[I225] Statistical Signal Processing(E) Office Hour 1

1. There are four boxes (Box 1 to Box 4), and each box contains a certain number of lottery tickets as described below:

- Box 1: 2000 tickets (5% of them are winning tickets)
- Box 2: 500 tickets (40% of them are winning tickets)
- Box 3: 1000 tickets (10% of them are winning tickets)
- Box 4: 1000 tickets (10% of them are winning tickets)

Now, one of the four boxes is chosen at random, and then one ticket is drawn at random from the selected box.

(a) What is the probability of drawing a winning ticket?

(b) Given that the ticket drawn is a winning ticket, what is the probability that it was drawn from Box 2?

Answer:

Let's define the events:

- B_i: The event that Box i was chosen, for $i = 1, 2, 3, 4$
- A: The event that the ticket drawn is a winning ticket

Since each box is equally likely to be chosen:

$$P(B_1) = P(B_2) = P(B_3) = P(B_4) = \frac{1}{4}$$

Conditional probabilities of drawing a winning ticket from each box:

$$P(A|B_1) = 0.05$$

$$P(A|B_2) = 0.40$$

$$P(A|B_3) = 0.10$$

$$P(A|B_4) = 0.10$$

(a) Total Probability of Drawing a Winning Ticket

Using the law of total probability:

$$P(A) = \Sigma\, P(B_i) * P(A|B_i)$$

$$= \frac{1}{4} * (0.05 + 0.40 + 0.10 + 0.10)$$

$$= \frac{1}{4} * 0.65$$

$$= 0.1625$$

The probability of drawing a winning ticket is 0.1625 (or 16.25%).

(b)Using Bayes' Theorem:

$$P(B2|A) = (P(B_2) * P(A|B_2)) / P(A)$$

$$= (1/4 * 0.40) / 0.1625$$

$$= 0.10 / 0.1625 \approx 0.615$$

The probability that the winning ticket came from Box 2 is approximately 0.615 (or 61.5%).

2. A box contains 12 new table tennis balls. For each match, 3 balls are randomly selected without replacement, used, and then returned to the box after the match. Let a ball be considered "used" once it has been drawn at least once.

(1) Let X be the number of new (unused) balls drawn in the second match. Find the probability distribution of X.
(2) Suppose that all 3 balls drawn in the third match are new (i.e., have not been drawn in either of the previous two matches). Given this condition, what is the probability that all 3 balls drawn in the second match were also new?

Answer:

Let's define the events:

- A_k: The event that exactly k of the 3 balls drawn in the second match were new, for k = 0,1,2,3
- B: The event that all 3 balls drawn in the third match were new.

Since each draw is random and the 3 used balls from the first match are placed back into the box, the second draw is from the full 12-ball set. However, only 9 of them are still new.

(1) Probability Distribution of X (Number of New Balls in Second Match)

To compute the probability that X = k balls in the second match were new (i.e., not used in the first match), we calculate:

$$P(X = k) = \frac{\binom{9}{k} \cdot \binom{3}{3-k}}{\binom{12}{3}}, \text{ for } k = 0,1,2,3$$

This expression counts the number of ways to choose k new balls from 9, and 3−k used balls from 3, over all ways to choose 3 balls from 12.

The probabilities are as follows:

$$P(X = 0) = \frac{\binom{9}{0} \cdot \binom{3}{3}}{\binom{12}{3}} = \frac{1 \cdot 1}{220} = \frac{1}{220}$$

$$P(X = 1) = \frac{\binom{9}{1} \cdot \binom{3}{2}}{\binom{12}{3}} = \frac{9 \cdot 3}{220} = \frac{27}{220}$$

$$P(X = 2) = \frac{\binom{9}{3} \cdot \binom{3}{1}}{\binom{12}{3}} = \frac{36 \cdot 3}{220} = \frac{108}{220}$$

$$P(X = 3) = \frac{\binom{9}{3} \cdot \binom{3}{0}}{\binom{12}{3}} = \frac{84 \cdot 1}{220} = \frac{84}{220}$$

The total probability sums to 1, as expected.

(2) Using Bayes' Theorem

We now calculate the conditional probability that all 3 balls in the second match were new, given that all 3 balls in the third match were new:

$$P(X = 3|B) = \frac{P(B|X = 3) \cdot P(X = 3)}{P(B)}$$

$P(B|X = 3) = \frac{\binom{6}{3}}{\binom{12}{3}} = \frac{20}{220}$, since 6 balls remain unused after two matches.

$P(X = 3) = \frac{84}{220}$, from part (a).

$P(B)$ is computed using the law of total probability:

$$P(B) = \sum_{k=0}^{3} P(X = k) \cdot P(B|X = k)$$

$$= \frac{1}{220} \cdot \frac{\binom{9}{3}}{220} + \frac{27}{220} \cdot \frac{\binom{8}{3}}{220} + \frac{108}{220} \cdot \frac{\binom{7}{3}}{220} + \frac{84}{220} \cdot \frac{\binom{6}{3}}{220}$$

$$\approx 0.148$$

Thus:

$$P(X = 3|B) = \frac{P(B|X = 3) \cdot P(X = 3)}{P(B)} \approx 0.234$$

3. A point (X, Y) is randomly and uniformly selected from the unit square region where $0 \leq X \leq 1$ and $0 \leq Y \leq 1$.

(1) Find the joint probability density function (PDF) of X and Y.
(2) Find the probability that the point lies within the triangle $T = \{(x, y) \mid 0 \leq y \leq x \leq 1\}$.
(3) Find the conditional probability that $X > 0.5$ given that $Y < 0.5$.

Answer:

(1) Since the point is uniformly distributed over the unit square, the joint PDF is:

$$f(x,y) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(2) The probability that the point lies in triangle $T = \{(x,y): 0 \leq y \leq x \leq 1\}$ is:

$$P\big((X,Y) \in T\big) = \iint_T f(x,y)dxdy = \int_0^1 \int_0^x 1dxdy = \int_0^1 xdx = \frac{1}{2}$$

(3) Conditional probability:

$$P(X > 0.5|Y < 0.5) = \frac{P(X > 0.5 \cap Y < 0.5)}{P(Y < 0.5)}$$

Numerator: Area of rectangle $[0.5,1]\times[0,0.5]=0.5\times0.5=0.25$

Denominator: Area of $Y<0.5$ over the square $= 1\times0.5=0.5$

So,

$$P(X > 0.5|Y < 0.5) = \frac{0.25}{0.5} = 0.5$$

4. Two people agree to meet at the east gate of a park sometime between 9:00 AM and 10:00 AM. The one who arrives first agrees to wait for at most 20 minutes (i.e., 1/3 hour). If the other person has not arrived by then, the first person will leave. What is the probability that the two people successfully meet?

Answer:

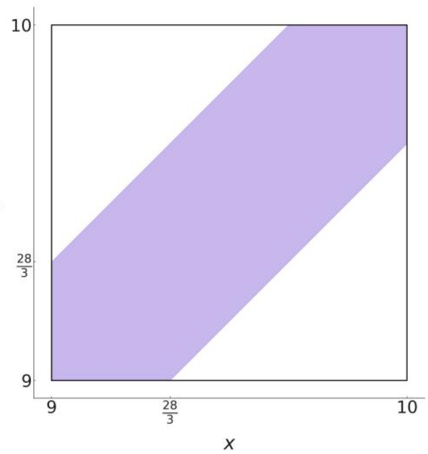
Let X and Y denote the arrival times of the two people.

Then X and Y are independent and uniformly distributed on the interval [0, 1].

- $X \sim U(9,\ 10)$ arrival time of person A
- $Y \sim U(9,\ 10)$ arrival time of person B
- X and Y are independent

They will meet if the absolute difference between their arrival times is at most 1/3 hour, i.e.:

$$|X - Y| \leq \frac{1}{3}$$



So the desired probability is:

$$P(|X - Y| \leq 1/3) = \iint_{|x-y|\leq 1/3} f(x,y)dxdy$$

$$= \iint_{|x-y|\leq 1/3} dxdy$$

$$= 1 - \iint_{|x-y|>1/3} dxdy$$

$$= 1 - 2 \times \frac{1}{2}\left(\frac{2}{3}\right)^2$$

$$= \frac{5}{9}$$

5. Let X be a random variable such that $X \sim N(\mu, \sigma^2)$. Derive the characteristic function of X, and use it to compute the mean and variance of X.

Answer:

The characteristic function $\varphi X(t)$ of a random variable X is defined as:

$\varphi X(t) = E[e^\wedge\{itX\}]$

where i is the imaginary unit and $t \in \mathbb{R}$.

For a normally distributed random variable $X \sim N(\mu, \sigma^2)$, the characteristic function is given by:

$$\varphi X(t) = exp(i\mu t - (1/2)\sigma^2 t^2)$$

The n^{th} moment of X can be obtained by differentiating the characteristic function:

$E[X^\wedge n] = (1/i^\wedge n) * d^n\varphi\_X(t)/dt^n$ evaluated at $t = 0$

Mean (first moment):

$E[X] = (1/i) * d\varphi\_X(t)/dt\ |\_\{t = 0\}$

$= (1/i) * d/dt\ [exp(i\mu t - (1/2)\sigma^2 t^2)]\ |\{t = 0\}$

$= \mu$

Variance (second central moment):

$Var(X) = E[X^2] - (E[X])^2$

$= (1/i^2) * d^2\varphi\_X(t)/dt^2\ |\{t = 0\} - \mu^2$

$= \sigma^2$