EE496 : COMPUTATIONAL INTELLIGENCE
FS03: FUZZY RELATIONS

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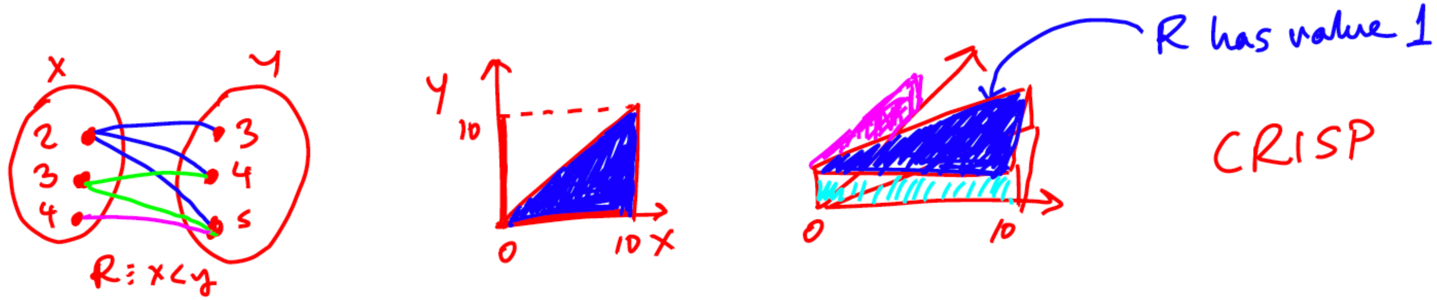
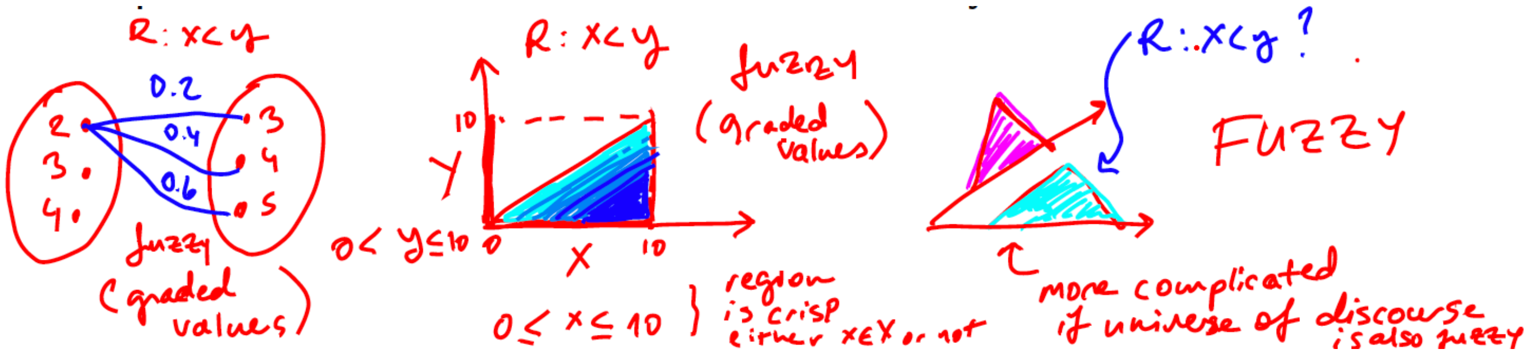
Crisp and Fuzzy Relations

A Crisp relation represents presence or absence of association, interaction or interconnection between elements of ≥ 2 sets.



This concept can be generalized to various degrees or strengths of association or interaction between elements.

A **fuzzy relation** generalizes these degrees to membership grades. So, a crisp relation is a restricted case of a fuzzy relation.



Definition of Relation (Crisp)

A relation among crisp sets $X_1, \ldots, X_n$ is a subset of $X_1 \times \ldots \times X_n$ denoted as $R(X_1, \ldots, X_n)$ or $R(X_i \mid 1 \le i \le n)$.

So, the relation $R(X1, \ldots, Xn) \subseteq X_1 \times \ldots \times X_n$ is set, too.

The basic concept of sets can be also applied to relations:

- containment, subset, union, intersection, complement

Each crisp relation can be defined by its characteristic function

$$R(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if and only if } (x_1, \ldots, x_n) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

The membership of $(x_1, \ldots, x_n)$ in R signifies that the elements of $(x_1, \ldots, x_n)$ are related to each other.

Relation as Ordered Set of Tuples (crisp)

A relation can be written as a set of ordered tuples.

Thus $R(X_1, \ldots, X_n)$ represents n-dim. membership array $R = [r_{i1,\ldots,in}]$.

- Each element of i_1 of R corresponds to exactly one member of X_1.
- Each element of i2 of R corresponds to exactly one member of X_2.
- And so on...

If $(x_1, \ldots, x_n) \in X1 \times \ldots \times Xn$ corresponds to $r_{i1,\ldots,in} \in R$, then

$$r_{i_1,\ldots,i_n} = \begin{cases} 1, & \text{if and only if } (x_1, \ldots, x_n) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

Fuzzy Relations

The characteristic function of a crisp relation can be generalized to allow tuples to have degrees of membership.

• Recall the generalization of the characteristic function of a crisp set!

Then a **fuzzy relation** is a fuzzy set defined on tuples $(x_1, \ldots, x_n)$ that may have varying degrees of membership within the relation.

The membership grade indicates strength of the present relation between elements of the tuple.

The fuzzy relation can also be represented by an n-dimensional membership array.

Example

Let R be a fuzzy relation between two sets X = {New York City, Paris} and Y = {Beijing, New York City, London}.

R shall represent relational concept “very far”.

This relation can be written in a list notation as

R(X,Y) =1/NYC, Beijing + 0/NYC, NYC + 0.6/NYC, London +
0.9/Paris, Beijing + 0.7/Paris, NYC + 0.3/Paris, London.

It can be also represented as two-dimensional membership array:

		x	
		NYC	Paris
y	Beijing	1	0.9
	NYC	1	0.7
	London	0.6	0.3

Cartesian Product of Fuzzy Sets: n Dimensions

Let $n \geq 2$ fuzzy sets $A_1, \ldots, A_n$ be defined in the universes of discourse $X_1, \ldots, X_n$, respectively.

The Cartesian product of $A_1, \ldots, A_n$ denoted by $A_1 \times \ldots \times A_n$ is a fuzzy relation in the product space $X_1 \times \ldots \times X_n$.

It is defined by its membership function

$$\mu_{A1} \times \ldots \times A_n(x_1, \ldots, x_n) = \top(\mu_{A1}(x_1), \ldots, \mu A_n(x_n))$$

whereas $x_i \in X_i$, $1 \leq i \leq n$.

Usually $\top$ is the minimum (sometimes also the product).

Cartesian Product of Fuzzy Sets: 2 Dimensions

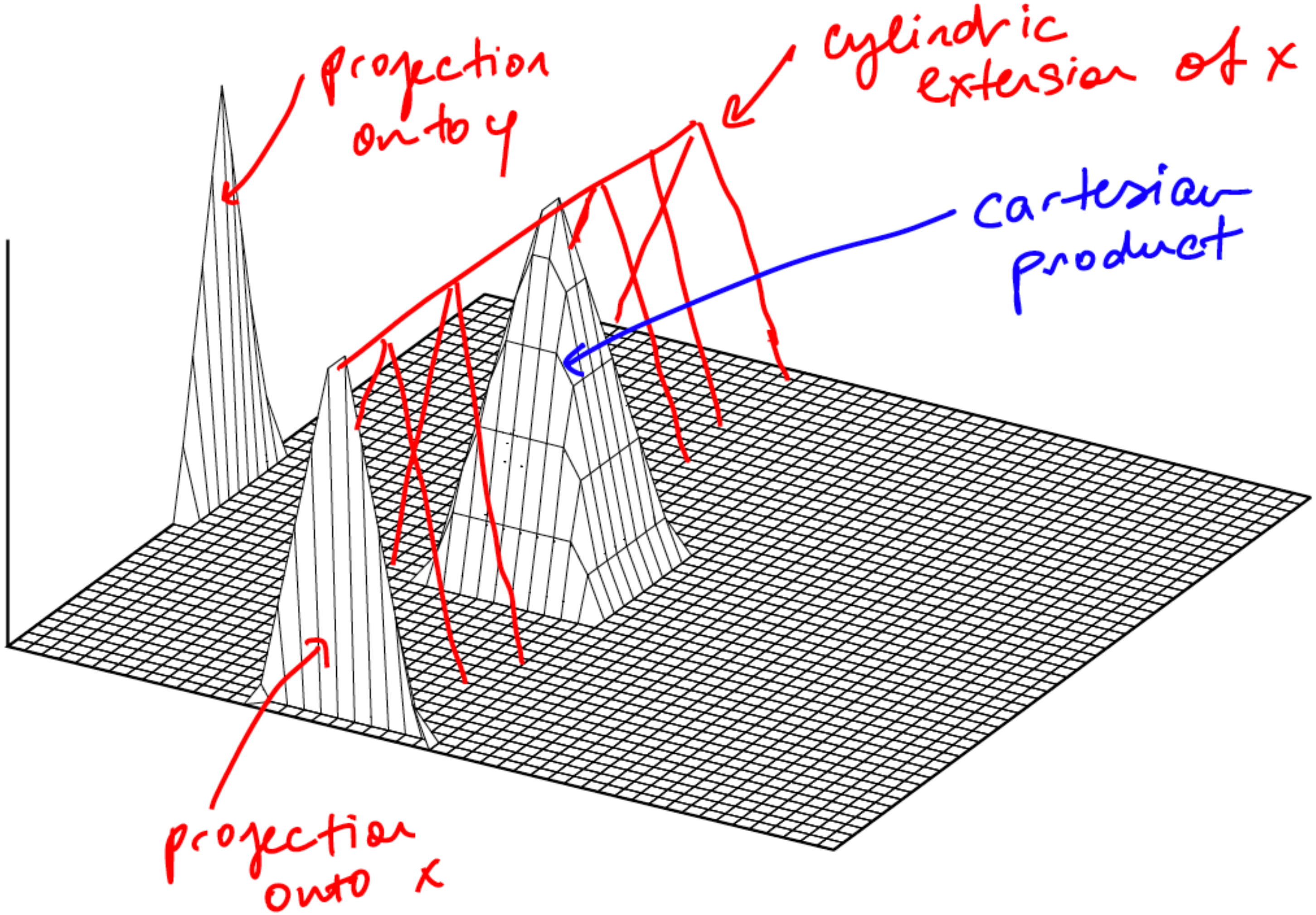
A special case of the Cartesian product is when n = 2.

Then the Cartesian product of fuzzy sets $A \in F(X)$ and $B \in F(Y)$ is a fuzzy relation $A \times B \in F(X \times Y)$ defined by

- μ_A x $B(x, y) = \top[\mu_{A(x)}, \mu_{B(y)}]$, $\forall x \in X, \forall y \in Y$.

As $\top$ operation usually "minimum" is used, "product" also can be used

Example: Cartesian Product in F(X) x (Y)



Subsequences

Consider the Cartesian product of all sets in the family

$\mathcal{X} = \{X_i \mid i \in N_n = \{1, 2, . . . , n\}\}$.

For each sequence (n-tuple) $\mathbf{x} = (x_1, . . . , x_n) \in \times_{i \in Nn} X_i$

and each sequence (r -tuple, $r \leq n$) $\mathbf{y} = (y1, . . . , yr) \in \times_{i \in J} X_j$

where $J \subseteq N_n$ and $|J| = r$

y is called s**ubsequence** of **x** if and only if $y_j = x_j$, $j \in J$.

$\mathbf{y} \prec \mathbf{x}$ denotes that **y** is subsequence of ***x***.

example : $\mathbf{x}=(x_1, x_2, x_3)$, $\mathbf{y}=(x_1, x_3)$, so $\mathbf{y} \prec \mathbf{x}$

Projection

Given a relation $R(x_1, \ldots, x_n)$.
Let $[R \downarrow Y]$ denote the projection of R on Y.
It disregards all sets in X except those in the family

$$Y = \{X_j \mid j \in J \subseteq N_n\} .$$

Then $[R \downarrow Y]$ is a fuzzy relation whose membership function is defined on the Cartesian product of the sets in Y

$$[R \downarrow Y](y) = \max_{x \succ y} R(x). \text{ (note: } y_i \text{ is a subsequence of } x)$$

Under special circumstances, this projection can be generalized by replacing the max operator by another t-conorm.

T: t-norm (for ex. min) $\perp$: t-conorm (for example max)

Example

Consider the sets $X_1 = \{0, 1\}$, $X_2 = \{0, 1\}$, $X_3 = \{0, 1, 2\}$ and the ternary fuzzy relation on X_1 x X2 x X3 defined as follows:

Let $R_{ij} = [R \downarrow \{Xi, Xj\}]$ and $Ri = [R \downarrow \{Xi\}]$ for all $i, j \in \{1, 2, 3\}$.

Using this notation, all possible projections of R are given below.

			GIVEN RELATION	PROJECTIONS OF THE RELATION					
$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$	$R_{13}(x_1, x_3)$	$R_{23}(x_2, x_3)$	$R_1(x_1)$	$R_2(x_2)$	$R_3(x_3)$
0	0	0	0.4	0.9	1.0	0.5	1.0	0.9	1.0
0	0	1	0.9	0.9	0.9	0.9	1.0	0.9	0.9
0	0	2	0.2	0.9	0.8	0.2	1.0	0.9	1.0
0	1	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0	1	1	0.0	1.0	0.9	0.5	1.0	1.0	0.9
0	1	2	0.8	1.0	0.8	1.0	1.0	1.0	1.0
1	0	0	0.5	0.5	0.5	0.5	1.0	0.9	1.0
1	0	1	0.3	0.5	0.5	0.9	1.0	0.9	0.9
1	0	2	0.1	0.5	1.0	0.2	1.0	0.9	1.0
1	1	0	0.0	1.0	0.5	1.0	1.0	1.0	1.0
1	1	1	0.5	1.0	0.5	0.5	1.0	1.0	0.9
1	1	2	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Example: Detailed Calculation

Here, only consider the projection R_{12}:

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$
0	0	0	0.4	
0	0	1	0.9	$\max[R(0,0,0), R(0,0,1), R(0,0,2)] = 0.9$
0	0	2	0.2	
0	1	0	1.0	
0	1	1	0.0	$\max[R(0,1,0), R(0,1,1), R(0,1,2)] = 1.0$
0	1	2	0.8	
1	0	0	0.5	
1	0	1	0.3	$\max[R(1,0,0), R(1,0,1), R(1,0,2)] = 0.5$
1	0	2	0.1	
1	1	0	0.0	
1	1	1	0.5	$\max[R(1,1,0), R(1,1,1), R(1,1,2)] = 1.0$
1	1	2	1.0	

Cylindric Extension

Another operation on relations is called **cylindric extension**.
Let X and Y denote the same families of sets as used for projection.
Let R be a relation defined on Cartesian product of sets in family Y.
Let $[R \uparrow \mathcal{X} \setminus \mathcal{Y}]$ denote the cylindric extension of R into sets
X_1, ($i \in N_n$) which are in X but not in Y.

It follows that for each x with $x \succ y$

$$[R \uparrow \mathcal{X} \setminus \mathcal{Y}](x) = R(\mathbf{y}).$$

(relation defined on Y extended to $\mathcal{X}$ U $\mathcal{Y}$)

The cylindric extension

- produces **largest** fuzzy relation that is compatible with projection,
- is the least specific of all relations compatible with projection,
- guarantees that no information not included in projection is used to determine extended relation.

Example

Consider again the example for the projection.
The membership functions of the cylindric extensions of all projections are already shown in the table under the assumption that their arguments are extended to (x_1, x_2, x_3) e.g.

$$[R_{23} \uparrow \{X_1\}](0, 0, 2) = [R_{23} \uparrow \{X_1\}](1, 0, 2) = R_{23}(0, 2) = 0.2.$$

In this example none of the cylindric extensions are equal to the original fuzzy relation.

This is identical with the respective projections.

Some information was lost when the given relation was replaced by any one of its projections.

Cylindric Closure

Relations that can be reconstructed from one of their projections by cylindric extension exist.

However, they are rather rare.

It is more common that relation can be exactly reconstructed

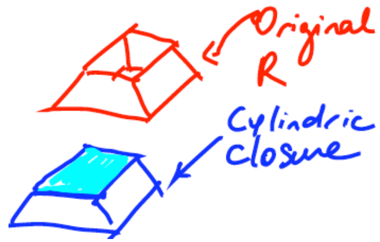
- from several of its projections (max),
- by taking set intersection of their cylindric extensions (min).

The resulting relation is usually called **cylindric closure**.

Let the set of projections $\{P_i \mid i \in I\}$ of a relation on $\mathcal{X}$ be given.

Then the cylindric closure $\mathrm{cyl}\{P_i\}$ is defined for each $\mathbf{x} \in \mathcal{X}$ as

$$\mathrm{cyl}\{P_i\}(\mathbf{x}) = \min_{i\in I} [Pi \uparrow \mathcal{X} \setminus \mathcal{Y}_i](x).$$



$\mathcal{Y}_i$ denotes the family of sets on which P_i is defined.

Example

Consider again the example for the projection.
The cylindric closures of three families of the projections are shown below (see page 12 for R and projections)

given relation

cylindrical closures

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$\text{cyl}(R_{12}, R_{13}, R_{23})$	$\text{cyl}(R_1, R_2, R_3)$	$\text{cyl}(R_{12}, R_3)$
0	0	0	0.4	0.5	0.9	0.9
0	0	1	0.9	0.9	0.9	0.9
0	0	2	0.2	0.2	0.9	0.9
0	1	0	1.0	1.0	1.0	1.0
0	1	1	0.0	0.5	0.9	0.9
0	1	2	0.8	0.8	1.0	1.0
1	0	0	0.5	0.5	0.9	0.5
1	0	1	0.3	0.5	0.9	0.5
1	0	2	0.1	0.2	0.9	0.5
1	1	0	0.0	0.5	1.0	1.0
1	1	1	0.5	0.5	0.9	0.9
1	1	2	1.0	1.0	1.0	1.0

None of them is the same as the original relation R.
So the relation R is not fully reconstructible from its projections.

Binary Fuzzy Relations

Motivation and Domain

Binary relations are significant among n-dimensional relations.

They are (in some sense) generalized mathematical functions.

On the contrary to functions from X to Y, binary relations R(X,Y) may assign to each element of X two or more elements of Y .

Some basic operations on functions, e.g. inverse and composition, are applicable to binary relations as well.

Given a fuzzy relation R(X,Y).

Its domain domR is the fuzzy set on X whose membership function is

defined for each $x \in X$ as

$$\text{dom}R(x) = \max_{y \in Y} R(x, y),$$

i.e. each element of X belongs to the domain of R to a degree equal to the strength of its strongest relation to any $y \in Y$.

Range and Height

The **range** ran of R(X,Y) is a fuzzy relation on Y whose membership function is defined for each $y \in Y$ as

$$\text{ran } R(y) = \max_{x \in X} R(x, y),$$

i.e. the strength of the strongest relation which each $y \in Y$ has to an $x \in X$ equals to the degree of membership of y in the range of R.

The **height** h of R(X,Y) is a number defined by

$$h(R) = \max_{y \in Y} \max_{x \in X} R(x, y).$$

h(R) is the largest membership grade obtained by any pair $(x, y) \in R$.

Representation and Inverse

Consider e.g. the **membership matrix R** = [rxy] with rxy = R(x, y).

Its **inverse** $R^{-1}(Y, X)$ of R(X,Y) is a relation on Y x X defined by

$R^{-1}(y, x) = R(x, y), \forall x \in X, \forall y \in Y$.

$R^{-1} = [r^{-1}_{xy}]$ representing $R^{-1}(y, x)$ is the transpose of **R** for R(X,Y)

$(R^{-1})^{-1} = R, \forall R$.

Standard Composition

Consider the binary relations $P(X,Y)$, $Q(Y, Z)$ with common set Y.

The s**tandard composition** of P and Q is defined as

$$(x, z) \in P \circ Q \Longleftrightarrow \exists y \in Y : \{(x, y) \in P \wedge (y, z) \in Q\}.$$

In the fuzzy case this is generalized by

$$[P \circ Q](x, z) = \sup_{y \in Y} \min\{P(x, y), Q(y, z)\}, \forall x \in X, \forall z \in Z.$$

If Y is finite, sup operator is replaced by max.

Then the standard composition is also called **max-min** composition.

Inverse of Standard Composition

The inverse of the max-min composition follows from its definition:

$$[P(X,Y) \circ Q(Y,Z)]^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y,X).$$

Its associativity also comes directly from its definition:

$$[P(X,Y)] \circ Q(Y,Z)] \circ R(Z,W) = P(X,Y) \circ [Q(Y,Z) \circ R(Z,W)].$$

Note that the standard composition is not commutative.

Matrix notation: $[r_{ij}] = [p_{ik}] \circ [q_{kj}]$ with $r_{ij} = \max_k \min(p_{ik}, q_{kj})$.

Example : Standart Composition

$$\boldsymbol{P} \circ \boldsymbol{Q} = \boldsymbol{R}$$

$$\begin{array}{cc} & \begin{array}{ccc}1 & 2 & 3\end{array}\\ \begin{array}{c}P_{1-}\rightarrow\\ \\ P_{3-}\rightarrow\end{array} & \begin{bmatrix}.3 & .5 & .8\\ 0 & .7 & 1\\ .4 & .6 & .5\end{bmatrix}\end{array} \circ \begin{array}{c}\begin{array}{cccc}q_{-1} & q_{-2} & & \end{array}\\ \begin{bmatrix}.9 & .5 & .7 & .7\\ .3 & .2 & 0 & .9\\ 1 & 0 & .5 & .5\end{bmatrix}\end{array} = \begin{array}{c}\begin{array}{cccc}r_{11} & & & \end{array}\\ \begin{bmatrix}.8 & .3 & .5 & .5\\ 1 & .2 & 5 & .7\\ .5 & .4 & .5 & .5\end{bmatrix}\\ \begin{array}{cccc} & r & & \end{array}\end{array}$$

For instance:

r_{11} = max{min(p_{11}, q_{11}), min(p_{12}, q_{21}), min(p_{13}, q_{31})}
= max{min(.3, .9), min(.5, .3), min(.8, 1)}
= .8

r_{32} = max{min(p_{31}, $q1_2$), min(p_{32}, q_{22}), min(p_{33}, q_{32})}
= max{min(.4, .5), min(.6, .2), min(.5, 0)}
= .4

Example: standart decomposition

Types of Airplanes (Speed, Height,Type)

Consider the following fuzzy relations for airplanes:

- relation A between maximal speed and maximal height,
- relation B between maximal height and the type.

maximal height

A	h_1	h_2	h_3
s_1	1	.2	0
s_2	.1	1	0
s_3	0	1	1
s_4	0	.3	1

maximal Speed

type

B	t_1	t_2
h_1	1	0
h_2	.9	1
h_3	0	.9

maximal height

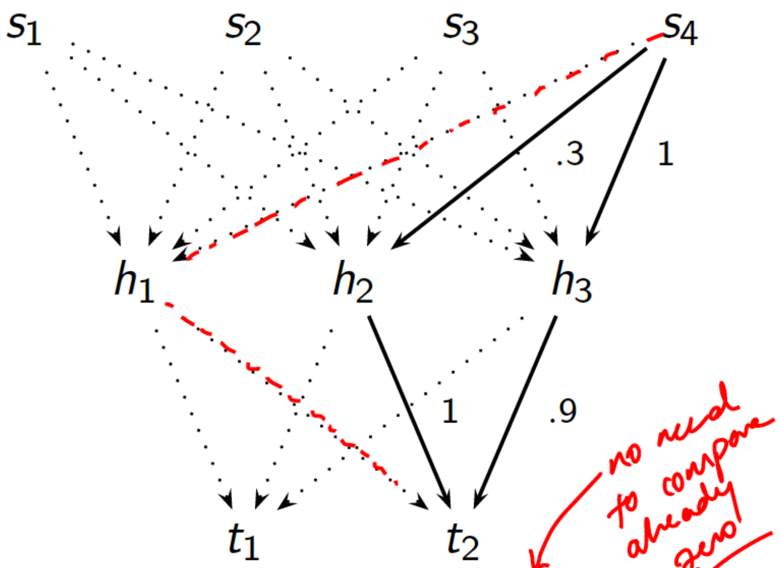
Example: standart decomposition (cont.)

matrix multiplication scheme

$$
\begin{array}{c|ccc|cc}
 & & & & t_1 & t_2 \\
 & & & h_1 & 1 & 0 \\
 & A & \circ & B\ h_2 & .9 & 1 \\
 & h_1 & h_2 & h_3\ h_3 & 0 & .9 \\
\hline
s_1 & 1 & .2 & 0 & 1 & .2 \\
s_2 & .1 & 1 & 0 & .9 & 1 \\
s_3 & 0 & 1 & 1 & .9 & 1 \\
s_4 & 0 & .3 & 1 & .3 & .9
\end{array}
$$

$A \circ B$ speed-type relation

flow scheme



no need to compare already zero

min(0,0)

$$
\begin{aligned}
(A \circ B)(s_4, t_2) &= \max\{\min\{.3,\ 1\},\ \min\{.1,\ .9\}\} \\
&= .9
\end{aligned}
$$

Relational Join

A similar operation on two binary relations is the **relational join**.

It yields triples (whereas composition returned pairs).

For P(X,Y) and Q(Y , Z), the relational join $P * Q$ is defined by

$$[P * Q](x, y, z) = \min\{P(x, y), Q(y, z)\}, \forall x \in X, \forall y \in Y, \forall z \in Z.$$

Then the max-min composition is obtained by aggregating the join by the maximum:

$$[P \circ Q](x, z) = \max_{y \in Y} [P * Q](x, y, z), \forall x \in X, \forall z \in Z.$$

A(x,y)	a	b
1	0.3	0.5
2	0.6	1

B(y,z)	α	β
a	0.7	0.2
b	0.4	0.1

x y z	A∗B(x,y,z)	
1 a α	0.3	min(0.3, 0.7)
1 a β	0.2	min(0.3, 0.2)
1 b α	0.5	
1 b β	0.2	
2 a α	0.6	min(0.6, 0.7)
2 a β	0.2	
2 b α	0.4	
2 b β	0.1	

Example : Relational Join

The join S = P ∗ Q of the relations P and Q has the following membership function (shown below on left-hand side).

To convert this join into its corresponding composition R = P ∘ Q (shown on right-hand side),

the two indicated pairs of S(x, y, z) are aggregated using max.

x	y	z	$\mu_S x, y, z$
1	a	α	.6
1	a	β	.7*
1	b	β	.5*
2	a	α	.6
2	a	β	.8
3	b	β	1
4	b	β	.4*
4	c	β	.3*

given
S=P*Q
is given

→ find
R=P∘Q

x	z	$\mu_R(x, z)$
1	α	.6
1	β	.7
2	α	.6
2	β	.8
3	β	.1
4	β	.4

For instance,

$$R(1,\beta) = \max\{S(1, a,\beta), S(1, b,\beta)\}$$
$$= \max\{.7, .5\} = .7$$

Binary Relations on a Single Set

It is also possible to define crisp or fuzzy binary relations among elements of a single set X.

Such a binary relation can be denoted by R(X,X) or R(X^2) which is a subset of X x X = X^2.

These relations are often referred to as **directed graphs** which is also arepresentation of them.

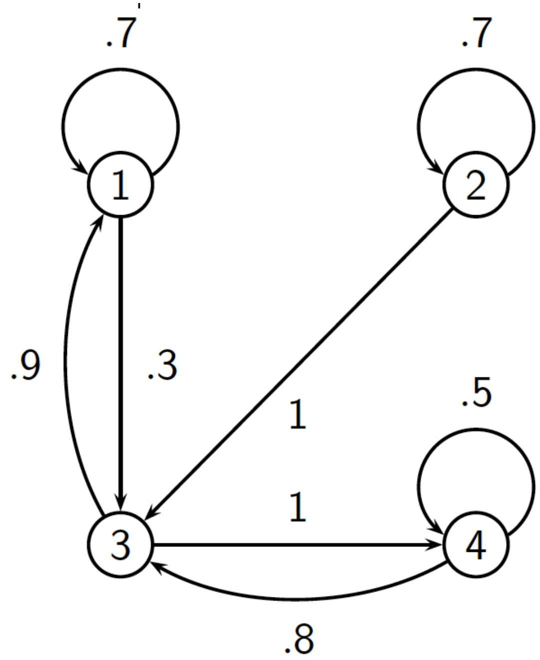
- Each element of X is represented as node.
- Directed connections between nodes indicate pairs of $x \in X$ for which the grade of the membership is nonzero.
- Each connection is labeled by its actual membership grade of the corresponding pair in R.

Example

An example of R(X,X) defined on X = 1, 2, 3, 4.

Two different representation are shown below.

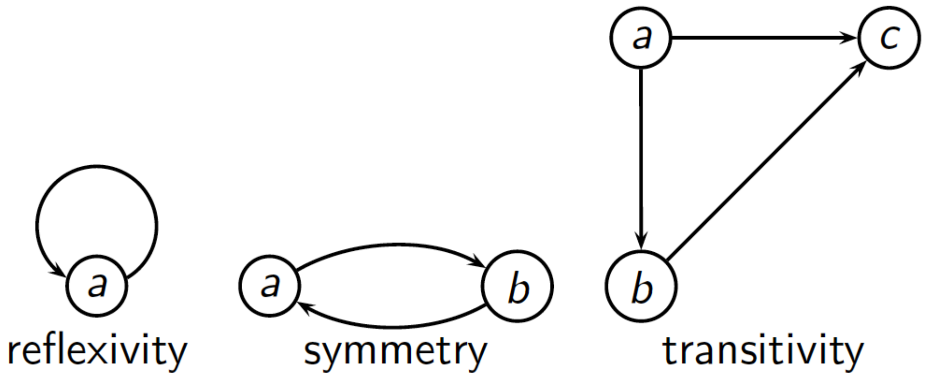
	1	2	3	4
1	.7	0	.3	0
2	0	.7	1	0
3	.9	0	0	1
3	0	0	.8	.5



Properties of Crisp Relations

A crisp relation R(X,X) is called

- reflexive if and only if 'x $\in$ X : (x, x) $\in$ R,
- symmetric if and only if 'x, y $\in$ X : (x, y) $\in$ R $\leftrightarrow$ (y, x) $\in$ R,
- transitive if and only if (x, z) $\in$ R whenever both (x, y) $\in$ R and (y, z) $\in$ R for at least one y $\in$ X.



All these properties are preserved under inversion of the relation.

Properties of Fuzzy Relations

These properties can be extended for fuzzy relations.

So one can define them in terms of the membership function of the relation.

A fuzzy relation $R(X,X)$ is called

- **reflexive** if and only if $\forall x \in X : R(x, x) = 1$,
- **symmetric** if and only if $\forall x, y \in X : R(x, y) = R(y, x)$,
- **transitive** if it satisfies

$$R(x, z) \geq \max_{y \in Y} \min\{R(x, y), R(y, z)\}, \forall (x, z) \in X^2.$$

Note that a fuzzy binary relation that is reflexive, symmetric and transitive is called **fuzzy equivalence relation**.