MAT 281E, Linear Algebra and Applications
Homework 2

Due: November 26, 2021

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- Scan or take photos of clearly handwritten solutions, combine them in a pdf file and submit it through Ninova.
- Late homework are not accepted!
- For your questions related with the homework email the teaching assistants.

Question 1

Evaluate the determinant of the following matrix by cofactor expansion along the row or the column of your choice. Explain your reasoning for choosing the row or the column that you used.

$$A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 4 & 3 & -6 & 0 \\ 0 & 3 & 2 & -1 \\ 3 & 2 & -1 & 0 \end{bmatrix}$$

Question 2

Compute the determinant of A

$$A = \begin{bmatrix} 2 & 2 & 4 & 6 \\ 1 & 3 & -2 & 1 \\ 2 & 8 & -4 & 2 \\ 1 & 3 & 6 & 7 \end{bmatrix}$$

1. Using only row reduction
2. Using both row reduction and cofactor expansion

Question 3

Evaluate A^{-1} using the method of adjoints

$$A = \begin{bmatrix} 3 & 2 & 7 & 6 \\ -3 & 3 & 4 & 1 \\ 2 & 1 & -4 & -3 \\ -4 & 1 & 2 & -2 \end{bmatrix}$$

Question 4

Let

$$\begin{aligned} 2x_1 + x_3 - 3x_4 + 4x_5 &= 3 \\ 3x_3 + 4x_5 &= 9 \\ x_1 - 4x_3 - 2x_4 &= 7 \\ 3x_1 + 4x_2 + 2x_5 &= 5 \\ 2x_2 + 3x_3 - x_4 + x_5 &= -4 \end{aligned}$$

Determine x_2 and x_4 using Cramer's method.

Question 5

Reduce the matrix below into upper triangle form. Find the value(s) of x that makes the matrix non-invertible.

$$\begin{bmatrix} 2 & 3 & 7 \\ -2 & x & -11 \\ 0 & -3 & x \end{bmatrix}$$

Question 6

The matrix A is given below. Determine A^{-1}. Which elements of A^{-1} requires almost no computation? (Hint: Do you expect A^{-1} to be triangular?)

$$A = \begin{bmatrix} 3 & 2 & 3 & 5 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 7

Evaluate the determinant of the matrix A using row reduction only.

$$A = \begin{bmatrix} 1 & 2 & 7 & 6 \\ -4 & -7 & -17 & -13 \\ 2 & 2 & 11 & 7 \\ -3 & 1 & -16 & -21 \end{bmatrix}$$

Question 8

For the following matrix A:

$$\begin{bmatrix} 5 & -1 & 0 \\ -10 & 2 & 7 \\ 0 & -3 & 4 \end{bmatrix}$$

calculate the determinant of $\left(\left(A^{-1}\right)^T\right)^3$

Question 9

By inspection, explain why the following matrix is not invertible. Do not try to compute the inverse!

$$A = \begin{bmatrix} 1 & 2 & 7 & 6 \\ 2 & 6 & 8 & 20 \\ -5 & 7 & 11 & 7 \\ 0 & 1 & -3 & 4 \end{bmatrix}$$

Question 10

Given that $\det(A) = -2$ and $\det(B) = 1$ find the determinant of $A^{-2}B^3$.

Question 11

Show the following by applying elementary row or column operations:

$$\begin{vmatrix} x & y & z & w \\ 2x & y & 2z & 2w \\ x & y & 2z & 2w \\ -x & 0 & 0 & 0 \end{vmatrix} = 0$$

Question 12

Let $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ be two vectors from the origin to the points P_1 and P_2. Let Q be the midpoint of the line between P_1 and P_2 respectively. Show that $\overrightarrow{OP_1}$, $\overrightarrow{OQ}$, and $\overrightarrow{OP_2}$ are coplanar. (Hint: Either show that the vector $\overrightarrow{OQ}$ and its projection onto the plane defined by $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ is the same as the vector $\overrightarrow{OQ}$ or show that the vector $\overrightarrow{OP_1} \times \overrightarrow{OP_2}$ is orthogonal to $\overrightarrow{OQ}$).

Question 13

In $\mathbb{R}^n$, use triangular inequality for two vectors to prove or disprove $\|x+y-z\| \leq \|x\|+\|y\|+\|z\|$. Demonstrate your answer in $\mathbb{R}^2$.

Question 14

Let a, b, and c be the vectors in $\mathbb{R}^3$. We can define a matrix $A = \begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix}$ using the vectors a, b, and c as the rows of the matrix A.

1. Show that $\det(A) = a \cdot (b \times c)$
2. Using the determinant show that $a \cdot (b \times c) = -b \cdot (a \times c)$

Question 15

In $\mathbb{R}^2$, construct two examples to demonstrate $|\|x\| - \|y\|| \leq \|x - y\|$ for both equality and inequality.

Question 16

Find a unit norm vector in the direction of the vector formed by the orthogonal projection of the vector $a = (1, 2, 1)$ onto the plane described by the equation $x - 3y + 4z = 5$.

Question 17

Find a unit norm vector that is orthogonal to both the vector $(1, 2, -1)$ and the line of intersection of the planes $x + 2y + 2z = 5$ and $x + y - 3z = 2$.

Question 18

Find the equation describing the set of all points $P \in \mathbb{R}^3$ such that the vector $\overrightarrow{P_0P}$ is orthogonal to the vectors $(1, 3, -2)$ and $(-1, 1, 2)$ where $P_0 = (2, -1, 3)$. What is this set?

Question 19

Find the equation of all points $P \in \mathbb{R}^3$ such that the vector $\overrightarrow{P_0P}$ is orthogonal to the vector $(2, 1, -5)$ where $P_0 = (1, 1, -1)$. What does this equation stand for?

Question 20

Let vectors $(2, 0, 0)$, $(2, 3, 0)$, and $(2, 3, 2)$ form the three edges of a parallelpiped. Determine the surface area and volume of the parallelepiped by using the determinant.

Question 21

Two triangles (T_1 corners: $(2, 1, -3)$,$(1, 2, 1)$,$(0, 1, 0)$) and (T_2 corners: $(2, 1, -3)$,$(1, 2, 1)$,$(2, 5, 1)$) share an edge in $\mathbb{R}^3$. Describe and apply a method to determine the dihedral angle between the two triangles. Dihedral angle is the angle between the planes of the triangles.

Question 22 Updated

Determine the distance between the point $(0, 1, -4)$ and the plane $x - 3y + z = 1$ in $\mathbb{R}^3$ using the method of orthogonal projections.

Question 23

Let a and b be the vectors of arbitrary dimension. Given that $(a+b) \perp (a-b)$, what is the ratio of their norms?

Question 24

Let a, b, and c be the vectors in $\mathbb{R}^n$. Are these vectors coplanar if $a + b + c = 0$? Explain your answer.

Question 25 Updated

Show that $\|a + b\|^2 + \|a - b\|^2 = 2\big(\|a\|^2 + \|b\|^2\big)$ for the vectors a and b of arbitrary dimension.

Question 26

Let P_0, P_1, P_2, P_3 be the noncollinear points in $\mathbb{R}^3$. Given that the $\overrightarrow{P_0P_1} \times \overrightarrow{P_2P_3} \neq 0$ and $\overrightarrow{P_0P_2} \cdot \big(\overrightarrow{P_0P_1} \times \overrightarrow{P_2P_3}\big) = 0$, explain why the line passing through points P_0 and P_1 must intersect the line passing through P_2 and P_3.