[I225] Statistical Signal Processing(E) Office Hour 5

1. The stochastic process $X(t)$ is white noise satisfying $E\{X(T)\} = 0$ and $R_{XX}(\tau) = 12\delta_{(\tau)}$. ow consider a system in which the input $X(t)$ and the output $Y(t)$ satisfy the following relationship: $Y''(t) - Y'(t) - 2Y(t) = X(t)$. In this case, find the autocorrelation function $R_{YY}(\tau)$ and the power spectral density $S_{YY}(\omega)$ of the output $Y(t)$.

Answer:

Given the equation:

$$Y''(t) - Y'(t) - 2Y(t) = X(t)$$

Let $t = t_2$, and multiply both sides of equation by $X(t_1)$.

$$X(t_1)Y''(t_2) - X(t_1)Y'(t_2) - 2X(t_1)Y(t_2) = X(t_1)X(t_2)$$

Taking expectations:

$$E[X(t_1)Y''(t_2)] - E[X(t_1)Y'(t_2)] - 2E[X(t_1)Y(t_2)] = E[X(t_1)X(t_2)]$$

This yields:

$$R''_{XY}(\tau) - R'_{XY}(\tau) - 2R_{XY}(\tau) = R_{XX}(\tau), \quad R_{XX}(\tau) = 12\delta_{(\tau)}$$

Applying the Fourier transform:

$$(j\omega)^2 S_{XY}(\omega) - j\omega S_{XY}(\omega) - 2S_{XY}(\omega) = 12$$

$$\Rightarrow \left(-\omega^2 - j\omega - 2\right)S_{XY}(\omega) = 12$$

Thus:

$$S_{XY}(\omega) = \frac{12}{-\omega^2 - j\omega - 2}$$

Let $t = t_1$, and multiply both sides of equation by $Y(t_2)$.

$$Y''(t_1)Y(t_2) - Y'(t_1)X(t_2) - 2Y(t_1)Y(t_2) = X(t_1)Y(t_2)$$

$$\Rightarrow R''_{YY}(\tau) - R'_{YY}(\tau) - 2R_{YY}(\tau) = R_{XY}(\tau)$$

$$\Rightarrow (-j\omega)^2 S_{YY}(\omega) - (-j\omega)S_{YY}(\omega) - 2S_{YY}(\omega) = S_{XY}(\omega)$$

$$\Rightarrow \left(-\omega^2 + j\omega - 2\right)S_{YY}(\omega) = S_{XY}(\omega)$$

Substitute $S_{XY}(\omega)$:

$$S_{YY}(\omega) = \frac{S_{XY}(\omega)}{-\omega^2 + j\omega - 2} = \frac{12}{\left(-\omega^2 + j\omega - 2\right)\left(-\omega^2 - j\omega - 2\right)}$$

$$= \frac{12}{(\omega^2 + j\omega + 2)(\omega^2 - j\omega + 2)}$$

$$= \frac{12}{(\omega^2 + 2)^2 + \omega^2}$$

$$= \frac{12}{\omega^4 + 5\omega^2 + 4}$$

$$= \frac{4}{1 + \omega^2} - \frac{4}{4 + \omega^2}$$

So, $R_{YY}(\tau) = 2e^{-|\tau|} - e^{-2|\tau|}$

2. Given the power spectral density

$$S_X(\omega) = \frac{\omega^4 + 64}{\omega^4 + 10\omega^2 + 9}$$

determine the innovation filter for the process $x(t)$.

Answer:

If a real-valued process x(t) has a valid spectrum $S(\omega)$, then since $S(-\omega) = S(\omega)$, $S(\omega)$ can be expressed as a ratio of two polynomials in ω^2:

$$S(\omega) = \frac{A(\omega^2)}{B(\omega^2)}$$

The corresponding function $S(s)$ is:

$$S(s) = \frac{A(-s^2)}{B(-s^2)}$$

The innovation filter $L(s)$ can be constructed by using the fact that its poles must have negative real parts (i.e., singularities with $\mathrm{Re}s < 0$).

Given:

$$S_x(\omega) = \frac{\omega^4 + 64}{\omega^4 + 10\omega^2 + 9}$$

Thus,

$$S_x(s) = \frac{s^4 + 64}{s^4 - 10s^2 + 9}$$

Factoring numerator and denominator:

$$S_x(s) = \frac{(s^2 + 4s + 8)(s^2 - 4s + 8]}{(1 - s^2)(9 - s^2)}$$

Therefore, the corresponding innovation filter $L_x(s)$ is:

$$L_x(s) = \frac{s^2 + 4s + 8}{(1 + s)(3 + s)}$$

Note: A simple method to find complex poles:

If $S(s)$ contains a factor of the form:

$$s^4 + bs^2 + c, \text{with } b^4 < 4c,$$

then the corresponding factor in $L(s)$ is:

$$s^2 + \beta s + \gamma, \text{where } \gamma = \sqrt{c}, \text{and } \beta = \sqrt{(2\gamma - b)}$$

Example: If

$$S(s) = \frac{3}{(s^4 + 64)}, \text{then } b = 0, c = 64$$

We get:

$$\gamma = 8, \beta = 4, \text{so } L(s) = \frac{\sqrt{3}}{s^2 + 4s + 8}$$

3. Given the observation vector and design matrix:

$$s = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Use the least squares method to find the optimal parameter vector $\widehat{\boldsymbol{\theta}} \in \mathbb{R}^{\mathbf{2}}$ that minimizes the squared error:

$$J(\theta) = \| s - H\theta \|^2$$

Answer:

We use the normal equation:

$$\hat{\theta} = (H^T H)^{-1} H^T s$$

compute $H^T H$

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$H^T H = \begin{bmatrix} 1+1+1 & 1+2+3 \\ 1+2+3 & 1^2+2^2+3^2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

compute $H^T s$

$$H^T s = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 23 \end{bmatrix}$$

compute $(H^T H)^{-1}$

$$det\left(H^T H\right) = 3 \cdot 14 - 6 \cdot 6 = 42 - 36 = 6$$

$$(H^T H)^{-1} = \frac{1}{6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$$

Solve for $\hat{\theta}$

$$\hat{\theta} = \frac{1}{6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 23 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1.5 \end{bmatrix}$$

4. Let the observation matrix H: and the observed vector s be:

$$H = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, s = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(1) Compute the least squares solution $\hat{\theta}$ for minimizing $\| s - H\theta \|^2$.

(2) Compute the residual vector $\epsilon = s - H\hat{\theta}$.

(3) Verify that ϵ is orthogonal to both column vectors of H i.e., show $h_1^T \epsilon = 0$ and $h_2^T \epsilon = 0$

Answer:

(1) Compute $\hat{\theta} = (H^T H)^{-1} H^T s$

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$H^T H = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$H^T s = \begin{bmatrix} 1 + 2 + 2 \\ 0 * 1 + 1 * 2 + 2 * 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\left(H^T H\right)^{-1} = \frac{1}{3 * 5 - 3 * 3}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\hat{\theta} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{7}{6} \\ \frac{1}{2} \end{bmatrix}$$

(2) Compute residual vector

$$\hat{s} = H\hat{\theta} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{7}{6} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{6} \\ \frac{10}{6} \\ \frac{13}{6} \end{bmatrix}$$

$$\epsilon = s - \hat{s} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{7}{6} \\ \frac{10}{6} \\ \frac{13}{6} \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/3 \\ -1/6 \end{bmatrix}$$

(3) Verify orthogonality

Column vectors of H

$$h_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, h_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Inner product with ϵ

$$\mathrm{h}_1^T \epsilon = -\frac{1}{6} + \frac{1}{3} - \frac{1}{6} = 0$$

$$\mathrm{h}_2^T \epsilon = 0 * \left(-\frac{1}{6}\right) + 1 * \frac{1}{3} + 2 * \left(-\frac{1}{6}\right) = 0$$

Confirmed: residual is orthogonal to both column vectors of H

5. Suppose the observed signal is given by:

$$X[n] = \theta + W[n], n = 0,1,\ldots,N-1$$

where:

θ is a random constant following a uniform distribution:

$$\theta \sim U(-1,1)$$

$W[n]$ are independent Gaussian noises with mean 0 and variance $\sigma^2 = 0.25$

θ and $W[n]$ are independent.

Let $N = 4$, and assume the observed values are:

$$X = [1.2, 0.8, 1.0, 0.6]^T$$

Compute the minimum mean square error (MMSE) estimate of θ.

Answer:

Prior variance of θ

Since

$$\theta \sim U(-1,1)$$

the variance is:

$$\sigma_\theta^2 = \frac{\left(1-(-1)\right)^2}{12} = \frac{1}{3}$$

Compute sample mean of X

$$\overline{X} = \frac{1.2+0.8+1.0+0.6}{4} = \frac{3.6}{4} = 0.9$$

MMSE estimation formula

$$\hat{\theta} = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma^2}{N}} \cdot \overline{X} = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{16}} \cdot 0.9 = \frac{14.4}{19} \approx 0.7579$$