[I225] Statistical Signal Processing(E) Office Hour 2

1. There are 80 identical machines working independently. Each machine has a probability of 0.01 to fail at any given moment. A failed machine can be repaired by a single maintenance worker. Two maintenance strategies are considered:

a) Strategy A: 4 workers, each assigned to 20 specific machines.

b) Strategy B: 3 workers jointly responsible for all 80 machines.

Compare the probability that a machine cannot be repaired in time under each strategy.

Answer:

For Strategy A: Let Ai denote the event that worker i has at least 2 machines breaking down simultaneously among their 20 assigned machines. Then, the probability that at least one of the four workers cannot handle all failures in time is:

$$P\left(\bigcup_{i=2}^{4} A_i\right) \geq P(A_1) = P(X \geq 2)$$

Where X~Binomial(20,0.01). Then:

$$P(X \geq 2) = 1 - \sum_{k=0}^{1} \binom{20}{k} (0.01)^k (0.99)^{20-k} \approx 0.0169$$

So the probability that at least one worker is overloaded is:

$$P\left(\bigcup_{i=0}^{4} A_i\right) \geq \ 0.0169$$

For Strategy B: Let Y denote the number of failures occurring simultaneously among all 80 machines. Then:

Y~Binomial(80,0.01)

The event that more than 3 machines fail at once (exceeding the capacity of the 3 available workers) has probability:

$$P(Y \geq 4) = 1 - \sum_{k=0}^{3} \binom{80}{k} (0.01)^k (0.99)^{80-k} \approx 0.0087$$

2. A total of balls are to be placed into distinct boxes. Consider the following four different cases:

a)Balls are distinguishable, and no limit on the number of balls per box.

How many different arrangements are possible?

b)Balls are distinguishable, and each box can contain at most one ball.

How many such arrangements exist?

c)Balls are indistinguishable, and no limit on the number of balls per box.

How many different combinations are possible?

d)Balls are indistinguishable, and each box can contain at most one ball.

How many such distributions are possible?

Please give the answer for each case using combinatorial notation such as .

Answer:

a) Balls are distinguishable, and no limit on the number of balls per box.

- Each ball is labeled (i.e., distinguishable), and every ball can be placed into any of the n boxes independently.
- This is a standard repetition-based choice problem, where each ball has n choices.
- Why: $n \times n \times n \times \ldots \times n = n^r$ (r times) : Follows the multiplication rule (each of the r balls has n choices).

b) Balls are distinguishable, and each box can contain at most one ball.

- We are choosing r boxes from n and assigning one distinct ball to each box.
- This is a permutation problem where we are choosing and ordering r out of n options.
- $\frac{n!}{(n-r)!}$: This is the number of permutations of r distinguishable balls in n distinct positions without repetition.

c) Balls are indistinguishable, and no limit on the number of balls per box.

- All balls are identical, and we only care about how many balls are in each box.
- This corresponds to the integer partition problem: find the number of non-negative integer solutions to: $x_1 + x_2 + \ldots + x_n = r, \ \ x_i \geq 0$.
- The solution uses the stars and bars method:

$$\frac{(n-1+r)!}{(n-1)!\,r!} = \binom{n-1+r}{r}$$

Classic result for distributing identical balls into distinct boxes allowing empty boxes.

d) Balls are indistinguishable, and each box can contain at most one ball.

- We choose r boxes from n, and place one indistinguishable ball in each.
- This is a combination problem without order:

$$\binom{n}{r}$$

We are selecting r boxes from n for identical balls – only the choice of boxes matters.

Summary:

Case	Assumption	Reasoning	Answer
a)	Distinct balls, unlimited boxes	Independent choices $\rightarrow$ multiplication	n^r
b)	Distinct balls, ≤ 1 per box	Ordered assignment $\rightarrow$ permutations	$\frac{n!}{(n-r)!}$
c)	Identical balls, unlimited boxes	Integer partition $\rightarrow$ stars and bars	$\binom{n+r-1}{r}$
d)	Identical balls, ≤ 1 per box	Unordered selection $\rightarrow$ combinations	$\binom{n}{r}$

3. A complex stochastic process Z(t) is defined as:

$$Z(t) = 2X(t) - iY(t)$$

Where:X(t) and Y(t) are real-valued random processes. i is the imaginary unit.

You are given the following information:

$$E[X(t)] = 0, E[Y(t)] = 1$$
$$Var[X(t)] = 2, Var[Y(t)] = 6$$
$$Cov\big(X(t), Y(t)\big) = 0$$

The autocorrelation functions:

$$R_{XX}(t_1, t_2) = 2 \cdot e^{-\frac{|t_1 - t_2|}{3}}$$

$$R_{YY}(t_1, t_2) = 6 \cdot \cos\big(\pi(t_1 - t_2)\big)$$

a) Calculate the mean function of $Z(t)$

b) Calculate the autocovariance function of $Z(t)$

c) Calculate the autocorrelation function of $Z(t)$

d) Calculate the correlation coefficient between $Z(t_1)$ and $Z(t_2)$

Answer:

a) Mean function:

$$E[Z(t)] = 2 \cdot E[X(t)] - i \cdot E[Y(t)] = 2 \cdot 0 - i \cdot 1 =- i$$

b) Autocovariance function:

$$C_{zz}(t_1, t_2) = E[(Z(t_1) - E[Z(t_1)])(\overline{Z(t_2)} - E[\overline{Z(t_2)}])$$

Because from the answer of (a), we know $E[Z(t)] =- i$

$$Z(t) - E[Z(t)] = \big(2X(t) - iY(t)\big) + i = 2X(t) - i(Y(t) - 1)$$

Expand

$$(Z(t_1) - E[Z(t_1)])\left(\overline{Z(t_2)} - E[\overline{Z(t_2)}]\right) = \big(2X(t_1) - i(Y(t_1) - 1)\big)\big(2X(t_2) + i(Y(t_2) - 1)\big)$$

$$= 2X(t_1)2X(t_2) + 2X(t_1)i(Y(t_2) - 1) - i(Y(t_1) - 1)2X(t_2) - i(Y(t_1) - 1)i(Y(t_2) - 1)$$

$$= 4X(t_1)X(t_2) + 2iX(t_1)(Y(t_2) - 1) - 2i(Y(t_1) - 1)X(t_2) + (Y(t_1) - 1)(Y(t_2) - 1)$$

Because

$$E[X(t_1)Y(t_2)] = E[X(t_1)]E[Y(t_2)] = 0$$

$$E[Y(t_1)X(t_2)] = E[Y(t_1)]E[X(t_2)] = 0$$

So

$$C_{zz}(t_1, t_2) = E[(Z(t_1) - E[Z(t_1)])(\overline{Z(t_2)} - E[\overline{Z(t_2)}])]$$

$$= E[4X(t_1)X(t_2)] + E[2iX(t_1)(Y(t_2) - 1)] - E[2i(Y(t_1) - 1)X(t_2)] + E[(Y(t_1) - 1)(Y(t_2) - 1)]$$

Since the covariance between $X(t)$ and $Y(t)$ is 0, and $E[X(t)] = 0, E[Y(t)] = 1$, any expected value involving one term of X and another term of Y is 0

So

$$C_{zz}(t_1, t_2) = E[4X(t_1)X(t_2)] + E[(Y(t_1) - 1)(Y(t_2) - 1)]$$

And the part

$$E[(Y(t_1) - 1)(Y(t_2) - 1)] = E[Y(t_1)Y(t_2)] - E[Y(t_1)] - E[Y(t_2)] + 1$$
$$= R_{YY}(t_1, t_2) - 1 - 1 + 1 = R_{YY}(t_1, t_2) - 1$$

so

$$C_{zz}(t_1, t_2) = 4E[X(t_1)X(t_2)] + R_{YY}(t_1, t_2) - 1$$

Therefore

$$C_{zz}(t_1, t_2) = 4 \cdot R_{XX}(t_1, t_2) + R_{YY}(t_1, t_2) - 1$$

$$R_{XX}(t_1, t_2) = 2 \cdot e^{-\frac{|t_1 - t_2|}{3}}, \ R_{YY}(t_1, t_2) = 6 \cdot \cos\left(\pi(t_1 - t_2)\right)$$

$$C_{zz}(t_1, t_2) = 8 \cdot e^{-\frac{|t_1 - t_2|}{3}} + 6 \cdot \cos\left(\pi(t_1 - t_2)\right) - 1$$

c) Autocorrelation function:

$$R_Z(t_1, t_2) = E\left[Z(t_1)\overline{Z(t_2)}\right]$$

Because

$$Z(t_1) = 2X(t_1) - iY(t_1), \ \overline{Z(t_2)} = 2X(t_2) + iY(t_2)$$

So

$$Z(t_1)\overline{Z(t_2)} = \big(2X(t_1) - iY(t_1)\big)\big(2X(t_2) + iY(t_2)\big)$$

$$= 4X(t_1)X(t_2) + 2iX(t_1)Y(t_2) - 2iY(t_1)X(t_2) + (-i^2)Y(t_1)Y(t_2)$$

$$= 4X(t_1)X(t_2) + 2i\big(X(t_1)Y(t_2) - Y(t_1)X(t_2)\big) + Y(t_1)Y(t_2)$$

So

$$R_Z(t_1, t_2) = E\left[Z(t_1)\overline{Z(t_2)}\right] =$$

$$= E[4X(t_1)X(t_2) + 2i\big(X(t_1)Y(t_2) - Y(t_1)X(t_2)\big) + Y(t_1)Y(t_2)]$$

And because

$$E[X(t_1)Y(t_2)] = E[X(t_1)]E[Y(t_2)] = 0$$

$$E[Y(t_1)X(t_2)] = E[Y(t_1)]E[X(t_2)] = 0$$

So

$$R_Z(t_1, t_2) = 4 \cdot E[X(t_1)X(t_2)] + E[Y(t_1)Y(t_2)]$$

$$= 4 \cdot R_{XX}(t_1, t_2) + R_{YY}(t_1, t_2)$$

$$R_{XX}(t_1, t_2) = 2 \cdot e^{-\frac{|t_1 - t_2|}{3}},\ R_{YY}(t_1, t_2) = 6 \cdot \cos\big(\pi(t_1 - t_2)\big)$$

Therefore

$$R_Z(t_1, t_2) = 8 \cdot e^{-\frac{|t_1 - t_2|}{3}} + 6 \cdot \cos\big(\pi(t_1 - t_2)\big)$$

d) Correlation coefficient:

$$\rho\big(Z(t_1)Z(t_2)\big) = \frac{C_z(t_1, t_2)}{\sqrt{C_z(t_1, t_1)}\sqrt{C_z(t_2, t_2)}}$$

Form the result of (b), we know

$$C_{zz}(t_1, t_2) = 8 \cdot e^{-\frac{|t_1 - t_2|}{3}} + 6 \cdot \cos\big(\pi(t_1 - t_2)\big) - 1$$

So

$$C_z(t, t) = 8 \cdot e^0 + 6 \cdot \cos(0) - 1 = 8 + 6 - 1 = 13$$

$$\rho\big(Z(t_1)Z(t_2)\big) = \frac{C_z(t_1, t_2)}{\sqrt{C_z(t_1, t_1)}\sqrt{C_z(t_2, t_2)}} = \frac{C_z(t_1, t_2)}{\sqrt{13}\sqrt{13}} = \frac{C_z(t_1, t_2)}{13}$$

$$= \frac{8 \cdot e^{-\frac{|t_1 - t_2|}{3}} + 6 \cdot \cos\big(\pi(t_1 - t_2)\big) - 1}{13}$$

4. Suppose you have a transformation that maps a point (u, v) in the uv-plane to a point (x, y) in the xy-plane, defined by the following equations:

$$x(u, v) = sin(u) + v^2$$
$$y(u, v) = uv + cos(v)$$

Calculate the Jacobian matrix of the output variables (x, y) with respect to the input variables (u, v).

Answer:

We are given:

$$x(u, v) = sin(u) + v^2$$
$$y(u, v) = uv + cos(v)$$

The Jacobian matrix is defined as:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Step 1: Compute the partial derivatives

For

$$x(u, v) = sin(u) + v^2:$$
$$\partial x/\partial u = cos(u)$$
$$\partial x/\partial v = 2v$$

For

$$y(u, v) = uv + cos(v):$$
$$\partial y/\partial u = v$$
$$\partial y/\partial v = u - sin(v)$$

Step 2: Construct the Jacobian matrix:

$$J = \begin{bmatrix} cos(u) & 2v \\ v & u - sin(v) \end{bmatrix}$$

5. Let the original audio signal be a simple sequence representing a short sound:

$$x[n] = \{1, \ -2, 0, 4\}$$

Let the impulse response be:

$$h[n] = \{3, 0, \ -1\}$$

Calculate the output signal $y[n]$, which is the convolution of the input signal $x[n]$ with the impulse response $h[n]$.

The linear convolution for discrete-time signals is defined as:

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Answer:

Input signal: $x[n] = 1, -2{,}0{,}4$ (length = 4)

$(x[0] = 1, x[1] =-2, x[2] = 0, x[3] = 4)$

Impulse response: $h[n] = 3{,}0, -1$ (length = 3)

$(h[0] = 3, h[1] = 0, h[2] =-1)$

Length of output:

$$len(y[n]) = len(x[n]) + len(h[n])$$

$$= 4 + 3 - 1 = 6$$

$$\rightarrow y[0], y[1], \ldots, y[5]$$

Now calculate each:

$y[0] = x[0] * h[0] = 1 * 3 = 3$

$y[1] = \ x[0] * h[1] + \ x[1] * h[0] = 1 * 0 + (-2) * 3 =-6$

$y[2] = x[0] * h[2] + x[1] * h[1] + x[2] * h[0] = 1 * (-1) + (-2) * 0 + 0 * 3 =-1$

$y[3] = x[0] * h[3] + x[1] * h[2] + x[2] * h[1] + x[3] * h[0]$

$= 0 + (-2) * (-1) + 0 + 4 * 3 = 14$

$y[4] = x[0] * h[4] + x[1] * h[3] + x[2] * h[2] + \ x[3] * h[1] = 0 + 0 + 0 * (-1) + 4 * 0$

$= 0$

$y[5] = x[0] * h[5] + x[1] * h[4] + x[2] * h[3] + x[3] * h[2] = 0 + 0 + 0 + 4 * (-1) =-4$

Final Output:

$$y[n] = 3, -6, \ -1, 14, 0, \ -4$$