Ergün Yaraneri
yaraneri@itu.edu.tr

Solutions of Question 4 in Homework 4

Only Question 4 will be solved and graded! Each part is worth 50 points.

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(4) (a) Let A, B and X be sets such that $A \subseteq X \subseteq B$ and $A \sim B$. Prove that $A \sim X$ and $B \sim X$.

We will use the Schröder-Berstein theorem stating that: "Let U and V be sets. If there is an injective map $f : U \to V$ and there is an injective map $g : V \to U$, then there is a bijective map $h : U \to V$ (and so $U \sim V$)".

As A is a subset of X and X is a subset of B, we may consider the inclusion maps

$$\iota : A \to X \quad \text{and} \quad \nu : X \to B$$

defined by $\iota(a) = a$ for all $a \in A$ and $\nu(x) = x$ for all $x \in X$. It is clear that both maps ι and ν are injective. As $A \sim B$, there is a bijective map $\phi : B \to A$. (Also there is a bijective map from A to B, for instance ϕ^{-1}, but we dont need here). As the composition of injective maps is injective, it follows that both of the two maps

$$\iota \circ \phi : B \to X \quad \text{and} \quad \phi \circ \nu : X \to A$$

are injective.

As there is an injective map $\iota : A \to X$ and there is an injective map $\phi \circ \nu : X \to A$, it follows from the Schröder-Berstein theorem that there is a bijective map $A \to X$. Therefore, $A \sim X$.

As there is an injective map $\iota \circ \phi : B \to X$ and there is an injective map $\nu : X \to B$, it follows from the Schröder-Berstein theorem that there is a bijective map $B \to X$. Therefore, $B \sim X$.

(b) Find a bijective map $\mathbb{R} \to \mathbb{R} - \{0\}$.

If we can find a countable infinite subset C of $\mathbb{R} - \{0\}$, then shifting elements of countable infinite sets we may construct a bijection $f : C \cup \{0\} \to C$. As $\mathbb{R} - (C \cup \{0\}) = \mathbb{R} - C$, it is clear that the map

$$\phi : \begin{cases} f(x), & \text{if } x \in C \cup \{0\} \\ x, & \text{if } x \in \mathbb{R} - (C \cup \{0\}) \end{cases}$$

gives a bijection $\phi : \mathbb{R} \to \mathbb{R} - \{0\}$.

We may take $C = \mathbb{N}^+$, the set of positive integers, which is an infinite countable subset of $\mathbb{R} - \{0\}$. Then $C \cup \{0\} = \mathbb{N}$ and we may take $f(x) = x + 1$, which defines a bijection $f : \mathbb{N} \to \mathbb{N}^+$. Therefore, the map

$$\phi : \mathbb{R} \to \mathbb{R} - \{0\} \quad \text{defined by} \quad \phi : \begin{cases} x + 1, & \text{if } x \in \mathbb{N} \\ x, & \text{if } x \in \mathbb{R} - \mathbb{N} \end{cases}$$

must be a bijection.

Indeed, for any $r \in \mathbb{R} - \{0\}$, we see that if $r \in \mathbb{N}$ then $r - 1 \in \mathbb{N}$ and $\phi(r - 1) = r$, and we see that if $r \in \mathbb{R} - \mathbb{N}$ then $\phi(r) = r$. Hence, ϕ is onto.

Suppose that $\phi(a) = \phi(b)$ for some real numbers a and b. It is clear from the definition of ϕ that the image of any natural number is natural and the image of any nonnatural number is nonnatural. So, a and b are both natural or both nonnatural. Therefore, $a + 1 = b + 1$ or $a = b$. Hence, $a = b$. Consequently, ϕ is one to one.