I225E Statistical Signal Processing

4. Stochastic Process and Systems I

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Agenda

- Review
- Practice Exercise Discussion
- Introduction to Stochastic System
- Memoryless System

REVIEW OF PROBABILITY THEORY & BASICS OF STOCHASTIC PROCESS

Review Terminology

Stochastic Process

Definition: A stochastic process is a family of random variables,

$$\{X(t) : t \in T\},$$

where t usually denotes time.

Discrete-time Process

Definition: if the set T is finite or countable.

In practice, this generally means $T = \{0, 1, 2, 3, \ldots\}$

Continuous-time Process

Definition: if T is not finite or countable.

In practice, this generally means $T = [0, \infty)$, or $T = [0, K]$ for some K.

Review Terminology

State Space

Definition: The state space, S, is the set of real values that $X(t)$ can take.

Every X(t) takes a value in $\mathbb{R}$, but S will often be a smaller set: $S \subseteq \mathbb{R}$.

For example, if $X(t)$ is the outcome of a coin tossed at time t, then the state space is $S = \{0, 1\}$.

Discrete and Continuous

The state space S is *discrete* if it is finite or countable.

Otherwise, it is *continuous.*

Review Terminology

- **Independent Random Variables**

Consider two discrete random variables X and Y. We say that X and Y are independent if

$$P(X = x, Y = y) = P(X = x)P(Y = y), \text{ for all } x, y.$$

Remember, the concept of independent events A and B.

$$P(A, B) = P(A \cap B) = P(A)P(B)$$

Review Terminology

- **Expected value (=mean=average)**

$$EX = E[X] = E(X) = \mu X$$

- Theorem:
  - $E[aX + b] \ = \ aE[X] + b$, for all $a, b \in \mathbb{R}$
  - $E[X_1 + X_2 + \cdots + X_n] = E[X_1] + \cdots + \mathrm{E}[X_n]$, for any set of random variables $X_1, X_2, \ldots, X_n$.
- Expected value of a function of a random variable:

$$E[g(X)] = \sum_{x_k \in R_x} g(x_k) P_X(x_k)$$

Review Terminology

- **Variance**

$$Var(X) = E[(X - \mu X)2].$$

  - $Var(X) = E[X^2] - [EX]^2$

- **Standard Deviation**

$$SD(X) = \sigma X = \sqrt{Var(X)}.$$

- Theorem:
  - For a random variable X and real numbers a and b, $\boldsymbol{Var(aX + b) = a^2 Var(X)}$
  - If $X_1, X_2, .., X_n$ are independent random variables and $X = X_1 + X_2 + \cdots + X_n$, then

$$\boldsymbol{Var(X) = Var(X_1) + Var(X_2) + \cdots + Var(X_n)}$$

Special Distribution (Discrete)

- **Bernouli distribution**
- **Notation:** $X \sim Bernouli(p)$.
- **Description:** A random variable X is a Bernoulli random variable with parameter p, where $0 < \text{p} < 1$.
- **Probability function:**

$$f_X(x) = \begin{cases} p & \text{for } x = 1 \\ 1 - p & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

- **Mean:** $E(X) = p$.
- **Variance:** $Var(X) = pq$, where $q = 1 - p$.

Special Distribution (Discrete)

- **Binomial distribution**
- **Notation:** $X \sim Binomial(n, p)$.
- **Description:** number of successes in n independent trials, each with probability p of success.
- **Probability function:**

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, \ldots, n.$$

- **Mean:** $E(X) = np$.
- **Variance:** $Var(X) = np(1-p) = npq$, where $q = 1 - p$.
- **Sum:** If $X \sim Binomial(n, p)$, $Y \sim Binomial(m, p)$, and X and Y are independent, then

$$X + Y \sim Binomial(n + m, p).$$

Special Distribution (Discrete)

- **Poisson distribution**
- **Notation:** $X \sim Poisson(\lambda)$.
- **Probability function:**

$$f_X(x) = P(X = x) = \frac{\lambda^x}{x!}e^{-\lambda}, \text{ for } x = 0, 1, 2, \ldots$$

- **Mean:** $E(X) = \lambda$.
- **Variance:** $Var(X) = \lambda$.
- **Sum:** If $X \sim Poisson(\lambda)$, $Y \sim Poisson(\mu)$, and X and Y are independent, then

$$X + Y \sim Poisson(\lambda + \mu).$$

Special Distribution (Continuous)

- **Uniform distribution**
- **Notation:** $X \sim Uniform(a, b)$.
- **Probability density function (PDF):**

$f_X(x) = \frac{1}{b-a}$, for $a < x < b$

- **Cumulative distribution function:**

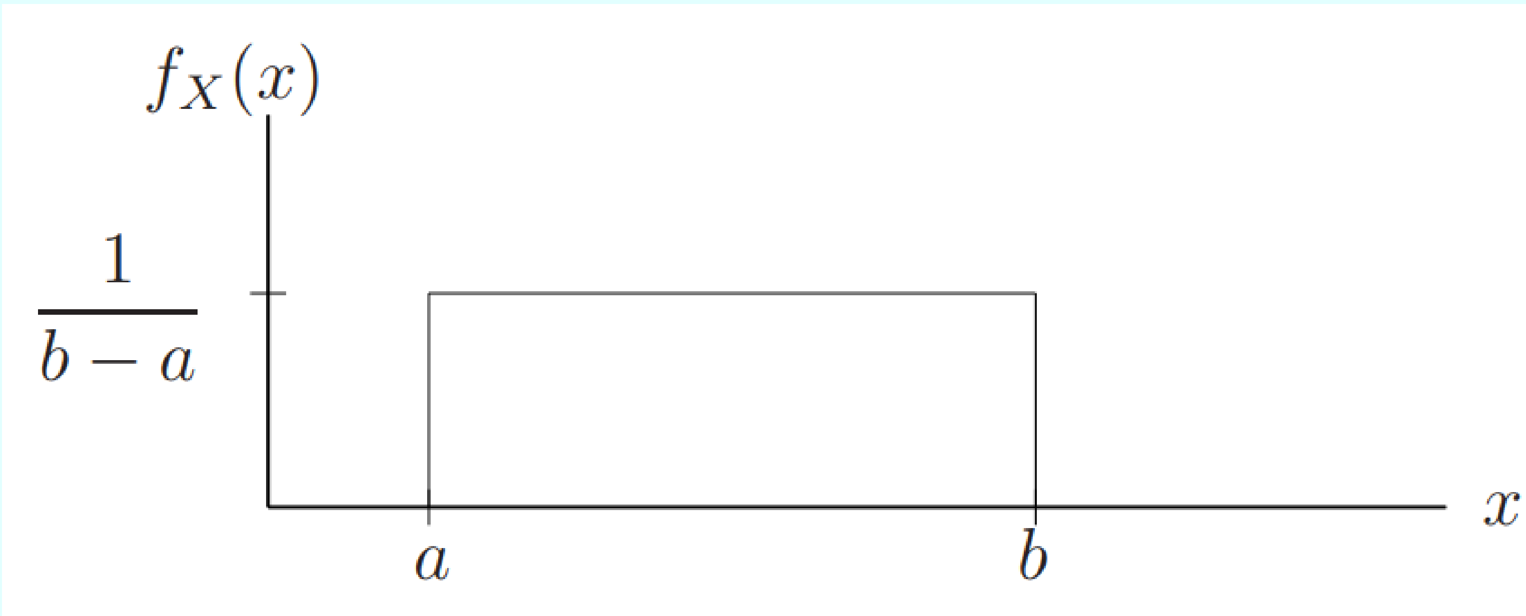
$F_X(x) = P(X \leq x) = \frac{x-a}{b-a}$, for $a < x < b$

$F_X(x) = 0$, for $x \leq a$, and $F_X(x) = 1$, $x \geq b$

- **Mean:** $E(x) = \frac{a+b}{2}$
- **Variance:** $Var(X) = \frac{(b-a)^2}{12}$

Special Distribution (Continuous)

- **Uniform distribution** (a, b)



Special Distribution (Continuous)

- **Exponential distribution**
- **Notation:** $X \sim Exponential(\lambda)$.
- **Probability density function (PDF):**

$f_X(x) = \lambda e^{-\lambda x}$, for $0 < x < \infty$

- **Cumulative distribution function:**

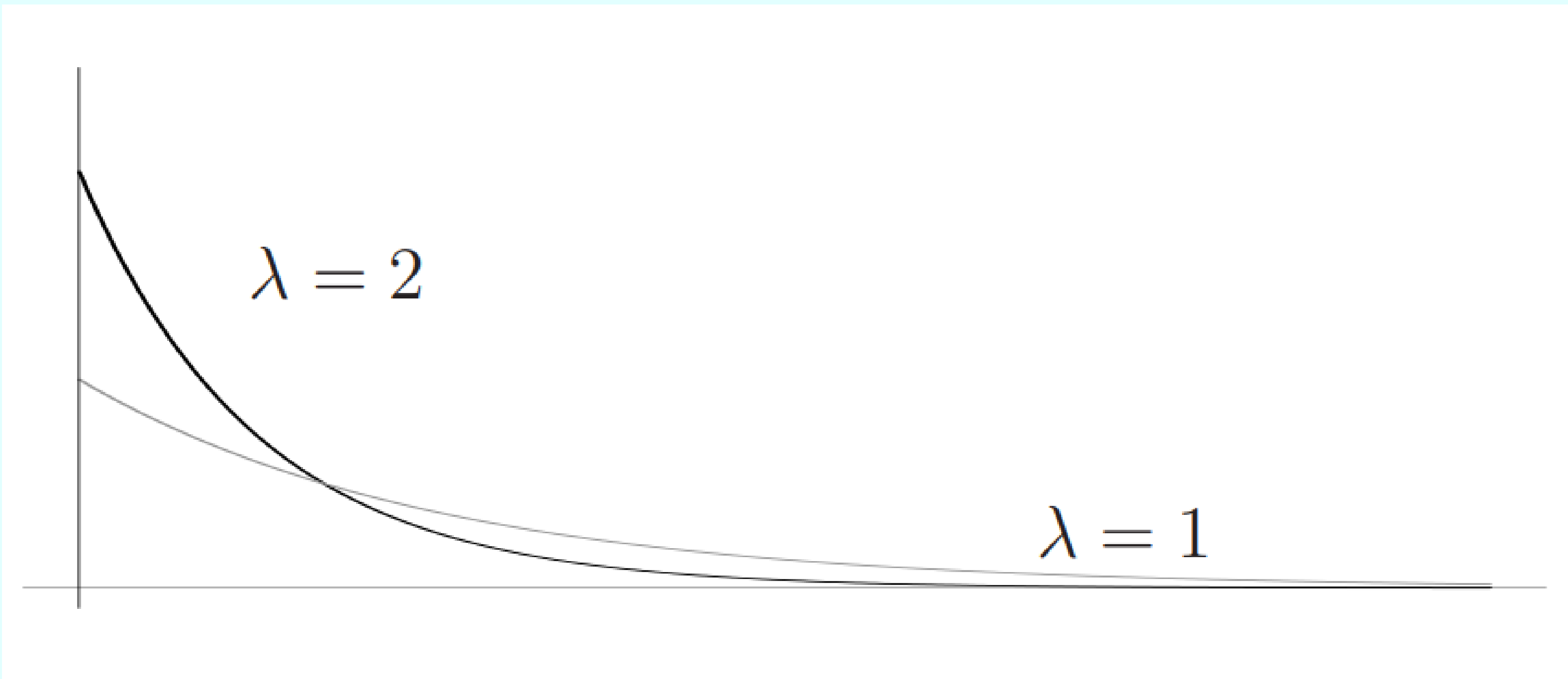
$F_X(x) = P(X \leq x) = 1 - e^{-\lambda x}$, for $0 < x < \infty$

$F_X(x) = 0$, for $x \leq 0$

- **Mean:** $E(x) = \frac{1}{\lambda}$
- **Variance:** $Var(X) = \frac{1}{\lambda^2}$

Special Distribution (Continuous)

- **Exponential distribution** (λ)



Special Distribution (Continuous)

- **Gamma distribution**
- **Notation:** $X \sim Gamma(k, \lambda)$.
- **Probability density function (PDF):**

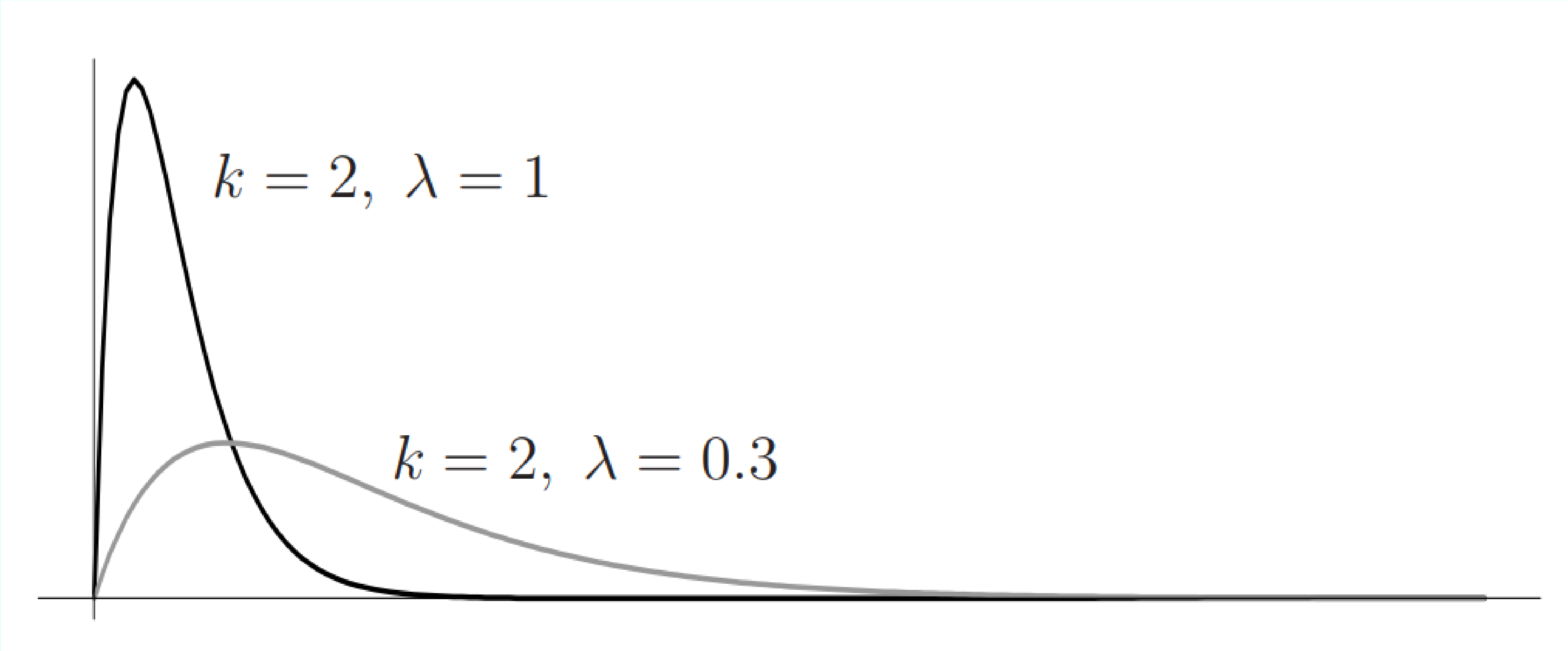
$$f_X(x) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}, \text{ for } 0 < x < \infty$$

where $\Gamma(k) = \int_0^\infty y^{k-1} e^{-y} dy$ (the Gamma function).

- **Cumulative distribution function:** no closed form.
- **Mean:** $E(x) = \frac{k}{\lambda}$
- **Variance:** $Var(X) = \frac{k}{\lambda^2}$

Special Distribution (Continuous)

- **Gamma distribution** (k, λ)



Special Distribution (Continuous)

- **Normal distribution**
- **Notation:** $X \ \sim Normal(\mu, \sigma^2)$.
- **Probability density function (PDF):**

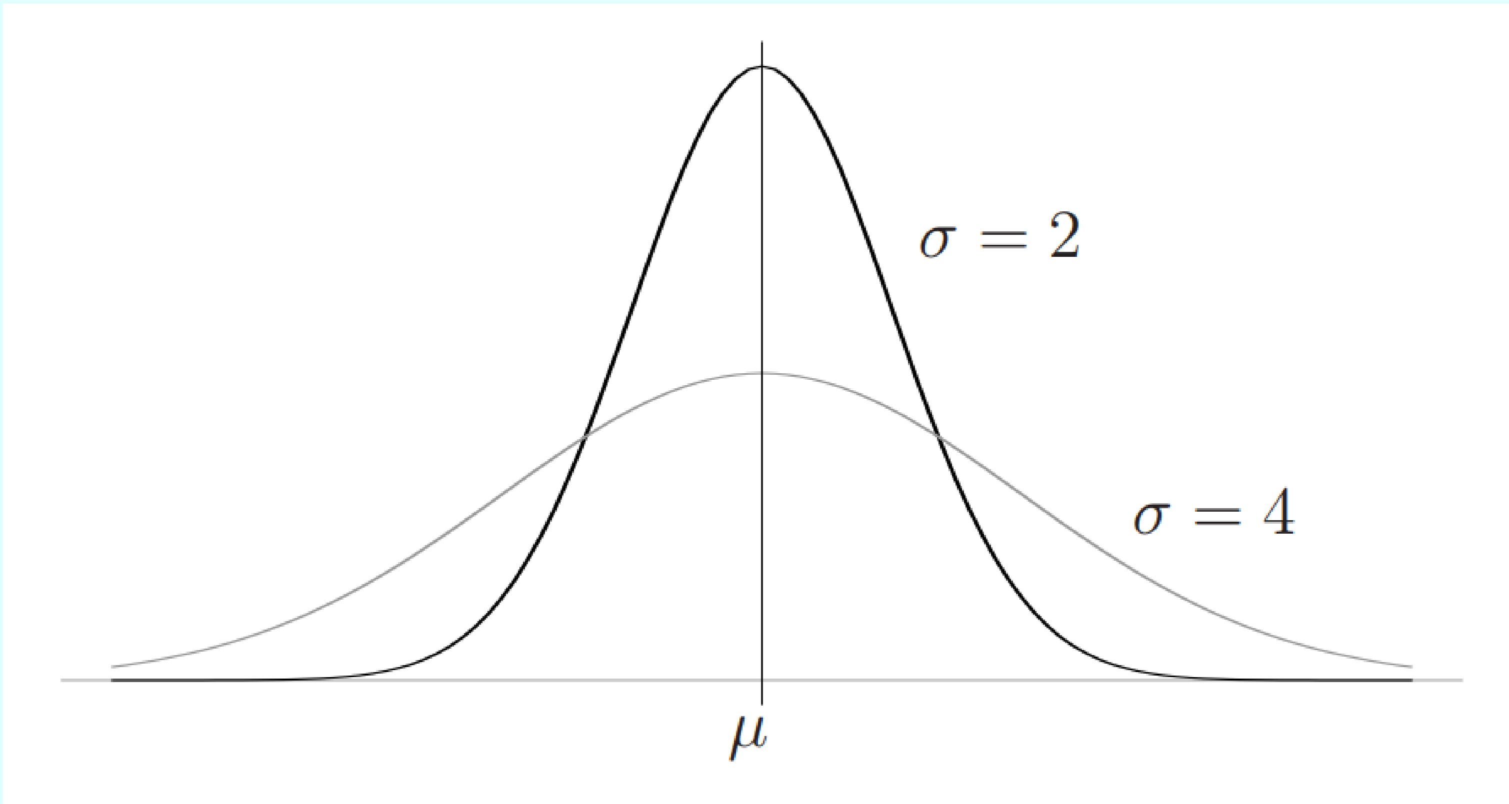
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\{-\frac{(x-\mu)^2}{2\sigma^2}\}}, \text{ for } -\infty < x < \infty$$

where $\Gamma(k)$=$\int_0^\infty y^{k-1}e^{-y}dy$ (the Gamma function).

- **Cumulative distribution function:** no closed form.
- **Mean:** $E(x) = \mu$
- **Variance:** $Var(X) \ = \sigma^2$

Special Distribution (Continuous)

- **Normal distribution** $(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$



Autocorrelation Function (ACF)

- **Autocorrelation** of $\boldsymbol{X}(t)$

$$\rho_X(t_1, t_2) = Corr\left(X_{t_1}, X_{t_2}\right) =$$
$$R_{\boldsymbol{XX}}(t_1, t_2) = E\{\boldsymbol{X}(t_1)\boldsymbol{X}(t_2)\} =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{\boldsymbol{X}}(x_1, x_2; t_1, t_2) dx_1 dx_2$$

where **sample autocorrelation** is

$$\bar{R}_{\boldsymbol{XX}}\,(t_1, t_2) = \frac{1}{n}\sum_{t=1}^{n} \boldsymbol{X}(t + t_1)\boldsymbol{X}(t + t_2)$$

Covariance Function

- The autocovariance function $\gamma_X(t_1, t_2)$ of a stochastic process $\{X_t\}$ measures the covariance between the process at two different time points t_1 and t_2:

$$\gamma_X(t_1, t_2) = Cov\left(X_{t_1}, X_{t_2}\right) = C_{\boldsymbol{XX}}(t_1, t_2)$$
$$= E\left[\left(X_{t_1} - \mu_X(t_1)\right)\left(X_{t_2} - \mu_X(t_2)\right)\right]$$
$$= R_{\boldsymbol{XX}}\,(t_1, t_2) - \eta_{\boldsymbol{X}}(t_1)\eta_{\boldsymbol{X}}(t_2)$$

- For a <u>weakly stationary </u>process, the autocovariance function depends only on the time difference (lag) $k = t_2 - t_1$.
- In the case of $t_1 = t_2 = t$, $C_{\boldsymbol{XX}}(t_1, t_2)$ is equal to variance of

$$\boldsymbol{X}(t) \rightarrow C_{\boldsymbol{XX}}(t, t) = E\{\boldsymbol{X}(t)\boldsymbol{X}(t)\} - \eta_{\boldsymbol{X}}^2(t) = Var\left(\boldsymbol{X}(t)\right)$$

Complex process

$\boldsymbol{X}(t) = \boldsymbol{Y}(t) + j\boldsymbol{Z}(t)$: complex variable $\boldsymbol{X}(t)$ is composed of real part $\boldsymbol{Y}(t)$ and imaginary part $\boldsymbol{Z}(t)$.

$$Corr\left(X_{t_1}, X_{t_2}\right) = R_{\boldsymbol{XX}}(t_1, t_2) = E\{\boldsymbol{X}(t_1)\boldsymbol{X}^*(t_2)\}$$

$$Corr\left(X_{t_1}, X_{t_2}\right) = R_{\boldsymbol{XX}}(t_2, t_1) = R^*_{\boldsymbol{XX}}(t_1, t_2)$$

$$Corr(X_t, X_t) = R_{\boldsymbol{XX}}(t, t) = E\{|\boldsymbol{X}(t)|^2\} \geq 0$$

$$Cov\left(X_{t_1}, X_{t_2}\right) = C_{\boldsymbol{XX}}(t_1, t_2) = R_{\boldsymbol{XX}}(t_1, t_2) - \eta_{\boldsymbol{X}}(t_1)\eta^*_{\boldsymbol{X}}(t_2)$$

Correlation coefficient

$$r(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)C_{XX}(t_2, t_2)}}$$

Previous Exercise

[Exercise]

A complex stochastic process Z(t) is given by:

$$Z(t) = X(t) + iY(t)$$

Where:

$X(t)$ and $Y(t)$ are real-valued random processes.

i is the imaginary unit.

$E[X(t)] = 1,\ E[Y(t)] = 2$, for all t.

$Var[X(t)] = 4,\ Var[Y(t)] = 9$, for all t.

$Cov(X(t), Y(t)) = 0$, for all t.

The autocorrelation function of $X(t)$ is $R_{XX}\ (t_1, t_2) = 4 \cdot \exp(-\frac{|t1 - t2|}{2})$.

The autocorrelation function of $Y(t)$ is $R_{YY}\ (t_1, t_2) = 9 \cdot \cos\big(\pi * (t1 - t2)\big)$.

Previous Exercise

[Exercise]

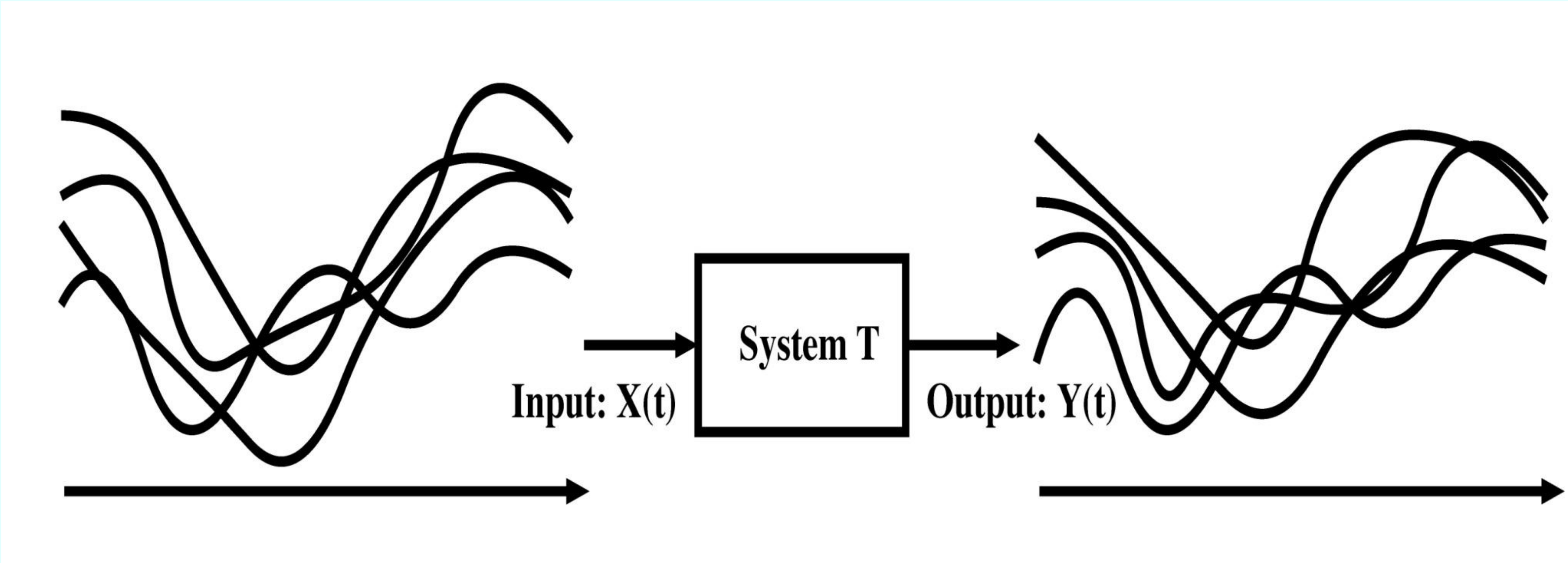
a) Calculate the mean function of the complex process $Z(t)$.

b) Calculate the autocovariance function of the complex process $Z(t)$.

c) Calculate the autocorrelation function of the complex process $Z(t)$.

d) Calculate the correlation coefficient between $Z(t_1)$ and $Z(t_2)$.

1. System with stochastic input



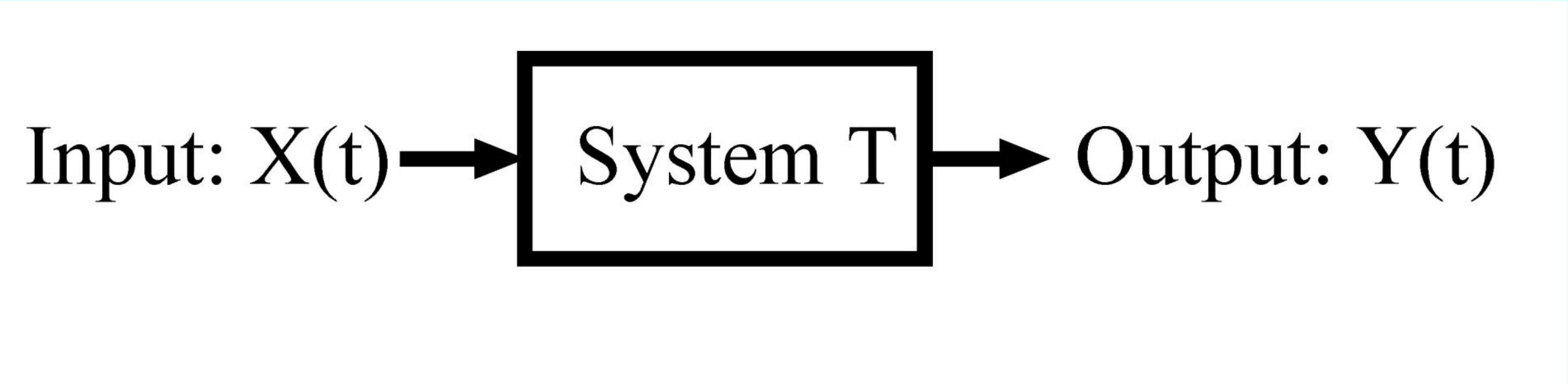
System

Given a stochastic process $\boldsymbol{X}(t)$ as input, $\boldsymbol{Y}(t)$ represents its output.

$$\boldsymbol{Y}(t) = T[\boldsymbol{X}(t)]$$

Purpose:

If statistical properties of input $\boldsymbol{X}(t)$ are known, study the statistical properties of output $\boldsymbol{Y}(t)$



$$\boldsymbol{Y}(t) = T[\boldsymbol{X}(t)]$$

System Dynamics

- **Deterministic system:**

  System operates only on variable t, treating outcome ω as a parameter. Namely,

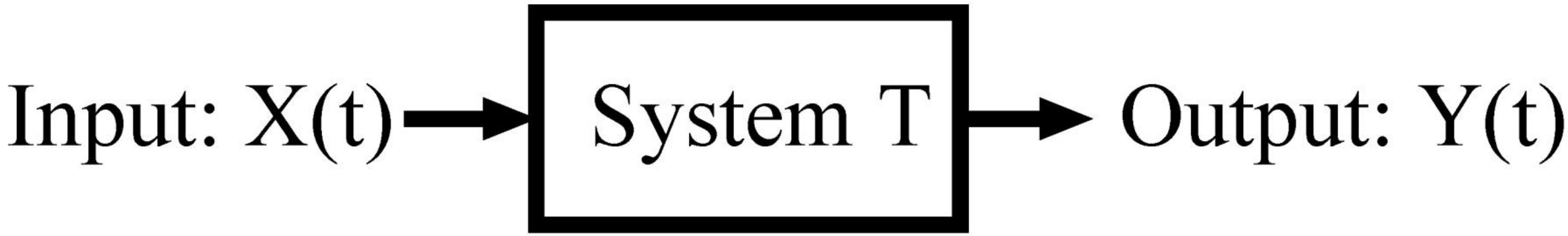
  If $\boldsymbol{X}(t,\omega_1) = \boldsymbol{X}(t,\omega_2)$, then $\boldsymbol{Y}(t,\omega_1) = \boldsymbol{Y}(t,\omega_2)$,

- **Stochastic system:**

  System operates on both t and ω. Namely,

  Even if $\boldsymbol{X}(t,\omega_1) = \boldsymbol{X}(t,\omega_2)$, $\boldsymbol{Y}(t,\omega_1) \neq \boldsymbol{Y}(t,\omega_2)$.

Example: Physical element of the system or coefficient of the system equation is stochastic.



$$\boldsymbol{Y}(t) = T[\boldsymbol{X}(t)]$$

- This lecture deals with only **deterministic systems**.
- In deterministic systems, transformation T may depend on t. To emphasize this, sometimes denoted as

$$\boldsymbol{Y}(t) = T_t[\boldsymbol{X}(t)]$$

referred to as a time-dependent system.

Deterministic System

Memoryless System

$$Y(t) = g[X(t)]$$

System with Memory

Time-Varying System

Time-Invariant System

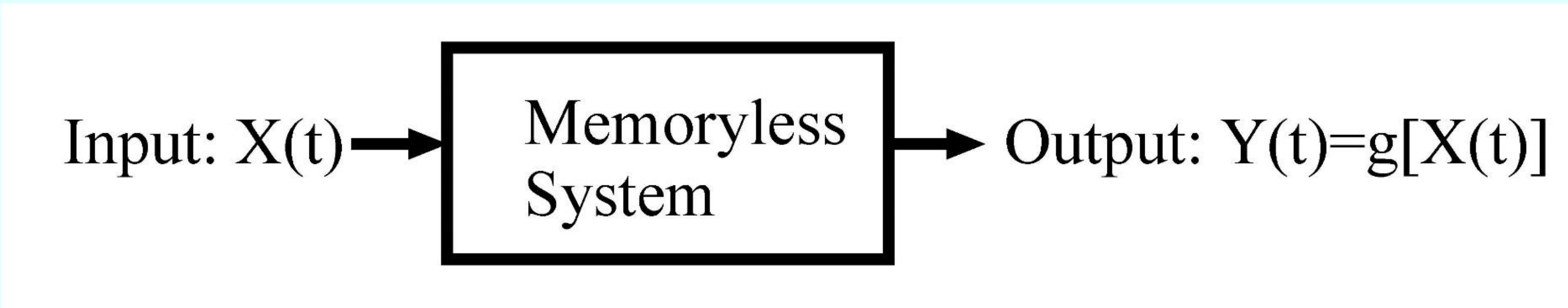
Linear System

$$Y(t) = L[X(t)]$$

Linear Time-Invariant system

$$Y(t)=X(t)*h(t)$$

2. Memoryless System



- **Output** $\boldsymbol{Y}(t_1)$ at time $t = t_1$ depends only upon the simultaneous state of input $\boldsymbol{X}(t_1)$, but not upon past or future state of $\boldsymbol{X}(t)$

$$\boldsymbol{Y}(t) = g[\boldsymbol{X}(t)]$$

Mean of the Output

- The mean (or expected value) of the output process $Y(t)$, denoted by $E[Y(t)]$ or $\mu_Y(t)$, can be found using the law of the unconscious statistician:

$$\text{(a1) } \mu_Y(t) = E\{\boldsymbol{Y}(t)\} = E\left[g\left(X(t)\right)\right] = \int_{-\infty}^{\infty} g(x) f_{\boldsymbol{X}}(x,t)\,dx$$

where $f_X(x,t)$ is the first-order probability density function (PDF) of the input process $X(t)$ at time t.

Correlation of the Output

- The autocorrelation function of the output process $Y(t)$, denoted by $R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)]$, measures the statistical dependence between the output at two different times t_1 and t_2 .
- For a memoryless system, $Y(t_1)$ depends only on $X(t_1)$, and $Y(t_2)$ depends only on $X(t_2)$. Therefore, the autocorrelation of the output is:

(a2) $E\{\boldsymbol{Y}(t_1)\boldsymbol{Y}(t_2)\} = E\{g\big(\boldsymbol{X}(t_1)\big)g(\boldsymbol{X}(t_2))\}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)g(x_2)f_{\boldsymbol{X}}(x_1, x_2; t_1, t_2)dx_1dx_2$$

(a3) *n*th-order density of $\boldsymbol{Y}(t)$, $f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n; t_1, t_2, \cdots, t_n)$

is obtained from nth-order density of $\boldsymbol{X}(t)$,

$f_{\boldsymbol{X}}(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n)$ through transformation

$\boldsymbol{Y}(t_1) = g[\boldsymbol{X}(t_1)]$, $\boldsymbol{Y}(t_2) = g[\boldsymbol{X}(t_2)]$, $\cdots, \boldsymbol{Y}(t_n) = g[\boldsymbol{X}(t_n)]$.

If the following system $y_1 = g[x_1], y_2 = g[x_2], \cdots, y_n = g[x_n]$ has a unique solution $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$, *n*th-order density of $\boldsymbol{Y}(t)$ is obtained as

$$f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n; t_1, t_2, \cdots, t_n) = \frac{f_{\boldsymbol{X}}(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n)}{|\boldsymbol{J}\ (x_1, x_2, \cdots, x_n)|}$$

where $\boldsymbol{J}$ is Jacobian $\boldsymbol{J} = [g'(x_1) g'(x_2) \cdots g'(x_n)]$.

When more than two solutions exist, summation of the corresponding terms $\frac{f_{\boldsymbol{X}}}{|\boldsymbol{J}|}$ gives the *n*th-order density.

Digression on coordinate transformation

- Let us consider *n*-dim variables $\mathbf{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ and $\mathbf{y} = (y_1, \cdots, y_n)^{\mathrm{T}}$ and the mapping $y_i = y_i(x_1, \cdots, x_n)$=$y_i(\mathbf{x})$. In this case, their infinitesimal volume are related as

$$dy_1 \cdots dy_n = |J(x_1, \cdots, x_n)| dx_1 \cdots dx_n$$

where the matrix is called the Jacobian:

$$J = \begin{pmatrix} \partial y_1 / \partial x_1 & \cdots & \partial y_1 / \partial x_n \\ \vdots & \ddots & \vdots \\ \partial y_n / \partial x_1 & \cdots & \partial y_n / \partial x_n \end{pmatrix}$$

- Accordingly, their probability densities are related as

$$f_{\mathbf{y}}(y_1, \cdots, y_n) = \frac{1}{|J(x_1, \cdots, x_n)|} f_{\mathbf{x}}(x_1, \cdots, x_n)$$

Appendix

Following has been used for the derivation of the density in (a3).

With respect to random variables $\boldsymbol{X} = [X_1, X_2, \cdots, X_n]$, n functions

$$Y_1 = g_1(\boldsymbol{X}),\ Y_2 = g_2(\boldsymbol{X}),\ \cdots,\ Y_n = g_n(\boldsymbol{X}),$$

are given. For *n* random numbers $\boldsymbol{Y} = [Y_1, Y_2, \cdots, Y_n]$, we determine their joint density $f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n)$, where $y_1, y_2, \cdots, y_n$ represent a specific set of numbers.

To find the density, we solve the system

$$g_1(\boldsymbol{X}) = y_1,\ g_2(\boldsymbol{X}) = y_2,\ \cdots,\ g_n(\boldsymbol{X}) = y_n.$$

If the system has no solution, then $f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n) = 0$. If the system has a single solution **x**$= [x_1, x_2, \cdots, x_n]$, the density can be obtained by substituting the solution into following formula

$$f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n) = \frac{f_{\boldsymbol{X}}(x_1, x_2, \cdots, x_n)}{|J(x_1, x_2, \cdots, x_n)|},$$

where $$J(x_1, x_2, \cdots, x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}$$

If more than two solutions exist, the density is given by summation of all the corresponding terms

$$f_{\boldsymbol{Y}} = \left.\frac{f_{\boldsymbol{X}}}{|J|}\right|_{\boldsymbol{X}=\boldsymbol{X}_1} + \left.\frac{f_{\boldsymbol{X}}}{|J|}\right|_{\boldsymbol{X}=\boldsymbol{X}_2} + \cdots$$

[Exercise]

Imagine you have a transformation that takes a point (u, v) in the uv-plane and maps it to a point (x, y) in the xy-plane. Let's say this transformation is defined by the following equations:

$$x(u, v) = u^2\, v$$

$$y(u, v) = u + v^2$$

Calculate the Jacobian matrix of the output variables (x and y) with respect to the input variables (u and v).

[Answer]

$$J(u,v) = \begin{pmatrix} 2uv & u^2 \\ 1 & 2v \end{pmatrix}$$

(a4) If input $\boldsymbol{X}(t)$ is strict sense stationary, output $\boldsymbol{Y}(t)$ is also strict sense stationary.

[Proof] According to (a3), *n*th-order density of $\boldsymbol{Y}(t)$ is given as

$$f_{\boldsymbol{Y}}(y_1, y_2, \cdots, y_n; t_1, t_2, \cdots, t_n) = \frac{f_{\boldsymbol{X}}(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n)}{|\boldsymbol{J}\,(x_1, x_2, \cdots, x_n)|}$$

Since $\boldsymbol{X}(t)$ is strict sense stationary, its density is invariant to a shift of the origin in time, denominator is independent of t. Therefore, $f_{\boldsymbol{Y}}$ is also invariant to time-shift. This proves that $\boldsymbol{Y}(t)$ is strict sense stationary.

- From the properties of strict sense stationary

(i) First-order density of $\boldsymbol{Y}(t)$ is independent of t

$$\rightarrow f_{\boldsymbol{Y}}(y;t) = f_{\boldsymbol{Y}}(y)$$

(ii) Second-order density is a function of time lag $\tau = t_1 - t_2$

$$\rightarrow f_{\boldsymbol{Y}}(y_1, y_2; t_1, t_2) = f_{\boldsymbol{Y}}(y_1, y_2; \tau)$$

Example of memoryless system

Square-law detector

Square-law detector is a memoryless system whose output equals

$$\boldsymbol{Y}(t) = \boldsymbol{X}^2(t).$$

Using the density $f_{\boldsymbol{X}}(x, t)$ of input $\boldsymbol{X}(t)$, find the density $f_{\boldsymbol{Y}}(y, t)$ of output $\boldsymbol{Y}(t)$.

First-order density

If $y > 0$, solutions of $y = x^2$ are $x = \pm\sqrt{y}$.

The corresponding Jacobian matrices are $J = \frac{dx^2}{dx} = 2x = \pm 2\sqrt{y}$.

Hence

$$f_Y(y;t) = \frac{f_X}{|J|}\bigg]_{x=\sqrt{y}} + \frac{f_X}{|J|}\bigg]_{x=-\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\left[f_X\left(\sqrt{y};t\right) + f_X\left(-\sqrt{y};t\right)\right]$$

- **Second-order density**:

If $y_1 > 0$, $y_2 > 0$, solutions of $y_1 = x_1^2$, $y_2 = x_2^2$ are $\left(\pm\sqrt{y_1}, \pm\sqrt{y_2}\right)$. Since the corresponding Jacobian matrices are $J = \pm 4\sqrt{y_1 y_2}$,

$$f_Y(y_1, y_2; t_1, t_2) = \frac{1}{4\sqrt{y_1 y_2}} \sum f_X\left(\pm\sqrt{y_1}, \pm\sqrt{y_2}; t_1, t_2\right)$$