Algorithmic Trading and Quantitative Strategies
Homework 4

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1 Cointegration

2 Kernel Impact Model

3 LQ Regulator

Denote the cost at time T to be $C_T(x) = x^T R_T x$. We will assume that it is quadratic in position x.

Start with the value function given in the question.

$$
\begin{aligned}
F(x,\tau) &= \inf_u \left\{ \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} R + A^T K_{\tau+1} A & S^T + A^T K_{\tau+1} B \\ S + B^T K_{\tau+1} A & Q + B^T K_{\tau+1} B \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\} \\
&= \inf_u \left\{ x^T R x + x^T A^T K_{\tau+1} A x + 2\left(u^T S x + u^T B^T K_{\tau+1} A x\right) + u^T Q u + u^T B^T K_{\tau+1} B u) \right\}
\end{aligned}
$$

Now we can substitute the optimal control

$$
u_t = -\left(Q + B^T K_{\tau+1} B\right)^{-1} \left(S + B^T K_{\tau+1} A\right) x_t
$$

We get

$$
\begin{aligned}
F(x,\tau) &= \inf_u \left\{ x^T \left[R + A^T K_{\tau+1} A\right] x + 2u^T \left[S + B^T K_{\tau+1} A\right] x + u^T \left[Q + B^T K_{\tau+1} B\right] u) \right\} \\
&= x^T \left[R + A^T K_{\tau+1} A - (S + B^T K_{\tau+1} A)^T (Q + B^T K_{\tau+1} B)^{-1} (S + B^T K_{\tau+1} A)\right] x \\
&= x^T K_\tau x
\end{aligned}
$$

where we use the result that $(Q + B^T K_{\tau+1} B)$ is symmetric and positive definite, so its inverse is also symmetric.

We finally have the **Riccati equation**.

$$
K_\tau = R + A^T K_{\tau+1} A - (S + B^T K_{\tau+1} A)^T (Q + B^T K_{\tau+1} B)^{-1} (S + B^T K_{\tau+1} A)
$$

with the boundary condition

$$
K_T = R_T
$$

NB: To be more rigorous, we need to show that $(Q + B^T K_\tau B)$ is invertible for all τ. I will not do this here and simply assume it to be true.

4 Stochastic LQ Regulator

1. First, I will present the optimization procedure used by Almgren and Chriss(2000).

   We need to liquidate a position from X to 0 between times 0 and T. We seek to minimize the expected shortfall $E(x)$ subject to certain constraints on the variance $V(x)$. We can rewrite this as a minimization problem with λ as a risk-aversion parameter.

$$\min_{x} \Big[E(x) + \lambda V(x)\Big]$$

   They adopt the following conventions.

   - We have N trading periods and x_i denotes the wealth at time $\frac{i}{N}T$
   - $\tau = \frac{T}{N}$ is the time between trading periods.
   - $n_k = x_{k-1} - x_k$ is the amount liquidated between two periods
   - The permanent impact $g(\frac{n_k}{\tau}) = \gamma \frac{n_k}{\tau}$
   - γ is the permanent price impact parameter
   - The temporary impact $h(\frac{n_k}{\tau}) = \epsilon \,\mathrm{sign}(n_k) + \eta \frac{n_k}{\tau}$
   - ϵ represents fixed linear trading costs (e.g. spread)
   - η represents quadratic trading costs coming from market impact (e.g temporary loss of liquidity in order book)

   The implementation shortfall and variance can thus be written as

$$\begin{aligned}
E(x) &= \sum_{k=1}^{N} \tau x_k g(\frac{n_k}{\tau}) + \sum_{i=1}^{N} n_k h(\frac{n_k}{\tau}) \\
&= \sum_{k=1}^{N} \gamma x_k n_k + \sum_{i=1}^{N} \epsilon |n_k| + \sum_{i=1}^{N} \frac{\eta}{\tau} n_k^2 \\
&\quad \vdots \qquad\quad \vdots \\
&= \frac{1}{2}\gamma X^2 + \epsilon \sum_{i=1}^{N} |n_k| + \frac{1}{\tau}\left(\eta - \frac{1}{2}\gamma\tau\right) \sum_{i=1}^{N} n_k^2 \\
V(x) &= \sigma^2 \sum_{k=1}^{N} \tau x_k^2
\end{aligned} \tag{1}$$

   They then proceed to relax the absolute value constraint by assuming that n_k never changes sign. This is the optimization problem they deal with for three quarters of the paper, where the stock has no expected drift.

   In Page 19 of the paper, Almgren and Chriss prove a theorem that a quadratic utility function has a "time-homogeneous" solution. This means the optimal solution from time t is simply the continuation of the optimal solution derived from time 0. This means their original formulation of the problem requires only a deterministic LQ regulator.

The minimization problem can be written as

$$\min_{x} \; \frac{1}{\tau}\left(\eta - \frac{1}{2}\gamma\tau\right)\sum_{k=1}^{N}(x_k - x_{k-1})^2 + \sigma^2 \sum_{k=0}^{N-1} \tau x_k^2$$

The LQ formulation of Almgren and Chriss' model is thus as follows:

Under **controls** $\mathbf{x_{k+1} = x_k + u_k}$,

$$F(x, t_k) = \inf_u \left\{ \left(\frac{\eta}{\tau} - \frac{\gamma}{2}\right) u^2 + \lambda\sigma^2\tau x^2 + F(x+u, t_{k+1}) \right\}$$

with the boundary condition

$$F(x, t_N) = \left(\frac{\eta}{\tau} - \frac{\gamma}{2}\right) x^2$$

Since we need to finish at a zero position, we are forced to liquidate at the end.

However, for this assignment we will **not** be using Almgren and Chriss' derivation. We will use Petter Kolm's modified approach found on slides 24-32 of his *'Dynamic Portfolio Analysis'* lecture slides.

Kolm starts by assuming stock returns follow

$$r_{t+1} = \mu_t + \alpha_t + \epsilon_{t+1}^r$$

and alpha is driven by mean-reverting factors and temporary and permanent impact.

$$r_{t+1} = \mu_t + \alpha_t + \epsilon_{t+1}^r$$

$$\alpha_t = \cancel{Bf_t + \epsilon_t^\alpha} + \Pi\Delta x_t + H\left(\Delta x_t - \Delta x_{t-1}\right) = \left(\Pi + H\right)\Delta x_t - H\Delta x_{t-1}$$

When the instructions say we can assume alpha is zero, we assume this refers to the alpha coming from the factors, not the price impact. Changing notations slightly, $u_{t-1} = (x_t - x_{t-1})$ are our controls.

Define $h_t = H\Delta x_{t-1}$ and consider the augmented state $s_t = (x_t, h_t)^T$.

$$\begin{pmatrix} x_t \\ h_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{t-1} \\ h_{t-1} \end{pmatrix} + \begin{pmatrix} 1 \\ H \end{pmatrix} u_{t-1} + \begin{pmatrix} 0 \\ \epsilon_t^h \end{pmatrix}$$

$$s_t = As_{t-1} + Bu_{t-1} + \epsilon_t \tag{2}$$

and we now have the updating equation after introducing noise to the temporary impact.

Our multiperiod optimization is given by

$$\max_{u_1, u_2, \ldots} E\left[\sum_{t=1}^{T-1}\left(\alpha_t x_t - \frac{\lambda}{2}x_t^T\Sigma x_t - \frac{1}{2}u_t^T\Lambda u_t\right) + \left(\alpha_T x_T - \frac{\lambda}{2}x_T^T\Sigma x_T\right)\right]$$

and we rewrite

$$\begin{aligned}\alpha_t x_t - \frac{\lambda}{2} x_t^T \Sigma x_t - \frac{1}{2} u_t^T \Lambda u_t &= x_t((\Pi + H)u_t - h_t) - \frac{\lambda}{2} x_t^T \Sigma x_t - \frac{1}{2} u_t^T \Lambda u_t \\ &= \begin{pmatrix} s_t \\ u_t \end{pmatrix}^T \begin{pmatrix} R & S \\ S^T & Q \end{pmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} \\ &= c(s_t, u_t)\end{aligned} \tag{3}$$

where

$$R = \begin{pmatrix} -\frac{\lambda}{2}\Sigma & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \qquad S = \begin{pmatrix} \frac{\Pi+H}{2} \\ 0 \end{pmatrix} \qquad Q = -\frac{1}{2}\Lambda$$

Combining Equations 2 and 3, our stochastic LQ problem is as follows ($\rho = 0$):

Process Dynamics:

$$s_t = As_{t-1} + Bu_{t-1} + \epsilon_t$$

Objective Function:

$$\max_{u_1, u_2, \ldots} E\left[\sum_{t=1}^{T-1} c(s_t, u_t) + C(s_T)\right]$$

where

$$c(s, u) = \begin{pmatrix} s_t \\ u_t \end{pmatrix}^T \begin{pmatrix} R & S \\ S^T & Q \end{pmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} \qquad\qquad C(s_T) = s_T^T R s_T$$

2. We will now guess a value function of the form $F(x, t) = x^T K_t x + \beta_t$. Assuming $E[\epsilon] = 0$, the dynamic programming equation becomes

$$\begin{aligned}F(s, t) &= \min_u E\left[c(s, u) + F(As + Bu + \epsilon, t + 1)\right] \\ &= \min_u s^T Rs + 2s^T Su + u^T Qu + (As + Bu)^T K_{t+1}(As + Bu) + E\left[\epsilon^T K_{t+1}\epsilon\right] + \beta_{t+1}\end{aligned}$$

Taking partial derivatives in u, the optimal control is given by

$$u = -\left(Q + B^T K_{t+1} B\right)^{-1}\left(S^T + B^T K_{t+1} A\right) s \tag{4}$$

with the Riccati equation given by

$$K_t = R + A^T K_{t+1} A - (S + A^T K_{t+1} B)(Q + B^T K_{t+1} B)^{-1}(S^T + B^T K_{t+1} A) \tag{5}$$

and the β_t update via

$$\beta_t = \beta_{t+1} + E[\epsilon^T K_{t+1}\epsilon]$$

We see that adding noise to the state has not affected the optimal controls in Equations 4 and 5. We proceed to optimize as if the problem was deterministic. However, the value function is larger than before with the introduction of the β_t term. Our system has to account for additional cost associated with correcting the noise. This is the certainty equivalence result of the Riccati recursion.

Kolm's approach and Almgren's are very similar. Referring to Equation 1 and setting $\epsilon = 0$, we remark the following equivalences to Equation 3.

- $\gamma \leftrightarrow \Pi$
- $\eta \leftrightarrow H$
- $\sigma^2\tau \leftrightarrow \Sigma$
- Kolm has introduced an additional $\frac{1}{2}u^T\Lambda u$ component for transaction costs, but this can be easily absorbed into the temporary impact $\frac{\eta}{\tau}$.

We would therefore expect both optimal trajectories to be very similar in shape when the noise is small (i.e. $E[\epsilon^2] << 1$). When the noise is large, the expected optimal trajectories would be similar in shape at time 0, but updating the optimizer at some time t in the future would result in a different path, destroying the "time homogeneity" found in Almgren and Chriss.

3. For the coding, we will use Almgren's values $\Pi = \gamma = 0.314$ and $H = \eta = 0.142$. Since it is unspecified in the question, we will take trading costs $\Lambda = 0$. The code *SingleStockLQ.py* contains the program. However, unlike the deterministic case, our code can only determine the optimal liquidation for the next step due to the stochastic nature of h_t. The user can input his current position x_t, current temporary impact h_t and the current time t and the code will return the optimal trade. In this problem, our optimal trade is to **sell 244,167 shares on Day 1**, assuming no prior temporary impact ($h_1 = 0$).

4. The larger the number of time periods used, the lower the temporary impact cost. Since it is not so easy to extract the full cost of the temporary impact from the code (since h_t is stochastic), I will instead provide justification using the structure of the problem.

   Start with N trading periods. Intuitively, doubling the number of time periods to 2N but trading on every even period corresponds to the original problem. Since we now have more optionality, the value function must be at least as low the N-period case.

   However, I argue that for very large N, the temporary impact costs are the same for a well-constructed model. Returning to Almgren and Chriss' model,

$$F(x, t_k) = \inf_u \left\{ \left(\frac{\eta}{\tau} - \frac{\gamma}{2}\right) u^2 + \lambda\sigma^2\tau x^2 + F(x + u, t_{k+1}) \right\}$$

   The number of terms terms double, but $\tau \to \frac{\tau}{2}$, so the total variance contribution will be approximately the same. The same result applies to the $\frac{\eta}{\tau}u^2$ temporary impact cost. This is because in a reasonable model, the temporary impact is a function of trading speed, hence

τ appearing in the denominator. It cannot go away instantaneously, so for very large N, additional intermediate periods will not significantly change the overall trading speed.