

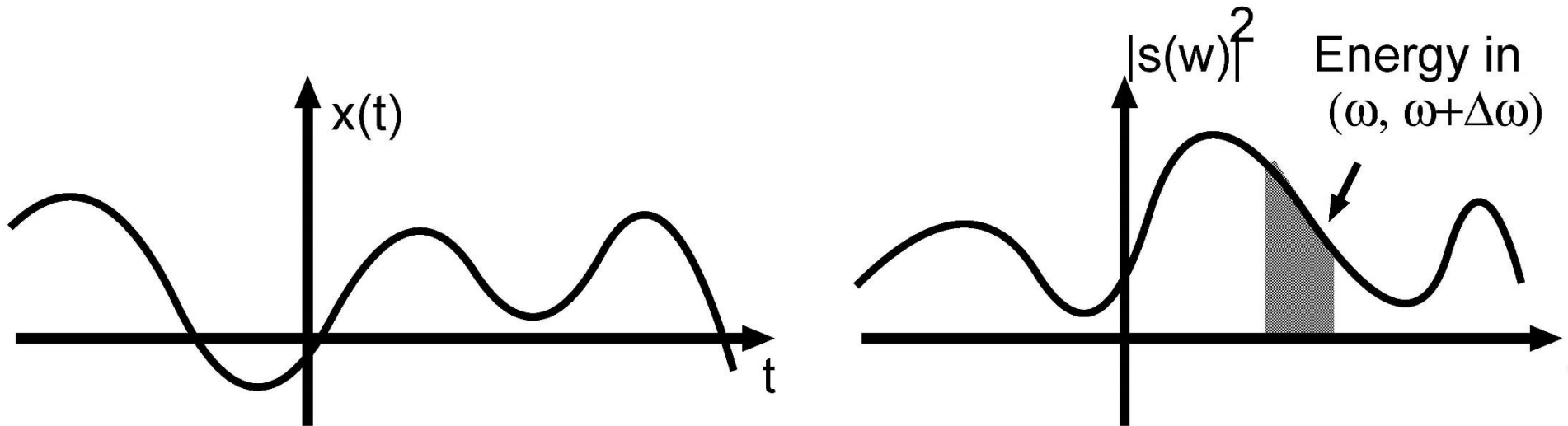
I225E Statistical Signal Processing

6. Spectral Analysis I

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1. Spectral Representation



For deterministic signal $x(t)$, its spectrum is computed by Fourier transform

$$s(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

Fourier transforms

- Fourier and inverse Fourier transformations:

$$x(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega t} s(\omega),$$
$$s(\omega) = \int_{-\infty}^{+\infty} dt e^{-i\omega t} x(t).$$

- A Fourier and a subsequent inverse Fourier transform form an identity mapping:

$$\begin{aligned} x(t) &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega t} s(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega t} \int_{-\infty}^{+\infty} dt' e^{-i\omega t'} x(t') \\ &= \int_{-\infty}^{+\infty} dt' \delta(t - t') x(t') = x(t). \end{aligned}$$

Parseval's theorem

- The power in the temporal domain is equal to the power in the frequency domain:

$$\begin{aligned}\int_{-\infty}^{+\infty} dt |x(t)|^2 &= \int_{-\infty}^{+\infty} dt x(t) x^*(t) \\&= \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} s(\omega) \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} e^{-i\omega' t} s^*(\omega') \\&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\omega d\omega'}{2\pi} \delta(\omega - \omega') s(\omega) s^*(\omega') \\&= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} |s(\omega)|^2\end{aligned}$$

■ From Parseval's theorem,

$$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |s(\omega)|^2 d\omega = E.$$

This implies that power spectrum $|s(\omega)|^2 \Delta\omega$ represents energy concentrated within frequency range of $[\omega, \omega + \Delta\omega]$.

How to define spectrum for stochastic process?

Problem

Fourier transform of stochastic signal $X(t)$ results in different spectra for every trial.

→ Definition necessary for sample average.

Derivation of power spectrum

For finite interval $[-T, T]$, Fourier transform of one realization of stochastic process $\mathbf{X}(t)$ is given by

$$\mathbf{X}_T(\omega) = \int_{-T}^T \mathbf{X}(t) e^{-i\omega t} dt$$

whereas its power is given by $\frac{|\mathbf{X}_T(\omega)|^2}{2T} = \frac{1}{2T} \left| \int_{-T}^T \mathbf{X}(t) e^{-i\omega t} dt \right|^2$.

Taking the expectation $E\{\cdot\}$, mean power spectra $\bar{\mathbf{X}}_T(\omega)$ can be calculated as

$$\begin{aligned} \bar{\mathbf{X}}_T(\omega) &= E \left\{ \frac{|\mathbf{X}_T(\omega)|^2}{2T} \right\} \\ &= \frac{1}{2T} \int_{-T}^T \int_{-T}^T E \{ \mathbf{X}(t_1) \mathbf{X}^*(t_2) \} e^{-i\omega(t_1 - t_2)} dt_1 dt_2 \\ &= \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{\mathbf{X}\mathbf{X}}(t_1, t_2) e^{-i\omega(t_1 - t_2)} dt_1 dt_2. \end{aligned}$$

Supposing $X(t)$ is a wide-sense stationary process, so that $R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2)$,

$$\bar{X}_T(\omega) = \frac{1}{2T} \int_{-T}^T \int_{-T}^T R_{XX}(t_1 - t_2) e^{-i\omega(t_1 - t_2)} dt_1 dt_2$$

Denoting $\tau = t_1 - t_2$,

$$\begin{aligned} \bar{X}_T(\omega) &= \frac{1}{2T} \int_{-2T}^{2T} R_{XX}(\tau) e^{-i\omega\tau} (2T - |\tau|) d\tau \\ &= \int_{-2T}^{2T} R_{XX}(\tau) e^{-i\omega\tau} \left(1 - \frac{|\tau|}{2T}\right) d\tau \geq 0 \end{aligned}$$

Finally, taking the limit of $T \rightarrow \infty$.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \geq 0$$

[Wiener-Khinchin Theorem]

- Autocorrelation $R_{XX}(\tau)$ and spectral density $S_{XX}(\omega)$ are related with each other via Fourier transform.

The inverse Fourier transform gives

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega \geq 0$$

In particular, the case of $\tau = 0$ gives the signal power

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega &= R_{XX}(0) \\ &= E\{|X(t)|^2\} \\ &= P \end{aligned}$$

Properties of power spectrum

(1) $S_{XX}(\omega)$ is a real function of ω .

(Because $R_{XX}(-\tau) = R_{XX}^*(\tau)$ and $S_{XX}(\omega) = S_{XX}^*(\omega)$)

(2) $S_{XX}(\omega) \geq 0$.

(3) If $X(t)$ is a real process,

$$\begin{aligned} R_{XX}(\tau) &= E\{X(t + \tau)X(t)\} \\ &= E\{X(s)X(s - \tau)\} = R_{XX}(-\tau). \text{ Therefore} \\ S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} R_{XX}(\tau) \cos \omega\tau d\tau + i \int_{-\infty}^{\infty} R_{XX}(\tau) \sin \omega\tau d\tau \\ &= 2 \int_0^{\infty} R_{XX}(\tau) \cos \omega\tau d\tau = S_{XX}(-\omega) \end{aligned}$$

Hence, $S_{XX}(\omega)$ is an even function and can be represented in terms of cos-transform.

Cross-power spectrum

Cross-power spectrum $S_{XY}(\omega)$ of two processes $X(t)$ and $Y(t)$ is

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau,$$

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\omega\tau} d\omega,$$

$$S_{YX}(\omega) = S_{XY}^*(\omega).$$

Example of power spectrum

Random variables a_i are mutually uncorrelated, and have mean 0 and variance σ_i^2 . Compute power spectrum of stochastic process:

$$X(t) = \sum_i a_i e^{i\omega_i t}$$

Auto-correlation of $X(t)$ is computed as

$$\begin{aligned} R_{XX}(\tau) &= E\{X(t + \tau)X^*(t)\} \\ &= E\left\{\sum_i a_i e^{i\omega_i(t+\tau)} \sum_k a_k^* e^{-i\omega_k t}\right\} \\ &= \sum_i \sum_k E\{a_i a_k^*\} e^{i(\omega_i - \omega_k)t + i\omega_i \tau} \\ &= \sum_i \sigma_i^2 e^{i\omega_i \tau} \end{aligned}$$

Here, we used $E\{a_i a_k^*\} = 0$ ($i \neq k$), σ_i^2 ($i = k$), due to uncorrelation property of a_i . The power spectrum is

$$\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \\ &= \sum_i \sigma_i^2 \int_{-\infty}^{\infty} e^{i\omega_i\tau} e^{-i\omega\tau} d\tau \\ &= 2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i) \end{aligned}$$

Here, we used $\int_{-\infty}^{\infty} e^{i\omega_i\tau} e^{-i\omega\tau} d\tau = 2\pi\delta(\omega - \omega_i)$.

Autocorrelations and the corresponding spectra

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \iff S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$\delta(\tau)$	\iff	1
1	\iff	$2\pi\delta(\omega)$
$e^{j\beta\tau}$	\iff	$2\pi\delta(\omega - \beta)$
$\cos \beta\tau$	\iff	$\pi\delta(\omega - \beta) + \pi\delta(\omega + \beta)$
$e^{-\alpha \tau }$	\iff	$\frac{2\alpha}{\alpha^2 + \omega^2}$
$e^{-\alpha\tau^2}$	\iff	$\sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}$

$$e^{-\alpha|\tau|} \cos \beta\tau \quad \Leftrightarrow \quad \frac{\alpha}{\alpha^2 + (\omega - \beta)^2} + \frac{\alpha}{\alpha^2 + (\omega + \beta)^2}$$

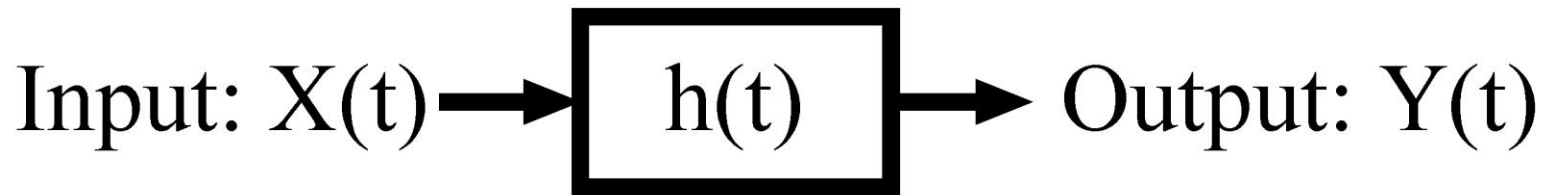
$$e^{-\alpha\tau^2} \cos \beta\tau \quad \Leftrightarrow \quad \sqrt{\frac{\pi}{\alpha}} \left[e^{-\frac{(\omega - \beta)^2}{4\alpha}} + e^{-\frac{(\omega + \beta)^2}{4\alpha}} \right]$$

$$\begin{cases} 1 - \frac{|\tau|}{T} & |\tau| < T \\ 0 & |\tau| > T \end{cases} \quad \Leftrightarrow \quad \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin \sigma\tau}{\pi\tau} \quad \Leftrightarrow \quad \begin{cases} 1 & |\omega| < \sigma \\ 0 & |\omega| > \sigma \end{cases}$$

2. Power-spectra and Linear System

- Wide sense stationary process $X(t)$ has autocorrelation of $R_{XX}(\tau)$ and its corresponding spectrum $S_{XX}(\omega)$. Suppose $X(t)$ is input to a linear system, whose impulse response is given by $h(t)$. How to compute the output spectrum $S_{YY}(\omega)$?



- From properties of autocorrelation function

$$R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau),$$

$$R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau),$$

In general, “If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$, then, $f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$.” Therefore,

$$\begin{aligned} S_{XY}(\omega) &= F\{R_{XY}(\tau)\} = F\{R_{XX}(\tau) * h^*(-\tau)\} \\ &= S_{XX}(\omega)H^*(\omega) \end{aligned}$$

Here, transfer function is defined as

$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt$. The following was used.

$$\begin{aligned} F\{h^*(-\tau)\} &= \int_{-\infty}^{\infty} h^*(-\tau)e^{-i\omega\tau} d\tau \\ &= \left(\int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt \right)^* = H^*(\omega) \end{aligned}$$

Hence,

$$\begin{aligned} S_{YY}(\omega) &= F\{R_{YY}(\tau)\} = S_{XY}(\omega)H(\omega) \\ &= S_{XX}(\omega)|H(\omega)|^2 \end{aligned}$$

■ **Appendix:** “If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$, then,
 $f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$.”

Proof:

$$\begin{aligned} F\{f(t) * g(t)\} &= \int_{-\infty}^{\infty} f(t) * g(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau \right\} e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} f(\tau) e^{-i\omega \tau} d\tau \int_{-\infty}^{\infty} g(t - \tau) e^{-i\omega(t - \tau)} d(t - \tau) \\ &= F(\omega)G(\omega) \end{aligned}$$

($F\{\cdot\}$ represents Fourier transform.)