

Manufactured Solution for the Compressible Transient Navier–Stokes Equations with a Passive Scalar and Sutherland Viscosity Model using Sympy*

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Abstract

The Method of Manufactured Solutions is a valuable approach for code verification, providing means to verify how accurately the numerical method solves the partial differential equations of interest. This document presents the source terms generated by the application of the Method of Manufactured Solutions (MMS) on 3D transient Navier–Stokes equations with transport of a passive scalar and Sutherland viscosity model using the analytical manufactured solutions presented by Roy et al. (2002) for density, velocity, pressure and a passive scalar.

1 Mathematical Model

The conservation of mass, momentum, and total energy for a compressible transient viscous fluid may be written as:

$$\frac{\partial(\rho)}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1)$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla p + \nabla \cdot (\boldsymbol{\tau}), \quad (2)$$

$$\frac{\partial(\rho e_t)}{\partial t} + \nabla \cdot (\rho \mathbf{u} H) = -\nabla \cdot (p \mathbf{u}) - \nabla \cdot \mathbf{q} + \nabla \cdot (\boldsymbol{\tau} \cdot \mathbf{u}). \quad (3)$$

Additionally, the equation for the passive transport of a generic scalar ϕ is given by:

$$\frac{\partial(\rho \phi)}{\partial t} + \nabla \cdot (\rho \mathbf{u} \phi) = \Gamma \nabla^2(\phi), \quad (4)$$

where Γ is the diffusion coefficient and it's taken as constant.

Equations (1)–(3) are known as Navier–Stokes equations and, ρ is the density, $\mathbf{u} = (u, v, w)$ is the velocity in x , y or z -direction, respectively, and p is the pressure. The total enthalpy, H , may be expressed in terms of the total energy per unit mass e_t , density, and pressure:

$$H = e_t + \frac{p}{\rho}.$$

For a calorically perfect gas, the Navier–Stokes equations are closed with two auxiliary relations for energy:

$$e_t = e + \frac{\mathbf{u} \cdot \mathbf{u}}{2}, \quad \text{and} \quad e = \frac{1}{\gamma - 1} RT, \quad (5)$$

and with the ideal gas equation of state:

$$p = \rho RT. \quad (6)$$

*SymPy is a free open-source Python library for symbolic mathematics. <http://sympy.org/>.

The shear stress tensor is:

$$\begin{aligned}\tau_{xx} &= \frac{2}{3}\mu \left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right), & \tau_{xy} &= \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \\ \tau_{yy} &= \frac{2}{3}\mu \left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right), & \tau_{yz} &= \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \\ \tau_{zz} &= \frac{2}{3}\mu \left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right), & \tau_{zx} &= \tau_{xz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right),\end{aligned}$$

where μ is the absolute viscosity. The heat flux vector $\mathbf{q} = (q_x, q_y, q_z)$ is given by:

$$q_x = -\kappa \frac{\partial T}{\partial x}, \quad q_y = -\kappa \frac{\partial T}{\partial y}, \quad \text{and} \quad q_z = -\kappa \frac{\partial T}{\partial z}$$

where κ is the thermal conductivity, which can be determined by choosing the Prandtl number:

$$\kappa = \frac{\gamma R \mu}{(\gamma - 1) \text{Pr}}, \quad (7)$$

where γ is the ratio of specific heats.

1.1 Sutherland viscosity model

Sutherland (1893) published a relationship between the absolute temperature of an ideal gas, T , and its dynamic (absolute) viscosity, μ . The model is based on the kinetic theory of ideal gases and an idealized intermolecular-force potential. The general equation is given as:

$$\mu = \frac{A_\mu T^{\frac{3}{2}}}{T + B_\mu} \quad (8)$$

$$A_\mu = \frac{\mu_{\text{ref}}}{T_{\text{ref}}^{\frac{3}{2}}} (T_{\text{ref}} + B_\mu), \quad (9)$$

where B_μ is the Sutherland temperature, T_{ref} is a reference temperature, and μ_{ref} is the viscosity at the reference temperature T_{ref} .

2 Manufactured Solution

The Method of Manufactured Solutions (MMS) applied to Navier–Stokes equations with the transport of a scalar consists in modifying Equations (1) – (4) by adding a source term to the right-hand side of each equation, so the modified set of equations conveniently has the analytical solution chosen *a priori*.

Roy et al. (2002) introduce the general form of the primitive manufactured solution variables to be a function of sines and cosines in x , y and z only. In this work, Roy et al. (2002)’s manufactured solutions are modified in order to address temporal accuracy as well:

$$\varphi(x, y, z, t) = \varphi_0 + \varphi_x f_s \left(\frac{a_{\varphi x} \pi x}{L} \right) + \varphi_y f_s \left(\frac{a_{\varphi y} \pi y}{L} \right) + \varphi_z f_s \left(\frac{a_{\varphi z} \pi z}{L} \right) + \varphi_t f_s \left(\frac{a_{\varphi t} \pi t}{L} \right), \quad (10)$$

where $\varphi = \rho, u, v, w, \phi$ or p , and $f_s(\cdot)$ functions denote either sine or cosine function. Note that in this case, φ_x , φ_y , φ_z and φ_t are constants and the subscripts do not denote differentiation.

Although Roy et al. (2002) provide the constants used in the manufactured solutions for the 2D supersonic and subsonic cases for Euler and Navier–Stokes equations, only the source term for the 2D mass conservation equation (1) is presented.

Source terms for mass conservation (Q_ρ), momentum (Q_u , Q_v and Q_w), total energy (Q_{e_t}) and scalar transport (Q_ϕ) equations are obtained by symbolic manipulations of compressible transient Navier–Stokes+transport equations above using Sympy and are presented in the following sections.

The manufactured analytical solution for for each one of the variables in the 3D transient Navier–Stokes equations are:

$$\begin{aligned}
\rho(x, y, z, t) &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x} \pi x}{L}\right) + \rho_y \cos\left(\frac{a_{\rho y} \pi y}{L}\right) + \rho_z \sin\left(\frac{a_{\rho z} \pi z}{L}\right) + \rho_t \sin\left(\frac{a_{\rho t} \pi t}{L_t}\right), \\
u(x, y, z, t) &= u_0 + u_x \sin\left(\frac{a_{ux} \pi x}{L}\right) + u_y \cos\left(\frac{a_{uy} \pi y}{L}\right) + u_z \cos\left(\frac{a_{uz} \pi z}{L}\right) + u_t \cos\left(\frac{a_{ut} \pi t}{L_t}\right), \\
v(x, y, z, t) &= v_0 + v_x \cos\left(\frac{a_{vx} \pi x}{L}\right) + v_y \sin\left(\frac{a_{vy} \pi y}{L}\right) + v_z \sin\left(\frac{a_{vz} \pi z}{L}\right) + v_t \sin\left(\frac{a_{vt} \pi t}{L_t}\right), \\
w(x, y, z, t) &= w_0 + w_x \sin\left(\frac{a_{wx} \pi x}{L}\right) + w_y \sin\left(\frac{a_{wy} \pi y}{L}\right) + w_z \cos\left(\frac{a_{wz} \pi z}{L}\right) + w_t \cos\left(\frac{a_{wt} \pi t}{L_t}\right), \\
p(x, y, z, t) &= p_0 + p_x \cos\left(\frac{a_{px} \pi x}{L}\right) + p_y \sin\left(\frac{a_{py} \pi y}{L}\right) + p_z \cos\left(\frac{a_{pz} \pi z}{L}\right) + p_t \cos\left(\frac{a_{pt} \pi t}{L_t}\right), \\
\phi(x, y, z, t) &= \phi_0 + \phi_x \cos\left(\frac{a_{\phi x} \pi x}{L}\right) + \phi_y \cos\left(\frac{a_{\phi y} \pi y}{L}\right) + \phi_z \sin\left(\frac{a_{\phi z} \pi z}{L}\right) + \phi_t \cos\left(\frac{a_{\phi t} \pi t}{L_t}\right).
\end{aligned} \tag{11}$$

The MMS applied to 3D Navier–Stokes equations with transport of a scalar and using Sutherland viscosity model consists in modifying Equations (1)–(4) by adding a source term to the right-hand side of each equation:

$$\begin{aligned}
\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} &= Q_\rho \\
\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} + \frac{\partial(p)}{\partial x} - \frac{\partial(\tau_{xx})}{\partial x} - \frac{\partial(\tau_{xy})}{\partial y} - \frac{\partial(\tau_{xz})}{\partial z} &= Q_u, \\
\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} + \frac{\partial(p)}{\partial y} - \frac{\partial(\tau_{yx})}{\partial x} - \frac{\partial(\tau_{yy})}{\partial y} - \frac{\partial(\tau_{yz})}{\partial z} &= Q_v, \\
\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} + \frac{\partial(p)}{\partial z} - \frac{\partial(\tau_{zx})}{\partial x} - \frac{\partial(\tau_{zy})}{\partial y} - \frac{\partial(\tau_{zz})}{\partial z} &= Q_w, \\
\frac{\partial(\rho e_t)}{\partial t} + \frac{\partial(\rho u e_t)}{\partial x} + \frac{\partial(\rho v e_t)}{\partial y} + \frac{\partial(\rho w e_t)}{\partial z} + \frac{\partial(pu)}{\partial x} + \frac{\partial(pv)}{\partial y} + \frac{\partial(pw)}{\partial z} + \frac{\partial(q_x)}{\partial x} + \frac{\partial(q_y)}{\partial y} + \frac{\partial(q_z)}{\partial z} &+ \\
- \frac{\partial(u\tau_{xx} + v\tau_{xy} + w\tau_{xz})}{\partial x} - \frac{\partial(u\tau_{yx} + v\tau_{yy} + w\tau_{yz})}{\partial y} - \frac{\partial(u\tau_{zx} + v\tau_{zy} + w\tau_{zz})}{\partial z} &= Q_{e_t}, \\
\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho u\phi)}{\partial x} + \frac{\partial(\rho v\phi)}{\partial y} + \frac{\partial(\rho w\phi)}{\partial z} - \Gamma \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) &= Q_\phi
\end{aligned} \tag{12}$$

so this modified set of equations has for analytical solution Equation (11).

The source terms Q_ρ , Q_u , Q_v , Q_w , Q_{e_t} and Q_ϕ are presented with the use of the following auxiliary variables:

$$\begin{aligned}
\text{Rho} &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x} \pi x}{L}\right) + \rho_y \cos\left(\frac{a_{\rho y} \pi y}{L}\right) + \rho_z \sin\left(\frac{a_{\rho z} \pi z}{L}\right) + \rho_t \sin\left(\frac{a_{\rho t} \pi t}{L_t}\right), \\
\text{U} &= u_0 + u_x \sin\left(\frac{a_{ux} \pi x}{L}\right) + u_y \cos\left(\frac{a_{uy} \pi y}{L}\right) + u_z \cos\left(\frac{a_{uz} \pi z}{L}\right) + u_t \cos\left(\frac{a_{ut} \pi t}{L_t}\right), \\
\text{V} &= v_0 + v_x \cos\left(\frac{a_{vx} \pi x}{L}\right) + v_y \sin\left(\frac{a_{vy} \pi y}{L}\right) + v_z \sin\left(\frac{a_{vz} \pi z}{L}\right) + v_t \sin\left(\frac{a_{vt} \pi t}{L_t}\right), \\
\text{W} &= w_0 + w_x \sin\left(\frac{a_{wx} \pi x}{L}\right) + w_y \sin\left(\frac{a_{wy} \pi y}{L}\right) + w_z \cos\left(\frac{a_{wz} \pi z}{L}\right) + w_t \cos\left(\frac{a_{wt} \pi t}{L_t}\right), \\
\text{P} &= p_0 + p_x \cos\left(\frac{a_{px} \pi x}{L}\right) + p_y \sin\left(\frac{a_{py} \pi y}{L}\right) + p_z \cos\left(\frac{a_{pz} \pi z}{L}\right) + p_t \cos\left(\frac{a_{pt} \pi t}{L_t}\right), \\
\Phi &= \phi_0 + \phi_x \cos\left(\frac{a_{\phi x} \pi x}{L}\right) + \phi_y \cos\left(\frac{a_{\phi y} \pi y}{L}\right) + \phi_z \sin\left(\frac{a_{\phi z} \pi z}{L}\right) + \phi_t \cos\left(\frac{a_{\phi t} \pi t}{L_t}\right), \\
\text{T} &= \frac{\text{P}}{R \text{Rho}}, \\
\text{Mu} &= \frac{A_\mu T^{\frac{3}{2}}}{T + B_\mu},
\end{aligned} \tag{13}$$

which simply are the manufactured solutions for ρ, u, v, w, p and ϕ , the temperature (6), and the fluid viscosity according to Sutherland model (8), respectively. The following derivatives are also used:

$$\begin{aligned}
\frac{\partial \text{Mu}}{\partial x} &= \frac{A_\mu \text{T}^{1/2}}{B_\mu + \text{T}} \left[-\frac{3}{2} \frac{\text{T} a_{\rho x} \pi \rho_x}{L \text{Rho}} \cos\left(\frac{a_{\rho x} \pi x}{L}\right) - \frac{3}{2} \frac{a_{px} \pi p_x}{LR \text{Rho}} \sin\left(\frac{a_{px} \pi x}{L}\right) \right] + \\
&+ \frac{A_\mu \text{T}^{3/2}}{(B_\mu + \text{T})^2} \left[\frac{\text{T} a_{\rho x} \pi \rho_x}{L \text{Rho}} \cos\left(\frac{a_{\rho x} \pi x}{L}\right) + \frac{a_{px} \pi p_x}{LR \text{Rho}} \sin\left(\frac{a_{px} \pi x}{L}\right) \right], \\
\frac{\partial \text{Mu}}{\partial y} &= \frac{A_\mu \text{T}^{1/2}}{B_\mu + \text{T}} \left[\frac{3}{2} \frac{\text{T} a_{\rho y} \pi \rho_y}{L \text{Rho}} \sin\left(\frac{a_{\rho y} \pi y}{L}\right) + \frac{3}{2} \frac{a_{py} \pi p_y}{LR \text{Rho}} \cos\left(\frac{a_{py} \pi y}{L}\right) \right] + \\
&+ \frac{A_\mu \text{T}^{3/2}}{(B_\mu + \text{T})^2} \left[-\frac{\text{T} a_{\rho y} \pi \rho_y}{L \text{Rho}} \sin\left(\frac{a_{\rho y} \pi y}{L}\right) - \frac{a_{py} \pi p_y}{LR \text{Rho}} \cos\left(\frac{a_{py} \pi y}{L}\right) \right], \\
\frac{\partial \text{Mu}}{\partial z} &= \frac{A_\mu \text{T}^{1/2}}{B_\mu + \text{T}} \left[-\frac{3}{2} \frac{\text{T} a_{\rho z} \pi \rho_z}{L \text{Rho}} \cos\left(\frac{a_{\rho z} \pi z}{L}\right) - \frac{3}{2} \frac{a_{pz} \pi p_z}{LR \text{Rho}} \sin\left(\frac{a_{pz} \pi z}{L}\right) \right] + \\
&+ \frac{A_\mu \text{T}^{3/2}}{(B_\mu + \text{T})^2} \left[\frac{\text{T} a_{\rho z} \pi \rho_z}{L \text{Rho}} \cos\left(\frac{a_{\rho z} \pi z}{L}\right) + \frac{a_{pz} \pi p_z}{LR \text{Rho}} \sin\left(\frac{a_{pz} \pi z}{L}\right) \right].
\end{aligned} \tag{14}$$

2.1 Mass Conservation Equation

The mass conservation equation may be written as an operator \mathcal{L}_ρ :

$$\mathcal{L}_\rho = \mathcal{L}_{\rho \text{ time}} + \mathcal{L}_{\rho \text{ convection}}$$

where:

$$\begin{aligned}
\mathcal{L}_{\rho \text{ time}} &= \frac{\partial(\rho)}{\partial t} \\
\mathcal{L}_{\rho \text{ convection}} &= \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}.
\end{aligned} \tag{15}$$

The operators defined in Eq. (15) are applied into Equation (11), providing respective source terms that will compound source term Q_ρ :

$$Q_\rho = Q_{\rho \text{ time}} + Q_{\rho \text{ convection}}.$$

They are:

$$\begin{aligned}
Q_{\rho \text{ time}} &= \frac{a_{\rho t} \pi \rho_t}{L_t} \cos\left(\frac{a_{\rho t} \pi t}{L_t}\right) \\
Q_{\rho \text{ convection}} &= \frac{\pi \text{Rho}}{L} \left[a_{ux} u_x \cos\left(\frac{a_{ux} \pi x}{L}\right) + a_{vy} v_y \cos\left(\frac{a_{vy} \pi y}{L}\right) - a_{wz} w_z \sin\left(\frac{a_{wz} \pi z}{L}\right) \right] + \\
&+ \frac{\text{U} a_{\rho x} \pi \rho_x}{L} \cos\left(\frac{a_{\rho x} \pi x}{L}\right) - \frac{\text{V} a_{\rho y} \pi \rho_y}{L} \sin\left(\frac{a_{\rho y} \pi y}{L}\right) + \frac{\text{W} a_{\rho z} \pi \rho_z}{L} \cos\left(\frac{a_{\rho z} \pi z}{L}\right),
\end{aligned}$$

where Rho , U , V and W are defined in Equation (13).

2.2 Momentum Conservation Equations

2.2.1 Velocity u

For the generation of the analytical source term Q_u , x -momentum equation (2) is written as an operator \mathcal{L}_u :

$$\mathcal{L}_u = \mathcal{L}_{u \text{ time}} + \mathcal{L}_{u \text{ convection}} + \mathcal{L}_{u \text{ grad } p} + \mathcal{L}_{u \text{ viscous}}$$

with each one of the sub-operators defined as follows:

$$\begin{aligned}
\mathcal{L}_{u \text{ time}} &= \frac{\partial(\rho u)}{\partial t} \\
\mathcal{L}_{u \text{ convection}} &= \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} \\
\mathcal{L}_{u \text{ grad} p} &= \frac{\partial(p)}{\partial x} \\
\mathcal{L}_{u \text{ viscous}} &= -\frac{\partial(\tau_{xx})}{\partial x} - \frac{\partial(\tau_{xy})}{\partial y} - \frac{\partial(\tau_{xz})}{\partial z}
\end{aligned}$$

Source term Q_u is obtained by operating \mathcal{L}_u on Equations (11) together with the use of the auxiliary relations given in Equations (13) and (14). It yields:

$$Q_u = Q_{u \text{ time}} + Q_{u \text{ convection}} + Q_{u \text{ grad} p} + Q_{u \text{ viscous}}$$

with

$$\begin{aligned}
Q_{u \text{ time}} &= \frac{U a_{\rho t} \pi \rho_t}{L_t} \cos\left(\frac{a_{\rho t} \pi t}{L_t}\right) - \frac{\text{Rho } a_{ut} \pi u_t}{L_t} \sin\left(\frac{a_{ut} \pi t}{L_t}\right), \\
Q_{u \text{ convection}} &= \frac{\pi \text{Rho } U}{L} \left[a_{vy} v_y \cos\left(\frac{a_{vy} \pi y}{L}\right) - a_{wz} w_z \sin\left(\frac{a_{wz} \pi z}{L}\right) + 2a_{ux} u_x \cos\left(\frac{a_{ux} \pi x}{L}\right) \right] + \\
&\quad - \frac{\text{Rho } V a_{uy} \pi u_y}{L} \sin\left(\frac{a_{uy} \pi y}{L}\right) - \frac{\text{Rho } W a_{uz} \pi u_z}{L} \sin\left(\frac{a_{uz} \pi z}{L}\right) + \\
&\quad + \frac{U^2 a_{\rho x} \pi \rho_x}{L} \cos\left(\frac{a_{\rho x} \pi x}{L}\right) - \frac{U V a_{\rho y} \pi \rho_y}{L} \sin\left(\frac{a_{\rho y} \pi y}{L}\right) + \frac{U W a_{\rho z} \pi \rho_z}{L} \cos\left(\frac{a_{\rho z} \pi z}{L}\right), \\
Q_{u \text{ grad} p} &= -\frac{a_{px} \pi p_x}{L} \sin\left(\frac{a_{px} \pi x}{L}\right), \\
Q_{u \text{ viscous}} &= \frac{\partial \text{Mu}}{\partial x} \frac{\pi}{L} \left[-4/3 a_{ux} u_x \cos\left(\frac{a_{ux} \pi x}{L}\right) - 2/3 a_{wz} w_z \sin\left(\frac{a_{wz} \pi z}{L}\right) + 2/3 a_{vy} v_y \cos\left(\frac{a_{vy} \pi y}{L}\right) \right] + \\
&\quad + \frac{\partial \text{Mu}}{\partial y} \frac{\pi}{L} \left[a_{uy} u_y \sin\left(\frac{a_{uy} \pi y}{L}\right) + a_{vx} v_x \sin\left(\frac{a_{vx} \pi x}{L}\right) \right] + \\
&\quad + \frac{\partial \text{Mu}}{\partial z} \frac{\pi}{L} \left[a_{uz} u_z \sin\left(\frac{a_{uz} \pi z}{L}\right) - a_{wx} w_x \cos\left(\frac{a_{wx} \pi x}{L}\right) \right] + \\
&\quad + \frac{\text{Mu} \pi^2}{L^2} \left[a_{uy}^2 u_y \cos\left(\frac{a_{uy} \pi y}{L}\right) + a_{uz}^2 u_z \cos\left(\frac{a_{uz} \pi z}{L}\right) + 4/3 a_{ux}^2 u_x \sin\left(\frac{a_{ux} \pi x}{L}\right) \right],
\end{aligned}$$

with Mu , Rho , U , V and W defined in Equation (13) and the derivatives $\frac{\partial \text{Mu}}{\partial x}$, $\frac{\partial \text{Mu}}{\partial y}$ and $\frac{\partial \text{Mu}}{\partial z}$ defined in Equation (14).

2.2.2 Velocity v

Analogously to the velocity- u case, for the generation of the analytical source term Q_v , y -momentum equation (2) is written as an operator \mathcal{L}_v :

$$\mathcal{L}_v = \mathcal{L}_{v \text{ time}} + \mathcal{L}_{v \text{ convection}} + \mathcal{L}_{v \text{ grad} p} + \mathcal{L}_{v \text{ viscous}}$$

with each one of the sub-operators defined as follows:

$$\begin{aligned}
\mathcal{L}_{v \text{ time}} &= \frac{\partial(\rho v)}{\partial t} \\
\mathcal{L}_{v \text{ convection}} &= \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} \\
\mathcal{L}_{v \text{ grad} p} &= \frac{\partial(p)}{\partial y} \\
\mathcal{L}_{v \text{ viscous}} &= -\frac{\partial(\tau_{yx})}{\partial x} - \frac{\partial(\tau_{yy})}{\partial y} - \frac{\partial(\tau_{yz})}{\partial z}
\end{aligned}$$

Source term Q_v is obtained by operating \mathcal{L}_v on Equations (11) together with the use of the auxiliary relations given in Equations (13) and (14). It yields:

$$Q_v = Q_{v \text{ time}} + Q_{v \text{ convection}} + Q_{v \text{ gradp}} + Q_{v \text{ viscous}}$$

with

$$\begin{aligned} Q_{v \text{ time}} &= \frac{V a_{\rho t} \pi \rho_t}{L_t} \cos\left(\frac{a_{\rho t} \pi t}{L_t}\right) + \frac{\text{Rho } a_{vt} \pi v_t}{L_t} \cos\left(\frac{a_{vt} \pi t}{L_t}\right), \\ Q_{v \text{ convection}} &= \frac{\pi \text{Rho } V}{L} \left[a_{ux} u_x \cos\left(\frac{a_{ux} \pi x}{L}\right) - a_{wz} w_z \sin\left(\frac{a_{wz} \pi z}{L}\right) + 2a_{vy} v_y \cos\left(\frac{a_{vy} \pi y}{L}\right) \right] + \\ &\quad - \frac{\text{Rho } U a_{vx} \pi v_x}{L} \sin\left(\frac{a_{vx} \pi x}{L}\right) + \frac{\text{Rho } W a_{vz} \pi v_z}{L} \cos\left(\frac{a_{vz} \pi z}{L}\right) + \\ &\quad + \frac{U V a_{\rho x} \pi \rho_x}{L} \cos\left(\frac{a_{\rho x} \pi x}{L}\right) - \frac{a_{\rho y} \pi \rho_y V^2}{L} \sin\left(\frac{a_{\rho y} \pi y}{L}\right) + \frac{V W a_{\rho z} \pi \rho_z}{L} \cos\left(\frac{a_{\rho z} \pi z}{L}\right), \\ Q_{v \text{ gradp}} &= \frac{a_{py} \pi p_y}{L} \cos\left(\frac{a_{py} \pi y}{L}\right), \\ Q_{v \text{ viscous}} &= \frac{\partial \text{Mu}}{\partial x} \frac{\pi}{L} \left[a_{uy} u_y \sin\left(\frac{a_{uy} \pi y}{L}\right) + a_{vx} v_x \sin\left(\frac{a_{vx} \pi x}{L}\right) \right] + \\ &\quad + \frac{\partial \text{Mu}}{\partial y} \frac{\pi}{L} \left[-4/3 a_{vy} v_y \cos\left(\frac{a_{vy} \pi y}{L}\right) - 2/3 a_{wz} w_z \sin\left(\frac{a_{wz} \pi z}{L}\right) + 2/3 a_{ux} u_x \cos\left(\frac{a_{ux} \pi x}{L}\right) \right] + \\ &\quad + \frac{\partial \text{Mu}}{\partial z} \frac{\pi}{L} \left[-a_{vz} v_z \cos\left(\frac{a_{vz} \pi z}{L}\right) - a_{wy} w_y \cos\left(\frac{a_{wy} \pi y}{L}\right) \right] + \\ &\quad + \frac{\text{Mu} \pi^2}{L^2} \left[a_{vx}^2 v_x \cos\left(\frac{a_{vx} \pi x}{L}\right) + a_{vz}^2 v_z \sin\left(\frac{a_{vz} \pi z}{L}\right) + 4/3 a_{vy}^2 v_y \sin\left(\frac{a_{vy} \pi y}{L}\right) \right], \end{aligned}$$

where Mu , Rho , U , V and W are defined in Equation (13) and the derivatives $\frac{\partial \text{Mu}}{\partial x}$, $\frac{\partial \text{Mu}}{\partial y}$ and $\frac{\partial \text{Mu}}{\partial z}$ are given in Equation (14).

2.2.3 Velocity w

Finally, for the generation of the analytical source term Q_w , z -momentum equation (2) is written as an operator \mathcal{L}_w :

$$\mathcal{L}_w = \mathcal{L}_{w \text{ time}} + \mathcal{L}_{w \text{ convection}} + \mathcal{L}_{w \text{ gradp}} + \mathcal{L}_{w \text{ viscous}}$$

with each one of the sub-operators defined as follows:

$$\begin{aligned} \mathcal{L}_{w \text{ time}} &= \frac{\partial(\rho w)}{\partial t} \\ \mathcal{L}_{w \text{ convection}} &= \frac{\partial(\rho u w)}{\partial x} + \frac{\partial(\rho v w)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} \\ \mathcal{L}_{w \text{ gradp}} &= \frac{\partial(p)}{\partial z} \\ \mathcal{L}_{w \text{ viscous}} &= -\frac{\partial(\tau_{zx})}{\partial x} - \frac{\partial(\tau_{zy})}{\partial y} - \frac{\partial(\tau_{zz})}{\partial z} \end{aligned}$$

Source term Q_w is obtained by operating \mathcal{L}_w on Equations (11) together with the use of the auxiliary relations given in Equations (13) and (14). It yields:

$$Q_w = Q_{w \text{ time}} + Q_{w \text{ convection}} + Q_{w \text{ gradp}} + Q_{w \text{ viscous}}$$

where

$$\begin{aligned}
Q_{w \text{ time}} &= \frac{W a_{\rho t} \pi \rho_t}{L_t} \cos\left(\frac{a_{\rho t} \pi t}{L_t}\right) - \frac{Rho a_{wt} \pi w_t}{L_t} \sin\left(\frac{a_{wt} \pi t}{L_t}\right), \\
Q_{w \text{ convection}} &= \frac{\pi Rho W}{L} \left[a_{ux} u_x \cos\left(\frac{a_{ux} \pi x}{L}\right) + a_{vy} v_y \cos\left(\frac{a_{vy} \pi y}{L}\right) - 2 a_{wz} w_z \sin\left(\frac{a_{wz} \pi z}{L}\right) \right] + \\
&\quad + \frac{Rho U a_{wx} \pi w_x}{L} \cos\left(\frac{a_{wx} \pi x}{L}\right) + \frac{Rho V a_{wy} \pi w_y}{L} \cos\left(\frac{a_{wy} \pi y}{L}\right) + \\
&\quad + \frac{U W a_{\rho x} \pi \rho_x}{L} \cos\left(\frac{a_{\rho x} \pi x}{L}\right) - \frac{V W a_{\rho y} \pi \rho_y}{L} \sin\left(\frac{a_{\rho y} \pi y}{L}\right) + \frac{W^2 a_{\rho z} \pi \rho_z}{L} \cos\left(\frac{a_{\rho z} \pi z}{L}\right), \\
Q_{w \text{ gradp}} &= -\frac{a_{pz} \pi p_z}{L} \sin\left(\frac{a_{pz} \pi z}{L}\right), \\
Q_{w \text{ viscous}} &= \frac{\partial Mu}{\partial x} \frac{\pi}{L} \left[a_{uz} u_z \sin\left(\frac{a_{uz} \pi z}{L}\right) - a_{wx} w_x \cos\left(\frac{a_{wx} \pi x}{L}\right) \right] + \\
&\quad + \frac{\partial Mu}{\partial y} \frac{\pi}{L} \left[-a_{vz} v_z \cos\left(\frac{a_{vz} \pi z}{L}\right) - a_{wy} w_y \cos\left(\frac{a_{wy} \pi y}{L}\right) \right] + \\
&\quad + \frac{\partial Mu}{\partial z} \frac{\pi}{L} \left[2/3 a_{ux} u_x \cos\left(\frac{a_{ux} \pi x}{L}\right) + 2/3 a_{vy} v_y \cos\left(\frac{a_{vy} \pi y}{L}\right) + 4/3 a_{wz} w_z \sin\left(\frac{a_{wz} \pi z}{L}\right) \right] + \\
&\quad + \frac{Mu \pi^2}{L^2} \left[a_{wx}^2 w_x \sin\left(\frac{a_{wx} \pi x}{L}\right) + a_{wy}^2 w_y \sin\left(\frac{a_{wy} \pi y}{L}\right) + 4/3 a_{wz}^2 w_z \cos\left(\frac{a_{wz} \pi z}{L}\right) \right],
\end{aligned}$$

where Mu , Rho , U , V and W are defined in Equation (13) and the derivatives $\frac{\partial Mu}{\partial x}$, $\frac{\partial Mu}{\partial y}$ and $\frac{\partial Mu}{\partial z}$ are given in Equation (14).

2.3 Total Energy Conservation Equation

The total energy equation is written as an operator:

$$\mathcal{L}_{e_t} = \mathcal{L}_{e_t \text{ time}} + \mathcal{L}_{e_t \text{ convection}} + \mathcal{L}_{e_t \text{ gradp}} + \mathcal{L}_{e_t \text{ viscous}} + \mathcal{L}_{e_t \text{ heat flux}}$$

with

$$\begin{aligned}
\mathcal{L}_{e_t \text{ time}} &= \frac{\partial(\rho e_t)}{\partial t}, \\
\mathcal{L}_{e_t \text{ convection}} &= \frac{\partial(\rho u e_t)}{\partial x} + \frac{\partial(\rho v e_t)}{\partial y} + \frac{\partial(\rho w e_t)}{\partial z}, \\
\mathcal{L}_{e_t \text{ gradp}} &= + \frac{\partial(pu)}{\partial x} + \frac{\partial(pv)}{\partial y} + \frac{\partial(pw)}{\partial z}, \\
\mathcal{L}_{e_t \text{ heat flux}} &= + \frac{\partial(q_x)}{\partial x} + \frac{\partial(q_y)}{\partial y} + \frac{\partial(q_z)}{\partial z}, \\
\mathcal{L}_{e_t \text{ viscous}} &= - \frac{\partial(u \tau_{xx} + v \tau_{xy} + w \tau_{xz})}{\partial x} - \frac{\partial(u \tau_{yx} + v \tau_{yy} + w \tau_{yz})}{\partial y} - \frac{\partial(u \tau_{zx} + v \tau_{zy} + w \tau_{zz})}{\partial z},
\end{aligned}$$

Therefore, source term Q_{e_t} is given by

$$Q_{e_t} = Q_{e_t \text{ time}} + Q_{e_t \text{ convection}} + Q_{e_t \text{ gradp}} + Q_{e_t \text{ viscous}} + Q_{e_t \text{ heat flux}},$$

where:

$$\begin{aligned}
Q_{e_t \text{ time}} &= \frac{\pi Rho}{L_t} \left[V a_{vt} v_t \cos\left(\frac{a_{vt} \pi t}{L_t}\right) - U a_{ut} u_t \sin\left(\frac{a_{ut} \pi t}{L_t}\right) - W a_{wt} w_t \sin\left(\frac{a_{wt} \pi t}{L_t}\right) \right] + \\
&\quad + \frac{E_t a_{\rho t} \pi \rho_t}{L_t} \cos\left(\frac{a_{\rho t} \pi t}{L_t}\right) + \frac{P a_{\rho t} \pi \rho_t}{L_t Rho (1 - \gamma)} \cos\left(\frac{a_{\rho t} \pi t}{L_t}\right) + \frac{a_{pt} \pi p_t}{L_t (1 - \gamma)} \sin\left(\frac{a_{pt} \pi t}{L_t}\right),
\end{aligned}$$

$$\begin{aligned}
Q_{e_t \text{ convection}} &= \frac{\pi \text{Rho}}{L} \left[a_{ux} u_x U^2 \cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy} v_y V^2 \cos\left(\frac{a_{vy}\pi y}{L}\right) - a_{wz} w_z W^2 \sin\left(\frac{a_{wz}\pi z}{L}\right) + \right. \\
&\quad + U W a_{wx} w_x \cos\left(\frac{a_{wx}\pi x}{L}\right) + V W a_{vz} v_z \cos\left(\frac{a_{vz}\pi z}{L}\right) + V W a_{wy} w_y \cos\left(\frac{a_{wy}\pi y}{L}\right) + \\
&\quad \left. - U V a_{uy} u_y \sin\left(\frac{a_{uy}\pi y}{L}\right) - U V a_{vx} v_x \sin\left(\frac{a_{vx}\pi x}{L}\right) - U W a_{uz} u_z \sin\left(\frac{a_{uz}\pi z}{L}\right) \right] + \\
&\quad + \frac{\pi}{L(1-\gamma)} \left[U a_{px} p_x \sin\left(\frac{a_{px}\pi x}{L}\right) + W a_{pz} p_z \sin\left(\frac{a_{pz}\pi z}{L}\right) - V a_{py} p_y \cos\left(\frac{a_{py}\pi y}{L}\right) \right] + \\
&\quad + \frac{\pi \text{E}_t \text{Rho}}{L} \left[a_{ux} u_x \cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy} v_y \cos\left(\frac{a_{vy}\pi y}{L}\right) - a_{wz} w_z \sin\left(\frac{a_{wz}\pi z}{L}\right) \right] + \\
&\quad + \frac{a_{\rho x} \pi \rho_x}{L} \left(\text{E}_t U + \frac{P U}{\text{Rho} (1-\gamma)} \right) \cos\left(\frac{a_{\rho x} \pi x}{L}\right) + \\
&\quad - \frac{a_{\rho y} \pi \rho_y}{L} \left(\text{E}_t V + \frac{P V}{\text{Rho} (1-\gamma)} \right) \sin\left(\frac{a_{\rho y} \pi y}{L}\right) + \\
&\quad + \frac{a_{\rho z} \pi \rho_z}{L} \left(\text{E}_t W + \frac{P W}{\text{Rho} (1-\gamma)} \right) \cos\left(\frac{a_{\rho z} \pi z}{L}\right), \\
Q_{e_t \text{ gradp}} &= \frac{\pi P}{L} \left[a_{ux} u_x \cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy} v_y \cos\left(\frac{a_{vy}\pi y}{L}\right) - a_{wz} w_z \sin\left(\frac{a_{wz}\pi z}{L}\right) \right] + \\
&\quad - \frac{U a_{px} \pi p_x}{L} \sin\left(\frac{a_{px}\pi x}{L}\right) + \frac{V a_{py} \pi p_y}{L} \cos\left(\frac{a_{py}\pi y}{L}\right) - \frac{W a_{pz} \pi p_z}{L} \sin\left(\frac{a_{pz}\pi z}{L}\right), \\
Q_{e_t \text{ heatflux}} &= \frac{k\pi^2}{RL^2 \text{Rho}} \left[a_{px}^2 p_x \cos\left(\frac{a_{px}\pi x}{L}\right) + a_{py}^2 p_y \sin\left(\frac{a_{py}\pi y}{L}\right) + a_{pz}^2 p_z \cos\left(\frac{a_{pz}\pi z}{L}\right) \right] + \\
&\quad + \frac{k\pi^2}{RL^2 \text{Rho}^2} \left[-2a_{px} a_{\rho x} p_x \rho_x \cos\left(\frac{a_{\rho x} \pi x}{L}\right) \sin\left(\frac{a_{px} \pi x}{L}\right) - 2a_{py} a_{\rho y} p_y \rho_y \cos\left(\frac{a_{\rho y} \pi y}{L}\right) \sin\left(\frac{a_{py} \pi y}{L}\right) + \right. \\
&\quad \left. - 2a_{pz} a_{\rho z} p_z \rho_z \cos\left(\frac{a_{\rho z} \pi z}{L}\right) \sin\left(\frac{a_{pz} \pi z}{L}\right) \right] + \\
&\quad + \frac{P k\pi^2}{RL^2 \text{Rho}^3} \left[-2a_{\rho x}^2 \rho_x^2 \cos^2\left(\frac{a_{\rho x} \pi x}{L}\right) - 2a_{\rho y}^2 \rho_y^2 \sin^2\left(\frac{a_{\rho y} \pi y}{L}\right) - 2a_{\rho z}^2 \rho_z^2 \cos^2\left(\frac{a_{\rho z} \pi z}{L}\right) \right] + \\
&\quad + \frac{P k\pi^2}{RL^2 \text{Rho}^2} \left[-\rho_x a_{\rho x}^2 \sin\left(\frac{a_{\rho x} \pi x}{L}\right) - \rho_y a_{\rho y}^2 \cos\left(\frac{a_{\rho y} \pi y}{L}\right) - \rho_z a_{\rho z}^2 \sin\left(\frac{a_{\rho z} \pi z}{L}\right) \right] + \\
&\quad - \frac{\partial \text{Mu}}{\partial x} \frac{\gamma a_{px} \pi p_x}{L \text{Pr} \text{Rho} (1-\gamma)} \sin\left(\frac{a_{px} \pi x}{L}\right) - \frac{\partial \text{Mu}}{\partial x} \frac{P \gamma a_{\rho x} \pi \rho_x}{L \text{Pr} \text{Rho}^2 (1-\gamma)} \cos\left(\frac{a_{\rho x} \pi x}{L}\right) + \\
&\quad + \frac{\partial \text{Mu}}{\partial y} \frac{\gamma a_{py} \pi p_y}{L \text{Pr} \text{Rho} (1-\gamma)} \cos\left(\frac{a_{py} \pi y}{L}\right) + \frac{\partial \text{Mu}}{\partial y} \frac{P \gamma a_{\rho y} \pi \rho_y}{L \text{Pr} \text{Rho}^2 (1-\gamma)} \sin\left(\frac{a_{\rho y} \pi y}{L}\right) + \\
&\quad - \frac{\partial \text{Mu}}{\partial z} \frac{\gamma a_{pz} \pi p_z}{L \text{Pr} \text{Rho} (1-\gamma)} \sin\left(\frac{a_{pz} \pi z}{L}\right) - \frac{\partial \text{Mu}}{\partial z} \frac{P \gamma a_{\rho z} \pi \rho_z}{L \text{Pr} \text{Rho}^2 (1-\gamma)} \cos\left(\frac{a_{\rho z} \pi z}{L}\right), \\
Q_{e_t \text{ viscous}} &= \frac{\text{Mu}\pi^2}{L^2} \left\{ U \left[u_y a_{uy}^2 \cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z a_{uz}^2 \cos\left(\frac{a_{uz}\pi z}{L}\right) + 4/3 u_x a_{ux}^2 \sin\left(\frac{a_{ux}\pi x}{L}\right) \right] + \right. \\
&\quad + V \left[v_x a_{vx}^2 \cos\left(\frac{a_{vx}\pi x}{L}\right) + v_z a_{vz}^2 \sin\left(\frac{a_{vz}\pi z}{L}\right) + 4/3 v_y a_{vy}^2 \sin\left(\frac{a_{vy}\pi y}{L}\right) \right] + \\
&\quad + W \left[w_x a_{wx}^2 \sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y a_{wy}^2 \sin\left(\frac{a_{wy}\pi y}{L}\right) + 4/3 w_z a_{wz}^2 \cos\left(\frac{a_{wz}\pi z}{L}\right) \right] + \\
&\quad - 2a_{uy} a_{vx} u_y v_x \sin\left(\frac{a_{uy}\pi y}{L}\right) \sin\left(\frac{a_{vx}\pi x}{L}\right) - 2a_{vz} a_{wy} v_z w_y \cos\left(\frac{a_{vz}\pi z}{L}\right) \cos\left(\frac{a_{wy}\pi y}{L}\right) + \\
&\quad + 2a_{uz} a_{wx} u_z w_x \cos\left(\frac{a_{wx}\pi x}{L}\right) \sin\left(\frac{a_{uz}\pi z}{L}\right) - 4/3 a_{ux} a_{wz} u_x w_z \cos\left(\frac{a_{ux}\pi x}{L}\right) \sin\left(\frac{a_{wz}\pi z}{L}\right) + \\
&\quad - 4/3 a_{vy} a_{wz} v_y w_z \cos\left(\frac{a_{vy}\pi y}{L}\right) \sin\left(\frac{a_{wz}\pi z}{L}\right) + 4/3 a_{ux} a_{vy} u_x v_y \cos\left(\frac{a_{ux}\pi x}{L}\right) \cos\left(\frac{a_{vy}\pi y}{L}\right) + \\
&\quad - a_{uy}^2 u_y^2 \sin^2\left(\frac{a_{uy}\pi y}{L}\right) - a_{uz}^2 u_z^2 \sin^2\left(\frac{a_{uz}\pi z}{L}\right) - a_{vx}^2 v_x^2 \sin^2\left(\frac{a_{vx}\pi x}{L}\right) - a_{vz}^2 v_z^2 \cos^2\left(\frac{a_{vz}\pi z}{L}\right) + \\
&\quad - a_{wx}^2 w_x^2 \cos^2\left(\frac{a_{wx}\pi x}{L}\right) - a_{wy}^2 w_y^2 \cos^2\left(\frac{a_{wy}\pi y}{L}\right) - 4/3 a_{ux}^2 u_x^2 \cos^2\left(\frac{a_{ux}\pi x}{L}\right) - 4/3 a_{vy}^2 v_y^2 \cos^2\left(\frac{a_{vy}\pi y}{L}\right) + \\
&\quad \left. - 4/3 a_{wz}^2 w_z^2 \sin^2\left(\frac{a_{wz}\pi z}{L}\right) \right\} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{\partial \text{Mu}}{\partial x} \frac{\pi \text{U}}{L} \left[-4/3 a_{ux} u_x \cos \left(\frac{a_{ux} \pi x}{L} \right) - 2/3 a_{wz} w_z \sin \left(\frac{a_{wz} \pi z}{L} \right) + 2/3 a_{vy} v_y \cos \left(\frac{a_{vy} \pi y}{L} \right) \right] + \\
& + \frac{\partial \text{Mu}}{\partial x} \frac{\pi \text{V}}{L} \left[a_{uy} u_y \sin \left(\frac{a_{uy} \pi y}{L} \right) + a_{vx} v_x \sin \left(\frac{a_{vx} \pi x}{L} \right) \right] + \\
& + \frac{\partial \text{Mu}}{\partial x} \frac{\pi \text{W}}{L} \left[a_{uz} u_z \sin \left(\frac{a_{uz} \pi z}{L} \right) - a_{wx} w_x \cos \left(\frac{a_{wx} \pi x}{L} \right) \right] + \\
& + \frac{\partial \text{Mu}}{\partial y} \frac{\pi \text{U}}{L} \left[a_{uy} u_y \sin \left(\frac{a_{uy} \pi y}{L} \right) + a_{vx} v_x \sin \left(\frac{a_{vx} \pi x}{L} \right) \right] + \\
& + \frac{\partial \text{Mu}}{\partial y} \frac{\pi \text{V}}{L} \left[-4/3 a_{vy} v_y \cos \left(\frac{a_{vy} \pi y}{L} \right) - 2/3 a_{wz} w_z \sin \left(\frac{a_{wz} \pi z}{L} \right) + 2/3 a_{ux} u_x \cos \left(\frac{a_{ux} \pi x}{L} \right) \right] + \\
& + \frac{\partial \text{Mu}}{\partial y} \frac{\pi \text{W}}{L} \left[-a_{vz} v_z \cos \left(\frac{a_{vz} \pi z}{L} \right) - a_{wy} w_y \cos \left(\frac{a_{wy} \pi y}{L} \right) \right] + \\
& + \frac{\partial \text{Mu}}{\partial z} \frac{\pi \text{U}}{L} \left[a_{uz} u_z \sin \left(\frac{a_{uz} \pi z}{L} \right) - a_{wx} w_x \cos \left(\frac{a_{wx} \pi x}{L} \right) \right] + \\
& + \frac{\partial \text{Mu}}{\partial z} \frac{\pi \text{V}}{L} \left[-a_{vz} v_z \cos \left(\frac{a_{vz} \pi z}{L} \right) - a_{wy} w_y \cos \left(\frac{a_{wy} \pi y}{L} \right) \right] + \\
& + \frac{\partial \text{Mu}}{\partial z} \frac{\pi \text{W}}{L} \left[2/3 a_{ux} u_x \cos \left(\frac{a_{ux} \pi x}{L} \right) + 2/3 a_{vy} v_y \cos \left(\frac{a_{vy} \pi y}{L} \right) + 4/3 a_{wz} w_z \sin \left(\frac{a_{wz} \pi z}{L} \right) \right]
\end{aligned}$$

Again, Mu , Rho , U , V , W and P are defined in Equation (13) and the derivatives $\frac{\partial \text{Mu}}{\partial x}$, $\frac{\partial \text{Mu}}{\partial y}$ and $\frac{\partial \text{Mu}}{\partial z}$ are given in Equation (14). The variable E_t is defined according to the definition of total energy (5) and it's given by:

$$\text{E}_t = \frac{\text{P}}{\text{Rho} (\gamma - 1)} + \frac{1}{2} \text{U}^2 + \frac{1}{2} \text{V}^2 + \frac{1}{2} \text{W}^2.$$

Accordingly,

$$\mathbf{k} = \frac{\gamma R \text{Mu}}{(\gamma - 1) \text{Pr}}.$$

2.4 Passive Transport

Similarly to the Navier-Stokes equations, source term Q_ϕ for the passive transport of a generic scalar ϕ is obtained by writing Equation (4) as an operator \mathcal{L}_ϕ :

$$\mathcal{L}_\phi = \mathcal{L}_{\phi \text{ time}} + \mathcal{L}_{\phi \text{ convection}} + \mathcal{L}_{\phi \text{ diffusion}}$$

with each one of the sub-operators defined as follows:

$$\begin{aligned}
\mathcal{L}_{\phi \text{ time}} &= \frac{\partial(\rho\phi)}{\partial t} \\
\mathcal{L}_{\phi \text{ convection}} &= \frac{\partial(\rho u \phi)}{\partial x} + \frac{\partial(\rho v \phi)}{\partial y} + \frac{\partial(\rho w \phi)}{\partial z} \\
\mathcal{L}_{\phi \text{ diffusion}} &= -\Gamma \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right)
\end{aligned}$$

Therefore,

$$Q_\phi = Q_{\phi \text{ time}} + Q_{\phi \text{ convection}} + Q_{\phi \text{ diffusion}},$$

where:

$$\begin{aligned}
Q_{\phi \text{ time}} &= \frac{\Phi a_{\rho t} \pi \rho_t}{L_t} \cos\left(\frac{a_{\rho t} \pi t}{L_t}\right) - \frac{\text{Rho } a_{\phi t} \pi \phi_t}{L_t} \sin\left(\frac{a_{\phi t} \pi t}{L_t}\right), \\
Q_{\phi \text{ convection}} &= \frac{\pi \Phi \text{ Rho}}{L} \left[a_{ux} u_x \cos\left(\frac{a_{ux} \pi x}{L}\right) + a_{vy} v_y \cos\left(\frac{a_{vy} \pi y}{L}\right) - a_{wz} w_z \sin\left(\frac{a_{wz} \pi z}{L}\right) \right] + \\
&\quad + \frac{\Phi \text{ U } a_{\rho x} \pi \rho_x}{L} \cos\left(\frac{a_{\rho x} \pi x}{L}\right) - \frac{\Phi \text{ V } a_{\rho y} \pi \rho_y}{L} \sin\left(\frac{a_{\rho y} \pi y}{L}\right) + \frac{\Phi \text{ W } a_{\rho z} \pi \rho_z}{L} \cos\left(\frac{a_{\rho z} \pi z}{L}\right) + \\
&\quad - \frac{\text{Rho } \text{ U } a_{\phi x} \pi \phi_x}{L} \sin\left(\frac{a_{\phi x} \pi x}{L}\right) - \frac{\text{Rho } \text{ V } a_{\phi y} \pi \phi_y}{L} \sin\left(\frac{a_{\phi y} \pi y}{L}\right) + \frac{\text{Rho } \text{ W } a_{\phi z} \pi \phi_z}{L} \cos\left(\frac{a_{\phi z} \pi z}{L}\right), \\
Q_{\phi \text{ diffusion}} &= \frac{\Gamma \pi^2}{L^2} \left[a_{\phi x}^2 \phi_x \cos\left(\frac{a_{\phi x} \pi x}{L}\right) + a_{\phi y}^2 \phi_y \cos\left(\frac{a_{\phi y} \pi y}{L}\right) + a_{\phi z}^2 \phi_z \sin\left(\frac{a_{\phi z} \pi z}{L}\right) \right],
\end{aligned} \tag{16}$$

with Rho , U , V , W and Φ defined in Equation (13).

3 Hierarchic MMS

The complexity, and consequently length, of the source terms increase with both dimension and physics handled by the governing equations. In some cases, such as the 3D energy equation, the final expression for Q_{e_t} may reach 40,800 characters, including parenthesis and mathematical operators, prior to factorization.

Applying commands in order to simplify such extensive expression is challenging even with a very good machine; thus, a suitable alternative to this issue is to simplify the equation by dividing it into a combination of sub-operators handling different physical phenomena. Then, each one of the operators may be applied to the manufactured solutions individually, and the resulting sub-source terms are combined back to represent the source term for the original equation.

For instance, instead of writing the three-dimensional Navier-Stokes energy equation using one single operator \mathcal{L} :

$$\mathcal{L} = \frac{\partial(\rho e_t)}{\partial t} + \nabla \cdot (\rho \mathbf{u} e_t) + \nabla \cdot \mathbf{q} + \nabla \cdot (p \mathbf{u}) - \nabla \cdot (\boldsymbol{\tau} \cdot \mathbf{u}), \tag{17}$$

to then be used in the MMS, let Equation (17) be written with five operators, according to their physical meaning:

$$\begin{aligned}
\mathcal{L}_{\text{time}} &= \frac{\partial(\rho e_t)}{\partial t}, \\
\mathcal{L}_{\text{convection}} &= \nabla \cdot (\rho \mathbf{u} e_t), \\
\mathcal{L}_{\text{heat flux}} &= \nabla \cdot \mathbf{q}, \\
\mathcal{L}_{\text{gradp}} &= \nabla \cdot (p \mathbf{u}), \\
\mathcal{L}_{\text{viscous}} &= -\nabla \cdot (\boldsymbol{\tau} \cdot \mathbf{u}),
\end{aligned} \tag{18}$$

where $\mathcal{L}_{\text{time}}$ denotes the rate of accumulation of inertial and kinetic energy, $\mathcal{L}_{\text{convection}}$ is the net rate of internal and kinetic energy increase by convection, $\mathcal{L}_{\text{heat flux}}$ is the net rate of heat addition due to heat conduction, $\mathcal{L}_{\text{gradp}}$ is the rate of work done on the fluid by external body forces, and $\mathcal{L}_{\text{viscous}}$ is the rate of work done on the fluid by viscous forces. Naturally:

$$\mathcal{L} = \mathcal{L}_{\text{time}} + \mathcal{L}_{\text{convection}} + \mathcal{L}_{\text{heat flux}} + \mathcal{L}_{\text{gradp}} + \mathcal{L}_{\text{viscous}}.$$

After the application of each sub-operator defined in (18), the corresponding sub-source terms are also simplified, factorized and sorted. Then, the final factorized version is checked against the original one, in order to assure that no human error has been introduced.

An advantage of this strategy is the possibility of inclusion and/or removal of other physical effects without the need of re-doing previous manipulations. For instance, in order to simplify this model by considering constant viscosity, changes should be made only in operator $\mathcal{L}_{\text{viscous}}$, in the total energy and momentum equations; the other terms on such equations, the continuity equation and the equation for the transport of the scalar all remain the same.

This strategy, named ‘‘Hierarchic MMS’’, results in less time spent in the manipulations, decreases the computational effort and occasional software crashes, and also increases the flexibility in the code verification procedure.

3.1 Boundary Conditions

Additionally to verifying code capability of solving the governing equations accurately in the interior of the domain of interest, one may also verify the software capability of correctly imposing boundary conditions. Therefore, the gradients of the analytical solutions (10) have been calculated and translated into *C* codes. For the 3D case, they are:

$$\nabla \rho = \begin{bmatrix} \frac{a_{\rho x} \pi \rho_x}{L} \cos\left(\frac{a_{\rho x} \pi x}{L}\right) \\ -\frac{a_{\rho y} \pi \rho_y}{L} \sin\left(\frac{a_{\rho y} \pi y}{L}\right) \\ \frac{a_{\rho z} \pi \rho_z}{L} \cos\left(\frac{a_{\rho z} \pi z}{L}\right) \end{bmatrix}, \quad \nabla p = \begin{bmatrix} -\frac{a_{p x} \pi p_x}{L} \sin\left(\frac{a_{p x} \pi x}{L}\right) \\ \frac{a_{p y} \pi p_y}{L} \cos\left(\frac{a_{p y} \pi y}{L}\right) \\ -\frac{a_{p z} \pi p_z}{L} \sin\left(\frac{a_{p z} \pi z}{L}\right) \end{bmatrix}, \quad \nabla \phi = \begin{bmatrix} -\frac{a_{\phi x} \pi \phi_x}{L} \sin\left(\frac{a_{\phi x} \pi x}{L}\right) \\ -\frac{a_{\phi y} \pi \phi_y}{L} \sin\left(\frac{a_{\phi y} \pi y}{L}\right) \\ \frac{a_{\phi z} \pi \phi_z}{L} \cos\left(\frac{a_{\phi z} \pi z}{L}\right) \end{bmatrix},$$

$$\nabla u = \begin{bmatrix} \frac{a_{u x} \pi u_x}{L} \cos\left(\frac{a_{u x} \pi x}{L}\right) \\ -\frac{a_{u y} \pi u_y}{L} \sin\left(\frac{a_{u y} \pi y}{L}\right) \\ -\frac{a_{u z} \pi u_z}{L} \sin\left(\frac{a_{u z} \pi z}{L}\right) \end{bmatrix}, \quad \nabla v = \begin{bmatrix} -\frac{a_{v x} \pi v_x}{L} \sin\left(\frac{a_{v x} \pi x}{L}\right) \\ \frac{a_{v y} \pi v_y}{L} \cos\left(\frac{a_{v y} \pi y}{L}\right) \\ \frac{a_{v z} \pi v_z}{L} \cos\left(\frac{a_{v z} \pi z}{L}\right) \end{bmatrix} \quad \text{and} \quad \nabla w = \begin{bmatrix} \frac{a_{w x} \pi w_x}{L} \cos\left(\frac{a_{w x} \pi x}{L}\right) \\ \frac{a_{w y} \pi w_y}{L} \cos\left(\frac{a_{w y} \pi y}{L}\right) \\ -\frac{a_{w z} \pi w_z}{L} \sin\left(\frac{a_{w z} \pi z}{L}\right) \end{bmatrix}.$$

3.2 C Files

The translation of the manufactured solutions, their gradients and all the auxiliary relations into C code is presented in the file `NS_Sutherland_scalar_transient_manuf_solutions_gradients.c`. Files containing *C* codes for the source terms have also been automatically generated. They are: `NS_Sutherland_transient_scalar_3d_e.c`, `NS_Sutherland_scalar_transient_3d_u.c`, `NS_Sutherland_scalar_transient_3d_v.c`, `NS_Sutherland_scalar_transient_3d_w.c`, `NS_Sutherland_scalar_transient_3d_rho.c` and `NS_Sutherland_scalar_transient_3d_phi.c`.

An example of the automatically generated *C* file from the source term for the 3D total energy source term Q_{e_t} is:

```

/*****
 *                               Code generated with sympy 0.6.7                               *
 *                               *                                                             *
 *                               See http://www.sympy.org/ for more information.                 *
 *                               *                                                             *
 *                               This file is part of 'project'                               *
 *****/

#include "NS_Sutherland_scalar_transient_3d_e.h"
#include <math.h>

double Rho(double L, double Lt, double a_rhot, double a_rhox, double a_rhox, double a_rhox, double a_rhox, double rho_0, double rho_t,
double rho_x, double rho_y, double rho_z, double t, double x, double y, double z) {
    return rho_0 + rho_t*sin(M_PI*a_rhot*t/Lt) + rho_x*sin(M_PI*a_rhox*x/L) + rho_y*cos(M_PI*a_rhox*y/L)
+ rho_z*sin(M_PI*a_rhox*z/L);
}

double U(double L, double Lt, double a_ut, double a_ux, double a_uy, double a_uz, double t, double u_0, double u_t,
double u_x, double u_y, double u_z, double x, double y, double z) {
    return u_0 + u_t*cos(M_PI*a_ut*t/Lt) + u_x*sin(M_PI*a_ux*x/L) + u_y*cos(M_PI*a_uy*y/L) + u_z*cos(M_PI*a_uz*z/L);
}

double V(double L, double Lt, double a_vt, double a_vx, double a_vy, double a_vz, double t, double v_0, double v_t,
double v_x, double v_y, double v_z, double x, double y, double z) {
    return v_0 + v_t*sin(M_PI*a_vt*t/Lt) + v_x*cos(M_PI*a_vx*x/L) + v_y*sin(M_PI*a_vy*y/L) + v_z*sin(M_PI*a_vz*z/L);
}

double W(double L, double Lt, double a_wt, double a_wx, double a_wy, double a_wz, double t, double w_0,
double w_t, double w_x, double w_y, double w_z, double x, double y, double z) {
    return w_0 + w_t*cos(M_PI*a_wt*t/Lt) + w_x*sin(M_PI*a_wx*x/L) + w_y*sin(M_PI*a_wy*y/L) + w_z*cos(M_PI*a_wz*z/L);
}

```

```

}

double P(double L, double Lt, double a_pt, double a_px, double a_py, double a_pz, double p_0, double p_t,
double p_x, double p_y, double p_z, double t, double x, double y, double z) {
    return p_0 + p_t*cos(M_PI*a_pt*t/Lt) + p_x*cos(M_PI*a_px*x/L) + p_y*sin(M_PI*a_py*y/L) + p_z*cos(M_PI*a_pz*z/L);
}

double T(double P, double R, double Rho) {
    return P/(R*Rho);
}

double E_t(double P, double Rho, double U, double V, double W, double gamma) {
    return -P/(Rho*(1 - gamma)) + pow(U,2)/2 + pow(V,2)/2 + pow(W,2)/2;
}

double A_mu(double B_mu, double T_ref, double mu_ref) {
    return mu_ref*pow(T_ref,-1.5)*(B_mu + T_ref);
}

double Mu(double A_mu, double B_mu, double T) {
    return A_mu*pow(T,1.5)/(B_mu + T);
}

double DMu_Dx(double A_mu, double B_mu, double L, double R, double Rho, double T, double a_px, double a_rhox,
double p_x, double rho_x, double x) {
    return A_mu*pow(T,0.5)*(-1.5*M_PI*T*a_rhox*rho_x*cos(M_PI*a_rhox*x/L)/(L*Rho)
- 1.5*M_PI*a_px*p_x*sin(M_PI*a_px*x/L)/(L*Rho))/(B_mu + T)
+ A_mu*pow(T,1.5)*(M_PI*T*a_rhox*rho_x*cos(M_PI*a_rhox*x/L)/(L*Rho)
+ M_PI*a_px*p_x*sin(M_PI*a_px*x/L)/(L*Rho))/pow((B_mu + T),2);
}

double DMu_Dy(double A_mu, double B_mu, double L, double R, double Rho, double T, double a_py, double a_rho_y,
double p_y, double rho_y, double y) {
    return A_mu*pow(T,0.5)*(1.5*M_PI*T*a_rho_y*rho_y*sin(M_PI*a_rho_y*y/L)/(L*Rho)
+ 1.5*M_PI*a_py*p_y*cos(M_PI*a_py*y/L)/(L*Rho))/(B_mu + T)
+ A_mu*pow(T,1.5)*(-M_PI*T*a_rho_y*rho_y*sin(M_PI*a_rho_y*y/L)/(L*Rho)
- M_PI*a_py*p_y*cos(M_PI*a_py*y/L)/(L*Rho))/pow((B_mu + T),2);
}

double DMu_Dz(double A_mu, double B_mu, double L, double R, double Rho, double T, double a_pz, double a_rhoz,
double p_z, double rho_z, double z) {
    return A_mu*pow(T,0.5)*(-1.5*M_PI*T*a_rhoz*rho_z*cos(M_PI*a_rhoz*z/L)/(L*Rho)
- 1.5*M_PI*a_pz*p_z*sin(M_PI*a_pz*z/L)/(L*Rho))/(B_mu + T)
+ A_mu*pow(T,1.5)*(M_PI*T*a_rhoz*rho_z*cos(M_PI*a_rhoz*z/L)/(L*Rho)
+ M_PI*a_pz*p_z*sin(M_PI*a_pz*z/L)/(L*Rho))/pow((B_mu + T),2);
}

double kappa(double Mu, double Pr, double R, double gamma) {
    return -Mu*R*gamma/(Pr*(1 - gamma));
}

double Q_et_convection(double E_t, double L, double P, double Rho, double U, double V, double W, double a_px,
double a_py, double a_pz, double a_rhox, double a_rho_y, double a_rhoz, double a_ux, double a_uy, double a_uz,
double a_vx, double a_vy, double a_vz, double a_wx, double a_wy, double a_wz, double gamma, double p_x, double p_y,
double p_z, double rho_x, double rho_y, double rho_z, double u_x, double u_y, double u_z, double v_x, double v_y,
double v_z, double w_x, double w_y, double w_z, double x, double y, double z) {
    return M_PI*Rho*(a_ux*u_x*pow(U,2)*cos(M_PI*a_ux*x/L) + a_vy*v_y*pow(V,2)*cos(M_PI*a_vy*y/L)
- a_wz*w_z*pow(W,2)*sin(M_PI*a_wz*z/L) + U*W*a_wx*w_x*cos(M_PI*a_wx*x/L) + V*W*a_vz*v_z*cos(M_PI*a_vz*z/L)
+ V*W*a_wy*w_y*cos(M_PI*a_wy*y/L) - U*V*a_uy*u_y*sin(M_PI*a_uy*y/L) - U*V*a_vx*v_x*sin(M_PI*a_vx*x/L)
- U*W*a_uz*u_z*sin(M_PI*a_uz*z/L))/L + M_PI*(U*a_px*p_x*sin(M_PI*a_px*x/L) + W*a_pz*p_z*sin(M_PI*a_pz*z/L)
- V*a_py*p_y*cos(M_PI*a_py*y/L))/(L*(1 - gamma)) + M_PI*E_t*Rho*(a_ux*u_x*cos(M_PI*a_ux*x/L)
+ a_vy*v_y*cos(M_PI*a_vy*y/L) - a_wz*w_z*sin(M_PI*a_wz*z/L))/L
+ M_PI*a_rhox*rho_x*(E_t*U + P*U/(Rho*(1 - gamma)))*cos(M_PI*a_rhox*x/L)/L
+ M_PI*a_rho_y*rho_y*(-E_t*V - P*V/(Rho*(1 - gamma)))*sin(M_PI*a_rho_y*y/L)/L
+ M_PI*a_rhoz*rho_z*(E_t*W + P*W/(Rho*(1 - gamma)))*cos(M_PI*a_rhoz*z/L)/L;
}

double Q_et_gradp(double L, double P, double U, double V, double W, double a_px, double a_py, double a_pz,
double a_ux, double a_vy, double a_wz, double p_x, double p_y, double p_z, double u_x, double v_y, double w_z,
double x, double y, double z) {

```

```

    return M_PI*P*(a_ux*u_x*cos(M_PI*a_ux*x/L) + a_vy*v_y*cos(M_PI*a_vy*y/L) - a_wz*w_z*sin(M_PI*a_wz*z/L))/L
+ M_PI*V*a_py*p_y*cos(M_PI*a_py*y/L)/L - M_PI*U*a_px*p_x*sin(M_PI*a_px*x/L)/L - M_PI*W*a_pz*p_z*sin(M_PI*a_pz*z/L)/L;
}

double Q_et_viscous(double DMu_Dx, double DMu_Dy, double DMu_Dz, double L, double Mu, double U, double V, double W,
    double a_ux, double a_uy, double a_uz, double a_vx, double a_vy, double a_vz, double a_wx, double a_wy, double a_wz,
    double u_x, double u_y, double u_z, double v_x, double v_y, double v_z, double w_x, double w_y, double w_z, double x,
    double y, double z) {
    return Mu*pow(M_PI,2)*(U*(u_y*pow(a_uy,2)*cos(M_PI*a_uy*y/L) + u_z*pow(a_uz,2)*cos(M_PI*a_uz*z/L)
+ 4*u_x*pow(a_ux,2)*sin(M_PI*a_ux*x/L)/3) + V*(v_x*pow(a_vx,2)*cos(M_PI*a_vx*x/L) + v_z*pow(a_vz,2)*sin(M_PI*a_vz*z/L)
+ 4*v_y*pow(a_vy,2)*sin(M_PI*a_vy*y/L)/3) + W*(w_x*pow(a_wx,2)*sin(M_PI*a_wx*x/L) + w_y*pow(a_wy,2)*sin(M_PI*a_wy*y/L)
+ 4*w_z*pow(a_wz,2)*cos(M_PI*a_wz*z/L)/3) - 2*a_uy*a_vx*u_y*v_x*sin(M_PI*a_uy*y/L)*sin(M_PI*a_vx*x/L)
- 2*a_vz*a_wy*v_z*w_y*cos(M_PI*a_vz*z/L)*cos(M_PI*a_wy*y/L) + 2*a_uz*a_wx*u_z*w_x*cos(M_PI*a_wx*x/L)*sin(M_PI*a_uz*z/L)
- 4*a_ux*a_wz*u_x*w_z*cos(M_PI*a_ux*x/L)*sin(M_PI*a_wz*z/L)/3 - 4*a_vy*a_wz*v_y*w_z*cos(M_PI*a_vy*y/L)*sin(M_PI*a_wz*z/L)/3
+ 4*a_ux*a_vy*u_x*v_y*cos(M_PI*a_ux*x/L)*cos(M_PI*a_vy*y/L)/3 - pow(a_uy,2)*pow(u_y,2)*pow(sin(M_PI*a_uy*y/L),2)
- pow(a_uz,2)*pow(u_z,2)*pow(sin(M_PI*a_uz*z/L),2) - pow(a_vx,2)*pow(v_x,2)*pow(sin(M_PI*a_vx*x/L),2)
- pow(a_vz,2)*pow(v_z,2)*pow(cos(M_PI*a_vz*z/L),2) - pow(a_wx,2)*pow(w_x,2)*pow(cos(M_PI*a_wx*x/L),2)
- pow(a_wy,2)*pow(w_y,2)*pow(cos(M_PI*a_wy*y/L),2) - 4*pow(a_ux,2)*pow(u_x,2)*pow(cos(M_PI*a_ux*x/L),2)/3
- 4*pow(a_vy,2)*pow(v_y,2)*pow(cos(M_PI*a_vy*y/L),2)/3 - 4*pow(a_wz,2)*pow(w_z,2)*pow(sin(M_PI*a_wz*z/L),2)/3)/pow(L,2)
+ M_PI*DMu_Dx*U*(-4*a_ux*u_x*cos(M_PI*a_ux*x/L)/3 - 2*a_wz*w_z*sin(M_PI*a_wz*z/L)/3 + 2*a_vy*v_y*cos(M_PI*a_vy*y/L)/3)/L
+ M_PI*DMu_Dx*V*(a_uy*u_y*sin(M_PI*a_uy*y/L) + a_vx*v_x*sin(M_PI*a_vx*x/L))/L + M_PI*DMu_Dx*W*(a_uz*u_z*sin(M_PI*a_uz*z/L)
- a_wx*w_x*cos(M_PI*a_wx*x/L))/L + M_PI*DMu_Dy*U*(a_uy*u_y*sin(M_PI*a_uy*y/L) + a_vx*v_x*sin(M_PI*a_vx*x/L))/L
+ M_PI*DMu_Dy*V*(-4*a_vy*v_y*cos(M_PI*a_vy*y/L)/3 - 2*a_wz*w_z*sin(M_PI*a_wz*z/L)/3 + 2*a_ux*u_x*cos(M_PI*a_ux*x/L)/3)/L
+ M_PI*DMu_Dy*W*(-a_vz*v_z*cos(M_PI*a_vz*z/L) - a_wy*w_y*cos(M_PI*a_wy*y/L))/L + M_PI*DMu_Dz*U*(a_uz*u_z*sin(M_PI*a_uz*z/L)
- a_wx*w_x*cos(M_PI*a_wx*x/L))/L + M_PI*DMu_Dz*V*(-a_vz*v_z*cos(M_PI*a_vz*z/L) - a_wy*w_y*cos(M_PI*a_wy*y/L))/L
+ M_PI*DMu_Dz*W*(2*a_ux*u_x*cos(M_PI*a_ux*x/L)/3 + 2*a_vy*v_y*cos(M_PI*a_vy*y/L)/3 + 4*a_wz*w_z*sin(M_PI*a_wz*z/L)/3)/L;
}

double Q_et_heatflux(double DMu_Dx, double DMu_Dy, double DMu_Dz, double L, double P, double Pr, double Rho, double Rho,
    double a_px, double a_py, double a_pz, double a_rhox, double a_rhoy, double a_rhoz, double gamma, double kappa,
    double p_x, double p_y, double p_z, double rho_x, double rho_y, double rho_z, double x, double y, double z) {
    return kappa*pow(M_PI,2)*(p_x*pow(a_px,2)*cos(M_PI*a_px*x/L) + p_y*pow(a_py,2)*sin(M_PI*a_py*y/L)
+ p_z*pow(a_pz,2)*cos(M_PI*a_pz*z/L))/(pow(L,2)*R*Rho)
+ kappa*pow(M_PI,2)*(-2*a_px*a_rhox*p_x*rho_x*cos(M_PI*a_rhox*x/L)*sin(M_PI*a_px*x/L)
- 2*a_py*a_rhoy*p_y*rho_y*cos(M_PI*a_py*y/L)*sin(M_PI*a_rhoy*y/L)
- 2*a_pz*a_rhoz*p_z*rho_z*cos(M_PI*a_rhoz*z/L)*sin(M_PI*a_pz*z/L))/(pow(L,2)*R*pow(Rho,2))
+ P*kappa*pow(M_PI,2)*(-2*pow(a_rhox,2)*pow(rho_x,2)*pow(cos(M_PI*a_rhox*x/L),2)
- 2*pow(a_rhoy,2)*pow(rho_y,2)*pow(sin(M_PI*a_rhoy*y/L),2)
- 2*pow(a_rhoz,2)*pow(rho_z,2)*pow(cos(M_PI*a_rhoz*z/L),2))/(pow(L,2)*R*pow(Rho,3))
+ P*kappa*pow(M_PI,2)*(-rho_x*pow(a_rhox,2)*sin(M_PI*a_rhox*x/L)
- rho_y*pow(a_rhoy,2)*cos(M_PI*a_rhoy*y/L) - rho_z*pow(a_rhoz,2)*sin(M_PI*a_rhoz*z/L))/(pow(L,2)*R*pow(Rho,2))
+ M_PI*DMu_Dy*a_py*gamma*p_y*cos(M_PI*a_py*y/L)/(L*Pr*Rho*(1 - gamma))
- M_PI*DMu_Dx*a_px*gamma*p_x*sin(M_PI*a_px*x/L)/(L*Pr*Rho*(1 - gamma))
- M_PI*DMu_Dz*a_pz*gamma*p_z*sin(M_PI*a_pz*z/L)/(L*Pr*Rho*(1 - gamma))
+ M_PI*DMu_Dy*P*a_rhoy*gamma*rho_y*sin(M_PI*a_rhoy*y/L)/(L*Pr*pow(Rho,2)*(1 - gamma))
- M_PI*DMu_Dx*P*a_rhox*gamma*rho_x*cos(M_PI*a_rhox*x/L)/(L*Pr*pow(Rho,2)*(1 - gamma))
- M_PI*DMu_Dz*P*a_rhoz*gamma*rho_z*cos(M_PI*a_rhoz*z/L)/(L*Pr*pow(Rho,2)*(1 - gamma));
}

double Q_et_time(double E_t, double Lt, double P, double Rho, double U, double V, double W, double a_pt, double a_rhot,
    double a_ut, double a_vt, double a_wt, double gamma, double p_t, double rho_t, double t, double u_t, double v_t,
    double w_t) {
    return M_PI*Rho*(V*a_vt*v_t*cos(M_PI*a_vt*t/Lt) - U*a_ut*u_t*sin(M_PI*a_ut*t/Lt)
- W*a_wt*w_t*sin(M_PI*a_wt*t/Lt))/Lt + M_PI*E_t*a_rhot*rho_t*cos(M_PI*a_rhot*t/Lt)/Lt
+ M_PI*a_pt*p_t*sin(M_PI*a_pt*t/Lt)/(Lt*(1 - gamma))
+ M_PI*P*a_rhot*rho_t*cos(M_PI*a_rhot*t/Lt)/(Lt*Rho*(1 - gamma));
}

double Q_et(double Q_et_convection, double Q_et_gradp, double Q_et_heatflux, double Q_et_time, double Q_et_viscous) {
    return Q_et_convection + Q_et_gradp + Q_et_heatflux + Q_et_time + Q_et_viscous;
}

```

References

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A List of model / manufactured solution parameters

There are a variety of parameters present in the flow described by Navier-Stokes equations with the passive transport of a generic scalar and the Power-law viscosity model, due to both fluid properties and the constants arising from the chosen manufactured solutions.

Table 1 shows the constants arising from fluid properties and their representation in the C code and Table 2 shows the constants present in the manufactured solutions.

Table 1: Relations between auxiliary variables, model documentation, and C code.

Variable	Description	Equation	Representation in C
B_μ	Sutherland temperature	(8)	<code>B_mu</code>
T_{ref}	reference temperature	(8), (9)	<code>T_ref</code>
μ_{ref}	reference viscosity	(9)	<code>mu_ref</code>
γ	ratio of specific heats	(7)	<code>gamma</code>
Γ	diffusion coefficient	(4)	<code>Gamma</code>
Pr	Prandtl number	(7)	<code>Pr</code>
R	gas constant	(6)	<code>R</code>

For air, the Sutherland’s law coefficients are: $B_\mu = 110.4 [K]$, $T_{ref} = 273.15 [K]$, $\mu_{ref} = 1.716 \times 10^{-5} [kg/ms]$ and, therefore, $A_\mu = 1.458 \times 10^{-6} [kg/ms\sqrt{K}]$. Additionally, $\gamma = 1.4$, $R = 287 [J/kgK]$, $Pr = 0.7$.

Table 2: Constants in the manufactured solutions and their representation in C codes.

Constant	Representation in C	Constant	Representation in C
L	L	L_t	Lt
$a_{\rho x}$	a_rhox	a_{ux}	a_ux
$a_{\rho y}$	a_rhoy	a_{uy}	a_uy
$a_{\rho z}$	a_rhoz	a_{uz}	a_uz
$a_{\rho t}$	a_rhot	a_{ut}	a_ut
ρ_0	rho_0	u_0	u_0
ρ_x	rho_x	u_x	u_x
ρ_y	rho_y	u_y	u_y
ρ_z	rho_z	u_z	u_z
ρ_t	rho_t	u_t	u_t
a_{px}	a_px	a_{vx}	a_vx
a_{py}	a_py	a_{vy}	a_vy
a_{pz}	a_pz	a_{vz}	a_vz
a_{pt}	a_pt	a_{vt}	a_vt
p_0	p_0	v_0	v_0
p_x	p_x	v_x	v_x
p_y	p_y	v_y	v_y
p_z	p_z	v_z	v_z
p_t	p_t	v_t	v_t
$a_{\phi x}$	a_phix	a_{wx}	a_wx
$a_{\phi y}$	a_phiy	a_{wy}	a_wy
$a_{\phi z}$	a_phiz	a_{wz}	a_wz
$a_{\phi t}$	a_phit	a_{wt}	a_wt
ϕ_0	phi_0	w_0	w_0
ϕ_x	phi_x	w_x	w_x
ϕ_y	phi_y	w_y	w_y
ϕ_z	phi_z	w_z	w_z
ϕ_t	phi_t	w_t	w_t