I225E Statistical Signal Processing

6. Spectral Analysis I

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Periodic Processes

- Consider a periodic process of the form

$$x(t) = A\,cos(2\pi\omega t + \varphi)$$

where:

A is called the amplitude;

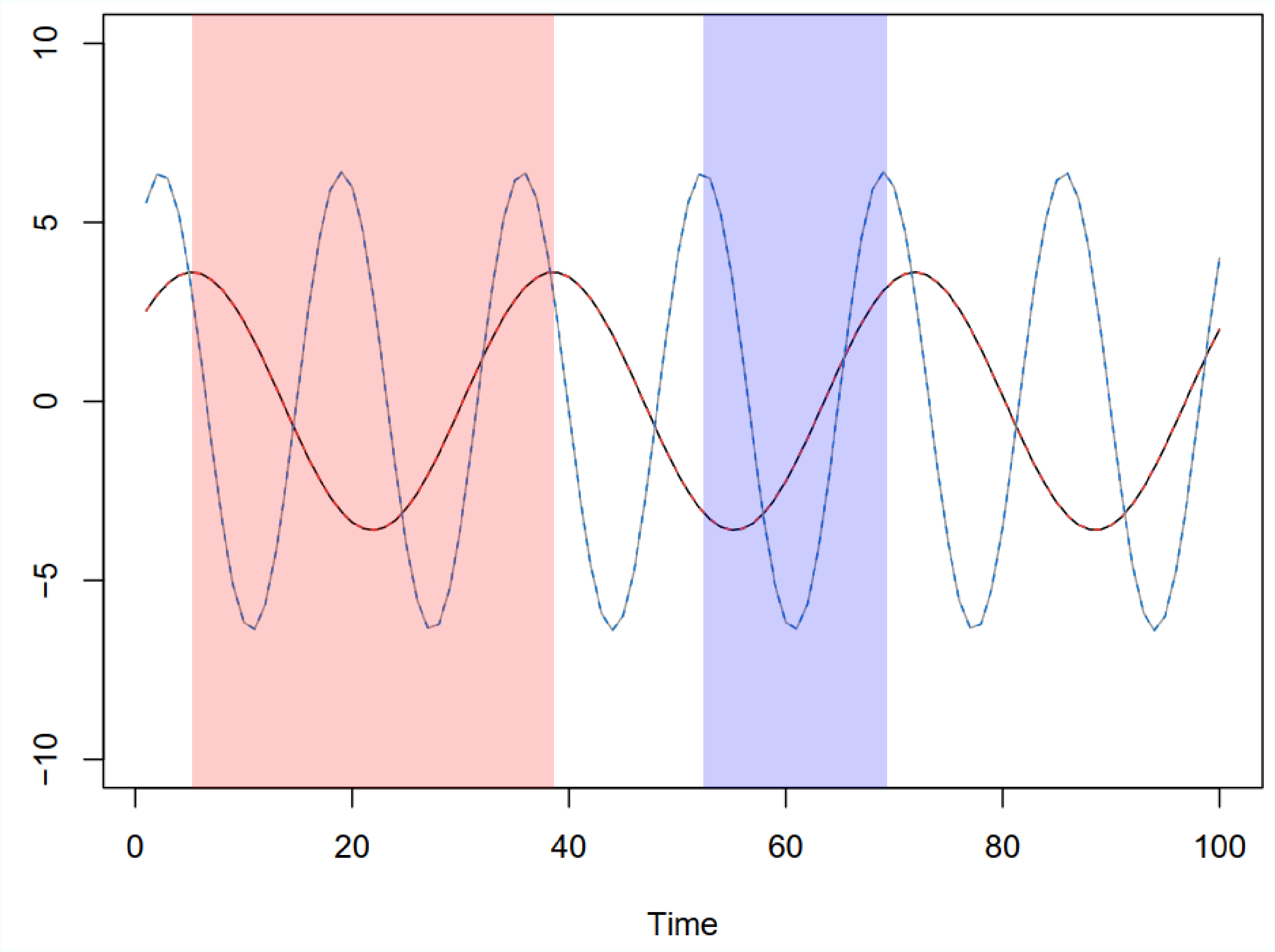
φ the phase of the process;

ω is called frequency of the process; and

$T = 1/\omega$ is called the period or cycle.

- As t varies from 0 to $1/\omega$, note that the process goes through one complete cycle.

Periodic Processes



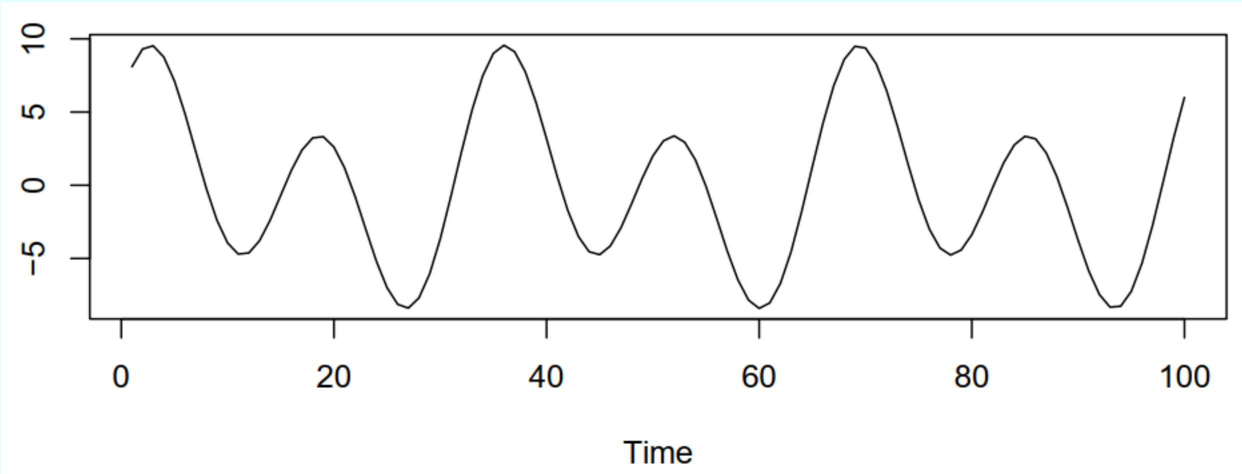
Example: General Mixtures

- We can mix together a total of p periodic processes, which can be expressed as follows:

$$x(t) = \sum_{i=1}^{p} \left(U_{j_1} \cos\left(2\pi\omega_j t\right) + U_{j_2} \sin\left(2\pi\omega_j t\right)\right)$$

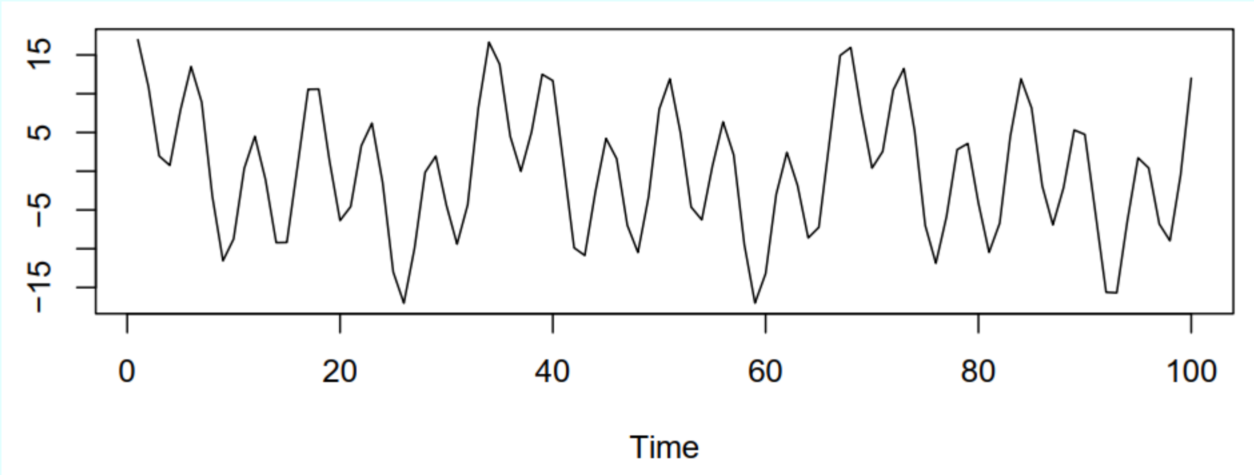
for $U_{j_1}, U_{j_2}, j = 1, \ldots, p$ all uncorrelated random variables with mean zero, where U_{j_1}, U_{j_2} have variance σ_j^2.

Example: General Mixtures



$p = 2$

$p = 3$



Fourier Decomposition

- A Fourier series is a way to represent a periodic function as a sum of simple sine and cosine waves (or complex exponentials) with different frequencies and amplitudes.
- For a signal $x(t)$ with a period T, its Fourier series representation is given by:

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency, and the coefficients a_0, a_n, b_n are calculated using integrals of the function over one period.

Fourier Decomposition

- Alternatively, using complex exponentials, the Fourier series can be written as:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where the complex Fourier coefficients c_n are given by:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt$$

- The Fourier series provides a discrete frequency spectrum of the periodic function, consisting of the amplitudes and phases of the fundamental frequency and its integer multiples (harmonics).

Fourier Series to Fourier Transform

- The Fourier transform extends the concept of frequency analysis to non-periodic functions defined over an infinite interval $(-\infty, \infty)$. Consider as the period T of a function approaches infinity.
  - The fundamental frequency becomes infinitesimally small (ω_0 approaches zero)
  - The discrete spectrum becomes continuous ($n\omega_0$ effectively merge into a continuous frequency variable ω)
  - The sum becomes an integral.
- Let's define a function $X(\omega)$ as a scaled version of complex Fourier coefficients c_n.

$$X(n\omega_0) = Tc_n = \int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}dt$$

Fourier Series to Fourier Transform

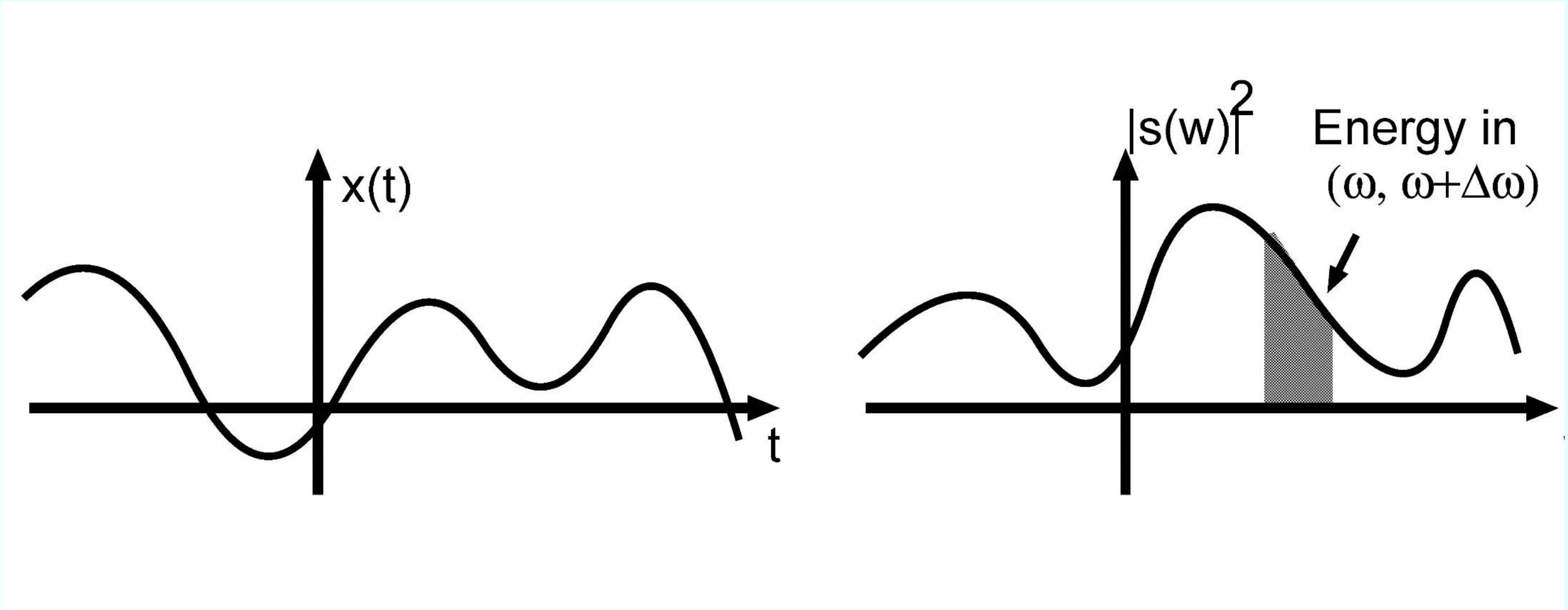
- As $T \to \infty, \omega_0 \to d\omega$ and $n\omega_0 \to \omega$. The summation in the synthesis equation of the Fourier series becomes an integral:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$
$$= \sum_{n=-\infty}^{\infty} \frac{1}{T} X(n\omega_0) e^{jn\omega_0 t}$$
$$\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

- The analysis equation for the Fourier series coefficients becomes the Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Spectral Representation



For deterministic signal $x(t)$, its spectrum is computed by Fourier transform

$$s(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

Fourier transforms

- Fourier and inverse Fourier transformations:

$$x(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega t} s(\omega),$$

$$s(\omega) = \int_{-\infty}^{+\infty} dt e^{-i\omega t} x(t).$$

- A Fourier and a subsequent inverse Fourier transform form an identity mapping:

$$x(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega t} s(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega t} \int_{-\infty}^{+\infty} dt' e^{-i\omega t'} x(t')$$

$$= \int_{-\infty}^{+\infty} dt' \delta(t - t') x(t') = x(t).$$

Parseval's theorem

- The power in the temporal domain is equal to the power in the frequency domain:

$$\int_{-\infty}^{+\infty} dt|x(t)|^2 = \int_{-\infty}^{+\infty} dt x(t)x^*(t)$$
$$= \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} s(\omega) \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} e^{-i\omega' t} s^*(\omega')$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\omega d\omega'}{2\pi} \delta(\omega - \omega') s(\omega) s^*(\omega')$$
$$= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} |s(\omega)|^2$$

- From Parseval's theorem,

$$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |s(\omega)|^2 d\omega = E.$$

This implies that power spectrum $|s(\omega)|^2 \Delta\omega$ represents energy concentrated within frequency range of $[\omega, \omega + \Delta\omega]$.

How to define spectrum for stochastic process?

Problem

Fourier transform of stochastic signal $\boldsymbol{X}(t)$ results in different spectra for every trial.

→Definition necessary for sample average.

Derivation of power spectrum

For finite interval $[-T, T]$, Fourier transform of one realization of stochastic process $\boldsymbol{X}(t)$ is given by

$$\boldsymbol{X}_T(\omega) = \int_{-T}^{T} \boldsymbol{X}(t) e^{-i\omega t} dt$$

whereas its power is given by $\frac{|\boldsymbol{X}_T(\omega)|^2}{2T} = \frac{1}{2T}\left|\int_{-T}^{T} \boldsymbol{X}(t) e^{-i\omega t} dt\right|^2$.

Taking the expectation $E\{\cdot\}$, mean power spectra $\bar{\boldsymbol{X}}_T(\omega)$ can be calculated as

$$\begin{aligned}\bar{\boldsymbol{X}}_T(\omega) &= E\left\{\frac{|X_T(\omega)|^2}{2T}\right\} \\ &= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E\{\boldsymbol{X}(t_1)\boldsymbol{X}^*(t_2)\} e^{-i\omega(t_1-t_2)} dt_1 dt_2 \\ &= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{\boldsymbol{XX}}(t_1, t_2) e^{-i\omega(t_1-t_2)} dt_1 dt_2.\end{aligned}$$

Supposing $\boldsymbol{X}(t)$ is a wide-sense stationary process, so that $R_{\boldsymbol{XX}}(t_1, t_2)$=$R_{\boldsymbol{XX}}(t_1 - t_2)$,

$$\overline{\boldsymbol{X}}_T(\omega) = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{\boldsymbol{XX}}(t_1 - t_2)e^{-i\omega(t_1 - t_2)}dt_1 dt_2$$

Denoting $\tau = t_1 - t_2$,

$$\overline{\boldsymbol{X}}_T(\omega) = \frac{1}{2T}\int_{-2T}^{2T} R_{\boldsymbol{XX}}(\tau)e^{-i\omega\tau}(2T - |\tau|)\, d\tau$$

$$= \int_{-2T}^{2T} R_{\boldsymbol{XX}}(\tau)e^{-i\omega\tau}\left(1 - \frac{|\tau|}{2T}\right) d\tau \geq 0$$

Finally, taking the limit of $T \to \infty$.

$$\boxed{S_{\boldsymbol{XX}}(\omega) = \int_{-\infty}^{\infty} R_{\boldsymbol{XX}}(\tau)e^{-i\omega\tau}d\tau \geq 0}$$

[Wiener-Khinchin Theorem]

- Autocorrelation $R_{\boldsymbol{XX}}(\tau)$ and spectral density $S_{\boldsymbol{XX}}(\omega)$ are related with each other via Fourier transform.

The inverse Fourier transform gives

$$R_{\boldsymbol{XX}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\boldsymbol{XX}}(\omega)e^{i\omega\tau}d\omega \geq 0$$

In particular, the case of $\tau = 0$ gives the signal power

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\boldsymbol{XX}}(\omega)\, d\omega = R_{\boldsymbol{XX}}(0) \\ = E\{|\boldsymbol{X}(t)|^2\} \\ = P\end{aligned}$$

Exercise

Consider a deterministic periodic signal $p(t)$ with a period of $T = 2$ seconds, defined over one period as:

$$p(t) = \begin{cases} 1, & -0.5 \leq t < 0.5 \\ 0, & 0.5 \leq t < 1.5 \end{cases}$$

A stochastic process $X(t) = p(t) + N(t)$, where $N(t)$ is a white Gaussian noise process with a mean of zero and a power spectral density $S_{NN}(\omega) = \sigma^2$.

Suppose you observe a single realization of this stochastic process over a finite time interval $-L \leq t \leq L$. Let this observation be $x_L(t) = p(t) + n(t)$, where $n(t)$ is a sample from the noise process $N(t)$.

a) Find the Fourier transform $P(j\omega)$ of one period of the deterministic signal $p(t)$.

b) Describe the expected Fourier transform $X_L(j\omega)$ of the observed signal $x_L(t)$.

Answer

a) $P_1(j\omega) = \frac{2\sin(0.5\omega)}{\omega}$

b) $X_L(j\omega) = \mathrm{F}\{p(t)\omega_L(t)\} + \mathrm{F}\{n(t)\omega_L(t)\}$, where $\omega_L(t)$ is a rectangular window function of width $2L$ centered at $t = 0$.

Exercise (Cont.)

- What is the autocorrelation function $R_{PP}(\tau)$ of the periodic signal $p(t)$? (Hint: You might find it easier to think about the time average of $p(t)p(t-\tau)$ over one period).
- What is the autocorrelation function $R_{NN}(\tau)$ of the white Gaussian noise $N(t)$ with power spectral density $S_{NN}(\omega) = \sigma^2$? (Recall the inverse Fourier transform relationship between PSD and autocorrelation).

Properties of power spectrum

(1) $S_{\boldsymbol{XX}}(\omega)$ is a real function of ω.

(Because $R_{XX}(-\tau) = R^*_{\boldsymbol{XX}}(\tau)$ and $S_{\boldsymbol{XX}}(\omega) = S^*_{\boldsymbol{XX}}(\omega)$)

(2) $S_{\boldsymbol{XX}}(\omega) \geq 0$.

(3) If $\boldsymbol{X}(t)$ is a real process,

$$R_{\boldsymbol{XX}}(\tau) = E\{\boldsymbol{X}(t+\tau)\boldsymbol{X}(t)\}$$

$$= E\{\boldsymbol{X}(s)\boldsymbol{X}(s-\tau)\} = R_{\boldsymbol{XX}}(-\tau).$$ Therefore

$$S_{\boldsymbol{XX}}(\omega) = \int_{-\infty}^{\infty} R_{\boldsymbol{XX}}(\tau)\, e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{\boldsymbol{XX}}(\tau) \cos\omega\tau\, d\tau + i \int_{-\infty}^{\infty} R_{\boldsymbol{XX}}(\tau) \sin\omega\tau\, d\tau$$

$$= 2\int_{0}^{\infty} R_{\boldsymbol{XX}}(\tau) \cos\omega\tau\, d\tau = S_{\boldsymbol{XX}}(-\omega)$$

Hence, $S_{\boldsymbol{XX}}(\omega)$ is an even function and can be represented in terms of cos-transform.

Cross-power spectrum

Cross-power spectrum $S_{\boldsymbol{XY}}(\omega)$ of two processes $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$ is

$$S_{\boldsymbol{XY}}(\omega) = \int_{-\infty}^{\infty} R_{\boldsymbol{XY}}(\tau)\, e^{-i\omega\tau} d\tau,$$

$$R_{\boldsymbol{XY}}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\boldsymbol{XY}}(\omega)\, e^{i\omega\tau} d\omega,$$

$$S_{\boldsymbol{YX}}(\omega) = S_{\boldsymbol{XY}}^{*}(\omega).$$

Example of power spectrum

Random variables a_i are mutually uncorrelated, and have mean 0 and variance σ_i^2. Compute power spectrum of stochastic process:

$$\boldsymbol{X}(t) = \sum_i a_i e^{i\omega_i t}$$

Auto-correlation of $\boldsymbol{X}(t)$ is computed as

$$\begin{aligned} R_{\boldsymbol{XX}}(\tau) &= E\{\boldsymbol{X}(t+\tau)\boldsymbol{X}^*(t)\} \\ &= E\left\{\sum_i a_i e^{i\omega_i(t+\tau)} \sum_k a_k^* e^{-i\omega_k t}\right\} \\ &= \sum_i \sum_k E\{a_i a_k^*\} e^{i(\omega_i-\omega_k)t + i\omega_i\tau} \\ &= \sum_i \sigma_i^2 e^{i\omega_i\tau} \end{aligned}$$

Here, we used $E\{a_i a_k^*\} = 0\ (i \neq k),\ \sigma_i^2\ (i = k)$, due to uncorrelation property of a_i. The power spectrum is

$$\begin{aligned} S_{\boldsymbol{XX}}(\omega) &= \int_{-\infty}^{\infty} R_{\boldsymbol{XX}}(\tau) e^{-i\omega\tau} d\tau \\ &= \sum_i \sigma_i^2 \int_{-\infty}^{\infty} e^{i\omega_i\tau} e^{-i\omega\tau} d\tau \\ &= 2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i) \end{aligned}$$

Here, we used $\int_{-\infty}^{\infty} e^{i\omega_i\tau} e^{-i\omega\tau} d\tau = 2\pi\delta(\omega - \omega_i)$.

Autocorrelations and the corresponding spectra

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}d\omega \iff S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$$

$$\delta(\tau) \iff 1$$

$$1 \iff 2\pi\delta(\omega)$$

$$e^{j\beta\tau} \iff 2\pi\delta(\omega-\beta)$$

$$\cos\beta\tau \iff \pi\delta(\omega-\beta)+\pi\delta(\omega+\beta)$$

$$e^{-\alpha|\tau|} \iff \frac{2\alpha}{\alpha^2+\omega^2}$$

$$e^{-\alpha\tau^2} \iff \sqrt{\frac{\pi}{\alpha}}e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|}\cos\beta\tau \quad\Longleftrightarrow\quad \frac{\alpha}{\alpha^2+(\omega-\beta)^2}+\frac{\alpha}{\alpha^2+(\omega+\beta)^2}$$

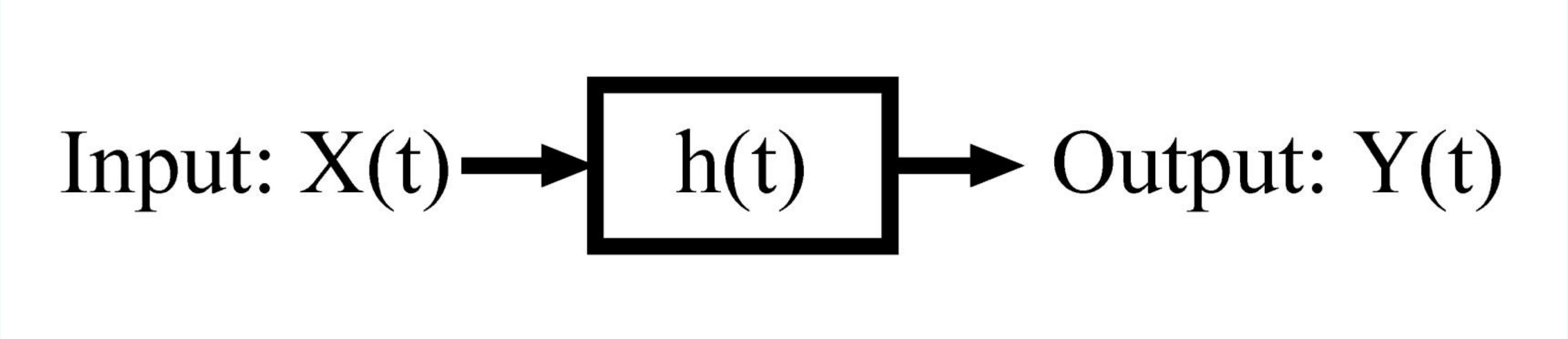
$$e^{-\alpha\tau^2}\cos\beta\tau \quad\Longleftrightarrow\quad \sqrt{\frac{\pi}{\alpha}}\left[e^{-\frac{(\omega-\beta)^2}{4\alpha}}+e^{-\frac{(\omega+\beta)^2}{4\alpha}}\right]$$

$$\begin{cases}1-\frac{|\tau|}{T} & |\tau|<T\\ 0 & |\tau|>T\end{cases} \quad\Longleftrightarrow\quad \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin\sigma\tau}{\pi\tau} \quad\Longleftrightarrow\quad \begin{cases}1 & |\omega|<\sigma\\ 0 & |\omega|>\sigma\end{cases}$$

Power-spectra and Linear System

- Wide sense stationary process $\boldsymbol{X}(t)$ has autocorrelation of $R_{\boldsymbol{XX}}(\tau)$ and its corresponding spectrum $S_{\boldsymbol{XX}}(\omega)$. Suppose $\boldsymbol{X}(t)$ is input to a linear system, whose impulse response is given by $h(t)$. How to compute the output spectrum $S_{\boldsymbol{YY}}(\omega)$?



- From properties of autocorrelation function

$$R_{\boldsymbol{XY}}(\tau) = R_{\boldsymbol{XX}}(\tau) * h^*(-\tau),$$
$$R_{\boldsymbol{YY}}(\tau) = R_{\boldsymbol{XX}}(\tau) * h^*(-\tau) * h(\tau),$$

In general, "If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$, then, $f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$." Therefore,

$$S_{\boldsymbol{XY}}(\omega) = F\{R_{\boldsymbol{XY}}(\tau)\} = F\{R_{\boldsymbol{XX}}(\tau) * h^*(-\tau)\}$$
$$= S_{\boldsymbol{XX}}(\omega)H^*(\omega)$$

Here, transfer function is defined as

$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$. The following was used.

$$F\{h^*(-\tau)\} = \int_{-\infty}^{\infty} h^*(-\tau)e^{-i\omega\tau}d\tau$$
$$= \left(\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt\right)^* = H^*(\omega)$$

Hence,

$$S_{\boldsymbol{YY}}(\omega) = F\{R_{\boldsymbol{YY}}(\tau)\} = S_{\boldsymbol{XY}}(\omega)H(\omega)$$
$$= S_{\boldsymbol{XX}}(\omega)|H(\omega)|^2$$

- **Appendix:** "If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$, then, $f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$."

Proof:

$$F\{f(t) * g(t))\} = \int_{-\infty}^{\infty} f(t) * g(t) e^{-i\omega t} dt$$
$$= \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau\} e^{-i\omega t} dt$$
$$= \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} d\tau \int_{-\infty}^{\infty} g(t-\tau) e^{-i\omega(t-\tau)} d(t-\tau)$$
$$= F(\omega)G(\omega)$$

($F\{\cdot\}$ represents Fourier transform.)

Example

- Consider white noise with a PSD $S_{XX}(\omega) = \frac{N_0}{2}$ passing through a low-pass filter with a frequency response:

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

- The PSD of the output noise $Y(t)$ will be:

$$S_{YY}(\omega) = S_{XX}(\omega)|H(j\omega)|^2 = \begin{cases} \frac{N_0}{2} \cdot 1^2 = N_0/2, & |\omega| \le \omega_c \\ \frac{N_0}{2} \cdot 0^2 = 0, & |\omega| > \omega_c \end{cases}$$

- The output noise has a PSD that is band-limited to the cutoff frequency ω_c of the low-pass filter. The total power of the output noise would be the integral of $S_{YY}(\omega)$ over all frequencies.

Applications

- **Filtering in the Frequency Domain:** This fundamental result shows that an LTI system acts as a filter on the power spectrum of the input signal.
- **Frequency Shaping:** The frequency response $H(j\omega)$ determines how different frequency components of the input signal are amplified or attenuated by the system. The power spectrum of the output reflects this shaping.
- **System Identification:** By analyzing the input and output power spectra of a system with a known input (e.g., white noise), we can potentially estimate the magnitude of the system's frequency response $|H(j\omega)|$.