MAT 175E

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Homework 3

Only one randomly chosen question (which is the same for all of you) will be graded!

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(1) (a) Let $A = \{1,2,3,4,5\}$, $B = \{3,4,5\}$ and

$$R = \big\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(1,5),(2,3),(2,4),(2,5),(3,5),(4,5)\big\}.$$

Explain briefly why R is a partial order on A. Find a chain in A containing three elements. Find lower bounds and upper bounds of the set B. Find minimal and maximal elements of B. Find the smallest and the greatest element of B.

(b) Let $(X, \leq)$ be a nonempty poset such that every chain in it has an upper bound, and let $F : X \to X$ be a function satisfying that $x \leq F(x)$ for all $x \in X$. Does $(X, \leq)$ have a maximal element? Show that there is an $x_0 \in X$ such that $F(x_0) = x_0$.

(2) (a) Let R be a symmetric relation on a set A and S be the relation on A defined by $S = (A \times A) - R$. Prove that S is symmetric.

(b) Let T be a symmetric and transitive relation on a set A. Show that T is reflexive if and ony if the domain of T is A.

(3) (a) Let R be an equivalence relation on a finite set A. Prove that $|R| - |A|$ is even.

(b) Let $A = \{1,2,3,4\}$ and $B = \{3,4\}$. Let E be the equivalence relation on $\mathcal{P}(A)$ defined for any elements X and Y of $\mathcal{P}(A)$ by

$$(X, Y) \in E \text{ if and only if } X \cap B = Y \cap B.$$

Find the equivalence classes and write the quotient set A/E.

(4) Let $f : A \to B$ be a function where A and B are sets. The kernel $\mathrm{Ker}f$ of f is the relation on A defined for any elements r and s of A by

$$(r, s) \in \mathrm{Ker}f \text{ if and only if } f(r) = f(s).$$

One may easily see that $\mathrm{Ker}f$ is an equivalence relation on A. Consider the map $F : A/\mathrm{Ker}f \to f(B)$ defined by

$$F([a]) = f(a)$$

for any $[a] \in A/\mathrm{Ker}f$ where $[a]$ denotes the $\mathrm{Ker}f$-equivalence class of $a \in A$ and $A/\mathrm{Ker}f = \{[a] \mid a \in A\}$ denotes the quotient set of A by the equivalence relation $\mathrm{Ker}f$ and $f(B)$ denotes the range of f. Consider also the following maps $\mu : A \to A/\mathrm{Ker}f$ and $\nu : f(B) \to B$ defined by

$$\mu(a) = [a] \qquad \text{and} \qquad \nu(x) = x$$

for any $a \in A$ and for any $x \in f(B)$.

(i) Show that F is well defined.

(ii) Explain why we do not need to justify that μ and ν are well defined.

(iii) Show that F is bijective.

(iv) Justify that μ is surjective and ν is injective.

(v) Justify that $f = \nu \circ F \circ \mu$.

(vi) Show that any function is a composition of three functions that are, from right to left, a surjection, a bijection and an injection.

(vii) Suppose that $A = \{1,2,3,4,5,6\}$ and $B = \{r,s,t,u\}$, and suppose that $f : A \to B$ is given by

$$f(1) = f(3) = f(6) = r, \quad f(2) = f(5) = t \quad \text{and} \quad f(4) = u.$$

Write explicitly each of $\mathrm{Ker}f$, $f(B)$, F, μ, ν and check that the previous parts are all true.