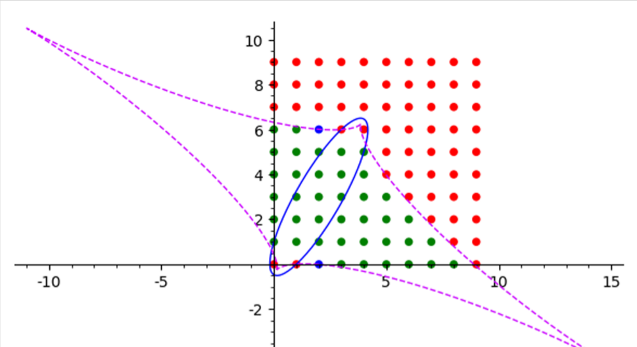
DEPARTMENT OF APPLIED MATHEMATICS

應 用 數 學 系

AMA1500

Foundation Mathematics for Accounting and Finance

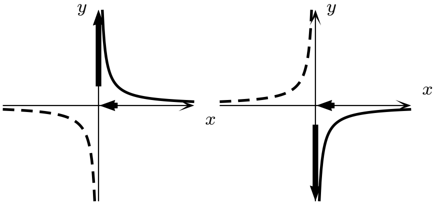
Part 02



Infinite limits

Consider the function $f(x) = 1/x$ near $x = 0$. If $x > 0$ anthe value of $f(x)$ becomes very big and can be made as big as we please (as indicated by the thick black vertical arrow).

To describe this property of $f(x)$ we introduce the following definitions.



(a) The graph of $y = 1/x$. (b) The graph of $y = -1/x$.

Figure: Functions tending to ∞ and $-\infty$ as $x \to 0^+$.

Definition

If the value of $f(x)$ can be made bigger than any prescribed positive and large number by taking $x > a$ and close enough to a, we say that $f(x)$ *approaches to infinity as x approaches a from the right* and we write

$$\lim_{x \to a^+} f(x) = \infty.$$

The situation in Figure (b) motivates the next definition.

Definition

If the value of $f(x)$ can be made smaller than any prescribed number (usually negative and numerically large) by taking $x > a$ and close enough to a, we say that $f(x)$ *approaches to negative infinity as x approaches a from the right* and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

0.3cm Clearly, by considering x approaching from the left, i.e. $x < a$ instead of $x > a$ in the above two definitions, we can define the two *infinite limits from the left*:

$$\lim_{x \to a^-} f(x) = \infty \quad \lim_{x \to a^-} f(x) = -\infty.$$

Definition

- If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \infty$, we write $\lim_{x\to a} f(x) = \infty$.
- If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = -\infty$, we write $\lim_{x\to a} f(x) = -\infty$.

The symbol ∞ stands for "infinity". It is not a real number and is used as a "quantity" greater than any real number. In the same way, $-\infty$ is a "quantity" smaller than any real number. Infinity or negative infinity is always associated with limits such as those defined above.

Formulas

By actually computing $1/x$ using small values of x or by observing the graph of $1/x$ in the Figure, we see that

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

On the other hand, direct substitution gives

$$\lim_{x\to 0^+} \frac{1}{x} = \frac{1}{0^+} \quad \lim_{x\to 0^-} \frac{1}{x} = \frac{1}{0^-}$$

which are undefined. Therefore, we can write

$$\frac{1}{0^+} = \infty \frac{1}{0^-} = -\infty$$

where 0^- means a function approaching 0 from the left and 0^+ means from the right. Although ∞ is not a number, the above formulas in are sometimes useful for finding limits. These, together with other useful formulas about infinity are listed in the following table.

Useful formulas involving infinity

(k is a constant, m and n are positive integers)

$\infty + k = \infty,$	$(-\infty) + k = -\infty.$	
$\infty + \infty = \infty,$	$(-\infty) - \infty = -\infty,$	$\infty - \infty =$ indeterminate.
$k \cdot \infty = \infty,$	$k \cdot (-\infty) = -\infty$	if $k > 0$.
$k \cdot \infty = -\infty,$	$k \cdot (-\infty) = \infty$	if $k < 0$.
$\infty \cdot \infty = \infty,$	$\infty \cdot (-\infty) = -\infty,$	$(-\infty) \cdot (-\infty) = \infty.$
$k \div \infty = 0,$	$k \div (-\infty) = 0,$	$\infty \div \infty =$ indeterminate.
$1 \div 0^+ = \infty,$	$1 \div 0^- = -\infty,$	$0 \cdot \infty =$ indeterminate.
$\sqrt[m]{\infty} = \infty,$	$\sqrt[n]{-\infty} = -\infty$	if n is odd.

Remark

The formula "$\infty + k = \infty$" means that if $\lim f(x) = \infty$ then $\lim(f(x) + k) = \infty$. The formula "$\infty + \infty = \infty$" means that if $\lim f(x) = \infty$ and $\lim g(x) = \infty$ then $\lim[f(x) + g(x)] = \infty$. Also the formula "$\infty - \infty =$ indeterminate" means that we cannot draw any conclusion on $\lim[f(x) - g(x)]$ if we only know that $\lim f(x) = \infty$ and $\lim g(x) = \infty$. Other formulas in the above table are to be interpreted similarly.

Based on the algebraic operations on infinity given in the above table, Theorem is also true if $\lim f(x)$ and $\lim g(x)$ are infinite.

Example

Is it true that $\lim_{x \to 0} \frac{1}{x} = \infty$?

Solution

No, because $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and they are not the same. $\square$

Example

Find $\displaystyle\lim_{x\to 0}\frac{1}{x^2}$.

Solution

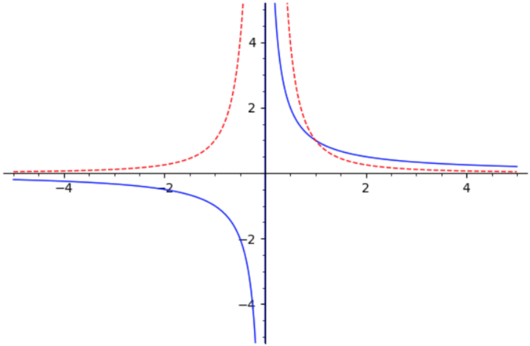
On substitution, $\displaystyle\lim_{x\to 0-}\frac{1}{x^2}=\frac{1}{0+}=\infty$ *and* $\displaystyle\lim_{x\to 0+}\frac{1}{x^2}=\frac{1}{0+}=\infty$.

Therefore $\displaystyle\lim_{x\to 0}\frac{1}{x^2}=\infty$. $\square$

In [2]:

```
f(x)=1/x
g(x)=1/x^2
p1=plot(f(x),x,-5,5)
p2=plot(g(x),x,-5,5,  rgbcolor="red",  linestyle = "dashed")
(p1+p2).show(xmin=-5, xmax=5, ymin=-5, ymax=5)
```

Out[2]:



Example

Find $\lim\limits_{x\to 1^-} \dfrac{x+2}{(x-1)(x+1)}$ *and* $\lim\limits_{x\to 1^+} \dfrac{x+2}{(x-1)(x+1)}$.

Solution

For x near to 1, the factor $x-1$ is negative if $x<1$, positive if $x>1$. Therefore we have on substitution,

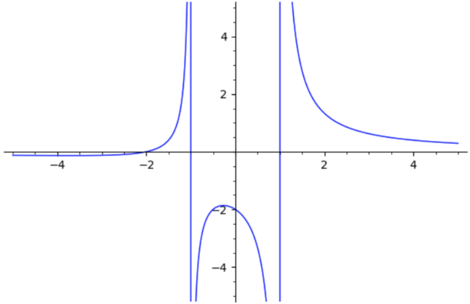
$$\lim_{x\to 1^-} \frac{x+2}{(x-1)(x+1)} = \frac{3}{(0^-)(2)} = -\infty \quad \lim_{x\to 1^+} \frac{x+2}{(x-1)(x+1)} = \frac{3}{(0^+)(2)} = \infty. \qquad \square$$

File Edit View Insert Cell Kernel Help

Run Stop Tab Code Format Validate

In [2]:
```
f(x)=(x+2)/((x-1)*(x+1))
p1=plot(f(x),x,-5,5)
(p1).show(xmin=-5, xmax=5, ymin=-5, ymax=5)
```

Out[2]:



In [3]:
```
limit(f(x),x=1,dir='-')
```

Out[3]: -Infinity

In [4]:
```
limit(f(x),x=1,dir='+')
```

Out[4]: +Infinity

Example

Find $\lim\limits_{x \to 1^-} \dfrac{1}{x^2 - 1}$ *and* $\lim\limits_{x \to 1^+} \dfrac{1}{x^2 - 1}$.

Solution

Similar to the previous example,

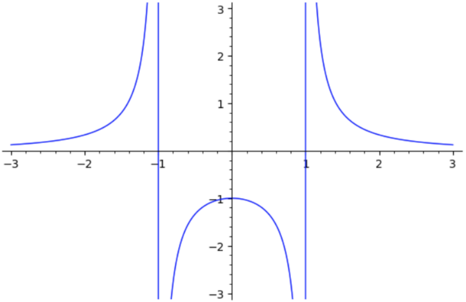
$$\lim_{x \to 1^-} \frac{1}{x^2 - 1} = \frac{1}{0^-} = -\infty \quad \lim_{x \to 1^+} \frac{1}{x^2 - 1} = \frac{1}{0^+} = \infty. \qquad \square$$

File Edit View Insert Cell Kernel Help

Run Stop Tab Code Format Validate

In [2]:
```
f(x)=1/(x^2-1)
p1=plot(f(x),x,-3,3)
(p1).show(xmin=-3, xmax=3, ymin=-3, ymax=3)
```

Out[2]:



In [3]:
```
limit(f(x),x=1,dir='-')
```

Out[3]: -Infinity

In [4]:
```
limit(f(x),x=1,dir='+')
```

Out[4]: +Infinity

Example

Find $\lim_{x \to \pi/2^-} \tan x$ *and* $\lim_{x \to \pi/2^+} \tan x$.

Solution

Using $\tan x = (\sin x)/\cos x$ and the fact that

$$\cos x \quad \text{is near to 0 and} \begin{cases} \text{positive} & \text{if } x < \pi/2 \text{ and near to } \pi/2, \\ \text{negative} & \text{if } x > \pi/2 \text{ and near to } \pi/2 \end{cases}$$

we get

$$\lim_{x \to \pi/2^-} \tan x = \frac{1}{0^+} = +\infty \quad \lim_{x \to \pi/2^+} \tan x = \frac{1}{0^-} = -\infty. \qquad \square$$

The results can also be seen from the graph of $\tan x$ directly.

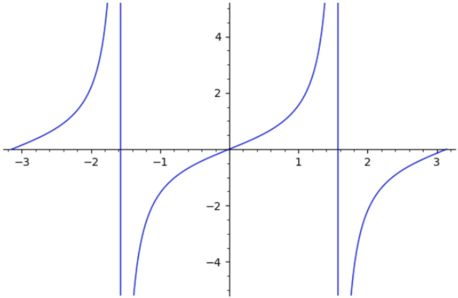
Save TimeTravel ChatGPT... Actions Zoom

File Edit View Insert Cell Kernel Help

Run Stop Tab Code Format Validate

In [1]:
```
f(x)=tan(x)
plot(f(x),x,-pi,pi).show(xmin=-pi, xmax=pi, ymin=-5, ymax=5)
```

Out[1]:



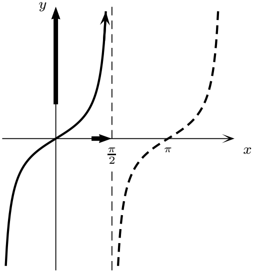
In [2]:
```
limit(f(x),x=pi/2,dir='-')
```

Out[2]: +Infinity

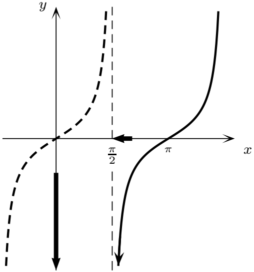
In [3]:
```
limit(f(x),x=pi/2,dir='+')
```

Out[3]: -Infinity

Remark The formula is *not* saying that $1/0 = \pm\infty$ is true. In fact there are examples of functions $f(x)$ which approach to 0 but the limits of their reciprocals $1/f(x)$ do not exist and are neither ∞ nor $-\infty$.



(a) $\tan x \to \infty$ as $x \to \pi/2^-$.



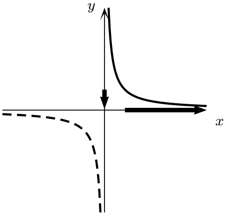
(b) $\tan x \to -\infty$ as $x \to \pi/2^+$.

Figure: Infinite limits of $\tan x$ at $x = \pi/2$.

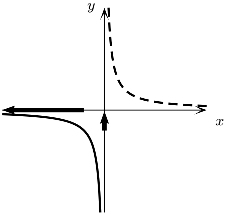
Limits at infinity

Consider again the function $f(x) = 1/x$. As x increases without bound ($x \to \infty$) or decreases without bound ($x \to -\infty$), the value of $f(x)$ approaches to 0. These facts can be seen from the graph of $1/x$ and are stated mathematically as

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \textit{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$



(a) $y \to 0$ as $x \to \infty$.



(a) $y \to 0$ as $x \to -\infty$.

Figure: Function $1/x$ tending to 0 as x tends to ∞ and $-\infty$.

Definition

Let A and B be real numbers.

We write $\lim\limits_{x \to \infty} f(x) = A$ if $f(x)$ approaches A as x increases without bound.

We write $\lim\limits_{x \to -\infty} f(x) = B$ if $f(x)$ approaches B as x decreases without bound.

Similar definitions are for

$$\lim_{x \to \infty} f(x) = \infty, \ \lim_{x \to \infty} f(x) = -\infty, \ \lim_{x \to -\infty} f(x) = \infty, \ \lim_{x \to -\infty} f(x) = -\infty.$$

The above six types of limits are called limits at infinity. By the formulas in the table (Useful formulas involving infinity), we see that Theorem (Limit Theorems for algebraic operations) is true also for limits at infinity.

Example

By inspecting the graph of $y = \tan^{-1} x$ we see that

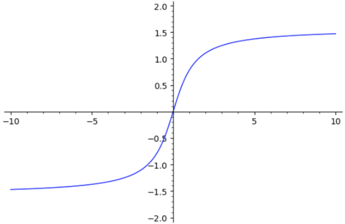
$$\lim_{x \to -\infty} \tan^{-1} x = -\pi/2 \ \ and \ \ \lim_{x \to \infty} \tan^{-1} x = \pi/2.$$

File Edit View Insert Cell Kernel Help

+ ↑ ↓ Run Stop Tab Code Format Validate

In [2]:
```
f(x)=arctan(x)
plot(f(x),x,-10,10).show(xmin=-10, xmax=10, ymin=-2, ymax=2)
```

Out[2]:



In [3]:
```
limit(f(x),x=infinity,dir='-')
```

Out[3]: 1/2*pi

In [4]:
```
limit(f(x),x=-infinity,dir='+')
```

Out[4]: -1/2*pi

Example

Find $\displaystyle\lim_{x\to\infty}(3x-2)$ and $\displaystyle\lim_{x\to-\infty}(3x-2)$.

Solution

Using formulas, we get

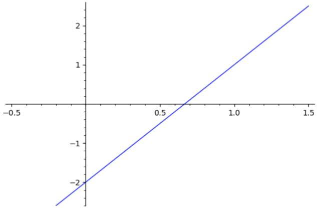
$$\lim_{x\to\infty}(3x-2)=3\cdot\infty-2=\infty-2=\infty.$$

$$\lim_{x\to-\infty}(3x-2)=3\cdot(-\infty)-2=-\infty-2=-\infty.$$

□

In [2]:
```
f(x)=3*x-2
plot(f(x),x,-0.5,1.5).show(xmin=-0.5, xmax=1.5, ymin=-2.5, ymax=2.5)
```

Out[2]:



In [3]:
```
limit(f(x),x=infinity,dir='-')
```

Out[3]: +Infinity

In [4]:
```
limit(f(x),x=-infinity,dir='+')
```

Out[4]: -Infinity

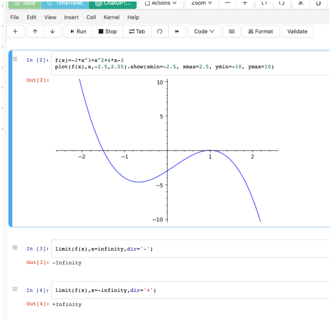
Example

Find $\lim\limits_{x\to\infty}(-2x^3+x^2+4x-3)$ and $\lim\limits_{x\to-\infty}(-2x^3+x^2+4x-3)$.

Solution

$$\lim_{x\to\infty}(-2x^3+3x^2+4x-5) = \lim_{x\to\infty} x^3\Big(-2+\frac{3}{x}+\frac{4}{x^2}-\frac{5}{x^3}\Big)$$
$$= \infty(-2+0+0-0) = -\infty.$$
$$\lim_{x\to-\infty}(-2x^3+3x^2+4x-5) = \lim_{x\to-\infty} x^3\Big(-2+\frac{3}{x}+\frac{4}{x^2}-\frac{5}{x^3}\Big)$$
$$= -\infty(-2+0+0-0) = \infty.$$

□



Polynomials at infinity

From these examples, we obtain the rule:

Theorem

If $P(x)$ is a polynomial of degree n with *positive* leading coefficient:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n > 0,$$

then

$$\lim_{x \to \infty} P(x) = \infty \ \textit{and} \ \lim_{x \to -\infty} P(x) = \begin{cases} \infty & \text{if } n \text{ is even,} \\ -\infty & \text{if } n \text{ is odd.} \end{cases}$$

Rational functions at infinity

In the following examples, we find limits of rational functions at infinity. For this, we first divide the numerator and denominator by the highest-degree term of the denominator.

Example

Find $\displaystyle\lim_{x\to\infty}\frac{4x-1}{x^2+1}$ and $\displaystyle\lim_{x\to-\infty}\frac{4x-1}{x^2+1}$.

Solution

$$\lim_{x\to\infty}\frac{4x-1}{x^2+1}=\lim_{x\to\infty}\frac{\frac{4}{x}-\frac{1}{x^2}}{1+\frac{1}{x^2}}=\frac{0-0}{1+0}=0.$$

Similarly we get

$$\lim_{x\to-\infty}\frac{4x-1}{x^2+1}=\lim_{x\to-\infty}\frac{\frac{4}{x}-\frac{1}{x^2}}{1+\frac{1}{x^2}}=\frac{0-0}{1+0}=0.\qquad\square$$

Example

Find $\lim_{x\to\infty} \frac{2x^2+3x-1}{4x^2+1}$ and $\lim_{x\to-\infty} \frac{2x^2+3x-1}{4x^2+1}$.

Solution

$$\lim_{x\to\infty} \frac{2x^2+3x-1}{4x^2+1} = \lim_{x\to\infty} \frac{2+\frac{3}{x}-\frac{1}{x^2}}{4+\frac{1}{x^2}} = \frac{2+0-0}{4+0} = \frac{1}{2}.$$

Clearly we get the same answer for the second limit. □

In [1]:

```
f(x)=(2*x^2+3*x-1)/(4*x^2+1)
show(f)
```

Out[1]:

$$x \mapsto \frac{2\,x^2+3\,x-1}{4\,x^2+1}$$

In [2]:

```
limit(f(x),x=+Infinity)
```

Out[2]:

1/2

In [3]:

```
limit(f(x),x=-Infinity)
```

Out[3]:

1/2

Example

Find $\displaystyle\lim_{x\to\infty}\frac{-3x^3+4x^2-1}{2x^2+x}$ and $\displaystyle\lim_{x\to-\infty}\frac{-3x^3+4x^2-1}{2x^2+x}$.

Solution

$$\lim_{x\to\infty}\frac{-3x^3+4x^2-1}{2x^2+x}=\lim_{x\to\infty}\frac{-3x+4-1x^2}{2+1x}=\frac{-\infty+4-0}{2+0}=-\infty.$$

$$\lim_{x\to-\infty}\frac{-3x^3+4x^2-1}{2x^2+x}=\lim_{x\to-\infty}\frac{-3x+4-1x^2}{2+1x}=\frac{-3(-\infty)+4-0}{2+0}=\infty.$$

$\square$

Limits of rational functions

From the above examples, we obtain the following rules for limits of rational functions $P(x)/Q(x)$ at infinity. Here we assume that $P(x)$ and $Q(x)$ are polynomials:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0,$$

$$Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0, \quad b_m \neq 0.$$

Theorem

Let $n = \deg P(x)$ and $m = \deg Q(x)$ so that $a_n \neq 0$ and $b_m \neq 0$.

If $n < m$, then both limits $\lim\limits_{x \to \pm\infty} P(x)/Q(x) = 0$.

If $n = m$, then both limits $\lim\limits_{x \to \pm\infty} P(x)/Q(x) = a_n/b_m$.

If $n > m$, then the limits $\lim\limits_{x \to \pm\infty} P(x)/Q(x) = -\infty$ or $+\infty$ depending on the signs of the ratio a_n/b_m, and whether $n - m$ is even or odd.

Example

Find $\lim_{x\to\infty} \dfrac{2x+3}{\sqrt{x^2+4}}$.

Solution

$$\lim_{x\to\infty} \frac{2x+3}{\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{\sqrt{1+\frac{4}{x^2}}} = \frac{2+0}{\sqrt{1+0}} = 2. \qquad \square$$

Example

Find $\lim_{x\to\infty} \dfrac{x+1}{\sqrt[3]{x^2+1}}$.

Solution

$$\begin{aligned}\lim_{x\to\infty} \frac{x+1}{\sqrt[3]{x^2+1}} &= \lim_{x\to\infty} \frac{(x+1) \div x^{2/3}}{(\sqrt[3]{x^2+1}) \div x^{2/3}} \\ &= \lim_{x\to\infty} \frac{x^{1/3}+x^{-2/3}}{\sqrt[3]{1+x^{-2}}} \\ &= \frac{\infty+0}{1+0} = \infty. \qquad \square\end{aligned}$$

Example

Find $\lim_{x\to\infty}\left(\sqrt{x^2+3}-x\right)$.

Solution

$$\lim_{x\to\infty}\left(\sqrt{x^2+3}-x\right)=\lim_{x\to\infty}\frac{(x^2+3)-x^2}{\sqrt{x^2+3}+x}=\lim_{x\to\infty}\frac{3}{\sqrt{x^2+3}+x}=\frac{3}{\infty}=0. \qquad \square$$

In [1]:

```
f(x)=sqrt(x^2+3)-x
show(f)
```

Out[1]:

$x \mapsto -x+\sqrt{x^2+3}$

In [2]:

```
show(limit(f(x),x=+Infinity))
```

Out[2]:

0

Limits of functions - a more rigorous approach

Definition

The set of all points x such that $|x - x_0| < \delta$ is called a δ neighborhood of the point x_0. The set of all points x such that $0 < |x - x_0| < \delta$ in which $x = x_0$ is excluded, is called a deleted δ neighborhood of x_0.

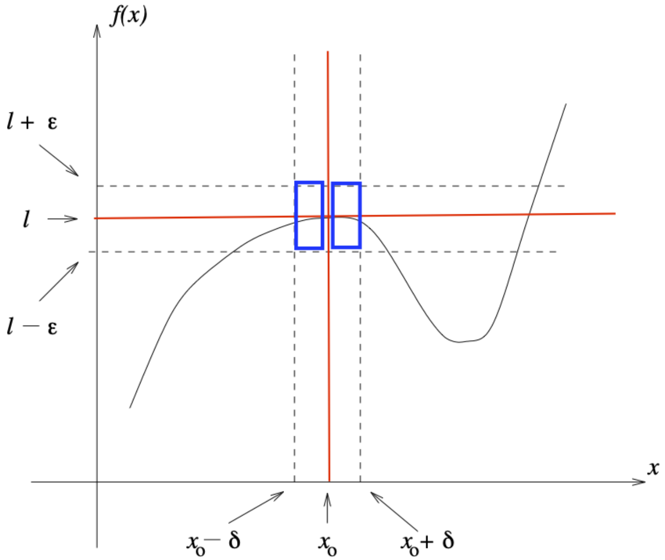
Definition

The number l is the limit of $f(x)$ as x approaches x_0 denoted by

$$\lim_{x \to x_0} f(x) = l,$$

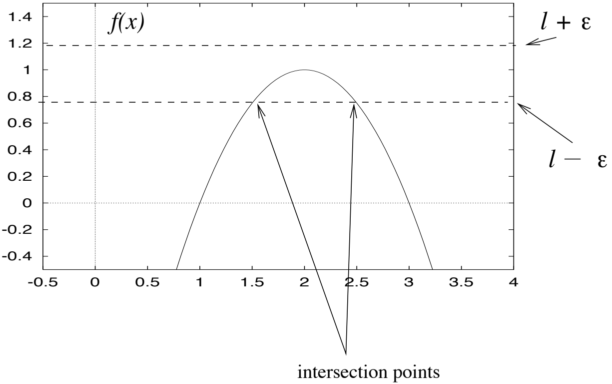
if $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $0 < |x - x_0| < \delta$.

This definition simply says that for any positive number ϵ (however small) we can find some positive number δ (usually depending on ϵ) such that whenever x in the deleted δ neighbourhood of x_0, $x \in (x_0 - \delta\,,\, x_0 + \delta) \backslash \{x_0\}$, then $f(x) \in (l - \epsilon\,,\, l + \epsilon)$.
Note that $f(x_0)$ may not equals the limit value l according to the definition.


f(x)
l + ε
l
l − ε
x
x0 − δ
x0
x0 + δ

Example

Consider the function $f(x) = -x^2 + 4x - 3$. We are going to show that $\lim_{x\to 2} f(x) = 1$ by definition.



Note that no matter how small we choose the number ϵ, the intersection points indicated in the diagram can be obtained by solving

$$\begin{aligned} -x^2 + 4x - 3 &= 1 - \epsilon \\ x^2 - 4x + 4 - \epsilon &= 0, \end{aligned}$$

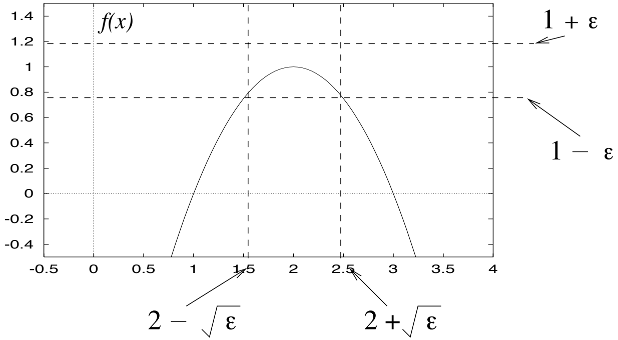
thus, the intersection points are given by

$$\frac{4 \pm \sqrt{16 - 4(4 - \epsilon)}}{2} = 2 \pm \sqrt{\epsilon}.$$

Let $\delta = \sqrt{\epsilon}$. Hence, no matter how small the number ϵ is, we are going to have

$$|f(x) - 1| < \epsilon$$

whenever $0 < |x - 2| < \sqrt{\epsilon} = \delta$. Therefore, by definition, $\lim_{x \to 2} f(x) = 1$.


f(x)
1 + ε
1 − ε
2 − √ε
2 + √ε
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4

Right and Left Hand Limits

We call l^+ the **right hand limit** of $f(x)$ at x_0 if $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that $|f(x) - l^+| < \epsilon$ whenever $0 < |x - x_0| < \delta$ **and** $x > x_0$. We write it as

$$\lim_{x \to x_0^+} f(x) = l^+ = f(x_0^+).$$

Similarly, the **Left hand limit** can be defined with the alternate condition $x < x_0$ the same way as the above.

Example

Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

What is $\lim\limits_{x \to 0^+} f(x) = f(0^+)$?

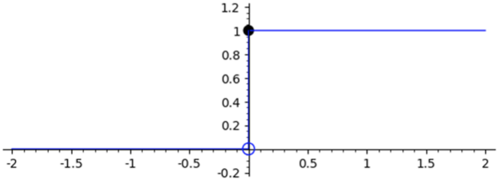
What is $\lim\limits_{x \to 0^-} f(x) = f(0^-)$?

In this example, both $f(0^+)$ and $f(0^-)$ exist, but $\lim\limits_{x \to 0} f(x)$ does not.

In [1]:

```
p=plot(unit_step(x),x,-2,2)
pt1=circle((0,0), 0.05)
pt2=point((0, 1), rgbcolor='black', pointsize=80)
(p+pt1+pt2).show(xmin=-2, xmax=2, ymin=-0.2, ymax=1.2)
```

Out[1]:



In [2]:

```
show(limit(unit_step(x),x=0,dir='+'))
```

Out[2]:

1

In [3]:

```
show(limit(unit_step(x),x=0,dir='-'))
```

Out[3]:

0