Manufactured Solution for the Compressible Steady Euler Equations using Maple

Kemelli C. Estacio-Hiroms

November 4, 2010

Abstract

The Method of Manufactured Solutions is a valuable approach for code verification, providing means to verify how accurately the numerical method solves the partial differential equations of interest. This document presents the source terms generated by the application of the Method of Manufactured Solutions (MMS) on the 1D, 2D, 3D steady Euler equations using the analytical manufactured solutions for density, velocity and pressure presented by Roy et al. (2002).

1 Mathematical Model

The conservation of mass, momentum, and total energy for a compressible steady inviscid fluid may be written as:

$$\nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1}$$

$$\nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p, \tag{2}$$

$$\nabla \cdot (\rho \boldsymbol{u} H) = 0. \tag{3}$$

Equations (1)–(3) are known as Euler equations and, ρ is the density, $\boldsymbol{u} = (u, v, w)$ is the velocity in x, y or z-direction, respectively, and p is the pressure. The total enthalpy, H, may be expressed in terms of the total energy per unit mass e_t, density, and pressure:

$$H = e_t + \frac{p}{\rho}.$$

For a calorically perfect gas, the Euler equations are closed with two auxiliary relations for energy:

$$e_t = e + \frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}, \quad \text{and} \quad e = \frac{1}{\gamma - 1} RT, \tag{4}$$

and with the ideal gas equation of state:

$$p = \rho R T. \tag{5}$$

2 Manufactured Solution

The Method of Manufactured Solutions (MMS) applied to Euler equations consists in modifying Equations (1) – (3) by adding a source term to the right-hand side of each equation, so the modified set of equations conveniently has the analytical solution chosen *a priori*.

Roy et al. (2002) propose the general form of the primitive manufactured solution variables to be a function of sines and cosines:

$$\phi(x,y,z) = \phi_0 + \phi_x\, f_s\left(\frac{a_{\phi x}\pi x}{L}\right) + \phi_y\, f_s\left(\frac{a_{\phi y}\pi y}{L}\right) + \phi_z\, f_s\left(\frac{a_{\phi z}\pi z}{L}\right), \tag{6}$$

where $\phi = \rho, u, v, w$ or p, and $f_s(\cdot)$ functions denote either sine or cosine function. Note that in this case, ϕ_x, ϕ_y and ϕ_z are constants and the subscripts do not denote differentiation.

Although Roy et al. (2002) provide the constants used in the manufactured solutions for the 2D supersonic and subsonic cases for Euler and Navier-Stokes equations, only the source term for the 2D mass conservation equation (1) is presented.

Source terms for mass conservation (Q_ρ), momentum (Q_u, Q_v and Q_w) and total energy (Q_{e_t}) equations are obtained by symbolic manipulations of compressible steady Euler equations above using Maple 13 (Maplesoft, 2009) and are presented in the following sections for the one, two and three-dimensional cases.

3 1D Steady Euler equations

The manufactured analytical solutions (6) for each one of the variables in one-dimensional case of Euler equations are:

$$\begin{aligned} \rho(x) &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right), \\ u(x) &= u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right), \\ p(x) &= p_0 + p_x \cos\left(\frac{a_{px}\pi x}{L}\right). \end{aligned} \tag{7}$$

The MMS applied to Euler equations consists in modifying the 1D Euler equations (1) – (3) by adding a source term to the right-hand side of each equation:

$$\begin{aligned} \frac{\partial(\rho u)}{\partial x} &= Q_\rho, \\ \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(p)}{\partial x} &= Q_u, \\ \frac{\partial(\rho u e_t)}{\partial x} + \frac{\partial(pu)}{\partial x} &= Q_{e_t}, \end{aligned} \tag{8}$$

so the modified set of equations (8) conveniently has the analytical solution given in Equation (7).

Source terms Q_ρ, Q_u and Q_{e_t} are obtained by symbolic manipulations of equations above using Maple and are presented in the following sections. The following auxiliary variables have been included in order to improve readability and computational efficiency:

$$\begin{aligned} \texttt{Rho}_1 &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right), \\ \texttt{U}_1 &= u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right), \\ \texttt{P}_1 &= p_0 + p_x \cos\left(\frac{a_{px}\pi x}{L}\right), \end{aligned}$$

where the subscripts in $\texttt{Rho}_1$, $\texttt{P}_1$ and $\texttt{U}_1$ refer to the 1D case.

3.1 1D Mass Conservation

The mass conservation equation written as an operator is:

$$\mathcal{L} = \frac{\partial(\rho u)}{\partial x}.$$

Analytically differentiating Equation (7) for ρ and u using operator $\mathcal{L}$ defined above gives the source term Q_ρ:

$$\begin{aligned} Q_\rho = &\frac{a_{\rho x}\pi\rho_x\,\texttt{U}_1}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\ &+ \frac{a_{ux}\pi u_x\,\texttt{Rho}_1}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right). \end{aligned} \tag{9}$$

3.2 1D Momentum

For the generation of the analytical source term Q_u for the x-momentum equation, Equation (2) is written as an operator $\mathcal{L}$:

$$\mathcal{L} = \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(p)}{\partial x},$$

which, when operated in Equation (7), provides source term Q_u:

$$\begin{aligned} Q_u = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U}_1^2}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\ & -\frac{a_{px}\pi p_x}{L}\sin\left(\frac{a_{px}\pi x}{L}\right) + \\ & +\frac{2a_{ux}\pi u_x\,\mathtt{Rho}_1\,\mathtt{U}_1}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right). \end{aligned} \tag{10}$$

3.3 1D Total Energy

The last component of Euler equations is written as an operator:

$$\mathcal{L} = \frac{\partial(\rho u e_t)}{\partial x} + \frac{\partial(pu)}{\partial x}.$$

Source term Q_e is obtained by operating $\mathcal{L}$ on Equation (7) together with the use of the auxiliary relations (4)–(5) for energy:

$$\begin{aligned} Q_e = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U}_1^3}{2L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\ & -\frac{\gamma}{\gamma-1}\frac{a_{px}\pi p_x\,\mathtt{U}_1}{L}\sin\left(\frac{a_{px}\pi x}{L}\right) + \\ & +\frac{\gamma}{\gamma-1}\frac{a_{ux}\pi u_x\,\mathtt{P}_1}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right) + \\ & +\frac{3a_{ux}\pi u_x\,\mathtt{Rho}_1\,\mathtt{U}_1^2}{2L}\cos\left(\frac{a_{ux}\pi x}{L}\right). \end{aligned} \tag{11}$$

4 2D Steady Euler equations

The manufactured analytical solutions (6) for each one of the variables in two-dimensional case of Euler equations are:

$$\begin{aligned} \rho(x,y) &= \rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right), \\ u(x,y) &= u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right), \\ v(x,y) &= v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right), \\ p(x,y) &= p_0 + p_x\cos\left(\frac{a_{px}\pi x}{L}\right) + p_y\sin\left(\frac{a_{py}\pi y}{L}\right). \end{aligned} \tag{12}$$

Analogously to the 1D case, MMS applied to the 2D steady Euler consists in modifying Equations (1) – (3) by adding a source term to the right-hand side of each equation:

$$\begin{aligned} &\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = Q_\rho \\ &\frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(p)}{\partial x} = Q_u \\ &\frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(p)}{\partial y} = Q_v \\ &\frac{\partial(\rho u e_t)}{\partial x} + \frac{\partial(\rho v e_t)}{\partial y} + \frac{\partial(pu)}{\partial x} + \frac{\partial(pv)}{\partial y} = Q_{e_t} \end{aligned} \tag{13}$$

so the modified set of Equations (13) has Equation (12) as analytical solution.

Source terms Q_ρ, Q_u, Q_v and Q_{e_t} are presented in the subsequent sessions with the use of the auxiliary variables:

$$\begin{aligned}
\mathtt{Rho}_2 &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y \cos\left(\frac{a_{\rho y}\pi y}{L}\right),\\
\mathtt{U}_2 &= u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y \cos\left(\frac{a_{uy}\pi y}{L}\right),\\
\mathtt{V}_2 &= v_0 + v_x \cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y \sin\left(\frac{a_{vy}\pi y}{L}\right),\\
\mathtt{P}_2 &= p_0 + p_x \cos\left(\frac{a_{px}\pi x}{L}\right) + p_y \sin\left(\frac{a_{py}\pi y}{L}\right),
\end{aligned}$$

where the subscripts in $\mathtt{Rho}_2$, $\mathtt{P}_2$, $\mathtt{U}_2$ and $\mathtt{V}_2$ refer to the 2D case.

4.1 2D Mass Conservation

The 2D mass conservation equation written as an operator is:

$$\mathcal{L} = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}.$$

Analytically differentiating Equation (12) for ρ, u and v using operator $\mathcal{L}$ defined above gives the source term Q_ρ:

$$\begin{aligned}
Q_\rho = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U}_2}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\
& - \frac{a_{\rho y}\pi\rho_y\,\mathtt{V}_2}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) + \\
& + \frac{\pi\,\mathtt{Rho}_2}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right].
\end{aligned} \tag{14}$$

4.2 2D Momentum

For the generation of the analytical source term Q_u for the x-momentum equation, the first component of Equation (2) is written as an operator $\mathcal{L}$:

$$\mathcal{L} = \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(p)}{\partial x},$$

which, when operated in Equation (12), provides source term Q_u:

$$\begin{aligned}
Q_u = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U}_2^2}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\
& - \frac{a_{\rho y}\pi\rho_y\,\mathtt{U}_2\,\mathtt{V}_2}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) + \\
& - \frac{a_{px}\pi p_x}{L}\sin\left(\frac{a_{px}\pi x}{L}\right) + \\
& + \frac{\pi\,\mathtt{Rho}_2\,\mathtt{U}_2}{L}\left[2a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right] + \\
& - \frac{a_{uy}\pi u_y\,\mathtt{Rho}_2\,\mathtt{V}_2}{L}\sin\left(\frac{a_{uy}\pi y}{L}\right).
\end{aligned} \tag{15}$$

Analogously, for the generation of the analytical source term Q_v for the y-momentum equation, the second component of Equation (2) is written as an operator $\mathcal{L}$:

$$\mathcal{L} = \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(p)}{\partial y},$$

and then applied to Equation (12). It yields:

$$
\begin{aligned}
Q_v = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U_2}\,\mathtt{V_2}}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)+\\
& -\frac{a_{\rho y}\pi\rho_y\,\mathtt{V_2^2}}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)+\\
& +\frac{a_{py}\pi p_y}{L}\cos\left(\frac{a_{py}\pi y}{L}\right)+\\
& -\frac{a_{vx}\pi v_x\,\mathtt{Rho_2}\,\mathtt{U_2}}{L}\sin\left(\frac{a_{vx}\pi x}{L}\right)+\\
& +\frac{\pi\,\mathtt{Rho_2}\,\mathtt{V_2}}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+2v_y a_{vy}\cos\left(\frac{a_{vy}\pi y}{L}\right)\right].
\end{aligned}
\tag{16}
$$

4.3 2D Total Energy

The operator for the 2D Euler total energy is:

$$
\mathcal{L}=\frac{\partial(\rho u e_t)}{\partial x}+\frac{\partial(\rho v e_t)}{\partial y}+\frac{\partial(pu)}{\partial x}+\frac{\partial(pv)}{\partial y}.
$$

Source term Q_{e_t} is obtained by operating $\mathcal{L}$ on Equation (12) together with the use of the auxiliary relations (4) – (5) for energy :

$$
\begin{aligned}
Q_{e_t} = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U_2}(\mathtt{U_2^2}+\mathtt{V_2^2})}{2L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)+\\
& -\frac{a_{\rho y}\pi\rho_y\,\mathtt{V_2}(\mathtt{U_2^2}+\mathtt{V_2^2})}{2L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)+\\
& -\frac{\gamma}{\gamma-1}\frac{a_{px}\pi p_x\,\mathtt{U_2}}{L}\sin\left(\frac{a_{px}\pi x}{L}\right)+\\
& +\frac{\gamma}{\gamma-1}\frac{a_{py}\pi p_y\,\mathtt{V_2}}{L}\cos\left(\frac{a_{py}\pi y}{L}\right)+\\
& +\frac{\pi\,\mathtt{Rho_2}\,\mathtt{U_2^2}}{2L}\left[3a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right]+\\
& -\frac{\pi\,\mathtt{Rho_2}\,\mathtt{U_2}\,\mathtt{V_2}}{L}\left[a_{uy}u_y\sin\left(\frac{a_{uy}\pi y}{L}\right)+a_{vx}v_x\sin\left(\frac{a_{vx}\pi x}{L}\right)\right]+\\
& +\frac{\pi\,\mathtt{Rho_2}\,\mathtt{V_2^2}}{2L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+3a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right]+\\
& +\frac{\gamma}{\gamma-1}\frac{\pi\,\mathtt{P_2}}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)\right].
\end{aligned}
\tag{17}
$$

5 3D Steady Euler equations

The manufactured analytical solution for for each one of the variables in steady Euler equations are:

$$
\begin{aligned}
\rho(x,y,z) &= \rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right),\\
u(x,y,z) &= u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right),\\
v(x,y,z) &= v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right),\\
w(x,y,z) &= w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right),\\
p(x,y,z) &= p_0+p_x\cos\left(\frac{a_{px}\pi x}{L}\right)+p_y\sin\left(\frac{a_{py}\pi y}{L}\right)+p_z\cos\left(\frac{a_{pz}\pi z}{L}\right).
\end{aligned}
\tag{18}
$$

The MMS applied to 3D Euler equations consists in modifying Equations (1)–(3) by adding a source term to the right-hand side of each equation:

$$
\begin{aligned}
&\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=Q_\rho\\
&\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}+\frac{\partial(p)}{\partial x}=Q_u,\\
&\frac{\partial(\rho vu)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}+\frac{\partial(p)}{\partial y}=Q_v,\\
&\frac{\partial(\rho wu)}{\partial x}+\frac{\partial(\rho wv)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}+\frac{\partial(p)}{\partial z}=Q_w,\\
&\frac{\partial(\rho ue_t)}{\partial x}+\frac{\partial(\rho ve_t)}{\partial y}+\frac{\partial(\rho we_t)}{\partial z}+\frac{\partial(pu)}{\partial x}+\frac{\partial(pv)}{\partial y}+\frac{\partial(pw)}{\partial z}=Q_{e_t},
\end{aligned}
\tag{19}
$$

so this modified set of equations has for analytical solution Equation (18).

Analogously to the 1D and 2D cases, the source terms Q_ρ, Q_u, Q_v, Q_w and Q_{e_t} are presented with the use of the following auxiliary variables:

$$
\begin{aligned}
\mathtt{Rho}_3&=\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right),\\
\mathtt{U}_3&=u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right),\\
\mathtt{V}_3&=v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right),\\
\mathtt{W}_3&=w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right),\\
\mathtt{P}_3&=p_0+p_x\cos\left(\frac{a_{px}\pi x}{L}\right)+p_y\sin\left(\frac{a_{py}\pi y}{L}\right)+p_z\cos\left(\frac{a_{pz}\pi z}{L}\right).
\end{aligned}
$$

where, again, the subscripts in $\mathtt{Rho}_3$, $\mathtt{P}_3$, $\mathtt{U}_3$, $\mathtt{V}_3$ and $\mathtt{W}_3$ refer to the 3D case.

5.1 3D Mass Conservation

The source term Q_ρ for the 3D mass conservation equation is:

$$
\begin{aligned}
Q_\rho=&\frac{a_{\rho x}\pi\rho_x\,\mathtt{U}_3}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)+\\
&-\frac{a_{\rho y}\pi\rho_y\,\mathtt{V}_3}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)+\\
&+\frac{a_{\rho z}\pi\rho_z\,\mathtt{W}_3}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right)+\\
&+\frac{\pi\,\mathtt{Rho}_3}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)-a_{wz}w_z\sin\left(\frac{a_{wz}\pi z}{L}\right)\right].
\end{aligned}
\tag{20}
$$

5.2 3D Momentum

The source terms Q_u, Q_v and Q_w for the 3D momentum equations the in x, y and z directions are, respectively:

$$
\begin{aligned}
Q_u = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U}_3^2}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\
& - \frac{a_{\rho y}\pi\rho_y\,\mathtt{U}_3\,\mathtt{V}_3}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) + \\
& + \frac{a_{\rho z}\pi\rho_z\,\mathtt{U}_3\,\mathtt{W}_3}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right) + \\
& - \frac{a_{px}\pi p_x}{L}\sin\left(\frac{a_{px}\pi x}{L}\right) + \\
& + \frac{\pi\,\mathtt{Rho}_3\,\mathtt{U}_3}{L}\left[2a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right) - a_{wz}w_z\sin\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
& - \frac{a_{uy}\pi u_y\,\mathtt{Rho}_3\,\mathtt{V}_3}{L}\sin\left(\frac{a_{uy}\pi y}{L}\right) + \\
& - \frac{a_{uz}\pi u_z\,\mathtt{Rho}_3\,\mathtt{W}_3}{L}\sin\left(\frac{a_{uz}\pi z}{L}\right),
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
Q_v = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U}_3\,\mathtt{V}_3}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\
& - \frac{a_{\rho y}\pi\rho_y\,\mathtt{V}_3^2}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) + \\
& + \frac{a_{\rho z}\pi\rho_z\,\mathtt{V}_3\,\mathtt{W}_3}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right) + \\
& + \frac{a_{py}\pi p_y}{L}\cos\left(\frac{a_{py}\pi y}{L}\right) + \\
& - \frac{a_{vx}\pi v_x\,\mathtt{Rho}_3\,\mathtt{U}_3}{L}\sin\left(\frac{a_{vx}\pi x}{L}\right) + \\
& + \frac{\pi\,\mathtt{Rho}_3\,\mathtt{V}_3}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + 2a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right) - a_{wz}w_z\sin\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
& + \frac{a_{vz}\pi v_z\,\mathtt{Rho}_3\,\mathtt{W}_3}{L}\cos\left(\frac{a_{vz}\pi z}{L}\right),
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
Q_w = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U}_3\,\mathtt{W}_3}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) + \\
& - \frac{a_{\rho y}\pi\rho_y\,\mathtt{V}_3\,\mathtt{W}_3}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) + \\
& + \frac{a_{\rho z}\pi\rho_z\,\mathtt{W}_3^2}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right) + \\
& - \frac{a_{pz}\pi p_z}{L}\sin\left(\frac{a_{pz}\pi z}{L}\right) + \\
& + \frac{a_{wx}\pi w_x\,\mathtt{Rho}_3\,\mathtt{U}_3}{L}\cos\left(\frac{a_{wx}\pi x}{L}\right) + \\
& + \frac{a_{wy}\pi w_y\,\mathtt{Rho}_3\,\mathtt{V}_3}{L}\cos\left(\frac{a_{wy}\pi y}{L}\right) + \\
& + \frac{\pi\,\mathtt{Rho}_3\,\mathtt{W}_3}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right) + a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right) - 2a_{wz}w_z\sin\left(\frac{a_{wz}\pi z}{L}\right)\right].
\end{aligned}
\tag{23}
$$

5.3 3D Total Energy

Finally, the source term Q_{e_t} for the 3D total energy equation is:

$$
\begin{aligned}
Q_{e_t} = & \frac{a_{\rho x}\pi\rho_x\,\mathtt{U}_3(\,\mathtt{U}_3^2+\,\mathtt{V}_3^2+\,\mathtt{W}_3^2)}{2L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)+\\
& -\frac{a_{\rho y}\pi\rho_y\,\mathtt{V}_3(\,\mathtt{U}_3^2+\,\mathtt{V}_3^2+\,\mathtt{W}_3^2)}{2L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)+\\
& +\frac{a_{\rho z}\pi\rho_z\,\mathtt{W}_3(\,\mathtt{U}_3^2+\,\mathtt{V}_3^2+\,\mathtt{W}_3^2)}{2L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right)+\\
& -\frac{\gamma}{\gamma-1}\frac{a_{px}\pi p_x\,\mathtt{U}_3}{L}\sin\left(\frac{a_{px}\pi x}{L}\right)+\\
& +\frac{\gamma}{\gamma-1}\frac{a_{py}\pi p_y\,\mathtt{V}_3}{L}\cos\left(\frac{a_{py}\pi y}{L}\right)+\\
& -\frac{\gamma}{\gamma-1}\frac{a_{pz}\pi p_z\,\mathtt{W}_3}{L}\sin\left(\frac{a_{pz}\pi z}{L}\right)+\\
& +\frac{\gamma}{\gamma-1}\frac{\mathtt{P}_3\pi}{L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)-a_{wz}w_z\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
& +\frac{\pi\,\mathtt{Rho}_3\,\mathtt{U}_3^2}{2L}\left[3a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)-a_{wz}w_z\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
& -\frac{\pi\,\mathtt{Rho}_3\,\mathtt{U}_3\,\mathtt{V}_3}{L}\left[a_{uy}u_y\sin\left(\frac{a_{uy}\pi y}{L}\right)+a_{vx}v_x\sin\left(\frac{a_{vx}\pi x}{L}\right)\right]+\\
& -\frac{\pi\,\mathtt{Rho}_3\,\mathtt{U}_3\,\mathtt{W}_3}{L}\left[a_{uz}u_z\sin\left(\frac{a_{uz}\pi z}{L}\right)-a_{wx}w_x\cos\left(\frac{a_{wx}\pi x}{L}\right)\right]+\\
& +\frac{\pi\,\mathtt{Rho}_3\,\mathtt{V}_3^2}{2L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+3a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)-a_{wz}w_z\sin\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
& +\frac{\pi\,\mathtt{Rho}_3\,\mathtt{V}_3\,\mathtt{W}_3}{L}\left[a_{vz}v_z\cos\left(\frac{a_{vz}\pi z}{L}\right)+a_{wy}w_y\cos\left(\frac{a_{wy}\pi y}{L}\right)\right]+\\
& +\frac{\pi\,\mathtt{Rho}_3\,\mathtt{W}_3^2}{2L}\left[a_{ux}u_x\cos\left(\frac{a_{ux}\pi x}{L}\right)+a_{vy}v_y\cos\left(\frac{a_{vy}\pi y}{L}\right)-3a_{wz}w_z\sin\left(\frac{a_{wz}\pi z}{L}\right)\right].
\end{aligned}
\tag{24}
$$

6 Comments

The complexity, and consequently length, of the source terms increase with both dimension and physics handled by the governing equations. In some cases, such as the 3D energy equation, the final expression for Q_{e_t} may reach 139,000 characters, including parenthesis and mathematical operators, prior to factorization.

Applying commands in order to simplify such extensive expression is challenging even with a very good machine; thus, a suitable alternative to this issue is to simplify the equation by dividing it into a combination of sub-operators handling different physical phenomena. Then, each one of the operators may be applied to the manufactured solutions individually, and the resulting sub-source terms are combined back to represent the source term for the original equation.

For instance, instead of writing the three-dimensional steady Euler energy equation using one single operator $\mathcal{L}$:

$$
\mathcal{L} = \nabla\cdot(\rho\mathbf{u}e_t)+\nabla\cdot(p\mathbf{u}),
\tag{25}
$$

to then be used in the MMS, let equation (25) be written with two distinct operators, according to their physical meaning:

$$
\begin{aligned}
\mathcal{L}_1 &= \nabla\cdot(\rho\mathbf{u}e_t),\\
\mathcal{L}_2 &= \nabla\cdot(p\mathbf{u}),
\end{aligned}
$$

where $\mathcal{L}_1$ is the net rate of internal and kinetic energy increase by convection and $\mathcal{L}_2$ is the rate of work done on the fluid by external body forces. Naturally:

$$
\mathcal{L} = \mathcal{L}_1+\mathcal{L}_2.
$$

After the application of $\mathcal{L}_1$ and $\mathcal{L}_2$, the corresponding sub-source terms are also simplified, factorized and sorted. Then, the final factorized version is checked against the original one, to assure that not human error has been introduced. This strategy allowed the original 139,000 character-long expression for Q_{e_t} to be reduced to 3,000, and expressed in Equation (24).

6.1 Boundary Conditions

Additionally to verifying code capability of solving the governing equations accurately in the interior of the domain of interest, one may also verify the software capability of correctly imposing boundary conditions. Therefore, the gradients of the analytical solutions (6) have been calculated and translated into C codes. For the 3D case, they are:

$$\nabla\rho = \begin{bmatrix} \frac{a_{\rho x}\pi\rho_x}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) \\ -\frac{a_{\rho y}\pi\rho_y}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) \\ \frac{a_{\rho z}\pi\rho_z}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right) \end{bmatrix}, \qquad \nabla p = \begin{bmatrix} -\frac{a_{px}\pi p_x}{L}\sin\left(\frac{a_{px}\pi x}{L}\right) \\ \frac{a_{py}\pi p_y}{L}\cos\left(\frac{a_{py}\pi y}{L}\right) \\ -\frac{a_{pz}\pi p_z}{L}\sin\left(\frac{a_{pz}\pi z}{L}\right) \end{bmatrix}, \quad \nabla u = \begin{bmatrix} \frac{a_{ux}\pi u_x}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right) \\ -\frac{a_{uy}\pi u_y}{L}\sin\left(\frac{a_{uy}\pi y}{L}\right) \\ -\frac{a_{uz}\pi u_z}{L}\sin\left(\frac{a_{uz}\pi z}{L}\right) \end{bmatrix},$$

$$\nabla v = \begin{bmatrix} -\frac{a_{vx}\pi v_x}{L}\sin\left(\frac{a_{vx}\pi x}{L}\right) \\ \frac{a_{vy}\pi v_y}{L}\cos\left(\frac{a_{vy}\pi y}{L}\right) \\ \frac{a_{vz}\pi v_z}{L}\cos\left(\frac{a_{vz}\pi z}{L}\right) \end{bmatrix} \quad \text{and} \quad \nabla w = \begin{bmatrix} \frac{a_{wx}\pi w_x}{L}\cos\left(\frac{a_{wx}\pi x}{L}\right) \\ \frac{a_{wy}\pi w_y}{L}\cos\left(\frac{a_{wy}\pi y}{L}\right) \\ -\frac{a_{wz}\pi w_z}{L}\sin\left(\frac{a_{wz}\pi z}{L}\right) \end{bmatrix}$$

6.2 C Files

The C files for both source terms and gradients of the manufactured solutions are:

- `Euler_1d_steady_codes.C`,
- `Euler_1d_manuf_solutions_grad_code.C`
- `Euler_2d_steady_e_code.C`
- `Euler_2d_steady_rho_code.C`
- `Euler_2d_steady_u_code.C`
- `Euler_2d_steady_v_code.C`
- `Euler_2d_manuf_solutions_grad_code.C`
- `Euler_3d_steady_e_code.C`
- `Euler_3d_steady_rho_code.C`
- `Euler_3d_steady_u_code.C`
- `Euler_3d_steady_v_code.C`
- `Euler_3d_steady_w_code.C`
- `Euler_3d_manuf_solutions_grad_code.C`

An example of the automatically generated C file from the source term for the 3D total energy source term Q_{e_t} is:

```
#include <math.h>

double SourceQ_e (double x, double y, double z)
{
  double Qe;
  double RHO;
```

```
  double P;
  double U;
  double V;
  double W;

  RHO = rho_0 + rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L)
    + rho_z * sin(a_rhoz * PI * z / L);
  U = u_0 + u_x * sin(a_ux * PI * x / L) + u_y * cos(a_uy * PI * y / L) + u_z * cos(a_uz * PI * z / L);
  V = v_0 + v_x * cos(a_vx * PI * x / L) + v_y * sin(a_vy * PI * y / L) + v_z * sin(a_vz * PI * z / L);
  W = w_0 + w_x * sin(a_wx * PI * x / L) + w_y * sin(a_wy * PI * y / L) + w_z * cos(a_wz * PI * z / L);
  P = p_0 + p_x * cos(a_px * PI * x / L) + p_y * sin(a_py * PI * y / L) + p_z * cos(a_pz * PI * z / L);

  Qe = -a_px * PI * p_x * Gamma * U * sin(a_px * PI * x / L) / (Gamma - 0.1e1) / L
    + a_py * PI * p_y * Gamma * V * cos(a_py * PI * y / L) / (Gamma - 0.1e1) / L
    - a_pz * PI * p_z * Gamma * W * sin(a_pz * PI * z / L) / (Gamma - 0.1e1) / L
    + (U * U + V * V + W * W) * a_rhox * PI * rho_x * U * cos(a_rhox * PI * x / L) / L / 0.2e1
    - (U * U + V * V + W * W) * a_rhoy * PI * rho_y * V * sin(a_rhoy * PI * y / L) / L / 0.2e1
    + (U * U + V * V + W * W) * a_rhoz * PI * rho_z * W * cos(a_rhoz * PI * z / L) / L / 0.2e1
    - (-0.3e1 * a_ux * u_x * cos(a_ux * PI * x / L) - a_vy * v_y * cos(a_vy * PI * y / L)
      + a_wz * w_z * sin(a_wz * PI * z / L)) * PI * RHO * U * U / L / 0.2e1
    - (a_uy * u_y * sin(a_uy * PI * y / L) + a_vx * v_x * sin(a_vx * PI * x / L)) * PI * RHO * U * V / L
    - (a_uz * u_z * sin(a_uz * PI * z / L) - a_wx * w_x * cos(a_wx * PI * x / L)) * PI * RHO * U * W / L
    - (-a_ux * u_x * cos(a_ux * PI * x / L) - 0.3e1 * a_vy * v_y * cos(a_vy * PI * y / L)
      + a_wz * w_z * sin(a_wz * PI * z / L)) * PI * RHO * V * V / L / 0.2e1
    + (a_vz * v_z * cos(a_vz * PI * z / L) + a_wy * w_y * cos(a_wy * PI * y / L)) * PI * RHO * V * W / L
    - (-a_ux * u_x * cos(a_ux * PI * x / L) - a_vy * v_y * cos(a_vy * PI * y / L)
      + 0.3e1 * a_wz * w_z * sin(a_wz * PI * z / L)) * PI * RHO * W * W / L / 0.2e1
    - (-a_ux * u_x * cos(a_ux * PI * x / L) - a_vy * v_y * cos(a_vy * PI * y / L)
      + a_wz * w_z * sin(a_wz * PI * z / L)) * PI * Gamma * P / (Gamma - 0.1e1) / L;
  return(Qe);
}
```

References

Maplesoft (2009, November). Maple: the essential tool for mathematics and modeling. http://www.maplesoft.com/Products/Maple/.

Roy, C., T. Smith, and C. Ober (2002). Verification of a compressible CFD code using the method of manufactured solutions. In *AIAA Fluid Dynamics Conference and Exhibit*, Number AIAA 2002-3110.