[I225] Statistical Signal Processing(E) Office Hour 3

1. Consider a discrete-time linear time-invariant (LTI) system with the following impulse response: $h[n] = \delta[n] + 0.6\,\delta[n-2] + 0.3\,\delta[n-4]$

The input signal is given by: $x[n] = \{1, -1, 2, 0, -2, 1\}$

Compute the output signal $y[n] = x[n] * h[n]$ using convolution.

Answer:

- $x[n] = \{1, -1, 2, 0, -2, 1\}$, n=0,1,2,3,4,5
- $h[n] = \delta[n] + 0.6\,\delta[n-2] + 0.3\,\delta[n-4]$, the impulse response has nonzero components at h[0]=1, h[2]=0.6, h[4]=0.3

Step 1: Determine Output Range

$L_y = L_x + L_h - 1 = 6 + 5 - 1 = 10 \rightarrow y[n]$ is defined for n=0,1,2,3,4,5,6,7,8,9

Step 2: The output is calculated by: $y[n] = \sum_{m-0}^{L_h-1} h[m] \cdot x[n-m]$,

$$y[n] = x[n] + 0.6\,x[n-2] + 0.3\,x[n-4]$$

Step 3: The output $y[n]$ from n = 0 to 9 is computed as follows:

n	x[n]	x[n-2]	x[n-4]	y[n] = x[n] + 0.6 x[n-2] + 0.3 x[n-4]
0	1	0	0	1
1	-1	0	0	-1
2	2	1	0	2.60
3	0	-1	0	-0.60
4	-2	2	1	-0.50
5	1	0	-1	0.70
6	0	-2	2	-0.60
7	0	1	0	0.60
8	0	0	-2	-0.60
9	0	0	1	0.30

So, $y[n] = \{1,\ -1,\ 2.6,\ -0.6,\ -0.5,\ 0.7,\ -0.6,\ 0.6,\ -0.6,\ 0.3\}$

2. Consider a discrete-time LTI system with an impulse response:

$$h[n] = \{1, 0.5, -0.25\}$$

where the value at n=0 is the first element. The input to this system is a zero-mean white noise process $\omega[n]$ with variance σ_ω^2. Recall that for zero-mean white noise:

- $\mathbb{E}[\omega[n]] = 0$ for all n
- The autocorrelation function $R_{\omega\omega}[k] = \mathbb{E}[\omega[n]\omega[n-k]] = \sigma_\omega^2 \delta[k]$

The output of the LTI system is:

$$y[n] = (\omega * h)[n] = \sum_{m=-\infty}^{\infty} h[m]\omega[n-m]$$

a) Determine the autocorrelation function of the output signal:

$$R_{yy}[k] = \mathbb{E}[y[n]y[n-k]]$$

b) Calculate the specific form for $R_{yy}[k]$ using the given impulse response

$h[n] = \{1, 0.5, -0.25\}$

c) Compute the power spectral density (PSD) of the output $S_{yy}[\omega]$, using the output signal that:

$$S_{yy}(\omega) = \sum_{k=-\infty}^{\infty} R_{yy}[k]e^{-j\omega k}$$

Answer:

a) We are given:

Input is zero-mean white noise $w[n]$ with: $R_{\omega\omega}[k] = \mathbb{E}[\omega[n]\omega[n-k]] = \sigma_\omega^2 \delta[k]$

Output is: $y[n] = (\omega * h)[n] = \sum_{m=-\infty}^{\infty} h[m]\omega[n-m]$

To compute: $R_{yy}[k] = \mathbb{E}[y[n]y[n-k]]$

Use the formula for output autocorrelation of an LTI system driven by white noise:

$$R_{yy}[k] = \sigma_\omega^2 \cdot (h * h^*)[k]$$

Since $h[n]$ is real-valued, $h^*[n] = h[n]$, so:

$R_{yy}[k] = \sigma_\omega^2 \cdot r_h[k]$, where $r_h[k] = h[n] * h[-n] = h * h[-k]$

b)

Given:

$$h[n] = \{1, 0.5, -0.25\} \quad \rightarrow \quad \begin{cases} h[0] = 1, h[1] = 0.5, h[2] = -0.2 \\ \text{all other } h[n] = 0 \end{cases}$$

We compute the autocorrelation of $h[n], r_h[k] = h * h[-k]$

This is a linear convolution of $h[n]$ and its time reversal:

Step 1: Flip $h[n]$ to get $h[-n] = \{-0.25, 0.5, 1\}$

Step 2: Convolve $h[n] * h[-n]$:

This results in a sequence of length 2N−1=5, centered at 0:

$$r_h[k] = h[k] * h[-k] = \begin{cases} r_h[-2] = 1 \cdot (-0.25) = -0.25 \\ r_h[-1] = 1 \cdot 0.5 + 0.5 \cdot (-0.25) = 0.375 \\ r_h[0] = 1^2 + 0.5^2 + (-0.25)^2 = 1.312 \\ r_h[1] = 0.5 \cdot 1 + (-0.25) \cdot 0.5 = 0.375 \\ r_h[2] = -0.25 \cdot 1 = -0.25 \end{cases}$$

So,

$$R_{yy}[k] = \sigma_\omega^2 \cdot r_h[k] = \sigma_\omega^2 \cdot \begin{cases} -0.25, & k = \pm 2 \\ 0.375, & k = \pm 1 \\ 1.3125, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

c)

Recall:

$$S_{yy}(\omega) = \mathcal{F}\{R_{yy}[k]\} = \sigma_\omega^2 \cdot \left|H\left(e^{j\omega}\right)\right|^2$$

Compute the frequency response $H\left(e^{j\omega}\right)$:

Given $h[n] = \{1, 0.5, -0.25\}$, then:

$$\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right) \cdot H^*\left(e^{j\omega}\right)$$

Instead of expanding, we can give a closed-form expression or plot this function.

Thus:

$$S_{yy}(\omega) = \sigma_\omega^2 \cdot \left|1 + 0.5e^{-j\omega} - 0.25e^{-j2\omega}\right|^2$$

This is the Power Spectral Density of the output.

3. A stationary input process $X(t)$ is passed through a linear system with the following transfer function:

$$H(\omega) = \frac{1}{(\omega - 2)^2 + 1}$$

Assume that the total input power is: $\mathbb{E}\{X^2(t)\} = 20$, Find the input power spectrum $S_{XX}(\omega)$ hat maximizes the output power $\mathbb{E}\{Y^2(t)\}$, where the output $Y(t)$ is the result of filtering $X(t)$ through the system $H(\omega)$.

Answer:

Step 1: Express Output Power

From LTI system theory, output power is:

$$\mathbb{E}\{Y^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)|H(\omega)|^2 d\omega$$

Step 2: Use Input Power Constraint

We are told:

$$\mathbb{E}[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = 20$$

Step 3: Apply Cauchy-Schwarz Type Inequality

We want to maximize:

$$\int S_{XX}(\omega)\cdot|H(\omega)|^2 d\omega$$

subject to:

$$\int S_{XX}(\omega)d\omega = 2\pi\cdot 20$$

The maximum is achieved when all power is concentrated at the frequency where $|H(\omega)|^2$ is maximized.

Step 4: Find Maximum of $|H(\omega)|^2$

Given: $H(\omega) = \frac{1}{(\omega-2)^2+1}$

This is a Lorentzian function, maximized when denominator is minimized:

$$\omega = 2 \rightarrow H(2) = \frac{1}{1} = 1 \rightarrow \max|H(\omega)|^2 = 1$$

Step 5: Compute Upper Bound of Output Power

$$\mathbb{E}\{Y^2(t)\} \leq \max|H(\omega)|^2\cdot\mathbb{E}[X^2(t)] = 1\cdot 20 = 20$$

Step 6: Find Input PSD that Achieves Equality

To achieve equality, all power must be concentrated at $\omega = 2$, so:

$$S_{XX}(\omega) = 2\pi\cdot 20\cdot\delta(\omega-2)$$

Step 7: Optional – Corresponding Autocorrelation Function

By inverse Fourier transform:

$$R_{XX}(\tau) = \int S_{XX}(\omega)e^{j\omega\tau}\frac{d\omega}{2\pi} = 20 \cdot e^{j2\tau} \quad \rightarrow \quad R_{XX}(\tau) = 20cos(2\tau)$$

4. A linear system is described by the following first-order differential equation driven by white noise input $X(t)$:

$$\frac{dZ(t)}{dt} + 2bZ(t) = X(t)$$

Where $b > 0$ is a positive constant. The input $X(t)$ is a white noise process with autocorrelation function

$$R_{xx}(\tau) = r\delta(\tau)$$

Where r is the power spectral density (PSD) level.

Find:

a) The transfer function $H(\omega)$ of the system.

b) The power spectral density $S_{zz}(\omega)$ of the output $Z(t)$.

c) The autocorrelation function $R_{zz}(\tau)$ of the output $Z(t)$.

Answer:

a) Transfer Function $H(\omega)$:

we tale the Fourier transform of both sides of the differential equation:

$$\frac{dZ(t)}{dt} + 2bZ(t) = X(t)$$

Using

$$\mathcal{F}\left\{\frac{dZ(t)}{dt}\right\} = j\omega Z(j\omega)$$

$$\mathcal{F}\{Z(t)\} = Z(j\omega)$$

So

$$\mathcal{F}\left\{\frac{dZ(t)}{dt} + 2bZ(t)\right\} = \mathcal{F}\{X(t)\}$$

$$j\omega Z(j\omega) + 2bZ(j\omega) = X(j\omega) \rightarrow$$

$$Z(j\omega)(j\omega + 2b) = X(j\omega)$$

Thus

$$H(j\omega) = \frac{Z(j\omega)}{X(j\omega)} = \frac{1}{j\omega + 2b}$$

b) Power Spectral Density $S_{zz}(\omega)$:

using

$$S_{zz}(\omega) = |H(j\omega)|^2 \cdot S_{XX}(\omega)$$

Given $X(t)$ is white noise with PSD $S_{XX}(\omega) = r$, and

$$|H(j\omega)|^2 = H(j\omega)H(j\omega)^* = \frac{1}{j\omega + 2b} \cdot \frac{1}{-j\omega + 2b} = \frac{1}{\omega^2 + 4b^2}$$

Then

$$S_{zz}(\omega) = \frac{r}{\omega^2 + 4b^2}$$

c) Autocorrelation Function $R_{zz}(\tau)$

By inverse Fourier transform:

$$R_{zz}(\tau) = \mathcal{F}^{-1}\{S_{zz}(\omega)\} = \mathcal{F}^{-1}\left\{\frac{r}{\omega^2 + 4b^2}\right\}$$

Because in Fourier Transform Pairs Table:

$$\mathcal{F}^{-1}\left\{\frac{1}{\omega^2 + a^2}\right\} = \frac{1}{2\alpha} e^{-\alpha|\tau|}, a > 0$$

So

$$R_{zz}(\tau) = \mathcal{F}^{-1}\left\{\frac{1}{\omega^2 + 4b^2}\right\} = \frac{r}{4b} e^{-2b|r|}$$

5. Suppose a temperature sensor is used to record ambient temperatures each minute, and the recorded values are modeled as

$$Y[n] \sim \mathcal{N}(\mu, \sigma^2), n = 0, 1, \ldots., N-1$$

Where μ is the true average ambient temperature, and the measurements are i.i.d. Gaussian. Consider the estimator:

$$\hat{\mu} = \frac{1}{N} \sum_{n=0}^{N-1} Y[n]$$

a) What is the bias of $\hat{\mu}$

b) Compute the variance of $\hat{\mu}$ and check whether it is a consistent estimator.

Answer:

a) Bias:

We compute the expected value:

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}\left[\frac{1}{N}\sum_{n=0}^{N-1} Y[n]\right] = \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}[Y[n]] = \frac{1}{N}\sum_{n=0}^{N-1} \mu = \mu$$

Therefore

$$Bias(\mu) = \mathbb{E}[\hat{\mu}] - \mu = 0$$

Hence, $\hat{\mu}$ is unbiased.

b) Variance and Consistency

$$Var(\hat{\mu}) = Var\left(\frac{1}{N}\sum_{n=0}^{N-1} Y[n]\right) = \frac{1}{N^2}\sum_{n=0}^{N-1} Var(Y[n]) = \frac{1}{N^2}\cdot N \cdot \sigma^2 = \frac{\sigma^2}{N}$$

Now we evaluate Mean Squared Error (MSE):

$$MSE(\hat{\mu}) = \mathbb{E}[(\hat{\mu} - \mu)^2]$$

Because of the results in (a), $\mathbb{E}[\hat{\mu}] = \mu$,

and $Var(\hat{\mu}) = \mathbb{E}[(\hat{\mu} - \mathbb{E}[\hat{\mu}])]$, so $\mathbb{E}[\hat{\mu} - \mu] = \mathbb{E}[(\hat{\mu} - \mathbb{E}[\hat{\mu}])] = Var(\hat{\mu})$

so

$$MSE(\hat{\mu}) = Var(\hat{\mu}) = \frac{\sigma^2}{N} \to 0 \; as \; N \to \infty$$

Hence, $\hat{\mu}$ is consistent