I225E Statistical Signal Processing

5. Stochastic Process and Systems II

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Deterministic System

Memoryless System

$$Y(t) = g[X(t)]$$

System with Memory

Time-Varying System

Time-Invariant System

Linear System

$$Y(t) = L[X(t)]$$

Linear Time-Invariant system

$$Y(t)=X(t)*h(t)$$

System with Memory

- **[Discrete time]** If the response of a system $y[k]$ at $k = k_0$ depends on the values of the input $x[k]$ in the past or in the future of time $k = k_0$.
- **[Continuous time]** If the response of a system $Y(t)$ at $t = t_0$ depends on the values of the input $X(t)$ in the past or in the future of time $t = t_0$.

Examples

	Memoryless systems	Systems with memory
Discrete-time	$y[k] = 3x[k] + 7$	$y[k] = 3x[k-5]$
	$y[k] = \sin(x[k]) + 5$	$y[k] = x[2k] + 5$
	$y[k] = x^2[k]$	$y[k] = x^2[k/2]$
Continuous-time	$y(t) = 3x(t) + 7$	$y(t) = 3x[t-5]$
	$y(t) = \sin(x(t)) + 5$	$y(t) = x[2t] + 5$
	$y(t) = x^2(t)$	$y(t) = x^2[t/2]$

Time-Invariant System

- **Definition:** A system where its characteristics and behavior **do not change** over time.
- Mathematically, if a system T produces an output $Y(t) = T\{X(t)\}$ for an input $X(t)$, then for a time-invariant system, a delayed input $X(t - t_0)$ will produce a delayed output:

$$Y(t - t_0) = T\{X(t - t_0)\}$$

Time-Invariant System

- **Example:**
  - Resistors, capacitors, and inductors in a linear circuit
  - A mass-spring-damper system with constant parameters
  - A simple amplifier with a fixed gain

Time-Variant System

- **Definition**

A system where its properties and behavior **do change** over time.

- Mathematically, if a system T produces an output $Y(t) = T\{X(t)\}$ for an input $X(t)$, then for a time-variant system, a delayed input $X(t - t_0)$ will produce an output $Y'(t)$ such that

$$Y'(t) \neq Y(t - t_0) \text{ in general.}$$

Time-Variant System

- **Example**
  - A rocket's pitch response as it burns fuel
  - Aging electronic components
  - Discrete Wavelet Transform
  - Amplitude modulation: The sinusoidal carrier signal introduces time-varying behavior. For example:

$$y(t) = x(t)\cos(\omega t)$$

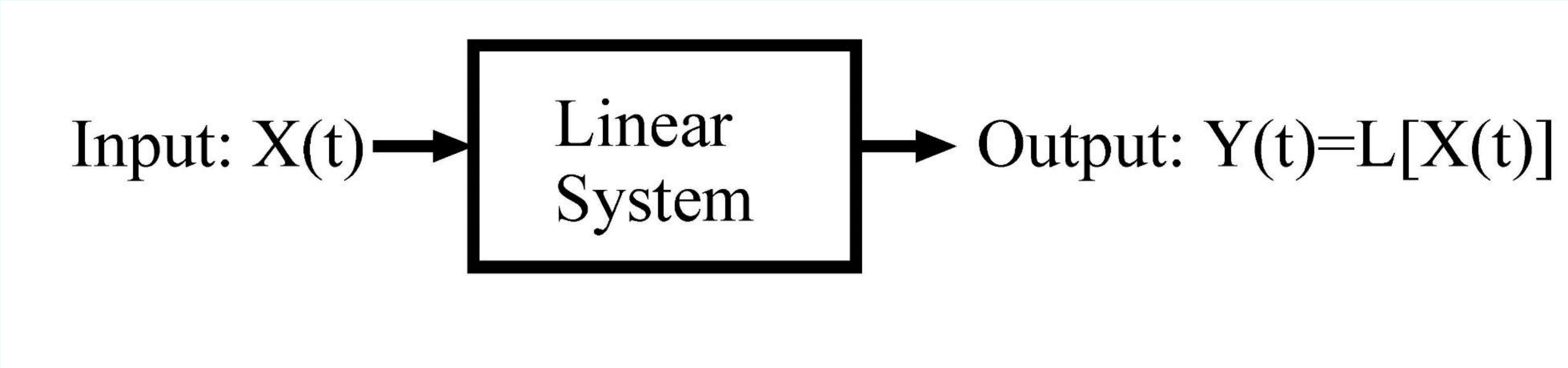
Recall: Linear Function

- **Definition:** A function $f(x)$ is linear if it is additive and homogeneous.
- **Additive**:

  $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all vectors x_1, x_2
- **Homogeneous**:

  $f(\alpha x) = \alpha f(x)$ for all vectors x and scalar α

Linear System



$$\boldsymbol{Y}(t) = L[\boldsymbol{X}(\mathrm{t})]$$

Properties of linear system

(b1) For any $a_1, a_2, \boldsymbol{X}_1(t), \boldsymbol{X}_2(t)$,

$$L[a_1\boldsymbol{X}_1(t) + a_2\boldsymbol{X}_2(t)] = a_1L[\boldsymbol{X}_1(t)] + a_2L[\boldsymbol{X}_2(t)].$$

Linear System

(b2) A system is called ***time-invariant***, if its response to $\boldsymbol{X}(t+c)$ equals $\boldsymbol{Y}(t+c)$.

(b3) Output of linear time-invariant system is a convolution

$$\boldsymbol{Y}(t) = \boldsymbol{X}(t) * \boldsymbol{h}(t) = \int_{-\infty}^{\infty} \boldsymbol{X}(t-\alpha)\boldsymbol{h}(\alpha)d\alpha$$

Here, $\boldsymbol{h}(t) = L[\delta(t)]$ represents impulse response.

Recall: Convolution

Exercise

Imagine you are working with audio signals and want to create a simple echo effect. You have an original audio signal, $x[n]$, and you want to add a delayed and attenuated version of it to create the echo.

a) Assume the echo effect produces a single echo that is delayed by 3 sample units and attenuated by a factor of 0.5. Determine the impulse response, $h[n]$, of the system that creates this echo.

Recall: Convolution

Exercise

b) Let the original audio signal be a simple sequence representing a short sound: $x[n] = \{2,1,-1,0,3\}$.

Calculate the output signal, $y[n]$, which is the convolution of the input signal $x[n]$ with the impulse response $h[n]$ you found in part a.

The linear convolution for discrete-time signals is:

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Recall: Convolution

- **Answer**

$$y[n] = \{2,1,-1,1,3.5,0.5,-0.5,1.5\}$$

Recall: Convolution

Exercise

c) What would the impulse response be if you wanted to create two distinct echoes: one delayed by 2 samples with an attenuation of 0.3, and another delayed by 5 samples with an attenuation of 0.2? How would the output of the convolution with $x[n]$ look in this case?

Recall: Convolution

- **Answer**

$$h'[n] = \delta[n] + 0.3\delta[n-2] + 0.2\delta[n-5]$$

$$y'[n] = \{2,1,-0.4,0.3,2.7,0.4,1.1,-0.2,0,0.6\}$$

(b4) Linear time-invariant system has following properties.

i) If input $\boldsymbol{X}(t)$ is a normal process, then output $\boldsymbol{Y}(t)$ is also a normal process.

ii) If input $\boldsymbol{X}(t)$ is strict sense stationary, then output $\boldsymbol{Y}(t)$ is also strict sense stationary.

Fundamental theorem

For any linear system, $E\{L[\boldsymbol{X}(t)]\} = L[E\{\boldsymbol{X}(t)\}]$

This means that output average $\eta_{\boldsymbol{Y}}(t)$ equals to output of a system, to which input average $\eta_{\boldsymbol{X}}(t)$ is input.

$$\eta_{\boldsymbol{Y}}(t) = L[\eta_{\boldsymbol{X}}(t)]$$

(Intuitive proof)

Supposing that *i*th trial is $\boldsymbol{Y}(t, \omega_i) = L[\boldsymbol{X}(t, \omega_i)]$

$$E\{\boldsymbol{Y}(t)\} \approx \frac{\boldsymbol{Y}(t,\omega_1)+\boldsymbol{Y}(t,\omega_2)+\cdots+\boldsymbol{Y}(t,\omega_n)}{n}$$

$$= \frac{L[\boldsymbol{X}(t,\omega_1)]+L[\boldsymbol{X}(t,\omega_2)]+\cdots+L[X(t,\omega_n)]}{n}$$

$$= L\left[\frac{\boldsymbol{X}(t,\omega_1)+\boldsymbol{X}(t,\omega_2)+\cdots+\boldsymbol{X}(t,\omega_n)}{n}\right]$$

$$\approx L[E\{\boldsymbol{X}(t)\}]$$

Exercise

- Consider a discrete-time linear system $T\{\cdot\}$. Let $x_1[n]$ and $x_2[n]$ be two input signals, and let $y_1[n] = T\{x_1[n]\}$ and $y_2[n] = T\{x_2[n]\}$ be their respective outputs.
- Now, suppose these input signals are random processes. Consider a new input signal $x[n]$ formed by a linear combination of $x_1[n]$ and $x_2[n]$:

$$x[n] = ax_1[n] + bx_2[n]$$

where a and b are constant scalars.

Exercise

a) Determine the output $y[n]$ of the linear system when the input is $x[n]$

b) Find the expected value of the output signal $E\{y[n]\}$

c) Verify the given statement: $E\{T\{x[n]\}\} = T\{E\{x[n]\}\}$

(b5) Using the convolution of (b3), the theorem is described as

$$E\{\boldsymbol{Y}(t)\} = \int_{-\infty}^{\infty} E\{\boldsymbol{X}(t-\alpha)\}h(\alpha)d\alpha$$
$$= \int_{-\infty}^{\infty} \eta_{\boldsymbol{X}}(t-\alpha)h(\alpha)d\alpha = \eta_{\boldsymbol{X}}(t) * h(t)$$

(b6) Setting $\widetilde{\boldsymbol{X}}(t) = \boldsymbol{X}(t) - \eta_{\boldsymbol{X}}(t)$ and $\widetilde{\boldsymbol{Y}}(t) = \boldsymbol{Y}(t) - \eta_{\boldsymbol{Y}}(t)$,

$$\widetilde{\boldsymbol{Y}}(t) = L[\widetilde{\boldsymbol{X}}(t)]$$

[Proof]

$$\begin{aligned}\widetilde{\boldsymbol{Y}}(t) &= \boldsymbol{Y}(t) - \eta_{\boldsymbol{Y}}(t) \\ &= L[\boldsymbol{X}(t)] - L[\eta_{\boldsymbol{X}}(t)] \\ &= L[\boldsymbol{X}(t) - \eta_{\boldsymbol{X}}(t)] \\ &= L[\widetilde{\boldsymbol{X}}(t)]\end{aligned}$$

(b7) With respect to input $\boldsymbol{X}(t) = f(t) + \nu(t)$ ($f(t)$ is a deterministic signal; $E\{\nu(t)\} = 0$), the output average is

$$\eta_{\boldsymbol{Y}}(t) = E\{\boldsymbol{Y}(t)\} = E\{L[\boldsymbol{X}(t)]\} = L[E\{\boldsymbol{X}(t)\}]$$
$$= L[E\{f(t) + \nu(t)\}]$$
$$= L[E\{f(t)\} + E\{\nu(t)\}]$$
$$= L[f(t)] = f(t) * h(t)$$

Autocorrelation of output

Consider input $\boldsymbol{X}(t)$ and output $\boldsymbol{Y}(t)$ for a linear system. We represent output autocorrelation $R_{\boldsymbol{YY}}(t_1, t_2)$ in terms of input autocorrelation $R_{\boldsymbol{XX}}(t_1, t_2)$.

■ Theorem

$$\text{(A) } R_{\boldsymbol{XY}}(t_1, t_2) = L_{t_2}[R_{\boldsymbol{XX}}(t_1, t_2)\ ].$$

$$\text{(B) } R_{\boldsymbol{YY}}(t_1, t_2) = L_{t_1}[R_{\boldsymbol{XY}}(t_1, t_2)\ ].$$

[**Proof**]

Multiplying both side of $\boldsymbol{Y}(t) = L_t[\boldsymbol{X}(t)]$ by $\boldsymbol{X}(t_1)$,

$$\boldsymbol{X}(t_1)\boldsymbol{Y}(t) = \boldsymbol{X}(t_1)L_t[\boldsymbol{X}(t)] = L_t[\boldsymbol{X}(t_1)\boldsymbol{X}(t)].$$

Taking expectation,

$$\begin{aligned} E\{\boldsymbol{X}(t_1)\boldsymbol{Y}(t)\} &= E\{L_t[\boldsymbol{X}(t_1)\boldsymbol{X}(t)]\} \\ &= L_t[E\{\boldsymbol{X}(t_1)\boldsymbol{X}(t)\}]. \end{aligned}$$

Substitution of $t = t_2$ yields (A).

In a similar manner, multiplication of $\boldsymbol{Y}(t) = L_t[\boldsymbol{X}(t)]$ by $\boldsymbol{Y}(t_2)$ and so on yields (B).

(c1) In the form of convolution,

(a) $R_{\boldsymbol{XY}}(t_1, t_2) = R_{\boldsymbol{XX}}(t_1, t_2) * h(t_2)$

$$= \int_{-\infty}^{\infty} R_{\boldsymbol{XX}}(t_1, t_2 - \alpha) h(\alpha) d\alpha$$

(b) $R_{\boldsymbol{YY}}(t_1, t_2) = R_{\boldsymbol{XY}}(t_1, t_2) * h(t_1)$

$$= \int_{-\infty}^{\infty} R_{\boldsymbol{XY}}(t_1 - \alpha, t_2) h(\alpha) d\alpha$$

(c) Combining (a) and (b),

$$R_{\boldsymbol{YY}}(t_1, t_2) =$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{\boldsymbol{XX}}(t_1 - \alpha, t_2 - \beta) h(\alpha) h(\beta) d\alpha d\beta$$

(c2) For cross-covariance, the following holds.

$$C_{\boldsymbol{XY}}(t_1, t_2) = C_{\boldsymbol{XX}}(t_1, t_2) * h(t_2)$$
$$C_{\boldsymbol{YY}}(t_1, t_2) = C_{\boldsymbol{XY}}(t_1, t_2) * h(t_1)$$

[**Proof**]

Denoting $\widetilde{\boldsymbol{X}}(t) = \boldsymbol{X}(t) - \eta_{\boldsymbol{X}}(t)$, $\widetilde{\boldsymbol{Y}}(t) = \boldsymbol{Y}(t) - \eta_{\boldsymbol{Y}}(t)$,

$$C_{\boldsymbol{XX}}(t_1, t_2) = E\left\{\widetilde{\boldsymbol{X}}(t_1)\widetilde{\boldsymbol{X}}(t_2)\right\} = R_{\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{X}}}(t_1, t_2)$$
$$C_{\boldsymbol{XY}}(t_1, t_2) = R_{\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{Y}}}(t_1, t_2),$$
$$C_{\boldsymbol{YY}}(t_1, t_2) = R_{\widetilde{\boldsymbol{Y}}\widetilde{\boldsymbol{Y}}}(t_1, t_2).$$

From (b6), $\widetilde{\boldsymbol{Y}}(t) = L[\widetilde{\boldsymbol{X}}(t)]$. Hence, (a) and (b) can be applied to $R_{\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{X}}}$, $R_{\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{Y}}}$, and $R_{\widetilde{\boldsymbol{Y}}\widetilde{\boldsymbol{Y}}}$.

(c3) Above results can be extended to the case the impulse response $\boldsymbol{h}(t)$ is complex function.

In this case,

$$R_{\boldsymbol{XY}}(t_1,t_2) = R_{\boldsymbol{XX}}(t_1,t_2) * \boldsymbol{h}^*(t_2)$$
$$R_{\boldsymbol{YY}}(t_1,t_2) = R_{\boldsymbol{XY}}(t_1,t_2) * \boldsymbol{h}(t_1)$$

White Noise

- **Definition:** A random signal containing all frequencies at equal intensity.
- **Key Characteristic:** It has a constant power spectral density. $S_x(f) = N_0/2$ (for a double-sided spectrum) or $S_x(f) = N_0$ (for a single-sided spectrum), where N_0 is a constant representing the noise power per unit bandwidth.
- In the time domain, ideal white noise has an autocorrelation function that is a Dirac delta function at zero lag:

$$R_x(\tau) = E[x(t)x(t-\tau)] = \frac{N_0}{2}\delta(\tau)$$

White Noise

Power Spectral Density of the Output

The Power Spectral Density of the output $S_y(f)$ is related to the Power Spectral Density of the input $S_x(f)$ and the magnitude squared of the system's frequency response $|\,H(f)\,|^2$ by the following equation:

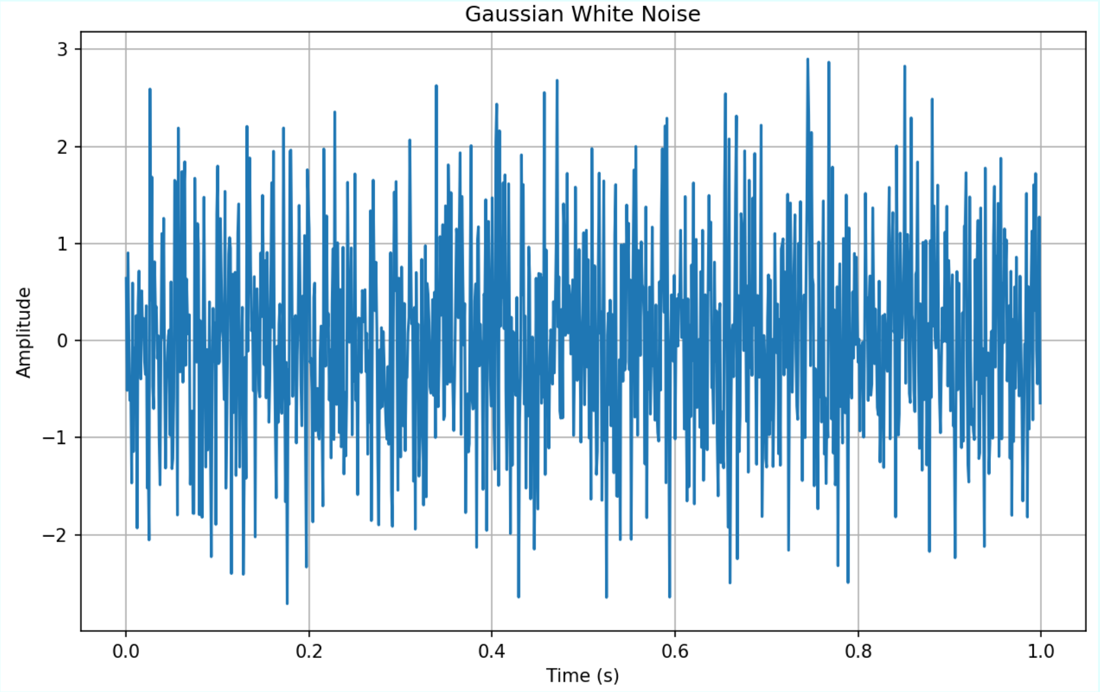
$$S_y(f) = |H(f)|^2 \, S_x(f)$$

If the input is white noise with $S_x(f) = N_0/2$, then the Power Spectral Density of the output becomes:

$$S_y(f) = |H(f)|^2 N_0/2$$

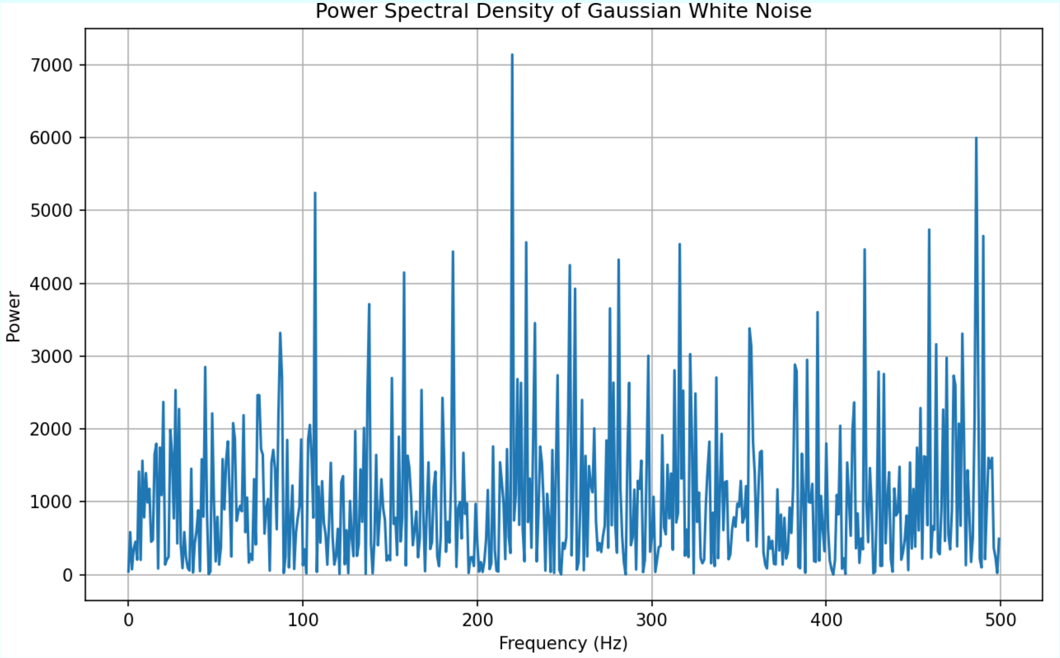
White Noise

```
# Define parameters
duration = 1  # seconds
sampling_rate = 1000  # Hz
num_samples = int(sampling_rate *
duration)

# Generate Gaussian white noise
mean = 0
std_dev = 1
noise = np.random.normal(mean,
std_dev, num_samples)

# Create time vector
time = np.linspace(0, duration,
num_samples, endpoint=False)

# Plot the noise
plt.figure(figsize=(10, 6))
plt.plot(time, noise)
plt.title('Gaussian White Noise')
plt.xlabel('Time (s)')
plt.ylabel('Amplitude')
plt.grid(True)
plt.show()
```



White Noise

```
# Plot the power spectral density
(PSD) to see the 'white' nature
frequencies =
np.fft.fftfreq(num_samples, 1 /
sampling_rate)
power_spectrum =
np.abs(np.fft.fft(noise))**2

plt.figure(figsize=(10, 6))
plt.plot(frequencies[:num_samples//2]
, power_spectrum[:num_samples//2])
plt.title('Power Spectral Density of
Gaussian White Noise')
plt.xlabel('Frequency (Hz)')
plt.ylabel('Power')
plt.grid(True)
plt.show()
```



Response to white noise

Theorem

Consider white noise, whose autocorrelation is given by $R_{\boldsymbol{XX}}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$, as input to linear system. The output power is given by

$$E\{\boldsymbol{Y}^2(t)\} = q(t) * h^2(t) = \int_{-\infty}^{\infty} q(t-\alpha)h^2(\alpha)d\alpha$$

[Proof]

Substitution of $R_{\boldsymbol{XX}}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$ into convolution form of (A), (B) yields,

$$R_{\boldsymbol{XY}}(t_1, t_2) = q(t_1)\delta(t_1 - t_2) * h(t_2) = q(t_1)h(t_2 - t_1)$$
$$R_{\boldsymbol{YY}}(t_1, t_2) = \int_{-\infty}^{\infty} q(t_1 - \alpha)h[t_2 - (t_1 - \alpha)]h(\alpha)d\alpha.$$

Substitution of $t_1 = t_2 = t$ gives the output power.

(d1) If input $\boldsymbol{X}(t)$ is stationary white noise, $q(t) = q$.

Therefore,

$$E\{\boldsymbol{Y}^2(t)\} = qE, \quad E = \int_{-\infty}^{\infty} h^2(\alpha)d\alpha$$

(d2) If correlation time of impulse response $h(t)$ is shorter than that of $q(t)$,

$$E\{\boldsymbol{Y}^2(t)\} \approx q(t)\int_{-\infty}^{\infty} h^2(\alpha)d\alpha = Eq(t).$$

(d3) If $R_{\boldsymbol{XX}}(\tau) = q\delta(\tau)$ and $\boldsymbol{X}(t)$ is injected at $t = 0$.

$$q(t) = qU(t)$$

Therefore,

$$E\{\boldsymbol{Y}^2(t)\} \approx q\int_{-\infty}^{t} h^2(\alpha)d\alpha$$

Digression on Dirac's delta function

- Properties of delta function $\delta(t)$:

$$\int_{-\infty}^{+\infty} dt\delta(t) = 1.$$

$$\int_{-\infty}^{+\infty} dt\delta(t)f(t) = f(0).$$

- Delta function as a limit of infinitesimally narrow Gaussian:

$$\delta(t) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

- Delta function has a uniform (white) spectrum:

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega e^{i\omega t}$$

Example 1

Consider a linear system having impulse response

$$h(t) = e^{-ct}U(t)$$

Input a white noise, whose autocorrelation is given by $R_{\boldsymbol{WW}}(\tau) = q\delta(\tau)$, at $t = 0$. Calculate autocorrelation $R_{\boldsymbol{YY}}(t_1, t_2)$ $(0 < t_1 < t_2)$ of the output $\boldsymbol{Y}(t)$.

Answer Denoting input as $\boldsymbol{X}(t)$, $\boldsymbol{X}(t) = \boldsymbol{W}(t)U(t)$.

(I) Case of $t_1 < 0$ or $t_2 < 0$: Since there is no input,

$$R_{\boldsymbol{XX}}(t_1, t_2) = R_{\boldsymbol{XY}}(t_1, t_2) = R_{\boldsymbol{YY}}(t_1, t_2) = 0$$

(II) Case of $0 < t_1 < t_2$:

$$R_{\boldsymbol{XX}}(t_1, t_2) = R_{\boldsymbol{WW}}(t_1 - t_2) = q\delta(t_1 - t_2)$$

Therefore,

$$R_{\boldsymbol{XY}}(t_1, t_2) = \int_{-\infty}^{t_2} R_{\boldsymbol{XX}}(t_1, t_2 - \alpha) h(\alpha) d\alpha$$

$$= \int_{-\infty}^{t_2} q\delta(t_1 - t_2 + \alpha) e^{-c\alpha} U(\alpha) d\alpha$$

$$= q e^{-c(t_2 - t_1)}$$

$$R_{\boldsymbol{YY}}(t_1, t_2) = \int_{-\infty}^{t_1} R_{\boldsymbol{XY}}(t_1 - \alpha, t_2) h(\alpha) d\alpha$$

$$= \int_{-\infty}^{t_1} q e^{-c(t_2 - t_1 + \alpha)} e^{-c\alpha} U(\alpha) d\alpha$$

$$= q e^{-c(t_2 - t_1)} \int_0^{t_1} e^{-2c\alpha} d\alpha$$

$$= \frac{q}{2c} e^{-c(t_2 - t_1)} (1 - e^{-2ct_1})$$

$$E\{\boldsymbol{Y}^2(t)\} = R_{\boldsymbol{YY}}(t, t) = \frac{q}{2c}(1 - e^{-2ct}).$$

Example 2: Differentiator

Consider the output properties (mean and autocorrelation) of differentiator:

$$\boldsymbol{Y}(t) = L[\boldsymbol{X}(t)] = \boldsymbol{X}'(t).$$

- **Mean** is,

$$\eta_{\boldsymbol{Y}}(t) = L[\eta_{\boldsymbol{X}}(t)] = \eta'_{\boldsymbol{X}}(t)$$

- **Autocorrelation** is

$$R_{\boldsymbol{XY}}(t_1, t_2) = L_{t_2}[R_{\boldsymbol{XX}}(t_1, t_2)] = \frac{\partial R_{\boldsymbol{XX}}(t_1,t_2)}{\partial t_2}$$

$$R_{\boldsymbol{YY}}(t_1, t_2) = L_{t_1}[R_{\boldsymbol{XY}}(t_1, t_2)] = \frac{\partial R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_1}$$

Hence

$$R_{\boldsymbol{YY}}(t_1, t_2) = L_{t_1}[R_{\boldsymbol{XY}}(t_1, t_2)] = \frac{\partial^2 R_{\boldsymbol{XX}}(t_1,t_2)}{\partial t_1 \partial t_2}$$

If $\boldsymbol{X}(t)$ is wide-sense stationary, its **mean**, $\eta_{\boldsymbol{X}}(t)$, is a constant. Therefore, $\eta_{\boldsymbol{Y}}(t) = 0$. Since **autocorrelation** is a function of $\tau = t_1 - t_2$,

$$\frac{\partial R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2} = -\frac{dR_{\boldsymbol{XX}}(\tau)}{d\tau}$$

$$\frac{\partial^2 R_{\boldsymbol{XX}}(t_1,t_2)}{\partial t_1 \partial t_2} = -\frac{d^2 R_{\boldsymbol{XX}}(\tau)}{d\tau^2}$$

Hence,

$$R_{\boldsymbol{XY}}(\tau) = -R'_{\boldsymbol{XX}}(\tau),$$

$$R_{\boldsymbol{YY}}(\tau) = -R''_{\boldsymbol{XX}}(\tau),$$

Example 3: Differential Equation

Consider output properties (mean and autocorrelation) of the following differential equation driven by noise $\boldsymbol{X}(t)$.

$$a_n \boldsymbol{Y}^{(n)}(t) + a_{n-1}\boldsymbol{Y}^{(n-1)}(t) + \cdots + a_0 \boldsymbol{Y}(t) = \boldsymbol{X}(t)$$

The initial condition is given by

$$\boldsymbol{Y}(0) = \boldsymbol{Y}^{(1)}(0) = \cdots = \boldsymbol{Y}^{(n)}(0) = 0$$

- **Mean**:

  Taking the expectation $E\{\cdot\}$ for both sides,

  $$a_n E\left\{\boldsymbol{Y}^{(n)}(t)\right\} + a_{n-1} E\left\{\boldsymbol{Y}^{(n-1)}(t)\right\} + \cdots + a_0 E\{\boldsymbol{Y}(t)\} = E\{\boldsymbol{X}(t)\}$$

  Since $\frac{d}{dx}$ and $E\{\cdot\}$ are commutative, $E\left\{\boldsymbol{Y}^{(k)}(t)\right\} = \eta_Y^{(k)}(t)$

$$a_n \eta_{\boldsymbol{Y}}^{(n)}(t) + a_{n-1}\eta_{\boldsymbol{Y}}^{(n-1)}(t) + \cdots + a_0 \eta_{\boldsymbol{Y}}(t) = \eta_{\boldsymbol{X}}(t)$$

Because of $\boldsymbol{Y}^{(k)}(0) = 0$, $\eta_{\boldsymbol{Y}}^{(k)}(0) = 0$. The mean $\eta_{\boldsymbol{Y}}^{(k)}(t)$ is obtained by integrating the differential equation for the following initial condition:

$$\eta_{\boldsymbol{Y}}(0) = \eta_{\boldsymbol{Y}}^{(1)}(0) = \cdots = \eta_{\boldsymbol{Y}}^{(n)}(0) = 0$$

- **Correlation**:

  By substituting $t = t_2$ and multiplying by $\boldsymbol{X}(t_1)$, the differential equation becomes

$$X(t_1)\left[a_n \boldsymbol{Y}^{(n)}(t_2) + a_{n-1}\boldsymbol{Y}^{(n-1)}(t_2) + \cdots + a_0 \boldsymbol{Y}(t_2)\right]$$
$$= \boldsymbol{X}(t_1)\boldsymbol{X}(t_2)$$

By taking the expectation and using the formula of

$$E\{\boldsymbol{X}(t_1)\boldsymbol{Y}^{(k)}(t_2)\} = \frac{\partial^k R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2^k},$$

$$a_n \frac{\partial^n R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2^n} + a_{n-1} \frac{\partial^{n-1} R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2^{n-1}} + \cdots$$

$$+a_0 R_{\boldsymbol{XY}}(t_1, t_2) = R_{\boldsymbol{XX}}(t_1, t_2),$$

Since $\boldsymbol{X}(t_1)\boldsymbol{Y}^{(k)}(0) = 0$, initial condition is given by

$$R_{\boldsymbol{XY}}(t_1, 0) = \frac{\partial R_{\boldsymbol{XY}}(t_1,0)}{\partial t_2} = \cdots = \frac{\partial^n R_{\boldsymbol{XY}}(t_1,0)}{\partial t_2^n} = 0$$

Autocorrelations $\frac{\partial^n R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2^n}$ are obtained by solving the differential equation for the above initial condition.

In a similar manner, by substituting $t = t_1$ and multiplying by $\boldsymbol{Y}(t_2)$,

$$\left[a_n \boldsymbol{Y}^{(n)}(t_1) + a_{n-1}\boldsymbol{Y}^{(n-1)}(t_1) + \cdots + a_0 \boldsymbol{Y}(t_1)\right]\boldsymbol{Y}(t_2)$$
$$= \boldsymbol{X}(t_1)\boldsymbol{Y}(t_2)$$

By taking the expectation and by using the formula

$$E\left\{\boldsymbol{Y}^{(k)}(t_1)\boldsymbol{Y}(t_2)\right\} = \frac{\partial^k R_{\boldsymbol{YY}}(t_1,t_2)}{\partial t_1^k},$$

$$a_n \frac{\partial^n R_{\boldsymbol{YY}}(t_1,t_2)}{\partial t_1^n} + a_{n-1}\frac{\partial^{n-1} R_{\boldsymbol{YY}}(t_1,t_2)}{\partial t_1^{n-1}} + \cdots$$
$$+ a_0 R_{\boldsymbol{YY}}(t_1,t_2) = R_{\boldsymbol{XY}}(t_1,t_2).$$

Since $\boldsymbol{Y}(0)\boldsymbol{Y}^{(k)}(t_2) = 0$, initial condition is given by

$$R_{\boldsymbol{YY}}(0, t_2) = \frac{\partial R_{\boldsymbol{YY}}(0,t_2)}{\partial t_1} = \cdots = \frac{\partial^n R_{\boldsymbol{XY}}(0,t_2)}{\partial t_1^n} = 0$$

Autocorrelations $\frac{\partial^n R_{\boldsymbol{YY}}(t_1,t_2)}{\partial t_1^n}$ are obtained by solving the differential equation for the above initial condition.

Exercise

- Consider a discrete-time LTI system with an impulse response $h[n] = \{1, -0.5\}$, where the value at $n = 0$ is the first element. The input to this system is a zero-mean white noise process $w[n]$ with a variance of ${\sigma_w}^2$. Recall that for a zero-mean white noise process:
  - $E[w[n]] = 0$ for all n.
  - The autocorrelation function $R_{ww}\,[k] = E[w[n]w[n-k]] = \sigma_W^2\,\delta[k]$, where $\delta[k]$ is the Kronecker delta function ($\delta[0] = 1$, $\delta[k] = 0$ for $k \neq 0$).
- The output of the LTI system is $y[n] = (w * h)[n] = \sum_{m=-\infty}^{\infty} h[m]w[n-m]$.

a) Determine the autocorrelation function of the output signal, $R_{yy}[k] = E\big[y[n]y[n-k]\big]$

b) Calculate the specific form of $R_{yy}[k]$ for the given impulse response $h[n] = \{1, -0.5\}$.

c) Find the power spectral density (PSD) of the input signal $S_{ww}(\omega)$ and the output signal $S_{yy}(\omega)$. PSD is the discrete-time Fourier transform of the autocorrelation function.

$$S_{xx}(\omega) = \sum_{k=-\infty}^{\infty} R_{xx}[k]e^{-j\omega k}$$