Symbolic Logic (the language of math)

Logic is the business of deciding whether a particular conclusion is a consequence of particular assumptions. Here, conclusion and assumptions are collection of statements, and each statement is a combination of some types of sentences. We will use symbols to denote the sentences, allowing us to translate the thing in plain English into something in mathematical language (i.e., in propositional logic).

By an assertion (or statement, or proposition) we mean a declaritive sentence which is either true or false, but not both. A sentence may be a combination of sentences.

Ex: Ahmet's hair is brown ← an assertion

$2+3=1$ ← an assertion

What a nice day ← not an assertion (it explains an opinion)

Are you okey? ← not an assertion (it is a question)

Sit down ← not an assertion

It is raining ← an assertion

This sentence is false ← not an assertion (It is an example of paradoxes) Think abouth its truth

I am sleepy ← an assertion

If you get up early, then you will catch the bus ← an assertion

It is raining and it is not raining ← an assertion

Combining Assertions (with logical connectives)

" not, and, or, if_then_, if and only if (iff for short) "

$\neg$ $\wedge\,(\&)$ $\vee$ $\rightarrow\,(\Rightarrow)$ $\leftrightarrow\,(\Leftrightarrow)$

New assertions can be obtained from existing ones by using the above connectives. The used existing assertions are called the component assertions of the obtained assertion.

Negation (Not, $\neg$): Given an assertion P, we define its negation $\neg P$ (read as "not P") to be the statement which is false when P is true, and which is true when P is false. We may illustrate this in a truth table as follows. (In plain English $\neg P$ can be written as "It is not the case that P". For instance, if P symbolizes the assertion "It is raining", then $\neg P$ symbolizes the assertion "It is not raining").

P	$\neg P$
T	F
F	T

<u>Conjunction</u> (And, $\wedge$, &): Given two assertions P and Q, we define the assertion $P \wedge Q$ (read as "P and Q") to be true precisely when both P and Q are true. We may illustrate the truth value of $P \wedge Q$ in the following truth table.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Note in plain English that the assertions "P, but Q" and "Although P, Q" can both symbolized as "$P \wedge Q$"

<u>Disjunction</u> (Or, $\vee$): Given two assertions P and Q we define the assertion $P \vee Q$ (read as "P or Q") to be true precisely when at least one of P and Q is true. So $P \vee Q$ can be defined by the following truth table.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Note that $P \vee Q$ is false precisely when both P and Q are false. In plain English the usage of "or" may be confusing, because sometimes it excludes the possibility that both P and Q are true in "P or Q". This is called <u>exclusive or</u>. However, in mathematics "or" always means <u>inclusive or</u> so that $P \vee Q$ will be true when both P and Q are true. So the sentence "Either P or Q" can be symbolized as $P \vee Q$.

Implication (Implies, $\to$, $\Rightarrow$): Given two assertions P and Q, we define the assertion $P \to Q$ (read as "P implies Q") to be false precisely when P is true but Q is false. The assertion $P \to Q$ is the symbolization of the sentence "If P, then Q". The truth table of $P \to Q$ is given by

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

You may find the last two rows confusing, but it is actually natural. "If P, then Q" means that Q is true under the condition that P is true. When P is false, the condition on P is not satisfied, and so "If P, then Q" says nothing about Q; and Q may be true or false and either possibility does not make "If P, then Q" wrong. For instance, saying that "If $2=1$, then my name is Ali" is not wrong, although my name is Ergün. This is because the sentence claims that my name is Ali under the condition that $2=1$ is true, and the condition $2=1$ is of course not true.

Note that each of the following sentences in plain English

If P then Q

P implies Q

Whenever P, Q

Q whenever P

P is sufficient for Q

Q is necessary for P

P guarantees that Q

P is stronger condition than Q

Q is weaker condition than P

P only if Q

Q $\underbrace{\text{unless}}_{\text{if not}}$ not P

can be symbolized as "$P \to Q$". (We will observe later that $P \to Q$ and $\neg Q \to \neg P$ are logically equivalent. Assuming that $\neg Q \to \neg P$ is true, for P to be true we see that Q must be true. In other words, for P to be true, it is necessary that Q is true)

Biconditional (If and only if, iff, $\leftrightarrow$, $\Leftrightarrow$): Given assertions P and Q, we define the assertion $P \leftrightarrow Q$ to be true precisely when P and Q have the same truth value (that is, either both are true or both are false). So the truth table for "iff" is

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex: Let P and Q be assertions for which the implication $P \to Q$ is false. Determine the truth values of the following assertions:

(1) $P \wedge Q$ (3) $Q \to P$ (5) $\neg Q \leftrightarrow P$

(2) $\neg P \vee Q$ (4) $\neg Q \to \neg P$

Sol: As $P \to Q$ is false, P is true and Q is false. ....

Ex: Find the truth table of the assertion $P \wedge (\neg Q \to R)$

Sol: The assertion has 3 component assertions (or atomic assertions or primitive assertions) and so the table will have $2^3 = 8$ rows. P, Q, R

P	Q	R	$\neg Q$	$\neg Q \to R$	$P \wedge (\neg Q \to R)$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	T	T	F
F	F	F	T	F	F

Tautology, Contradiction, Contingent

An assertion is called a tautology if it is true for all truth value assignments of its component assertions.

An assertion is called a contradiction if it is false for all truth value assignments of its component assertions.

An assertion is called a contingent assertion if it is not a tautology and it is not a contradiction.

For instance, the assertion $P \wedge (\neg Q \to R)$ is a contingent assertion because there are truth values of P, Q, R making $P \wedge (\neg Q \to R)$ true (for instance, P = True, Q = True, R = True) and there are truth values of P, Q, R making $P \wedge (\neg Q \to R)$ false (for instance, P = False, Q = True, R = True).

Ex: For any assertion P

(1) $P \wedge \neg P$ is a contradiction: The component assertions of $P \wedge \neg P$ is P only. We need to check that $P \wedge \neg P$ is false for all truth value assignment of P. We have two cases. P is either True or False.

Case 1: P is true: In this case $\neg P$ is false, so $P \wedge \neg P$ becomes $T \wedge F = F$

Case 2: P is false: In this case $\neg P$ is true, so $P \wedge \neg P$ becomes $F \wedge T = F$

As in both cases $P \wedge \neg P$ is false, $P \wedge \neg P$ is a contradiction.

(2) $P \vee \neg P$ is a tautology: We may write the truth table to see the truth values of the assertion under each of the truth value assignments of the component assertions.

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

As all are true, $P \vee \neg P$ is a tautology

Ex: Consider the assertion: "If Yasemin goes out tonight then I eat cake, and I don't eat cake, and Yasemin goes out tonight".

$\underbrace{\text{Symbolize it}}$ and determine whether it is a contradiction.

(or, translate it into propositional logic)

Sol: Let $P =$ Yasemin goes out tonight
$Q =$ I eat cake.

The given assertion is $(P \to Q) \wedge \neg Q \wedge P$. The component assertions are P and Q and we need to check the truth value of the assertion by considering all possible assignments to P and Q. So there are $2^2 = 4$ cases to check. (If you wish, you may also write the truth table, which will have 4 rows too). Instead you may argue as follows: Let us check is there any assignment for P and Q making $(P \to Q) \wedge \neg Q \wedge P$ true. As we have "and"s, for this each of

$$P \to Q, \quad \neg Q, \quad P$$

must be true. So $P =$ True, $Q =$ False. But then $P \to Q$ will be false. Hence, there is no way of making $(P \to Q) \wedge \neg Q \wedge P$ true. In other words, $(P \to Q) \wedge \neg Q \wedge P$ is false for all truth value assignment of P and Q. Hence the given assertion is a contradiction.

Converse, Inverse, and Contrapositive of an implication

Definition: Given an implication $P \to Q$,

by its converse we mean the implication $Q \to P$

by its inverse we mean the implication $\neg P \to \neg Q$

by its contrapositive we mean the implication $\neg Q \to \neg P$

Ex: Given the implication "If it rains, then I go out",

its converse is "If I go out, then it rains"

its inverse is "If it doesn't rain, then I don't go out"

its contrapositive is "If I don't go out, then it doesn't rain"

Note that converse of inverse and inverse of converse are both contrapositive.

Logical Equivalence

Two assertions S_1 and S_2 are called <u>logically equivalent</u> if S_1 and S_2 have the same truth value for every assignment of true or false to the component assertions of S_1 and S_2. (In other words, S_1 and S_2 are called logically equivalent if they have the same truth tables). Recalling the biconditional we may state that: Two assertions S_1 and S_2 are called logically equivalent if $S_1 \longleftrightarrow S_2$ is a tautology. We will use the notation $S_1 \equiv S_2$ to indicate that the assertions S_1 and S_2 are logically equivalent.

<u>Ex</u>: $P \to Q \not\equiv P \vee Q$ (That is, $P \to Q$ and $P \vee Q$ are not logically equivalent)

<u>Sol</u>: We should find some assignments of P and Q making the truth values of $P \to Q$ and $P \vee Q$ different. For instance, letting P = false and Q = false,

$$P \to Q = F \to F = T \quad \text{but} \quad P \vee Q = F \vee F = F.$$

<u>Fact</u>: $P \to Q \equiv \neg P \vee Q$ (That is, $P \to Q$ and $\neg P \vee Q$ are logically equivalent).

<u>Proof</u>: Let us check their truth tables. Component assertions are P, Q.

P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Same. So $P \to Q \equiv \neg P \vee Q$

<u>Fact</u>: (Any implication is logically equivalent to its contrapositive)

$$P \to Q \equiv \neg Q \to \neg P$$

<u>Proof</u>: We consider all the possible values of the component assertions P and Q.

Case 1: Assume P is true.

Subcase 1.1: Assume Q is true. We have

$$P \to Q = T \to T = T$$

$$\neg Q \to \neg P = \neg T \to \neg T = F \to F = T$$

Both assertions are true. So they have the same truth value.

Subcase 1.2: Assume Q is false. We have

$$P \to Q = T \to F = F$$

$$\neg Q \to \neg P = \neg F \to \neg T = T \to F = F$$

Both assertions are false. So they have the same truth value.

Case 2: Assume P is false.

Subcase 2.1: Assume Q is true.

⋮

Subcase 2.2: Assume Q is false.

⋮

In all cases, we see that two assertions have the same truth value, so they are logically equivalent.

Exercise: Justify the following logical equivalences

(1) (Commutativity of $\wedge, \vee, \leftrightarrow$)

$$P \wedge Q \equiv Q \wedge P, \quad P \vee Q \equiv Q \vee P, \quad P \leftrightarrow Q \equiv Q \leftrightarrow P$$

(2) (Associativity of $\wedge, \vee$)

$$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R), \quad (P \vee Q) \vee R \equiv P \vee (Q \vee R)$$

(3) $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$

(4) (DeMorgan's Laws or Rules of Negation)

important!

$$\neg\neg P \equiv P \quad \text{(Double Negation)}$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

no need to memorize!

$$\neg(P \to Q) \equiv P \wedge \neg Q$$

$$\neg(P \leftrightarrow Q) \equiv (P \wedge \neg Q) \vee (Q \wedge \neg P)$$

(5) (Distributivity)

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

(6) (Idempotent Laws)

$$P \wedge P \equiv P \quad , \quad P \vee P \equiv P$$

(7) (Identity Laws)

$P \wedge T_0 \equiv P$, $\quad P \vee F_0 \equiv P$ where T_0 is a tautology and F_0 is a contradiction.

(8) (Inverse Laws)

$$P \wedge \neg P \equiv F_0 \quad , \quad P \vee \neg P \equiv T_0$$

(9) (Domination Laws)

$$P \wedge F_0 \equiv F_0 , \quad P \vee T_0 \equiv T_0$$

(10) (Absorption Laws)

$$P \wedge (P \vee Q) \equiv P \quad , \quad P \vee (P \wedge Q) \equiv P$$

Ex: Simplify the assertion $\neg\left[\neg\left((P \vee Q) \wedge R\right) \vee \neg Q\right]$

Sol:

$$\neg\left[\neg\left((P \vee Q) \wedge R\right) \vee \neg Q\right] \underset{\substack{\uparrow \\ \text{De Morgan}}}{\equiv} \neg\neg\left((P \vee Q) \wedge R\right) \wedge \neg\neg Q \underset{\substack{\uparrow \\ \text{Double Negation}}}{\equiv} \left((P \vee Q) \wedge R\right) \wedge Q$$

$$\underset{\substack{\uparrow \\ \text{associativity}}}{\equiv} (P \vee Q) \wedge (R \wedge Q) \underset{\substack{\uparrow \\ \text{Commutativity}}}{\equiv} (P \vee Q) \wedge (Q \wedge R) \underset{\substack{\uparrow \\ \text{associativity}}}{\equiv} \left((P \vee Q) \wedge Q\right) \wedge R$$

$$\underset{\substack{\uparrow \\ \text{absorption}}}{\equiv} Q \wedge R$$

Substitution Rule

If we replace each occurence of component assertion in a tautology or in a logical equivalence, then what we get will be another tautology or logical equivalence. For instance, we observed before that $P \vee \neg P$ is a tautology. If we replace P by $S \to T$, then we get $(S \to T) \vee \neg (S \to T)$ which is also a tautology.

Duality

As $P \to Q \equiv \neg P \vee Q$ and $(P \leftrightarrow Q) \equiv (P \to Q) \wedge (Q \to P)$, any assertion can be obtained from its component assertions by using the connectives $\neg, \wedge, \vee$ only.

Recall from the De Morgan's Laws that the negation changes $\wedge$ to $\vee$ & $\vee$ to $\wedge$.

Noting also that $S_1 \equiv S_2$ if and only if $\neg S_1 \equiv \neg S_2$, we conclude the principle of duality:

$$S_1 \equiv S_2 \text{ if and only if } S_1^d \equiv S_2^d$$

where the dual A^d of an assertion A is obtained as follows: Write first A as a combination of component assertions using only the connectives $\neg, \wedge, \vee$, and then in A replace

$$\left\{ \begin{array}{l} \wedge \text{ with } \vee \\ \vee \text{ with } \wedge \\ T_0 \text{ with } F_0 \\ F_0 \text{ with } T_0 \end{array} \right\}$$

For instance, by the commutativity we know that $A \wedge B \equiv B \wedge A$, using the principal of duality $A \vee B \equiv B \vee A$, and then using the logical equivalence "$P \to Q \equiv \neg P \vee Q$" we see that $\neg A \to B \equiv \neg B \to A$, finally substituting $B = \neg C$ we get the equivalence $\neg A \to \neg C \equiv C \to A$

Deduction, Valid dedections, Counterexample to an invalid deduction

Any implication of the form $P_1 \wedge P_2 \wedge \dots \wedge P_n \longrightarrow Q$ is called a deduction (or an argument) where n is a natural number. The assertions $P_1, P_2, \dots, P_n$ are called the hypotheses (or premises) of the deduction and the assertion

Q is called the conclusion of the deduction. We usually write the above deduction as

$$\left.\begin{array}{l} P_1 \\ P_2 \\ \vdots \\ P_n \end{array}\right\} \text{hypotheses} \qquad \text{or} \qquad \underbrace{P_1, P_2, \ldots, P_n}_{\text{hypotheses}}, \therefore \underbrace{Q}_{\text{conclission}}$$
$$\overline{}$$
$$Q \; \} \text{conclusion}$$

The symbol " $\therefore$ " means "therefore"

A deduction $P_1, P_2, \ldots, P_n, \therefore Q$ is called valid if $P_1 \wedge P_2 \wedge \cdots \wedge P_n \rightarrow Q$ is a tautology (that is, whenever each P_i of the hypoteses is true, the conclusion is true). A deduction which is not valid is called invalid. Thus, to show that a deduction is invalid (i.e., not valid), it is enough to find an assignment of true and false to component assertions that making each hypotesis true but the conclusion false (i.e., making the implacition $P_1 \wedge P_2 \wedge \cdots \wedge P_n \longrightarrow Q$ false). Any such assignment is called a counter example to the deduction

Ex (1)
$$\begin{array}{l} P \vee Q \\ \quad \neg P \\ \hline Q \end{array}$$

is a valid deduction: We want to justify that Q is true for all true and false assignments to P and Q such that the hypotheses $P \vee Q$ and $\neg P$ are both true. Assume that the hypotheses $P \vee Q$ and $\neg P$ are both true. As $\neg P$ is true, P must be false. As $P \vee Q$ is true and P is false, Q must be true. Alternatively, we may write the truth table of $(P \vee Q) \wedge \neg P \longrightarrow Q$ to see that it is a tautology

(2) $P \to Q$

$Q \to S$

$R \to \neg S$

$\overline{\qquad\qquad}$

$\neg P$

is an invalid deduction: We want to find only one assignment of true and falso to P, Q, R, S such that all the hypothes $P \to Q$, $Q \to S$, $R \to \neg S$ are true but the conclusion $\neg P$ is false. In other words, we want to find a counterexample As we want $\neg P$ to be false, P must be true. To make first hypothesis true, Q must be true. Similarly for the second hypothesis $Q \to S$ to be true, S must be true. Consider now the third hypothesis $R \to \neg S$. As S is true, to make $R \to \neg S$ true we must let R be false. Consequently, we see that P=true, Q=true, S=true, R=false is a counterexample (i.e., this assignment makes that all the hypotheses are true but the conclusion is false).

Any valid deduction can be called as a <u>theorem</u>.

<u>Remark</u>: Any deduction whose hypotheses form a contradiction is a valid deduction That is, for any deduction $P_1, P_2, \ldots, P_n, \therefore Q$ if $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ is a contradiction then the deduction is valid. This is because the implication $A \to B$ is true when A is false