Solutions Of Midterm

1. Determine truth values of the following quantified assertions:

   (a) (5 points) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + 2y = 7)$
   TRUE. For any given real number x, the real number $y = (7 - 2x)/2$ satisfies that $x + 2y = 7$.

   (b) (5 points) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x + 2y = 7)$
   FALSE. Suppose for a moment that it is true. There must be a fixed real number y such that $x+2y = 7$ for all real numbers x. Choosing, for instance $x = 1$ and $x = 3$, we see that $y = 3$ and $y = 2$. A contradiction.

2. (10 points) Write the negation of the following quantified assertion about a real sequence in plain English:

   "For every positive real number ε, there is a positive integer N such that, for all natural numbers m and n, if $m > N$ and $n > N$ then $|a_n - a_m| < \varepsilon$"

   The given assertion can be symbolized as

   $$\text{“}(\forall \varepsilon \in \mathbb{R}^+)(\exists N \in \mathbb{Z}^+)(\forall m, n \in \mathbb{N})\Big(\big((m > N) \wedge (n > M)\big) \to |a_n - a_m| < \varepsilon\Big)\text{”}$$

   So its negation is

   $$\text{“}(\exists \varepsilon \in \mathbb{R}^+)(\forall N \in \mathbb{Z}^+)(\exists m, n \in \mathbb{N})\Big(\big((m > N) \wedge (n > M)\big) \wedge |a_n - a_m| \geq \varepsilon\Big)\text{”}$$

   where we used the logical equivalence "$P \to Q \equiv \neg P \vee Q$". So the negation can be written in plain English as follows

   "There is a positive real number ε such that, for every positive integer N, there are natural numbers m and n satisfying $m > N$ and $n > N$ and $|a_n - a_m| \geq \varepsilon$"

3. Prove the following results about sets:

   (a) (5 points) $\emptyset \subseteq A$ for any set A.
   We need to show that "$\forall x\big((x \in \emptyset) \to (x \in A)\big)$" is true. As the empty set $\emptyset$ has no elements, "$x \in \emptyset$" is false for any x, and so "$(x \in \emptyset) \to (x \in A)$" is true for any x. Thus "$\forall x\big((x \in \emptyset) \to (x \in A)\big)$" is true, as desired.

   Or we may argue as follows. Suppose for a contradiction that $\emptyset \not\subseteq A$ for some set A. There must be an x such that $x \in \emptyset$ but $x \notin A$. This is a contradiction because the empty set $\emptyset$ has no elements.

   (b) (5 points) $\emptyset \cup A = A$ for any set A.

   $$\emptyset \cup A = \big\{x \mid x \in \emptyset \text{ or } x \in A\big\} = \big\{x \mid x \in A\big\} = A$$

   where we used the logical equivalence "$(x \in \emptyset \text{ or } x \in A) \equiv x \in A$", which is true for any x, because "$x \in \emptyset$" is false.

   Or we may argue as follows to justify that $\emptyset \cup A \subseteq A$ and $A \subseteq \emptyset \cup A$. Take any $x \in \emptyset \cup A$. Then $x \in \emptyset$ or $x \in A$. As the the empty set $\emptyset$ has no elements, we must have that $x \in A$. Therefore $\emptyset \cup A \subseteq A$. Conversely, take any $y \in A$. By the definition of the union we see that $y \in B \cup A$ for any set B. In particular $y \in \emptyset \cup A$. Therefore $A \subseteq \emptyset \cup A$.

   (c) (5 points) For any set A, if $A \subseteq \emptyset$ then $A = \emptyset$.
   As $A \subseteq \emptyset$, any element of A must be an element of the empty set $\emptyset$. Since the empty set has no elements, we conclude that A has no elements. Thus $A = \emptyset$.

4. (10 points) Suppose that the assertions

$$\forall n\big(P(n) \to Q(n)\big) \quad \text{and} \quad P(3)$$

are both true where the universe for the variable n is $\mathbb{N}$. Prove that the assertion $\exists n Q(n)$ is true.

As "$\forall n\big(P(n) \to Q(n)\big)$" is true, "$P(n) \to Q(n)$" is true for every $n \in \mathbb{N}$. In particular, as $3 \in \mathbb{N}$, it follows that "$P(3) \to Q(3)$" is true. Recalling the truth values of an implication, we see from the trueness of "$P(3) \to Q(3)$" and "$P(3)$" that "$Q(3)$" is true. So "$Q(n)$" is true for $n = 3$. In particular, "$Q(n)$" is true for some n. As a result, the assertion "$\exists n Q(n)$" is true.

We may also write a two column proof:

1)	$\forall n\big(P(n) \to Q(n)\big)$	hypothesis
2)	$P(3)$	hypothesis
3)	$P(3) \to Q(3)$	1), $\forall$-elimination
4)	$Q(3)$	2), 3), $\Rightarrow$-elimination
5)	$\exists n Q(n)$	4), $\exists$-introduction

5. (15 points) Let A and B be sets. Show that $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ if and only if $A \subseteq B$ or $B \subseteq A$. (Here $\mathcal{P}(\ )$ denotes the power set of its argument).

$(\Rightarrow)$: (We assume that $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$. We want to prove that $A \subseteq B$ or $B \subseteq A$.) The proof is by contradiction. Assume for a contradiction that "$A \subseteq B$ or $B \subseteq A$" is not true. So we assume that $A \not\subseteq B$ and $B \not\subseteq A$. As $A \not\subseteq B$, there is an $a \in A$ such that $a \notin B$. As $B \not\subseteq A$, there is a $b \in B$ such that $b \notin A$. Consider the set $S = \{a, b\}$. Note that $S \subseteq A \cup B$ (because $a, b \in A \cup B$) and $S \not\subseteq A$ (because $b \notin A$) and $S \not\subseteq B$ (because $a \notin B$). Thus, $S \in \mathcal{P}(A \cup B)$ and $S \notin \mathcal{P}(A)$ and $S \notin \mathcal{P}(B)$. Consequently, $S \in \mathcal{P}(A \cup B)$ but $S \notin \mathcal{P}(A) \cup \mathcal{P}(B)$. This is a contradiction to the assumption "$\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$".

$(\Rightarrow)$: Or we may argue as follows. As $A \cup B \in \mathcal{P}(A \cup B)$, it follows that $A \cup B \in \mathcal{P}(A) \cup \mathcal{P}(B)$ because $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$. From $A \cup B \in \mathcal{P}(A) \cup \mathcal{P}(B)$, we conclude that $A \cup B \in \mathcal{P}(A)$ or $A \cup B \in \mathcal{P}(B)$. Recalling the definition of the power set, we see that $A \cup B \subseteq A$ or $A \cup B \subseteq B$. Therefore, $B \subseteq A$ (because $B \subseteq A \cup B$) or $A \subseteq B$ (because $A \subseteq A \cup B$).

$(\Leftarrow)$: (We assume that $A \subseteq B$ or $B \subseteq A$. We want to prove that $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.) As $A \subseteq B$ or $B \subseteq A$, we see that $A \cup B = B$ or $A \cup B = A$. Therefore, $\mathcal{P}(A \cup B) = \mathcal{P}(B)$ or $\mathcal{P}(A \cup B) = \mathcal{P}(A)$. Consequently, $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ in both cases.

6. (15 points) For any natural number m the m th Fermat number F_m is defined by

$$F_m = 2^{(2^m)} + 1.$$

Prove by induction that

$$F_n = (F_0 \cdot F_1 \cdot F_2 \cdots F_{n-1}) + 2$$

for all natural numbers $n \geq 1$.

Base case: Note that $F_0 = 2^{(2^0)} + 1 = 3$ and $F_1 = 2^{(2^1)} + 1 = 5$, implying that $F_1 = F_0 + 2$. So the result is true for $n = 1$.

Induction step: Let $k \geq 1$ be an arbitrary natural number. Assume that the result is true for $n = k$. Thus we assume that

$$F_k = (F_0 \cdot F_1 \cdot F_2 \cdots F_{k-1}) + 2.$$

We want to justify that the result is true for $n = k + 1$. That is, we want to justify that

$$F_{k+1} = (F_0 \cdot F_1 \cdot F_2 \cdots F_k) + 2.$$

Now

$$\begin{aligned}
(F_0 \cdot F_1 \cdot F_2 \cdots F_k) + 2 &= (F_0 \cdot F_1 \cdot F_2 \cdots F_{k-1}) \cdot F_k + 2 \\
&= (F_k - 2)F_k + 2, \qquad (F_0 \cdot F_1 \cdot F_2 \cdots F_{k-1} = F_k - 2 \text{ by the induction hypothesis}) \\
&= F_k^2 - 2F_k + 2 \\
&= (F_k - 1)^2 + 1 \\
&= \left(2^{(2^k)}\right)^2 + 1, \qquad (F_k = 2^{(2^k)} + 1) \\
&= 2^{(2^{k+1})} + 1 \\
&= F_{k+1},
\end{aligned}$$

as desired.

Unfortunately, almost all of you do not how to prove that a function is one to one!

(Although we have solved many such problems in our lectures)

7. Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

   (a) (7 points) Show that if f and g are both one to one functions, then $g \circ f$ is a one to one function.
   (Recall first how to prove that a function $h : A \to B$ is one to one. We need to show that "for any $a_1, a_2 \in A$, if $h(a_1) = h(a_2)$ then $a_1 = a_2$".) Suppose that f and g are both one to one functions. We want to prove that $g \circ f$ is a one to one function. Note that the domain of $g \circ f$ is X. Let x_1 and x_2 be arbitrary elements of X such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. As $(g \circ f)(x) = g(f(x))$ for any $x \in X$, we see that $g(f(x_1)) = g(f(x_1))$. From the equation $g(f(x_1)) = g(f(x_1))$, it follows that $f(x_1) = f(x_2)$ because g is one to one. Since $f(x_1) = f(x_2)$ and f is one to one, we obtain that $x_1 = x_2$. Hence, $g \circ f$ is one to one.

   (b) (8 points) Show that if $g \circ f$ is a one to one function, then f is a one to one function.
   Let $x_1, x_2 \in X$. Assume that $f(x_1) = f(x_2)$. (We want to show that $x_1 = x_2$). As $f(x_1) = f(x_2)$, we see that $g(f(x_1)) = g(f(x_1))$. Using the definition of the composition, we see from the last equation that $(g \circ f)(x_1) = (g \circ f)(x_2)$. As $(g \circ f)(x_1) = (g \circ f)(x_2)$ and $g \circ f$ is one to one, it follows that $x_1 = x_2$, as desired. Hence, f is one to one.

8. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$f(x) = \begin{cases} 3x, & \text{if } x \geq 0 \\ x + 3, & \text{if } x < 0 \end{cases}$$

Find the function $f \circ f$.

Note first that

$$(f \circ f)(x) = f(f(x)) = \begin{cases} 3f(x), & \text{if } f(x) \geq 0 \\ f(x) + 3, & \text{if } f(x) < 0 \end{cases}$$

So we should find all x values for which $f(x)$ is negative and positive. It is clear from the definition (rule) of f that

$$f(x) \geq 0 \iff (x \geq 0 \text{ and } 3x \geq 0) \text{ or } (x < 0 \text{ and } x + 3 \geq 0) \iff x \geq -3$$

Thus

$$(f \circ f)(x) = \begin{cases} 3f(x), & \text{if } x \geq -3 \\ f(x) + 3, & \text{if } x < -3 \end{cases} = \begin{cases} 3\begin{cases} 3x, & \text{if } x \geq 0 \\ x + 3, & \text{if } x < 0 \end{cases}, & \text{if } x \geq -3 \\ (x+3) + 3, & \text{if } x < -3 \end{cases} = \begin{cases} 9x, & \text{if } x \geq 0 \\ 3x + 9, & \text{if } -3 \leq x < 0 \\ x + 6, & \text{if } x < -3 \end{cases}$$