Matrices and Gaussian Elimination

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Coefficient Matrices

A general system of m linear equations in the n variables $x_1, x_2, \ldots, x_n$ may be written in the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
\tag{5}
$$

The **coefficient matrix** of the linear system in (5) is the $m \times n$ matrix

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}. \tag{6}
$$

$$\mathbf{A} = [\,a_{ij}\,] = \begin{array}{cl} \begin{bmatrix} & \vdots & \\ \cdots\cdots & a_{ij} & \cdots\cdots \\ & \vdots & \end{bmatrix} & (i\text{th row}) \\ (j\text{th column}) & \end{array}$$

The first subscript i specifies the row and the second subscript j the column of $\mathbf{A}$ in which the element a_{ij} appears:

First subscript	Second subscript
Row	Column

Let us write

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{8}$$

for the column of constants in the general system in (5). An $m \times 1$ matrix—that is, one with a single column—is often called a (**column**) **vector** and is denoted by a boldface letter. When we adjoin the constant vector $\mathbf{b}$ to the coefficient matrix $\mathbf{A}$ (as a final column), we get the matrix

$$[\,\mathbf{A} \mid \mathbf{b}\,] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]. \tag{9}$$

This $m \times (n+1)$ matrix is called the **augmented coefficient matrix**, or simply the augmented matrix, of the $m \times n$ system in (5).

The augmented coefficient matrix of the system

$$\begin{aligned} 2x_1 + 3x_2 - 7x_3 + 4x_4 &= 6 \\ x_2 + 3x_3 - 5x_4 &= 0 \\ -x_1 + 2x_2 \qquad\quad - 9x_4 &= 17 \end{aligned}$$

of three equations in 4 variables is the 3x5 matrix

$$\left[\begin{array}{cccc|c} 2 & 3 & -7 & 4 & 6 \\ 0 & 1 & 3 & -5 & 0 \\ -1 & 2 & 0 & -9 & 17 \end{array}\right]_{3\times 5}$$

Elementary Row Operations

DEFINITION Elementary Row Operations

The following are the three types of **elementary row operations** on the matrix **A**:

1. Multiply any (single) row of **A** by a nonzero constant. $(R_i \to aR_i)$
2. Interchange two rows of **A**. $(R_i \leftrightarrow R_j)$
3. Add a constant multiple of one row of **A** to another row. $(R_i \to R_i + aR_j)$

Example: Solve the system

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 4 \\
3x_1 + 8x_2 + 7x_3 &= 20 \\
2x_1 + 7x_2 + 9x_3 &= 23,
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right] \xrightarrow{R_2 \to (-3)R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 2 & 7 & 9 & 23 \end{array}\right]
$$

$$
\xrightarrow{R_3 \to (-2)R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right]
$$

$$
\xrightarrow{R_3 \to (-3)R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The final matrix is the augmented coeffient matrix of the system.

$$x_1 + 2x_2 + 1x_3 = 4$$
$$x_2 + 2x_3 = 4$$
$$1 \cdot x_3 = 3$$

(Back substitution)

$$x_3 = 3$$
$$x_2 = 4 - 2x_3 = -2$$
$$x_1 = 5$$

unique solution!

DEFINITION Row-Equivalent Matrices

Two matrices are called **row equivalent** if one can be obtained from the other by a (finite) sequence of elementary row operations.

THEOREM 1 Equivalent Systems and Equivalent Matrices

If the augmented coefficient matrices of two linear systems are row equivalent, then the two systems have the same solution set.

DEFINITION Echelon Matrix

The matrix **E** is called an **echelon matrix** provided it has the following two properties:

1. Every row of **E** that consists entirely of zeros (if any) lies *beneath* every row that contains a nonzero element.
2. In each row of **E** that contains a nonzero element, the *first* nonzero element lies strictly to the *right* of the first nonzero element in the preceding row (if there is a preceding row).

Echelon matrices are sometimes called *row-echelon matrices*. Property 1 says that if **E** has any all-zero rows, then they are grouped together at the bottom of the matrix. The first (from the left) *nonzero* element in each of the other rows is called its **leading entry**. Property 2 says that the leading entries form a "descending staircase" pattern from upper left to lower right, as in the following echelon matrix.

$$\mathbf{E} = \begin{bmatrix} 2 & -1 & 0 & 4 & 7 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

not echelon matrix

not echelon matrix

Suppose that a linear system is in **echelon form**—its augmented matrix is an echelon matrix. Then those variables that correspond to *columns* containing leading entries are called **leading variables**; all the other variables are called **free variables**.

The augmented coeffient matrix of the system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \quad \text{(E1)} \\ x_3 + 2x_5 &= -3 \quad \text{(E2)} \\ x_4 - 4x_5 &= 7 \quad \text{(E3)} \end{aligned}$$

is echelon matrix.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$

The leading enteries are in the first, third and fourth columns. Hence x_1, x_3, x_4 are the leading variables and x_2 and x_5 are the free variables. To solve the system by back substitution, we first write

$x_2 = s$, $x_5 = t$, where s and t are arbitrary parameters.

(E3) $x_4 = 7 + 4x_5 = 7 + 4t$

(E2) $x_3 = -3 - 2x_5 = 3 - 2t$

(E1) $x_1 = 10 + 2x_2 - 3x_3 - 2x_4 - x_5$
$= 5 + 2s - 3t.$

Thus the system has an infinite solution set consisting of all $(x_1, x_2, x_3, x_4, x_5)$ given in terms of the two parameters s and t.

$$\left.\begin{aligned} x_1 &= 5+2s-3t \\ x_2 &= s \\ x_3 &= -3-2t \\ x_4 &= 7+4t \\ x_5 &= t \end{aligned}\right\}$$

with $s=2$, $t=1$

the solution

$x_1 = 6$ $\quad$ $x_4 = 11$

$x_2 = 2$ $\quad$ $x_5 = 1$.

$x_3 = -5$

We can transform any matrix (using elementary row operations) into an echelon matrix. The procedure is known as Gaussian elimination.

ALGORITHM Gaussian Elimination

1. Locate the first column of $\mathbf{A}$ that contains a nonzero element.
2. If the first (top) entry in this column is zero, interchange the first row of $\mathbf{A}$ with a row in which the corresponding entry is nonzero.
3. Now the first entry in our column is nonzero. Replace the entries below it in the same column with zeros by adding appropriate multiples of the first row of $\mathbf{A}$ to lower rows.
4. After Steps 1–3, the matrix looks like the matrix below, although there may be several initial columns of zeros or even none at all. Perform Steps 1–3 on the indicated lower right matrix $\mathbf{A}_1$.

Example: Solve the system

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\
2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\
3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27,
\end{aligned}
$$

$$
\left[\begin{array}{ccccc|c}
1 & -2 & 3 & 2 & 1 & 10 \\
2 & -4 & 8 & 3 & 10 & 7 \\
3 & -6 & 10 & 6 & 5 & 27
\end{array}\right]
$$

$$
\xrightarrow[R_3 \to (-3)R_1 + R_3]{R_2 \to (-2)R_1 + R_2}
\left[\begin{array}{ccccc|c}
1 & -2 & 3 & 2 & 1 & 10 \\
0 & 0 & 2 & -1 & 8 & -13 \\
0 & 0 & 1 & 0 & 2 & -3
\end{array}\right]
$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -3 \\ 0 & 0 & 1 & 0 & 2 & 7 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & 7 \\ 0 & 0 & 2 & -1 & 8 & -3 \end{array}\right]$$

$$\xrightarrow{R_3 \to (-2)R_2 + R_3} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{array}\right]$$

$$\xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ x_3 + 2x_5 &= -3 \\ x_4 - 4x_5 &= 7 \end{aligned}$$

$m = 3$ equations $n = 5$ unknowns

$|m-n| = |3-5| = 2$ free variables.

We have solved!.

Example: Solve the system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ x_2 + x_3 &= 1 \\ 2x_1 - 3x_2 + 7x_3 &= 3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & -3 & 7 & 3 \end{array}\right] \xrightarrow{R_3 \to (-2)R_1 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 \to (-1)R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$x_1 - 2x_2 + 3x_3 = 0$

$x_2 + x_3 = 1$

$0 \neq 2$

no solution!

Example: Solve the system

$$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ x_1 - x_2 &= 0 \\ 3x_1 + 2x_2 &= 1 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{array}\right] \xrightarrow[R_3 \to (-3)R_1 + R_3]{R_2 \to (-2)R_1 + R_2}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 5 & 1 \\ 0 & 5 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

2 leading variables: x_1, x_2

0 free variable

$$\begin{aligned} x_1 - x_2 &= 0 \\ 5x_2 &= 1 \\ 0 &= 0 \end{aligned}$$

$x_2 = 1/5$, $x_1 = \frac{1}{5}$, unique solution!

Remark

A linear system is said to be consistent if it has at least one solution and inconsistent if it has none.

- If the reduction of the augmented matrix to echelon form leads to a row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & * \end{array}\right],$$

where $*$ denotes a nonzero entry in the last column, then we have an inconsistent equation, so the system has no solution.

- If there is no such a row in the augmented matrix, the system is consistent. In this case, there are two possibilities:
  1. If number of leading variables is equal to number of unknowns, the system has a unique solution.
  2. If number of leading variables is less than number of unknowns, the system has infinitely many solutions.

Example:

Under what condition on the constants a, b, and c does the system

$$\begin{aligned} 2x - \ \ y + 3z &= a \\ x + 2y + \ \ z &= b \\ 7x + 4y + 9z &= c \end{aligned}$$

have a unique solution? No solution? Infinitely many solutions?

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & a \\ 1 & 2 & 1 & b \\ 7 & 4 & 9 & c \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 2 & -1 & 3 & a \\ 7 & 4 & 9 & c \end{array}\right]$$

$$\xrightarrow[R_3 \to (-7)R_1 + R_3]{R_2 \to (-2)R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 0 & -5 & 1 & a-2b \\ 0 & -10 & 2 & c-7b \end{array}\right] \xrightarrow{R_3 \to (-2)R_2 + R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 0 & -5 & 1 & a-2b \\ 0 & 0 & 0 & c-3b-2a \end{array}\right]$$

The system has no solution when $c-3b-2a \neq 0$.

The system has infinitely many solutions when $c-3b-2a=0$.

The system does not have a unique solution.