Choice of Manufactured Analytical Solution for Code Verification of Axisymmetric Euler Equations

Kemelli C. Estacio-Hiroms

May 19, 2010

Abstract

The Method of Manufactured Solutions is a valuable approach for code verification, providing means to verify how accurately the numerical method solves the equations of interest. The method generates a related set of governing equations by adding a source term to the RHS of the original set of equations, making use of analytical solutions chosen a priori. In this document, a choice of analytical solutions for the flow variables together with their respective source terms is presented for the Axisymmetric Euler equations.

1 Axisymmetric Euler equations

Euler equations may be written in cylindrical coordinates for (r, θ, z), where r is the radial coordinate, θ is the angular coordinate, and z is the axial coordinate. In axisymmetrical flows, the pressure and the velocity fields are independent of the angular variable θ, and the problem depends exclusively on r and z. Therefore, Euler equations for axisymmetric flows, in conservative form, are:

$$\frac{\partial(\rho)}{\partial t} + \frac{1}{r}\frac{\partial(\rho r u)}{\partial r} + \frac{1}{r}\frac{\partial(\rho r w)}{\partial z} = 0 \tag{1}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{1}{r}\frac{\partial(r\rho u^2)}{\partial r} + \frac{1}{r}\frac{\partial(r\rho u w)}{\partial z} + \frac{\partial p}{\partial r} = 0, \tag{2}$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{1}{r}\frac{\partial(r\rho w u)}{\partial r} + \frac{1}{r}\frac{\partial(r\rho w^2)}{\partial z} + \frac{\partial p}{\partial z} = 0, \tag{3}$$

$$\frac{\partial(\rho e_t)}{\partial t} + \frac{1}{r}\frac{\partial(r(\rho e_t + p)u)}{\partial r} + \frac{1}{r}\frac{\partial(r(\rho e_t + p)w)}{\partial z} = 0. \tag{4}$$

For a calorically perfect gas, Euler equations are closed with two auxiliary relations for energy:

$$e_t = e + \frac{u^2 + w^2}{2} \quad \text{and} \quad e = \frac{1}{\gamma - 1} RT, \tag{5}$$

where γ is the ratio of specific heats, and with the ideal gas equation of state:

$$p = \rho R T. \tag{6}$$

2 Manufactured Solution

The choice of cylindrical coordinates, like any system that contains a symmetry axis, introduces singular terms in the governing equation of the type r^{-n}, being r the radial coordinate and n a positive exponent, although the flow is continuous and regular at the axis [1].

Accordingly, the representation of fluid flows in cylindrical coordinates requires the definition of appropriate boundary conditions at $r = 0$, despite the fact that it is not a physical boundary, that would guarantee the regularity of the flow:

$$\begin{aligned} u\Big|_{r=0} &= 0, \\ \frac{\partial u}{\partial r}\Big|_{r=0} &= 0, \\ \frac{\partial w}{\partial r}\Big|_{r=0} &= 0. \end{aligned} \tag{7}$$

The strategy to deal with this difficulty in analytical approaches is commonly that of discarding the singular solutions among all the admissible ones. Consequently, a suitable form of each one of the primitive solution variables is a function of sines and cosines:

$$\begin{aligned} \rho(r,z) &= \rho_0 + \rho_1 \cos\left(\frac{a_{\rho r}\pi r}{L}\right) \sin\left(\frac{a_{\rho z}\pi z}{L}\right), \\ u(r,z) &= u_1 \left[\cos\left(\frac{a_{ur}\pi r}{L}\right) - 1\right] \sin\left(\frac{a_{uz}\pi z}{L}\right), \\ w(r,z) &= w_0 + w_1 \cos\left(\frac{a_{wr}\pi r}{L}\right) \sin\left(\frac{a_{wz}\pi z}{L}\right), \\ p(r,z) &= p_0 + p_1 \sin\left(\frac{a_{pr}\pi r}{L}\right) \cos\left(\frac{a_{pz}\pi z}{L}\right), \end{aligned} \tag{8}$$

where ρ_0, ρ_1, p_0, p_1, u_1, w_0 and w_1 are pre-defined constants.

The MMS applied to Euler equations consists in modifying Equations (1) – (4) by adding a source term to the right-hand side of each equation:

$$\begin{aligned} \frac{\partial(\rho)}{\partial t} + \frac{1}{r}\frac{\partial(\rho r u)}{\partial r} + \frac{1}{r}\frac{\partial(\rho r w)}{\partial z} &= Q_\rho, \\ \frac{\partial(\rho u)}{\partial t} + \frac{1}{r}\frac{\partial(r\rho u^2)}{\partial r} + \frac{1}{r}\frac{\partial(r\rho u w)}{\partial z} + \frac{\partial p}{\partial r} &= Q_u, \\ \frac{\partial(\rho w)}{\partial t} + \frac{1}{r}\frac{\partial(r\rho w u)}{\partial r} + \frac{1}{r}\frac{\partial(r\rho w^2)}{\partial z} + \frac{\partial p}{\partial z} &= Q_w, \\ \frac{\partial(\rho e_t)}{\partial t} + \frac{1}{r}\frac{\partial(r(\rho e_t + p)u)}{\partial r} + \frac{1}{r}\frac{\partial(r(\rho e_t + p)w)}{\partial z} &= Q_e, \end{aligned} \tag{9}$$

so the modified set of equations conveniently has the analytical solution given in Equation (8). This is achieved by simply applying Equations (1) – (4) as operators on Equation (8).

Terms Q_ρ, Q_u, Q_w and Q_e are obtained by symbolic manipulations of equations above using Maple and are presented in the following sections.

2.1 Source term for mass conservation equation

$$
\begin{aligned}
Q_\rho =& - \frac{a_{\rho r}\pi\rho_1}{L}\sin\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&+\frac{a_{\rho z}\pi\rho_1}{L}\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&-\frac{a_{ur}\pi u_1}{L}\sin\left(\frac{a_{ur}\pi r}{L}\right)\sin\left(\frac{a_{uz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]+\\
&+\frac{a_{wz}\pi w_1}{L}\cos\left(\frac{a_{wr}\pi r}{L}\right)\cos\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]+\\
&+\frac{1}{r}\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right].
\end{aligned}
\tag{10}
$$

2.2 Source term for radial velocity

$$
\begin{aligned}
Q_u =& - \frac{a_{\rho r}\pi\rho_1}{L}\sin\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]^2+\\
&+\frac{a_{\rho z}\pi\rho_1}{L}\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&-\frac{2a_{ur}\pi u_1}{L}\sin\left(\frac{a_{ur}\pi r}{L}\right)\sin\left(\frac{a_{uz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&+\frac{a_{uz}\pi u_1}{L}\cos\left(\frac{a_{uz}\pi z}{L}\right)\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&+\frac{a_{wz}\pi w_1}{L}\cos\left(\frac{a_{wr}\pi r}{L}\right)\cos\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&+\frac{a_{pr}\pi p_1}{L}\cos\left(\frac{a_{pr}\pi r}{L}\right)\cos\left(\frac{a_{pz}\pi z}{L}\right)\quad+\quad\frac{1}{r}\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]^2.
\end{aligned}
\tag{11}
$$

2.3 Source term for axial velocity

$$
\begin{aligned}
Q_w =& - \frac{a_{\rho r}\pi\rho_1}{L}\sin\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&+\frac{a_{\rho z}\pi\rho_1}{L}\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]^2+\\
&-\frac{a_{ur}\pi u_1}{L}\sin\left(\frac{a_{ur}\pi r}{L}\right)\sin\left(\frac{a_{uz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&-\frac{a_{wr}\pi w_1}{L}\sin\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&+\frac{2a_{wz}\pi w_1}{L}\cos\left(\frac{a_{wr}\pi r}{L}\right)\cos\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&-\frac{a_{pz}\pi p_1}{L}\sin\left(\frac{a_{pr}\pi r}{L}\right)\sin\left(\frac{a_{pz}\pi z}{L}\right)\quad+\quad\frac{1}{r}\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right].
\end{aligned}
\tag{12}
$$

2.4 Source term for energy

$$
\begin{aligned}
Q_{e_t} =& -\frac{a_{\rho r}\pi\rho_1}{2L}\sin\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]^2+\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]^2\right]+\\
&+\frac{a_{\rho z}\pi\rho_1}{2L}\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]\left[\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]^2+\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]^2\right]+\\
&-\frac{a_{ur}\pi u_1}{2L}\sin\left(\frac{a_{ur}\pi r}{L}\right)\sin\left(\frac{a_{uz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]^2+3\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]^2\right]+\\
&-\frac{a_{ur}\pi u_1}{L}\frac{\gamma}{\gamma-1}\sin\left(\frac{a_{ur}\pi r}{L}\right)\sin\left(\frac{a_{uz}\pi z}{L}\right)\left[p_0+p_1\sin\left(\frac{a_{pr}\pi r}{L}\right)\cos\left(\frac{a_{pz}\pi z}{L}\right)\right]+\\
&+\frac{a_{uz}\pi u_1}{L}\cos\left(\frac{a_{uz}\pi z}{L}\right)\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&-\frac{a_{wr}\pi w_1}{L}\sin\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&+\frac{a_{wz}\pi w_1}{2L}\cos\left(\frac{a_{wr}\pi r}{L}\right)\cos\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[3\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]^2+\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]^2\right]+\\
&+\frac{a_{wz}\pi w_1}{L}\frac{\gamma}{\gamma-1}\cos\left(\frac{a_{wr}\pi r}{L}\right)\cos\left(\frac{a_{wz}\pi z}{L}\right)\left[p_0+p_1\sin\left(\frac{a_{pr}\pi r}{L}\right)\cos\left(\frac{a_{pz}\pi z}{L}\right)\right]+\\
&+\frac{a_{pr}\pi p_1}{L}\frac{\gamma}{\gamma-1}\cos\left(\frac{a_{pr}\pi r}{L}\right)\cos\left(\frac{a_{pz}\pi z}{L}\right)\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&-\frac{a_{pz}\pi p_1}{L}\frac{\gamma}{\gamma-1}\sin\left(\frac{a_{pr}\pi r}{L}\right)\sin\left(\frac{a_{pz}\pi z}{L}\right)\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&+\frac{1}{2r}\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]^2+\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]^2\right]+\\
&+\frac{\gamma}{r(\gamma-1)}\left[p_0+p_1\sin\left(\frac{a_{pr}\pi r}{L}\right)\cos\left(\frac{a_{pz}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right].
\end{aligned}
\tag{13}
$$

3 Comments

Source terms Q_ρ, Q_u, Q_w and Q_e have been generated by replacing the analytical expressions (8) into respective equations (1) – (4), followed by the usage of Maple commands for collecting, sorting and factorizing the terms. Files containing C codes for the source terms have also been generated. They are: `Euler_axi_rho_code.C`, `Euler_axi_u_code.C`, `Euler_axi_w_code.C` and `Euler_axi_e_code.C`.

An example of the automatically generated C file from the source term for the radial velocity u equation is:

```
#include <math.h>

double SourceQ_u ( double r, double z, double p_0, double p_1, double rho_0, double rho_1, double u_1, double w_0, double w_1,
                   double a_pr, double a_pz, double a_rhor, double a_rhoz, double a_ur, double a_uz, double a_wr, double a_wz,
                   double PI, double L)
{
  double Q_u;
  Q_u = p_1 * cos(a_pr * PI * r / L) * cos(a_pz * PI * z / L) * a_pr * PI / L -
        u_1 * u_1 * pow(sin(a_uz * PI * z / L), 0.2e1) * pow(cos(a_ur * PI * r / L) - 0.1e1, 0.2e1) *
        rho_1 * sin(a_rhor * PI * r / L) * sin(a_rhoz * PI * z / L) * a_rhor * PI / L +
        u_1 * (cos(a_ur * PI * r / L) - 0.1e1) * rho_1 * cos(a_rhor * PI * r / L) * cos(a_rhoz * PI * z / L) *
        sin(a_uz * PI * z / L) * (w_0 + w_1 * cos(a_wr * PI * r / L) * sin(a_wz * PI * z / L)) * a_rhoz * PI / L
        - 0.2e1 * u_1 * u_1 * pow(sin(a_uz * PI * z / L), 0.2e1) * (cos(a_ur * PI * r / L) - 0.1e1) * sin(a_ur * PI * r / L) *
        (rho_0 + rho_1 * cos(a_rhor * PI * r / L) * sin(a_rhoz * PI * z / L)) * a_ur * PI / L +
        u_1 * (cos(a_ur * PI * r / L) - 0.1e1) * cos(a_uz * PI * z / L) *
        (rho_0 + rho_1 * cos(a_rhor * PI * r / L) * sin(a_rhoz * PI * z / L)) *
        (w_0 + w_1 * cos(a_wr * PI * r / L) * sin(a_wz * PI * z / L)) * a_uz * PI / L +
        u_1 * (cos(a_ur * PI * r / L) - 0.1e1) * w_1 * cos(a_wr * PI * r / L) * cos(a_wz * PI * z / L) * sin(a_uz * PI * z / L) *
        (rho_0 + rho_1 * cos(a_rhor * PI * r / L) * sin(a_rhoz * PI * z / L)) * a_wz * PI / L +
        u_1 * u_1 * pow(sin(a_uz * PI * z / L), 0.2e1) * pow(cos(a_ur * PI * r / L) - 0.1e1, 0.2e1) *
        (rho_0 + rho_1 * cos(a_rhor * PI * r / L) * sin(a_rhoz * PI * z / L)) / r;
  return(Q_u);
}
```

Finally the gradients of the analytical solutions (8) have also been computed and their respective C codes are presented in `Euler_manuf_solutions_grad_and_code_axisymmetric.C`. Therefore,

$$\nabla\rho = \begin{bmatrix} -\dfrac{a_{\rho r}\pi\rho_1}{L}\sin\left(\dfrac{a_{\rho r}\pi r}{L}\right)\sin\left(\dfrac{a_{\rho z}\pi z}{L}\right) \\ \dfrac{a_{\rho z}\pi\rho_1}{L}\cos\left(\dfrac{a_{\rho r}\pi r}{L}\right)\cos\left(\dfrac{a_{\rho z}\pi z}{L}\right) \end{bmatrix}, \qquad \nabla p = \begin{bmatrix} \dfrac{a_{pr}\pi p_1}{L}\cos\left(\dfrac{a_{pr}\pi r}{L}\right)\cos\left(\dfrac{a_{pz}\pi z}{L}\right) \\ -\dfrac{a_{pz}\pi p_1}{L}\sin\left(\dfrac{a_{pr}\pi r}{L}\right)\sin\left(\dfrac{a_{pz}\pi z}{L}\right) \end{bmatrix},$$

$$\nabla u = \begin{bmatrix} -\dfrac{a_{ur}\pi u_1}{L}\sin\left(\dfrac{a_{ur}\pi r}{L}\right)\sin\left(\dfrac{a_{uz}\pi z}{L}\right) \\ \dfrac{a_{uz}\pi u_1}{L}\left(\cos\left(\dfrac{a_{ur}\pi r}{L}\right)-1\right)\cos\left(\dfrac{a_{uz}\pi z}{L}\right) \end{bmatrix}, \qquad \nabla w = \begin{bmatrix} -\dfrac{a_{wr}\pi w_1}{L}\sin\left(\dfrac{a_{wr}\pi r}{L}\right)\sin\left(\dfrac{a_{wz}\pi z}{L}\right) \\ \dfrac{a_{wz}\pi w_1}{L}\cos\left(\dfrac{a_{wr}\pi r}{L}\right)\cos\left(\dfrac{a_{wz}\pi z}{L}\right) \end{bmatrix},$$

are written in C language as:

```
grad_rho_an[0] = -rho_1 * sin(a_rhor * pi * r / L) * a_rhor * pi / L * sin(a_rhoz * pi * z / L);
grad_rho_an[1] = 0;
```

```
grad_rho_an[2] = rho_1 * cos(a_rhor * pi * r / L) * cos(a_rhoz * pi * z / L) * a_rhoz * pi / L;
grad_p_an[0] = p_1 * cos(a_pr * pi * r / L) * a_pr * pi / L * cos(a_pz * pi * z / L);
grad_p_an[1] = 0;
grad_p_an[2] = -p_1 * sin(a_pr * pi * r / L) * sin(a_pz * pi * z / L) * a_pz * pi / L;
grad_u_an[0] = -u_1 * sin(a_ur * pi * r / L) * a_ur * pi / L * sin(a_uz * pi * z / L);
grad_u_an[1] = 0;
grad_u_an[2] = u_1 * (cos(a_ur * pi * r / L) - 0.1e1) * cos(a_uz * pi * z / L) * a_uz * pi / L;
grad_w_an[0] = -w_1 * sin(a_wr * pi * r / L) * a_wr * pi / L * sin(a_wz * pi * z / L);
grad_w_an[1] = 0;
grad_w_an[2] = w_1 * cos(a_wr * pi * r / L) * cos(a_wz * pi * z / L) * a_wz * pi / L;
```

References

[1] F. Domenichini and B. Baccani, "A formulation of Navier–Stokes problem in cylindrical coordinates applied to the 3D entry jet in a duct," *Journal of Computational Physics*, vol. 200, p. 177–191, 2004.