I225E Statistical Signal Processing

13. Signal Processing I

MAWALIM and UNOKI

candylim@jaist.ac.jp and unoki@jaist.ac.jp

School of Information Science

JAPAN
ADVANCED INSTITUTE OF
SCIENCE AND TECHNOLOGY
HOKURIKU 1990

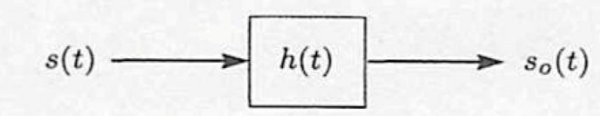
Signal processing

- Continuous and discrete-time systems
- Nyquist-Shannon sampling theorem
- Wiener filter
- Kalman filter

1. Continuous-Time System

- **Continuous-time signal** $s(t)$ is the input to a linear time invariant system with **impulse response** $h(t)$.
- The output $s_o(t)$ is

$$s_o(t) = \int_{-\infty}^{\infty} h(t-\tau)s(\tau)d\tau$$



(a) Original continuous-time system

- **Assumptions**
  - $s(t)$: bandlimited to B Hz
  - $s_o(t)$: bandlimited to B Hz
  - Frequency response $\mathcal{F}\{h(t)\}$: bandlimited to B Hz

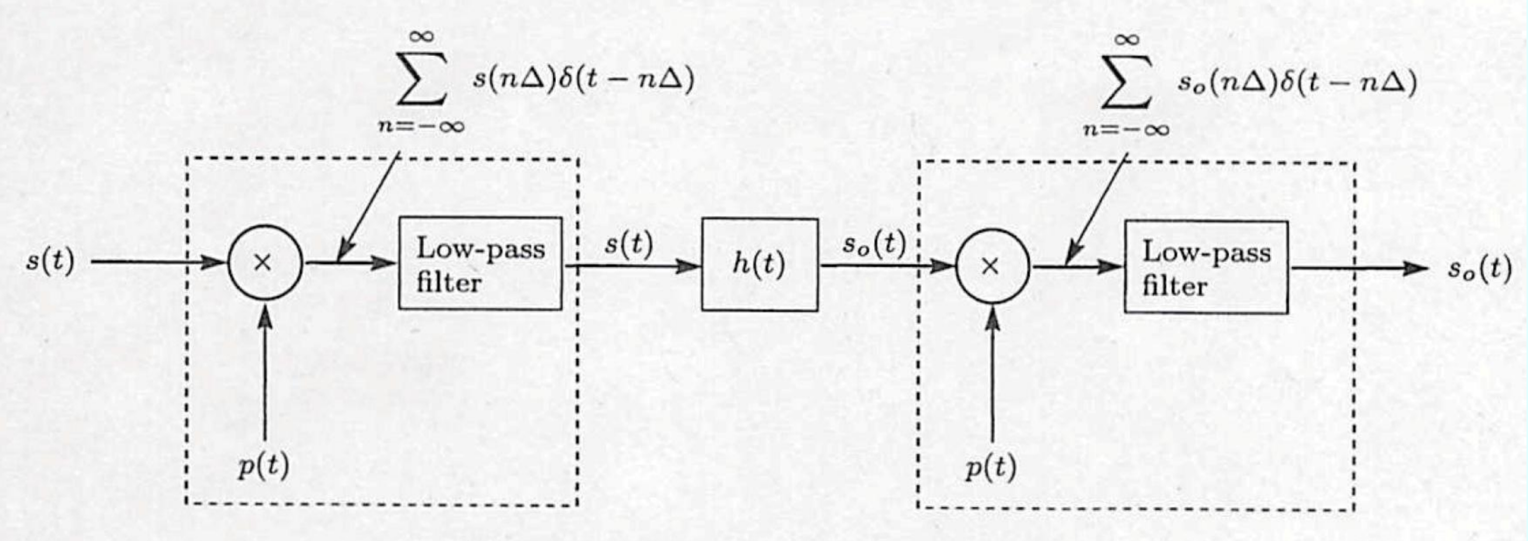
Sampling

- Sampling function is

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta).$$

- Frequency response of the low-pass filter is

$$H_{\text{lpf}}(F) = \begin{cases} \Delta & |F| < B \\ 0 & |F| > B \end{cases}.$$

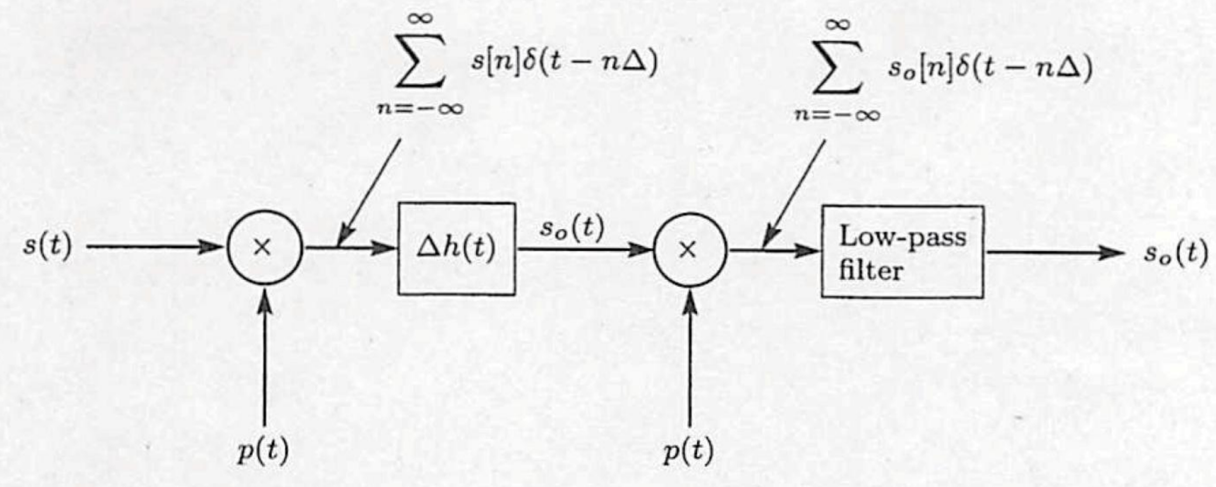


(b) Conversion to sampled data system

2. Discrete-Time System

- Reconstruction of $s_o(t)$

$$s_o(t) = \int_{-\infty}^{\infty} \Delta h(t-\tau) \sum_{m=-\infty}^{\infty} s(m\Delta)\delta(\tau - m\Delta) d\tau$$
$$= \Delta \sum_{m=-\infty}^{\infty} s(m\Delta) \int_{-\infty}^{\infty} h(t-\tau)\, \delta(\tau - m\Delta) d\tau$$
$$= \Delta \sum_{m=-\infty}^{\infty} s(m\Delta) h(t - m\Delta)$$



(c) Sampled data system

- Output samples

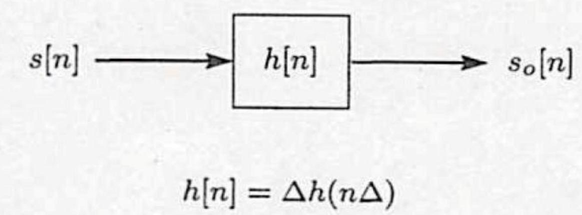
$$s_o(n\Delta) = \sum_{m=-\infty}^{\infty} \Delta h(n\Delta - m\Delta) s(m\Delta)$$

$$s_o[n] = s_o(n\Delta),\ h[n] = \Delta h(n\Delta),\ s[n] = s(n\Delta)$$

- Discrete-time output signal

$$s_o[n] = \sum_{m=-\infty}^{\infty} h[n-m] s[m]$$

$$\sum_{n=-\infty}^{\infty} s_o[n] \delta(t - n\Delta)$$



(d) Discrete-time system

Nyquist-Shannon sampling theorem (1/2)

- **Sampling (Definition)**
  The process of converting a signal (a function of continuous time or space) into a numeric sequence (a function of discrete time or space).
- A function is called bandlimited to *B* Hz if it Fourier transform

$$X(\omega) = F\left[x(t)\right] = \int \frac{d\omega}{2\pi} x(t) e^{-i\omega t}$$

  satisfies

$$\left|X(\omega)\right| = 0 \text{ for } |\omega| > \frac{B}{2\pi}$$

- Shannon's sampling theorem
  If a function *x*(*ω*) contains no frequency higher than *B* Hz (bandlimited to *B* Hz), it is completely determined by giving its ordinates at a series of points spaced 1/2*B* seconds apart.

Nyquist-Shannon sampling theorem (2/2)

- Whittaker-Shannon interpolation formula
  If a continuous-time signal $x(t)$ is bandlimited to B Hz and $x[n]'s$ are samples from $x(t)$ at interval $1/2B$, then $x(t)$ is reconstructed by interpolating $x[n]$:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \pi(2Bt-n)}{\pi(2Bt-n)} = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc} \pi(2Bt-n)$$

Note on the definition of frequency

- Angular frequency

$$\begin{cases} x(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} X(\omega) \\ X(\omega) = \int dt e^{-i\omega t} x(t) \end{cases}$$

$$\delta(t) = \int \frac{d\omega}{2\pi} e^{i\omega t}$$

- Frequency

$$\begin{cases} x(t) = \int df e^{2\pi i f t} X(f) \\ X(\omega) = \int dt e^{-2\pi i f t} x(t) \end{cases}$$

$$\delta(t) = \int df e^{2\pi i f t}$$

$$\boxed{\omega = 2\pi f}$$

Exercise

Consider a continuous-time signal $x(t) = \cos(2\pi \cdot 10t)$.

a) What is the regular frequency (f) of this signal in Hertz (Hz)?

b) What is the angular frequency (ω) of this signal in radians per second (rad/s)?

c) Find the Fourier Transform $X(f)$ of $x(t)$

(Hint: You might find Euler's formula helpful: $\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}$)

a) Now, find the Fourier Transform X(ω) of $x(t)$.

3. Signal Processing Example

- There are many signal processing problems that fit the Bayesian linear model.

- **Problem:** transmit a signal $s(t)$ through a channel with impulse response $h(t)$.

- To estimate $s(t)$ over the interval $0 \leq t \leq T_s$.

  Channel will distort and lengthen the signal and $x(t)$ is observed over the longer interval $0 \leq t \leq T$.

- ***Deconvolution problem (blind deconvolution)***

  To deconvolve $s(t)$ from a noise corrupted version of $s_o(t) = s(t) * h(t)$.

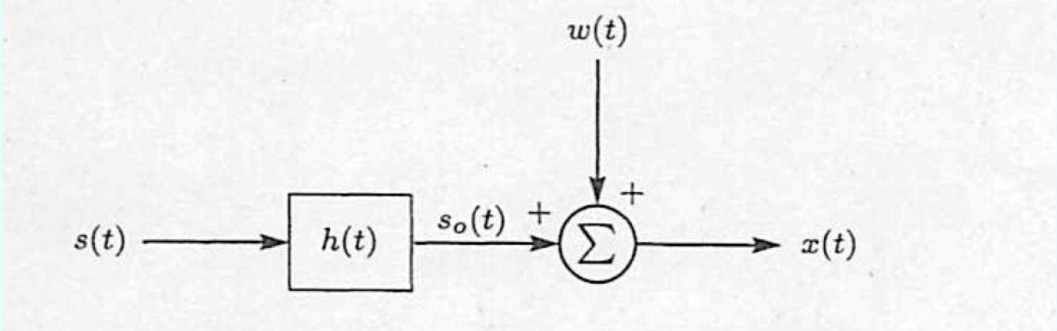
- $h(t)$: known
- $s(t)$: realization of a random

$$x(t) = \int_0^{T_s} h(t-\tau)s(\tau)d\tau + w(t), \qquad 0 \le t \le T$$

$s(t)$ has nonzero over the interval $[0, T_s]$

$h(t)$ has nonzero over the interval $[0, T_h]$

$T = T_s + T_h$



(a) System model

- Discrete time signals for $s(t),\ s_o(t)$: bandlimitted to B Hz
- $w(t)$: WSS (wide-sense stationary) Gaussian random process with $P_{ww}(F) = \begin{cases} \frac{N_0}{2} & |F| < B \\ 0 & |F| > B \end{cases}$
- Sampling

  $t = n\Delta$, $\Delta = 1/(2B)$, $n = 0,1,\cdots,N-1$

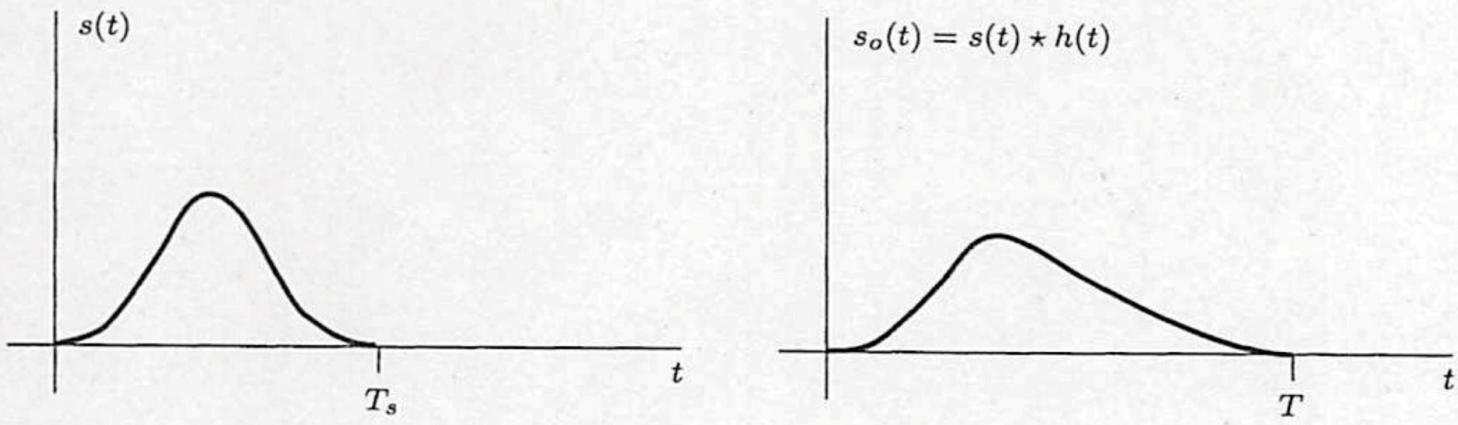
  $x[n] = \sum_{m=0}^{n_s-1} h[n-m]s[m] + w[n]$

  $s[n]$ has nonzero over the interval $[0, n_s - 1]$

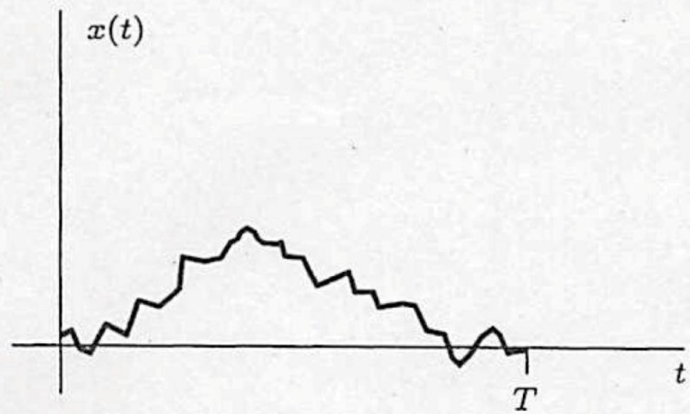
  $h[n]$ has nonzero over the interval $[0, n_h - 1]$

  $N = n_s + n_h - 1$

  $w[n]$ is White Gaussian Noise (WGN) with $\sigma^2 = N_0 B$



(b) Typical signals



(c) Typical data waveform

$$x[n] = \sum_{m=0}^{n_s-1} h[n-m]s[m] + w[n]$$

Component form:

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & \cdots & 0 \\ h[1] & h[0] & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ h[N-1] & h[N-2] & \cdots & h[N-n_s] \end{bmatrix} \begin{bmatrix} s[0] \\ s[1] \\ \vdots \\ s[n_s-1] \end{bmatrix} + \begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{bmatrix}$$

Matrix form:

$$\mathbf{x} = \mathbf{H}\mathbf{s} + \mathbf{w}$$

Note

- $h[n] = 0$ for $n > n_h - 1$
- $\mathbf{s} \sim N(\mathbf{0}, \mathbf{C}_s)$
- $[\mathbf{C}_s]_{ij} = r_{ss}[i-j]$, $r_{ss}[k]$: Autocorrelation

- MMSE estimator

$$\hat{\mathbf{s}} = \mathbf{C}_s \mathbf{H}^{\mathrm{T}} \left(\mathbf{H} \mathbf{C}_s \mathbf{H}^{\mathrm{T}} + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{x}$$

MAP derivation of MMSE estimate

- Find **s** that maximizes the posterior distribution:

$$\begin{aligned}\hat{\mathbf{s}} &= \arg\max_{\mathbf{s}}\ p(\mathbf{s}\,|\,\mathbf{x}) \\ &= \arg\max_{\mathbf{s}}\ p(\mathbf{x}\,|\,\mathbf{s})\,p(\mathbf{s}) \\ &= \arg\max_{\mathbf{s}}\ \log\left(p(\mathbf{x}\,|\,\mathbf{s})\,p(\mathbf{s})\right) \\ &= \arg\max_{\mathbf{s}}\left[-\frac{1}{2}(\mathbf{x}-\mathbf{H}\mathbf{s})^{\mathrm{T}}\sigma^{-2}(\mathbf{x}-\mathbf{H}\mathbf{s})-\frac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{C}_{\mathbf{s}}^{-1}\mathbf{s}\right]\end{aligned}$$

  by taking the derivative and setting it to zero:

$$0=\frac{\partial}{\partial\mathbf{s}}\left[\frac{1}{2}(\mathbf{x}-\mathbf{H}\mathbf{s})^{\mathrm{T}}\sigma^{-2}(\mathbf{x}-\mathbf{H}\mathbf{s})+\frac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{C}_{\mathbf{s}}^{-1}\mathbf{s}\right]_{\hat{\mathbf{s}}}$$

$$\boxed{\begin{aligned}\hat{\mathbf{s}} &= \left(\mathbf{H}^{\mathrm{T}}\sigma^{-2}\mathbf{H}+\mathbf{C}_{\mathbf{s}}^{-1}\right)^{-1}\mathbf{H}^{\mathrm{T}}\sigma^{-2}\mathbf{x} \\ &= \mathbf{C}_{\mathbf{s}}\mathbf{H}^{\mathrm{T}}\left(\mathbf{H}\mathbf{C}_{\mathbf{s}}\mathbf{H}^{\mathrm{T}}+\sigma^{2}\mathbf{I}\right)^{-1}\mathbf{x}\end{aligned}}$$

(another) Woodbury formula

$$(\mathbf{A}+\mathbf{BCD})^{-1}\mathbf{BC}=\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}^{-1}+\mathbf{D}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}$$

Wiener filter

$$\hat{\mathbf{s}} = \mathbf{A}\mathbf{x} = \mathbf{C_s}\mathbf{H}^{\mathrm{T}}\left(\mathbf{H}\mathbf{C_s}\mathbf{H}^{T} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{x}$$

Scalar case

$$\hat{s}[0] = \frac{C_s}{C_s+\sigma^2}x[0] = \frac{r_{ss}[0]}{r_{ss}[0]+\sigma^2}x[0] = \frac{\eta}{\eta+1}x[0]$$

$$\eta = r_{ss}[0]/\sigma^2$$

$$s[n] = -a[1]s[n-1] + u[n]$$

$u[n]$: WGN with variance σ^2

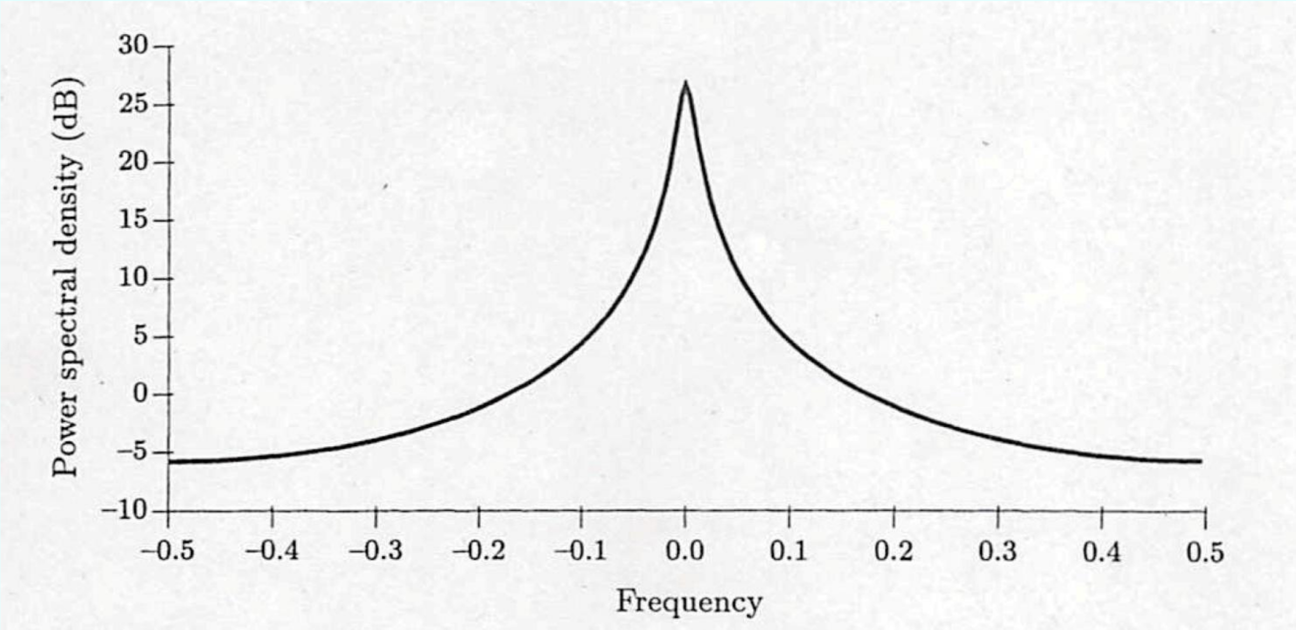
$$r_{ss}[k] = \frac{\sigma_u^2}{1-a^2[1]}(-a[1])^{|k|}$$

$$P_{ss}(f) = \frac{\sigma_u^2}{|1+a[1]\exp(-2\pi i f)|^2}$$

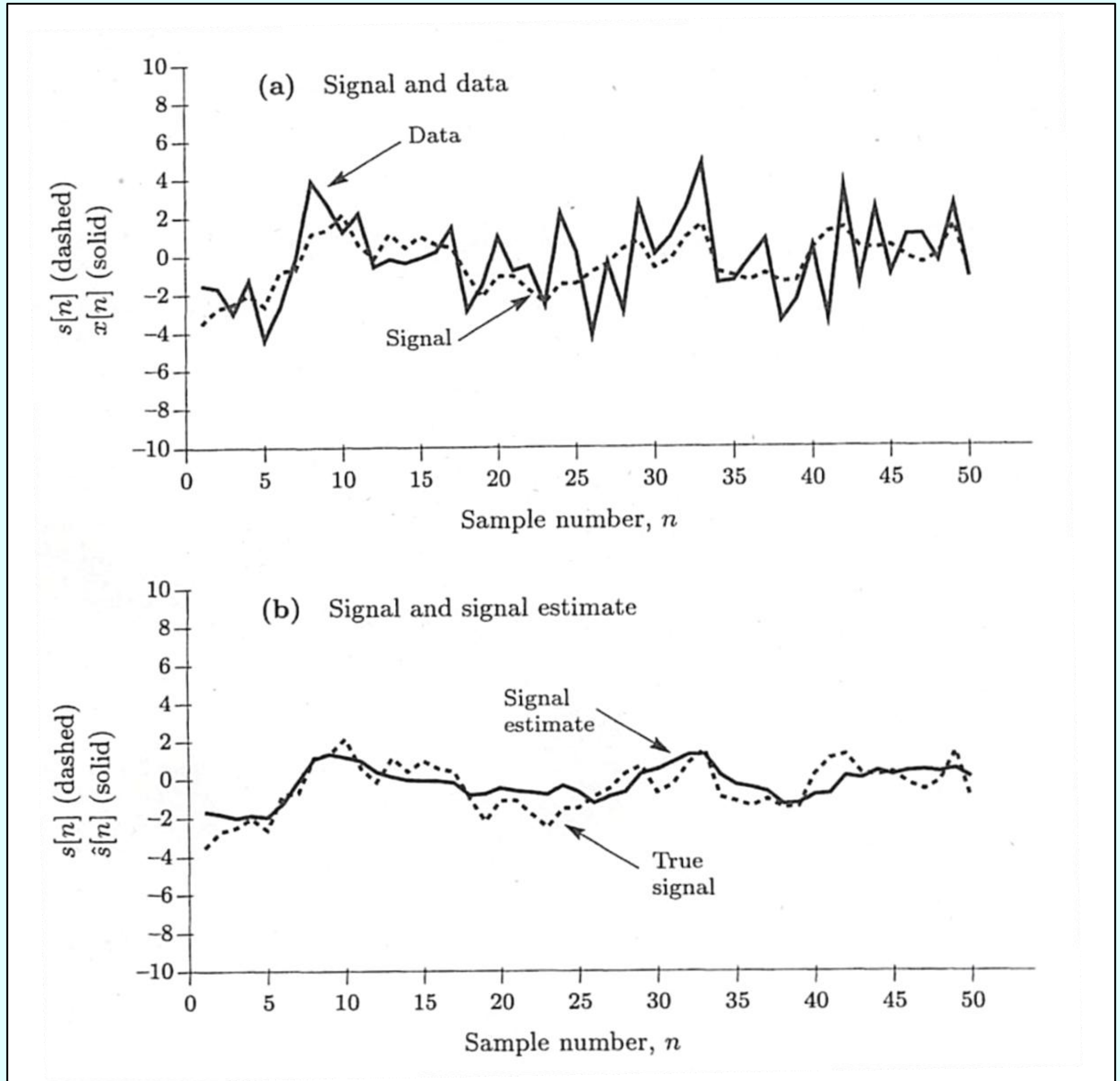
Example

- $a[1] < 0$, the power density function is that of a low-pass process.

- In case of $a[1] = -0.95$ and $\sigma_u^2 = 1$



- Realization of $s[n]$



4. Wiener Filtering

- **LMMSE estimator**
- Data $\{x[0], x[1], \cdots, x[n-1]\}$ is WSS with zero mean.
- **Covariance matrix:**

$$\boldsymbol{C}_{xx} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[N-1] \\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_{xx}[N-1] & r_{xx}[N-2] & \cdots & r_{xx}[0] \end{bmatrix} = \boldsymbol{R}_{xx}$$

(=Autocorrelation matrix)

Wiener filters:

Filtering

$\theta = s[n]$ is to be estimated based on

$$x[m] = s[m] + w[m] \text{ for } m = 0,1,\cdots,n.$$

Smoothing

$\theta = s[n]$ is to be estimated for $n = 0,1,\cdots,N-1$

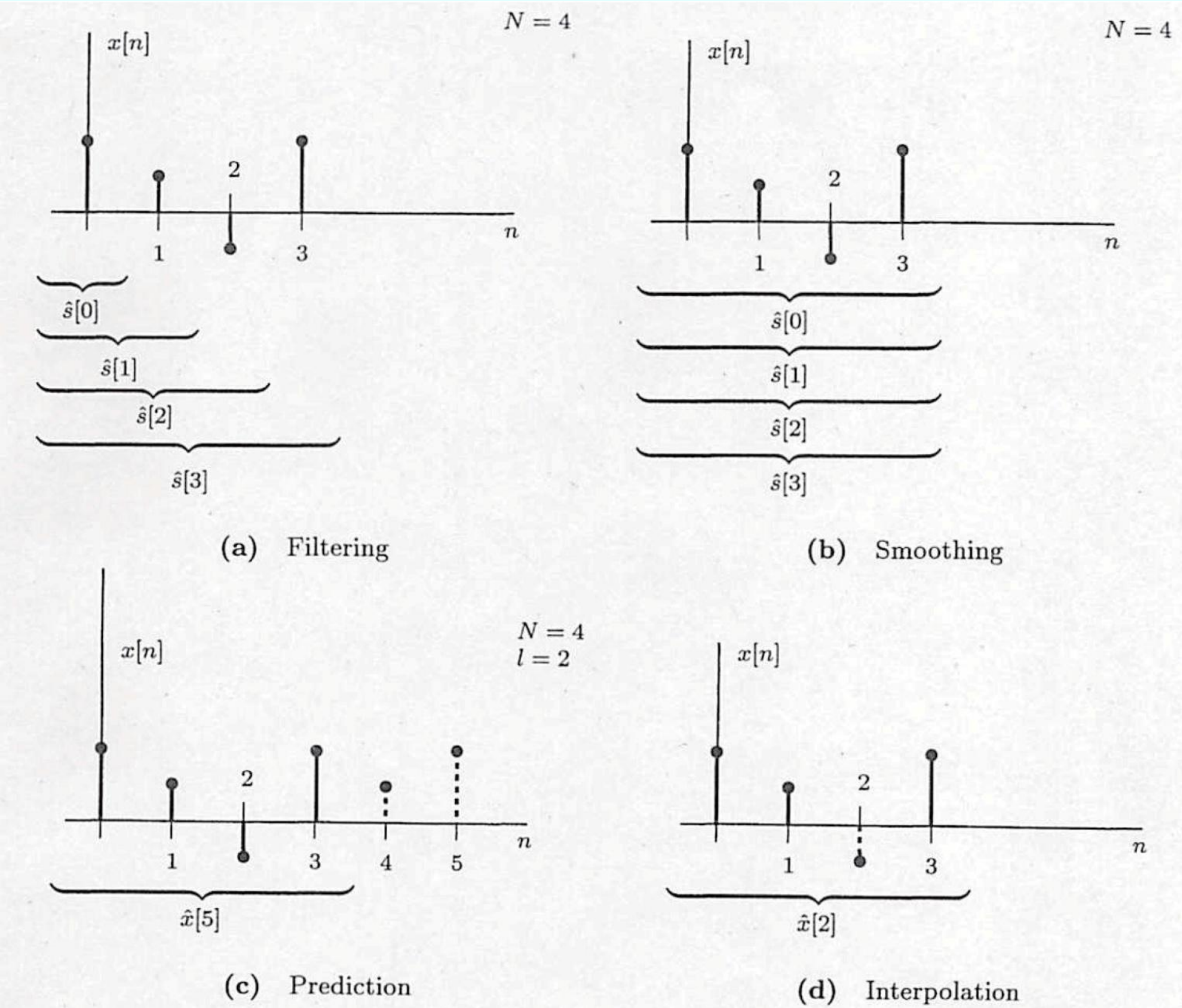
based on the data set $\{x[0], x[1], \cdots, x[N-1]\}$,

where $x[n] = s[n] + w[n]$.

Prediction

$\theta = x[N-1+l]$ for l a positive integer is to be estimated base on $\{x[0], x[1], \cdots, x[N-1]\}$.

$\rightarrow$ l***-step prediction***



$N = 4$
$x[n]$
n
$\hat{s}[0]$
$\hat{s}[1]$
$\hat{s}[2]$
$\hat{s}[3]$
(a) Filtering
(b) Smoothing
$l = 2$
$\hat{x}[5]$
$\hat{x}[2]$
(c) Prediction
(d) Interpolation

Wiener filter and Kalman filter

Kalman filter is an important generalization of Wiener filter.

- ***Wiener Filter***
  - WSS (Wide-sense-stationary) Process
  - Data from infinite past
  - Scalar signals
  - Non-adaptive
- ***Kalman Filter***
  - Gauss-Markov Process
  - Data from a specific point in time
  - Vector signals
  - Adaptive (model may evolve over time)