Manufactured Solution for the 2D Favre-Averaged Navier–Stokes Equations with Spalart-Allmaras turbulence model using Maple

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February 11, 2011

Abstract

The Method of Manufactured Solutions is a valuable approach for code verification, providing means to verify how accurately the numerical method solves the partial differential equations of interest. This document presents the source terms generated by the application of the Method of Manufactured Solutions on the 2D transient Favre-Averaged Navier–Stokes Equations with Spalart-Allmaras turbulence model using the analytical manufactured solutions for density, velocity and pressure presented by Roy et al. (2002).

1 Mathematical Model

Turbulent flows occur at high Reynolds numbers, when the inertia of the fluid overwhelms the viscosity of the fluid, causing the laminar flow motions to become unstable. Under these conditions, the flow is characterized by rapid fluctuations in pressure and velocity which are inherently three dimensional and unsteady. Turbulent flow is composed of large eddies that migrate across the flow generating smaller eddies as they go. These smaller eddies in turn generates smaller eddies until they become small enough that their energy is dissipated due to the presence of molecular viscosity.

In practice, the effect of this sensitivity is to make the value of any flow quantity at any particular point in time and space uncertain. Thus, these quantities may be viewed as random variables with associated probability density functions, allowing the use of statistical techniques in the description and analysis of the flow. Or, in other words, the full influence of the turbulent fluctuations on the mean flow must be modelled.

For flows with significant density variations it is possible to capture the turbulent effects using the Favre averaged Navier-Stokes equations (FANS), together with the one-equation Spalart-Allmaras (SA) turbulent model (Oliver, 2010):

Mass conservation:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\bar{\rho}\tilde{u}_i) = 0, \tag{1}$$

Momentum conservation:

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{u}_i\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\tilde{u}_i\right) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2(\bar{\mu}+\mu_t)\tilde{S}_{ji}\right), \tag{2}$$

Total energy conservation:

$$\frac{\partial}{\partial t}\left[\bar{\rho}\left(\tilde{e}+\frac{1}{2}\tilde{u}_i\tilde{u}_i\right)\right] + \frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_j\left(\tilde{h}+\frac{1}{2}\tilde{u}_i\tilde{u}_i\right)\right] = \frac{\partial}{\partial x_j}\left(2(\bar{\mu}+\mu_t)\tilde{S}_{ji}\tilde{u}_i\right) + \frac{\partial}{\partial x_j}\left[\left(\frac{\bar{\mu}}{\mathrm{Pr}}+\frac{\mu_t}{\mathrm{Pr}_t}\right)\frac{\partial \tilde{h}}{\partial x_j}\right], \tag{3}$$

Baseline compressible Spalart-Allmaras equation:

$$\frac{\partial}{\partial t}(\bar{\rho}\nu_{\mathrm{sa}}) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j\nu_{\mathrm{sa}}) = c_{b1}S_{\mathrm{sa}}\bar{\rho}\nu_{\mathrm{sa}} - c_{w1}f_w\bar{\rho}\left(\frac{\nu_{\mathrm{sa}}}{d}\right)^2 + \frac{1}{\sigma}\frac{\partial}{\partial x_k}\left[(\bar{\mu}+\bar{\rho}\nu_{\mathrm{sa}})\frac{\partial \nu_{\mathrm{sa}}}{\partial x_k}\right] + \frac{c_{b2}}{\sigma}\bar{\rho}\frac{\partial \nu_{\mathrm{sa}}}{\partial x_k}\frac{\partial \nu_{\mathrm{sa}}}{\partial x_k}, \tag{4}$$

where $[\tilde{\ }]$ denotes a Favre-averaging variable and $[\bar{\ }]$ denotes Reynolds averaging.

To close the equations, many additional relationships are necessary—e.g., a constitutive relation for the viscous stress, an equation of state, etc. In this work, the gas is considered calorically perfect and:

$$
\begin{aligned}
&\bar{\mu} = \mu_0 \left(\frac{\tilde{T}}{T_0}\right)^{3/2} \frac{T_0 + S}{\tilde{T} + S}, \quad \tilde{S}_{ij} = \tilde{s}_{ij} - \frac{1}{3}\tilde{s}_{kk}\delta_{ij}, \quad \tilde{s}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right), \\
&\bar{p} = \bar{\rho} R \tilde{T}, \quad \tilde{e} = c_v \tilde{T}, \quad \tilde{h} = c_p \tilde{T} = \tilde{e} + \frac{\bar{p}}{\bar{\rho}} \\
&\mu_t = \bar{\rho}\nu_t = \bar{\rho}\nu_{\rm sa} f_{v1}, \quad S_{\rm sa} = \Omega + \frac{\nu_{\rm sa}}{\kappa^2 d^2} f_{v2}, \quad \Omega = \sqrt{2\tilde{\Omega}_{ij}\tilde{\Omega}_{ij}}, \quad \tilde{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i}\right), \\
&f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}, \quad f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}, \quad \chi = \frac{\nu_{\rm sa}}{\tilde{\nu}}, \\
&f_w = g\left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right)^{1/6}, \quad g = r + c_{w2}\left(r^6 - r\right), \quad r = \frac{\nu_{\rm sa}}{S_{\rm sa}\kappa^2 d^2}.
\end{aligned}
\tag{5}
$$

where d is the distance to the nearest no slip wall.

The constants c_v and c_p are fluid properties. The constants μ_0, T_0, S, are the calibration parameters appearing in Sutherland's law, and c_{b1}, c_{b2}, c_{v1}, σ, c_{w1}, c_{w2}, c_{w3}, and κ are the SA model calibration parameters.

Note: In this work the averaged absolute viscosity is considered constant, $\bar{\mu} = \texttt{constant}$ and the distance to the nearest no-slip wall is assumed to be infinite, $d \to \infty$. Due to such assumptions, the simplifications carried out in the governing equations as well as in the closure expressions are presented in Section 2.1.

2 Manufactured Solution

The Method of Manufactured Solutions (MMS) applied to Favre-Averaged Navier–Stokes equations with baseline compressible Spalart-Allmaras turbulence model consists in modifying Equations (1) – (4) by adding a source term to the right-hand side of each equation, so the modified set of equations conveniently has the analytical solution chosen *a priori*.

Roy et al. (2002) introduce the general form of the two-dimensional primitive manufactured solution variables to be a function of sines and cosines in x and y. In this work, Roy et al. (2002)'s manufactured solutions are modified in order to address temporal accuracy as well:

$$
\phi(x,y,t) = \phi_0 + \phi_x\, f_s\left(\frac{a_{\phi x}\pi x}{L}\right) + \phi_y\, f_s\left(\frac{a_{\phi y}\pi y}{L}\right) + \phi_t\, f_s\left(\frac{a_{\phi t}\pi t}{L}\right), \tag{6}
$$

where $\phi = \rho, u, v, p$ or $\nu_{\rm sa}$, and $f_s(\cdot)$ functions denote either sine or cosine function. Note that in this case, ϕ_x, ϕ_y and ϕ_t are constants and the subscripts do not denote differentiation.

The manufactured analytical solutions (6) for each one of the variables in two-dimensional case of FANS equations with SA turbulence model are:

$$
\begin{aligned}
\rho\,(x,y,t) &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y \cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_t \sin\left(\frac{a_{\rho t}\pi t}{L}\right), \\
u\,(x,y,t) &= u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y \cos\left(\frac{a_{uy}\pi y}{L}\right) + u_t \cos\left(\frac{a_{ut}\pi t}{L}\right), \\
v\,(x,y,t) &= v_0 + v_x \cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y \sin\left(\frac{a_{vy}\pi y}{L}\right) + v_t \sin\left(\frac{a_{vt}\pi t}{L}\right), \\
p\,(x,y,t) &= p_0 + p_x \cos\left(\frac{a_{px}\pi x}{L}\right) + p_y \sin\left(\frac{a_{py}\pi y}{L}\right) + p_t \cos\left(\frac{a_{pt}\pi t}{L}\right), \\
\nu_{\rm sa}(x,y,t) &= \nu_{\rm sa0} + \nu_{{\rm sa}x} \cos\left(\frac{a_{\nu_{\rm sa}x}\pi x}{L}\right) + \nu_{{\rm sa}y} \cos\left(\frac{a_{\nu_{\rm sa}y}\pi y}{L}\right) + \nu_{{\rm sa}t} \cos\left(\frac{a_{\nu_{\rm sa}t}\pi t}{L}\right).
\end{aligned}
\tag{7}
$$

Source terms for mass conservation (Q_ρ), momentum (Q_u, and Q_v), total energy (Q_E) and SA variable ($Q_{\nu_{\rm sa}}$) equations are obtained by symbolic manipulations of FANS equations with SA turbulence model above using Maple 13 (Maplesoft, 2010) and are presented in the following sections.

2.1 2D FANS equations and SA turbulence model

MMS applied to the 2D transient FANS equations with SA turbulent model simply consists in modifying Equations (1) – (4) by adding a source term to the right-hand side of each equation:

$$
\begin{aligned}
&\frac{\partial(\bar{\rho})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{\boldsymbol{u}}) = Q_{\bar{\rho}},\\
&\frac{\partial(\bar{\rho}\tilde{\boldsymbol{u}})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}) + \nabla\bar{p} - \nabla \cdot (2(\bar{\mu}+\mu_t)\tilde{\boldsymbol{S}}) = Q_{\tilde{\boldsymbol{u}}},\\
&\frac{\partial(\bar{\rho}\tilde{E})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{\boldsymbol{u}}\tilde{H}) - \nabla \cdot \bar{\boldsymbol{q}} - \nabla \cdot (2(\bar{\mu}+\mu_t)\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{u}}) = Q_{\tilde{E}},\\
&\frac{\partial(\bar{\rho}\nu_{\rm sa})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{\boldsymbol{u}}\nu_{\rm sa}) - c_{b1}S_{\rm sa}\bar{\rho}\nu_{\rm sa} - \frac{1}{\sigma}\nabla \cdot ((\bar{\mu}+\bar{\rho}\nu_{\rm sa})\nabla\nu_{\rm sa}) - \frac{c_{b2}\bar{\rho}}{\sigma}\nabla\nu_{\rm sa}\cdot\nabla\nu_{\rm sa} = Q_{\nu_{\rm sa}}
\end{aligned}
\tag{8}
$$

so the modified set of Equations (8) has Equation (7) as analytical solution. Note that in Equation (8) it is assumed that the distance to the neartest no slip wall is sufficiently large, i.e., $d \to \infty$.

Recall that the averaged kinematic viscosity, total energy per unit mass and the total enthalpy per unit mass are give, respectively, by:

$$
\tilde{\nu} = \frac{\bar{\mu}}{\bar{\rho}}, \qquad \tilde{E} = \tilde{e} + \frac{\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}}{2}, \quad \tilde{H} = \tilde{h} + \frac{\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}}{2} \tag{9}
$$

with $\tilde{e}$ and $\tilde{h}$ defined in Equation (5) and $\bar{\mu}$ is the averaged absolute viscosity. The laminar mean heat-flux vector $\bar{\boldsymbol{q}} = (\bar{q}_x, \bar{q}_y)$ is given by:

$$
\bar{q}_x = \left(\frac{\bar{\mu}}{\mathrm{Pr}} + \frac{\mu_t}{\mathrm{Pr}_t}\right)\frac{\partial\tilde{h}}{\partial x} \quad \text{and} \quad \bar{q}_y = \left(\frac{\bar{\mu}}{\mathrm{Pr}} + \frac{\mu_t}{\mathrm{Pr}_t}\right)\frac{\partial\tilde{h}}{\partial y} \tag{10}
$$

where the Prandtl number Pr and the turbulent Prandtl number Pr_t are assumed to be constant.

Since the fluid is considered to have constant viscosity and the distance to the neartest no slip wall is assumed to be sufficient large, i.e., $d \to \infty$, the expressions in (5) are simplified accordingly:

$$
\bar{\mu} = \text{constant}, \quad \bar{p} = \bar{\rho}R\tilde{T}, \quad \tilde{e} = c_v\tilde{T}, \quad \tilde{h} = c_p\tilde{T} = \tilde{e} + \frac{\bar{p}}{\bar{\rho}} \tag{11}
$$

$$
\mu_t = \bar{\rho}\nu_t = \bar{\rho}\nu_{\rm sa}f_{v1}, \quad f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}, \quad \chi = \frac{\nu_{\rm sa}}{\tilde{\nu}}, \tag{12}
$$

$$
S_{\rm sa} = \Omega, \quad \Omega = \sqrt{\left(\frac{\partial\tilde{u}}{\partial y} - \frac{\partial\tilde{v}}{\partial x}\right)^2} \tag{13}
$$

$$
\tilde{S}_{xx} = \frac{\partial\tilde{u}}{\partial x} - \frac{1}{3}\nabla\cdot\tilde{\boldsymbol{u}}, \quad \tilde{S}_{yy} = \frac{\partial\tilde{v}}{\partial y} - \frac{1}{3}\nabla\cdot\tilde{\boldsymbol{u}}, \quad \tilde{S}_{xy} = \tilde{S}_{yx} = \left(\frac{\partial\tilde{u}}{\partial y} + \frac{\partial\tilde{v}}{\partial x}\right), \tag{14}
$$

with

$$
\tilde{\boldsymbol{S}} = \left[\begin{array}{cc} \tilde{S}_{xx} & \tilde{S}_{xy} \\ \tilde{S}_{yx} & \tilde{S}_{yy} \end{array}\right]. \tag{15}
$$

Source terms $Q_{\bar{\rho}}$, $Q_{\tilde{u}}$, $Q_{\tilde{v}}$, $Q_{\tilde{E}}$ and $Q_{\nu_{\rm sa}}$ are presented in the subsequent sessions with the use of the auxiliary variables:

$$
\begin{aligned}
\texttt{Rho} &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y \cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_t \sin\left(\frac{a_{\rho t}\pi t}{L}\right),\\
\texttt{U} &= u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y \cos\left(\frac{a_{uy}\pi y}{L}\right) + u_t \cos\left(\frac{a_{ut}\pi t}{L}\right),\\
\texttt{V} &= v_0 + v_x \cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y \sin\left(\frac{a_{vy}\pi y}{L}\right) + v_t \sin\left(\frac{a_{vt}\pi t}{L}\right),\\
\texttt{P} &= p_0 + p_x \cos\left(\frac{a_{px}\pi x}{L}\right) + p_y \sin\left(\frac{a_{py}\pi y}{L}\right) + p_t \cos\left(\frac{a_{pt}\pi t}{L}\right),\\
\texttt{Nu}_{\texttt{sa}} &= \nu_{\rm sa0} + \nu_{{\rm sa}x} \cos\left(\frac{a_{\nu_{\rm sa}x}\pi x}{L}\right) + \nu_{{\rm sa}y} \cos\left(\frac{a_{\nu_{\rm sa}y}\pi y}{L}\right) + \nu_{{\rm sa}t} \cos\left(\frac{a_{\nu_{\rm sa}t}\pi t}{L}\right).
\end{aligned}
\tag{16}
$$

2.1.1 2D FANS Mass Conservation

The 2D mass conservation equation written as an operator is:

$$\mathcal{L} = \frac{\partial(\bar{\rho})}{\partial t} + \frac{\partial(\bar{\rho}\tilde{u})}{\partial x} + \frac{\partial(\bar{\rho}\tilde{v})}{\partial y}.$$

Analytically differentiating Equation (7) for $\bar{\rho}$, $\tilde{u}$ and $\tilde{v}$ using operator $\mathcal{L}$ defined above gives the source term $Q_{\bar{\rho}}$:

$$\begin{aligned}
Q_{\bar{\rho}} = &\frac{a_{\rho x}\pi\rho_x\,\mathtt{U}}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\
&- \frac{a_{\rho y}\pi\rho_y\,\mathtt{V}}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) + \\
&+ \frac{\pi\,\mathtt{Rho}}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
&+ \frac{a_{\rho t}\pi\rho_t}{L}\cos\left(\frac{a_{\rho t}\pi t}{L}\right).
\end{aligned} \tag{17}$$

where $\mathtt{Rho}$, $\mathtt{U}$ and $\mathtt{V}$ are given in Equation (16).

2.1.2 2D FANS Momentum Conservation

For the generation of the analytical source term $Q_{\tilde{u}}$ for the x-momentum equation, the first component of Equation (2) is written as an operator $\mathcal{L}$:

$$\mathcal{L} = \frac{\partial(\bar{\rho}\tilde{u})}{\partial t} + \frac{\partial(\bar{\rho}\tilde{u}^2)}{\partial x} + \frac{\partial(\bar{\rho}\tilde{u}\tilde{v})}{\partial y} + \frac{\partial(\bar{p})}{\partial x} - \frac{\partial(2(\bar{\mu}+\mu_t)\tilde{S}_{xx})}{\partial x} - \frac{\partial(2(\bar{\mu}+\mu_t)\tilde{S}_{xy})}{\partial y},$$

which, when operated in Equation (7), provides source term $Q_{\tilde{u}}$:

$$\begin{aligned}
Q_{\tilde{u}} = &\frac{a_{\rho x}\pi\rho_x\,\mathtt{U}^2}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\
&- \frac{a_{\rho y}\pi\rho_y\,\mathtt{U}\,\mathtt{V}}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) + \\
&- \frac{a_{uy}\pi u_y\,\mathtt{Rho}\,\mathtt{V}}{L}\sin\left(\frac{a_{uy}\pi y}{L}\right) + \\
&+ \frac{\pi\,\mathtt{Rho}\,\mathtt{U}}{L}\left[2a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
&- \frac{a_{px}\pi p_x}{L}\sin\left(\frac{a_{px}\pi x}{L}\right) + \\
&+ \frac{f_{v1}\pi^2\,\mathtt{Rho}\,\mathtt{Nu_{sa}}}{L^2}\left[{}^4\!/_3\,a_{ux}^2u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + a_{uy}^2u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)\right] + \\
&+ \frac{f_{v1}\pi^2\,\mathtt{Rho}}{L^2}\left[{}^4\!/_3\,a_{ux}a_{\nu_{sa}x}u_x\nu_{sax}\cos\left(\frac{a_{ux}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{sa}x}\pi x}{L}\right) - a_{uy}a_{\nu_{sa}y}u_y\nu_{say}\sin\left(\frac{a_{uy}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{sa}y}\pi y}{L}\right) + \right.\\
&\qquad \left. - a_{vx}a_{\nu_{sa}y}v_x\nu_{say}\sin\left(\frac{a_{vx}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{sa}y}\pi y}{L}\right) - {}^2\!/_3\,a_{vy}a_{\nu_{sa}x}v_y\nu_{sax}\cos\left(\frac{a_{vy}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{sa}x}\pi x}{L}\right)\right] + \\
&+ \frac{f_{v1}\pi^2\,\mathtt{Nu_{sa}}}{L^2}\left[-{}^4\!/_3\,a_{\rho x}a_{ux}\rho_x u_x\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\cos\left(\frac{a_{ux}\pi x}{L}\right) + {}^2\!/_3\,a_{\rho x}a_{vy}\rho_x v_y\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\cos\left(\frac{a_{vy}\pi y}{L}\right) + \right.\\
&\qquad \left. - a_{\rho y}a_{uy}\rho_y u_y\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\sin\left(\frac{a_{uy}\pi y}{L}\right) - a_{\rho y}a_{vx}\rho_y v_x\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\sin\left(\frac{a_{vx}\pi x}{L}\right)\right] + \\
&+ \frac{\pi^2\bar{\mu}}{3L^2}\left[\frac{c_{v1}^3}{\chi^3+c_{v1}^3} + f_{v1}\right]\left[4a_{ux}^2u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + 3a_{uy}^2u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)\right] + \\
&+ \frac{a_{\rho t}\pi\rho_t\,\mathtt{U}}{L}\cos\left(\frac{a_{\rho t}\pi t}{L}\right) + \\
&- \frac{a_{ut}\pi u_t\,\mathtt{Rho}}{L}\sin\left(\frac{a_{ut}\pi t}{L}\right),
\end{aligned} \tag{18}$$

where

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \quad \text{and} \quad \chi = \frac{\texttt{Nu}_{\texttt{sa}}}{\tilde{\nu}} = \frac{\texttt{Rho}\,\texttt{Nu}_{\texttt{sa}}}{\bar{\mu}}, \tag{19}$$

and Rho, U, V and $\texttt{Nu}_{\texttt{sa}}$ are given in Equation (16).

Analogously, for the generation of the analytical source term $Q_{\tilde{v}}$ for the y-momentum equation, the second component of Equation (2) is written as an operator $\mathcal{L}$:

$$\mathcal{L} = \frac{\partial(\bar{\rho}\tilde{v})}{\partial t} + \frac{\partial(\bar{\rho}\tilde{u}\tilde{v})}{\partial x} + \frac{\partial(\bar{\rho}\tilde{v}^2)}{\partial y} + \frac{\partial(\bar{p})}{\partial y} - \frac{\partial(2(\bar{\mu}+\mu_t)\tilde{S}_{yx})}{\partial x} - \frac{\partial(2(\bar{\mu}+\mu_t)\tilde{S}_{yy})}{\partial y},$$

and then applied to Equation (7). It yields:

$$\begin{aligned}
Q_{\tilde{v}} = & \frac{a_{\rho x}\pi\rho_x\,\texttt{U}\,\texttt{V}}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\
& - \frac{a_{\rho y}\pi\rho_y\,\texttt{V}^2}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) + \\
& - \frac{a_{vx}\pi v_x\,\texttt{Rho}\,\texttt{U}}{L}\sin\left(\frac{a_{vx}\pi x}{L}\right) + \\
& + \frac{\pi\,\texttt{Rho}\,\texttt{V}}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + 2a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& + \frac{a_{py}\pi p_y}{L}\cos\left(\frac{a_{py}\pi y}{L}\right) + \\
& + \frac{f_{v1}\pi^2\,\texttt{Rho}\,\texttt{Nu}_{\texttt{sa}}}{L^2}\left[a_{vx}^2 v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + {}^4\!/_3\, a_{vy}^2 v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& + \frac{f_{v1}\pi^2\,\texttt{Rho}}{L^2}\Bigg[-{}^2\!/_3\, a_{ux}a_{\nu_{sa}y}u_x\nu_{say}\cos\left(\frac{a_{ux}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{sa}y}\pi y}{L}\right) - a_{uy}a_{\nu_{sa}x}u_y\nu_{sax}\sin\left(\frac{a_{uy}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{sa}x}\pi x}{L}\right) + \\
& \quad - a_{vx}a_{\nu_{sa}x}v_x\nu_{sax}\sin\left(\frac{a_{vx}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{sa}x}\pi x}{L}\right) + {}^4\!/_3\, a_{vy}a_{\nu_{sa}y}v_y\nu_{say}\cos\left(\frac{a_{vy}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{sa}y}\pi y}{L}\right)\Bigg] + \\
& + \frac{f_{v1}\pi^2\,\texttt{Nu}_{\texttt{sa}}}{L^2}\Bigg[a_{\rho x}a_{uy}\rho_x u_y\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\sin\left(\frac{a_{uy}\pi y}{L}\right) + a_{\rho x}a_{vx}\rho_x v_x\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\sin\left(\frac{a_{vx}\pi x}{L}\right) + \\
& \quad -{}^2\!/_3\, a_{\rho y}a_{ux}\rho_y u_x\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\cos\left(\frac{a_{ux}\pi x}{L}\right) + {}^4\!/_3\, a_{\rho y}a_{vy}\rho_y v_y\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\cos\left(\frac{a_{vy}\pi y}{L}\right)\Bigg] + \\
& + \frac{\pi^2\bar{\mu}}{3L^2}\left[\frac{c_{v1}^3}{\chi^3 + c_{v1}^3} + f_{v1}\right]\left[3a_{vx}^2 v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + 4a_{vy}^2 v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& + \frac{a_{\rho t}\pi\rho_t\,\texttt{V}}{L}\cos\left(\frac{a_{\rho t}\pi t}{L}\right) + \\
& + \frac{a_{vt}\pi v_t\,\texttt{Rho}}{L}\cos\left(\frac{a_{vt}\pi t}{L}\right),
\end{aligned} \tag{20}$$

where χ and f_{v1} are given in Equation (19), and Rho, U, V and $\texttt{Nu}_{\texttt{sa}}$ are given in Equation (16).

2.1.3 2D FANS Total Energy Conservation

The operator for the 2D Navier–Stokes total energy is:

$$\begin{aligned}
\mathcal{L} = & \frac{\partial(\bar{\rho}\tilde{E})}{\partial t} + \frac{\partial(\bar{\rho}\tilde{u}\tilde{E})}{\partial x} + \frac{\partial(\bar{\rho}\tilde{v}\tilde{E})}{\partial y} + \frac{\partial(\bar{p}\tilde{u})}{\partial x} + \frac{\partial(\bar{p}\tilde{v})}{\partial y} + \frac{\partial(\bar{q}_x)}{\partial x} + \frac{\partial(\bar{q}_y)}{\partial y} + \\
& - \frac{\partial(2(\bar{\mu}+\mu_t)\tilde{u}\tilde{S}_{xx})}{\partial x} - \frac{\partial(2(\bar{\mu}+\mu_t)\tilde{v}\tilde{S}_{xy})}{\partial x} - \frac{\partial(2(\bar{\mu}+\mu_t)\tilde{u}\tilde{S}_{yx})}{\partial y} - \frac{\partial(2(\bar{\mu}+\mu_t)\tilde{v}\tilde{S}_{yy})}{\partial y}.
\end{aligned}$$

Source term $Q_{\tilde{E}}$ is obtained by operating $\mathcal{L}$ on Equation (7) together with the use of the auxiliary relations for energy given in Equations (11), (12) and (14). It yields:

$$
\begin{aligned}
Q_{\tilde{E}} = {} & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U}(\mathtt{U}^2+\mathtt{V}^2)}{2L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) - \frac{a_{\rho y}\pi\rho_y\,\mathtt{V}(\mathtt{U}^2+\mathtt{V}^2)}{2L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) + \frac{a_{\rho t}\pi\rho_t(\mathtt{U}^2+\mathtt{V}^2)}{2L}\cos\left(\frac{a_{\rho t}\pi t}{L}\right) + \\
& - \frac{c_p a_{px}\pi p_x\,\mathtt{U}}{LR}\sin\left(\frac{a_{px}\pi x}{L}\right) + \frac{c_p a_{py}\pi p_y\,\mathtt{V}}{LR}\cos\left(\frac{a_{py}\pi y}{L}\right) + \\
& + \frac{c_p\pi\,\mathtt{P}}{LR}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& - \frac{a_{ut}\pi u_t\,\mathtt{Rho}\,\mathtt{U}}{L}\sin\left(\frac{a_{ut}\pi t}{L}\right) + \frac{a_{vt}\pi v_t\,\mathtt{Rho}\,\mathtt{V}}{L}\cos\left(\frac{a_{vt}\pi t}{L}\right) + \\
& + \frac{\pi^2\,\mathtt{U}\mu_t}{L^2}\left[{}^4\!/_3\, a_{ux}^2 u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + a_{uy}^2 u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)\right] + \\
& + \frac{\pi^2\,\mathtt{V}\mu_t}{L^2}\left[a_{vx}^2 v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + {}^4\!/_3\, a_{vy}^2 v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& + \frac{\pi^2\mu_t}{L^2}\left[-{}^4\!/_3\, a_{ux}^2 u_x^2\cos\left(\frac{a_{ux}\pi x}{L}\right)^2 + {}^4\!/_3\, a_{ux}a_{vy}u_x v_y\cos\left(\frac{a_{ux}\pi x}{L}\right)\cos\left(\frac{a_{vy}\pi y}{L}\right) - a_{uy}^2 u_y^2\sin\left(\frac{a_{uy}\pi y}{L}\right)^2 + \right. \\
& \qquad \left. -2a_{uy}a_{vx}u_y v_x\sin\left(\frac{a_{uy}\pi y}{L}\right)\sin\left(\frac{a_{vx}\pi x}{L}\right) - a_{vx}^2 v_x^2\sin\left(\frac{a_{vx}\pi x}{L}\right)^2 - {}^4\!/_3\, a_{vy}^2 v_y^2\cos\left(\frac{a_{vy}\pi y}{L}\right)^2\right] + \\
& + \frac{\pi^2 f_{v1}\,\mathtt{Rho}\,\mathtt{U}}{L^2}\left[{}^4\!/_3\, a_{ux}a_{\nu_{\mathrm{sa}}x}u_x\nu_{sax}\cos\left(\frac{a_{ux}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}x}\pi x}{L}\right) - a_{uy}a_{\nu_{\mathrm{sa}}y}u_y\nu_{say}\sin\left(\frac{a_{uy}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}y}\pi y}{L}\right) + \right. \\
& \qquad \left. -a_{vx}a_{\nu_{\mathrm{sa}}y}v_x\nu_{say}\sin\left(\frac{a_{vx}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}y}\pi y}{L}\right) - {}^2\!/_3\, a_{vy}a_{\nu_{\mathrm{sa}}x}v_y\nu_{sax}\cos\left(\frac{a_{vy}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}x}\pi x}{L}\right)\right] + \\
& + \frac{\pi^2 f_{v1}\,\mathtt{Rho}\,\mathtt{V}}{L^2}\left[-{}^2\!/_3\, a_{ux}a_{\nu_{\mathrm{sa}}y}u_x\nu_{say}\cos\left(\frac{a_{ux}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}y}\pi y}{L}\right) - a_{uy}a_{\nu_{\mathrm{sa}}x}u_y\nu_{sax}\sin\left(\frac{a_{uy}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}x}\pi x}{L}\right) + \right. \\
& \qquad \left. -a_{vx}a_{\nu_{\mathrm{sa}}x}v_x\nu_{sax}\sin\left(\frac{a_{vx}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}x}\pi x}{L}\right) + {}^4\!/_3\, a_{vy}a_{\nu_{\mathrm{sa}}y}v_y\nu_{say}\cos\left(\frac{a_{vy}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}y}\pi y}{L}\right)\right] + \\
& + \frac{\pi^2 f_{v1}\,\mathtt{U}\,\mathtt{Nu_{sa}}}{L^2}\left[-{}^4\!/_3\, a_{\rho x}a_{ux}\rho_x u_x\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\cos\left(\frac{a_{ux}\pi x}{L}\right) + {}^2\!/_3\, a_{\rho x}a_{vy}\rho_x v_y\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\cos\left(\frac{a_{vy}\pi y}{L}\right) + \right. \\
& \qquad \left. -a_{\rho y}a_{uy}\rho_y u_y\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\sin\left(\frac{a_{uy}\pi y}{L}\right) - a_{\rho y}a_{vx}\rho_y v_x\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\sin\left(\frac{a_{vx}\pi x}{L}\right)\right] + \\
& + \frac{\pi^2 f_{v1}\,\mathtt{V}\,\mathtt{Nu_{sa}}}{L^2}\left[a_{\rho x}a_{uy}\rho_x u_y\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\sin\left(\frac{a_{uy}\pi y}{L}\right) + a_{\rho x}a_{vx}\rho_x v_x\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\sin\left(\frac{a_{vx}\pi x}{L}\right) + \right. \\
& \qquad \left. -{}^2\!/_3\, a_{\rho y}a_{ux}\rho_y u_x\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\cos\left(\frac{a_{ux}\pi x}{L}\right) + {}^4\!/_3\, a_{\rho y}a_{vy}\rho_y v_y\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\cos\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& + \frac{c_v a_{\rho t}\pi\rho_t\,\mathtt{P}}{LR\,\mathtt{Rho}}\cos\left(\frac{a_{\rho t}\pi t}{L}\right) + \\
& - \frac{\pi^2\bar{\mu}}{3L^2}\left[\frac{c_{v1}^3}{\chi^3+c_{v1}^3} + f_{v1}\right]\left[4a_{ux}^2u_x^2\cos\left(\frac{a_{ux}\pi x}{L}\right)^2 - 4a_{ux}a_{vy}u_x v_y\cos\left(\frac{a_{ux}\pi x}{L}\right)\cos\left(\frac{a_{vy}\pi y}{L}\right) + 3a_{uy}^2u_y^2\sin\left(\frac{a_{uy}\pi y}{L}\right)^2 + \right. \\
& \qquad \left. +6a_{uy}a_{vx}u_y v_x\sin\left(\frac{a_{uy}\pi y}{L}\right)\sin\left(\frac{a_{vx}\pi x}{L}\right) + 3a_{vx}^2v_x^2\sin\left(\frac{a_{vx}\pi x}{L}\right)^2 + 4a_{vy}^2v_y^2\cos\left(\frac{a_{vy}\pi y}{L}\right)^2\right] + \\
& + \frac{\pi^2\bar{\mu}\,\mathtt{U}}{3L^2}\left[\frac{c_{v1}^3}{\chi^3+c_{v1}^3} + f_{v1}\right]\left[4a_{ux}^2u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + 3a_{uy}^2u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)\right] + \\
& + \frac{\pi^2\bar{\mu}\,\mathtt{V}}{3L^2}\left[\frac{c_{v1}^3}{\chi^3+c_{v1}^3} + f_{v1}\right]\left[3a_{vx}^2v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + 4a_{vy}^2v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& + \frac{\pi\,\mathtt{Rho}\,\mathtt{U}^2}{2L}\left[3a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& - \frac{\pi\,\mathtt{Rho}\,\mathtt{U}\,\mathtt{V}}{L}\left[a_{uy}u_y\sin\left(\frac{a_{uy}\pi y}{L}\right) + a_{vx}v_x\sin\left(\frac{a_{vx}\pi x}{L}\right)\right] + \\
& + \frac{\pi\,\mathtt{Rho}\,\mathtt{V}^2}{2L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + 3a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{c_p\pi^2}{L^2 R\,\mathtt{Rho}\,\mathtt{Nu_{sa}}}\frac{\mu_t}{\mathrm{Pr}_t}\left[a_{px}a_{\nu_{\mathrm{sa}}x}p_x\nu_{sax}\sin\left(\frac{a_{px}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}x}\pi x}{L}\right)-a_{py}a_{\nu_{\mathrm{sa}}y}p_y\nu_{say}\cos\left(\frac{a_{py}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}y}\pi y}{L}\right)\right]+\\
&-\frac{c_p\pi^2\,\mathtt{P}}{L^2 R\,\mathtt{Rho}^2\,\mathtt{Nu_{sa}}}\frac{\mu_t}{\mathrm{Pr}_t}\left[a_{\rho x}a_{\nu_{\mathrm{sa}}x}\rho_x\nu_{sax}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}x}\pi x}{L}\right)-a_{\rho y}a_{\nu_{\mathrm{sa}}y}\rho_y\nu_{say}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{\mathrm{sa}}y}\pi y}{L}\right)\right]+\\
&-\frac{c_p\pi^2}{L^2 R\,\mathtt{Rho}^2}\left[\frac{\mu_t}{\mathrm{Pr}_t}+\frac{2\bar\mu}{\mathrm{Pr}}\right]\left[a_{px}a_{\rho x}\rho_x p_x\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\sin\left(\frac{a_{px}\pi x}{L}\right)+a_{py}a_{\rho y}\rho_y p_y\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\cos\left(\frac{a_{py}\pi y}{L}\right)\right]+\\
&-\frac{c_p\pi^2\,\mathtt{P}}{L^2 R\,\mathtt{Rho}^3}\left[\frac{\mu_t}{\mathrm{Pr}_t}+\frac{2\bar\mu}{\mathrm{Pr}}\right]\left[a_{\rho x}^2\rho_x^2\cos\left(\frac{a_{\rho x}\pi x}{L}\right)^2+a_{\rho y}^2\rho_y^2\sin\left(\frac{a_{\rho y}\pi y}{L}\right)^2\right]+\\
&+\frac{c_p\pi^2}{L^2 R\,\mathtt{Rho}}\left[\frac{\mu_t}{\mathrm{Pr}_t}+\frac{\bar\mu}{\mathrm{Pr}}\right]\left[a_{px}^2p_x\cos\left(\frac{a_{px}\pi x}{L}\right)+a_{py}^2p_y\sin\left(\frac{a_{py}\pi y}{L}\right)\right]+\\
&-\frac{c_p\pi^2\,\mathtt{P}}{L^2 R\,\mathtt{Rho}^2}\left[\frac{\mu_t}{\mathrm{Pr}_t}+\frac{\bar\mu}{\mathrm{Pr}}\right]\left[a_{\rho x}^2\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+a_{\rho y}^2\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)\right],
\end{aligned}
\tag{21}
$$

where `Rho`, `P`, `U`, `V` and $\mathtt{Nu_{sa}}$ are given in Equation (16), χ and f_{v1} are given in Equation (19), and

$$\mu_t = f_{v1}\,\mathtt{Rho}\,\mathtt{Nu_{sa}}.$$

2.1.4 2D SA Transport Equation

The operator for the viscosity-like baseline compressible Spalart-Allmaras equation is:

$$
\begin{aligned}
\mathcal{L} = &\frac{\partial(\bar\rho\nu_{\mathrm{sa}})}{\partial t}+\frac{\partial(\bar\rho\tilde u\nu_{\mathrm{sa}})}{\partial x}+\frac{\partial(\bar\rho\tilde v\nu_{\mathrm{sa}})}{\partial y}-c_{b1}S_{\mathrm{sa}}\bar\rho\nu_{\mathrm{sa}}+\\
&+\frac{1}{\sigma}\left[\frac{\partial}{\partial x}\left((\bar\mu+\bar\rho\nu_{\mathrm{sa}})\frac{\partial\nu_{\mathrm{sa}}}{\partial x}\right)+\frac{\partial}{\partial y}\left((\bar\mu+\bar\rho\nu_{\mathrm{sa}})\frac{\partial\nu_{\mathrm{sa}}}{\partial y}\right)\right]+\frac{c_{b2}\bar\rho}{\sigma}\left[\left(\frac{\partial\nu_{\mathrm{sa}}}{\partial x}\right)^2+\left(\frac{\partial\nu_{\mathrm{sa}}}{\partial y}\right)^2\right]
\end{aligned}
$$

Source term $Q_{\nu_{\mathrm{sa}}}$ is obtained by operating $\mathcal{L}$ on Equation (7) together with the use of the auxiliary relations for energy given in Equations (11) – (13). It yields:

$$
\begin{aligned}
Q_{\nu_{\mathrm{sa}}} = &\frac{a_{\rho x}\pi\rho_x\,\mathtt{U}\,\mathtt{Nu_{sa}}}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)+\\
&-\frac{a_{\rho y}\pi\rho_y\,\mathtt{V}\,\mathtt{Nu_{sa}}}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)+\\
&-\frac{a_{\nu_{sa}x}\pi\nu_{sax}\,\mathtt{Rho}\,\mathtt{U}}{L}\sin\left(\frac{a_{\nu_{sa}x}\pi x}{L}\right)+\\
&-\frac{a_{\nu_{sa}y}\pi\nu_{say}\,\mathtt{Rho}\,\mathtt{V}}{L}\sin\left(\frac{a_{\nu_{sa}y}\pi y}{L}\right)+\\
&-\frac{(1+c_{b2})\pi^2\,\mathtt{Rho}}{L^2\sigma}\left[a_{\nu_{sa}x}^2\nu_{sax}^2\sin\left(\frac{a_{\nu_{sa}x}\pi x}{L}\right)^2+a_{\nu_{sa}y}^2\nu_{say}^2\sin\left(\frac{a_{\nu_{sa}y}\pi y}{L}\right)^2\right]+\\
&+\frac{\pi^2\,\mathtt{Nu_{sa}}}{L^2\sigma}\left[a_{\rho x}a_{\nu_{sa}x}\rho_x\nu_{sax}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\sin\left(\frac{a_{\nu_{sa}x}\pi x}{L}\right)-a_{\rho y}a_{\nu_{sa}y}\rho_y\nu_{say}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\sin\left(\frac{a_{\nu_{sa}y}\pi y}{L}\right)\right]+\\
&-c_{b1}\pi\,\mathtt{Rho}\,\mathtt{Nu_{sa}}\sqrt{\frac{1}{L^2}\left[-a_{uy}u_y\sin\left(\frac{a_{uy}\pi y}{L}\right)+a_{vx}v_x\sin\left(\frac{a_{vx}\pi x}{L}\right)\right]^2}+\\
&+\frac{\pi\,\mathtt{Rho}\,\mathtt{Nu_{sa}}}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right]+\\
&+\frac{(\mathtt{Rho}\,\mathtt{Nu_{sa}}+\bar\mu)\pi^2}{L^2\sigma}\left[a_{\nu_{sa}x}^2\nu_{sax}\cos\left(\frac{a_{\nu_{sa}x}\pi x}{L}\right)+a_{\nu_{sa}y}^2\nu_{say}\cos\left(\frac{a_{\nu_{sa}y}\pi y}{L}\right)\right]+\\
&+\frac{a_{\rho t}\pi\rho_t\,\mathtt{Nu_{sa}}}{L}\cos\left(\frac{a_{\rho t}\pi t}{L}\right)+\\
&-\frac{a_{\nu_{sa}t}\pi\nu_{sa_t}}{L}\,\mathtt{Rho}\sin\left(\frac{a_{\nu_{sa}t}\pi t}{L}\right),
\end{aligned}
\tag{22}
$$

with `Rho`, `U`, `V` and $\mathtt{Nu_{sa}}$ defined in (16).

3 Comments

The complexity, and consequently length, of the source terms increase with both dimension and physics handled by the governing equations. In some cases, such as the 2D energy equation, the final expression for $Q_{\tilde{E}}$ may reach 107,600 characters, including parenthesis and mathematical operators, prior to factorization.

Applying commands in order to simplify such extensive expression is challenging even with a very good machine; thus, a suitable alternative to this issue is to simplify the equation by dividing it into a combination of sub-operators handling different physical phenomena. Then, each one of the operators may be applied to the manufactured solutions individually, and the resulting sub-source terms are combined back to represent the source term for the original equation.

For instance, instead of writing the two-dimensional FANS energy equation using one single operator $\mathcal{L}$:

$$\mathcal{L} = \frac{\partial(\bar{\rho}\tilde{E})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{\boldsymbol{u}}\tilde{H}) - \nabla \cdot \bar{\boldsymbol{q}} - \nabla \cdot (2(\bar{\mu} + \mu_t)\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{u}}) \tag{23}$$

to then be used in the MMS, let Equation (23) be written with four operators, according to their physical meaning:

$$\begin{aligned}
\mathcal{L}_1 &= \frac{\partial(\bar{\rho}\tilde{E})}{\partial t}, \\
\mathcal{L}_2 &= \nabla \cdot (\bar{\rho}\tilde{\boldsymbol{u}}\tilde{H}), \\
\mathcal{L}_3 &= -\nabla \cdot \bar{\boldsymbol{q}}, \\
\mathcal{L}_4 &= -\nabla \cdot (2(\bar{\mu} + \mu_t)\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{u}}),
\end{aligned} \tag{24}$$

where $\mathcal{L}_1$ denotes the rate of accumulation of inertial and kinetic energy, $\mathcal{L}_2$ is the net rate of internal and kinetic energy increase by convection together with the work done on the fluid by external body forces, $\mathcal{L}_3$ is the net rate of heat addition due to heat conduction, and $\mathcal{L}_4$ is the rate of work done on the fluid by viscous forces. Naturally:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4.$$

In fact, due to the extremely high complexity of operator $\mathcal{L}_3$ (see Eq. (10) and (12)), further subdivision had to be carried out in order to allow the algebraic manipulations:

$$\begin{aligned}
&\mathcal{L}_3 = \mathcal{L}_{3a} \cdot (\mathcal{L}_{3b} + \mathcal{L}_{3c}) + \mathcal{L}_{3d} \cdot \mathcal{L}_{3e} + \mathcal{L}_{3f} \cdot \mathcal{L}_{3g} \\
&\mathcal{L}_{3a} = -\left(\frac{\bar{\mu}}{\mathrm{Pr}} + \frac{\mu_t}{\mathrm{Pr}_t}\right), \quad \mathcal{L}_{3b} = \frac{\partial^2 \tilde{h}}{\partial x^2}, \quad \mathcal{L}_{3c} = \frac{\partial^2 \tilde{h}}{\partial y^2} \\
&\mathcal{L}_{3d} = \frac{\partial \mu_t}{\partial x}, \quad \mathcal{L}_{3e} = -\frac{1}{\mathrm{Pr}_t}\frac{\partial \tilde{h}}{\partial x}, \\
&\mathcal{L}_{3f} = \frac{\partial \mu_t}{\partial y}, \quad \mathcal{L}_{3g} = -\frac{1}{\mathrm{Pr}_t}\frac{\partial \tilde{h}}{\partial y}.
\end{aligned} \tag{25}$$

After the application of each sub-operator defined in (24) and (25), the corresponding sub-source terms are also simplified, factorized and sorted. Then, the final factorized version is checked against the original one, in order to assure that not human error has been introduced. This strategy allowed the original 107,600 character-long expression for $Q_{\tilde{E}}$ to be reduced to less than 12,400, and expressed in Equation (21).

3.1 Boundary Conditions

Additionally to verifying code capability of solving the governing equations accurately in the interior of the domain of interest, one may also verify the software capability of correctly imposing boundary conditions. Therefore, the gradients of the analytical solutions (7) have been calculated and translated into C codes. They are:

$$\nabla\bar{\rho} = \begin{bmatrix} \frac{a_{\rho x}\pi\rho_x}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) \\ -\frac{a_{\rho y}\pi\rho_y}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) \end{bmatrix}, \quad \nabla\bar{p} = \begin{bmatrix} -\frac{a_{px}\pi p_x}{L}\sin\left(\frac{a_{px}\pi x}{L}\right) \\ \frac{a_{py}\pi p_y}{L}\cos\left(\frac{a_{py}\pi y}{L}\right) \end{bmatrix}, \quad \nabla\tilde{u} = \begin{bmatrix} \frac{a_{ux}\pi u_x}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right) \\ -\frac{a_{uy}\pi u_y}{L}\sin\left(\frac{a_{uy}\pi y}{L}\right) \end{bmatrix},$$

$$\nabla\tilde{v} = \begin{bmatrix} -\frac{a_{vx}\pi v_x}{L}\sin\left(\frac{a_{vx}\pi x}{L}\right) \\ \frac{a_{vy}\pi v_y}{L}\cos\left(\frac{a_{vy}\pi y}{L}\right) \end{bmatrix} \quad \text{and} \quad \nabla\nu_{\mathrm{sa}} = \begin{bmatrix} -\frac{a_{\nu_{\mathrm{sa}}x}\pi\nu_{\mathrm{sa}x}}{L}\sin\left(\frac{a_{\nu_{\mathrm{sa}}x}\pi x}{L}\right) \\ -\frac{a_{\nu_{\mathrm{sa}}y}\pi\nu_{\mathrm{sa}x}}{L}\sin\left(\frac{a_{\nu_{\mathrm{sa}}y}\pi y}{L}\right) \end{bmatrix}.$$

3.2 C Files

Files containing C codes for the source terms have also been automatically generated. They are: FANS_SA_transient_2d_rho_code.C, FANS_SA_transient_2d_u_code.C, FANS_SA_transient_2d_v_code.C, FANS_SA_transient_2d_E_code.C and FANS_SA_transient_2d_nu_code.C.

An example of the C file from the source term for the 2D total energy source term $Q_{\tilde{E}}$ is:

```
#include <math.h>

double SourceQ_e (double x, double y, double t, double mu, double c_v1, double cp, double cv, double Pr_t, double Pr)
{
  double Q_E;
  double RHO;
  double U;
  double V;
  double P;
  double NU_SA;
  double chi;
  double f_v1;
  double R;
  double mu_t;

  NU_SA = nu_sa_0 + nu_sa_x * cos(a_nusax * PI * x / L) + nu_sa_y * cos(a_nusay * PI * y / L)
    + nu_sa_t * cos(a_nusat * PI * t / L);
  RHO = r * rho_0 + rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L)
    + rho_t * sin(a_rhot * PI * t / L);
  U = u_0 + u_x * sin(a_ux * PI * x / L) + u_y * cos(a_uy * PI * y / L) + u_t * cos(a_ut * PI * t / L);
  V = v_0 + v_x * cos(a_vx * PI * x / L) + v_y * sin(a_vy * PI * y / L) + v_t * sin(a_vt * PI * t / L);
  P = p_0 + p_x * cos(a_px * PI * x / L) + p_y * sin(a_py * PI * y / L) + p_t * cos(a_pt * PI * t / L);
  chi = RHO * NU_SA / mu;
  f_v1 = pow(chi, 0.3e1) / (pow(chi, 0.3e1) + pow(c_v1, 0.3e1));
  mu_t = RHO * NU_SA * f_v1;
  R = cp - cv;

  Q_E = -a_ut * PI * u_t * RHO * U * sin(a_ut * PI * t / L) / L
  + a_vt * PI * v_t * RHO * V * cos(a_vt * PI * t / L) / L
  + (U * U + V * V) * a_rhox * PI * rho_x * U * cos(a_rhox * PI * x / L) / L / 0.2e1
  - (U * U + V * V) * a_rhoy * PI * rho_y * V * sin(a_rhoy * PI * y / L) / L / 0.2e1
  + (0.4e1 / 0.3e1 * a_ux * a_nusax * u_x * nu_sa_x * cos(a_ux * PI * x / L) * sin(a_nusax * PI * x / L)
    - a_uy * a_nusay * u_y * nu_sa_y * sin(a_uy * PI * y / L) * sin(a_nusay * PI * y / L)
    - a_vx * a_nusay * v_x * nu_sa_y * sin(a_vx * PI * x / L) * sin(a_nusay * PI * y / L)
    - 0.2e1 / 0.3e1 * a_vy * a_nusax * v_y * nu_sa_x * cos(a_vy * PI * y / L) * sin(a_nusax * PI * x / L))
    * PI * PI * f_v1 * RHO * U * pow(L, -0.2e1)
  + (-0.2e1 / 0.3e1 * a_ux * a_nusay * u_x * nu_sa_y * cos(a_ux * PI * x / L) * sin(a_nusay * PI * y / L)
    - a_uy * a_nusax * u_y * nu_sa_x * sin(a_uy * PI * y / L) * sin(a_nusax * PI * x / L)
    - a_vx * a_nusax * v_x * nu_sa_x * sin(a_vx * PI * x / L) * sin(a_nusax * PI * x / L)
    + 0.4e1 / 0.3e1 * a_vy * a_nusay * v_y * nu_sa_y * cos(a_vy * PI * y / L) * sin(a_nusay * PI * y / L))
    * PI * PI * f_v1 * RHO * V * pow(L, -0.2e1)
  + (-0.4e1 / 0.3e1 * a_rhox * a_ux * rho_x * u_x * cos(a_rhox * PI * x / L) * cos(a_ux * PI * x / L)
    + 0.2e1 / 0.3e1 * a_rhox * a_vy * rho_x * v_y * cos(a_rhox * PI * x / L) * cos(a_vy * PI * y / L)
    - a_rhoy * a_uy * rho_y * u_y * sin(a_rhoy * PI * y / L) * sin(a_uy * PI * y / L)
    - a_rhoy * a_vx * rho_y * v_x * sin(a_rhoy * PI * y / L) * sin(a_vx * PI * x / L))
    * PI * PI * f_v1 * U * NU_SA * pow(L, -0.2e1)
  + (a_rhox * a_uy * rho_x * u_y * cos(a_rhox * PI * x / L) * sin(a_uy * PI * y / L)
    + a_rhox * a_vx * rho_x * v_x * cos(a_rhox * PI * x / L) * sin(a_vx * PI * x / L)
    - 0.2e1 / 0.3e1 * a_rhoy * a_ux * rho_y * u_x * sin(a_rhoy * PI * y / L) * cos(a_ux * PI * x / L)
    + 0.4e1 / 0.3e1 * a_rhoy * a_vy * rho_y * v_y * sin(a_rhoy * PI * y / L) * cos(a_vy * PI * y / L))
    * PI * PI * f_v1 * V * NU_SA * pow(L, -0.2e1)
  + (U * U + V * V) * a_rhot * PI * rho_t * cos(a_rhot * PI * t / L) / L / 0.2e1
  + (f_v1 + pow(c_v1, 0.3e1) / (pow(chi, 0.3e1) + pow(c_v1, 0.3e1)))
    * (0.4e1 * a_ux * a_ux * u_x * sin(a_ux * PI * x / L) + 0.3e1 * a_uy * a_uy * u_y * cos(a_uy * PI * y / L))
    * PI * PI * mu * U * pow(L, -0.2e1) / 0.3e1
  + (f_v1 + pow(c_v1, 0.3e1) / (pow(chi, 0.3e1) + pow(c_v1, 0.3e1)))
    * (0.3e1 * a_vx * a_vx * v_x * cos(a_vx * PI * x / L) + 0.4e1 * a_vy * a_vy * v_y * sin(a_vy * PI * y / L))
    * PI * PI * mu * V * pow(L, -0.2e1) / 0.3e1
  - (a_uy * u_y * sin(a_uy * PI * y / L) + a_vx * v_x * sin(a_vx * PI * x / L)) * PI * RHO * U * V / L
  + (a_ux * u_x * cos(a_ux * PI * x / L) + a_vy * v_y * cos(a_vy * PI * y / L)) * cp * PI * P / L / R
  - (0.4e1 * a_ux * a_ux * u_x * u_x * pow(cos(a_ux * PI * x / L), 0.2e1)
    - 0.4e1 * a_ux * a_vy * u_x * v_y * cos(a_ux * PI * x / L) * cos(a_vy * PI * y / L)
```

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    + 0.3e1 * a_uy * a_uy * u_y * u_y * pow(sin(a_uy * PI * y / L), 0.2e1)
    + 0.6e1 * a_uy * a_vx * u_y * v_x * sin(a_uy * PI * y / L) * sin(a_vx * PI * x / L)
    + 0.3e1 * a_vx * a_vx * v_x * v_x * pow(sin(a_vx * PI * x / L), 0.2e1)
    + 0.4e1 * a_vy * a_vy * v_y * v_y * pow(cos(a_vy * PI * y / L), 0.2e1))
    * PI * PI * mu_t * pow(L, -0.2e1) / 0.3e1
  + (0.4e1 * a_ux * a_ux * u_x * sin(a_ux * PI * x / L) + 0.3e1 * a_uy * a_uy * u_y * cos(a_uy * PI * y / L))
    * PI * PI * mu_t * U * pow(L, -0.2e1) / 0.3e1
  + (0.3e1 * a_vx * a_vx * v_x * cos(a_vx * PI * x / L) + 0.4e1 * a_vy * a_vy * v_y * sin(a_vy * PI * y / L))
    * PI * PI * mu_t * V * pow(L, -0.2e1) / 0.3e1
  + (0.3e1 * a_ux * u_x * cos(a_ux * PI * x / L) + a_vy * v_y * cos(a_vy * PI * y / L)) * PI * RHO * U * U / L / 0.2e1
  + (a_ux * u_x * cos(a_ux * PI * x / L) + 0.3e1 * a_vy * v_y * cos(a_vy * PI * y / L)) * PI * RHO * V * V / L / 0.2e1
  - (f_v1 + pow(c_v1, 0.3e1) / (pow(chi, 0.3e1) + pow(c_v1, 0.3e1)))
    * (0.4e1 * a_ux * a_ux * u_x * u_x * pow(cos(a_ux * PI * x / L), 0.2e1)
    - 0.4e1 * a_ux * a_vy * u_x * v_y * cos(a_ux * PI * x / L) * cos(a_vy * PI * y / L)
    + 0.3e1 * a_uy * a_uy * u_y * u_y * pow(sin(a_uy * PI * y / L), 0.2e1)
    + 0.6e1 * a_uy * a_vx * u_y * v_x * sin(a_uy * PI * y / L) * sin(a_vx * PI * x / L)
    + 0.3e1 * a_vx * a_vx * v_x * v_x * pow(sin(a_vx * PI * x / L), 0.2e1)
    + 0.4e1 * a_vy * a_vy * v_y * v_y * pow(cos(a_vy * PI * y / L), 0.2e1)) * PI * PI * mu * pow(L, -0.2e1) / 0.3e1
  - (mu_t / Pr_t + mu / Pr) * (-(a_px * a_px * p_x * cos(a_px * PI * x / L)
    + a_py * a_py * p_y * sin(a_py * PI * y / L)) * cp * PI * PI * pow(L, -0.2e1) / R / RHO
  + (a_rhox * a_rhox * rho_x * sin(a_rhox * PI * x / L) + a_rhoy * a_rhoy * rho_y * cos(a_rhoy * PI * y / L))
    * cp * PI * PI * P * pow(L, -0.2e1) / R * pow(RHO, -0.2e1))
  + cv * a_rhot * PI * rho_t * P * cos(a_rhot * PI * t / L) / L / R / RHO
  - (a_px * a_nusax * p_x * nu_sa_x * sin(a_px * PI * x / L) * sin(a_nusax * PI * x / L)
    - a_py * a_nusay * p_y * nu_sa_y * cos(a_py * PI * y / L) * sin(a_nusay * PI * y / L))
    * cp * PI * PI * mu_t * pow(L, -0.2e1) / Pr_t / R / RHO / NU_SA
  - (a_rhox * a_nusax * rho_x * nu_sa_x * cos(a_rhox * PI * x / L) * sin(a_nusax * PI * x / L)
    - a_rhoy * a_nusay * rho_y * nu_sa_y * sin(a_rhoy * PI * y / L) * sin(a_nusay * PI * y / L))
    * cp * PI * PI * mu_t * P * pow(L, -0.2e1) / Pr_t / R * pow(RHO, -0.2e1) / NU_SA
  - (a_rhox * a_px * rho_x * p_x * cos(a_rhox * PI * x / L) * sin(a_px * PI * x / L)
    + a_rhoy * a_py * rho_y * p_y * sin(a_rhoy * PI * y / L) * cos(a_py * PI * y / L))
    * (Pr * mu_t + 0.2e1 * Pr_t * mu) * cp * PI * PI / Pr / Pr_t * pow(L, -0.2e1) / R * pow(RHO, -0.2e1)
  - (a_rhox * a_rhox * rho_x * rho_x * pow(cos(a_rhox * PI * x / L), 0.2e1)
    + a_rhoy * a_rhoy * rho_y * rho_y * pow(sin(a_rhoy * PI * y / L), 0.2e1))
    * (Pr * mu_t + 0.2e1 * Pr_t * mu) * cp * PI * PI * P / Pr / Pr_t * pow(L, -0.2e1) / R * pow(RHO, -0.3e1)
  + cp * a_py * PI * p_y * V * cos(a_py * PI * y / L) / L / R
  - cp * a_px * PI * p_x * U * sin(a_px * PI * x / L) / L / R;
}
```

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