Given $\ln VIX = Y$ .... (1)

& $dY_t = \kappa(\mu - Y_t)\,dt + \sigma\,dW_t$ ...(2)

Now, consider

$$\int_0^z e^{\kappa t} Y_t\,dt$$

~~Dividing~~ Integrating by parts, we get

$$\int_0^z e^{\kappa t} Y_t\,dt = \frac{e^{\kappa t}}{\kappa} Y_t \Big|_0^z - \frac{1}{\kappa}\int_0^z e^{\kappa t}\,dY_t$$

Substituting the value of dY_t from (2)

$$\Rightarrow \int_0^z e^{\kappa t} Y_t\,dt = \frac{1}{\kappa}\left[e^{\kappa z} Y_z - Y_0\right] - \frac{1}{\kappa}\int_0^z e^{\kappa t}\left[\kappa(\mu - Y_t)\,dt + \sigma\,dW_t\right]$$

$$\Rightarrow \int_0^z e^{\kappa t}\cancel{Y_t}\,dt = \frac{1}{\kappa}\left[e^{\kappa z} Y_z - Y_0\right] - \mu\int_0^z e^{\kappa t}\,dt + \int_0^z e^{\kappa t}\cancel{Y_t}\,dt - \frac{\sigma}{\kappa}\int_0^z e^{\kappa t}\,dW_t$$

$$\Rightarrow 0 = \frac{1}{\kappa}\left[e^{\kappa z} Y_z - Y_0\right] - \frac{\mu}{\kappa}\left[e^{\kappa z} - 1\right] - \frac{\sigma}{\kappa}\int_0^z e^{\kappa t}\,dW_t$$

$$\Rightarrow e^{k\tau} Y_\tau = Y_0 + \mu\left(e^{k\tau} - 1\right) + \sigma \int_0^\tau e^{kt}\, dW_t$$

~~or $Y_t = Y_0 e^-$~~

$$\Rightarrow \boxed{Y_\tau = e^{-k\tau} Y_0 + \mu\left(1 - e^{-k\tau}\right) + \sigma \int_0^\tau e^{k(\tau - t)}\, dW_t} \quad \ldots (3)$$

Now, consider

$$\sigma \int_0^\tau e^{k(\tau - t)}\, dW_t = Z$$

$$Z \sim N(0, \sigma_*^2)$$

where $\sigma_*^2 \rightarrow Var(Z)$

$$Var(Z) = E[Z^2] - (E[Z])^2$$

$$= E[Z^2] - 0 \quad [\because Z \sim N(0, \sigma_*)]$$

$$\Rightarrow \sigma_*^2 = E\left[\left(\sigma \int_0^\tau e^{k(\tau - t)}\, dW_t\right)^2\right]$$

Using Ito's Isometry:

$$E\left[\left(\int_0^T X_t\, dW_t\right)^2\right] = E\left[\int_0^T X^2\, dt\right]$$

$$\Rightarrow \sigma_*^2 = \sigma^2 E\left[\int_0^\tau e^{-2k(t - \tau)}\, dt\right.$$

$$\Rightarrow \sigma_x^2 = \sigma^2 E\left[\frac{e^{-2k(t-\tau)}}{2k}\Big|_0^{\tau}\right]$$

$$\Rightarrow \boxed{\sigma_x^2 = \frac{\sigma^2}{2k}\left(1 - e^{-2k\tau}\right)} \quad \ldots (4)$$

Now, we know that

$$V^{\tau} = E[V \mid X_{\tau}] = E[e^{Y_t}]$$

Using (3) →

$$\Rightarrow V^{\tau} = E\left[\exp\left\{e^{-k\tau} Y_0 + \mu(1 - e^{-k\tau}) + \sigma \int_0^{\tau} e^{-k(t-\tau)} dW_t\right\}\right]$$

$$= \exp\left\{e^{-k\tau} Y_0 + \cancel{\mu(k} \mu - \mu e^{-k\tau}\right\} \cdot E\left[\exp\left\{\sigma \int_0^{\tau} e^{-k(t-\tau)} dW_t\right\}\right]$$

[∵ since the first 2 terms are constants]

we know that if X_t follows a normal distribution (μ, σ),

$$E[e^{X_t}] = e^{\mu + \frac{\sigma^2}{2}}$$

$$\Rightarrow \quad V^z = \exp\{e^{-kz}y_0 + \mu - \mu e^{-kz}\}$$

$$\cdot \exp\{0 + \tfrac{1}{2}\sigma_x^2\}$$

$$\Rightarrow \quad V^z = \exp\{e^{-kz}y_0 + \mu - \mu e^{-kz}\}$$

$$\cdot \exp\left\{\frac{\sigma^2}{4k}(1 - e^{-2kz})\right\}$$

$$= \exp\left\{e^{-kz}y_0 + \mu - \mu e^{-kz} + \frac{\sigma^2}{4k} - \frac{\sigma^2}{4k}e^{-2kz}\right\}$$

Now, $V^\infty = \lim_{z \to \infty} V^z$

$$= \exp\left\{\mu + \frac{\sigma^2}{4k}\right\}$$

$$\Rightarrow \quad V^z = V^\infty \exp\left\{e^{-kz}y_0 - \mu e^{-kz} - \frac{\sigma^2}{4k}e^{-2kz}\right\}$$

$$\Rightarrow \boxed{V^z = V^\infty \exp\left\{e^{-kz}(y_0 - \mu) - \frac{\sigma^2}{4k}e^{-2kz}\right\}}$$

$$\boxed{\text{or } V^\infty \exp\left\{e^{-kz}X - \frac{\sigma^2}{4k}e^{-2kz}\right\}}$$

(5)

Now $V^z = V^\infty \exp\left\{ e^{-kz} X - \frac{\sigma^2}{4k} e^{-2kz} \right\}$

$$\boxed{VIX = V^0 = V^\infty \exp\left\{ X - \frac{\sigma^2}{4k} \right\}} \quad \ldots (6)$$

Also, $\ln(V^z) = \ln V^\infty + e^{-kz} X - \frac{\sigma^2}{4k} e^{-2kz} \quad \ldots (7)$

From (6) we have that

$$\ln VIX = \ln V^\infty + X - \frac{\sigma^2}{4k}$$

or $X = \ln VIX - \ln V^\infty + \frac{\sigma^2}{4k}$

Substituting X in (7)

$$\Rightarrow \ln(V^z) = \ln V^\infty + e^{-kz}\left(\ln VIX - \ln V^\infty + \frac{\sigma^2}{4k} \right) - \frac{\sigma^2}{4k} e^{-2kz}$$

$$\Rightarrow \boxed{\ln(V^z) = e^{-kz} \ln VIX + (1 - e^{-kz}) \ln V^\infty + \frac{\sigma^2}{4k}\left(e^{-kz} - e^{-2kz} \right)}$$