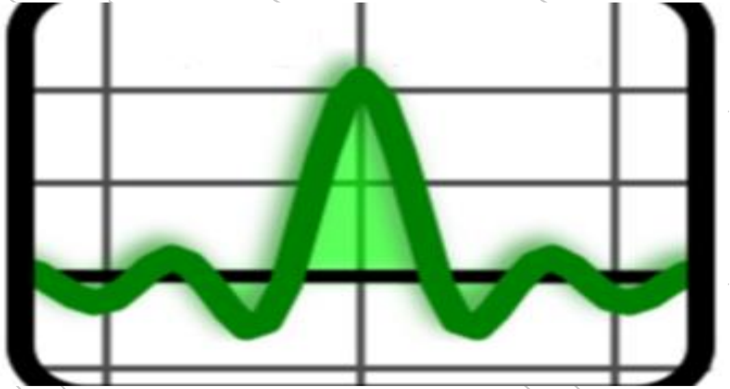


Signals & Systems For Computer Engineering

Prof.Dr. B.Berk ÜSTÜNDAĞ
Istanbul Technical University
Faculty of Computer Engineering and Informatics
bustundag@itu.edu.tr

BLG354E

12th Week Lecture

Fourier Analysis of Discrete-Time Signals and Systems

The Fourier analysis plays the same fundamental role in discrete time as in continuous time.
There are many similarities between the discrete-time Fourier analysis techniques and their continuous-time counterparts.

x[n] to be periodic if there is a positive integer N for which $x[n+N]=x[n] \quad \forall n$

Here, the fundamental period N_0 of x[n] is the smallest positive integer N

For the complex exponential DT signal $x[n]=e^{j(2\pi/N_0)n}=e^{j\Omega_0 n} \qquad \Omega_0=2\pi/N_0$

$$e^{j(\Omega_0+2\pi k)n}=e^{j\Omega_0 n}\,e^{j2\pi kn}=e^{j\Omega_0 n} \longleftarrow$$

This differs between the discrete-time and the continuous- time complex exponential

(In contrary to the CT signals where $e^{j\omega_0 t}$ are distinct for distinct values of ω_0 , the sequences $e^{j\Omega_0 n}$ are identical for multiples of 2π)

Discrete Fourier series representation of a periodic sequence x[n] with fundamental period N_0 is,

$$x[n]=\sum_{k=0}^{N_0-1} c_k\, e^{jk\Omega_0 n} \qquad \Omega_0=\frac{2\pi}{N_0}$$

where the DT Fourier coefficients are $c_k=\dfrac{1}{N_0}\displaystyle\sum_{n=0}^{N_0-1} x[n]\,e^{-jk\Omega_0 n}$

Fourier series representation can be rearranged as $$\boxed{x[n] = \sum_{k=\langle N_0 \rangle} c_k \, e^{jk\Omega_0 n}}$$

where $$c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n] \, e^{-jk\Omega_0 n} \qquad c_0 = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n] \qquad \Omega_0 = \frac{2\pi}{N_0}$$

since $$\Psi_k[n] = \Psi_{k+mN_0}[n]$$ for m=integer

because if $$\Psi_k[n] = e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0} \qquad k = 0, \pm 1, \pm 2, \dots$$

then $$\Psi_0[n] = \Psi_{N_0}[n] \qquad \Psi_1[n] = \Psi_{N_0+1}[n] \qquad \dots \qquad \Psi_k[n] = \Psi_{N_0+k}[n] \qquad \dots$$

Properties of Discrete Fourier Series:

1- As different from the continuous-time Fourier Series, there are no convergence issues with discrete Fourier series (DFS) since DFS is a finite series

2- Fourier series coefficients c, are periodic with fundamental period N_0 → $c_{k+N_0} = c_k$

3- Since the Fourier coefficients c_k form a periodic sequence with fundamental period N_0 , c[k] can be stated as

$$c[k] = \sum_{n=\langle N_0\rangle} \frac{1}{N_0} x[n]\, e^{-jk\Omega_0 n} \qquad \text{Let } n=-m \;\rightarrow\; c[k] = \sum_{m=\langle N_0\rangle} \frac{1}{N_0} x[-m]\, e^{jk\Omega_0 m}$$

$$\text{Let } k=n \text{ and } m=k \rightarrow c[n] = \sum_{k=\langle N_0\rangle} \frac{1}{N_0} x[-k]\, e^{jk\Omega_0 n}$$

$$(1/N_0)x[-k] \text{ are the Fourier coefficients of } c[n] \;\rightarrow\; \begin{cases} x[n] \overset{\text{DFS}}{\longleftrightarrow} c_k = c[k] \\ c[n] \overset{\text{DFS}}{\longleftrightarrow} \dfrac{1}{N_0} x[-k] \end{cases} \rightarrow \textbf{Duality}$$

4- When x[n] is real: $c_{-k} = c_{N_0-k} = c_k^*$

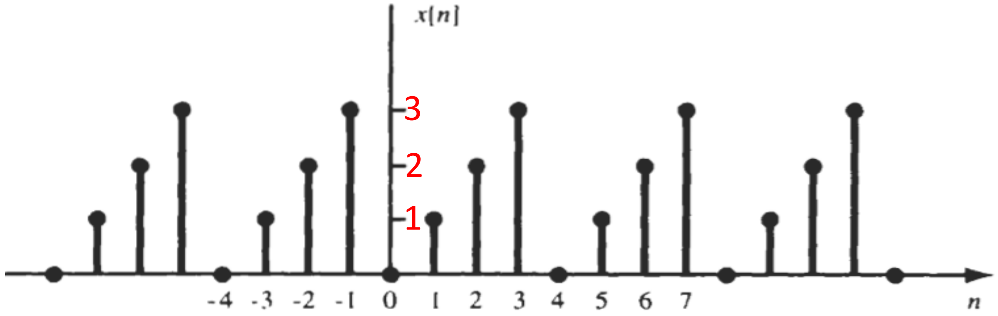
5- If x[n] is real and even, then its Fourier coefficients are real, while if x[n] is odd, its Fourier coefficients are imaginary

$$x[n] = x_e[n] + x_o[n] \;\rightarrow\; \begin{cases} x_e[n] \overset{\text{DFS}}{\longleftrightarrow} \text{Re}[c_k] \\ x_o[n] \overset{\text{DFS}}{\longleftrightarrow} j\,\text{Im}[c_k] \end{cases}$$

$$x[n] \overset{\text{DFS}}{\longleftrightarrow} c_k$$

Example:

Find the Fourier coefficients for the periodic sequence x[n] shown below



Since x[n] is the periodic extension of {0,1,2,3} the fundamental period N_0= 4 $\rightarrow$ $\Omega_0 = \frac{2\pi}{4}$

$$e^{-j\Omega_0} = e^{-j2\pi/4} = e^{-j\pi/2} = -j$$

$$c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n]\, e^{-jk\Omega_0 n}$$

$$c_0 = \frac{1}{4} \sum_{n=0}^{3} x[n] = \frac{1}{4}(0+1+2+3) = \frac{3}{2}$$

$$c_1 = \frac{1}{4} \sum_{n=0}^{3} x[n](-j)^n = \frac{1}{4}(0-j1-2+j3) = -\frac{1}{2} + j\frac{1}{2}$$

$$c_2 = \frac{1}{4} \sum_{n=0}^{3} x[n](-j)^{2n} = \frac{1}{4}(0-1+2-3) = -\frac{1}{2}$$

$$c_3 = c_{4-1} = c_1^* \quad \rightarrow \quad c_3 = \frac{1}{4} \sum_{n=0}^{3} x[n](-j)^{3n} = \frac{1}{4}(0+j1-2-j3) = -\frac{1}{2} - j\frac{1}{2}$$

Example:

Determine the discrete Fourier series representation of $x[n] = \cos^2\left(\frac{\pi}{8}n\right)$

Reminder: $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$x[n] = \cos^2\frac{\pi}{8}n = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{4}n = x_1[n] + x_2[n]$$

$x_1[n] = \frac{1}{2} = \frac{1}{2}(1)^n$ is periodic with fundamental period $N_1 = 1$

$x_2[n] = \frac{1}{2}\cos(\pi/4)n = \frac{1}{2}\cos\Omega_2 n \longrightarrow \Omega_2 = \pi/4$

$$\Omega_2/2\pi = \tfrac{1}{8} \quad N_2 = 8$$

$x[n]$ is periodic with fundamental period $N_0 = 8$ due to the least common multiple of N_1 and N_2

The fundamental period of x[n] is $N_0 = 8$, $\Omega_0 = 2\pi/N_0 = \pi/4$.

By the Euler's formula, $x[n] = \left(\frac{1}{2}e^{j(\pi/8)n} + \frac{1}{2}e^{-j(\pi/8)n}\right)^2$

$$= \frac{1}{4}e^{j(\pi/4)n} + \frac{1}{2} + \frac{1}{4}e^{-j(\pi/4)n}$$

$$= \frac{1}{4}e^{j\Omega_0 n} + \frac{1}{2} + \frac{1}{4}e^{-j\Omega_0 n} \rightarrow c_0 = \frac{1}{2}, c_1 = \frac{1}{4}, c_{-1} = c_{-1+8} = c_7 = \frac{1}{4}$$

$\forall$ other $c_k = 0$

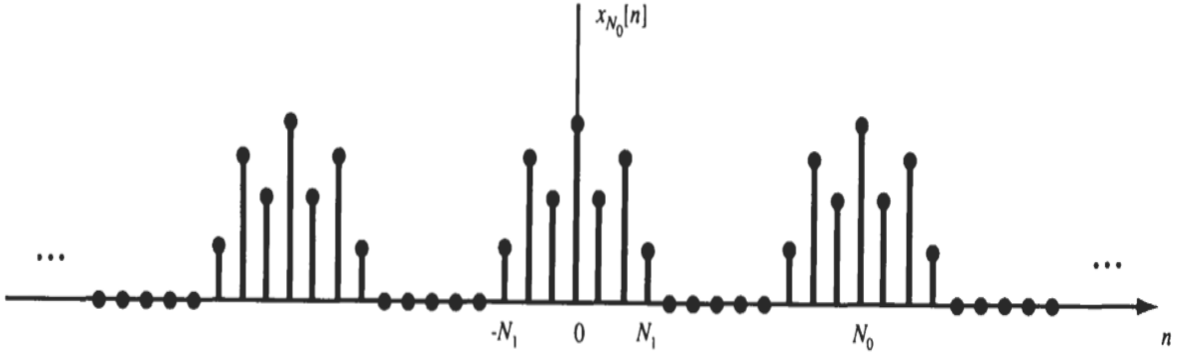
Discrete Fourier Series of x[n] : $x[n] = \frac{1}{2} + \frac{1}{4}e^{j\Omega_0 n} + \frac{1}{4}e^{j7\Omega_0 n} \qquad \Omega_0 = \frac{\pi}{4}$

FOURIER TRANSFORM DISCRETE TIME SIGNALS

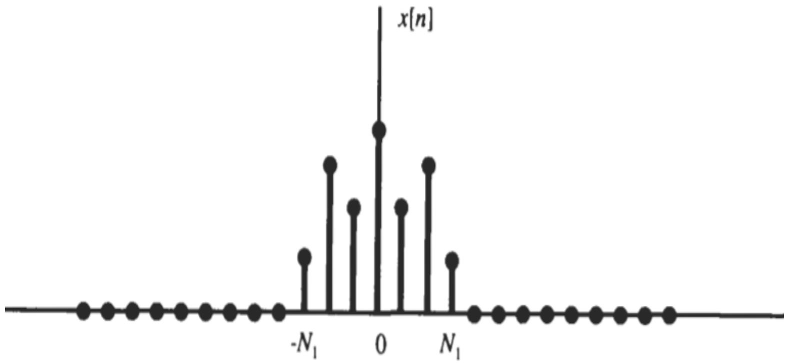
If x[n] is a periodic sequence then there will be a positive integer N_0 that satisfies $x[n]=x[n+N_0]$

Let x[n] be a nonperiodic sequence of finite duration. for some positive integer N_1,

$$x[n] = 0 \quad \text{for } |n| > N_1$$



$N_0 \to \infty$



$$\lim_{N_0 \to \infty} x_{N_0}[n] = x[n]$$

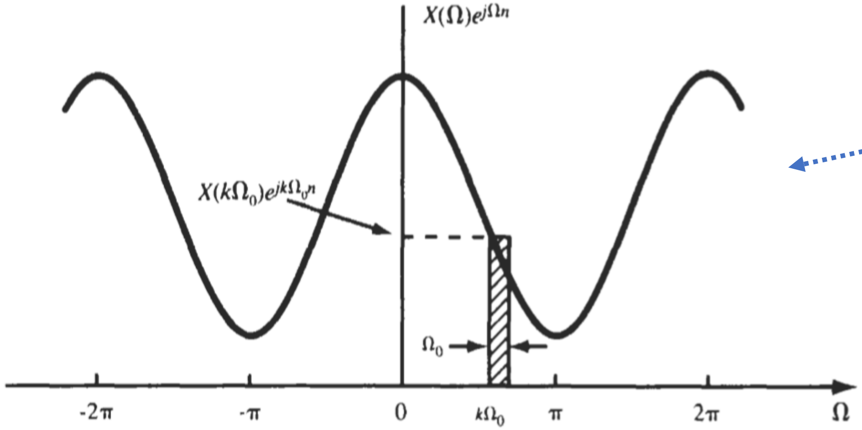
DFS of the periodic signal: $x[n] = \sum_{k=\langle N_0 \rangle} c_k\, e^{jk\Omega_0 n}$ $\quad c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n]\, e^{-jk\Omega_0 n}$ $\quad \Omega_0 = \frac{2\pi}{N_0}$

When $N_0 \to \infty$ since $x[n] = 0$ for $|n| > N_1$ $\rightarrow$ $c_k = \frac{1}{N_0} \sum_{n=-N_1}^{N_1} x[n]\, e^{-jk\Omega_0 n} = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\Omega_0 n}$

Let $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$ then $c_k = \frac{1}{N_0} X(k\Omega_0)$

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} \frac{1}{N_0} X(k\Omega_0)\, e^{jk\Omega_0 n}$$

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=\langle N_0 \rangle} X(k\Omega_0)\, e^{jk\Omega_0 n} \Omega_0$$



$X(\Omega)\, e^{j\Omega n}$ is periodic with the period 2π

$N_0 \to \infty, \Omega_0 = 2\pi / N_0$ becomes infinitesimal $(\Omega_0 \to 0)$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

Properties of the Discrete Fourier Transform Pair

FT

$$X(\Omega)=\mathscr{F}\{x[n]\}=\sum_{n=-\infty}^{\infty}x[n]\,e^{-j\Omega n}$$

Inverse FT

$$x[n]=\mathscr{F}^{-1}\{X(\Omega)\}=\frac{1}{2\pi}\int_{2\pi}X(\Omega)\,e^{j\Omega n}\,d\Omega$$

$$x[n]\leftrightarrow X(\Omega)$$

Fourier Spectra: $X(\Omega)=|X(\Omega)|e^{j\phi(\Omega)}$

Magnitude spectrum

Phase spectrum

$X(\Omega)$ is convergent if $\sum_{n=-\infty}^{\infty}|x[n]|<\infty$

FT: $X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]\,e^{-j\Omega n}$

z-Transform: $X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$

$$X(\Omega)=X(z)|_{z=e^{j\Omega}}$$

$\mathcal{Z}\{\delta[n]\}=1 \rightarrow \mathscr{F}\{\delta[n]\}=\sum_{n=-\infty}^{\infty}\delta[n]\,e^{-j\Omega n}=1$

$\mathcal{Z}\{u[n]\}=\frac{1}{1-z^{-1}}$ $\quad |z|>1$ $\rightarrow \mathscr{F}\{u[n]\}=\pi\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}$ $\quad |\Omega|\le\pi$

Exception for u[n]:
ROC of Z{u[n]} does not
include unit circle

$\mathcal{Z}\{a^n u[n]\}=\frac{1}{1-az^{-1}}\rightarrow X(e^{j\Omega})=\frac{1}{1-ae^{-j\Omega}}$ $\quad |a|<1$

proof

$$X(\Omega)=\sum_{n=-\infty}^{\infty}a^n u[n]\,e^{-j\Omega n}=\sum_{n=0}^{\infty}a^n e^{-j\Omega n}=\sum_{n=0}^{\infty}\left(ae^{-j\Omega}\right)^n=\frac{1}{1-ae^{-j\Omega}}$$

$$|ae^{-j\Omega}|=|a|<1$$

Common Fourier Transform Pairs

x[n]	X[Ω]
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \lvert\Omega\rvert \le \pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\cos \Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\sin \Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$u[n]$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \lvert a\rvert > 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$(n+1)a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{\left(1-ae^{-j\Omega}\right)^2}$
$a^{\lvert n\rvert}, \lvert a\rvert < 1$	$\dfrac{1-a^2}{1-2a\cos\Omega+a^2}$
$x[n]=\begin{cases}1 & \lvert n\rvert \le N_1 \\ 0 & \lvert n\rvert > N_1\end{cases}$	$\dfrac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$
$\dfrac{\sin Wn}{\pi n}, 0 < W < \pi$	$X(\Omega)=\begin{cases}1 & 0 \le \lvert\Omega\rvert \le W \\ 0 & W < \lvert\Omega\rvert \le \pi\end{cases}$
$\displaystyle\sum_{k=-\infty}^{\infty} \delta[n-kN_0]$	$\displaystyle\Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega-k\Omega_0), \Omega_0=\frac{2\pi}{N_0}$

Fourier Transform Properties

Property	Sequence	Fourier Transform
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega + 2\pi) = X(\Omega)$
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(\Omega) + a_2 X_2(\Omega)$
Time shifting	$x[n - n_0]$	$e^{-j\Omega n_0} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n} x[n]$	$X(\Omega - \Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n = km \\ 0 & \text{if } n \neq km \end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-j\Omega}) X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega) + \dfrac{1}{1 - e^{-j\Omega}} X(\Omega)$
		$\lvert\Omega\rvert \leq \pi$
Convolution	$x_1[n] * x_2[n]$	$X_1(\Omega) X_2(\Omega)$
Multiplication	$x_1[n] x_2[n]$	$\dfrac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$
Real sequence	$x[n] = x_e[n] + x_o[n]$	$X(\Omega) = A(\Omega) + jB(\Omega)$
		$X(-\Omega) = X^*(\Omega)$
Even component	$x_e[n]$	$\mathrm{Re}\{X(\Omega)\} = A(\Omega)$
Odd component	$x_o[n]$	$j\,\mathrm{Im}\{X(\Omega)\} = jB(\Omega)$

Frequency response of DT LTI systems

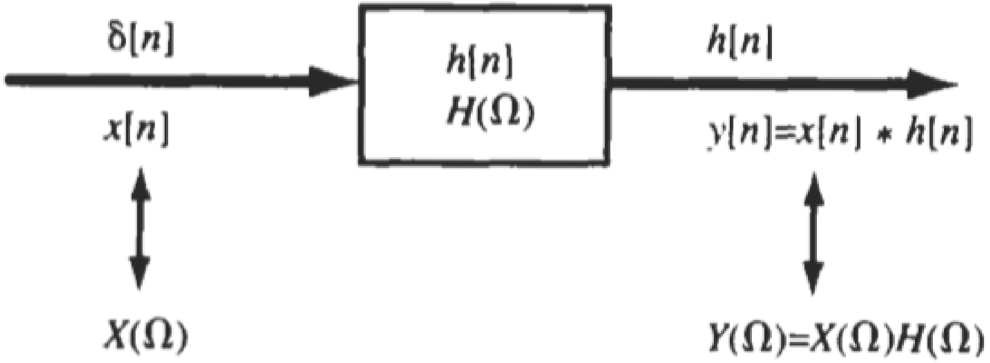
$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

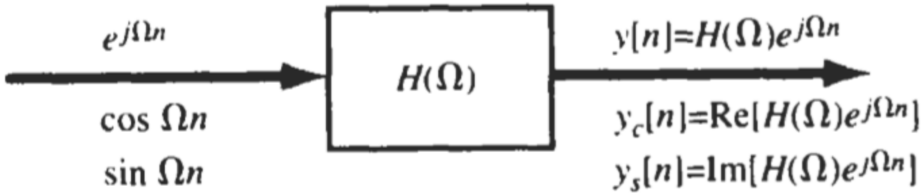
$$H(\Omega) = |H(\Omega)| e^{j\theta_H(\Omega)}$$

$$y[n] = y_c[n] + jy_s[n] = H(\Omega)\, e^{j\Omega n}$$

$$y_c[n] = \mathrm{Re}\{y[n]\} = \mathrm{Re}\{H(\Omega)\, e^{j\Omega n}\}$$

$$y_s[n] = \mathrm{Im}\{y[n]\} = \mathrm{Im}\{H(\Omega)\, e^{j\Omega n}\}$$





$$y[n] = \mathbf{T}\{z^n\} = H(z)z^n$$

$$z = e^{j\Omega_0}$$

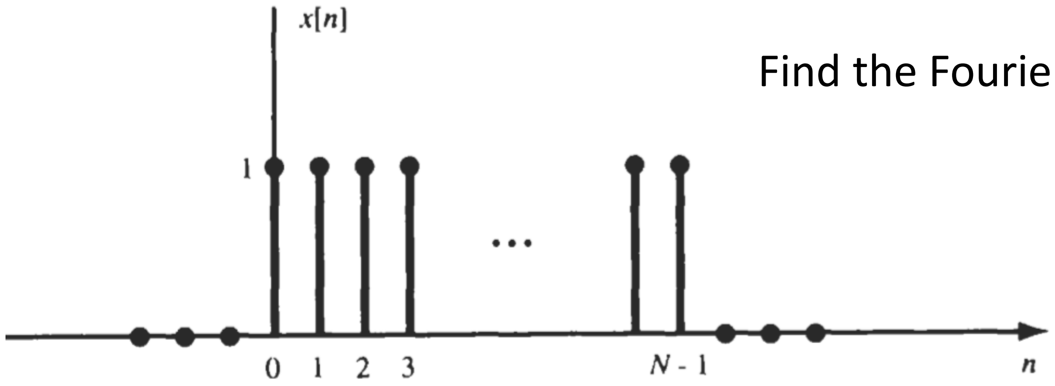
$$y[n] = H(e^{j\Omega_0})\, e^{j\Omega_0 n} = H(\Omega_0)\, e^{j\Omega_0 n}$$

where $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$ $\qquad x[n] = \sum_{k=\langle N_0 \rangle} c_k\, e^{jk\Omega_0 n}$

$$x[n] = e^{j\Omega_0 n}$$

The complex exponential sequence $e^{j\Omega_0 n}$ is an eigenfunction of the LTI system with corresponding eigenvalue $H(\Omega_0)$

Example:



Find the Fourier transform of the rectangular pulse sequence x[n]

$$x[n] = u[n] - u[n-N]$$

$$X(z) = \sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z} \qquad |z| > 0$$

$$X(\Omega) = X(e^{j\Omega}) = \frac{1-e^{-j\Omega N}}{1-e^{-j\Omega}} = \frac{e^{-j\Omega N/2}(e^{j\Omega N/2} - e^{j\Omega N/2})}{e^{-j\Omega/2}(e^{j\Omega/2} - e^{-j\Omega/2})} = e^{-j\Omega(N-1)/2}\,\frac{\sin(\Omega N/2)}{\sin(\Omega/2)}$$

Prove the time-shifting property

$$x[n-n_0] \leftrightarrow e^{-j\Omega n_0} X(\Omega)$$

$$\mathscr{F}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0]\, e^{-j\Omega n}$$

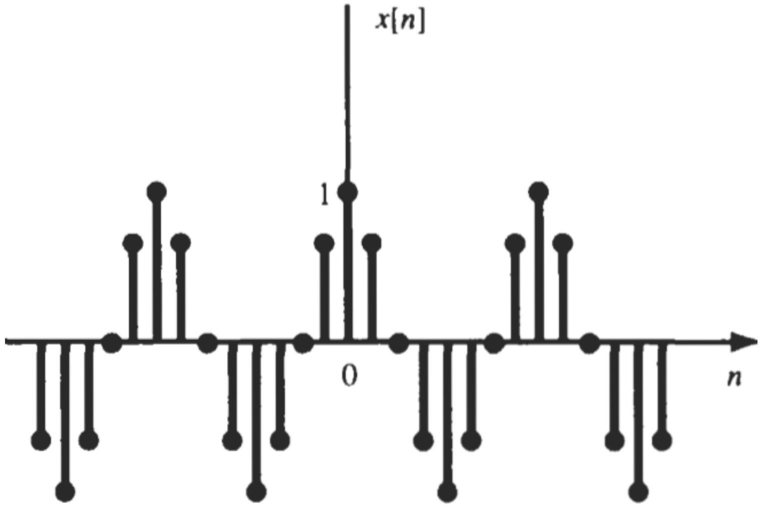
By the change of variable $m = n - n_0$
$$\mathscr{F}\{x[n-n_0]\} = \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega(m+n_0)}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega m} = e^{-j\Omega n_0} X(\Omega)$$

Hence
$$x[n-n_0] \leftrightarrow e^{-j\Omega n_0} X(\Omega)$$

Example:

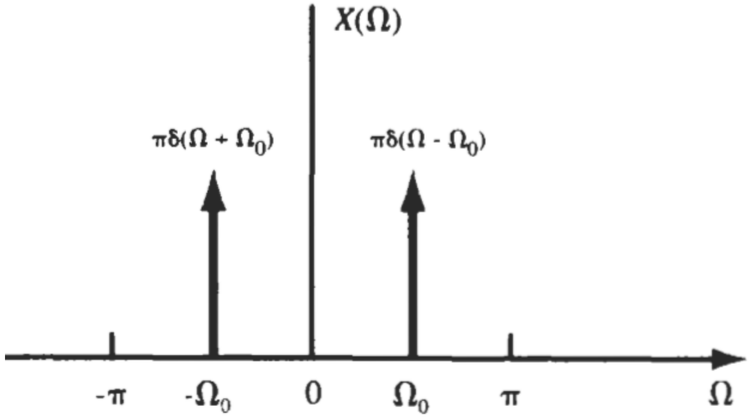
Find the Fourier transform of the sinusoidal sequence $x[n] = \cos \Omega_0 n \qquad |\Omega_0| \le \pi$



$$\cos \Omega_0 n = \tfrac{1}{2}(e^{j\Omega_0 n} + e^{-j\Omega_0 n})$$

$$X(\Omega) = \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \qquad |\Omega|, |\Omega_0| \le \pi$$

$$\cos \Omega_0 n \longleftrightarrow \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \qquad |\Omega|, |\Omega_0| \le \pi$$



DISCRETE FOURIER TRANSFORM

The DFT is the appropriate Fourier representation for digital computer realization because it is discrete and of finite length in both the time and frequency domains.

Let x[n] be a finite-length sequence of length N, that is, x[n]=0 outside the range 0≤n≤ N-1

The DFT of x[n], denoted as X[k], is defined by $$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

where W_N is the Nth root of unity given by $$W_N = e^{-j(2\pi/N)}$$

The inverse DFT (IDFT) is given by $$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

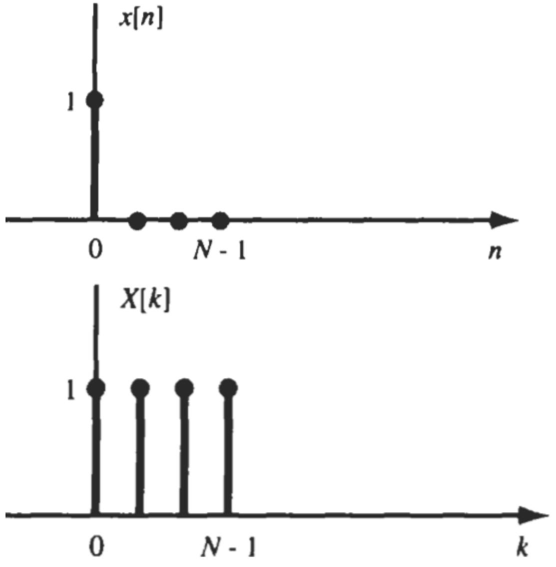
$$x[n] \overset{\textbf{DFT}}{\longleftrightarrow} X[k]$$

* If x[n] has length N, < N, we want to assume that x[n] has length N by simply adding (N - Nl) samples with a value of 0. This addition of dummy samples is known as zero padding. Then the resultant x[n] is often referred to as an N-point sequence, and X[k] is referred to as an N-point DFT.

* Fast Fourier transform (FFT) is an extremely fast algorithm calculation of the DFT

DFT of some common sequences:

$$x[n] = \delta[n]$$



By the definition,

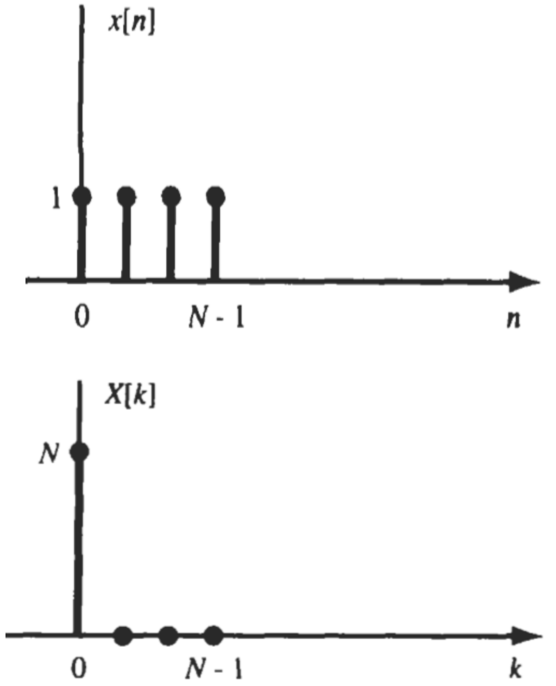
$$X[k] = \sum_{n=0}^{N-1} \delta[n] w_N^{kn} = 1 \qquad k = 0, 1, \ldots, N-1$$

$$x[n] = u[n] - u[n-N]$$

$$X[k] = \sum_{n=0}^{N-1} W_N^{kn} = \frac{1 - W_N^{kN}}{1 - W_N^{k}} = 0 \qquad k \neq 0$$

$$W_N^{kN} = e^{-j(2\pi/N)kN} = e^{-jk2\pi} = 1$$

$$X[0] = \sum_{n=0}^{N-1} W_N^0 = \sum_{n=0}^{N-1} 1 = N$$



Example:

Calculate the convolution $y[n]=x[n]\otimes h[n]$

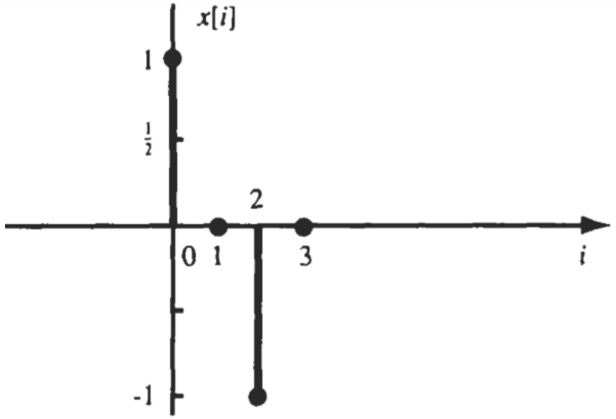
where $x[n]=\cos\left(\frac{\pi}{2}n\right)$ $n=0,1,2,3$ and $h[n]=\left(\frac{1}{2}\right)^n$ $n=0,1,2,3$

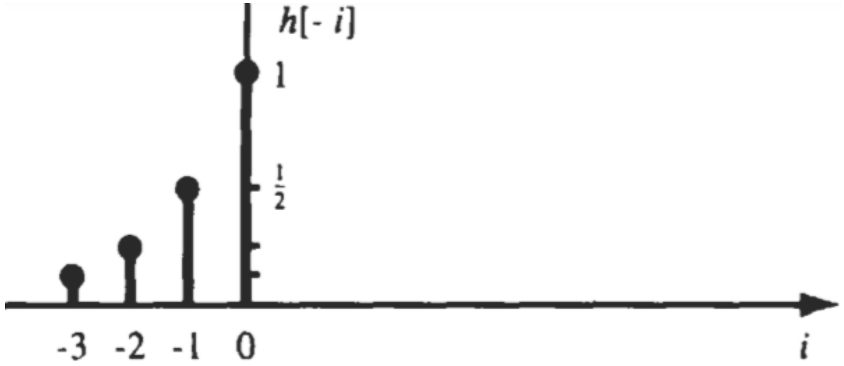
by using the circular convolution and DFT methods separately

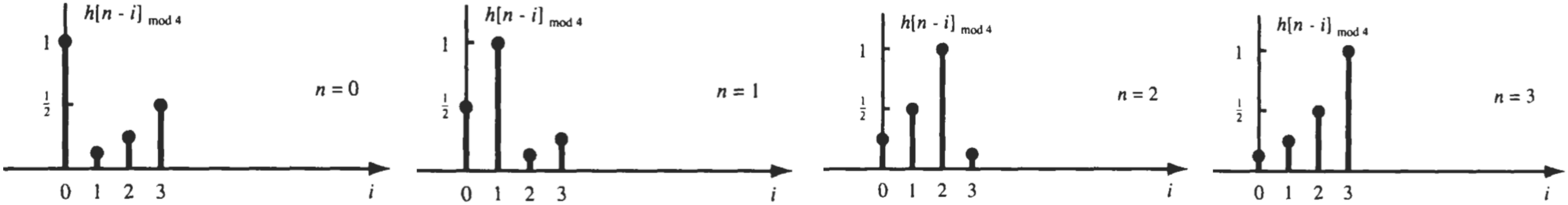
For n=0,1,2,3 $x[n]=\{1,0,-1,0\}$ $h[n]=\{1,\frac{1}{2},\frac{1}{4},\frac{1}{8}\}$

$$y[n]=x[n]\otimes h[n]=\sum_{i=0}^{3}x[i]h[n-i]_{\text{mod}\,4}$$

x[i] and $h[n-i]_{mod4}$ sequences for n=0,1,2,3 can be plotted as shown below





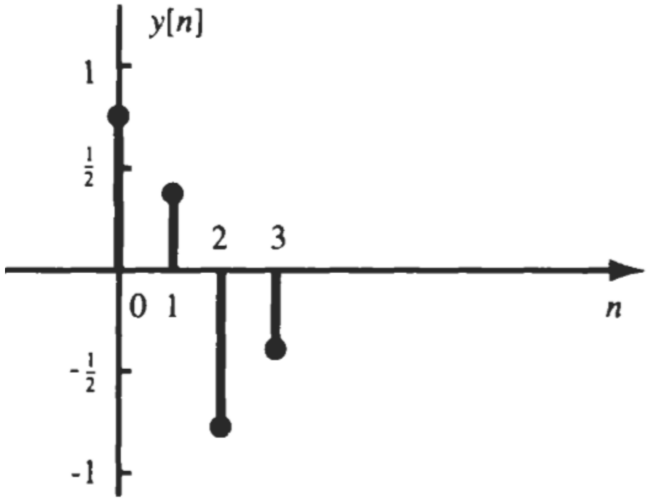


$$n = 0 \qquad y[0] = 1(1) + (-1)\left(\tfrac{1}{4}\right) = \tfrac{3}{4}$$

$$n = 1 \qquad y[1] = 1\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{8}\right) = \tfrac{3}{8}$$

$$n = 2 \qquad y[2] = 1\left(\tfrac{1}{4}\right) + (-1)(1) = -\tfrac{3}{4}$$

$$n = 3 \qquad y[3] = 1\left(\tfrac{1}{8}\right) + (-1)\left(\tfrac{1}{2}\right) = -\tfrac{3}{8}$$



Hence we get $y[n] = \left\{\frac{3}{4}, \frac{3}{8}, -\frac{3}{4}, -\frac{3}{8}\right\}$

By the DFT definition,
$$X[k] = \sum_{n=0}^{3} x[n]W_4^{kn} = 1 - W_4^{2k} \qquad k = 0, 1, 2, 3$$

$$H[k] = \sum_{n=0}^{3} h[n]W_4^{kn} = 1 + \tfrac{1}{2}W_4^{k} + \tfrac{1}{4}W_4^{2k} + \tfrac{1}{8}W_4^{3k} \qquad k = 0, 1, 2, 3$$

The DFT of y[n],

$$Y[k] = X[k]H[k] = \left(1 - W_4^{2k}\right)\left(1 + \tfrac{1}{2}W_4^{k} + \tfrac{1}{4}W_4^{2k} + \tfrac{1}{8}W_4^{3k}\right)$$
$$= 1 + \tfrac{1}{2}W_4^{k} - \tfrac{3}{4}W_4^{2k} - \tfrac{3}{8}W_4^{3k} - \tfrac{1}{4}W_4^{4k} - \tfrac{1}{8}W_4^{5k}$$

$$W_N = e^{-j(2\pi/N)}$$

$$\left.\begin{aligned} W_4^{4k} &= (W_4^4)^k = 1^k \\ W_4^{5k} &= W_4^{(4+1)k} = W_4^{k} \end{aligned}\right\} \quad Y[k] = \tfrac{3}{4} + \tfrac{3}{8}W_4^{k} - \tfrac{3}{4}W_4^{2k} - \tfrac{3}{8}W_4^{3k} \qquad k = 0, 1, 2, 3$$

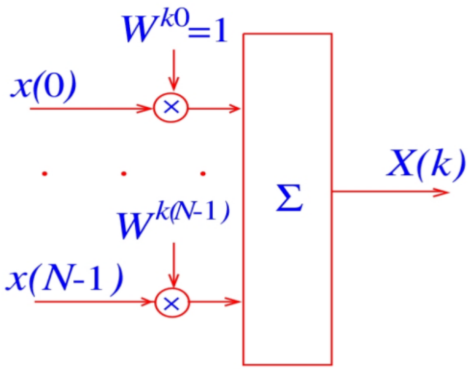
$$y[n] = \left\{\tfrac{3}{4}, \tfrac{3}{8}, -\tfrac{3}{4}, -\tfrac{3}{8}\right\}$$

Matrix Representation of N-point DFT

The DFT definition $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$ can be expressed in a matrix operation form as $\mathbf{X} = \mathbf{W}_N \mathbf{x}$

DFT Matrix:

$$\mathbf{X} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} \qquad \mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$



$$\mathbf{W}_N^T = \mathbf{W}_N \qquad \mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^* \qquad \boldsymbol{x} = \frac{1}{N}\mathcal{W}^H \boldsymbol{X}$$ ←Inverse DFT

$$W_{n+1,k+1} = W_4^{nk} = e^{-j(2\pi/4)nk} = e^{-j(\pi/2)nk} = (-j)^{nk}$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \mathbf{W}_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Example:

$$f(t) = 5 + 2\cos(2\pi t - 90^o) + 3\cos 4\pi t$$ Find the 4 points DFT of this signal if it is sampled at 4Hz

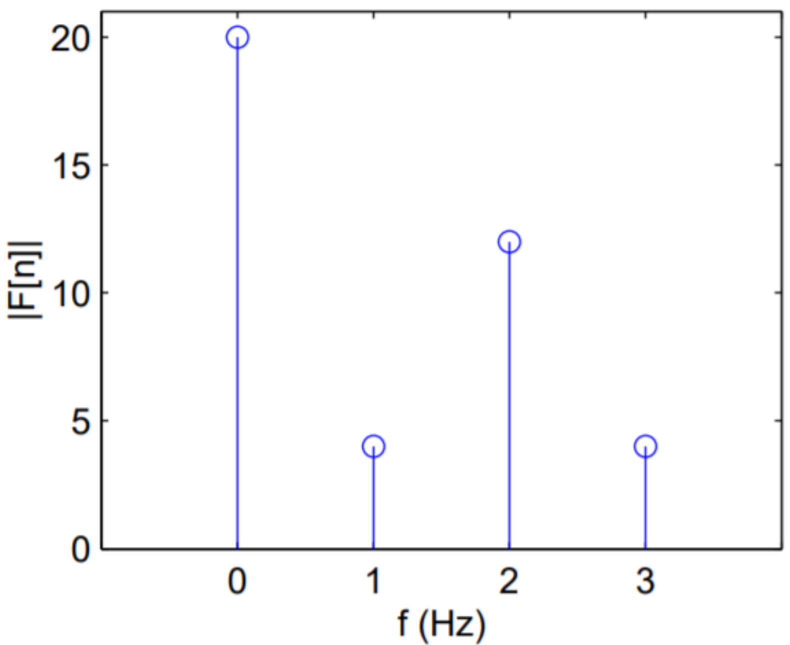
If we sample f(t) at 4Hz from t=0 to t=¾ then the values of the discrete samples can be given as,

$$t = kT_s = \frac{k}{4} \quad \rightarrow \quad f[k] = 5 + 2cos(\tfrac{\pi}{2}k - 90^o) + 3cos\pi k$$

$$f[0] = 8,\ f[1] = 4,\ f[2] = 8,\ f[3] = 0 \qquad (N = 4)$$

$$F[n] = \sum_{0}^{3} f[k]e^{-j\frac{\pi}{2}nk} = \sum_{k=0}^{3} f[k](-j)^{nk}$$

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \end{pmatrix} = \begin{pmatrix} 20 \\ -j4 \\ 12 \\ j4 \end{pmatrix}$$



Example:

$x[n]=\{0, 1, 2, 3\}$ Show that IDFT of x[k] recovers DFT of x[k]

$$X[k] = \mathrm{DFT}\{x[n]\} \rightarrow x[n] = \frac{1}{N}\left[\sum_{n=0}^{N-1} X[k]\, e^{j(2\pi/N)kn}\right] = \frac{1}{N}\left[\overbrace{\sum_{n=0}^{N-1} X^*[k]\, e^{-j(2\pi/N)nk}}^{\text{DFT of } X^*[k]}\right]^*$$

$$x[n] = \mathrm{IDFT}\{X[k]\} = \frac{1}{N}[\mathrm{DFT}\{X^*[k]\}]^*$$

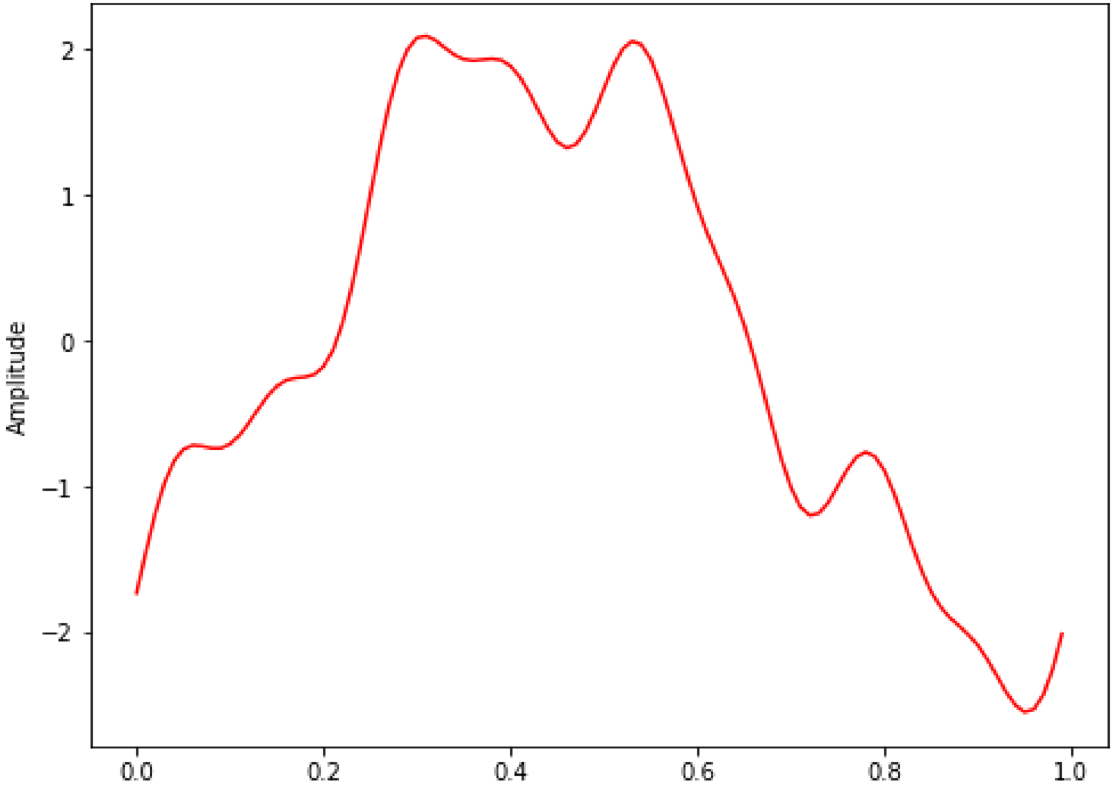
$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}}^{W_4} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2+j2 \\ -2 \\ -2-j2 \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \frac{1}{4} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}}_{W_4^{-1}} \begin{bmatrix} 6 \\ -2+j2 \\ -2 \\ -2-j2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 4 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

N=4

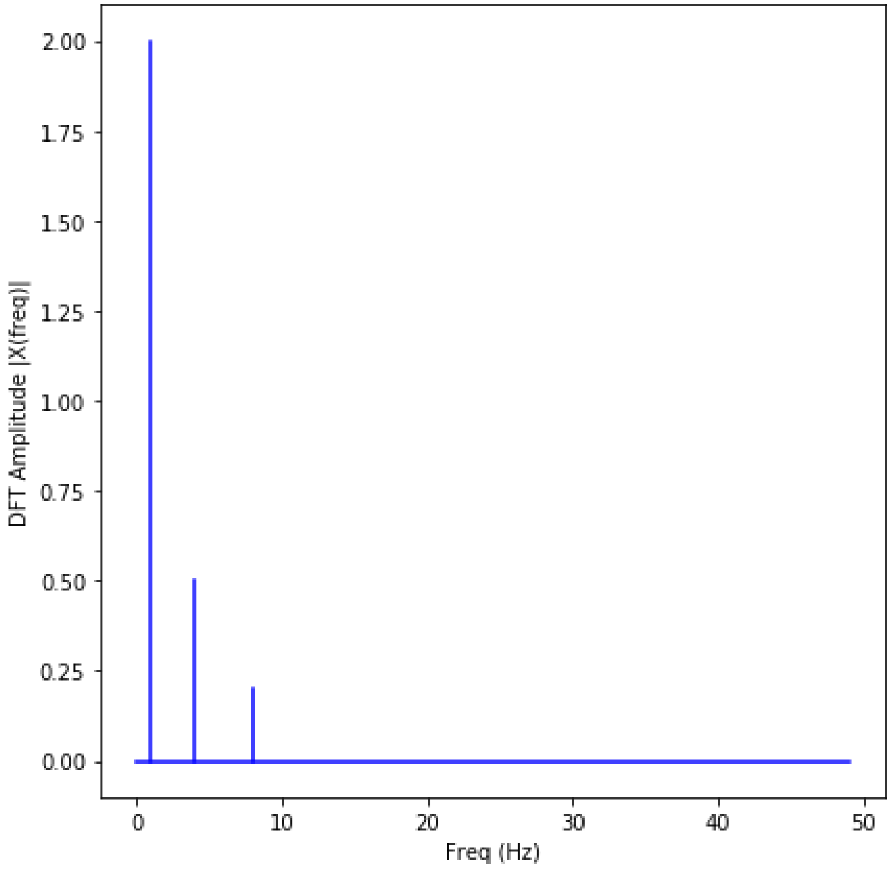
Python code for DFT

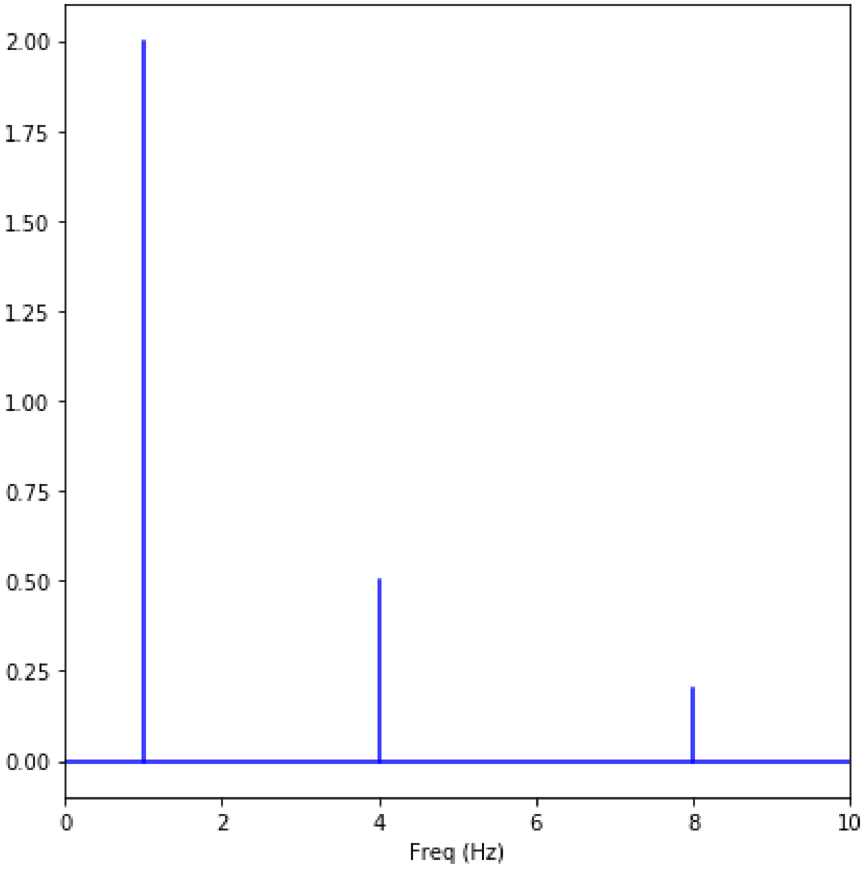
$$A_k = \sum_{m=0}^{n-1} a_m \exp\left\{-2\pi i \frac{mk}{n}\right\} \qquad k = 0, \ldots, n-1$$



$$a_m = \frac{1}{n} \sum_{k=0}^{n-1} A_k \exp\left\{2\pi i \frac{mk}{n}\right\} \qquad m = 0, \ldots, n-1.$$

```
import numpy as np
import matplotlib.pyplot as plt

def DFT(x): # To calculate DFT of a 1D real-valued signal x
    N = len(x)
    n = np.arange(N)
    k = n.reshape((N, 1))
    e = np.exp(-2j * np.pi * k * n / N)
    X = np.dot(e, x)
    return X

sr = 100    # sampling rate
ts = 1.0/sr # sampling interval
t = np.arange(0,1,ts)

freq = 1.
x = 2*np.sin(2*np.pi*freq*t-np.pi/3)

freq = 4
x += 0.5*np.sin(2*np.pi*freq*t)

freq = 8
x += 0.2*np.sin(2*np.pi*freq*t)

plt.figure(figsize = (8, 6))
plt.plot(t, x, 'r')
plt.ylabel('Amplitude')

plt.show()
```






```
X = DFT(x)

# calculate the frequency
N = len(X)
n = np.arange(N)
T = N/sr
freq = n/T

n_oneside = N//2
# get the one side frequency
f_oneside = freq[:n_oneside]

# normalize the amplitude
X_oneside =X[:n_oneside]/n_oneside

plt.figure(figsize = (12, 6))
plt.subplot(121)
plt.stem(f_oneside, abs(X_oneside), 'b', \
      markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.ylabel('DFT Amplitude |X(freq)|')

plt.subplot(122)
plt.stem(f_oneside, abs(X_oneside), 'b', \
      markerfmt=" ", basefmt="-b")
plt.xlabel('Freq (Hz)')
plt.xlim(0, 10)
plt.tight_layout()
plt.show()
```

For higher order DFT:

$$A_{kl} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} a_{mn} \exp\left\{-2\pi i \left(\frac{mk}{M} + \frac{nl}{N}\right)\right\} \qquad k = 0, \ldots, M-1; \quad l = 0, \ldots, N-1$$