

# Choice of Manufactured Analytical Solution for Code Verification of Axisymmetric Navier–Stokes Equations

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## Abstract

The Method of Manufactured Solutions is a valuable approach for code verification, providing means to verify how accurately the numerical method solves the equations of interest. The method generates a related set of governing equations by applying the differential operator for the governing equations to an analytical solution chosen *a priori* to generate source terms, which are added the RHS of the original set of equations. Then, the modified set of equations may be discretized and solved numerically, so its solution may be compared to the exact solution. A choice of analytical solutions for the flow variables of the axisymmetric Navier–Stokes and their respective source terms are presented in this document.

## 1 Axisymmetric Navier–Stokes equations

Navier–Stokes equations may be written in cylindrical coordinates for  $(r, \theta, z)$ , where  $r$  is the radial coordinate,  $\theta$  is the angular coordinate, and  $z$  is the axial coordinate. In axisymmetrical flows, the pressure and the velocity fields are independent of the angular variable  $\theta$ , and the problem depends exclusively on  $r$  and  $z$ . Therefore, Navier–Stokes equations for axisymmetric flows, in conservation form, are:

$$\frac{\partial(\rho)}{\partial t} + \frac{1}{r} \frac{\partial(r\rho u)}{\partial r} + \frac{1}{r} \frac{\partial(r\rho w)}{\partial z} = 0, \quad (1)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{1}{r} \frac{\partial(r\rho u^2)}{\partial r} + \frac{1}{r} \frac{\partial(r\rho uw)}{\partial z} = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r} \frac{\partial(r\tau_{rz})}{\partial z}, \quad (2)$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{1}{r} \frac{\partial(r\rho wu)}{\partial r} + \frac{1}{r} \frac{\partial(r\rho w^2)}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial(r\tau_{zr})}{\partial r} + \frac{1}{r} \frac{\partial(r\tau_{zz})}{\partial z}, \quad (3)$$

$$\frac{\partial(\rho e_t)}{\partial t} + \frac{1}{r} \frac{\partial(r(\rho e_t + p)u)}{\partial r} + \frac{1}{r} \frac{\partial(r(\rho e_t + p)w)}{\partial z} = \frac{1}{r} \frac{\partial(r(\tau_{rr}u + \tau_{rz}w - q_r))}{\partial r} + \frac{1}{r} \frac{\partial(r(\tau_{zr}u + \tau_{zz}w - q_z))}{\partial z}. \quad (4)$$

where the Equation (1) is mass conservation, Equations (2) – (3) are momentum in the radial and axial directions, respectively, and Equation (4) is the energy. Notice that Equations (2) and (3) include viscous effects.

For a calorically perfect gas, Navier–Stokes equations are closed with two auxiliary relations for energy:

$$e_t = e + \frac{u^2 + w^2}{2} \quad \text{and} \quad e = \frac{1}{\gamma - 1} RT, \quad (5)$$

where  $\gamma$  is the ratio of specific heats, and with the ideal gas equation of state:

$$p = \rho RT. \quad (6)$$

The shear stress tensor is:

$$\tau_{rr} = \mu \left( 2 \frac{\partial u}{\partial r} - \frac{2}{3} \nabla \cdot V \right), \quad \tau_{zz} = \mu \left( 2 \frac{\partial w}{\partial z} - \frac{2}{3} \nabla \cdot V \right), \quad \tau_{rz} = \tau_{zr} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right), \quad (7)$$

where  $\mu$  is the absolute viscosity,  $V = (u, 0, w)$  and

$$\nabla \cdot V = \frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{1}{r} \frac{\partial(rw)}{\partial z}.$$

The heat flux vector is given by:

$$q_r = -k \frac{\partial T}{\partial r}, \quad \text{and} \quad q_z = -k \frac{\partial T}{\partial z}, \quad (8)$$

where  $k$  is the thermal conductivity, which can be determined by choosing the Prandtl number:

$$k = \frac{\gamma R \mu}{(\gamma - 1) \text{Pr}}.$$

## 2 Manufactured Solution

The choice of cylindrical coordinates, like any system that contains a symmetry axis, introduces singular terms in the governing equation of the type  $r^{-n}$ , being  $r$  the radial coordinate and  $n$  a positive exponent, although the flow is continuous and regular at the axis [1].

Accordingly, the representation of fluid flows in cylindrical coordinates requires the definition of appropriate boundary conditions at  $r = 0$ , despite the fact that it is not a physical boundary, that would guarantee the regularity of the flow:

$$\begin{aligned} u|_{r=0} &= 0, \\ \frac{\partial u}{\partial r}|_{r=0} &= 0, \\ \frac{\partial w}{\partial r}|_{r=0} &= 0. \end{aligned} \quad (9)$$

The strategy to deal with this difficulty in analytical approaches is commonly that of discarding the singular solutions among all the admissible ones. Consequently, a suitable form of each one of the primitive solution variables is a function of sines and cosines:

$$\begin{aligned} \rho(r, z) &= \rho_0 + \rho_1 \cos\left(\frac{a_{\rho r} \pi r}{L}\right) \sin\left(\frac{a_{\rho z} \pi z}{L}\right), \\ u(r, z) &= u_1 \left[ \cos\left(\frac{a_{ur} \pi r}{L}\right) - 1 \right] \sin\left(\frac{a_{uz} \pi z}{L}\right), \\ w(r, z) &= w_0 + w_1 \cos\left(\frac{a_{wr} \pi r}{L}\right) \sin\left(\frac{a_{wz} \pi z}{L}\right), \\ p(r, z) &= p_0 + p_1 \sin\left(\frac{a_{pr} \pi r}{L}\right) \cos\left(\frac{a_{pz} \pi z}{L}\right), \end{aligned} \quad (10)$$

where  $\rho_0, \rho_1, p_0, p_1, u_1, w_0$  and  $w_1$  are pre-defined constants.

The MMS applied to Navier–Stokes equations consists in modifying Equations (1) – (4) by adding a source term to the right-hand side of each equation:

$$\begin{aligned}
\frac{\partial(\rho)}{\partial t} + \frac{1}{r} \frac{\partial(\rho r u)}{\partial r} + \frac{1}{r} \frac{\partial(\rho r w)}{\partial z} &= Q_\rho, \\
\frac{\partial(\rho u)}{\partial t} + \frac{1}{r} \frac{\partial(r \rho u^2)}{\partial r} + \frac{1}{r} \frac{\partial(r \rho u w)}{\partial z} + \frac{\partial p}{\partial r} - \left( \frac{1}{r} \frac{\partial(r \tau_{rr})}{\partial r} + \frac{1}{r} \frac{\partial(r \tau_{rz})}{\partial z} \right) &= Q_u, \\
\frac{\partial(\rho w)}{\partial t} + \frac{1}{r} \frac{\partial(r \rho w u)}{\partial r} + \frac{1}{r} \frac{\partial(r \rho w^2)}{\partial z} + \frac{\partial p}{\partial z} - \left( \frac{1}{r} \frac{\partial(r \tau_{zr})}{\partial r} + \frac{1}{r} \frac{\partial(r \tau_{zz})}{\partial z} \right) &= Q_w, \\
\frac{\partial(\rho e_t)}{\partial t} + \frac{1}{r} \frac{\partial(r(\rho e_t + p)u)}{\partial r} + \frac{1}{r} \frac{\partial(r(\rho e_t + p)w)}{\partial z} - \left( \frac{1}{r} \frac{\partial(r(\tau_{rr}u + \tau_{rz}w - q_r))}{\partial r} + \frac{\partial(\tau_{zr}u + \tau_{zz}w - q_z)}{\partial z} \right) &= Q_e,
\end{aligned} \tag{11}$$

so the modified set of equations (11) conveniently has the analytical solution given in Equation (10). This is achieved by simply applying Equations (1) – (4) as operators on Equation (10). Terms  $Q_\rho, Q_u, Q_w$  and  $Q_e$  are presented in the following sections.

## 2.1 Source term for mass conservation equation

$$\begin{aligned}
Q_\rho = & -\frac{a_{\rho r} \pi \rho_1}{L} \sin\left(\frac{a_{\rho r} \pi r}{L}\right) \sin\left(\frac{a_{\rho z} \pi z}{L}\right) \left[ u_1 \left[ \cos\left(\frac{a_{ur} \pi r}{L}\right) - 1 \right] \sin\left(\frac{a_{uz} \pi z}{L}\right) \right] + \\
& + \frac{a_{\rho z} \pi \rho_1}{L} \cos\left(\frac{a_{\rho r} \pi r}{L}\right) \cos\left(\frac{a_{\rho z} \pi z}{L}\right) \left[ w_0 + w_1 \cos\left(\frac{a_{wr} \pi r}{L}\right) \sin\left(\frac{a_{wz} \pi z}{L}\right) \right] + \\
& - \frac{a_{ur} \pi u_1}{L} \sin\left(\frac{a_{uz} \pi z}{L}\right) \sin\left(\frac{a_{ur} \pi r}{L}\right) \left[ \rho_0 + \rho_1 \cos\left(\frac{a_{\rho r} \pi r}{L}\right) \sin\left(\frac{a_{\rho z} \pi z}{L}\right) \right] + \\
& + \frac{a_{wz} \pi w_1}{L} \cos\left(\frac{a_{wr} \pi r}{L}\right) \cos\left(\frac{a_{wz} \pi z}{L}\right) \left[ \rho_0 + \rho_1 \cos\left(\frac{a_{\rho r} \pi r}{L}\right) \sin\left(\frac{a_{\rho z} \pi z}{L}\right) \right] + \\
& + \frac{1}{r} \left[ \rho_0 + \rho_1 \cos\left(\frac{a_{\rho r} \pi r}{L}\right) \sin\left(\frac{a_{\rho z} \pi z}{L}\right) \right] \left[ u_1 \left[ \cos\left(\frac{a_{ur} \pi r}{L}\right) - 1 \right] \sin\left(\frac{a_{uz} \pi z}{L}\right) \right].
\end{aligned} \tag{12}$$

## 2.2 Source term for radial velocity

$$\begin{aligned}
Q_u = & -\frac{a_{\rho r}\pi\rho_1}{L}\sin\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]^2 + \\
& +\frac{a_{\rho z}\pi\rho_1}{L}\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
& +\frac{a_{pr}\pi p_1}{L}\cos\left(\frac{a_{pr}\pi r}{L}\right)\cos\left(\frac{a_{pz}\pi z}{L}\right) + \\
& -\frac{2a_{ur}\pi u_1}{L}\sin\left(\frac{a_{ur}\pi r}{L}\right)\sin\left(\frac{a_{uz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right] + \frac{2a_{ur}\pi u_1\mu}{3Lr}\sin\left(\frac{a_{ur}\pi r}{L}\right)\sin\left(\frac{a_{uz}\pi z}{L}\right) + \\
& +\frac{a_{uz}\pi u_1}{L}\left(\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right)\cos\left(\frac{a_{uz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
& +\frac{a_{wz}\pi w_1}{L}\cos\left(\frac{a_{wr}\pi r}{L}\right)\cos\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right] + \frac{2a_{wz}\pi w_1\mu}{3Lr}\cos\left(\frac{a_{wr}\pi r}{L}\right)\cos\left(\frac{a_{wz}\pi z}{L}\right) + \\
& +\frac{4a_{ur}^2\pi^2\mu u_1}{3L^2}\cos\left(\frac{a_{ur}\pi r}{L}\right)\sin\left(\frac{a_{uz}\pi z}{L}\right) + \frac{a_{uz}^2\pi^2\mu u_1}{L^2}\left(\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right)\sin\left(\frac{a_{uz}\pi z}{L}\right) - \frac{2a_{wr}a_{wz}\pi^2\mu w_1}{3L^2}\sin\left(\frac{a_{wr}\pi r}{L}\right)\cos\left(\frac{a_{wz}\pi z}{L}\right) + \\
& +\frac{1}{r}\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]^2.
\end{aligned} \tag{13}$$

## 2.3 Source term for axial velocity

$$\begin{aligned}
Q_w = & -\frac{a_{\rho r}\pi\rho_1}{L}\sin\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
& +\frac{a_{\rho z}\pi\rho_1}{L}\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]^2 + \\
& -\frac{a_{pz}\pi p_1}{L}\sin\left(\frac{a_{pr}\pi r}{L}\right)\sin\left(\frac{a_{pz}\pi z}{L}\right) + \\
& -\frac{a_{ur}\pi u_1}{L}\sin\left(\frac{a_{uz}\pi z}{L}\right)\sin\left(\frac{a_{ur}\pi r}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
& -\frac{a_{uz}\pi\mu u_1}{3Lr}\left(\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right)\cos\left(\frac{a_{uz}\pi z}{L}\right) + \\
& -\frac{a_{wr}\pi w_1}{L}\sin\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right] + \\
& +\frac{2a_{wz}\pi w_1}{L}\cos\left(\frac{a_{wr}\pi r}{L}\right)\cos\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
& +\frac{4a_{wz}^2\pi^2\mu w_1}{3L^2}\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right) + \frac{a_{ur}a_{uz}\pi^2\mu u_1}{3L^2}\sin\left(\frac{a_{ur}\pi r}{L}\right)\cos\left(\frac{a_{uz}\pi z}{L}\right) + \\
& +\frac{1}{r}\left[\rho_0+\rho_1\cos\left(\frac{a_{\rho r}\pi r}{L}\right)\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_1\left[\cos\left(\frac{a_{ur}\pi r}{L}\right)-1\right]\sin\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[w_0+w_1\cos\left(\frac{a_{wr}\pi r}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)\right].
\end{aligned} \tag{14}$$

## 2.4 Source term for energy

[illegible]

$$\begin{aligned}
& - \frac{2a_{\rho z}^2 \pi^2 k \rho_1^2}{RL^2 \left[ \rho_0 + \rho_1 \cos \left( \frac{a_{\rho r} \pi r}{L} \right) \sin \left( \frac{a_{\rho z} \pi z}{L} \right) \right]^3} \cos \left( \frac{a_{\rho r} \pi r}{L} \right)^2 \cos \left( \frac{a_{\rho z} \pi z}{L} \right)^2 \left[ p_0 + p_1 \sin \left( \frac{a_{pr} \pi r}{L} \right) \cos \left( \frac{a_{pz} \pi z}{L} \right) \right] + \\
& + \frac{a_{pr}^2 \pi^2 k p_1}{RL^2 \left[ \rho_0 + \rho_1 \cos \left( \frac{a_{\rho r} \pi r}{L} \right) \sin \left( \frac{a_{\rho z} \pi z}{L} \right) \right]} \sin \left( \frac{a_{pr} \pi r}{L} \right) \cos \left( \frac{a_{pz} \pi z}{L} \right) + \\
& + \frac{a_{pz}^2 \pi^2 k p_1}{RL^2 \left[ \rho_0 + \rho_1 \cos \left( \frac{a_{\rho r} \pi r}{L} \right) \sin \left( \frac{a_{\rho z} \pi z}{L} \right) \right]} \sin \left( \frac{a_{pr} \pi r}{L} \right) \cos \left( \frac{a_{pz} \pi z}{L} \right) + \\
& - \frac{4a_{ur}^2 \pi^2 \mu}{3L^2} \left[ u_1 \left[ \cos \left( \frac{a_{ur} \pi r}{L} \right) - 1 \right] \sin \left( \frac{a_{uz} \pi z}{L} \right) \right]^2 + \frac{4a_{ur}^2 \pi^2 \mu u_1^2}{3L^2} \sin \left( \frac{a_{uz} \pi z}{L} \right)^2 \sin \left( \frac{a_{ur} \pi r}{L} \right)^2 - \frac{4a_{ur}^2 \pi^2 \mu u_1}{3L^2} \sin \left( \frac{a_{uz} \pi z}{L} \right) \left[ u_1 \left[ \cos \left( \frac{a_{ur} \pi r}{L} \right) - 1 \right] \sin \left( \frac{a_{uz} \pi z}{L} \right) \right] + \\
& - \frac{a_{uz}^2 \pi^2 \mu}{L^2} \left[ u_1 \left[ \cos \left( \frac{a_{ur} \pi r}{L} \right) - 1 \right] \sin \left( \frac{a_{uz} \pi z}{L} \right) \right]^2 + \frac{a_{uz}^2 \pi^2 \mu u_1^2}{L^2} \left( \cos \left( \frac{a_{ur} \pi r}{L} \right) - 1 \right)^2 \cos \left( \frac{a_{uz} \pi z}{L} \right)^2 + \\
& - \frac{4a_{wz}^2 \pi^2 \mu w_1}{3L^2} \cos \left( \frac{a_{wr} \pi r}{L} \right) \sin \left( \frac{a_{wz} \pi z}{L} \right) \left[ w_0 + w_1 \cos \left( \frac{a_{wr} \pi r}{L} \right) \sin \left( \frac{a_{wz} \pi z}{L} \right) \right] + \frac{4a_{wz}^2 \pi^2 \mu w_1^2}{3L^2} \cos \left( \frac{a_{wr} \pi r}{L} \right)^2 \cos \left( \frac{a_{wz} \pi z}{L} \right)^2 + \\
& - \frac{2a_{\rho r} a_{pr} \pi^2 k p_1 \rho_1}{RL^2 \left[ \rho_0 + \rho_1 \cos \left( \frac{a_{\rho r} \pi r}{L} \right) \sin \left( \frac{a_{\rho z} \pi z}{L} \right) \right]^2} \sin \left( \frac{a_{\rho r} \pi r}{L} \right) \sin \left( \frac{a_{\rho z} \pi z}{L} \right) \cos \left( \frac{a_{pr} \pi r}{L} \right) \cos \left( \frac{a_{pz} \pi z}{L} \right) + \\
& - \frac{2a_{\rho z} a_{pz} \pi^2 k p_1 \rho_1}{RL^2 \left[ \rho_0 + \rho_1 \cos \left( \frac{a_{\rho r} \pi r}{L} \right) \sin \left( \frac{a_{\rho z} \pi z}{L} \right) \right]^2} \cos \left( \frac{a_{\rho r} \pi r}{L} \right) \cos \left( \frac{a_{\rho z} \pi z}{L} \right) \sin \left( \frac{a_{pr} \pi r}{L} \right) \sin \left( \frac{a_{pz} \pi z}{L} \right) + \\
& - \frac{a_{ur} a_{uz} \pi^2 \mu u_1}{3L^2} \sin \left( \frac{a_{ur} \pi r}{L} \right) \cos \left( \frac{a_{uz} \pi z}{L} \right) \left[ w_0 + w_1 \cos \left( \frac{a_{wr} \pi r}{L} \right) \sin \left( \frac{a_{wz} \pi z}{L} \right) \right] + \\
& + \frac{4a_{ur} a_{wz} \pi^2 \mu u_1 w_1}{3L^2} \sin \left( \frac{a_{ur} \pi r}{L} \right) \cos \left( \frac{a_{wr} \pi r}{L} \right) \cos \left( \frac{a_{wz} \pi z}{L} \right) \sin \left( \frac{a_{uz} \pi z}{L} \right) + \\
& - \frac{a_{uz} a_{wr} \pi^2 \mu u_1 w_1}{L^2} \cos \left( \frac{a_{uz} \pi z}{L} \right) \sin \left( \frac{a_{wr} \pi r}{L} \right) \sin \left( \frac{a_{wz} \pi z}{L} \right) \left( \cos \left( \frac{a_{ur} \pi r}{L} \right) - 1 \right) + \\
& + \frac{2a_{wr} a_{wz} \pi^2 \mu w_1}{3L^2} \sin \left( \frac{a_{wr} \pi r}{L} \right) \cos \left( \frac{a_{wz} \pi z}{L} \right) \left[ u_1 \left[ \cos \left( \frac{a_{ur} \pi r}{L} \right) - 1 \right] \sin \left( \frac{a_{uz} \pi z}{L} \right) \right] + \\
& + \frac{1}{2r} \left[ \rho_0 + \rho_1 \cos \left( \frac{a_{\rho r} \pi r}{L} \right) \sin \left( \frac{a_{\rho z} \pi z}{L} \right) \right] \left[ u_1 \left[ \cos \left( \frac{a_{ur} \pi r}{L} \right) - 1 \right] \sin \left( \frac{a_{uz} \pi z}{L} \right) \right] \left( \left[ u_1 \left[ \cos \left( \frac{a_{ur} \pi r}{L} \right) - 1 \right] \sin \left( \frac{a_{uz} \pi z}{L} \right) \right]^2 + \left[ w_0 + w_1 \cos \left( \frac{a_{wr} \pi r}{L} \right) \sin \left( \frac{a_{wz} \pi z}{L} \right) \right]^2 \right) + \\
& + \frac{\gamma}{r(\gamma - 1)} \left[ p_0 + p_1 \sin \left( \frac{a_{pr} \pi r}{L} \right) \cos \left( \frac{a_{pz} \pi z}{L} \right) \right] \left[ u_1 \left[ \cos \left( \frac{a_{ur} \pi r}{L} \right) - 1 \right] \sin \left( \frac{a_{uz} \pi z}{L} \right) \right].
\end{aligned}$$

(16)

### 3 Comments

Source terms  $Q_\rho$ ,  $Q_u$ ,  $Q_w$  and  $Q_e$  have been generated by replacing the analytical expressions (10) into respective equations (1) – (4), followed by the usage of Maple commands for collecting, sorting and factorizing the terms. Files containing C codes for the source terms have also been generated. They are: NavierStokes\_axi\_rho\_code.C, NavierStokes\_axi\_u\_code.C, NavierStokes\_axi\_w\_code.C and NavierStokes\_axi\_e\_code.C.

An example of the automatically generated C file from the source term for the axial velocity  $w$  equation is:

```
#include <math.h>

double SourceQ_w ( double r, double z, double p_0, double p_1, double rho_0, double rho_1, double u_1, double w_0, double w_1,
double a_pr, double a_pz, double a_rhor, double a_rhoz, double a_ur, double a_uz, double a_wr, double a_wz,
double PI, double L)
{
double Q_w;
Q_w = sin(a_ur * PI * r / L) * cos(a_uz * PI * z / L) * a_ur * a_uz * PI * PI * mu * u_1 * pow(L, -0.2e1) / 0.3e1 +
0.4e1 / 0.3e1 * cos(a_wr * PI * r / L) * sin(a_wz * PI * z / L) * a_wz * a_wz * PI * PI * mu * w_1 * pow(L, -0.2e1) -
sin(a_pr * PI * r / L) * sin(a_pz * PI * z / L) * a_pz * PI * p_1 / L - sin(a_rhor * PI * r / L) *
sin(a_rhoz * PI * z / L) * u_1 * (cos(a_ur * PI * r / L) - 0.1e1) * sin(a_uz * PI * z / L) *
(w_0 + w_1 * cos(a_wr * PI * r / L) * sin(a_wz * PI * z / L)) * a_rhor * PI * rho_1 / L + cos(a_rhor * PI * r / L) *
cos(a_rhoz * PI * z / L) * pow(w_0 + w_1 * cos(a_wr * PI * r / L) * sin(a_wz * PI * z / L), 0.2e1) *
a_rhoz * PI * rho_1 / L - sin(a_uz * PI * z / L) * sin(a_ur * PI * r / L) * (rho_0 + rho_1 * cos(a_rhor * PI * r / L) *
sin(a_rhoz * PI * z / L)) * (w_0 + w_1 * cos(a_wr * PI * r / L) * sin(a_wz * PI * z / L)) * a_ur * PI * u_1 / L -
sin(a_wr * PI * r / L) * sin(a_wz * PI * z / L) * (rho_0 + rho_1 * cos(a_rhor * PI * r / L) *
sin(a_rhoz * PI * z / L)) * u_1 * (cos(a_ur * PI * r / L) - 0.1e1) * sin(a_uz * PI * z / L) * a_wr * PI * w_1 / L +
0.2e1 * cos(a_wz * PI * z / L) * cos(a_wr * PI * r / L) * (rho_0 + rho_1 * cos(a_rhor * PI * r / L) *
sin(a_rhoz * PI * z / L)) * (w_0 + w_1 * cos(a_wr * PI * r / L) * sin(a_wz * PI * z / L)) * a_wz * PI * w_1 / L -
(cos(a_ur * PI * r / L) - 0.1e1) * cos(a_uz * PI * z / L) * a_uz * PI * mu * u_1 / L / r / 0.3e1 +
sin(a_uz * PI * z / L) * u_1 * (w_0 + w_1 * cos(a_wr * PI * r / L) * sin(a_wz * PI * z / L)) *
(cos(a_ur * PI * r / L) - 0.1e1) * (rho_0 + rho_1 * cos(a_rhor * PI * r / L) * sin(a_rhoz * PI * z / L)) / r;
return(Q_w);
}
```

Finally the gradients of the analytical solutions (10) have also been computed and their respective C codes are presented in NavierStokes\_manuf\_solutions\_grad\_and\_code\_axisymmetric.C. Therefore,

$$\nabla \rho = \begin{bmatrix} -\frac{a_{pr}\pi\rho_1}{L} \sin\left(\frac{a_{pr}\pi r}{L}\right) \sin\left(\frac{a_{pz}\pi z}{L}\right) \\ \frac{a_{pz}\pi\rho_1}{L} \cos\left(\frac{a_{pr}\pi r}{L}\right) \cos\left(\frac{a_{pz}\pi z}{L}\right) \end{bmatrix}, \quad \nabla p = \begin{bmatrix} \frac{a_{pr}\pi p_1}{L} \cos\left(\frac{a_{pr}\pi r}{L}\right) \cos\left(\frac{a_{pz}\pi z}{L}\right) \\ -\frac{a_{pz}\pi p_1}{L} \sin\left(\frac{a_{pr}\pi r}{L}\right) \sin\left(\frac{a_{pz}\pi z}{L}\right) \end{bmatrix},$$

$$\nabla u = \begin{bmatrix} -\frac{a_{ur}\pi u_1}{L} \sin\left(\frac{a_{ur}\pi r}{L}\right) \sin\left(\frac{a_{uz}\pi z}{L}\right) \\ \frac{a_{uz}\pi u_1}{L} \left(\cos\left(\frac{a_{ur}\pi r}{L}\right) - 1\right) \cos\left(\frac{a_{uz}\pi z}{L}\right) \end{bmatrix}, \quad \nabla w = \begin{bmatrix} -\frac{a_{wr}\pi w_1}{L} \sin\left(\frac{a_{wr}\pi r}{L}\right) \sin\left(\frac{a_{wz}\pi z}{L}\right) \\ \frac{a_{wz}\pi w_1}{L} \cos\left(\frac{a_{wr}\pi r}{L}\right) \cos\left(\frac{a_{wz}\pi z}{L}\right) \end{bmatrix},$$

are written in C language as:

```
grad_rho_an[0] = -rho_1 * sin(a_rhor * pi * r / L) * a_rhor * pi / L * sin(a_rhoz * pi * z / L);
grad_rho_an[1] = 0;
grad_rho_an[2] = rho_1 * cos(a_rhor * pi * r / L) * cos(a_rhoz * pi * z / L) * a_rhoz * pi / L;
grad_p_an[0] = p_1 * cos(a_pr * pi * r / L) * a_pr * pi / L * cos(a_pz * pi * z / L);
grad_p_an[1] = 0;
grad_p_an[2] = -p_1 * sin(a_pr * pi * r / L) * sin(a_pz * pi * z / L) * a_pz * pi / L;
grad_u_an[0] = -u_1 * sin(a_ur * pi * r / L) * a_ur * pi / L * sin(a_uz * pi * z / L);
grad_u_an[1] = 0;
grad_u_an[2] = u_1 * (cos(a_ur * pi * r / L) - 0.1e1) * cos(a_uz * pi * z / L) * a_uz * pi / L;
grad_w_an[0] = -w_1 * sin(a_wr * pi * r / L) * a_wr * pi / L * sin(a_wz * pi * z / L);
grad_w_an[1] = 0;
grad_w_an[2] = w_1 * cos(a_wr * pi * r / L) * cos(a_wz * pi * z / L) * a_wz * pi / L;
```

## References

- [1] F. Domenichini and B. Baccani, “A formulation of Navier–Stokes problem in cylindrical coordinates applied to the 3D entry jet in a duct,” *Journal of Computational Physics*, vol. 200, p. 177–191, 2004.