# The Eigenvalue Method for Linear Systems

Consider the homogeneous first-order system with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

where $\mathbf{A} = [\, a_{ij} \,]$. When we substitute the trial solution $\mathbf{x} = \mathbf{v}e^{\lambda t}$ (having derivative $\mathbf{x}' = \lambda \mathbf{v} e^{\lambda t}$) in Eq. (4), the result is

$x = v e^{\lambda t} \Rightarrow x' = \lambda v e^{\lambda t}$

$$\lambda \mathbf{v} e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t}.$$

$x' = \lambda v e^{\lambda t} = Ax = A(v e^{\lambda t})$

We cancel the nonzero scalar factor $e^{\lambda t}$ to get

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}.$$

$\lambda v e^{\lambda t} = A(v e^{\lambda t})$

$\underbrace{e^{\lambda t}}_{\neq 0}(\underbrace{\lambda v - Av}_{=0}) = 0$

$Av = \lambda v$

### THEOREM 1 Eigenvalue Solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Let $\lambda$ be an eigenvalue of the [constant] coefficient matrix $\mathbf{A}$ of the first-order linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}.$$

If $\mathbf{v}$ is an eigenvector associated with $\lambda$, then

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

is a nontrivial solution of the system.

# Distinct Real Eigenvalues

Example: a) Find a general solution of the system

$$x_1' = 4x_1 + 2x_2,$$
$$x_2' = 3x_1 - x_2$$

b) Find a particular solution of the system in (a) that satisfies the initial conditions $x_1(0) = 1$ and $x_2(0) = -1$

(a) The matrix form of the system is

$$x' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} x$$

step 1: **1.** First, we solve the characteristic equation in (6) for the eigenvalues $\lambda_1$, $\lambda_2$, $\ldots$, $\lambda_n$ of the matrix $\mathbf{A}$.

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = -4 - 4\lambda + \lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$$

$$\lambda_1 = -2, \quad \lambda_2 = 5.$$

Step 2:

**2.** Next, we attempt to find $n$ *linearly independent* eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with these eigenvalues.

Case 1: $\lambda_1 = -2$

$$(A - \lambda_1 I)v_1 = 0$$

$$(A + 2I)v_1 = 0$$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left.\begin{matrix} 6a+2b=0 \\ 3a+b=0 \end{matrix}\right\} \quad \begin{matrix} 3a+b=0 \\ a=1 \\ \Rightarrow b-3 \end{matrix}$$

$$v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

case 2: $\lambda_2 = 5$

$$(A - 5I)v_2 = 0$$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left.\begin{matrix} -a+2b=0 \\ 3a-6b=0 \end{matrix}\right\} \quad \begin{matrix} a=2b \\ b=1 \Rightarrow a=2 \end{matrix}$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

step 3:

**3.** Step 2 is not always possible, but, when it is, we get $n$ linearly independent solutions

$$\mathbf{u}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \quad \mathbf{u}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \quad \ldots, \quad \mathbf{u}_n(t) = \mathbf{v}_n e^{\lambda_n t}. \tag{8}$$

In this case the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a linear combination

$$\mathbf{x}(t) = c_1 \mathbf{u}_1(t) + c_2 \mathbf{u}_2(t) + \cdots + c_n \mathbf{u}_n(t)$$

of these $n$ solutions.

$$x(t) = c_1 u_1(t) + c_2 u_2(t)$$

$$= c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 = c_1 e^{-2t} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{5t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{-2t} + 2c_2 e^{5t} \\ -3c_1 e^{-2t} + c_2 e^{5t} \end{bmatrix}$$

$$x_1(t) = c_1 e^{-2t} + 2c_2 e^{5t}$$

$$x_2(t) = -3c_1 e^{-2t} + c_2 e^{5t}$$

(b)

$$x_1(t) = c_1 e^{-2t} + 2c_2 e^{5t}$$

$$x_2(t) = -3c_1 e^{-2t} + c_2 e^{5t}$$

$$X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \left( \begin{array}{l} x_1(0) = 1 \\ x_2(0) = -1 \end{array} \right)$$

$$\left. \begin{array}{l} c_1 + 2c_2 = 1 \\ -3c_1 + c_2 = -1 \end{array} \right\} \quad c_2 = \frac{2}{7}, \quad c_1 = \frac{3}{7}$$

$$x_1(t) = \frac{3}{7} e^{-2t} + \frac{4}{7} e^{5t}$$

$$x_2(t) = -\frac{6}{7} e^{-2t} + \frac{2}{7} e^{5t}$$

## Complex Eigenvalues

Let $\lambda = p + qi$ and $\overline{\lambda} = p - qi$ are such a pair of eigenvalues. If v is an eigenvector associated with $\lambda$, so that Thus the conjugate $\overline{v}$ of v is an eigenvector associated with $\overline{\lambda}$. If $v = a + ib$, then $\overline{v} = a - ib$.

The complex-valued solution associated with $\lambda$ and v is

then

$$x(t) = e^{\lambda t} v = e^{(p+iq)t} \cdot v = e^{pt}(\cos qt + i.\sin qt)v$$

$$= e^{pt}(\cos qt + i.\sin qt)(a+ib)$$

$$= e^{pt}((\cos qt).a - (\sin qt).b) + i.e^{pt}.((\sin qt).a + (\cos qt).b)$$

Because the real and imaginary parts of a complex-valued solution are also solutions. We that get two real valued solutions

$$x_1(t) = Re[x(t)] = e^{pt}((\cos qt).a - (\sin qt).b)$$

$$x_2(t) = Im[x(t)] = e^{pt}.((\sin qt).a + (\cos qt).b)$$

Example:

Find a general solution of the system

$$\frac{dx_1}{dt} = 4x_1 - 3x_2,$$

$$\frac{dx_2}{dt} = 3x_1 + 4x_2.$$

$$X' = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} X \quad , \quad A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = 0 \qquad (4-\lambda)^2 = -9$$

$$4 - \lambda = \mp 3i$$

$$\lambda_1 = 4 + 3i, \quad \lambda_2 = 4 - 3i$$

$$\lambda_1 = 4 + 3i \Rightarrow (A - \lambda_1 I) V = 0$$

$$\begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} -3ia - 3b = 0 \\ 3a - 3ib = 0 \end{matrix}$$

$$-ia = b$$

$$a = ib$$

$$a = 1 \quad , \quad b = -i$$

$$v = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$x(t) = e^{\lambda_1 t} v = e^{(4+3i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{4t}(\cos 3t + i \sin 3t) \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} e^{4t}\cos 3t + i e^{4t} . \sin 3t \\ e^{4t} \sin 3t - i e^{4t} \cos 3t \end{bmatrix}$$

$$= \begin{bmatrix} e^{4t}\cos 3t \\ e^{4t}\sin 3t \end{bmatrix} + i . \begin{bmatrix} e^{4t}\sin 3t \\ -e^{4t}\cos 3t \end{bmatrix}$$

$$u_1(t) = Re[X(t)] = \begin{bmatrix} e^{4t}\cos 3t \\ e^{4t}\sin 3t \end{bmatrix}$$

$$u_2(t) = Im[X(t)] = \begin{bmatrix} e^{4t}\sin 3t \\ -e^{4t}\cos 3t \end{bmatrix}$$

$$X(t) = c_1 u_1(t) + c_2 u_2(t)$$

$$= \begin{bmatrix} c_1 e^{4t}\cos 3t + c_2 e^{4t}\sin 3t \\ c_1 e^{4t}\sin 3t - c_2 e^{4t}\cos 3t \end{bmatrix}$$

$$X_1(t) = c_1 e^{4t}\cos 3t + c_2 e^{4t}\sin 3t$$

$$X_2(t) = c_1 e^{4t}\sin 3t - c_2 e^{4t}\cos 3t .$$

# Multiple Eigenvalue Solutions

In this section we discuss the situation when the characteristic equation

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{3}$$

does *not* have $n$ distinct roots, and thus has at least one repeated root.

An eigenvalue is of **multiplicity** $k$ if it is a $k$-fold root of Eq. (3). For each eigenvalue $\lambda$, the eigenvector equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \tag{4}$$

has at least one nonzero solution $\mathbf{v}$, so there is at least one eigenvector associated with $\lambda$. But an eigenvalue of multiplicity $k > 1$ may have *fewer* than $k$ linearly independent associated eigenvectors. In this case we are unable to find a "complete set" of $n$ linearly independent eigenvectors of $\mathbf{A}$, as needed to form the general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$$

Let us call an eigenvalue of multiplicity $k$ **complete** if it has $k$ linearly independent associated eigenvectors. If every eigenvalue of the matrix $\mathbf{A}$ is complete, then—because eigenvectors associated with different eigenvalues are linearly independent—it follows that $\mathbf{A}$ does have a complete set of $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (each repeated with its multiplicity). In this case a general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is still given by the usual combination

Example: Find a general solution of the system

$$x' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)^2 \Rightarrow \lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 2$$

$\lambda_1 = \lambda_2 = 1$ :

$(A-\lambda I)v = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$a+b=0$

$c=0 \quad b=-1 \Rightarrow a=1$

$c=1 \quad b=0 \Rightarrow a=0$

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$u_2(t) = e^{\lambda_1 t} v_1 = e^{t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad u_2(t) = e^{\lambda_2 t} v_2 = e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 2: \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} b = 0 \\ -b = 0 \\ -c = 0 \end{matrix}, \quad a = 1$$

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_3(t) = e^{\lambda_3 t} v_3 = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X(t) = c_1 u_1(t) + c_2 u_2(t) + c_3 u_3(t) = c_1 e^{t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{t} + c_3 e^{2t} \\ -c_1 e^{t} \\ c_2 e^{t} \end{bmatrix}$$

$$x_1(t) = c_1 e^{t} + c_3 e^{2t}$$

$$x_2(t) = -c_1 e^{t}$$

$$x_3(t) = c_2 e^{t}.$$

An eigenvalue $\lambda$ of multiplicity $k > 1$ is called **defective** if it is not complete. If $\lambda$ has only $p < k$ linearly independent eigenvectors, then the number

$$d = k - p$$

of "missing" eigenvectors is called the **defect** of the defective eigenvalue $\lambda$.

Defective Multiplicity 2 eigenvalues :

Suppose that there is only a single eigenvector $v_1$ associated with the defective eigenvalues $\lambda$. Then at this point there is only the single solution

$$u_1(t) = e^{\lambda t} v_1$$

of $x' = AX$.

By anology with the case of repeated characteristic root for a single linear differential equation, we might hope to find a second solution of the form

$$u_2(t) = t e^{\lambda t} v_2$$

$$u_2'(t) = (e^{\lambda t} + \lambda t e^{\lambda t}) v_2$$

$$u_2' = A u_2 \Rightarrow e^{\lambda t}(1 + \lambda t) v_2 = A(t e^{\lambda t} v_2)$$

$$e^{\lambda t}(1 + \lambda t) v_2 = e^{\lambda t} t A v_2$$

But because the coefficients of bot $e^{\lambda t}$ and $te^{\lambda t}$ must balance, it follows that $v_2 = 0 \Rightarrow u_2(t) = 0$.

Thus we explore the possibility of a second solution of the form $u_2(t) = t e^{\lambda t} v_1 + e^{\lambda t} v_2$,

where $v_1$ and $v_2$ are nonzero constant vectors.

$$u_2' = (e^{\lambda t} + \lambda t e^{\lambda t}) v_1 + \lambda e^{\lambda t} v_2$$

$$u_2' = Au_2 \Rightarrow (e^{\lambda t} + \lambda t e^{\lambda t})v_1 + \lambda e^{\lambda t} v_2 = A(te^{\lambda t} v_1 + e^{\lambda t} v_2)$$

$$e^{\lambda t}(v_1 + \lambda v_2) + te^{\lambda t}(\lambda v_1) = e^{\lambda t} A v_2 + te^{\lambda t}(Av_1)$$

$$\Rightarrow Av_1 = \lambda v_1 \quad \Rightarrow \quad (A - \lambda I)v_1 = 0$$

$$v_1 + \lambda v_2 = Av_2 \quad \Rightarrow \quad (A - \lambda I)v_2 = v_1$$

Note that $v_1$ is an eigenvector of $A$ associated with the eigenvalue $\lambda$. Then

$$(A - \lambda I)^2 v_2 = (A - \lambda I)v_1 = 0$$

## ALGORITHM Defective Multiplicity 2 Eigenvalues

1. First find a nonzero solution $\mathbf{v}_2$ of the equation

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0} \tag{16}$$

such that

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 \tag{17}$$

is nonzero, and therefore is an eigenvector $\mathbf{v}_1$ associated with $\lambda$.

2. Then form the two independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t} \tag{18}$$

and

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} \tag{19}$$

of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to $\lambda$.

Example:

Find a general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \mathbf{x}.$$

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = 7 - \lambda - 7\lambda + \lambda^2 + 9$$

$$= (\lambda - 4)^2 = 0$$

$$\lambda_1 = \lambda_2 = 4$$

$$(A - 4I).V = 0 \Rightarrow \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left. \begin{matrix} -3a - 3b = 0 \\ 3a + 3b = 0 \end{matrix} \right\} \begin{matrix} a = -b \\ b = -1 \\ \Rightarrow a = 1 \end{matrix}$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u_1(t) = e^{\lambda_1 t} v_1 = e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$(A - \lambda I)v_2 = v_1$ $\quad (A - 4I)v_2 = v_1$

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \left.\begin{aligned} -3a-3b &= 1 \\ 3a+3b &= -1 \end{aligned}\right\} \quad \begin{aligned} &3a+3b=-1 \\ &b=0 \\ &\Rightarrow a = -1/3 \end{aligned}$$

$$v_2 = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$$

$$u_2(t) = e^{\lambda_1 t}(tv_1 + v_2) = e^{4t}\cdot\left(t\cdot\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}\right)$$

$$x(t) = c_1 u_1(t) + c_2 u_2(t) = c_1 e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\cdot\left(t\cdot\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}\right)$$

$$= e^{4t}(c_1 + c_2 t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$$

$$x_1(t) = e^{4t}(c_1 + c_2 t) - \tfrac{1}{3} c_2 e^{4t}$$

$$x_2(t) = -e^{4t}(c_1 + c_2 t).$$