MAT 271E: PROBABILITY AND STATISTICS

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WEEK 6

COUNTING TECHNIQUES : PROBABILITY TREE

PERMUTATIONS COMBINATIONS

Counting Techniques

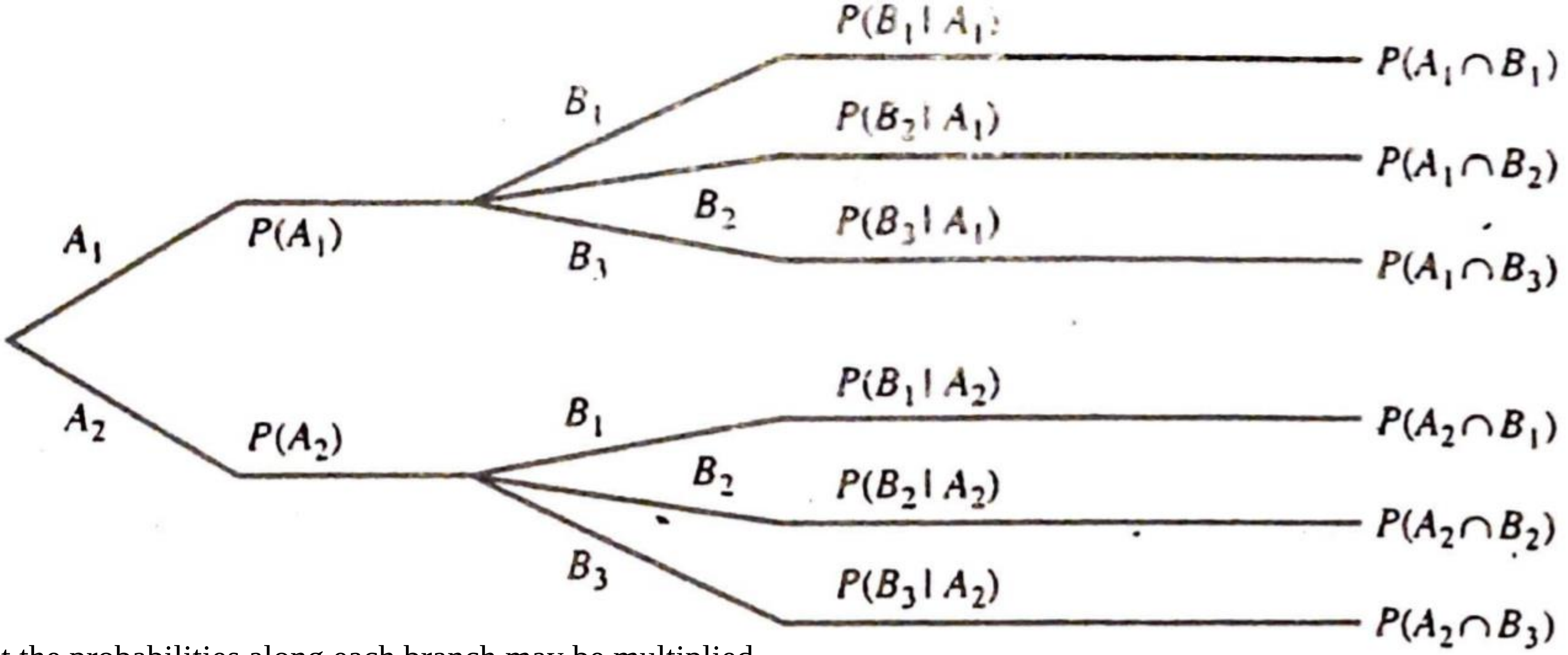
- ✓ Probability tree
- ✓ Permutation
- ✓ Combination

Probability tree

Often, in physical applications, repeatable experiments can be quite complex, sometimes consisting of several stages. in evaluating probabilities associated with such experiments, the **probability tree** is a convenient pictorial tool.

- ✓ Suppose in stage 1, one of two events can occur, A1 and A2
- ✓ Further in stage 2, one of three events can occur, B1, B2 or B3
- ✓ A probability tree depicting the sample space for this two stage experiment is given as:

Probability tree



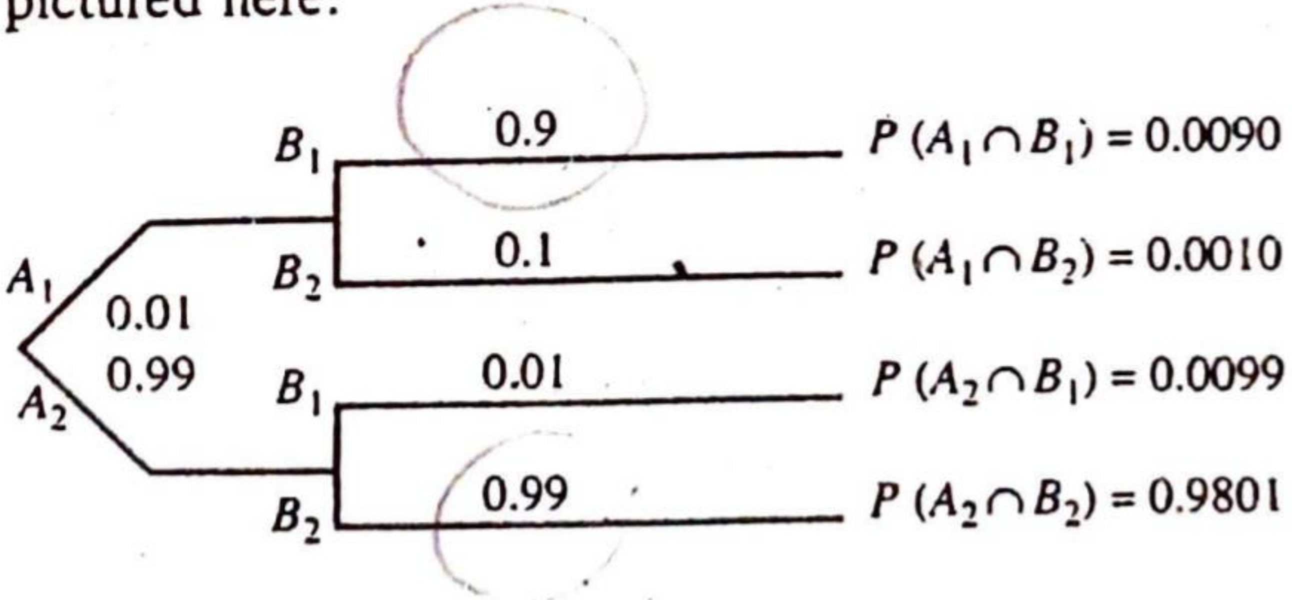
Note that the probabilities along each branch may be multiplied to obtain the probabilities at the far right of this branch; $P(A_1 \cap B_1) = P(A_1)P(B_1/A_1)$

Example 4.16

- A sensitive electronic component is a part of guidance system on a military vehicle.Because it fails on the average about once every hundred flights, it is important to have a reliable performance test for it. Quality engineers have devised a test that indicates that a component is defective is 90% of the time, when the component is defective. Further, the test will indicate that a component is good 99% of the time when the component is good.
- The guidance system on one of the vehicles failed. The component in that system was tested and the test indicate that the component was efective. What is the probability that the component was indeed defective.

Example 1 - solution:

Solution. In this problem we have two stages, each with two possible events, as pictured here.



where A_1 = component is defective, A_2 = component is good, B_1 = test says component is defective, and B_2 = test says component is good. We *know* that the test said the component was defective. Therefore, only outcomes containing B_1 may form part of the reduced sample space.

Example 1 - solution:

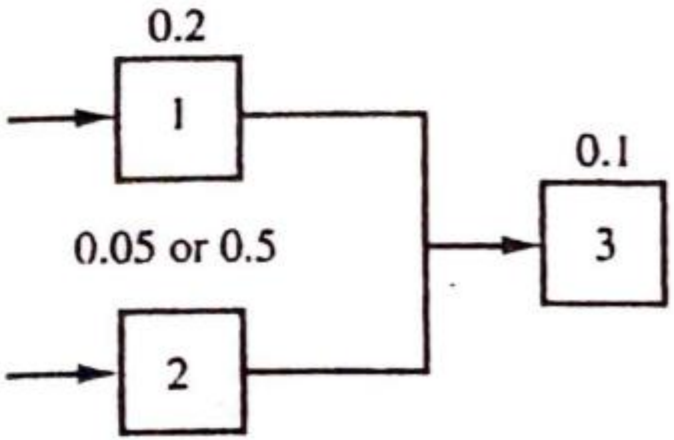
$$P(A_1 \mid B_1) = \frac{P(A_1 \cap B_1)}{P(B_1)} = \frac{P(A_1 \cap B_1)}{P(A_1 \cap B_1) + P(A_2 \cap B_1)} = 0.476$$

This implies that when the test says that the component is defective the component is more likely to be good! This apparently nonsensical result stems from the relatively

high probability (0.01) of the test saying the component is defective when it is indeed good.

Example 2:

Example 4.17. Two electronic components are in parallel and one is in series.



Event	Description	failure Probability
I	component 1 fails	0.20
II|I′	component 2 fails|1 succeeds	0.05
III	component 3 fails	0.10
II|I	component 2 fails|1 fails	0.50

Example 2:

The probability of failure is given for components 1 and 3. In addition, because component 1 will give off considerable heat when it fails, it is estimated that if component 1 fails there is a 0.5 probability that component 2 will also fail. If component 1 does not fail, component 2 will fail with a probability of only 0.05. The parallel components 1 and 2 are redundant, so they must both fail before current cannot flow to component 3. Compute the probability that

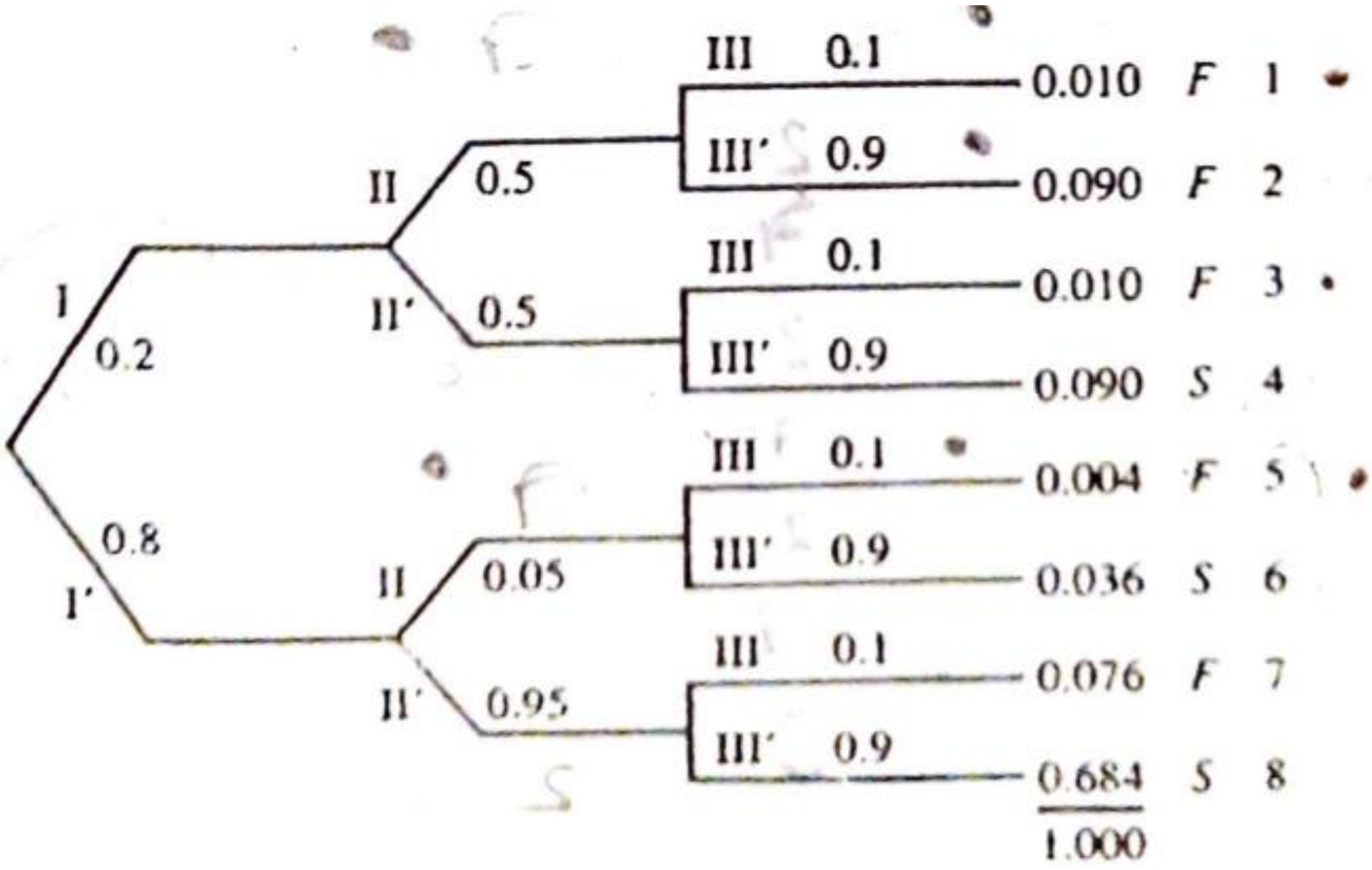
(a) the device does not fail

(b) the device fails

(c) either component 1 or 2 fails and component 3 fails

(d) component 2 failed given that the device fails

Example 2 - Solution



Example 2 - Solution

(a) The probability of success, $P(S)$, is the sum of results 4, 6, and 8, $P(S) = 0.090 + 0.036 + 0.684 = 0.810$.

(b) The probability of failure, $P(F)$, is found by $P(F) = 1 - P(S) = 1 - 0.810 = 0.190$.

(c) If the event $(\mathrm{I} \cup \mathrm{II}) \cap \mathrm{III}$ occurs, the probability tree results 1, 3, and 5 apply. Therefore, $P[(\mathrm{I} \cup \mathrm{II}) \cap \mathrm{III}] = 0.01 + 0.01 + 0.004 = 0.024$. Another way to reach this result is to notice that events I and II are not mutually exclusive. First we compute $P(\mathrm{I} \cup \mathrm{II}) = P(\mathrm{I}) + P(\mathrm{II}) - P(\mathrm{II}|\mathrm{I})P(\mathrm{I}) = 0.20 + 0.14 - (0.5)0.2 = 0.24$. To obtain $P(\mathrm{II}) = 0.14$, we add the probabilities of the events where II occurs in results 1, 2, 5, and 6. Since the events I, II, and III are independent, $P[(\mathrm{I} \cup \mathrm{II}) \cap \mathrm{III}] = P(\mathrm{I} \cup \mathrm{II})P(\mathrm{III}) = 0.24(0.10) = 0.024$.

(d) If failure results, $P(\mathrm{II}|\mathrm{F})$ is a probability that II was, in part, a cause of the failure. Recall that $P(\mathrm{II}|\mathrm{F}) = P(\mathrm{II} \cap F)/P(F)$. Failure and II occur in results 1, 2, and 5; $P(F)$ was computed in (b). Therefore,

$$P(\mathrm{II} \mid F) = \frac{0.010 + 0.090 + 0.004}{0.190} = \frac{0.104}{0.190} = 0.547$$

EXTRA EXERCISE

Because a new medical procedure has been shown to be effective in the early detection of an illness, a medical screening of the population is proposed. The probability that the test correctly identifies someone with the illness as positive is 0.99, and the probability that the test correctly identifies someone without the illness as negative is 0.95. The incidence of the illness in the general population is 0.0001. You take the test, and the result is positive. What is the probability that you have the illness?

EXTRA EXERCISE

Let D denote the event that you have the illness, and let S denote the event that the test signals positive. The probability requested can be denoted as $P(D|S)$. The probability that the test correctly signals someone without the illness as negative is 0.95. Consequently, the probability of a positive test without the illness is

$$P(S|D') = 0.05$$

From Bayes' Theorem,

$$\begin{aligned} P(D|S) &= P(S|D)P(D)/[P(S|D)P(D) + P(S|D')P(D')] \\ &= 0.99(0.0001)/[0.99(0.0001) + 0.05(1 - 0.0001)] \\ &= 1/506 = 0.002 \end{aligned}$$

Surprisingly, even though the test is effective, in the sense that $P(S|D)$ is high and $P(S|D')$ is low, because the incidence of the illness in the general population is low, the chances are quite small that you actually have the disease even if the test is positive.

Permutations

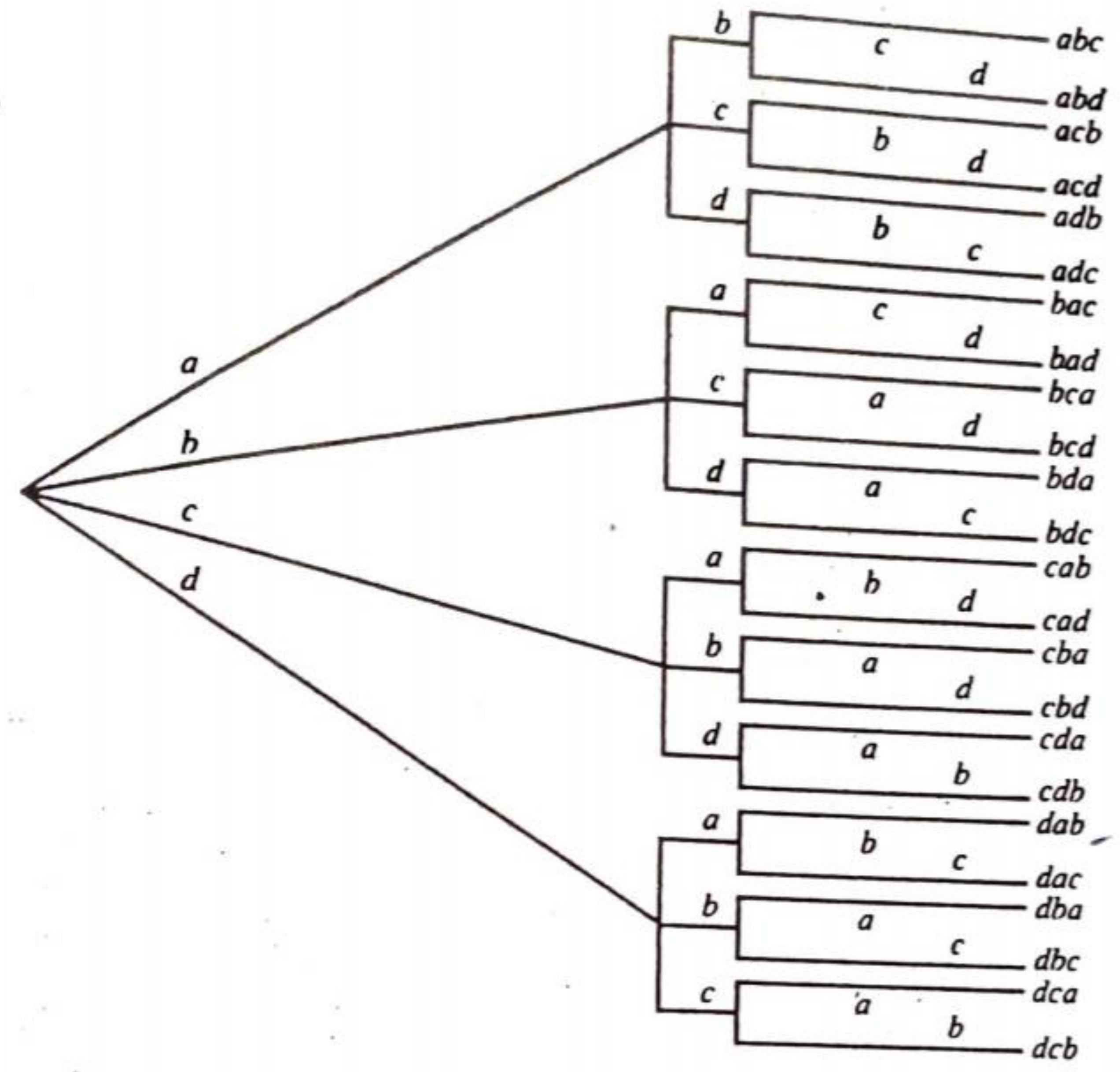
The probability tree approach will quickly become cumbersome in the presence of a large number of stages and events at each stage. Therefore, probability trees must be viewed as useful conceptual tools and not ends in themselves. Clearly, an algebraic and systematic method of counting elements of the sample space is needed.

Consider a general, 2-stage system with r and s possibilities at stages 1 and 2, respectively. The experiment would have (r)(s) outcomes. A third stage with t possibilities would yield (r)(s)(t) outcomes for the experiment. Such a general tree may be used to assist in answering the question, How many ways can r thıngs be selected from n distinct things?

Permutations

To clarify this consider how many three-letter "words" can be made from the letters **a, b, c and d**, if each letter may be used only once. Thinking of this as a 3 stage decision problem, the answer is easily seen to be (4).(3).(2) = 24, since the first stage has 4 selectable letters whereas the second and third stages have only 3 and 2 letters respectively. The specific tree for this problem is:

Permutations



Permutations

The answer is that r things may be selected from n things in

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

ways.

... permutations of n things taken r at a time. Formally speaking, a permutation is an ordered arrangement of all or some of the elements of a set. The notation $n!$ is read *n factorial* and is defined to mean the product of all integers from 1 up to n. For example,

$$4! = (1)(2)(3)(4) = 24 \quad \text{and} \quad {}_4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = \frac{(4)(3)(2)(1)}{1} = 24$$

Zero factorial (0!) is a special case and is defined to have a value of 1.

Combinations

Permutation is an ordered arrangement of all or some of the elements of a set.

Permutation is used when the order is important.

Combination is used when the order is NOT important

Number of permutations of n things taken r at a time: $${}_nP_r = \frac{n!}{(n-r)!}$$

Number of combinations of n things taken r at a time: $${}_nC_r = \frac{n!}{(n-r)!r!}$$

Extra Example

- ✓ In a group of 10 people, three $5 prizes will be given. How many ways can the prizes be distributed?

- ✓ There are 12 horses in a horseshow competition. The top three winning horses receive Money. How many possible money winning orders are there for a competition with 12 horses.

Extra Example

In a group of 10 people, three $5 prizes will be given. How many ways can the prizes be distributed?

$${}_{10}C_3 = \frac{10 \quad 9 \quad 8}{} = \frac{720}{6} = 120$$

$$= \frac{10!}{7!}$$

Extra Example

There are 12 horses in a horseshow competition. The top three winning horses receive Money. How many possible money winning orders are there for a competition with 12 horses.

$${}_{12}P_{3}: 12 \text{ x } 11 \text{ x } 10=1320$$

COUNTING TECHNIQUES: COMBINATIONS

Example 4.19: Ten electronic components are available. However, it is known that 3 are defective. A set of 3 of the 10 components is selected at random.

A.) What is the probability that at least 2 of the components are defective

B.) What is the probability that at least 1 component is defective

Example 4.19-Solution

- A.)

$$\frac{{}_3C_2 \ * {}_7C_1 \ + {}_3C_3 \ * {}_7C_0}{{}_{10}C_3} = \frac{11}{60}$$

- B.)

$$1 - \frac{{}_3C_0 * {}_7C_3}{{}_{10}C_3} = 1 - \frac{1 * 35}{120} = \frac{85}{120}$$

EXTRA EXERCISE

A bin of 50 manufactured parts contains three defective parts and 47 nondefective parts. A sample of six parts is selected from the 50 parts. Selected parts are not replaced. That is, each part can only be selected once and the sample is a subset of the 50 parts. How many different samples are there of size six that contain exactly two defective parts?

EXTRA EXERCISE

A bin of 50 manufactured parts contains three defective parts and 47 nondefective parts. A sample of six parts is selected from the 50 parts. Selected parts are not replaced. That is, each part can only be selected once and the sample is a subset of the 50 parts. How many different samples are there of size six that contain exactly two defective parts?

- A. How many different samples are there of size six that contain two defective parts

$$\underline{{}_{3}C_{2} * {}_{47}C_{4}} = \underline{3 * 178365} = 535095$$

- B.What is the probability that a randomly selected sample of size six contain exactly two defective parts.

$$\frac{{}_{3}C_{2} * {}_{47}C_{4}}{{}_{50}C_{6}} = \frac{3 * 178365}{15890700} = 0.034$$

Extra Example

<u>**Problem 4:**</u> A batch of 140 semiconductor chips is inspected by choosing a sample of 5 chips. Assume 10 of the chips do not conform to customer requirements.

(a) How many different samples are possible?

b.) What is the probability that a randomly selecte sample of five contain exactly one conforming chip

c.)What is the probability that a randomly selected sample of five contain at least one nonforming chip

Extra Example

Problem 4: A batch of 140 semiconductor chips is inspected by choosing a sample of 5 chips. Assume 10 of the chips do not conform to customer requirements.

(a) How many different samples are possible?

${}_{140}C_5$

b.) What is the probability that a randomly selected sample of five contain exactly one conforming chip

$({}_{10}C_4 * {}_{130}C_1)/{}_{140}C_5$

c.)What is the probability that a randomly selected sample of five contain at least one nonforming chip

$1-(({}_{130}C_5 * {}_{10}C_0)/ {}_{140}C_5)$