Manufactured Solution for 3D Navier-Stokes equation using Maple[*]

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Abstract

This document presents the source terms generated by the application of the Method of Manufactured Solutions (MMS) on the the 3D Navier-Stokes equations using the analytical manufactured solutions for ρ, u, v, w and p presented by Roy et al. (2002).

1 3D Navier-Stokes Equations

The 3D Navier-Stokes equations in conservation form are:

$$\frac{\partial(\rho)}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{1}$$

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+p-\tau_{xx})}{\partial x}+\frac{\partial(\rho uv-\tau_{xy})}{\partial y}+\frac{\partial(\rho uw-\tau_{xz})}{\partial z}=0 \tag{2}$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho vu-\tau_{yx})}{\partial x}+\frac{\partial(\rho v^2+p-\tau_{yy})}{\partial y}+\frac{\partial(\rho vw-\tau_{yz})}{\partial z}=0 \tag{3}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho wu-\tau_{zx})}{\partial x}+\frac{\partial(\rho wv-\tau_{zy})}{\partial y}+\frac{\partial(\rho w^2+p-\tau_{zz})}{\partial z}=0 \tag{4}$$

$$\frac{\partial(\rho e_t)}{\partial t}+\frac{\partial(\rho ue_t+pu-u\tau_{xx}-v\tau_{xy}-w\tau_{xz}+q_x)}{\partial x}+\frac{\partial(\rho ve_t+pv-u\tau_{yx}-v\tau_{yy}-w\tau_{yz}+q_y)}{\partial y}+\frac{\partial(\rho we_t+pw-u\tau_{zx}-v\tau_{zy}-w\tau_{zz}+q_z)}{\partial z}=0 \tag{5}$$

where the Equation (1) is mass conservation, Equations (2) – (4) are momentum, and Equation (5) is the energy. Notice that Equations (2)–(4) include viscous effects.

For a calorically perfect gas, the Navier-Stokes equations are closed with two auxiliary relations for energy:

$$e_t=e+\frac{u^2+v^2+w^2}{2} \quad \text{and} \quad e=\frac{1}{\gamma-1}RT, \tag{6}$$

[*]The manufactured solution has been presented by Roy, Smith, and Ober (2002).

where γ is the ratio of specific heats, and with the ideal gas equation of state:

$$p = \rho R T. \tag{7}$$

The shear stress tensor is:

$$\begin{aligned}
\tau_{xx} &= \tfrac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right), \quad & \tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\
\tau_{yy} &= \tfrac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right), \quad & \tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \\
\tau_{zz} &= \tfrac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right), \quad & \tau_{xz} &= \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),
\end{aligned} \tag{8}$$

where μ is the absolute viscosity. The heat flux vector is given by:

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}, \quad \text{and} \quad q_z = -k\frac{\partial T}{\partial z} \tag{9}$$

where k is the thermal conductivity, which can be determined by choosing the Prandtl number:

$$k = \frac{\gamma R \mu}{(\gamma - 1) Pr}.$$

2 Manufactured Solution

Roy et al. (2002) introduce the general form of the primitive solution variables to be a function of sines and cosines:

$$\phi(x, y) = \phi_0 + \phi_x\, f_s\left(\frac{a_{\phi x}\pi x}{L}\right) + \phi_y\, f_s\left(\frac{a_{\phi y}\pi y}{L}\right) + \phi_z\, f_s\left(\frac{a_{\phi z}\pi z}{L}\right), \tag{10}$$

where $\phi = \rho, u, v, w$ or p, and $f_s(\cdot)$ functions denote either sine or cosine function. Note that in this case, ϕ_x, ϕ_y and ϕ_z are constants and the subscripts do not denote differentiation.

Therefore, the manufactured analytical solution for each one of the variables in Navier-Stokes equations are:

$$\begin{aligned}
\rho\,(x, y, z) &= \rho_0 + \rho_x \sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y \cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z \sin\left(\frac{a_{\rho z}\pi z}{L}\right), \\
u\,(x, y, z) &= u_0 + u_x \sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y \cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z \cos\left(\frac{a_{uz}\pi z}{L}\right), \\
v\,(x, y, z) &= v_0 + v_x \cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y \sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z \sin\left(\frac{a_{vz}\pi z}{L}\right), \\
w\,(x, y, z) &= w_0 + w_x \sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y \sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z \cos\left(\frac{a_{wz}\pi z}{L}\right), \\
p\,(x, y, z) &= p_0 + p_x \cos\left(\frac{a_{px}\pi x}{L}\right) + p_y \sin\left(\frac{a_{py}\pi y}{L}\right) + p_z \cos\left(\frac{a_{pz}\pi z}{L}\right).
\end{aligned} \tag{11}$$

The MMS applied to Navier-Stokes equations consists in modifying Equations (1) – (5) by adding a source term to the right-hand side of each equation:

$$
\begin{aligned}
&\frac{\partial(\rho)}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=Q_\rho\\
&\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+p-\tau_{xx})}{\partial x}+\frac{\partial(\rho uv-\tau_{xy})}{\partial y}+\frac{\partial(\rho uw-\tau_{xz})}{\partial z}=Q_u,\\
&\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho vu-\tau_{yx})}{\partial x}+\frac{\partial(\rho v^2+p-\tau_{yy})}{\partial y}+\frac{\partial(\rho vw-\tau_{yz})}{\partial z}=Q_v,\\
&\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho wu-\tau_{zx})}{\partial x}+\frac{\partial(\rho wv-\tau_{zy})}{\partial y}+\frac{\partial(\rho w^2+p-\tau_{zz})}{\partial z}=Q_w,\\
&\frac{\partial(\rho e_t)}{\partial t}+\frac{\partial(\rho ue_t+pu-u\tau_{xx}-v\tau_{xy}-w\tau_{xz}+q_x)}{\partial x}+\frac{\partial(\rho ve_t+pv-u\tau_{yx}-v\tau_{yy}-w\tau_{yz}+q_y)}{\partial y}+\frac{\partial(\rho we_t+pw-u\tau_{zx}-v\tau_{zy}-w\tau_{zz}+q_z)}{\partial z}=Q_{e_t},
\end{aligned}
\tag{12}
$$

so the modified set of equations has known, analytical solution.

In the case of Q_ρ, Q_u, Q_v, Q_w and Q_{e_t} are conveniently obtained by analytical differentiation of Equation (11) using Equations (1) – (5) as differential operators, the solution of Equation (12) is Equation (11).

Such terms are obtained by symbolic manipulations of equations above using Maple and are presented in the following sections.

3 Source term for Navier-Stokes mass conservation equation

$$
\begin{aligned}
Q_\rho = & \frac{a_{\rho x}\pi\rho_x}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&-\frac{a_{\rho y}\pi\rho_y}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\left[v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]+\\
&+\frac{a_{\rho z}\pi\rho_z}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&+\frac{a_{ux}\pi u_x}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]+\\
&+\frac{a_{vy}\pi v_y}{L}\cos\left(\frac{a_{vy}\pi y}{L}\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]+\\
&-\frac{a_{wz}\pi w_z}{L}\sin\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]
\end{aligned}
\tag{13}
$$

4 Source term for Navier-Stokes momentum equation

$$
\begin{aligned}
Q_u =& -\frac{a_{px}\pi p_x}{L}\sin\left(\frac{a_{px}\pi x}{L}\right)+\\
&+\frac{a_{\rho x}\pi\rho_x}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]^2+\\
&-\frac{a_{\rho y}\pi\rho_y}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\left[v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&+\frac{a_{\rho z}\pi\rho_z}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&+\frac{2a_{ux}\pi u_x}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&-\frac{a_{uy}\pi u_y}{L}\sin\left(\frac{a_{uy}\pi y}{L}\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]+\\
&-\frac{a_{uz}\pi u_z}{L}\sin\left(\frac{a_{uz}\pi z}{L}\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&+\frac{a_{vy}\pi v_y}{L}\cos\left(\frac{a_{vy}\pi y}{L}\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&-\frac{a_{wz}\pi w_z}{L}\sin\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]+\\
&+\frac{4a_{ux}^2\pi^2\mu u_x}{3L^2}\sin\left(\frac{a_{ux}\pi x}{L}\right)\quad+\quad\frac{a_{uy}^2\pi^2\mu u_y}{L^2}\cos\left(\frac{a_{uy}\pi y}{L}\right)\quad+\quad\frac{a_{uz}^2\pi^2\mu u_z}{L^2}\cos\left(\frac{a_{uz}\pi z}{L}\right)
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
Q_v =& + \frac{a_{py}\pi p_y}{L}\cos\left(\frac{a_{py}\pi y}{L}\right) + \\
&+ \frac{a_{\rho x}\pi \rho_x}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right] + \\
&- \frac{a_{\rho y}\pi \rho_y}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]^2 + \\
&+ \frac{a_{\rho z}\pi \rho_z}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{ux}\pi u_x}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right] + \\
&- \frac{a_{vx}\pi v_x}{L}\sin\left(\frac{a_{vx}\pi x}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right] + \\
&+ \frac{2a_{vy}\pi v_y}{L}\cos\left(\frac{a_{vy}\pi y}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{vz}\pi v_z}{L}\cos\left(\frac{a_{vz}\pi z}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
&- \frac{a_{wz}\pi w_z}{L}\sin\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{vx}^2\pi^2\mu v_x}{L^2}\cos\left(\frac{a_{vx}\pi x}{L}\right) \quad + \quad \frac{4a_{vy}^2\pi^2\mu v_y}{3L^2}\sin\left(\frac{a_{vy}\pi y}{L}\right) \quad + \quad \frac{a_{vz}^2\pi^2\mu v_z}{L^2}\sin\left(\frac{a_{vz}\pi z}{L}\right)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
Q_w =& - \frac{a_{pz}\pi p_z}{L}\sin\left(\frac{a_{pz}\pi z}{L}\right) + \\
&+ \frac{a_{\rho x}\pi \rho_x}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right] + \\
&- \frac{a_{\rho y}\pi \rho_y}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{\rho z}\pi \rho_z}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]^2 + \\
&+ \frac{a_{ux}\pi u_x}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{vy}\pi v_y}{L}\cos\left(\frac{a_{vy}\pi y}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{wx}\pi w_x}{L}\cos\left(\frac{a_{wx}\pi x}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{wy}\pi w_y}{L}\cos\left(\frac{a_{wy}\pi y}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right] + \\
&- \frac{2a_{wz}\pi w_z}{L}\sin\left(\frac{a_{wz}\pi z}{L}\right)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{wx}^2\pi^2\mu w_x}{L^2}\sin\left(\frac{a_{wx}\pi x}{L}\right) \quad + \quad \frac{a_{wy}^2\pi^2\mu w_y}{L^2}\sin\left(\frac{a_{wy}\pi y}{L}\right) \quad + \quad \frac{4a_{wz}^2\pi^2\mu w_z}{3L^2}\cos\left(\frac{a_{wz}\pi z}{L}\right)
\end{aligned}
\tag{16}
$$

5 Source term for Navier-Stokes energy equation

$$
\begin{aligned}
Qe = &- \frac{a_{px}\pi p_x}{L}\frac{\gamma}{\gamma-1}\sin\left(\frac{a_{px}\pi x}{L}\right)\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{py}\pi p_y}{L}\frac{\gamma}{\gamma-1}\cos\left(\frac{a_{py}\pi y}{L}\right)\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right] + \\
&- \frac{a_{pz}\pi p_z}{L}\frac{\gamma}{\gamma-1}\sin\left(\frac{a_{pz}\pi z}{L}\right)\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right] + \\
&+ \frac{a_{\rho x}\pi \rho_x}{2L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right)\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]\Bigg(\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]^2 + \\
&\qquad + \left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]^2 + \left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]^2\Bigg) + \\
&- \frac{a_{\rho y}\pi \rho_y}{2L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right)\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]\Bigg(\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]^2 + \\
&\qquad + \left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]^2 + \left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]^2\Bigg) + \\
&+ \frac{a_{\rho z}\pi \rho_z}{2L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right)\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\Bigg(\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]^2 + \\
&\quad + \left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]^2 + \left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]^2\Bigg) + \\
&+ \frac{a_{ux}\pi u_x}{2L}\cos\left(\frac{a_{ux}\pi x}{L}\right)\Bigg\{\Bigg(\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]^2 + \left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]^2 + \\
&\qquad +3\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]^2\Bigg)\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right] + \\
&\qquad + \left[p_0 + p_x\cos\left(\frac{a_{px}\pi x}{L}\right) + p_y\sin\left(\frac{a_{py}\pi y}{L}\right) + p_z\cos\left(\frac{a_{pz}\pi z}{L}\right)\right]\frac{2\gamma}{(\gamma-1)}\Bigg\} + \\
&- \frac{a_{uy}\pi u_y}{L}\sin\left(\frac{a_{uy}\pi y}{L}\right)\left[v_0 + v_x\cos\left(\frac{a_{vx}\pi x}{L}\right) + v_y\sin\left(\frac{a_{vy}\pi y}{L}\right) + v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\cdot \\
&\qquad \cdot\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right] + \\
&- \frac{a_{uz}\pi u_z}{L}\sin\left(\frac{a_{uz}\pi z}{L}\right)\left[w_0 + w_x\sin\left(\frac{a_{wx}\pi x}{L}\right) + w_y\sin\left(\frac{a_{wy}\pi y}{L}\right) + w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\left[\rho_0 + \rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right) + \rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right) + \rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\cdot \\
&\qquad \cdot\left[u_0 + u_x\sin\left(\frac{a_{ux}\pi x}{L}\right) + u_y\cos\left(\frac{a_{uy}\pi y}{L}\right) + u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right] +
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&-\frac{a_{vx}\pi v_x}{L}\sin\left(\frac{a_{vx}\pi x}{L}\right)\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\cdot\\
&\qquad\cdot\left[v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]+\\
&+\frac{a_{vy}\pi v_y}{2L}\cos\left(\frac{a_{vy}\pi y}{L}\right)\left\{\left(\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]^2+\left[w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]^2+\right.\right.\\
&\qquad\left.+3\left[v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]^2\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]+\\
&\qquad\left.+\left[p_0+p_x\cos\left(\frac{a_{px}\pi x}{L}\right)+p_y\sin\left(\frac{a_{py}\pi y}{L}\right)+p_z\cos\left(\frac{a_{pz}\pi z}{L}\right)\right]\frac{2\gamma}{\gamma-1}\right\}+\\
&+\frac{a_{vz}\pi v_z}{L}\cos\left(\frac{a_{vz}\pi z}{L}\right)\left[w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\cdot\\
&\qquad\cdot\left[v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]+\\
&+\frac{a_{wx}\pi w_x}{L}\cos\left(\frac{a_{wx}\pi x}{L}\right)\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\cdot\\
&\qquad\cdot\left[w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&+\frac{a_{wy}\pi w_y}{L}\cos\left(\frac{a_{wy}\pi y}{L}\right)\left[v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\cdot\\
&\qquad\cdot\left[w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]+\\
&-\frac{a_{wz}\pi w_z}{2L}\sin\left(\frac{a_{wz}\pi z}{L}\right)\left\{\left(3\left[w_0+w_x\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_y\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_z\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]^2+\left[v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_z\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]^2+\right.\right.\\
&\qquad\left.+\left[u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_z\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]^2\right)\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]+\\
&\qquad\left.+\left[p_0+p_x\cos\left(\frac{a_{px}\pi x}{L}\right)+p_y\sin\left(\frac{a_{py}\pi y}{L}\right)+p_z\cos\left(\frac{a_{pz}\pi z}{L}\right)\right]\frac{2\gamma}{(\gamma-1)}\right\}+\\
&+\frac{a_{px}^2\pi^2 k p_x\cos\left(\frac{a_{px}\pi x}{L}\right)}{L^2R\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]}+\\
&+\frac{a_{py}^2\pi^2 k p_y\sin\left(\frac{a_{py}\pi y}{L}\right)}{L^2R\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]}+\\
&+\frac{a_{pz}^2\pi^2 k p_z\cos\left(\frac{a_{pz}\pi z}{L}\right)}{L^2R\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]}+
\end{aligned}
$$

$$
-\frac{a_{\rho x}^{2}\pi^{2}k\rho_{x}\left[2\cos\left(\frac{a_{\rho x}\pi x}{L}\right)^{2}\rho_{x}+\sin\left(\frac{a_{\rho x}\pi x}{L}\right)\left[\rho_{0}+\rho_{x}\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_{y}\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_{z}\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\right]\left[p_{0}+p_{x}\cos\left(\frac{a_{px}\pi x}{L}\right)+p_{y}\sin\left(\frac{a_{py}\pi y}{L}\right)+p_{z}\cos\left(\frac{a_{pz}\pi z}{L}\right)\right]}{RL^{2}\left[\rho_{0}+\rho_{x}\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_{y}\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_{z}\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]^{3}}+
$$

$$
-\frac{a_{\rho y}^{2}\pi^{2}k\rho_{y}\left[2\sin\left(\frac{a_{\rho y}\pi y}{L}\right)^{2}\rho_{y}+\cos\left(\frac{a_{\rho y}\pi y}{L}\right)\left[\rho_{0}+\rho_{x}\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_{y}\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_{z}\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\right]\left[p_{0}+p_{x}\cos\left(\frac{a_{px}\pi x}{L}\right)+p_{y}\sin\left(\frac{a_{py}\pi y}{L}\right)+p_{z}\cos\left(\frac{a_{pz}\pi z}{L}\right)\right]}{RL^{2}\left[\rho_{0}+\rho_{x}\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_{y}\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_{z}\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]^{3}}+
$$

$$
-\frac{a_{\rho z}^{2}\pi^{2}k\rho_{z}\left[2\cos\left(\frac{a_{\rho z}\pi z}{L}\right)^{2}\rho_{z}+\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\left[\rho_{0}+\rho_{x}\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_{y}\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_{z}\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]\right]\left[p_{0}+p_{x}\cos\left(\frac{a_{px}\pi x}{L}\right)+p_{y}\sin\left(\frac{a_{py}\pi y}{L}\right)+p_{z}\cos\left(\frac{a_{pz}\pi z}{L}\right)\right]}{RL^{2}\left[\rho_{0}+\rho_{x}\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_{y}\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_{z}\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]^{3}}+
$$

$$
-\frac{4}{3}\frac{a_{ux}^{2}\pi^{2}\mu u_{x}}{L^{2}}\left[\cos\left(\frac{a_{ux}\pi x}{L}\right)^{2}u_{x}-\sin\left(\frac{a_{ux}\pi x}{L}\right)\left[u_{0}+u_{x}\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_{y}\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_{z}\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]\right]+
$$

$$
-\frac{a_{uy}^{2}\pi^{2}\mu u_{y}}{L^{2}}\left[\sin\left(\frac{a_{uy}\pi y}{L}\right)^{2}u_{y}-\cos\left(\frac{a_{uy}\pi y}{L}\right)\left[u_{0}+u_{x}\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_{y}\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_{z}\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]\right]+
$$

$$
-\frac{a_{uz}^{2}\pi^{2}\mu u_{z}}{L^{2}}\left[\sin\left(\frac{a_{uz}\pi z}{L}\right)^{2}u_{z}-\cos\left(\frac{a_{uz}\pi z}{L}\right)\left[u_{0}+u_{x}\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_{y}\cos\left(\frac{a_{uy}\pi y}{L}\right)+u_{z}\cos\left(\frac{a_{uz}\pi z}{L}\right)\right]\right]+
$$

$$
-\frac{a_{vx}^{2}\pi^{2}\mu v_{x}}{L^{2}}\left[\sin\left(\frac{a_{vx}\pi x}{L}\right)^{2}v_{x}-\cos\left(\frac{a_{vx}\pi x}{L}\right)\left[v_{0}+v_{x}\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_{y}\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_{z}\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]\right]+
$$

$$
-\frac{4}{3}\frac{a_{vy}^{2}\pi^{2}\mu v_{y}}{L^{2}}\left[\cos\left(\frac{a_{vy}\pi y}{L}\right)^{2}v_{y}-\sin\left(\frac{a_{vy}\pi y}{L}\right)\left[v_{0}+v_{x}\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_{y}\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_{z}\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]\right]+
$$

$$
-\frac{a_{vz}^{2}\pi^{2}\mu v_{z}}{L^{2}}\left[\cos\left(\frac{a_{vz}\pi z}{L}\right)^{2}v_{z}-\sin\left(\frac{a_{vz}\pi z}{L}\right)\left[v_{0}+v_{x}\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_{y}\sin\left(\frac{a_{vy}\pi y}{L}\right)+v_{z}\sin\left(\frac{a_{vz}\pi z}{L}\right)\right]\right]+
$$

$$
-\frac{a_{wx}^{2}\pi^{2}\mu w_{x}}{L^{2}}\left[\cos\left(\frac{a_{wx}\pi x}{L}\right)^{2}w_{x}-\sin\left(\frac{a_{wx}\pi x}{L}\right)\left[w_{0}+w_{x}\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_{y}\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_{z}\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\right]+
$$

$$
-\frac{a_{wy}^{2}\pi^{2}\mu w_{y}}{L^{2}}\left[\cos\left(\frac{a_{wy}\pi y}{L}\right)^{2}w_{y}-\sin\left(\frac{a_{wy}\pi y}{L}\right)\left[w_{0}+w_{x}\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_{y}\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_{z}\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\right]+
$$

$$
-\frac{4}{3}\frac{a_{wz}^{2}\pi^{2}\mu w_{z}}{L^{2}}\left[\sin\left(\frac{a_{wz}\pi z}{L}\right)^{2}w_{z}-\cos\left(\frac{a_{wz}\pi z}{L}\right)\left[w_{0}+w_{x}\sin\left(\frac{a_{wx}\pi x}{L}\right)+w_{y}\sin\left(\frac{a_{wy}\pi y}{L}\right)+w_{z}\cos\left(\frac{a_{wz}\pi z}{L}\right)\right]\right]+
$$

$$
\begin{aligned}
&-\frac{2a_{px}a_{\rho x}\pi^2 k p_x \rho_x \cos\left(\frac{a_{\rho x}\pi x}{L}\right)\sin\left(\frac{a_{px}\pi x}{L}\right)}{L^2 R\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]^2}+\\
&-\frac{2a_{py}a_{\rho y}\pi^2 k p_y \rho_y \sin\left(\frac{a_{\rho y}\pi y}{L}\right)\cos\left(\frac{a_{py}\pi y}{L}\right)}{L^2 R\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]^2}+\\
&-\frac{2a_{pz}a_{\rho z}\pi^2 k p_z \rho_z \cos\left(\frac{a_{\rho z}\pi z}{L}\right)\sin\left(\frac{a_{pz}\pi z}{L}\right)}{L^2 R\left[\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)+\rho_z\sin\left(\frac{a_{\rho z}\pi z}{L}\right)\right]^2}+\\
&+\frac{4}{3}\frac{a_{ux}a_{vy}\pi^2\mu u_x v_y}{L^2}\cos\left(\frac{a_{ux}\pi x}{L}\right)\cos\left(\frac{a_{vy}\pi y}{L}\right)+\\
&-\frac{4}{3}\frac{a_{ux}a_{wz}\pi^2\mu u_x w_z}{L^2}\cos\left(\frac{a_{ux}\pi x}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)+\\
&-\frac{2a_{uy}a_{vx}\pi^2\mu u_y v_x}{L^2}\sin\left(\frac{a_{uy}\pi y}{L}\right)\sin\left(\frac{a_{vx}\pi x}{L}\right)+\\
&+\frac{2a_{uz}a_{wx}\pi^2\mu u_z w_x}{L^2}\cos\left(\frac{a_{wx}\pi x}{L}\right)\sin\left(\frac{a_{uz}\pi z}{L}\right)+\\
&-\frac{4}{3}\frac{a_{vy}a_{wz}\pi^2\mu v_y w_z}{L^2}\cos\left(\frac{a_{vy}\pi y}{L}\right)\sin\left(\frac{a_{wz}\pi z}{L}\right)+\\
&-\frac{2a_{vz}a_{wy}\pi^2\mu v_z w_y}{L^2}\cos\left(\frac{a_{vz}\pi z}{L}\right)\cos\left(\frac{a_{wy}\pi y}{L}\right).
\end{aligned}
$$

6 Comments

Source terms Q_ρ, Q_u, Q_v and Q_w have been generated automatically by replacing the analytical expressions (11) into respective operators, followed by the usage of Maple commands for collecting, sorting and factorizing the terms. Yet, to achieve the final form of source term Q_{e_t}, due to its higher complexity, a slightly different procedure has been applied. First, Q_{e_t} has manually been split into several subterms, to facilitate and improve the symbolic manipulation; then Maple commands for collecting, sorting and factorizing have been employed. Finally, the resulting source term has been compared to its original form to ensure correctness. As result, the initially 41–page long expression for Q_{e_t} has been reduced to Equation (17). Please see file `NavierStokes_equation_3d_e_Maple.pdf` for the original expression of Q_{e_t}.

Files containing C codes for the source terms have also been generated. They are: `NavierStokes_3d_e_code.C`, `NavierStokes_3d_rho_code.C`, `NavierStokes_3d_u_code.C`, `NavierStokes_3d_v_code.C` and `NavierStokes_3d_w_code.C`.

An example of the automatically generated C file from the source term for mass conservation equation is:

```
double SourceQ_rho (  double x,  double y,  double z,  double u_0,  double u_x,  double u_y,  double u_z,  double v_0,
                      double v_x,  double v_y,  double v_z,  double rho_0,  double rho_x,  double rho_y,  double rho_z,
                      double p_0,  double p_x,  double p_y,  double p_z,  double a_px,  double a_py,  double a_pz,
                      double a_rhox,  double a_rhoy,  double a_rhoz,  double a_ux, double a_uy,  double a_uz,  double a_vx,
                      double a_vy,  double a_vz,  double a_wx,  double a_wy,  double a_wz,  double mu,  double L)
{
  double Q_rho;
  Q_rho = rho_x * cos(a_rhox * PI * x / L) * (u_0 + u_x * sin(a_ux * PI * x / L) + u_y * cos(a_uy * PI * y / L) +
          u_z * cos(a_uz * PI * z / L)) * a_rhox * PI / L - rho_y * sin(a_rhoy * PI * y / L) * (v_0 +
          v_x * cos(a_vx * PI * x / L) + v_y * sin(a_vy * PI * y / L) + v_z * sin(a_vz * PI * z / L)) * a_rhoy * PI / L +
          rho_z * cos(a_rhoz * PI * z / L) * (w_0 + w_x * sin(a_wx * PI * x / L) + w_y * sin(a_wy * PI * y / L) +
          w_z * cos(a_wz * PI * z / L)) * a_rhoz * PI / L + u_x * cos(a_ux * PI * x / L) * (rho_0 +
          rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L) + rho_z * sin(a_rhoz * PI * z / L)) * a_ux * PI / L +
          v_y * cos(a_vy * PI * y / L) * (rho_0 + rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L) +
          rho_z * sin(a_rhoz * PI * z / L)) * a_vy * PI / L - w_z * sin(a_wz * PI * z / L) * (rho_0 +
          rho_x * sin(a_rhox * PI * x / L) + rho_y * cos(a_rhoy * PI * y / L) + rho_z * sin(a_rhoz * PI * z / L)) * a_wz * PI / L;
  return(Q_rho);
}
```

Finally the gradients of the analytical solutions (11) have also been computed and their respective C codes are presented in `NavierStokes_manuf_solutions_grad_and_code_3d.C`. Therefore,

$$\nabla\rho = \begin{bmatrix} \frac{a_{\rho x}\pi\rho_x}{L}\cos\left(\frac{a_{\rho x}\pi x}{L}\right) \\ -\frac{a_{\rho y}\pi\rho_y}{L}\sin\left(\frac{a_{\rho y}\pi y}{L}\right) \\ \frac{a_{\rho z}\pi\rho_z}{L}\cos\left(\frac{a_{\rho z}\pi z}{L}\right) \end{bmatrix}, \qquad \nabla p = \begin{bmatrix} -\frac{a_{px}\pi p_x}{L}\sin\left(\frac{a_{px}\pi x}{L}\right) \\ \frac{a_{py}\pi p_y}{L}\cos\left(\frac{a_{py}\pi y}{L}\right) \\ -\frac{a_{pz}\pi p_z}{L}\sin\left(\frac{a_{pz}\pi z}{L}\right) \end{bmatrix}, \qquad \nabla u = \begin{bmatrix} \frac{a_{ux}\pi u_x}{L}\cos\left(\frac{a_{ux}\pi x}{L}\right) \\ -\frac{a_{uy}\pi u_y}{L}\sin\left(\frac{a_{uy}\pi y}{L}\right) \\ -\frac{a_{uz}\pi u_z}{L}\sin\left(\frac{a_{uz}\pi z}{L}\right) \end{bmatrix}, \tag{18}$$

$$\nabla v = \begin{bmatrix} -\frac{a_{vx}\pi v_x}{L}\sin\left(\frac{a_{vx}\pi x}{L}\right) \\ \frac{a_{vy}\pi v_y}{L}\cos\left(\frac{a_{vy}\pi y}{L}\right) \\ \frac{a_{vz}\pi v_z}{L}\cos\left(\frac{a_{vz}\pi z}{L}\right) \end{bmatrix} \quad \text{and} \quad \nabla w = \begin{bmatrix} \frac{a_{wx}\pi w_x}{L}\cos\left(\frac{a_{wx}\pi x}{L}\right) \\ \frac{a_{wy}\pi w_y}{L}\cos\left(\frac{a_{wy}\pi y}{L}\right) \\ -\frac{a_{wz}\pi w_z}{L}\sin\left(\frac{a_{wz}\pi z}{L}\right) \end{bmatrix} \tag{19}$$

are written in C language as:

```
grad_rho_an[0] = rho_x * cos(a_rhox * pi * x / L) * a_rhox * pi / L;
grad_rho_an[1] = -rho_y * sin(a_rhoy * pi * y / L) * a_rhoy * pi / L;
grad_rho_an[2] = rho_z * cos(a_rhoz * pi * z / L) * a_rhoz * pi / L;
grad_p_an[0] = -p_x * sin(a_px * pi * x / L) * a_px * pi / L;
grad_p_an[1] = p_y * cos(a_py * pi * y / L) * a_py * pi / L;
```

```
grad_p_an[2] = -p_z * sin(a_pz * pi * z / L) * a_pz * pi / L;
grad_u_an[0] = u_x * cos(a_ux * pi * x / L) * a_ux * pi / L;
grad_u_an[1] = -u_y * sin(a_uy * pi * y / L) * a_uy * pi / L;
grad_u_an[2] = -u_z * sin(a_uz * pi * z / L) * a_uz * pi / L;
grad_v_an[0] = -v_x * sin(a_vx * pi * x / L) * a_vx * pi / L;
grad_v_an[1] = v_y * cos(a_vy * pi * y / L) * a_vy * pi / L;
grad_v_an[2] = v_z * cos(a_vz * pi * z / L) * a_vz * pi / L;
grad_w_an[0] = w_x * cos(a_wx * pi * x / L) * a_wx * pi / L;
grad_w_an[1] = w_y * cos(a_wy * pi * y / L) * a_wy * pi / L;
grad_w_an[2] = -w_z * sin(a_wz * pi * z / L) * a_wz * pi / L;
```

References

Roy, C., T. Smith, and C. Ober (2002). Verification of a compressible cfd code using the method of manufactured solutions. In *AIAA FLuid Dynamics Conference and Exhibit*, Number AIAA 2002-3110.