[I225] Statistical Signal Processing(E) Office Hour 6

1. There are four boxes (Box 1 to Box 4), and each box contains a certain number of lottery tickets as described below:

Box 2: 500 tickets (50% of them are winning tickets)

Box 3: 200 tickets (20% of them are winning tickets)

Box 4: 1000 tickets (5% of them are winning tickets)

Now, one of the four boxes is chosen at random, and then one ticket is drawn at random from the selected box.

(a) What is the probability of drawing a winning ticket?

(b) Given that the ticket drawn is a winning ticket, what is the probability that it was drawn from Box 2?

Answer:

Let's define the events:

- B_i: The event that Box i was chosen, for $i = 1, 2, 3, 4$
- A: The event that the ticket drawn is a winning ticket

Since each box is equally likely to be chosen:

$$P(B_1) = P(B_2) = P(B_3) = P(B_4) = \frac{1}{4}$$

Conditional probabilities of drawing a winning ticket from each box:

$$P(A|B_1) = 0.10$$

$$P(A|B_2) = 0.50$$

$$P(A|B_3) = 0.20$$

$$P(A|B_4) = 0.05$$

(a) Total Probability of Drawing a Winning Ticket

Using the law of total probability:

$$P(A) = \sum_{i=1}^{4} P(B_i) * P(A|B_i)$$

$$=\frac{1}{4}(0.10 + 0.50 + 0.20 + 0.05)$$

$$=\frac{1}{4}0.85$$
$$= 0.2125$$

The probability of drawing a winning ticket is 0.2125.

(b)Using Bayes' Theorem:

$$P(B2|A) = (P(B_2) * P(A|B_2)) / P(A)$$
$$= (1/4 * 0.50) / 0.2125$$
$$= 0.125 / 0.2125 \approx 0.5882$$

the probability that the winning ticket came from Box 2 is approximately 0.5882

2. A complex stochastic process Z(t) is defined as:

$$Z(t) = 3X(t) - iY(t)$$

Where:X(t) and Y(t) are real-valued random processes. i is the imaginary unit.

You are given the following information:

$$E[X(t)] = 1, E[Y(t)] = 2$$
$$Var[X(t)] = 3, Var[Y(t)] = 4$$
$$Cov\big(X(t), Y(t)\big) = 0$$

The autocorrelation functions:

$$R_{XX}(t_1, t_2) = 3 + \sin\left(\omega_0(t_1 - t_2)\right)$$

$$R_{YY}(t_1, t_2) = 4 \cdot \cos\big(\omega_0(t_1 - t_2)\big)$$

(a) Calculate the mean function of $Z(t)$

(b) Calculate the autocovariance function of $Z(t)$

(c) Calculate the autocorrelation function of $Z(t)$

(d) Calculate the correlation coefficient between $Z(t_1)$ and $\overline{Z(t_2)}$

Answer:

(a) Mean function:

$$\mathrm{E[Z(t)] = 3 \cdot E[X(t)] - i \cdot E[Y(t)] = 3 \cdot 1 - i \cdot 2 = 3 - 2i}$$

(b) Autocovariance function:

$$\mathrm{C_{zz}(t_1, t_2) = E[(Z(t_1) - E[Z(t_1)])(\overline{Z(t_2)} - E[\overline{Z(t_2)}])}$$

$$\mathrm{Z(t) = 3X(t) - iY(t),\ \overline{Z(t)} = 3X(t) + iY(t)}$$

$$C_{zz}(t_1, t_2) = E[3X(t_1) - iY(t_1) - 3 + 2i)(3X(t_1) + iY(t_1) - 3 - 2i)$$

Because

$$E[X(t_1)X(t_2)] = R_{XX}(t_1, t_2)$$

$$E[Y(t_1)Y(t_2)] = R_{YY}(t_1, t_2)$$

$$Cov\big(X(t), Y(t)\big) = 0$$

So

$C_{zz}(t_1, t_2) = 9R_{XX}(t_1, t_2) + +R_{YY} - 9 - 4 = 14 + 9\sin\big(\omega_0(t_1 - t_2)\big) + 4\cos\left(\omega_0(t_1 - t_2)\right)$

(c) Autocorrelation function:

$$R_{ZZ}(t_1, t_2) = E\left[Z(t_1)\overline{Z(t_2)}\right]$$

$Z(t_1)\overline{Z(t_2)} = (3X(t_1) - iY(t_1))(3X(t_2) + iY(t_2)) = 9\,X(t_1)X(t_2) + 3iX(t_1)\,Y(t_2) - 3iY(t_1)\,X(t_2) + Y(t_1)Y(t_2)$

Because

$$E[X(t_1)Y(t_2)] = E[X(t_1)]E[Y(t_2)] = 0$$

$$E[Y(t_1)X(t_2)] = E[Y(t_1)]E[X(t_2)] = 0$$

So

$$R_{ZZ}(t_1, t_2) = 9R_{XX}(t_1, t_2) + R_{YY}(t_1, t_2) = 27 + 9\sin\big(\omega_0(t_1 - t_2)\big) + 4\cos\big(\omega_0(t_1 - t_2)\big)$$

(d) Correlation coefficient:

$$\rho\big(Z(t_1)Z(t_2)\big) = \frac{C_z(t_1, t_2)}{\sqrt{C_z(t_1, t_1)}\sqrt{C_z(t_2, t_2)}}$$

Because when $t_1 = t_2$,
$C_{zz}(t_1, t_2) = 14 + 9\sin(0) + 4\cos(0) = 18 = Var[Z(t)]$

So

$$\rho\big(Z(t_1)Z(t_2)\big) = \frac{C_z(t_1, t_2)}{\sqrt{C_z(t_1, t_1)}\sqrt{C_z(t_2, t_2)}} = \frac{14 + 9\sin\big(\omega_0(t_1 - t_2)\big) + 4\cos\big(\omega_0(t_1 - t_2)\big)}{\sqrt{18}\sqrt{18}}$$

$$= \frac{14 + 9\sin(\cdot) + 4\cos(\cdot)}{18}$$

3. A call center records the number of calls received during 5 different one-hour intervals: Data: [1, 2, 3, 2, 2]

Assume the number of calls received in one hour follows a Poisson distribution with rate λ (calls per hour), and each hour is independent.
Answer the following:

(a) Write the likelihood function for this data given the parameter λ

(b) Write the log-likelihood function.

(c) Find the maximum likelihood estimate $\hat{\lambda}$ for the rate of calls per hour.

Answer:

(a) Likelihood Function:

The Poisson distribution's probability mass function is:

$$P(x_i; \lambda) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

For independent samples $x_1, x_2, x_3, ..., x_n$, the likelihood is:

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod x_i!}$$

In our data [1, 2, 3, 2, 2], we have:

$$\sum x_i = 10,$$

$$n = 5$$

So

$$L(\lambda) = \frac{\lambda^{14} e^{-5\lambda}}{1! \cdot 2! \cdot 3! \cdot 2! \cdot 2!}$$

We write as

$$L(\lambda) = \frac{\lambda^{10} e^{-5\lambda}}{C}, \text{where } C = \prod x_i!$$

(b) Log-Likelihood Function

Take the logarithm:

$$\ell(\lambda) = log\mathrm{L}(\lambda) = 10log\lambda - 5\lambda + \text{const}$$

(c) Maximum Likelihood Estimate

Differentiate and set to zero:

$$\frac{d\ell}{d\lambda} = \frac{10}{\lambda} - 5 = 0 \rightarrow \frac{10}{\lambda} = 5$$

$$\lambda = 2$$

4. Consider a continuous-time signal $x(t) = cos(2\pi \cdot 50t)$.

(a) What is the regular frequency f of this signal in Hertz (Hz)?

(b) What is the angular frequency ω of this signal in radians per second (rad/s)?

(Hint: You might find Euler's formula helpful: $\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}$)

(d) Now, find the Fourier Transform $X(\omega)$of $x(t)$.

Answer:

(a) From the signal $x(t) = \cos(2\pi \cdot 50t)$, we observe that:

$$f = 50Hz$$

(b) The angular frequency is:

$$\omega = 2\pi f = 2\pi \cdot 50 = 100\pi rad/s$$

(c) Using the identity:

$$\cos(2\pi \cdot 50t) = \frac{1}{2}\left(e^{j2\pi \cdot 50t} + e^{-j2\pi \cdot 50t}\right)$$

So the Fourier Transform in the frequency domain f is:

$$X(f) = \frac{1}{2}\delta(f - 50) + \frac{1}{2}\delta(f + 50)$$

(d) In the angular frequency domain ω, recall that $\omega = 2\pi f$, so:

$$X(\omega) = \frac{1}{2}\delta(\omega - 100\pi) + \frac{1}{2}\delta(\omega + 100\pi)$$

5. Consider a modulated signal defined as:

$$x(t) = \cos(2\pi \cdot 10t) + \cos(2\pi \cdot 40t)$$

(a) What are the regular frequencies f_1 and f_2 (in Hz) of the two cosine components?

(b) What are the angular frequencies ω_1 and ω_2 (in rad/s) of these components?

(c) Sketch (or describe) the frequency-domain representation $X(f)$ of $x(t)$.

(d) Express the Fourier transform $X(f)$ using delta functions.

(e) Repeat (d) using angular frequency ω and write $X(\omega)$.

Answers:

(a) The signal consists of two cosine terms.

Each term corresponds to a sinusoidal signal with a specific frequency. From the expression, we can read off the frequencies:

$$f_1 = 10Hz, f_2 = 40Hz$$

(b) The angular frequency ω is related to regular frequency f by the formula$\omega = 2\pi f$. Thus, the angular frequencies are:

$$\omega_1 = 2\pi \times 10 = 20\pi rad/s$$

$$\omega_2 = 2\pi \times 40 = 80\pi rad/s$$

(c) In the frequency domain, each cosine contributes two delta functions, one at $+f$ and one at $-f$. Therefore, the spectrum consists of spikes at ±10 Hz and ±40 Hz. These are symmetric about 0 Hz, which is typical for real-valued signals.

$$\delta(f - 10), \delta(f + 10)$$

$$\delta(f - 40), \delta(f + 40)$$

(d) Using the Fourier Transform property for$\cos(2\pi f t) = \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$, we can write the spectrum as:

$$X(f) = \frac{1}{2}\delta(f - 10) + \frac{1}{2}\delta(f + 10) + \frac{1}{2}\delta(f - 40) + \frac{1}{2}\delta(f + 40)$$

(e) By converting to angular frequency domain using $\omega = 2\pi f$, the delta locations are scaled accordingly. Each $\delta(f \pm a)$ becomes $\delta(\omega \pm 2\pi a)$, so we write:

$$X(\omega) = \frac{1}{2}\delta(\omega - 20\pi) + \frac{1}{2}\delta(\omega + 20\pi) + \frac{1}{2}\delta(\omega - 80\pi) + \frac{1}{2}\delta(\omega + 80\pi)$$