

# I225E Statistical Signal Processing

## 5. Stochastic Process and Systems II

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# Deterministic System

**Memoryless System**

$$Y(t) = g[X(t)]$$

**System with Memory**

**Time-Varying  
System**

**Time-Invariant  
System**

**Linear System**

$$Y(t) = L[X(t)]$$

**Linear Time-Invariant system**

$$Y(t) = X(t) * h(t)$$

# System with Memory

- **[Discrete time]** If the response of a system  $y[k]$  at  $k = k_0$  depends on the values of the input  $x[k]$  in the past or in the future of time  $k = k_0$ .
- **[Continuous time]** If the response of a system  $Y(t)$  at  $t = t_0$  depends on the values of the input  $X(t)$  in the past or in the future of time  $t = t_0$ .

# Examples

	Memoryless systems	Systems with memory
Discrete-time	$y[k] = 3x[k] + 7$	$y[k] = 3x[k - 5]$
	$y[k] = \sin(x[k]) + 5$	$y[k] = x[2k] + 5$
	$y[k] = x^2[k]$	$y[k] = x^2[k/2]$
Continuous-time	$y(t) = 3x(t) + 7$	$y(t) = 3x[t - 5]$
	$y(t) = \sin(x(t)) + 5$	$y(t) = x[2t] + 5$
	$y(t) = x^2(t)$	$y(t) = x^2[t/2]$

# Time-Invariant System

- **Definition:** A system where its characteristics and behavior **do not change** over time.
- Mathematically, if a system  $T$  produces an output  $Y(t) = T\{X(t)\}$  for an input  $X(t)$ , then for a time-invariant system, a delayed input  $X(t - t_0)$  will produce a delayed output:

$$Y(t - t_0) = T\{X(t - t_0)\}$$

# Time-Invariant System

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## ■ Example:

- Resistors, capacitors, and inductors in a linear circuit
- A mass-spring-damper system with constant parameters
- A simple amplifier with a fixed gain

# Time-Variant System

## ■ Definition

A system where its properties and behavior **do change** over time.

■ Mathematically, if a system  $T$  produces an output  $Y(t) = T\{X(t)\}$  for an input  $X(t)$ , then for a time-variant system, a delayed input  $X(t - t_0)$  will produce an output  $Y'(t)$  such that

$$Y'(t) \neq Y(t - t_0) \text{ in general.}$$

# Time-Variant System

## ■ Example

- A rocket's pitch response as it burns fuel
- Aging electronic components
- Discrete Wavelet Transform
- Amplitude modulation: The sinusoidal carrier signal introduces time-varying behavior. For example:

$$y(t) = x(t)\cos(\omega t)$$



# Recall: Linear Function

■ **Definition:** A function  $f(x)$  is linear if it is additive and homogeneous.

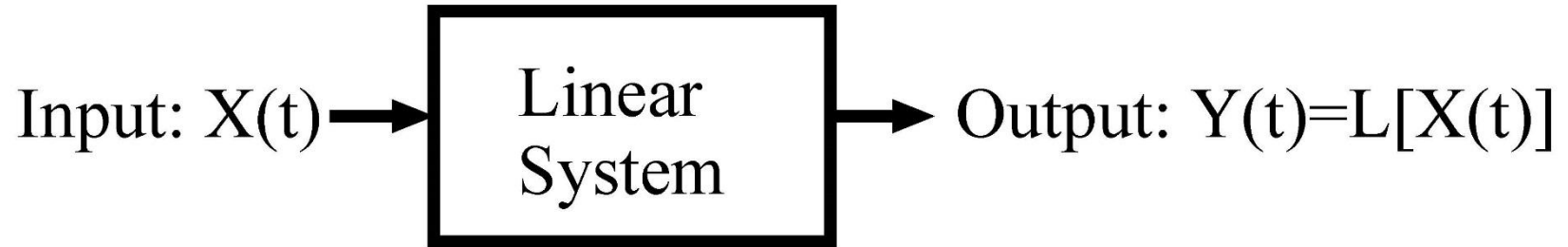
■ **Additive:**

$$f(x_1 + x_2) = f(x_1) + f(x_2) \text{ for all vectors } x_1, x_2$$

■ **Homogeneous:**

$$f(\alpha x) = \alpha f(x) \text{ for all vectors } x \text{ and scalar } \alpha$$

# Linear System



$$Y(t) = L[X(t)]$$

## ■ Properties of linear system

(b1) For any  $a_1, a_2, X_1(t), X_2(t)$ ,

$$L[a_1 X_1(t) + a_2 X_2(t)] = a_1 L[X_1(t)] + a_2 L[X_2(t)].$$

# Linear System

(b2) A system is called ***time-invariant***, if its response to  $X(t + c)$  equals  $Y(t + c)$ .

(b3) Output of linear time-invariant system is a convolution

$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(t - \alpha)h(\alpha)d\alpha$$

Here,  $h(t) = L[\delta(t)]$  represents impulse response.

# Recall: Convolution

## ■ Exercise

Imagine you are working with audio signals and want to create a simple echo effect. You have an original audio signal,  $x[n]$ , and you want to add a delayed and attenuated version of it to create the echo.

- a) Assume the echo effect produces a single echo that is delayed by 3 sample units and attenuated by a factor of 0.5. Determine the impulse response,  $h[n]$ , of the system that creates this echo.

# Recall: Convolution

## ■ Answer

Recall that the impulse response is the output of the system when the input is a unit impulse,  $\delta[n]$ , where:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

The echo is delayed by 3 sample units, meaning the impulse response will have a component at  $n = 3$ . The echo is also attenuated by a factor of 0.5.

The original sound passes through with no delay or attenuation, which corresponds to the impulse at  $n = 0$  with a gain of 1.

Therefore, the impulse response  $h[n]$  is:

$$h[n] = \delta[n] + 0.5\delta[n - 3]$$

# Recall: Convolution

## ■ Exercise

- b) Let the original audio signal be a simple sequence representing a short sound:  $x[n] = \{2, 1, -1, 0, 3\}$ . Calculate the output signal,  $y[n]$ , which is the convolution of the input signal  $x[n]$  with the impulse response  $h[n]$  you found in part a.

The linear convolution for discrete-time signals is:

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

# Recall: Convolution

## ■ Answer

$$y[n] = \{2, 1, -1, 1, 3.5, -0.5, 0, 1.5\}$$

# Recall: Convolution

## ■ Exercise

- c) What would the impulse response be if you wanted to create two distinct echoes: one delayed by 2 samples with an attenuation of 0.3, and another delayed by 5 samples with an attenuation of 0.2? How would the output of the convolution with  $x[n]$  look in this case?



# Recall: Convolution

## ■ Answer

$$h'[n] = \delta[n] + 0.3\delta[n - 2] + 0.2\delta[n - 5]$$

$$y'[n] = \{2, 1, -0.4, 0.3, 2.7, 0.4, 1.1, -0.2, 0, 0.6\}$$



(b4) Linear time-invariant system has following properties.

- i) If input  $X(t)$  is a normal process, then output  $Y(t)$  is also a normal process.
- ii) If input  $X(t)$  is strict sense stationary, then output  $Y(t)$  is also strict sense stationary.

## ■ Fundamental theorem

For any linear system,  $E\{L[X(t)]\} = L[E\{X(t)\}]$

This means that output average  $\eta_Y(t)$  equals to output of a system, to which input average  $\eta_X(t)$  is input.

$$\eta_Y(t) = L[\eta_X(t)]$$

(Intuitive proof)

Supposing that  $i$ th trial is  $Y(t, \omega_i) = L[X(t, \omega_i)]$

$$\begin{aligned} E\{Y(t)\} &\approx \frac{Y(t, \omega_1) + Y(t, \omega_2) + \cdots + Y(t, \omega_n)}{n} \\ &= \frac{L[X(t, \omega_1)] + L[X(t, \omega_2)] + \cdots + L[X(t, \omega_n)]}{n} \\ &= L \left[ \frac{X(t, \omega_1) + X(t, \omega_2) + \cdots + X(t, \omega_n)}{n} \right] \\ &\approx L[E\{X(t)\}] \end{aligned}$$

## ■ Exercise

- Consider a discrete-time linear system  $T\{\cdot\}$ . Let  $x_1[n]$  and  $x_2[n]$  be two input signals, and let  $y_1[n] = T\{x_1[n]\}$  and  $y_2[n] = T\{x_2[n]\}$  be their respective outputs.
- Now, suppose these input signals are random processes. Consider a new input signal  $x[n]$  formed by a linear combination of  $x_1[n]$  and  $x_2[n]$  :
$$x[n] = ax_1[n] + bx_2[n]$$
where  $a$  and  $b$  are constant scalars.

## ■ Exercise

- a) Determine the output  $y[n]$  of the linear system when the input is  $x[n]$
- b) Find the expected value of the output signal  $E\{y[n]\}$
- c) Verify the given statement:  $E\{T\{x[n]\}\} = T\{E\{x[n]\}\}$

(b5) Using the convolution of (b3), the theorem is described as

$$\begin{aligned} E\{Y(t)\} &= \int_{-\infty}^{\infty} E\{X(t - \alpha)\}h(\alpha)d\alpha \\ &= \int_{-\infty}^{\infty} \eta_X(t - \alpha)h(\alpha)d\alpha = \eta_X(t) * h(t) \end{aligned}$$

(b6) Setting  $\tilde{X}(t) = X(t) - \eta_X(t)$  and  $\tilde{Y}(t) = Y(t) - \eta_Y(t)$ ,

$$\tilde{Y}(t) = L[\tilde{X}(t)]$$

**[Proof]**

$$\begin{aligned} \tilde{Y}(t) &= Y(t) - \eta_Y(t) \\ &= L[X(t)] - L[\eta_X(t)] \\ &= L[X(t) - \eta_X(t)] \\ &= L[\tilde{X}(t)] \end{aligned}$$

(b7) With respect to input  $X(t) = f(t) + v(t)$  ( $f(t)$  is a deterministic signal;  $E\{v(t)\} = 0$ ), the output average is

$$\begin{aligned}\eta_Y(t) &= E\{Y(t)\} = E\{L[X(t)]\} = L[E\{X(t)\}] \\ &= L[E\{f(t) + v(t)\}] \\ &= L[E\{f(t)\} + E\{v(t)\}] \\ &= L[f(t)] = f(t) * h(t)\end{aligned}$$

## ■ Autocorrelation of output

Consider input  $X(t)$  and output  $Y(t)$  for a linear system. We represent output autocorrelation  $R_{YY}(t_1, t_2)$  in terms of input autocorrelation  $R_{XX}(t_1, t_2)$ .

## ■ Theorem

$$(A) R_{XY}(t_1, t_2) = L_{t_2}[R_{XX}(t_1, t_2)].$$

$$(B) R_{YY}(t_1, t_2) = L_{t_1}[R_{XY}(t_1, t_2)].$$

### [Proof]

Multiplying both side of  $Y(t) = L_t[X(t)]$  by  $X(t_1)$ ,

$$X(t_1)Y(t) = X(t_1)L_t[X(t)] = L_t[X(t_1)X(t)].$$

Taking expectation,

$$\begin{aligned} E\{X(t_1)Y(t)\} &= E\{L_t[X(t_1)X(t)]\} \\ &= L_t[E\{X(t_1)X(t)\}]. \end{aligned}$$

Substitution of  $t = t_2$  yields (A).

In a similar manner, multiplication of  $Y(t) = L_t[X(t)]$  by  $Y(t_2)$  and so on yields (B).



(c1) In the form of convolution,

$$\begin{aligned} \text{(a)} \quad R_{XY}(t_1, t_2) &= R_{XX}(t_1, t_2) * h(t_2) \\ &= \int_{-\infty}^{\infty} R_{XX}(t_1, t_2 - \alpha) h(\alpha) d\alpha \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad R_{YY}(t_1, t_2) &= R_{XY}(t_1, t_2) * h(t_1) \\ &= \int_{-\infty}^{\infty} R_{XY}(t_1 - \alpha, t_2) h(\alpha) d\alpha \end{aligned}$$

(c) Combining (a) and (b),

$$\begin{aligned} R_{YY}(t_1, t_2) &= \\ &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(t_1 - \alpha, t_2 - \beta) h(\alpha) h(\beta) d\alpha d\beta \end{aligned}$$

(c2) For cross-covariance, the following holds.

$$C_{XY}(t_1, t_2) = C_{XX}(t_1, t_2) * h(t_2)$$

$$C_{YY}(t_1, t_2) = C_{XY}(t_1, t_2) * h(t_1)$$

**[Proof]**


Denoting  $\tilde{X}(t) = X(t) - \eta_X(t)$ ,  $\tilde{Y}(t) = Y(t) - \eta_Y(t)$ ,

$$C_{XX}(t_1, t_2) = E\{\tilde{X}(t_1)\tilde{X}(t_2)\} = R_{\tilde{X}\tilde{X}}(t_1, t_2)$$

$$C_{XY}(t_1, t_2) = R_{\tilde{X}\tilde{Y}}(t_1, t_2),$$

$$C_{YY}(t_1, t_2) = R_{\tilde{Y}\tilde{Y}}(t_1, t_2).$$

From (b6),  $\tilde{Y}(t) = L[\tilde{X}(t)]$ . Hence, (a) and (b) can be applied to  $R_{\tilde{X}\tilde{X}}$ ,  $R_{\tilde{X}\tilde{Y}}$ , and  $R_{\tilde{Y}\tilde{Y}}$ .



(c3) Above results can be extended to the case the impulse response  $\mathbf{h}(t)$  is complex function.

In this case,

$$R_{XY}(t_1, t_2) = R_{XX}(t_1, t_2) * \mathbf{h}^*(t_2)$$

$$R_{YY}(t_1, t_2) = R_{XY}(t_1, t_2) * \mathbf{h}(t_1)$$

# White Noise

- **Definition:** A random signal containing all frequencies at equal intensity.
- **Key Characteristic:** It has a constant power spectral density.  $S_x(f) = N_0/2$  (for a double-sided spectrum) or  $S_x(f) = N_0$  (for a single-sided spectrum), where  $N_0$  is a constant representing the noise power per unit bandwidth.
- In the time domain, ideal white noise has an autocorrelation function that is a Dirac delta function at zero lag:

$$R_x(\tau) = E[x(t)x(t - \tau)] = \frac{N_0}{2} \delta(\tau)$$

# White Noise

## ■ Power Spectral Density of the Output

The Power Spectral Density of the output  $S_y(f)$  is related to the Power Spectral Density of the input  $S_x(f)$  and the magnitude squared of the system's frequency response  $|H(f)|^2$  by the following equation:

$$S_y(f) = |H(f)|^2 S_x(f)$$

If the input is white noise with  $S_x(f) = N_0/2$ , then the Power Spectral Density of the output becomes:

$$S_y(f) = |H(f)|^2 N_0/2$$

# White Noise

## # Define parameters

```
duration = 1 # seconds
sampling_rate = 1000 # Hz
num_samples = int(sampling_rate *
duration)
```

## # Generate Gaussian white noise

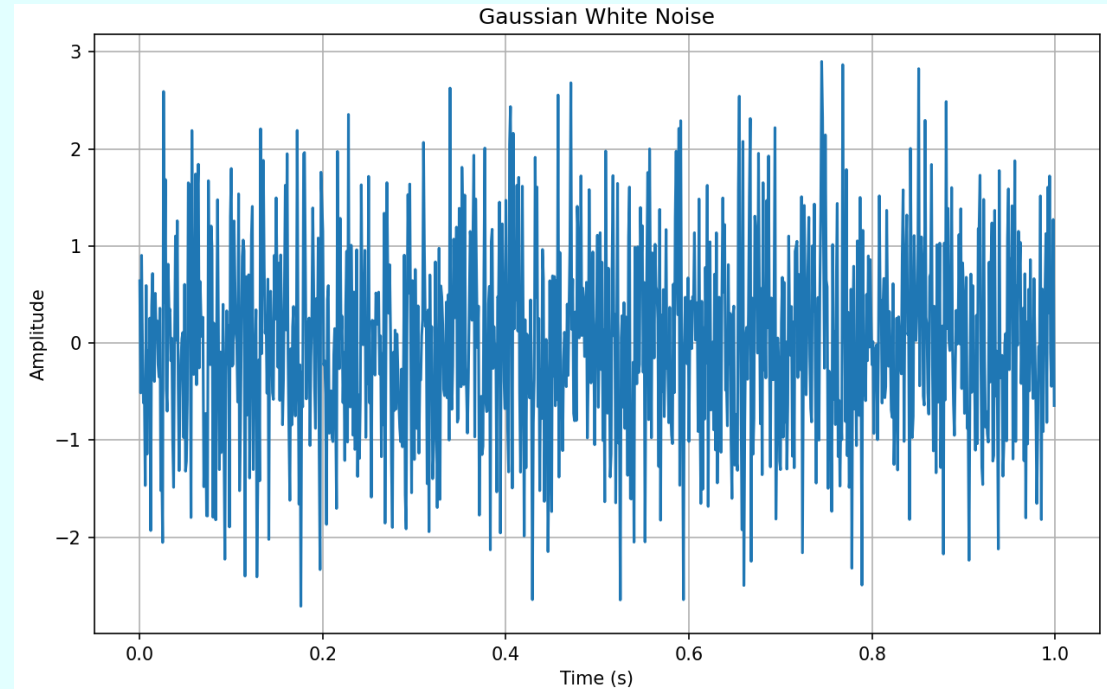
```
mean = 0
std_dev = 1
noise = np.random.normal(mean,
std_dev, num_samples)
```

## # Create time vector

```
time = np.linspace(0, duration,
num_samples, endpoint=False)
```

## # Plot the noise

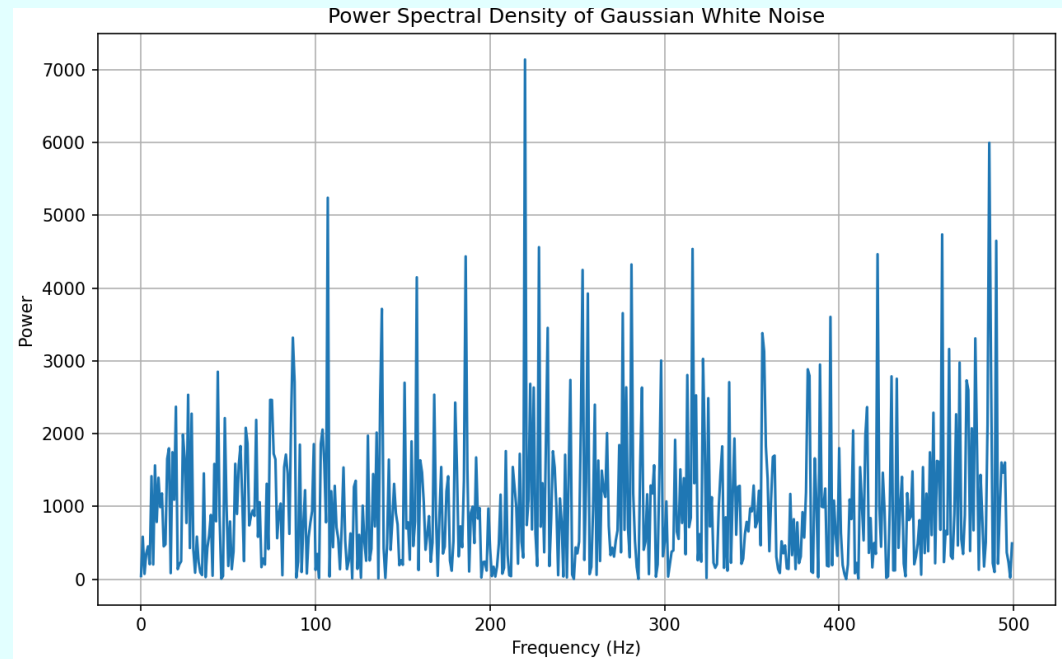
```
plt.figure(figsize=(10, 6))
plt.plot(time, noise)
plt.title('Gaussian White Noise')
plt.xlabel('Time (s)')
plt.ylabel('Amplitude')
plt.grid(True)
plt.show()
```



# White Noise

```
# Plot the power spectral density
(PSD) to see the 'white' nature
frequencies =
np.fft.fftfreq(num_samples, 1 /
sampling_rate)
power_spectrum =
np.abs(np.fft.fft(noise))**2

plt.figure(figsize=(10, 6))
plt.plot(frequencies[:num_samples//2],
power_spectrum[:num_samples//2])
plt.title('Power Spectral Density of
Gaussian White Noise')
plt.xlabel('Frequency (Hz)')
plt.ylabel('Power')
plt.grid(True)
plt.show()
```



# Response to white noise

## Theorem

Consider white noise, whose autocorrelation is given by  $R_{XX}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$ , as input to linear system. The output power is given by

$$E\{Y^2(t)\} = q(t) * h^2(t) = \int_{-\infty}^{\infty} q(t - \alpha)h^2(\alpha)d\alpha$$

## [Proof]

Substitution of  $R_{XX}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$  into convolution form of (A), (B) yields,

$$\begin{aligned} R_{XY}(t_1, t_2) &= q(t_1)\delta(t_1 - t_2) * h(t_2) = q(t_1)h(t_2 - t_1) \\ R_{YY}(t_1, t_2) &= \int_{-\infty}^{\infty} q(t_1 - \alpha)h[t_2 - (t_1 - \alpha)]h(\alpha)d\alpha. \end{aligned}$$

Substitution of  $t_1 = t_2 = t$  gives the output power.



(d1) If input  $X(t)$  is stationary white noise,  $q(t) = q$ .

Therefore,

$$E\{Y^2(t)\} = qE, \quad E = \int_{-\infty}^{\infty} h^2(\alpha) d\alpha$$

(d2) If correlation time of impulse response  $h(t)$  is shorter than that of  $q(t)$ ,

$$E\{Y^2(t)\} \approx q(t) \int_{-\infty}^{\infty} h^2(\alpha) d\alpha = Eq(t).$$

(d3) If  $R_{XX}(\tau) = q\delta(\tau)$  and  $X(t)$  is injected at  $t = 0$ .

$$q(t) = qU(t)$$

Therefore,

$$E\{Y^2(t)\} \approx q \int_{-\infty}^t h^2(\alpha) d\alpha$$

# Digression on Dirac's delta function

- Properties of delta function  $\delta(t)$ :

$$\int_{-\infty}^{+\infty} dt \delta(t) = 1.$$

$$\int_{-\infty}^{+\infty} dt \delta(t) f(t) = f(0).$$

- Delta function as a limit of infinitesimally narrow Gaussian:

$$\delta(t) = \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$$

- Delta function has a uniform (white) spectrum:

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{i\omega t}$$

# Example 1

Consider a linear system having impulse response

$$h(t) = e^{-ct}U(t)$$

Input a white noise, whose autocorrelation is given by  $R_{WW}(\tau) = q\delta(\tau)$ , at  $t = 0$ . Calculate autocorrelation  $R_{YY}(t_1, t_2)$  ( $0 < t_1 < t_2$ ) of the output  $Y(t)$ .

**Answer** Denoting input as  $X(t)$ ,  $X(t) = W(t)U(t)$ .

(I) Case of  $t_1 < 0$  or  $t_2 < 0$ : Since there is no input,

$$R_{XX}(t_1, t_2) = R_{XY}(t_1, t_2) = R_{YY}(t_1, t_2) = 0$$

(II) Case of  $0 < t_1 < t_2$ :

$$R_{XX}(t_1, t_2) = R_{WW}(t_1 - t_2) = q\delta(t_1 - t_2)$$

Therefore,

$$\begin{aligned} R_{XY}(t_1, t_2) &= \int_{-\infty}^{t_2} R_{XX}(t_1, t_2 - \alpha) h(\alpha) d\alpha \\ &= \int_{-\infty}^{t_2} q \delta(t_1 - t_2 + \alpha) e^{-c\alpha} U(\alpha) d\alpha \\ &= q e^{-c(t_2 - t_1)} \end{aligned}$$

$$\begin{aligned} R_{YY}(t_1, t_2) &= \int_{-\infty}^{t_1} R_{XY}(t_1 - \alpha, t_2) h(\alpha) d\alpha \\ &= \int_{-\infty}^{t_1} q e^{-c(t_2 - t_1 + \alpha)} e^{-c\alpha} U(\alpha) d\alpha \\ &= q e^{-c(t_2 - t_1)} \int_0^{t_1} e^{-2c\alpha} d\alpha \\ &= \frac{q}{2c} e^{-c(t_2 - t_1)} (1 - e^{-2ct_1}) \end{aligned}$$

$$E\{Y^2(t)\} = R_{YY}(t, t) = \frac{q}{2c} (1 - e^{-2ct}).$$

## Example 2: Differentiator

Consider the output properties (mean and autocorrelation) of differentiator:

$$Y(t) = L[X(t)] = X'(t).$$

■ **Mean** is,

$$\eta_Y(t) = L[\eta_X(t)] = \eta'_X(t)$$

■ **Autocorrelation** is

$$R_{XY}(t_1, t_2) = L_{t_2}[R_{XX}(t_1, t_2)] = \frac{\partial R_{XX}(t_1, t_2)}{\partial t_2}$$

$$R_{YY}(t_1, t_2) = L_{t_1}[R_{XY}(t_1, t_2)] = \frac{\partial R_{XY}(t_1, t_2)}{\partial t_1}$$

Hence

$$R_{YY}(t_1, t_2) = L_{t_1}[R_{XY}(t_1, t_2)] = \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2}$$

If  $X(t)$  is wide-sense stationary, its **mean**,  $\eta_X(t)$ , is a constant. Therefore,  $\eta_Y(t) = 0$ . Since **autocorrelation** is a function of  $\tau = t_1 - t_2$ ,

$$\frac{\partial R_{XY}(t_1, t_2)}{\partial t_2} = - \frac{dR_{XX}(\tau)}{d\tau}$$
$$\frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2} = - \frac{d^2 R_{XX}(\tau)}{d\tau^2}$$

Hence,

$$R_{XY}(\tau) = -R'_{XX}(\tau),$$
$$R_{YY}(\tau) = -R''_{XX}(\tau),$$

# Example 3: Differential Equation

Consider output properties (mean and autocorrelation) of the following differential equation driven by noise  $X(t)$ .

$$a_n Y^{(n)}(t) + a_{n-1} Y^{(n-1)}(t) + \cdots + a_0 Y(t) = X(t)$$

The initial condition is given by

$$Y(0) = Y^{(1)}(0) = \cdots = Y^{(n)}(0) = 0$$

## ■ Mean:

Taking the expectation  $E\{\cdot\}$  for both sides,

$$\begin{aligned} a_n E\{Y^{(n)}(t)\} + a_{n-1} E\{Y^{(n-1)}(t)\} + \cdots + a_0 E\{Y(t)\} \\ = E\{X(t)\} \end{aligned}$$

Since  $\frac{d}{dx}$  and  $E\{\cdot\}$  are commutative,  $E\{Y^{(k)}(t)\} = \eta_Y^{(k)}(t)$

$$a_n \eta_Y^{(n)}(t) + a_{n-1} \eta_Y^{(n-1)}(t) + \cdots + a_0 \eta_Y(t) = \eta_X(t)$$

Because of  $Y^{(k)}(0) = 0$ ,  $\eta_Y^{(k)}(0) = 0$ . The mean  $\eta_Y^{(k)}(t)$  is obtained by integrating the differential equation for the following initial condition:

$$\eta_Y(0) = \eta_Y^{(1)}(0) = \cdots = \eta_Y^{(n)}(0) = 0$$

## ■ Correlation:

By substituting  $t = t_2$  and multiplying by  $X(t_1)$ , the differential equation becomes

$$\begin{aligned} X(t_1) [a_n Y^{(n)}(t_2) + a_{n-1} Y^{(n-1)}(t_2) + \cdots + a_0 Y(t_2)] \\ = X(t_1) X(t_2) \end{aligned}$$



By taking the expectation and using the formula of

$$E\{X(t_1)Y^{(k)}(t_2)\} = \frac{\partial^k R_{XY}(t_1, t_2)}{\partial t_2^k},$$

$$a_n \frac{\partial^n R_{XY}(t_1, t_2)}{\partial t_2^n} + a_{n-1} \frac{\partial^{n-1} R_{XY}(t_1, t_2)}{\partial t_2^{n-1}} + \dots \\ + a_0 R_{XY}(t_1, t_2) = R_{XX}(t_1, t_2),$$

Since  $X(t_1)Y^{(k)}(0) = 0$ , initial condition is given by

$$R_{XY}(t_1, 0) = \frac{\partial R_{XY}(t_1, 0)}{\partial t_2} = \dots = \frac{\partial^n R_{XY}(t_1, 0)}{\partial t_2^n} = 0$$

Autocorrelations  $\frac{\partial^n R_{XY}(t_1, t_2)}{\partial t_2^n}$  are obtained by solving the differential equation for the above initial condition.

In a similar manner, by substituting  $t = t_1$  and multiplying by  $Y(t_2)$ ,

$$\begin{aligned} & [a_n Y^{(n)}(t_1) + a_{n-1} Y^{(n-1)}(t_1) + \cdots + a_0 Y(t_1)] Y(t_2) \\ & = X(t_1) Y(t_2) \end{aligned}$$

By taking the expectation and by using the formula

$$\begin{aligned} E\{Y^{(k)}(t_1) Y(t_2)\} &= \frac{\partial^k R_{YY}(t_1, t_2)}{\partial t_1^k}, \\ a_n \frac{\partial^n R_{YY}(t_1, t_2)}{\partial t_1^n} &+ a_{n-1} \frac{\partial^{n-1} R_{YY}(t_1, t_2)}{\partial t_1^{n-1}} + \cdots \\ &+ a_0 R_{YY}(t_1, t_2) = R_{XY}(t_1, t_2). \end{aligned}$$

Since  $Y(0)Y^{(k)}(t_2) = 0$ , initial condition is given by

$$R_{YY}(0, t_2) = \frac{\partial R_{YY}(0, t_2)}{\partial t_1} = \dots = \frac{\partial^n R_{YY}(0, t_2)}{\partial t_1^n} = 0$$

Autocorrelations  $\frac{\partial^n R_{YY}(t_1, t_2)}{\partial t_1^n}$  are obtained by solving the differential equation for the above initial condition.

## ■ Exercise

- Consider a discrete-time LTI system with an impulse response  $h[n] = \{1, -0.5\}$ , where the value at  $n = 0$  is the first element. The input to this system is a zero-mean white noise process  $w[n]$  with a variance of  $\sigma_w^2$ . Recall that for a zero-mean white noise process:
  - $E[w[n]] = 0$  for all  $n$ .
  - The autocorrelation function  $R_{ww}[k] = E[w[n]w[n-k]] = \sigma_w^2 \delta[k]$ , where  $\delta[k]$  is the Kronecker delta function ( $\delta[0] = 1$ ,  $\delta[k] = 0$  for  $k \neq 0$ ).
- The output of the LTI system is  $y[n] = (w * h)[n] = \sum_{m=-\infty}^{\infty} h[m]w[n-m]$ .

- a) Determine the autocorrelation function of the output signal,  $R_{yy}[k] = E[y[n]y[n-k]]$
- b) Calculate the specific form of  $R_{yy}[k]$  for the given impulse response  $h[n] = \{1, -0.5\}$ .
- c) Find the power spectral density (PSD) of the input signal  $S_{ww}(\omega)$  and the output signal  $S_{yy}(\omega)$ . PSD is the discrete-time Fourier transform of the autocorrelation function.

$$S_{xx}(\omega) = \sum_{k=-\infty}^{\infty} R_{xx}[k]e^{-j\omega k}$$