EE496 : COMPUTATIONAL INTELLIGENCE
FS02: FUZZY SET THEORY

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Definition of a "set"

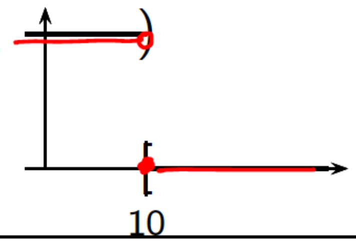
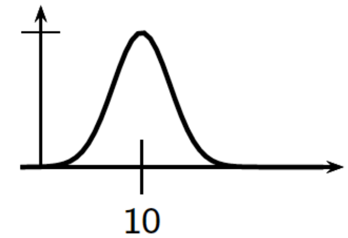
"By a set we understand every collection made into a whole of definite, distinct objects of our intuition or of our thought." (Georg Cantor).

For a set in Cantor's sense, the following properties hold:

- $x \neq \{x\}$.
- If $x \in X$ and $X \in Y$, then $x \notin Y$.
- The Set of all subsets of X is denoted as 2^X.
- $\varnothing$ is the empty set and thus very important.

Extension to a fuzzy set

- Extension to a fuzzy set

ling. description		model	
all numbers smaller than 10	→ *objective*		characteristic function of a set
all numbers almost equal to 10	→ *subjective*		membership function of a "fuzzy set"

Definition

A fuzzy set μ of $X \neq \varnothing$ is a function from the **reference set X** to the unit interval, i.e. $\mu : X \rightarrow [0, 1]$. $F(X)$ represents the set of all fuzzy sets of X, i.e. $F(X) \text{ def} = \{\mu \mid \mu : X \rightarrow [0, 1]\}$

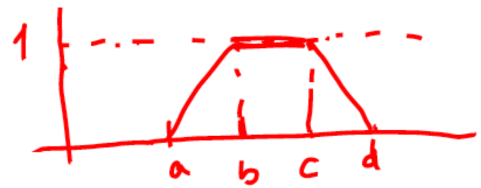
Vertical Representation

So far, fuzzy sets were described by
their characteristic/membership function and
assigning degree of membership μ(x) to each element x ∈ X.
That is the **vertical representation** of the corresponding fuzzy set,
e.g. linguistic expression like "about m"

$$\mu_{m,d}(x) = \begin{cases} 1 - \left|\frac{m-x}{d}\right|, & \text{if } m - d \leq x \leq m + d \\ 0, & \text{otherwise,} \end{cases}$$

or "approximately between b and c"

$$\mu_{a,b,c,d}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x < b \\ 1, & \text{if } b \leq x \leq c \\ \frac{x-d}{c-d}, & \text{if } c < x \leq d \\ 0, & \text{if } x < a \text{ or } x > d. \end{cases}$$



Horizontal Representation

Another representation is very often applied as follows:

For all membership degrees α belonging to chosen subset of $[0, 1]$, human expert lists elements of X that fulfill vague concept of fuzzy set with degree $\geq \alpha$.

That is the **horizontal representation** of fuzzy sets by their **α-cuts**.

Definition

Let $\mu \in F(X)$ and $\alpha \in [0, 1]$. Then the α-cut and } strict α-cut of μ are defined as

α-cut: $[\mu]_{\alpha} = \{x \in X \mid \mu(x) \geq \alpha\}$

strict α-cut: $[\mu]_{\underline{\alpha}} = \{x \in X \mid \mu(x) > \alpha\}$ of μ.

A Simple Example

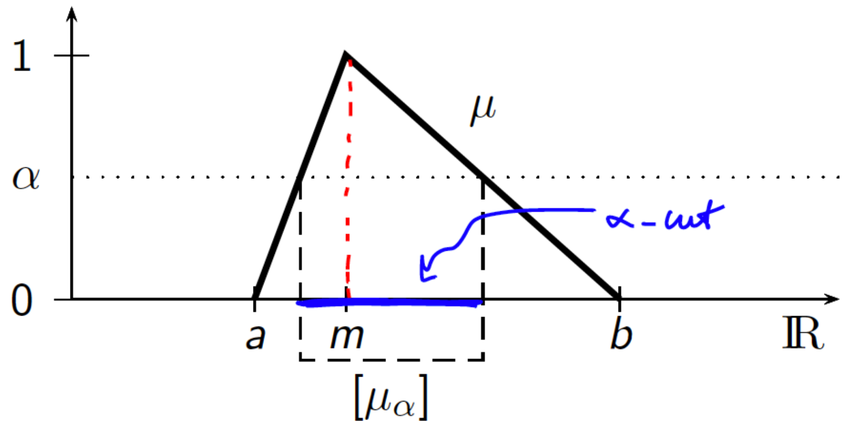
Let $A \subseteq X$, $\chi A : X \rightarrow [0, 1]$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise } 0 < \alpha < 1. \end{cases}$$

Then $[\chi A]\alpha = A$.

χ_A is called **indicator function** or **characteristic function** of A.

An Example



Let μ be triangular function on R as shown above.

α-cut of μ can be constructed by

1. drawing horizontal line parallel to x-axis through point (0,α),
2. projecting this section onto x-axis.

$$[\mu]_\alpha = \begin{cases} [a + \alpha(m - a), b - \alpha(b - m)], & \text{if } 0 < \alpha \leq 1, \\ \mathbb{R}, & \text{if } \alpha = 0. \end{cases}$$

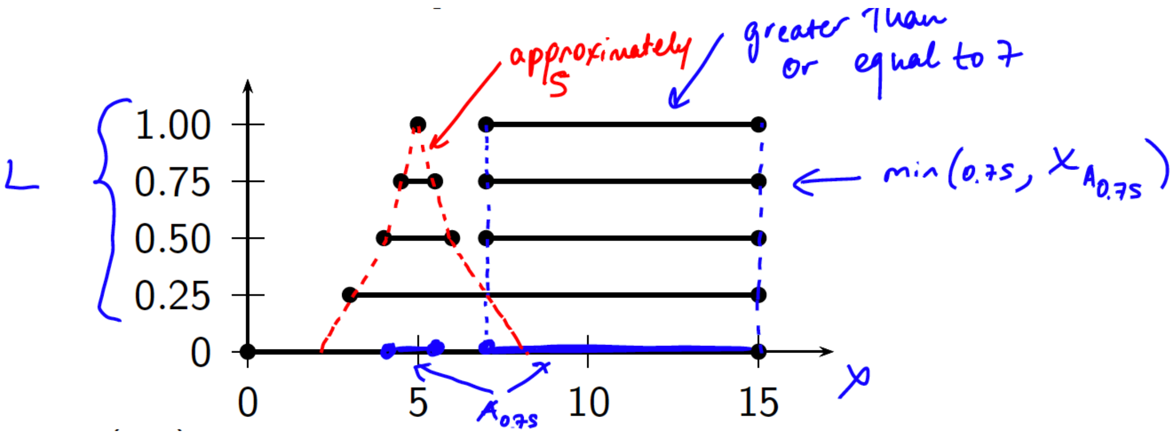
An Exemplary Horizontal View

Approximately 5 or greater than or equal to 7

Suppose that X = [0, 15].

An expert chooses L = {0, 0.25, 0.5, 0.75, 1} and α-cuts:

- $A_0 = [0, 15]$,
- $A_{0.25} = [3, 15]$,
- $A_{0.5} = [4, 6] \cup [7, 15]$,
- $A_{0.75} = [4.5, 5.5] \cup [7, 15]$,
- $A_1 = \{5\} \cup [7, 15]$



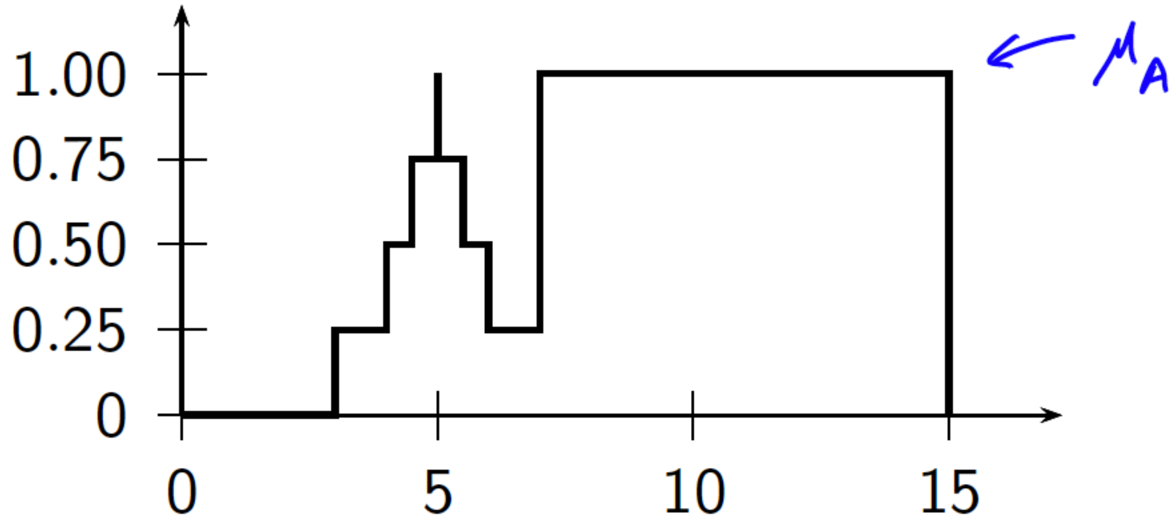
The family $(A_\alpha)_{\alpha \in L}$

An Exemplary Vertical View

"Approximately 5 or greater than or equal to 7"

μ_A is obtained as upper envelope of the family A of sets.

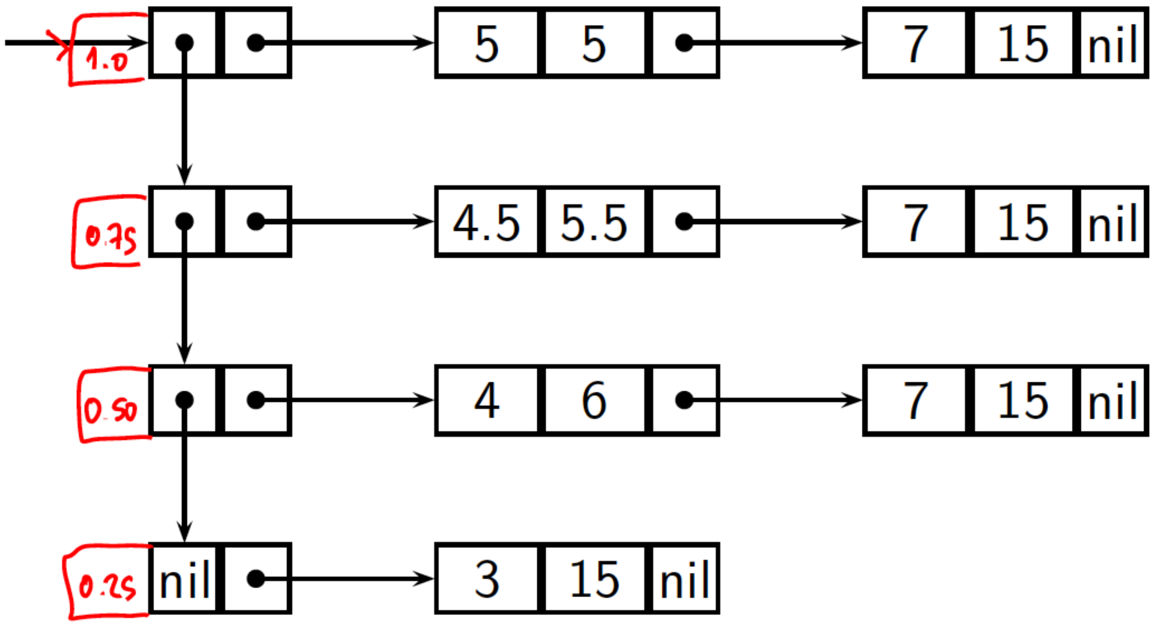
The difference between horizontal and vertical view is obvious:



The horizontal representation is easier to process in computers.

Also, restricting the domain of x-axis to a discrete set is usually done.

Horizontal Representation in the Computer



Fuzzy sets are usually stored as chain of linear lists.

For each α-level, α≠0.

A finite union of closed intervals is stored by their bounds.

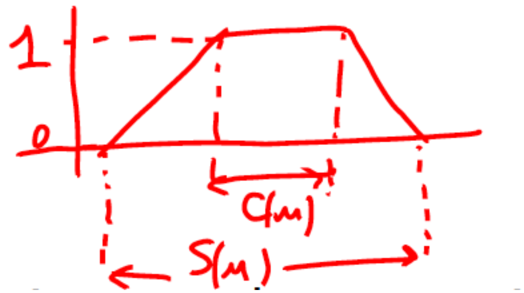
This data structure is appropriate for arithmetic operators.

Support and Core of a Fuzzy Set

Definition

The support $S(\mu)$ of a fuzzy set $\mu \in F(X)$ is the crisp set that contains all elements of X that have nonzero membership. Formally

$$S(\mu) = [\mu]_{\underline{0}} = \{x \in X \mid \mu(x) > 0\}\ .$$



Definition

The core $C(\mu)$ of a fuzzy set $\mu \in F(X)$ is the crisp set that contains all elements of X that have membership of one. Formally,

$$C(\mu) = [\mu]_1 = \{x \in X \mid \mu(x) = 1\}\ .$$

Definition

The **height** $h(\mu)$ of a fuzzy set $\mu \in F(X)$ is the largest membership grade obtained by any element in that set. Formally,

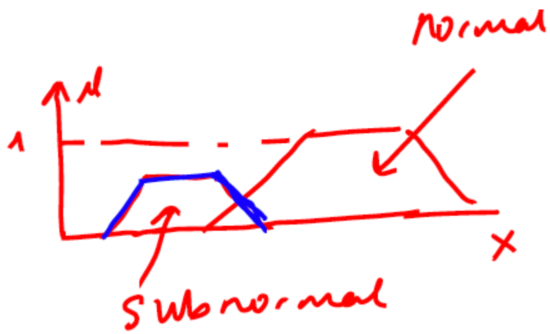
$$h(\mu) = \overset{\text{max}}{\sup_{x \in X}} \mu(x).$$

$h(\mu)$ may also be viewed as supremum (maximum) of α for which $[\mu]_\alpha \neq \emptyset$.

Definition

A fuzzy set μ is called **normal** when $h(\mu) = 1$.

It is called **subnormal** when $h(\mu) < 1$.



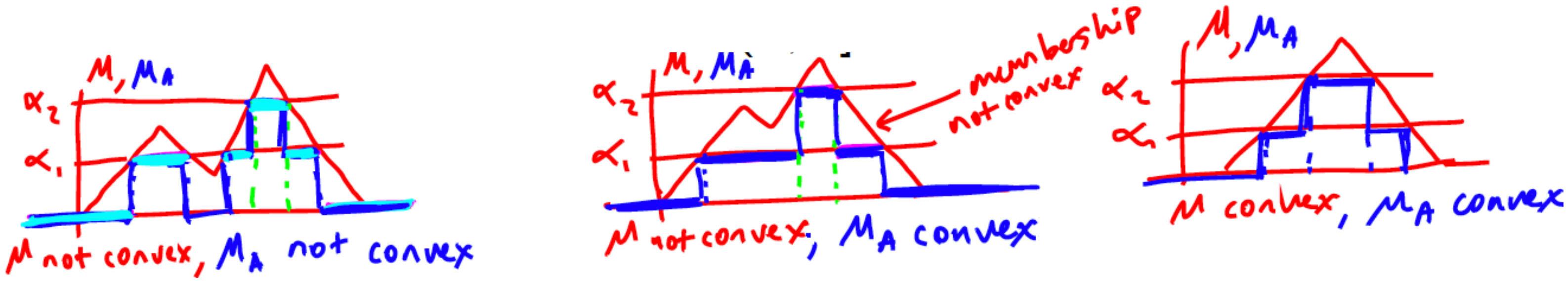
Convex Fuzzy Sets

Convex Crisp set:

A set S is **convex** iff for $\forall$ $x_1,x_2 \in S$, $\lambda x_1 + (1-\lambda)\, x_2 \in S$, $\lambda \in [0,1]$

Convex Fuzzy Set: Definition

Let X be a vector space. A fuzzy set $\mu \in F(X)$ is called fuzzy convex if its α-cuts are convex for all $\alpha \in (0, 1]$.



Fuzzy Numbers

Definition

μ is a fuzzy number if and only if μ is normal and $[\mu]_\alpha$ is bounded, closed, and convex $\forall\ \alpha \in (0, 1]$.

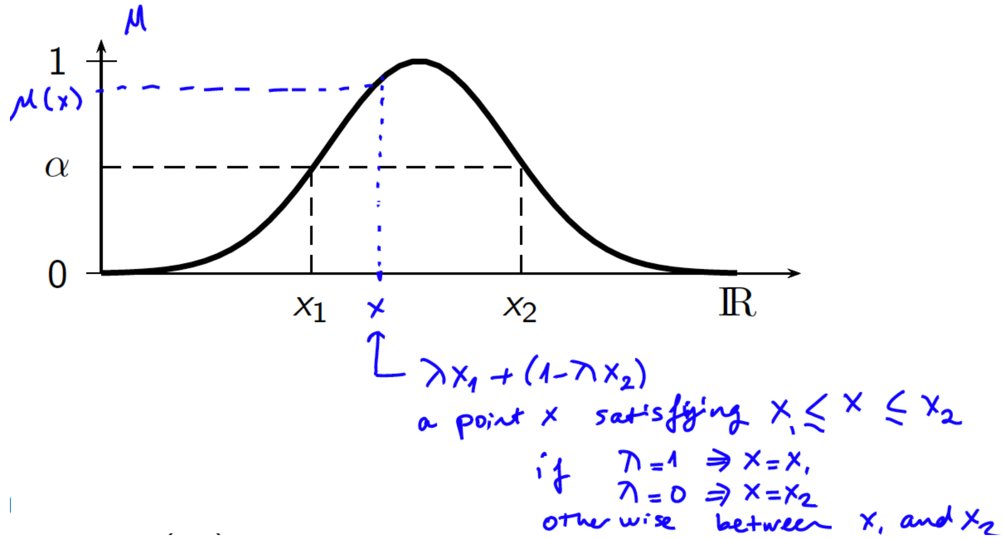
Example:

The term approximately x_0 is often described by a parametrized class of membership functions, e.g.

$$\mu_1(x) = \max\{0, 1 - c_1|x - x0|\}, \quad c_1 > 0,$$

$$\mu_2(x) = \exp(-c_2|x - x0|), \quad c_2 > 0.$$

Convex Fuzzy Sets



Theorem

A fuzzy set $\mu \in F(R)$ is convex if and only if

$$\mu(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu(x_1), \mu(x2)\}$$

for all $x_1, x_2 \in R$ and all $\lambda \in [0, 1]$.

Set Operators

Set Operators are defined by using traditional logics operator

Let X be universe of discourse (universal set):

$A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$

$A \cup B = \{x \in X \mid x \in A \vee x \in B\}$

$A^c = \{x \in X \mid x \notin A\} = \{x \in X \mid \neg(x \in A)\}$

$A \subseteq B$ if and only if $(x \in A) \rightarrow (x \in B)$ for all $x \in X$

One idea to define fuzzy set operators: use fuzzy logics.

Classical Logic: An Overview

Classical logic deals with propositions (either true or false).

The propositional logic handles combination of logical variables.

Key idea: how to express n-ary logic functions with logic primitives, e.g. $\neg, \wedge, \vee, \rightarrow$.

A set of logic primitives is complete if any logic function can be composed by a finite number of these primitives, e.g. $\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$, $\{\downarrow\}$ (NOR), $\{\uparrow\}$ (NAND)

Inference Rules

When a variable represented by logical formula is:

- true for all possible truth values, i.e. it is called **tautology**,
- false for all possible truth values, i.e. it is called **contradiction**.

Various forms of tautologies exist to perform deductive inference

They are called inference rules:

$(a \wedge (a \rightarrow b)) \rightarrow b$	(modus ponens)
$(\neg b \wedge (a \rightarrow b)) \rightarrow \neg a$	(modus tollens)
$((a \rightarrow b) \wedge (b \rightarrow c)) \rightarrow (a \rightarrow c)$	(hypothetical syllogism)

(note: $a \rightarrow b \equiv a' + b$)

e.g. modus ponens: given two true propositions a and $a \rightarrow b$

(premises), truth of proposition b (conclusion) can be inferred.

Every tautology remains a tautology when any of its variables is replaced with an arbitrary logic formula.

Boolean Algebra

The propositional logic based on finite set of logic variables is isomorphic (having same structure) to **finite set theory**.

Both of these systems are isomorphic to a finite Boolean algebra.

Definition

A Boolean algebra on a set B is defined as quadruple B = (B,+, ·, ’) where B has at least two elements (bounds) 0 and 1, + and · are binary operators on B, and ‘ is a unary operator on B for which the following properties hold.

Properties of Boolean Algebras

(B1) Idempotence	$a + a = a$	$a \cdot a = a$
(B2) Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
(B3) Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(B4) Absorption	$a + (a \cdot b) = a$	$a \cdot (a + b) = a$
(B5) Distribution	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	$a + (b \cdot c) = (a + b) \cdot (a + c)$
(B6) Universal Bounds	$a + 0 = a$, $a + 1 = 1$	$a \cdot 1 = a$, $a \cdot 0 = 0$
(B7) Complementary	$a + a' = 1$	$a \cdot a' = 0$
(B8) Involution	$a'' = a$	
(B9) Dualization	$(a + b)' = a' \cdot b'$	$(a \cdot b)' = a' + b'$

Boolean algebra can be characterized by a partial ordering on a set, i.e. $a \leq b$ if $a \cdot b = a$ or, alternatively, if $a + b = b$.

Set Theory, Boolean Algebra, Propositional Logic

Every theorem in one theory has a counterpart in each other theory.

Counterparts can be obtained applying the following substitutions:

Meaning	Set Theory	Boolean Algebra	Prop. Logic
values	2^X	B	$\mathcal{L}(V)$
"meet"/"and"	$\cap$	$\cdot$	$\wedge$
"join"/"or"	$\cup$	$+$	$\vee$
"complement"/"not"	^{c}	$^-$	$\neg$
identity element	X	1	1
zero element	$\emptyset$	0	0
partial order	$\subseteq$	$\leq$	$\rightarrow$

power set 2^X, set of logic variables V , set of all combinations L(V) of truth values of V

The Basic Principle of Classical Logic

The Principle of Bivalence:

"Every proposition is either true or false."

It has been formally developed by Tarski.

Łukasiewicz suggested to replace it by The Principle of Valence:

"Every proposition has a truth value."

Propositions can have intermediate truth value, expressed by a number from the unit interval [0, 1].

The Traditional or Aristotlelian Logic

Aristotle introduced a logic of terms and drawing conclusion from two premises.

The great Greeks (Chrisippus) also developed logic of propositions.

Jan Łukasiewicz founded the multi-valued logic.

The multi-valued logic is to fuzzy set theory
what classical logic is to set theory.

Three-valued Logics

A 2-valued logic can be extended to a 3-valued logic in several ways, i.e. different three-valued logics have been well established:

truth, falsity, indeterminacy are denoted by 1, 0, and 1/2, resp.

The negation $\neg a$ is defined as $1 - a$, i.e. $\neg 1 = 0, \neg 0 = 1$ and $\neg 1/2 = 1/2$.

Other primitives, e.g. $\wedge, \vee, \rightarrow, \leftrightarrow$, differ from logic to logic.

Five well-known three-valued logics (named after their originators) are defined in the following.

Primitives of Some Three-valued Logics

min(1,1+b-a) 1-|a-b|

a	b	Łukasiewicz (min) $\wedge$	(max) $\vee$	$\rightarrow$	$\leftrightarrow$	Bochvar $\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	Kleene $\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	Heyting $\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	Reichenbach $\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

All of them fully conform the usual definitions for a, b ∈ {0, 1}.

They differ from each other only in their treatment of 1/2.

Question: Do they satisfy the law of contradiction ($a \wedge \neg a = 0$) and the law of excluded middle ($a \vee \neg a = 1$)?

n-valued Logics

After the three-valued logics: generalizations to n-valued logics for arbitrary number of truth values n ≥ 2.

In the 1930s, various n-valued logics were developed.

Usually truth values are assigned by rational number in [0, 1].

Key idea: uniformly divide [0, 1] into n truth values.

Definition

The set Tn of truth values of an n-valued logic is defined as

$$T_n = \left\{ 0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1 \right\}.$$

These values can be interpreted as degree of truth.

Primitives in n-valued Logics

Łukasiewicz proposed first series of n-valued logics for $n \geq 2$.

In the early 1930s, he simply generalized his three-valued logic.

It uses truth values in T_n and defines primitives as follows:

$$\neg a = 1 - a$$

$$a \wedge b = \min(a, b) \qquad a \rightarrow b = \min(1, 1 + b - a)$$

$$a \vee b = \max(a, b) \qquad a \leftrightarrow b = 1 - |a - b|$$

The n-valued logic of Łukasiewicz is denoted by L_n.

The sequence $(L_2, L_3, \ldots, L_\infty)$ contains the classical two-valued logic L_2 and an infinite-valued logic L_∞ (rational **countable** values T_∞).

The infinite-valued logic L_1 (**standard Łukasiewicz logic**) is the logic with all real numbers in [0, 1] (1 = cardinality of continuum)

Zadeh's fuzzy logic

Zadeh's fuzzy logic proposal was much simpler

In 1965, he proposed a logic with values in [0, 1]:

$\neg a = 1 - a,$

$a \wedge b = \min(a, b),$

$a \vee b = \max(a, b).$

The set operators are defined pointwise as follows for μ, μ':

$\neg\mu : X \rightarrow X, \neg\mu(x) = 1 - \mu(x),$

$\mu \wedge \mu' : X \rightarrow X (\mu \wedge \mu')(x) = \min\{\mu(x), \mu'(x)\},$

$\mu \vee \mu' : X \rightarrow X (\mu \vee \mu')(x) = \max\{\mu(x), \mu'(x)\}.$

Standard Fuzzy Set Operators – Example