Manufactured Solution for the Incompressible Transient Flow Equations using Sympy

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Abstract

The Method of Manufactured Solutions is a valuable approach for code verification, providing means to verify how accurately the numerical method solves the partial differential equations of interest. This document presents the source terms generated by the application of the Method of Manufactured Solutions (MMS) for an incompressible turbulence flow using the analytical manufactured solutions for velocity presented by Ulerich (2011).

1 Mathematical model

The governing equations for a incompressible flow can be written as (Kim et al., 1987):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{H} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2}$$

Here, all the variables are non-dimensionalized by the channel half-width δ, and the wall shear velocity u_τ. $\boldsymbol{u} = (u, v, w)$, $\boldsymbol{H} = \boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and Re denotes the Reynolds number defined as $Re = u_\tau \delta / \nu$.

Equation (1) may be reduced to yield a fourth-order equation for v, and a second-order equation for the normal component of vorticity, g, as follows:

$$\frac{\partial}{\partial t}\nabla^2 v + h_v = \frac{1}{Re}\nabla^4 v \tag{3}$$

$$\frac{\partial g}{\partial t} + h_g = \frac{1}{Re}\nabla^2 g \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5}$$

where

$$\begin{aligned} g &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \\ h_v &= -\frac{\partial}{\partial y}\left(\frac{\partial H_1}{\partial x} + \frac{\partial H_3}{\partial z}\right) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) H_2, \\ h_g &= \frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x}. \end{aligned} \tag{6}$$

2 Manufactured solution

MMS applied to Navier–Stokes equations consists in modifying Equations (3) – (5) by adding a source term to the right-hand side of each equation, so the modified set of equations conveniently has the analytical solution chosen *a priori*.

We choose the general form of the primitive manufactured solution variables to be a function of cosines in x, y, z and t:

$$
\begin{aligned}
\phi(x,y,z,t) = {} & a_{\phi 0} \cos\left(f_{\phi 0} t + g_{\phi 0}\right) \\
& + a_{\phi x} \cos\left(\frac{b_{\phi x} 2\pi x}{L_x} + c_{\phi x}\right) \cos\left(f_{\phi x} t + g_{\phi x}\right) \\
& + a_{\phi y} \cos\left(\frac{b_{\phi y} 2\pi y}{L_y} + c_{\phi y}\right) \cos\left(f_{\phi y} t + g_{\phi y}\right) \\
& + a_{\phi z} \cos\left(\frac{b_{\phi z} 2\pi z}{L_z} + c_{\phi z}\right) \cos\left(f_{\phi z} t + g_{\phi z}\right) \\
& + a_{\phi xy} \cos\left(\frac{b_{\phi xy} 2\pi x}{L_x} + c_{\phi xy}\right) \cos\left(\frac{d_{\phi xy} 2\pi y}{L_y} + e_{\phi xy}\right) \cos\left(f_{\phi xy} t + g_{\phi xy}\right) \\
& + a_{\phi xz} \cos\left(\frac{b_{\phi xz} 2\pi x}{L_x} + c_{\phi xz}\right) \cos\left(\frac{d_{\phi xz} 2\pi z}{L_z} + e_{\phi xz}\right) \cos\left(f_{\phi xz} t + g_{\phi xz}\right) \\
& + a_{\phi yz} \cos\left(\frac{b_{\phi yz} 2\pi y}{L_y} + c_{\phi yz}\right) \cos\left(\frac{d_{\phi yz} 2\pi z}{L_z} + e_{\phi yz}\right) \cos\left(f_{\phi yz} t + g_{\phi yz}\right)
\end{aligned}
\tag{7}
$$

where $\phi = u, v$ or w, and a, b, c, d, e, f, and g are constant coefficient collections indexed by ϕ and one or more directions (Ulerich, 2011). To aid in providing reusable, physically realizable coefficients for Cartesian domains of arbitrary size, domain extents L_x, L_y, L_z have been introduced.

Partial derivatives ϕ_t, ϕ_x, ϕ_y, ϕ_z, ϕ_{xx}, ϕ_{xy}, ϕ_{xz}, ϕ_{yy}, ϕ_{yz}, and ϕ_{zz} may be computed directly from the chosen solutions. Though they increase the solution's complexity significantly, mixed partial spatial derivatives are included to improve code coverage. Each term has an adjustable amplitude, frequency, and phase for all spatial dimensions. Cosines were chosen so all terms can be "turned off" by employing zero coefficients. It is suggested that users gradually "turn on" the more complicated features of the solution (i.e. use non-zero coefficients) after ensuring simpler usage has been successful.

The MMS consists in modifying the reduced incompressible transient flow equations (3) – (5) by adding a source term to the right-hand side of each equation:

$$
\begin{aligned}
& \frac{\partial}{\partial t} \nabla^2 v + h_v - \frac{1}{Re} \nabla^4 v = Q_v \\
& \frac{\partial g}{\partial t} + h_g - \frac{1}{Re} \nabla^2 g = Q_g \\
& \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = Q_{\text{cont}}
\end{aligned}
\tag{8}
$$

so the modified set of equations (8) conveniently has the analytical solution given in Equation (7).

Source terms for equations for v and g are obtained by symbolic manipulations of reduced incompressible transient Navier–Stokes equations above using Sympy[1] and are presented in the following sections. The following auxiliary variables have been included in order to improve readability and computational efficiency:

[1] http://sympy.org

$$
\begin{aligned}
\mathtt{U} = {} & a_{u0}\cos\left(f_{u0}t+g_{u0}\right) + a_{ux}\cos\left(\frac{b_{ux}2\pi x}{L_x}+c_{ux}\right)\cos\left(f_{ux}t+g_{ux}\right) + a_{uy}\cos\left(\frac{b_{uy}2\pi y}{L_y}+c_{uy}\right)\cos\left(f_{uy}t+g_{uy}\right) + \\
& + a_{uz}\cos\left(\frac{b_{uz}2\pi z}{L_z}+c_{uz}\right)\cos\left(f_{uz}t+g_{uz}\right) + a_{uxy}\cos\left(\frac{b_{uxy}2\pi x}{L_x}+c_{uxy}\right)\cos\left(\frac{d_{uxy}2\pi y}{L_y}+e_{uxy}\right)\cos\left(f_{uxy}t+g_{uxy}\right) \\
& + a_{uxz}\cos\left(\frac{b_{uxz}2\pi x}{L_x}+c_{uxz}\right)\cos\left(\frac{d_{uxz}2\pi z}{L_z}+e_{uxz}\right)\cos\left(f_{uxz}t+g_{uxz}\right) + a_{uyz}\cos\left(\frac{b_{uyz}2\pi y}{L_y}+c_{uyz}\right)\cos\left(\frac{d_{uyz}2\pi z}{L_z}+e_{uyz}\right)\cos\left(f_{uyz}t+g_{uyz}\right), \\
\mathtt{V} = {} & a_{v0}\cos\left(f_{v0}t+g_{v0}\right) + a_{vx}\cos\left(\frac{b_{vx}2\pi x}{L_x}+c_{vx}\right)\cos\left(f_{vx}t+g_{vx}\right) + a_{vy}\cos\left(\frac{b_{vy}2\pi y}{L_y}+c_{vy}\right)\cos\left(f_{vy}t+g_{vy}\right) \\
& + a_{vz}\cos\left(\frac{b_{vz}2\pi z}{L_z}+c_{vz}\right)\cos\left(f_{vz}t+g_{vz}\right) + a_{vxy}\cos\left(\frac{b_{vxy}2\pi x}{L_x}+c_{vxy}\right)\cos\left(\frac{d_{vxy}2\pi y}{L_y}+e_{vxy}\right)\cos\left(f_{vxy}t+g_{vxy}\right) \qquad (9) \\
& + a_{vxz}\cos\left(\frac{b_{vxz}2\pi x}{L_x}+c_{vxz}\right)\cos\left(\frac{d_{vxz}2\pi z}{L_z}+e_{vxz}\right)\cos\left(f_{vxz}t+g_{vxz}\right) + a_{vyz}\cos\left(\frac{b_{vyz}2\pi y}{L_y}+c_{vyz}\right)\cos\left(\frac{d_{vyz}2\pi z}{L_z}+e_{vyz}\right)\cos\left(f_{vyz}t+g_{vyz}\right), \\
\mathtt{W} = {} & a_{w0}\cos\left(f_{w0}t+g_{w0}\right) + a_{wx}\cos\left(\frac{b_{wx}2\pi x}{L_x}+c_{wx}\right)\cos\left(f_{wx}t+g_{wx}\right) + a_{wy}\cos\left(\frac{b_{wy}2\pi y}{L_y}+c_{wy}\right)\cos\left(f_{wy}t+g_{wy}\right) \\
& + a_{wz}\cos\left(\frac{b_{wz}2\pi z}{L_z}+c_{wz}\right)\cos\left(f_{wz}t+g_{wz}\right) + a_{wxy}\cos\left(\frac{b_{wxy}2\pi x}{L_x}+c_{wxy}\right)\cos\left(\frac{d_{wxy}2\pi y}{L_y}+e_{wxy}\right)\cos\left(f_{wxy}t+g_{wxy}\right) \\
& + a_{wxz}\cos\left(\frac{b_{wxz}2\pi x}{L_x}+c_{wxz}\right)\cos\left(\frac{d_{wxz}2\pi z}{L_z}+e_{wxz}\right)\cos\left(f_{wxz}t+g_{wxz}\right) + a_{wyz}\cos\left(\frac{b_{wyz}2\pi y}{L_y}+c_{wyz}\right)\cos\left(\frac{d_{wyz}2\pi z}{L_z}+e_{wyz}\right)\cos\left(f_{wyz}t+g_{wyz}\right),
\end{aligned}
$$

which simply are the manufactured solutions for u, v and w, respectively.

2.1 Suggested coefficients for isothermal channels

Employing the manufactured solution requires fixing nearly two hundred coefficients appearing in Equations (7), i.e., Equations (9); selecting usable values is not difficult but it can be time consuming. Ulerich (2011) presents reasonable coefficient choices for testing channel (and flat plate codes), which is of interest for this work. The streamwise, wall-normal, and spanwise directions are labeled x, y, and z respectively. Both x and z are periodic while $y \in \{0, L_y\}$ is not. Transient tests should likely take place within the duration $0 \leq t \leq 1/10$ seconds as the time phase offsets (e.g. g_{Tyz}) have been chosen for appreciable transients to occur throughout this time window.

For isothermal channel flow code verification Ulerich (2011) recommends testing using:

$$L_x = 4\pi, \quad L_y = 2, \quad L_z = 4\pi/3 \quad \text{and} \quad b_{uy} = b_{vy} = b_{wy} = \frac{1}{2},$$

and the coefficients given in Table 2.1. With these choices the manufactured solution satisfies isothermal, no-slip conditions at $y = 0$ and $y = L_y$. Unlisted coefficients should be set to zero.

Table 1: Coefficient recommendations from Section 2.1. Standard MKS units are implied with each value; e.g. R is given in $\mathrm{J\,kg^{-1}\,K^{-1}}$ and μ_r is given in Pa s. Unlisted coefficients should be set to zero.

Coefficients, ϕ	u	v	w
$a_{\phi xy}$	53/37	3	11
$b_{\phi xy}$	3	3	3
$c_{\phi xy}$	- π/2	- π/2	-π/2
$d_{\phi xy}$	3	3	3
$e_{\phi xy}$	- π/2	- π/2	-π/2
$f_{\phi xy}$	3	3	3
$g_{\phi xy}$	π/4	π/4	π/4
$a_{\phi y}$	53	2	7
$b_{\phi y}$	*see § 2.1*	*see § 2.1*	*see § 2.1*
$c_{\phi y}$	-π/2	- π/2	-π/2
$f_{\phi y}$	1	1	1
$g_{\phi y}$	π/4 - 1/20	π/4 - 1/20	π/4 - 1/20
$a_{\phi yz}$	53/41	5	13
$b_{\phi yz}$	2	2	2
$c_{\phi yz}$	- π/2	-π/2	- π/2
$d_{\phi yz}$	2	2	2
$e_{\phi yz}$	- π/2	-π/2	- π/2
$f_{\phi yz}$	2	2	2
$g_{\phi yz}$	π/4 + 1/20	π/4 + 1/20	π/4 + 1/20

2.2 Continuity equation

The mass conservation equation written as an operator is:

$$\mathcal{L} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

Analytically differentiating Equation (7) for u, v and w using operator $\mathcal{L}$ defined above provides the source term Q_{cont}, given by:

$$
\begin{aligned}
Q_{\mathrm{cont}} =& -\frac{2\pi a_{ux} b_{ux}}{L_x}\cos\left(g_{ux}+f_{ux}t\right)\sin\left(c_{ux}+b_{ux}\frac{2\pi x}{L_x}\right) - \frac{2\pi a_{vy} b_{vy}}{L_y}\cos\left(g_{vy}+f_{vy}t\right)\sin\left(c_{vy}+b_{vy}\frac{2\pi y}{L_y}\right) + \\
& -\frac{2\pi a_{wz} b_{wz}}{L_z}\cos\left(g_{wz}+f_{wz}t\right)\sin\left(c_{wz}+b_{wz}\frac{2\pi z}{L_z}\right) - \frac{2\pi a_{uxy} b_{uxy}}{L_x}\cos\left(g_{uxy}+f_{uxy}t\right)\cos\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right) + \\
& -\frac{2\pi a_{uxz} b_{uxz}}{L_x}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right) + \\
& -\frac{2\pi a_{vxy} d_{vxy}}{L_y}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right) + \\
& -\frac{2\pi a_{vyz} b_{vyz}}{L_y}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right) + \\
& -\frac{2\pi a_{wxz} d_{wxz}}{L_z}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right) + \\
& -\frac{2\pi a_{wyz} d_{wyz}}{L_z}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right).
\end{aligned}
\tag{10}
$$

2.3 Equation for g

For the generation of the analytical source term Q_g for the second-order equation for the normal component of vorticity, Equation (4) is written as an operator $\mathcal{L}$:

$$
\mathcal{L} = \frac{\partial g}{\partial t} + h_g - \frac{1}{Re}\nabla^2 g,
$$

which, when operated in Equation (7), provides source term Q_g, given in Eq. (11).

$$
\begin{aligned}
Q_g = \frac{2\pi}{L_x} \Bigg[& -a_{wx} b_{wx} f_{wx} \sin\left(g_{wx} + f_{wx} t\right) \sin\left(c_{wx} + b_{wx} \frac{2\pi x}{L_x}\right) - a_{wxy} b_{wxy} f_{wxy} \sin\left(g_{wxy} + f_{wxy} t\right) \cos\left(e_{wxy} + d_{wxy} \frac{2\pi y}{L_y}\right) \sin\left(c_{wxy} + b_{wxy} \frac{2\pi x}{L_x}\right) + \\
& - a_{wxz} b_{wxz} f_{wxz} \sin\left(g_{wxz} + f_{wxz} t\right) \cos\left(e_{wxz} + d_{wxz} \frac{2\pi z}{L_z}\right) \sin\left(c_{wxz} + b_{wxz} \frac{2\pi x}{L_x}\right) \Bigg] + \\
+ \frac{2\pi}{L_z} \Bigg[& a_{uz} b_{uz} f_{uz} \sin\left(g_{uz} + f_{uz} t\right) \sin\left(c_{uz} + b_{uz} \frac{2\pi z}{L_z}\right) + a_{uxz} d_{uxz} f_{uxz} \sin\left(g_{uxz} + f_{uxz} t\right) \cos\left(c_{uxz} + b_{uxz} \frac{2\pi x}{L_x}\right) \sin\left(e_{uxz} + d_{uxz} \frac{2\pi z}{L_z}\right) + \\
& + a_{uyz} d_{uyz} f_{uyz} \sin\left(g_{uyz} + f_{uyz} t\right) \cos\left(c_{uyz} + b_{uyz} \frac{2\pi y}{L_y}\right) \sin\left(e_{uyz} + d_{uyz} \frac{2\pi z}{L_z}\right) \Bigg] + \\
- \frac{8\pi^3}{Re} \Bigg[& \frac{a_{uz} b_{uz}^3}{L_z^3} \cos\left(g_{uz} + f_{uz} t\right) \sin\left(c_{uz} + b_{uz} \frac{2\pi z}{L_z}\right) - \frac{a_{wx} b_{wx}^3}{L_x^3} \cos\left(g_{wx} + f_{wx} t\right) \sin\left(c_{wx} + b_{wx} \frac{2\pi x}{L_x}\right) + \\
& + \frac{a_{uxz} d_{uxz}^3}{L_z^3} \cos\left(g_{uxz} + f_{uxz} t\right) \cos\left(c_{uxz} + b_{uxz} \frac{2\pi x}{L_x}\right) \sin\left(e_{uxz} + d_{uxz} \frac{2\pi z}{L_z}\right) + \\
& + \frac{a_{uyz} d_{uyz}^3}{L_z^3} \cos\left(g_{uyz} + f_{uyz} t\right) \cos\left(c_{uyz} + b_{uyz} \frac{2\pi y}{L_y}\right) \sin\left(e_{uyz} + d_{uyz} \frac{2\pi z}{L_z}\right) + \\
& - \frac{a_{wxy} b_{wxy}^3}{L_x^3} \cos\left(g_{wxy} + f_{wxy} t\right) \cos\left(e_{wxy} + d_{wxy} \frac{2\pi y}{L_y}\right) \sin\left(c_{wxy} + b_{wxy} \frac{2\pi x}{L_x}\right) + \\
& - \frac{a_{wxz} b_{wxz}^3}{L_x^3} \cos\left(g_{wxz} + f_{wxz} t\right) \cos\left(e_{wxz} + d_{wxz} \frac{2\pi z}{L_z}\right) \sin\left(c_{wxz} + b_{wxz} \frac{2\pi x}{L_x}\right) + \\
& + \frac{a_{uxz} d_{uxz} b_{uxz}^2}{L_z L_x^2} \cos\left(g_{uxz} + f_{uxz} t\right) \cos\left(c_{uxz} + b_{uxz} \frac{2\pi x}{L_x}\right) \sin\left(e_{uxz} + d_{uxz} \frac{2\pi z}{L_z}\right) + \\
& + \frac{a_{uyz} d_{uyz} b_{uyz}^2}{L_z L_y^2} \cos\left(g_{uyz} + f_{uyz} t\right) \cos\left(c_{uyz} + b_{uyz} \frac{2\pi y}{L_y}\right) \sin\left(e_{uyz} + d_{uyz} \frac{2\pi z}{L_z}\right) + \\
& - \frac{a_{wxy} b_{wxy} d_{wxy}^2}{L_x L_y^2} \cos\left(g_{wxy} + f_{wxy} t\right) \cos\left(e_{wxy} + d_{wxy} \frac{2\pi y}{L_y}\right) \sin\left(c_{wxy} + b_{wxy} \frac{2\pi x}{L_x}\right) + \\
& - \frac{a_{wxz} b_{wxz} d_{wxz}^2}{L_x L_z^2} \cos\left(g_{wxz} + f_{wxz} t\right) \cos\left(e_{wxz} + d_{wxz} \frac{2\pi z}{L_z}\right) \sin\left(c_{wxz} + b_{wxz} \frac{2\pi x}{L_x}\right) \Bigg] + \\
+ \frac{4\pi^2\, \mathtt{U}}{L_x^2} \Bigg[& a_{wx} b_{wx}^2 \cos\left(g_{wx} + f_{wx} t\right) \cos\left(c_{wx} + b_{wx} \frac{2\pi x}{L_x}\right) + a_{wxy} b_{wxy}^2 \cos\left(g_{wxy} + f_{wxy} t\right) \cos\left(c_{wxy} + b_{wxy} \frac{2\pi x}{L_x}\right) \cos\left(e_{wxy} + d_{wxy} \frac{2\pi y}{L_y}\right) + \\
& + a_{wxz} b_{wxz}^2 \cos\left(g_{wxz} + f_{wxz} t\right) \cos\left(c_{wxz} + b_{wxz} \frac{2\pi x}{L_x}\right) \cos\left(e_{wxz} + d_{wxz} \frac{2\pi z}{L_z}\right) \Bigg] + \\
+ \frac{4\pi^2\, \mathtt{V}}{L_y} \Bigg[& \frac{a_{uyz} b_{uyz} d_{uyz}}{L_z} \cos\left(g_{uyz} + f_{uyz} t\right) \sin\left(c_{uyz} + b_{uyz} \frac{2\pi y}{L_y}\right) \sin\left(e_{uyz} + d_{uyz} \frac{2\pi z}{L_z}\right) + \\
& - \frac{a_{wxy} b_{wxy} d_{wxy}}{L_x} \cos\left(g_{wxy} + f_{wxy} t\right) \sin\left(c_{wxy} + b_{wxy} \frac{2\pi x}{L_x}\right) \sin\left(e_{wxy} + d_{wxy} \frac{2\pi y}{L_y}\right) \Bigg] +
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&+\frac{4\pi^2\,\mathtt{W}}{L_z^2}\Bigg[-a_{uz}b_{uz}^2\cos\left(g_{uz}+f_{uz}t\right)\cos\left(c_{uz}+b_{uz}\frac{2\pi z}{L_z}\right)+-a_{uxz}d_{uxz}^2\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\cos\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)+\\
&\quad -a_{uyz}d_{uyz}^2\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\Bigg]+\\
&+\frac{4\pi^2}{L_x^2}\Bigg[a_{ux}b_{ux}\cos\left(g_{ux}+f_{ux}t\right)\sin\left(c_{ux}+b_{ux}\frac{2\pi x}{L_x}\right)+a_{uxy}b_{uxy}\cos\left(g_{uxy}+f_{uxy}t\right)\cos\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)+\\
&\quad +a_{uxz}b_{uxz}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\Bigg]\cdot\\
&\quad \cdot\Bigg[-a_{wx}b_{wx}\cos\left(g_{wx}+f_{wx}t\right)\sin\left(c_{wx}+b_{wx}\frac{2\pi x}{L_x}\right)-a_{wxy}b_{wxy}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)+\\
&\quad -a_{wxz}b_{wxz}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\Bigg]+\\
&+\frac{4\pi^2}{L_z^2}\Bigg[-a_{uz}b_{uz}\cos\left(g_{uz}+f_{uz}t\right)\sin\left(c_{uz}+b_{uz}\frac{2\pi z}{L_z}\right)-a_{uxz}d_{uxz}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)+\\
&\quad -a_{uyz}d_{uyz}\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\Bigg]\cdot\\
&\quad \cdot\Bigg[-a_{wz}b_{wz}\cos\left(g_{wz}+f_{wz}t\right)\sin\left(c_{wz}+b_{wz}\frac{2\pi z}{L_z}\right)-a_{wxz}d_{wxz}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)+\\
&\quad -a_{wyz}d_{wyz}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\Bigg]+\\
&+\frac{4\pi^2}{L_xL_y}\Bigg[a_{vx}b_{vx}\cos\left(g_{vx}+f_{vx}t\right)\sin\left(c_{vx}+b_{vx}\frac{2\pi x}{L_x}\right)+a_{vxy}b_{vxy}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+\\
&\quad +a_{vxz}b_{vxz}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\Bigg]\cdot\\
&\quad \cdot\Bigg[-a_{wy}b_{wy}\cos\left(g_{wy}+f_{wy}t\right)\sin\left(c_{wy}+b_{wy}\frac{2\pi y}{L_y}\right)-a_{wxy}d_{wxy}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)+\\
&\quad -a_{wyz}b_{wyz}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\Bigg]+\\
&+\frac{4\pi^2}{L_xL_z}\Bigg[a_{wx}b_{wx}\cos\left(g_{wx}+f_{wx}t\right)\sin\left(c_{wx}+b_{wx}\frac{2\pi x}{L_x}\right)++a_{wxy}b_{wxy}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)+\\
&\quad +a_{wxz}b_{wxz}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\Bigg]\cdot\\
&\quad \cdot\Bigg[-a_{wz}b_{wz}\cos\left(g_{wz}+f_{wz}t\right)\sin\left(c_{wz}+b_{wz}\frac{2\pi z}{L_z}\right)-a_{wxz}d_{wxz}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)+\\
&\quad -a_{wyz}d_{wyz}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\Bigg]+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4\pi^2}{L_xL_z}\left[-a_{ux}b_{ux}\cos\left(g_{ux}+f_{ux}t\right)\sin\left(c_{ux}+b_{ux}\frac{2\pi x}{L_x}\right)-a_{uxy}b_{uxy}\cos\left(g_{uxy}+f_{uxy}t\right)\cos\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)+\right.\\
&\quad\left.-a_{uxz}b_{uxz}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\right]\cdot\\
&\quad\cdot\left[-a_{uz}b_{uz}\cos\left(g_{uz}+f_{uz}t\right)\sin\left(c_{uz}+b_{uz}\frac{2\pi z}{L_z}\right)-a_{uxz}d_{uxz}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)+\right.\\
&\quad\left.-a_{uyz}d_{uyz}\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\right]+\\
&+\frac{4\pi^2}{L_yL_z}\left[-a_{vz}b_{vz}\cos\left(g_{vz}+f_{vz}t\right)\sin\left(c_{vz}+b_{vz}\frac{2\pi z}{L_z}\right)-a_{vxz}d_{vxz}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)+\right.\\
&\quad\left.-a_{vyz}d_{vyz}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\right]\cdot\\
&\quad\cdot\left[-a_{uy}b_{uy}\cos\left(g_{uy}+f_{uy}t\right)\sin\left(c_{uy}+b_{uy}\frac{2\pi y}{L_y}\right)-a_{uxy}d_{uxy}\cos\left(g_{uxy}+f_{uxy}t\right)\cos\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)+\right.\\
&\quad\left.-a_{uyz}b_{uyz}\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\right]+\\
&+\frac{4\pi^2a_{uxz}b_{uxz}d_{uxz}\,\mathtt{U}}{L_xL_z}\cos\left(g_{uxz}+f_{uxz}t\right)\sin\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)+\\
&-\frac{4\pi^2a_{wxz}b_{wxz}d_{wxz}\,\mathtt{W}}{L_xL_z}\cos\left(g_{wxz}+f_{wxz}t\right)\sin\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right).
\end{aligned}
$$

2.4 Equation for v

For the generation of the analytical source term Q_v for the velocity in the y-direction, Equation (3) is written as an operator $\mathcal{L}$:

$$
\mathcal{L}=\frac{\partial}{\partial t}\nabla^2 v+h_v-\frac{1}{Re}\nabla^4 v
$$

which, when operated in Equation (7), provides source term Q_v, given in Eq. (12).

$$
\begin{aligned}
Q_v = \mathtt{U}\Bigg[& -\frac{8\pi^3 a_{vx} b_{vx}^3}{L_x^3}\cos\left(g_{vx}+f_{vx}t\right)\sin\left(c_{vx}+b_{vx}\frac{2\pi x}{L_x}\right) - \frac{8\pi^3 a_{vxy} b_{vxy}^3}{L_x^3}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+ \\
& -\frac{8\pi^3 a_{vxz} b_{vxz}^3}{L_x^3}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)+ \\
& +\frac{8\pi^3 a_{uxy} d_{uxy} b_{uxy}^2}{L_y L_x^2}\cos\left(g_{uxy}+f_{uxy}t\right)\cos\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)+ \\
& -\frac{8\pi^3 a_{vxz} b_{vxz} d_{vxz}^2}{L_x L_z^2}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\Bigg]+ \\
+\mathtt{V}\Bigg[& \frac{8\pi^3 a_{uxy} b_{uxy} d_{uxy}^2}{L_x L_y^2}\cos\left(g_{uxy}+f_{uxy}t\right)\cos\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)+ \\
& +\frac{8\pi^3 a_{wyz} d_{wyz} b_{wyz}^2}{L_z L_y^2}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)+ \\
& -\frac{8\pi^3 a_{vxy} d_{vxy} b_{vxy}^2}{L_y L_x^2}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)+ \\
& -\frac{8\pi^3 a_{vyz} b_{vyz} d_{vyz}^2}{L_y L_z^2}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\Bigg]+ \\
+\mathtt{W}\Bigg[& -\frac{8\pi^3 a_{vz} b_{vz}^3}{L_z^3}\cos\left(g_{vz}+f_{vz}t\right)\sin\left(c_{vz}+b_{vz}\frac{2\pi z}{L_z}\right) - \frac{8\pi^3 a_{vxz} d_{vxz}^3}{L_z^3}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)+ \\
& -\frac{8\pi^3 a_{vyz} d_{vyz}^3}{L_z^3}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)+ \\
& +\frac{8\pi^3 a_{wyz} b_{wyz} d_{wyz}^2}{L_y L_z^2}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)+ \\
& -\frac{8\pi^3 a_{vxz} d_{vxz} b_{vxz}^2}{L_z L_x^2}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\Bigg]+ \\
+\frac{4\pi^2}{L_x^2}\Bigg[& a_{vx} f_{vx} b_{vx}^2 \sin\left(g_{vx}+f_{vx}t\right)\cos\left(c_{vx}+b_{vx}\frac{2\pi x}{L_x}\right) + a_{vxy} f_{vxy} b_{vxy}^2 \sin\left(g_{vxy}+f_{vxy}t\right)\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+ \\
& +a_{vxz} f_{vxz} b_{vxz}^2 \sin\left(g_{vxz}+f_{vxz}t\right)\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\Bigg]+ \\
+\frac{4\pi^2}{L_z^2}\Bigg[& a_{vz} f_{vz} b_{vz}^2 \sin\left(g_{vz}+f_{vz}t\right)\cos\left(c_{vz}+b_{vz}\frac{2\pi z}{L_z}\right) + a_{vxz} f_{vxz} d_{vxz}^2 \sin\left(g_{vxz}+f_{vxz}t\right)\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)+ \\
& +a_{vyz} f_{vyz} d_{vyz}^2 \sin\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\Bigg]+
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&+\Bigg\{\frac{4\pi^2}{L_z^2}\Bigg[-a_{wy}b_{wy}\frac{2\pi}{L_y}\cos\left(g_{wy}+f_{wy}t\right)\sin\left(c_{wy}+b_{wy}\frac{2\pi y}{L_y}\right)-a_{wxy}d_{wxy}\frac{2\pi}{L_y}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)+\\
&\quad -a_{wyz}b_{wyz}\frac{2\pi}{L_y}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\Bigg]+\\
&\quad -\frac{4\pi^2}{L_z^2}\Bigg[-a_{vz}b_{vz}\frac{2\pi}{L_z}\cos\left(g_{vz}+f_{vz}t\right)\sin\left(c_{vz}+b_{vz}\frac{2\pi z}{L_z}\right)-a_{vxz}d_{vxz}\frac{2\pi}{L_z}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)+\\
&\quad -a_{vyz}d_{vyz}\frac{2\pi}{L_z}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\Bigg]\Bigg]\Bigg\}\cdot\\
&\quad\cdot\Bigg[-a_{wz}b_{wz}^2\cos\left(g_{wz}+f_{wz}t\right)\cos\left(c_{wz}+b_{wz}\frac{2\pi z}{L_z}\right)-a_{wxz}d_{wxz}^2\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\cos\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)+\\
&\quad -a_{wyz}d_{wyz}^2\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\Bigg]+\\
&+\Bigg\{-\frac{4\pi^2}{L_x^2}\Bigg[-a_{vy}b_{vy}\frac{2\pi}{L_y}\cos\left(g_{vy}+f_{vy}t\right)\sin\left(c_{vy}+b_{vy}\frac{2\pi y}{L_y}\right)-a_{vxy}d_{vxy}\frac{2\pi}{L_y}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)+\\
&\quad -a_{vyz}b_{vyz}\frac{2\pi}{L_y}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\Bigg]+\\
&\quad -\frac{8\pi^2}{L_x^2}\Bigg[-a_{ux}b_{ux}\frac{2\pi}{L_x}\cos\left(g_{ux}+f_{ux}t\right)\sin\left(c_{ux}+b_{ux}\frac{2\pi x}{L_x}\right)-a_{uxy}b_{uxy}\frac{2\pi}{L_x}\cos\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)\cos\left(g_{uxy}+f_{uxy}t\right)\sin\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)+\\
&\quad -a_{uxz}b_{uxz}\frac{2\pi}{L_x}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\Bigg]\Bigg]\Bigg\}\cdot\\
&\quad\cdot\Bigg[-a_{vx}b_{vx}^2\cos\left(c_{vx}+b_{vx}\frac{2\pi x}{L_x}\right)\cos\left(g_{vx}+f_{vx}t\right)-a_{vxy}b_{vxy}^2\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+\\
&\quad -a_{vxz}b_{vxz}^2\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\Bigg]+\\
&+\Bigg\{\frac{4\pi^2}{L_x^2}\frac{2\pi}{L_y}\Bigg[-a_{uy}b_{uy}\cos\left(g_{uy}+f_{uy}t\right)\sin\left(c_{uy}+b_{uy}\frac{2\pi y}{L_y}\right)-a_{uxy}d_{uxy}\cos\left(g_{uxy}+f_{uxy}t\right)\cos\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)+\\
&\quad -a_{uyz}b_{uyz}\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\Bigg]+\\
&\quad -\frac{4\pi^2}{L_x^2}\frac{2\pi}{L_x}\Bigg[-a_{vx}b_{vx}\cos\left(g_{vx}+f_{vx}t\right)\sin\left(c_{vx}+b_{vx}\frac{2\pi x}{L_x}\right)-a_{vxy}b_{vxy}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+\\
&\quad -a_{vxz}b_{vxz}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\Bigg]\Bigg]\Bigg\}\cdot\\
&\quad\cdot\Bigg[-a_{ux}b_{ux}^2\cos\left(g_{ux}+f_{ux}t\right)\cos\left(c_{ux}+b_{ux}\frac{2\pi x}{L_x}\right)-a_{uxy}b_{uxy}^2\cos\left(g_{uxy}+f_{uxy}t\right)\cos\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)\cos\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)+\\
&\quad -a_{uxz}b_{uxz}^2\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\cos\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)\Bigg]+
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{-\frac{4\pi^2}{L_z^2}\frac{2\pi}{L_y}\left[-a_{vy}b_{vy}\cos\left(g_{vy}+f_{vy}t\right)\sin\left(c_{vy}+b_{vy}\frac{2\pi y}{L_y}\right)-a_{vxy}d_{vxy}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)+\right.\right.\\
&\quad\left.-a_{vyz}b_{vyz}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\right]+\\
&\quad-2\frac{4\pi^2}{L_z^2}\frac{2\pi}{L_z}\left[-a_{wz}b_{wz}\cos\left(g_{wz}+f_{wz}t\right)\sin\left(c_{wz}+b_{wz}\frac{2\pi z}{L_z}\right)-a_{wxz}d_{wxz}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)+\right.\\
&\quad\left.\left.-a_{wyz}d_{wyz}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\right]\right\}\cdot\\
&\quad\cdot\left[-a_{vz}b_{vz}^2\cos\left(g_{vz}+f_{vz}t\right)\cos\left(c_{vz}+b_{vz}\frac{2\pi z}{L_z}\right)-a_{vxz}d_{vxz}^2\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)+\right.\\
&\quad\left.-a_{vyz}d_{vyz}^2\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\right]+\\
&+\frac{16\pi^4}{L_x^4Re}\left[-a_{vx}b_{vx}^4\cos\left(g_{vx}+f_{vx}t\right)\cos\left(c_{vx}+b_{vx}\frac{2\pi x}{L_x}\right)+-a_{vxy}b_{vxy}^4\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+\right.\\
&\quad\left.-a_{vxz}b_{vxz}^4\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\right]+\\
&+\frac{16\pi^4}{L_y^4Re}\left[-a_{vy}b_{vy}^4\cos\left(g_{vy}+f_{vy}t\right)\cos\left(c_{vy}+b_{vy}\frac{2\pi y}{L_y}\right)+-a_{vxy}d_{vxy}^4\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+\right.\\
&\quad\left.-a_{vyz}b_{vyz}^4\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\right]+\\
&+\frac{16\pi^4}{L_z^4Re}\left[-a_{vz}b_{vz}^4\cos\left(g_{vz}+f_{vz}t\right)\cos\left(c_{vz}+b_{vz}\frac{2\pi z}{L_z}\right)+-a_{vxz}d_{vxz}^4\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)+\right.\\
&\quad\left.-a_{vyz}d_{vyz}^4\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\right]+\\
&+\frac{4\pi^2}{L_y^2}\frac{2\pi}{L_x}\left[-a_{uy}b_{uy}^2\cos\left(g_{uy}+f_{uy}t\right)\cos\left(c_{uy}+b_{uy}\frac{2\pi y}{L_y}\right)-a_{uxy}d_{uxy}^2\cos\left(g_{uxy}+f_{uxy}t\right)\cos\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)\cos\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)+\right.\\
&\quad\left.-a_{uyz}b_{uyz}^2\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\right]\cdot\\
&\quad\cdot\left[-a_{vx}b_{vx}\cos\left(g_{vx}+f_{vx}t\right)\sin\left(c_{vx}+b_{vx}\frac{2\pi x}{L_x}\right)-a_{vxy}b_{vxy}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+\right.\\
&\quad\left.-a_{vxz}b_{vxz}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\right]+
\end{aligned}
$$

$$+\frac{4\pi^2}{L_y^2}\frac{2\pi}{L_z}\left[-a_{wy}b_{wy}^2\cos\left(g_{wy}+f_{wy}t\right)\cos\left(c_{wy}+b_{wy}\frac{2\pi y}{L_y}\right)-a_{wxy}d_{wxy}^2\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)\cos\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)+\right.$$

$$\left.-a_{wyz}b_{wyz}^2\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\right]\cdot\left[-a_{vz}b_{vz}\cos\left(g_{vz}+f_{vz}t\right)\sin\left(c_{vz}+b_{vz}\frac{2\pi z}{L_z}\right)\right.$$

$$\left.-a_{vxz}d_{vxz}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)-a_{vyz}d_{vyz}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\right]+$$

$$-\frac{4\pi^2}{L_x^2}\frac{2\pi}{L_z}\left[-a_{wx}b_{wx}^2\cos\left(g_{wx}+f_{wx}t\right)\cos\left(c_{wx}+b_{wx}\frac{2\pi x}{L_x}\right)-a_{wxy}b_{wxy}^2\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)\cos\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)+\right.$$

$$\left.-a_{wxz}b_{wxz}^2\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\cos\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)\right]\cdot\left[-a_{vz}b_{vz}\cos\left(g_{vz}+f_{vz}t\right)\sin\left(c_{vz}+b_{vz}\frac{2\pi z}{L_z}\right)+\right.$$

$$\left.-a_{vxz}d_{vxz}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)-a_{vyz}d_{vyz}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\right]+$$

$$-\frac{4\pi^2}{L_z^2}\frac{2\pi}{L_x}\left[a_{uz}b_{uz}^2\cos\left(g_{uz}+f_{uz}t\right)\cos\left(c_{uz}+b_{uz}\frac{2\pi z}{L_z}\right)+a_{uxz}d_{uxz}^2\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\cos\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)+\right.$$

$$\left.+a_{uyz}d_{uyz}^2\cos\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\right]\cdot\left[a_{vx}b_{vx}\cos\left(g_{vx}+f_{vx}t\right)\sin\left(c_{vx}+b_{vx}\frac{2\pi x}{L_x}\right)+\right.$$

$$\left.+a_{vxy}b_{vxy}\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\cos\left(g_{vxy}+f_{vxy}t\right)\sin\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+a_{vxz}b_{vxz}\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\cos\left(g_{vxz}+f_{vxz}t\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\right]+$$

$$+a_{vy}f_{vy}b_{vy}^2\frac{4\pi^2}{L_y^2}\sin\left(g_{vy}+f_{vy}t\right)\cos\left(c_{vy}+b_{vy}\frac{2\pi y}{L_y}\right)+a_{vxy}f_{vxy}d_{vxy}^2\frac{4\pi^2}{L_y^2}\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\sin\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+$$

$$+a_{vyz}f_{vyz}b_{vyz}^2\frac{4\pi^2}{L_y^2}\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\sin\left(g_{vyz}+f_{vyz}t\right)+$$

$$+a_{uxy}b_{uxy}d_{uxy}\frac{2\pi}{L_x}\frac{2\pi}{L_y}\left[-a_{vy}b_{vy}\frac{2\pi}{L_y}\cos\left(g_{vy}+f_{vy}t\right)\sin\left(c_{vy}+b_{vy}\frac{2\pi y}{L_y}\right)-a_{vxy}d_{vxy}\frac{2\pi}{L_y}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)+\right.$$

$$\left.-a_{vyz}b_{vyz}\frac{2\pi}{L_y}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\right]\cos\left(g_{uxy}+f_{uxy}t\right)\sin\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)+$$

$$+a_{uxz}b_{uxz}d_{uxz}\frac{2\pi}{L_x}\frac{2\pi}{L_z}\left[-a_{wy}b_{wy}\frac{2\pi}{L_y}\cos\left(g_{wy}+f_{wy}t\right)\sin\left(c_{wy}+b_{wy}\frac{2\pi y}{L_y}\right)-a_{wxy}d_{wxy}\frac{2\pi}{L_y}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)+\right.$$

$$\left.-a_{wyz}b_{wyz}\frac{2\pi}{L_y}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\right]\cos\left(g_{uxz}+f_{uxz}t\right)\sin\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)+$$

$$+a_{vxy}b_{vxy}d_{vxy}\frac{2\pi}{L_x}\frac{2\pi}{L_y}\left[-a_{uy}b_{uy}\frac{2\pi}{L_y}\cos\left(g_{uy}+f_{uy}t\right)\sin\left(c_{uy}+b_{uy}\frac{2\pi y}{L_y}\right)-a_{uxy}d_{uxy}\frac{2\pi}{L_y}\cos\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)\cos\left(g_{uxy}+f_{uxy}t\right)\sin\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)+\right.$$

$$\left.-a_{uyz}b_{uyz}\frac{2\pi}{L_y}\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\right]\cos\left(g_{vxy}+f_{vxy}t\right)\sin\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+$$

$$+ a_{vyz}b_{vyz}d_{vyz}\frac{2\pi}{L_y}\frac{2\pi}{L_z}\Big[-a_{wy}b_{wy}\frac{2\pi}{L_y}\cos\left(g_{wy}+f_{wy}t\right)\sin\left(c_{wy}+b_{wy}\frac{2\pi y}{L_y}\right)-a_{wxy}d_{wxy}\frac{2\pi}{L_y}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)+$$

$$-a_{wyz}b_{wyz}\frac{2\pi}{L_y}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\Big]\cos\left(g_{vyz}+f_{vyz}t\right)\sin\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)+$$

$$+ a_{wxz}b_{wxz}d_{wxz}\frac{2\pi}{L_x}\frac{2\pi}{L_z}\Big[-a_{uy}b_{uy}\frac{2\pi}{L_y}\cos\left(g_{uy}+f_{uy}t\right)\sin\left(c_{uy}+b_{uy}\frac{2\pi y}{L_y}\right)-a_{uxy}d_{uxy}\frac{2\pi}{L_y}\cos\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)\cos\left(g_{uxy}+f_{uxy}t\right)\sin\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)+$$

$$-a_{uyz}b_{uyz}\frac{2\pi}{L_y}\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\Big]\cos\left(g_{wxz}+f_{wxz}t\right)\sin\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)+$$

$$+ a_{wyz}b_{wyz}d_{wyz}\frac{2\pi}{L_y}\frac{2\pi}{L_z}\Big[-a_{vy}b_{vy}\frac{2\pi}{L_y}\cos\left(g_{vy}+f_{vy}t\right)\sin\left(c_{vy}+b_{vy}\frac{2\pi y}{L_y}\right)-a_{vxy}d_{vxy}\frac{2\pi}{L_y}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)+$$

$$-a_{vyz}b_{vyz}\frac{2\pi}{L_y}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\Big]\cos\left(g_{wyz}+f_{wyz}t\right)\sin\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)+$$

$$-2a_{vxy}b_{vxy}d_{vxy}\frac{2\pi}{L_x}\frac{2\pi}{L_y}\Big[-a_{vx}b_{vx}\frac{2\pi}{L_x}\cos\left(g_{vx}+f_{vx}t\right)\sin\left(c_{vx}+b_{vx}\frac{2\pi x}{L_x}\right)-a_{vxy}b_{vxy}\frac{2\pi}{L_x}\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\cos\left(g_{vxy}+f_{vxy}t\right)\sin\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+$$

$$-a_{vxz}b_{vxz}\frac{2\pi}{L_x}\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\cos\left(g_{vxz}+f_{vxz}t\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\Big]\cos\left(g_{vxy}+f_{vxy}t\right)\sin\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+$$

$$-2\frac{a_{vxy}b_{vxy}^2d_{vxy}^2}{Re}\frac{4\pi^2}{L_x^2}\frac{4\pi^2}{L_y^2}\cos\left(e_{vxy}+d_{vxy}\frac{2\pi y}{L_y}\right)\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi x}{L_x}\right)+$$

$$-2a_{vxz}b_{vxz}d_{vxz}\frac{2\pi}{L_x}\frac{2\pi}{L_z}\Big[-a_{wx}b_{wx}\frac{2\pi}{L_x}\cos\left(g_{wx}+f_{wx}t\right)\sin\left(c_{wx}+b_{wx}\frac{2\pi x}{L_x}\right)-a_{wxy}b_{wxy}\frac{2\pi}{L_x}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)+$$

$$-a_{wxz}b_{wxz}\frac{2\pi}{L_x}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\Big]\cos\left(g_{vxz}+f_{vxz}t\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)+$$

$$-2a_{vxz}b_{vxz}d_{vxz}\frac{2\pi}{L_x}\frac{2\pi}{L_z}\Big(-a_{uz}b_{uz}\frac{2\pi}{L_z}\cos\left(g_{uz}+f_{uz}t\right)\sin\left(c_{uz}+b_{uz}\frac{2\pi z}{L_z}\right)-a_{uxz}d_{uxz}\frac{2\pi}{L_z}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)+$$

$$-a_{uyz}d_{uyz}\frac{2\pi}{L_z}\cos\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\cos\left(g_{uyz}+f_{uyz}t\right)\sin\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\Big)\cos\left(g_{vxz}+f_{vxz}t\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)+$$

$$-2\frac{a_{vxz}b_{vxz}^2d_{vxz}^2}{Re}\frac{4\pi^2}{L_x^2}\frac{4\pi^2}{L_z^2}\cos\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)+$$

$$-2a_{vyz}b_{vyz}d_{vyz}\frac{2\pi}{L_y}\frac{2\pi}{L_z}\Big[-a_{vz}b_{vz}\frac{2\pi}{L_z}\cos\left(g_{vz}+f_{vz}t\right)\sin\left(c_{vz}+b_{vz}\frac{2\pi z}{L_z}\right)-a_{vxz}d_{vxz}\frac{2\pi}{L_z}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi z}{L_z}\right)+$$

$$-a_{vyz}d_{vyz}\frac{2\pi}{L_z}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)\Big]\cos\left(g_{vyz}+f_{vyz}t\right)\sin\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)+$$

$$-2\frac{a_{vyz}b_{vyz}^2d_{vyz}^2}{Re}\frac{4\pi^2}{L_y^2}\frac{4\pi^2}{L_z^2}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi y}{L_y}\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi z}{L_z}\right)$$

$$
\begin{aligned}
&+ 2a_{uxy}b_{uxy}d_{uxy}\frac{2\pi}{L_x}\frac{2\pi}{L_y}\left[-a_{ux}b_{ux}\frac{2\pi}{L_x}\cos\left(g_{ux}+f_{ux}t\right)\sin\left(c_{ux}+b_{ux}\frac{2\pi x}{L_x}\right) - a_{uxy}b_{uxy}\frac{2\pi}{L_x}\cos\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right)\cos\left(g_{uxy}+f_{uxy}t\right)\sin\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right) +\right.\\
&\quad\left. -a_{uxz}b_{uxz}\frac{2\pi}{L_x}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\right]\cos\left(g_{uxy}+f_{uxy}t\right)\sin\left(c_{uxy}+b_{uxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxy}+d_{uxy}\frac{2\pi y}{L_y}\right) +\\
&+ 2a_{uyz}b_{uyz}d_{uyz}\frac{2\pi}{L_y}\frac{2\pi}{L_z}\left[-a_{wx}b_{wx}\frac{2\pi}{L_x}\cos\left(g_{wx}+f_{wx}t\right)\sin\left(c_{wx}+b_{wx}\frac{2\pi x}{L_x}\right) - a_{wxy}b_{wxy}\frac{2\pi}{L_x}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right)\sin\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right) +\right.\\
&\quad\left. -a_{wxz}b_{wxz}\frac{2\pi}{L_x}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right)\sin\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\right]\cos\left(g_{uyz}+f_{uyz}t\right)\sin\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right) +\\
&+ 2a_{wxy}b_{wxy}d_{wxy}\frac{2\pi}{L_x}\frac{2\pi}{L_y}\left[-a_{uz}b_{uz}\frac{2\pi}{L_z}\cos\left(g_{uz}+f_{uz}t\right)\sin\left(c_{uz}+b_{uz}\frac{2\pi z}{L_z}\right) - a_{uxz}d_{uxz}\frac{2\pi}{L_z}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(c_{uxz}+b_{uxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{uxz}+d_{uxz}\frac{2\pi z}{L_z}\right) +\right.\\
&\quad\left. -a_{uyz}d_{uyz}\frac{2\pi}{L_z}\cos\left(c_{uyz}+b_{uyz}\frac{2\pi y}{L_y}\right)\cos\left(g_{uyz}+f_{uyz}t\right)\sin\left(e_{uyz}+d_{uyz}\frac{2\pi z}{L_z}\right)\right]\cos\left(g_{wxy}+f_{wxy}t\right)\sin\left(c_{wxy}+b_{wxy}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxy}+d_{wxy}\frac{2\pi y}{L_y}\right) +\\
&+ 2a_{wyz}b_{wyz}d_{wyz}\frac{2\pi}{L_y}\frac{2\pi}{L_z}\left[-a_{wz}b_{wz}\frac{2\pi}{L_z}\cos\left(g_{wz}+f_{wz}t\right)\sin\left(c_{wz}+b_{wz}\frac{2\pi z}{L_z}\right) - a_{wxz}d_{wxz}\frac{2\pi}{L_z}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(c_{wxz}+b_{wxz}\frac{2\pi x}{L_x}\right)\sin\left(e_{wxz}+d_{wxz}\frac{2\pi z}{L_z}\right) +\right.\\
&\quad\left. -a_{wyz}d_{wyz}\frac{2\pi}{L_z}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right)\right]\cos\left(g_{wyz}+f_{wyz}t\right)\sin\left(c_{wyz}+b_{wyz}\frac{2\pi y}{L_y}\right)\sin\left(e_{wyz}+d_{wyz}\frac{2\pi z}{L_z}\right).
\end{aligned}
$$

3 Comments

The complexity, and consequently length, of the source terms increase with both dimension and physics handled by the governing equations. In some cases, such as the modified equation for v, the expression for Q_v almost reaches 21,000 characters, including parenthesis and mathematical operators, prior to factorization.

Applying commands in order to simplify such extensive expression is challenging even with a very good machine; thus, a suitable alternative to this issue is to simplify the equation by dividing it into a combination of sub-operators handling different physical phenomena. Then, each one of the operators may be applied to the manufactured solutions individually, and the resulting sub-source terms are combined back to represent the source term for the original equation.

For instance, instead of writing the fourth-order equation for v (3) using one single operator $\mathcal{L}$:

$$
\mathcal{L} = \frac{\partial}{\partial t}\nabla^2 v + h_v - \frac{1}{Re}\nabla^4 v \tag{13}
$$

to then be used in the MMS, let Equation (3) be written with five operators, according to their physical meaning:

$$
\mathcal{L}_1 = \frac{\partial}{\partial t}\nabla^2 v, \qquad \mathcal{L}_2 = h_v, \qquad \mathcal{L}_3 = \frac{1}{Re}\nabla^4 v. \tag{14}
$$

After the application of $\mathcal{L}_i$, $i = 1,..,3$, the corresponding sub-source terms are also simplified, factorized and sorted. Then, the final factorized version is checked against the original one, to assure that not human error has been introduced. This strategy allowed the original 21,000 character-long expression for Q_v to be reduced to less than 15,100, and expressed in Equation (12).

3.1 Boundary Conditions

Additionally to verifying code capability of solving the governing equations accurately in the interior of the domain of interest, one may also verify the software capability of correctly imposing boundary conditions. Therefore, the gradients of the analytical solutions (7) for u, v and w have been calculated and translated into C codes. They are:

$$
\begin{aligned}
\frac{\partial u}{\partial x} =& \frac{2\pi}{L_x}\Bigg[-a_{ux}b_{ux}\cos\left(g_{ux}+f_{ux}t\right)\sin\left(c_{ux}+b_{ux}\frac{2\pi}{L_x}x\right) - a_{uxy}b_{uxy}\cos\left(e_{uxy}+d_{uxy}\frac{2\pi}{L_y}y\right)\cos\left(g_{uxy}+f_{uxy}t\right)\sin\left(c_{uxy}+b_{uxy}\frac{2\pi}{L_x}x\right) + \\
&-a_{uxz}b_{uxz}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(e_{uxz}+d_{uxz}\frac{2\pi}{L_z}z\right)\sin\left(c_{uxz}+b_{uxz}\frac{2\pi}{L_x}x\right)\Bigg] \\
\frac{\partial u}{\partial y} =& \frac{2\pi}{L_y}\Bigg[-a_{uy}b_{uy}\cos\left(g_{uy}+f_{uy}t\right)\sin\left(c_{uy}+b_{uy}\frac{2\pi}{L_y}y\right) - a_{uxy}d_{uxy}\cos\left(c_{uxy}+b_{uxy}\frac{2\pi}{L_x}x\right)\cos\left(g_{uxy}+f_{uxy}t\right)\sin\left(e_{uxy}+d_{uxy}\frac{2\pi}{L_y}y\right) + \\
&-a_{uyz}b_{uyz}\cos\left(g_{uyz}+f_{uyz}t\right)\cos\left(e_{uyz}+d_{uyz}\frac{2\pi}{L_z}z\right)\sin\left(c_{uyz}+b_{uyz}\frac{2\pi}{L_y}y\right)\Bigg] \\
\frac{\partial u}{\partial z} =& \frac{2\pi}{L_z}\Bigg[-a_{uz}b_{uz}\cos\left(g_{uz}+f_{uz}t\right)\sin\left(c_{uz}+b_{uz}\frac{2\pi}{L_z}z\right) - a_{uxz}d_{uxz}\cos\left(g_{uxz}+f_{uxz}t\right)\cos\left(c_{uxz}+b_{uxz}\frac{2\pi}{L_x}x\right)\sin\left(e_{uxz}+d_{uxz}\frac{2\pi}{L_z}z\right) + \\
&-a_{uyz}d_{uyz}\cos\left(c_{uyz}+b_{uyz}\frac{2\pi}{L_y}y\right)\cos\left(g_{uyz}+f_{uyz}t\right)\sin\left(e_{uyz}+d_{uyz}\frac{2\pi}{L_z}z\right)\Bigg] \\
\frac{\partial v}{\partial x} =& \frac{2\pi}{L_x}\Bigg[-a_{vx}b_{vx}\cos\left(g_{vx}+f_{vx}t\right)\sin\left(c_{vx}+b_{vx}\frac{2\pi}{L_x}x\right) - a_{vxy}b_{vxy}\cos\left(e_{vxy}+d_{vxy}\frac{2\pi}{L_y}y\right)\cos\left(g_{vxy}+f_{vxy}t\right)\sin\left(c_{vxy}+b_{vxy}\frac{2\pi}{L_x}x\right) + \\
&-a_{vxz}b_{vxz}\cos\left(e_{vxz}+d_{vxz}\frac{2\pi}{L_z}z\right)\cos\left(g_{vxz}+f_{vxz}t\right)\sin\left(c_{vxz}+b_{vxz}\frac{2\pi}{L_x}x\right)\Bigg] \\
\frac{\partial v}{\partial y} =& \frac{2\pi}{L_y}\Bigg[-a_{vy}b_{vy}\cos\left(g_{vy}+f_{vy}t\right)\sin\left(c_{vy}+b_{vy}\frac{2\pi}{L_y}y\right) - a_{vxy}d_{vxy}\cos\left(g_{vxy}+f_{vxy}t\right)\cos\left(c_{vxy}+b_{vxy}\frac{2\pi}{L_x}x\right)\sin\left(e_{vxy}+d_{vxy}\frac{2\pi}{L_y}y\right) + \\
&-a_{vyz}b_{vyz}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(e_{vyz}+d_{vyz}\frac{2\pi}{L_z}z\right)\sin\left(c_{vyz}+b_{vyz}\frac{2\pi}{L_y}y\right) \\
\frac{\partial v}{\partial z} =& \frac{2\pi}{L_z}\Bigg[-a_{vz}b_{vz}\cos\left(g_{vz}+f_{vz}t\right)\sin\left(c_{vz}+b_{vz}\frac{2\pi}{L_z}z\right) - a_{vxz}d_{vxz}\cos\left(g_{vxz}+f_{vxz}t\right)\cos\left(c_{vxz}+b_{vxz}\frac{2\pi}{L_x}x\right)\sin\left(e_{vxz}+d_{vxz}\frac{2\pi}{L_z}z\right) + \\
&-a_{vyz}d_{vyz}\cos\left(g_{vyz}+f_{vyz}t\right)\cos\left(c_{vyz}+b_{vyz}\frac{2\pi}{L_y}y\right)\sin\left(e_{vyz}+d_{vyz}\frac{2\pi}{L_z}z\right)\Bigg] \\
\frac{\partial w}{\partial x} =& \frac{2\pi}{L_x}\Bigg[-a_{wx}b_{wx}\cos\left(g_{wx}+f_{wx}t\right)\sin\left(c_{wx}+b_{wx}\frac{2\pi}{L_x}x\right) - a_{wxy}b_{wxy}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(e_{wxy}+d_{wxy}\frac{2\pi}{L_y}y\right)\sin\left(c_{wxy}+b_{wxy}\frac{2\pi}{L_x}x\right) + \\
&-a_{wxz}b_{wxz}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(e_{wxz}+d_{wxz}\frac{2\pi}{L_z}z\right)\sin\left(c_{wxz}+b_{wxz}\frac{2\pi}{L_x}x\right)\Bigg] \\
\frac{\partial w}{\partial y} =& \frac{2\pi}{L_y}\Bigg[-a_{wy}b_{wy}\cos\left(g_{wy}+f_{wy}t\right)\sin\left(c_{wy}+b_{wy}\frac{2\pi}{L_y}y\right) - a_{wxy}d_{wxy}\cos\left(g_{wxy}+f_{wxy}t\right)\cos\left(c_{wxy}+b_{wxy}\frac{2\pi}{L_x}x\right)\sin\left(e_{wxy}+d_{wxy}\frac{2\pi}{L_y}y\right) + \\
&-a_{wyz}b_{wyz}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(e_{wyz}+d_{wyz}\frac{2\pi}{L_z}z\right)\sin\left(c_{wyz}+b_{wyz}\frac{2\pi}{L_y}y\right)\Bigg]
\end{aligned}
$$

$$\frac{\partial w}{\partial z} = \frac{2\pi}{L_z}\left[-a_{wz}b_{wz}\cos\left(g_{wz}+f_{wz}t\right)\sin\left(c_{wz}+b_{wz}\frac{2\pi}{L_z}z\right) - a_{wxz}d_{wxz}\cos\left(g_{wxz}+f_{wxz}t\right)\cos\left(c_{wxz}+b_{wxz}\frac{2\pi}{L_x}x\right)\sin\left(e_{wxz}+d_{wxz}\frac{2\pi}{L_z}z\right) + \right.$$
$$\left. -a_{wyz}d_{wyz}\cos\left(g_{wyz}+f_{wyz}t\right)\cos\left(c_{wyz}+b_{wyz}\frac{2\pi}{L_y}y\right)\sin\left(e_{wyz}+d_{wyz}\frac{2\pi}{L_z}z\right)\right]$$

3.2 C Files

The C files for both source terms and gradients of the manufactured solutions are: `incompressible_flow_source_Qg.c`, `incompressible_flow_source_Qv.c`, `incompressible_flow_source_Qcontinuity.c`, and `incompressible_flow_manuf_solutions_gradients.c`

An example of the automatically generated C file from the source term for the 3D total energy source term Q_g is:

```
double Q_g(double Re, double U, double V, double W, double a_ux, double a_uxy, double a_uxz, double a_uy, double a_uyz, double a_uz, double a_vx, double a_vxy,
  double a_vxz, double a_vyz, double a_vz, double a_wx, double a_wxy, double a_wxz, double a_wy, double a_wyz, double a_wz, double b_ux, double b_uxy,
  double b_uxz, double b_uy, double b_uyz, double b_uz, double b_vx, double b_vxy, double b_vxz, double b_vyz, double b_vz, double b_wx, double b_wxy,
  double b_wxz, double b_wy, double b_wyz, double b_wz, double c_ux, double c_uxy, double c_uxz, double c_uy, double c_uyz, double c_uz, double c_vx,
  double c_vxy, double c_vxz, double c_vyz, double c_vz, double c_wx, double c_wxy, double c_wxz, double c_wy, double c_wyz, double c_wz, double d_uxy,
  double d_uxz, double d_uyz, double d_vxy, double d_vxz, double d_vyz, double d_wxy, double d_wxz, double d_wyz, double e_uxy, double e_uxz, double e_uyz,
  double e_vxy, double e_vxz, double e_vyz, double e_wxy, double e_wxz, double e_wyz, double f_ux, double f_uxy, double f_uxz, double f_uy, double f_uyz,
  double f_uz, double f_vx, double f_vxy, double f_vxz, double f_vyz, double f_vz, double f_wx, double f_wxy, double f_wxz, double f_wy, double f_wyz,
  double f_wz, double g_ux, double g_uxy, double g_uxz, double g_uy, double g_uyz, double g_uz, double g_vx, double g_vxy, double g_vxz, double g_vyz,
  double g_vz, double g_wx, double g_wxy, double g_wxz, double g_wy, double g_wyz, double g_wz, double t, double twopi_invLx, double twopi_invLy,
  double twopi_invLz, double x, double y, double z)
{
  return twopi_invLx*(-a_wx*b_wx*f_wx*sin(g_wx + f_wx*t)*sin(c_wx + b_wx*twopi_invLx*x)
    - a_wxy*b_wxy*f_wxy*cos(e_wxy + d_wxy*twopi_invLy*y)*sin(g_wxy + f_wxy*t)*sin(c_wxy + b_wxy*twopi_invLx*x)
    - a_wxz*b_wxz*f_wxz*cos(e_wxz + d_wxz*twopi_invLz*z)*sin(g_wxz + f_wxz*t)*sin(c_wxz + b_wxz*twopi_invLx*x))
    + twopi_invLz*(a_uz*b_uz*f_uz*sin(g_uz + f_uz*t)*sin(c_uz + b_uz*twopi_invLz*z)
      + a_uxz*d_uxz*f_uxz*cos(c_uxz + b_uxz*twopi_invLx*x)*sin(g_uxz + f_uxz*t)*sin(e_uxz + d_uxz*twopi_invLz*z)
      + a_uyz*d_uyz*f_uyz*cos(c_uyz + b_uyz*twopi_invLy*y)*sin(g_uyz + f_uyz*t)*sin(e_uyz + d_uyz*twopi_invLz*z))
    - (a_uz*pow(b_uz,3)*pow(twopi_invLz,3)*cos(g_uz + f_uz*t)*sin(c_uz + b_uz*twopi_invLz*z)
      - a_wx*pow(b_wx,3)*pow(twopi_invLx,3)*cos(g_wx + f_wx*t)*sin(c_wx + b_wx*twopi_invLx*x)
      + a_uxz*pow(d_uxz,3)*pow(twopi_invLz,3)*cos(g_uxz + f_uxz*t)*cos(c_uxz + b_uxz*twopi_invLx*x)*sin(e_uxz + d_uxz*twopi_invLz*z)
      + a_uyz*pow(d_uyz,3)*pow(twopi_invLz,3)*cos(c_uyz + b_uyz*twopi_invLy*y)*cos(g_uyz + f_uyz*t)*sin(e_uyz + d_uyz*twopi_invLz*z)
      - a_wxy*pow(b_wxy,3)*pow(twopi_invLx,3)*cos(g_wxy + f_wxy*t)*cos(e_wxy + d_wxy*twopi_invLy*y)*sin(c_wxy + b_wxy*twopi_invLx*x)
      - a_wxz*pow(b_wxz,3)*pow(twopi_invLx,3)*cos(g_wxz + f_wxz*t)*cos(e_wxz + d_wxz*twopi_invLz*z)*sin(c_wxz + b_wxz*twopi_invLx*x)
      + a_uxz*d_uxz*twopi_invLz*pow(b_uxz,2)*pow(twopi_invLx,2)*cos(g_uxz + f_uxz*t)*cos(c_uxz + b_uxz*twopi_invLx*x)*sin(e_uxz + d_uxz*twopi_invLz*z)
      + a_uyz*d_uyz*twopi_invLz*pow(b_uyz,2)*pow(twopi_invLy,2)*cos(c_uyz + b_uyz*twopi_invLy*y)*cos(g_uyz + f_uyz*t)*sin(e_uyz + d_uyz*twopi_invLz*z)
      - a_wxy*b_wxy*twopi_invLx*pow(d_wxy,2)*pow(twopi_invLy,2)*cos(g_wxy + f_wxy*t)*cos(e_wxy + d_wxy*twopi_invLy*y)*sin(c_wxy + b_wxy*twopi_invLx*x)
      - a_wxz*b_wxz*twopi_invLx*pow(d_wxz,2)*pow(twopi_invLz,2)*cos(g_wxz + f_wxz*t)*cos(e_wxz + d_wxz*twopi_invLz*z)*sin(c_wxz + b_wxz*twopi_invLx*x))/Re
    + U*pow(twopi_invLx,2)*(a_wx*pow(b_wx,2)*cos(g_wx + f_wx*t)*cos(c_wx + b_wx*twopi_invLx*x)
      + a_wxy*pow(b_wxy,2)*cos(g_wxy + f_wxy*t)*cos(c_wxy + b_wxy*twopi_invLx*x)*cos(e_wxy + d_wxy*twopi_invLy*y)
      + a_wxz*pow(b_wxz,2)*cos(g_wxz + f_wxz*t)*cos(c_wxz + b_wxz*twopi_invLx*x)*cos(e_wxz + d_wxz*twopi_invLz*z))
    + V*twopi_invLy*(a_uyz*b_uyz*d_uyz*twopi_invLz*cos(g_uyz + f_uyz*t)*sin(c_uyz + b_uyz*twopi_invLy*y)*sin(e_uyz + d_uyz*twopi_invLz*z)
      - a_wxy*b_wxy*d_wxy*twopi_invLx*cos(g_wxy + f_wxy*t)*sin(c_wxy + b_wxy*twopi_invLx*x)*sin(e_wxy + d_wxy*twopi_invLy*y))
    + W*pow(twopi_invLz,2)*(-a_uz*pow(b_uz,2)*cos(g_uz + f_uz*t)*cos(c_uz + b_uz*twopi_invLz*z)
      - a_uxz*pow(d_uxz,2)*cos(g_uxz + f_uxz*t)*cos(c_uxz + b_uxz*twopi_invLx*x)*cos(e_uxz + d_uxz*twopi_invLz*z)
      - a_uyz*pow(d_uyz,2)*cos(c_uyz + b_uyz*twopi_invLy*y)*cos(g_uyz + f_uyz*t)*cos(e_uyz + d_uyz*twopi_invLz*z))
```

```
    + pow(twopi_invLx,2)*(a_ux*b_ux*cos(g_ux + f_ux*t)*sin(c_ux + b_ux*twopi_invLx*x)
      + a_uxy*b_uxy*cos(e_uxy + d_uxy*twopi_invLy*y)*cos(g_uxy + f_uxy*t)*sin(c_uxy + b_uxy*twopi_invLx*x)
      + a_uxz*b_uxz*cos(g_uxz + f_uxz*t)*cos(e_uxz + d_uxz*twopi_invLz*z)*sin(c_uxz + b_uxz*twopi_invLx*x))
      *(-a_wx*b_wx*cos(g_wx + f_wx*t)*sin(c_wx + b_wx*twopi_invLx*x)
- a_wxy*b_wxy*cos(g_wxy + f_wxy*t)*cos(e_wxy + d_wxy*twopi_invLy*y)*sin(c_wxy + b_wxy*twopi_invLx*x)
- a_wxz*b_wxz*cos(g_wxz + f_wxz*t)*cos(e_wxz + d_wxz*twopi_invLz*z)*sin(c_wxz + b_wxz*twopi_invLx*x))
    + pow(twopi_invLz,2)*(-a_uz*b_uz*cos(g_uz + f_uz*t)*sin(c_uz + b_uz*twopi_invLz*z)
      - a_uxz*d_uxz*cos(g_uxz + f_uxz*t)*cos(c_uxz + b_uxz*twopi_invLx*x)*sin(e_uxz + d_uxz*twopi_invLz*z)
      - a_uyz*d_uyz*cos(c_uyz + b_uyz*twopi_invLy*y)*cos(g_uyz + f_uyz*t)*sin(e_uyz + d_uyz*twopi_invLz*z))
      *(-a_wz*b_wz*cos(g_wz + f_wz*t)*sin(c_wz + b_wz*twopi_invLz*z)
- a_wxz*d_wxz*cos(g_wxz + f_wxz*t)*cos(c_wxz + b_wxz*twopi_invLx*x)*sin(e_wxz + d_wxz*twopi_invLz*z)
- a_wyz*d_wyz*cos(g_wyz + f_wyz*t)*cos(c_wyz + b_wyz*twopi_invLy*y)*sin(e_wyz + d_wyz*twopi_invLz*z))
    + twopi_invLx*twopi_invLy*(a_vx*b_vx*cos(g_vx + f_vx*t)*sin(c_vx + b_vx*twopi_invLx*x)
      + a_vxy*b_vxy*cos(e_vxy + d_vxy*twopi_invLy*y)*cos(g_vxy + f_vxy*t)*sin(c_vxy + b_vxy*twopi_invLx*x)
      + a_vxz*b_vxz*cos(e_vxz + d_vxz*twopi_invLz*z)*cos(g_vxz + f_vxz*t)*sin(c_vxz + b_vxz*twopi_invLx*x))
      *(-a_wy*b_wy*cos(g_wy + f_wy*t)*sin(c_wy + b_wy*twopi_invLy*y)
- a_wxy*d_wxy*cos(g_wxy + f_wxy*t)*cos(c_wxy + b_wxy*twopi_invLx*x)*sin(e_wxy + d_wxy*twopi_invLy*y)
- a_wyz*b_wyz*cos(g_wyz + f_wyz*t)*cos(e_wyz + d_wyz*twopi_invLz*z)*sin(c_wyz + b_wyz*twopi_invLy*y))
    + twopi_invLx*twopi_invLz*(a_wx*b_wx*cos(g_wx + f_wx*t)*sin(c_wx + b_wx*twopi_invLx*x)
      + a_wxy*b_wxy*cos(g_wxy + f_wxy*t)*cos(e_wxy + d_wxy*twopi_invLy*y)*sin(c_wxy + b_wxy*twopi_invLx*x)
      + a_wxz*b_wxz*cos(g_wxz + f_wxz*t)*cos(e_wxz + d_wxz*twopi_invLz*z)*sin(c_wxz + b_wxz*twopi_invLx*x))
      *(-a_wz*b_wz*cos(g_wz + f_wz*t)*sin(c_wz + b_wz*twopi_invLz*z)
- a_wxz*d_wxz*cos(g_wxz + f_wxz*t)*cos(c_wxz + b_wxz*twopi_invLx*x)*sin(e_wxz + d_wxz*twopi_invLz*z)
- a_wyz*d_wyz*cos(g_wyz + f_wyz*t)*cos(c_wyz + b_wyz*twopi_invLy*y)*sin(e_wyz + d_wyz*twopi_invLz*z))
    + twopi_invLx*twopi_invLz*(-a_ux*b_ux*cos(g_ux + f_ux*t)*sin(c_ux + b_ux*twopi_invLx*x)
      - a_uxy*b_uxy*cos(e_uxy + d_uxy*twopi_invLy*y)*cos(g_uxy + f_uxy*t)*sin(c_uxy + b_uxy*twopi_invLx*x)
      - a_uxz*b_uxz*cos(g_uxz + f_uxz*t)*cos(e_uxz + d_uxz*twopi_invLz*z)*sin(c_uxz + b_uxz*twopi_invLx*x))
      *(-a_uz*b_uz*cos(g_uz + f_uz*t)*sin(c_uz + b_uz*twopi_invLz*z)
- a_uxz*d_uxz*cos(g_uxz + f_uxz*t)*cos(c_uxz + b_uxz*twopi_invLx*x)*sin(e_uxz + d_uxz*twopi_invLz*z)
- a_uyz*d_uyz*cos(c_uyz + b_uyz*twopi_invLy*y)*cos(g_uyz + f_uyz*t)*sin(e_uyz + d_uyz*twopi_invLz*z))
    + twopi_invLy*twopi_invLz*(-a_vz*b_vz*cos(g_vz + f_vz*t)*sin(c_vz + b_vz*twopi_invLz*z)
      - a_vxz*d_vxz*cos(g_vxz + f_vxz*t)*cos(c_vxz + b_vxz*twopi_invLx*x)*sin(e_vxz + d_vxz*twopi_invLz*z)
      - a_vyz*d_vyz*cos(g_vyz + f_vyz*t)*cos(c_vyz + b_vyz*twopi_invLy*y)*sin(e_vyz + d_vyz*twopi_invLz*z))
      *(-a_uy*b_uy*cos(g_uy + f_uy*t)*sin(c_uy + b_uy*twopi_invLy*y)
- a_uxy*d_uxy*cos(c_uxy + b_uxy*twopi_invLx*x)*cos(g_uxy + f_uxy*t)*sin(e_uxy + d_uxy*twopi_invLy*y)
- a_uyz*b_uyz*cos(g_uyz + f_uyz*t)*cos(e_uyz + d_uyz*twopi_invLz*z)*sin(c_uyz + b_uyz*twopi_invLy*y))
    + U*a_uxz*b_uxz*d_uxz*twopi_invLx*twopi_invLz*cos(g_uxz + f_uxz*t)*sin(c_uxz + b_uxz*twopi_invLx*x)*sin(e_uxz + d_uxz*twopi_invLz*z)
    - W*a_wxz*b_wxz*d_wxz*twopi_invLx*twopi_invLz*cos(g_wxz + f_wxz*t)*sin(c_wxz + b_wxz*twopi_invLx*x)*sin(e_wxz + d_wxz*twopi_invLz*z);
}
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