EE496 : COMPUTATIONAL INTELLIGENCE
FS05: FUZZY RULE BASES

UGUR HALICI
METU: Department of Electrical and Electronics Engineering (EEE)
METU-Hacettepe U: Neuroscience and Neurotechnology (NSNT)

Approximate Reasoning with Fuzzy Rules

General schema

Rule 1: **if** X is M_1, **then** Y is N_1

Rule 2: **if** X is M_2, then Y is N_2

...

Rule r : if X is M_r , then Y is N_r

Fact: X is M’

____

Conclusion: Y is N’

Given **r if-then rules** and fact **X is M’**, we conclude **Y is N’**.

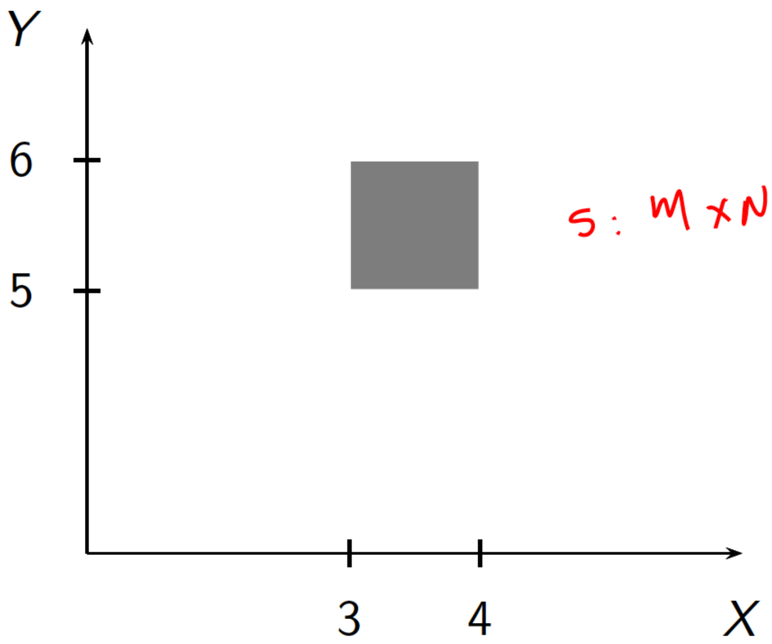
Typically used in **fuzzy controllers**.

Approximate Reasoning

Disjunctive Imprecise Rule

Imprecise rule: if X = $\overbrace{[3, 4]}^{M}$ then Y = $\overbrace{[5, 6]}^{N}$.

Interpretation: values coming from [3, 4] x [5, 6].



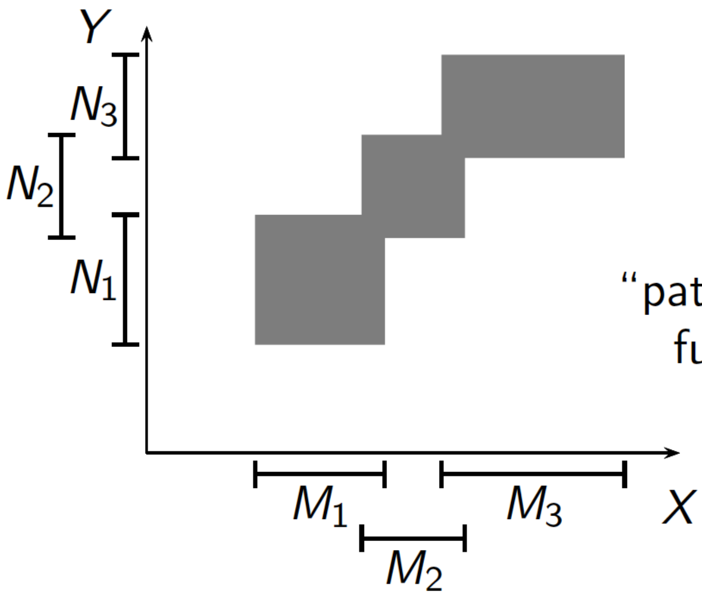
S : M x N

Approximate Reasoning

Disjunctive Imprecise Rules

Several imprecise rules: if X = M1 then Y = N1,
if X = M_2 then Y = N_2, if X = M3 then Y = N_3.

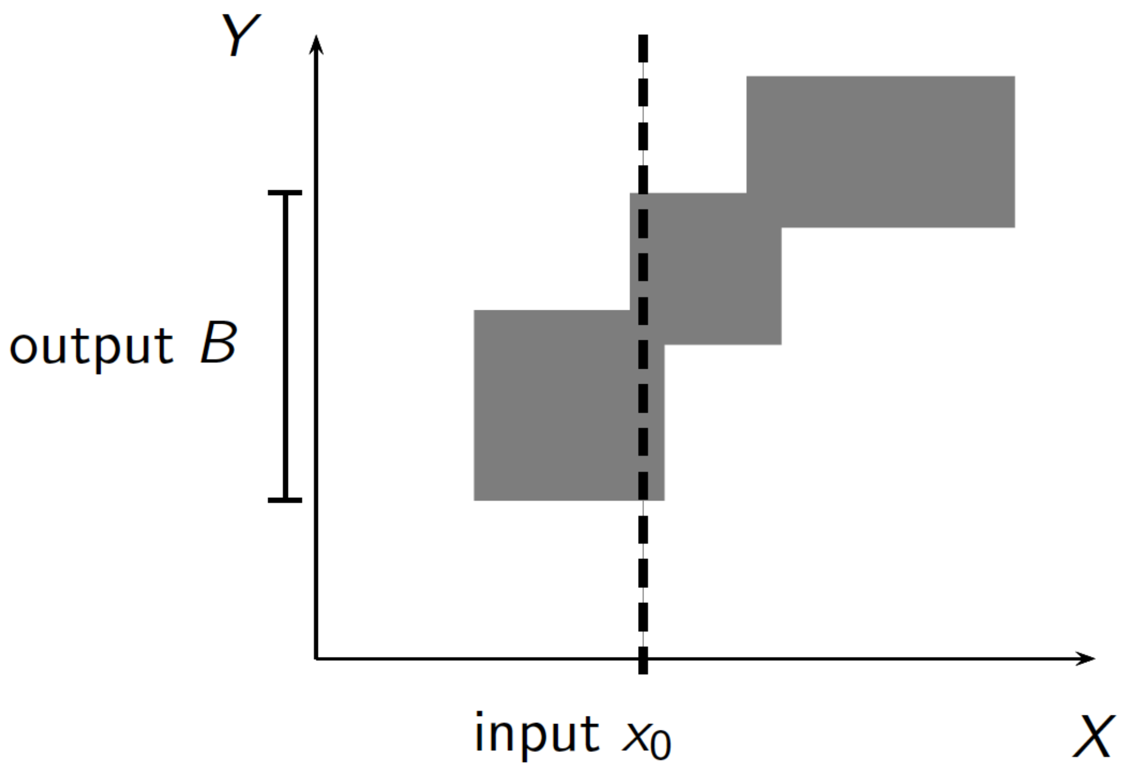
Interpretation: rule 1 as well as rule 2 as well as rule 3 hold true.



$$S = \bigcup_{i=1}^{r} M_i \times N_i$$

"patchwork rug" describing function's behavior as indicator function

Conclusion

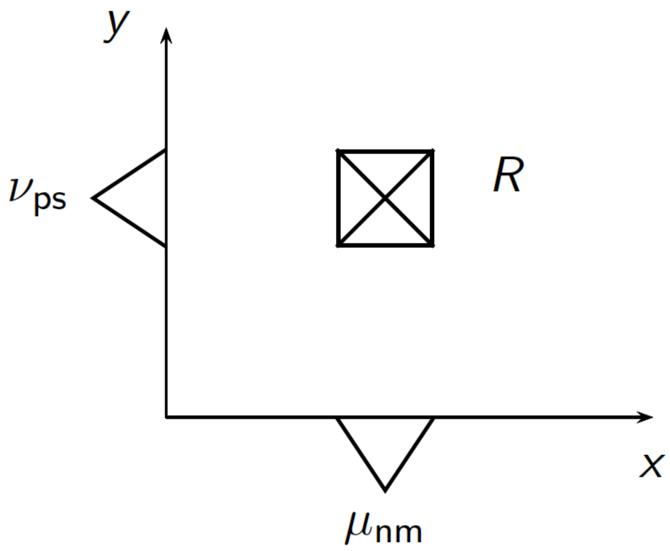


$$\{x_0\} \circ S = B$$

Disjunctive Fuzzy Rules

one fuzzy rule:

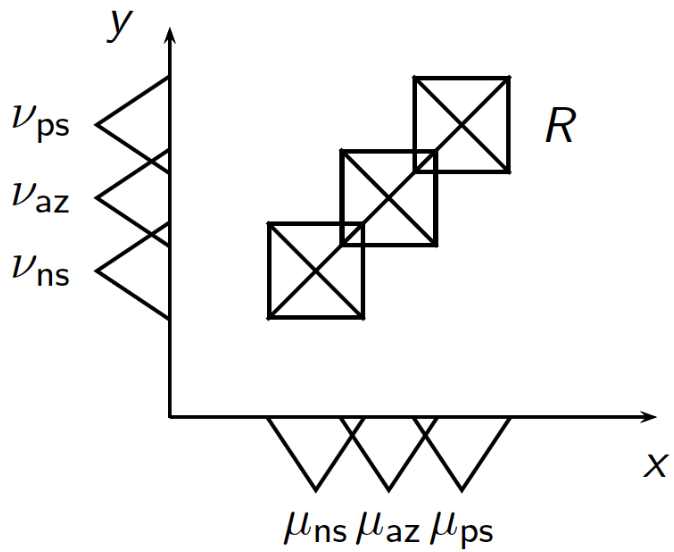
if $X = \text{nm}$ **then** $Y = \text{ps}$



$$R = \mu_{\text{nm}} \times \nu_{\text{ps}}$$

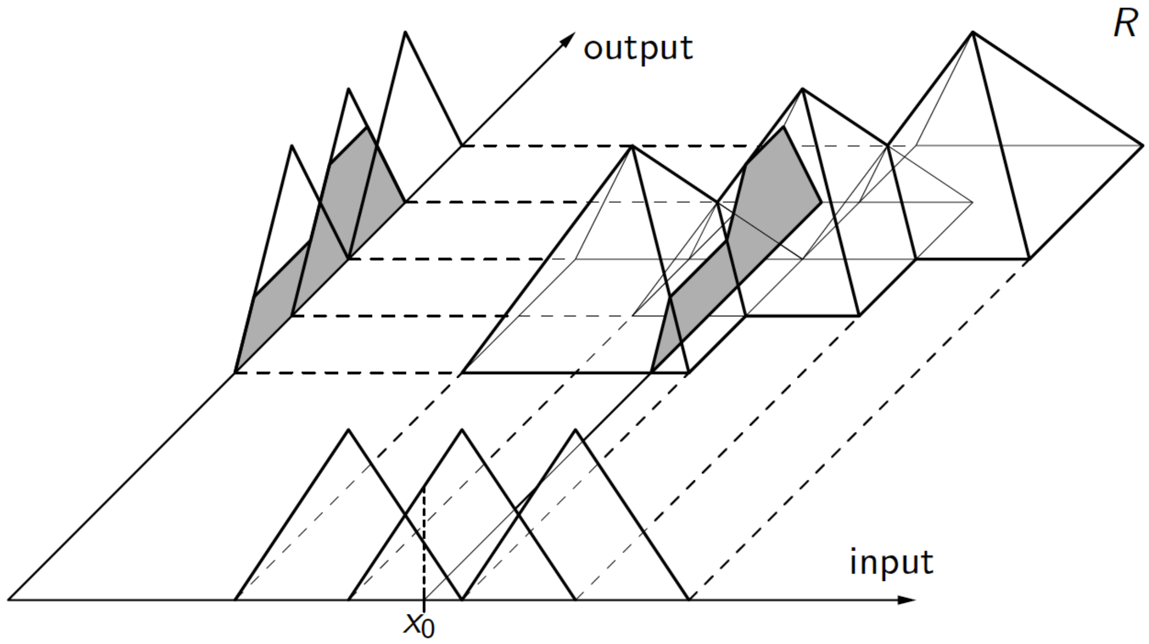
several fuzzy rules:

ns $\rightarrow$ ns', az $\rightarrow$ az', ps $\rightarrow$ ps'



$$R = \mu_{\text{ns}} \times \nu_{\text{ns'}} \cup$$
$$\mu_{\text{az}} \times \nu_{\text{az'}} \cup \mu_{\text{ps}} \times \nu_{\text{ps'}}$$

Conclusion



3 fuzzy rules.

Every pyramid is specified by 1 fuzzy rule (Cartesian product).

Input x_0 leads to gray-shaded fuzzy output $\{x_0\} \circ R$.

Disjunctive or Conjunctive?

Fuzzy relation R employed in reasoning is obtained as follows.

For each rule i, we determine relation Ri by

$$R_i(x, y) = \min[M_i(x), N_i(y)]$$

for all $x \in X$, $y \in Y$.

Then, R is defined by union of R_i , i.e.

$$R = \bigcup_{1 \le i \le r} R_i.$$

That is, if-then rules are treated disjunctive.

If-then rules can be also treated conjunctive by

$$R = \bigcap_{1 \le i \le r} R_i.$$

Disjunctive or Conjunctive?

Decision depends on intended use and how Ri are obtained.

For both interpretations, two possible ways of applying composition:

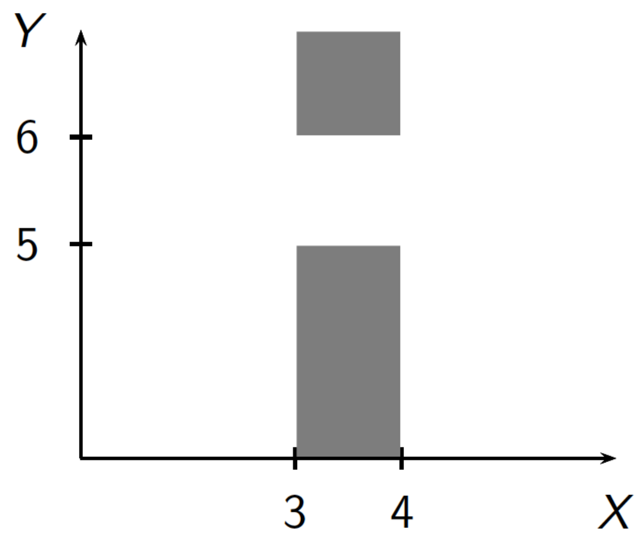
$$B'_1 = A' \circ \left(\bigcup_{1 \le i \le r} R_i \right) \qquad B'_2 = A' \circ \left(\bigcap_{1 \le i \le r} R_i \right)$$

$$B'_3 = \bigcup_{1 \le i \le r} A' \circ R_i \qquad B'_4 = \bigcap_{1 \le i \le r} A' \circ R_i$$

Conjunctive Imprecise Rules

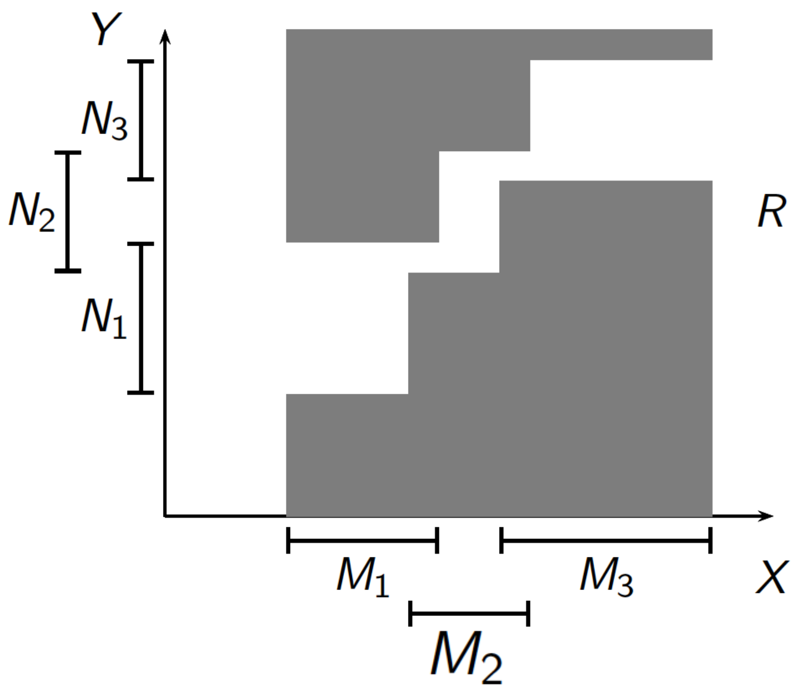
if X = [3, 4] **then** Y = [5, 6]

Gray-shaded values are impossible, white ones are possible.



Conjunctive Imprecise Rules

Several imprecise rules: if X = M_1 then Y = N_1, if X = M_2 then Y = N_2, if X = M_3 then Y = N_3.

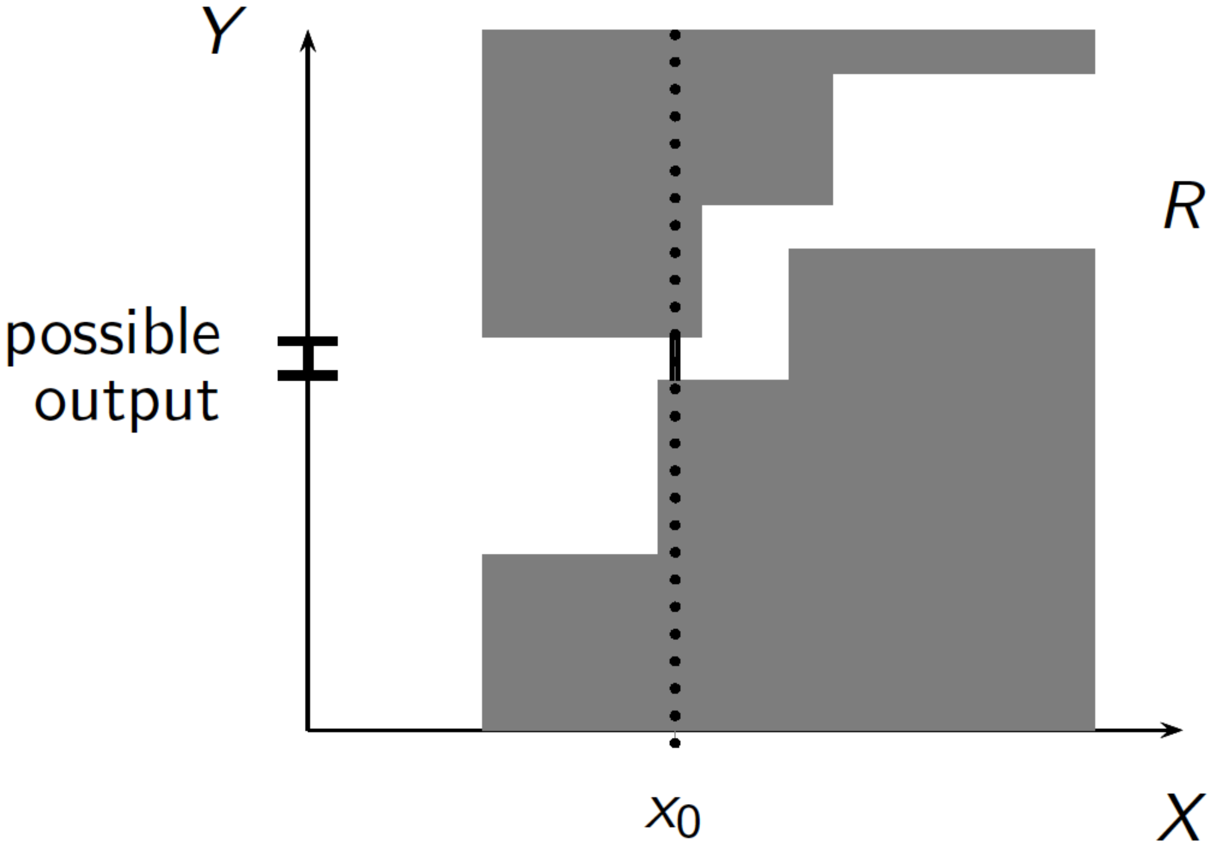


still possible are

$$R = \bigcap_{i=1}^{r} (M_i \times N_i) \cup (M_i^C \times Y)$$

"corridor" describing function's behavior

Approximate Reasoning with Crisp Input



output $= \{x_0\} \circ R$

Example: Fuzzy Relation

Classes of cars X = {s,m, h} (small, medium, high quality).

Possible maximum speeds Y = {140, 160, 180, 200, 220} (in km/h).

Gor any (x, y) $\in X \times Y$, fuzzy relation ϱ states possibility that maximum speed of car of class x is y.

ϱ	140	160	180	200	220
s	1	.5	.1	0	0
m	0	.5	1	.5	0
h	0	0	.4	.8	1

Fuzzy Relational Equations

Given $\mu_1, \ldots, \mu_r$ of X and $\nu_1, \ldots, \nu_r$ of Y and r rules if μ_i then ν_i .

What is a **fuzzy relation** ϱ that fits the rule system?

One solution is to find a relation ϱ such that

$\forall\, i \in \{1, \ldots, r\} : \nu i = \mu i \circ \varrho,$

$\mu \circ \varrho : Y \to [0, 1], y \to \sup_{x \in X} \min\{\mu(x), \varrho(x, y)\}.$

Solution of a Relational Equation

Single Rule Largest Solution

Theorem

i) Let "if A then B" be a rule with $\mu_A \in F(X)$ and $\nu_B \in F(Y)$.

ii) Then the relational equation $\nu B = \mu A \circ \varrho$ can be solved iff the Gödel relation $\varrho_{A\sim B}$ is a solution.

$\varrho_{A\sim B} : X \times Y \rightarrow [0, 1]$ is defined by

$$(x, y) \rightarrow \begin{cases} 1 & \text{if } \mu_A(x) \leq \nu_B(y), \\ \nu_B(y) & \text{otherwise.} \end{cases}$$

ii) If ϱ is a solution, then the set of solutions

$$R = \{\varrho_S \in F(X \times Y) \mid \nu_B = \mu_A \circ \varrho_S\}$$

has the following property:

If $\varrho_{S'}, \varrho_{S''} \in R$, then $\varrho_{S' \cup S'}' \in R$.

iii) If $\varrho_{A\sim B}$ is a solution, then $\varrho_{A\sim B}$ is the largest solution w.r.t. $\subseteq$.

Example

Single Rule Largest Solution

$$\mu_A = (\ .9 \quad 1 \quad .7\ ) \qquad \nu_B = (\ 1 \quad .4 \quad .8 \quad .7\ )$$

$$\varrho_{A\circledS B} \quad (x,y) \to \begin{cases} 1 & \text{if } \mu_A(x) \le \nu_B(y) \\ \nu_B(y) & \text{otherwise} \end{cases}$$

$$\varrho_{A\circledS B} = \begin{pmatrix} 1 & .4 & .8 & .7 \\ 1 & .4 & .8 & .7 \\ 1 & .4 & 1 & 1 \end{pmatrix} \qquad \varrho_1 = \begin{pmatrix} 0 & 0 & 0 & .7 \\ 1 & .4 & .8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} 0.9 \\ 1 \\ .7 \end{matrix}$$

max min

$$(0.9, 1, 0.7) \circ \varrho_1 = (1 \quad 0.4 \quad 0.8 \quad 0.7) \ \checkmark \quad \varrho_1 \text{ is a solution}$$

$$\underbrace{(\overset{x}{0.9}, \overset{y}{1}) \to 1} \qquad \underbrace{(\overset{x}{0.9}, \overset{y}{0.8}) \to 0.8}$$

$$\begin{array}{ccc|cccc} & & .9 & 1 & .4 & .8 & .7 \\ & & 1 & 1 & .4 & .8 & .7 \\ & & .7 & 1 & .4 & 1 & 1 \\ \hline .9 & 1 & .7 & 1 & .4 & .8 & .7 \end{array}$$

$$\varrho_2 = \begin{pmatrix} 0 & .4 & .8 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7 \end{pmatrix}$$

$$\varrho_1 \cup \varrho_2 = \begin{pmatrix} 0 & .4 & .8 & .7 \\ 1 & .4 & .8 & 0 \\ 0 & 0 & 0 & .7 \end{pmatrix}$$

$\varrho_{A\circledS B}$ largest solution, ϱ_1, ϱ_2 are two minimal solutions.

Solution space forms upper semilattice.

Solution of a Set of Relational Equations

Multiple rules, Largest Solution

Generalization of this result to system of r relational equations:

Theorem

Let $\nu_{Bi} = \mu_{Ai} \circ \varrho$ for i = 1, . . . ,r be a system of relational equations.

i) There is a solution iff

$\bigcap^r_{i=1} \varrho_{Ai\sim Bi}$, is a solution. (intersection of the largest solutions)

ii) If $\bigcap^r_{i=1} \varrho_{Ai\sim Bi}$ is a solution, then this solution is the biggest solution w.r.t. $\subseteq$.

Remark: if there is no solution, then Gödel relation is at least a good approximation.

Solving a System of Relational Equations

Single Rule Smallest Solution

Sometimes it is a good choice not to use the largest but one of the smallest solutions.

i.e. the Cartesian product $\varrho_{A \times B}(x, y) = \min\{\mu_A(x), \nu_B(y)\}$.

If a solution of $\nu B = \mu A \circ \varrho$ exists, then $\varrho_{A \times B}$ is a solution, too. (smallest solution)

Theorem

Let $\mu A \in F(X)$, $\nu_B \in F(Y)$. Furthermore, let $\varrho \in F(X \times Y)$ be a fuzzy relation which satisfies the relational equation $\nu_B = \mu_A \circ \varrho$.

Then $\nu_B = \mu_A \circ \varrho_{A \times B}$ holds.

(i.e. if a solution exists then cartesian product is a solution)

Multiple rules, Smallest Solution

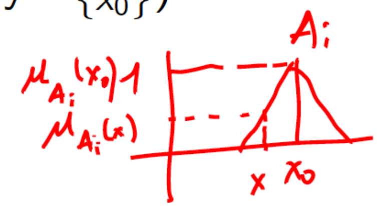
Using Cartesian product

$\mu_{A_i} = \nu_{B_i} \circ \varrho,\ 1 \leq i \leq r$ can be reasonably solved with $A \times B$ by

$$\varrho = \max\left\{\varrho_{A_i \times B_i} \mid 1 \leq i \leq r\right\}.$$

consider value over rules and then take maximum.

For crisp value $x_0 \in X$ (represented by $\mathbb{1}_{\{x_0\}}$):



if normal

$$\begin{aligned}\nu(y) &= \left(\mathbb{1}_{\{x_0\}} \circ \varrho\right)(y)\\ &= \max_{1\leq i\leq r}\left\{\sup_{x\in X}\min\{\mathbb{1}_{\{x_0\}}(x), \varrho_{A_i\times B_i}(x,y)\}\right\}\\ &= \max_{1\leq i\leq r}\{\min\{\mu_{A_i}(x_0), \nu_{B_i}(y)\}.\}\end{aligned}$$

valid also for For not normal case

if not normal

That is Mamdani-Assilian fuzzy control (to be discussed).