# I225E Statistical Signal Processing

## 2. Review of Probability Theory


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# Review of probability theory

- Probability theory
  Sample space, Borel set, conditional probability, Bayes' theorem
- Random variables
- Distribution functions, density functions
- Joint distributions
- Moments
- Characteristic functions
- Law of large numbers
- Central limit theorem

# 1. Review of Probability

All possible outcomes that may result from a trial (experiment or observation) are known ***a priori***.

However, it is impossible to predict which outcome to occur.

- Trial ***S***: Doing experiment or observation
- Sample Point $\omega$: Individual outcome that results from each trial
- Sample Space $\boldsymbol{\Omega}$: All sets of sample points
- Event ***A***: Subset of sample space

When $\omega \in \boldsymbol{\Omega}$ for $\omega \in \boldsymbol{A}$ , we say event A took place.

## [Example]

- Trial ***S***: Throw a dice
- Sample Point $\omega$ : 1, 2, 3, …
- Sample Space $\boldsymbol{\Omega}$ : {1, 2, 3, 4, 5, 6}
- Event ***A***: Odd {1,3,5} and Even {2,4,6}

- Trial ***S***: Twice coin-toss
- Sample Point $\omega$ : head (h) or tail (t)
- Sample Space $\boldsymbol{\Omega}$ : {hh, ht, th, tt}
- Event ***A***: only one head showed

# Event

(E1) Complementary Event: $\boldsymbol{A}^C = \{\omega \in \boldsymbol{\Omega}: \omega \notin \boldsymbol{A}\}$

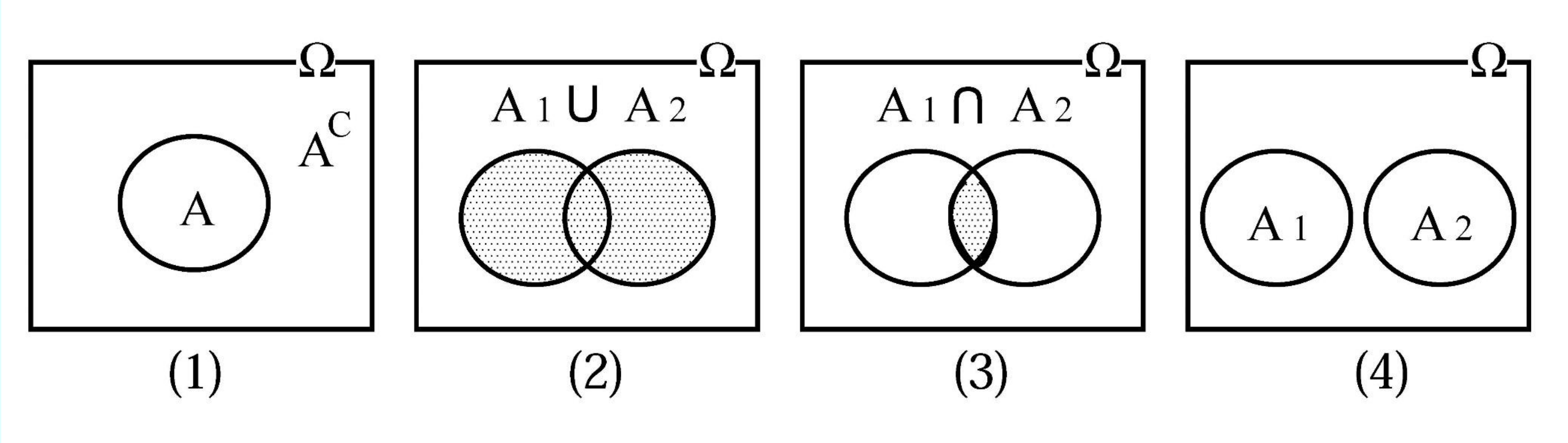
(E2) Sum Event (Union, OR):

$\boldsymbol{A}_1 \cup \boldsymbol{A}_2 = \{\omega \in \boldsymbol{\Omega}: \omega \in \boldsymbol{A}_1 \text{ or } \omega \in \boldsymbol{A}_2\}$ ("$\boldsymbol{A}_1$ or $\boldsymbol{A}_2$")

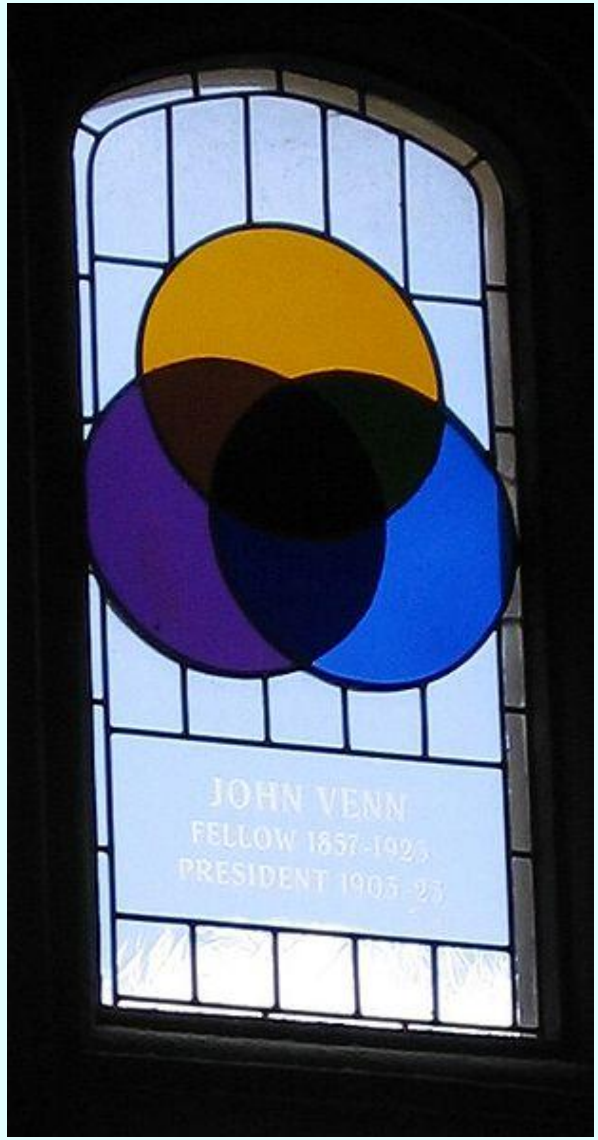
(E3) Product Event (Intersection, AND):

$\boldsymbol{A}_1 \cap \boldsymbol{A}_2 = \{\omega \in \boldsymbol{\Omega}: \omega \in \boldsymbol{A}_1 \text{ and } \omega \in \boldsymbol{A}_2\}$ ("$\boldsymbol{A}_1$ or $\boldsymbol{A}_2$")

(E4) Exclusive Event: $\boldsymbol{A}_1 \cap \boldsymbol{A}_2 = \boldsymbol{0}$



# Draw Venn diagrams!



# Borel set: *B*

A Borel set ***B*** is any set in a topological space that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement.

(B1) $\boldsymbol{\Omega} \in \boldsymbol{B}$.

(B2) If $\boldsymbol{A} \in \boldsymbol{B}$ then $\boldsymbol{A}^C \in \boldsymbol{B}$.

(B3) If $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots \in \boldsymbol{B}$ then $\cup_{i=1}^{\infty} \boldsymbol{A}_i \;(= \boldsymbol{A}_1 \cup \boldsymbol{A}_2 \cup \cdots) \in \boldsymbol{B}$.

From (B1)-(B3),

(B4) $\mathbf{0} \in \boldsymbol{B}$ .

(B5) If $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots \in \boldsymbol{B}$, then $\cap_{i=1}^{\infty} \boldsymbol{A}_i \;(= \boldsymbol{A}_1 \cap \boldsymbol{A}_2 \cap \cdots) \in \boldsymbol{B}$ .

# Borel set: Example

- Dice throwing

$$\boldsymbol{\Omega} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$$

$$\begin{aligned}\boldsymbol{B} = \{&\emptyset, \{\mathbf{1}\}, \cdots, \{\mathbf{6}\},\\ &\{\mathbf{1}, \mathbf{2}\}, \cdots \{\mathbf{5}, \mathbf{6}\},\\ &\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}, \cdots, \{\mathbf{4}, \mathbf{5}, \mathbf{6}\},\\ &\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}, \{\mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\},\\ &\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}, \cdots \{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\},\\ &\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}\}\end{aligned}$$

Borel set = “a set of all possible events”

# Probability

- Probability of event ***A***: *P*(***A***)
  mapping from event ***A*** to some numbers *P*(***A***).
- **Properties of** *P*(***A***) (Axioms of Probability theory)

  (P1) $0 \leq P(\boldsymbol{A}) \leq 1$

  (P2) $P(\boldsymbol{\Omega}) = 1$

  (P3) If $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots$ are exclusive events, then

  $P(\cup_{i=1}^{\infty} \boldsymbol{A}_i) = \sum_{i=1}^{\infty} P(\boldsymbol{A}_i)$ (Complete Additiveness)

- **Definition of Probability Space**:

  Sample space $\boldsymbol{\Omega}$ , Borel set ***B***, and probability *P* define probability space.

## Basic properties

(P4) $P(\mathbf{0}) = 0$

[Proof] Assuming $\boldsymbol{A}_i = \mathbf{0}\ (i = 1, 2, \cdots)$, then

$$\cup_{i=1}^{\infty} \boldsymbol{A}_i = \cup_{i=1}^{\infty} \mathbf{0} = \mathbf{0}.$$

On the other hand, from the assumption,

$$\boldsymbol{A}_i \cap \boldsymbol{A}_j = \mathbf{0} \cap \mathbf{0} = \mathbf{0}\ (i \neq \mathrm{j}).$$

According to Axiom (P3),

$$P(\mathbf{0}) = P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\mathbf{0}).$$

Since $\mathbf{0} \in \boldsymbol{\Omega}, P(\mathbf{0}) \geq 0$ (due to Axiom (P1)). The above equation holds only if $P(\mathbf{0}) = 0$.

(P5) If events $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots$ are mutually exclusive, then

$$P(\cup_{i=1}^{n} \boldsymbol{A}_i) = \sum_{i=1}^{n} P(\boldsymbol{A}_i).$$

[Proof] Setting $\boldsymbol{A}_{n+1} = \boldsymbol{A}_{n+2}, = \cdots = \mathbf{0}$,

$\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n, \boldsymbol{A}_{n+1}, \cdots$ are mutually exclusive.

Moreover, $\cup_{i=1}^{n} \boldsymbol{A}_i = \cup_{i=1}^{\infty} \boldsymbol{A}_i$.

According to Axiom (P3) and Property (P4),

$$P(\cup_{i=1}^{n} \boldsymbol{A}_i) = P(\cup_{i=1}^{\infty} \boldsymbol{A}_i)$$
$$= \sum_{i=1}^{n} P(\boldsymbol{A}_i) + \sum_{i=n+1}^{\infty} P(\boldsymbol{A}_i)$$
$$= \sum_{i=1}^{n} P(\boldsymbol{A}_i) + \sum_{i=n+1}^{\infty} P(\mathbf{0}) = \sum_{i=1}^{n} P(\boldsymbol{A}_i)$$

(P6) $P(\boldsymbol{A}^C) = 1 - P(\boldsymbol{A})$

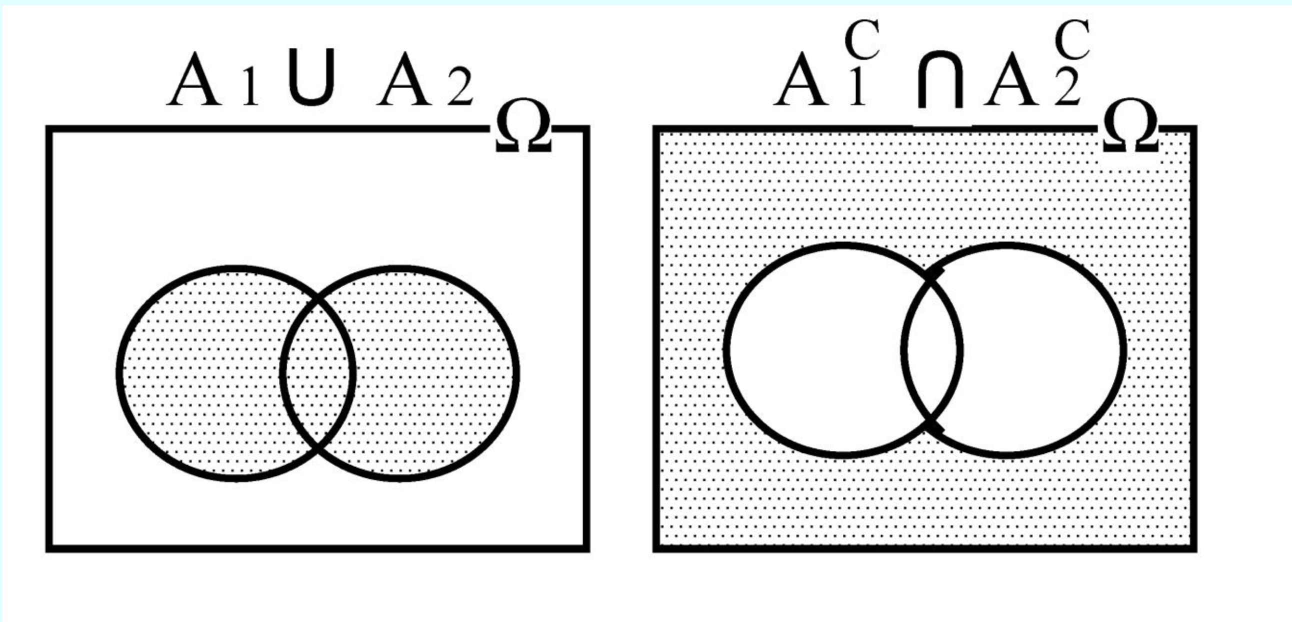
(P7) $P(\cup_i \boldsymbol{A}_i) = 1 - P(\cap_i \boldsymbol{A}_i^C)$

**De Morgan's laws**:

$$(\cap_i \boldsymbol{A}_i)^C = \cup_i \boldsymbol{A}_i^C,\ (\cup_i \boldsymbol{A}_i)^C = \cap_i \boldsymbol{A}_i^C$$

De Morgan's laws in case of 2 sets

$$(\boldsymbol{A}_1 \cup \boldsymbol{A}_2)^C = \boldsymbol{A}_1^C \cap \boldsymbol{A}_2^C$$

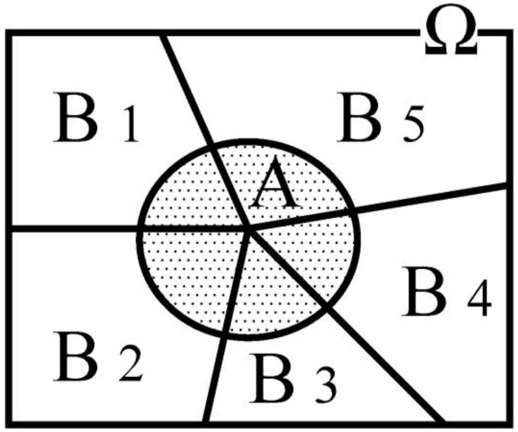


(P8) If sequence of mutually exclusive events, $\boldsymbol{B}_1, \boldsymbol{B}_2, \cdots$, is such that :

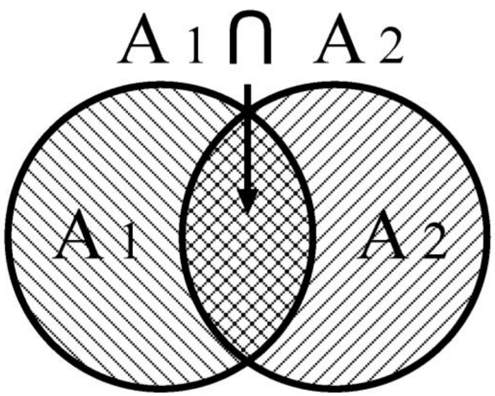
$\cup_i \boldsymbol{B}_i = \boldsymbol{\Omega}$, then $P(\boldsymbol{A}) = \sum_i P(\boldsymbol{A} \cap B_i)$.

(P9) If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are not exclusive, then $P(\boldsymbol{A}_1 \cup \boldsymbol{A}_2) = P(\boldsymbol{A}_1) + P(\boldsymbol{A}_2) - P(\boldsymbol{A}_1 \cap \boldsymbol{A}_2)$.

(8)




(9)



(P10) For any $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3$,

$$
\begin{aligned}
&\mathrm{P}(\boldsymbol{A}_1 \cup \boldsymbol{A}_2 \cup \boldsymbol{A}_3) \\
&= P(\boldsymbol{A}_1) + P(\boldsymbol{A}_2) + P(\boldsymbol{A}_3) \\
&- \{P(\boldsymbol{A}_1 \cap \boldsymbol{A}_2) + P(\boldsymbol{A}_2 \cap \boldsymbol{A}_3) + P(\boldsymbol{A}_3 \cap \boldsymbol{A}_1)\} \\
&\qquad + P(\boldsymbol{A}_1 \cap \boldsymbol{A}_2 \cap \boldsymbol{A}_3)
\end{aligned}
$$

(P11) General case: Denoting

$S_m = \sum_{i_1 \le i_2 \le \cdots \le i_m} P(\boldsymbol{A}_{i_1} \cap \boldsymbol{A}_{i_2} \cap \cdots \cap \boldsymbol{A}_{i_m})$ , then

$$P(\cup_{i=1}^{n} \boldsymbol{A}_i) = S_1 - S_2 + S_3 - S_4 + \cdots + (-1)^{n-1} S_n.$$

Note: Additive terms on the right side are subject to all combination of $S_n$ choosing *n* events from $1, 2, \cdots, n$.

The combination number is ${}_nC_m = \binom{n}{m} = \frac{n!}{(n-m)!m!}$.

## Continuity of Probability

(P12) Consider infinite sequence of events $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots$ such that $\boldsymbol{A}_1 \subset \boldsymbol{A}_2 \subset \cdots$. For $\boldsymbol{A} = \bigcup_{i=1}^{\infty} \boldsymbol{A}_i$,

$$P(\boldsymbol{A}) = \lim_{i\to\infty} P(\boldsymbol{A}_i)$$

(P13) Consider infinite sequence of events $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots$ such that $\boldsymbol{A}_1 \supset \boldsymbol{A}_2 \supset \cdots$. For $\boldsymbol{A} = \bigcap_{\boldsymbol{i=1}}^{\infty} \boldsymbol{A}_i$,

$$P(\boldsymbol{A}) = \lim_{i\to\infty} P(\boldsymbol{A}_i)$$

# Conditional Probability

## Conditional Probability

First event has an influence on probability of next event.

***B***: First event

***A***: Next event

Conditional probability $P(\boldsymbol{A}|\boldsymbol{B})$ of event ***A*** assuming event ***B*** is

$$P(\boldsymbol{A}|\boldsymbol{B}) = \frac{P(\boldsymbol{A} \cap \boldsymbol{B})}{P(\boldsymbol{B})}$$

# Properties of Conditional Probability

(C1) If we fix $\boldsymbol{B}$ and denote as $P(\boldsymbol{A}|\boldsymbol{B}) \equiv P^*(\boldsymbol{A})$,

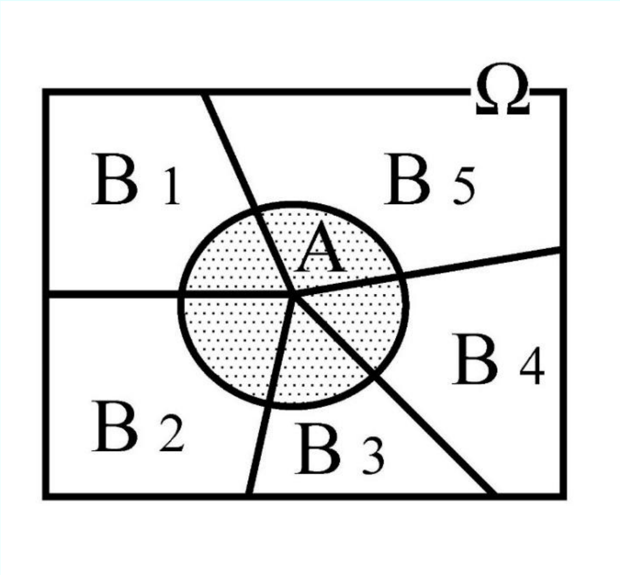
(P1) - (P13) hold for $P^*(\boldsymbol{A})$.

(C2) $P(\boldsymbol{A} \cap \boldsymbol{B}) = P(\boldsymbol{A}|\boldsymbol{B})P(\boldsymbol{B})$,

(C3) If events $\boldsymbol{B}_1, \boldsymbol{B}_2, \cdots$ are mutually exclusive and $\bigcup_i \boldsymbol{B}_i = \boldsymbol{\Omega}$, then, for any event $\boldsymbol{A}$,

$$P(\boldsymbol{A}) = \sum_i P(\boldsymbol{B}_i)P(\boldsymbol{A}|\boldsymbol{B}_i)$$

(Total Probability Theorem)




(C4) If $P(\boldsymbol{A}) \geq 0$,

$$P(\boldsymbol{B}_i|\boldsymbol{A}) = \frac{P(\boldsymbol{A}|\boldsymbol{B}_i)P(\boldsymbol{B}_i)}{\sum_j P(\boldsymbol{A}|\boldsymbol{B}_j)P(\boldsymbol{B}_j)}$$

(Bayes' Theorem)

**[Exercise]** Company C buys the same product from three companies ($B_1$, $B_2$, $B_3$). The purchase ratio is $B_1 = 0.5$, $B_2 = 0.3$, and $B_3 = 0.2$. Probability that a new product is broken within one year is given for each company as $B_1 = 0.015$, $B_2 = 0.025$, and $B_3 = 0.035$.

- Suppose now that one product is broken within one year of purchase. We denote this event by $A$. What is the probability that the broken product was purchased from company $B_1$, $B_2$, or $B_3$.

**[Answer]** $P(B_1) = 0.5$, $P(B_2) = 0.3$, $P(B_3) = 0.2$

$P(A|B_1) = 0.015$, $P(A|B_2) = 0.025$,

$P(A|B_3) = 0.035$

From Total Probability Theorem (C3), $P(A) = \sum_{i=1}^{3} P(A|B_i)P(B_i) = 0.0075 + 0.0075 + 0.007$

$= 0.022$

Using Bayes' Theorem (C4)

$$P(B_1|A) = \frac{0.0075}{0.022} \approx 0.34,\ P(B_2|A) = \frac{0.0075}{0.022} \approx 0.34$$

$$P(B_3|A) = \frac{0.0070}{0.022} \approx 0.32$$

# Independency

(1) If events ***A*** and ***B*** are mutually independent, then

$$P(\boldsymbol{A} \cap \boldsymbol{B}) = P(\boldsymbol{A})P(\boldsymbol{B}).$$

(2) Necessary and sufficient condition for mutual independence of *n* events, $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n$, is that, for a set of arbitrarily chosen events, $\boldsymbol{A}_{i(1)}, \boldsymbol{A}_{i(2)}, \cdots, \boldsymbol{A}_{i(k)}$, the following holds

$$P\left(\boldsymbol{A}_{i(1)} \cap \boldsymbol{A}_{i(2)} \cap \cdots \cap \boldsymbol{A}_{i(k)}\right) = P\left(\boldsymbol{A}_{i(1)}\right)P\left(\boldsymbol{A}_{i(2)}\right) \cdots P\left(\boldsymbol{A}_{i(k)}\right)$$

# 2. Random variable

## Definition:

If real valued function $\boldsymbol{X}(\omega)$ $(\omega \in \boldsymbol{\Omega})$ defined on probability space satisfies $\{\omega : \boldsymbol{X}(\omega) \leq x\} \in \boldsymbol{B}$ for any real value *x*, $\boldsymbol{X}(\boldsymbol{\Omega})$ is called ***random variables***.

## Example

From a box with many balls with different color, pick up one ball. One obtains a coupon card corresponding to the color of the ball.

| Color (Even) $\omega$ | White | Green | Yellow | Blue | Red |
|---|---|---|---|---|---|
| Coupon (Random variable) $X(\boldsymbol{\Omega})$ | 500 | 1000 | 2000 | 4000 | 6000 |
| Probability | 0.72 | 0.15 | 0.1 | 0.02 | 0.01 |

P(white)=$\{\omega : X(\boldsymbol{\Omega}) \leq 500\}$

# 3. Distribution function

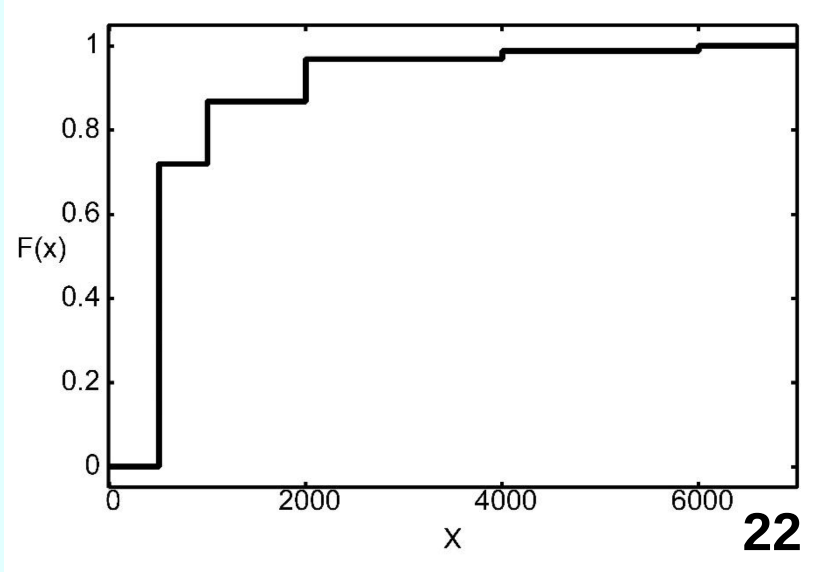
- **Definition**:

$$F_{\boldsymbol{X}}(x) = P(\{\omega : \boldsymbol{X}(\omega) \leq x\}), -\infty \leq x \leq \infty$$

Left-hand-side can be simply described as *F(x)*
Right-hand-side as $P(\boldsymbol{X} \leq x)$.

- **Example**

$$\begin{aligned} F&(2000) \\ &= P(\{\omega : X(\omega) \leq 2000\}) \\ &= P(\{\text{Yellow} \cup \text{Green} \cup \text{White}\}) \\ &= 0.72 + 0.15 + 0.1 \\ &= 0.97 \end{aligned}$$



## Properties of the distribution function

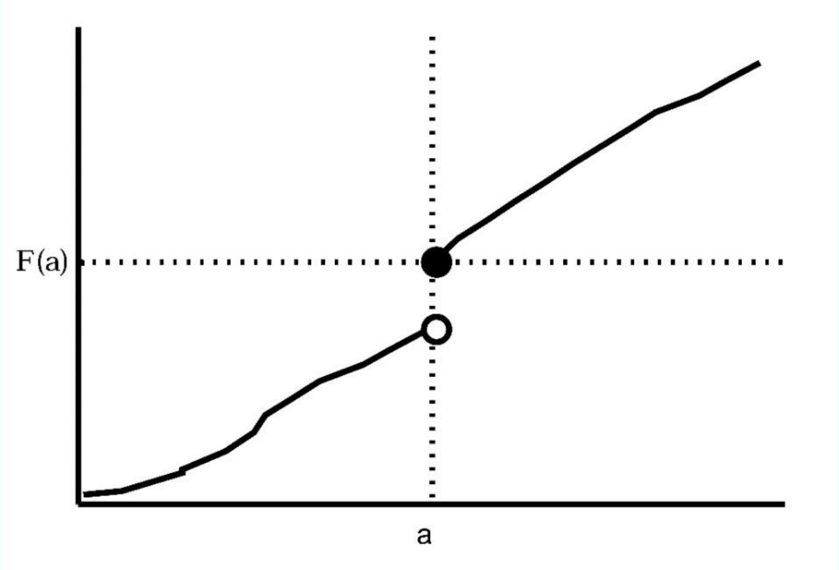
(PD1) Non-decreasing

If $a < b$,

then $F(a) \leq F(b)$.

(PD2) Continuous from the right

$$\lim_{x \to a+0} F(x) = F(a)$$




(PD3) Limit

$$\lim_{x \to -\infty} F(x) = 0$$

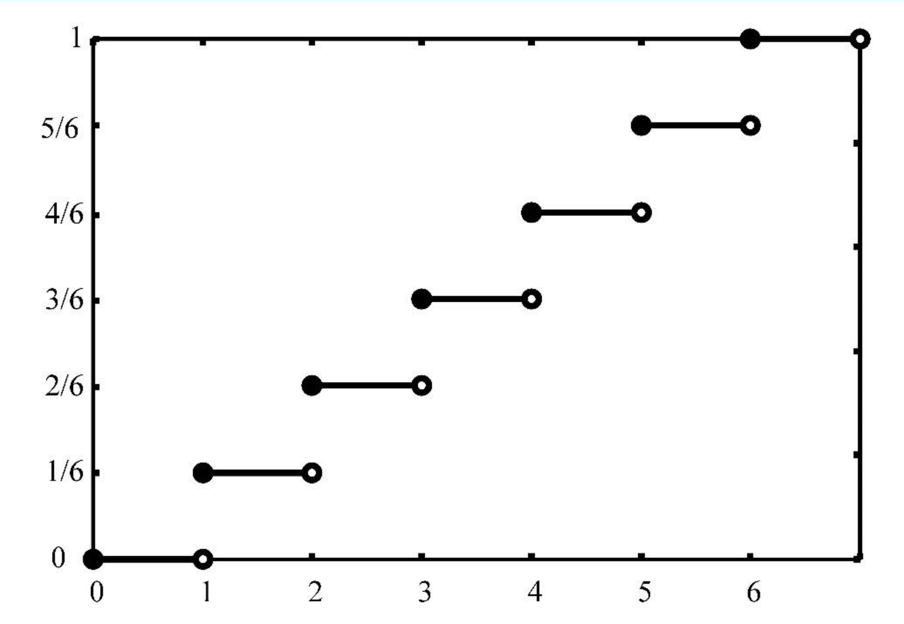
$$\lim_{x \to \infty} F(x) = 1$$

# [Example]

- Trial ***S***: Throw dice
- Sample Point $\omega$: 1, 2, 3, …
- Sample Space $\boldsymbol{\Omega}$: {1, 2, 3, 4, 5, 6}

- $F(\{1 \leq \omega < 2\}) = \frac{1}{6}$
- $F(2) = \frac{1}{3}$
- $F(3) = \frac{1}{2}$

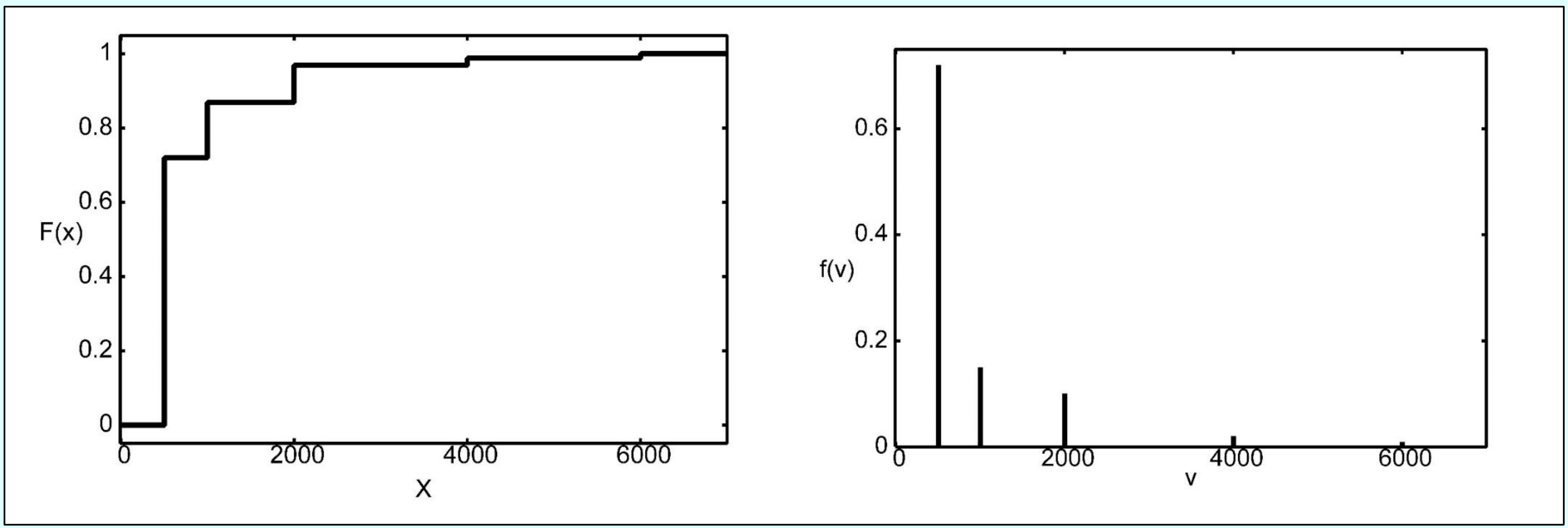
  …
- $F(6) = 1$



# 4. Discrete-type Distribution Function

- Possible values $\boldsymbol{V} = \{v_0, v_1, v_2, \cdots\}$ that ***X*** can take is finite or countable. Taking $\varepsilon > 0$ in a way that no elements except $v_k$ are included in the interval $[v_k - \varepsilon, v_k]$, the probability function (probability that ***X*** takes a value of ):

$$f(v_k) = P(X = v_k) \text{ is } f(v_k) = F(v_k) - F(v_k - \varepsilon).$$



## Properties

(DT1) $F(x) = \sum_{v_k \leq x} f(v_k)$

(DT2) For all $k$, $f(v_k) \geq 0$

(DT3) $\sum_k f(v_k) = 1$

## Example of discrete-type distribution function

(Ex1) **Bernoulli distribution**: $B(1;p), 0 < p < 1$

Random variable that can take only two values (i.e., biased coin)

$$V = \{0, 1\}$$

$$f(0) = 1 - p$$

$$f(1) = p$$

(Ex2) **Binominal distribution**: $B(n;p), 0 < p < 1$

Consider a lottery that has *p* percent of winning tickets. Try the lottery for *n* times. Probability to draw the winning ticket for k times yields binominal distribution. Probability to win *k* times and loose $(n-k)$ times is $p^k = (1-p)^{n-k}$. Combination of choosing *k* objects among *n* is ${}_nC_k = \binom{n}{k}$.

Hence, $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$

(Ex3) **Poisson distribution**: $P_o(\lambda)$

Binominal distribution in the limit of $n \to \infty,\ p \to 0$,

$$np = \lambda,\ f(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Stirling's formula $n! \approx \sqrt{2\pi n}\ n^n e^{-n}$

# 5. Continuous-type Distribution Function

- **Distribution function:**

  $F(x)$ can be described as

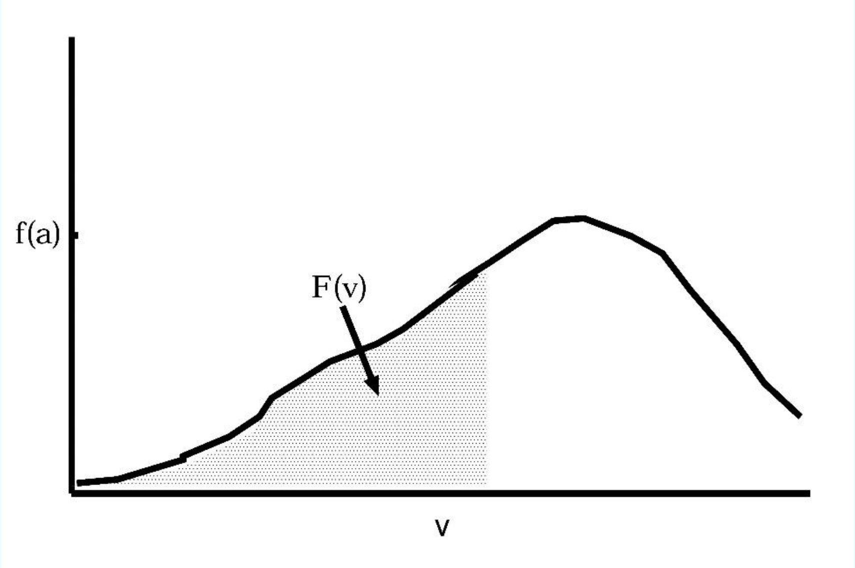
$$F(x) = \int_{-\infty}^{x} f(v)dv$$

- **Probability density** is

$$f(x) = \frac{d}{dx}F(x)$$

(CT1) $f(x) > 0, -\infty < x < \infty$

(CT2) $\int_{-\infty}^{\infty} f(x)dx = 1$



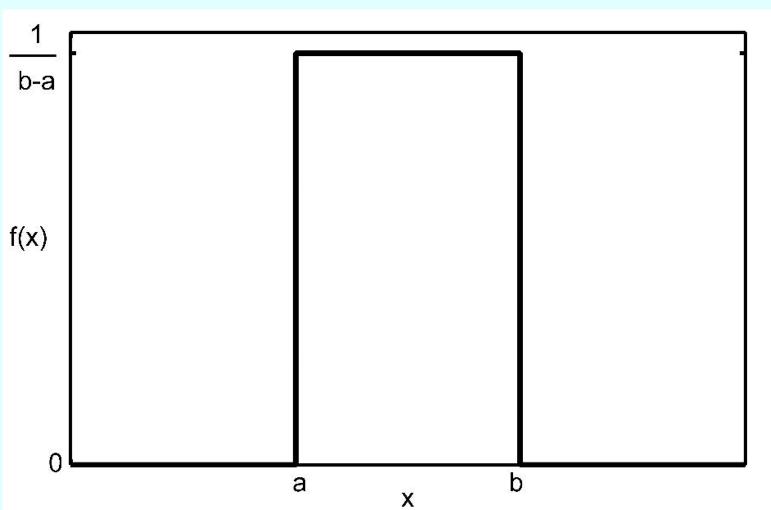
# Example of continuous-type distribution function

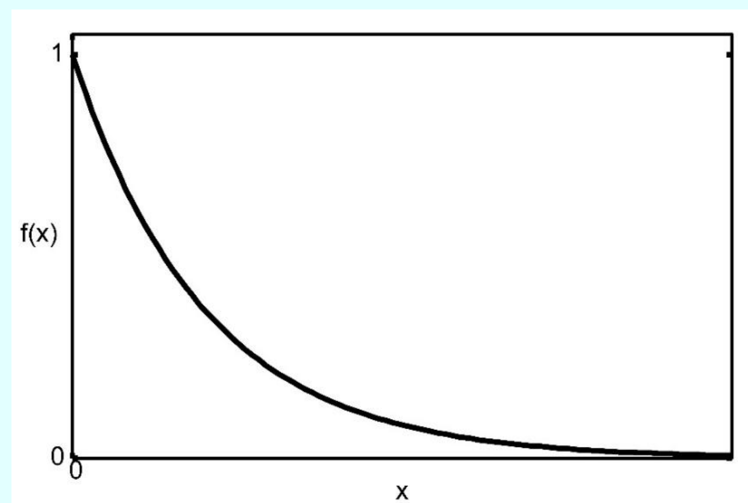
(Ex1) **Uniform distribution**: $U(a,b), a < b$

$$U(a,b) = f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

(Ex2) **Exponential distribution**: $E_x(\alpha), \alpha > 0$

$$E_x(\alpha) = f(x) = \begin{cases} \alpha e^{-\alpha x} & x \ge 0 \\ 0, & x < 0 \end{cases}$$



(Ex3) **Gamma distribution**: $G(\alpha, \nu)$

$$G(\alpha,\nu) = f(x) = \begin{cases} \frac{1}{\Gamma(\nu)} \alpha^{\nu} x^{\nu-1} e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

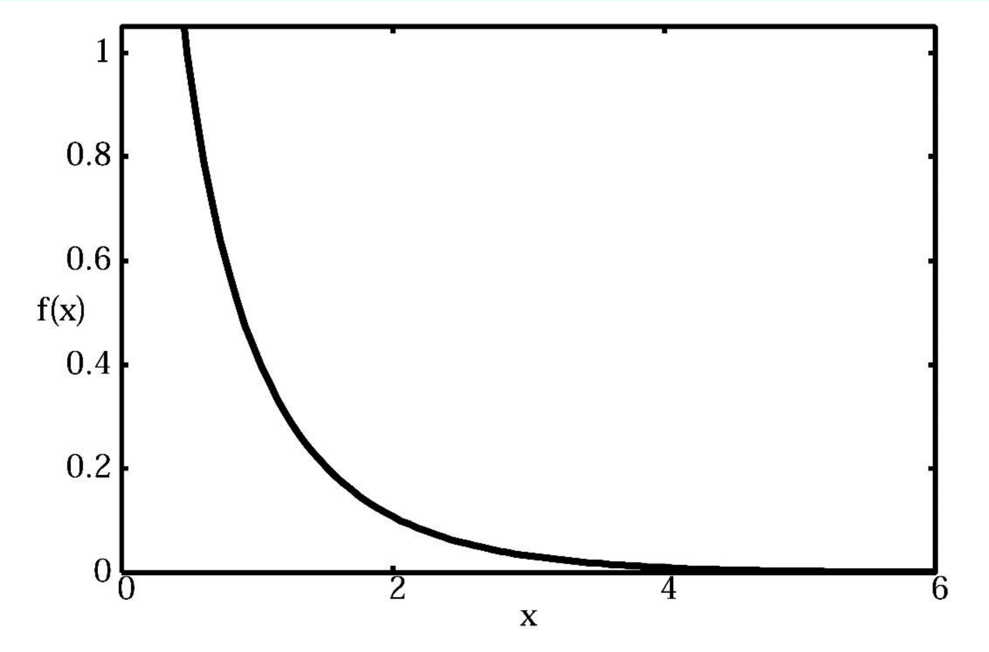
where $\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt.$

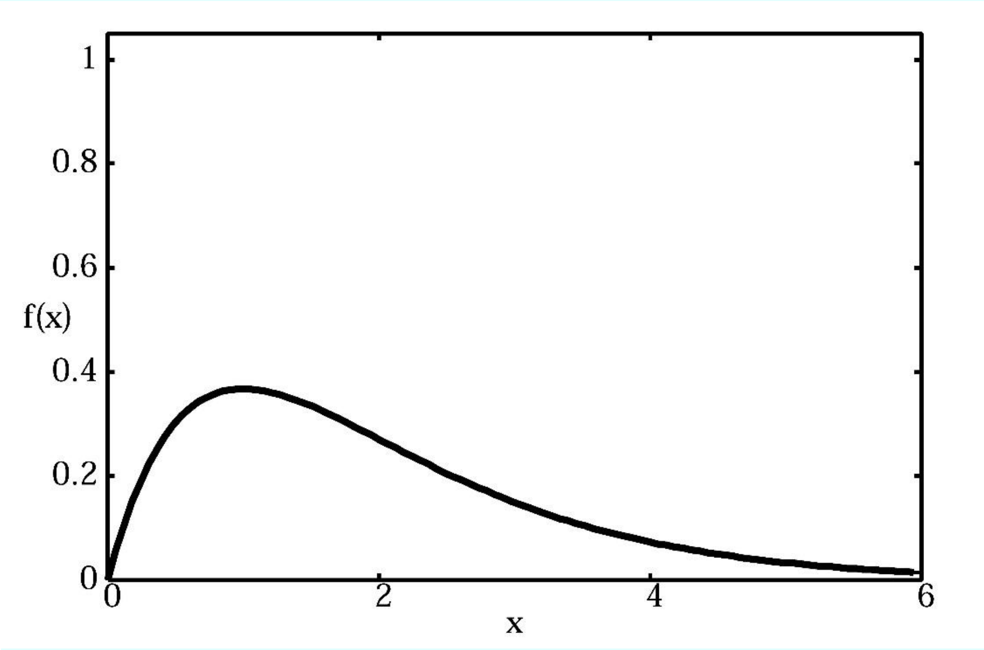
- Properties of $\Gamma$ :
  - $\Gamma(\nu) = (\nu-1)\Gamma(\nu-1),\ \Gamma(1) = 1$
  - $\Gamma(1/2) = \sqrt{\pi}$

  Especially, if *n* is an integer,
  - $\Gamma(n) = (n-1)!$
  - $\Gamma\left(n + \frac{1}{2}\right) = \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right)\cdots\frac{1}{2}\sqrt{\pi}$

When $\nu = 1$, equivalent to $E_x(\alpha)$.

- $G\left(\frac{1}{2}, \frac{n}{2}\right)$: $\chi^2$ (chi-square) distribution with *n* degree of freedom (often used for ANOVA in statistics).



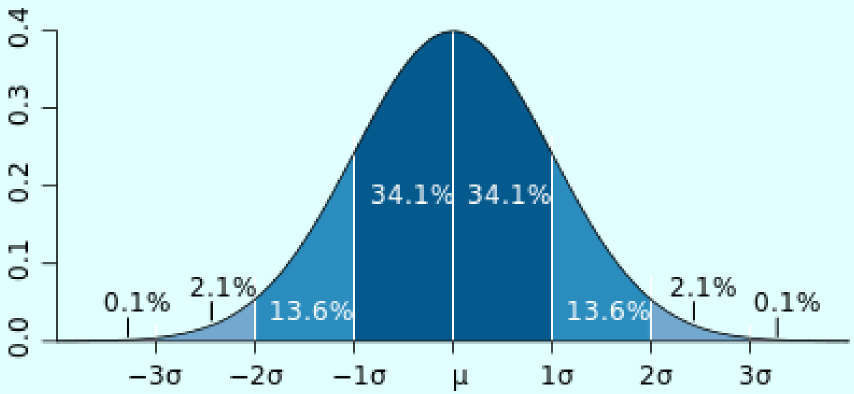



(Ex4) **Gaussian distribution**:

$$N(\mu, \sigma^2), -\infty < \mu < \infty, \sigma > 0$$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], -\infty < x < \infty$$

- In case of $N(0,1)$, **standard normal distribution**
- Probability density: $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2\right]$
- Distribution function: $\boldsymbol{\Phi}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}t^2\right] dt$
- Properties of $\phi(x)$
  - $\phi(x) = \phi(-x)$
  - $\boldsymbol{\Phi}(x) = 1 - \boldsymbol{\Phi}(-x)$
  - $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$



**[Exercise]** Suppose $\boldsymbol{X}$ is a random variable of $N(0,1)$. With respect to $\boldsymbol{Y}=\boldsymbol{X}^2$, find its distribution function $F_{\boldsymbol{Y}}(y)$ and the probability density $f_{\boldsymbol{Y}}(y)$.

**[Answer]** If $y<0, F_{\boldsymbol{Y}}(y)=0$, $f_{\boldsymbol{Y}}(y)=0$.

If $y \geq 0$,

$$F_{\boldsymbol{Y}}(y)=P(\boldsymbol{Y} \leq y)=P(\boldsymbol{X}^2 \leq y)=P(-\sqrt{y} \leq \boldsymbol{X} \leq \sqrt{y})$$

$$=\int_{-\sqrt{y}}^{\sqrt{y}} \phi(x)dx=2\int_0^{\sqrt{y}} \phi(x)dx,$$

$$f_{\boldsymbol{Y}}(y)=\frac{dF_{\boldsymbol{Y}}(y)}{dy}=\frac{d}{dy}\left\{2\int_0^{\sqrt{y}} \phi(x)dx\right\}=2\phi\left(\sqrt{y}\right)\frac{1}{2\sqrt{y}}$$

$$=\frac{1}{\sqrt{2\pi}}y^{-\frac{1}{2}}\exp\left(-\frac{1}{2}y\right)=G\left(\frac{1}{2},\frac{1}{2}\right)$$

This is $\chi^2$ distribution with $n=1$ degrees of freedom.

# 6. Joint Distribution

- Consider more than $n$ random variables $\boldsymbol{X}_1(\omega), \boldsymbol{X}_2(\omega), \cdots, \boldsymbol{X}_n(\omega)$ simultaneously, and examine their interrelation.
- With regard to event $\{\omega : \boldsymbol{X}_1(\omega) \leq x_1, \boldsymbol{X}_2(\omega) \leq x_2, \cdots, \boldsymbol{X}_n(\omega) \leq x_n\} = \bigcap_{k=1}^{n} \{\omega : \boldsymbol{X}_k(\omega) \leq x_k\}$ the distribution function is defined as

$$F(x_1, x_2, \cdots, x_n) = P(\bigcap_{k=1}^{n} \{\omega : \boldsymbol{X}_k(\omega) \leq x_k\})$$

Case of $n = 2$: $F(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u, v) du dv$

- Marginal distribution function of $\boldsymbol{X}_1$: $F_1(x) = F(x, +\infty)$
- Marginal distribution function of $\boldsymbol{X}_1$: $F_2(x) = F(+\infty, x)$
- Conditional distribution function given

$$\boldsymbol{X}_2 < c : F(x, c)/F(+\infty, c)$$

## [Example] Two-Dimensional Gaussian Distribution

**Density function**

$$f(u,v)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2}Q(u,v)\right\}$$

where

$$Q(u,v)=\frac{1}{1-\rho^2}\left\{\left(\frac{u-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{u-\mu_1}{\sigma_1}\right)\left(\frac{v-\mu_2}{\sigma_2}\right)+\left(\frac{v-\mu_2}{\sigma_2}\right)^2\right\}$$

$$\rho=\frac{C}{\sigma_1\sigma_2},$$

$C=E\{(u-\mu_1)(v-\mu_2)\}$ (**Covariance**)

# 7. Independency of random variables

Consider *n* random variables $\boldsymbol{X_1}(\omega), \boldsymbol{X}_2(\omega), \cdots, \boldsymbol{X}_n(\omega)$. If their joint distribution function $F(x_1, x_2, \cdots, x_n)$ is equal to the product of distribution functions of each random variable

$$F(x_1, x_2, \cdots, x_n) = F(x_1)F(x_2)\cdots F(x_n).$$

- *n* random variables are mutually independent.

- The same holds for probability density as

$$f(x_1, x_2, \cdots, x_n) = f(x_1)f(x_2)\cdots f(x_n)$$

Case of 2-dimentional normal distribution function.
Suppose $\rho = 0$ (covariance is zero).

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$
$$= f_1(x_1) f_2(x_2)$$

where

$$f_i(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left\{-\frac{1}{2}\left[\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right]\right\} \quad (i = 1, 2)$$

# 8. Mean

- **Discrete-type Distribution Function**:

  When probability function $f(v_k)$ is given by
  $$f(v_k) = P(\boldsymbol{X} = v_k),$$
  $$E(\boldsymbol{X}) = \sum_k v_k f(v_k)$$
  is called ***mean*** or ***expected value***.

- **Remark**

  Strictly speaking, the mean exists when
  $$\sum_k |v_k| f(v_k) < \infty$$

- **Example  Dice**

  $$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5$$

## Continuous-type Distribution Function:

When the probability function of $\boldsymbol{X}$ is given by $f(x)$,

$$E(\boldsymbol{X}) = \int_{-\infty}^{\infty} x f(x) dx$$

## Remark

Strictly speaking, the mean exists when

$$\int_{-\infty}^{\infty} |x| f(x) dx < \infty$$

## Example

(Ex1) **Bernoulli distribution** $B(1; p)$

$E(\boldsymbol{X}) = 0 \times f(0) + 1 \times f(1) = 0 \times 1(1-p) + 1 \times p = p$

(Ex2) **Binomial distribution** $B(n; p)$

$$
\begin{aligned}
E(\boldsymbol{X}) &= \textstyle\sum_{k=0}^{n} k f(k) = \sum_{k=1}^{n} k f(k) \\
&= \textstyle\sum_{k=1}^{n} k \binom{n}{k} p^{k} (1-p)^{n-k} \\
&= \textstyle\sum_{k=1}^{n} n \binom{n-1}{k-1} p^{k} (1-p)^{n-k} \\
&= np \textstyle\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \textstyle\sum_{l=0}^{m} \binom{m}{l} p^{l} (1-p)^{m-l} \\
&= np
\end{aligned}
$$

## (Ex3) **Poisson distribution** $P_0(\lambda)$

$$E(\boldsymbol{X}) = \sum_{k=0}^{\infty} k f(k) = \sum_{k=1}^{\infty} k f(k)$$

$$= \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}$$

$$\{\text{substituting } m = k - 1\}$$

$$= \lambda \sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!}$$

$$\left\{\text{from } \sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!} = \sum_{m=0}^{\infty} f(m) = 1\right\}$$

$$= \lambda$$

(Ex4) **Uniform distribution** $U(a,b)$

$$E(\boldsymbol{X}) = \int_a^b x\frac{1}{b-a}dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

(Ex5) **Exponential distribution** $E_x(\alpha)$

$$E(\boldsymbol{X}) = \int_0^\infty x\alpha e^{-\alpha x}dx$$

$$= \left[x\alpha\frac{1}{-\alpha}e^{-\alpha x}\right]_0^\infty - \int_0^\infty \alpha\frac{1}{-\alpha}e^{-\alpha x}dx$$

$$= \int_0^\infty e^{-\alpha x}dx = \left[\frac{1}{-\alpha}e^{-\alpha x}\right]_0^\infty$$

$$= \frac{1}{\alpha}$$

(Ex6) **Gamma distribution** $G(\alpha,\nu)$

$$E(\boldsymbol{X}) = \int_0^\infty x\,\frac{1}{\Gamma(\nu)}\alpha^\nu x^{\nu-1}e^{-\alpha x}dx = \int_0^\infty \frac{1}{\Gamma(\nu)}t^\nu e^{-t}\frac{dt}{\alpha}$$

$$= \frac{1}{\alpha\Gamma(\nu)}\int_0^\infty t^\nu e^{-t}dt = \frac{\Gamma(\nu+1)}{\alpha\Gamma(\nu)} = \frac{\nu}{\alpha}$$

(Ex7) **Normal distribution** $N(\mu,\sigma^2)$

$$E(\boldsymbol{X}) = \int_{-\infty}^\infty x\,\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]dx$$

$$= \int_{-\infty}^\infty (\mu+\sigma t)\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}t^2\right]dt$$

$$= \mu\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}t^2\right]dt$$

$$+ \int_{-\infty}^\infty \frac{\sigma t}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}t^2\right]dt = \mu$$

# Properties of the mean value

(M1) For any real-value function $g(x)$, $g(\boldsymbol{X})$ is also a random variable. Mean value for $g(\boldsymbol{X})$ is given as follows.

- Case of discrete-type random variable

$$E\left(g(\boldsymbol{X})\right) = \sum_k g(v_k) f(v_k)$$

- Case of continuous-type random variable

$$E\left(g(\boldsymbol{X})\right) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

(M2) Conditional expectation

- Case of discrete-type random variable

$$f(v|y) = P(\boldsymbol{X} = v|\boldsymbol{Y} = y) \frac{P(\boldsymbol{X}=v,\boldsymbol{Y}=y)}{P(\boldsymbol{Y}=y)}$$

yields

$$E(\boldsymbol{X}|\boldsymbol{Y} = y) = \sum_k v_k f(v_k|y)$$

$$E(\boldsymbol{X}) = \sum_y E(\boldsymbol{X}|\boldsymbol{Y} = y) P(\boldsymbol{Y} = y)$$

- Case of continuous-type random variable

$$f(v|y) = \frac{f(x,y)}{f(y)}$$

yields
$$E(\boldsymbol{X}|\boldsymbol{Y} = y) = \int_{-\infty}^{\infty} x f(x|y)\, dx$$

$$E(\boldsymbol{X}) = \int_{-\infty}^{\infty} E(\boldsymbol{X}|\boldsymbol{Y} = y) f(y)\, dy$$

(M3) To unify the framework of computing the expected value for discrete-type and continuous-type random variables,

$$E(\boldsymbol{X}) = \int_{-\infty}^{\infty} x dF(x)$$

(**Reimann-Stieltjes integral**)

(a) $E(a\boldsymbol{X}+b)=aE(\boldsymbol{X})+b$

(b) When joint distribution function of $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ is given by $F(x_1, x_2, \cdots, x_n)$,

$$E\big(g(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n)\big) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2, \cdots, x_n) dF(x_1, x_2, \cdots, x_n)$$

(c) $E(\boldsymbol{X}_1+\boldsymbol{X}_2+\cdots+\boldsymbol{X}_n)=E(\boldsymbol{X}_1)+E(\boldsymbol{X}_2)+\cdots+E(\boldsymbol{X}_n)$

(d) If $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ are mutually independent,

$$E(\boldsymbol{X}_1\boldsymbol{X}_2\cdots\boldsymbol{X}_n)=E(\boldsymbol{X}_1)E(\boldsymbol{X}_2)\cdots E(\boldsymbol{X}_n)$$

This can be extended to

$$E(g_1(\boldsymbol{X}_1)g_2(\boldsymbol{X}_2)\cdots g_n(\boldsymbol{X}_n)) = E(g_1(\boldsymbol{X}_1))E(g_2(\boldsymbol{X}_2))\cdots E(g_n(\boldsymbol{X}_n))$$

# 9. Variance

## Definition

Suppose mean of random variable $\boldsymbol{X}$ is given by $E(\boldsymbol{X}) = \mu$, $(\boldsymbol{X} - \mu)^2$ gives also a random variable. Mean of $(\boldsymbol{X} - \mu)^2$ is called ***variance*** of $\boldsymbol{X}$, which is denoted as $Var(\boldsymbol{X})$.

$$Var(\boldsymbol{X}) = E\{(\boldsymbol{X} - \mu)^2\} = \int_{-\infty}^{\infty} (x - \mu)^2 dF(x)$$
$$= \sum_k (v_k - \mu)^2 f(v_k)$$
$$= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

## Remark

$$\sum_k |v_k|^2 f(v_k) < \infty, \int_{-\infty}^{\infty} |x|^2 f(x) dx < \infty$$

## Standard deviation

$$\sqrt{Var(\boldsymbol{X})}$$

# Properties of variance

(V1) $Var(\boldsymbol{X}) \geq 0$

(V2) $Var(\boldsymbol{X}) = E(\boldsymbol{X}^2) - \{E(\boldsymbol{X})\}^2$

[Proof] $Var(\boldsymbol{X}) = E(\boldsymbol{X}^2 - 2\mu\boldsymbol{X} + \mu^2)$

$$= E(\boldsymbol{X}^2) - 2E(\mu\boldsymbol{X}) + E(\mu^2)$$

$$= E(\boldsymbol{X}^2) - 2\mu E(\boldsymbol{X}) + \mu^2$$

$$= E(\boldsymbol{X}^2) - 2\mu\mu + \mu^2 = E(\boldsymbol{X}^2) - \mu^2$$

(V3) $Var(a\boldsymbol{X} + b) = a^2 Var(\boldsymbol{X})$

[Proof] $Var(a\boldsymbol{X} + b) = E\left\{\left((a\boldsymbol{X} + b) - (a\mu + b)\right)^2\right\}$

$$= E\{a^2(\boldsymbol{X} - \mu)^2\}$$

$$= a^2 E\{(\boldsymbol{X} - \mu)^2\}$$

$$= a^2 Var(\boldsymbol{X})$$

# Properties of variance

(V4) $Var(\boldsymbol{X}_1 + \boldsymbol{X}_2) = Var(\boldsymbol{X}_1) + Var(\boldsymbol{X}_2) + 2Cov(\boldsymbol{X}_1, \boldsymbol{X}_2)$

Covariance: $Cov(\boldsymbol{X}_1, \boldsymbol{X}_2) = E\{(\boldsymbol{X}_1 - \mu_1)(\boldsymbol{X}_2 - \mu_2)\}$

Correlation coefficient: $r(\boldsymbol{X}_1, \boldsymbol{X}_2) = \frac{Cov(\boldsymbol{X}_1, \boldsymbol{X}_2)}{\sqrt{Var(\boldsymbol{X}_1)Var(\boldsymbol{X}_2)}}$

If $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are mutually independent, $Cor(\boldsymbol{X}_1, \boldsymbol{X}_2) = 0$.

$\mathrm{r}(\boldsymbol{X}_1, \boldsymbol{X}_2) = 0$: No correlation

(V5) $Var(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \cdots \boldsymbol{X}_n) =$

$$\sum_{i=1}^{n} Var(\boldsymbol{X}_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Cor(\boldsymbol{X}_i, \boldsymbol{X}_j)$$

In particular, if $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ are mutually independent,

$$Var(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \cdots + \boldsymbol{X}_n) = \sum_{i=1}^{n} Var(\boldsymbol{X}_i)$$

# 10. Moment

## Definition

- *m*-th central moment (moment about mean)

$$\mu_m = E\{(\boldsymbol{X} - \mu)^m\}$$

- *m*-th moment about zero

$$\mu'_m = E\{\boldsymbol{X}^m\}$$

## Remark

- If there exists an *m*-th moment, there exists all the moments lower than *m*-th order.
- If there exists no *m*-th moment, there exists no moment higher than *m*-th order.

# Table of distribution functions

| Name of the distribution and range of the parameters | Probability density function | Mean and Variance | Characteristic function |
|---|---|---|---|
| Bernoulli distribution $B(1;p)$ <br> $(0 < p < 1, q = 1-p)$ | $p^k q^{1-k}$ <br> $k = 0, 1$ | $p, pq$ | $pe^{jt} + q$ |
| Binominal distribution $B(n;p)$ <br> ($n$: Integer, $0 < p < 1, q = 1-p$) | $\binom{n}{k} p^k q^{n-k}$ <br> $k = 0, 1, 2, \cdots, n$ | $np, npq$ | $(pe^{jt} + q)^n$ |
| Poisson distribution $Po(\lambda)$ <br> $(\lambda > 0)$ | $e^{-\lambda} \frac{\lambda^k}{k!}$ <br> $k = 1, 2, \cdots$ | $\lambda, \lambda$ | $\exp[\lambda(e^{jt} - 1]$ |
| Uniform distribution $U(a, b)$ <br> $(-\infty < a < b < \infty)$ | $\frac{1}{b-a}$ <br> $a \leq x \leq b$ | $(a+b)/2$ <br> $(b-a)^2/12$ | $\frac{e^{jbt} - e^{jat}}{j(b-a)t}$ |
| Exponential distribution $Ex(\alpha)$ <br> $(\alpha > 0)$ | $\alpha e^{-\alpha x}$ <br> $x \geq 0$ | $1/\alpha, 1/\alpha^2$ | $(1 - \frac{jt}{\alpha})^{-1}$ |
| Gamma distribution $G(\alpha, \nu)$ <br> $(\alpha, \nu > 0)$ | $\frac{1}{\Gamma(\nu)} \alpha^\nu x^{\nu-1} e^{-\alpha x}$ <br> $x \geq 0$ | $\nu/\alpha$ <br> $\nu/\alpha^2$ | $(1 - \frac{jt}{\alpha})^{-\nu}$ |
| Normal distribution $N(\mu, \sigma^2)$ <br> $(-\infty < \mu < \infty, \sigma > 0)$ | $\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ <br> $-\infty < x < \infty$ | $\mu$ <br> $\sigma^2$ | $\exp[j\mu t - \frac{\sigma^2}{2} t^2]$ |

# 11. Characteristic Function

- **Definition** (Characteristic Function)

$$\varphi(t) \equiv E\left(e^{it\boldsymbol{X}}\right) = \int_{-\infty}^{\infty} e^{itx} dF(\boldsymbol{X})$$

- **Remark**

$$\begin{aligned}|\varphi(t)| &= \left|\int_{-\infty}^{\infty} e^{itx} dF(\boldsymbol{X})\right| \\ &\leq \int_{-\infty}^{\infty} \left|e^{itx}\right| dF(\boldsymbol{X}) \\ &= \int_{-\infty}^{\infty} dF(\boldsymbol{X}) = 1\end{aligned}$$

Hence, characteristic function exists for arbitrary distribution.

# Properties of characteristic function

(CF1) Moments can be obtained from derivatives of the characteristic function.

$$\varphi^{(m)}(t) = E\left\{(i\boldsymbol{X})^m e^{it\boldsymbol{X}}\right\}$$

$$\varphi^{(m)}(0) = E\{(i\boldsymbol{X})^m\}$$

Hence $$\mu'_m = E(\boldsymbol{X}^m) = \frac{\varphi^{(m)}(0)}{i^m}$$

$$E(\boldsymbol{X}) = \frac{\varphi'(0)}{i}$$

$$Var(\boldsymbol{X}) = \frac{\varphi''(0)}{i^2} - \left(\frac{\varphi'(0)}{i}\right)^2$$

(CF2) If there exist central moment lower than *m*-th order $\mu'_k\ (1 \le k \le m)$, characteristic function $\varphi(t)$ can be expanded around $t = 0$ into Taylor series:

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \varphi^{(k)}(0) t^k = 1 + \sum_{k=1}^{m} \frac{(it)^k}{k!} \mu'_k + O(t^{m+1})$$

(CF3) Suppose random variables $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ are mutually independent and characteristic function of $\boldsymbol{X}_i$ is given by $\varphi_i(t)$. Then, characteristic function of the following variable

$$S = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \cdots + \boldsymbol{X}_n$$

is given by

$$\varphi_S(t) = \varphi_1(t)\ \varphi_2(t) \cdots \varphi_n(t)$$

(CF4) Consider two random variables $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. Suppose their distribution functions are given respectively by $F_1(x)$ and $F_2(x)$ and their characteristic functions are given by $\varphi_1(t)$ and $\varphi_2(t)$.

The necessary and sufficient condition of $F_1(x) = F_2(x)$ is $\varphi_1(t) = \varphi_2(t)$.

Probability density function is obtained by inversion formula.

- Discrete-type distribution function:

$$f(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\varphi(t)dt$$

- Continuous-type distribution function:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(t)dt$$

(CF5) (Continuity theorem)

Suppose $F_1(x), F_2(x), \cdots$ represent a sequence of distributions and $\varphi_1(t), \varphi_2(t), \cdots$ represent a sequence of corresponding characteristic functions. If the series $\varphi_1(t), \varphi_2(t), \cdots$ converges for any constant $t$

$$\lim_{n\to\infty} \varphi_n(t) = \varphi(t),$$

and the convergence value $\varphi(t)$ is continuous at $t = 0$, $\varphi(t)$ gives the characteristic function.

For a distribution function $F(x)$ that corresponds to $\varphi(t)$.

$$\lim_{n\to\infty} F_n(x) = F(x).$$

(CF6) Characteristic function for joint distribution of random variables $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ is defined by

$$\varphi(t_1, t_2, \cdots, t_n) = E\{\exp[i(t_1\boldsymbol{X}_1 + t_2\boldsymbol{X}_2 + \cdots t_n\boldsymbol{X}_n)]\}$$

Necessary and sufficient condition for mutual independency of $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ is given by

$$\varphi(t_1, t_2, \cdots, t_n) = \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n).$$

Where:

$$\varphi_i(t_i) = \varphi_1(0, \cdots, 0, t_i, 0, \cdots, 0)\ .$$

# 12. Application of characteristic function

## Law of Large Numbers

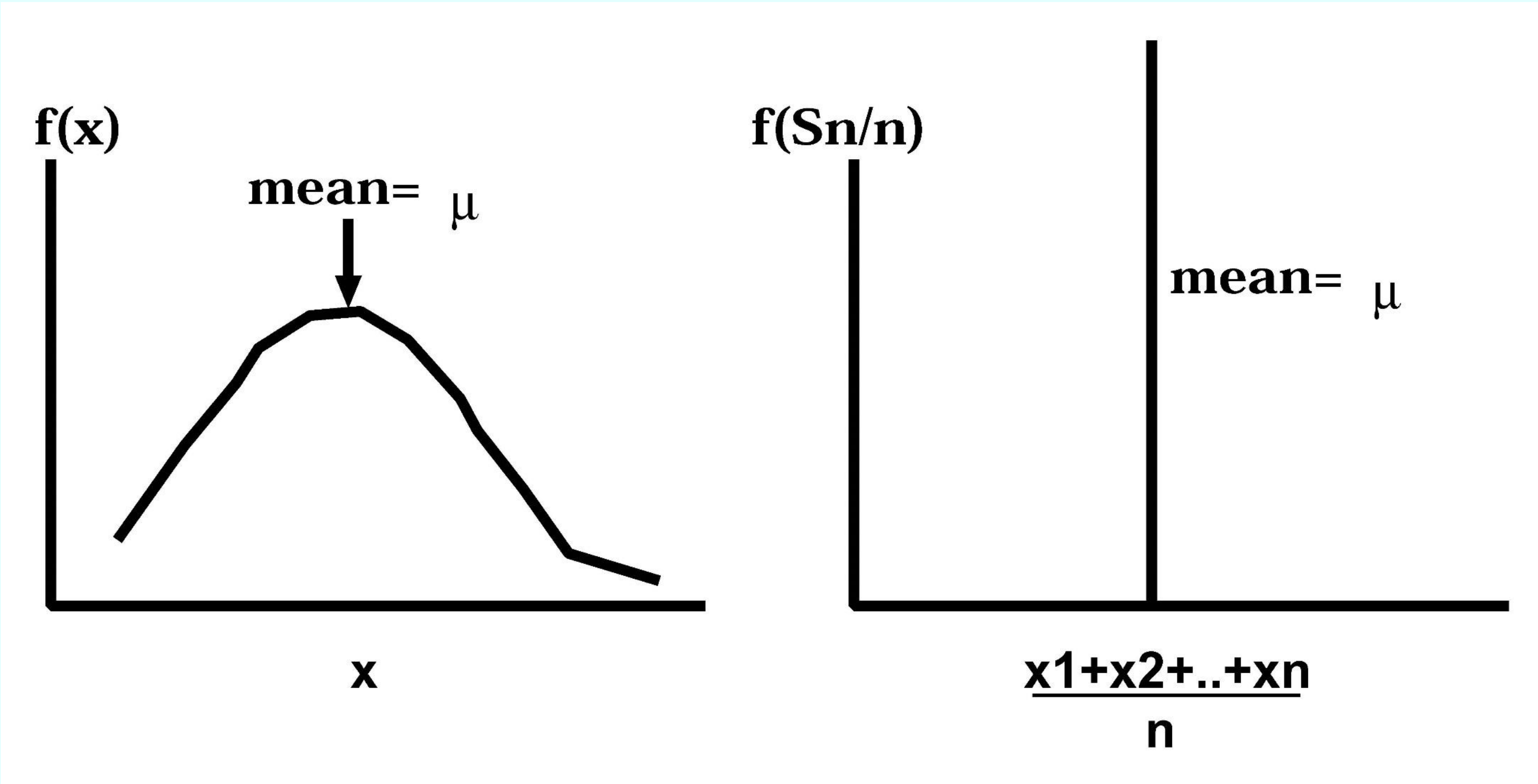
Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ be mutually independent random variables which obey the same distribution. Suppose that the distribution has a finite expectation value $\mu = E(\boldsymbol{X}_j)$, where its characteristic function is given by $\varphi(t)$.

Let

$$S_n = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \cdots + \boldsymbol{X}_n$$

Then for any $\epsilon > 0$,

$$\lim_{n \to \infty} P\left( \left| \frac{S_n}{n} - \mu \right| > \epsilon \right) = 0$$


f(x)
mean= μ
x
f(Sn/n)
mean= μ
x1+x2+..+xn
n

- [Proof] Let characteristic functions of $\boldsymbol{X}_i$ and $\frac{S_n}{n}$ be $\varphi(t)$ and $\varphi_n(t)$, respectively. $\varphi_n(t) = \left[\varphi\left(\frac{t}{n}\right)\right]^n$

From (CF2), Taylor's expansion of $\varphi(t)$ around $t = 0$ yields $\varphi(t) = 1 + i\mu t + O(t^2)$. On the other hand, Taylor's expansion of $e^{i\mu t}$ around $t = 0$ gives $e^{i\mu t} = 1 + i\mu t + O(t^2)$.

Hence, $\varphi(t) = e^{i\mu t} + O(t^2)$. This leads to

$$\varphi_n(t) = \left[\varphi\left(\frac{t}{n}\right)\right]^n = \left[e^{\frac{i\mu t}{n}} + O(t^2)\right]^n \to e^{i\mu t} \ (n \to \infty)$$

$e^{j\mu t}$ represents characteristic function of a random variable that takes a value of $\mu$ with a probability of 1. From the continuity theorem, $\varphi_n(t) \to e^{i\mu t}, \quad \frac{S_n}{n} \to \mu \ \ (n \to \infty).$

## Central Limit Theorem

$\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ be a sequence of *n* independent and identically distributed random variables, each having finite values of expectation $\mu$ and variance $\sigma^2$ $(> 0)$.

Let $S_n^*$ be

$$S_n^* = \sum_{j=1}^{n} \frac{X_j - \mu}{\sqrt{n}\sigma}.$$

Then, as *n* approaches infinity $(n \to \infty)$, $S_n^*$ will converge in distribution to the standard normal distribution $N(0,1)$.

## Example of central limit theorem: Summation of uniform random variables