MAT 175E **Solutions Of Midterm** 17.11.2022

Q1	Q2	Q3	Q4	Total

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(1) Is the argument below valid? If it is valid give a proof, if not provide a counter-example.

$$\begin{array}{l} \forall x\big(\neg B(x) \to A(x)\big) \\ \neg\big(\forall x A(x)\big) \\ \forall x\big(\neg B(x) \vee \neg C(x)\big) \\ \hline \neg\big(\forall x C(x)\big) \end{array}$$

It is valid, and a proof may be given as follows.

(1)	$\forall x\big(\neg B(x) \to A(x)\big)$	Premise
(2)	$\neg\big(\forall x A(x)\big)$	Premise
(3)	$\forall x\big(\neg B(x) \vee \neg C(x)\big)$	Premise
(4)	$\exists x\big(\neg A(x)\big)$	(2), Negation
(5)	$\neg A(a)$	(4), Existential Instantiation
(6)	$\neg B(a) \to A(a)$	(1), Universal Instantiation
(7)	$\neg B(a) \vee \neg C(a)$	(3), Universal Instantiation
(8)	$\neg\big(\neg B(a)\big)$	(5), (6), Modus Tollens
(9)	$\neg C(a)$	(7), (8), Modus Tollendo Ponens
(10)	$\exists x\big(\neg C(x)\big)$	(9), Existential Generalization
(11)	$\neg\big(\forall x C(x)\big)$	(10), Negation

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(2) Let A, B, C be sets such that $A\Delta C = B\Delta C$. Show that $A = B$.

We will show that $A \subseteq B$ and $B \subseteq A$.

We first want to show that $A \subseteq B$: Let $a \in A$. As either $a \in C$ or $a \notin C$, we have two cases to consider.

Case I: Assume that $a \in C$.
As $a \in C$, we see that $a \notin A - C$. As $a \in A$, we see that $a \notin C - A$. So $a \notin (A - C) \cup (C - A) = A\Delta C$. Since $A\Delta C = B\Delta C$, it follows that $a \notin B\Delta C = (B - C) \cup (C - B)$. In particular, $a \notin (C - B)$. As $a \in C$, it follows from $a \notin (C - B)$ that $a \in B$. Therefore $A \subseteq B$.

Case II: Assume that $a \notin C$.
As $a \in A$ and $a \notin C$, we see that $a \in A - C$, implying that $a \in (A - C) \cup (C - A) = A\Delta C$. Since $A\Delta C = B\Delta C$, it follows that $a \in B\Delta C = (B - C) \cup (C - B)$. So, $a \in B - C$ or $a \in C - B$. As $a \notin C$, the case $a \in C - B$ cannot be true. Hence, we must have that $a \in B - C$. In particular, $a \in B$. Therefore $A \subseteq B$.

The reverse containment $B \subseteq A$ can be proved similarly. (Indeed, with the different notations for the sets, we have proved above that if $X \cap Z = X \cap Z$ then $X \subseteq Y$. Since $A\Delta C = B\Delta C$ can be written as $B\Delta C = A\Delta C$, it follows that $B \subseteq A$).

(3) Let A and B be sets such that $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ where $\mathcal{P}(\ )$ denotes the power set of its argument. Show that $A \subseteq B$ or $B \subseteq A$.

Note that $A \cup B \in \mathcal{P}(A \cup B)$. As $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$, we see that $A \cup B \in \mathcal{P}(A)$ or $A \cup B \in \mathcal{P}(B)$. If $A \cup B \in \mathcal{P}(A)$, then $A \cup B \subseteq A$ and so $B \subseteq A$. If $A \cup B \in \mathcal{P}(B)$, then $A \cup B \subseteq B$ and so $A \subseteq B$.

Second Solution: The proof is by contradiction. Suppose for a moment that the conclusion "$A \subseteq B$ or $B \subseteq A$" is not true. So $A \not\subseteq B$ and $B \not\subseteq A$. As $A \not\subseteq B$, there is an $a \in A$ such that $a \notin B$. As $B \not\subseteq A$, there is a $b \in B$ such that $b \notin A$. Consider the set

$$X = \{a, b\}.$$

We note that $X \subseteq A \cup B$, but $X \not\subseteq A$ and $X \not\subseteq B$. Therefore, $X \in \mathcal{P}(A \cup B)$, but $X \notin \mathcal{P}(A)$ and $X \notin \mathcal{P}(B)$. Consequently, $X \in \mathcal{P}(A \cup B)$ but $X \notin \mathcal{P}(A) \cup \mathcal{P}(B)$. This is a contradiction to the given condition $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

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(4) Let $B \subseteq A$ be sets such that $A \neq B$. Consider the function $\psi : \mathcal{P}(A) \to \mathcal{P}(B)$ defined by $\psi(X) = B \cap X$ for all $X \in \mathcal{P}(A)$ where $\mathcal{P}(\ )$ denotes the power set of its argument.

(a) Is ψ injective?

Note, for instance, that $\psi(A) = B = \psi(B)$ but $A \neq B$. Therefore, ψ is not injective.

(b) Is ψ surjective?

Let Y be any element of the codomain $\mathcal{P}(B)$ of ψ. Then $Y \subseteq B$. As $B \subseteq A$, we see that $Y \subseteq A$ and $B \cap Y = Y$. Thus, $Y \in \mathcal{P}(A)$ and $\psi(Y) = Y$. Hence ψ is surjective.

(c) If possible, find two distinct right inverse of ψ.

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\begin{aligned}
\text{A function } \phi : \mathcal{P}(B) \to \mathcal{P}(A) \text{ is a right inverse of } \psi \quad &\iff \quad \psi \circ \phi = 1_{\mathcal{P}(B)} \\
&\iff \quad (\psi \circ \phi)(Y) = 1_{\mathcal{P}(B)}(Y) \text{ for all } Y \in \mathcal{P}(B) \\
&\iff \quad \psi\big(\phi(Y)\big) = Y \text{ for all } Y \subseteq B \\
&\iff \quad B \cap \phi(Y) = Y \text{ for all } Y \subseteq B
\end{aligned}
$$

Now it is easy to find examples of functions ϕ satisfying the condition $B \cap \phi(Y) = Y$ for all $Y \subseteq B$.

For instance, the inclusion function, $\phi(Y) = Y$ for all $Y \in \mathcal{P}(B)$, satisfies the above condition and so it forms an example of a right inverse of ψ.

For other examples, for any element $a \in A - B$, consider the functions $\phi_a : \mathcal{P}(B) \to \mathcal{P}(A)$ defined for any $Y \in \mathcal{P}(B)$ by $\phi_a(Y) = Y \cup \{a\}$. We easily check that $B \cap \phi_a(Y) = B \cap (Y \cup \{a\}) = Y$ for all $Y \subseteq B$. Hence, ϕ_a is an example of a right inverse of ψ.