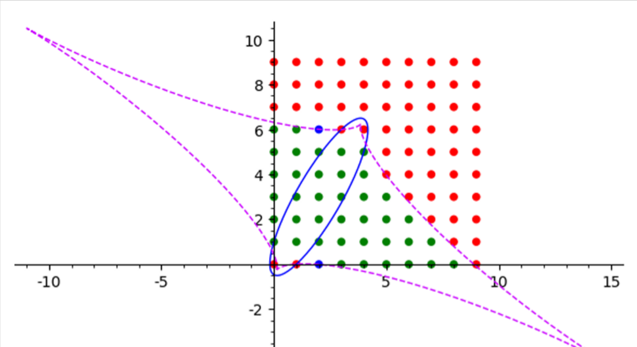
DEPARTMENT OF APPLIED MATHEMATICS

應 用 數 學 系

AMA1500

Foundation Mathematics for Accounting and Finance

Part 03



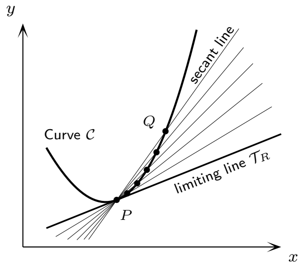
Secant line and Tangent line

(a) Let P and Q be two distinct points on a curve $\mathcal{C}$. The straight line which passes through P and Q is called a *secant line* (or simply a *secant*) of the curve $\mathcal{C}$.

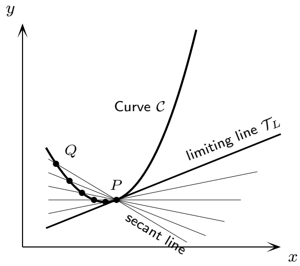
(b) If P is fixed and we allow Q to move along the curve towards P from both sides of P, and if the secants PQ approaches to the same limiting straight line, we call this limiting straight line the *tangent line* (or simply the *tangent*) to the curve at P.

Tangent to a curve at a point

Tangent to a curve at a point



(a) $Q \to P$ from the right along $\mathcal{C}$



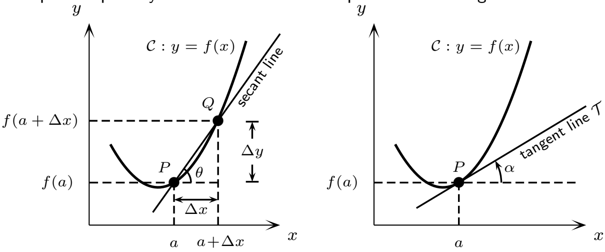
(b) $Q \to P$ from the left along $\mathcal{C}$

Figure: Tangent line as the limit of secant lines.

In the figure, we see that as Q approaches P from the left and from the right, the limiting lines $\mathcal{T}_R$ and $\mathcal{T}_L$ so obtained are the same straight line and therefore $\mathcal{T}_R (= \mathcal{T}_L)$ is the tangent line to the curve at P.

Slope of a curve at a point

The slope is a quantity which measures the steepness of the straight line.



(a) The slope of the secant line PQ is $\tan\theta = \Delta y/\Delta x$

(b) The slope of the tangent line is $\tan\alpha = \lim_{\Delta x \to 0} \Delta y/\Delta x$

Figure: The slope of a curve as a limit.

Definition

Let P be a point on a curve $\mathcal{C}$ and let $\mathcal{T}$ be the tangent to $\mathcal{C}$ at P. Then the *slope* of the curve $\mathcal{C}$ at P is the slope of $\mathcal{T}$, if $\mathcal{T}$ is not vertical.

Let $\mathcal{C}$ be a given curve whose equation is $y = f(x)$ and let $P(a, f(a))$ be a point on $\mathcal{C}$. Let Q be another point on $\mathcal{C}$ with coordinates $(a + \Delta x, f(a + \Delta x))$. Here Δx is called an *increment in x*. The point $x = a + \Delta x$ is on the left or on the right of the point $x = a$ according as Δx is negative or positive. In the figure, Q is on the right of P and therefore Δx is positive. The *increment in y* is defined by

$$\Delta y = f(a + \Delta x) - f(a).$$

As $Q \to P$ from both sides along $\mathcal{C}$, $\Delta x \to 0$ from both sides. Consequently, if the curve $\mathcal{C}$ has a tangent line at P, we have:

$$\begin{aligned}
\text{secant line } PQ &\quad\longrightarrow\quad \text{tangent line at } P \\
\theta &\quad\longrightarrow\quad \alpha \\
\frac{f(a+\Delta x) - f(a)}{\Delta x} = \frac{\Delta y}{\Delta x} = \tan\theta &\quad\longrightarrow\quad \tan\alpha \\
&\qquad\qquad = \text{the slope of the tangent line at } P \\
&\qquad\qquad = \text{the slope of the curve at } P.
\end{aligned}$$

The above limiting values can be seen intuitively from the Figure though it shows only the case when $\Delta x \to 0^+$. The result is stated as follows:

Theorem

The slope of the tangent line of the curve $\mathcal{C} : y = f(x)$ at $x = a$ is equal to the limit

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where Δy is defined by (5).

Example

Find the slope of the tangent line of the curve $y = x^2$ at $x = a$.

Solution

Let $f(x) = x^2$. Then the required slope is equal to

$$\lim_{\Delta x \to 0} \frac{(a + \Delta x)^2 - a^2}{\Delta x} = \lim_{\Delta x \to 0} (2a + \Delta x) = 2a.$$

Example

Show that the absolute value function $f(x) = |x|$ has no slope at $x = 0$.

Solution

$f(x) = x$ if $x > 0$; $f(x) = -x$ if $x < 0$ and $f(0) = 0$. Therefore, for $\Delta x > 0$, we have $f(0 + \Delta x) = f(\Delta x) = \Delta x$ and hence

$$\lim_{\Delta x \to 0^+} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1.$$

However, for $\Delta x < 0$, we have $f(0 + \Delta x) = f(\Delta x) = -\Delta x$ and hence

$$\lim_{\Delta x \to 0^-} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1.$$

Since the left-hand and right-hand limits are not equal, $f(x)$ has no slope at $x = 0$.

Derivative of a function

The slope of the graph of $y = f(x)$ at $x = a$ is the limit (6). This is a number dependent on the function $f(x)$ and on the constant a. If the limit (6) exists, we say that the function $f(x)$ is *differentiable* at $x = a$.

If we consider $f(x)$ as a given function and replace a by the variable x, then the limit (6), if exists, becomes a function of x called the *derivative* of the function $f(x)$. The derivative of $f(x)$, which is defined by the formula

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where $\Delta y = f(x + \Delta x) - f(x)$, is denoted by the symbols

$$f'(x) \quad \text{or} \quad y' \quad \text{or} \quad \frac{dy}{dx} \quad \text{or} \quad \frac{df}{dx}.$$

Note that all these symbols represent the same function of x.

Differentiation

We often say that we *differentiate* the function $f(x)$ to get its derivative $f'(x)$, and *differentiation* means the process of getting the derivative $f'(x)$ from $f(x)$. We also say that the function $f(x)$ is *differentiable* on an interval J if $f'(x)$ exists at every x in J.

The value of the derivative at a particular point $x = a$ is denoted by

$$f'(a) \quad \text{or} \quad y'(a) \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{df}{dx}\right|_{x=a}.$$

This value is the slope of the curve $y = f(x)$ at $x = a$.

Summary

We summarize the above definitions and notations in the following table:

	Derivative of $f(x)$	Derivative of $f(x)$ at $x = a$
Definition:	$\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$	$\lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a)}{\Delta x}$
Nature:	A function of x	A real number
Notation:	$f'(x)$ or y' or $\dfrac{dy}{dx}$ or $\dfrac{df}{dx}$	$f'(a)$ or $y'(a)$ or $\left.\dfrac{dy}{dx}\right\|_{x=a}$ or $\left.\dfrac{df}{dx}\right\|_{x=a}$
Geometric meaning:	The slope of the curve $y = f(x)$ at a general point x.	The slope of the curve $y = f(x)$ at a particular point where $x = a$. This is equal to $\tan\alpha$ in Fig. 6.

Example

Ddifferentiate $y = x^2$.

Solution

Let $f(x) = x^2$. Then

$$\begin{aligned} y' &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x. \end{aligned}$$

Note that the above process of getting the derivative is exactly the same as that in example (for slope at point a) except that we have replaced the constant a by the variable x. In the forthcoming section, we shall list formulas of derivatives in a table so that we may use the formulas to get derivatives directly without spending time in evaluating limits.

Differentiability implies continuity

$$Differentiability \quad \Rightarrow \quad Continuity$$

Using the previous setup, we can prove that:

If $f'(a)$ exists then the function $f(x)$ is continuous at $x = a$.

The **converse**, however, of the theorem **is not true**.

The function $f(x) = |x|$ gives a counter-example.
This $f(x)$ is continuous at $x = 0$ but $f(x)$ is not differentiable at $x = 0$.

Rate of change

Suppose an object moves along a straight line. Its distance from a certain fixed point O on the line at time t is given by $y = F(t)$. Over the time interval $[t_0, t_0 + \Delta t]$, the object covers a distance equal to $\Delta y = F(t_0 + \Delta t) - F(t_0)$. The *difference quotient*

$$\frac{\Delta y}{\Delta t} = \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t}$$

is the average *velocity* of the object over the time interval $[t_0, t_0 + \Delta t]$.

As a result, the derivative $f'(t_0)$ is simply the *instantaneous velocity* of the object at the instant t_0.

More generally, for $y = f(x)$, the *difference quotient*

$$\frac{f(b) - f(a)}{b - a}$$

is called the *average rate of change of* y *with respect to* x *over the interval* $[a, b]$, and the derivative $f'(a)$ at $x = a$ is called the *rate of change of* y *with respect to* x *at* $x = a$.

Differentiation by the first principle

In addition to the Example, we give more examples to show how differentiation formulas can be derived from the first principle, i.e. from the definition of derivatives.

Example

Let $y = f(x) = C$, a constant. Show that $\dfrac{dy}{dx} = 0$ for every x.

Proof

Since

$$\Delta y = f(x + \Delta x) - f(x) = C - C = 0,$$

we get

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$$

Table of differentiation formulas

	$f(x)$	$f'(x)$	
1.	constant	0	
2.	x^n	nx^{n-1}	$n =$ real constant
3.	$\sin x$	$\cos x$	
4.	$\cos x$	$-\sin x$	
5.	$\tan x$	$\sec^2 x$	
6.	$\cot x$	$-\csc^2 x$	
7.	$\sec x$	$\sec x \tan x$	
8.	$\csc x$	$-\csc x \cot x$	
9.	e^x	e^x	$e = 2.718281828\cdots$
10.	a^x	$a^x \ln a$	$a > 0,$ real constant
11.	$\ln x$	$1/x$	$x > 0$
12.	$\log_a x$	$1/(x \ln a)$	$a > 0,$ real constant
13.	$\sin^{-1} x$	$1/\sqrt{1-x^2}$	
14.	$\cos^{-1} x$	$-1/\sqrt{1-x^2}$	
15.	$\tan^{-1} x$	$1/(1+x^2)$	

Example

If $y = 1/x^3$, find y'.

Solution

Since we can write $y = x^{-3}$, we get, using formula 2 with $n = -3$,

$$y' = (-3)x^{-3-1} = -3x^{-4}.$$

Example

If $y = \sqrt[3]{x}$, find y'.

Solution

Since we can write $y = x^{1/3}$, we get , using formula 2 with $n = 1/3$,

$$y' = \frac{1}{3}x^{1/3-1} = \frac{1}{3x^{2/3}}.$$

Example

The point $P(\pi/4, 1)$ lies on the curve $y = \tan x$. Find the slope of the curve at this point.

Solution

Using formula from the table (the 5th item), we have $y' = \sec^2 x$. Therefore at $x = \pi/4$, the slope of the curve $y = \tan x$ is given by

$$m = \sec^2(\pi/4) = 2.$$

Example

The point $A(2, 8)$ lies on the curve $y = x^3$. Find the equation of the tangent line at A.

Solution

Using formula 2 with $n = 3$, we have $y' = 3x^2$. Therefore at $x = 2$, the slope of the curve $y = x^3$ is $m = 3 \cdot 2^2 = 12$. Hence the equation of the tangent line through $A(2, 8)$ is

$$y - 8 = 12(x - 2) \quad \text{or} \quad y = 12x - 16.$$

Basic rules of differentiation

(Scalar multiplication) $y = kf(x)$, k is a constant : $\dfrac{dy}{dx} = kf = kf'(x)$

(Sum) $y = f(x) + g(x)$: $\dfrac{dy}{dx} = f + g = f'(x) + g'(x)$

(Difference) $y = f(x) - g(x)$: $\dfrac{dy}{dx} = f - g = f'(x) - g'(x)$

(Product) $y = f(x)g(x)$: $\dfrac{dy}{dx} = f(x)g + g(x)f = f(x)g'(x) + g(x)f'(x)$

(Quotient) $y = \dfrac{f(x)}{g(x)}$: $\dfrac{dy}{dx} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

(Composite function) $y = f(u)$ & $u = g(x)$: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = f'(u)g'(x)$

(Inverse function) $y = f(x) \& x = f^{-1}(y)$: $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}} = \dfrac{1}{f'(x)}$ if $f'(x) \neq 0$

Example

If $y = 4x^3$, find y'.

Solution

$y' = 4 \cdot 3x^2 = 12x^2$.

Example

If $y = x^3 + x^2 - 2x + 3$, find y'.

Solution

$y' = 3x^2 + 2x - 2$.

Example

If $y = 5x^3 + x + 4/x$, find y'.

Solution

$y' = 5 \cdot 3x^2 + 1 - 4/x^2 = 15x^2 + 1 - 4/x^2$.

Example

If $y = \dfrac{3}{x^2} - \dfrac{2}{x}$, find y'.

Solution

$$y' = \frac{3 \cdot (-2)}{x^3} - \frac{2 \cdot (-1)}{x^2} = \frac{2}{x^2} - \frac{6}{x^3}$$

Example

If $y = x^n + bx^3 + c$ where n, b, c are constants. Find y'.

Solution

$y' = nx^{n-1} + 3bx^2.$

Example

If $y = 2\cos x + \sin x$, find y'.

Solution

$y' = -2\sin x + \cos x.$

Example

If $y = 2^x + 3\sin x$, find y'.

Solution

$y' = 2^x ln(2) + 3\cos x.$

Example

If $y = 4x + \ln x$, find y'.

Solution

$y' = 4 + 1/x.$

Example

If $y = 3x^2 - 2\tan x$, find y'.

Solution

$y' = 6x - 2\sec^2 x$.

Example

If $y = 2x^3 + 3^x - \sin x$, find y'.

Solution

$y' = 6x^2 + 3^x ln(3) - \cos x$.

Example

If $y = 1 + x + x^2 + x^3 + x^4 + x^5$, find y'.

Solution

$y' = 1 + 2x + 3x^2 + 4x^3 + 5x^4$.

Examples: Products and quotients

Example

If $y = x^3 \sin x$, find y'.

Solution

$$y' = x^3 \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x^3 = x^3 \cos x + 3x^2 \sin x.$$

Example

If $y = x^2 e^x$, find y'.

Solution

$$y' = x^2 \frac{d}{dx} e^x + e^x \frac{d}{dx} x^2 = x^2 e^x + 2x e^x = e^x x(x+2).$$

Example

If $y = (x^2 + x - 2)e^x$, find y'.

Solution

If $y = (x^2 + x - 2)e^x$, find y'.

$$\begin{aligned} y' &= (x^2 + x - 2)\frac{d}{dx}e^x + e^x\frac{d}{dx}(x^2 + x - 2) \\ &= (x^2 + x - 2)e^x + e^x(2x + 1) \\ &= e^x(x^2 + 3x - 1). \end{aligned}$$

Example

If $y = (x^2 + 3x - 2)\cos x$, find y'.

Solution

$$\begin{aligned} y' &= (x^2 + 3x - 2)\frac{d}{dx}\cos x + \cos x\frac{d}{dx}(x^2 + 3x - 2) \\ &= (x^2 + 3x - 2)(-\sin x) + (\cos x)(2x + 3) \\ &= -(x^2 + 3x - 2)\sin x + (2x + 3)\cos x. \end{aligned}$$

Example

If $y = \dfrac{x^2}{\sin x}$, find y'.

Solution

$$y' = \frac{(\sin x)\dfrac{d}{dx}(x^2) - x^2\dfrac{d}{dx}\sin x}{[\sin x]^2} = \frac{2x\sin x - x^2\cos x}{\sin^2 x}.$$

In [1]:

```
f(x)=x^2/sin(x)
show(f)
```

Out[1]:

$$x \mapsto \frac{x^2}{\sin(x)}$$

In [2]:

```
fdash=derivative(f(x),x)
show(fdash)
```

Out[2]:

$$-\frac{x^2\cos(x)}{\sin(x)^2} + \frac{2\,x}{\sin(x)}$$

Example

If $y = \dfrac{x^2+3x+2}{x^2+2}$, find y'.

Solution

$$\begin{aligned} y' &= \frac{(x^2+2)\dfrac{d}{dx}(x^2+3x+2) - (x^2+3x+2)\dfrac{d}{dx}(x^2+2)}{(x^2+2)^2} \\ &= \frac{(x^2+2)(2x+3) - (x^2+3x+2)(2x)}{(x^2+2)^2} = \frac{-3(x^2-2)}{(x^2+2)^2}. \end{aligned}$$

Example

If $y = \dfrac{e^x}{\cos x}$, find y'.

Solution

$$y' = \frac{(\cos x)\dfrac{d}{dx}(e^x) - e^x\dfrac{d}{dx}\cos x}{\cos^2 x} = \frac{e^x(\cos x + \sin x)}{\cos^2 x}.$$

Chain Rule:

Example

If $y = \sin 3x$, find y'.

Solution

Let $y = \sin u$ and $u = 3x$. Then

$$y' = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u)(3) = 3\cos 3x.$$

Example

If $y = \cos x^3$, find y'.

Solution

Let $y = \cos u$ and $u = x^3$. Then

$$y' = \frac{dy}{du} \cdot \frac{du}{dx} = (-\sin u)(3x^2) = -3x^2 \sin x^3.$$

Example

If $y = (x^2 - 3x + 2)^6$, find y'.

Solution

Let $y = u^6$ and $u = x^2 - 3x + 2$. Then

$$y' = \frac{dy}{du} \cdot \frac{du}{dx} = (6u^5)(2x - 3) = 6(2x - 3)(x^2 - 3x + 2)^5.$$

Example

If $y = e^{-4x^2+3x-1}$, find y'.

Solution

Let $y = e^u$ and $u = -4x^2 + 3x - 1$. Then

$$y' = (e^u)(-8x+3) = (-8x+3)\exp(-4x^2+3x-1).$$

Example

If $y = \ln(2x^2 + 1)$, find y'.

Solution

Let $y = \ln u$ and $u = 2x^2 + 1$. Then

$$y' = \frac{1}{u} \cdot (4x) = \frac{4x}{2x^2 + 1}.$$

Example

Let $y = \sqrt{x^2 + a^2}$ where a is a nonzero constant. Find y'.

Solution

Let $y = u^{1/2}$ and $u = x^2 + a^2$. Then

$$y' = \frac{1}{2}u^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + a^2}}.$$

Example

If $y = \dfrac{1}{x^2 + 4x + 2}$, find y'.

Solution

Let $y = u^{-1}$ and $u = x^2 + 4x + 2$. Then

$$y' = -u^{-2}(2x + 4) = -\frac{2(x+2)}{(x^2 + 4x + 2)^2}.$$

Example

If $y = \sin \ln x^2$, find y'.

Solution

Let $y = \sin u$, $u = \ln v$ and $v = x^2$. Then

$$y' = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = (\cos u)\frac{1}{v}(2x) = \frac{2(\cos \ln x^2)}{x}.$$

Example

If $y = (\sin x^2)\exp(\cos x)$, find y'.

Solution

$$
\begin{aligned}
y' &= \exp(\cos x)\frac{d}{dx}(\sin x^2) + (\sin x^2)\frac{d}{dx}\exp(\cos x) \\
&= 2x\exp(\cos x)(\cos x^2) - (\sin x)(\sin x^2)\exp(\cos x).
\end{aligned}
$$

In [1]:

```
f(x)=sin(x^2)*e^(cos(x))
show(f)
```

Out[1]:

$x \mapsto e^{\cos(x)}\sin(x^2)$

In [2]:

```
fdash=derivative(f(x),x)
show(fdash)
```

Out[2]:

$2\,x\cos(x^2)e^{\cos(x)} - e^{\cos(x)}\sin(x^2)\sin(x)$

Example

If $y = \ln\left(\dfrac{2x+3}{x+2}\right)$, find y'.

Solution

$$\begin{aligned} y' &= \frac{x+2}{2x+3} \cdot \frac{2(x+2)-(2x+3)}{(x+2)^2} \\ &= \frac{1}{(2x+3)(x+2)}. \end{aligned}$$

Example

Let f and g be two differentiable functions. If $y = f(x^2)g(3x+2)$, find y' in terms of f, g, f' and g'.

Solution

$$
\begin{aligned}
y' &= f(x^2)\frac{d}{dx}g(3x+2) + g(3x+2)\frac{d}{dx}f(x^2) \\
&= 3\,f(x^2)\,g'(3x+2) + 2x\,g(3x+2)\,f'(x^2).
\end{aligned}
$$

Derivative of Inverse functions

Example

Let $y = \sin x^2$, $0 < x < 1$. Find $\dfrac{dx}{dy}$ in terms of x.

Solution

$$\frac{dx}{dy} = 1 \Big/ \frac{dy}{dx} = \frac{1}{2x \cos x^2}.$$

Example

Show that $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$ for $|x| < 1$.

Proof

If $y = f(x) = \sin x$, $-\pi/2 < x < \pi/2$, then $x = g(y) = \sin^{-1} y$. Since $\cos x > 0$ whenever $-\pi/2 < x < \pi/2$, we have

$$\frac{dx}{dy} \quad \text{or} \quad g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}},$$

for every $y \in (-1,\ 1)$. Changing the dummy variable y to x, we have

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \quad \text{for} \quad |x| < 1.$$

Remark Similarly, we can establish the formula

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}} \quad \text{for} \quad |x| < 1.$$

Derivatives of Implicit functions

In the xy-plane, the unit circle with centre at the origin O can be represented by the equation

$$x^2 + y^2 = 1. \tag{1}$$

We see that the circle describes two functions of x. One of these can be represented by the upper half of the circle while the other by the lower half.

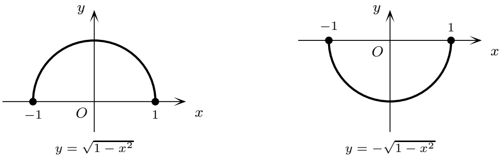


Figure: Two functions defined by the equation $x^2 + y^2 = 1$.

Each of these functions is called an *implicit function* defined by the equation. In fact, the functions in this special case can be defined explicitly by:

$$y = \sqrt{1 - x^2}, \ \ -1 \leq x \leq 1$$

and

$$y = -\sqrt{1 - x^2}, \ \ -1 \leq x \leq 1.$$

However, not all implicit functions can be expressed in explicit forms. For example, the equation

$$y + \sin(xy) = 2\pi \tag{1}$$

defines y as one or more functions of x implicitly, but not explicitly.

Example

Find y' if y is the function defined implicitly by the equation $y + \sin(xy) = 2\pi$. Show that the point $P(1,\ 2\pi)$ lies on the curve defined by the equation and find the slope of the curve at P.

Solution

Regarding y as a function of x, we can differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(y + \sin xy) = \frac{d}{dx}(2\pi).$$

It follows that $y' + (\cos xy)(xy' + y) = 0$. Solving for y', we get

$$y' = \frac{-y\cos xy}{1 + x\cos xy}.$$

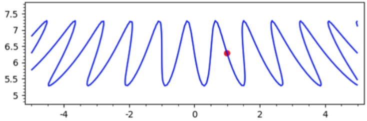
At $x = 1, y = 2\pi$, the LHS of the equation is $2\pi + \sin 2\pi$ which is equal to the RHS. Therefore the point $P(1,\ 2\pi)$ lies on the curve defined bythe equation. At this point, $x = 1$ and $y = 2\pi$. Therefore the slope of the curve at P is

$$y' = \frac{-y\cos xy}{1 + x\cos xy} = \frac{-2\pi\cos(2\pi)}{1 + \cos(2\pi)} = -\pi.$$

In [1]:

```
var('x,y')
implicit_plot(y+sin(x*y) == 2*pi, (x,-5,5), (y,2*pi-1.5,2*pi+1.5)) + point((1,
2*pi), rgbcolor='red', pointsize=50)
```

Out[1]:



In [2]:

```
y=function('y')(x)
dydx(x)=solve(derivative(y+sin(x*y) == 2*pi,x),derivative(y,x))
show(dydx)
```

Out[2]:

$$x \mapsto \left(\frac{\partial}{\partial x}y(x) = -\frac{\cos(xy(x))y(x)}{x\cos(xy(x)) + 1}\right)$$

In [3]:

```
# evaluate the value at x=1
show(dydx(1))
```

Out[3]:

$$\left(\mathrm{D}_0(y)(1) = -\frac{\cos(y(1))y(1)}{\cos(y(1)) + 1}\right)$$

In [4]:

```
# given that y(1)=2*pi, we put it in the expression
show(-cos(2*pi)*2*pi/(cos(2*pi) + 1))
```

Out[4]:

$-\pi$

Higher derivatives

If $y = f(x)$ has a derivative $f'(x)$ and if $f'(x)$ also has a derivative, this derivative of the derivative of $f(x)$ is called the *second order derivative* of $f(x)$. The second order derivative is denoted by

$$y'' \quad \text{or} \quad f''(x) \quad \text{or} \quad \frac{d^2y}{dx^2}.$$

If we differentiate the second order derivative, we get the *third order derivative* denoted by:

$$y''' \quad \text{or} \quad f'''(x) \quad \text{or} \quad \frac{d^3y}{dx^3}.$$

In this way, the nth order derivative is defined and is denoted by

$$y^{(n)} \quad \text{or} \quad f^{(n)}(x) \quad \text{or} \quad \frac{d^ny}{dx^n}.$$

For convenience, we also define

$$y^{(0)} = f^{(0)}(x) = f(x)$$

so that $y^{(n)}$ or $f^{(n)}(x)$ is defined for $n = 0, 1, 2, 3, \ldots$.

Example

Let $y = x^3 - 4\ln x$. Find y', y'' and y'''.

Solution

$$y' = 3x^2 - 4/x, \quad y'' = 6x + 4/x^2 \quad \text{and} \quad y''' = 6 - 8/x^3.$$

Leibniz's rule (a useful rule for differentiating a product n times:)

For differentiable functions $u(x)$ and $v(x)$,

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

where $\binom{n}{k}$ denotes the coefficient of t^k in the binomial expansion of $(1+t)^n$.

The formula for the binomial coefficient is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for } n = 1, 2, 3, \ldots; \quad k = 0, 1, 2, \ldots, n$$

and the rules for $n = 1, 2, 3, 4$ are:

$$\begin{aligned}
(uv)' &= u'v + uv' \\
(uv)'' &= u''v + 2u'v' + uv'' \\
(uv)''' &= u'''v + 3u''v' + 3u'v'' + uv''' \\
(uv)^{(4)} &= u^{(4)}v + 4u'''v' + 6u''v'' + 4u'v''' + uv^{(4)}
\end{aligned}$$

Example

Find y'' if $y = x^3 \sin 2x$.

Solution

$$
\begin{aligned}
y'' &= x^3(-4\sin 2x) + 2(3x^2)(2\cos 2x) + 6x\sin 2x \\
&= (-4x^3 + 6x)\sin 2x + 12x^2 \cos 2x.
\end{aligned}
$$

Example

Let $y = x^2 e^{2x}$. Find $y^{(n)}$ for $n \geq 0$.

Solution

Let $u = x^2$ and $v = e^{2x}$. Then $u^{(k)} = 0$ for $k \geq 3$. By Leibniz's rule,

$$\begin{aligned} y^{(n)} &= x^2(2^n e^{2x}) + n(2x)(2^{n-1}e^{2x}) + \frac{n(n-1)}{2}(2)(2^{n-2}e^{2x}) + 0 + \cdots \\ &= 2^{n-2}e^{2x}[4x^2 + 4nx + n(n-1)] \end{aligned}$$

for $n \geq 2$. Furthermore, direct differentiation gives

$$y = x^2 e^{2x} \quad \text{and} \quad y' = e^{2x}(2x + 2x^2)$$

showing that it is also true for $n = 0$ and 1.

In [1]:

```
f(x)=x^2*e^(2*x)
show(f)
```

Out[1]:

$x \mapsto x^2 e^{(2\,x)}$

In [2]:

```
show(derivative(f(x),x,1))
```

Out[2]:

$2\,x^2 e^{(2\,x)} + 2\,x e^{(2\,x)}$

In [3]:

```
show(derivative(f(x),x,2))
```

Out[3]:

$4\,x^2 e^{(2\,x)} + 8\,x e^{(2\,x)} + 2\,e^{(2\,x)}$

In [4]:

```
show(derivative(f(x),x,3))
```

Out[4]:

$8\,x^2 e^{(2\,x)} + 24\,x e^{(2\,x)} + 12\,e^{(2\,x)}$

In [5]:

```
show(derivative(f(x),x,4))
```

Out[5]:

$16\,x^2 e^{(2\,x)} + 64\,x e^{(2\,x)} + 48\,e^{(2\,x)}$

In [6]:

```
show(derivative(f(x),x,5))
```

Out[6]:

$32\,x^2 e^{(2\,x)} + 160\,x e^{(2\,x)} + 160\,e^{(2\,x)}$