I225E Statistical Signal Processing

7. Spectral Analysis II

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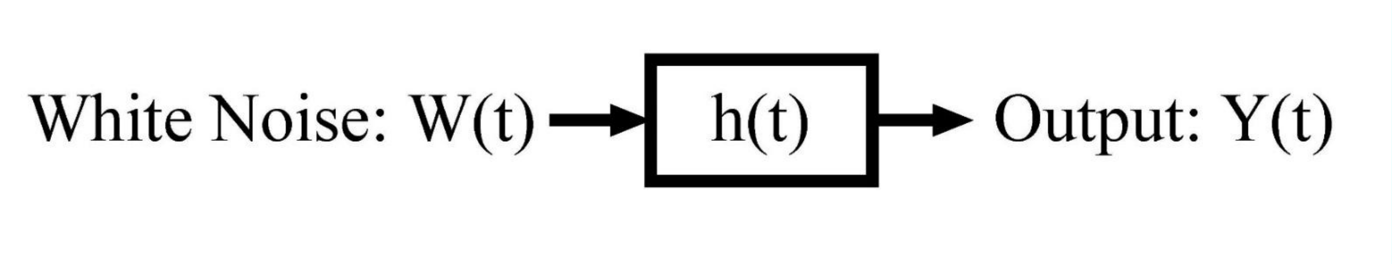
3. White Noise and Linear System

Consider white noise $\boldsymbol{W}(t)$ as input to linear system, whose impulse response is given by $h(t)$. Find the output spectrum.



Autocorrelation and power-spectrum of white noise $\boldsymbol{W}(t)$ are

$$R_{\boldsymbol{WW}}(\tau) = q\delta(\tau),\ S_{\boldsymbol{WW}}(\omega) = q$$

Therefore, the output spectrum $S_{YY}(\omega)$ is

$$S_{\boldsymbol{YY}}(\omega) = q|H(\omega)|^2$$

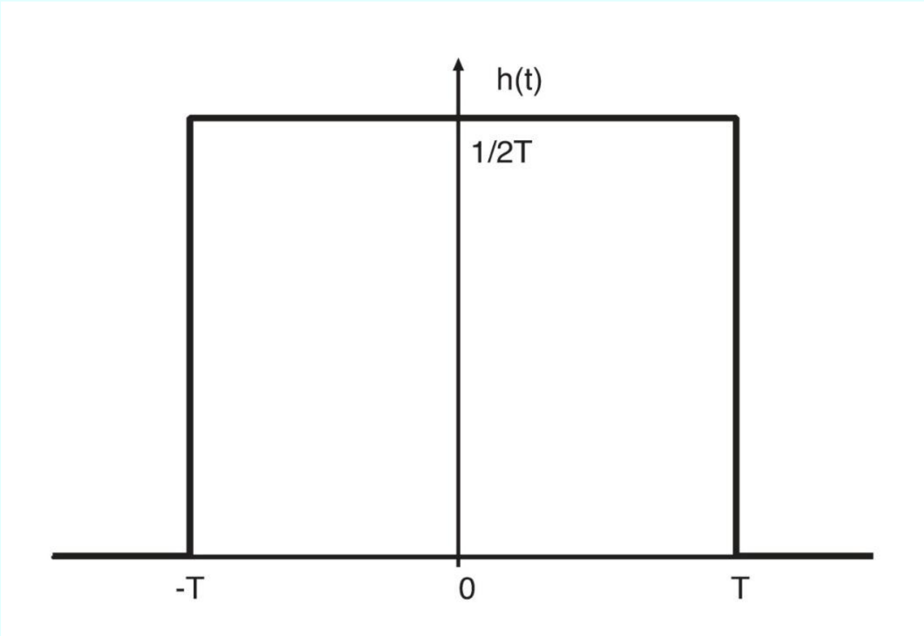
This implies that transfer function $H(\omega)$ can be obtained by injecting a white noise into linear system and then by computing the output spectrum. →***System Identification***

Example 1: Smoothing Filter

- Linear system that takes moving average of input $\boldsymbol{X}(t)$

$$\boldsymbol{Y}(t) = \frac{1}{2T}\int_{t-T}^{t+T} \boldsymbol{X}(\tau)d\tau$$

is called ***smoothing filter***. Compute power-spectrum of output $\boldsymbol{Y}(t)$.



Denoting impulse response of linear system by $h(t)$,

$$\boldsymbol{Y}(t) = \int_{-\infty}^{\infty} h(t-\tau)\boldsymbol{X}(\tau)d\tau = h(t) * \boldsymbol{X}(t)$$

$$H(\omega) = \int_{-T}^{T} \frac{1}{2T} e^{-i\omega t} dt = \frac{\sin T\omega}{T\omega}.$$

Therefore,

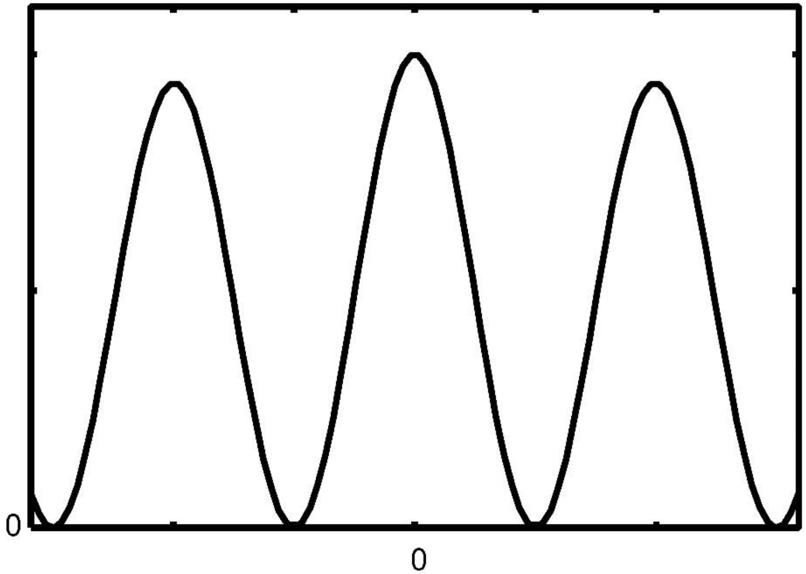
$$S_{\boldsymbol{YY}}(\omega) = S_{\boldsymbol{XX}}(\omega)|H(\omega)|^2 = S_{\boldsymbol{XX}}(\omega)\frac{\sin^2 T\omega}{T^2\omega^2}.$$

Smoothing filter functions as low-pas filter.

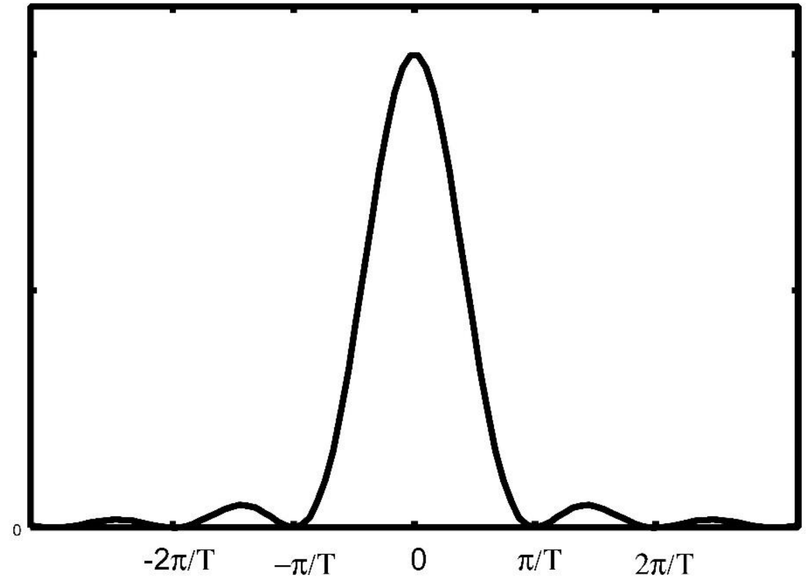
On the contrary, by subtracting the moving average from input $\boldsymbol{X}(t)$ as

$$\mathbf{Z}(t) = \boldsymbol{X}(t) - \frac{1}{2T}\int_{t-T}^{t+T} \boldsymbol{X}(\tau)d\tau$$

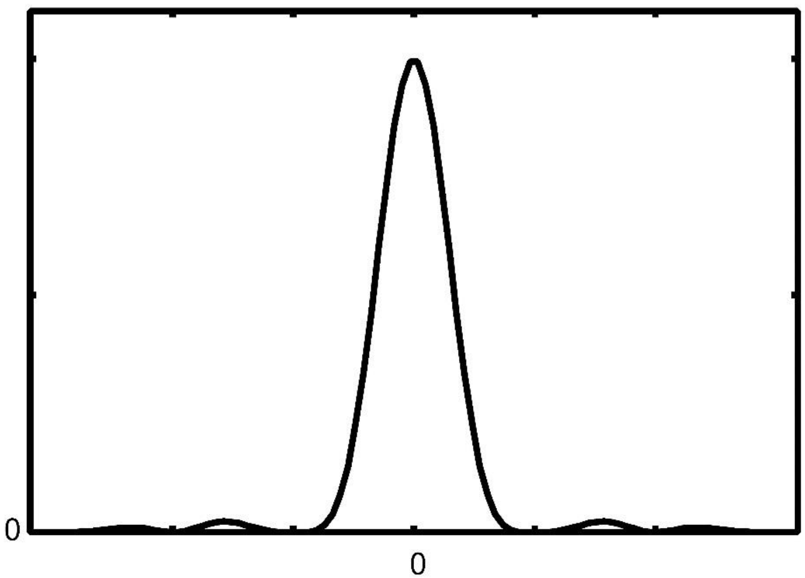
transfer function $H(\omega) = 1 - \frac{\sin T\omega}{T\omega}$ provides high-pass filter.



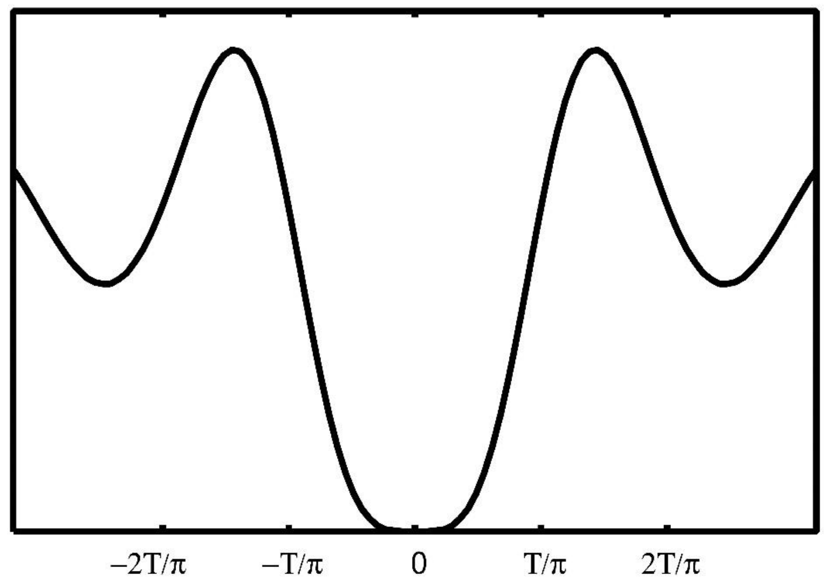
Input spectrum $S_{\boldsymbol{XX}}(\omega)$



Transfer function $|H(\omega)|^2$



Output spectrum $S_{\boldsymbol{YY}}(\omega)$



Example 2: Stochastic Resonance

- Input stationary process $\boldsymbol{X}(t)$ into a linear system having the following transfer function:

$$H(\omega) = \frac{1}{\omega^2 - 2\omega + 5}.$$

Suppose that input power is $E\{\boldsymbol{X}^2(t)\} = 10$. Find input spectrum $S_{\boldsymbol{XX}}(\omega)$ that maximizes power $E\{\boldsymbol{Y}^2(t)\}$ of output $\boldsymbol{Y}(t)$.

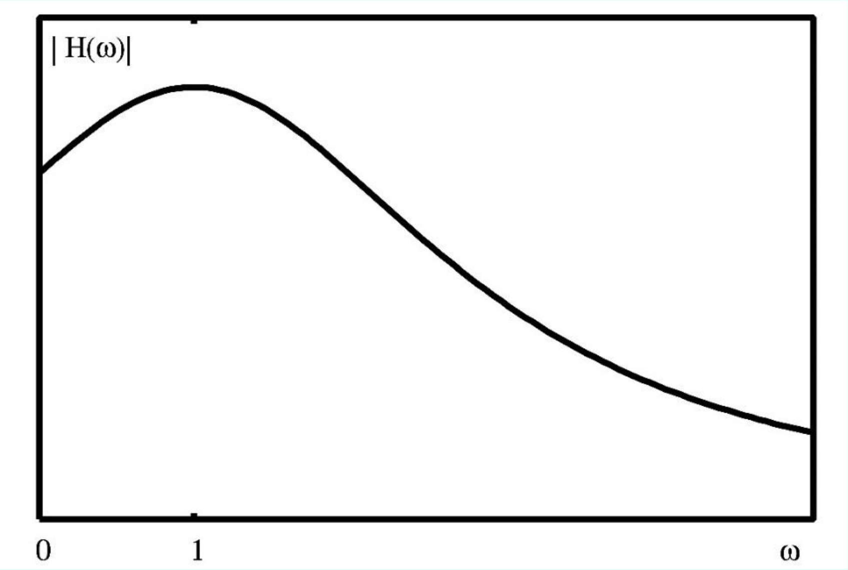
$$\begin{aligned} E\{\boldsymbol{Y}^2(t)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\boldsymbol{YY}}(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\boldsymbol{XX}}(\omega)|H(\omega)|^2 d\omega \\ &\leq |H(\omega_n)|^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\boldsymbol{XX}}(\omega)d\omega \\ &= |H(\omega_n)|^2\ E\{\boldsymbol{X}^2(t)\} \end{aligned}$$

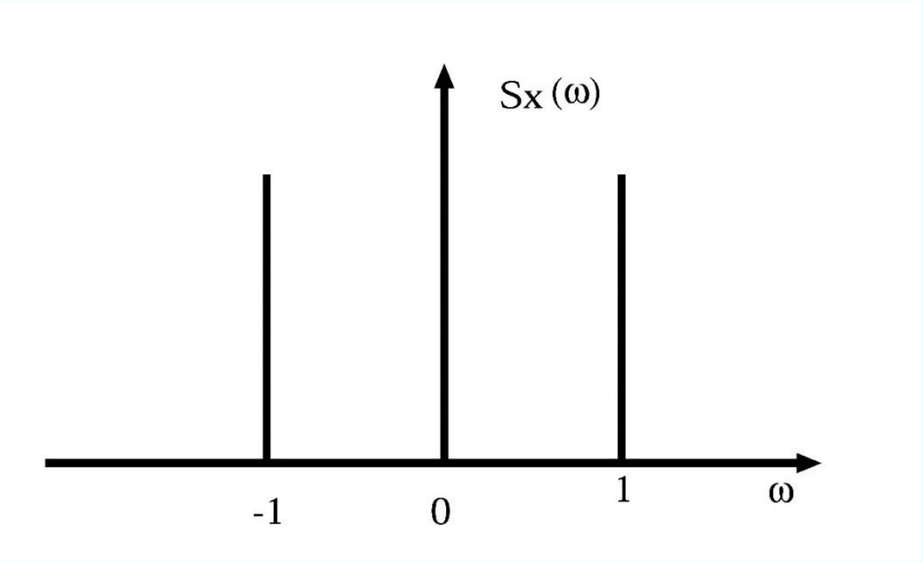
Here, $|H(\omega_n)|$ stands for maximal value of $|H(\omega)|$.

Because $H(\omega) = \frac{1}{(\omega-1)^2+4}$, $|H(\omega_m)| = \frac{1}{4}$ when $\omega_m = 1$.

Hence,

$$E\{\boldsymbol{Y}^2(t)\} \le \frac{10}{16}.$$ Equality holds for $R_{\boldsymbol{XX}}(\tau) = 10\cos\tau$.





Example 3: Differential Equation

For a system of differential equation with white noise input, find auto-correlation and power-spectrum of the system.

$$\boldsymbol{Y}''(t) + 5\boldsymbol{Y}'(t) + 6Y(t) = \boldsymbol{X}(t),$$

$$R_{\boldsymbol{XX}}(\tau) = 60\delta(\tau).$$

Solution: By substituting $t = t_2$, multiplying by $\boldsymbol{X}(t_1)$ from left hand side, and taking expectation,

$$R_{\boldsymbol{XY}''}(t_1, t_2) + 5R_{\boldsymbol{XY}'}(t_1, t_2) + 6R_{\boldsymbol{XY}}(t_1, t_2) = 60\delta(\tau)$$

By denoting $\tau = t_1 - t_2$ and taking Fourier transform

$$S_{\boldsymbol{XY}}(\omega) = \int_{-\infty}^{\infty} R_{\boldsymbol{XY}}(\tau)e^{-i\omega\tau}d\tau,$$

$$S_{XY'}(\omega) = \int_{-\infty}^{\infty} R_{XY'}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} (-1) \frac{dR_{XY}(\tau)}{d\tau} e^{-i\omega\tau} d\tau$$

$$= (-i\omega) \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau$$

$$= (-i\omega) S_{XY}(\omega)$$

$$S_{XY''}(\omega) = (-i\omega) S_{XY'}(\omega) = (-i\omega)^2 S_{XY}(\omega)$$

Therefore,

$$(-i\omega)^2 S_{XY}(\omega) + (-i\omega) 5 S_{XY}(\omega) + 6 S_{XY}(\omega) = 60$$

$$\rightarrow \; S_{XY}(\omega) = \frac{60}{-\omega^2 - 5i\omega + 6}$$

By taking the Fourier transform of

$$R_{\boldsymbol{Y''Y}}(\tau) + 5R_{\boldsymbol{Y'Y}}(\tau) + 6R_{\boldsymbol{YY}}(\tau) = R_{\boldsymbol{XY}}(\tau),$$

$$(i\omega)^2 S_{\boldsymbol{YY}}(\omega)(j\omega)5S_{\boldsymbol{YY}}(\omega) + 6S_{\boldsymbol{YY}}(\omega) = \frac{60}{-\omega^2 - 5j\omega + 6}.$$

Hence,

$$S_{\boldsymbol{YY}}(\omega) = \frac{60}{(-\omega^2+5i\omega+6)(-\omega^2-5i\omega+6)}$$

$$= 3\frac{2\times 2}{2^2+\omega^2} - 2\frac{2\times 3}{3^2+\omega^2}$$

$$R_{\boldsymbol{YY}}(\tau) = 3e^{-2|\tau|} - 2e^{-3|\tau|}$$

Application of Differential Equation

- $\boldsymbol{X}(t)$: Location of particle,
- m: Mass
- f: Coefficient of friction
- $c\boldsymbol{X}(t)$: External force

  (c is constant and constrained motion if $c \neq 0$)
- $\boldsymbol{F}(t)$: Collision force
- T: Temperature
- k: Boltzmann constant

$$m\boldsymbol{X}''(t) + f\boldsymbol{X}'(t) + c\boldsymbol{X}(t) = \boldsymbol{F}(t)$$

Suppose $F(t)$ has mean $E\{F(t)\} = 0$ and power-spectrum $S_F(\omega) = 2kTf$.

Solution:

The system function is

$$\frac{1}{|H(\omega)|^2} = (ms^2 + fs + c)(ms^2 - fs + c)\ |_{s=i\omega}$$
$$= (c - m\omega^2)^2 + f^2\omega^2.$$

Therefore,

$$S_{\boldsymbol{X}}(\omega) = S_{\boldsymbol{F}}(\omega)|H(\omega)|^2$$
$$= \frac{2kTf}{(c - m\omega^2)^2 + f^2\omega^2}.$$

If equation $ms^2 + fs + c = 0$ has complex conjugate

$$s_{1,2} = -\alpha \pm i\beta,$$

$$\alpha = \frac{f}{2m},\;\; \alpha^2 + \beta^2 = \frac{c}{m},$$

autocorrelation of $\boldsymbol{X}(\mathrm{t})$ is

$$R_{\boldsymbol{X}}(\tau) = \frac{kT}{c} e^{-\alpha|\tau|}(\cos\beta\tau + \frac{\beta}{\alpha}\sin\beta|\tau|).$$

Since $\boldsymbol{X}(t)$ is a normal process with mean 0 and variance $R_{\boldsymbol{X}}(0) = \frac{kT}{C}$ for fixed t, its density is

$$f_{\boldsymbol{X}}(x) = \sqrt{\frac{c}{2\pi kT}} e^{-\frac{cx^2}{2kT}}$$

In case of Free Motion ($c = 0$):

Denoting $\boldsymbol{V}(t) = \boldsymbol{X}'(t)$, $m\boldsymbol{V}'(t) + f\boldsymbol{V}(t) = \boldsymbol{F}(t)$.

The system function is

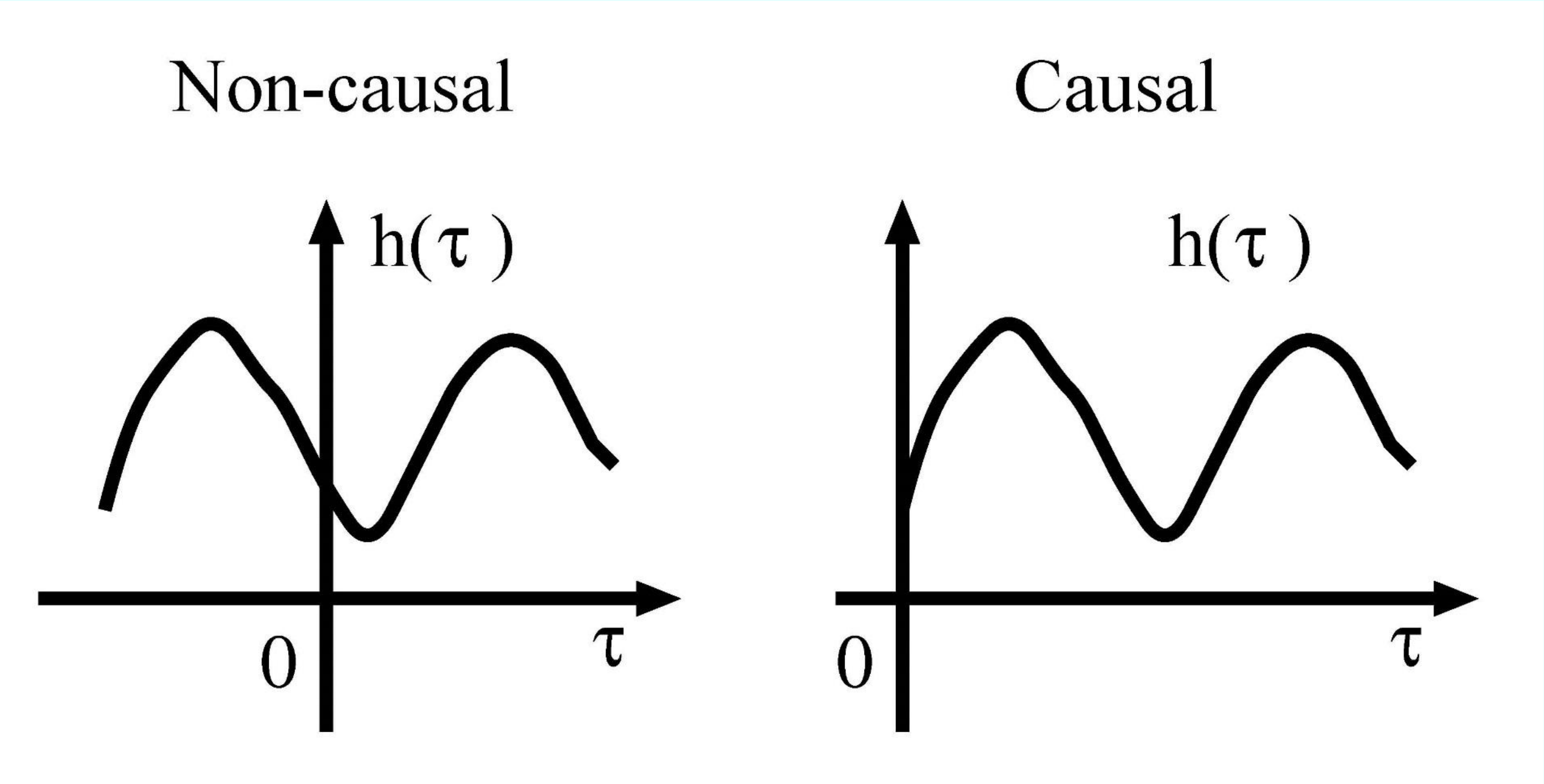
$$\frac{1}{|H(\omega)|^2} = (ms+f)(ms-f)\,|_{s=j\omega} = m^2\omega^2 + f^2.$$

Therefore, $S_{\boldsymbol{V}}(\omega) = S_{\boldsymbol{F}}(\omega)|H(\omega)|^2 = \frac{2kTf}{m^2\omega^2+f^2}$,

$$R_{\boldsymbol{V}}(\tau) = \frac{kT}{m} e^{-\frac{f|\tau|}{m}}.$$

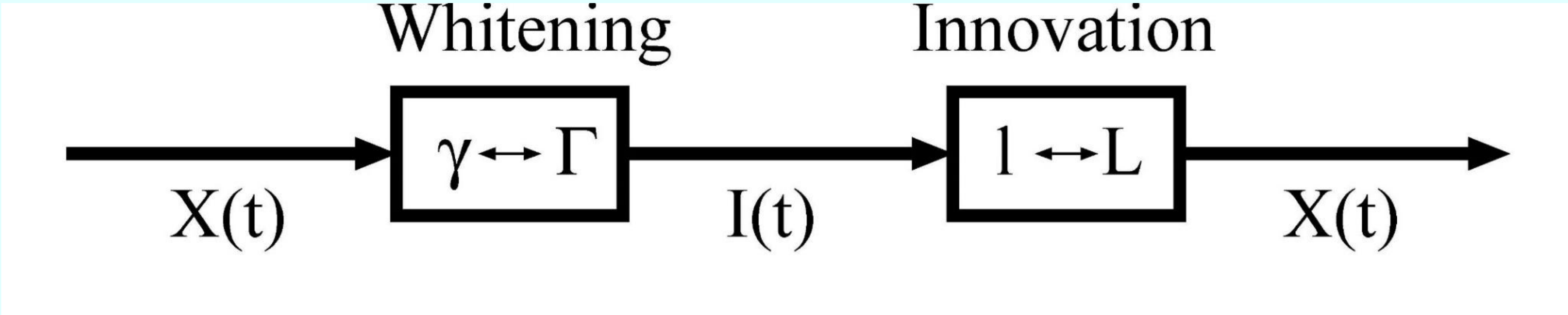
Since $\boldsymbol{V}(t)$ is a normal process with mean 0 and variance $R_{\boldsymbol{V}}(0) = \frac{kT}{m}$, its density is $f_{\boldsymbol{V}}(v) = \sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{mv^2}{2kT}}$

4. Causality



- Non-causal: For $\tau < 0$, $h(\tau) \neq 0$.
- Causal: For $\tau < 0$, $h(\tau) = 0$.

5. Factorization and Innovation



For stationary process $\boldsymbol{X}(t)$, consider system $\Gamma(s)$ that satisfies

$$\boldsymbol{I}(t) = \int_0^\infty \gamma(\alpha)\boldsymbol{X}(t-\alpha)d\alpha$$

$$R_{\boldsymbol{II}}(\tau) = E\{\boldsymbol{I}(t+\tau)\boldsymbol{I}(t)\} = \delta(\tau)$$

Since $L(s) = 1/\Gamma(s)$ is stable and causal,

$$\boldsymbol{X}(t) = \int_0^\infty l(\alpha)\boldsymbol{I}(t-\alpha)d\alpha.$$

$\boldsymbol{X}(t)$ is called ***regular***.

- $\boldsymbol{\Gamma}(s)$: Whitening filter
- $\boldsymbol{I}(t)$: Innovations of $X(t)$
- $\boldsymbol{L}(s)$: Innovation filter

Note Laplace transform:

$$\boldsymbol{\Gamma}(s) = \int_0^{\infty} \gamma(\tau) e^{-s\tau} d\tau$$

$$\boldsymbol{L}(s) = \int_0^{\infty} l(\tau) e^{-s\tau} d\tau$$

Necessary and sufficient condition of regular process

- Denoting power-spectrum of $\boldsymbol{X}(t)$ by $S_{\boldsymbol{XX}}(s)$ ($s = i\omega$),

$$
\begin{aligned}
S_{\boldsymbol{XX}}(i\omega) = S_{\boldsymbol{II}}(i\omega)|L(i\omega)|^2 \\
&= S_{\boldsymbol{II}}(i\omega)L(i\omega)L^*(i\omega) \\
&= S_{\boldsymbol{II}}(i\omega)L(i\omega)L(-i\omega).
\end{aligned}
$$

Since power-spectrum of $\boldsymbol{I}(t)$ is $S_{\boldsymbol{II}}(i\omega) = 1$, substitution of $s = i\omega$ yields $S_{\boldsymbol{XX}}(s) = L(s)L(-s)$.

On the other hand, if power-spectrum of $\boldsymbol{X}(t)$ can be written as the above product, power spectrum of $\boldsymbol{I}(t)$ is

$$
\begin{aligned}
S_{\boldsymbol{II}}(s) &= S_{\boldsymbol{XX}}(s)\Gamma(s)\Gamma(-s) \\
&= L(s)L(-s)\Gamma(s)\Gamma(-s) = 1.
\end{aligned}
$$

Hence, necessary and sufficient condition for $\boldsymbol{X}(t)$ to be regular is $S_{\boldsymbol{XX}}(s)$ can be decomposed as $S_{\boldsymbol{XX}}(s) = L(s)L(-s)$.

Paley-Wiener Condition

Necessary and sufficient condition for $\boldsymbol{X}(t)$ to be regular is

$$\int_{-\infty}^{\infty} \frac{\ln S_{\boldsymbol{XX}}(\omega)}{1+\omega^2} d\omega < \infty$$

Rational Spectra

Any positive rational spectrum $S(\omega) = \frac{A(\omega)}{B(\omega)}$ (where $A(\omega)$ and $B(\omega)$ are polynomials of ω) satisfies the Paley-Wiener condition as

$$\int_{-\infty}^{\infty} \frac{\ln S_{\boldsymbol{XX}}(\omega)}{1+\omega^2} d\omega = \int_{-\infty}^{\infty} \frac{\ln A(\omega) - \ln B(\omega)}{1+\omega^2} d\omega < \infty$$

Corresponding process $\boldsymbol{X}(t)$ is regular.

- In case $\boldsymbol{X}(t)$ is real and has rational spectrum

Since $S_{\boldsymbol{XX}}(-\omega) = S_{\boldsymbol{XX}}(\omega)$, $S_{\boldsymbol{XX}}(\omega) = A(\omega^2)/B(\omega^2)$.
Substitution of $s = j\omega$ yields

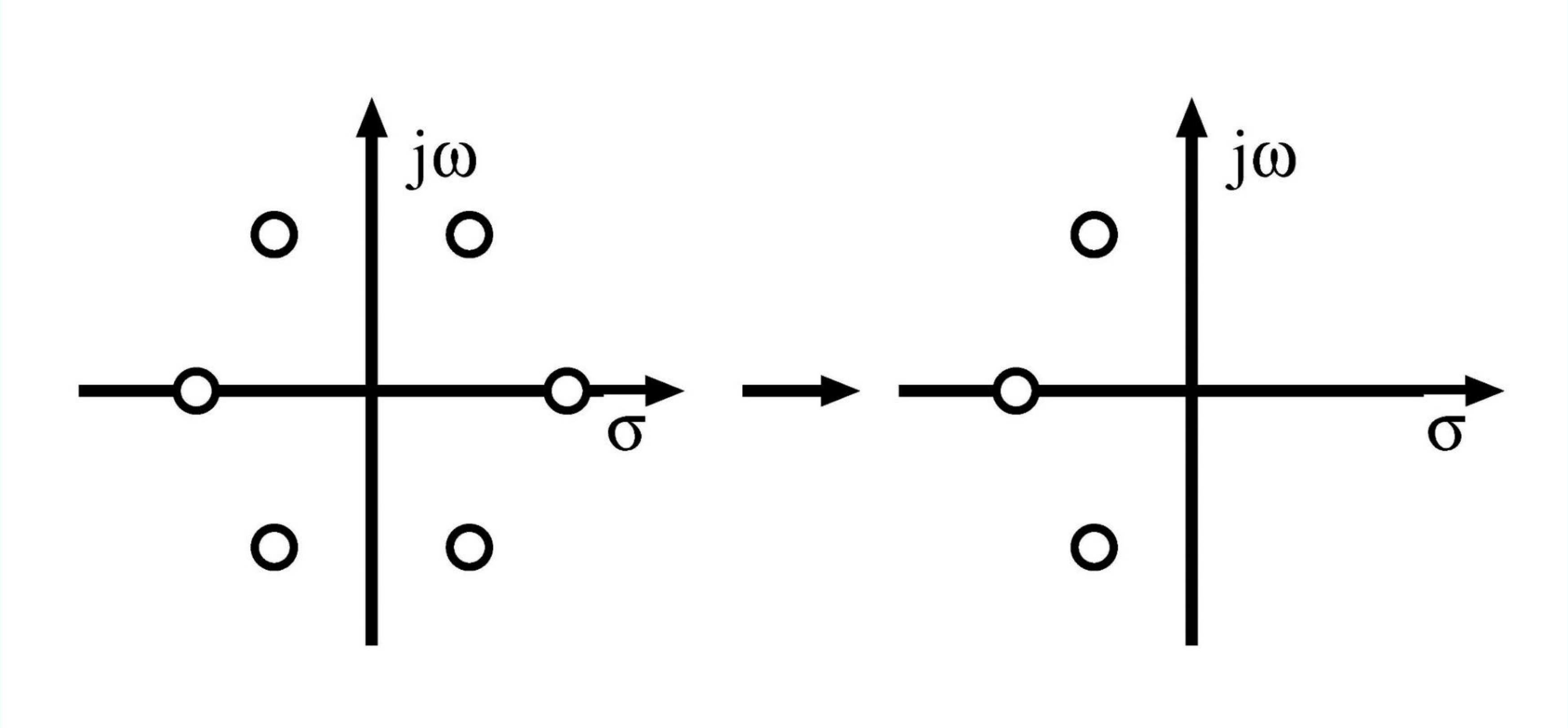
$$S_{\boldsymbol{XX}}(s) = A(-s^2)/B(-s^2).$$

Since $A(-s^2)$ and $B(-s^2)$ have real coefficients, their roots s_i are real or complex conjugate. If s_i is pole or zero, $-s_i$ is also pole or zero.

All roots (pole or zero) can be separated into two groups: negative real part ($\Re s_i < 0$) and positive real part ($\Re s_i > 0$). $L(s)$ can be constructed using roots with negative real part as

$$S_{\boldsymbol{XX}}(s) = \frac{N(s)N(-s)}{D(s)D(-s)}, \;\; L(s) = \frac{N(s)}{D(s)}.$$

If $L(s)$ is analytic in space of negative real part ($\Re s_i < 0$), we say "$L(s)$ has ***minimum-phase property***."



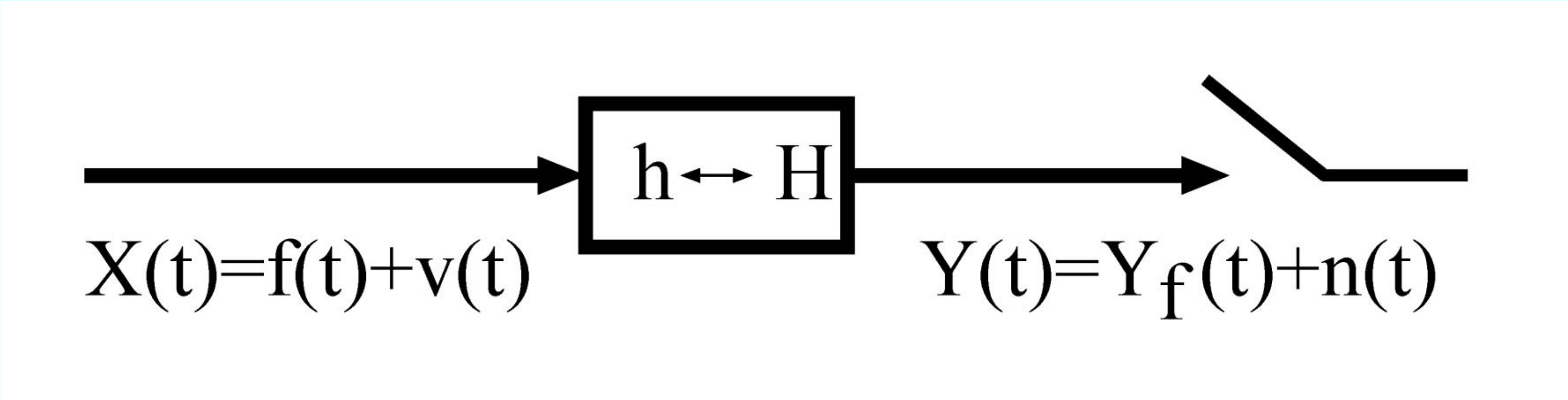
Matched Filter

Consider deterministic signal $f(t)$ with additive noise $\boldsymbol{v}(t)$:

$$\boldsymbol{X}(t) = f(t) + \boldsymbol{v}(t)$$

We detect whether signal $f(t)$ is included in observation data $\boldsymbol{X}(t)$. The function $f(t)$ is known; $\boldsymbol{v}(t)$ is stationary noise and its power-spectrum $S_{\boldsymbol{vv}}(\Omega)$ is known.

(Example: Detection of reflection wave from radar)



Consider $\boldsymbol{X}(t)$ as input to system $H(\omega)$. Denoting the output by $\boldsymbol{Y}(t)$,

$$\boldsymbol{Y}(t) = Y_f(t) + \boldsymbol{n}(t),$$

$$Y_f(t) = f(t) * h(t),$$

$$\boldsymbol{n}(t) = \boldsymbol{v}(t) * h(t).$$

Signal-to-noise ratio (SNR) r_0 is given by

$$r_0 = \frac{\left|Y_f(t)\right|^2}{E\{n^2(t)\}} = \frac{\left|Y_f(t)\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{nn}(\omega)d\omega}$$

$$= \frac{\left|Y_f(t)\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{vv}(\omega)|H(\omega)|^2 d\omega}$$

Design filter $H(\omega)$ in such a way that SNR is maximized.

Case of white noise

- White noise $\boldsymbol{v}(t)$

Because of white noise, $S_{\boldsymbol{vv}}(\omega) = S_0$.

By inverse Fourier transform, $y_f(t)$ can be written as

$$y_f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{i\omega t}d\omega.$$

From Cauchy-Schwarz inequality,

$$r_0 = \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{i\omega t}d\omega\right|^2}{\frac{S_0}{2\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega} \leq \frac{\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega \int_{-\infty}^{\infty}|H(\omega)|^2 d\omega}{2\pi S_0 \int_{-\infty}^{\infty}|H(\omega)|^2 d\omega}$$

The equality holds in the following case:

$$H(\omega) = F^*(\omega)e^{-i\omega t},$$

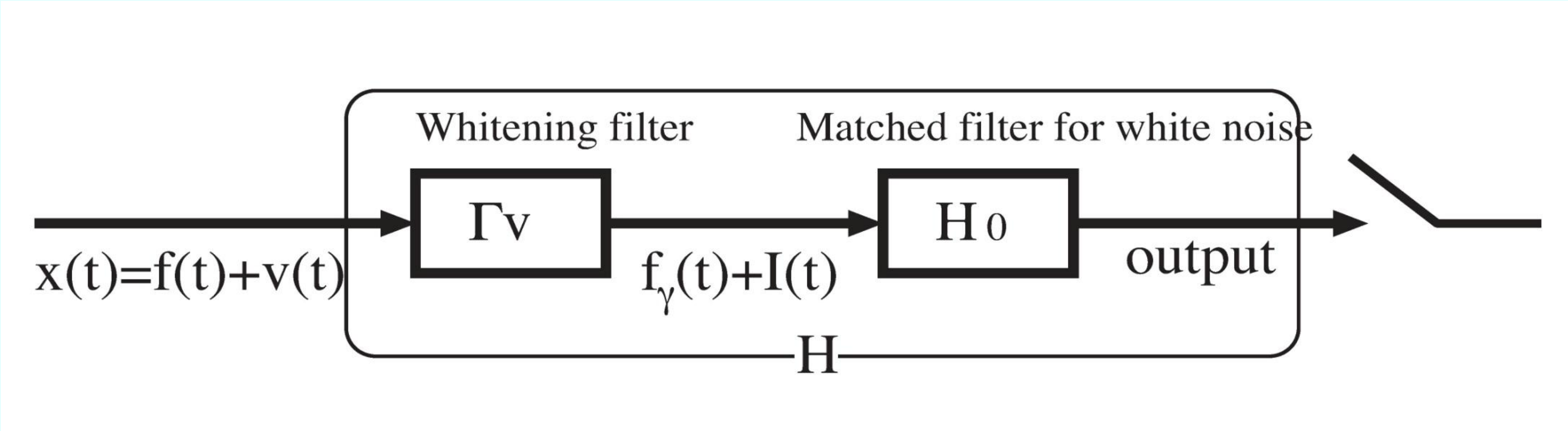
$$h(\tau) = f(t - \tau).$$

$H(\omega)$ that maximizes cross-correlation between $x(t) = f(t) + \boldsymbol{v}(t)$ and $kf(t)$ is called ***Matched Filter***.

Remark:

$h(\tau) = f(t - \tau)$ is not necessarily causal. In that case, optimal causal filter is given by $h(\tau) = f(t - \tau)U(\tau)$ (where $U(\tau) = 1\ (\tau \geq 0); 0\ (\tau < 0)$).

Case of colored noise



- Colored noise: $\boldsymbol{v}(t)$

Consider a whitening filter $\Gamma_v(j\omega)$ for $\boldsymbol{v}(t)$. If we input $\boldsymbol{X}(t) = f(t) + \boldsymbol{v}(t)$ into $\Gamma_v(j\omega)$, the output is given by

$$\boldsymbol{Y}(t) = \big(f(t) + \boldsymbol{v}(t)\big) * \gamma_v(t)$$

$$= f(t) * \gamma_v(t) + \boldsymbol{v}(t) * \gamma_v(t) = f_\gamma(t) + \boldsymbol{I}(t)$$

The output $\boldsymbol{Y}(t)$ is a transformed deterministic signal $f_\gamma(t) = f(t) * \gamma_v(t)$ with additive white noise $\boldsymbol{I}(t)$.

Using the results obtained with white noise, the optimal filter is given by

$$h_0(\tau) = f_\gamma(t-\tau),$$

$$H_0(\omega) = F_\gamma^*(\omega)e^{-i\omega t} = F^*(\omega)\Gamma_v^*(j\omega)e^{-i\omega t}.$$

Finally, the total filter is obtained as

$$H(\omega) = H_0(\omega)\Gamma_v\ (i\omega) = F^*(\omega)\Gamma_v^*(i\omega)\Gamma_v\ (i\omega)e^{-i\omega t}$$

$$= F^*(\omega)\left|\Gamma_v\ (i\omega)\right|^2 e^{-i\omega t}$$

$$= \frac{F^*(\omega)e^{-i\omega t}}{S_{vv}(\omega)}.$$