Manufactured Solution for 2D Burgers equations using Maple*

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Abstract

This document describes the usage of the Method of Manufactured Solutions (MMS) for Code Verification of Burgers solvers for two-dimensional, viscous and inviscid flows in both steady and unsteady regimen. The manufactured solutions chosen for the flow variables are time-dependent, smooth and have non-trivial derivatives, so they may exercise all the terms of the governing equations without imposing special coordinate systems or particular boundary conditions. By the application of the MMS, a related set of governing equations is generated. The analytical solutions of the MMS-modified governing equations are, in turn, the manufactured solutions chosen a priori. Thus, the modified set of equations genetared by the MMS may be discretized and solved numerically, convergence studies may be conducted, and the code may be verified. A choice of analytical solutions for the flow variables of the 2D Burgers equations and their respective source terms are presented in this document.

1 2D Burgers Equations

Burgers' equation is a useful test case for numerical methods due to its simplicity and predictable dynamics, together with its non-linearity and multidimensionality. The various kinds of Burgers equation constitute a good benchmark to modelling traffic flows, shock waves and acoustic transmission, and they are also considered a basic model of nonlinear convective-diffusive phenomena such as those that arise in Navier–Stokes equations.

The 2D Burgers equations are:

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{1}$$

and

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \tag{2}$$

where u and v are the velocity in the x and y directions, respectively, and ν is the viscosity.

2 Manufactured Solution

The Method of Manufactured Solutions (MMS) provides a general procedure for code accuracy verification (Roache, 2002; Bond et al., 2007). The MMS constructs a non-trivial but analytical solution for the flow variables; this manufactured solution usually does not satisfy the governing equations, since the choice is somewhat arbitrary. However, by passing the solution through the governing equations gives the production terms Q. A modified set of equations formed by adding these source terms to the right-hand-side of the original governing equations is forced to become a model for the constructed solution, i.e., the manufactured solutions chosen *a priori* are the analytical solutions of the MMS-modified equations.

Although the form of the manufactured solution is slightly arbitrary, it should be chosen to be smooth, infinitely differentiable and realizable (solutions should be avoided which have negative densities, pressures,

*Work based on Salari and Knupp (2000).

temperatures, etc.)(Salari and Knupp, 2000; Roy et al., 2004). Solutions should also be chosen that are sufficiently general so as to exercise all terms in the governing equations. Examples of manufactured solutions and convergence studies for Burgers, Euler and/or Navier–Stokes equations may be found in Salari and Knupp (2000); Roy et al. (2002, 2004); Bond et al. (2007); Orozco et al. (2010).

Salari and Knupp (2000) propose the general form of the primitive solution variables to be a function of sines or cosines:

$$\phi(x, y, t) = \phi_0 \Big(f_s(x^2 + y^2 + \omega t) + \varepsilon \Big), \tag{3}$$

where $\phi = u$ or v, and $f_s(\cdot)$ functions denote either sine or cosine function. Note that ϕ_0, ω and ε are constants.

Therefore, a suitable set of time-dependent manufactured analytical solutions for each one of the variables in Burgers equations is:

$$\begin{aligned} u(x, y, t) &= u_0 \Big(\sin(x^2 + y^2 + \omega t) + \varepsilon \Big), \\ v(x, y, t) &= v_0 \Big(\cos(x^2 + y^2 + \omega t) + \varepsilon \Big), \end{aligned} \tag{4}$$

The source terms for the 2D Burger equations (1) and (2) using time-dependent manufactured solutions for u and v, described in Equations (4) are presented in the following sections. The inviscid and the steady state cases are also presented.

3 Burgers equation

For the generation of the analytical source term Q_u for the velocity in the x-direction, Equation (1) is written as an operator L:

$$L = \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} - \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{5}$$

which, when operated in Equation (4), provides source term Q_u:

$$\begin{aligned} Q_u = {} & 2u_0^2\, x \left[\sin\big(2(x^2 + y^2 + \omega t)\big) + 2\varepsilon \cos(x^2 + y^2 + \omega t)\right] + \\ & + 2u_0\, v_0\, y \left[-2\sin^2(x^2 + y^2 + \omega t) - \varepsilon \sin(x^2 + y^2 + \omega t) + \varepsilon \cos(x^2 + y^2 + \omega t) + 1\right] + \\ & + 4u_0\, \nu \left[(x^2 + y^2) \sin(x^2 + y^2 + \omega t) - \cos(x^2 + y^2 + \omega t)\right] + \\ & + u_0\, \omega\, \cos(x^2 + y^2 + \omega t). \end{aligned} \tag{6}$$

Analogously, for the generation of the analytical source term Q_v for the y-velocity, Equation (2) is written as an operator L:

$$L = \frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} - \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \tag{7}$$

and then applied to Equation (4). It yields:

$$\begin{aligned} Q_v = {} & - 2v_0^2\, y \left[2\varepsilon \sin(x^2 + y^2 + \omega t) + \sin\big(2(x^2 + y^2 + \omega t)\big)\right] + \\ & + 2v_0\, u_0\, x \left[-2\sin^2(x^2 + y^2 + \omega t) - \varepsilon \sin(x^2 + y^2 + \omega t) + \varepsilon \cos(x^2 + y^2 + \omega t) + 1\right] + \\ & + 4v_0\, \nu \left[(x^2 + y^2) \cos(x^2 + y^2 + \omega t) + \sin(x^2 + y^2 + \omega t)\right] \\ & - v_0\, \omega\, \sin(x^2 + y^2 + \omega t). \end{aligned} \tag{8}$$

3.1 Steady Burgers Equation

For the steady Burgers flow:

$$\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{9}$$

$$\frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \tag{10}$$

ω is set to zero in the manufactured solution (4) and the resulting source terms are:

$$\begin{aligned} Q_u = & \, 2u_0^2 x[\sin\left(2(x^2+y^2)\right) + 2\varepsilon\cos(x^2+y^2)] + \\ & + 2u_0 v_0 y[-2\sin^2(x^2+y^2) - \varepsilon\sin(x^2+y^2) + \varepsilon\cos(x^2+y^2) + 1] + \\ & + 4u_0\nu[(x^2+y^2)\sin(x^2+y^2) - 4\cos(x^2+y^2)], \end{aligned} \tag{11}$$

and

$$\begin{aligned} Q_v = & -2v_0^2 y[2\varepsilon\sin(x^2+y^2) + \sin\left(2(x^2+y^2)\right)] + \\ & + 2v_0 u_0 x[-2\sin^2(x^2+y^2) - \varepsilon\sin(x^2+y^2) + \varepsilon\cos(x^2+y^2) + 1] + \\ & + 4v_0\nu[(x^2+y^2)\cos(x^2+y^2) + 4\sin(x^2+y^2)]. \end{aligned} \tag{12}$$

3.2 Inviscid Burgers equation

The 2D inviscid Burgers equations are:

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = 0, \tag{13}$$

and

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = 0. \tag{14}$$

For the inviscid case, source terms Q_u and Q_v are

$$\begin{aligned} Q_u = & \, 2u_0^2\, x\, [\sin\left(2(x^2+y^2+\omega t)\right) + 2\varepsilon\cos(x^2+y^2+\omega t)] + \\ & + 2u_0\, v_0\, y\, [-2\sin^2(x^2+y^2+\omega t) - \varepsilon\sin(x^2+y^2+\omega t) + \varepsilon\cos(x^2+y^2+\omega t) + 1] + \\ & + u_0\, \omega\, \cos(x^2+y^2+\omega t) \end{aligned} \tag{15}$$

$$\begin{aligned} Q_v = & -2v_0^2\, y\, [2\varepsilon\sin(x^2+y^2+\omega t) + \sin\left(2(x^2+y^2+\omega t)\right)] + \\ & + 2v_0\, u_0\, x\, [-2\sin^2(x^2+y^2+\omega t) - \varepsilon\sin(x^2+y^2+\omega t) + \varepsilon\cos(x^2+y^2+\omega t) + 1] + \\ & - v_0\omega\sin(x^2+y^2+\omega t) \end{aligned} \tag{16}$$

3.3 Steady inviscid Burgers equation

The 2D steady inviscid Burgers equations are:

$$\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = 0, \tag{17}$$

and

$$\frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = 0. \tag{18}$$

For this case, source terms Q_u and Q_v are obtained by setting ω to zero in the manufactured solutions:

$$\begin{aligned} Q_u = & \, 2u_0^2\, x\, [\sin\left(2(x^2+y^2)\right) + 2\varepsilon\cos(x^2+y^2)] + \\ & + 2u_0\, v_0\, y\, [-2\sin^2(x^2+y^2) - \varepsilon\sin(x^2+y^2) + \varepsilon\cos(x^2+y^2) + 1], \end{aligned} \tag{19}$$

and

$$\begin{aligned} Q_v = & -2v_0^2\, y\, [2\varepsilon\sin(x^2+y^2) + \sin\left(2(x^2+y^2)\right)] + \\ & + 2v_0\, u_0\, x\, [-2\sin^2(x^2+y^2) - \varepsilon\sin(x^2+y^2) + \varepsilon\cos(x^2+y^2) + 1]. \end{aligned} \tag{20}$$

4 Comments

Source terms Q_u and Q_v have been generated by replacing the analytical Expressions (4) into respective set of Equations (1) – (2), (9) – (10), (13) – (14) and (17) – (18), followed by the usage of Maple commands for collecting, sorting and factorizing the terms. Files containing C codes for the source terms have also been generated. They are: `Burgers_2d_u_v_code.C`, `Burgers_2d_steady_u_v_code.C`, `Burgers_2d_inviscid_u_v_code.C` and `Burgers_2d_steady_inviscid_u_v_code.C`.

An example of the automatically generated C file from the source term for velocity u (Equation (6)) is:

```
#include <math.h>

double SourceQ_u (double x, double y, double t, double u_0, double v_0, double omega,
                  double epsilon, double nu){
  double Q_u;
  Q_u = (0.2e1 * sin(0.2e1 * x * x + 0.2e1 * y * y + 0.2e1 * omega * t) +
    0.4e1 * epsilon * cos(x * x + y * y + omega * t)) * u_0 * u_0 * x +
    (-0.4e1 * pow(sin(x * x + y * y + omega * t), 0.2e1) -
    0.2e1 * epsilon * sin(x * x + y * y + omega * t) +
    0.2e1 * epsilon * cos(x * x + y * y + omega * t) + 0.2e1) * u_0 * v_0 * y +
    u_0 * cos(x * x + y * y + omega * t) * omega +
    (0.4e1 * x * x * sin(x * x + y * y + omega * t) +
    0.4e1 * y * y * sin(x * x + y * y + omega * t) -
    0.4e1 * cos(x * x + y * y + omega * t)) * u_0 * nu;
  return(Q_u);
}
```

Finally, the gradients of the analytical solutions (4) have also been computed and their respective C codes are presented in `Burgers_2d_transient_manuf_solutions_grad_code.C`. Therefore,

$$\nabla u = \begin{bmatrix} 2u_0 x\cos(x^2+y^2+\omega t) \\ 2u_0 y\cos(x^2+y^2+\omega t) \end{bmatrix} \quad \text{and} \quad \nabla v = \begin{bmatrix} -2v_0 x\sin(x^2+y^2+\omega t) \\ -2v_0 y\sin(x^2+y^2+\omega t) \end{bmatrix} \tag{21}$$

are written in C language as:

```
grad_u_an[0] = 0.2e1 * u_0 * cos(x * x + y * y + omega * t) * x;
grad_u_an[1] = 0.2e1 * u_0 * cos(x * x + y * y + omega * t) * y;
grad_v_an[0] = -0.2e1 * v_0 * sin(x * x + y * y + omega * t) * x;
grad_v_an[1] = -0.2e1 * v_0 * sin(x * x + y * y + omega * t) * y;
```

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