Manufactured Solution for 3D Euler equation using Maple*

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April 13, 2010

Abstract

This document presents the source terms for code verification for the numerical solutions of the 3D Euler equations. Such source terms are obtained by the Method of Manufactured Solutions (MMS) using the analytical solution for density, velocity and pressure, proposed by (Roy et al., 2002).

1 3D Euler Equations

The 3D Euler equations in conservation form are:

$$\frac{\partial(\rho)}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{1}$$

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+p)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=0 \tag{2}$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho vu)}{\partial x}+\frac{\partial(\rho v^2+p)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}=0 \tag{3}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho wu)}{\partial x}+\frac{\partial(\rho wv)}{\partial y}+\frac{\partial(\rho w^2+p)}{\partial z}=0 \tag{4}$$

$$\frac{\partial(\rho e_t)}{\partial t}+\frac{\partial(\rho ue_t+pu)}{\partial x}+\frac{\partial(\rho ve_t+pv)}{\partial y}+\frac{\partial(\rho we_t+pw)}{\partial z}=0 \tag{5}$$

where the Equation (1) is the unsteady term (mass conservation), Equations (2)–(4) are the nonlinear convection term in the x, y and z direction (momentum), and Equation (5) is the energy. For a calorically perfect gas, the Euler equations are closed with two auxiliary relations for energy:

$$e=\frac{1}{\gamma-1}RT, \tag{6}$$

*Work based on Roy, Smith, and Ober (2002).

$$e_t = e + \frac{u^2 + v^2 + w^2}{2}, \tag{7}$$

and with the ideal gas equation of state:

$$p = \rho R T. \tag{8}$$

2 Manufactured Solution

Roy et al. (2002) propose the general form of the primitive solution variables to be a function of sines and cosines:

$$\phi(x, y) = \phi_0 + \phi_x \, f_s\left(\frac{a_{\phi x}\pi x}{L}\right) + \phi_y \, f_s\left(\frac{a_{\phi y}\pi y}{L}\right) + \phi_z \, f_s\left(\frac{a_{\phi z}\pi z}{L}\right), \tag{9}$$

where $\phi = \rho, u, v, w$ or p, and $f_s(\cdot)$ functions denote either sine or cosine function. Note that in this case, ϕ_x, ϕ_y and ϕ_z are constants and the subscripts do not denote differentiation.

Therefore, the manufactured analytical solution for for each one of the variables in Euler equations are:

$$\begin{aligned}
\rho\,(x, y, z) &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y \cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z \sin\left(\frac{a_{\rho z}\pi z}{L}\right),\\
u\,(x, y, z) &= u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y \cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z \cos\left(\frac{a_{uz}\pi z}{L}\right),\\
v\,(x, y, z) &= v_0 + v_x \cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y \sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z \sin\left(\frac{a_{vz}\pi z}{L}\right),\\
w\,(x, y, z) &= w_0 + w_x \sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y \sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z \cos\left(\frac{a_{wz}\pi z}{L}\right),\\
p\,(x, y, z) &= p_0 + p_x \cos\left(\frac{a_{px}\pi x}{L}\right) + p_y \sin\left(\frac{a_{py}\pi y}{L}\right) + p_z \cos\left(\frac{a_{pz}\pi z}{L}\right).
\end{aligned} \tag{10}$$

Constants ϕ_0, ϕ_x, ϕ_y and ϕ_z with $\phi = u, v, w, p, \rho$ for the manufactured solutions (10) for the 3D supersonic and subsonic cases are presented in Roy et al. (2002).

3 Source terms for 3D Euler equations

The source terms Q_ρ, Q_u, Q_v, Q_w and Q_{e_t} obtained through symbolic manipulation of 3D Euler equations (1) – (5), respectively, over the analytical solutions (10) are:

$$
\begin{aligned}
Q_\rho = {} & \frac{\pi a_{\rho x}\rho_x}{L}\cos\left(\frac{\pi x a_{\rho x}}{L}\right)\left[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\right] \\
& -\frac{\pi a_{\rho y}\rho_y}{L}\sin\left(\frac{\pi y a_{\rho y}}{L}\right)\left[v_x\cos\left(\frac{\pi x a_{vx}}{L}\right)+v_y\sin\left(\frac{\pi y a_{vy}}{L}\right)+v_z\sin\left(\frac{\pi z a_{vz}}{L}\right)+v_0\right] \\
& +\frac{\pi a_{\rho z}\rho_z}{L}\cos\left(\frac{\pi z a_{\rho z}}{L}\right)\left[w_x\sin\left(\frac{\pi x a_{wx}}{L}\right)+w_y\sin\left(\frac{\pi y a_{wy}}{L}\right)+w_z\cos\left(\frac{\pi z a_{wz}}{L}\right)+w_0\right] \\
& +\frac{\pi a_{ux}u_x}{L}\cos\left(\frac{\pi x a_{ux}}{L}\right)\left[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\right] \\
& +\frac{\pi a_{vy}v_y}{L}\cos\left(\frac{\pi y a_{vy}}{L}\right)\left[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\right] \\
& -\frac{\pi a_{wz}w_z}{L}\sin\left(\frac{\pi z a_{wz}}{L}\right)\left[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\right]
\end{aligned}
\tag{11}
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\begin{aligned}
Q_u = {} & -\frac{\pi a_{px}p_x}{L}\sin\left(\frac{\pi x a_{px}}{L}\right) \\
& +\frac{\pi a_{\rho x}\rho_x}{L}\cos\left(\frac{\pi x a_{\rho x}}{L}\right)\left[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\right]^2 \\
& -\frac{\pi a_{\rho y}\rho_y}{L}\sin\left(\frac{\pi y a_{\rho y}}{L}\right)\left[v_x\cos\left(\frac{\pi x a_{vx}}{L}\right)+v_y\sin\left(\frac{\pi y a_{vy}}{L}\right)+v_z\sin\left(\frac{\pi z a_{vz}}{L}\right)+v_0\right]\left[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\right] \\
& +\frac{\pi a_{\rho z}\rho_z}{L}\cos\left(\frac{\pi z a_{\rho z}}{L}\right)\left[w_x\sin\left(\frac{\pi x a_{wx}}{L}\right)+w_y\sin\left(\frac{\pi y a_{wy}}{L}\right)+w_z\cos\left(\frac{\pi z a_{wz}}{L}\right)+w_0\right]\left[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\right] \\
& +\frac{2\pi a_{ux}u_x}{L}\cos\left(\frac{\pi x a_{ux}}{L}\right)\left[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\right]\left[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\right] \\
& -\frac{\pi a_{uy}u_y}{L}\sin\left(\frac{\pi y a_{uy}}{L}\right)\left[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\right]\left[v_x\cos\left(\frac{\pi x a_{vx}}{L}\right)+v_y\sin\left(\frac{\pi y a_{vy}}{L}\right)+v_z\sin\left(\frac{\pi z a_{vz}}{L}\right)+v_0\right] \\
& -\frac{\pi a_{uz}u_z}{L}\sin\left(\frac{\pi z a_{uz}}{L}\right)\left[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\right]\left[w_x\sin\left(\frac{\pi x a_{wx}}{L}\right)+w_y\sin\left(\frac{\pi y a_{wy}}{L}\right)+w_z\cos\left(\frac{\pi z a_{wz}}{L}\right)+w_0\right] \\
& +\frac{\pi a_{vy}v_y}{L}\cos\left(\frac{\pi y a_{vy}}{L}\right)\left[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\right]\left[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\right] \\
& -\frac{\pi a_{wz}w_z}{L}\sin\left(\frac{\pi z a_{wz}}{L}\right)\left[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\right]\left[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\right]
\end{aligned}
\tag{12}
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\begin{aligned}
Q_v = \ & \frac{\pi a_{py} p_y}{L} \cos\left(\frac{\pi y a_{py}}{L}\right) \\
& + \frac{\pi a_{\rho x} \rho_x}{L} \cos\left(\frac{\pi x a_{\rho x}}{L}\right) \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right] \left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right] \\
& - \frac{\pi a_{\rho y} \rho_y}{L} \sin\left(\frac{\pi y a_{\rho y}}{L}\right) \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right]^2 \\
& + \frac{\pi a_{\rho z} \rho_z}{L} \cos\left(\frac{\pi z a_{\rho z}}{L}\right) \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right] \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right] \\
& + \frac{\pi a_{ux} u_x}{L} \cos\left(\frac{\pi x a_{ux}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right] \\
& - \frac{\pi a_{vx} v_x}{L} \sin\left(\frac{\pi x a_{vx}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right] \\
& + \frac{2\pi a_{vy} v_y}{L} \cos\left(\frac{\pi y a_{vy}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right] \\
& + \frac{\pi a_{vz} v_z}{L} \cos\left(\frac{\pi z a_{vz}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right] \\
& - \frac{\pi a_{wz} w_z}{L} \sin\left(\frac{\pi z a_{wz}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right]
\end{aligned}
\tag{13}
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\begin{aligned}
Q_w = & -\pi a_{pz} p_z \sin\left(\frac{\pi z a_{pz}}{L}\right) \\
& + \frac{\pi a_{\rho x} \rho_x}{L} \cos\left(\frac{\pi x a_{\rho x}}{L}\right) \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right] \left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right] \\
& - \frac{\pi a_{\rho y} \rho_y}{L} \sin\left(\frac{\pi y a_{\rho y}}{L}\right) \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right] \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right] \\
& + \frac{\pi a_{\rho z} \rho_z}{L} \cos\left(\frac{\pi z a_{\rho z}}{L}\right) \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right]^2 \\
& + \frac{\pi a_{ux} u_x}{L} \cos\left(\frac{\pi x a_{ux}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right] \\
& + \frac{\pi a_{vy} v_y}{L} \cos\left(\frac{\pi y a_{vy}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right] \\
& + \frac{\pi a_{wx} w_x}{L} \cos\left(\frac{\pi x a_{wx}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right] \\
& + \frac{\pi a_{wy} w_y}{L} \cos\left(\frac{\pi y a_{wy}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right] \\
& - \frac{2\pi a_{wz} w_z}{L} \sin\left(\frac{\pi z a_{wz}}{L}\right) \left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right]
\end{aligned}
\tag{14}
$$

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\begin{aligned}
Q_{e_t} = &- \frac{\gamma}{\gamma-1}\frac{\pi a_{px} p_x}{L}\sin\left(\frac{\pi x a_{px}}{L}\right)\left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right] + \\
&+ \frac{\gamma}{\gamma-1}\frac{\pi a_{py} p_y}{L}\cos\left(\frac{\pi y a_{py}}{L}\right)\left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right] + \\
&- \frac{\gamma}{\gamma-1}\frac{\pi a_{pz} p_z}{L}\sin\left(\frac{\pi z a_{pz}}{L}\right)\left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right] + \\
&+ \frac{\pi a_{\rho x} \rho_x}{2L}\cos\left(\frac{\pi x a_{\rho x}}{L}\right)\left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right]\left(\left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right]^2\right. \\
&\qquad \left. + \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right]^2 + \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right]^2\right) + \\
&- \frac{\pi a_{\rho y} \rho_y}{2L}\sin\left(\frac{\pi y a_{\rho y}}{L}\right)\left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right]\left(\left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right]^2\right. \\
&\qquad \left. + \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right]^2 + \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right]^2\right) + \\
&+ \frac{\pi a_{\rho z} \rho_z}{2L}\cos\left(\frac{\pi z a_{\rho z}}{L}\right)\left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right]\left(\left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right]^2\right. \\
&\qquad \left. + \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right]^2 + \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right]^2\right) + \\
&+ \frac{\pi a_{ux} u_x}{2L}\cos\left(\frac{\pi x a_{ux}}{L}\right)\left\{\left(3\left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right]^2 + \left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right]^2 +\right.\right. \\
&\qquad \left. + \left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right]^2\right)\left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] \\
&\qquad\quad \left. + \left[p_x \cos\left(\frac{\pi x a_{px}}{L}\right) + p_y \sin\left(\frac{\pi y a_{py}}{L}\right) + p_z \cos\left(\frac{\pi z a_{pz}}{L}\right) + p_0\right]\frac{2\gamma}{\gamma-1}\right\} + \\
&- \frac{\pi a_{uy} u_y}{L}\sin\left(\frac{\pi y a_{uy}}{L}\right)\left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right]\left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right]\cdot \\
&\qquad \cdot\left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] + \\
&- \frac{\pi a_{uz} u_z}{L}\sin\left(\frac{\pi z a_{uz}}{L}\right)\left[w_x \sin\left(\frac{\pi x a_{wx}}{L}\right) + w_y \sin\left(\frac{\pi y a_{wy}}{L}\right) + w_z \cos\left(\frac{\pi z a_{wz}}{L}\right) + w_0\right]\left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right]\cdot \\
&\qquad \cdot\left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] + \\
&- \frac{\pi a_{vx} v_x}{L}\sin\left(\frac{\pi x a_{vx}}{L}\right)\left[v_x \cos\left(\frac{\pi x a_{vx}}{L}\right) + v_y \sin\left(\frac{\pi y a_{vy}}{L}\right) + v_z \sin\left(\frac{\pi z a_{vz}}{L}\right) + v_0\right]\left[u_x \sin\left(\frac{\pi x a_{ux}}{L}\right) + u_y \cos\left(\frac{\pi y a_{uy}}{L}\right) + u_z \cos\left(\frac{\pi z a_{uz}}{L}\right) + u_0\right]\cdot \\
&\qquad \cdot\left[\rho_x \sin\left(\frac{\pi x a_{\rho x}}{L}\right) + \rho_y \cos\left(\frac{\pi y a_{\rho y}}{L}\right) + \rho_z \sin\left(\frac{\pi z a_{\rho z}}{L}\right) + \rho_0\right] +
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&+\frac{\pi a_{vy} v_y}{2L}\cos\left(\frac{\pi y a_{vy}}{L}\right)\Bigg\{\Bigg(\bigg[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\bigg]^2+3\bigg[v_x\cos\left(\frac{\pi x a_{vx}}{L}\right)+v_y\sin\left(\frac{\pi y a_{vy}}{L}\right)+v_z\sin\left(\frac{\pi z a_{vz}}{L}\right)+v_0\bigg]^2+\\
&\quad+\bigg[w_x\sin\left(\frac{\pi x a_{wx}}{L}\right)+w_y\sin\left(\frac{\pi y a_{wy}}{L}\right)+w_z\cos\left(\frac{\pi z a_{wz}}{L}\right)+w_0\bigg]^2\Bigg)\bigg[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\bigg]+\\
&\quad+\bigg[p_x\cos\left(\frac{\pi x a_{px}}{L}\right)+p_y\sin\left(\frac{\pi y a_{py}}{L}\right)+p_z\cos\left(\frac{\pi z a_{pz}}{L}\right)+p_0\bigg]\frac{2\gamma}{\gamma-1}\Bigg\}+\\
&+\frac{\pi a_{vz} v_z}{L}\cos\left(\frac{\pi z a_{vz}}{L}\right)\bigg[w_x\sin\left(\frac{\pi x a_{wx}}{L}\right)+w_y\sin\left(\frac{\pi y a_{wy}}{L}\right)+w_z\cos\left(\frac{\pi z a_{wz}}{L}\right)+w_0\bigg]\bigg[v_x\cos\left(\frac{\pi x a_{vx}}{L}\right)+v_y\sin\left(\frac{\pi y a_{vy}}{L}\right)+v_z\sin\left(\frac{\pi z a_{vz}}{L}\right)+v_0\bigg]\cdot\\
&\quad\cdot\bigg[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\bigg]+\\
&+\frac{\pi a_{wx} w_x}{L}\cos\left(\frac{\pi x a_{wx}}{L}\right)\bigg[w_x\sin\left(\frac{\pi x a_{wx}}{L}\right)+w_y\sin\left(\frac{\pi y a_{wy}}{L}\right)+w_z\cos\left(\frac{\pi z a_{wz}}{L}\right)+w_0\bigg]\bigg[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\bigg]\cdot\\
&\quad\cdot\bigg[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\bigg]+\\
&+\frac{\pi a_{wy} w_y}{L}\cos\left(\frac{\pi y a_{wy}}{L}\right)\bigg[w_x\sin\left(\frac{\pi x a_{wx}}{L}\right)+w_y\sin\left(\frac{\pi y a_{wy}}{L}\right)+w_z\cos\left(\frac{\pi z a_{wz}}{L}\right)+w_0\bigg]\bigg[v_x\cos\left(\frac{\pi x a_{vx}}{L}\right)+v_y\sin\left(\frac{\pi y a_{vy}}{L}\right)+v_z\sin\left(\frac{\pi z a_{vz}}{L}\right)+v_0\bigg]\cdot\\
&\quad\cdot\bigg[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\bigg]+\\
&-\frac{\pi a_{wz} w_z}{2L}\sin\left(\frac{\pi z a_{wz}}{L}\right)\Bigg\{\Bigg(\bigg[u_x\sin\left(\frac{\pi x a_{ux}}{L}\right)+u_y\cos\left(\frac{\pi y a_{uy}}{L}\right)+u_z\cos\left(\frac{\pi z a_{uz}}{L}\right)+u_0\bigg]^2+\bigg[v_x\cos\left(\frac{\pi x a_{vx}}{L}\right)+v_y\sin\left(\frac{\pi y a_{vy}}{L}\right)+v_z\sin\left(\frac{\pi z a_{vz}}{L}\right)+v_0\bigg]^2+\\
&\quad+3\bigg[w_x\sin\left(\frac{\pi x a_{wx}}{L}\right)+w_y\sin\left(\frac{\pi y a_{wy}}{L}\right)+w_z\cos\left(\frac{\pi z a_{wz}}{L}\right)+w_0\bigg]^2\Bigg)\bigg[\rho_x\sin\left(\frac{\pi x a_{\rho x}}{L}\right)+\rho_y\cos\left(\frac{\pi y a_{\rho y}}{L}\right)+\rho_z\sin\left(\frac{\pi z a_{\rho z}}{L}\right)+\rho_0\bigg]+\\
&\quad+\bigg[p_x\cos\left(\frac{\pi x a_{px}}{L}\right)+p_y\sin\left(\frac{\pi y a_{py}}{L}\right)+p_z\cos\left(\frac{\pi z a_{pz}}{L}\right)+p_0\bigg]\frac{2\gamma}{\gamma-1}\Bigg\}
\end{aligned}
\tag{16}
$$

4 C codes

Files containing C codes for the source terms have also been generated. They are: `Euler_3d_e_code.C`, `Euler_3d_rho_code.C`, `Euler_3d_u_code.C`, `Euler_3d_v_code.C` and `Euler_3d_w_code.C`. The gradients of the analytical solutions have also been computed and their respective C codes are presented in `Euler_manuf_solutions_grad_and_code_3d.C`

An example of the automatically generated C file from the source term for mass conservation equation is:

```
 #include <math.h>

double SourceQ_u (double x, double y, double z, double u_0, double u_x, double u_y, double u_z, double v_0, double v_x,
                  double v_y, double v_z, double rho_0, double rho_x, double rho_y, double rho_z, double p_0, double p_x,
                  double p_y, double p_z, double a_px, double a_py, double a_pz, double a_rhox, double a_rhoy,
                  double a_rhoz, double a_ux, double a_uy, double a_uz, double a_vx, double a_vy, double a_vz, double a_wx,
```

```
                  double a_wy, double a_wz, double L)
{
  double Q_u;
  Q_u = -p_x * sin(a_px * PI * x / L) * a_px * PI / L +
  rho_x * cos(a_rhox * PI * x / L) * pow(u_0 + u_x * sin(a_ux * PI * x / L) + u_y * cos(a_uy * PI * y / L) +
    u_z * cos(a_uz * PI * z / L), 0.2e1) * a_rhox * PI / L -
  rho_y * sin(a_rhoy * PI * y / L) * (v_0 + v_x * cos(a_vx * PI * x / L) + v_y * sin(a_vy * PI * y / L) +
    v_z * sin(a_vz * PI * z / L)) * (u_0 + u_x * sin(a_ux * PI * x / L) + u_y * cos(a_uy * PI * y / L) +
    u_z * cos(a_uz * PI * z / L)) * a_rhoy * PI / L +
  rho_z * cos(a_rhoz * PI * z / L) * (w_0 + w_x * sin(a_wx * PI * x / L) + w_y * sin(a_wy * PI * y / L) +
    w_z * cos(a_wz * PI * z / L)) * (u_0 + u_x * sin(a_ux * PI * x / L) + u_y * cos(a_uy * PI * y / L) +
    u_z * cos(a_uz * PI * z / L)) * a_rhoz * PI / L +
  0.2e1 * u_x * cos(a_ux * PI * x / L) * (rho_0 + rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L) +
    rho_z * sin(a_rhoz * PI * z / L)) * (u_0 + u_x * sin(a_ux * PI * x / L) +  u_y * cos(a_uy * PI * y / L) +
    u_z * cos(a_uz * PI * z / L)) * a_ux * PI / L -
  u_y * sin(a_uy * PI * y / L) * (rho_0 + rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L) +
    rho_z * sin(a_rhoz * PI * z / L)) * (v_0 + v_x * cos(a_vx * PI * x / L) + v_y * sin(a_vy * PI * y / L) +
    v_z * sin(a_vz * PI * z / L)) * a_uy * PI / L -
  u_z * sin(a_uz * PI * z / L) * (rho_0 + rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L) +
    rho_z * sin(a_rhoz * PI * z / L)) * (w_0 + w_x * sin(a_wx * PI * x / L) + w_y * sin(a_wy * PI * y / L) +
    w_z * cos(a_wz * PI * z / L)) * a_uz * PI / L +
  v_y * cos(a_vy * PI * y / L) * (rho_0 + rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L) +
    rho_z * sin(a_rhoz * PI * z / L)) *
    (u_0 + u_x * sin(a_ux * PI * x / L) + u_y * cos(a_uy * PI * y / L) + u_z * cos(a_uz * PI * z / L)) * a_vy * PI / L -
  w_z * sin(a_wz * PI * z / L) * (rho_0 + rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L) +
     rho_z * sin(a_rhoz * PI * z / L)) * (u_0 + u_x * sin(a_ux * PI * x / L) + u_y * cos(a_uy * PI * y / L) +
      u_z * cos(a_uz * PI * z / L)) * a_wz * PI / L;
  return(Q_u);
}
```

Finally the gradients of the analytical solutions (10) have also been computed and their respective C codes are presented in `Euler_manuf_solutions_grad_and_code_3d.C`. Therefore,

$$\nabla\rho = \begin{bmatrix} \dfrac{a_{\rho x}\pi\rho_x}{L}\cos\left(\dfrac{a_{\rho x}\pi x}{L}\right) \\ -\dfrac{a_{\rho y}\pi\rho_y}{L}\sin\left(\dfrac{a_{\rho y}\pi y}{L}\right) \\ \dfrac{a_{\rho z}\pi\rho_z}{L}\cos\left(\dfrac{a_{\rho z}\pi z}{L}\right) \end{bmatrix}, \quad \nabla p = \begin{bmatrix} -\dfrac{a_{px}\pi p_x}{L}\sin\left(\dfrac{a_{px}\pi x}{L}\right) \\ \dfrac{a_{py}\pi p_y}{L}\cos\left(\dfrac{a_{py}\pi y}{L}\right) \\ -\dfrac{a_{pz}\pi p_z}{L}\sin\left(\dfrac{a_{pz}\pi z}{L}\right) \end{bmatrix}, \quad \nabla u = \begin{bmatrix} \dfrac{a_{ux}\pi u_x}{L}\cos\left(\dfrac{a_{ux}\pi x}{L}\right) \\ -\dfrac{a_{uy}\pi u_y}{L}\sin\left(\dfrac{a_{uy}\pi y}{L}\right) \\ -\dfrac{a_{uz}\pi u_z}{L}\sin\left(\dfrac{a_{uz}\pi z}{L}\right) \end{bmatrix}, \tag{17}$$

$$\nabla v = \begin{bmatrix} -\frac{a_{vx}\pi v_x}{L}\sin\left(\frac{a_{vx}\pi x}{L}\right) \\ \frac{a_{vy}\pi v_y}{L}\cos\left(\frac{a_{vy}\pi y}{L}\right) \\ \frac{a_{vz}\pi v_z}{L}\cos\left(\frac{a_{vz}\pi z}{L}\right) \end{bmatrix} \quad \text{and} \quad \nabla w = \begin{bmatrix} \frac{a_{wx}\pi w_x}{L}\cos\left(\frac{a_{wx}\pi x}{L}\right) \\ \frac{a_{wy}\pi w_y}{L}\cos\left(\frac{a_{wy}\pi y}{L}\right) \\ -\frac{a_{wz}\pi w_z}{L}\sin\left(\frac{a_{wz}\pi z}{L}\right) \end{bmatrix} \tag{18}$$

are written in C language as:

```
grad_rho_an[0] = rho_x * cos(a_rhox * pi * x / L) * a_rhox * pi / L;
grad_rho_an[1] = -rho_y * sin(a_rhoy * pi * y / L) * a_rhoy * pi / L;
grad_rho_an[2] = rho_z * cos(a_rhoz * pi * z / L) * a_rhoz * pi / L;
grad_p_an[0] = -p_x * sin(a_px * pi * x / L) * a_px * pi / L;
grad_p_an[1] = p_y * cos(a_py * pi * y / L) * a_py * pi / L;
grad_p_an[2] = -p_z * sin(a_pz * pi * z / L) * a_pz * pi / L;
grad_u_an[0] = u_x * cos(a_ux * pi * x / L) * a_ux * pi / L;
grad_u_an[1] = -u_y * sin(a_uy * pi * y / L) * a_uy * pi / L;
grad_u_an[2] = -u_z * sin(a_uz * pi * z / L) * a_uz * pi / L;
grad_v_an[0] = -v_x * sin(a_vx * pi * x / L) * a_vx * pi / L;
grad_v_an[1] = v_y * cos(a_vy * pi * y / L) * a_vy * pi / L;
grad_v_an[2] = v_z * cos(a_vz * pi * z / L) * a_vz * pi / L;
grad_w_an[0] = w_x * cos(a_wx * pi * x / L) * a_wx * pi / L;
grad_w_an[1] = w_y * cos(a_wy * pi * y / L) * a_wy * pi / L;
grad_w_an[2] = -w_z * sin(a_wz * pi * z / L) * a_wz * pi / L;
```

References

Roy, C., T. Smith, and C. Ober (2002). Verification of a compressible cfd code using the method of manufactured solutions. In *AIAA FLuid Dynamics Conference and Exhibit*, Number AIAA 2002-3110.