Student ID : ________________

Name : ______________________

1. Observed data $\{X[0], X[1], \cdots, X[N-1]\}$ are mutually independent and normally distributed as $\sim N(0, \sigma^2)$. We estimate the unknown variance σ^2 as

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{n=0}^{N-1} X^2[n].$$

Judge whether this estimator is unbiased or not. Furthermore, compute the variance of $\hat{\sigma}^2$ to study its property in the limit of $N \to \infty$.
[Hint: $\mathrm{E}\left[X^4[\mathrm{n}]\right] = 3\sigma^4$.]

2. Observed data $\{X[0], X[1], \cdots, X[N-1]\}$ are mutually independent and uniformly distributed as $\sim U(0, \theta)$, where $0 < \theta < \infty$. Construct an unbiased estimator for the unknown parameter θ and show that it is unbiased.

[Hint: $\mathrm{E}\left[\mathrm{X}[\mathrm{n}]\right] = \frac{\theta}{2}$]

3. Consider the following observation

$$X[n] = \mathrm{A}r^n + W[n] \qquad (n = 0, 1, \cdots, N-1),$$

where $\{W[n]\}$ are mutually independent and normally distributed as $N(0, \sigma^2)$. r is a known constant. To estimate the unknown parameter A, compute the Cramér-Rao lower bound (CRLB) and derive the efficient estimator for A.

4. Observed data $\{X[0], X[1], \cdots, X[N-1]\}$ are mutually independent and exponentially distributed as $E_x(\lambda)$. Obtain the maximum likelihood estimator of the unknown parameter λ. [Note that the exponential distribution has the probability density function of $p(x;\lambda) = \lambda \exp(-\lambda x) \ \ (x \geq 0); 0 \ \ (x < 0)$.]

5. X is a random variable uniformly distributed as $X \sim U\left[-\frac{1}{2}, \frac{1}{2}\right]$. Let us estimate a variable $\theta = \cos 2\pi X$ by using the following function

$$\hat{\theta} = aX^2 + b.$$

(1) Compute the expectations for $\mathrm{E}[X^2], \mathrm{E}[X^4], E[\theta], E[\theta^2], E[\theta X^2]$.
(2) Derive the parameter values of *a*, *b* based on the linear minimum mean square error (LMMSE) estimation.
(3) Compute the minimum mean square error *Bmse* corresponding to the parameter values *a*, *b* obtained in (2).

6. We predict λ-period future state $S(t+\lambda)$ of a stationary process by using its current state

$S(t)$ and T-period previous state $S(t-T)$ as

$$\hat{S}(t+\lambda) = aS(t) + bS(t-T).$$

Derive the optimal parameter values for a and b. Note that $R_{SS}(\tau) = \exp\left(-\frac{|\tau|}{T}\right)$, $0 < \lambda < \mathrm{T}$.

Note:

- Please provide your answer in **clear handwriting**
- Dead-line for submission: **3 June 2025**