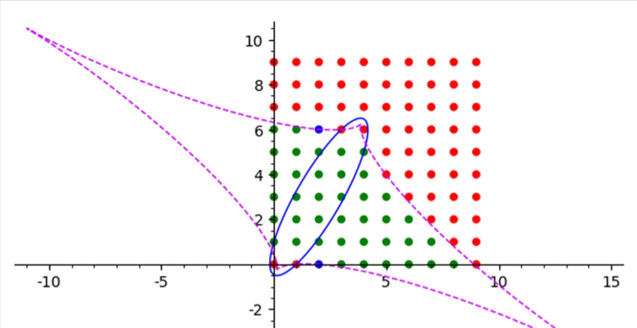
DEPARTMENT OF APPLIED MATHEMATICS
應 用 數 學 系

AMA1500

Foundation Mathematics for Accounting and Finance

Part 01



Subject Lecturer:
Dr. LEE Heung Wing Joseph 李向榮博士
Office: TU702
Email address: joseph.lee@polyu.edu.hk
(can also be contacted interactively via the Chat function of Zoom)

For tutorial materials and assignment matters, please contact your tutor.

Grading Policy

Continuous Assessment :	Assignments	10%
	Test	30%
Examination :		60%

Midterm Test and Examination rubric

A-/A /A+	80 - 100 (out of 100)
B-/ B /B+	65 - 79 (out of 100)
C-/ C /C+	50 - 64 (out of 100)
D /D+	40 - 49 (out of 100)
F	0 - 39 (out of 100)

Assignments:
There are 4 assignment sets. Solutions with detailed workings and explanations should be submitted by 5pm of the corresponding due dates. Students should submit their solutions of the assignments via Blackboard.

- Solutions must be scanned into one single clear and readable PDF file, but
- with file size no bigger than 3MB, and
- the file name must be the student name with surname first.

Midterm Test:
The Mid-term Test could be scheduled in one of the lecture between Week 8 to Week 11 within normal lecture time. Date and Venue TBA. There are 15 multiple choice questions in the test.

Learning Outcomes

This is a subject to provide students with a solid foundation in Differential and Integral Calculus. Upon satisfactory completion of the subject, students are expected to be able to:

- solve problems using the concept of functions and inverse functions
- apply mathematical reasoning to analyse essential features of different mathematical problems such as differentiation and integration
- apply appropriate mathematical techniques to model and solve problems in science and engineering
- extend their knowledge of mathematical techniques and adapt known solutions in different situations

Reading List and References

- *A Short Course in Calculus and Matrices* by **Kwok-Chiu Chung**, McGraw Hill 2008.
- *Calculus. 7th ed.* by **James Stewart**, Brooks/Cole 2012.
- *Thomas' Calculus 12th ed.* by **George B. Thomas Jr., Maurice D. Weir, Joel Hass**, Brooks/Cole 2012.

CoCalc Computational Mathematics with SageMath
https://cocalc.com/

How to begin using CoCalc

Step 1 Use any device with network connection (e.g. desktop, tablet)

Step 2 Use any web browser (e.g. Chrome), and get to https://cocalc.com/

Step 3 Open an account (set your own user name and password)

Step 4 Open a new Project (project is basically a folder, input a "project title")

Step 5 Open a new Jupyter notebook in the Project (yes, it is that same editor used by most python users)) (again, you choose a name for it)

Step 6 Use the latest version of SageMath (SageMath 9.5 or 9.8)

Whenever you have made a syntax error, you can use the built-in **ChatGPT** of CoCalc for suggestions on how to correct it.

Your Projects | Store | Features | Docs | Share | Support | News | Account ▼

Signed in as Anonymous User

Store | Licenses | Billing | Vouchers | Projects | Features | Software | Pricing | Config | Share | Support | Docs

2023-05-24 **API Keys!** api All news...

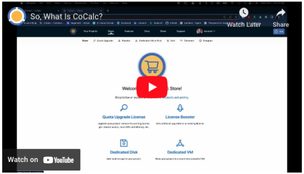
Ask ChatGPT: how can I do this on CoCalc? | Ask ChatGPT

Mission and Features of CoCalc

COCALC

Collaborative Calculation and Data Science

by Sagemath, Inc.



Open your CoCalc projects...

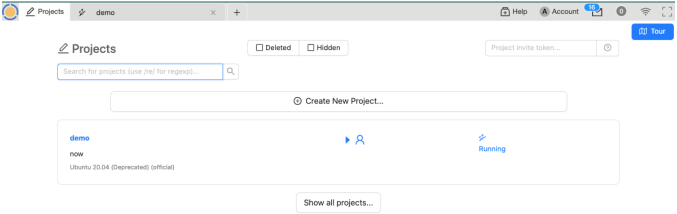
Edit | Copy | Run | SageMath 9.8

```
print('Welcome to CoCalc from Sage!', factor(2023))
```

```
Welcome to CoCalc from Sage! 7 * 17^2
```

Improve your research, teaching, and publishing using realtime collaborative Jupyter notebooks, LaTeX, Markdown, and Linux.

Mission: Enable better collaboration in science, engineering and mathematics by providing easily accessible and sustainable tools for computing, teaching, and publishing.


Projects
demo
Help
Account
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Tour
Projects
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Search for projects (use /re/ for regexp)...
Create New Project...
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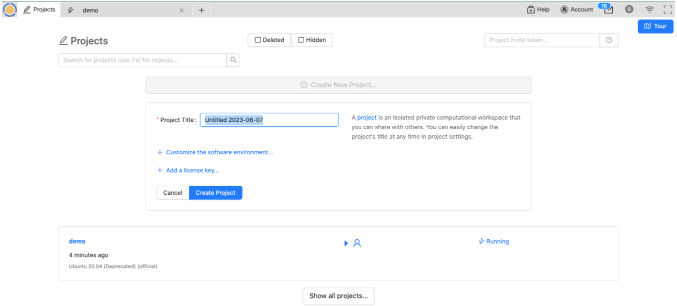
now
Ubuntu 20.04 (Deprecated) (official)
Running
Show all projects...


Projects
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Help
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Project invite token...
Search for projects (use /re/ for regexp)...
Create New Project...
* Project Title: Untitled 2023-06-07
A project is an isolated private computational workspace that you can share with others. You can easily change the project's title at any time in project settings.
Customize the software environment...
Add a license key...
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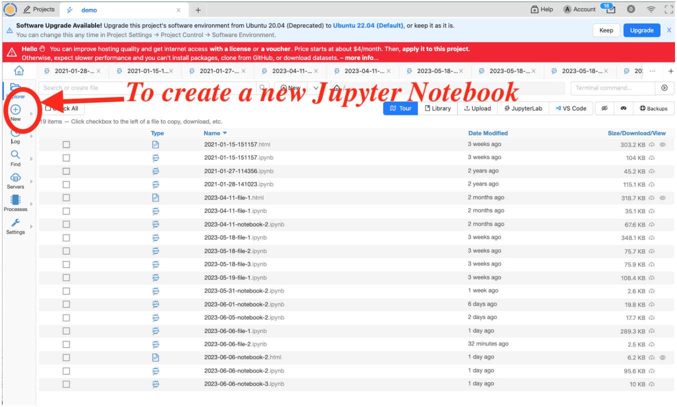
Show all projects...


To create a new Jupyter Notebook
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Software Upgrade Available! Upgrade this project's software environment from Ubuntu 20.04 (Deprecated) to Ubuntu 22.04 (Default), or keep it as it is.
You can change this any time in Project Settings → Project Control → Software Environment.
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Upgrade
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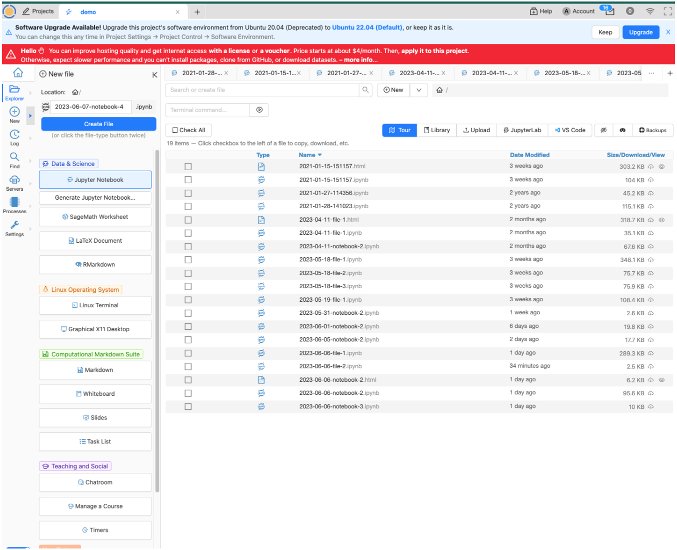
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Hello You can improve hosting quality and get internet access with a license or a voucher. Price starts at about $4/month. Then, apply it to this project.
Otherwise, expect slower performance and you can't install packages, clone from GitHub, or download datasets. – more info...
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Data & Science
Jupyter Notebook
Generate Jupyter Notebook...
SageMath Worksheet
LaTeX Document
RMarkdown
Linux Operating System
Linux Terminal
Graphical X11 Desktop
Computational Markdown Suite
Markdown
Whiteboard
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Task List
Teaching and Social
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Manage a Course
Timers
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Check All
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Upload
JupyterLab
VS Code
Backups
19 items — Click checkbox to the left of a file to copy, download, etc.
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3 weeks ago
108.4 KB
2023-05-31-notebook-2.ipynb
1 week ago
2.6 KB
2023-06-01-notebook-2.ipynb
6 days ago
19.8 KB
2023-06-05-notebook-2.ipynb
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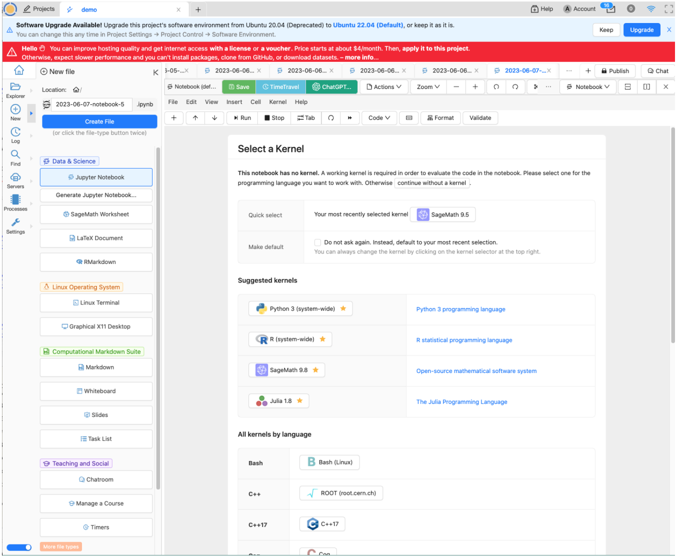
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Data & Science
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SageMath Worksheet
LaTeX Document
RMarkdown
Linux Operating System
Linux Terminal
Graphical X11 Desktop
Computational Markdown Suite
Markdown
Whiteboard
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Manage a Course
Timers
More file types
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TimeTravel
ChatGPT...
Actions
Zoom
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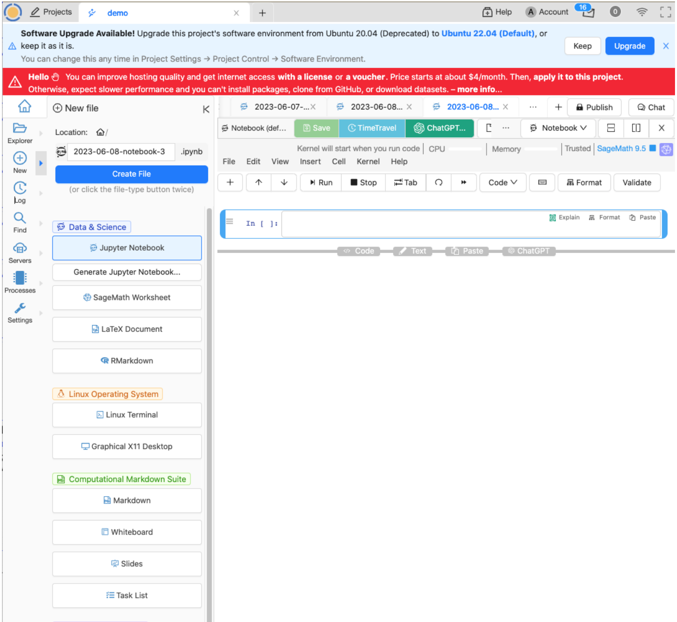
File Edit View Insert Cell Kernel Help
Run
Stop
Tab
Code
Format
Validate
Select a Kernel
This notebook has no kernel. A working kernel is required in order to evaluate the code in the notebook. Please select one for the programming language you want to work with. Otherwise continue without a kernel.
Quick select
Your most recently selected kernel
SageMath 9.5
Make default
Do not ask again. Instead, default to your most recent selection.
You can always change the kernel by clicking on the kernel selector at the top right.
Suggested kernels
Python 3 (system-wide)
Python 3 programming language
R (system-wide)
R statistical programming language
SageMath 9.8
Open-source mathematical software system
Julia 1.8
The Julia Programming Language
All kernels by language
Bash
Bash (Linux)
C++
ROOT (root.cern.ch)
C++17
C++17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You can change this any time in Project Settings → Project Control → Software Environment.
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Create File
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Data & Science
Jupyter Notebook
Generate Jupyter Notebook...
SageMath Worksheet
LaTeX Document
RMarkdown
Linux Operating System
Linux Terminal
Graphical X11 Desktop
Computational Markdown Suite
Markdown
Whiteboard
Slides
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ChatGPT

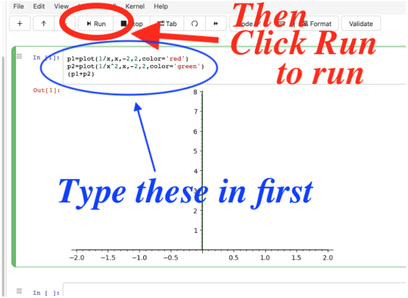
Consider the following example, that a student would like to plot of $\frac{1}{x}$ and $\frac{1}{x^2}$ on the same graph from $x = -2$ to $x = 2$. The student decided to make the plot of $\frac{1}{x}$ in red and $\frac{1}{x^2}$ in green. The input text are:

10/06/2023, 19:09 2023-06-10-notebook-3

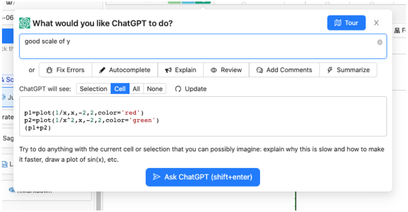
In [0]:

```
p1=plot(1/x,x,-2,2,color='red')
p2=plot(1/x^2,x,-2,2,color='green')
(p1+p2)
```

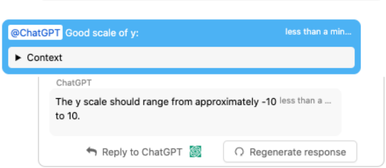
However, the student would have no idea on the range of y values in the plot. Thus, the plot is not quite useful.



Use the built-in **ChatGPT** function in CoCalc and ask for *good scale of* y



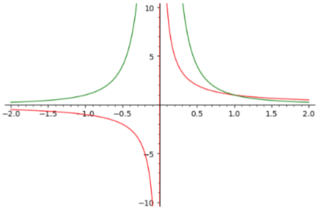
ChatGPT will make suggestions, and it will get smarter. In this particular response, it is suggesting to set $ymax = 10$ and $ymin = -10$



Inspired by the **ChatGPT** answer, thus, the student try $ymax = 10$ and $ymin = -10$, and get the desired plot.

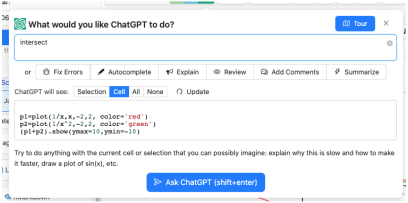
In [5]:
```
p1=plot(1/x,x,-2,2,color='red')
p2=plot(1/x^2,x,-2,2,color='green')
(p1+p2).show(ymin=-10, ymax=10)
```

Out[5]:

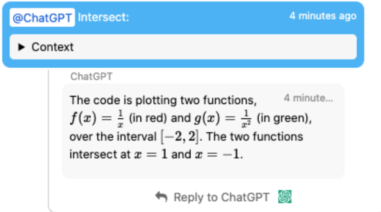


It is often needed to improvised (add some educated guesses) on the suggestions made by **ChatGPT**.

Suppose the student ask **ChatGPT** to check out the intersecting point.



In this case, **ChatGPT** gives two answers: $x = 1$ (which is correct), and $x = -1$ (which is not correct).



It will get smarter.
One can always ask CoCalc to compute the intersection point directly.

```
In [2]: show(solve(1/x-1/x^2==0,x))
Out[2]: [x = 1]
```

Set notations

- A set is a collection of objects.
- An element of a set is an object in the set.
- Objects — lower case; Sets — Upper case.
- $x \in A$ means "x is an element of the set A".
- $x \notin A$ means x is not an element of A.

Sets are described by:

- listing the elements, e.g. $A = \{2, 3, 4, 5\}$.
- stating what special property a typical element x of the set has, e.g.
  $A = \{x : x \text{ is an integer and } 2 \leq x \leq 5\}$.

Universal and empty sets:

- $\emptyset$ denotes the empty set, the set that contains no element (in some other texts, symbol ϕ is used instead).
- Ω denotes the universal set.

Sets sometimes are represented by Venn diagrams. A Venn diagram is an oval drawn on the plane so that all elements of the set are considered to be inside the oval. In the following, A and B are sets.

- A is a subset of B (written $A \subset B$) if every element of A is an element of B.
- A and B are equal (written $A = B$) if they contain the same elements, i.e. $A \subset B$ and $B \subset A$. The following sets S, T, U are equal: $S = \{1, 2, 3, 4\}$=,$T = \{2, 4, 3, 1\}$, $U = \{2, 1, 4, 4, 2, 3\}$.
- The intersection $A \cap B$ is the set $\{x : x \in A$ and $x \in B\}$.
- The union $A \cup B$ is the set $\{x : x \in A$ or $x \in B\}$.
- The relative complement $A \setminus B$ is the set $\{x : x \in A$ and $x \notin B\}$.
- Absolute complement: $A^c = \Omega \setminus A$, (denoted by $\overline{A}$ in some other texts).
- Disjoint Sets, A and B are disjoint if $A \cap B = \emptyset$.
- Product of 2 sets $S \times T = \{(s, t) : s \in S \quad \text{and} \quad t \in T\}$.

Note that the empty set is different from the singleton sets $\{0\}$ or $\{\emptyset\}$. Of course $\emptyset \in \{\emptyset\}$. Quick Questions : Is $\emptyset = \{\emptyset\}$?
Note that in some texts the symbol $\subseteq$ is used to denote subset instead of the commonly used symbol $\subset$.
The followings are not part of the formal mathematical language. We only use them in informal occasions.

- For all $\forall$
- there exists $\exists$
- there exists a unique .... $\exists!$
- implies $\Rightarrow$

Example Consider the following statement about the density of real numbers:

$$\forall\ x > 0,\ \exists\ y > 0 \text{ such that } x > y > 0.$$

Laws of Algebra of Sets

$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative Laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws
$A \cup \emptyset = A, \quad A \cup \Omega = \Omega$ $A \cap \emptyset = \emptyset, \quad A \cap \Omega = A$	Identity Laws
$(A^c)^c = A$	Double complementation
$A \cup A^c = \Omega, \quad A \cap A^c = \emptyset$	
$\Omega^c = \emptyset, \quad \emptyset^c = \Omega$	
$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$	DeMorgan Laws

Power Set: The symbol $|S|$ denotes the number of elements in the set S. For example, $|\emptyset| = 0$.

$\mathcal{P}(S)$ denotes the **Power set** of the set S, it contains all possible subsets of S. If $S = \{a, b, c\}$, then
$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, S\}$.

Quick Question : If there are n elements in S, how many elements are there in $\mathcal{P}(S)$? In other words, if $|S| = n$, what is $|\mathcal{P}(S)|$?

Note that $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

Note also that S is **not** a subset of $\mathcal{P}(S)$. Rather, S is an element of $\mathcal{P}(S)$. For example, $a \in S$ but $a \notin \mathcal{P}(S)$. Instead, we have $\{a\} \subset S$ and $\{a\} \in \mathcal{P}(S)$.

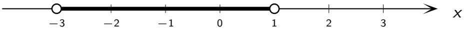
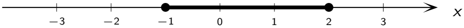
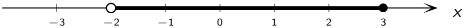
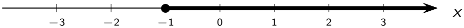
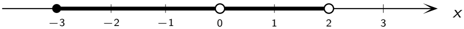
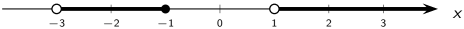
Real numbers and intervals: Real numbers are numbers represented as points on a straight line which extends indefinitely on both sides. The set of all real numbers are usually denoted by the symbol $\mathbb{R}$. Intervals are subsets of $\mathbb{R}$ described in the following table. The real numbers a and b (with $a < b$) for defining the intervals are the endpoints of the intervals.

Notation	Set description	Type
(a, b)	$\{\, x \in \mathbb{R} : a < x < b \,\}$	open
$(a, b]$	$\{\, x \in \mathbb{R} : a < x \leq b \,\}$	half-open
$[a, b)$	$\{\, x \in \mathbb{R} : a \leq x < b \,\}$	half-open
$[a, b]$	$\{\, x \in \mathbb{R} : a \leq x \leq b \,\}$	closed
(a, ∞)	$\{\, x \in \mathbb{R} : a < x \,\}$	open
$[a, \infty)$	$\{\, x \in \mathbb{R} : a \leq x \,\}$	closed
$(-\infty, b)$	$\{\, x \in \mathbb{R} : x < b \,\}$	open
$(-\infty, b]$	$\{\, x \in \mathbb{R} : x \leq b \,\}$	closed
$(-\infty, \infty)$	$\mathbb{R}$	open and closed

Question: Why $\mathbb{R} = (-\infty, \infty)$ could be considered closed?

- Observe that the open interval (a, a) is empty $\emptyset$.
- So $(a, a) = \emptyset$ is open.
- Now observe that the sets $\emptyset$ and $\Omega = \mathbb{R}$ are complementing each other.
- The complement of a closed interval is open, and the complement of an open interval is closed.
- Since $(a, a) = \emptyset$ is open. Therefore, $\mathbb{R}$ could be considered closed.

The following table shows a few examples of subsets of $\mathbb{R}$ as well as their graphs drawn on the x-axis.

Subsets	**Diagrams for the subsets**
$(-3,\ 1)$	
$[-1,\ 2]$	
$(-2,\ 3]$	
$[-1,\ \infty)$	
$[-3,\ 0) \cup (0, 2)$	
$(-3, -1] \cup (1,\ \infty)$	

Absolute values: If x is a real number, the *absolute value* of x is its distance from the origin O. We use the symbol $|x|$ to denote the absolute value. Mathematically,

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Therefore we have $|3| = 3$, $|-4| = 4$ and $|0| = 0$.

Properties. Let $a, b \in \mathbb{R}$. Then

- $|a - b| =$ distance between a and b on the real line.
- $|ab| = |a||b|$, $\quad |a \pm b| \leq |a| + |b|$ (the triangle inequality).
- $|a| < b$ iff[1] $-b < a < b$. Also, $|a| \leq b$ iff $-b \leq a \leq b$.

[1] iff means "if and only if"

Suppose a and $b > 0$ are constants. Consider all $x \in \mathbb{R}$ satisfies the following

$$|x - a| < b.$$

$\Longleftrightarrow$

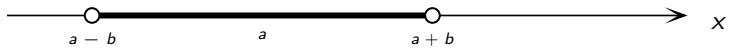
$$-b < x - a < b$$

$\Longleftrightarrow$

$$a - b < x < a + b$$

We say that, these values of x are in the neighbourhood of a with neighbourhood size b.

The following shows the interval of $x \in \mathbb{R}$ that satisfies $|x - a| < b$ where $b > 0$.

Subsets	**Diagrams for the subsets**
$(a - b, a + b)$	

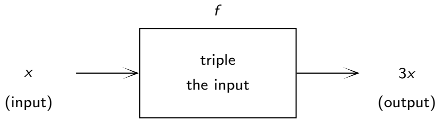
Basic concepts of functions: A function is a rule which, when given a number (input), produces a *single* number (output). Consider the function (or the rule) by which the output is three times the input.

$$\begin{array}{rcl} \text{input} & & \text{output} \\ 2 & \longmapsto & 6 \\ x & \longmapsto & 3x \\ t & \longmapsto & 3t \\ s-4 & \longmapsto & 3(s-4). \end{array}$$

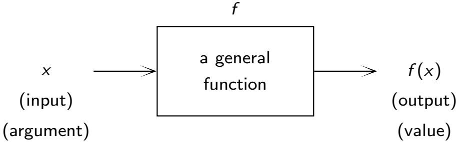
If we denote this function by f, the function can be represented by

$$f : x \longmapsto 3x \qquad \text{or} \qquad f(x) = 3x \qquad \text{or simply} \qquad y = 3x$$

The above function f can be thought of as a machine that gives an output $3x$ if we input x to it.



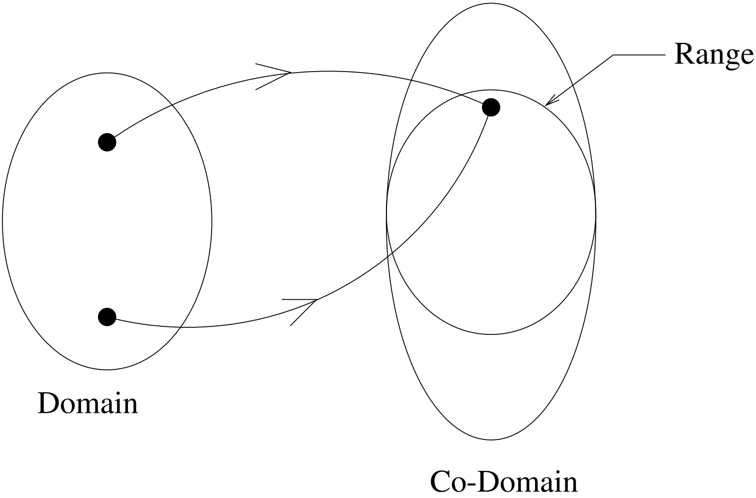
For a general function f, we have



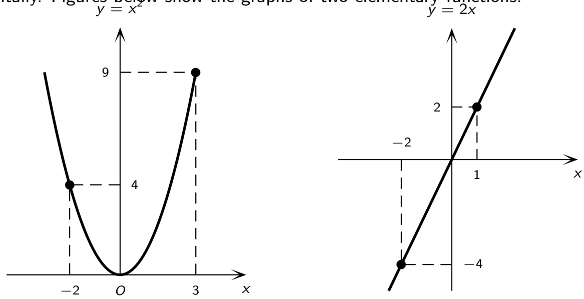
- The input to a function f is called the *argument* and the corresponding output the *value* of the function.
- If the argument is a given number x (so x is given and fixed), the value is denoted by $f(x)$.
- If the argument is a variable number x and $y = f(x)$, then x is called the *independent variable*, y is the *dependent variable*.

To indicate the symbol (x here) being used as the independent variable, we sometimes denote the function by $f(x)$, rather than just by f, and say that $f(x)$ is a function.

Function: Map



Graph of a function: Given a function $y = f(x)$, we can plot the points (x, y) on the xy-plane so that the function values y are plotted vertically and the x-values horizontally. Figures below show the graphs of two elementary functions.



Graph of $f(x) = x^2$ Graph of $f(x) = 2x$

Figure: Two examples of graphs of functions.

Domain and range:

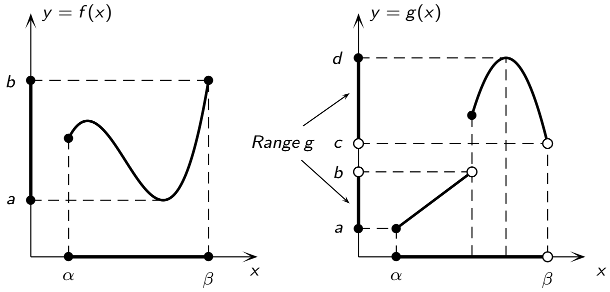
Consider a given function $y = f(x)$. The set of values that x is allowed to take is called the *domain of* f, written for short as $Dom\ f$. The domain is sometimes given when a function is defined.

For instance, we can define a function f in the following way:
$f(x) = x + 2, \quad 1 \leq x < 3$. This indicates that the domain of the function is the interval $[1, 3)$ so that f is not defined for x lies outside $[1, 3)$. However, if the domain is not explicitly given, it is taken to be the largest set possible.

For example, consider the function g defined by $g(x) = \sqrt{x - 2}$, where no domain is explicitly given. However, we understand that the domain of g is $[2, \infty)$ because it is the largest possible set of real numbers x for which $\sqrt{x - 2}$ are real.

The domain of $F(x) = x^2, \quad -\infty < x < \infty$ is $\mathbb{R}$. If we restrict the domain of F to $x > 2$ we get a new function G so that $G(x) = x^2, \quad x > 2$.

The set of values that the function f takes on is called the *range* of the function, written for short as *Range* f. To find *Range* f we ask the following question: What are the values of $y = f(x)$ where $x \in$ *Dom* f?



$$Dom\ f = [\alpha, \beta]$$
$$Range\ f = [a, b]$$

$$Dom\ g = [\alpha, \beta)$$
$$Range\ g = [a, b) \cup (c, d]$$

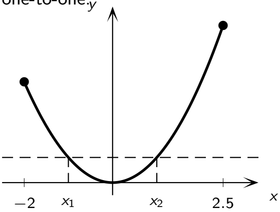
Figure: The domain and the range of functions.

Example:
Find the domains and ranges of the functions f, g, F and G defined previously.
Solution: The answers are shown in the following table:

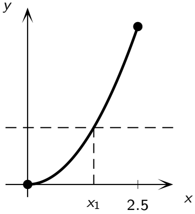
Function	Domain	Range
$f(x) = x + 2,\ 1 \le x < 3$	$1 \le x < 3$	$3 \le y < 5$
$g(x) = \sqrt{x-2}$	$2 \le x < \infty$	$0 \le y < \infty$
$F(x) = x^2,\ x \in \mathbb{R}$	$\mathbb{R}$	$0 \le y < \infty$
$G(x) = x^2,\ x > 2$	$2 < x < \infty$	$4 < y < \infty$

The ranges can be found by considering the graphs of the functions.

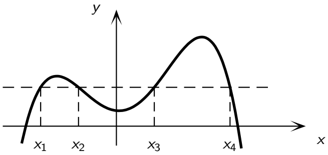
One-to-one functions Consider the function $f(x) = x^2, \quad -2 \le x \le 2.5$. As -1 and 1 are in the domain and $f(-1) = 1 = f(1)$, we see that two different inputs produce the same output. This is demonstrated in the figure below where there are two distinct numbers x_1, x_2 in the domain with $f(x_1) = f(x_2)$. In this case, we say that the function f is *many-to-one*. A function is *one-to-one* or *injective* if different inputs produce different outputs. If we change the domain of the above function to $0 \le x \le 2.5$, we have a new function $g(x) = x^2$, $0 \le x \le 2.5$. This function g is one-to-one.



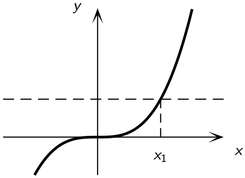
(a) $f(x) = x^2, \ -2 \le x \le 2.5$



(b) $g(x) = x^2, \ 0 \le x \le 2.5$



(a) A many-to-one function.



(b) A one-to-one function.

Composition of functions

Consider the function $y = 3x^2$. The value y of this function can be obtained in two stages: first we square the input x, then we triple the result.

$$x \longmapsto x^2 \longmapsto 3x^2$$

If g and h are functions defined by

$$g(x) = x^2 \quad \text{and} \quad h(x) = 3x$$

we can write

$$y = 3x^2 = 3g(x) = h(g(x)).$$

The function that is defined in terms of two functions g and h this way is denoted by $h \circ g$. That is,

$$\boxed{h \circ g(x) = h(g(x))}$$

Consider functions g and h defined by

$$g(x) = x + 1 \quad \text{and} \quad h(x) = x^2 \quad \text{for all } x \in \mathbb{R}.$$

Show that $g \circ h \neq h \circ g$ and $g \circ h(0) = h \circ g(0)$.

Solution:
We find that

$$g \circ h(x) = g(h(x)) = g(x^2) = x^2 + 1$$

and

$$h \circ g(x) = h(g(x)) = h(x + 1) = (x + 1)^2.$$

Therefore $g \circ h \neq h \circ g$ and $g \circ h(0) = h \circ g(0)$.

Inverse functions

For a given function f, suppose that the input x produces the output y, i.e. $y = f(x)$. We ask: Is there a function g (which depends on the given f) such that

- $Dom\ g = Range\ f$, and that
- $g(f(x)) = x$ (i.e. if $y = f(x)$ then $g(y) = x$) for all $x \in Dom\ f$?

If such a function g exists, we call this the *inverse function* of f and write $g = f^{-1}$. In this case, we have

$$Dom\ f = Range\ f^{-1}\ and\ Dom\ f^{-1} = Range\ f.$$

Theorem
If the function f is one-to-one, the inverse of f exists so that

$$f^{-1}(f(x)) = x \quad \text{for all } x \in \mathrm{Dom}\, f$$

and

$$f(f^{-1}(y)) = y \quad \text{for all } y \in \mathrm{Dom}\, f^{-1}.$$

Example Find the inverse function of $f(x) = \sqrt{x} + 1,\ 0 \le x \le 4.$

Solution
Write $y = \sqrt{x} + 1$ and solve x in terms of y. We obtain

$$x = (y-1)^2$$

from which we see that the given f is one-to-one. The range of f is obviously (perhaps from the graph of f) $1 \le y \le 3$. Therefore

$$f^{-1}(y) = (y-1)^2, \quad 1 \le y \le 3.$$

If we wish to use x rather than y as the independent valuable, we can replace y by x in the above to get another form of the solution:

$$f^{-1}(x) = (x-1)^2, \quad 1 \le x \le 3.$$

Example: Consider the function f defined by $f(x) = x^2, \ -3 \leq x \leq 3$. Show that this function is many-to-one and hence has no inverse.

Solution:
If we solve $y = x^2$ for x, we get two results

$$x = \sqrt{y} \quad \text{and} \quad x = -\sqrt{y}.$$

Since $\sqrt{y} \neq -\sqrt{y}$ if $y \neq 0$, we see that there are two different x-values taking the same nonzero y-value. This shows that the function is many-to-one.

However, if we restrict the domain to say the interval $[0, 3]$, the function f becomes a new function F which is one-to-one and whose range is $[0, 9]$.

Periodic functions

A function $f(x)$ defined on $\mathbb{R}$ is said to be *periodic* if there is a positive constant T (called a *period*) such that

$$f(x+T) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

Clearly, if T is a period of f, so are $2T$, $3T$, etc. Usually, when we say *the period* of a function, we mean the smallest period.

Example

For the functions shown below, each has a period of 2.

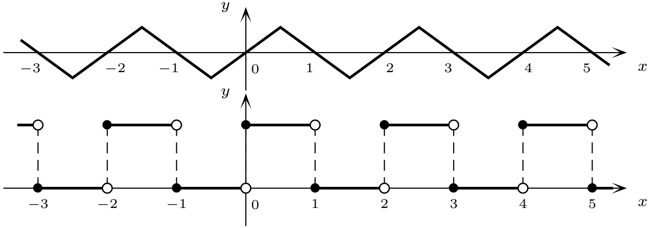
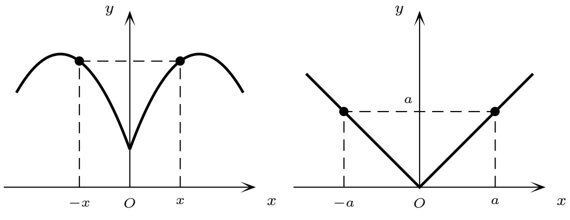


Figure: Two examples of periodic functions.

Even and odd functions

A function f is an *even* function if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. The graph of an even function is symmetrical about the y-axis.



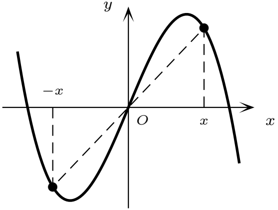
(a) An even function (b) The function $|x|$

Example

$|x|$, x^{2k} (k is an integer) and $\cos x$ are even functions.

Note that $|x|$ is an even function.

A function f is an *odd* function if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. The graph of an odd function is symmetrical about the origin.



(c) An odd function

Example

x^{2k+1} (k is an integer) and $\sin x$ are odd functions.

Polynomials

A polynomial is a function of the form

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where $a_0, \ a_1, \ldots, \ a_n$ are given constants (called the *coefficients*) and x is the independent variable. The domain of $P(x)$ is $\mathbb{R}$. If $a_n \neq 0$, n is the *degree* of $P(x)$. We sometimes write $\deg P$ for the degree of $P(x)$.

If all the coefficients $a_0, \ a_1, \ldots, \ a_n$ are zero, the polynomial reduces to *the zero polynomial*. The degree of the zero polynomial is regarded as 0 in this book. A *zero* of $P(x)$ is a root (or a solution) of the equation $P(x) = 0$.

Polynomial	Degree	Name
a_0	0	constant
$a_0 + a_1x, \ (a_1 \neq 0)$	1	linear
$a_0 + a_1x + a_2x^2, \ (a_2 \neq 0)$	2	quadratic
$a_0 + a_1x + a_2x^2 + a_3x^3, \ (a_3 \neq 0)$	3	cubic

The graphs of polynomials are continuous curves.

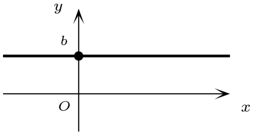
The following theorems are fundamental.

Remainder Theorem If we divide a polynomial $P(x)$ by $x - a$, the remainder is $P(a)$.

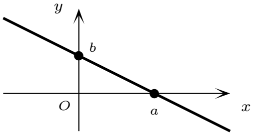
Fundamental theorem of algebra
If $P(z)$ is a polynomial of degree n (with real or complex coefficients, $n \neq 0$), the equation $P(z) = 0$ has exactly n roots (counting real roots, complex roots and their multiplicities).

This theorem involves complex numbers[2] and is the rare occasion where complex numbers are mentioned in this set of notes.

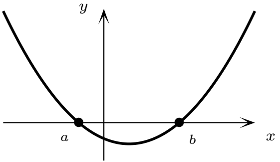
[2]See for example, Weir, et al., *Thomas' Calculus*, 8th edition, 2005



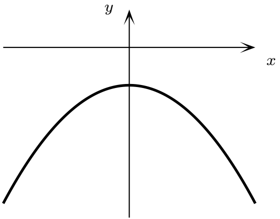
(a) Degree 0: $y = b$



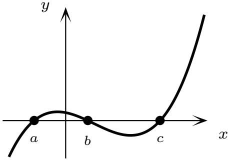
(b) Degree 1: $x/a + y/b = 1$



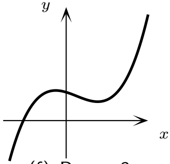
(c) Degree 2: $y = (x - a)(x - b)$



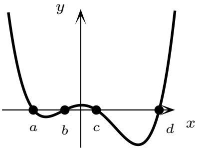
(d) Degree 2: $y = -x^2 - 2$



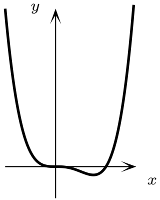
(e) Degree 3:
$y = (x-a)(x-b)(x-c)$



(f) Degree 3:
$y = x^3 - x^2 - x + 2$



(g) Degree 4:
$y = (x-a)(x-b)(x-c)(x-d)$



(h) Degree 4:
$y = x^4 - x^3$

In [2]:
```
f(x)=x^2+3*x+2
show(f)
```

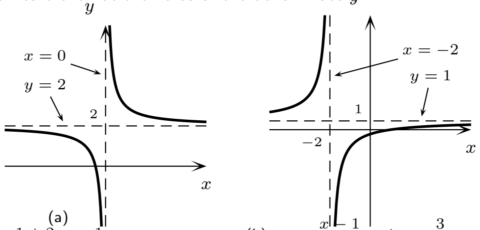
Out[2]: $x \mapsto x^2 + 3\,x + 2$

In [3]:
```
show(f.factor())
```

Out[3]: $x \mapsto (x + 2)(x + 1)$

Rational functions

A *rational function* $f(x)$ is the quotient of two polynomials: $f(x) = \frac{P(x)}{Q(x)}$.
$f(x)$ is not defined when $Q(x) = 0$. The graph of a rational function is formed by continuous curves broken at the zeros of the denominator.



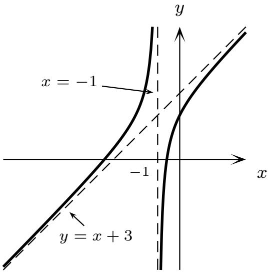
(a) $y = \dfrac{1+2x}{x} = \dfrac{1}{x} + 2$

(b) $y = \dfrac{x-1}{x+2} = 1 - \dfrac{3}{x+2}$

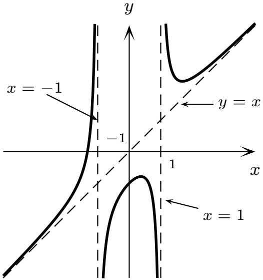
A rational function is *proper* if the degree of the numerator is less than that of the denominator. Otherwise it is *improper*.

By direct division, we can write a given improper rational function in the form:

$$\begin{pmatrix} \text{An improper} \\ \text{rational function} \end{pmatrix} = \text{a polynomial} + \begin{pmatrix} \text{a proper} \\ \text{rational function} \end{pmatrix} \tag{1}$$



(c) $y = x + 3 - \dfrac{1}{x+1}$



(d) $y = x + \dfrac{1}{(x-1)(x+1)}$

Figure: Examples of rational functions.

In [1]:

```
numerator(x)=x^4+4*x^3+3*x^2-3
denominator(x)=x^2+3*x+2
f(x)=numerator(x)/denominator(x)
show(f)
```

Out[1]:

$$x \mapsto \frac{x^4 + 4\,x^3 + 3\,x^2 - 3}{x^2 + 3\,x + 2}$$

In [2]:

```
[quotient,remainder]=(numerator(x)).maxima_methods().divide(denominator(x))
show(quotient)
```

Out[2]:

$$x^2 + x - 2$$

In [3]:

```
show(remainder)
```

Out[3]:

$$4\,x + 1$$

In [4]:

```
show(factor(denominator(x)))
```

Out[4]:

$$(x + 2)(x + 1)$$

In [5]:

```
show((remainder(x)/denominator(x)).partial_fraction())
```

Out[5]:

$$\frac{7}{x + 2} - \frac{3}{x + 1}$$

In [6]:

```
# double check
show(f.partial_fraction())
```

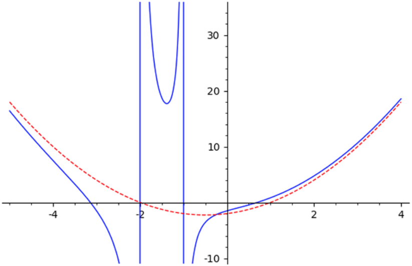
Out[6]:

$$x \mapsto x^2 + x + \frac{7}{x+2} - \frac{3}{x+1} - 2$$

In [7]:

```
p1=plot(f(x),x,-5,4)
p2=plot(quotient(x),x,-5,4,  rgbcolor="red",  linestyle = "dashed")
(p1+p2).show(xmin=-5, xmax=4, ymin=-10, ymax=35)
```

Out[7]:



Example:

Express the improper rational function $\dfrac{x^4+4x^3+3x^2-3}{x^2+3x+2}$ in the form of an improper rational function.

Solution

By long division, we get

$$
\begin{array}{rrrrrr}
 & x^2 & +x & -2 & & \\ \cline{2-6}
x^2 \quad +3x \quad +2 \;\big) & x^4 & +4x^3 & +3x^2 & +0 & -3 \\
 & x^4 & +3x^3 & +2x^2 & & \\ \cline{2-5}
 & & x^3 & +x^2 & +0 & \\
 & & x^3 & +3x^2 & +2x & \\ \cline{3-6}
 & & & -2x^2 & -2x & -3 \\
 & & & -2x^2 & -6x & -4 \\ \cline{4-6}
 & & & & 4x & +1
\end{array}
$$

and hence
$$\frac{x^4+4x^3+3x^2-3}{x^2+3x+2} = x^2+x-2+\frac{4x+1}{x^2+3x+2}.$$

In [2]:
```
f(x)=x^4+4*x^3+3*x^2-3
g(x)=x^2+3*x+2
show(f)
show(g)
```

Out[2]:

$$x \mapsto x^4 + 4\,x^3 + 3\,x^2 - 3$$

$$x \mapsto x^2 + 3\,x + 2$$

In [3]:
```
show(f.maxima_methods().divide(g))
```

Out[3]:

$$\left[x^2 + x - 2, 4\,x + 1\right]$$

In [4]:
```
show((f/g).partial_fraction())
```

Out[4]:

$$x \mapsto x^2 + x + \frac{7}{x+2} - \frac{3}{x+1} - 2$$

Asymptotes of rational functions

We see that the graph of a rational function consists of two or more continuous branches. Each of these branches approaches to a straight line (drawn as a dashed line) as the point on the branch moves towards infinity in a certain direction. Such a straight line is called an *asymptote* of the graph. The equations of the asymptotes of a given rational function can be found using the following theorem.

Theorem Let $P(x)$ and $Q(x)$ be nonzero polynomials having no common factor. Let $f(x) = P(x)/Q(x)$ be a rational function and suppose that $(x - c_1)$, $(x - c_2)$, etc. are factors of $Q(x)$ where c_1, c_2, etc. are distinct real constants.

- Then the vertical lines $x = c_1$, $x = c_2$, etc. are asymptotes of the graph of $f(x)$.
- Furthermore, if $\deg P \leq \deg Q + 1$ so that $f(x)$ can be resolved in the following special form of:

$$f(x) = ax + b + \frac{S(x)}{Q(x)}, \ \ \deg(S) < \deg(Q).$$

then the line $y = ax + b$ is also an asymptote of the graph.

Remark
In the first part of the theorem, the asymptotes are vertical. In the second part of the theorem, if $a \neq 0$, the asymptote is oblique, while if $a = 0$ the asymptote is horizontal. For a proper rational function $(\deg P < \deg Q)$, we have $a = b = 0$ and therefore the x-axis $(y = 0)$ is an asymptote of the graph.

Example

Find the asymptotes of the rational function $f(x) = \dfrac{x^3 + 2x^2 + 1}{(x-1)(x+2)}$.

Solution By long division, $f(x) = x + 1 + \dfrac{x+3}{(x-1)(x+2)}$. Therefore the asymptotes are the lines

$$x = 1, \quad x = -2 \quad y = x + 1.$$

Partial fractions

A proper rational function, with real coefficients, can sometimes be expressed as a sum of two or more proper rational functions, with real coefficients, called *partial fractions*.
For example,

$$\frac{x-3}{(2x-1)(x^2+1)} = \frac{-2}{2x-1} + \frac{x+1}{x^2+1}.$$

In Chapter 5 of the text book, we have to resolve a rational function into partial fractions this way to do integration, an important topic in calculus.

Each factor of the denominator of a given rational function, is associated with a partial fraction or a sum of partial fractions. The rule of association is shown in the table next page for a linear factor and an irreducible quadratic factor (that cannot be factorized into a product of real linear factors).

Rule	Factor of denominator	Form of the partial fractions
1	$ax+b$	$\dfrac{A_1}{ax+b}$
2	$(ax+b)^2$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}$
3	$(ax+b)^3$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\dfrac{A_3}{(ax+b)^3}$
4	ax^2+bx+c	$\dfrac{A_1x+B_1}{ax^2+bx+c}$
5	$(ax^2+bx+c)^2$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}$

Note: $a, b, c, A_1, A_2, A_3, B_1, B_2$ are real constants, $a \neq 0$.

Example

Resolve $f(x) = \dfrac{x+3}{(x-1)(x-3)}$ into partial fractions.

Solution

First we observe that the given $f(x)$ is a proper rational function. Next we consider each factor of the denominator of $f(x)$. There are two linear factors $x-1$ and $x-3$. By Rule 1 of the table next page, we can assume partial fractions of the forms $\frac{A}{x-1}$, $\frac{B}{x-3}$ (where A, B

are real constants) and get the identity

$$\frac{x+3}{(x-1)(x-3)} \equiv \frac{A}{x-1} + \frac{B}{x-3}$$

To find the constants A and B, we remove the denominators and get $x+3 \equiv A(x-3) + B(x-1)$.

Comparing the coefficient of x and the constant term, we get two equations $1 = A + B$, $3 = -3A - B$. Solving these equations we get $A = -2$, $B = 3$. Therefore

$$\frac{x+3}{(x-1)(x-3)} \equiv \frac{-2}{x-1} + \frac{3}{x-3}$$

%vspace0.3cm The above method for finding the coefficients A and B is called the *method of undetermined coefficients*.

In [2]:
```
numerator(x)=x+3
denominator(x)=(x-1)*(x-3)
f(x)=numerator(x)/denominator(x)
show(f)
```

Out[2]: $x \mapsto \frac{x+3}{(x-1)(x-3)}$

In [3]:
```
show(f.partial_fraction())
```

Out[3]: $x \mapsto -\frac{2}{x-1} + \frac{3}{x-3}$

Example

Resolve $f(x) = \dfrac{7x+5}{(x+1)^2(x-1)}$ into partial fractions.

First we observe that the given $f(x)$ is a proper rational function. Next we consider each factor of the denominator of $f(x)$. There are two linear factors $x+1$ (with power 2) and $x-1$. By Rule 1 and Rule 2 of the above table, we can assume partial fractions of the forms $\frac{A}{x+1} + \frac{B}{(x+1)^2} \quad \frac{C}{x-1}$ (where A, B, C are real constants) and get the identity $\frac{7x+5}{(x+1)^2(x-1)} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$.
Therefore we have $7x+5 \equiv A(x+1)(x-1) + B(x-1) + C(x+1)^2$.

Comparing the coefficient of

$$\begin{array}{rcl} x^2: & 0 & = A + C \\ x: & 7 & = B + 2C \\ \text{constant term}: & 5 & = -A - B + C. \end{array}$$

Solving these equations we get $A = -3$, $B = 1$, $C = 3$. Therefore

$$\frac{7x+5}{(x+1)^2(x-1)} \equiv \frac{-3}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x-1}$$

In [2]:
```
numerator(x)=7*x+5
denominator(x)=(x+1)^2*(x-1)
f(x)=numerator(x)/denominator(x)
show(f)
```

Out[2]: $x \mapsto \frac{7\,x+5}{(x+1)^2(x-1)}$

In [3]:
```
show(f.partial_fraction())
```

Out[3]: $x \mapsto -\frac{3}{x+1} + \frac{3}{x-1} + \frac{1}{(x+1)^2}$

Example

Resolve $f(x) = \dfrac{x-3}{(2x-1)(x^2+1)}$ into partial fractions.

The denominator has two factors: one is $2x-1$ and the other is x^2+1. By Rule 1 and Rule 4, $f(x)$ has partial fractions in the forms $\frac{A}{2x-1}$ $\frac{Bx+C}{x^2+1}$ where A, B, C are real constants. Therefore we have the identity

$$\frac{x-3}{(2x-1)(x^2+1)} \equiv \frac{A}{2x-1} + \frac{Bx+C}{x^2+1}$$

and hence $x - 3 \equiv A(x^2+1) + (Bx+C)(2x-1)$. Comparing coefficients of

$$\begin{aligned} x^2: &\qquad 0 &&= A + 2B \\ x: &\qquad 1 &&= -B + 2C \\ \text{constant term:} &\qquad -3 &&= A - C. \end{aligned}$$

Solving the equations, we get $A = -2$, $B = 1$, $C = 1$. Therefore

$$\frac{x-3}{(2x-1)(x^2+1)} \equiv \frac{-2}{2x-1} + \frac{x+1}{x^2+1}$$

In [1]:
```
numerator(x)=x-3
denominator(x)=(2*x-1)*(x^2+1)
f(x)=numerator(x)/denominator(x)
show(f)
```

Out[1]: $x \mapsto \frac{x-3}{(x^2+1)(2\,x-1)}$

In [2]:
```
show(f.partial_fraction())
```

Out[2]: $x \mapsto \frac{x+1}{x^2+1} - \frac{2}{2\,x-1}$

Trigonometric functions

Consider the xy-plane in rectangular coordinates such that the scales on both axes are the same. Let P be an arbitrary point on the unit circle (with centre at the origin O and unit radius). If the straight line OP makes an angle θ (in radian) with the positive x-axis and if P has coordinates (x, y), we define the *sine*, *cosine* and *tangent* functions by

$$\cos\theta = x, \quad \sin\theta = y \quad \text{and} \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

so that

$$\cos^2\theta + \sin^2\theta = 1.$$

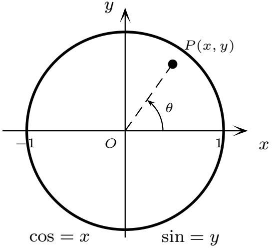


Figure: Definitions of $\cos$ and $\sin$.

Both $\cos$ and $\sin$ are continuous periodic functions with a period of 2π. $\cos$ is even while $\sin$ is odd. $\tan x$ is discontinuous at $x = \pm\pi/2, \pm 3\pi/2, \ldots$ where $\cos x = 0$.

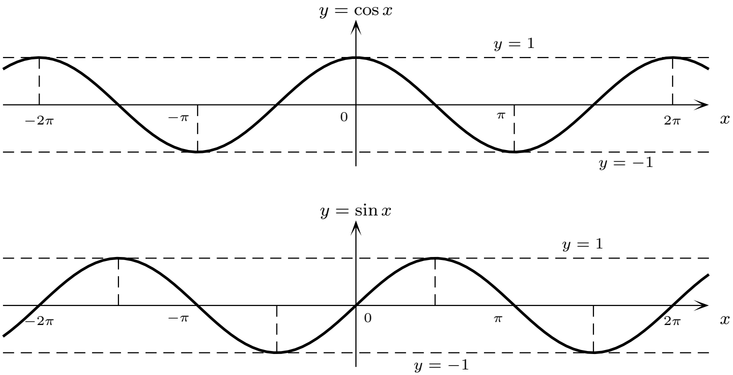


Figure: The graphs of $\cos x$ and $\sin x$.

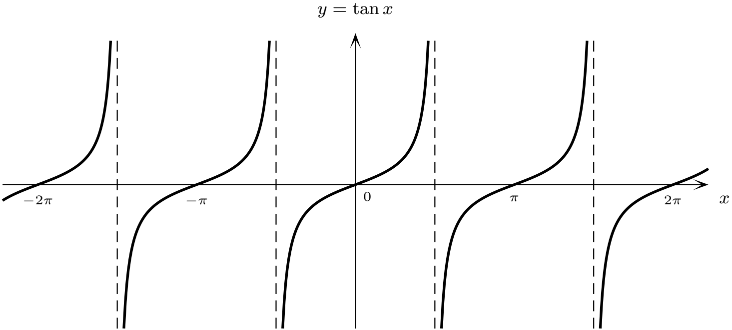


Figure: The graph of $\tan x$

Compound angle formulas

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B,\\
\sin(A-B) &= \sin A\cos B - \cos A\sin B\\
\cos(A+B) &= \cos A\cos B - \sin A\sin B\\
\cos(A-B) &= \cos A\cos B + \sin A\sin B\\
\tan(A+B) &= \frac{\tan A + \tan B}{1-\tan A\tan B}\\
\tan(A-B) &= \frac{\tan A - \tan B}{1+\tan A\tan B}
\end{aligned}$$

Double angle formulas:

$$
\begin{aligned}
\sin 2A &= 2\sin A\cos A \\
\cos 2A &= \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 \\
\tan 2A &= \frac{2\tan A}{1-\tan^2 A} \\
\cos^2 A &= \frac{1+\cos 2A}{2} \\
\sin^2 A &= \frac{1-\cos 2A}{2}
\end{aligned}
$$

Conversion formulas

$\sin(x+y)+\sin(x-y)=2\sin x\cos y$
$\sin(x+y)-\sin(x-y)=2\cos x\sin y$
$\cos(x+y)+\cos(x-y)=2\cos x\cos y,$
$\cos(x+y)-\cos(x-y)=-2\sin x\sin y.$

$$\begin{aligned}
\sin A+\sin B &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\
\sin A-\sin B &= 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\\
\cos A+\cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\
\cos A-\cos B &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)
\end{aligned}$$

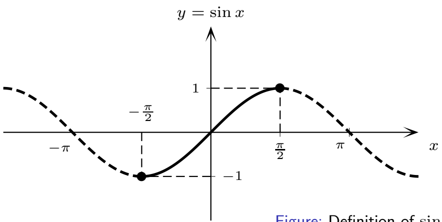
Inverse trigonometric functions

- **Arcsine.**

The function $\sin x$ is many-to-one. However, if we restrict the domain to the interval $[-\pi/2, \pi/2]$, the function becomes one-to-one and its range is $[-1,\ 1]$. With this special domain restriction, the inverse function of $\sin$ exists. It is called the *arcsine* function and is denoted by $\sin^{-1}$ or $\arcsin$. Thus,

$$\boxed{x = \sin^{-1} y \quad \text{iff} \quad y = \sin x \quad \text{and} \quad x \in [-\pi/2, \pi/2].}$$

Note that $\sin(\sin^{-1} y) = y$ for all $-1 \leq y \leq 1$ but $\sin^{-1}(\sin x) = x$ iff $\dfrac{-\pi}{2} \leq x \leq \dfrac{\pi}{2}$.



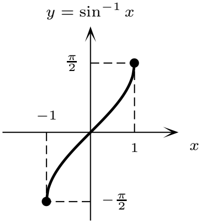


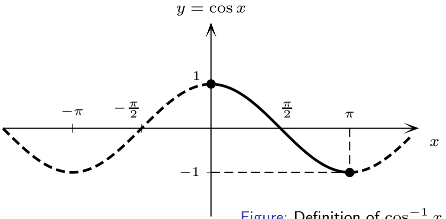
Figure: Definition of $\sin^{-1} x$.

Arccosine.

The function $\cos x$ is many-to-one. However, if we restrict the domain to the interval $[0, \pi]$, the function becomes one-to-one and its range is $[-1,\ 1]$. With this special domain restriction, the inverse function of $\cos$ exists. It is called the *arccosine* function and is denoted by $\cos^{-1}$ or $\arccos$. Thus,

$$\boxed{x = \cos^{-1} y \quad \text{iff} \quad y = \cos x \ \text{ and } \ x \in [0, \pi].}$$

Note that $\cos(\cos^{-1} y) = y$ for all $-1 \leq y \leq 1$ but $\cos^{-1}(\cos x) = x$ iff $0 \leq x \leq \pi$.



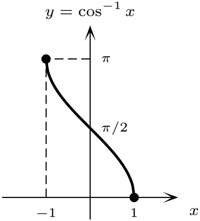


Figure: Definition of $\cos^{-1} x$.

Arctangent.

The function $\tan x$ is many-to-one. However, if we restrict the domain to the interval $(-\pi/2, \pi/2)$, the function becomes one-to-one and its range is $\mathbb{R}$. With this special domain restriction, the inverse function of $\tan$ exists. It is called the *arctangent* function and is denoted by $\tan^{-1}$ or $\arctan$. Thus,

$$\boxed{x = \tan^{-1} y \quad \text{iff} \quad y = \tan x \ \text{ and } \ x \in (-\pi/2, \pi/2).}$$

Note that $\tan(\tan^{-1} y) = y$ for all $-\infty < y < \infty$ but $\tan^{-1}(\tan x) = x$ iff $\frac{-\pi}{2} < x < \frac{\pi}{2}$.

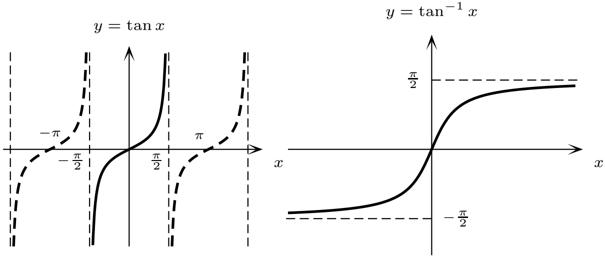


Figure: Definition of $\tan^{-1} x$.

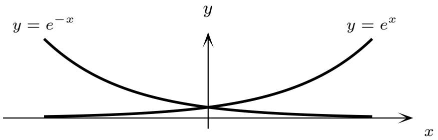
Exponential functions

The function $y = a^x$ is called an *exponential function*. The number a is the *base* and x the *exponent* (or *index*, or *power*). In order that $y = a^x$ takes on real values for all real x, we must assume $a > 0$.

Law of indices

$a^m a^n = a^{m+n}$	$a^m/a^n = a^{m-n}$	$(a^m)^n = a^{mn}$
$a^0 = 1$	$a^{-1} = 1/a$	$a^{-m} = 1/a^m$

If the base is the number e $= 2.718281828459\cdots$, the exponential function is denoted by $\exp$ ($\exp(x) \equiv \mathrm{e}^x$).



The graph of $y = e^x$ shows the exponential growth while that of $y = e^{-x}$ shows the exponential decay.

Logarithmic functions

Let a be a positive constant and consider the exponential function $y = a^x$. This function is one-to-one and its range is $(0, \infty)$. The inverse function of this exponential function is called a *logarithmic* function and is defined by

$$x = \log_a y \;\; (0 < y < \infty) \quad \text{iff} \quad y = a^x \;\; (-\infty < x < \infty)$$

The number $\log_a y$, where $y > 0$, is called the *logarithm* of y to the *base* a.
'cm

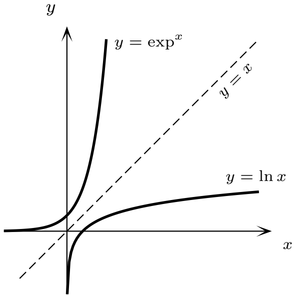


Figure: Graphs of $\exp x$ and $\ln x$.

Rules of logarithm Let a, b, x, y be positive real numbers.

$\log_a(xy) = \log_a x + \log_a y$	$\log_a 1 = 0$
$\log_a(x/y) = \log_a x - \log_a y$	$\log_a x^m = m\log_a x$
$\log_a x = \dfrac{\log_b x}{\log_b a}$	where m is real

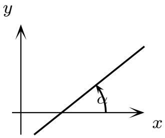
If the base is e $= 2.718281828\cdots$, the logarithm function is denoted by $\ln$ or $\log$. Thus

$$\boxed{y = \ln x \ \ (0 < x < \infty) \quad \text{iff} \quad x = \exp^y \ \ (-\infty < y < \infty).}$$

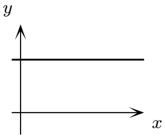
Slope of a straight line

Consider any straight line on the xy-plane. If the line is *not* parallel to the x-axis, the *angle of inclination* is the angle between the line and the positive x-axis. If the line is parallel to the x-axis, the angle of inclination is 0. If the angle of inclination of a straight line is α and $\alpha \neq \pi/2$, then the *slope* of the straight line is the real number

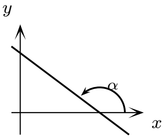
$$\boxed{\text{slope } = \tan\alpha.}$$



Inclined line
$0 < \alpha < \pi/2$
Slope $(= \tan\alpha)$ is positive



Horizontal line.
Angle of incl. $= 0$
Slope $= \tan 0 = 0$



Inclined line
$\pi/2 < \alpha < \pi$
Slope $(= \tan\alpha)$ is negative

The slope gives a measure of the steepness of the straight line. Also the sign of the slope tells us in which direction the straight line is running. Using the following theorem, we can find the slope

of a non-vertical line based on the coordinates of two distinct points on the line.

Theorem Let (x_0, y_0) and (x_1, y_1) be two points on a straight line with $x_0 \neq x_1$, then the slope m of the straight line is given by the formula $m = (y_1 - y_0)/(x_1 - x_0)$.

Example Find the slope the line containing the points $(3, -2)$ and $(-4, 1)$.

Solution The slope is $(1 - (-2))/(-4 - 3) = -3/7$.

One-sided limits

Consider a function $f(x)$ which is defined in $\mathbb{R}$ so that

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 2, \\ g(x) & \text{otherwise} \end{cases}$$

where $g(x)$ is a function which is not important (in fact not considered) in the following discussion.

x	0.0	1.0	1.90	1.99	1.999	1.9999
y	0.0	1.0	3.61	3.96	3.996	3.9996

The table shows that the values of $f(x)$ approaches the number 4 as x increases and approaches 2. This number 4 is called the limit of $f(x)$ as x approaches 2 from the left.

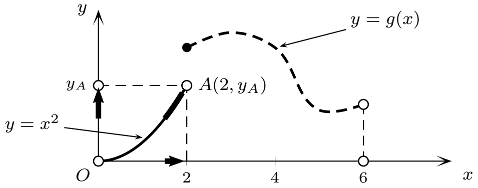
Intuitive definition

Definition
Let a be a point on the real axis such that $f(x)$ is defined when x is on the left of a $(x < a)$ and near to a and let L be a real number. If the values of $f(x)$ approaches L as x increases and approaches a, we call L *the limit of* $f(x)$ *as* x *approaches* a *from the left* and write

$$\lim_{x \to a^-} f(x) = L.$$

Remark The symbol $x \to a^-$ represents "x approaches a from the left" which means "x is getting closer and closer to a though it is always on the left of a". The number L, usually dependent on $f(x)$ and a, is also called *the left-hand limit of* $f(x)$ *at* a. The value $f(a)$, whether it is defined or not, plays no part in the definition of the left-hand limit at a.

Graphical demonstration



On an interval to the left of 2, say $(0, 2)$, the graph of the function $f(x)$ is a portion of the parabola $y = x^2$ as shown. On the graph, we see that as x moves towards 2 from the left the points on the parabola moves towards the point A and hence y moves towards the y-coordinate y_A of A. Since the curve OA here is part of the parabola $y = x^2$, we see that $y_A = 2^2 = 4$ and hence $\lim_{x \to 2^-} f(x) = 4$. Note that the value of $f(2)$ and the function $g(x)$ do not play any part in the definition of the left-hand limit at $x = 2$.

Right-hand limit

Analogous to the left-hand limit, we define the right-hand limit as follows:

Definition

Let a be a point on the real axis such that $f(x)$ is defined when x is on the right of a $(x > a)$ and near to a and let R be a real number. If the values of $f(x)$ approaches R as x decreases and approaches a, we call R *the limit of* $f(x)$ *as* x *approaches* a *from the right* and write

$$\lim_{x \to a^+} f(x) = R.$$

Remark The symbol $x \to a^+$ represents "x approaches a from the right." The number R, usually dependent on $f(x)$ and a, is also called *the right-hand limit of* $f(x)$ *at* a. The value $f(a)$, whether it is defined or not, plays no part in the definition of the right-hand limit at a.

Examples Let $F(x)$ be the piecewise-defined function defined by the graph shown below.Find the left-hand limits and the right-hand limits of $F(x)$ at the points $x = 1$, 2, 3, 4, 5 and 6.

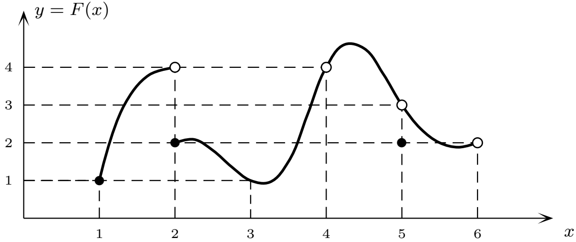


Figure: The graph of $F(x)$ defined on $[1, 4) \cup (4, 6)$.

Solution From the graph we see that

$$\lim_{x\to 1^-} F(x) \text{ is not defined}, \qquad \lim_{x\to 1^+} F(x) = 1,$$

$$\lim_{x\to 2^-} F(x) = 4, \qquad \lim_{x\to 2^+} F(x) = 2,$$

$$\lim_{x\to 3^-} F(x) = 1, \qquad \lim_{x\to 3^+} F(x) = 1,$$

$$\lim_{x\to 4^-} F(x) = 4, \qquad \lim_{x\to 4^+} F(x) = 4,$$

$$\lim_{x\to 5^-} F(x) = 3, \qquad \lim_{x\to 5^+} F(x) = 3,$$

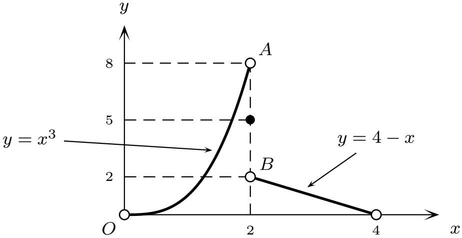
$$\lim_{x\to 6^-} F(x) = 2, \qquad \lim_{x\to 6^+} F(x) \text{ is not defined}.$$

Example:

Let $f(x)$ be the function on $(0, 4)$ defined by

$$f(x) = \begin{cases} x^3 & \text{if} \quad 0 < x < 2, \\ 5 & \text{if} \quad x = 2, \\ 4 - x & \text{if} \quad 2 < x < 4. \end{cases}$$

Sketch the graph of this function and find the limits $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.



From the graph we see that

$$\lim_{x\to 2^-} f(x) = \text{the } y\text{-coordinate of } A = \left[x^3\right]_{x=2} = 8.$$

$$\lim_{x\to 2^+} f(x) = \text{the } y\text{-coordinate of } B = \left[4-x\right]_{x=2} = 2.$$

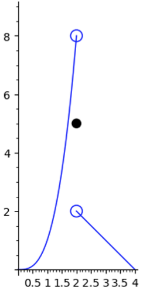
Note that, without referring to the graph, we can simply work out the limits as follows:

$$\lim_{x\to 2^-} f(x) = \left[x^3\right]_{x=2} = 8, \quad \lim_{x\to 2^+} f(x) = \left[4-x\right]_{x=2} = 2.$$

In [1]:

```
f = piecewise( [ [(0,2),x^3],[[2,2],5], [(2,4),4-x] ] )
p1= plot( x^3, 0, 2 )
p2= plot( 4-x, 2, 4 )
pt1 = circle((2,8), 0.2)
pt2 = circle((2,2), 0.2)
pt3 = point((2,5), rgbcolor='black', pointsize=80)
(p1 + p2 + pt1 + pt2 + pt3).show(xmin=0, xmax=4, ymin=0, ymax=9)
```

Out[1]:



In [2]:

```
f(2)
```

Out[2]:

5

In [3]:

```
# alternatively, we can use unit_step function
# but we CANNOT define a separate point g(2)=5 using this way
g(x)=x^3*(unit_step(x)-unit_step(x-2)) + (4-x)*(unit_step(x-2)-unit_step(x-4))
#but we can take limit of g(x) at x=2 from positive or negative side
# g is a right-continuous function
```

In [4]:

```
limit(g(x),x=2,dir='+')
```

Out[4]:

2

In [5]:

```
limit(g(x),x=2,dir='-')
```

Out[5]:

8

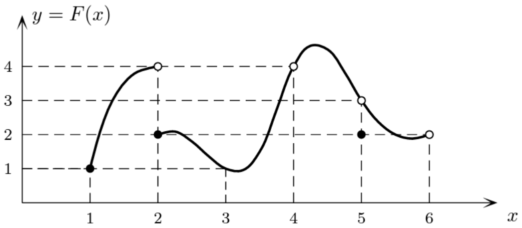
In [6]:

```
# since g is right-continuous, g(2)=2
g(2)
```

Out[6]:

2

Limits of functions:



Let

$$L = \lim_{x \to a^-} F(x) \quad R = \lim_{x \to a^+} F(x).$$

Using the results of Example of $F(x)$, we have

- At $a = 1$: $L \neq R$ since L is not defined.
- At $a = 2$: $L \neq R$.
- At $a = 3$: $L = R = f(a)$.
- At $a = 4$: $L = R \neq f(a)$ since $f(a)$ is not defined.
- At $a = 5$: $L = R \neq f(a)$.
- At $a = 6$: $L \neq R$ since R is not defined.

These results shows that for the equality and inequality of L and R at a general point a, there are three possibilities:

Three possibilities at a

1. $L \neq R$.	a is an endpoint of the domain of $f(x)$, or there is a vertical gap (or jump) at a.
2. $L = R \neq f(a)$.	There is no vertical gap but there is a hole on the graph. $f(a)$ may or may not be defined in this case.
3. $L = R = f(a)$.	There is neither a vertical gap nor a hole on the graph.

When Case **2** or Case **3** in the table occurs we call the common values of L and R simply the limit of the function.

Definition of Limit of a function:

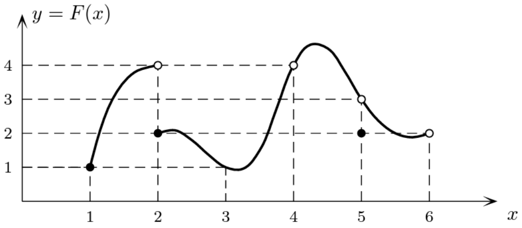
If the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and both are equal to L, we say that *the limit of* $f(x)$ *as* x *approaches* a *is* L and write

$$\lim_{x\to a} f(x) = L.$$

If either of the one-sided limits does not exist or, if they exist but are not equal, we say that *the limit of* $f(x)$ *as* x *approaches* a *does not exist.*

Example:

Consider again the piecewise-defined function $F(x)$. Find the limits of $F(x)$ (if exist) at the points $x = 1,\ 2,\ 3,\ 4,\ 5$ and 6.



$\lim\limits_{x\to 1} F(x)$ does not exist.

$\lim\limits_{x\to 2} F(x)$ does not exist.

$\lim\limits_{x\to 3} F(x) = 1.$

$\lim\limits_{x\to 4} F(x) = 4.$

$\lim\limits_{x\to 5} F(x) = 3.$

$\lim\limits_{x\to 6} F(x)$ does not exist.

Figure below shows the graph of $f(x) = (\sin x)/x$. Based on the graph, we see that the limit $\lim_{x\to 0} f(x)$ exists. Find this limit by evaluating the values of $f(x)$ near $x = 0$.

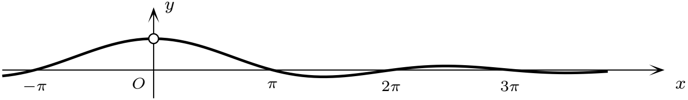


Figure: The graph of $(\sin x)/x$.

Solution.

The values of $f(x) = (\sin x)/x$ are computed by a desk calculator at some (numerically) small values of x approaching 0. table:

x	± 0.100	± 0.080	± 0.060	± 0.040	± 0.020	± 0.010	± 0.008
y	0.9983	0.9989	0.9994	0.9997	0.9999	1.0000	1.0000

From the table, we see that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. □

In [1]:

```
f(x)=sin(x)/x
show(f)
```

Out[1]:

$$x \mapsto \frac{\sin(x)}{x}$$

In [2]:

```
show(limit(f(x), x=0))
```

Out[2]:

1

In [3]:

```
p = plot(f(x),x,-5*pi,5*pi)
pt = point((0, 1), rgbcolor="red", pointsize=50, faceted=True)
(p+pt).show(xmin=-5*pi, xmax=5*pi, ymin=-0.5, ymax=1)
```

Out[3]:

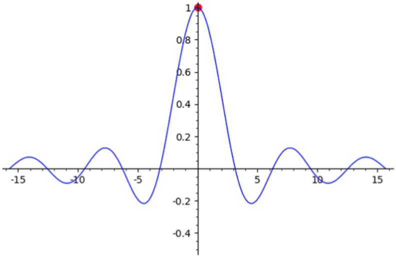


Figure below shows the graph of $f(x) = x\sin(1/x)$ near $x = 0$. Based on the graph, find $\lim_{x\to 0}[x\sin(1/x)]$ if it exists.

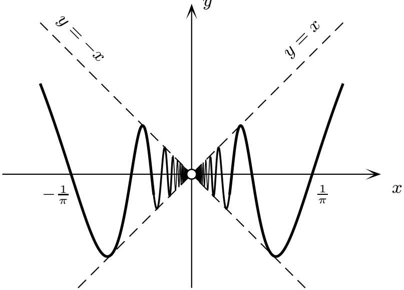


Figure: The graph of $x\sin(1/x)$ near $x = 0$.

From the graph, we see that $\lim_{x\to 0} x\sin(1/x) = 0$.

Example:
Figure below shows the graph of $f(x) = \sin(1/x)$ near $x = 0$. Based on the graph, find $\lim_{x \to 0} \sin(1/x)$ if it exists.

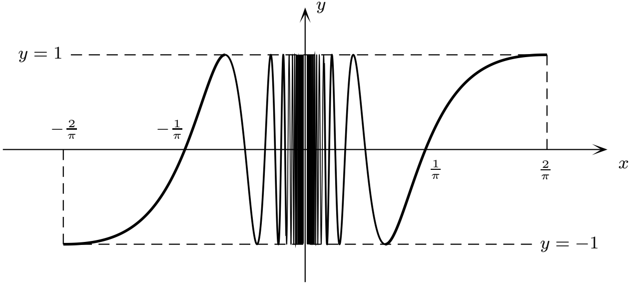


Figure: The graph of $\sin(1/x)$ near $x = 0$.

Solution:
From the graph, we see that $\lim_{x \to 0} \sin(1/x)$ does not exist.

The following theorems are important as they help us find limits of functions derived by algebraic operations on elementary functions.

Theorem: Let n be a positive integer and k a constant. Assume that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

- $\lim_{x\to a} k = k$
- $\lim_{x\to a} x = a$
- $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$
- $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- $\lim_{x\to a} [f(x)/g(x)] = \lim_{x\to a} f(x) / \lim_{x\to a} g(x)$ if $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ (assume $f(x) \geq 0$ near $x = a$ if n is even.)

Composite function

If $\lim_{x \to a} f(x) = A$ and $\lim_{u \to A} g(u) = B$ then $\lim_{x \to a} g(f(x)) = B$.

Squeeze Theorem

Let $f(x)$, $g(x)$, $h(x)$ be functions such that $f(x) \leq g(x) \leq h(x)$ for all x near a, except possibly at a itself. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $\lim_{x \to a} g(x) = L$.

Definition

Let S be a subset of the domain of a function $f(x)$. We say that $f(x)$ is *bounded on* S if there is a constant K such that

$$|f(x)| \leq K \quad \text{for all } x \in S.$$

In particular, if S is an open interval (p, q) where $p < a < q$, we say that $f(x)$ is *bounded near* a.

The composite functions of the form $\sin F(x)$ and $\cos F(x)$ are bounded functions as $|\sin F(x)| \leq 1$ and $|\cos F(x)| \leq 1$. These functions are bounded on the domain of $F(x)$.

The following theorem follows directly from the Squeeze Theorem.

Theorem:

If $g(x)$ is bounded near a, except possibly at a itself, and if $\lim_{x\to a} f(x) = 0$ then $\lim_{x\to a} f(x)g(x) = 0$.

Remark: Obviously, the above theorems are true if we replace the limits by the left-hand limits (or by the right-hand limits).

Continuity of functions

We know that at a general point a there are three possibilities. If the third case in the table is true, i.e. if on the graph there is no gap or hole at a, we say that the function is continuous at a.

Definition: Continuity at a point

If $\lim_{x \to a} f(x) = f(a)$, we say that $f(x)$ *is continuous at* a.

Definition: Discontinuity at a point

We say that $f(x)$ is discontinuous (i.e. not continuous) at a if any one of the following holds:

$f(a)$ is not defined $\quad \lim_{x \to a} f(x) \neq f(a) \quad \lim_{x \to a} f(x)$ does not exist

If the function is defined only on one side of a, then the associated one-sided limit is used instead of two-sided limits in the above definitions.

Properties of continuity

Theorem Let n be a positive integer and k a constant. If $f(x)$ and $g(x)$ are continuous at a then the following functions are also continuous at a:

1. (Scalar multiple) $kf(x)$ where k is a constant.
2. (Sum and difference) $f(x)+g(x)$ and $f(x)-g(x)$.
3. (Product) $f(x)g(x)$.
4. (Quotient) $f(x)/g(x)$ if $g(a)\neq 0$.
5. (Power) $[f(x)]^n$.
6. (Root) $\sqrt[n]{f(x)}$ (assume $f(a)\geq 0$ if n is even.)

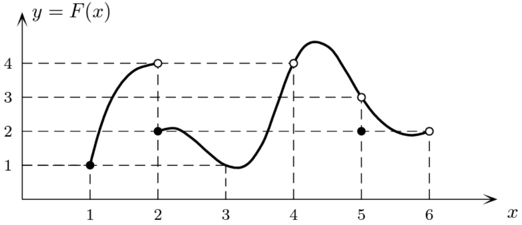
Theorem If $f(x)$ is continuous at $x=a$ and $g(u)$ is continuous at $u=f(a)$ then $g(f(x))$ is continuous at $x=a$.

Continuity on an interval

Definition Let J be an interval. If $f(x)$ is continuous at every point of J, we say that $f(x)$ *is continuous on* J, otherwise $f(x)$ is *not continuous on* J.

Continuity on an interval means that the graph of the function is a continuous or one-piece curve, i.e. the graph can be drawn without lifting the pencil from the paper.

Example Consider again the function $F(x)$. Find all the x-values at which the function is discontinuous. Find the intervals on which the function is continuous.



Solution The function is not continuous at $x = 2$ (where there is a gap), $x = 4$ (where there is a hole), $x = 5$ (where there is a hole) and at all points outside the interval $[1, 6)$ (where $F(x)$ is not defined). The function is continuous at every point in the intervals $[1, 2)$, $[2, 4)$, $(4, 5)$ and $(5, 6)$.

Continuous elementary functions

Elementary functions continuous on the interval J

Function $f(x)$	Interval J
Polynomials, $\exp kx$ ($k =$ const.)	$\mathbb{R}$
$\sin kx$, $\cos kx$ ($k =$ const.)	$\mathbb{R}$
$\tan kx$ ($k =$ const., $k \neq 0$)	$\ldots, \left(\frac{-3\pi}{2k}, \frac{-\pi}{2k}\right), \left(\frac{-\pi}{2k}, \frac{\pi}{2k}\right), \left(\frac{\pi}{2k}, \frac{3\pi}{2k}\right), \ldots$
$\ln kx$ ($k =$ const., $k > 0$)	$(0, \infty)$
Rational functions $P(x)/Q(x)$	$(-\infty, x_0), (x_0, x_1), (x_1, x_2), \ldots, (x_n, \infty)$ where $x_0, x_1, \ldots, x_n$ are distinct zeros of $Q(x)$.

The above table can be further extended by Theorem (Properties of Continuity) to include many other functions generated from elementary functions by algebraic operations.

Examples of functions continuous on the interval J

$f(x)$	J	Remarks
$x^3 - 2x^2 + x - 2$	$\mathbb{R}$	Polynomial
$5\sin 3x$	$\mathbb{R}$	Scalar multiple a of trigonometric function
$\exp 2x$	$\mathbb{R}$	Exponential function
$\sin 2x + \exp 3x$	$\mathbb{R}$	Sum of continuous functions
$e^x \cos 3x$	$\mathbb{R}$	Product of continuous functions
$\cos(x^2 + x + 1)$	$\mathbb{R}$	Composition of continuous functions
$\sin \ln x$	$(0, \infty)$	Composition of continuous functions
$\sqrt{\ln 2x}$	$[1/2, \infty)$	Square-root of a continuous function
$\dfrac{\cos x}{(x-1)(x-2)}$	$(-\infty, 1)$, $(1, 2)$, $(2, \infty)$	Quotient of continuous functions

Methods of finding limits: Substitution

We have seen that for function $f(x)$ continuous at a, we have the formula:

$$\lim_{x \to a} f(x) = f(a),$$

i.e. the limit can be found *by substitution*. If $f(x)$ is generated from elementary functions by algebraic operations and if $f(a)$ is defined then $f(x)$ is continuous at a and hence the substitution formula works.

Example By substitution, we get

1. $\lim_{x\to 2}(x^3-4x^2+x+5)=2^3-4\times 2^2+2+5=-1.$
2. $\lim_{x\to 2}\dfrac{x+1}{x^2-1}=\dfrac{2+1}{2^2-1}=1.$
3. $\lim_{x\to 0}\dfrac{x}{1+^{2x}}=\dfrac{0}{1+^{0}}=\dfrac{1}{1+1}=\dfrac{1}{2}.$
4. $\lim_{x\to 2}\sin^2 3x=\sin^2(3\times 2)=\sin^2 6.$
5. $\lim_{x\to 0}[(x^2-2)\cos 3x]=(0-2)\cos 0=-2.$
6. $\lim_{x\to 1}\dfrac{\sin x}{1+x^2}=\dfrac{\sin 1}{2}.$
7. $\lim_{t\to 2}\tan(3t^2-2)=\tan(3\times 2^2-2)=\tan 10.$
8. $\lim_{y\to 1}\cos\exp(y^2+1)=\cos\exp(1^1+1)=\cos^2.$
9. $\lim_{u\to\pi/2}\sqrt{u-\sin u}=\sqrt{\pi/2-\sin(\pi/2)}=\sqrt{\pi/2-1}.$

Example The method of substitution fails in the following cases:

1. $\lim_{x\to 1} \dfrac{1}{x^2-1} = \dfrac{1}{0}$ which is undefined.
2. $\lim_{x\to \pi/2} \tan x = \tan(\pi/2)$ which is undefined.
3. $\lim_{x\to 0} \dfrac{\cos x}{1-\cos x} = \dfrac{1}{0}$ which is undefined.
4. $\lim_{x\to 1} \ln(x^2-3) = \ln(-2)$ which is undefined.
5. $\lim_{x\to 0} \dfrac{\sin x}{x} = \dfrac{0}{0}$ which is undefined.
6. $\lim_{x\to 0} x\sin\dfrac{1}{x} = 0\cdot\sin(1/0)$ which is undefined.

Cancellation of factors

Example Find $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$.

Solution On substitution, we get $0/0$. The difficulty can be removed by observing that we do not have to consider $x = 2$ for the limit. Since

$$\frac{x^2 - 4}{x - 2} = x + 2 \qquad \text{for } x \neq 2,$$

we have

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4.$$

Example:

Let n be a positive integer. Find $\lim_{x \to a} \dfrac{x^n - a^n}{x - a}$.

Solution Using the factorization

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

we get on substitution

$$\begin{aligned} \lim_{x \to a} \frac{x^n - a^n}{x - a} &= \lim_{x \to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\ &= a^{n-1} + a^{n-1} + \cdots + a^{n-1} + a^{n-1} \quad (n \text{ terms}) \\ &= na^{n-1}. \end{aligned}$$

A trigonometric formula

Theorem If θ is in radians, then

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.}$$

Remark The above limit formula holds for θ in radians, not in degrees. Unless otherwise stated, the unit for angles in this book is radian.

Example

Find $\displaystyle\lim_{x\to 1}\frac{\sin(x^3+2x-3)}{x^3+2x-3}$.

Solution

Put $u = x^3 + 2x - 3$. Then $\displaystyle\lim_{x\to 1} u = 1 + 2 - 3 = 0$. Applying the Theorems above, we get

$$\lim_{x\to 1}\frac{\sin(x^3+2x-3)}{x^3+2x-3} = \lim_{u\to 0}\frac{\sin u}{u} = 1. \qquad \square$$

Example

Find $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x}$.

Solution

Using the formula $1 - \cos 2\theta = 2\sin^2\theta$ and the Theorem, we get

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \left(\frac{2}{x} \sin^2 \frac{x}{2} \right) = \lim_{x \to 0} \left(\sin \frac{x}{2} \right) \cdot \lim_{x \to 0} \frac{\sin(x/2)}{(x/2)} = 0 \times 1 = 0. \quad \square$$

Example

Find $\displaystyle\lim_{x\to 0}\sqrt{\frac{7x+\sin 2x}{x}}$

Solution

By the Theorems, we get

$$\lim_{x\to 0}\sqrt{\frac{7x+\sin 2x}{x}} = \sqrt{\lim_{x\to 0}\frac{7x+\sin 2x}{x}}$$

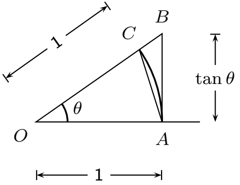
$$= \sqrt{\lim_{x\to 0}\left(7+\frac{\sin 2x}{x}\right)} = \sqrt{7+2\times 1} = 3.$$

$\square$

Proof

First consider the case when $\theta > 0$. Since θ is approaching 0, we can assume that $0 < \theta < \pi/2$ and construct a right-angled triangle OAB with $OA = 1$ and $\angle BOA = \theta$ as shown in the diagram.

The arc $\overset{\frown}{AC}$ is part of the circle centred at O with unit radius.



Comparing areas, we have $\triangle OAC <$ sector $OAC < \triangle OAB$, i.e. $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$. Therefore, $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$ and hence, for $0 < \theta < \pi/2$,

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $(\sin\theta)/\theta$ and $\cos\theta$ are even functions, the inequalities are true also for $-\pi/2 < \theta < 0$.

As $\lim_{\theta\to 0}\cos\theta = 1$, we get by squeeze theorem

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

For the case $\theta < 0$, we write $\phi = -\theta > 0$ and get

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = \lim_{\phi\to 0^+}\frac{\sin\phi}{\phi} = 1.$$

Combining the two, we get the required result

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$