I225E Statistical Signal Processing

4. Stochastic Process and Systems I

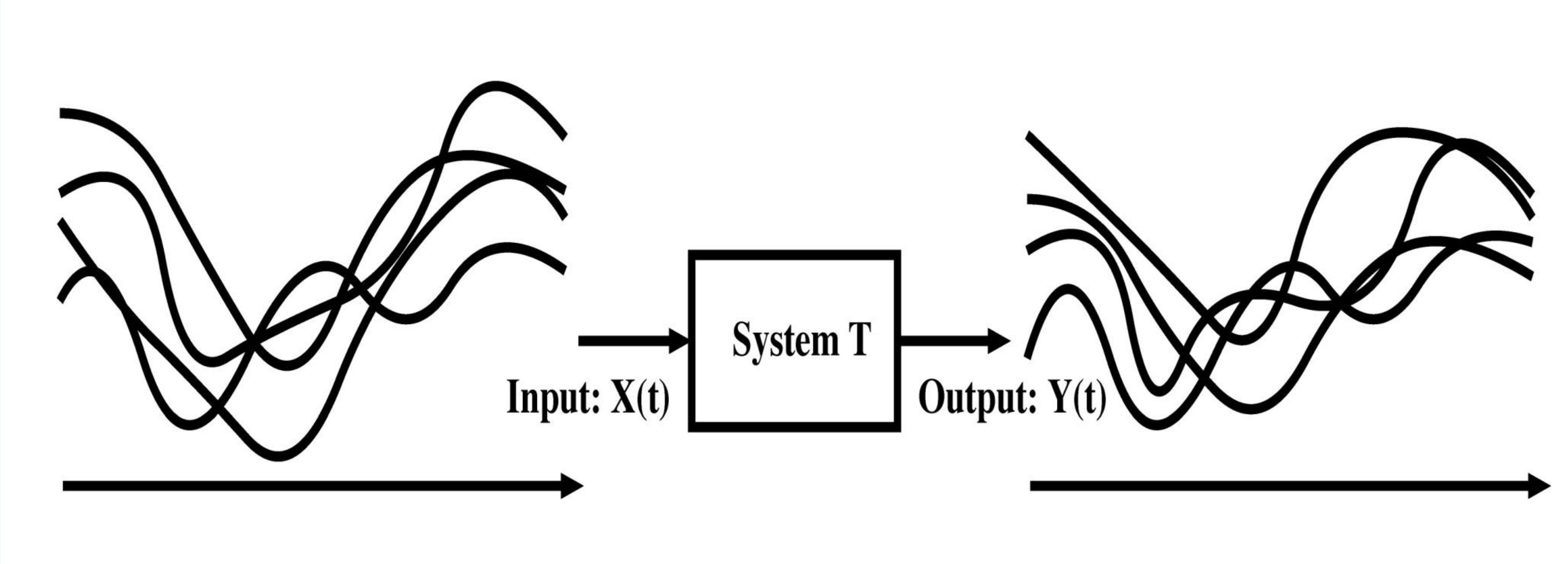
Masashi Unoki and Candy Olivia Mawalim

unoki@jaist.ac.jp and candylim@jaist.ac.jp

School of Information Science

JAPAN
ADVANCED INSTITUTE OF
SCIENCE AND TECHNOLOGY
HOKURIKU 1990

1. System with stochastic input



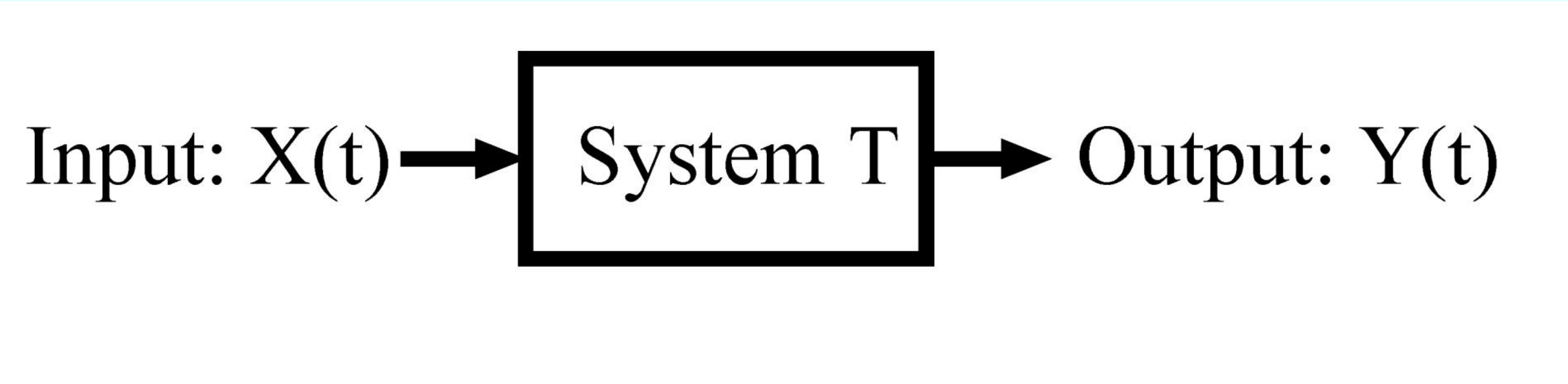
System

Given a stochastic process $\boldsymbol{X}(t)$ as input, $\boldsymbol{Y}(t)$ represents its output.

$$\boldsymbol{Y}(t) = T[\boldsymbol{X}(t)]$$

Purpose:

If statistical properties of input $\boldsymbol{X}(t)$ are known, study the statistical properties of output $\boldsymbol{Y}(t)$



$$\boldsymbol{Y}(t) = T[\boldsymbol{X}(t)]$$

Deterministic system:

System operates only on variable t, treating outcome ω as a parameter. Namely,

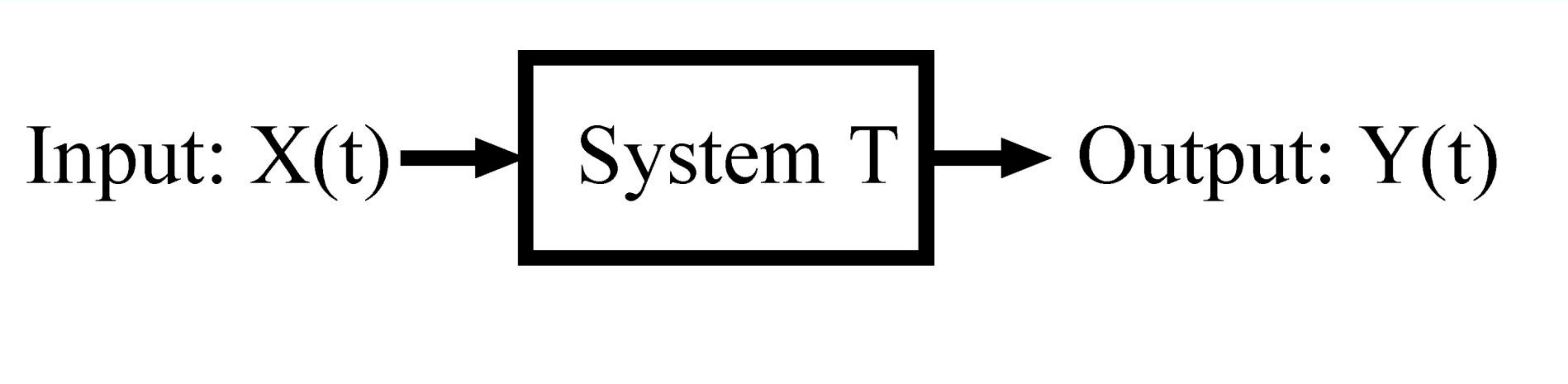
If $\boldsymbol{X}(t,\omega_1)=\boldsymbol{X}(t,\omega_2)$, then $\boldsymbol{Y}(t,\omega_1)=\boldsymbol{Y}(t,\omega_2)$,

Stochastic system:

System operates on both t and ω. Namely,

Even if $\boldsymbol{X}(t,\omega_1)=\boldsymbol{X}(t,\omega_2)$, $\boldsymbol{Y}(t,\omega_1)\neq\boldsymbol{Y}(t,\omega_2)$.

Example: Physical element of the system or coefficient of the system equation is stochastic.



$$\boldsymbol{Y}(t) = T[\boldsymbol{X}(t)]$$

- This lecture deals with only deterministic systems.
- In deterministic systems, transformation T may depend on t. To emphasize this, sometimes denoted as

$$\boldsymbol{Y}(t) = T_t[\boldsymbol{X}(t)]$$

referred to as a time-dependent system.

Deterministic System

Memoryless System

$$Y(t) = g[X(t)]$$

System with Memory

Time-Varying System

Time-Invariant System

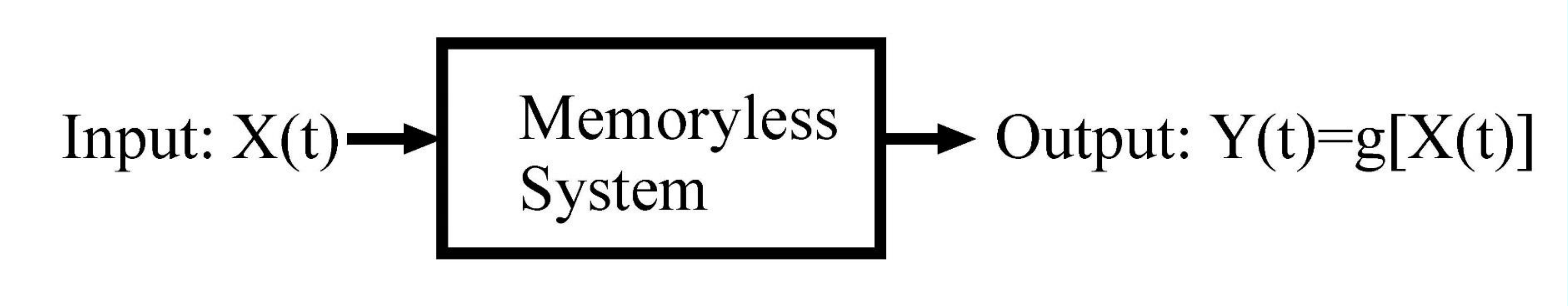
Linear System

$$Y(t) = L[X(t)]$$

Linear Time-Invariant system

$$Y(t)=X(t)*h(t)$$

2. Memoryless System



- **Output** $\boldsymbol{Y}(t_1)$ at time $t = t_1$ depends only upon the simultaneous state of input $\boldsymbol{X}(t_1)$, but not upon past or future state of $\boldsymbol{X}(t)$

$$\boldsymbol{Y}(t) = g[\boldsymbol{X}(t)]$$

(a1) Output mean $E\{\boldsymbol{Y}(t)\} = \int_{-\infty}^{\infty} g(x) f_{\boldsymbol{X}}(x;t)dx$

(a2) Output correlation

$$E\{\boldsymbol{Y}(t_1)\boldsymbol{Y}(t_2)\} = E\left\{g\big(\boldsymbol{X}(t_1)\big)g(\boldsymbol{X}(t_2))\right\}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)g(x_2)f_{\boldsymbol{X}}(x_1,x_2;t_1,t_2)dx_1dx_2$$

(a3) *n*th-order density of $\boldsymbol{Y}(t)$, $f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n; t_1, t_2, \cdots, t_n)$ is obtained from nth-order density of $\boldsymbol{X}(t)$, $f_{\boldsymbol{X}}(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n)$ through transformation

$$\boldsymbol{Y}(t_1) = g[\boldsymbol{X}(t_1)], \boldsymbol{Y}(t_2) = g[\boldsymbol{X}(t_2)], \cdots, \boldsymbol{Y}(t_n) = g[\boldsymbol{X}(t_n)].$$

If the following system $y_1 = g[x_1], y_2 = g[x_2], \cdots, y_n = g[x_n]$ has a unique solution $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$, *n*th-order density of $\boldsymbol{Y}(t)$ is obtained as

$$f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n; t_1, t_2, \cdots, t_n) = \frac{f_{\boldsymbol{X}}(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n)}{|\boldsymbol{J}\,(x_1, x_2, \cdots, x_n)|}$$

where $\boldsymbol{J}$ is Jacobian $\boldsymbol{J} = [g'(x_1) g'(x_2) \cdots g'(x_n)]$.

When more than two solutions exist, summation of the corresponding terms $\frac{f_{\boldsymbol{X}}}{|\boldsymbol{J}|}$ gives the *n*th-order density.

Digression on coordinate transformation

- Let us consider *n*-dim variables $\mathbf{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ and $\mathbf{y} = (y_1, \cdots, y_n)^{\mathrm{T}}$ and the mapping $y_i = y_i(x_1, \cdots, x_n)$=$y_i(\mathbf{x})$. In this case, their infinitesimal volume are related as

$$dy_1 \cdots dy_n = |J(x_1, \cdots, x_n)| dx_1 \cdots dx_n$$

where the matrix is called the Jacobian:

$$J = \begin{pmatrix} {\partial y_1}/{\partial x_1} & \cdots & {\partial y_1}/{\partial x_n} \\ \vdots & \ddots & \vdots \\ {\partial y_n}/{\partial x_1} & \cdots & {\partial y_n}/{\partial x_n} \end{pmatrix}$$

- Accordingly, their probability densities are related as

$$f_{\mathbf{y}}(y_1, \cdots, y_n) = \frac{1}{|J(x_1, \cdots, x_n)|} f_{\mathbf{x}}(x_1, \cdots, x_n)$$

Appendix

Following has been used for the derivation of the density in (a3).

With respect to random variables $\boldsymbol{X} = [X_1, X_2, \cdots, X_n]$, n functions

$$Y_1 = g_1(\boldsymbol{X}),\ Y_2 = g_2(\boldsymbol{X}),\ \cdots,\ Y_n = g_n(\boldsymbol{X}),$$

are given. For *n* random numbers $\boldsymbol{Y} = [Y_1, Y_2, \cdots, Y_n]$, we determine their joint density $f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n)$, where $y_1, y_2, \cdots, y_n$ represent a specific set of numbers.

To find the density, we solve the system

$$g_1(\boldsymbol{X}) = y_1,\ g_2(\boldsymbol{X}) = y_2,\ \cdots,\ g_n(\boldsymbol{X}) = y_n.$$

If the system has no solution, then $f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n) = 0$. If the system has a single solution **x**$= [x_1, x_2, \cdots, x_n]$, the density can be obtained by substituting the solution into following formula

$$f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n) = \frac{f_{\boldsymbol{X}}(x_1, x_2, \cdots, x_n)}{|J(x_1, x_2, \cdots, x_n)|},$$

where
$$J(x_1, x_2, \cdots, x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}$$

If more than two solutions exist, the density is given by summation of all the corresponding terms

$$f_{\boldsymbol{Y}} = \left.\frac{f_{\boldsymbol{X}}}{|J|}\right|_{\boldsymbol{X}=\boldsymbol{X}_1} + \left.\frac{f_{\boldsymbol{X}}}{|J|}\right|_{\boldsymbol{X}=\boldsymbol{X}_2} + \cdots$$

(a4) If input $\boldsymbol{X}(t)$ is strict sense stationary, output $\boldsymbol{Y}(t)$ is also strict sense stationary.

[Proof] According to (a3), *n*th-order density of $\boldsymbol{Y}(t)$ is given as

$$f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n; t_1, t_2, \cdots, t_n) = \frac{f_{\boldsymbol{X}}(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n)}{|\boldsymbol{J}\,(x_1, x_2, \cdots, x_n)|}$$

Since $\boldsymbol{X}(t)$ is strict sense stationary, its density is invariant to a shift of the origin in time, denominator is independent of t. Therefore, $f_{\boldsymbol{Y}}$ is also invariant to time-shift. This proves that $\boldsymbol{Y}(t)$ is strict sense stationary.

- From the properties of strict sense stationary

(i) First-order density of $\boldsymbol{Y}(t)$ is independent of t

$$\rightarrow f_{\boldsymbol{Y}}(y; t) = f_{\boldsymbol{Y}}(y)$$

(ii) Second-order density is a function of time lag $\tau = t_1 - t_2$

$$\rightarrow f_{\boldsymbol{Y}}(y_1, y_2; t_1, t_2) = f_{\boldsymbol{Y}}(y_1, y_2; \tau)$$

Example of memoryless system

Square-law detector

Square-law detector is a memoryless system whose output equals

$$\boldsymbol{Y}(t) = \boldsymbol{X}^2(t).$$

Using the density $f_{\boldsymbol{X}}(x,t)$ of input $\boldsymbol{X}(t)$, find the density $f_{\boldsymbol{Y}}(y,t)$ of output $\boldsymbol{Y}(t)$.

First-order density

If $y > 0$, solutions of $y = x^2$ are $x = \pm\sqrt{y}$.

The corresponding Jacobian matrices are $J = \frac{dx^2}{dx} = 2x = \pm 2\sqrt{y}$. Hence

Hence

$$f_Y(y;t) = \frac{f_X}{|J|}\Bigg]_{x=\sqrt{y}} + \frac{f_X}{|J|}\Bigg]_{x=-\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\left[f_X\left(\sqrt{y};t\right) + f_X\left(-\sqrt{y};t\right)\right]$$

- **Second-order density**:

If $y_1 > 0$, $y_2 > 0$, solutions of $y_1 = x_1^2$, $y_2 = x_2^2$ are $\left(\pm\sqrt{y_1}, \pm\sqrt{y_2}\right)$. Since the corresponding Jacobian matrices are $J = \pm 4\sqrt{y_1 y_2}$,

$$f_Y(y_1, y_2; t_1, t_2) = \frac{1}{4\sqrt{y_1 y_2}} \sum f_X\left(\pm\sqrt{y_1}, \pm\sqrt{y_2}; t_1, t_2\right)$$