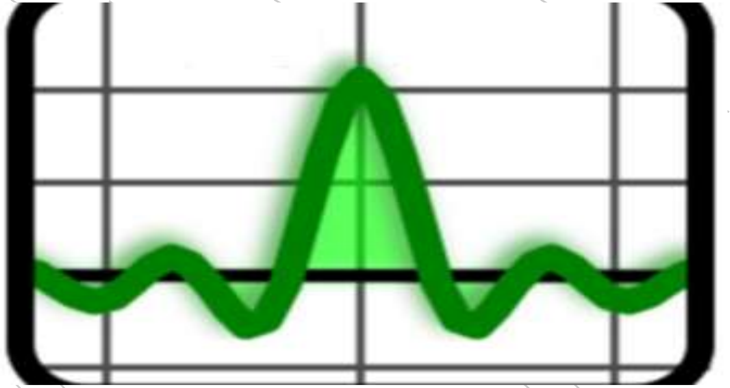

Signals & Systems For Computer Engineering

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BLG354E / CRN: 21560
8^{th} Week Lecture

Frequency response of the systems

Transfer function of an LTI sytem

$$H(s) = H_0 \frac{(s - z_1)(s - z_2)(s - z_3) \cdots (s - z_m)}{(s - p_1)(s - p_2)(s - p_3) \cdots (s - p_n)}$$

Steady state response of a system H(s) can be found as H(jω) by s= jω

$$H(j\omega) = |H(j\omega)|\, e^{j\theta(\omega)} = H(s)\big|_{s=j\omega}$$

Magnitude response of the system:

$$|H(j\omega)| = (\{\mathrm{Re}[H(j\omega)]\}^2 + \{\mathrm{Im}[H(j\omega)]\}^2)^{1/2}$$

$$|H(j\omega)|^2 = H(s)H(-s)\big|_{s=j\omega}$$

$$|H(j\omega)| = H_0 \frac{|j\omega - z_1| \cdot |j\omega - z_2| \cdot |j\omega - z_3| \cdot \cdots \cdot |j\omega - z_m|}{|j\omega - p_1| \cdot |j\omega - p_2| \cdot |j\omega - p_3| \cdot \cdots \cdot |j\omega - p_n|}$$

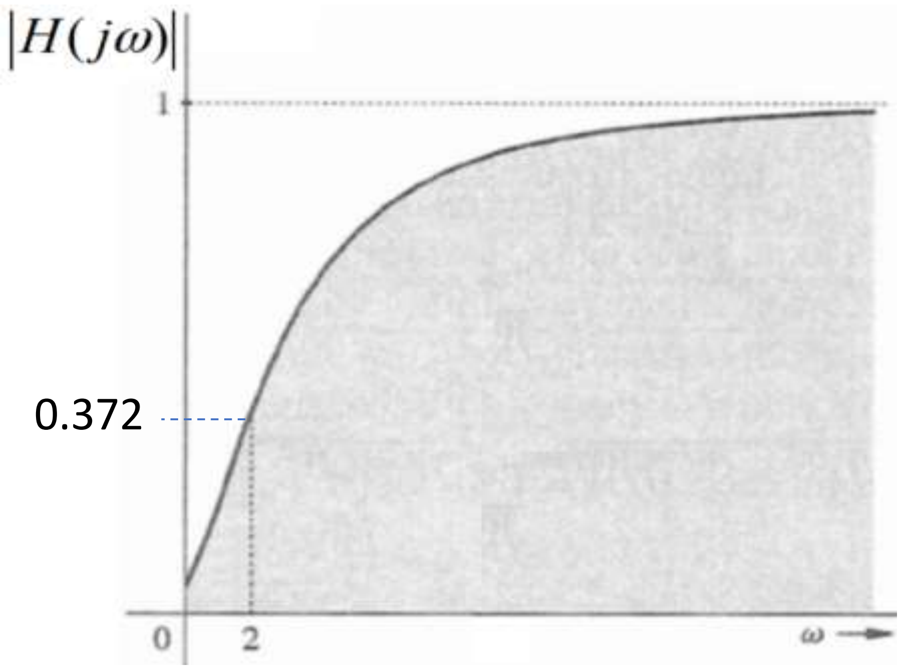
Phase response of the system:

$$\theta(\omega) = \tan^{-1}\left\{\frac{\mathrm{Im}[H(j\omega)]}{\mathrm{Re}[H(j\omega)]}\right\}$$

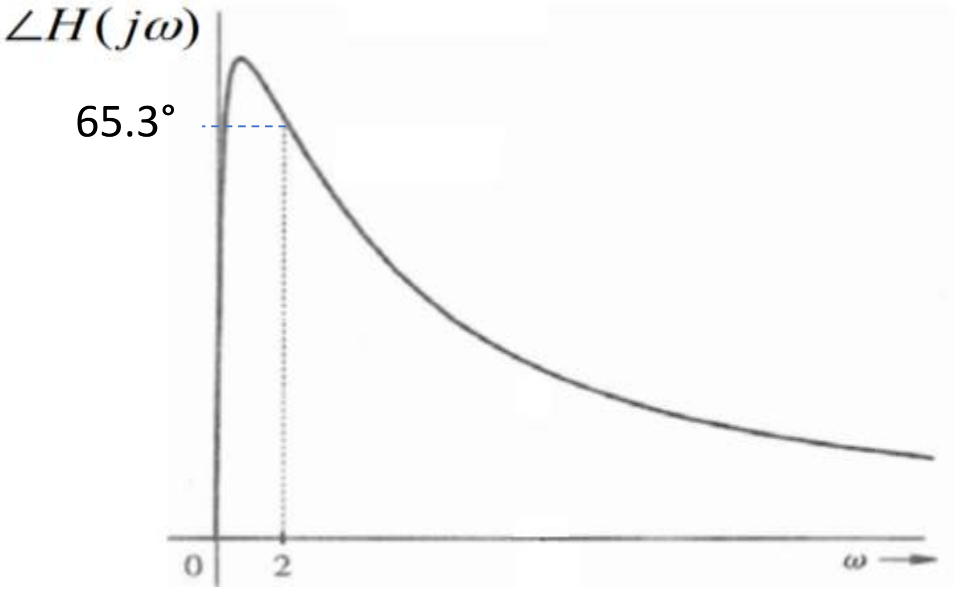
Example: Find the frequency response of the system $H(s)=\frac{s+0.1}{s+5}$ and its output y(t) for input x(t)=cos2t

When we substitute s= jω $\rightarrow$ $H(j\omega)=\frac{j\omega+0.1}{j\omega+5}$

Magnitude response: $|H(j\omega)|=\frac{\sqrt{\omega^2+0.01}}{\sqrt{\omega^2+25}}$



Phase response: $\angle H(j\omega)=\Phi(j\omega)=\tan^{-1}\left(\frac{\omega}{0.1}\right)-\tan^{-1}\left(\frac{\omega}{5}\right)$

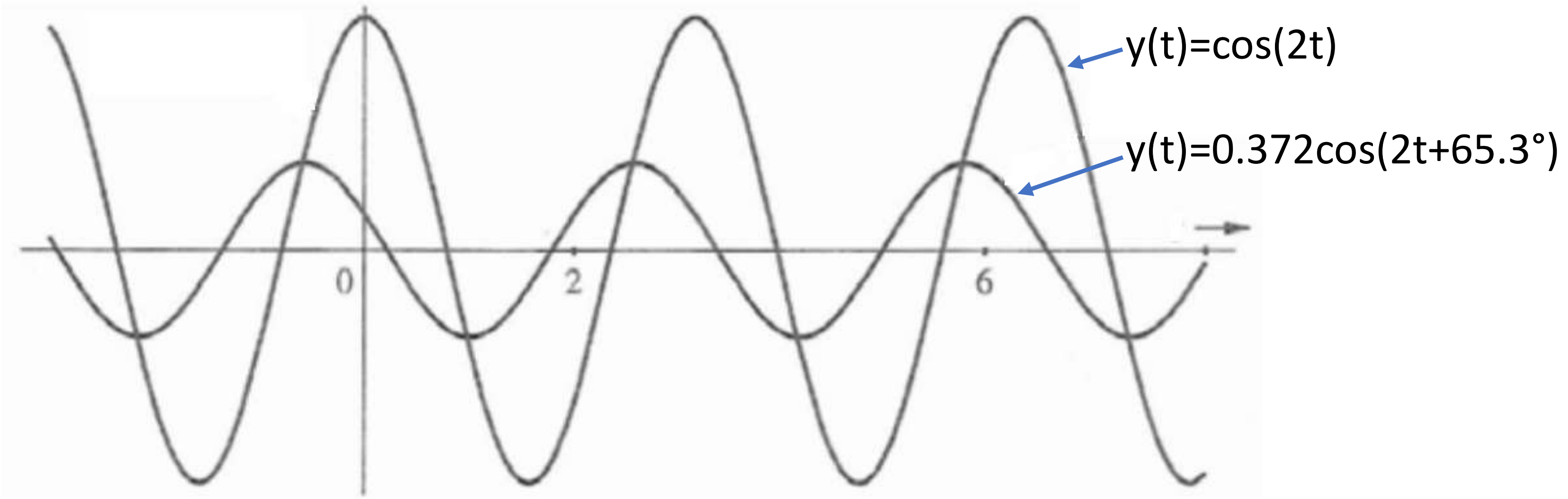


for input x(t)=cos2t

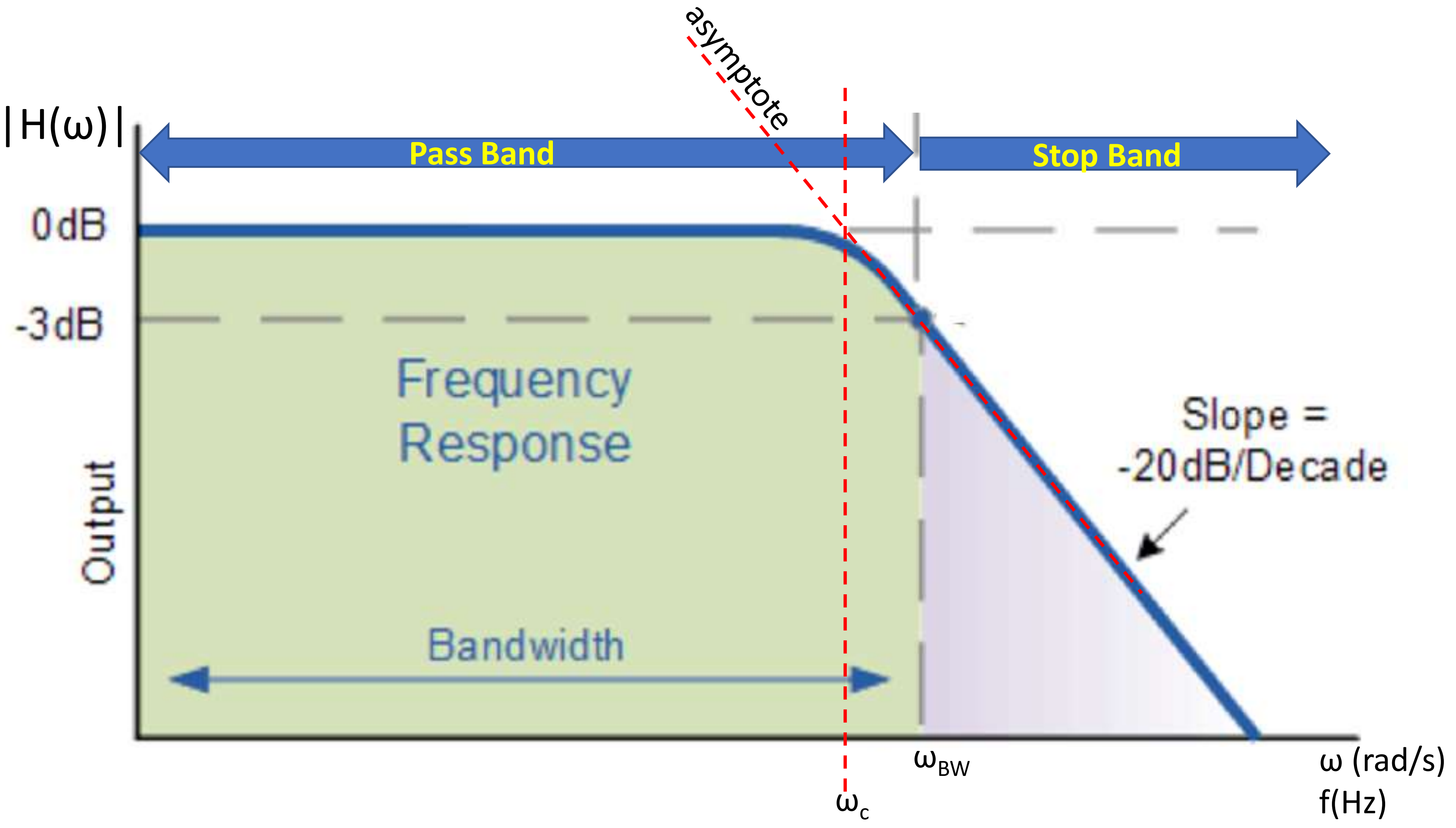
Amplitude: $$|H(j2)| = \frac{\sqrt{2^2+0.01}}{\sqrt{2^2+25}} = 0.372$$

Phase: $$\Theta(j2) = \tan^{-1}\left(\frac{2}{0.1}\right) - \tan^{-1}\left(\frac{2}{5}\right) = 65.3^{\circ}$$

Output: $$y(t)=0.372\cos(2t+65.3°)$$

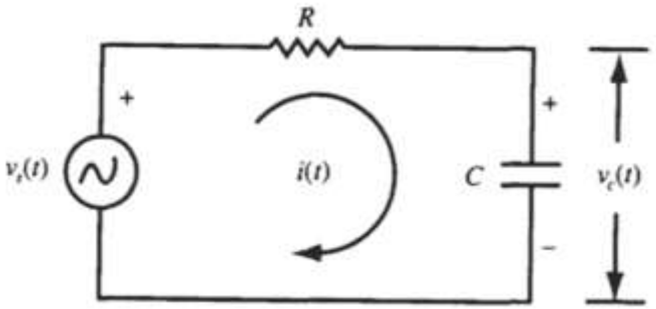


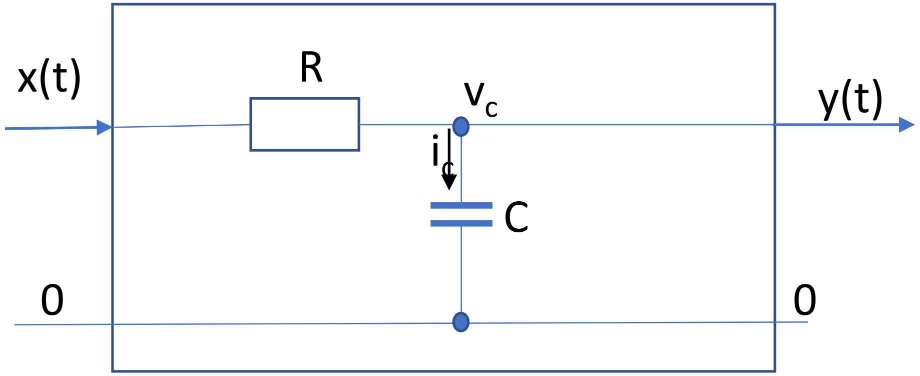
Bandwidth of a 1st order low pass system



Example:

Find the frequency response R=10kΩ, C=1µF





$v_c(t)$ $v_s(t)$

$$RC\frac{dy(t)}{dt}+y(t)=x(t)$$

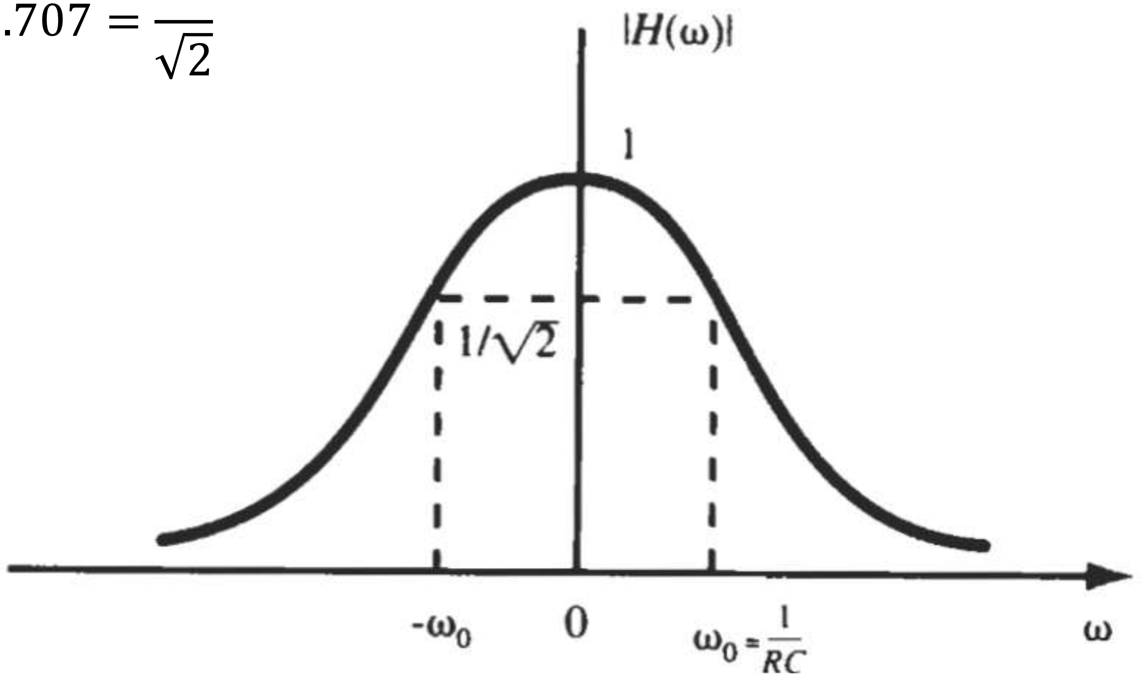
$$X(s) \rightarrow \boxed{\frac{1}{\tau s+1}} \rightarrow Y(s)$$

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{1}{1+j\omega RC}=\frac{1}{1+j\omega/\omega_0}$$

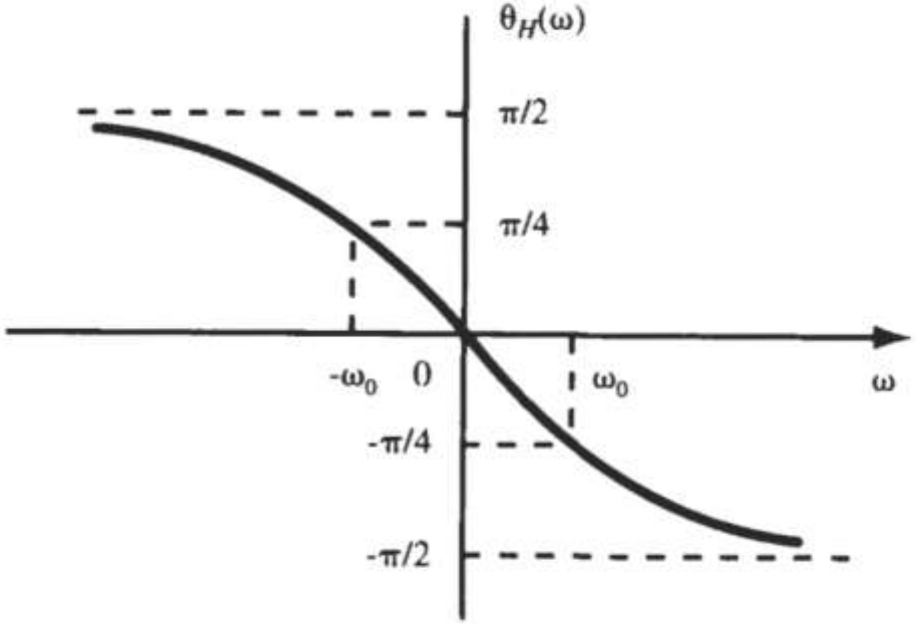
$\tau=RC=1/\omega_0$

$$H(\omega)=\frac{1}{1+j\omega/100}$$

Bandwidth: $20log\frac{U_2}{U_1} = -3dB \rightarrow \frac{U_2}{U_1} = 10^{\frac{-3}{20}} = 0.707 = \frac{1}{\sqrt{2}}$



$$|H(\omega)| = \frac{1}{|1 + j\omega/\omega_0|} = \frac{1}{\left[1 + (\omega/\omega_0)^2\right]^{1/2}}$$



$$\theta_H(\omega) = -\tan^{-1}\frac{\omega}{\omega_0}$$

Magnitude response of the RC Low Pass Filter:

$$|H(\omega)|_{\mathrm{dB}} = 20\log_{10}\left|\frac{1}{1 + j\omega/100}\right| = -20\log_{10}\left|1 + j\frac{\omega}{100}\right|$$

Phase response of the RC Low Pass Filter:

$$\theta_H(\omega) = -\tan^{-1}\frac{\omega}{100}$$

For ω≪100 :

as ω→0

$$|H(\omega)|_{\mathrm{dB}} = -20\log_{10}\left|1 + j\frac{\omega}{100}\right| \longrightarrow -20\log_{10} 1 = 0$$

$$\theta_H(\omega) = -\tan^{-1}\frac{\omega}{100} \longrightarrow 0$$

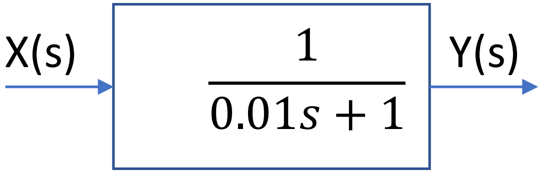
For ω≫100 :

as ω→∞

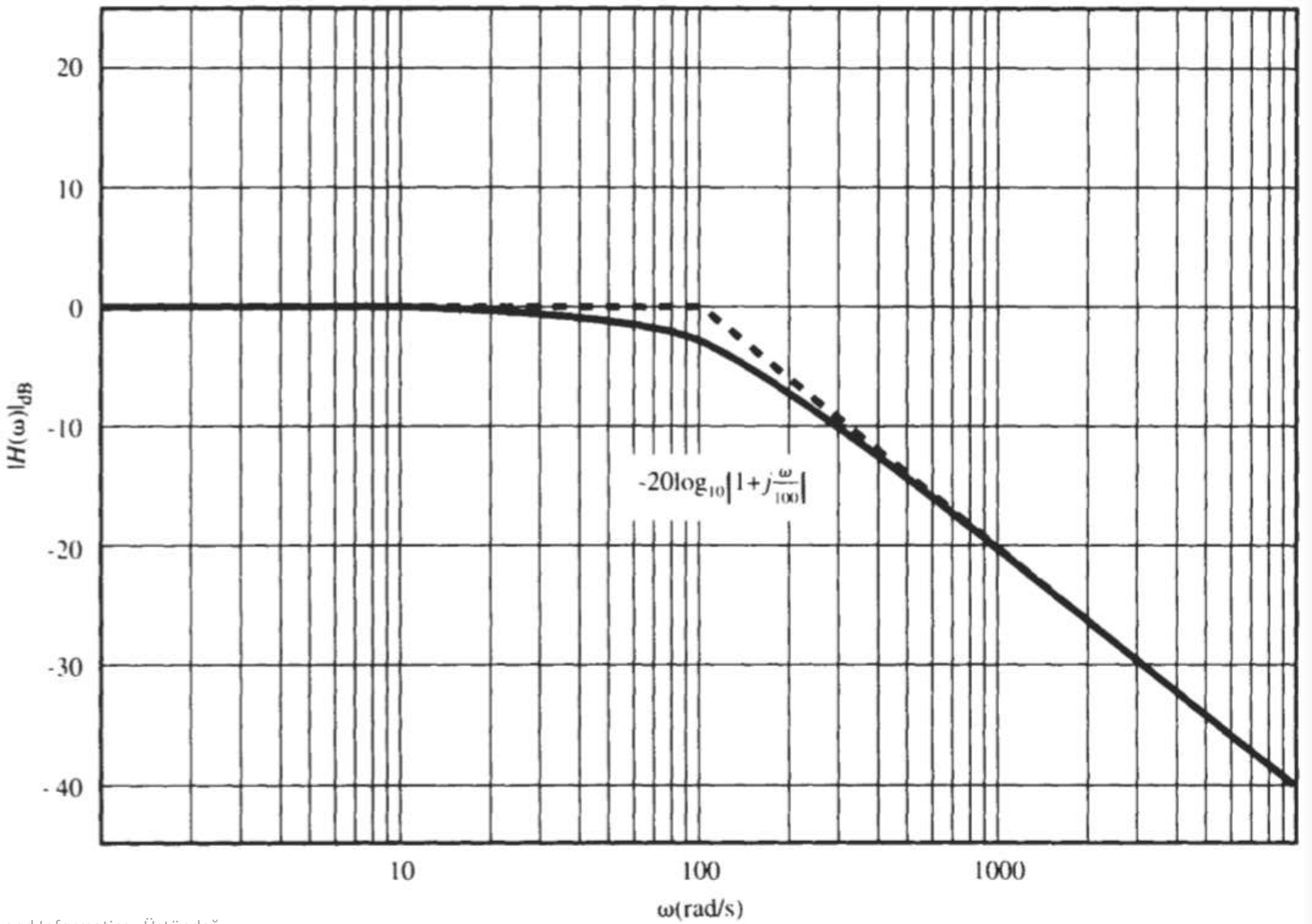
$$|H(\omega)|_{\mathrm{dB}} = -20\log_{10}\left|1 + j\frac{\omega}{100}\right| \longrightarrow -20\log_{10}\left(\frac{\omega}{100}\right)$$

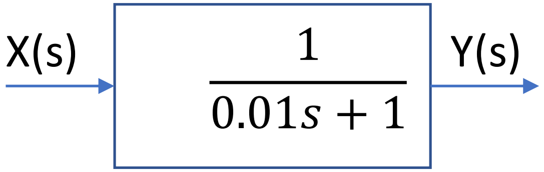
$$\theta_H(\omega) = -\tan^{-1}\frac{\omega}{100} \longrightarrow -\frac{\pi}{2}$$

$$H(100)|_{\mathrm{dB}} = -20\log_{10}\sqrt{2} \approx -3\ \mathrm{dB}$$

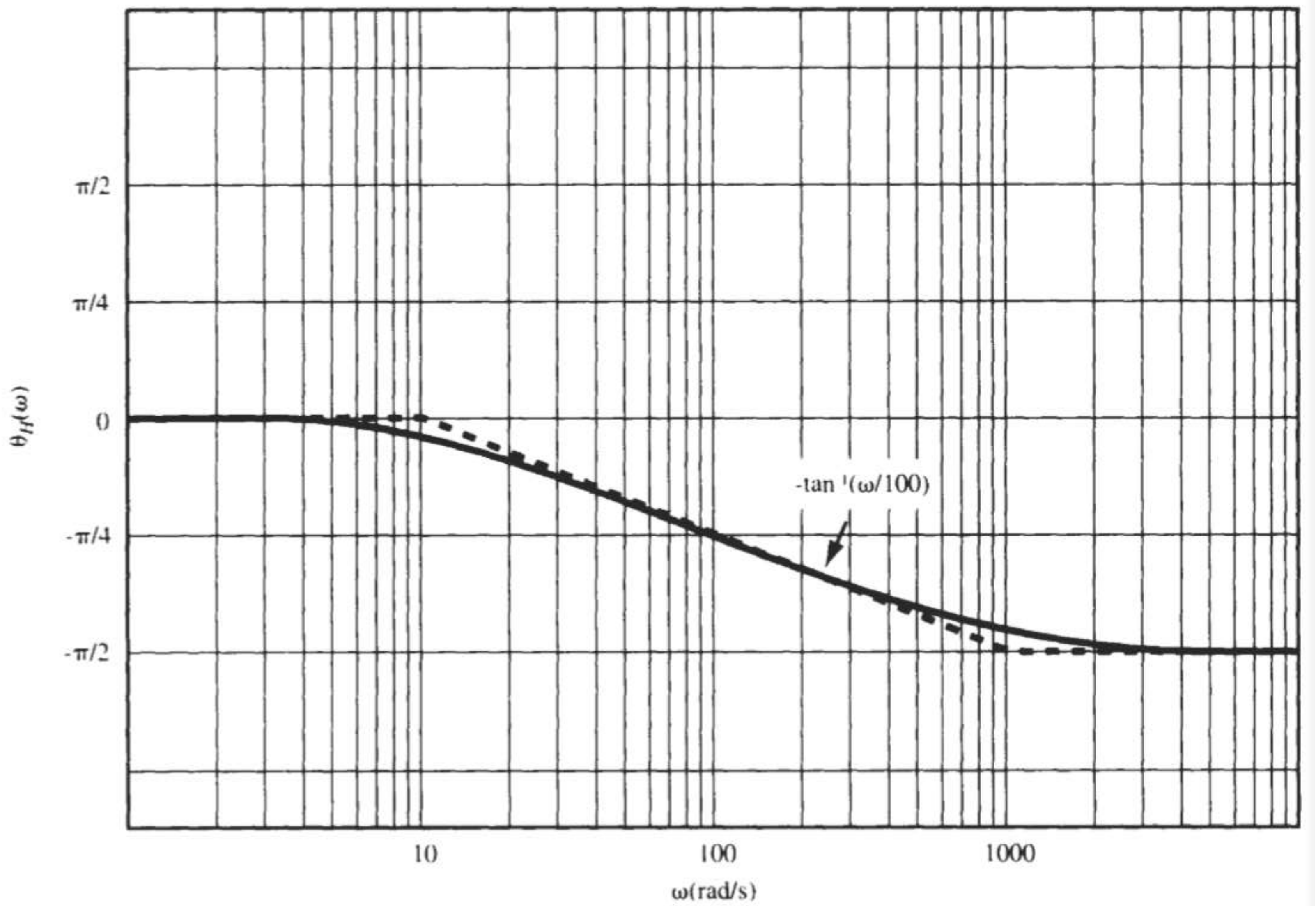


$$|H(\omega)|_{\mathrm{dB}} = 20\log_{10}\left|\frac{1}{1+j\omega/100}\right| = -20\log_{10}\left|1+j\frac{\omega}{100}\right|$$

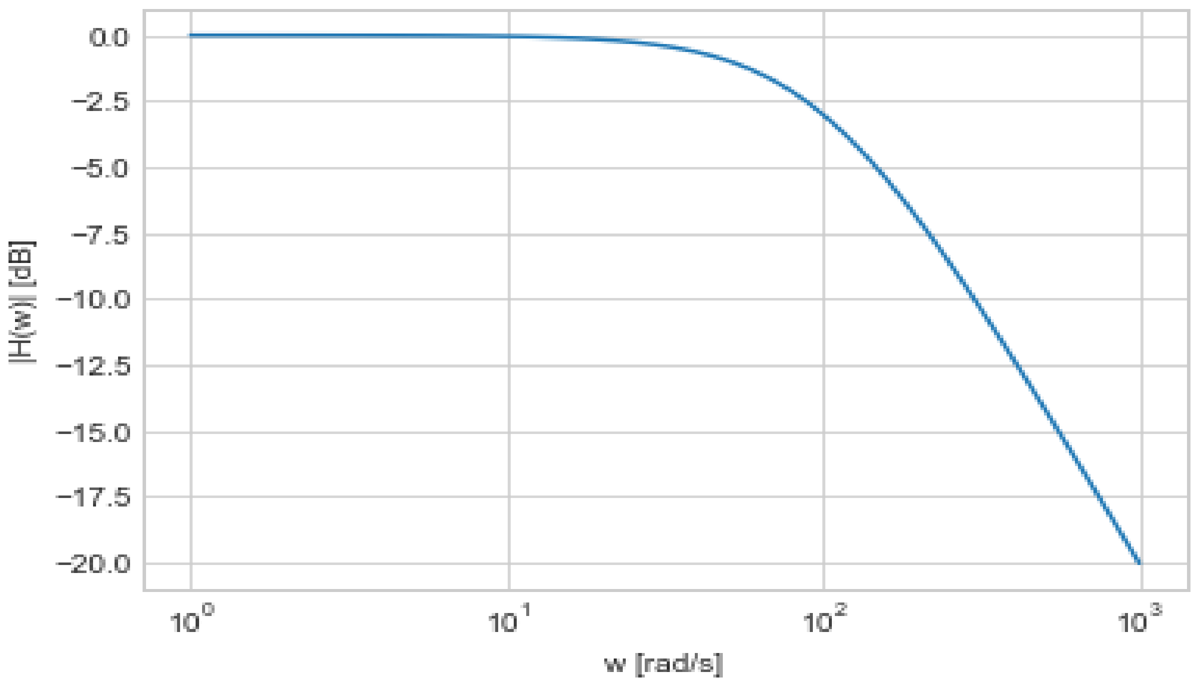




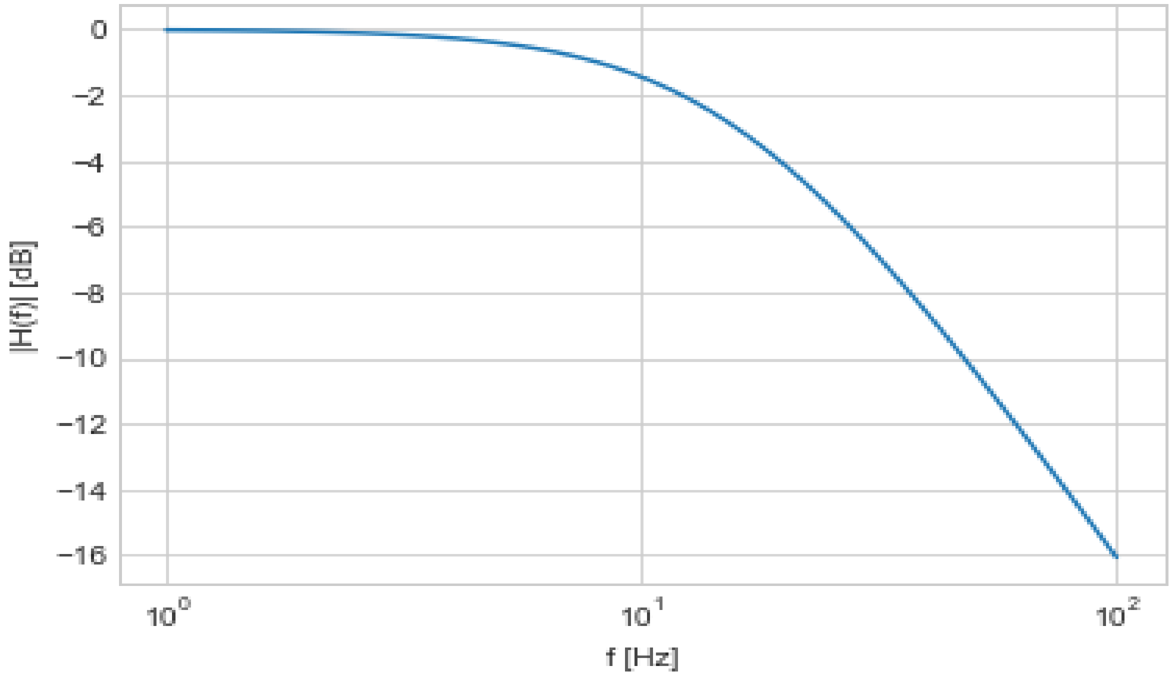
$$\theta_H(\omega) = -\tan^{-1}\frac{\omega}{100}$$



```
from pylab import *

def H(w):
  wc = 100
  G = 1.0 / (1.0 + 1j * w / wc)
  return G

w = logspace(0,3) # frequencies from 10**0 to 10**3
xscale('log')
ylabel('|H(w)| [dB]')
xlabel('w [rad/s]')
plot(w, 20*log10(abs(H(w))))
```




```
from pylab import *

def H(f):
  fc = 100/(2*pi)
  G = 1.0 / (1.0 + 1j * f / fc)
  return G

f = logspace(0,2) # frequencies from 10**-1 to 10**2
xscale('log')
ylabel('|H(f)| [dB]')
xlabel('f [Hz]')
plot(f, 20*log10(abs(H(f))))
```



$$X(s) \longrightarrow \boxed{\frac{1}{0.01s+1}} \longrightarrow Y(s)$$

a) If v_s=x(t)=10sin(100πt) → ω=100π rad/s, f=50Hz

$$v_c = y(t) = 10\sqrt{\frac{1}{1+(\frac{100\pi}{100})^2}}\sin\left(100\pi t - tan^{-1}(\frac{100\pi}{100})\right) = 10\sqrt{\frac{1}{1+\pi^2}}\sin(100\pi t - tan^{-1}(\pi))$$

$$y(t) = 3.03\sin(100\pi t - 1.26)$$

-72.3°

b) If v_s=x(t)=10sin(10πt) → ω=10π rad/s, f=5Hz

$$v_c = y(t) = 10\sqrt{\frac{1}{1+(\frac{10\pi}{100})^2}}\sin\left(10\pi t - tan^{-1}(\frac{10\pi}{100})\right) = 10\sqrt{\frac{1}{1+0.01\pi^2}}\sin(10\pi t - tan^{-1}(0.1\pi))$$

$$y(t) = 9.85\sin(10\pi t - 0.3)$$

-17.4°

c) If v_s=x(t)=10sin(1000πt) → ω=1000π rad/s, f=500Hz

$$v_c = y(t) = 10\sqrt{\frac{1}{1+(\frac{1000\pi}{100})^2}}\sin\left(1000\pi t - tan^{-1}(\frac{1000\pi}{100})\right) = 10\sqrt{\frac{1}{1+100\pi^2}}\sin(1000\pi t - tan^{-1}(10\pi))$$

$$y(t) = 0.318\sin(1000\pi t - 1.54)$$

$$20log\frac{U_2}{U_1} = -30dB \rightarrow \frac{U_2}{U_1} = 10^{\frac{-30}{20}} = 0.0316 \rightarrow \text{x10}$$

-88.2°

Example:

Transfer function of a system is $H(s) = \dfrac{10^4(s+1)}{(s+10)(s+100)}$. Find the frequency response of this system

$$H(\omega) = \frac{10^4(1+j\omega)}{(10+j\omega)(100+j\omega)} = \frac{10(1+j\omega)}{(1+j\omega/10)(1+j\omega/100)}$$

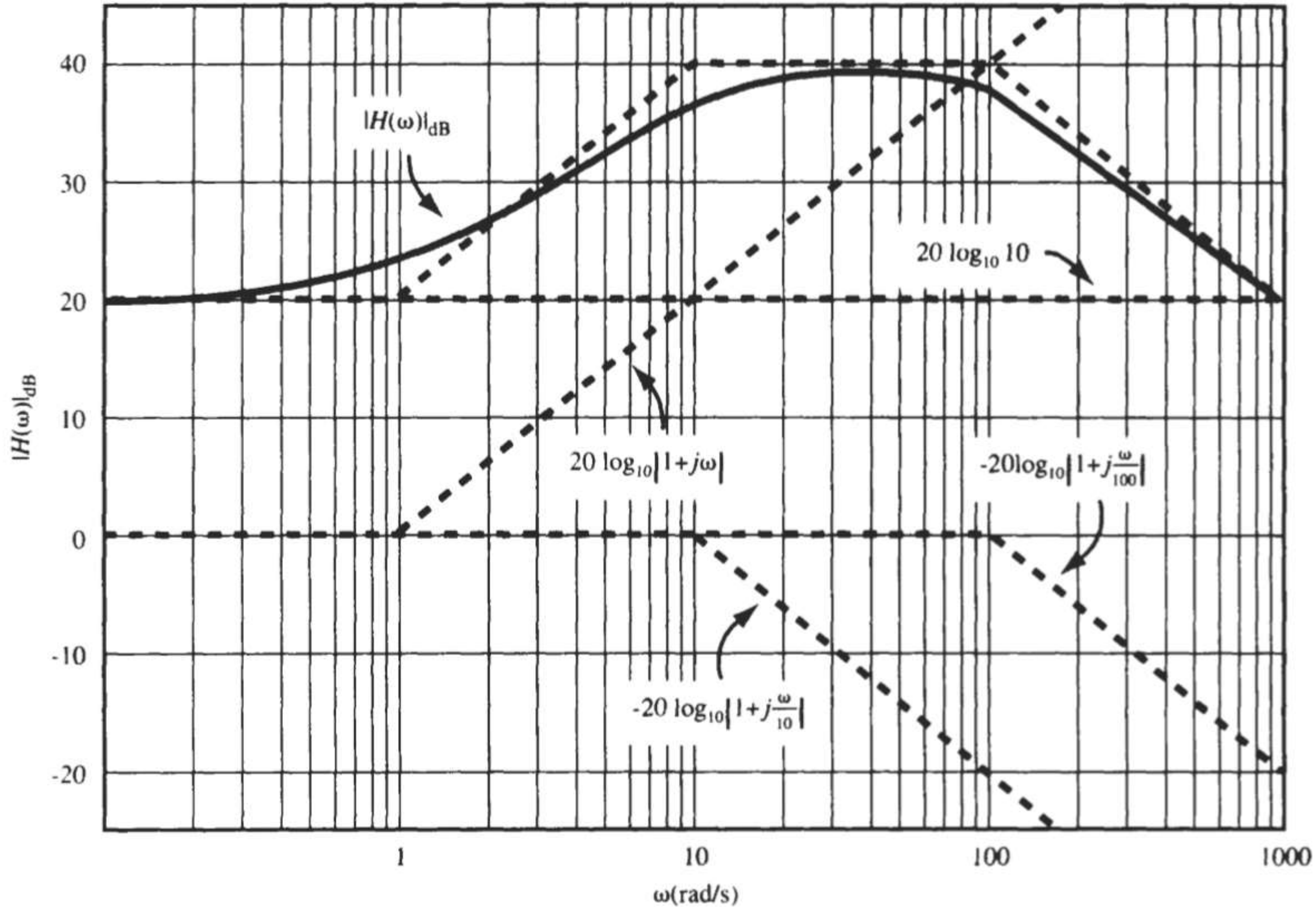
$$|H(\omega)|_{\mathrm{dB}} = 20\log_{10} 10 + 20\log_{10}|1+j\omega| - 20\log_{10}\left|1+j\frac{\omega}{10}\right| - 20\log_{10}\left|1+j\frac{\omega}{100}\right|$$

Corner frequencies: ω=1, ω=10, ω=1 , ω=100

For ω=1 → $H(1)|_{\mathrm{dB}} = 20 + 20\log_{10}\sqrt{2} - 20\log_{10}\sqrt{1.01} - 20\log_{10}\sqrt{1.0001} \approx 23\ \mathrm{dB}$

For ω=10 → $H(10)|_{\mathrm{dB}} = 20 + 20\log_{10}\sqrt{101} - 20\log_{10}\sqrt{2} - 20\log_{10}\sqrt{1.01} \approx 37\ \mathrm{dB}$

For ω=100 → $H(100)|_{\mathrm{dB}} = 20 + 20\log_{10}\sqrt{10{,}001} - 20\log_{10}\sqrt{101} - 20\log_{10}\sqrt{2} \approx 37\ \mathrm{dB}$


40
30
20
10
0
-10
-20
$|H(\omega)|_{dB}$
$|H(\omega)|_{dB}$
$20\log_{10}10$
$20\log_{10}|1+j\omega|$
$-20\log_{10}|1+j\frac{\omega}{100}|$
$-20\log_{10}|1+j\frac{\omega}{10}|$
1
10
100
1000
ω(rad/s)

$$\theta_H(\omega) = \tan^{-1}\omega - \tan^{-1}\frac{\omega}{10} - \tan^{-1}\frac{\omega}{100}$$

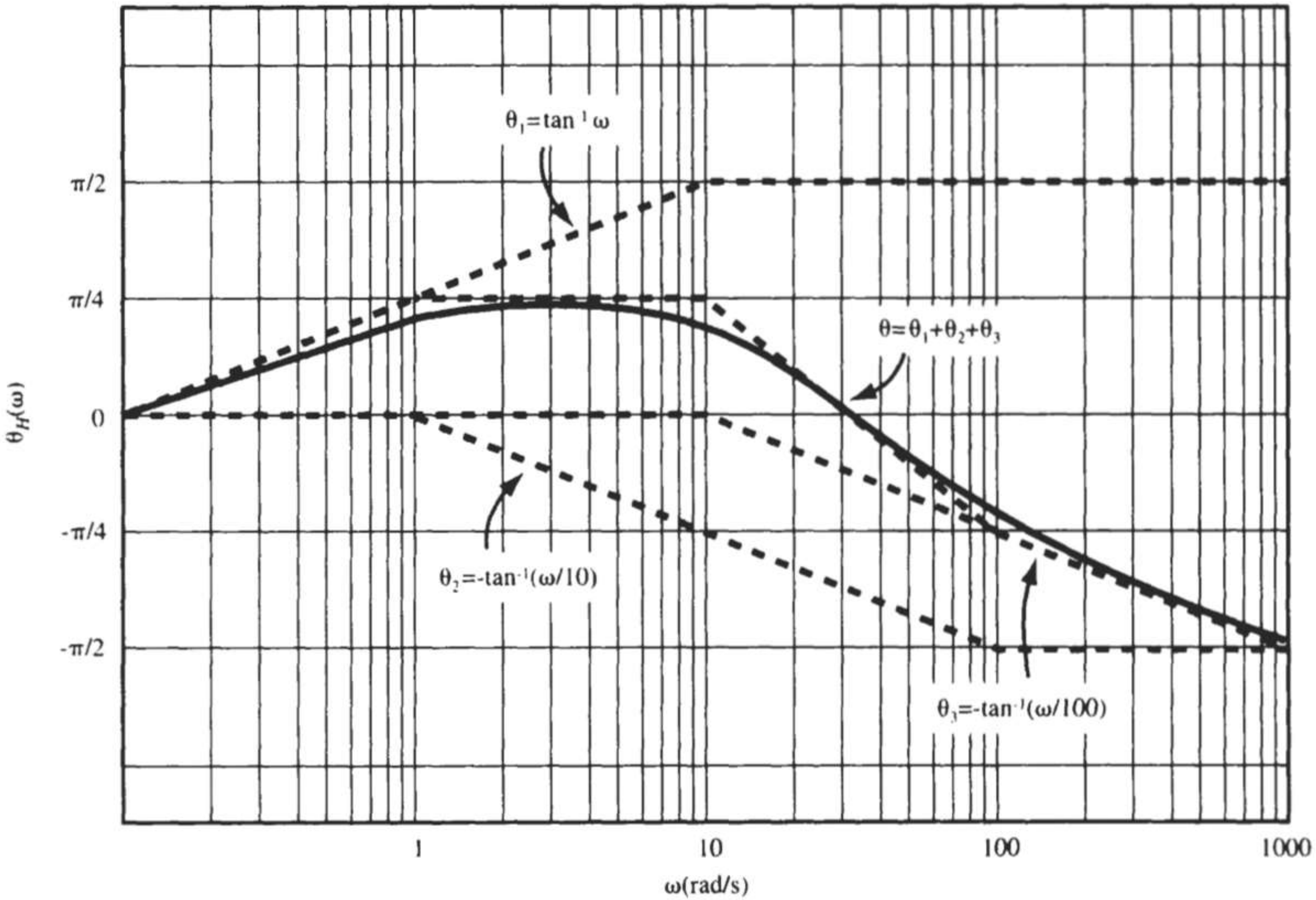
For ω → 0 : $\theta_H(\omega) = \longrightarrow 0 - 0 - 0 = 0$

For ω → ∞ : $\theta_H(\omega) = \longrightarrow \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2}$

For ω=1 → $\theta_H(1) = \tan^{-1}(1) - \tan^{-1}(0.1) - \tan^{-1}(0.01) = 0.676 \text{ rad}$

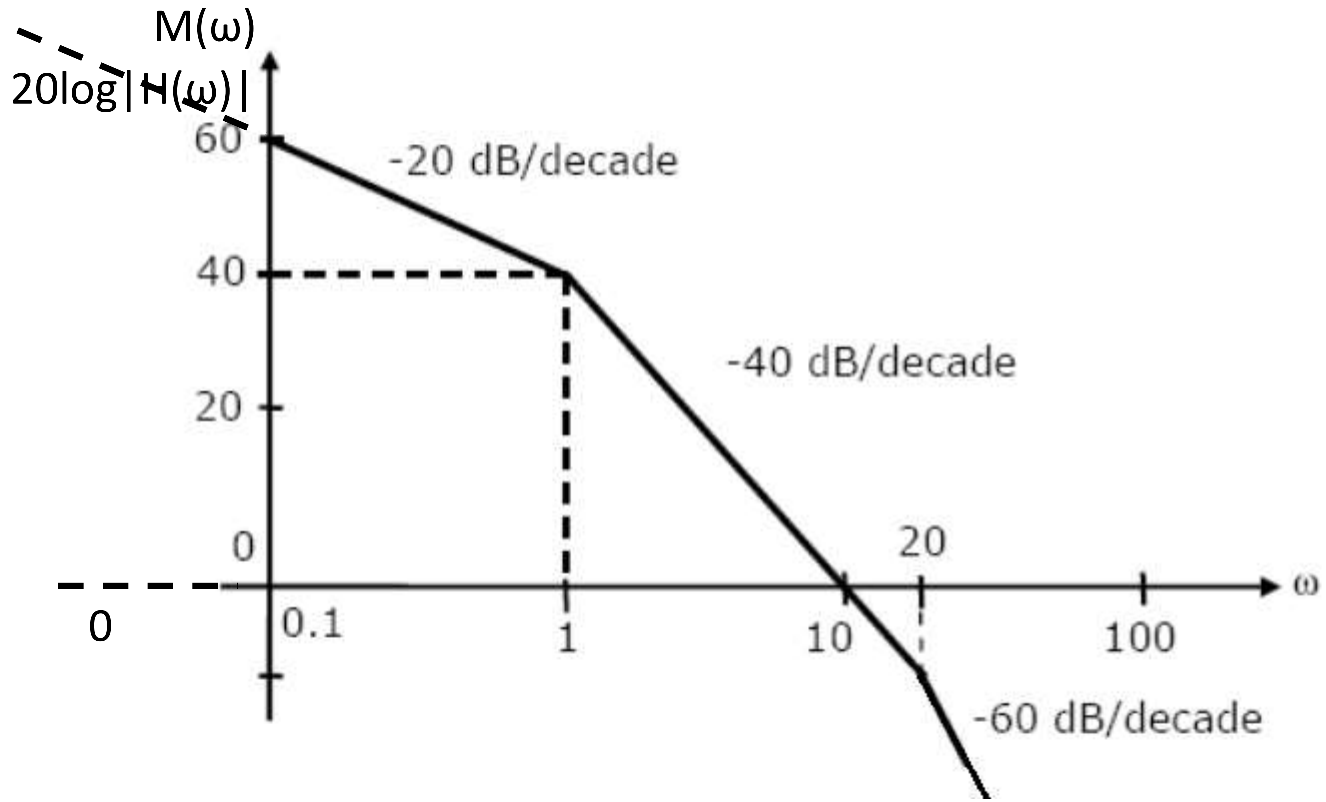
For ω=10 → $\theta_H(10) = \tan^{-1}(10) - \tan^{-1}(1) - \tan^{-1}(0.1) = 0.586 \text{ rad}$

For ω=100 → $\theta_H(100) = \tan^{-1}(100) - \tan^{-1}(10) - \tan^{-1}(1) = -0.696 \text{ rad}$



$\theta_1=\tan^{-1}\omega$
$\pi/2$
$\pi/4$
$\theta=\theta_1+\theta_2+\theta_3$
$\theta_H(\omega)$
0
$-\pi/4$
$\theta_2=-\tan^{-1}(\omega/10)$
$-\pi/2$
$\theta_3=-\tan^{-1}(\omega/100)$
1
10
100
1000
ω(rad/s)

Example: Magnitude response of H(s) is given in the below Bode plot. Find the transfer function H(s)



$$H(s)=\frac{K(1+\frac{s}{z_1})(1+\frac{s}{z_2})(1+\frac{s}{z_3})\cdots}{(1+\frac{s}{p_1})(1+\frac{s}{p_2})(1+\frac{s}{p_3})\cdots}$$

$$H(s)=\frac{K}{s(1+\frac{s}{1})(1+\frac{s}{20})}$$

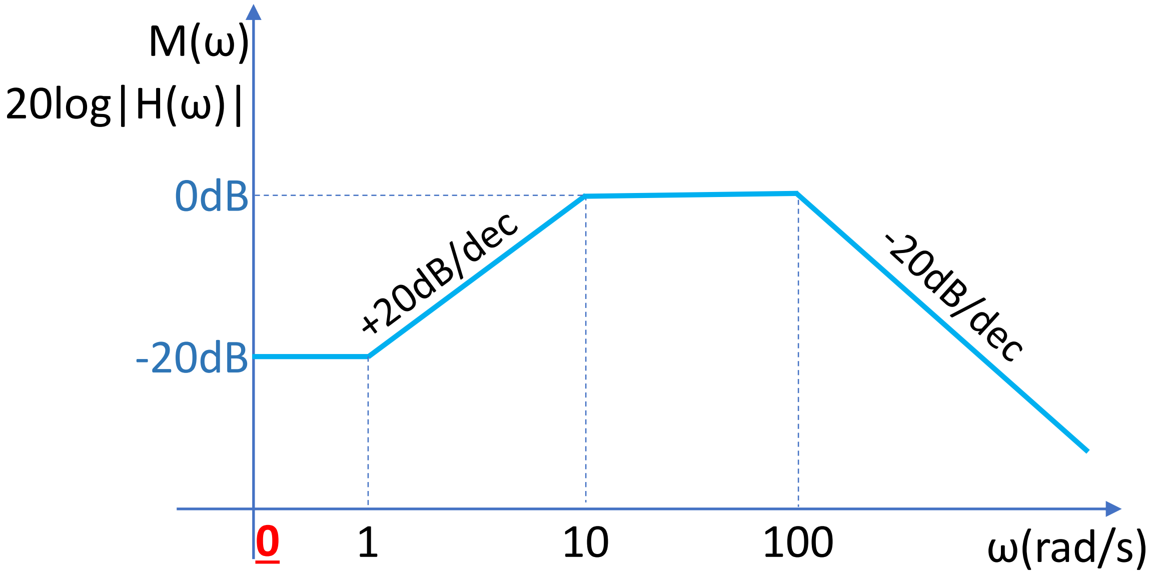
$$H(j\omega)=\left.\frac{K}{s(1+j\omega)(1+\frac{j\omega}{20})}\right|_{\omega=0.1}=60dB$$

$$|H(j\omega)|=\left.\frac{K}{\omega\sqrt{1+\omega^2}\left(\sqrt{1+\frac{\omega}{20}^2}\right)}\right|_{\omega=0.1}=10^{\frac{60}{20}}=1000$$

For ω=0.1 → $\sqrt{1+\omega^2}\cong 1$ $\sqrt{1+\frac{\omega}{20}^2}\cong 1$ $\frac{K}{\omega}=\frac{K}{0.1}\cong 1000$ → K=100

$$H(s)=\frac{100}{s(1+s)(1+0.05s)}$$

Example: Frequency response (magnitude) of H(s) is given in the below Bode plot. Find the transfer function H(s)



$$H(s) = \frac{K(1+\frac{s}{z_1})}{(1+\frac{s}{p_1})(1+\frac{s}{p_2})}$$

$$H(s) = \frac{K(1+\frac{s}{1})}{(1+\frac{s}{10})(1+\frac{s}{100})} = \frac{K(1+s)}{\frac{1}{10}(10+s)\frac{1}{100}(100+s)}$$

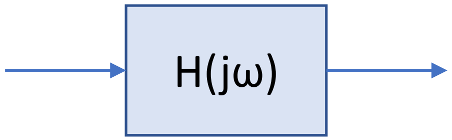
$$H(s) = \frac{1000K(1+s)}{(10+s)(100+s)}$$

$$H(j\omega) = \frac{1000K(1+j\omega)}{(10+j\omega)(100+j\omega)}$$

For ω=0 → 20log|H(ω)|)=-20 → $|H(j\omega)|_{\omega=0} = \frac{1000K}{10\cdot 100} = 10^{\frac{-20}{20}} = 0.1$ → K=0.1

$$H(s) = \frac{100(1+s)}{(10+s)(100+s)}$$

Bode Plot Application by Python:



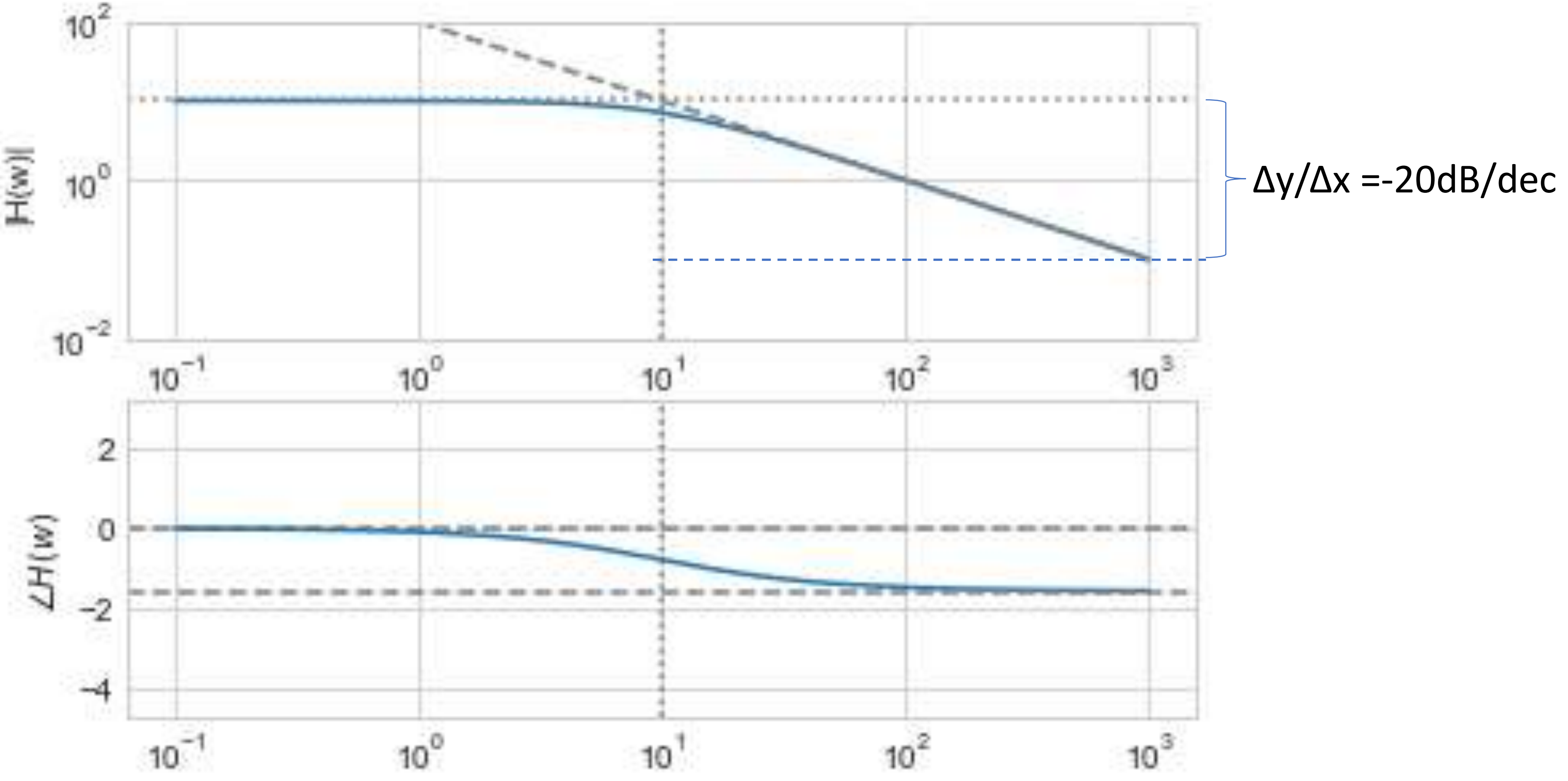
$$H(s) = \frac{K}{(\tau s + 1)^n}$$

$$H(j\omega) = \frac{K}{(\tau j\omega + 1)^n}$$

```
import numpy
import matplotlib.pyplot as plt
from ipywidgets import interact
omega = numpy.logspace(-1, 3, 1000); s = 1j*omega

def annotated_bode(ax_gain, ax_phase, H, K, tau, order):
    high_freq_asymptote = K/(tau*omega)**order
    # Gain part
    ax_gain.loglog(omega, numpy.abs(H))
    ax_gain.axhline(K, color='grey', linestyle=':')
    ax_gain.loglog(omega, high_freq_asymptote, color='grey',linestyle='--')
    ax_gain.axvline(1/tau, color='grey',linestyle=':')
    ax_gain.set_ylim([1e-2, 1e+2])
    ax_gain.set_ylabel('|H(w)|')
    # Phase part
    ax_phase.axhline(0, color='grey', linestyle='--')
    ax_phase.semilogx(omega, numpy.unwrap(numpy.angle(H)))
    ax_phase.axhline(-numpy.pi/2*order, color='grey',linestyle='--')
    ax_phase.axvline(1/tau, color='grey',linestyle=':')
    ax_phase.set_ylim([-3*numpy.pi/2, 2*numpy.pi/2])
    ax_phase.set_ylabel(r'$\angle H(w)$')

def plotresponse(order, tau, K):
    H = K/(tau*s + 1)**order
    fig, [ax_gain, ax_phase] = plt.subplots(2, 1)
    annotated_bode(ax_gain, ax_phase, H, K, tau, order)

interact(plotresponse, order=2, tau=0.1, K=10)
```

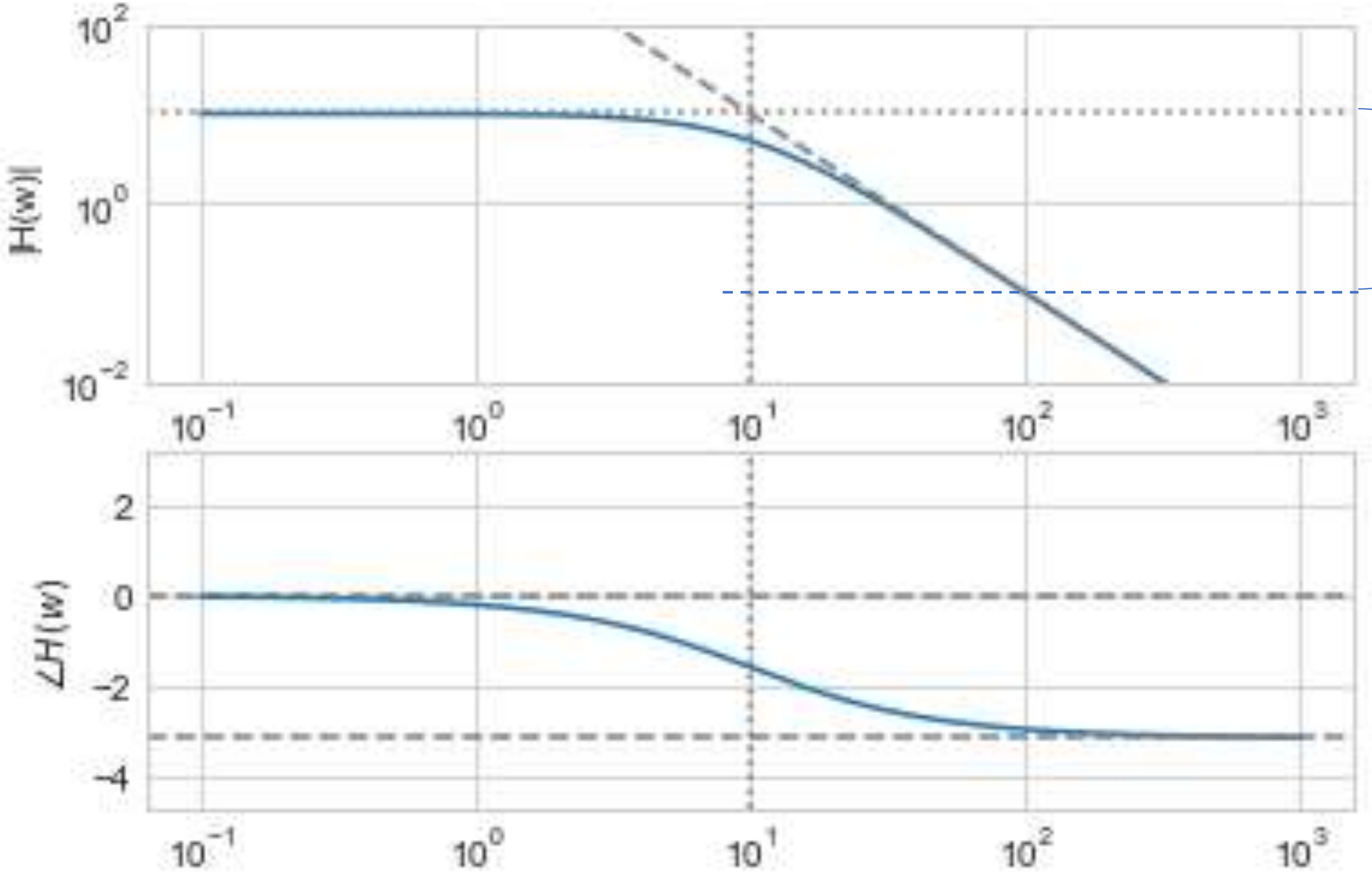
n=1, K=10, τ=0.1

$$H(s) = \frac{1}{0.1s + 1}$$



n=2, K=10, τ=0.1

$$H(s) = \frac{10}{(0.1s+1)^2}$$



Δy/Δx=-40dB/dec

Bilinear transformation

$$s = \frac{2(1-z^{-1})}{T(1+z^{-1})}$$

If a transfer function is known in s domain then its discrete equivalent can be found in z domain. It can also be converted into difference equations for a specified sampling time T . Therefore it can be used for real time code implementation

$$s = \frac{1}{T}\ln(z) = \frac{2}{T}\left[\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \frac{1}{7}\left(\frac{z-1}{z+1}\right)^7 + \cdots\right]$$
$$\approx \frac{2}{T}\frac{z-1}{z+1} = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

Inverse bilinear transformation can be used finding the frequency response of discrete time LTI systems

$$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} \approx \frac{1+sT/2}{1-sT/2}$$

Response to Weighted Sum of Two Sinusoids

Assume that an LTI system transfer function is H(s) and its frequency response is expressed in terms of its magnitude response $|H(\omega)|$ and the phase response $\Theta_H(\omega)$ (its also denoted as $\angle H(\omega)$). If a signal x(t) consisting of two sinusoidal components is applied to the system then output of the system y(t) can be determined by superposition of the frequency responses for each sinusoidal input signal component.

$$x(t)=A_1 \sin(\omega_1 t+ \Theta_1)+A_2 \sin(\omega_2 t+ \Theta_2)$$

$$y(t)=A_1 \, |H(\omega_1)| \sin(\omega_1 t+\Theta_1+\Theta_H(\omega_1))+A_2 \, |H(\omega_2)| \sin(\omega_2 t+ \Theta_2+\Theta_H(\omega_1))$$

This property is valid for discrete time systems too:

$$X[n]=A_1 \sin(\Omega_1 n+ \Theta_1)+A_2 \sin(\Omega_2 n + \Theta_2)$$

$$y[n]=A_1 \, |H(\Omega_1)| \sin(\Omega_1 n+\Theta_1+\Theta_H(\Omega_1)) + A_2 \, |H(\Omega_2)| \sin(\Omega_2 n+\Theta_2+\Theta_H(\Omega_2))$$

How can we find the frequency response if the input signal x(t) is not sinusoidal ?

Example:

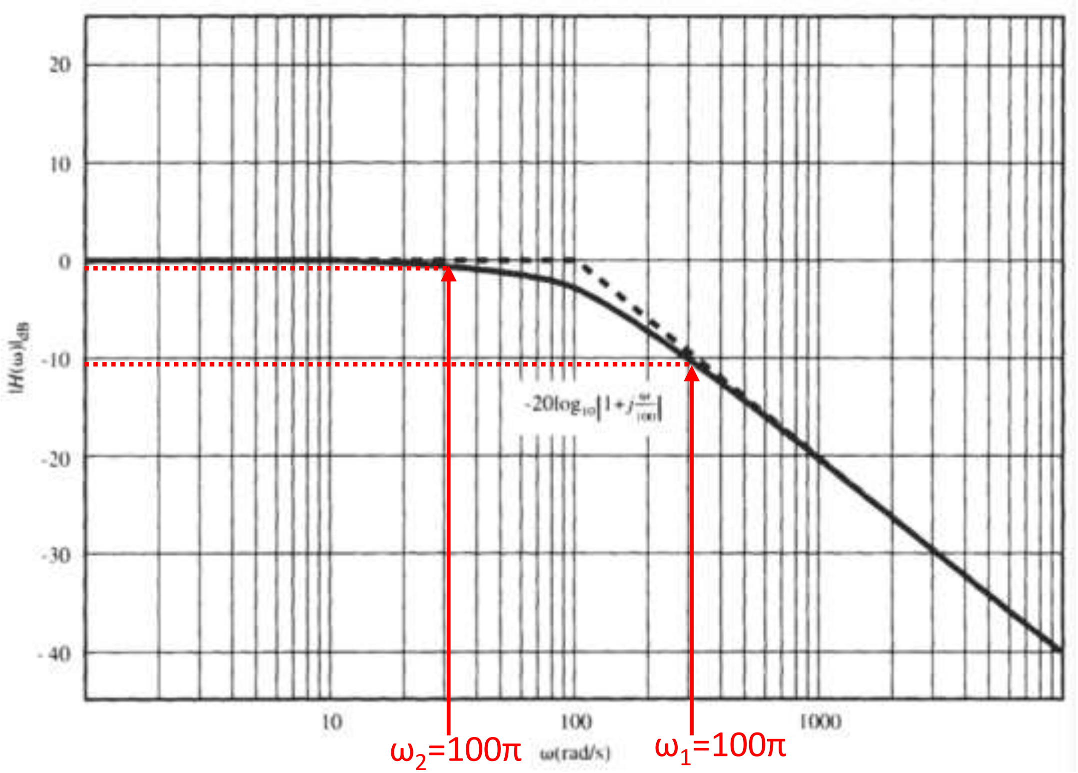
$x(t)= 5\sin(100\pi t+\pi/6) +10\sin(10\pi t)$

$X(s) \xrightarrow{U_1} \boxed{\dfrac{1}{0.01s+1}} \xrightarrow{U_2} Y(s)$ $\quad y(t)= ?$

$$y(t)=A_1\ |H(\omega_1)|\ \sin(\omega_1 t+\Theta_1+\Theta_H(\omega_1))+A_2\ |H(\omega_2)|\ \sin(\omega_2 t+ \Theta_2+\Theta_H(\omega_1))$$

$x_1(t)=5\sin(100\pi t +\pi/6)$ $\rightarrow$ $\omega_1=100\pi$ rad/s, f=50Hz, $A_1= 5$, $\Theta_1= \pi/6$ $\rightarrow$ $|H(\omega_1)| = \sqrt{\frac{1}{1+\pi^2}}=0.303$, $\Theta_H(\omega_1)= \tan^{-1}(\pi)=-1.26$

$x_2(t)=10\sin(10\pi t)$ $\rightarrow$ $\omega_2=10\pi$ rad/s, f=5Hz, $A_2= 10$, $\Theta_2= 0$ $\rightarrow$ $|H(\omega_1)| = \sqrt{\frac{1}{1+0.01\pi^2}}=0.985$, $\Theta_H(\omega_1)= \tan^{-1}(0.1\pi)=-0.3$



$y(t)=5\cdot 0.303\ \sin(100\pi t+0.523-1.26)+10\cdot 0.985\ \sin(10\pi t-0.3)$

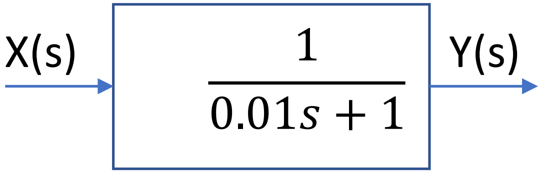
$\boxed{y(t)=1.52\sin(100\pi t-0.74)+9.85\sin(10\pi t-0.3)}$

$|H(\omega_1)| =20\log\frac{U_2}{U_1} = -10.2dB \rightarrow \frac{U_2}{U_1} = 10^{\frac{-10.2}{20}} \cong 0.3$

$|H(\omega_1)| =20\log\frac{U_2}{U_1} = -0.2dB \rightarrow \frac{U_2}{U_1} = 10^{\frac{-0.2}{20}} \cong 0.98$

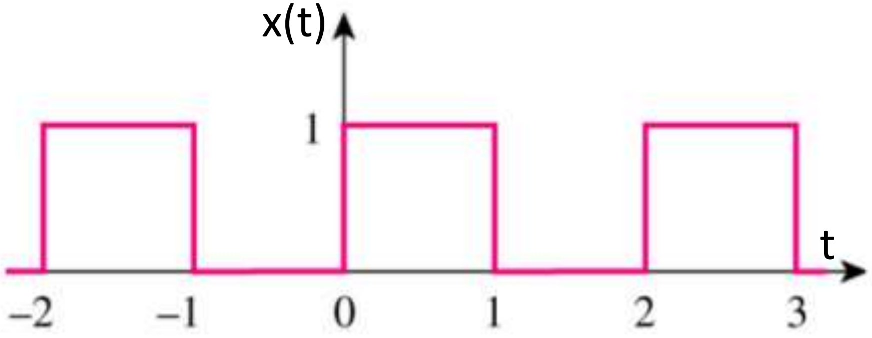
Example:

$$x(t)\begin{cases} 1, & kT \le t < kT + \frac{T}{2} \\ 0, & kT + \frac{T}{2} \le t < (k+1)T \end{cases}$$



y(t)= ?

k=0, 1, 2, …
T=2

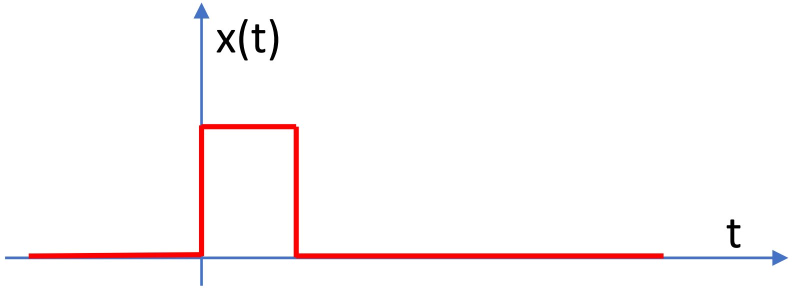


Since the signal is periodic then we can apply Fourier Series here

Fourier Series

∑sin

If the signal were non-periodic then we would apply Fourier Transform



Frequency response of a system if the input is not sinusoidal

Fourier Series: Decomposes periodic signals into sinusoidal harmonics