I225E Statistical Signal Processing

5. Stochastic Process and Systems II

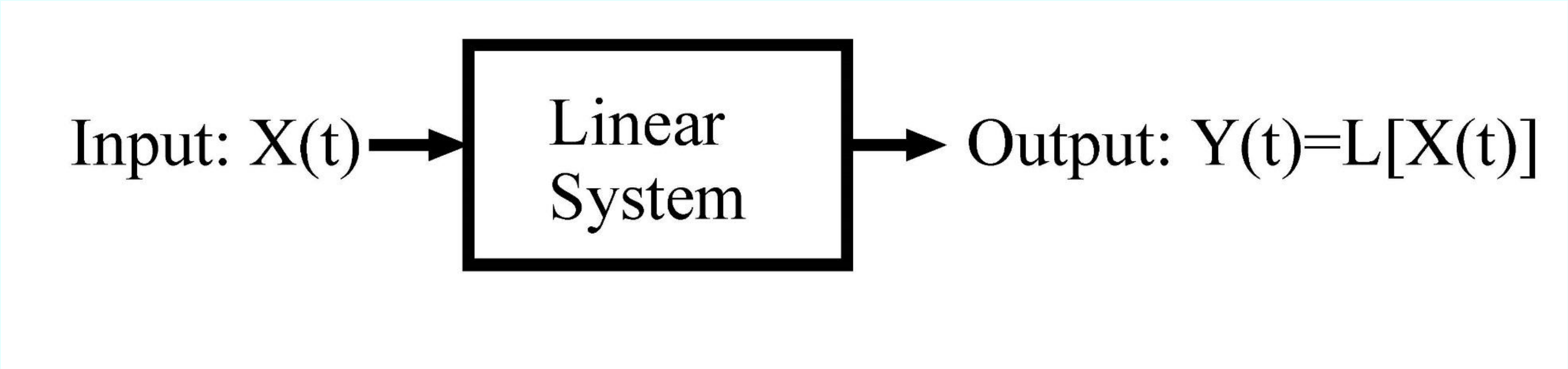
MAWALIM and UNOKI

candylim@jaist.ac.jp and unoki@jaist.ac.jp

School of Information Science

JAPAN
ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY
HOKURIKU 1990

3. Linear System



$$\boldsymbol{Y}(t) = L[\boldsymbol{X}(\mathrm{t})]$$

- **Properties of linear system**

(b1) For any $a_1, a_2, \boldsymbol{X}_1(t), \boldsymbol{X}_2(t)$,

$$L[a_1\boldsymbol{X}_1(t) + a_2\boldsymbol{X}_2(t)] = a_1L[\boldsymbol{X}_1(t)] + a_2L[\boldsymbol{X}_2(t)].$$

(b2) A system is called ***time-invariant***, if its response to $\boldsymbol{X}(t+c)$ equals $\boldsymbol{Y}(t+c)$.

(b3) Output of linear time-invariant system is a convolution

$$\boldsymbol{Y}(t) = \boldsymbol{X}(t) * \boldsymbol{h}(t) = \int_{-\infty}^{\infty} \boldsymbol{X}(t-\alpha)\boldsymbol{h}(\alpha)d\alpha$$

Here, $\boldsymbol{h}(t) = L[\delta(t)]$ represents impulse response.

(b4) Linear time-invariant system has following properties.

i) If input $\boldsymbol{X}(t)$ is a normal process, then output $\boldsymbol{Y}(t)$ is also a normal process.

ii) If input $\boldsymbol{X}(t)$ is strict sense stationary, then output $\boldsymbol{Y}(t)$ is also strict sense stationary.

Fundamental theorem

For any linear system, $E\{L[\boldsymbol{X}(t)]\} = L[E\{\boldsymbol{X}(t)\}]$

This means that output average $\eta_{\boldsymbol{Y}}(t)$ equals to output of a system, to which input average $\eta_{\boldsymbol{X}}(t)$ is input.

$$\eta_{\boldsymbol{Y}}(t) = L[\eta_{\boldsymbol{X}}(t)]$$

(Intuitive proof)

Supposing that *i*th trial is $\boldsymbol{Y}(t, \omega_i) = L[\boldsymbol{X}(t, \omega_i)]$

$$E\{\boldsymbol{Y}(t)\} \approx \frac{\boldsymbol{Y}(t,\omega_1)+\boldsymbol{Y}(t,\omega_2)+\cdots+\boldsymbol{Y}(t,\omega_n)}{n}$$

$$= \frac{L[\boldsymbol{X}(t,\omega_1)]+L[\boldsymbol{X}(t,\omega_2)]+\cdots+L[X(t,\omega_n)]}{n}$$

$$= L\left[\frac{\boldsymbol{X}(t,\omega_1)+\boldsymbol{X}(t,\omega_2)+\cdots+\boldsymbol{X}(t,\omega_n)}{n}\right]$$

$$\approx L[E\{\boldsymbol{X}(t)\}]$$

(b5) Using the convolution of (b3), the theorem is described as

$$E\{\boldsymbol{Y}(t)\} = \int_{-\infty}^{\infty} E\{\boldsymbol{X}(t-\alpha)\}h(\alpha)d\alpha$$
$$= \int_{-\infty}^{\infty} \eta_{\boldsymbol{X}}(t-\alpha)h(\alpha)d\alpha = \eta_{\boldsymbol{X}}(t) * h(t)$$

(b6) Setting $\widetilde{\boldsymbol{X}}(t) = \boldsymbol{X}(t) - \eta_{\boldsymbol{X}}(t)$ and $\widetilde{\boldsymbol{Y}}(t) = \boldsymbol{Y}(t) - \eta_{\boldsymbol{Y}}(t)$,

$$\widetilde{\boldsymbol{Y}}(t) = L[\widetilde{\boldsymbol{X}}(t)]$$

[**Proof**]

$$\begin{aligned}\widetilde{\boldsymbol{Y}}(t) &= \boldsymbol{Y}(t) - \eta_{\boldsymbol{Y}}(t)\\ &= L[\boldsymbol{X}(t)] - L[\eta_{\boldsymbol{X}}(t)]\\ &= L[\boldsymbol{X}(t) - \eta_{\boldsymbol{X}}(t)]\\ &= L[\widetilde{\boldsymbol{X}}(t)]\end{aligned}$$

(b7) With respect to input $\boldsymbol{X}(t) = f(t) + \nu(t)$ ($f(t)$ is a deterministic signal; $E\{\nu(t)\} = 0$), the output average is

$$\eta_{\boldsymbol{Y}}(t) = E\{\boldsymbol{Y}(t)\} = E\{L[\boldsymbol{X}(t)]\} = L[E\{\boldsymbol{X}(t)\}]$$
$$= L[E\{f(t) + \nu(t)\}]$$
$$= L[E\{f(t)\} + E\{\nu(t)\}]$$
$$= L[f(t)] = f(t) * h(t)$$

Autocorrelation of output

Consider input $\boldsymbol{X}(t)$ and output $\boldsymbol{Y}(t)$ for a linear system. We represent output autocorrelation $R_{\boldsymbol{YY}}(t_1, t_2)$ in terms of input autocorrelation $R_{\boldsymbol{XX}}(t_1, t_2)$.

Theorem

$$\text{(A) } R_{\boldsymbol{XY}}(t_1, t_2) = L_{t_2}[R_{\boldsymbol{XX}}(t_1, t_2)\ ].$$

$$\text{(B) } R_{\boldsymbol{YY}}(t_1, t_2) = L_{t_1}[R_{\boldsymbol{XY}}(t_1, t_2)\ ].$$

[**Proof**]

Multiplying both side of $\boldsymbol{Y}(t) = L_t[\boldsymbol{X}(t)]$ by $\boldsymbol{X}(t_1)$,

$$\boldsymbol{X}(t_1)\boldsymbol{Y}(t) = \boldsymbol{X}(t_1)L_t[\boldsymbol{X}(t)] = L_t[\boldsymbol{X}(t_1)\boldsymbol{X}(t)].$$

Taking expectation,

$$\begin{aligned} E\{\boldsymbol{X}(t_1)\boldsymbol{Y}(t)\} &= E\{L_t[\boldsymbol{X}(t_1)\boldsymbol{X}(t)]\} \\ &= L_t[E\{\boldsymbol{X}(t_1)\boldsymbol{X}(t)\}]. \end{aligned}$$

Substitution of $t = t_2$ yields (A).

In a similar manner, multiplication of $\boldsymbol{Y}(t) = L_t[\boldsymbol{X}(t)]$ by $\boldsymbol{Y}(t_2)$ and so on yields (B).

(c1) In the form of convolution,

(a) $R_{\boldsymbol{XY}}(t_1, t_2) = R_{\boldsymbol{XX}}(t_1, t_2) * h(t_2)$

$$= \int_{-\infty}^{\infty} R_{\boldsymbol{XX}}(t_1, t_2 - \alpha) h(\alpha) d\alpha$$

(b) $R_{\boldsymbol{YY}}(t_1, t_2) = R_{\boldsymbol{XY}}(t_1, t_2) * h(t_1)$

$$= \int_{-\infty}^{\infty} R_{\boldsymbol{XY}}(t_1 - \alpha, t_2) h(\alpha) d\alpha$$

(c) Combining (a) and (b),

$$R_{\boldsymbol{YY}}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{\boldsymbol{XX}}(t_1 - \alpha, t_2 - \beta) h(\alpha) h(\beta) d\alpha d\beta$$

(c2) For cross-covariance, the following holds.

$$C_{\boldsymbol{XY}}(t_1, t_2) = C_{\boldsymbol{XX}}(t_1, t_2) * h(t_2)$$
$$C_{\boldsymbol{YY}}(t_1, t_2) = C_{\boldsymbol{XY}}(t_1, t_2) * h(t_1)$$

[**Proof**]

Denoting $\widetilde{\boldsymbol{X}}(t) = \boldsymbol{X}(t) - \eta_{\boldsymbol{X}}(t)$, $\widetilde{\boldsymbol{Y}}(t) = \boldsymbol{Y}(t) - \eta_{\boldsymbol{Y}}(t)$,

$$C_{\boldsymbol{XX}}(t_1, t_2) = E\left\{\widetilde{\boldsymbol{X}}(t_1)\widetilde{\boldsymbol{X}}(t_2)\right\} = R_{\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{X}}}(t_1, t_2)$$
$$C_{\boldsymbol{XY}}(t_1, t_2) = R_{\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{Y}}}(t_1, t_2),$$
$$C_{\boldsymbol{YY}}(t_1, t_2) = R_{\widetilde{\boldsymbol{Y}}\widetilde{\boldsymbol{Y}}}(t_1, t_2).$$

From (b6), $\widetilde{\boldsymbol{Y}}(t) = L[\widetilde{\boldsymbol{X}}(t)]$. Hence, (a) and (b) can be applied to $R_{\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{X}}}$, $R_{\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{Y}}}$, and $R_{\widetilde{\boldsymbol{Y}}\widetilde{\boldsymbol{Y}}}$.

(c3) Above results can be extended to the case the impulse response $\boldsymbol{h}(t)$ is complex function.

In this case,

$$R_{\boldsymbol{XY}}(t_1, t_2) = R_{\boldsymbol{XX}}(t_1, t_2) * \boldsymbol{h}^*(t_2)$$
$$R_{\boldsymbol{YY}}(t_1, t_2) = R_{\boldsymbol{XY}}(t_1, t_2) * \boldsymbol{h}(t_1)$$

Response to white noise

Theorem

Consider white noise, whose autocorrelation is given by $R_{\boldsymbol{XX}}(t_1,t_2)=q(t_1)\delta(t_1-t_2)$, as input to linear system. The output power is given by

$$E\{\boldsymbol{Y}^2(t)\}=q(t)*h^2(t)=\int_{-\infty}^{\infty}q(t-\alpha)h^2(\alpha)d\alpha$$

[Proof]

Substitution of $R_{\boldsymbol{XX}}(t_1,t_2)=q(t_1)\delta(t_1-t_2)$ into convolution form of (A), (B) yields,

$$R_{\boldsymbol{XY}}(t_1,t_2)=q(t_1)\delta(t_1-t_2)*h(t_2)=q(t_1)h(t_2-t_1)$$
$$R_{\boldsymbol{YY}}(t_1,t_2)=\int_{-\infty}^{\infty}q(t_1-\alpha)h[t_2-(t_1-\alpha)]h(\alpha)d\alpha.$$

Substitution of $t_1=t_2=t$ gives the output power.

(d1) If input $\boldsymbol{X}(t)$ is stationary white noise, $q(t) = q$.

Therefore,

$$E\{\boldsymbol{Y}^2(t)\} = qE, \quad E = \int_{-\infty}^{\infty} h^2(\alpha)d\alpha$$

(d2) If correlation time of impulse response $h(t)$ is shorter than that of $q(t)$,

$$E\{\boldsymbol{Y}^2(t)\} \approx q(t)\int_{-\infty}^{\infty} h^2(\alpha)d\alpha = Eq(t).$$

(d3) If $R_{\boldsymbol{XX}}(\tau) = q\delta(\tau)$ and $\boldsymbol{X}(t)$ is injected at $t = 0$.

$$q(t) = qU(t)$$

Therefore,

$$E\{\boldsymbol{Y}^2(t)\} \approx q\int_{-\infty}^{t} h^2(\alpha)d\alpha$$

Digression on Dirac's delta function

- Properties of delta function $\delta(t)$:

$$\int_{-\infty}^{+\infty} dt\delta(t) = 1.$$

$$\int_{-\infty}^{+\infty} dt\delta(t) f(t) = f(0).$$

- Delta function as a limit of infinitesimally narrow Gaussian:

$$\delta(t) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

- Delta function has a uniform (white) spectrum:

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{i\omega t}$$

Example 1

Consider a linear system having impulse response

$$h(t) = e^{-ct}U(t)$$

Input a white noise, whose autocorrelation is given by $R_{\boldsymbol{WW}}(\tau) = q\delta(\tau)$, at $t = 0$. Calculate autocorrelation $R_{\boldsymbol{YY}}(t_1, t_2)$ $(0 < t_1 < t_2)$ of the output $\boldsymbol{Y}(t)$.

Answer Denoting input as $\boldsymbol{X}(t)$, $\boldsymbol{X}(t) = \boldsymbol{W}(t)U(t)$.

(I) Case of $t_1 < 0$ or $t_2 < 0$: Since there is no input,

$$R_{\boldsymbol{XX}}(t_1, t_2) = R_{\boldsymbol{XY}}(t_1, t_2) = R_{\boldsymbol{YY}}(t_1, t_2) = 0$$

(II) Case of $0 < t_1 < t_2$:

$$R_{\boldsymbol{XX}}(t_1, t_2) = R_{\boldsymbol{WW}}(t_1 - t_2) = q\delta(t_1 - t_2)$$

Therefore,

$$R_{\boldsymbol{XY}}(t_1,t_2)=\int_{-\infty}^{t_2}R_{\boldsymbol{XX}}(t_1,t_2-\alpha)h(\alpha)d\alpha$$
$$=\int_{-\infty}^{t_2}q\delta(t_1-t_2+\alpha)e^{-c\alpha}U(\alpha)d\alpha$$
$$=qe^{-c(t_2-t_1)}$$
$$R_{\boldsymbol{YY}}(t_1,t_2)=\int_{-\infty}^{t_1}R_{\boldsymbol{XY}}(t_1-\alpha,t_2)h(\alpha)d\alpha$$
$$=\int_{-\infty}^{t_1}qe^{-c(t_2-t_1+\alpha)}e^{-c\alpha}U(\alpha)d\alpha$$
$$=qe^{-c(t_2-t_1)}\int_{0}^{t_1}e^{-2c\alpha}d\alpha$$
$$=\frac{q}{2c}e^{-c(t_2-t_1)}(1-e^{-2ct_1})$$
$$E\{\boldsymbol{Y}^2(t)\}=R_{\boldsymbol{YY}}(t,t\ )=\frac{q}{2c}(1-e^{-2ct}).$$

Example 2: Differentiator

Consider the output properties (mean and autocorrelation) of differentiator:

$$\boldsymbol{Y}(t) = L[\boldsymbol{X}(t)] = \boldsymbol{X}'(t).$$

- **Mean** is,

$$\eta_{\boldsymbol{Y}}(t) = L[\eta_{\boldsymbol{X}}(t)] = \eta'_{\boldsymbol{X}}(t)$$

- **Autocorrelation** is

$$R_{\boldsymbol{XY}}(t_1, t_2) = L_{t_2}[R_{\boldsymbol{XX}}(t_1, t_2)] = \frac{\partial R_{\boldsymbol{XX}}(t_1,t_2)}{\partial t_2}$$

$$R_{\boldsymbol{YY}}(t_1, t_2) = L_{t_1}[R_{\boldsymbol{XY}}(t_1, t_2)] = \frac{\partial R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_1}$$

Hence

$$R_{\boldsymbol{YY}}(t_1, t_2) = L_{t_1}[R_{\boldsymbol{XY}}(t_1, t_2)] = \frac{\partial^2 R_{\boldsymbol{XX}}(t_1,t_2)}{\partial t_1 \partial t_2}$$

If $\boldsymbol{X}(t)$ is wide-sense stationary, its **mean**, $\eta_{\boldsymbol{X}}(t)$, is a constant. Therefore, $\eta_{\boldsymbol{Y}}(t) = 0$. Since **autocorrelation** is a function of $\tau = t_1 - t_2$,

$$\frac{\partial R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2} = -\frac{dR_{\boldsymbol{XX}}(\tau)}{d\tau}$$

$$\frac{\partial^2 R_{\boldsymbol{XX}}(t_1,t_2)}{\partial t_1 \partial t_2} = -\frac{d^2 R_{\boldsymbol{XX}}(\tau)}{d\tau^2}$$

Hence,

$$R_{\boldsymbol{XY}}(\tau) = -R'_{\boldsymbol{XX}}(\tau),$$

$$R_{\boldsymbol{YY}}(\tau) = -R''_{\boldsymbol{XX}}(\tau),$$

Example 3: Differential Equation

Consider output properties (mean and autocorrelation) of the following differential equation driven by noise $\boldsymbol{X}(t)$.

$$a_n\boldsymbol{Y}^{(n)}(t) + a_{n-1}\boldsymbol{Y}^{(n-1)}(t) + \cdots + a_0\boldsymbol{Y}(t) = \boldsymbol{X}(t)$$

The initial condition is given by

$$\boldsymbol{Y}(0) = \boldsymbol{Y}^{(1)}(0) = \cdots = \boldsymbol{Y}^{(n)}(0) = 0$$

- **Mean**:

  Taking the expectation $E\{\cdot\}$ for both sides,

  $$a_n E\left\{\boldsymbol{Y}^{(n)}(t)\right\} + a_{n-1}E\left\{\boldsymbol{Y}^{(n-1)}(t)\right\} + \cdots + a_0 E\{\boldsymbol{Y}(t)\} = E\{\boldsymbol{X}(t)\}$$

  Since $\frac{d}{dx}$ and $E\{\cdot\}$ are commutative, $E\left\{\boldsymbol{Y}^{(k)}(t)\right\} = \eta_Y^{(k)}(t)$

$$a_n\eta_{\boldsymbol{Y}}^{(n)}(t)+a_{n-1}\eta_{\boldsymbol{Y}}^{(n-1)}(t)+\cdots+a_0\eta_{\boldsymbol{Y}}(t)=\eta_{\boldsymbol{X}}(t)$$

Because of $\boldsymbol{Y}^{(k)}(0)=0$, $\eta_{\boldsymbol{Y}}^{(k)}(0)=0$. The mean $\eta_{\boldsymbol{Y}}^{(k)}(t)$ is obtained by integrating the differential equation for the following initial condition:

$$\eta_{\boldsymbol{Y}}(0)=\eta_{\boldsymbol{Y}}^{(1)}(0)=\cdots=\eta_{\boldsymbol{Y}}^{(n)}(0)=0$$

- **Correlation**:

By substituting $t=t_2$ and multiplying by $\boldsymbol{X}(t_1)$, the differential equation becomes

$$X(t_1)\left[a_n\boldsymbol{Y}^{(n)}(t_2)+a_{n-1}\boldsymbol{Y}^{(n-1)}(t_2)+\cdots+a_0\boldsymbol{Y}(t_2)\right]$$
$$=\boldsymbol{X}(t_1)\boldsymbol{X}(t_2)$$

By taking the expectation and using the formula of

$$E\{\boldsymbol{X}(t_1)\boldsymbol{Y}^{(k)}(t_2)\} = \frac{\partial^k R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2^k},$$

$$a_n \frac{\partial^n R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2^n} + a_{n-1} \frac{\partial^{n-1} R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2^{n-1}} + \cdots$$

$$+a_0 R_{\boldsymbol{XY}}(t_1, t_2) = R_{\boldsymbol{XX}}(t_1, t_2),$$

Since $\boldsymbol{X}(t_1)\boldsymbol{Y}^{(k)}(0) = 0$, initial condition is given by

$$R_{\boldsymbol{XY}}(t_1, 0) = \frac{\partial R_{\boldsymbol{XY}}(t_1,0)}{\partial t_2} = \cdots = \frac{\partial^n R_{\boldsymbol{XY}}(t_1,0)}{\partial t_2^n} = 0$$

Autocorrelations $\frac{\partial^n R_{\boldsymbol{XY}}(t_1,t_2)}{\partial t_2^n}$ are obtained by solving the differential equation for the above initial condition.

In a similar manner, by substituting $t = t_1$ and multiplying by $\boldsymbol{Y}(t_2)$,

$$\left[a_n \boldsymbol{Y}^{(n)}(t_1) + a_{n-1}\boldsymbol{Y}^{(n-1)}(t_1) + \cdots + a_0 \boldsymbol{Y}(t_1)\right]\boldsymbol{Y}(t_2)$$
$$= \boldsymbol{X}(t_1)\boldsymbol{Y}(t_2)$$

By taking the expectation and by using the formula

$$E\left\{\boldsymbol{Y}^{(k)}(t_1)\boldsymbol{Y}(t_2)\right\} = \frac{\partial^k R_{\boldsymbol{YY}}(t_1,t_2)}{\partial t_1^k},$$

$$a_n \frac{\partial^n R_{\boldsymbol{YY}}(t_1,t_2)}{\partial t_1^n} + a_{n-1}\frac{\partial^{n-1} R_{\boldsymbol{YY}}(t_1,t_2)}{\partial t_1^{n-1}} + \cdots$$
$$+ a_0 R_{\boldsymbol{YY}}(t_1,t_2) = R_{\boldsymbol{XY}}(t_1,t_2).$$

Since $\boldsymbol{Y}(0)\boldsymbol{Y}^{(k)}(t_2) = 0$, initial condition is given by

$$R_{\boldsymbol{YY}}(0, t_2) = \frac{\partial R_{\boldsymbol{YY}}(0,t_2)}{\partial t_1} = \cdots = \frac{\partial^n R_{\boldsymbol{XY}}(0,t_2)}{\partial t_1^n} = 0$$

Autocorrelations $\frac{\partial^n R_{\boldsymbol{YY}}(t_1,t_2)}{\partial t_1^n}$ are obtained by solving the differential equation for the above initial condition.