[I225] Statistical Signal Processing(E) Office Hour 4

1. There are three machines, A, B, and C, that manufacture a certain product. Machines A, B, and C produce 20%, 30%, and 50% of the total products, respectively. It is known from experience that 5%, 4%, and 2% of the products from machines A, B, and C, respectively, are defective.

a) What is the probability that a randomly selected product from the total production is defective?

b) Given that a product is found to be defective, what is the probability that it was produced by machine A, B, or C?

2. In a lottery, 6 numbers are drawn from a pool of 33 red balls numbered from 1 to 33. If all selected numbers match the drawn numbers, the player wins the second prize.

Suppose the probability of winning the second prize with a single ticket is

$$p = \frac{1}{\binom{33}{6}} \approx 9.0288 \times 10^{-7}$$

Assume the following:

- The lottery is held 3 times per week,
- The player buys 10 tickets per draw,
- The player keeps buying tickets for 5 years (52 weeks per year).

Question: What is the probability that the player wins the second prize at least once during the 5 years?

3. Let X be a random variable uniformly distributed on the interval $(0,1)$, and let Y be a random variable uniformly distributed on $(0, X)$. Find the probability density function (PDF) of Y, and compute the conditional cumulative distribution function $F_{X|Y}(0.5|0.25)$.

4. A call center records the number of calls received during 5 different one-hour intervals:
Data: [3, 2, 4, 3, 2]

Assume the number of calls received in one hour follows a Poisson distribution with rate λ (calls per hour), and each hour is independent.
Answer the following:

a) Write the likelihood function for this data given the parameter λ

b) Write the log-likelihood function.

c) Find the maximum likelihood estimate $\hat{\lambda}$ for the rate of calls per hour.

5. A researcher is studying the lifetimes of batteries used in remote sensors. Suppose the lifetimes are modeled as a vector

$$L = [L[0], L[1], \ldots, L(N-1)]^T,$$

where each observation is an independent and identically distributed sample from an exponential distribution with unknown parameter λ (the rate parameter):

$$L[n] \sim \mathrm{Exponential}(\lambda), \mathrm{n} = 0, \ldots, N-1$$

Find the maximum likelihood estimator (MLE) $\hat{\lambda}$ for λ.