Manufactured Solution for 1D Euler equation using Maple*

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Abstract

The Method of Manufactured Solutions is a valuable approach for code verification, providing means to verify how accurately the numerical method solves the equations of interest. The method generates a related set of governing equations by adding a source term to the RHS of the original set of equations, making use of analytical solutions chosen a priori. This document presents the source terms generated by the application of the Method of Manufactured Solutions (MMS) on the the 1D Euler equations using the analytical manufactured solutions for ρ, u and p presented by Roy et al. (2002).

1 1D Euler Equations

The 1D Euler equations in conservation form are:

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} = 0, \tag{2}$$

$$\frac{\partial(\rho e_t)}{\partial t} + \frac{\partial(\rho u e_t + pu)}{\partial x} = 0, \tag{3}$$

where the Equation (1) is the unsteady term (mass conservation), Equation (2) represents the nonlinear convection term in the x direction (momentum), and Equation (3) is the energy. For a calorically perfect gas, the Euler equations are closed with two auxiliary relations for energy:

$$e = \frac{1}{\gamma - 1} RT, \tag{4}$$

$$e_t = e + \frac{u^2}{2}, \tag{5}$$

and with the ideal gas equation of state:

$$p = \rho RT. \tag{6}$$

2 Manufactured Solution

Roy et al. (2002) propose the general form of the primitive solution variables to be a function of sines and cosines, which to the one-dimensional case is reduced to:

$$\phi(x) = \phi_0 + \phi_x f_s\left(\frac{a_{\phi x}\pi x}{L}\right), \tag{7}$$

where $\phi = \rho$, u or p, and $f_s(\cdot)$ functions denote either sine or cosine function. Note that in this case, ϕ_x is a constant and the subscript do not denote differentiation.

*Work based on Roy, Smith, and Ober (2002).

Therefore, the manufactured analytical solution for for each one of the variables in Euler equations are:

$$\begin{aligned}
\rho(x,y) &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right), \\
u(x,y) &= u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right), \\
p(x,y) &= p_0 + p_x \cos\left(\frac{a_{px}\pi x}{L}\right).
\end{aligned} \tag{8}$$

The MMS applied to Euler equations consists in modifying Equations (1) – (3) by adding a source term to the right-hand side of each equation:

$$\begin{aligned}
\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho u)}{\partial x} &= Q_\rho, \\
\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} &= Q_u, \\
\frac{\partial(\rho e_t)}{\partial t} + \frac{\partial(\rho u e_t + p u)}{\partial x} &= Q_e,
\end{aligned} \tag{9}$$

so the modified set of equations (9) conveniently has the analytical solution given in Equation (8). This is achieved by simply applying Equations (1) – (3) as operators on Equation (7). Terms Q_ρ, Q_u, and Q_e are obtained by symbolic manipulations of equations above using Maple and are presented in the following sections.

3 Euler mass conservation equation

The mass conservation equation written as an operator is:

$$L = \frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho u)}{\partial x}. \tag{10}$$

Analytically differentiating Equation (8) for ρ and u using operator L defined above gives the source term Q_ρ:

$$\begin{aligned}
Q_\rho = &\frac{a_{\rho x}\pi \rho_x}{L} \cos\left(\frac{a_{\rho x}\pi x}{L}\right) \left[u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right)\right] + \\
&+ \frac{a_{ux}\pi u_x}{L} \cos\left(\frac{a_{ux}\pi x}{L}\right) \left[\rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right)\right].
\end{aligned} \tag{11}$$

4 Euler momentum equation

For the generation of the analytical source term Q_u for the x momentum equation, Equation (2) is written as an operator L:

$$L = \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x}, \tag{12}$$

which, when operated in Equation (8), provides source term Q_u:

$$\begin{aligned}
Q_u = &\frac{a_{\rho x}\pi \rho_x}{L} \cos\left(\frac{a_{\rho x}\pi x}{L}\right) \left[u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right)\right]^2 + \\
&- \frac{a_{px}\pi p_x}{L} \sin\left(\frac{a_{px}\pi x}{L}\right) + \\
&+ \frac{2a_{ux}\pi u_x}{L} \cos\left(\frac{a_{ux}\pi x}{L}\right) \left[\rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right)\right] \left[u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right)\right].
\end{aligned} \tag{13}$$

5 Euler energy equation

The last component of Euler equations is written as an operator:

$$L = \frac{\partial(\rho e_t)}{\partial t} + \frac{\partial(\rho u e_t + p u)}{\partial x}. \tag{14}$$

Source term Q_e is obtained by operating L on Equation (8) together with the use of the auxiliary relations (4)–(6) for energy :

$$\begin{aligned} Q_e = & \frac{a_{\rho x}\pi\rho_x}{2L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)\right]^3 + \\ & -\frac{a_{px}\pi p_x}{L}\frac{\gamma}{\gamma-1}\sin\left(\frac{a_{px}\pi x}{L}\right)\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)\right] + \\ & +\frac{a_{ux}\pi u_x}{L}\frac{\gamma}{\gamma-1}\cos\left(\frac{a_{ux}\pi x}{L}\right)\left[p_0 + p_x\cos\left(\frac{a_{px}\pi x}{L}\right)\right] + \\ & +\frac{3a_{ux}\pi u_x}{2L}\cos\left(\frac{a_{ux}\pi x}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)\right]\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)\right]^2 . \end{aligned} \tag{15}$$

6 Comments

Source terms Q_ρ, Q_u and Q_e have been generated automatically by replacing the analytical expressions (8) into respective operators (10), (12) and (14), followed by the usage of Maple commands for collecting, sorting and factorizing the terms.

One file containing C codes for the source terms has also been generated: `Euler_1d_e_codes.C`. As an example, the automatically generated C file from the source term for the total energy is:

```
double SourceQ_e (double x, double y, double u_0, double u_x, double rho_0, double rho_x,
                  double p_0, double p_x, double a_px, double a_rhox, double a_ux,
                  double Gamma, double mu, double L)
{
  double Q_e;
  Q_e = cos(a_rhox * PI * x / L) * rho_x * pow(u_0 + u_x * sin(a_ux * PI * x / L), 0.3e1) *
        a_rhox * PI / L / 0.2e1 + cos(a_ux * PI * x / L) * (p_0 + p_x * cos(a_px * PI * x / L)) *
        a_ux * PI * u_x * Gamma / L / (Gamma - 0.1e1) - Gamma * p_x * sin(a_px * PI * x / L) *
        (u_0 + u_x * sin(a_ux * PI * x / L)) * a_px * PI / L / (Gamma - 0.1e1) + 0.3e1 / 0.2e1 *
        cos(a_ux * PI * x / L) * (rho_0 + rho_x * sin(a_rhox * PI * x / L)) *
        pow(u_0 + u_x * sin(a_ux * PI * x / L), 0.2e1) * a_ux * PI * u_x / L;
  return(Q_e);
}
```

Finally the gradients of the analytical solutions have also been computed and their respective C codes are presented in `Euler_manuf_solutions_grad_and_code_1d.C`. Therefore, the gradients of the analytical solution (8):

$$\nabla\rho = \frac{a_{\rho x}\pi\rho_x}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right), \qquad \nabla p = -\frac{a_{px}\pi p_x}{L}\sin\left(\frac{a_{px}\pi x}{L}\right), \qquad \nabla u = \frac{a_{ux}\pi u_x}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right), \tag{16}$$

are written in C language as:

```
grad_rho_an[0] = rho_x * cos(a_rhox * pi * x / L) * a_rhox * pi / L;
grad_rho_an[1] = 0;
grad_rho_an[2] = 0;
grad_p_an[0] = -p_x * sin(a_px * pi * x / L) * a_px * pi / L;
grad_p_an[1] = 0;
grad_p_an[2] = 0;
grad_u_an[0] = u_x * cos(a_ux * pi * x / L) * a_ux * pi / L;
grad_u_an[1] = 0;
grad_u_an[2] = 0;
```

References

Roy, C., T. Smith, and C. Ober (2002). Verification of a compressible CFD code using the method of manufactured solutions. In *AIAA Fluid Dynamics Conference and Exhibit*, Number AIAA 2002-3110.