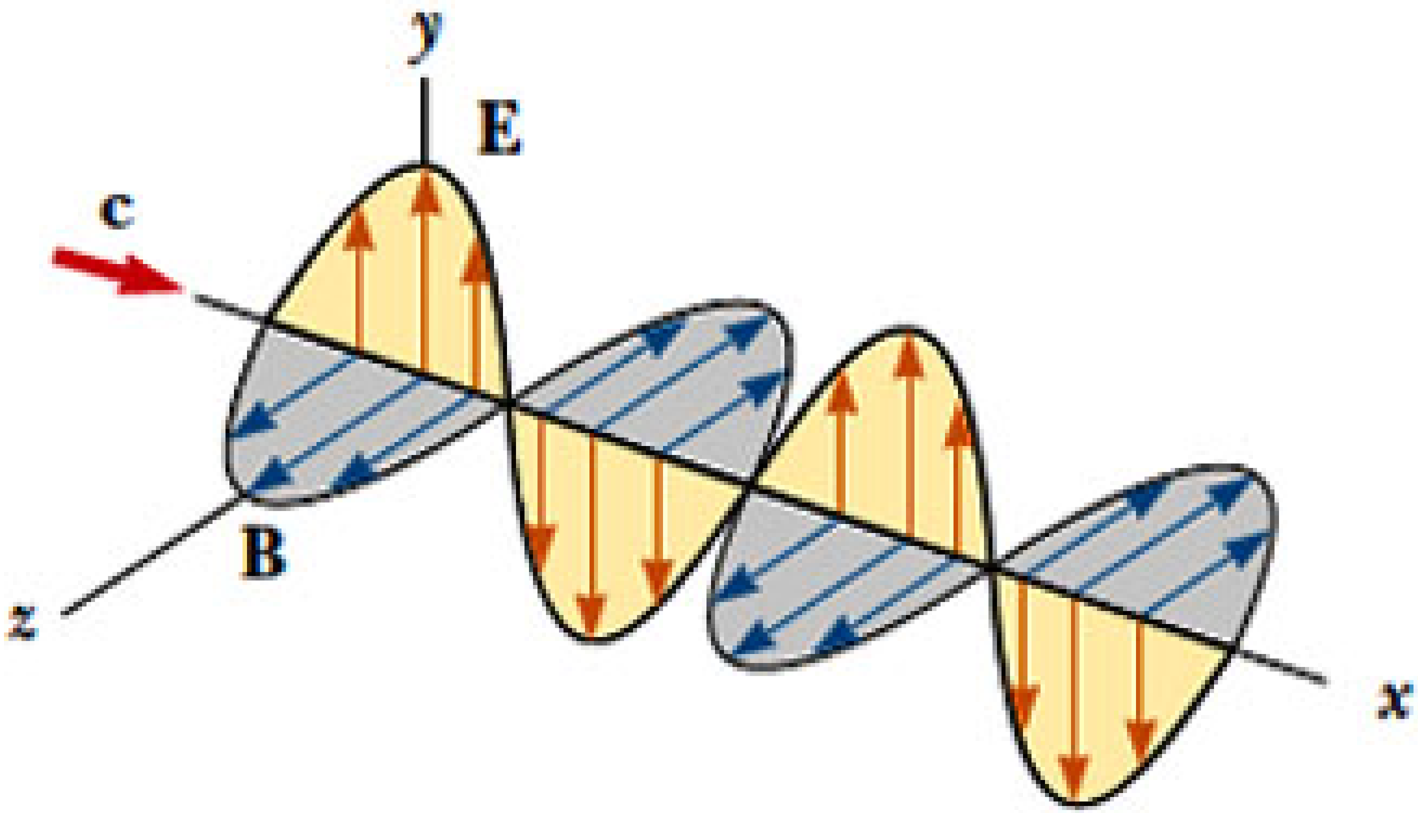
Also notice that:

- the wave propagates on the x-axis,
- the electric field oscillates on the y-axis, and
- the magnetic field oscillates on the z-axis

Therefore the **electric and magnetic fields are**:

- **perpendicular to each other**, and
- **perpendicular to the direction of motion**.

Worked example



Energy carried by electromagnetic waves

We know that an e/m wave carries energy.

How can we express the **energy transported by an e/m wave?**

The energy stored in the electric field is:

$$U_e = \frac{\epsilon_0}{2} \int_\tau d\tau |\vec{E}|^2$$

while the energy stored in the magnetic field is:

$$U_m = \frac{1}{2\mu_0} \int_\tau d\tau |\vec{B}|^2$$

Therefore the energy stored in an electromagnetic field is:

$$U_{em} = \frac{1}{2} \int_\tau d\tau \Big(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2 \Big)$$

Energy carried by electromagnetic waves

Consider fields $\vec{E}$ and $\vec{B}$ within some volume τ. A charge q moves within that volume with velocity $\vec{u}$ and it is displaced by a short distance $d\vec{\ell}$.

The rate of change of the work W done on the charge q by the e/m field (see Optional Reading) is given by:

$$\frac{dW}{dt} = -\oint_S d\vec{S} \cdot \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) - \frac{\partial}{\partial t} \int_\tau d\tau \left(\frac{1}{2\mu_0}\vec{B}^2 + \frac{\epsilon_0}{2}\vec{E}^2\right)$$

As could be anticipated, the **work done on the charge q** is related with the **decrease of the energy stored in the e/m field** (2nd term on right).

The additional "surface" term above, can be interpreted as the *energy carried away from the volume τ, through its surface S, by the e/m field*.

The above expression is known as the **Poynting theorem**.

The Poynting vector

We define the **Poynting vector** $\vec{N}$ as:

$$\vec{N} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$$

It represents the **energy flux density (rate energy transfer per unit area)** and it has units of $W(att)/m^2$.

Recall that the cross product $\vec{A} \times \vec{B}$ of 2 vectors $\vec{A}$ and $\vec{B}$ is perpendicular both to $\vec{A}$ and $\vec{B}$. So, in an electromagnetic wave, the energy flows in a direction which is perpendicular to both the electric and magnetic field.

Worked example

An e/m wave is given by the time-dependent fields:

$$\vec{E}(\vec{r},t) = E_0 sin\Big(k(z-ct)\Big)\hat{x}$$

$$\vec{B}(\vec{r},t) = \frac{E_0}{c} sin\Big(k(z-ct)\Big)\hat{y}$$

This gives the Poynting vector:

$$\vec{N} = \frac{1}{\mu_0}\Big(\vec{E}\times\vec{B}\Big) = \frac{E_0^2}{\mu_0 c} sin^2\Big(k(z-ct)\Big)\Big(\hat{x}\times\hat{y}\Big)$$

One can easily see that:

$$\hat{x}\times\hat{y} = \hat{z}$$

Therefore, the Poynting vector is:

$$\vec{N} = \frac{E_0^2}{\mu_0 c} sin^2\Big(k(z-ct)\Big)\hat{z}$$

Worked example

Therefore, the Poynting vector is:

$$\vec{N} = \frac{E_0^2}{\mu_0 c} sin^2\Big(k(z - ct)\Big)\hat{z}$$

The above vector describes the power transmitted by the e/m wave per unit area perpendicular to its propagation.
The average power transmitted by the e/m wave, is given by the **average of $\vec{N}$ over a period T**:

$$< \vec{N} >= \frac{1}{T}\int_0^T \vec{N} dt$$

$$< \vec{N} >= \frac{1}{T}\int_0^T \Big(\frac{E_0^2}{\mu_0 c} sin^2\Big(k(z - ct)\Big)\hat{z}\Big) dt$$

Worked example

$$< \vec{N} >= \frac{1}{T} \int_0^T \Big(\frac{E_0^2}{\mu_0 c} sin^2 \Big(k(z - ct) \Big) \hat{z} \Big) dt$$

$$< \vec{N} >= \hat{z} \frac{1}{T} \frac{E_0^2}{\mu_0 c} \int_0^T sin^2 \Big(k(z - ct) \Big) dt$$

The average of $sin^2\theta$ over a period is $1/2$, therefore:

$$< \vec{N} >= \hat{z} \frac{E_0^2}{2\mu_0 c}$$

Radiation Pressure

E/M waves **carry momentum**: Can **exert pressure** on an object when shining on it!

Assume that an object is illuminated by radiation for a time interval Δt and that the radiation is entirely **absorbed** by the object. If its energy gain is ΔU, then the magnitude of its momentum change Δp is given by:

$$\Delta p = \frac{\Delta U}{c} \quad \textit{(total absorption)}$$

where c is the speed of light.

If, instead of being fully absorbed, the radiation is fully reflected back by the object along its original path, the momentum change of the object is twice the one given above:

$$\Delta p = \frac{2\Delta U}{c} \quad \textit{(total reflection along original path)}$$

In realistic situations the momentum change is between $\frac{\Delta U}{c}$ and $\frac{2\Delta U}{c}$.

Radiation Pressure

From Newton's second law, a change in momentum is related to a force

$$F = \frac{\Delta p}{\Delta t}$$

If the radiation has intensity I (power per area, or energy per time per area), then over a time interval Δt, the energy intercepted by an area A is:

$$\Delta U = IA\Delta t$$

Therefore, the force exerted by radiation on an object is given by:

$$F = \frac{IA}{c} \quad (\textit{total absorption})$$

$$F = \frac{2IA}{c} \quad (\textit{total reflection along original path})$$

If the radiation is partially absorbed and partially reflected, the exerted force will be between $\frac{IA}{c}$ and $\frac{2IA}{c}$.

Radiation Pressure

The force per unit area on an object due to radiation is the **radiation pressure** P_r.

From the previous expressions, we obtain:

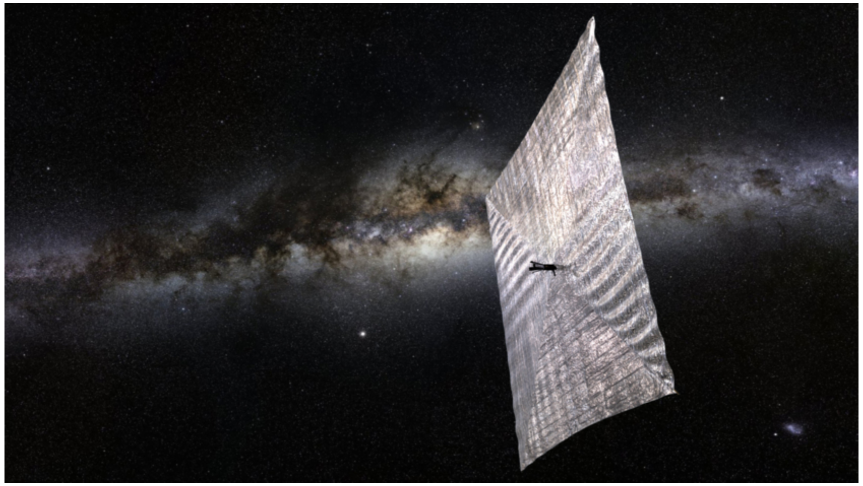
$$P_r = \frac{I}{c} \quad (\textit{total absorption})$$

$$P_r = \frac{2I}{c} \quad (\textit{total reflection along original path})$$

Radiation Pressure

Solar sail a form of spacecraft propulsion!

- Using radiation pressure from sunlight (see tomorrow's workshop tasks)
- See https://en.wikipedia.org/wiki/Solar_sail



LighSail 2

- Area: 32 m^2
- Launch date: March 2017!
- See http://sail.planetary.org

What if we replace sunlight with powerful directed laser beams?
See: "NASA thinks there's a way to get to Mars in three days"
http://phys.org/news/2016-02-nasa-mars-days.html

Radiation Pressure

Who cares about Mars!

Shoot for **Alpha Centauri** and its recently-discovered planet Proxima b!

See **Breakthrough Starshot**:
http://breakthroughinitiatives.org/Initiative/3

Worked example

Question

A small laser emits light at power $P = 5.00$ mW and wavelength $\lambda = 633$ nm. The laser beam is focused (narrowed) until its diameter d matches the 1266 nm diameter of a sphere placed in its path. The sphere is perfectly absorbing and has density $\rho = 5.00 \times 10^3$ kg/m^3. What are

- the beam intensity I at the sphere's location,
- the radiation pressure P_r on the sphere,
- the magnitude of the corresponding force F_r, and
- the magnitude of the acceleration α that force alone would give the sphere?

We note that cross-section of the beam is $\pi d^2/4$. Therefore, the beam intensity is:

$$I = \frac{P}{\pi d^2/4} = \frac{5 \times 10^{-3}\ W}{3.14(1266 \times 10^{-9}\ m)^2/4} = 3.97 \times 10^9\ W/m^2$$

Worked example

The rediation pressure is:

$$P_r = \frac{I}{c} = \frac{3.97 \times 10^9 \ W/m^2}{3.0 \times 10^8 \ m/s} = 13.2 \ Pa$$

The corresponding force is:

$$F_r = P_r \Big(\pi d^2/4 \Big) = 13.2 \Big(3.14 (1266 \times 10^{-9} \ m)^2/4 \Big) = 1.67 \times 10^{-11} \ N$$

We note that volume of the sphere is $4\pi(d/2)^3/3 = \pi d^3/6$ and, therefore its mass m, is given by $m = \rho \pi d^3/6$. The acceleration due to the force computed previously is

$$\alpha = \frac{F_r}{m} = \frac{F_r}{\rho \pi d^3/6} = \frac{6F_r}{\rho \pi d^3} \Rightarrow$$

$$\alpha = \frac{6(1.67 \times 10^{-11} \ N)}{(5 \times 10^3 \ kg/m^3) 3.14 (1266 \times 10^{-9} \ m)^3} = 3.14 \times 10^3 \ m/s^2$$

Worked example

Question

A small space shuttle with a mass of only 1.5×10^3 kg (including crew) is drifting in outer space with negligible gravitational forces acting on it. If the astronaut turns on a 10 kW laser beam, what speed will the shuttle attain in 1.0 day because of the momentum carried away by the beam?

Since the total momentum of the spaceship and light is conserved, this is the magnitude of the momentum acquired by the spaceship is equal to the magnitude of the momentum carried away by the laser beam.

If P is the power of the laser, then the energy U carried away in time t is:

$$U = P \cdot t$$

Thus the magntitude of the momentum is given by:

$$p = P \cdot t/c$$

Worked example

If m is the mass of the spaceship, its speed u is:

$$u = \frac{p}{m} = \frac{P \cdot t}{m \cdot c}$$

There are 86400 seconds in a day, therefore:

$$u = \frac{(10 \times 10^3 \ W) \cdot (86400 \ s)}{(1.5 \times 10^3 \ kg) \cdot (2.998 \times 10^8 \ m/s)} = 1.9 \times 10^{-3} \ m/s$$

Question

The magnetic field in a traveling EM wave has an rms strength of 22.5 nT. How long does it take to deliver 335 J of energy to 1 cm^2 of a wall that it hits perpendicularly?

The energy per unit area per unit time is given by the magnitude of the Poynting vector $\vec{S}$.

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \Rightarrow$$

Worked example

$$< S >= \frac{E_0 B_0}{2\mu_0} = \frac{cB_0^2}{2\mu_0} = \frac{cB_{rms}^2}{\mu_0}$$

Let ΔU represent the energy that crosses area A in a time Δt. Then:

$$< S >= \frac{\Delta U}{A \Delta t}$$

Therefore:

$$\frac{\Delta U}{A \Delta t} = \frac{cB_{rms}^2}{\mu_0} \Rightarrow \Delta t = \frac{\Delta U \mu_0}{AcB_{rms}^2} \Rightarrow$$

$$\Delta t = \frac{(335 \text{ J})(4\pi \times 10^{-7} \text{ Tm/A})}{(10^{-4} \text{ m}^2)(3 \times 10^8 \text{ m/s})(22.5 \times 10^{-9} \text{ T})^2} = 2.66 \times 10^7 \text{s} (\approx 321 \text{ days})$$

Worked example

Question

Estimate the rms electric field in the sunlight that hits Mars, knowing that the Earth receives about 1350 W/m^2 and that Mars is 1.52 times farther from the Sun (on average) than is the Earth.

The radiation from the Sun is isotropic, so the rate P at which energy passes through a sphere with radius R centered at the Sun is:

$$P = < S > \left(4\pi R^2\right)$$

This rate is the same at any distance from the Sun, so the above expression is valid on Earth:

$$P = < S_{Earth} > \left(4\pi R^2_{Earth}\right)$$

and on Mars:

$$P = < S_{Mars} > \left(4\pi R^2_{Mars}\right)$$

Worked example

Therefore, from above:

$$< S_{Mars} > = < S_{Earth} > \frac{R^2_{Earth}}{R^2_{Mars}}$$

But, we know that:

$$< S > = \frac{E_0 B_0}{2\mu_0} = \frac{cB_0^2}{2\mu_0} = \frac{cB^2_{rms}}{\mu_0} = \frac{E_0^2}{2\mu_0 c} = \frac{E^2_{rms}}{\mu_0 c}$$

Therefore, on Mars:

$$< S_{Mars} > = \frac{E^2_{rms;Mars}}{\mu_0 c} \Rightarrow E_{rms;Mars} = \sqrt{< S_{Mars} > \mu_0 c}$$

Using the earlier expression we found for $< S_{Mars} >$, we have:

$$E_{rms;Mars} = \sqrt{< S_{Earth} > \mu_0 c} \frac{R_{Earth}}{R_{Mars}} \Rightarrow$$

$$E_{rms;Mars} = \sqrt{(1350 \text{ W/m}^2)(4\pi \times 10^{-7} \text{ Tm/A})(3 \times 10^8 \text{ m/s})} \frac{1}{1.52} = 469 \; V/m$$

Lecture 9 - Main points to remember

- We looked at Maxwell's equation in vacuum in the absence of sources:

$$\vec{\nabla} \cdot \vec{E} = \cancel{\frac{\rho}{\epsilon_0}}^{\,0} \Rightarrow \vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \cancel{\mu_0 \vec{j}}^{\,0} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

- We saw that $\vec{E}$ and $\vec{B}$ **satisfy a wave equation**:

$$\vec{\nabla}^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \textit{and} \quad \vec{\nabla}^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

- EM waves have a velocity of $\frac{1}{\sqrt{\mu_0 \epsilon_0}} = \mathbf{299,792,458\ m/s}$

Lecture 9 - Main points to remember (cont'd)

- The conclusion that **light is an electromagnetic** wave is astounding!
  - Recall the "humble" and unrelated origin of ϵ_0 and μ_0.
- An electromagnetic wave is a **most peculiar wave**!
  - An electromagnetic wave **can not die off**!
    - A time-varying electric field generates a time-varying magnetic field.
    - A time-varying magnetic field generates a time-varying electric field.

    The change of one field feeds the other!
  - An electric wave can't exist without a magnetic one (and vice versa) They are part of the same phenomenon: **electromagnetic waves**.
  - The electric and magnetic fields are perpendicular to each other an perpendicular to the direction of wave propagation.
  - Electromagnetic waves **don't need a medium to propagate**!

Lecture 9 - Main points to remember (cont'd)

- **Poynting theorem**:

$$\frac{dW}{dt} = -\oint_S d\vec{S} \cdot \frac{1}{\mu_0}\Big(\vec{E} \times \vec{B}\Big) - \frac{\partial}{\partial t}\int_\tau d\tau\Big(\frac{1}{2\mu_0}\vec{B}^2 + \frac{\epsilon_0}{2}\vec{E}^2\Big)$$

- The **Poynting vector** $\vec{N}$ was defined as:

$$\vec{N} = \frac{1}{\mu_0}\Big(\vec{E} \times \vec{B}\Big)$$

  It represents the **energy flux density (rate energy transfer per unit area)** and it has units of $W(att)/m^2$.

- The average power transmitted by the e/m wave, is given by the **average of $\vec{N}$ over a period T**:

$$< \vec{N} >= \frac{1}{T}\int_0^T \vec{N} dt$$

Lecture 9 - Main points to remember (cont'd)

- E/M waves **carry momentum**: Can **exert pressure** on an object when shining on it!
- If an object is illuminated by radiation for a time interval Δt, during which it absorbs energy $\Delta U = \text{I}\text{A}\Delta t$, where I is the intensity I (power per area, or energy per time per area) of the radiation, and A is the area of the object, then:
  - The momentum change Δp of the object is given by:

$$\Delta p = \frac{\Delta U}{c} \quad (\textit{total absorption}) \quad \Delta p = \frac{2\Delta U}{c} \quad (\textit{total reflection})$$

  - The radiation pressure P_r exterted on the object is given by:

$$P_r = \frac{I}{c} \quad (\textit{total absorption}) \quad P_r = \frac{2I}{c} \quad (\textit{total reflection})$$

At the next lecture (Lecture 10)

- Electromagnetic waves in matter

Optional reading for Lecture 9

Energy carried by electromagnetic waves

We will see that the *energy carried away from the volume* τ *by an e/m wave through the surface S* is given by:

$$\oint_S d\vec{S} \cdot \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$$

Let's assume that I have an electric field $\vec{E}$ and a magnetic field $\vec{B}$ within some volume τ. A charge q moves within that volume with velocity $\vec{u}$ and it is displaced by a short distance $d\vec{\ell}$.

- There is work done by the e/m force.
- The energy has to come from somewhere.
  - Where from? Well, there is the energy $\displaystyle\frac{1}{2}\int_\tau d\tau\left(\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right)$ stored in the e/m field.
  - So, if there is work done on the charge, the energy stored in the e/m field must be decreased.

Energy carried by electromagnetic waves

The work done by the Lorentz force is:

$$dW = \vec{F} \cdot d\vec{\ell} = q\left(\vec{E} + \vec{u} \times \vec{B}\right) \cdot \vec{u}dt$$

We have already seen that magnetic forces do no work: The $\vec{u} \times \vec{B}$ vector is perpendicular to $\vec{u}$ so the dot product $\left(\vec{u} \times \vec{B}\right) \cdot \vec{u}$ is 0.
So the work done can be written just as

$$dW = \vec{F} \cdot d\vec{\ell} = q\vec{E} \cdot \vec{u}dt$$

To move from the discrete to the continuous case, take q to be the amount of charge contained within a volume $d\tau$ within a region characterised by charge density ρ. Then

$$dW = q\vec{E} \cdot \vec{u}dt \rightarrow \left(\rho d\tau\right)\vec{E} \cdot \vec{u}dt = dtd\tau\vec{E} \cdot \left(\rho\vec{u}\right)$$

Energy carried by electromagnetic waves

The work done per unit time, in the infinitesimal volume $d\tau$ is:

$$\frac{dW}{dt} = d\tau \vec{E} \cdot \left(\rho \vec{u}\right)$$

As we have seen, the product $\rho\vec{u}$ is the current density $\vec{j}$. Therefore, the work done per unit time and per unit volume can be written as:

$$\frac{dW}{dt} = d\tau \vec{E} \cdot \vec{j}$$

The total work (integrated over all space) per unit time is:

$$\frac{dW}{dt} = \int_{\tau} d\tau \vec{E} \cdot \vec{j}$$

To proceed, we need to examine the dot product $\vec{E} \cdot \vec{j}$ of the electric field $\vec{E}$ and the current density $\vec{j}$.

Energy carried by electromagnetic waves

Starting from Ampere's law, we can write $\vec{j}$ in terms of $\vec{\nabla} \times \vec{B}$ and $\frac{\partial \vec{E}}{\partial t}$:

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Big) \Rightarrow \vec{j} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Therefore:

$$\vec{E} \cdot \vec{j} = \vec{E} \cdot \Big(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Big) = \frac{1}{\mu_0} \vec{E} \cdot \Big(\vec{\nabla} \times \vec{B} \Big) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

The cross-product term can be written as:

$$\vec{E} \cdot \Big(\vec{\nabla} \times \vec{B} \Big) = -\vec{\nabla} \cdot \Big(\vec{E} \times \vec{B} \Big) - \vec{B} \cdot \frac{\partial \vec{B}}{\partial t}$$

Try to prove it at home! (Details on the next slide.)

Further details (skip on a first read)

Starting from $\vec{\nabla} \cdot \left(\vec{E} \times \vec{B} \right)$, we can write using the *CAB-BAC* identity:

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{B} \right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{E} \right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{B} \right)$$

The $\vec{E} \cdot \left(\vec{\nabla} \times \vec{B} \right)$ term is what appears in the $\vec{E} \cdot \vec{j}$ dot product.
Solving for it:

$$\vec{E} \cdot \left(\vec{\nabla} \times \vec{B} \right) = -\vec{\nabla} \cdot \left(\vec{E} \times \vec{B} \right) + \vec{B} \cdot \left(\vec{\nabla} \times \vec{E} \right)$$

From Faraday's law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Therefore:

$$\vec{E} \cdot \left(\vec{\nabla} \times \vec{B} \right) = -\vec{\nabla} \cdot \left(\vec{E} \times \vec{B} \right) - \vec{B} \cdot \frac{\partial \vec{B}}{\partial t}$$

Energy carried by electromagnetic waves

$$\vec{E} \cdot \vec{j} = -\frac{1}{\mu_0}\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) - \frac{1}{\mu_0}\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

For a vector field $\vec{F}$, we can easily see that:

$$\vec{F} \cdot \frac{\partial \vec{F}}{\partial t} = \frac{1}{2}\frac{\partial |\vec{F}|^2}{\partial t}$$

and, therefore:

$$\vec{E} \cdot \vec{j} = -\frac{1}{\mu_0}\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) - \frac{1}{2\mu_0}\frac{\partial |\vec{B}|^2}{\partial t} - \frac{\epsilon_0}{2}\frac{\partial |\vec{E}|^2}{\partial t} \Rightarrow$$

$$\vec{E} \cdot \vec{j} = -\frac{1}{\mu_0}\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) - \frac{\partial}{\partial t}\left(\frac{1}{2\mu_0}|\vec{B}|^2 + \frac{\epsilon_0}{2}|\vec{E}|^2\right)$$

Energy carried by electromagnetic waves

The total work done per unit time is given, as we have seen, by integrating $\vec{E} \cdot \vec{j}$ over the whole volume τ:

$$\frac{dW}{dt} = \int_\tau d\tau \vec{E} \cdot \vec{j} = -\int_\tau d\tau \frac{1}{\mu_0} \vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) - \frac{\partial}{\partial t} \int_\tau d\tau \left(\frac{1}{2\mu_0} |\vec{B}|^2 + \frac{\epsilon_0}{2} |\vec{E}|^2 \right)$$

Using Gauss' theorem, the volume integral of the divergence of a vector field can be replaced by the surface integral of that field:

$$\frac{dW}{dt} = -\oint_S d\vec{S} \cdot \frac{1}{\mu_0} \left(\vec{E} \times \vec{B}\right) - \frac{\partial}{\partial t} \int_\tau d\tau \left(\frac{1}{2\mu_0} |\vec{B}|^2 + \frac{\epsilon_0}{2} |\vec{E}|^2 \right)$$

As anticipated, the above tells you that the **work done on the charge q** is, indeed, related with the **decrease of the energy stored in the e/m field** (2nd term on right-hand side).

Energy carried by electromagnetic waves

But there is another term on the right-hand side of:

$$\frac{dW}{dt} = -\oint_S d\vec{S} \cdot \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) - \frac{\partial}{\partial t}\int_\tau d\tau \left(\frac{1}{2\mu_0}|\vec{B}|^2 + \frac{\epsilon_0}{2}|\vec{E}|^2\right)$$

It is the surface term:

$$\oint_S d\vec{S} \cdot \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$$

What does that term correspond to?

This term describes the *energy carried away from the volume τ, through its surface S, by the electromagnetic fields*.

So the energy stored in the e/m field in volume τ is decreased, because of the work done on the charge, but also because energy is flowing out.

This is the **Poynting theorem**.

PHYS 201 / Lecture 10
The most general case of Maxwell's equations (for time-dependent fields in matter); Electromagnetic waves in matter

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Rutherford Appleton Laboratory

- We looked at Maxwell's equation in vacuum in the absence of sources:

$$\vec{\nabla} \cdot \vec{E} = \cancelto{0}{\frac{\rho}{\epsilon_0}} \Rightarrow \vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \cancelto{0}{\mu_0 \vec{j}} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

- We saw that $\vec{E}$ and $\vec{B}$ **satisfy a wave equation**:

$$\vec{\nabla}^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \textit{and} \quad \vec{\nabla}^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

- EM waves have a velocity of $\frac{1}{\sqrt{\mu_0 \epsilon_0}} = \mathbf{299,792,458\ m/s}$

Lecture 9 - Revision (cont'd)

- The conclusion that **light is an electromagnetic** wave is astounding!
  - Recall the "humble" and unrelated origin of ϵ_0 and μ_0.
- An electromagnetic wave is a **most peculiar wave**!
  - An electromagnetic wave **can not die off**!
    - A time-varying electric field generates a time-varying magnetic field.
    - A time-varying magnetic field generates a time-varying electric field.

  The change of one field feeds the other!
  - An electric wave can't exist without a magnetic one (and vice versa) They are part of the same phenomenon: **electromagnetic waves**.
  - The electric and magnetic fields are perpendicular to each other an perpendicular to the direction of wave propagation.
  - Electromagnetic waves **don't need a medium to propagate**!

- **Poynting theorem**:

$$\frac{dW}{dt} = -\oint_S d\vec{S} \cdot \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) - \frac{\partial}{\partial t}\int_\tau d\tau\left(\frac{1}{2\mu_0}\vec{B}^2 + \frac{\epsilon_0}{2}\vec{E}^2\right)$$

- The **Poynting vector** $\vec{N}$ was defined as:

$$\vec{N} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$$

It represents the **energy flux density (rate energy transfer per unit area)** and it has units of $W(att)/m^2$.

- The average power transmitted by the e/m wave, is given by the **average of $\vec{N}$ over a period T**:

$$< \vec{N} >= \frac{1}{T}\int_0^T \vec{N} dt$$

Lecture 9 - Revision (cont'd)

- E/M waves **carry momentum**: Can **exert pressure** on an object when shining on it!
- If an object is illuminated by radiation for a time interval Δt, during which it absorbs energy $\Delta U = \mathrm{IA}\Delta t$, where I is the intensity I (power per area, or energy per time per area) of the radiation, and A is the area of the object, then:
  - The momentum change Δp of the object is given by:

$$\Delta p = \frac{\Delta U}{c} \quad \textit{(total absorption)} \quad \Delta p = \frac{2\Delta U}{c} \quad \textit{(total reflection)}$$

  - The radiation pressure P_r exterted on the object is given by:

$$P_r = \frac{I}{c} \quad \textit{(total absorption)} \quad P_r = \frac{2I}{c} \quad \textit{(total reflection)}$$

Plan for Lecture 10

- Complex representation of E/M waves
- Polarization and applications
- The most general set of Maxwell equations: Time-dependent case in materials
- E/M waves in matter
- E/M waves in the boundary of transparent media
  - Boundary conditions
  - Reflection and transmission
  - Deriving the laws of geometric optics

Imaginary number that when squared gives a negative result.

The concept appeared in the work of Gerolamo Cardano

- Ars Magna (1545) gave solutions for cubic (Targaglia) and quartic (Ferrari) equations.

The term *imaginary* originates from Rene Descartes who considered these numbers fictitious and useless!

Imaginary numbers became widely accepted through the work of Cauchy, Euler and Gauss.

Reminder: Complex numbers

We define a **complex** number as

$$z = x + iy$$

where x, y are real numbers and $i = \sqrt{-1}$.

The real part of z, written as Re(z) (or $\Re(z)$), is x.

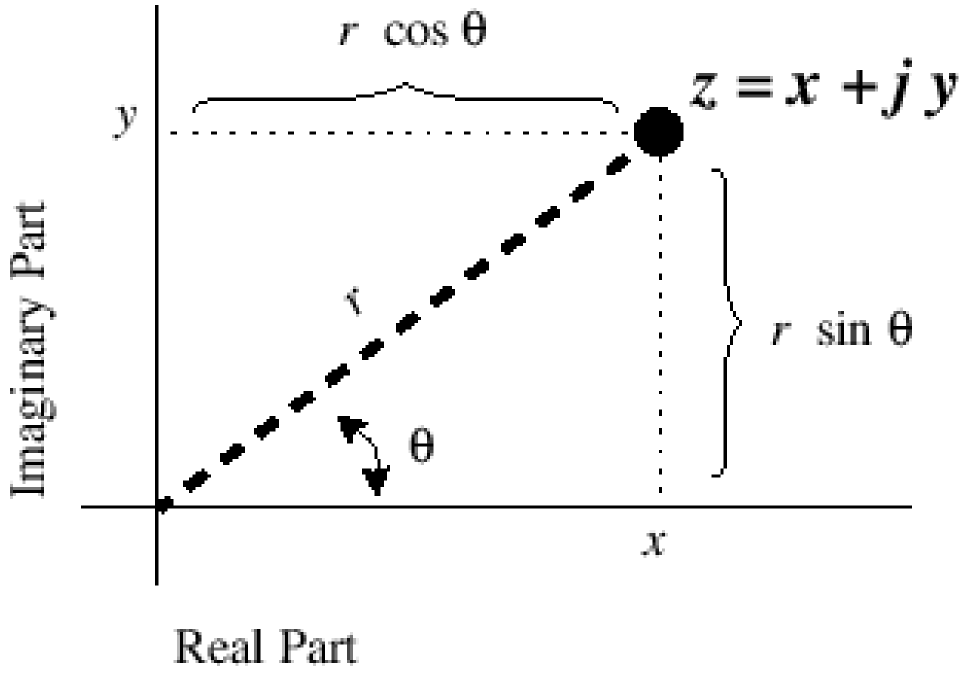
The imaginary part of z, written as Re(z) (or $\Im(z)$), is y.

- Note: The imaginary part is a real number.

Therefore, z can be written as

$$z = \Re(z) + i\Im(z)$$

Reminder: Complex plane



$$z = x + iy = rcos\theta + irsin\theta = re^{i\theta}$$

Reminder: Complex numbers in polar coordinates

A complex number z can be represented in polar coordinates as

$$z = re^{i\theta}$$

where r is the magnitude of z and θ is the angle of z with respect to the real axis.

Using polar coordinates simplifies multiplication and division:

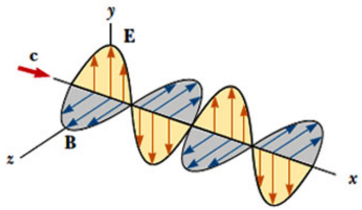
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Complex representation of EM waves

Previously, we studied monochromatic (single frequency = single "colour") EM waves propagating in the +z direction, that had no x or y dependence:

$$\vec{E}(\vec{r},t) = E_0 cos\Big(kz - \omega t\Big)\hat{x} \quad and \quad \vec{B}(\vec{r},t) = B_0 cos\Big(kz - \omega t\Big)\hat{y}$$



In the above expressions, E_0 and B_0 are the wave amplitudes, ω is the wave's angular frequency, and k is the wave number.

- k is related to the wavelength λ via $k = 2\pi/\lambda$
- ω and k are related to the wave velocity c via $c = \omega/k$

These are called **plane waves**: Have no dependence on the position on the plane perpendicular to the direction of travel.

Complex representation of EM waves

Previously, we studied monochromatic (single frequency = single "colour") EM waves propagating in the +z direction, that had no x or y dependence:

$$\vec{E}(\vec{r},t) = E_0 cos\Big(kz - \omega t\Big)\hat{x} \quad and \quad \vec{B}(\vec{r},t) = B_0 cos\Big(kz - \omega t\Big)\hat{y}$$

The above are the **real parts** of:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(kz-\omega t)} \quad and \quad \vec{B}(\vec{r},t) = \vec{B}_0 e^{i(kz-\omega t)}$$

Recall that

$$z = x + iy = rcos\theta + irsin\theta = re^{i\theta}$$

EM waves are transverse

In vacuum and away from sources ($\rho = 0$)

$$\vec{\nabla} \cdot \vec{E} = 0 \xRightarrow{\vec{E}(\vec{r},t)=\vec{E}_0 e^{i(kz-\omega t)}}$$

therefore

$$\left(E_{0x}\frac{\partial}{\partial x} + E_{0y}\frac{\partial}{\partial y} + E_{0z}\frac{\partial}{\partial z}\right)e^{i(kz-\omega t)} = 0 \Rightarrow ikE_{0z}e^{i(kz-\omega t)} = 0 \Rightarrow E_{0z} = 0$$

Similarly, because $\vec{\nabla} \cdot \vec{B} = 0$, $B_{0z} = 0$.

Therefore, the electromagnetic waves are **transverse** (i.e. perpendicular to the direction of propagation).

For a wave propagating in the $+$z direction:

$$\vec{E}_0 = \left(E_{0x}, E_{0y}, 0\right) \quad \mathit{and} \quad \vec{B}_0 = \left(B_{0x}, B_{0y}, 0\right)$$

EM waves are in phase and mutually perpendicular

Faraday's law:

$$\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t \xrightarrow{\vec{E}(\vec{r},t)=\vec{E}_0 e^{i(kz-\omega t)} \quad and \quad \vec{B}(\vec{r},t)=\vec{B}_0 e^{i(kz-\omega t)}}$$

$$\vec{\nabla} \times \left(\vec{E}_0 e^{i(kz-\omega t)}\right) = -\frac{\partial}{\partial t}\left(\vec{B}_0 e^{i(kz-\omega t)}\right) \xrightarrow{\vec{\nabla}\times(\lambda\vec{A})=\lambda\vec{\nabla}\times\vec{A}+(\vec{\nabla}\lambda)\times\vec{A}}$$

$$\left(\vec{\nabla} e^{i(kz-\omega t)}\right) \times \vec{E}_0 = -\vec{B}_0 \frac{\partial}{\partial t}\left(e^{i(kz-\omega t)}\right) \Rightarrow$$

$$\left(ik\hat{z} \times \vec{E}_0\right) e^{i(kz-\omega t)} = \left(i\omega \vec{B}_0\right) e^{i(kz-\omega t)} \Rightarrow$$

$$k\hat{z} \times \vec{E}_0 = \omega \vec{B}_0 \Rightarrow \vec{B}_0 = \frac{k}{\omega}\hat{z} \times \vec{E}_0$$

Therefore, $\vec{E}$ and $\vec{B}$ are **mutually perpendicular**.

Complex representation of EM waves

Let's start with fields of the general form:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(kz-\omega t)} \quad and \quad \vec{B}(\vec{r},t) = \vec{B}_0 e^{i(kz-\omega t)}$$

As we have seen:

$$\vec{B}_0 = \frac{k}{\omega}\left(\hat{z} \times \vec{E}_0\right)$$

Hence amplitudes B_0 and E_0 are related by:

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

Embedding the above constraints, the fields can be written as:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(kz-\omega t)} \quad and \quad \vec{B}(\vec{r},t) = \frac{1}{c}\left(\hat{z} \times \vec{E}_0\right) e^{i(kz-\omega t)} = \frac{1}{c}\left(\hat{z} \times \vec{E}\right)$$

Generalization for waves travelling in arbitrary direction

There is nothing special about the z-axis.

We can generalise the previous description:

Assume a monochromatic plane EM wave traveling along the propagation vector $\vec{k}$. Then $\vec{k}\vec{r}$ is the generalisation of kz and we can write the electric and magnetic waves as:

$$\vec{E}(\vec{r}, t) = E_0 e^{i(\vec{k}\vec{r} - \omega t)} \hat{n}$$

and

$$\vec{B}(\vec{r}, t) = \frac{1}{c}\left(\hat{k} \times \vec{E}\right) = \frac{E_0}{c} e^{i(\vec{k}\vec{r} - \omega t)} \left(\hat{k} \times \hat{n}\right)$$

where $\hat{n}$ is the **polarization vector**, which determines the direction of oscillation of the electric field.

Polarization

The expression

$$\vec{E}(\vec{r}, t) = E_0 e^{i(\vec{k}\vec{r} - \omega t)} \hat{n}$$

describes wave with its electric field vector always along the direction of $\hat{n}_a = \hat{n}$: This is a **linearly polarised** wave.

Evidently, there exists another electric wave, with direction $\hat{n}_b \neq \hat{n}_a$ that is linearly independent.

Therefore, the most general plane wave can be written as

$$\vec{E}(\vec{r}, t) = \Big(E_{0a}(\vec{r}, t)\hat{n}_a + E_{0b}(\vec{r}, t)\hat{n}_b \Big) e^{i(\vec{k}\vec{r} - \omega t)}$$

In principle, the amplitudes E_{0a} and E_{0b} are complex numbers, allowing the possibility of a phase difference between the 2 linearly independent waves.

Polarization

The most general plane wave can be written as

$$\vec{E}(\vec{r},t) = \Big(E_{0a}(\vec{r},t)\hat{n}_a + E_{0b}(\vec{r},t)\hat{n}_b \Big) e^{i(\vec{k}\vec{r}-\omega t)}$$

If E_{0a} and E_{0b} have the **same phase**, they yield a **linearly polarised** wave. The wave has a polarization vector forming an angle θ with respect to $\hat{n}_a$

$$tan\theta = E_{0b}/E_{0a}$$

The magnitude of that wave is given by

$$E_0 = \sqrt{E_{0b}^2 + E_{0a}^2}$$

Polarization

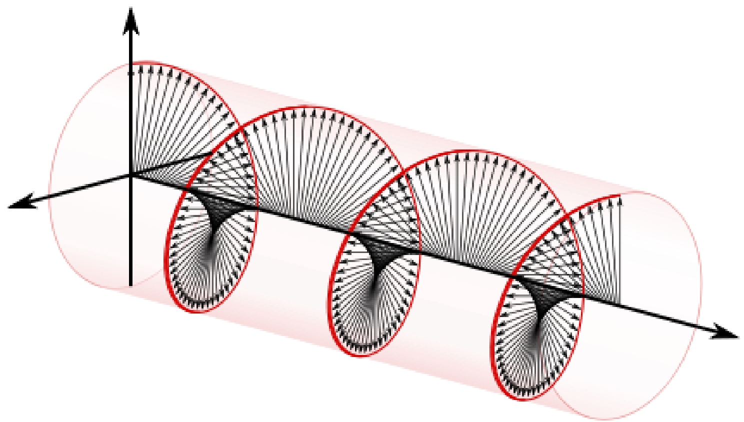
The most general plane wave can be written as

$$\vec{E}(\vec{r},t) = \Big(E_{0a}(\vec{r},t)\hat{n}_a + E_{0b}(\vec{r},t)\hat{n}_b \Big) e^{i(\vec{k}\vec{r}-\omega t)}$$

If the coplex amplitudes E_{0a} and E_{0b} have a **phase difference**, they yield an **elliptically polarised** wave.

In the special case where the phase difference is 90^o, and $E_{0a} = E_{0b}$, we have a **circularly polarised** wave.

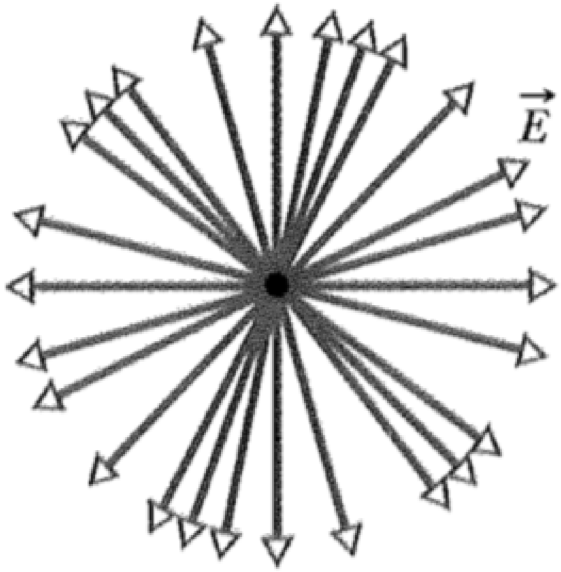
$$\vec{E}(\vec{r},t) = E_{0a}(\vec{r},t) \Big(\hat{n}_a + i\hat{n}_b \Big) e^{i(\vec{k}\vec{r}-\omega t)}$$



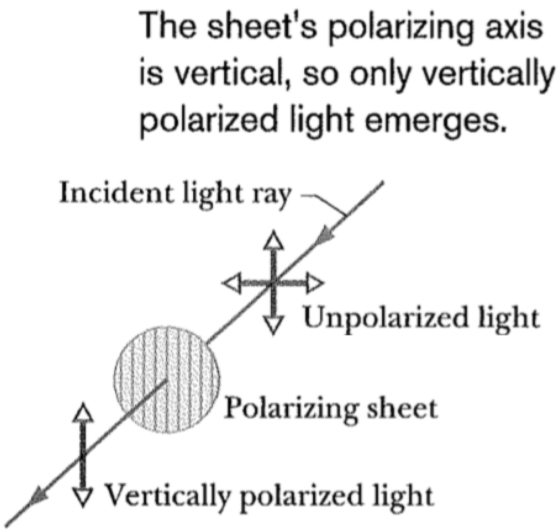
Polarization

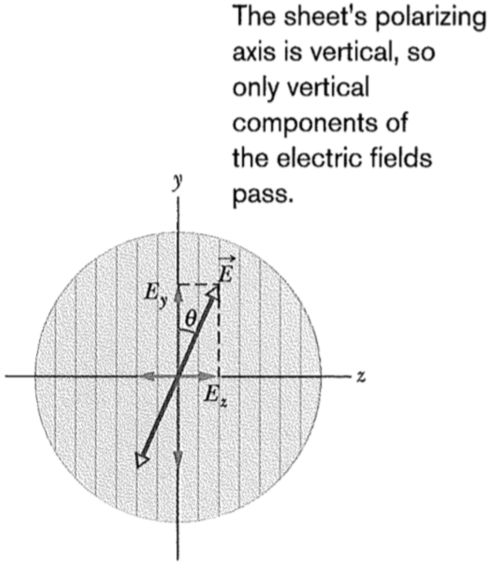
If it also possible that the direction of the polarization vector is **random in time**: In this case we have an **unpolarized** EM wave.

Unpolarized light headed toward you–the electric fields are in all directions in the plane.



Intensity of transmitted polarized light





We can transform unpolarized visible light into polarized light by passing it through a **polarizing sheet**.

If I_0 is the intensity of the unpolarized light, the intensity I of the transmitted light is:

$$I = \frac{1}{2} I_0$$

If the light reaching the filter is already polarized, the intensity I of the transmitted light is:

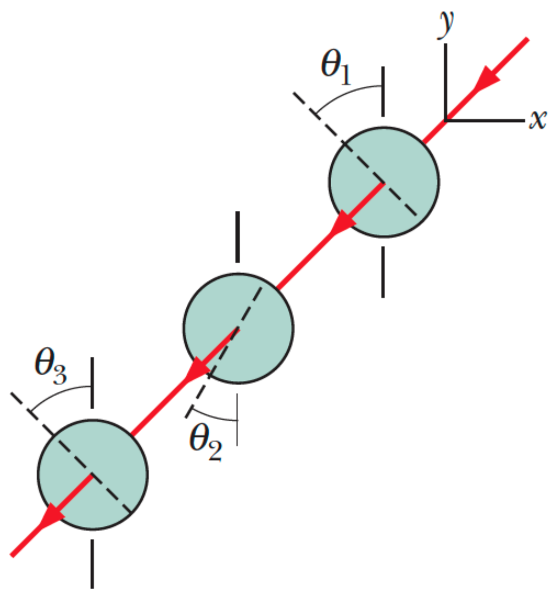
$$I = I_0 cos^2\theta$$

where θ is the angle between the electric field $\vec{E}$ and the polarizing direction of the sheet.

Worked example

Question

Initially unpolarized light is sent into a system of three polarizing sheets, as shown in the figure below, whose polarizing directions make angles of $\theta_1 = \theta_2 = \theta_3 = 50^o$ with the direction of the y axis. What percentage of the initial intensity is transmitted by the system?



Worked example

After passing through the first polarizer the initial intensity I_0 reduces by a factor of $1/2$.
After passing through the second one it is further reduced by a factor of

$$cos^2(\pi - \theta_1 - \theta_2) = cos^2(\theta_1 + \theta_2).$$

Finally, after passing through the third one it is again reduced by a factor of

$$cos^2(\pi - \theta_2 - \theta_3) = cos^2(\theta_2 + \theta_3).$$

Therefore, the ratio of the final intensity I_f to the the initial intensity I_0 is given by:

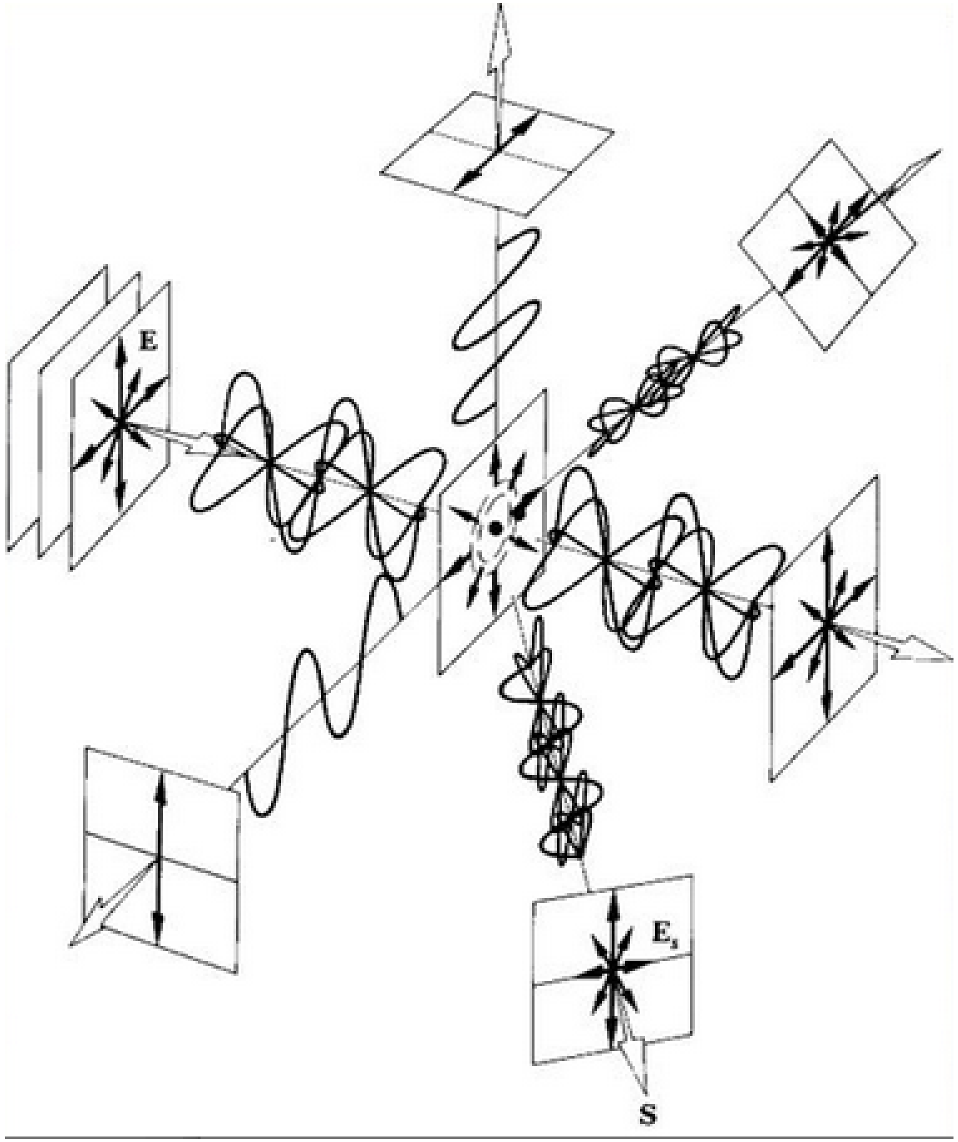
$$\frac{I_f}{I_0} = \frac{1}{2} cos^2(\theta_1 + \theta_2) cos^2(\theta_2 + \theta_3) = 4.5 \times 10^{-4}$$

Applications of polarized light

Many practical applications in science, industry and every-day life.

- in geology:
  - mineral identification
- in chemistry:
  - determining chirality of organic compounds,
- in astronomy:
  - information on the source of radiation, interstellar dust clouds and magnetic fields.
  - physics of early universe
- in 3D films, LCD TVs, sunglasses

Skylight polarization - Polarization by scattering



Skylight polarization

You can easily observe the polarization of the skylight using a simple linear polarizer.

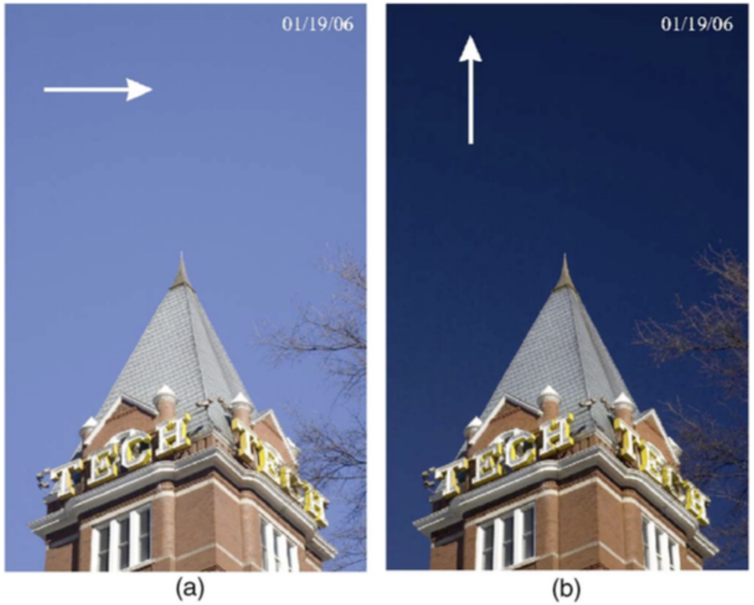
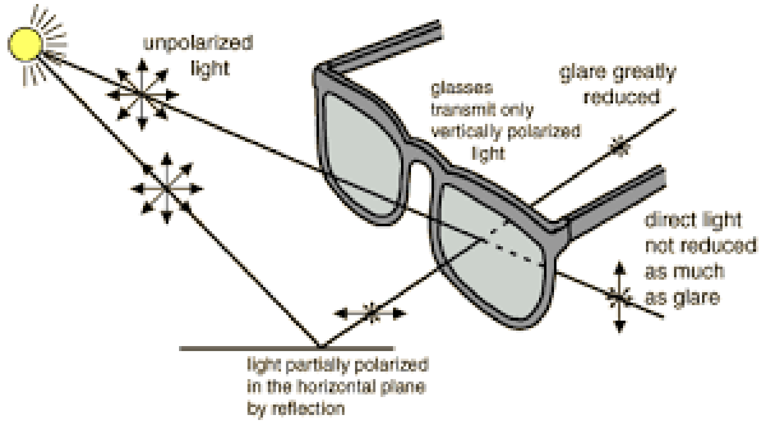


Fig. 1. A clear blue sky viewed through a linear polarizer. (a) The polarizer is oriented for maximum transmission (transmission axis, white arrow, is normal to the principal plane). (b) The polarizer is oriented for minimum transmission (transmission axis is parallel to the principal plane). From the light meter readings that go with these photographs, the degree of linear polarization is $d_l \approx 0.5$.

from G.S. Smith, Am. J. Phys 75 (1), January 2007

- Sunlight is scattered by aerosols as it propagates through the atmosphere.
- Scattered light is partially polarised.
- Effect stronger at points in the sky at a $\pi/2$ angle wrt the Sun.
- Insects can detect polarized light and use it for orientation.
- The human eye has marginal sensitivity to polarization.
- It is alleged that Vikings used naturally occurring (mineral) polarizing filters to navigate using the Sun even with an overcast sky.

Light polarization & sunglasses



hyperphysics.phy-astr.gsu.edu

Polarized sunglass lenses reduce reflected glare.

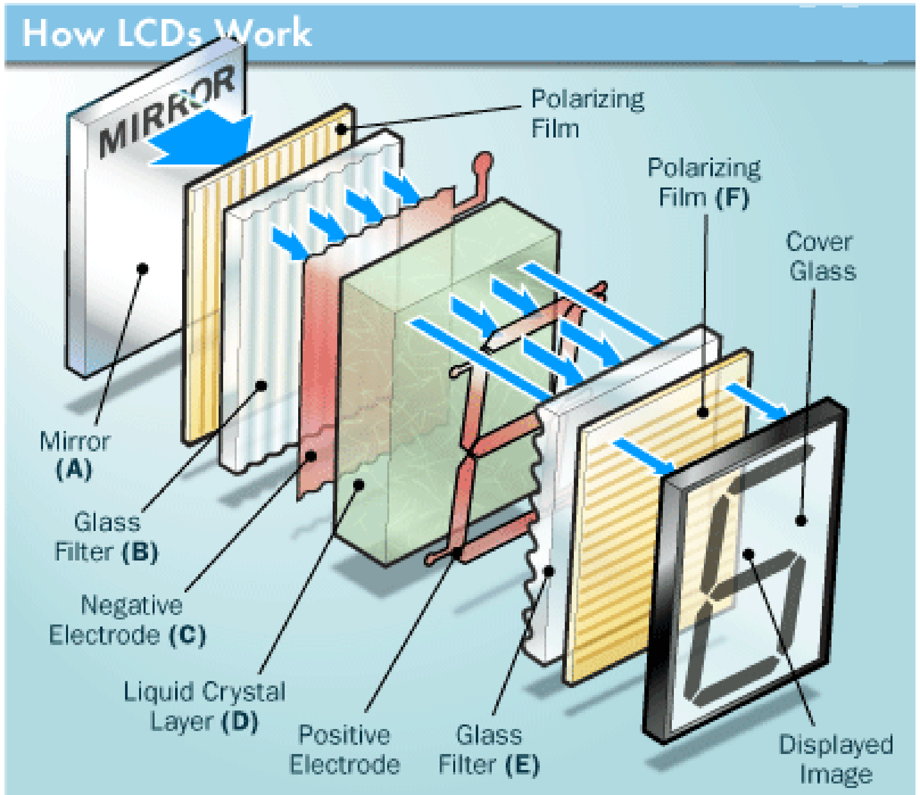
Without polarized lenses.

With polarized lenses.

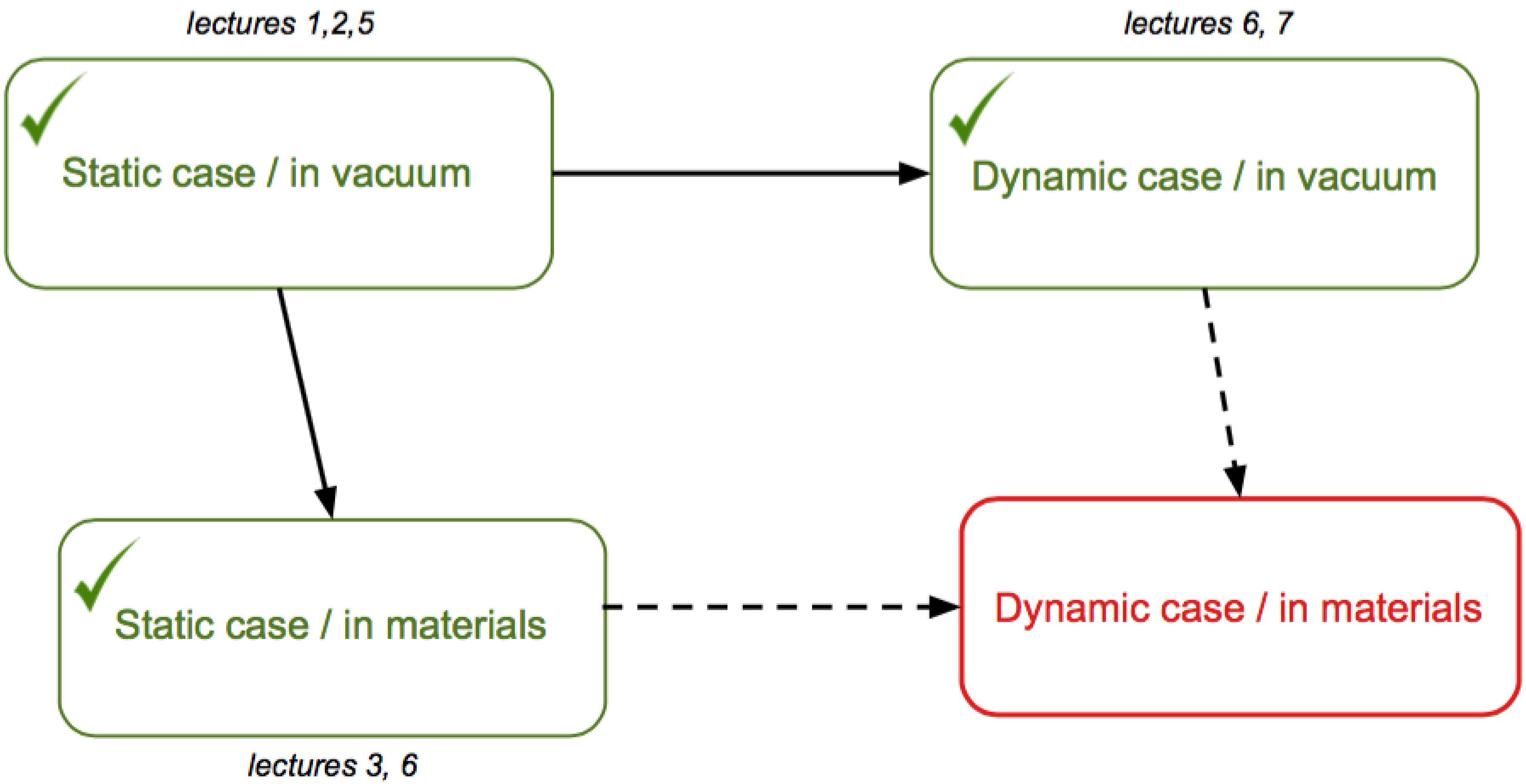
Light polarization & LCD TVs

Pixels between polarization filters, normally appear dark.

Liquid crystal that can be turned on electronically rotates the light by $\pi/2$ allowing light to flow through the two filters and making the pixel appear bright.



Working towards the most general set of equations



Maxwell's equations

For a linear medium:

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{H} = \frac{\vec{B}}{\mu}$$

Static case in matter

Gauss's law	$\oint \vec{D} \cdot d\vec{S} = \int \rho_f d\tau$	$\vec{\nabla} \cdot \vec{D} = \rho_f$
Circuital law	$\oint \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{H} \cdot d\vec{\ell} = \int \vec{j}_f \cdot d\vec{S}$	$\vec{\nabla} \times \vec{H} = \vec{j}_f$

Dynamic case in vacuum

Gauss's law	$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int \rho d\tau$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Faraday's law	$\oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int \vec{B} d\vec{S}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \cdot d\vec{S}$	$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$

Dynamic case in matter

Gauss's law	$\oint \vec{D} \cdot d\vec{S} = \int \rho_f d\tau = Q_f$	$\vec{\nabla} \cdot \vec{D} = \rho_f$
Faraday's law	$\oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S} \Rightarrow$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
	$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$	
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{j} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S} \Rightarrow$	$\vec{\nabla} \times \vec{H} = \vec{j}_f + \frac{\partial \vec{D}}{\partial t}$
	$\oint \vec{H} \cdot d\vec{\ell} = I_f + \frac{d\Phi_D}{dt}$	

EM waves in matter

In an earlier lecture, I solved the time-dependent Maxwell's equations in vacuum and in absence of sources. We saw that the solutions were (EM) waves!

Now I plan to do something similar, but starting from the time-dependent Maxwell's equations in matter. In absence of sources we have:

$$\vec{\nabla} \cdot \vec{D} = \cancel{\rho_f} \Rightarrow \vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \cancel{\vec{j}_f} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

EM waves in matter

For a linear medium:

$$\vec{D} = \epsilon \vec{E} \quad \textit{and} \quad \vec{H} = \frac{\vec{B}}{\mu}$$

Maxwell's equations become:

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

This system of equations is similar with the one we studied in an earlier lecture, with $\mu_0\epsilon_0$ replaced by $\mu\epsilon$.

EM waves in matter

So, through a linear medium electromagnetic waves propagate with velocity:

$$u = \frac{1}{\sqrt{\epsilon\mu}}$$

This velocity can be written as:

$$u = \frac{1}{\sqrt{\epsilon_0\mu_0}} \cdot \frac{\sqrt{\epsilon_0\mu_0}}{\sqrt{\epsilon\mu}} = \frac{c}{n}$$

where

$$n = \frac{\sqrt{\epsilon\mu}}{\sqrt{\epsilon_0\mu_0}}$$

is the **index of refraction** of the material.

EM waves in matter

The index of refraction:

$$n = \frac{\sqrt{\epsilon\mu}}{\sqrt{\epsilon_0\mu_0}}$$

can also be written as:

$$n = \frac{\sqrt{\epsilon_r\epsilon_0\mu_r\mu_0}}{\sqrt{\epsilon_0\mu_0}} = \sqrt{\epsilon_r\mu_r}$$

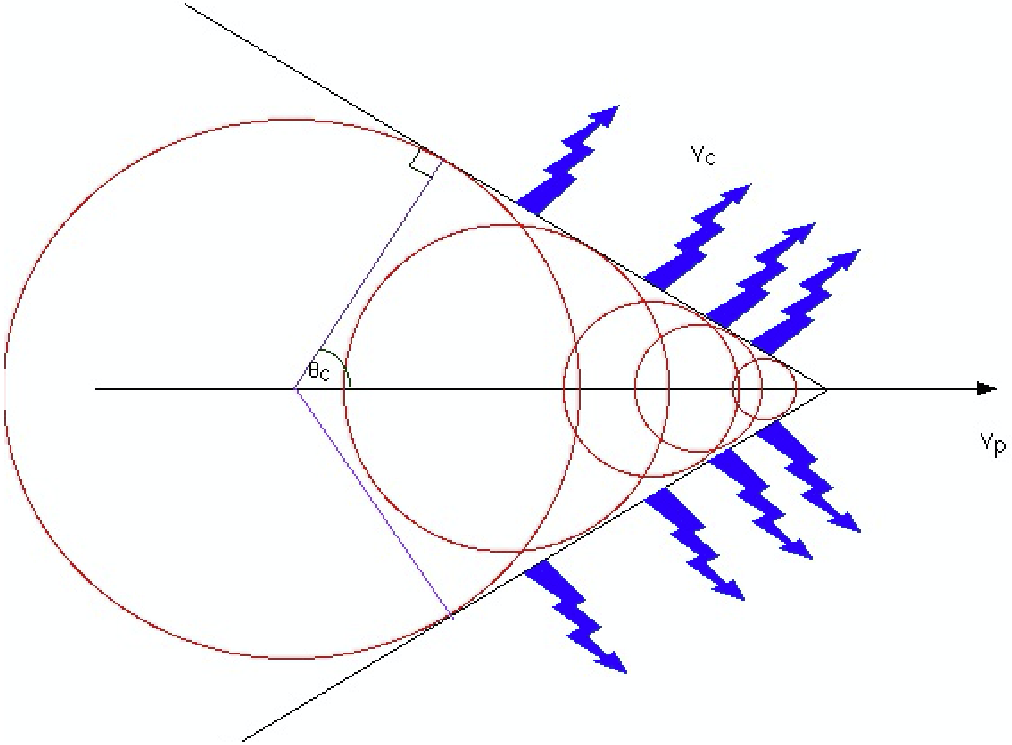
For most materials μ_r is close to 1 and ϵ_r is larger than one and, therefore, almost always $n > 1$.

So **light travels in matter more slowly** than it does in the vacuum!

For example, for water n $\approx$ 1.33 so light slows down by 25%.

Cherenkov radiation

So, within a medium (but not in vacuum), particles can travel faster than light!
When a charged particle travels faster than the speed of light in a medium, it generates an **EM shockwave** called **Cherenkov radiation**.

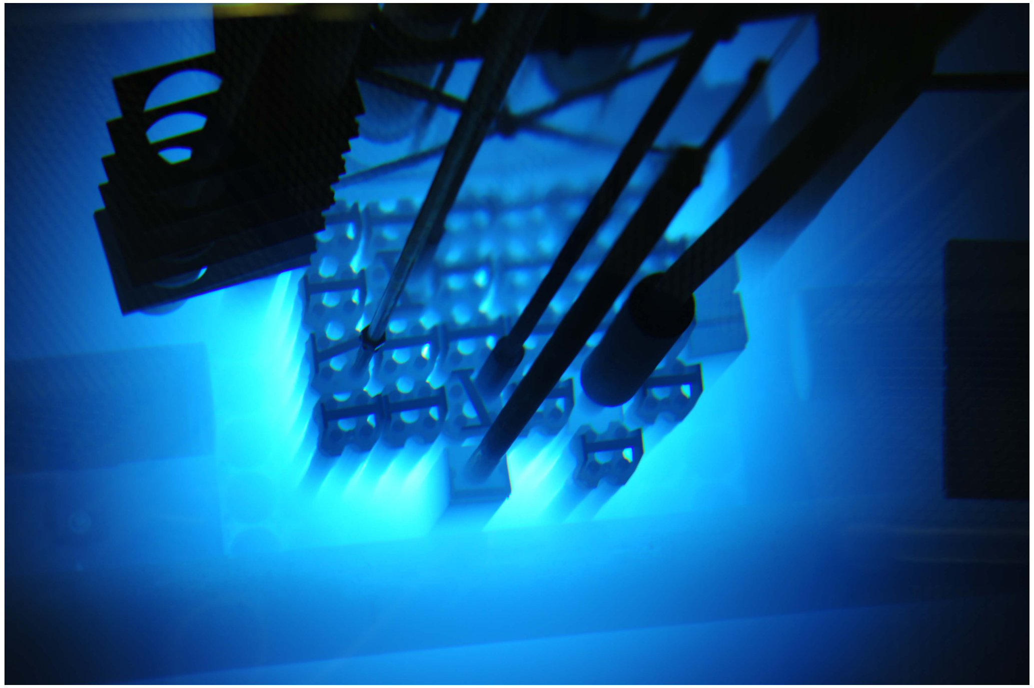


Cherenkov radiation is emitted in a cone with opening angle

$$\theta_C = cos^{-1}\left(\frac{1}{\beta n}\right)$$

where n is the index of refraction of the medium and β is the particle velocity ($\beta = u/c$).

Cherenkov radiation

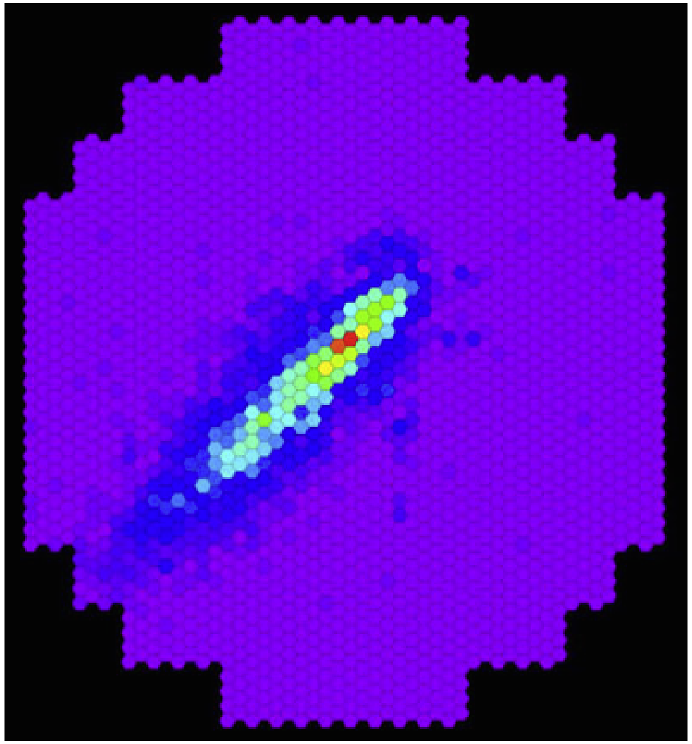


Cherenkov glow around a pool-type nuclear reactor.

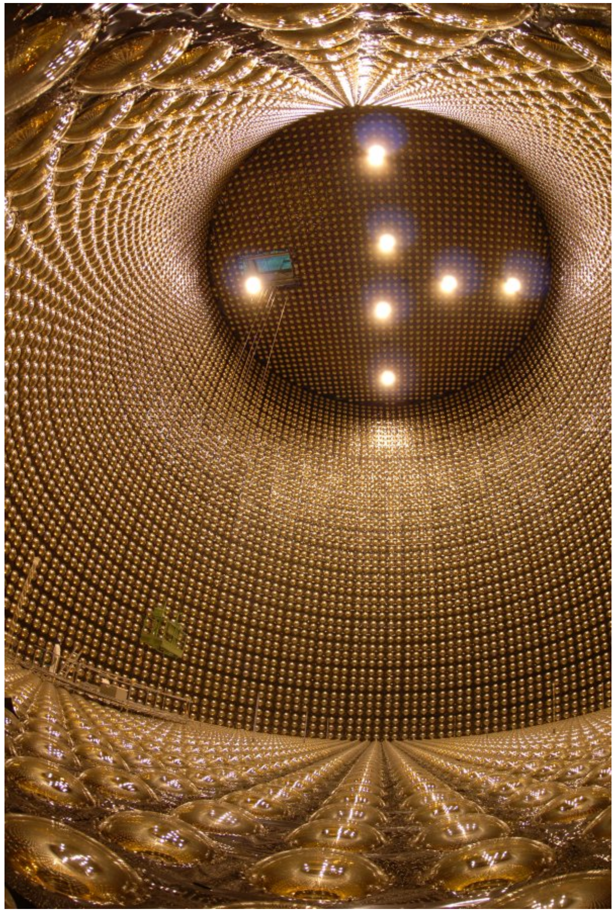
Cherenkov radiation

Cherenkov radiation is being used to detect high-energy gamma ray sources in the universe.

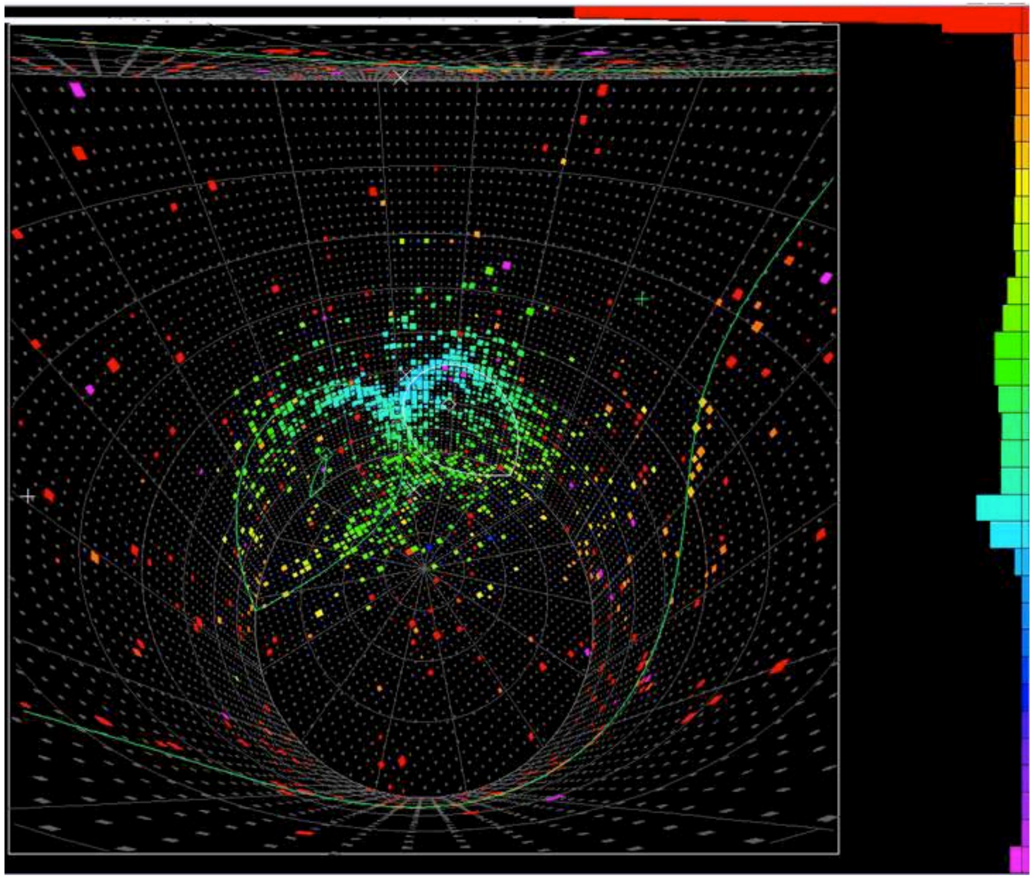
(HESS - High Energy Stereoscopic System)



Cherenkov radiation is also being used to detect man-made (accelerator), atmospheric and solar neutrinos [Nobel prize in Physics 2002 and 2015]



(Super-Kamiokande detector)



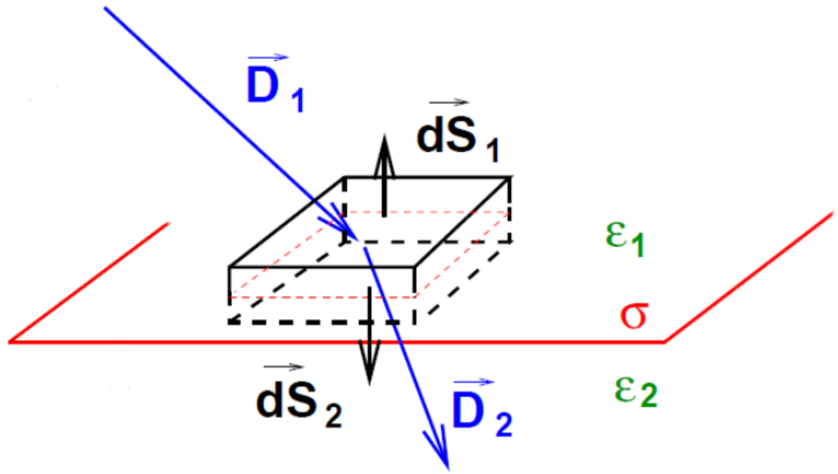
EM waves at the boundary of transparent media

What happens when a wave crosses the **boundary between two transparent media**?

To examine what exactly happens we need to understand the electrodynamic boundary conditions.

Boundary conditions for the electric field

Assume that the boundary between two dielectrics (with permittivities ϵ_1 and ϵ_2) is carrying surface charge density σ_f.
Consider the volume shown around the surface and assume that the height is infinitesimally small so the side surfaces can be ignored.



$$\oint \vec{D} \cdot d\vec{S} = Q_f \Rightarrow D_2^{\perp} S - D_1^{\perp} S = Q_f \xrightarrow{Q_f = \sigma_f S}$$

$$D_2^{\perp} - D_1^{\perp} = \sigma_f$$

The electric displacement $\vec{D}$ is given by $\vec{D} = \epsilon \vec{E}$, therefore:

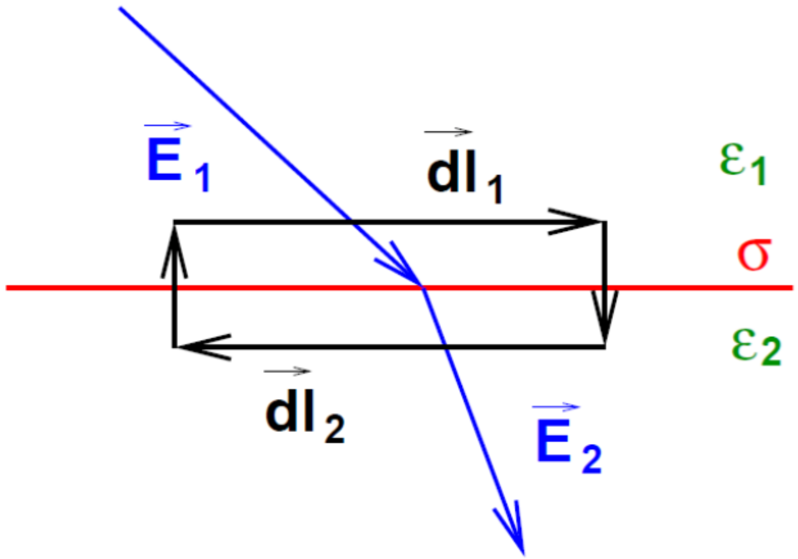
$$\epsilon_2 \vec{E}_2^{\perp} - \epsilon_1 \vec{E}_1^{\perp} = \sigma$$

In the absence of free charges ($\sigma_f = 0$):

$$\epsilon_1 \vec{E}_1^{\perp} = \epsilon_2 \vec{E}_2^{\perp}$$

Boundary conditions for the electric field

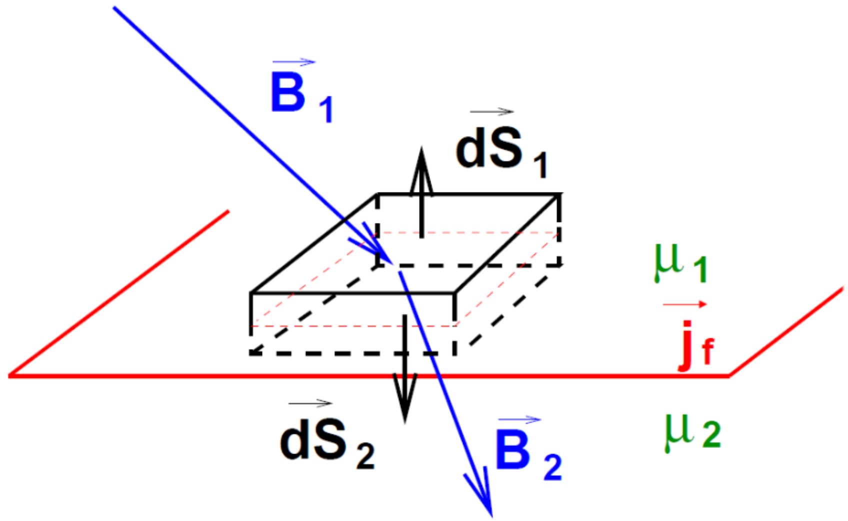
Again, assume that the boundary between two dielectrics (with permittivities ϵ_1 and ϵ_2) is carrying surface charge density σ.
Consider the path shown around the surface and assume that the height is infinitesimally small so the perpendicular sides can be ignored.



$$\oint \vec{E} \cdot d\vec{\ell} = 0 \Rightarrow E_1^{||}\ell - E_2^{||}\ell = 0 \Rightarrow E_1^{||} = E_2^{||}$$

Boundary conditions for the magnetic field

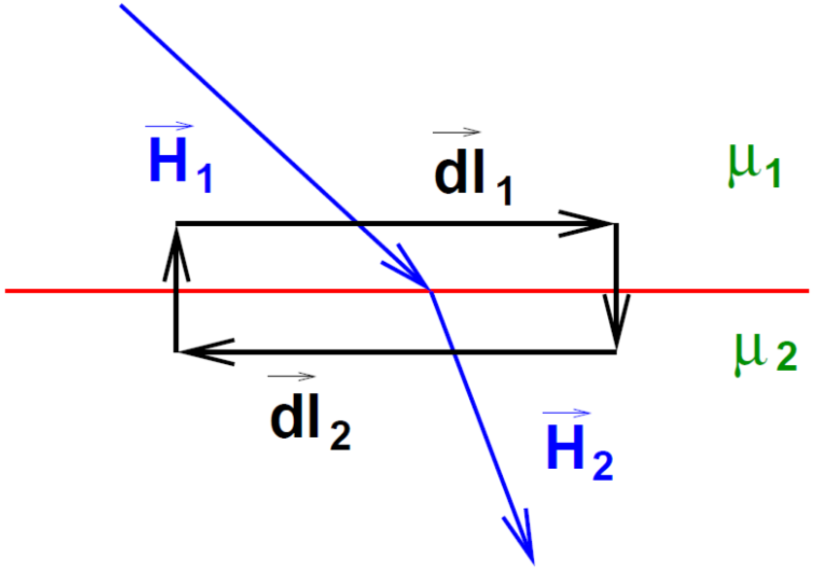
Assume that the boundary between two materials (with permeabilities μ_1 and μ_2) is carrying surface current density $\vec{j}_f$. Consider the volume shown around the surface and assume that the height is infinitesimally small so the side surfaces can be ignored.



$$\oint \vec{B} \cdot d\vec{S} = 0 \Rightarrow B_2^{\perp} S - B_1^{\perp} S = 0 \Rightarrow B_1^{\perp} = B_2^{\perp}$$

Boundary conditions for the magnetic field

Assume that the boundary between two materials (with permeabilities μ_1 and μ_2) is carrying surface current density $\vec{j}_f$. Consider the path shown around the surface and assume that the height is infinitesimally small so the perpendicular sides can be ignored.



$$\oint \vec{H} \cdot d\vec{\ell} = 0 \Rightarrow H_1^{||}\ell - H_2^{||}\ell = 0 \Rightarrow H_1^{||} = H_2^{||}$$

The magnetizing field $\vec{H}$ is given by $\vec{H} = \frac{1}{\mu}\vec{B}$, therefore:

$$\frac{1}{\mu_1} B_1^{||} = \frac{1}{\mu_2} B_2^{||}$$

Boundary conditions: Summary

For the electric field:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$$

$$E_1^{\parallel} = E_2^{\parallel}$$

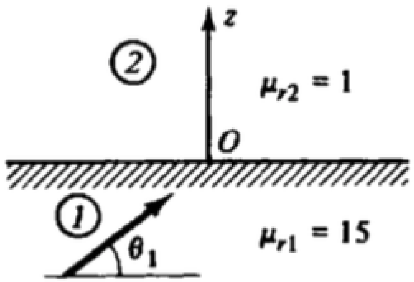
For the magnetic field:

$$B_1^{\perp} = B_2^{\perp}$$

$$\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}$$

Worked example

Question



In region 1, $\vec{B}_1 = \Big(1.2, 0.8, 0.4\Big)$ T. Find $\vec{H}_2$ at z=0. Calculate the angles between the B field vectors and a tangent to the interface between the two regions.

Recall that: $H = \dfrac{B}{\mu} = \dfrac{B}{\mu_r \mu_0}$

The relevenant electrodynamic boundary conditions are:

$$B_1^{\perp} = B_2^{\perp} \quad and \quad H_1^{||} = H_2^{||} \Rightarrow \frac{1}{\mu_1} B_1^{||} = \frac{1}{\mu_2} B_2^{||}$$

Worked example

Therefore:

- $\vec{B}_1 = \left(1.2, 0.8, 0.4\right)$ T
- $\vec{H}_1 = \frac{1}{\mu_0}\left(8.0, 5.33, 2.67\right) \cdot 10^{-2}$ A/m
- $\vec{H}_2 = \frac{1}{\mu_0}\left(8.0, 5.33, 10^2\mu_0 H_{z2}\right) \cdot 10^{-2}$ A/m
- $\vec{B}_2 = \left(B_{x2}, B_{y2}, 0.4\right)$ T

The remaining unknown terms follow directly:

$$B_{x2} = \mu_0 \cdot \mu_{r2} \cdot H_{x2} = 8.0 \times 10^{-2}\ T$$

$$B_{y2} = 5.33 \times 10^{-2}\ T$$

$$H_{z2} = \frac{H_{z2}}{\mu_0 \cdot \mu_{r2}} = \frac{0.4}{\mu_0}\ A/m$$

Worked example

Angle θ_1 is $90^o - \alpha_1$ where α_1 is the angle between B_1 and $\hat{z} = \Big(0, 0, 1\Big)$.

$$cos\alpha_1 = \frac{\vec{B_1} \cdot \hat{z}}{|\vec{B_1}|} = 0.27 \Rightarrow \alpha_1 = 74.5^o$$

and therefore:

$$\theta_1 = 90^o - 74.5^o \Rightarrow \theta_1 = 15.5^o.$$

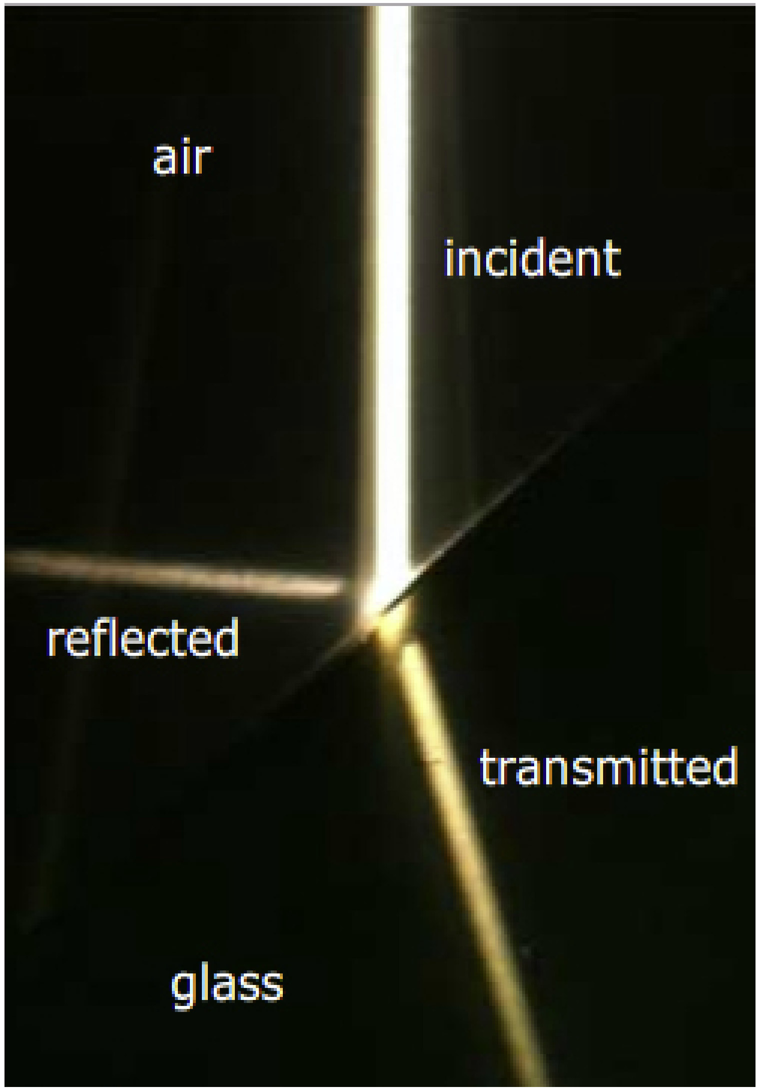
Similarly, $\theta_2 = 76.5^o$.

Reflection and transmission (refraction)

Consider a wave (incident wave) hitting the boundary between two linear transparent media.

This gives rise to a **reflected** and a **transmitted** wave.

I would like to calculate the intensity of the reflected and transmitted waves.



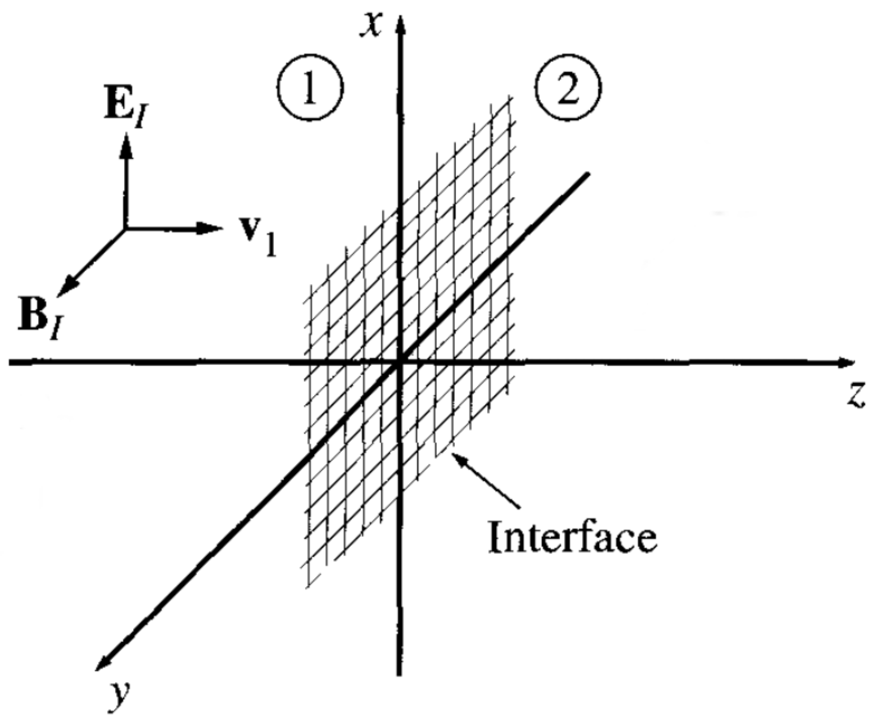
Reflection and transmission at normal incidence

Suppose the xy plane is the boundary between two linear media.

A plane wave of frequency ω, travelling in the $+z$ direction and polarized in the x direction approached the boundary from the left.

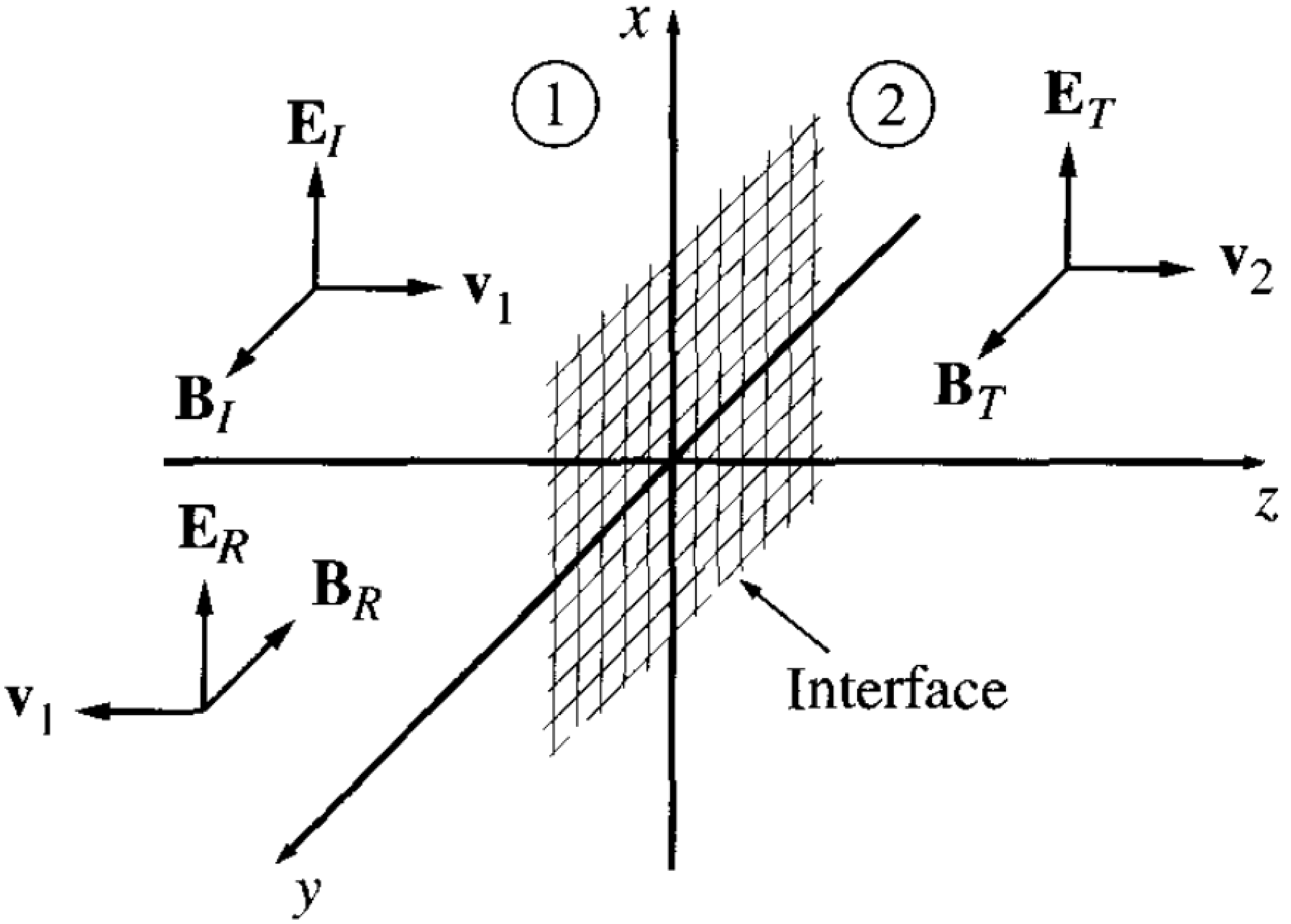
$$\vec{E}_I(z,t) = E_{I0} e^{i(k_1 z - \omega t)} \hat{x}$$

$$\vec{B}_I(z,t) = \frac{E_{I0}}{u_1} e^{i(k_1 z - \omega t)} \hat{y}$$



Reflection and transmission at normal incidence

This gives rise to a **reflected** and a **transmitted** wave.



Reflection and transmission at normal incidence

Reflected wave:

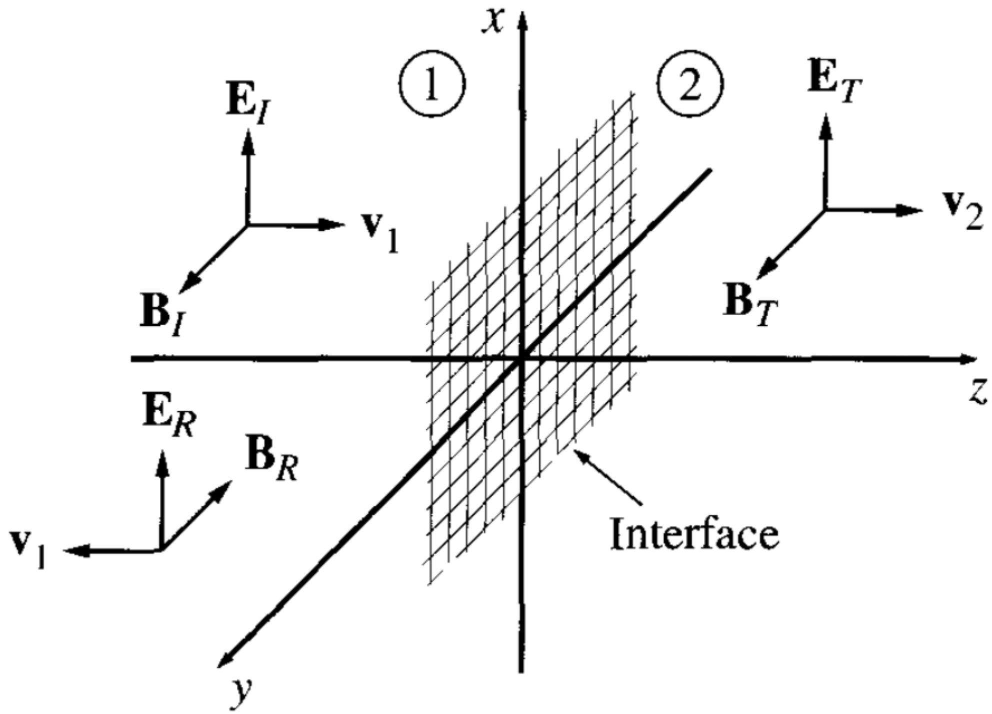
$$\vec{E}_R(z,t) = E_{R0} e^{i(-k_1 z - \omega t)} \hat{x}$$

$$\vec{B}_R(z,t) = -\frac{E_{R0}}{u_1} e^{i(-k_1 z - \omega t)} \hat{y}$$

Transmitted wave:

$$\vec{E}_T(z,t) = E_{T0} e^{i(k_2 z - \omega t)} \hat{x}$$

$$\vec{B}_T(z,t) = \frac{E_{T0}}{u_2} e^{i(k_2 z - \omega t)} \hat{y}$$



Notice that if the incident wave travels along the propagation vector $\vec{k}$, the reflected wave (for normal incidence) travels along $-\vec{k}$. Also, notice the change of sign of $\vec{B}_R$ as required by $\vec{B}_0 = \frac{1}{\omega}\left(\vec{k} \times \vec{E}_0\right)$ (see earlier in this lecture).

Reflection and transmission at normal incidence

- On the right of the boundary (2) there is only the transmitted wave::

$$\vec{E}_2 = \vec{E}_T \quad \textit{and} \quad \vec{B}_2 = \vec{B}_T$$

- On the left of the boundary (1) we have both the incoming and reflected waves:

$$\vec{E}_1 = \vec{E}_I + \vec{E}_R \quad \textit{and} \quad \vec{B}_1 = \vec{B}_I + \vec{B}_R$$

At z=0 (boundary between the two linear media), the following boundary conditions should be satisfied:

$$\cancel{\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}} \quad \textit{and} \quad E_1^{||} = E_2^{||}$$

$$\cancel{B_1^{\perp} = B_2^{\perp}} \quad \textit{and} \quad \frac{1}{\mu_1} B_1^{||} = \frac{1}{\mu_2} B_2^{||}$$

In this case there are **no field components perpendicular to the surface**, so the corresponding boundary conditions are trivially satisfied.

Reflection and transmission at normal incidence

Will examine what we learn from the $E_1^{||} = E_2^{||}$ boundary condition:

$$E_1^{||}(z=0,t) = E_2^{||}(z=0,t) \Rightarrow$$

$$\vec{E}_I(z=0,t) + \vec{E}_R(z=0,t) = \vec{E}_T(z=0,t) \Rightarrow$$

$$E_{I0} e^{-i\omega t} \hat{x} + E_{R0} e^{-i\omega t} \hat{x} = E_{T0} e^{-i\omega t} \hat{x} \Rightarrow$$

$$\Big(E_{I0} + E_{R0}\Big) \cancel{e^{-i\omega t} \hat{x}} = E_{T0} \cancel{e^{-i\omega t} \hat{x}} \Rightarrow$$

$$E_{I0} + E_{R0} = E_{T0}$$

Reflection and transmission at normal incidence

Will also examine what we learn from the $\frac{1}{\mu_1} B_1^{\|} = \frac{1}{\mu_2} B_2^{\|}$ condition:

$$\frac{1}{\mu_1} B_1^{\|}(z=0,t) = \frac{1}{\mu_2} B_2^{\|}(z=0,t) \Rightarrow$$

$$\frac{1}{\mu_1}\Big(\vec{B}_I(z=0,t) + \vec{B}_R(z=0,t)\Big) = \frac{1}{\mu_2}\vec{B}_T(z=0,t) \Rightarrow$$

$$\frac{1}{\mu_1}\left(\frac{E_{I0}}{u_1} e^{-i\omega t}\hat{y} - \frac{E_{R0}}{u_1} e^{-i\omega t}\hat{y}\right) = \frac{1}{\mu_2}\frac{E_{T0}}{u_2} e^{-i\omega t}\hat{y} \Rightarrow$$

$$\frac{1}{\mu_1 u_1}\Big(E_{I0} - E_{R0}\Big)\cancel{e^{-i\omega t}\hat{y}} = \frac{1}{\mu_2 u_2} E_{T0}\cancel{e^{-i\omega t}\hat{y}} \Rightarrow$$

$$\frac{1}{\mu_1 u_1}\Big(E_{I0} - E_{R0}\Big) = \frac{1}{\mu_2 u_2} E_{T0}$$

Reflection and transmission at normal incidence

The two boundary conditions led to the following system of equations:

$$E_{I0} + E_{R0} = E_{T0} \quad and \quad \frac{1}{\mu_1 u_1}\Big(E_{I0} - E_{R0}\Big) = \frac{1}{\mu_2 u_2} E_{T0}$$

Solving that system for E_{R0} and E_{T0} we get:

$$E_{R0} = \frac{1-\beta}{1+\beta} E_{I0}$$

$$E_{T0} = \frac{2}{1+\beta} E_{I0}$$

where

$$\beta = \frac{\mu_1 u_1}{\mu_2 u_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

Reflection and transmission coefficients

We define the reflection (transmission) coefficient using the ratio of the reflected (transmitted) intensity with the incident amplitude.

Reflection coefficient R:

$$R = \frac{I_R}{I_I}$$

Transmission coefficient T:

$$T = \frac{I_T}{I_I}$$

For a monochromatic plane wave with an electric field amplitude E_0, propagating with velocity u within a medium with permittivity (permeability) ϵ (μ), the intensity I (average power per unit area) is given by

$$I = \frac{1}{2\mu u} E_0^2 = \frac{1}{2} \epsilon u E_0^2$$

Reflection and transmission coefficients

$$R = \frac{I_R}{I_I} = \frac{\cancel{\frac{1}{2}\epsilon_1 u_1} E_{R0}^2}{\cancel{\frac{1}{2}\epsilon_1 u_1} E_{I0}^2} = \left(\frac{E_{R0}}{E_{I0}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \xRightarrow{\beta \approx n_2/n_1}$$

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\cancel{\frac{1}{2}\epsilon_2 u_2} E_{T0}^2}{\cancel{\frac{1}{2}\epsilon_1 u_1} E_{I0}^2} = \frac{\epsilon_2 u_2}{\epsilon_1 u_1}\left(\frac{E_{T0}}{E_{I0}}\right)^2 = \frac{\epsilon_2 u_2}{\epsilon_1 u_1}\left(\frac{2}{1+\beta}\right)^2 \xRightarrow{\beta \approx n_2/n_1}$$

$$T = \frac{\epsilon_2 u_2}{\epsilon_1 u_1}\frac{4n_1^2}{(n_1+n_2)^2} \xRightarrow{\frac{\epsilon_2 u_2}{\epsilon_1 u_1} = \frac{\mu_1 u_1^2 u_2}{\mu_2 u_2^2 u_1} \approx u_1/u_2 = n_2/n_1}$$

$$T = \frac{4 n_1 n_2}{(n_1+n_2)^2}$$

Reflection and transmission coefficients

So, for example, for an EM wave crossing the boundary between a volume of air (n_1=1) and a glass (n_2=1.5):

- $R = \Big(\frac{n_1 - n_2}{n_1 + n_2}\Big)^2 = \Big(\frac{1 - 1.5}{1 + 1.5}\Big)^2 = 4\%$
- $T = \frac{4 n_1 n_2}{(n_1 + n_2)^2} = \frac{4 \cdot 1.5}{(1 + 1.5)^2} = 96\%$

We can easily confirm that

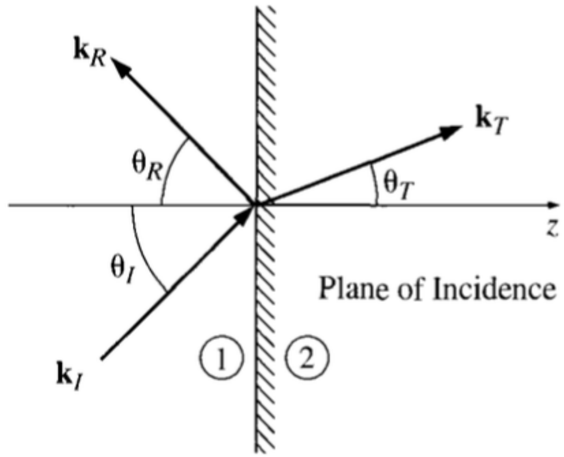
$$R + T = 1$$

as a consequence of the conservation of energy.

Reflection and transmission at oblique incidence

General case of *oblique* incidence:

- Incoming wave meets the boundary plane at an arbitrary angle θ_I.
- The angle of transmission (or refraction) is θ_T.
- The angle of reflection is θ_R.



The analysis is **similar as before**, but the algebra is slightly more complex.

Study this case on your own.

Reflection and transmission at oblique incidence

The analysis of reflection and transmission at oblique incidence leads to the three known laws of geometrical optics:

- **1st Law**:
  The incident, reflected, and transmitted wave vectors form a plane (plane of incidence), which also includes the normal to the surface.

$$k_I sin\theta_I = k_R sin\theta_R = k_T sin\theta_T$$

- **2nd Law** (**Law of reflection**):
  The angle of incidence is equal to the angle of reflection.

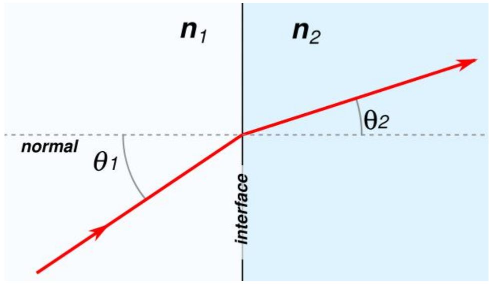
$$\theta_I = \theta_R$$

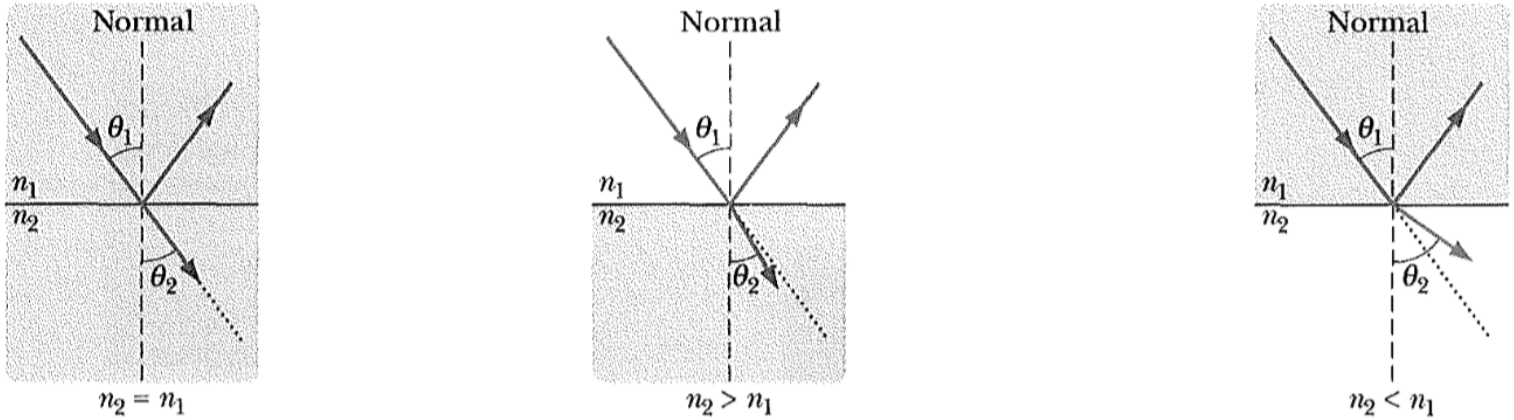
- **3rd Law** (**Law of refraction** or **Snell's law**):

$$\frac{sin\theta_T}{sin\theta_I} = \frac{n_1}{n_2}$$

Law of refraction or **Snell's law**:

$$n_1 sin\theta_1 = n_2 sin\theta_2$$





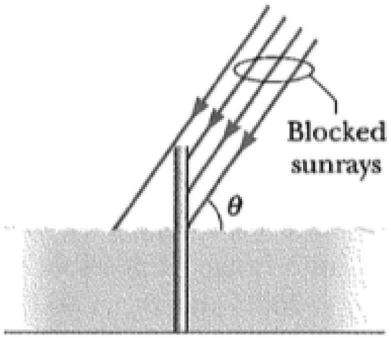
(*a*) If the indexes match, there is no direction change.

(*b*) If the next index is greater, the ray is bent *toward* the normal.

(*c*) If the next index is less, the ray is bent *away from* the normal.

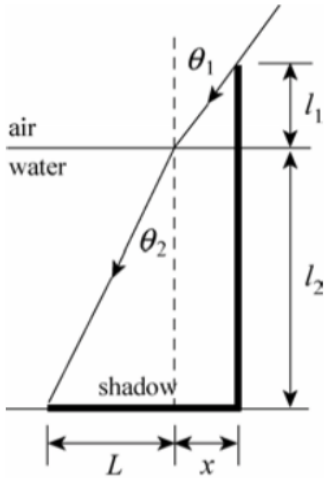
Worked example

Question



A 2.0-m-long vertical rod is extending from the bottom of a water tank to a point 50 cm above the surface of the water. The bottom of the water tank is level. Sunlight is incident at angle θ=55^o. What is the length of the shadow of the rod at the bottom of the tank?

The index of refraction for air (water) is 1 (1.33).



Consider a ray that grazes the top of the rod, as shown.

Here $\theta_1 = 90^o - \theta = 35^o$, l_1 =0.50 m, and l_2=1.50 m.

The length of the shadow is x + L.

Worked example

The distance x is given by:

$$x = l_1 \cdot tan\theta_1 = (0.50\ m) \cdot tan35^o = 0.35m$$

According to the law of refraction:

$$n_2 \cdot sin\theta_2 = n_1 \cdot sin\theta_1$$

Therefore:

$$\theta_2 = sin^{-1}\left(\frac{n_1}{n_2} \cdot sin\theta_1\right) = sin^{-1}\left(\frac{1}{1.33} \cdot sin35.0^o\right) \Rightarrow \theta_2 = 25.54^o$$

The distance L is given by:

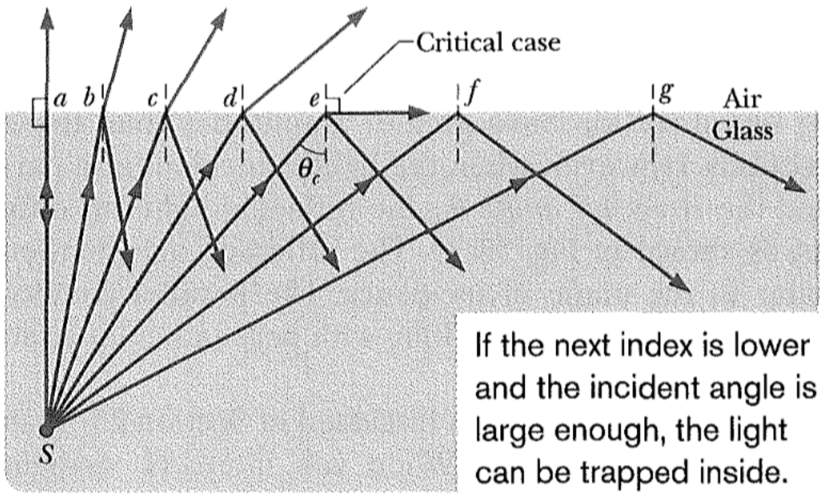
$$L = l_2 \cdot tan\theta_2 = (1.50\ m) \cdot tan25.54^o = 0.72\ m$$

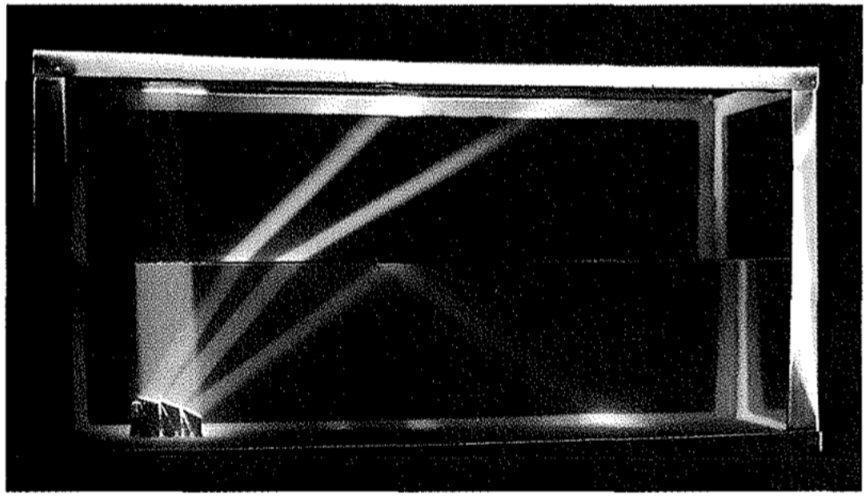
The length of the shadow is: 0.35 m + 0.72 m = 1.07 m.

As the angle of incidence increases, the angle of refraction increases: For some critical value θ_c, the angle of refraction becomes 90^o.

For angles of incidence larger than θ_c, such as for rays f and g below, there is no refracted ray and all the light is reflected; this effect is called **total internal reflection**.

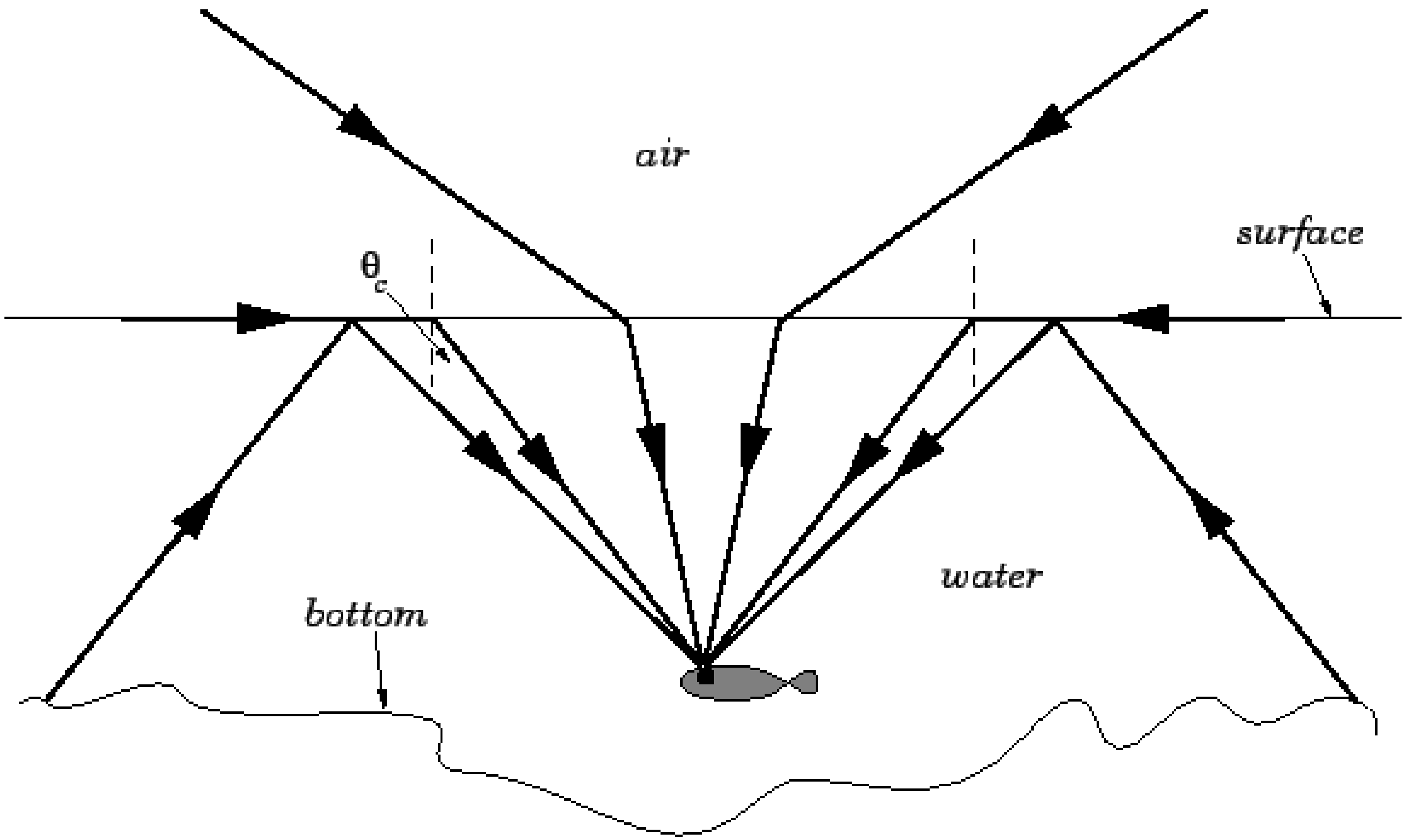
$$n_1 sin\theta_c = n_2 sin90^o \Rightarrow \theta_c = sin^{-1}\frac{n_2}{n_1} \quad (n_2 < n_1)$$





Total internal reflection

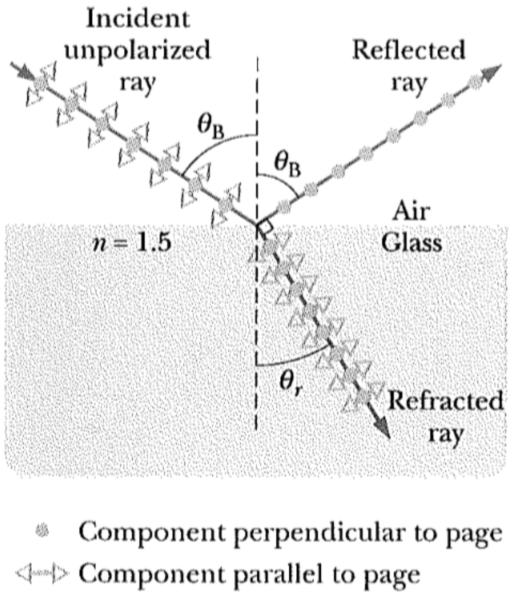
Consider how a fish sees the world outside the water: A 'circle of light'



Use this schematic for the workshop tomorrow!

Polarisation by reflection

- A randomly polarised wave: the sum of a wave polarised on the incidence plane and one polarised perpendicularly to it.
  - For unpolarized light, these two components are of equal magnitude.
- In general, the reflected light also has both components but with unequal magnitudes: the **reflected light is partially polarized**



- When the light is incident at a particular incident angle, called **Brewster angle** given by:

$$tan\theta_B \approx \frac{n_2}{n_1}$$

  The reflected light has only perpendicular components: It is **fully polarized** perpendicular to the plane of incidence.
- For light incident at that angle, the reflected and refracted rays are perpendicular to each other.

Lecture 10 - Main points to remember

We studied the most general case of Maxwell's equations:

Dynamic case in matter

Gauss's law	$\oint \vec{D} \cdot d\vec{S} = \int \rho_f d\tau = Q_f$	$\vec{\nabla} \cdot \vec{D} = \rho_f$
Faraday's law	$\oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S} \Rightarrow$ $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Gauss's law (magn.)	$\oint \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
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Lecture 10 - Main points to remember (cont'd)

- As we did in vacuum, we studied Maxwell's equations in matter (for time-dependent fields) and in the absence of sources and we saw that they give rise to EM waves:

$$\vec{\nabla}^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \textit{and} \quad \vec{\nabla}^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

- Speed of EM waves in matter: $u = \dfrac{1}{\sqrt{\epsilon\mu}}$
    - EM waves in matter propagate slower than EM waves in vacuum
    - $u = \dfrac{c}{n}$ where $n = \dfrac{\sqrt{\epsilon\mu}}{\sqrt{\epsilon_0\mu_0}}$ is the index of refraction of the material.
- We introduced a complex representation of EM waves starting from de Moivre's theorem and embedding the known EM wave properties (EM waves are always **transverse** and **mutually perpendicular**):

$$\vec{E}(\vec{r}, t) = E_0 e^{i(\vec{k}\vec{r}-\omega t)} \hat{n} \quad \textit{and} \quad \vec{B}(\vec{r}, t) = \frac{E_0}{c} e^{i(\vec{k}\vec{r}-\omega t)} \left(\hat{k} \times \hat{n}\right) = \frac{1}{c}\left(\hat{k} \times \vec{E}\right)$$

Lecture 10 - Main points to remember (cont'd)

- We also studied EM wave polarization and practical applications.
- We can transform unpolarized visible light into polarized light by passing it through a **polarizing sheet**.
- If I_0 is the intensity of the unpolarized light, the intensity I of the transmitted light is:

$$I = \frac{1}{2} I_0$$

- If the light reaching the filter is already polarized, the intensity I of the transmitted light is:

$$I = I_0 cos^2\theta$$

  where θ is the angle between the electric field $\vec{E}$ and the polarizing direction of the sheet.

Lecture 10 - Main points to remember (cont'd)

- Finally, we studied the electrodynamic boundary conditions:

  For the electric field:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \quad \textit{and} \quad E_1^{||} = E_2^{||}$$

  For the magnetic field:

$$B_1^{\perp} = B_2^{\perp} \quad \textit{and} \quad \frac{1}{\mu_1} B_1^{||} = \frac{1}{\mu_2} B_2^{||}$$

- We used the above conditions to study what happens when an EM wave crosses the **boundary between two transparent media**

  Two cases:
  - Normal incidence
  - Oblique incidence (general case / home study)

  We reproduced the laws of geometric optics!

Plan for Lecture 11

- Mutual and self inductance
- Inductance in circuits
    - RL circuits
    - LC circuits
    - RLC circuits

PHYS 201 / Lecture 11
Mutual and self inductance; Inductance in circuits; RL, LC and RLC circuits

Professor Costas Andreopoulos[1,2], *FHEA*

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Lectures delivered at the University of Liverpool, 2021-22

December 15, 2021

We studied the most general case of Maxwell's equations:

Dynamic case in matter

Gauss's law	$\oint \vec{D} \cdot d\vec{S} = \int \rho_f d\tau = Q_f$	$\vec{\nabla} \cdot \vec{D} = \rho_f$
Faraday's law	$\oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S} \Rightarrow$ $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
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Lecture 10 - Revision (cont'd)

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$$B_1^{\perp} = B_2^{\perp} \quad \textit{and} \quad \frac{1}{\mu_1} B_1^{||} = \frac{1}{\mu_2} B_2^{||}$$

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Plan for Lecture 11

- Mutual and self inductance
- Inductance in circuits
    - RL circuits
    - LC circuits
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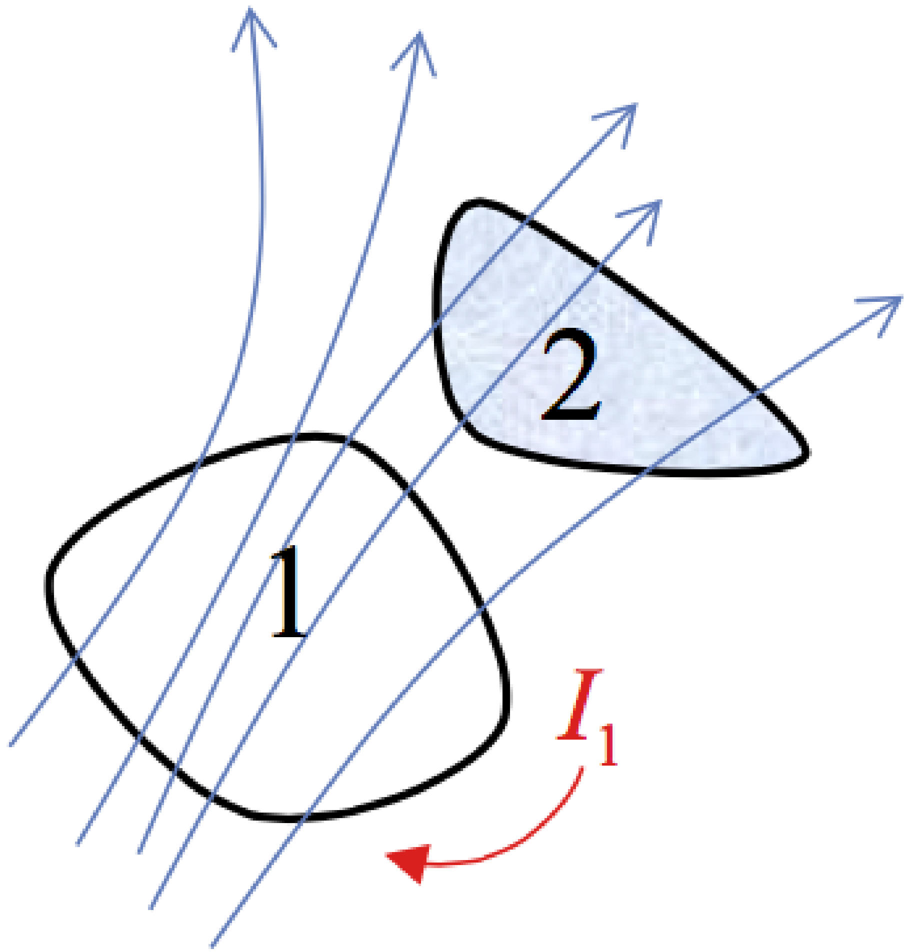
Mutual inductance

Consider two closed conductor loops (1 and 2) at rest, as shown.



- A steady current I_1 circulates in loop 1.
- The current I_1 generates a magnetic field B_1.
- Some of the flux of the magnetic field B_1 goes through the surface S_2

Mutual inductance

The magnetic field due to the current I_1 flowing in loop 1 is given by the Biot-Savart law:

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint_{L_1} \frac{d\vec{\ell_1} \times \vec{r}}{r^3}$$

The flux through the surface of loop 2, of the magnetic field $\vec{B}_1$ produced by loop 1 is:

$$\Phi_2 = \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2$$

Therefore:

$$\Phi_2 = \int_{S_2} \left\{ \frac{\mu_0}{4\pi} I_1 \oint_{L_1} \frac{d\vec{\ell_1} \times \vec{r}}{r^3} \right\} \cdot d\vec{S}_2$$

Mutual inductance

$$\Phi_2 = \int_{S_2} \left\{ \frac{\mu_0}{4\pi} I_1 \oint_{L_1} \frac{d\vec{\ell_1} \times \vec{r}}{r^3} \right\} \cdot d\vec{S_2} \Rightarrow$$

$$\Phi_2 = \left\{ \frac{\mu_0}{4\pi} \int_{S_2} \left\{ \oint_{L_1} \frac{d\vec{\ell_1} \times \vec{r}}{r^3} \right\} \cdot d\vec{S_2} \right\} I_1 \Rightarrow$$

$$\Phi_2 = M_{21} I_1$$

where

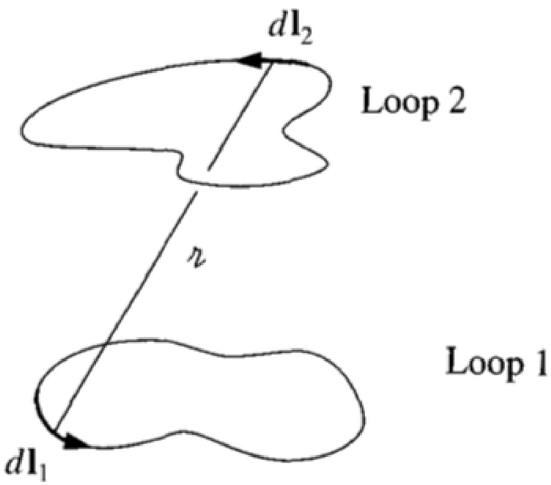
$$M_{21} = \frac{\mu_0}{4\pi} \int_{S_2} \left\{ \oint_{L_1} \frac{d\vec{\ell_1} \times \vec{r}}{r^3} \right\} \cdot d\vec{S_2}$$

The flux Φ_2 is proportional to I_1.

The constant of proportionality (M_{21}) is known as **mutual inductance**. M_{21} is a **purely geometrical** factor.

Mutual inductance

A simpler formula for M_{21} can be derived by expressing the magnetic field $\vec{B}$ in terms of its vector potential $\vec{A}$ and using Stokes's theorem.



$$M_{21} = \frac{\mu_0}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\vec{\ell_1} \cdot d\vec{\ell_2}}{r}$$

This is the **Neumann formula** (without proof): It involves a double line integral - one integration is around loop 1 and the other integration is around loop 2.

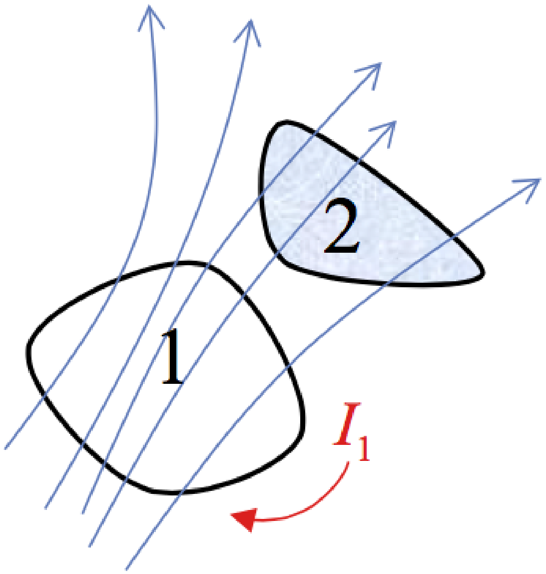
It reveals 2 important things about the mutual inductance:

- It is purely geometrical, as we have already discussed. (It has to do only with the shape, size and relative positions of the 2 loops.)
- It is **unchanged if one switches the roles of loop 1 and 2**
  - So $M_{21} = M_{12}$!

Mutual inductance

Mutual inductance unchanged if one switches the roles of loop 1 and 2.

This is a **remarkable conclusion**!



Whatever the shapes and positions of the loops, the flux through loop 2 when we run a current I around loop 1 is identical to the flux through loop 1 when we run the same current around loop 2.

$$\Phi_2 = M_{21} I_1 \xRightarrow{I_1 = I} \Phi_2 = M_{21} I$$

$$\Phi_1 = M_{12} I_2 \xRightarrow{I_2 = I;\, M_{12} = M_{21}} \Phi_1 = M_{12} I \Rightarrow \Phi_1 = \Phi_2$$

J.Henry, 1797-1878

From now on I will drop the indices: $M_{21} = M_{12} = M$
The SI unit of the mutual inductance is the Henry (H)

- Named in honour of Joseph Henry.
- $1 \text{ H} = \dfrac{Wb}{A} = \dfrac{V \cdot s}{A}$

Mutual inductance

Before, I_1 was a steady current, so the field $\vec{B}_1$ did not change with time. The conducting loops were fixed in their positions so Φ_2 was also constant.

Now, let's consider **what happens if I_1 varies**:

- time-varying current $\rightarrow$ time-varying magnetic field
- time-varying magnetic field $\rightarrow$ time-varying flux through loop 2.

Faraday's law tells us that this time-varying flux through loop 2 will generate an EMF in loop 2:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -\frac{d}{dt}\Big(MI_1\Big) = -M\frac{dI_1}{dt}$$

Varying the current in a loop, **causes an EMF** (and a current) **in a nearby loop**, although the two loops are not connected electrically.

- They are connected "magnetically"

Self inductance

Varying the current in loop 1, varies the magnetic field and thus the magnetic flux passing through loop 1 itself!

Again, we can write that:

$$\mathcal{E}_1 = -L\frac{dI_1}{dt}$$

where the constant of proportionality is called **self-inductance of the loop** (or simply inductance) and it is denoted with L (in honour of H.Lenz).

Self inductance / "Back EMF"

As we have seen, if you try to change the current in a loop, there will be an EMF developed in that same loop as a result of the current change.

$$\mathcal{E} = -L\frac{dI}{dt}$$

Remember Lenz's law: **Nature dislikes changes in the magnetic flux.**

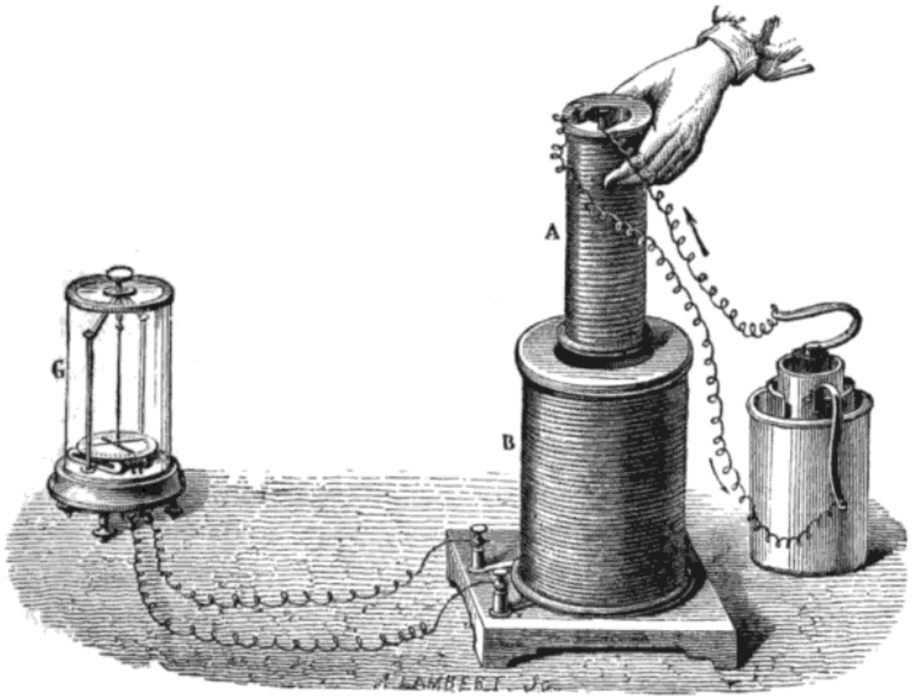
The EMF developed in the loop, as a result of the current change, will be such so as to **oppose the current change**.
For that reason, it is called **back EMF** (or **counter EMF**).

Inductance may be thought of as electromagnetic inertia.

- You can think that inductance plays in electromagnetism a role similar to that of mass in mechanics.
- In the same way that it is more difficult to change the velocity of a body that has a larger mass, it is more difficult to change the current in a circuit that has a larger inductance.

Faraday's experiment

Faraday's experiment showing induction between coils of wire.



The liquid battery (right) provides a current which flows through the small coil (A), creating a magnetic field. When the coils are stationary, no current is induced. But when the small coil is moved in or out of the large coil (B), the magnetic flux through the large coil changes, inducing a current which is detected by the galvanometer (G) [Wikipedia].

Inductance

We have seen the concepts of **mutual** and **self inductance**.

A change in current flow in a conductor induces a voltage (EMF)

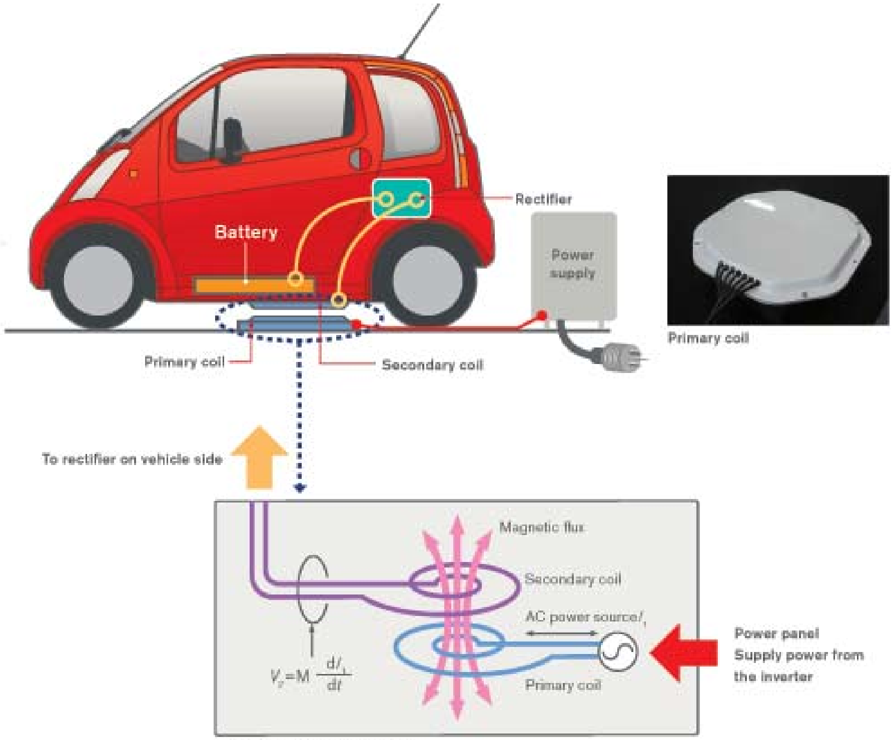
- in the same conductor (self-inductance):
  $$\mathcal{E} = -L\frac{dI}{dt}$$
- and in neighbouring conductors (mutual inductance):
  $$\mathcal{E}_{neighbouring\ loop} = -M\frac{dI}{dt}$$

In both cases the inductance (mutual or self) is the **constant of proportionality** between the EMF developed and the rate of current change.

Inductance, like capacitance, is an **intrinsically positive quantity**.

Inductance: An application from everyday life

Inductive wireless charging!



Inductance: An application from everyday life

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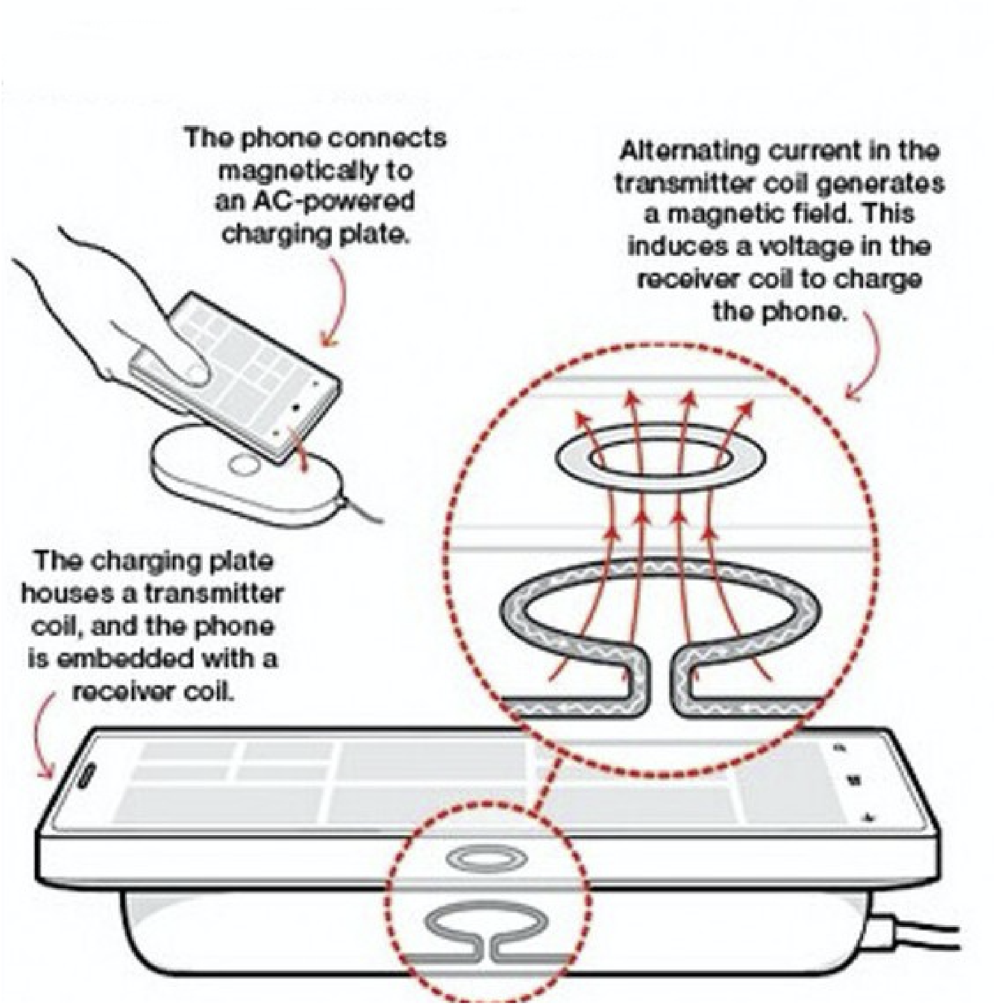


Inductors and inductance

An inductor is an electrical component which **resists changes in electric current passing through it**.

An inductor is **characterized by its inductance**

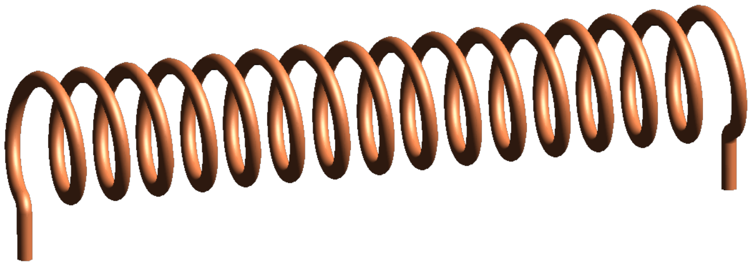
- Typical values in the μH to H range.

It typically consists of a conductor such as a wire wound into a coil.

Solenoids

A solenoid is a coil wound into a tightly packed helix, and whose length is substantially greater than its diameter.

- Often wrapped around a metallic core which produces a uniform magnetic field.



In a previous lecture we calculated the magnetic field produced by a solenoid within its volume:

$$B = \mu_0 \cdot n \cdot I$$

where n is the number of windings per unit length and I is the current flowing in the coil.

Inductance of solenoid

If a current I produces a magnetic flux Φ_B in a loop, the inductance is defined to be:

$$L = \frac{\Phi_B}{I}$$

If the inductor is a coil made of N loops:

$$L = \frac{N \cdot \Phi_B}{I}$$

All the windings are linked by the shared flux Φ_B. The product $N \cdot \Phi_B$ is called the **magnetic flux linkage**.

Inductance of solenoid

Consider a solenoid with n turns per unit length and area A.

The magnetic field is:

$$B = \mu_0 \cdot n \cdot I$$

Over length x (away from its ends) the magnetic flux linkage is:

$$N \cdot \Phi_B = \Big(n \cdot x\Big) \cdot \Big(B \cdot A\Big)$$

Hence:

$$L = \frac{N \cdot \Phi_B}{I} = \frac{\Big(n \cdot x\Big) \cdot \Big(B \cdot A\Big)}{I} = \frac{\Big(n \cdot x\Big) \cdot \Big((\mu_0 \cdot n \cdot I) \cdot A\Big)}{I} = \mu_0 \cdot n^2 \cdot x \cdot A$$

The inductance per unit length (far from the ends of the solenoid) is:

$$L = \mu_0 \cdot n^2 \cdot A$$

Worked example

Question

Two coils are at fixed locations. When coil 1 has no current and the current in coil 2 increases at the rate 15.0 A/s, the emf in coil 1 is 25.0 mV.

- What is their mutual inductance?
- When coil 2 has no current and coil 1 has a current of 3.60 A, what is the flux linkage in coil 2?

$$M = \frac{\mathcal{E}_1}{|dI_2/dt|} \Rightarrow M = \frac{25.0\ mV}{15.0\ A/s} = 1.67\ mH$$

$$N_2\Phi_{21} = MI_1 \Rightarrow N_2\Phi_{21} = (1.67\ mH)(3.60\ mWb) = 6.0\ mWb$$

Worked example

Question

A solenoid that is 85.0 cm long has a cross-sectional area of 17.0 cm^2. There are 950 turns of wire carrying a current of 6.60 A.

- Calculate the energy density of the magnetic field inside the solenoid.
- Find the total energy stored in the magnetic field there.

At any point in the solenoid the magnetic energy density is given by:

$$u_B = \frac{B^2}{2\mu_0}$$

where B is the magnitude of the magnetic field, given by:

$$B = \mu_0 n I$$

where n for the solenoid of this problem is:

$$n = (950\ turns)/(0.850\ m) = 1.118 \times 10^3 m^{-1}$$

Worked example

The magnetic energy density is:

$$u_B = \frac{(\mu_0 n I)^2}{2\mu_0} = \frac{1}{2}\mu_0 n^2 I^2 \Rightarrow$$

$$u_B = \frac{1}{2}(4\pi \times 10^{-7}\ T \cdot m/A)(1.118 \times 10^3\ m^{-1})^2(6.60\ A)^2 = 34.2\ J/m^3$$

Since the magnetic field is uniform inside an ideal solenoid, the total energy stored in the field is

$$U_B = u_B V$$

where V is the volume of the solenoid. V is calculated as the product of the cross-sectional area and the length:

$$V = (17.0 \times 10^{-4}\ m^2)(0.850\ m) = 1.445 \times 10^{-3}\ m^3$$

Thus:

$$U_B = (34.2\ J/m^3)(1.445 \times 10^{-3}\ m^3) = 4.942 \times 10^{-2}\ J$$

Worked example

Question

A 12 H inductor carries a current of 2.0 A. At what rate must the current be changed to produce a 60 V emf in the inductor?

$$\mathcal{E} = -L\frac{dI}{dt} \Rightarrow$$

$$\frac{dI}{dt} = -\frac{\mathcal{E}}{L} = -\frac{60\ V}{12\ H} = -5.0\ A/s$$

We might, for example, uniformly reduce the current from 2.0 A to zero in 40 ms.

Worked example

Question

Two coils are fixed in place in close proximity.
Coil 1 has self-inductance $L_1 = 25$ mH and $N_1 = 100$ turns.
Coil 2 has self-inductance $L_2 = 40$ mH and $N_2 = 200$ turns.
Their mutual inductance M is 3.0 mH. A 6.0 mA current in coil 1 is changing at the rate of 4.0 A/s.

- What magnetic flux Φ_{11} links coil 1?
- What self-induced EMF appears in coil 1?
- What magnetic flux Φ_{21} links coil 2?
- What mutually induced EMF appears in coil2?

Worked example

The flux in coil 1 is:

$$\frac{L_1 I_1}{N_1} = \frac{(25\ mH)(6.0\ mA)}{100} = 1.5 \mu Wb$$

The magnitude of the self-induced EMF in coil 1 is:

$$L_1 \frac{dI_1}{dt} = (25\ mH)(4.0\ A/s) = 1.0 \times 10^2 mV$$

The flux in coil 2 is:

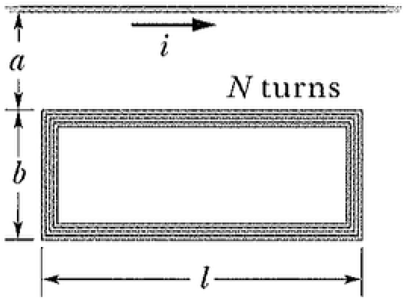
$$\frac{M I_1}{N_2} = \frac{(3\ mH)(6.0\ mA)}{200} = 90 nWb$$

The mutually induced EMF in coil 2 is:

$$\frac{M dI_1}{dt} = (3\ mH)(4.0\ A/s) = 12 mV$$

Worked example

Question



A rectangular loop of N closely packed turns is near a long straight wire as shown below. What is the mutual inductance M for the loop-wire system if N=100, a=1.0 cm, b = 8.0 cm, and ℓ = 30 cm?

The flux through the loop due to the field $\vec{B}$ of the current i is given by:

$$\Phi = \int_{S_{loop}} \vec{B} \cdot d\vec{S} = \int_a^b B\ell dr = \int_a^b \frac{\mu_0 I}{2\pi r}\ell dr = \frac{\mu_0 I \ell}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 i \ell}{2\pi} ln(1+\frac{b}{a})$$

Thus:

$$M = \frac{N\Phi}{i} \Rightarrow M = \frac{N\mu_0\ell}{2\pi} ln(1+\frac{b}{a}) \Rightarrow$$

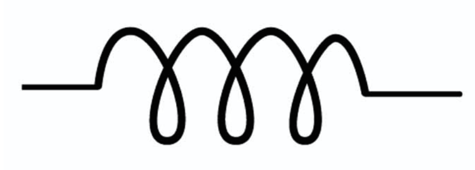
$$M = \frac{(100)(4\pi \times 10^{-7}\ H/m)(0.30\ m)}{2\pi} ln(1+\frac{8.0}{1.0}) = 1.3 \times 10^{-5}\ H$$

Inductance in a circuit

Now we will study the behaviour of electrical circuits which contain an element with inductance L and, in particular, we will study

- RL circuits,
- LC circuits and
- RLC circuits

An inductor is represented in circuit diagrams with the following symbol:

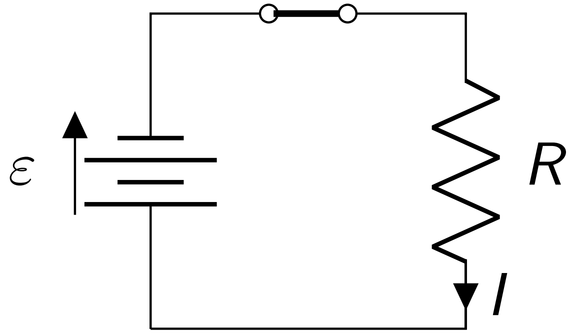


The thing to remember is that, if there is a change in the current flowing through the inductor, there will be a voltage developed across the conductor (back EMF) which is given by:

$$V_L = -L\frac{dI}{dt}$$

A simple circuit

First, let's examine a very simple DC circuit with an EMF and a resistor.



Kirchoff's voltage rule (the sum of EMFs in a closed loop equals the sum of potential drops):

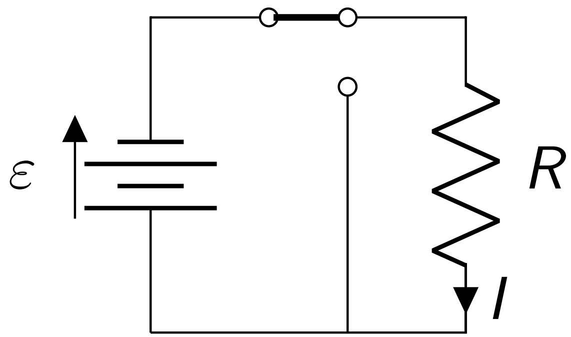
$$\sum_i \mathcal{E}_i = \sum_j I_j \cdot R_j$$

Applying Kirchoff's voltage rule for the above circuit we get:

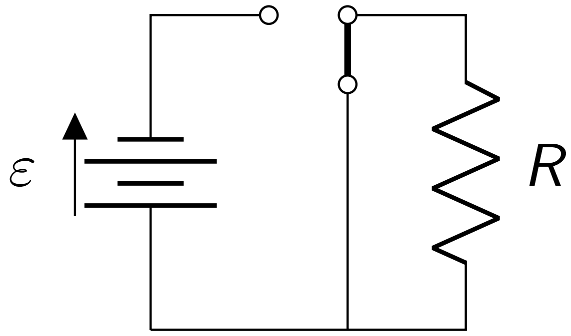
$$\mathcal{E} = I \cdot R \Rightarrow I = \frac{\mathcal{E}}{R}$$

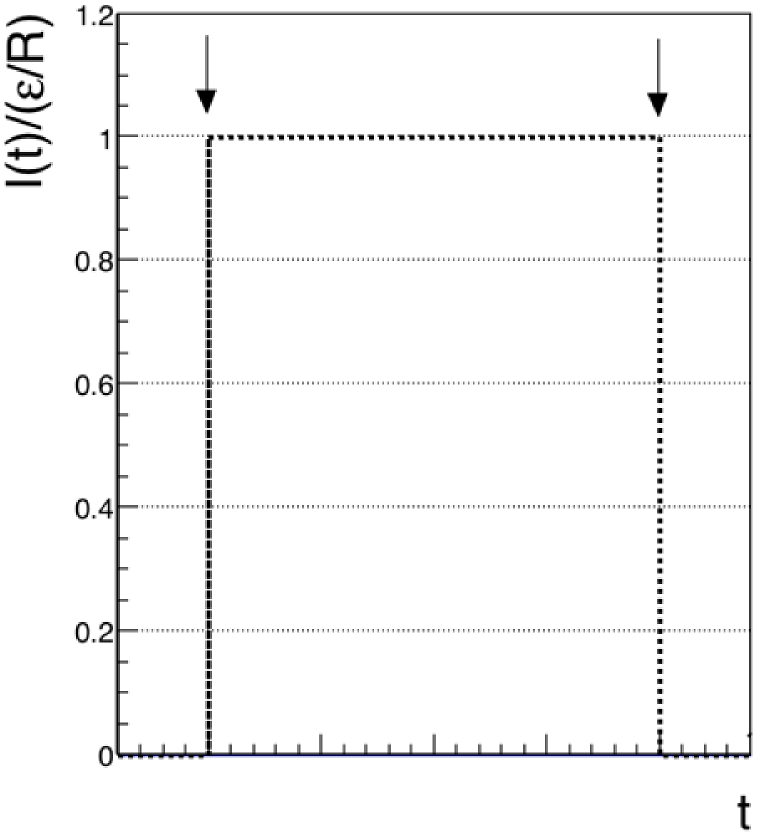
A simple circuit

Connecting the EMF: The current I takes **instantaneously** the value $\mathcal{E}/R$ and remains constant.



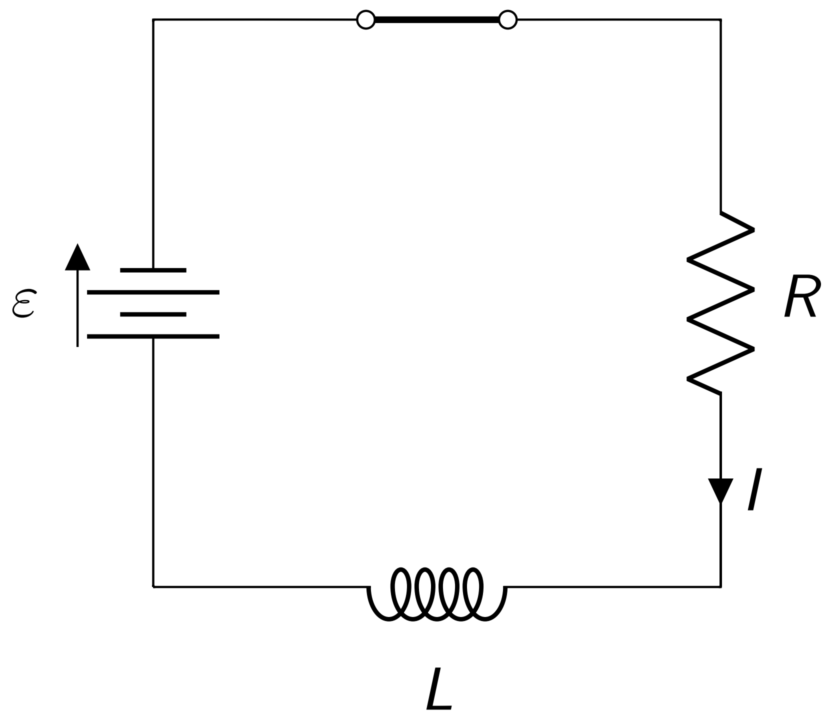
Disconnecting the EMF: the current stops (I=0) instananeously.





Adding inductance (The RL circuit)

What happens if we **add inductance in the circuit?** Does your physics intuition tells you anything about the behaviour of the RL circuit?

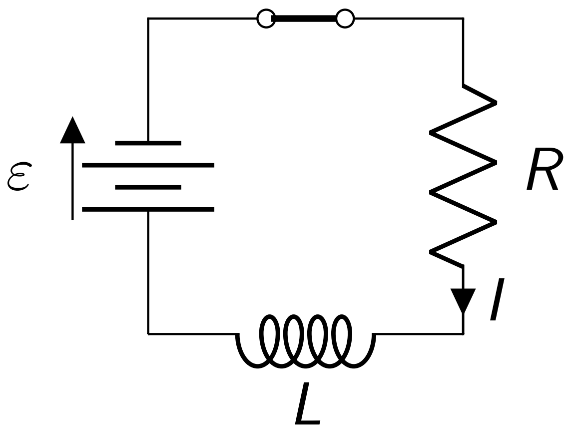


Inductance is a kind of inertia in the circuit.

So it is **no longer possible to just change the current instantaneously**.

RL circuit analysis

Let's study the RL circuit more quantitatively.



Using Kirchoff's voltage rule (the sum of EMFs in a closed loop equals the sum of potential drops):

$$\sum_i \mathcal{E}_i = \sum_j I_j \cdot R_j$$

This time I have to include the back-EMF $(-L\dfrac{dI}{dt})$, therefore:

$$\mathcal{E} - L \cdot \frac{dI}{dt} = I \cdot R$$

To figure out what the current I is as a function of time, we need to solve this first order differential equation above. Luckily, this is easy!

RL circuit analysis

$$\mathcal{E} - L \cdot \frac{dI}{dt} = I \cdot R \Rightarrow L \cdot \frac{dI}{dt} = \mathcal{E} - I \cdot R \Rightarrow L \cdot \frac{dI}{\mathcal{E} - I \cdot R} = dt \Rightarrow$$

$$L \cdot \int \frac{dI}{\mathcal{E} - I \cdot R} = \int dt \Rightarrow -\frac{L}{R} \cdot \int \frac{d\Big(\mathcal{E} - I \cdot R\Big)}{\mathcal{E} - I \cdot R} = \int dt \Rightarrow$$

$$-\frac{L}{R} \cdot ln\Big(\mathcal{E} - I \cdot R\Big) = t + const \Rightarrow ln\Big(\mathcal{E} - I \cdot R\Big) = -\frac{R}{L} t + const' \Rightarrow$$

$$\mathcal{E} - I \cdot R = const'' \cdot exp^{-\frac{R}{L} t} \Rightarrow$$

$$I(t) = \frac{\mathcal{E}}{R} + I_0 \cdot exp^{-\frac{R}{L} t}$$

where I_0 a constant defined by the initial conditions.

RL circuit analysis

We have the following solution to the $\mathcal{E} - L \cdot \frac{dI}{dt} = I \cdot R$ differential equation:

$$I(t) = \frac{\mathcal{E}}{R} + I_0 \cdot exp^{-\frac{R}{L}t}$$

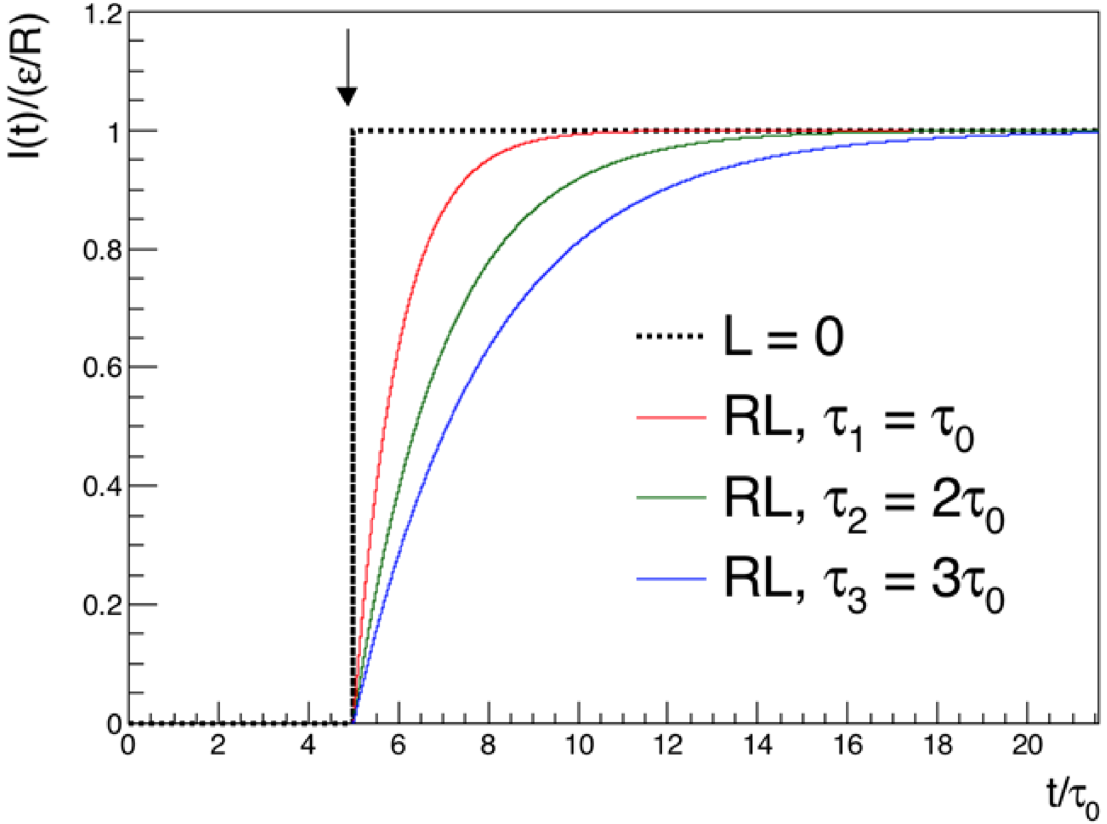
Initial condition:

$$I(t=0) = 0 \Rightarrow \frac{\mathcal{E}}{R} + I_0 = 0 \Rightarrow I_0 = -\frac{\mathcal{E}}{R}$$

Therefore, the solution to the above differential equation can be written as:

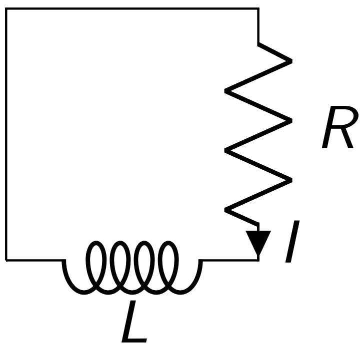
$$I(t) = \frac{\mathcal{E}}{R}\left(1 - exp^{-\frac{R}{L}t}\right) \Rightarrow I(t) = \frac{\mathcal{E}}{R}\left(1 - exp^{-\frac{t}{\tau}}\right)$$

where $\tau = L/R$ is a **characteristic time constant for the RL circuit**.



$$I(t) = \frac{\mathcal{E}}{R}\left(1 - exp^{-\frac{t}{\tau}}\right)$$

What happens if I disconnect the EMF?
The answer should be obvious by now. The current I will stop flowing, but the inertial effect of inductance will prevent that from happening instantaneously. We expect the current to drop exponentially towards zero.



Once again, I need to start from Kirchhoff's voltage rule. It gives me:

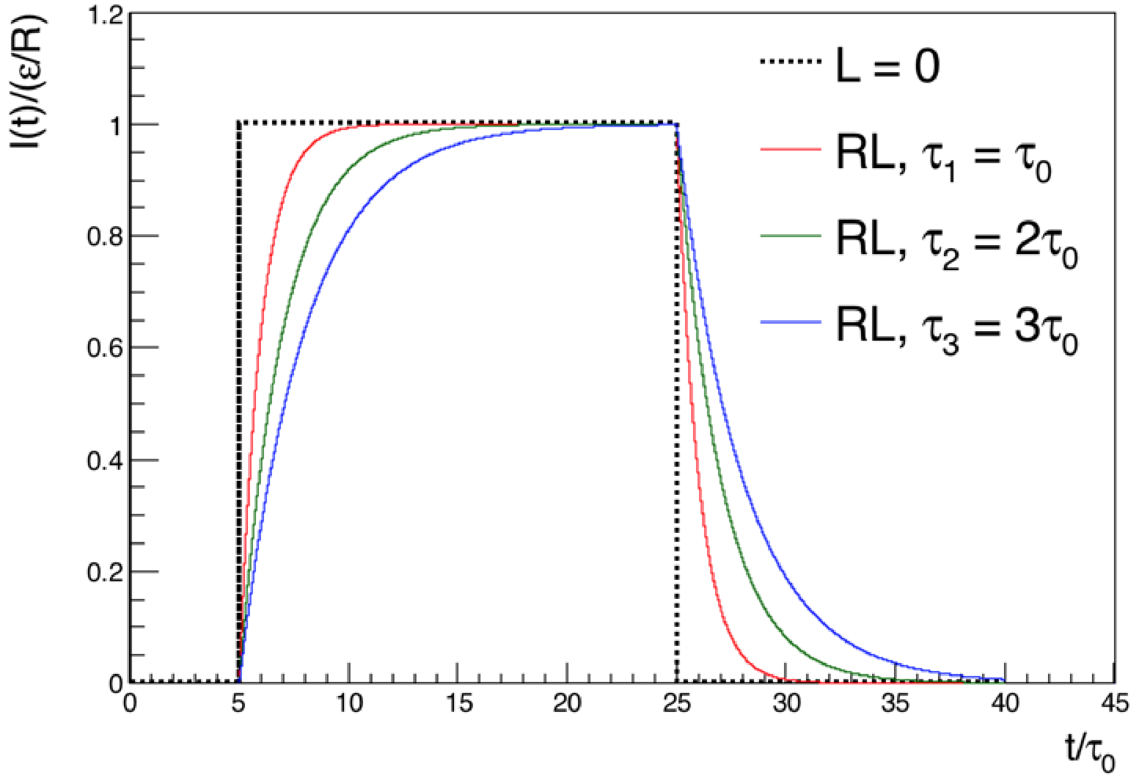
$$-L \cdot \frac{dI}{dt} = I \cdot R$$

Again, we need to solve the first order differential equation above to determine I(t). Using the ssame procedure as before, and applying the asymptotic condition $I(t \rightarrow \infty) = 0$, we get:

$$I(t) = \frac{\mathcal{E}}{R} \cdot exp^{-\frac{t}{\tau}}$$

where $\tau = L/R$ is a **characteristic time constant for the RL circuit**.

RL circuit analysis



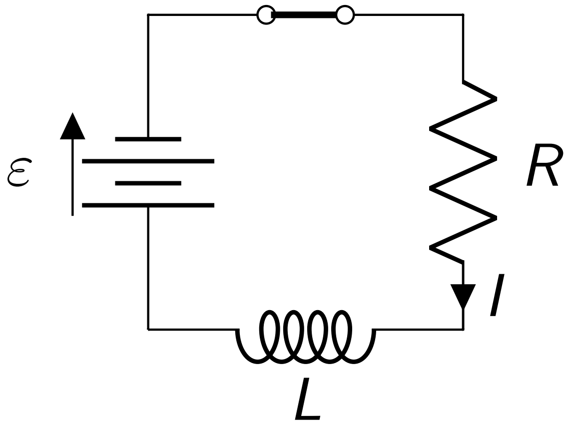
$$I(t) = \frac{\mathcal{E}}{R}\left(1 - exp^{-\frac{t}{\tau}}\right)$$

$$I(t) = \frac{\mathcal{E}}{R} \cdot exp^{-\frac{t}{\tau}}$$

Note that times are measured from the corresponding point of connecting or disconnecting the EMF.

Energy stored in the magnetic field of an inductor

Energy is stored in the magnetic field $\vec{B}$: $U_B = \dfrac{1}{2\mu_0} \displaystyle\int_\tau d\tau |\vec{B}|^2$.



The differential equation describing the RL circuit was:

$$\mathcal{E} = L \cdot \frac{dI}{dt} + I \cdot R$$

Multiplying each side by I we have:

$$\mathcal{E} \cdot I = L \cdot I \cdot \frac{dI}{dt} + I^2 \cdot R$$

As you know:

- $\mathcal{E} \cdot I$ is the rate at which energy is provided by the EMF source, and
- $I^2 \cdot R$ is the rate at which energy is dissipated in the resistor.

Energy stored in the magnetic field of an inductor

$$\mathcal{E} \cdot I = L \cdot I \cdot \frac{dI}{dt} + I^2 \cdot R$$

(rate at which energy is provided by the EMF source) =
? + (rate at which energy is dissipated in the resistor)

Therefore, the remaining term must describe the rate $\frac{dU_B}{dt}$ at which energy is stored in the magnetic field

$$\frac{dU_B}{dt} = L \cdot I \cdot \frac{dI}{dt} \Rightarrow \int dU_B = L \cdot \int I \cdot dI \Rightarrow U_B = \frac{1}{2} L \cdot I^2 + const$$

U_B=0 if I=0, hence:

$$U_B = \frac{1}{2} L \cdot I^2$$

Energy stored in the magnetic field of an inductor

A sanity check:

Consider the solenoid studied earlier in this lecture. Assume a solenoid with length x, cross-sectional area A and n turns per unit length.

The magnetic energy stored in the solenoid is:

$$U_B = \frac{1}{2} L I^2 \xRightarrow{\frac{L}{x} = \mu_0 n^2 A} U_B = \frac{1}{2} \mu_0 (Ax) n^2 I^2$$

The magnetic energy per unit volume is:

$$u_B = \frac{U_B}{Ax} = \frac{\frac{1}{2}\mu_0 (Ax) n^2 I^2}{Ax} = \frac{1}{2} \mu_0 n^2 I^2 \xRightarrow{B = \mu_0 n I} u_B = \frac{B^2}{2\mu_0}$$

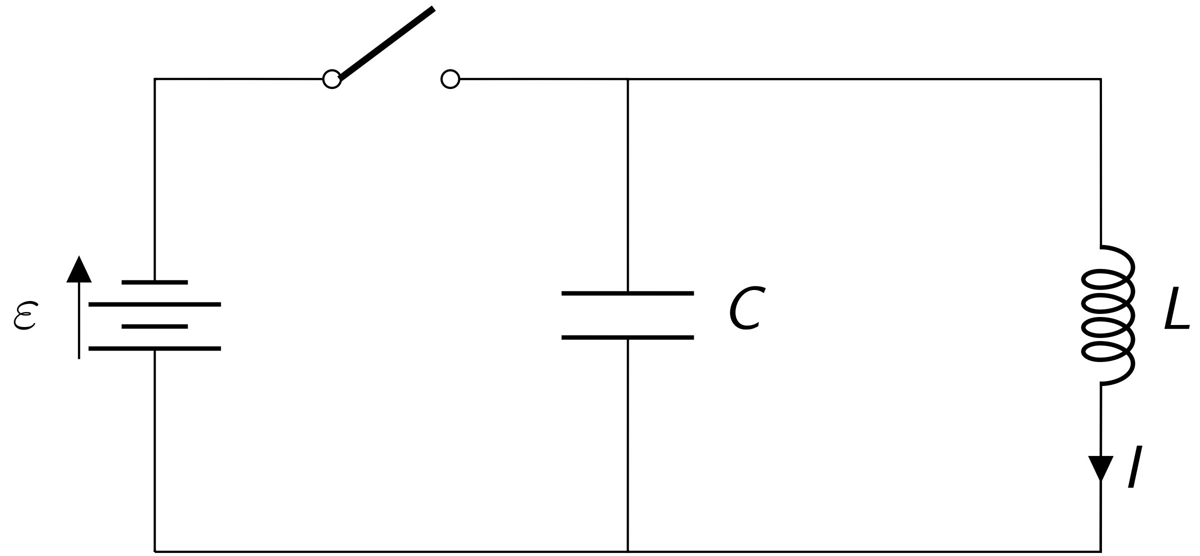
as expected!

LC electrical circuits

Now, we are going to examine the so-called **LC electrical circuits** that consist of an inductor L and a capacitor C.

- LC circuits are also called **resonant circuits** for reasons that will become clear in the next lecture.

Let's assume that at some initial time, the **capacitor is fully charged** and then it is left to discharge via the inductor.
Here is a simple representation of such a circuit.



LC electrical circuits

The **capacitor stores energy in the electric field** between its plates.

As we have seen the energy stored in a charged capacitor is:

$$U_E = \frac{1}{2} C V^2$$

The capacitor is connected to an inductor and it is discharged.

The current flowing will **build up a magnetic field in the inductor**.

The **inductor stores energy in its magnetic field**.

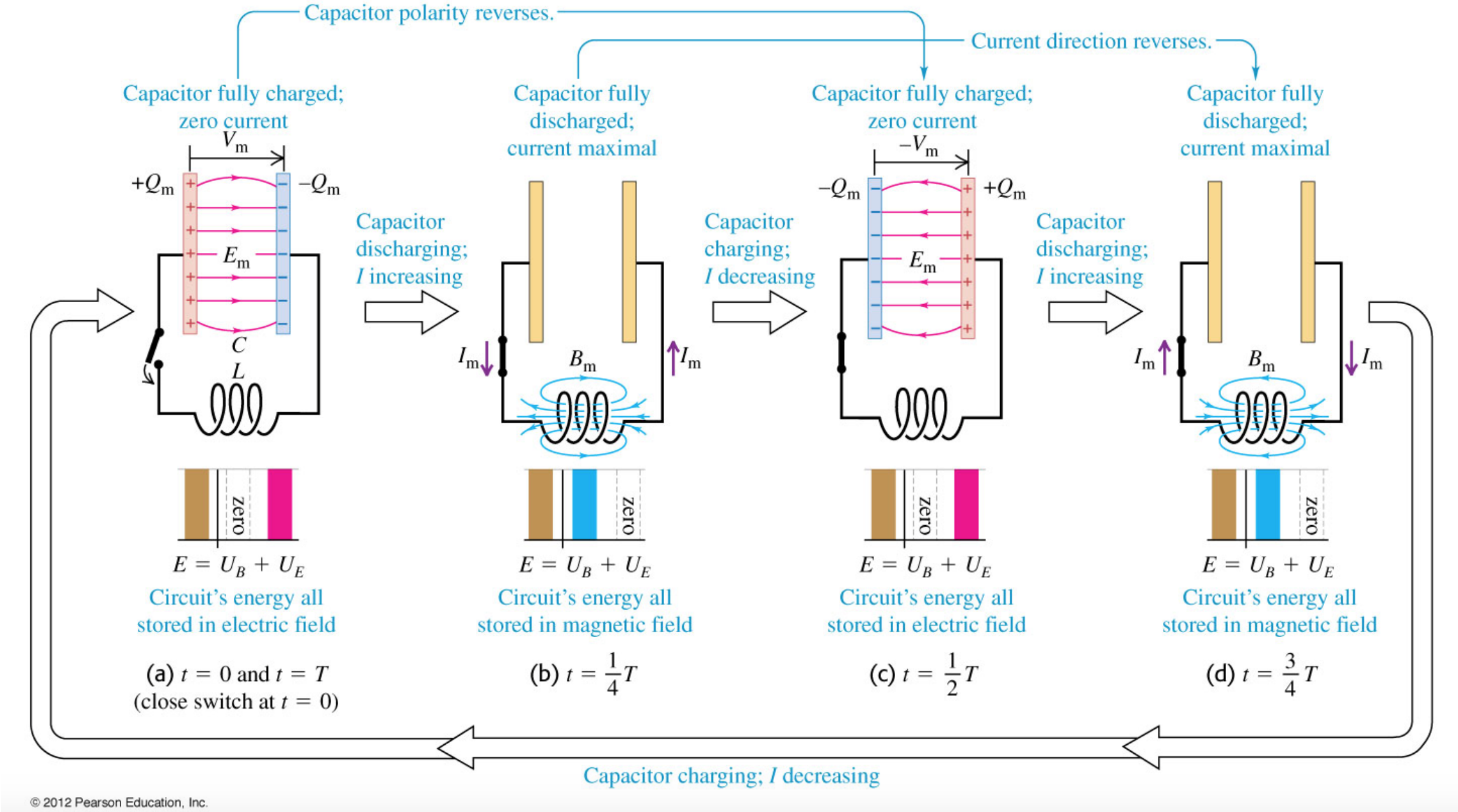
As we have seen the energy stored in an inductor is:

$$U_B = \frac{1}{2} L I^2$$

LC electrical circuits

- At some initial time, the **capacitor is fully charged** so **all energy is stored in the electric field**.
- As it is discharged, the current builds up the the magnetic field in the inductor so the **energy is transferred from the electric to magnetic field**.
- Once the **capacitor is fully discharged, all energy is stored in the magnetic field.**

- Even though the capacitor is now fully discharged the current will continue because inductors resist changes in the current.
- The current will start charging up the capacitor again, but with a voltage of opposite polarity.
- So energy starts moving back again from the magnetic field to the newly re-created electric field of the capacitor!


Capacitor polarity reverses.
Current direction reverses.
Capacitor fully charged; zero current
Capacitor fully discharged; current maximal
Capacitor fully charged; zero current
Capacitor fully discharged; current maximal
V_m
$+Q_m$
$-Q_m$
E_m
C
L
Capacitor discharging; I increasing
I_m
B_m
Capacitor charging; I decreasing
$-V_m$
$-Q_m$
$+Q_m$
E_m
Capacitor discharging; I increasing
I_m
B_m
zero
zero
zero
zero
$E = U_B + U_E$
Circuit's energy all stored in electric field
$E = U_B + U_E$
Circuit's energy all stored in magnetic field
$E = U_B + U_E$
Circuit's energy all stored in electric field
$E = U_B + U_E$
Circuit's energy all stored in magnetic field
(a) $t = 0$ and $t = T$ (close switch at $t = 0$)
(b) $t = \frac{1}{4}T$
(c) $t = \frac{1}{2}T$
(d) $t = \frac{3}{4}T$
Capacitor charging; I decreasing
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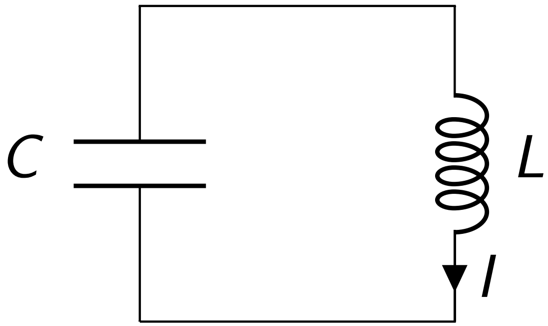
LC electrical circuit analysis

Let's do a more quantitative analysis of what we described above:



Using Kirchhoff's voltage rule: We take into account is the back-EMF because of the inductance and the voltage across the capacitor plates.

Therefore:

$$-L\frac{dI}{dt} = \frac{q}{C} \xRightarrow{I=\frac{dq}{dt}} -L\frac{d}{dt}\frac{dq}{dt} = \frac{q}{C} \Rightarrow$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

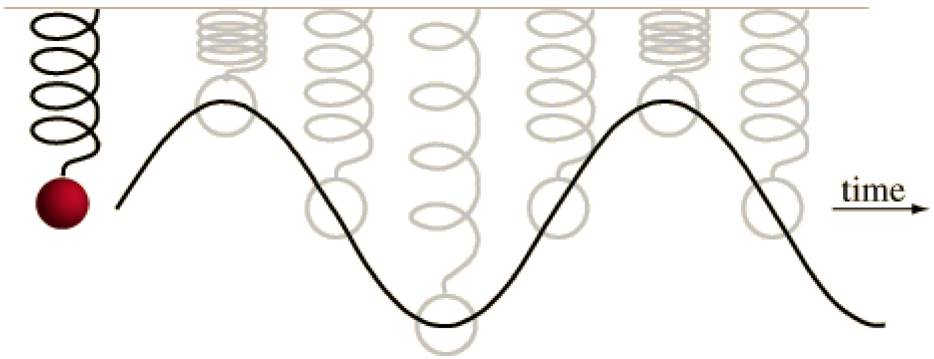
LC electrical circuit analysis

I am sure that you have seen a similar equation before as it is commonly derived in many different physical settings.

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

Do you recognize it?

This **second order differential equation** is similar to the equation for a simple harmonic oscillator, that is the equation for an oscillator that is neither driven nor damped.



The harmonic oscillator equation

For example, assume a mass m, which experiences a force F which pulls the mass towards x = 0.

$$F = -kx$$

Using Newton's 2^{nd} law, we have:

$$F = m\frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The motion of the mass m is periodic and it is described by the function:

$$x(t) = x_0 cos\Big(\omega t + \phi\Big)$$

where x_0 is the constant amplitude of the periodic motion, ω the angular frequency of the oscillation defined by

$$\omega = \sqrt{\frac{k}{m}}\Big(= \frac{2\pi}{T}\Big)$$

and ϕ is a phase determined by the initial conditions.

An LC circuit executes simple harmonic oscillations

Now contrast

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0 \qquad with \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

We can conclude that the **charge q(t) in an LC circuit oscillates harmonically**:

$$q(t) = q_0 cos\Big(\omega t + \phi\Big)$$

where q_0 is the constant amplitude of the oscillation (the max quantity of charge accumulated on the capacitor plates) and ω the angular frequency of the oscillation defined by

$$\omega = \frac{1}{\sqrt{LC}}$$

As before, ϕ is a phase determined by the initial conditions.

$$I(t) = \frac{dq(t)}{dt} = \frac{d}{dt}\Big\{q_0 cos\Big(\omega t + \phi\Big)\Big\} = -q_0 \omega sin\Big(\omega t + \phi\Big)$$

Energy oscillations in an LC circuit

The energy stored in the capacitor is:

$$U_E(t) = \frac{1}{2}CV(t)^2 = \frac{1}{2}C\left(\frac{q(t)}{C}\right)^2 = \frac{1}{2C}q(t)^2 \xrightarrow{q(t)=q_0 cos\left(\omega t+\phi\right)}$$

$$U_E(t) = \frac{1}{2C}\left\{q_0 cos\left(\omega t + \phi\right)\right\}^2 \Rightarrow U_E(t) = \frac{q_0^2}{2C}cos^2\left(\omega t + \phi\right)$$

The energy stored in the inductor is:

$$U_B(t) = \frac{1}{2}LI(t)^2 \xrightarrow{I(t)=-q_0\omega sin\left(\omega t+\phi\right)} U_B(t) = \frac{1}{2}L\left\{-q_0\omega sin\left(\omega t + \phi\right)\right\}^2$$

$$U_B(t) = \frac{1}{2}Lq_0^2\omega^2 sin^2\left(\omega t + \phi\right) \xrightarrow{\omega=\frac{1}{\sqrt{LC}}}$$

$$U_B(t) = \frac{1}{2}Lq_0^2\frac{1}{LC}sin^2\left(\omega t + \phi\right) \Rightarrow U_B(t) = \frac{q_0^2}{2C}sin^2\left(\omega t + \phi\right)$$

Energy oscillations in LC an circuit

So $U_E(t)$ and $U_B(t)$ oscillate!

$$U_E(t) = \frac{q_0^2}{2C} cos^2\left(\omega t + \phi\right)$$

$$U_B(t) = \frac{q_0^2}{2C} sin^2\left(\omega t + \phi\right)$$

Notice that:

- $U_E(t)$ and $U_B(t)$ have the same amplitude $\frac{q_0^2}{2C}$
  - The stored energy is fully transferred between the electric and magnetic fields (i.e. between the capacitor and the inductor).
- They have a phase difference of $pi/2$ ($cos^2 x \rightarrow sin^2 x$)
  - The energy stored in the electric field at some particular time t, is the same amount of energy stored in the magnetic field at $\mathrm{t}' = \mathrm{t} + \mathrm{T}/4$

Energy oscillations in LC an circuit

How about the total energy U(t)?

$$U(t) = U_E(t) + U_B(t) \Rightarrow$$

$$U(t) = \frac{q_0^2}{2C} cos^2\Big(\omega t + \phi\Big) + \frac{q_0^2}{2C} sin^2\Big(\omega t + \phi\Big) \Rightarrow$$

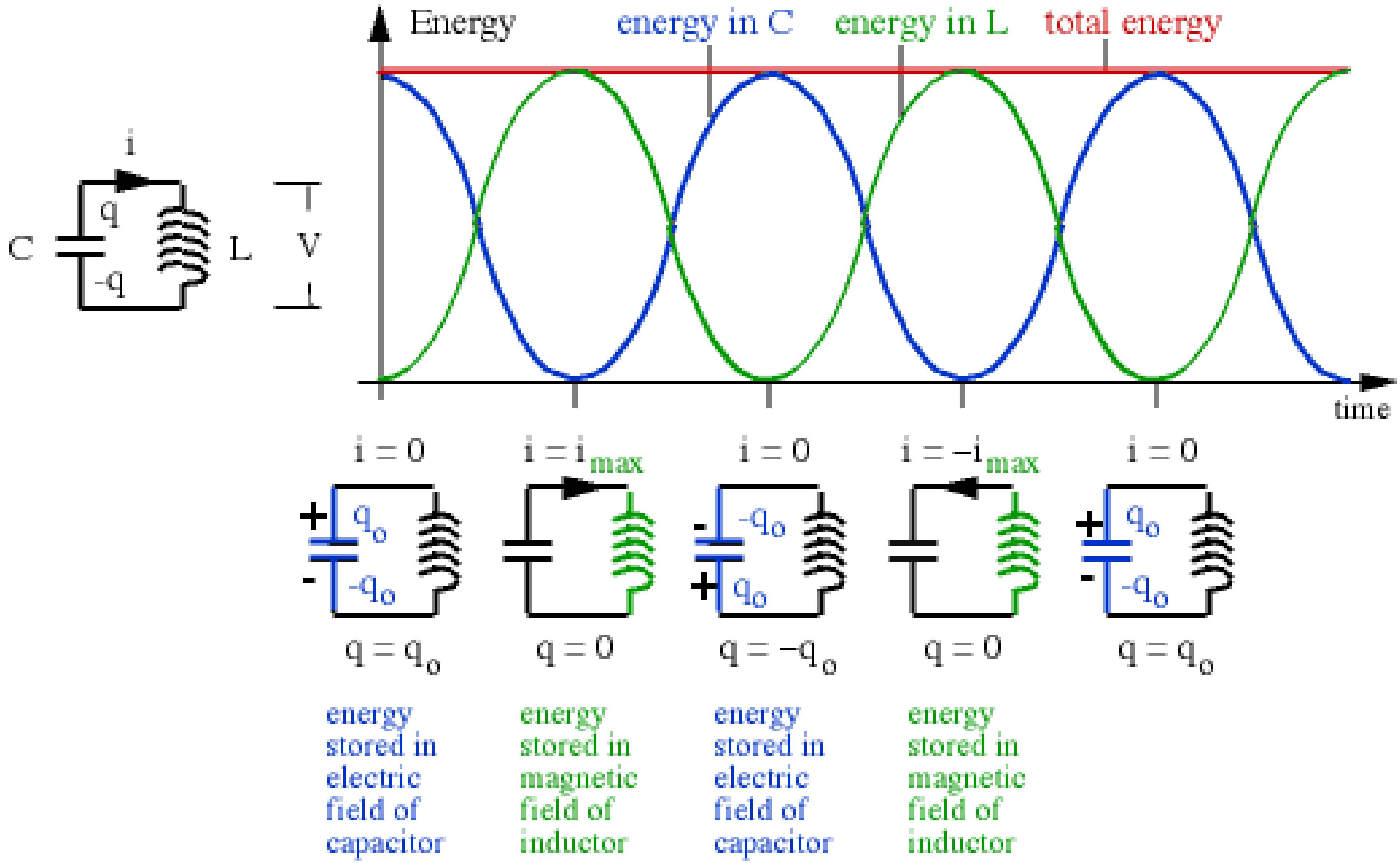
$$U(t) = \frac{q_0^2}{2C} \Big\{ cos^2\Big(\omega t + \phi\Big) + sin^2\Big(\omega t + \phi\Big) \Big\} \Rightarrow$$

$$U(t) = \frac{q_0^2}{2C}$$

The total energy stored is constant (as it should!)

- In the above system there was no mechanism for dissipating energy.

Energy oscillations in LC circuit

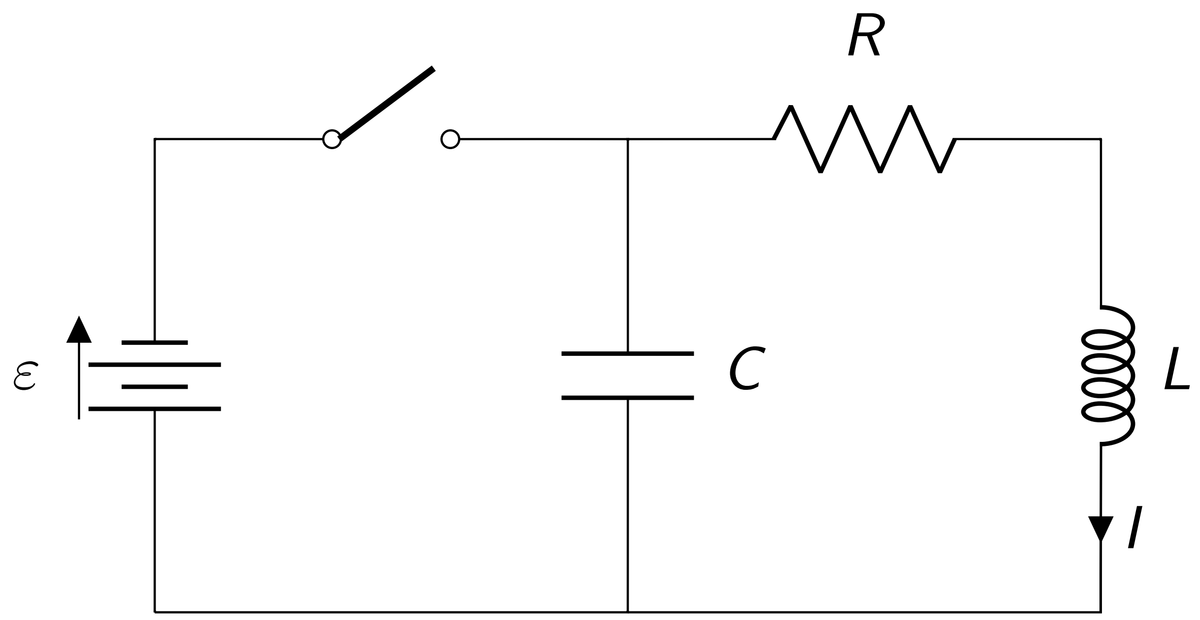


RLC electrical circuit

In the LC circuit, the stored energy was **fully transferred** between the electric and magnetic fields (i.e. between the capacitor and the inductor).

In absence of a mechanism for dissipating energy, the total energy stored in the LC circuit was constant.

Now, we will study a more realistic system: A system with capacitance (C), inductance (L) and resistance (R).



RLC electrical circuit

Based on what we discuss, we can guess the behaviour of the RLC circuit:

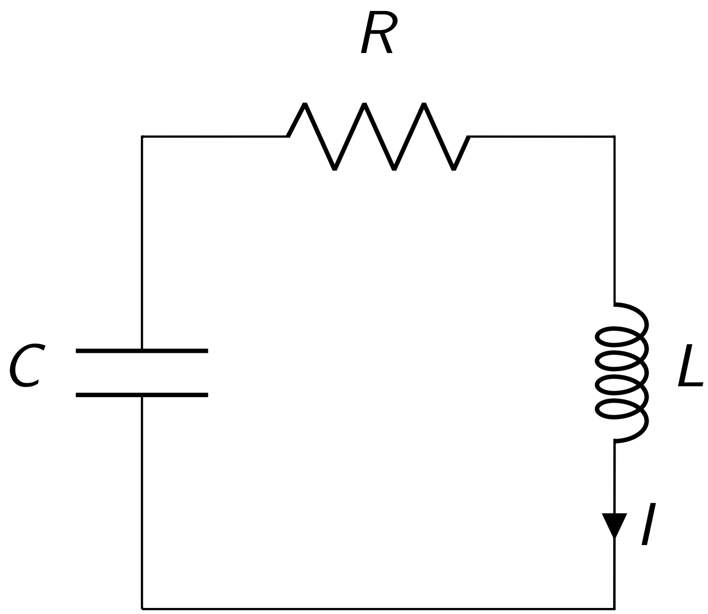
- As with the LC circuit, the energy will oscillated between the electric field of the capacitor and the magnetic field of the inductor.
- But we also have a resistor in the circuit, so the energy will not be fully transferred between the electric and magnetic field.
- Indeed, with every cycle, a fraction of the available stored energy would be lost (transformed to heat).

So, again, we will have energy oscillations.

But, in this case, the oscillations will be **damped**.

RLC circuit analysis

Let's do a more quantitative analysis of what we described above:



Using Kirchoff's voltage rule:

$$-L\frac{dI}{dt} = RI + \frac{q}{C} \xRightarrow{I=dq/dt}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

To obtain q=q(t) we have to solve the 2^{nd} order differential equation.

We will not do this here.

The differential equation:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

has the following solution:

$$q = q_0 e^{-\frac{R}{2L}t} cos\left(\omega' t + \phi\right)$$

where

$$\omega' = \sqrt{\omega^2 - \left(\frac{R}{2L}\right)^2} \quad and \quad \omega = \frac{1}{\sqrt{LC}}$$

You should confirm (on your own) that the above is indeed a solution of the differential equation.

RLC circuit analysis

Notice that the **frequency of oscillation is now different**:

$$\omega \rightarrow \omega' = \sqrt{\omega^2 - \left(\frac{R}{2L}\right)^2} \quad and \quad \omega = \frac{1}{\sqrt{LC}}$$

If R $= 0$, then $\omega' = \omega = \frac{1}{\sqrt{LC}}$ as expected.

But the most striking feature is that now the **amplitude is not constant, but it falls exponentially as time increases**.

$$q_0 \rightarrow q_0 e^{-\frac{R}{2L}t}$$

The oscillation is damped:

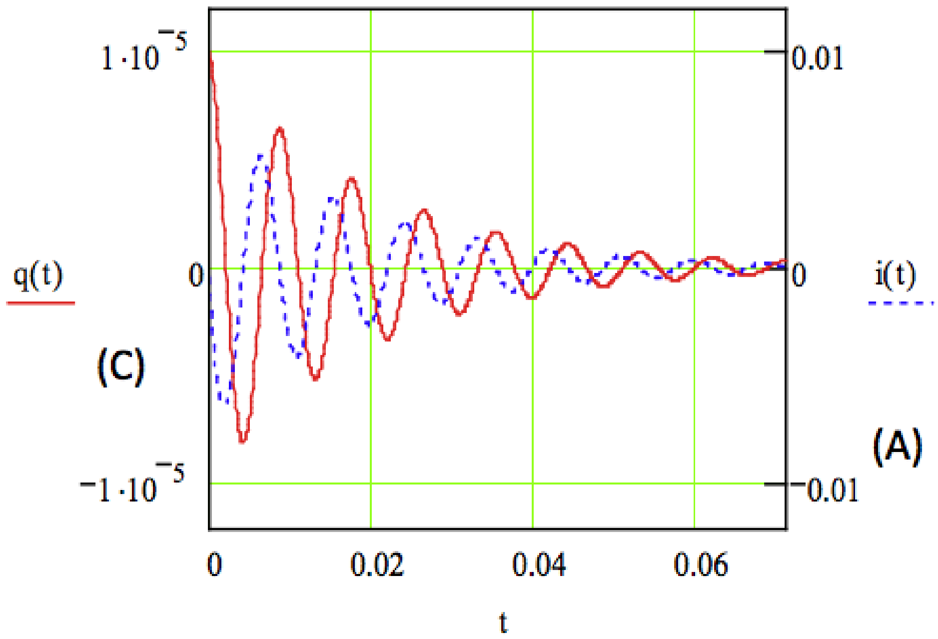
- Energy not fully oscillating between the electric and magnetic field.
- In every period, an amount of energy is converted to heat by the resistor and it is lost.

RLC circuit: Damped current oscillations

The current is given by:

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}\left\{q_0 e^{-\frac{R}{2L}t} cos\Big(\omega' t + \phi\Big)\right\} \Rightarrow$$

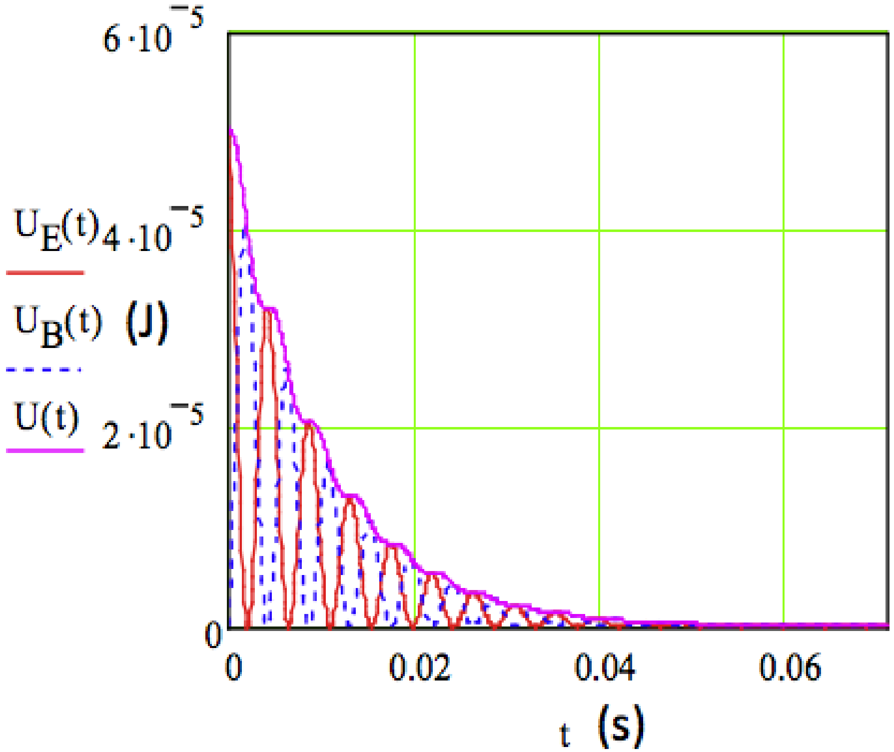
$$I(t) = -\frac{R}{2L} q_0 e^{-\frac{R}{2L}t} cos\Big(\omega' t + \phi\Big) - \omega' q_0 e^{-\frac{R}{2L}t} sin\Big(\omega' t + \phi\Big)$$



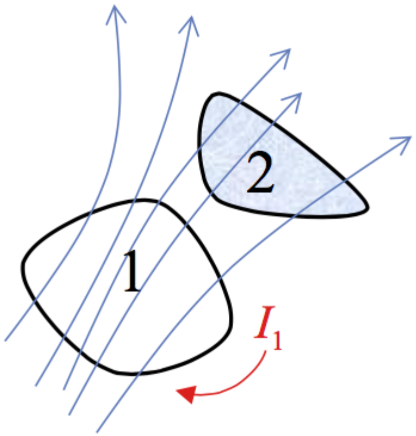
RLC circuit: Damped energy oscillations

$$U_E(t) = \frac{q^2}{2C} = \frac{q_0}{2C} e^{-\frac{R}{L}t} cos^2\left(\omega' t + \phi\right)$$

$$U_B(t) = \frac{1}{2}LI^2 = \frac{1}{2}L\left\{\frac{R}{2L} q_0 e^{-\frac{R}{2L}t} cos\left(\omega' t + \phi\right) + \omega' q_0 e^{-\frac{R}{2L}t} sin\left(\omega' t + \phi\right)\right\}^2$$



Lecture 11 - Main points to remember



The flux through the surface of loop 2, of the magnetic field $\vec{B}_1$ produced by the current I_1 in loop 1 is:

$$\Phi_2 = M_{21} I_1$$

The constant of proportionality (M_{21}) is known as **mutual inductance**. It is:

- purely geometrical, and
- unchanged if one switches the roles of loop 1 and 2.

So <u>whatever</u> the shapes and positions of the loops, the flux through loop 2 when we run a current I around loop 1 is identical to the flux through loop 1 when we run the same current around loop 2.

$$M_{21} = M_{12} = M$$

The SI unit of the mutual inductance is the Henry (H)

- A derived unit.
- $1 \text{ H} = \frac{Wb}{A} = \frac{V \cdot s}{A}$

Lecture 11 - Main points to remember (cont'd)

We also considered what happens if the current varies with time.

- time-varying current $\rightarrow$ time-varying magnetic field
- time-varying magnetic field $\rightarrow$ time-varying flux.
- time-varying flux $\rightarrow$ EMF (Faraday's law)

A change in current flow in a conductor induces a voltage (EMF)

- in the same conductor (self-inductance):
  $\mathcal{E} = -L\dfrac{dI}{dt}$
- and in neighbouring conductors (mutual inductance):
  $\mathcal{E}_{neighbouring\ loop} = -M\dfrac{dI}{dt}$

In both cases the inductance (mutual or self) is the **constant of proportionality** between the EMF developed and the rate of current change.

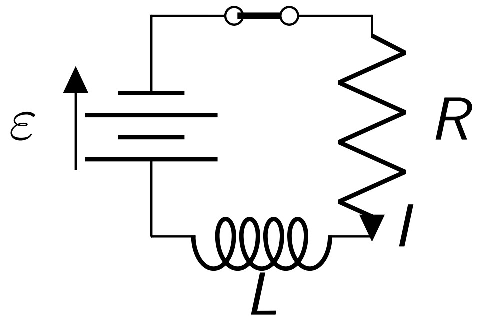
We also studied the solenoid and its inductance per unit length (far from the ends of the solenoid) is:

$$L = \mu_0 \cdot n^2 \cdot A$$

where A is the area of each winding, and n the number of turns per unit length.

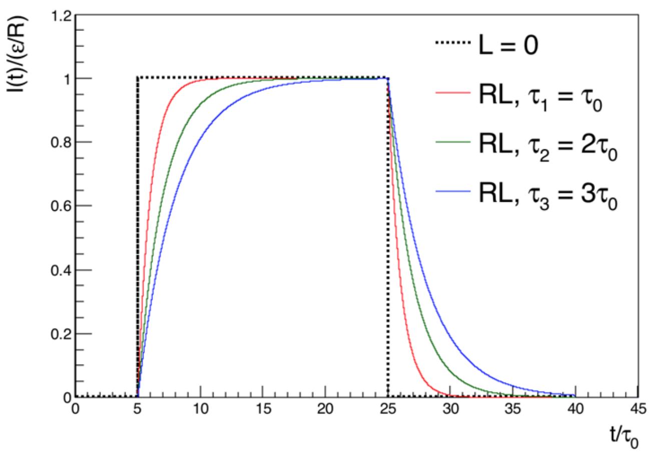
Lecture 11 - Main points to remember (cont'd)

Then we studied DC circuits with resistors, capacitors and inductors.



We studied an RL circuit and we saw that its behaviour is determined by the following differential equation:

$$\mathcal{E} - L \cdot \frac{dI}{dt} = I \cdot R$$



We solved that equation which gave as the following solutions for the current after connecting or disconnecting the EMF:

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - exp^{-\frac{t}{\tau}}\right) \quad and \quad I(t) = \frac{\mathcal{E}}{R} \cdot exp^{-\frac{t}{\tau}}$$

Note that times are measured from the corresponding point of connecting or disconnecting the EMF.

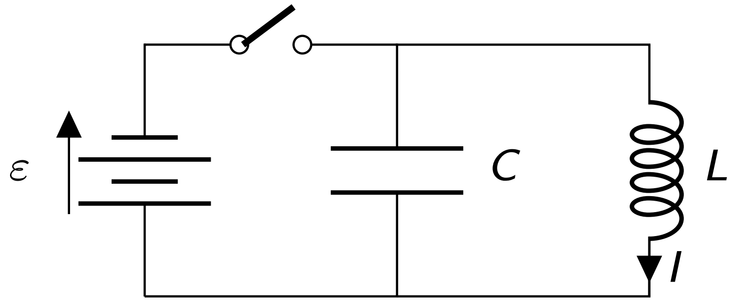
Inductance is a kind of inertia in the circuit.

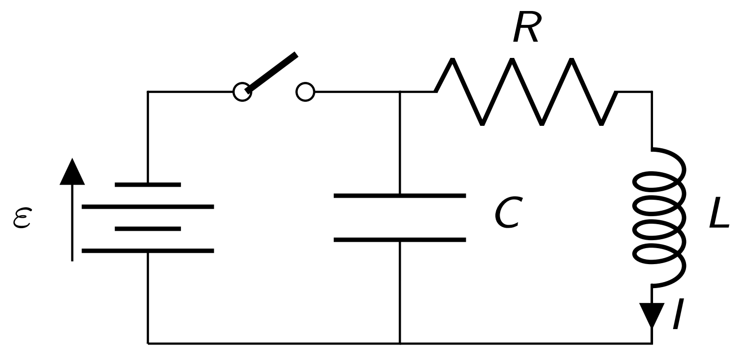
So it is **no longer possible to just change the current instantaneously** (as when L=0).

Lecture 11 - Main points to remember (cont'd)

We saw that the energy stored in the magnetic field of an inductor is: $U_B = \frac{1}{2}LI^2$.

Then we studied LC and RLC circuits both qualitatively and quantitatively:





RL and RLC are described by the following differential equation (with R=0 for LC):

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

- Which saw that for R=0 we have undamped oscillations of charge, current and voltage and that the stored energy is transferred fully between the capacitor (electric field) and the inductor (magnetic field).
- For R$\neq$0 we have damped oscillations as, on every iteration, a fraction of the available energy is converted to heat.

At the next lecture (Lecture 12)

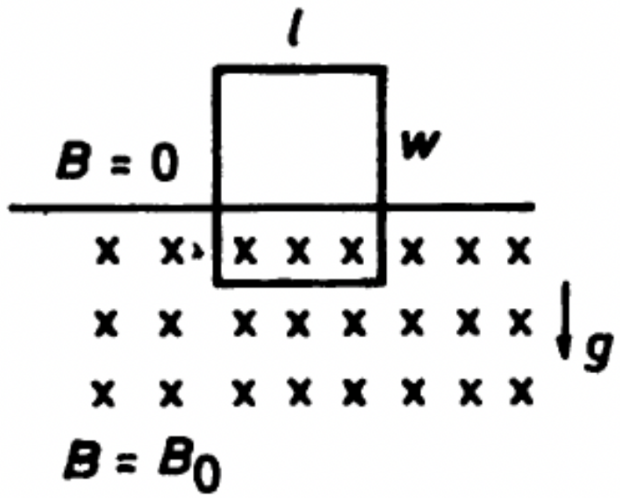
- Alternating Current (AC)
- AC circuits that include resistors, capacitors or inductors.
- Resonant (RLC) circuit

Optional reading for Lecture 11

Worked example: Wire loop falling in magnetic field

Question

A rectangular loop of wire with dimensions ℓ and w is released at $t = 0$ from rest, just above a region in which the magnetic field is $\vec{B}_0$, as shown in the figure below. $\vec{B}_0$ is perpendicular to the loop of wire.



The loop has resistance R, self-inductance L, and mass m. Consider the loop during the time that it has its upper edge in the zero-field region.

Worked example: Wire loop falling in magnetic field

Question (cont'd)

- Discuss the forces acting on the loop and write Newton's equation of motion for the loop.
- Find expressions for the electromotive force induced by the motion of the loop in the magnetic field, as well as for the voltage drops due to the resistance and self-inductance of the loop and relate them via Kirchhoff's voltage law.
- Assuming that you can ignore the self-inductance of the loop but not the resistance, find the current and velocity of the loop as functions of time.
- Assuming that you can ignore the resistance of the loop but not the self-inductance, find the current and velocity of the loop as functions of time.

Worked example: Wire loop falling in magnetic field

The loop has mass m and the gravitational force that will be exerted upon would be:

$$F_g = mg$$

At t=0, the horizontal (bottom) segment of the loop enters the magnetic field. At $t > 0$, the magnetic force exerted upon that segment is given by:

$$|\vec{F}_B| = |I \int_\ell d\vec{\ell} \times B| = B_0 \ell I$$

The magnetic force will point upwards and will oppose the gravitational force with points downwards.

Newton's law of motion for the loop is:

$$mg - B_0 \ell I = m\frac{du}{dt}$$

Worked example: Wire loop falling in magnetic field

The electromotive force (EMF) that will be developed along the horizontal (bottom) segment of the loop is:

$$\varepsilon = \left| \int_\ell (\vec{u} \times \vec{B}) \cdot d\vec{\ell} \right| = B_0 \ell u$$

The back-EMF due to the inductance L of the loop is:

$$\varepsilon_{back} = V_L = -L\frac{dI}{dt}$$

The voltage drop due to the resistance R of the loop is:

$$V_R = RI$$

Kirchoff's voltage law relates the directed sum of EMFs with the sum of voltage drops along the loop

$$\varepsilon + \varepsilon_{back} = V_R \Rightarrow B_0 \ell u - L\frac{dI}{dt} = RI$$

Worked example: Wire loop falling in magnetic field

Assuming that $L = 0$, Kirchoff's voltage law relates I and u as follows:

$$B_0 \ell u - \cancelto{0}{L} \frac{dI}{dt} = RI \Rightarrow B_0 \ell u = RI \Rightarrow I = \frac{B_0 \ell}{R} u$$

Substituting the above expression for I in Newton's law yields:

$$mg - B_0 \ell \Big(\frac{B_0 \ell}{R} u \Big) = m \frac{du}{dt} \Rightarrow \frac{du}{dt} = g - \frac{B_0^2 \ell^2}{mR} u \Rightarrow$$

$$\frac{d(u/g)}{dt} = 1 - \frac{B_0^2 \ell^2}{mR} (u/g)$$

By making the following substitutions:

$$C = \frac{B_0^2 \ell^2}{mR} \quad and \quad u' = u/g$$

the previous differential equation simplifies to:

$$\frac{du'}{dt} = 1 - Cu'$$

Worked example: Wire loop falling in magnetic field

Solving that differential equation:

$$\frac{du'}{1-Cu'} = dt \Rightarrow -\frac{1}{C}\frac{d(1-Cu')}{1-Cu'} = dt \Rightarrow$$

$$\int_{u'(t=0)=0}^{u'(t>0)} \frac{d(1-Cu')}{1-Cu'} = -C\int_{\tau=0}^{\tau=t} d\tau \Rightarrow$$

$$ln\Big(1-Cu'\Big)\Big|_{0}^{u'(t)} = -C\tau\Big|_{0}^{t} \Rightarrow$$

$$ln\Big(1-Cu'(t)\Big) - \cancelto{0}{ln(1)} = -Ct \Rightarrow$$

$$1 - Cu'(t) = e^{-Ct} \Rightarrow$$

$$u'(t) = \frac{1}{C}\Big(1-e^{-Ct}\Big)$$

Worked example: Wire loop falling in magnetic field

Replacing C and u' with their definitions, the above equation yields the sought after expression for u as a function of time:

$$u(t)/g = \frac{mR}{B_0^2 \ell^2} \left(1 - e^{-\frac{B_0^2 \ell^2}{mR} t} \right) \Rightarrow$$

$$u(t) = \frac{mgR}{B_0^2 \ell^2} \left(1 - e^{-\frac{B_0^2 \ell^2}{mR} t} \right)$$

Finally, substituting the above expression for $u(t)$, in the expression that resulted from Kirchoff's voltage law (relating I and u), we find:

$$I(t) = \frac{B_0 \ell}{R} \frac{mgR}{B_0^2 \ell^2} \left(1 - e^{-\frac{B_0^2 \ell^2}{mR} t} \right) \Rightarrow$$

$$I(t) = \frac{mg}{B_0 \ell} \left(1 - e^{-\frac{B_0^2 \ell^2}{mR} t} \right)$$

Worked example: Wire loop falling in magnetic field

Assuming that $R = 0$, Kirchoff's voltage law relates I and u as follows:

$$B_0 \ell u - L\frac{dI}{dt} = \cancelto{0}{R} I \Rightarrow B_0 \ell u = L\frac{dI}{dt} \Rightarrow \frac{dI}{dt} = \frac{B_0 \ell}{L} u$$

Differentiating with respect to time both sides of the previous expression of Newton's law of motion, yields:

$$-B_0 \ell \frac{dI}{dt} = m\frac{d^2 u}{dt^2}$$

Combining the two expressions above, we find:

$$-B_0 \ell \left(\frac{B_0 \ell}{L} u\right) = m\frac{d^2 u}{dt^2} \Rightarrow -\frac{B_0^2 \ell^2}{L} u = m\frac{d^2 u}{dt^2} \Rightarrow$$

$$\frac{d^2 u}{dt^2} + \frac{B_0^2 \ell^2}{mL} u = 0$$

Worked example: Wire loop falling in magnetic field

Making the following substitution:

$$\omega^2 = \frac{B_0^2 \ell^2}{mL}$$

the previous above differential equation becomes:

$$\frac{d^2 u}{dt^2} + \omega^2 u = 0$$

This is the well-known differential equation for the harmonic oscillator that has solutions of the form:

$$u(t) = a_1 cos(\omega t) + a_2 sin(\omega t)$$

where a_1 and a_2 are determined from the given boundary conditions:

$$u(t=0) = 0 \quad \text{and} \quad I(t=0) = 0$$

Worked example: Wire loop falling in magnetic field

The first of these boundary conditions yields:

$$u(t=0) = a_1 \cancelto{1}{cos(0)} + a_2 \cancelto{0}{sin(0)} = a_1 \Rightarrow a_1 = 0$$

Therefore, the general solution for $u(t)$ is reduced to:

$$u(t) = a_2 sin(\omega t)$$

From the expression of Newton's law of motion at $t = 0$, we find:

$$mg - B_0 \ell \cancelto{0}{I(0)} = m \frac{du}{dt}\bigg|_{t=0}$$

Substituting the above expression for $u(t)$, we have:

$$\cancel{m}g = \cancel{m}\frac{d}{dt}\Big(a_2 sin(\omega t)\Big)\bigg|_{t=0} \Rightarrow g = a_2 \omega cos(\omega t)\bigg|_{t=0} \Rightarrow a_2 = \frac{g}{\omega}$$

Worked example: Wire loop falling in magnetic field

So, finally, the required expression for $u(t)$ is:

$$u(t) = \frac{g}{\omega} sin(\omega t)$$

where ω is given by:

$$\omega = \frac{B_0 \ell}{\sqrt{mL}}$$

To find the required expression for $I(t)$, we substitute the above exression for $u(t)$ in Kirchoff's voltage law:

$$\frac{dI}{dt} = \frac{B_0 \ell}{L} \Big(\frac{g}{\omega} sin(\omega t) \Big) = \frac{B_0 \ell g}{L\omega} sin(\omega t)$$

Worked example: Wire loop falling in magnetic field

Solving this differential equation:

$$\int_{I(0)}^{I(t)} dI = \frac{B_0 \ell g}{L\omega} \int_0^t sin(\omega t')dt' \Rightarrow$$

$$I \Big|_{I(0)}^{I(t)} = -\frac{B_0 \ell g}{L\omega^2} cos(\omega t')\Big|_0^t \Rightarrow I(t) - \cancelto{0}{I(0)} = -\frac{B_0 \ell g}{L\omega^2}\Big(cos(\omega t) - 1\Big) \Rightarrow$$

we find:

$$I(t) = \frac{B_0 \ell g}{L\omega^2}\Big(1 - cos(\omega t)\Big) \xRightarrow{\omega^2 = \frac{B_0^2 \ell^2}{mL}}$$

$$I(t) = \frac{mg}{B_0 \ell}\Big(1 - cos(\omega t)\Big)$$

PHYS 201 / Lecture 12
Alternating Currents (AC); Resistor, Capacitor and Inductor in an AC circuit; The RLC series AC circuit

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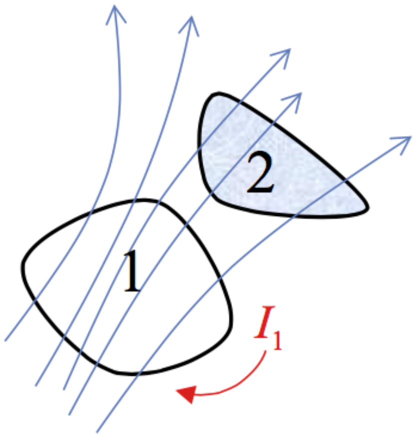


The flux through the surface of loop 2, of the magnetic field $\vec{B}_1$ produced by the current I_1 in loop 1 is:

$$\Phi_2 = M_{21} I_1$$

The constant of proportionality (M_{21}) is known as **mutual inductance**. It is:

- purely geometrical, and
- unchanged if one switches the roles of loop 1 and 2.

So whatever the shapes and positions of the loops, the flux through loop 2 when we run a current I around loop 1 is identical to the flux through loop 1 when we run the same current around loop 2.

$$M_{21} = M_{12} = M$$

The SI unit of the mutual inductance is the Henry (H)

- A derived unit.
- $1 \text{ H} = \dfrac{Wb}{A} = \dfrac{V \cdot s}{A}$

Lecture 11 - Revision (cont'd)

We also considered what happens if the current varies with time.

- time-varying current $\rightarrow$ time-varying magnetic field
- time-varying magnetic field $\rightarrow$ time-varying flux.
- time-varying flux $\rightarrow$ EMF (Faraday's law)

A change in current flow in a conductor induces a voltage (EMF)

- in the same conductor (self-inductance):
  $\mathcal{E} = -L\dfrac{dI}{dt}$
- and in neighbouring conductors (mutual inductance):
  $\mathcal{E}_{neighbouring\ \ loop} = -M\dfrac{dI}{dt}$

In both cases the inductance (mutual or self) is the **constant of proportionality** between the EMF developed and the rate of current change.

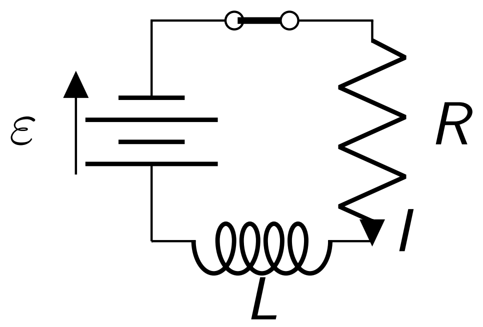
We also studied the solenoid and its inductance per unit length (far from the ends of the solenoid) is:

$$L = \mu_0 \cdot n^2 \cdot A$$

where A is the area of each winding, and n the number of turns per unit length.

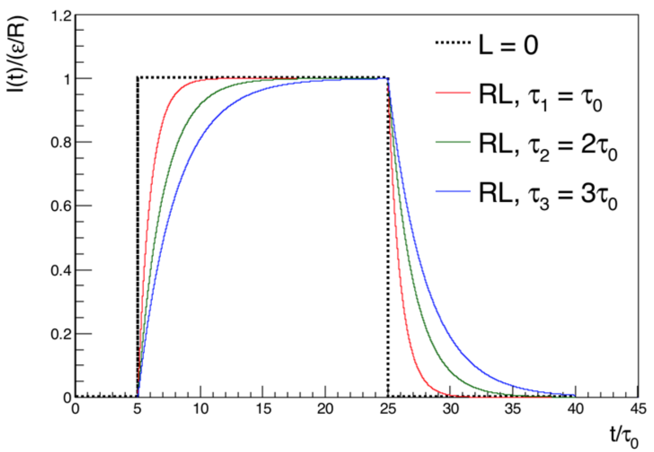
Lecture 11 - Revision (cont'd)

Then we studied DC circuits with resistors, capacitors and inductors.



We studied an RL circuit and we saw that its behaviour is determined by the following differential equation:

$$\mathcal{E} - L \cdot \frac{dI}{dt} = I \cdot R$$



We solved that equation which gave as the following solutions for the current after connecting or disconnecting the EMF:

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - exp^{-\frac{t}{\tau}}\right) \quad and \quad I(t) = \frac{\mathcal{E}}{R} \cdot exp^{-\frac{t}{\tau}}$$

Note that times are measured from the corresponding point of connecting or disconnecting the EMF.

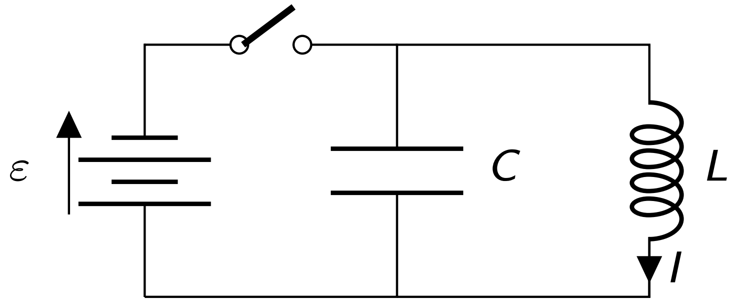
Inductance is a kind of inertia in the circuit.

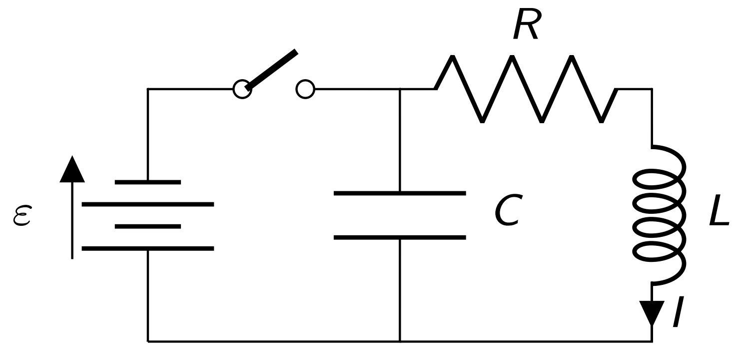
So it is **no longer possible to just change the current instantaneously** (as when L=0).

Lecture 11 - Revision (cont'd)

We saw that the energy stored in the magnetic field of an inductor is: $U_B = \frac{1}{2}LI^2$.

Then we studied LC and RLC circuits both qualitatively and quantitatively:





RL and RLC are described by the following differential equation (with R=0 for LC):

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

- Which saw that for R=0 we have undamped oscillations of charge, current and voltage and that the stored energy is transferred fully between the capacitor (electric field) and the inductor (magnetic field).
- For R$\neq$0 we have damped oscillations as, on every iteration, a fraction of the available energy is converted to heat.

Plan for Lecture 12

- Alternating Currents (AC)
- AC generation: Simple alternator
- Forced (or driven) oscillations
  - Resistor (R) in an AC circuit
  - Capacitor (C) in an AC circuit
  - Inductor (L) in an AC circuit
  - The RLC series AC circuit

Alternating Currents (AC)

So far we studied **Direct Currents (DC)** meaning

- voltage that maintains constant polarity over time, or
- current that maintains constant direction over time.

DC is the kind of electricity made by a solar cell or a battery.

Other sources of electricity (most notably rotary electro-mechanical generators) produce voltages that **alternate their polarity** over time.

Either as a voltage switching polarity or as a current switching direction, this kind of electricity is known as **Alternating Current (AC)**.

AC is the kind of electricity that comes to your home.
Typically in Europe we get a voltage of 220-240 V which alternates its polarity 50 times per second.

Alternating Currents (AC)

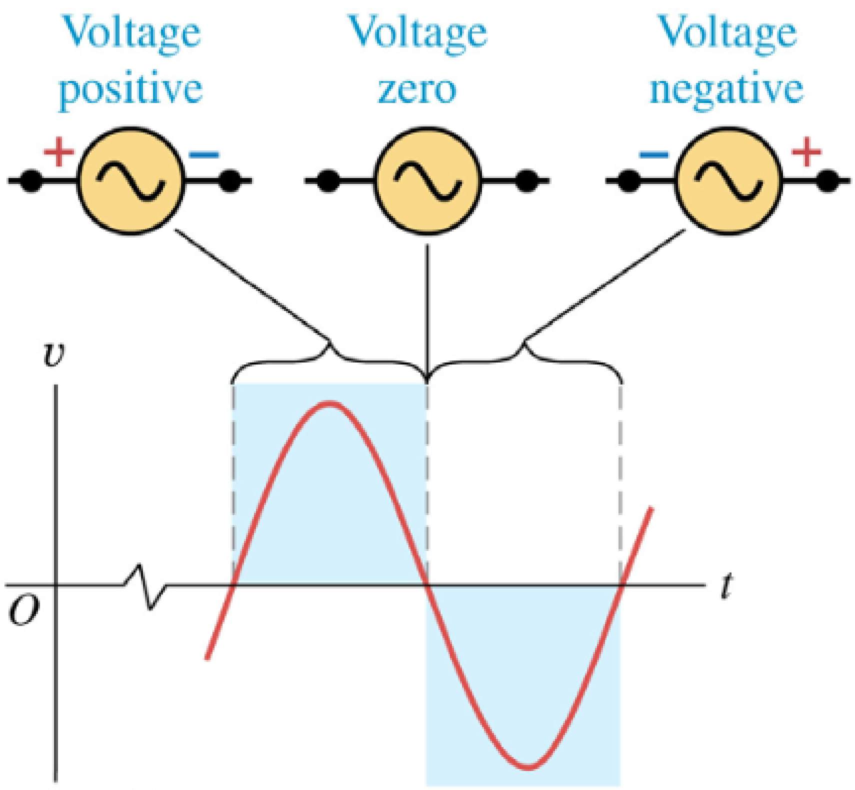
- In some cases AC holds no practical advantage over DC.
  - For example, in applications where electricity is used to dissipate energy in the form of heat, the polarity or direction of current is irrelevant.

- However, with AC it is possible to build **long-distance power distribution systems** that are far **more efficient than DC**
  - AC is used predominately across the world in high power applications.

Read the "*War of Currents*" article in Wikipedia.

Alternating Currents (AC)

Let $\mathcal{E} = \mathcal{E}_0 sin(\omega t)$ be an AC voltage, where $\mathcal{E}_0$ is its magnitude and, ω is the angular frequency of the alternating voltage source.



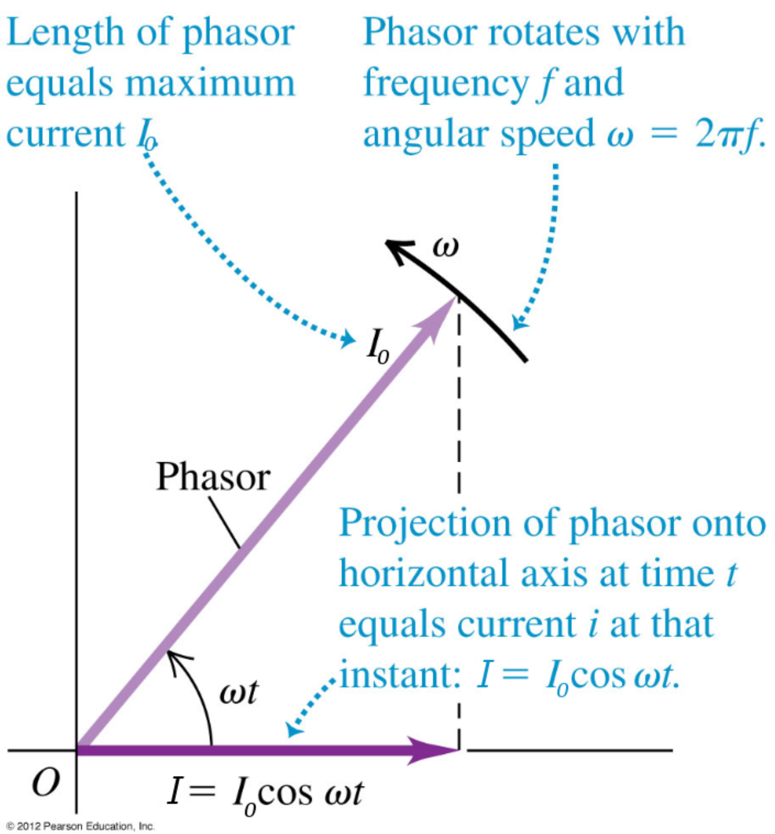
This voltage will induce a current which, generally, is given by:

$$I(t) = I_0 sin(\omega t + \phi)$$

where ϕ is the phase difference between the AC voltage and AC current.

We will see that in an AC circuit that includes only resistors, the voltage and current have no phase difference ($\phi = 0$) but in circuits with capacitors and inductors ϕ is generally not 0.

Phasor diagrams



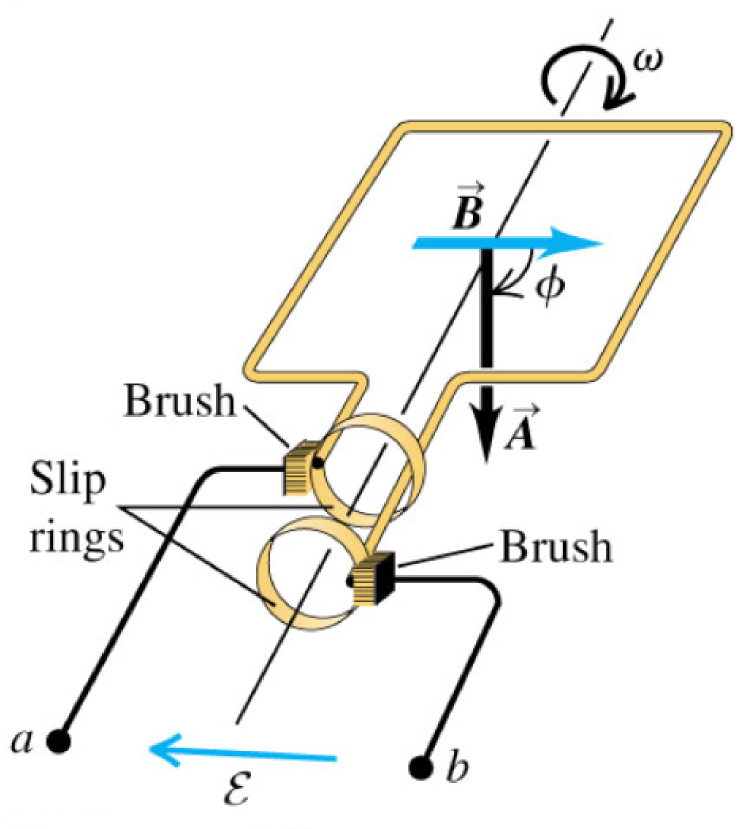
It is convenient to represent the AC voltage V(t) and the AC current I(t) using *"phasors"*. A phasor is a vector taken to be rotating counter-clockwisely with constant angular frequency ω.

- The length of the phasor represents the magnitude of the quantity represented.
- The angle of the phasor represents its phase. The usual reference for zero phase is taken to be the positive x-axis.

Assume that the quantity represented is an AC current I: At time t, the phase is ωt, and the projection of the phasor on the horizontal x axis is $I_0 cos(\omega t)$, the value of I at time t (I(t)).

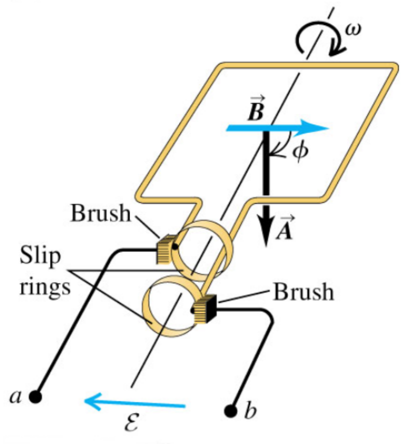
AC generation: Simple alternator

An **alternator** is an electrical generator that converts mechanical energy to electrical energy in the form of alternating current.



- A device that **uses Faraday's law**: A conductor moving relative to a magnetic field develops an electromotive force (EMF) in it.
- Typically, a **rotating magnet (called the rotor)** turns within a **stationary set of conductors wound in coils (called the stator)**.
  - the $\vec{B}$ field vector and the surface vector $\vec{S}$ rotate wrt each other.

AC generation: Simple alternator



Let's call ω the angular frequency of that rotation. Assume that at t=0, the $\vec{B}$ and $\vec{S}$ vectors point towards the same direction. After time t, the opening angle ϕ between $\vec{B}$ and $\vec{S}$ will be:

$$\phi = \omega t$$

The magnetic flux through the conductor is:

$$\Phi_B = \int_S \vec{B} \cdot d\vec{S} = \vec{B} \cdot \vec{S} = BScos\phi = BScos(\omega t) \xRightarrow{\Phi_{B0}=BS} \Phi_B = \Phi_{B0} cos(\omega t)$$

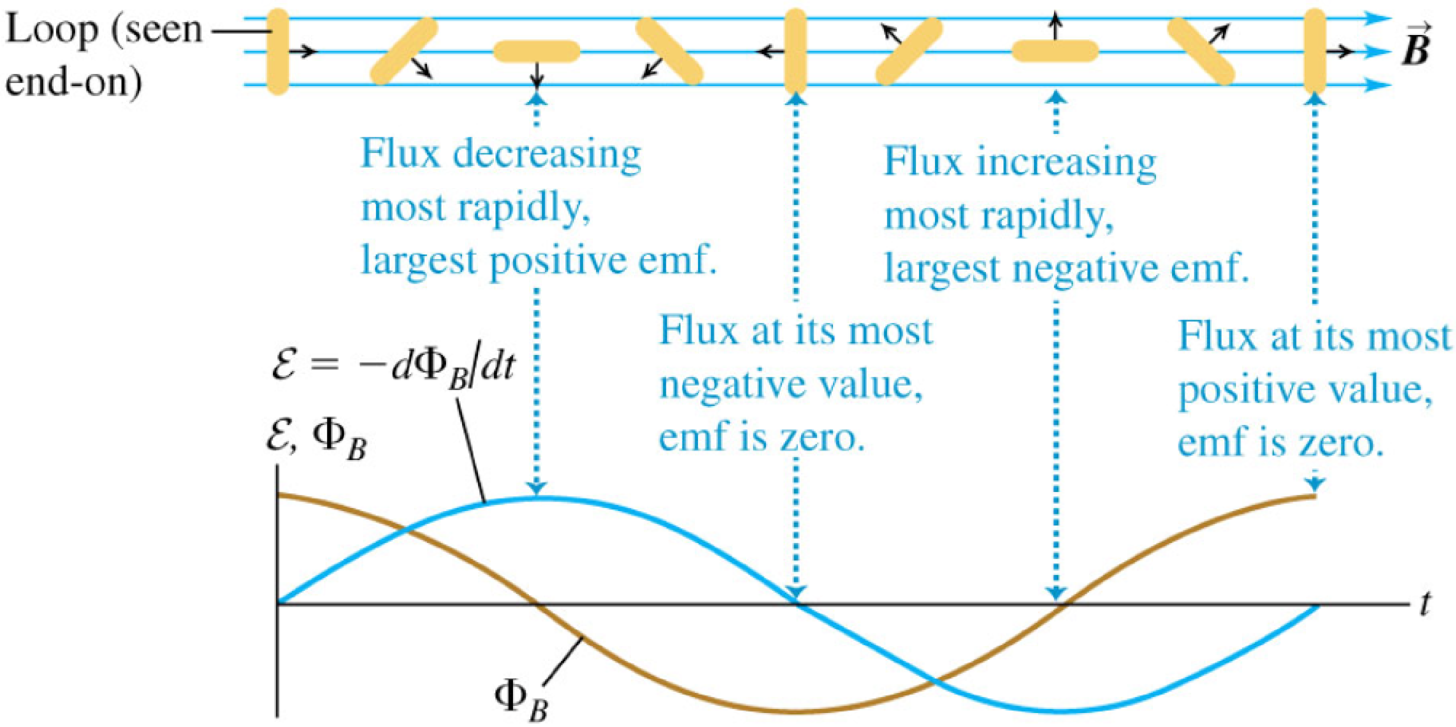
The rate of change of the flux gives us the EMF:

$$\mathcal{E} = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}\Big(BScos(\omega t)\Big) = \omega BSsin(\omega t) \xRightarrow{\mathcal{E}_0=\omega BS} \mathcal{E} = \mathcal{E}_0 sin(\omega t)$$

AC generation: Simple alternator

The rotating $\vec{B}$ field **induces an AC voltage** in the stator windings:

$$\Phi_B = \Phi_{B0} cos(\omega t) \quad \rightarrow \quad \mathcal{E} = \mathcal{E}_0 sin(\omega t)$$



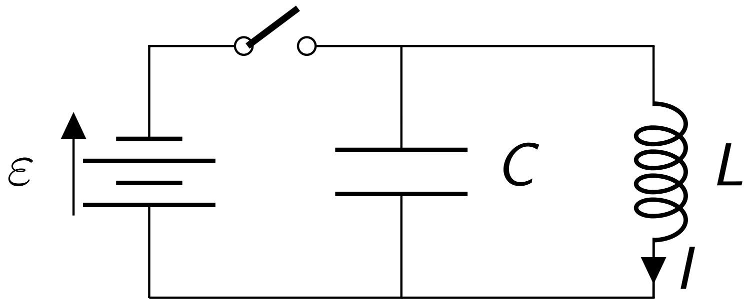
Notice that there is a phase of $\pi/2$ between the flux and the voltage:

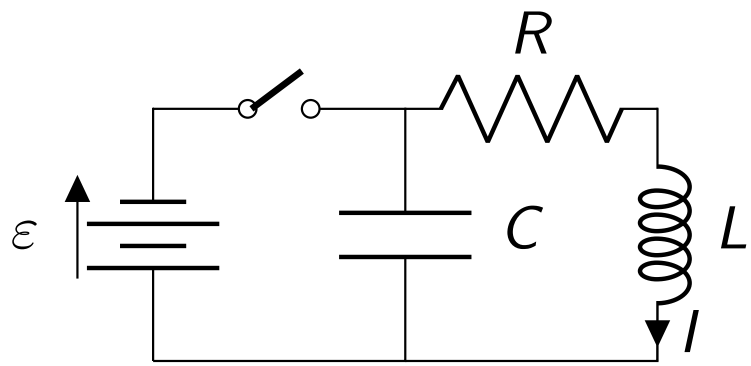
- $cos(\omega t) = sin(\omega t + \frac{\pi}{2}) \rightarrow$ The flux leads

Forced (or driven) oscillations

Before, we studied **undamped LC** and **damped RLC** circuits.

- They oscillated at angular frequency $\omega_0 = \dfrac{1}{\sqrt{LC}}$
- Such oscillations are **free oscillations**
- ω_0 is the circuit's **natural angular frequency**.





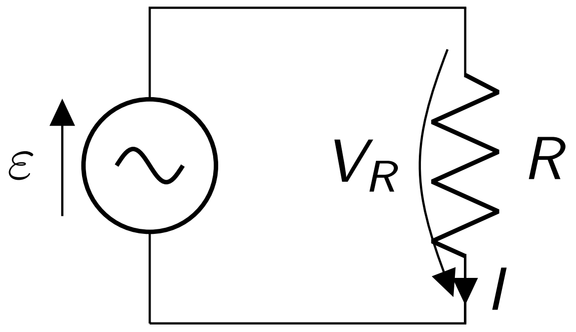
In this lecture we will consider **forced (or driven) oscillations**.

When the RLC circuit is connected to an alternating EMF $\mathcal{E} = \mathcal{E}_0 sin(\omega t)$, the oscillations of charge, voltage and current can occur at the driving frequency ω.

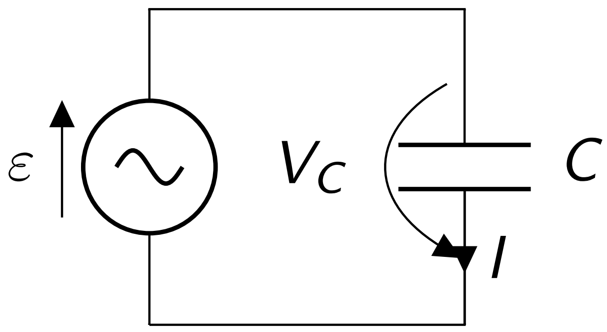
Resistor, Capacitor and Inductor in an AC circuit

We will study the **forced (or driven) oscillation** of the RLC circuit.

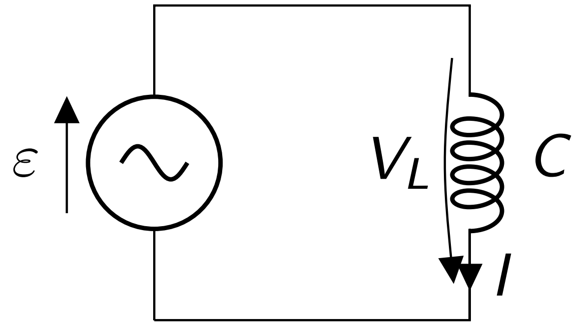
First, we will study the behaviour of different simple AC circuits:



Purely resistive load



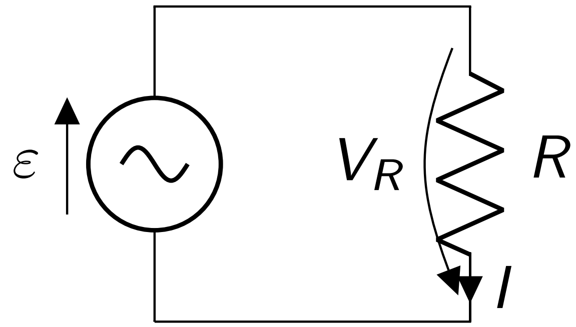
Purely capacitive load



Purely inductive load

In particular we will study what is the current $I(t)$ and the voltage $V_i(t)$ across the two ends of the resistor, capacitor and inductor (i = R,L,C) connected in the circuit with an AC voltage $\mathcal{E} = \mathcal{E}_0 sin(\omega t)$.

Resistor in an AC circuit



$V_R(t)$ is given by Kirchoff's voltage rule:

$$\mathcal{E}(t) = V_R(t) \Rightarrow V_R(t) = \mathcal{E}_0 sin(\omega t) \Rightarrow$$

$$V_R(t) = V_{R0} sin(\omega t)$$

where $V_{R0} = \mathcal{E}_0$.

The current $I(t)$ is given by:

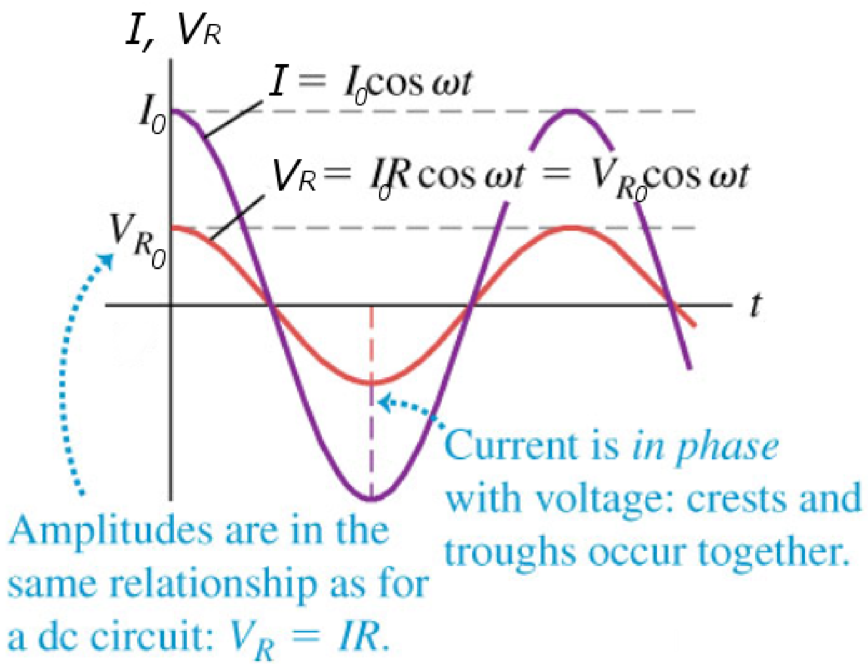
$$V_R(t) = I(t)R \Rightarrow I(t) = \frac{V_R(t)}{R} \Rightarrow I(t) = \frac{V_0}{R} sin(\omega t) \Rightarrow I(t) = I_0 sin(\omega t)$$

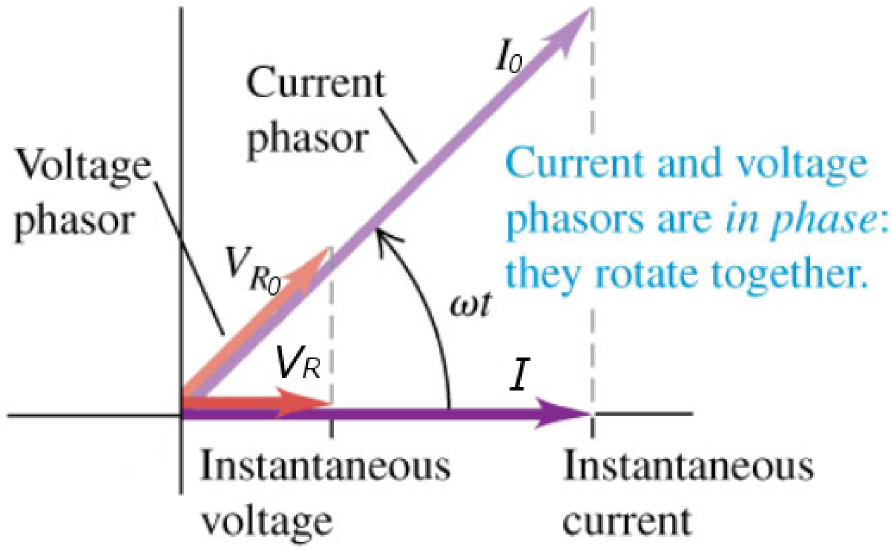
where $I_0 = V_{R0}/R$.

Resistor in an AC circuit

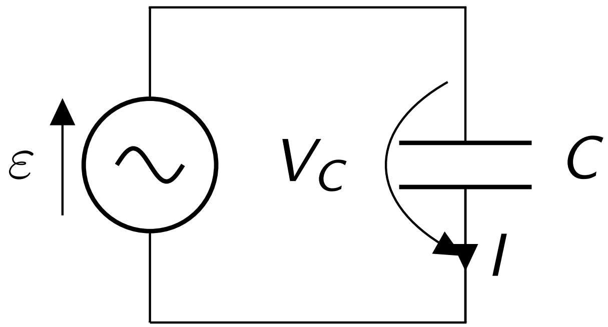
$$I(t) = I_0 sin(\omega t) \text{ and } V_R(t) = V_{R0} sin(\omega t) \text{ where } I_0 = \frac{V_{R0}}{R}, \text{ and } V_{R0} = \mathcal{E}_0$$

Current and voltage are always in phase.





Capacitor in an AC circuit



$V_C(t)$ is given by Kirchoff's voltage rule:

$$\mathcal{E}(t) = V_C(t) \Rightarrow V_C(t) = V_{C0} sin(\omega t)$$

where $V_{C0} = \mathcal{E}_0$.

The current $I(t)$ is calculated as follows:

$$V_C(t) = \frac{Q(t)}{C} \Rightarrow Q(t) = CV_C(t) \Rightarrow Q(t) = CV_{C0} sin(\omega t) \xrightarrow{I=dQ/dt}$$

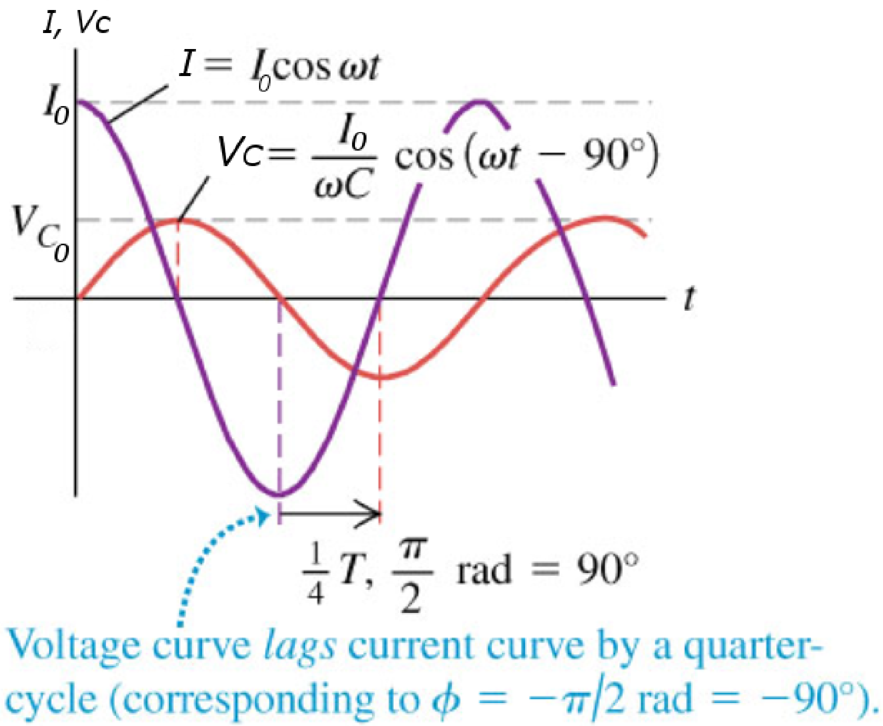
$$I(t) = \omega C V_{C0} cos(\omega t) \Rightarrow \xrightarrow{cos\theta = sin\left(\theta + \frac{\pi}{2}\right)} I(t) = I_0 sin\left(\omega t + \frac{\pi}{2}\right)$$

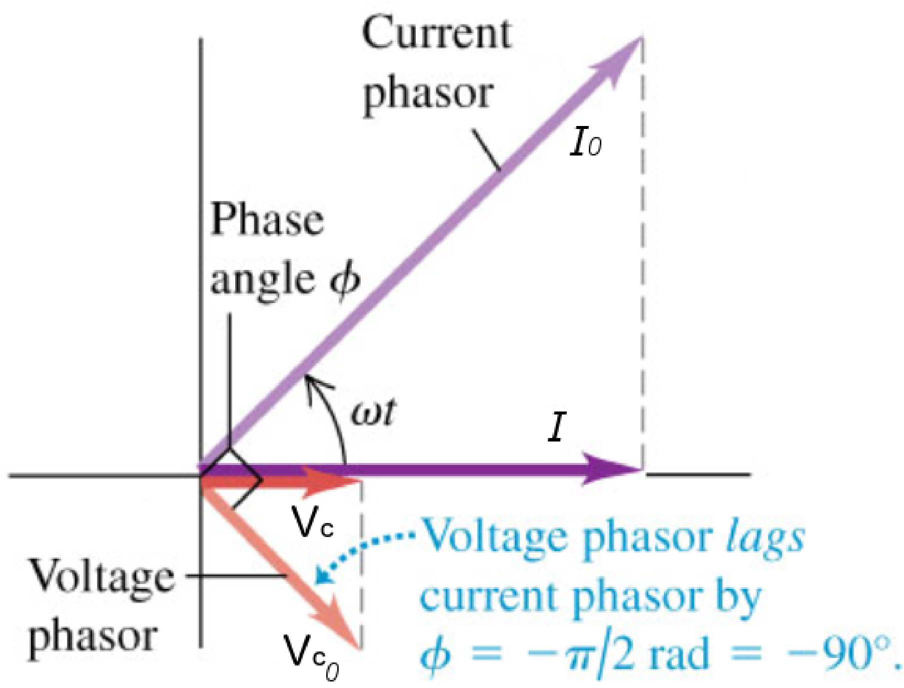
where $I_0 = \omega C V_{C0} = \dfrac{V_{C0}}{X_C}$.

The quantity $X_C = \dfrac{1}{\omega C}$ is the so-called **capacitive reactance** .

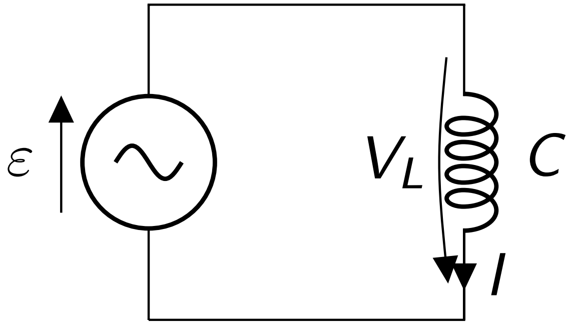
$$I(t) = I_0 sin\left(\omega t + \frac{\pi}{2}\right) \text{ and } V_C(t) = V_{C0} sin(\omega t) \text{ where } I_0 = \frac{V_{C0}}{X_C},\ V_{C0} = \mathcal{E}_0,\ \text{and } X_C = \frac{1}{\omega C}$$

Current leads voltage by $\pi/2$.





Inductor in an AC circuit



$V_L(t)$ is given by Kirchoff's voltage rule:

$$\mathcal{E}(t) - V_L(t) = 0 \Rightarrow V_L(t) = V_{L0} sin(\omega t)$$

where $V_{L0} = \mathcal{E}_0$.

The current $I(t)$ is calculated as follows:

$$V_L(t) = L\frac{dI(t)}{dt} \Rightarrow I(t) = \frac{1}{L}\int V_L(t)dt \Rightarrow I(t) = \frac{V_{L0}}{L}\int sin(\omega t)dt \Rightarrow$$

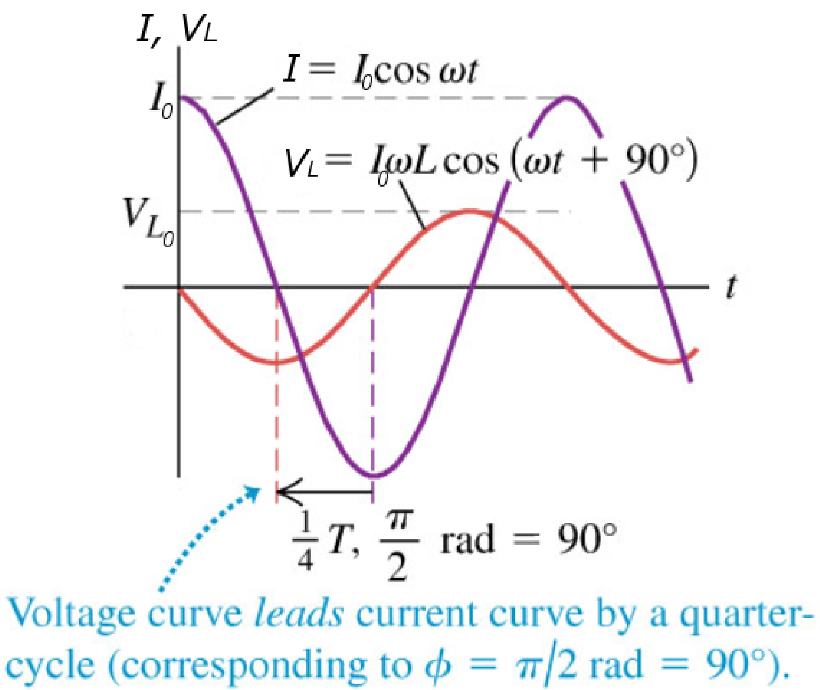
$$I(t) = -\frac{V_{L0}}{\omega L}cos(\omega t) \xrightarrow{-cos\theta = sin\left(\theta - \frac{\pi}{2}\right)} I(t) = I_0 sin\left(\omega t - \frac{\pi}{2}\right)$$

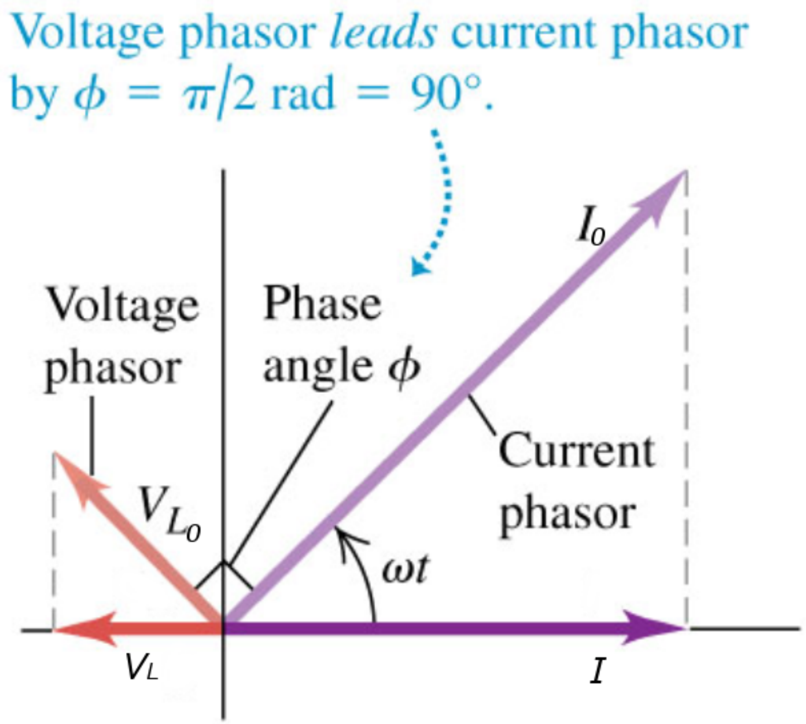
where $I_0 = \dfrac{V_{L0}}{\omega L} = \dfrac{V_{L0}}{X_L}$.

The quantity $X_L = \omega L$ is the so-called **inductive reactance** .

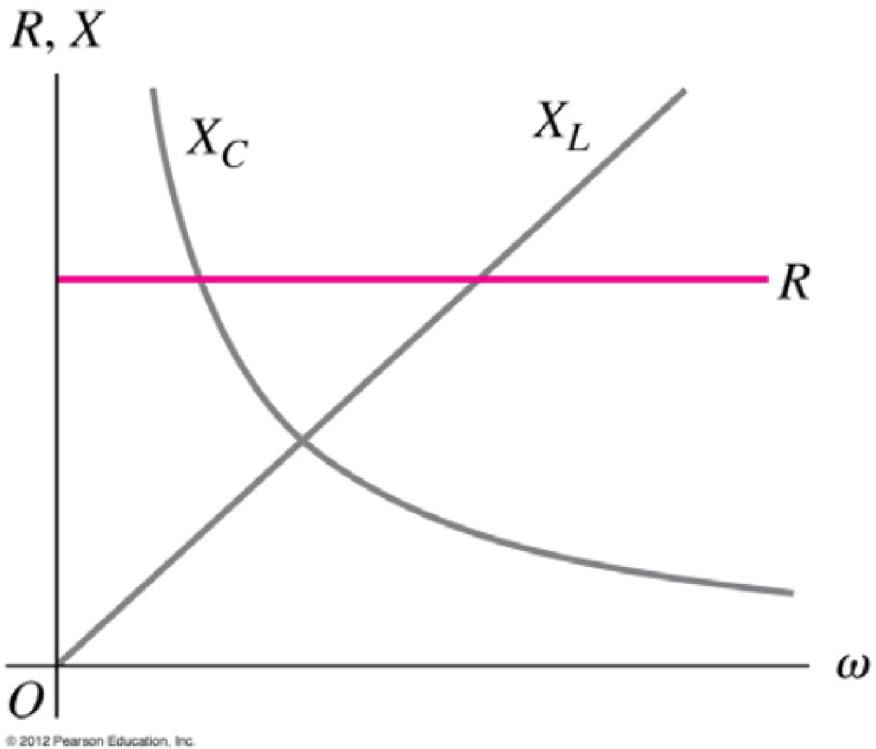
$$I(t) = I_0 sin\left(\omega t - \frac{\pi}{2}\right) \text{ and } V_L(t) = V_{L0} sin(\omega t) \text{ where } I_0 = \frac{V_{L0}}{X_L},\ V_{L0} = \mathcal{E}_0 \text{ and } X_L = \omega L$$

Voltage leads current by $\pi/2$.





Summary: R, L, C elements in an AC circuit



Resistance / Reactance

- Resistor: R
- Capacitor: $X_C = \frac{1}{\omega C}$
- Inductor: $X_L = \omega L$

- Capacitance causes poor response to low frequencies.
- Inductance causes poor response to high frequencies.
- For an RLC circuit, the response is maximised around some intermediate resonant frequency.

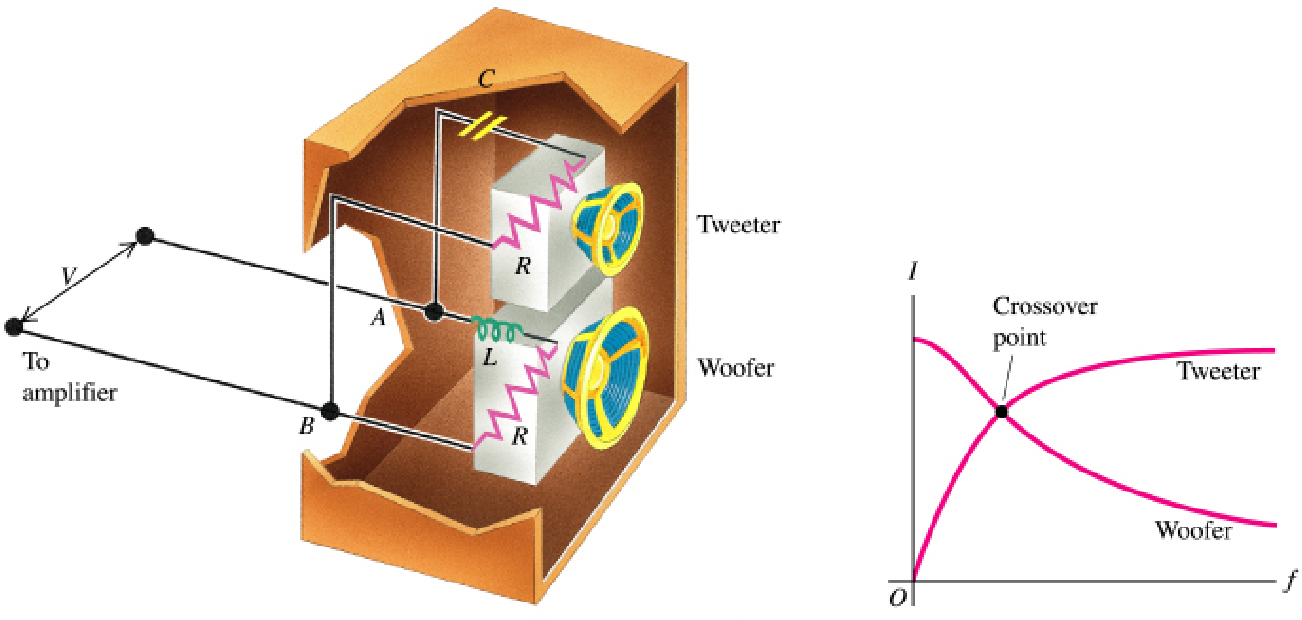
Summary: R, L, C elements in an AC circuit

$$V_i(t) = V_{i0} sin(\omega t) \quad [i = R, L, C] \quad and \quad I(t) = \frac{V_{i0}}{X} sin(\omega t + \phi)$$

Element	**Resistance/ Reactance**	**Current phase**	**Frequency response**
Resistor	R	$\phi = 0$	DC, AC: all frequencies
Capacitor	$X_C = \frac{1}{\omega C}$	$\phi = +\frac{\pi}{2}$	no DC, AC: high-pass filter
Inductor	$X_L = \omega L$	$\phi = -\frac{\pi}{2}$	DC, AC: low-pass filter

Application: Audio crossover

Audio crossovers are a class of electronic filter used in audio applications. Most individual loudspeaker drivers are incapable of covering the entire audio spectrum from low frequencies to high frequencies with acceptable relative volume and absence of distortion so most hi-fi speaker systems use a combination of multiple loudspeaker drivers, each catering to a different frequency band. Crossovers split the audio signal into separate frequency bands that can be separately routed to loudspeakers optimized for those bands [from Wikipedia].

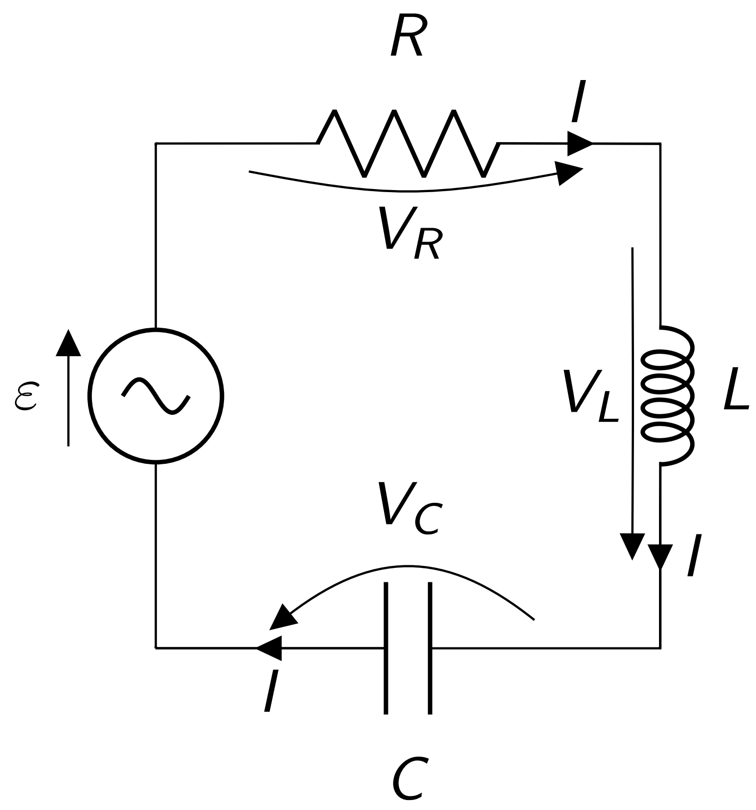


(a) (b)

The crossover frequency is determined by $X_L = X_C$

The RLC series AC circuit

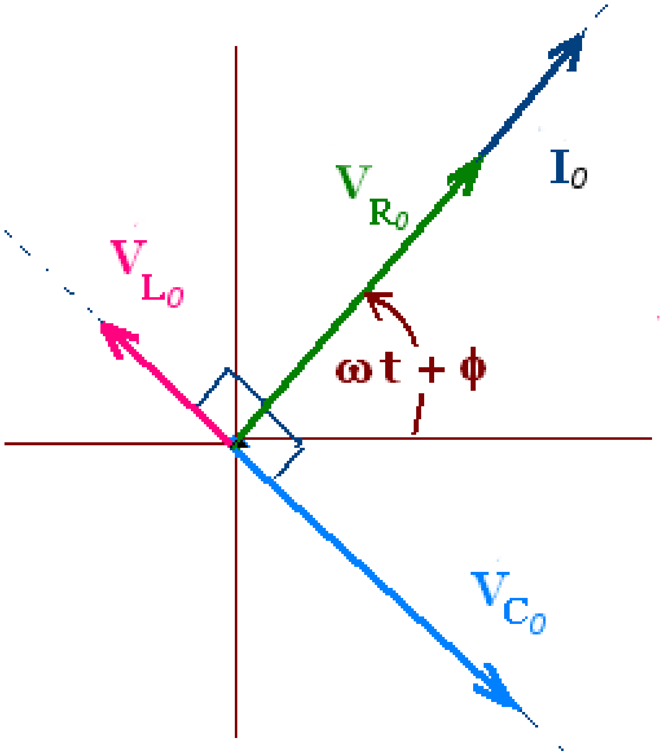
Now are now ready to study a circuit with a resistor, inductor and capacitor connected in series and connected to an external alternating EMF.



The **current is common**:

$$I_R(t) = I_L(t) = I_C(t) = I(t)$$

Assume: $I(t) = I_0 sin(\omega t)$.



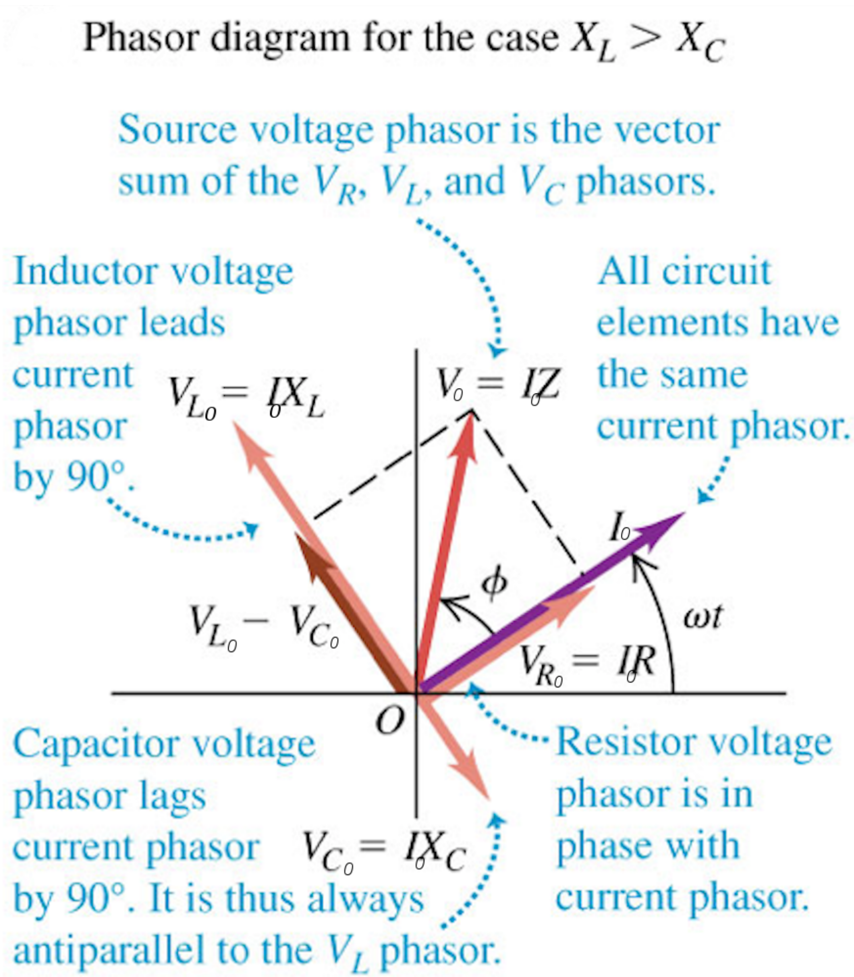
The voltages across the R, L, C elements can be written as:

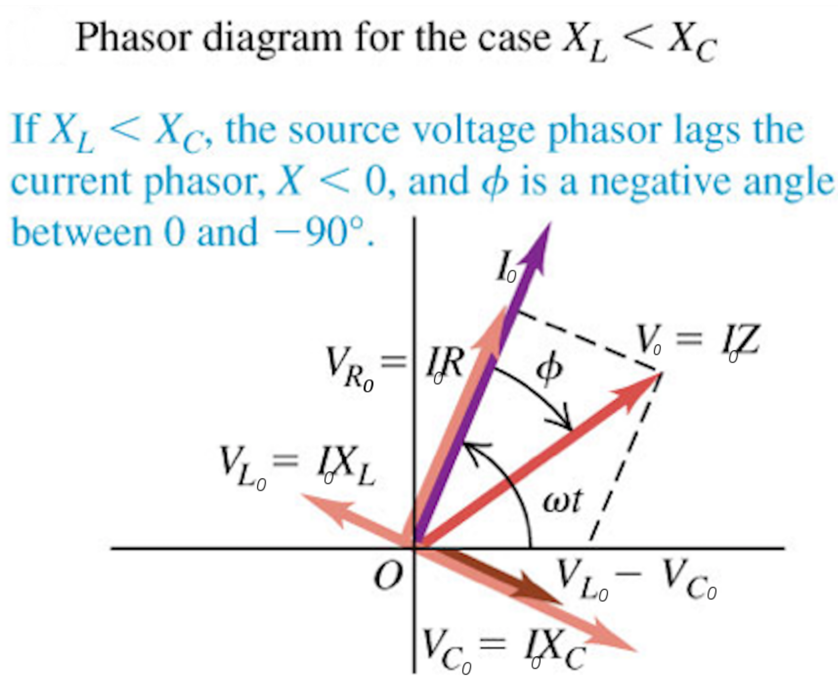
$$V_R(t) = I_0 R sin(\omega t);$$

$$V_L(t) = I_0 X_L sin\left(\omega t + \frac{\pi}{2}\right);$$

$$V_C(t) = I_0 X_C sin\left(\omega t - \frac{\pi}{2}\right)$$

The RLC series AC circuit





The RLC series AC circuit

Kirchoff's voltage rule is valid at all times t as the phasors rotate together:

$$\mathcal{E}(t) = V_R(t) + V_L(t) + V_C(t)$$

Because the phasors representing the voltages in the inductor and the capacitor have opposite directions and they are perpendicular to the phasor representing the voltage in the resistor:

$$\mathcal{E}_0^2 = V_{R0}^2 + \Big(V_{L0} - V_{C0}\Big)^2 \Rightarrow$$

$$\mathcal{E}_0^2 = I_0^2 R^2 + \Big(I_0 X_L - I_0 X_C\Big)^2 = I_0^2 \Big\{R^2 + \Big(X_L - X_C\Big)^2\Big\} \Rightarrow$$

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + \Big(X_L - X_C\Big)^2}} = \frac{\mathcal{E}_0}{\sqrt{R^2 + \Big(\omega L - \frac{1}{\omega C}\Big)^2}} = \frac{\mathcal{E}_0}{Z}$$

where $Z = \sqrt{R^2 + \Big(\omega L - \frac{1}{\omega C}\Big)^2}$ is the **impedance** of the RLC circuit.

The RLC series AC circuit

The value of I_0 depends on the difference between ωL and $\dfrac{1}{\omega C}$ (i.e. the difference between X_L and X_R)

- Doesn't matter which one is greater as the difference is squared.

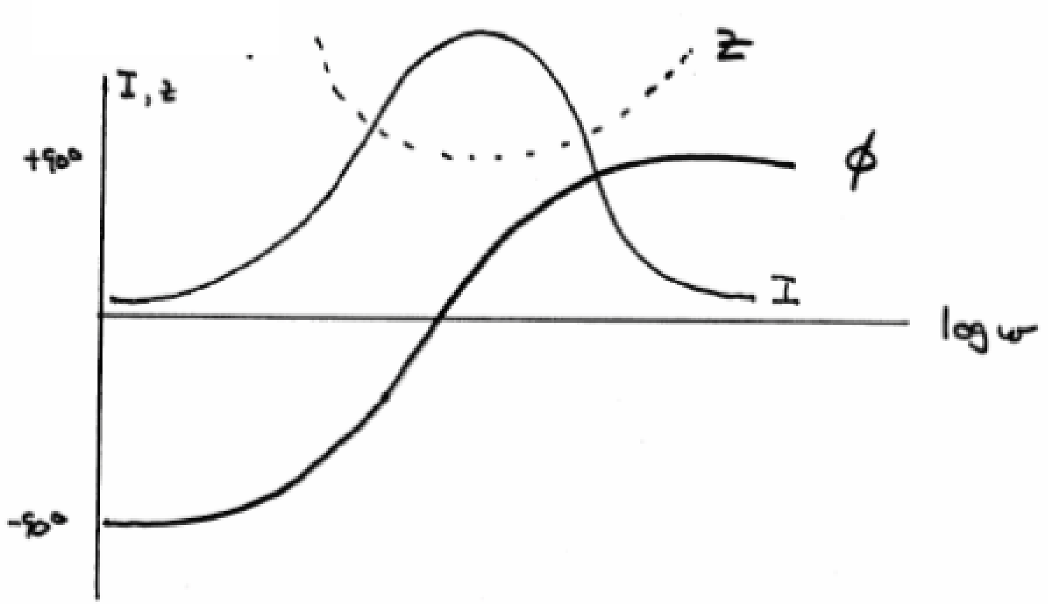
$$I_0 = \frac{\mathcal{E}_0}{Z} = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

But which one is greater determines the phase ϕ by which the voltage leads the current:

$$tan\phi = \frac{V_L - V_C}{V_R} \Rightarrow tan\phi = \frac{X_L - X_C}{R}$$

- $X_L > X_C$: The circuit is **more inductive than capacitive**.
  - The current trails behind the voltage (voltage leads).
- $X_L < X_C$: The circuit is **more capacitive then inductive**.
  - The current leads.
- $X_L = X_C$: ?

If $X_L = X_C$, the circuit is **in resonance**.



At resonance, we have

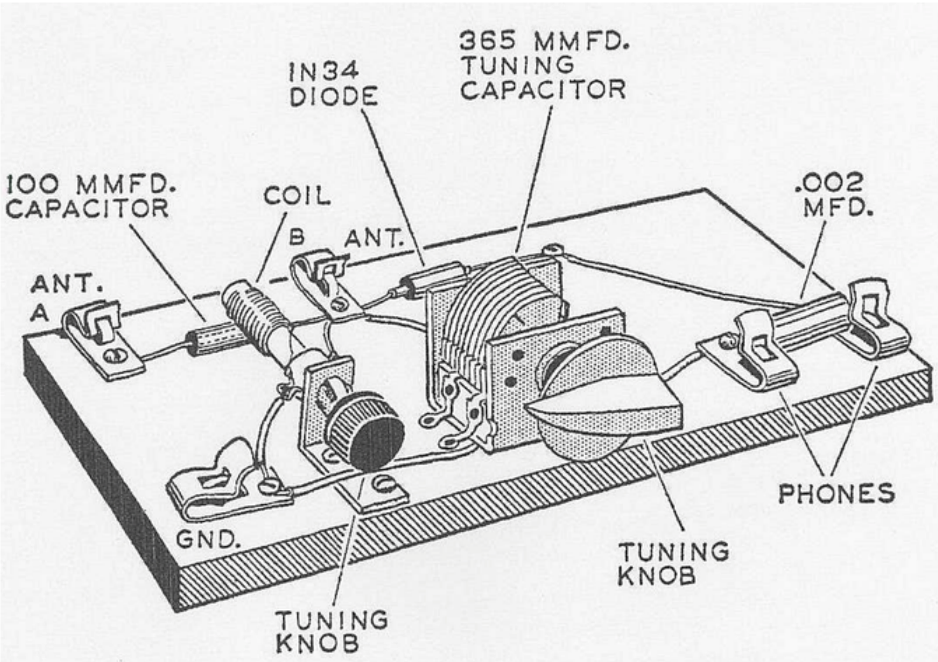
- minimum impedance,
- maximal current, and
- current in phase with voltage in ($\phi = 0$)

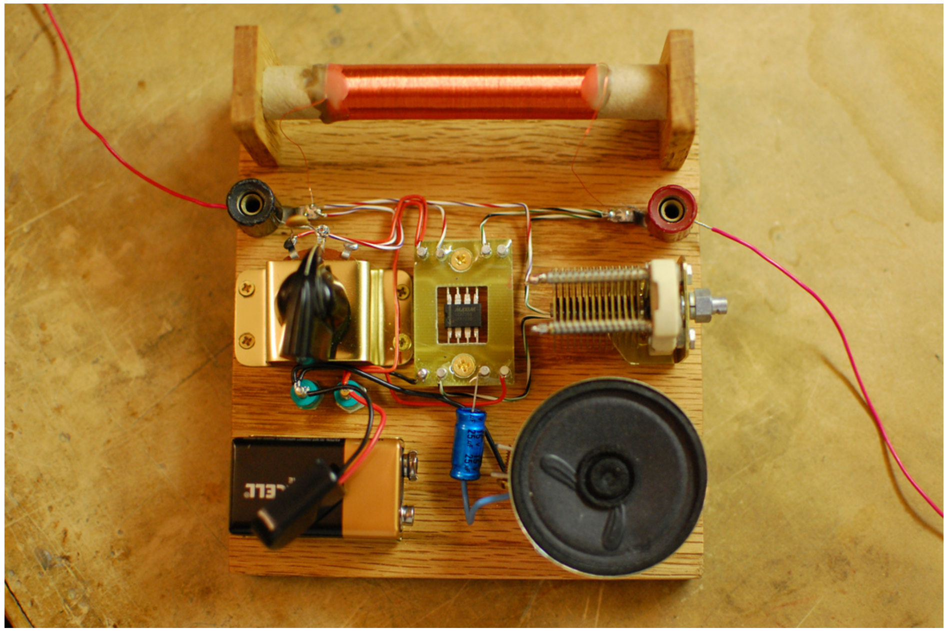
$$X_L = X_C \Rightarrow \omega L - \frac{1}{\omega C} = 0 \Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

The resonance happens when the **driving angular frequency matches the natural angular frequency**.

Resonant circuits are used to **respond selectively to signals of a given frequency** while **discriminating against signals of different frequencies**.

An example of the application of resonant circuits: selection of AM radio stations.





Power in the RLC series circuit

- Average power dissipated on resistor:

$$P_{avg} = \frac{1}{T}\int_0^T V_R(t)I(t) = \frac{1}{2}V_{R0}I_0 = V_{R\,rms}I_{rms} = I_{rms}^2 R = \frac{V_{R\,rms}^2}{R}$$

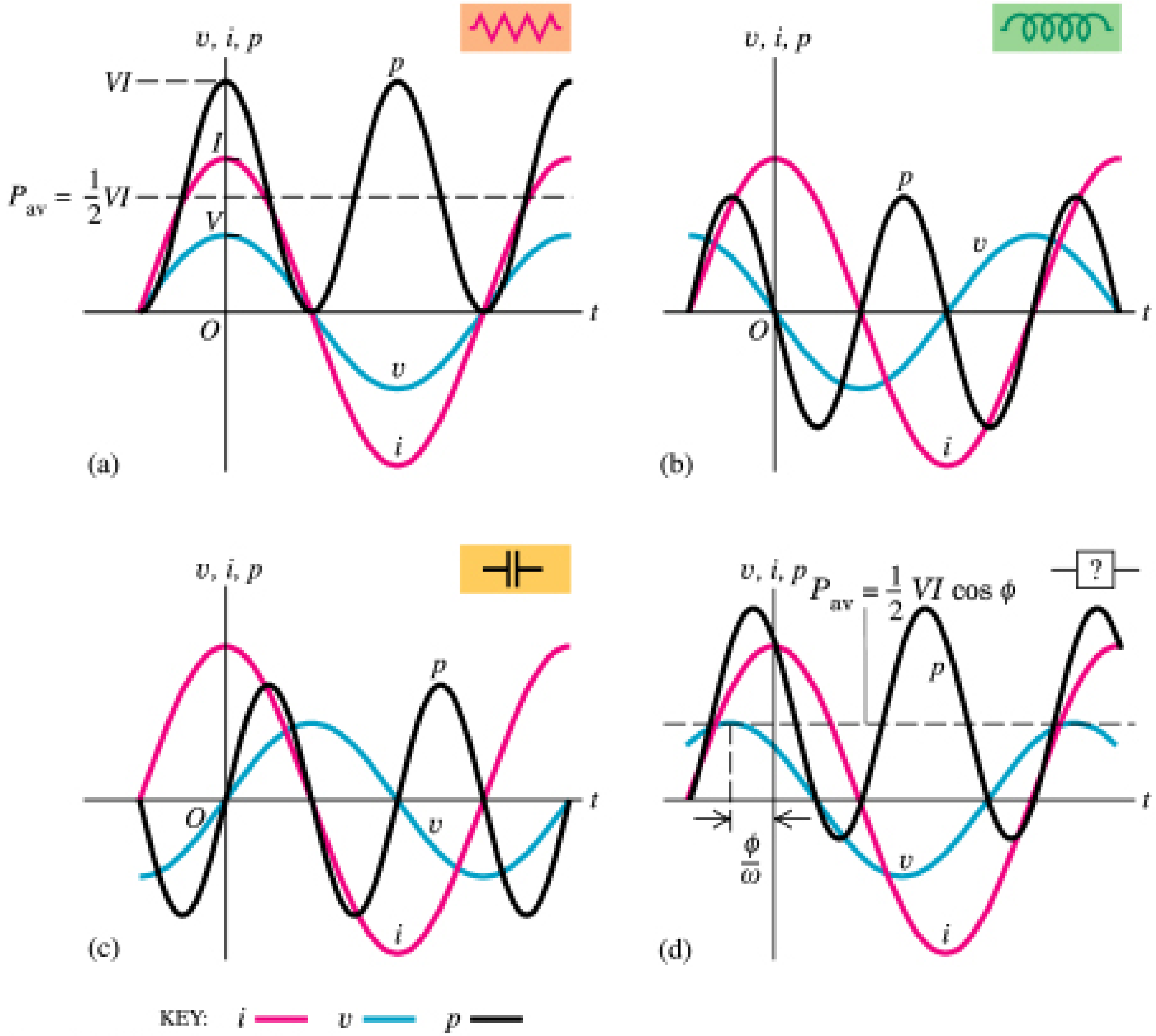
- The inductor stores and releases energy periodically in the $\vec{B}$ field.
- The capacitor stores and releases energy periodically in the $\vec{E}$ field.

Average power provided by a generator:

$$P_{avg} = \frac{1}{2}V_0 I_0 cos\phi = V_{rms}I_{rms}cos\phi$$

Note that $\int_0^{2\pi} sin^2 x dx = \frac{1}{2}$ leads to $I_{rms} = \frac{I_0}{\sqrt{2}}$ and $V_{R\,rms} = \frac{V_0}{\sqrt{2}}$

Power in the RLC series circuit



Lecture 12 - What to remember

When the RLC circuit is connected to an alternating EMF $\mathcal{E} = \mathcal{E}_0 sin(\omega t)$, the oscillations of charge, voltage and current can occur at the driving frequency ω.

$$V_i(t) = V_{i0} sin(\omega t) \quad [i = R, L, C] \quad and \quad I(t) = \frac{V_{i0}}{X} sin(\omega t + \phi)$$

Element	**Resistance/ Reactance**	**Current phase**	**Frequency response**
Resistor	R	$\phi = 0$	DC, AC: all frequencies
Capacitor	$X_C = \frac{1}{\omega C}$	$\phi = +\frac{\pi}{2}$	no DC, AC: high-pass filter
Inductor	$X_L = \omega L$	$\phi = -\frac{\pi}{2}$	DC, AC: low-pass filter

In an RLC circuit:

$$I_0 = \frac{\mathcal{E}_0}{Z} = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

where Z is the impedance of the circuit.

Lecture 12 - Main points to remember

In an RLC circuit, the phase ϕ by which the voltage leads the current is given by:

$$tan\phi = \frac{V_L - V_C}{V_R} \Rightarrow tan\phi = \frac{X_L - X_C}{R}$$

- $X_L > X_C$: The circuit is **more inductive than capacitive** (voltage leads).
- $X_L < X_C$: The circuit is **more capacitive then inductive** (current leads).
- $X_L = X_C$: the circuit is **in resonance**
  - min. impedance, max. current, and current in phase with voltage in ($\phi = 0$)

The resonance happens when the **driving angular frequency matches the natural angular frequency**:

$$X_L = X_C \Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

PHYS201 (Electromagnetism)
Formula Sheet

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University of Liverpool & U.K. Research and Innovation,
Science and Technology Facilities Council, Rutherford Appleton Laboratory

Coulomb's law - Force between two discrete charges q_1, q_2

- The force $\vec{F}_{12}$ exerted on test charge q_1 (placed in $\vec{r}_1$) by charge q_2 (placed in $\vec{r}_2$) is:

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}\hat{r}_{12} \quad or \quad \vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2)$$

  In the above expression, $\hat{r}_{12}$ is a unit vector in the direction of $\vec{r}_1 - \vec{r}_2$:

$$\hat{r}_{12} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

- Note that the force $\vec{F}_{21}$ exerted on test charge q_2 by charge q_1 is:

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3}(-\vec{r}_1 + \vec{r}_2) = -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) = -\vec{F}_{12}$$

Superposition principle - Force on charge Q due to an array of discrete charges

- Allows the calculation of the total force on a charge Q from an array of other charges q_1, q_2, ..., q_n

$$\vec{F}_Q = \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0}\frac{Q q_i}{|\vec{r}_Q - \vec{r}_{q_i}|^3}(\vec{r}_Q - \vec{r}_{q_i})$$

- Total force is the vector sum of forces.

Generalization of Coulomb's law - Force on Q due to a continuous distribution of charge

In general,the force on a charge Q, placed at position $\vec{r}$, due to a continuous distribution of charge, is given by:

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0}\int_\tau \frac{dq(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}(\vec{r} - \vec{r}\,')$$

where dq is the infinitesimal amount of charge in the viscinity of point $\vec{r}\,'$.

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The continuous distribution of charge can be described by a volume charge density $\rho(\vec{r}')$ (charge per unit volume), in which case the above expression becomes:

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0}\int_\tau d\tau' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

If one or two spatial dimensions of a problem can be ignored, the distribution of charge can be described with a surface charge density $\sigma(\vec{r}')$ (charge per unit area) or a linear charge density $\lambda(\vec{r}')$ (charge per unit length) respectively. In these cases, $\vec{F}_Q$ is determined from the following integral over the corresponding surface S or line L:

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0}\int_S dS' \frac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}') \quad \mathit{or} \quad \vec{F}_Q = \frac{Q}{4\pi\epsilon_0}\int_L d\ell' \frac{\lambda(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

Electric field

- A more fundamental way to think about electric forces in terms of a field that permeates space.
- Defined the electric field $\vec{E}$ as the force exerted on a test charge Q, placed in position $\vec{r}$, per unit charge.

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q(\vec{r})}{Q}$$

Electric flux

- The electric flux Φ_E is the number of field lines of the electric field $\vec{E}$ flowing through a surface S

$$\Phi_E = \int_S \vec{E}\cdot d\vec{S}$$

- On many occassions in this lecture series, we study the flux through a closed surface S

$$\Phi_E = \oint_S \vec{E}\cdot d\vec{S}$$

- The distinction is always clear from context. When I want to empasize the difference (see section on Maxwell's equations below) I will denote the former as Φ_E^{open}

Work

- A force is said to **do work** (denoted with W) if, when it is acting on a body, there is a **displacement of the point of application in the direction of the force**.
- The work dW done by a force $\vec{F}$ displacing the point of application by $d\vec{\ell}$, is given by the dot product:

$$dW = \vec{F}\cdot d\vec{\ell}$$

- Notice that work is a **scalar**.
- Note that **a force perpendicular to the direction of motion does no work**.
- The **work can be positive or negative**. By convention, we take work to be negative if it opposes the motion, i.e. $\theta > 90^o$.
- The total work along a trajectory is given by integrating (summing up) the work done for each infinitesimal displacement $d\vec{\ell}$:

$$W = \int dW = \int \vec{F}\cdot d\vec{\ell}$$

Electrostatic potential energy

The work W done to assemble a system of charges becomes electrostatic potential energy U ($U = W$) and it is stored in the field produced by the charges. The formulas below show how to calculate U for a:

- system of 2 charges:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{12}|}$$

- system of 3 charges:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{12}|} + \frac{q_1 q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{13}|} + \frac{q_2 q_3}{4\pi\epsilon_0} \frac{1}{|\vec{r}_{23}|}$$

- system of N charges:

$$U = \frac{1}{2} \sum_{i,j=1;i\neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

- continuous charge distribution (with density ρ over a volume τ):

$$U = \frac{1}{2} \int_\tau \int_{\tau'} \frac{dq(\vec{r})dq'(\vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|} = \frac{1}{2} \int_\tau d\tau \int_{\tau'} d\tau' \frac{\rho(\vec{r})\rho(\vec{r'})}{4\pi\epsilon_0 |\vec{r} - \vec{r'}|}$$

Electrostatic potential

The electrostatic potential $V(\vec{r})$ is a scalar field, and it represents the amount of work W required to bring a charge Q in position $\vec{r}$, divided by the charge Q:

$$V = \frac{W}{Q}$$

The expresions below show how to compute $V(\vec{r})$ due to:

- a point charge q placed at $\vec{r'}$:

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r'}|}$$

- charges q_1, q_2,..., q_N placed at positions $\vec{r}_1$, $\vec{r}_2$,..., $\vec{r}_N$:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|}$$

- a continuous charge distribution represented by $\rho(r')$:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\tau'} \frac{dq(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = \frac{1}{4\pi\epsilon_0} \int_{\tau'} \frac{\rho(\vec{r}\,')d\tau'}{|\vec{r} - \vec{r}\,'|}$$

Definition of electric current

An **electric current is a flow of electric charge.** It is represented by the amount of charge passing though per unit time.

$$I = \frac{dQ}{dt}$$

Current density $\vec{j}$

$\vec{j}$ represents the amount of current per unit area **perpendicular** to the direction of the current.

$$I = \int_S \vec{j} \cdot d\vec{S}$$

Microscopic view of electric current

The current density $\vec{j}$ is given by

$$\vec{j} = nq\vec{u}_d$$

where n is the density of the carriers of charge q, and $\vec{u}_d$ their average (drift) velocity.

In general:

$$\vec{j} = \sigma \vec{E}$$

where σ is the **conductivity** of the material (SI unit: $1/(\Omega \cdot m)$). The inverse quantity $\rho = 1/\sigma$ is called **resistivity**.

Continuity equation

Expresses local conservation of charge:

$$\vec{\nabla} \cdot \vec{j} + \frac{d\rho}{dt} = 0$$

Electric force on a charge

$$\vec{F}_E = q\vec{E}$$

Magnetic force on a (moving) charge

$$\vec{F}_M = q\vec{u} \times \vec{B}$$

Lorentz force on a charge

The total force felt by a charged body in the presence of both electric and magnetic fields is called the **Lorentz force**.

$$\vec{F} = q\Big(\vec{E} + \vec{u} \times \vec{B}\Big)$$

Magnetic force on a steady current

$$\vec{F}_M = I \int_L d\vec{\ell} \times \vec{B}$$

Generalization of magnetic force on a current

$$\vec{F}_M = \int_\tau \vec{j} \times \vec{B} d\tau$$

Magnetic force between two steady currents

At distance ρ, the force between two parallel straight conductors of length L carrying current I_1 and I_2, respectively, is:

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi\rho}$$

The force is attractive if both currents flow in the same direction, and repulsive if the two currents flow in opposite directions.

Cyclotron motion

The equation

$$mu = qBr \Rightarrow \mathbf{p = qBr}$$

where p is the particle momentum is known as the **cyclotron formula** and describes the motion of a particle with charge q perpendicularly to a uniform magnetic field B. The particle moves in a circle of radius r.

The period of rotation is given by:

$$T = \frac{2\pi r}{u} \xRightarrow{mu=qBr} T = \frac{2\pi m}{qB}$$

Biot-Savart law

Expresses $\vec{B}$ in terms of a steady current I:

$$\vec{B} = \int_L d\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

where the integral is over the elements $d\vec{\ell}$ along the conductor, and $\vec{r}$ is the distance from $d\vec{\ell}$ to the point where we want to know the field.

Generalization of Biot-Savart law

$$\vec{B} = \frac{\mu_0}{4\pi} \int_{\tau'} \frac{\vec{j}(\vec{r}') \times \vec{r}}{r^3} d\tau'$$

Magnetic field around an infinitely-long straight wire with current I

$$|\vec{B}| = \frac{\mu_0 I}{2\pi\rho}$$

where ρ is the distance from the wire. The field is azimuthal.

Magnetic field inside a solenoid with n windings per unit length, each carrying current I

$$|\vec{B}| = \mu_0 n I$$

The field is in the direction of the axis of the solenoid.

Magnetic field of a toroidal coil with N windings, each carrying current I

$$|\vec{B}| = \begin{cases} 0, \text{for } r < a, \\ \frac{\mu_0 N I}{2\pi r}, \text{for } a < r < b, \text{ and} \\ 0, \text{for } r > b \end{cases}$$

where a (b) is the inner (outer) radius.

Wave equation

A **wave equation** describes how a *disturbance* propagates in time.

Let $\phi(\vec{r}, t)$ be a function that describes that disturbance as a function of position in space and time. Then it satisfies the following equation:

$$\vec{\nabla}^2 \phi(\vec{r}, t) = \frac{1}{u^2} \frac{\partial^2 \phi(\vec{r}, t)}{\partial t^2}$$

where **u is the wave velocity**.

The electric and magnetic fields (in vacuum) satisfy a wave equation

$$\vec{\nabla}^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Speed of electromagnetic waves in vacuum

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

The electric and magnetic fields (in matter) satisfy a wave equation

$$\vec{\nabla}^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{B} = \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

Speed of electromagnetic waves in matter

$$u = \frac{1}{\sqrt{\mu \epsilon}} < c$$

Index of refraction of a material

$$n = \frac{c}{u} = \frac{\sqrt{\mu \epsilon}}{\sqrt{\mu_0 \epsilon_0}} = \frac{\sqrt{\epsilon_r \epsilon_0 \mu_r \mu_0}}{\sqrt{\epsilon_0 \mu_0}} = \sqrt{\epsilon_r \mu_r}$$

Energy stored in the electric field

$$U_E = \frac{\epsilon_0}{2} \int_{all\ space} |\vec{E}(\vec{r})|^2 d\tau$$

The energy density (energy per unit volume) is:

$$u_E = \frac{\epsilon_0}{2} |\vec{E}(\vec{r})|^2$$

Energy stored in the magnetic field

$$U_M = \frac{1}{2\mu_0} \int_{all\ space} |\vec{B}(\vec{r})|^2 d\tau$$

The energy density (energy per unit volume) is:

$$u_M = \frac{1}{2\mu_0} |\vec{B}(\vec{r})|^2$$

Energy stored in the electromagnetic field

$$U_{EM} = U_E + U_M = \int_{all\ space} \left(\frac{\epsilon_0}{2} |\vec{E}(\vec{r})|^2 + \frac{1}{2\mu_0} |\vec{B}(\vec{r})|^2 \right) d\tau$$

Poynting theorem

$$\frac{dW}{dt} = - \oint_S d\vec{S} \cdot \frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right) - \frac{\partial}{\partial t} \int_\tau d\tau \left(\frac{\epsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2 \right)$$

Poynting vector $\vec{N}$

$$\vec{N} = \frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right)$$

It represents the **energy flux density (rate energy transfer per unit area)** and it has units of $W(att)/m^2$.

The average power transmitted by an electromagnetic wave, is given by the **average of $\vec{N}$ over a period T**:

$$< \vec{N} >= \frac{1}{T} \int_0^T \vec{N} dt$$

The magnitude of the averaged Poynting vector can be written in terms of the amplitudes of the electric field, E_0, and magnetic field, B_0, as

$$< N >= \frac{E_0 B_0}{2\mu_0}$$

Considering that $B_0 = E_0/c$, and introducing the root mean square amplitudes $E_{rms} = E_0/\sqrt{2}$ and $B_{rms} = B_0/\sqrt{2}$, the above can be rewritten as

$$< N >= \frac{cB_0}{2\mu_0} = \frac{cB_{rms}^2}{\mu_0} = \frac{cE_0}{2\mu_0 c} = \frac{cE_{rms}^2}{\mu_0 c}$$

Radiation pressure

If an object is illuminated by radiation for a time interval Δt, during which it absorbs energy

$$\Delta U = IA\Delta t$$

where I is the intensity I (power per area, or energy per time per area) of the radiation, and A is the area of the object.

- The momentum change Δp of the object is given by:

$$\Delta p = \frac{\Delta U}{c} \quad (total\ absorption) \quad \Delta p = \frac{2\Delta U}{c} \quad (total\ reflection)$$

- The radiation pressure P_r exterted on the object is given by:

$$P_r = \frac{I}{c} \quad (total\ absorption) \quad P_r = \frac{2I}{c} \quad (total\ reflection)$$

Boundary conditions for the electric field

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \Rightarrow D_1^{\perp} = D_2^{\perp}$$

$$E_1^{||} = E_2^{||}$$

Boundary conditions for the magnetic field

$$B_1^{\perp} = B_2^{\perp}$$

$$\frac{1}{\mu_1} B_1^{||} = \frac{1}{\mu_2} B_2^{||} \Rightarrow H_1^{||} = H_2^{||}$$

Intensity of unpolarized light passing through a polarizing sheet

If I_0 is the intensity of the unpolarized light, the intensity I of the transmitted light is:

$$I = \frac{1}{2} I_0$$

Intensity of polarized light passing through a polarizing sheet

If the light reaching the filter is already polarized, the intensity I of the transmitted light is:

$$I = I_0 cos^2\theta$$

where θ is the angle between the electric field $\vec{E}$ and the polarizing direction of the sheet.

Laws of geometrical optics

- **1st Law**:
  The incident, reflected, and transmitted wave vectors form a plane (plane of incidence), which also includes the normal to the surface.

$$k_I sin\theta_I = k_R sin\theta_R = k_T sin\theta_T$$

- **2nd Law** (**Law of reflection**):
  The angle of incidence is equal to the angle of reflection.

$$\theta_I = \theta_R$$

- **3rd Law** (**Law of refraction** or **Snell's law**):

$$\frac{sin\theta_T}{sin\theta_I} = \frac{n_1}{n_2}$$

Total internal reflection

As the angle of incidence increases, the angle of refraction increases. For some critical value θ_c, the angle of refraction becomes 90^o.
For angles of incidence larger than θ_c, such as for rays f and g below, there is no refracted ray and all the light is reflected; this effect is called **total internal reflection**.

$$n_1 sin\theta_c = n_2 sin 90^o \Rightarrow \theta_c = sin^{-1}\frac{n_2}{n_1} \qquad (n_2 < n_1)$$

Polarization by reflection / Brewster angle

When the light is incident at a particular incident angle, called **Brewster angle** given by:

$$tan\theta_B \approx \frac{n_2}{n_1}$$

the reflected light has only perpendicular components: It is **fully polarized** perpendicular to the plane of incidence. For light incident at that angle, the reflected and refracted rays are perpendicular to each other.

Maxwell's equations - Static case (time-independent fields) in vacuum

Gauss's law	$\Phi_E = \oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_{\tau(S)} \rho d\tau = \frac{Q}{\epsilon_0}$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Circuital law	$\oint_L \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn)	$\Phi_B = \oint_S \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 \int_{S(L)} \vec{j} \cdot d\vec{S} = \mu_0 I$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

Static fields in matter - Polarization (magnetization) and induced charges (currents)

In materials, macroscopic polarization induces surface and volume charges that contribute to the total electic field and need to be taken into account. Similarly, macroscopic magnetization induces surface and volume currents that contribute to the total magnetic field and need to be taken into account. The relevant expressions for the induced charges and currents are given below.

Electrostatics		**Magnetostatics**	
electric dipole moment	$\vec{p} = q\vec{d}$	$\vec{m} = I\vec{S}$	magnetic dipole moment
torque within $\vec{E}$ field	$\vec{T} = \vec{p} \times \vec{E}$	$\vec{T} = \vec{m} \times \vec{B}$	torque within a $\vec{B}$ field
polarization	$\vec{P} = \frac{(e.d.m)}{volume}$	$\vec{M} = \frac{(m.d.m)}{volume}$	magnetization
surface charge density	$\sigma_P = \vec{P} \cdot \hat{n}$	$j_m^{surf} = \vec{M} \times \hat{n}$	surface current density
volume charge density	$\rho_P = -\vec{\nabla} \cdot \vec{P}$	$j_m^{vol} = \vec{\nabla} \times \vec{M}$	volume current density

Auxiliary fields $\vec{D}$ and $\vec{H}$

The introduction of auxiliary fields $\vec{D}$ (electric displacement) and $\vec{H}$ (H field, or magnetizing field), given below, simplifies then description of electric and magnetic phenomena in materials, since they allow one to write Maxwell's equation only in terms of free charges and currents.

Electric displacement $\vec{D}$:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Magnetizing field $\vec{H}$:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

Relationship between polarization (magnetization) and the electric (magnetic) field

The polarisation vector $\vec{P}$ can be expressed in terms of $\vec{E}$:

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

where χ_e is the so-called **electric susceptibility** (dimensionless).

If the analogy between electrostatics and magnetostatics was exact, we would write $\vec{M}$ in terms of $\vec{B}$. However, this is where the analogy breaks. Instead we express $\vec{M}$ in terms of $\vec{H}$:

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is the **magnetic susceptibility**.

Polarization and magnetization for linear materials

For linear dielectrics (and low intensity fields) χ_e is a constant that does not depend on $\vec{E}$. Therefore, the displacement vector $\vec{D}$ can be written as:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \chi_e \epsilon_0 \vec{E} = (1 + \chi_e)\epsilon_0 \vec{E} = \rho_f = \epsilon_r \epsilon_0 \vec{E} = \rho_f = \epsilon \vec{E}$$

where the factor $\epsilon_r = 1 + \chi_e$ is the **relative permittivity** or **dielectric constant** (dimensionless) and $\epsilon = \epsilon_r \epsilon_0$ is the **permittivity** of the dielectric

Similarly, for linear materials, χ_m is a constant independent of the value of $\vec{H}$. Expressing $\vec{B}$ in terms of $\vec{H}$:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \vec{B} = \mu_0\Big(\vec{H} + \vec{M}\Big) \xrightarrow{\vec{M}=\chi_m \vec{H}} \vec{B} = \Big(1 + \chi_m\Big)\mu_0 \vec{H} \Rightarrow$$

$$\vec{B} = \mu_r \mu_0 \vec{H} \Rightarrow \vec{B} = \mu \vec{H}$$

where $\mu_r = 1 + \chi_\mu$ is the **relative permeability** (dimensionless) and $\mu = \mu_r \mu_0$ is the **permeability** of the material

Maxwell's equations - Static case (time-independent fields) in matter

Gauss's law	$\Phi_D = \oint_S \vec{D} \cdot d\vec{S} = \int_{\tau(S)} \rho_{free} d\tau = Q_{free}$	$\vec{\nabla} \cdot \vec{D} = \rho_{free}$
Circuital law	$\oint_L \vec{E} \cdot d\vec{\ell} = 0$	$\vec{\nabla} \times \vec{E} = 0$
Gauss's law (magn.)	$\Phi_B = \oint_S \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint_L \vec{H} \cdot d\vec{\ell} = \int_{S(L)} \vec{j}_{free} \cdot d\vec{S} = I_{free}$	$\vec{\nabla} \times \vec{H} = \vec{j}_{free}$

Maxwell's equations - Dynamic case (time-dependent fields) in vacuum

Gauss's law	$\Phi_E = \oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_{\tau(S)} \rho d\tau = \frac{Q}{\epsilon_0}$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Faraday's law	$\oint_L \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_{S(L)} \vec{B} \cdot d\vec{S} = -\frac{\partial \Phi_B^{open}}{\partial t}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Gauss's law (magn)	$\Phi_B = \oint_S \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint_L \vec{B} \cdot d\vec{\ell} = \mu_0 \int_{S(L)} \Big(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\Big) \cdot d\vec{S}$	$\vec{\nabla} \times \vec{B} = \mu_0 \Big(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\Big)$

Maxwell's equations - Dynamic case (time-dependent fields) in matter

Gauss's law	$\Phi_D = \oint_S \vec{D} \cdot d\vec{S} = \int_{\tau(S)} \rho_{free} d\tau = Q_{free}$	$\vec{\nabla} \cdot \vec{D} = \rho_{free}$
Faraday's law	$\oint_L \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_{S(L)} \vec{B} \cdot d\vec{S} = -\frac{d\Phi_B^{open}}{dt}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Gauss's law (magn.)	$\Phi_B = \oint_S \vec{B} \cdot d\vec{S} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Ampere's law	$\oint_L \vec{H} \cdot d\vec{\ell} = \int_{S(L)} \left(\vec{j}_{free} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} = I_{free} + \frac{d\Phi_D^{open}}{dt}$	$\vec{\nabla} \times \vec{H} = \vec{j}_{free} + \frac{\partial \vec{D}}{\partial t}$

The scalar potential V - The Poisson and Laplace equations

In electrostatics, the circuital law:

$$\vec{\nabla} \times \vec{E} = 0$$

allowed us to write the electric field $\vec{E}$ as the gradient of a scalar field (recall that $\vec{\nabla} \times \left(\vec{\nabla}\phi\right) = 0$ for *any* scalar function ϕ). That scalar field was the potential V:

$$\vec{E} = -\vec{\nabla} V$$

By substituting the above into Gauss's law, we found that the scalar potential V satisfies the so-called *Poisson* equation:

$$\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$$

In absence of sources (ρ=0), the above equation is known as the *Laplace* equation:

$$\vec{\nabla}^2 V = 0$$

The vector potential $\vec{A}$

The relation

$$\vec{\nabla} \cdot \vec{B} = 0$$

allows us to express the magnetic field $\vec{B}$ as the rotation of a vector field (recall that $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$ for *any* vector field $\vec{A}$):

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

We call $\vec{A}$ the **vector potential**.

Substituting the above definition into Ampere's law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \Rightarrow \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 \vec{j} \Rightarrow$$

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{j}$$

We can add to $\vec{A}$ *any function* $\vec{\Lambda}$ whose curl vanishes ($\vec{\nabla} \times \vec{\Lambda} = 0$):

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\Lambda}$$

and the physics would remain unchanged! We can use this freedom to eliminate the divergence of $\vec{A}$ ($\vec{\nabla}\vec{A} = 0$). With this condition, $\vec{A}$ satisfies the following p.d.e.:

$$\vec{\nabla}^2\vec{A} = -\mu_0\vec{j}$$

This expression is very similar to our known *Poisson* equation.

Capacitance

Capacitance (C) denotes the ability of a body to store electric charge. For a system of two conductors, one with charge +Q held at potential V_+ and one with charge -Q held at potential V_-, the capacitance is a positive quantity defined as:

$$C = \frac{Q}{V_+ - V_-}$$

Calculating the capacitance for simple systems

- Use Gauss' law to calculate the electric field $\vec{E}$ in terms of the charge Q stored in one of the conductors: $\oint_S \vec{E} \cdot d\vec{S} = Q/\epsilon_0$
- Once $\vec{E}$ is known, calculate the potential difference V between the two conductors as: $V := \Delta V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{\ell}$
- From the known charge Q in the positive conductor and the potential difference V between the conductors, calculate $C = Q/V$

The parallel plate capacitor

- The electric field between the plates was found to be:

$$E = \frac{\sigma}{\epsilon_0}$$

- The parallel plate capacitor has capacitance:

$$C = \epsilon_0 \frac{A}{d}$$

  It depends only on the geometrical characteristics of the capacitor.

- Energy stored in the parallel plate capacitor:

$$U = \frac{Q^2}{2C} \quad or \quad U = \frac{1}{2}CV^2 \quad or \quad U = \frac{1}{2}QV$$

Mutual and self inductance / Back-EMF

Consider two closed loops 1 and 2. The magnetic flux Φ_2 through the surface of loop 2, of the magnetic field $\vec{B}_1$ produced by a steady current I_1 in loop 1 is

$$\Phi_2 = MI_1$$

where M is the mutual inductance of the two loops. Similarly, the magnetic flux Φ_1 through the surface of loop 1, of the magnetic field $\vec{B}_2$ produced by a steady current I_2 in loop 2 is

$$\Phi_1 = MI_2$$

Note: The constant M is the same (mutual inductuance)

If there is a change in the flow of current in one of the conductors, then a voltage (EMF) is induced both

- in itself (self-inductance):
  $\mathcal{E} = -L\frac{dI}{dt}$
- and the neighbouring conductor (mutual inductance):
  $\mathcal{E}_{neighbouring\ loop} = -M\frac{dI}{dt}$

The $\vec{\nabla}$ (*nabla*) operator

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

Just as we have 3 kinds of vector multiplications (with scalar, dot product, cross product), we have 3 ways the nabla operator can act

- $\vec{\nabla}$ (a scalar field) $\rightarrow$ **gradient**
- $\vec{\nabla}\cdot$ (a vector field) $\rightarrow$ **divergence**
- $\vec{\nabla}\times$ (a vector field) $\rightarrow$ **curl**

Gradient

Let $f(x, y, z)$ be a scalar field in a 3-dimensional space. Its gradient is:

$$\vec{\nabla} f(x, y, z) = \Big(\frac{\partial f(x, y, z)}{\partial x}, \frac{\partial f(x, y, z)}{\partial y}, \frac{\partial f(x, y, z)}{\partial z}\Big)$$

Notice that $\vec{\nabla} f(x, y, z)$ is a vector. It tells us how fast $f(x, y, z)$ changes as we move in space and it has the **direction of the steepest ascent**.

Divergence

Let $\vec{F}(x, y, z) = \Big(F_x(x, y, z),\ F_y(x, y, z),\ F_z(x, y, z)\Big)$ be a vector field in a 3-dimensional space. Its divergence is given by:

$$\vec{\nabla}\cdot\vec{F}(x, y, z) = \frac{\partial F_x(x, y, z)}{\partial x} + \frac{\partial F_y(x, y, z)}{\partial y} + \frac{\partial F_z(x, y, z)}{\partial z}$$

Notice that the divergence of a vector field is a scalar.
Its value for a particular point expresses the magnitude of the vector field's source (or sink) at that point.

Curl

The curl ($\vec{\nabla} \times \vec{A}$) of a vector field $\vec{A} = \Big(A_x, A_y, A_z\Big)$ is defined as:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \Big(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\Big)\hat{x} - \Big(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\Big)\hat{y} + \Big(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\Big)\hat{z}$$

Notice that the curl of a vector field is a vector.
Its value for a particular point expresses the rotation of the vector field around that point.

The Laplace operator

The Laplace operator is a 2^{nd} order differential operator defined as follows:

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplacian

So the Laplacian of a scalar function f is:

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Laplacian **is the divergence of the gradient** of the scalar function f:

$$\vec{\nabla}^2 f = \vec{\nabla} \cdot \Big(\vec{\nabla} f\Big)$$

It represents a quantity which is important in several physical processes:
The Laplacian $\vec{\nabla}^2 f(\vec{r})$ of a scalar function f at a point $\vec{r}$ tells you **how much $f(\vec{r})$ differs from its average over a small volume around $\vec{r}$.**

Difference between total and partial derivatives

$$\frac{df(x,y,z,...)}{dx} = \frac{\partial f(x,y,z,...)}{\partial x} + \frac{\partial f(x,y,z,...)}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial f(x,y,z,...)}{\partial z} \cdot \frac{\partial z}{\partial x} + ...$$

Gauss's theorem (Divergence theorem)

Let τ be a volume of space whose boundary is the closed surface S, and let $\vec{F}$ be a vector field which is defined anywhere in τ and it is continuous / differentiable anywhere in τ.
The integral of the flux of the field $\vec{F}$ over the closed surface S is equal to the integral of the divergence of $\vec{F}$ in the volume τ

$$\oint_S \vec{F} \cdot d\vec{S} = \int_{\tau(S)} \vec{\nabla} \cdot \vec{F} d\tau$$

Stokes's theorem

Stokes' theorem **relates the circulation of a vector field $\vec{F}$ around a closed line C with the flux of the curl of the vector field $\vec{F}$ through the open surface S** defined by the closed line C.

$$\oint_L \vec{F} \cdot d\vec{\ell} = \int_{S(L)} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$