MAT281E Linear Algebra and Applications HW 1

Instructions: Turn in your solutions (hardcopy) no later than November 1st, 2021, 16:00. (Scan or take photos of your solutions, organize them in order and submit as one pdf file). Late homeworks will not be accepted. 4-5 problems will be checked in detail which will contribute 80% to the final mark. The rest will be checked for completeness which will contribute 20% to the final mark.

1. For each of the following i) Express the linear system in matrix form, i.e. as $Ax = b$. Indicate dimensions of A, x, b. Determine the solution for $\underline{x}$ (if any) using Gauss-Jordan Elimination (i.e. row reduction).

a.

$$\begin{aligned}&x + y - 3z = 4\\&y - z = 3\\&-x + y + z = 2\end{aligned}$$

b.

$$\begin{aligned}&2x + y - 3z = 1\\&x + z = 1\\&-x + 2y - 4z = 5\end{aligned}$$

c.

$$\begin{aligned}&x + y + z + w = 1\\&2x + z + 2w = 5\end{aligned}$$

2. Bring the following matrix to row echelon form and then to reduced row echelon form by applying elementary row operations

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -1 & 2 & 1 \\ -2 & -1 & 4 \end{bmatrix}$$

3. Consider $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Are these matrices in row echelon form, reduced row echelon form or neither? Explain.

4. Explain why the products AA^T, A^TA always exist.

5. If matrices A is 3x4, $B$is 2x4 and C is 2x3 what are the dimensions (size) of the matrix resulting from $(CAB^T)^{-1}$ if it exists?

6. Let A be a 3x3 matrix. If $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of the matrix-vector equation $Ax = 0$ is A invertible? Why or why not?

7. Perform the matrix products AB and BA by first expressing the result in terms of the submatrices.

$$A = [A_1 \vdots A_2] = \begin{bmatrix} 1 & 0 & \vdots & 0 & 0 & 3 \\ 0 & 1 & \vdots & 0 & 1 & 1 \end{bmatrix}. \; B = \begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ \cdots & \\ -1 & 2 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$$

8. Write down a 4x4 matrix A for which the elements satisfy $a_{ij} = 0$ if $i - j \geq 1$. Determine the inverse of this matrix. What should you care about in specifying A?

8. Prove that the inverse of matrix $\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\underline{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$by solving a linear system of the form $AX = I$ where I is a 2x2 matrix.

9. Let $(A - I)^2 + A = 0$, and suppose that the inverse of the square matrix A exists. Write down a formula for A^{-1} in terms of A.

10. Let a matrix have a row of zeroes. Does its inverse exist? Explain by using $\underline{AA}^{-1} = \underline{I}$.

11. Suppose that $(A+B)^2 = A^2 + 2AB + B^2$ Is it necessarily true that A and B are inverses of each other? Could these matrices be inverses of each other?

12. Find shortest sequence of row operations that will turn $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}$ into $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.Write down the elementary matrix for each operation. Is the sequence unique? Explain.

13. Is it possible to apply elementary row operations to turn an invertible matrix into a matrix with whose two rows add up to another row? Explain.

14. Try to find the inverse of the following matrices by applying Gauss Jordan elimination on augmented matrices of the form $[A|I]$

a) $A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ c) $A = \begin{bmatrix} c & 0 & a \\ 0 & b & 0 \\ c & b & a \end{bmatrix}$

15. Can you state the elementary row operations that are needed to turn $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ into $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Repeat for turning $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ into $B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$?

16. Is the given triangular matrix invertible? Explain by saying what happens when the matrix is reduced to a row echelon or reduced row echelon form. Do you get a row of zeroes or not? (Do not try to compute the inverse.)

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 0 & 4 & -4 & -1 \\ 0 & 3 & 0 & -2 & 2 \\ 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

17. Determine if there are value(s) of x for which the inverse of

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & x & 3 & -1 \\ 0 & 0 & 1 & 0 \\ x+1 & 2 & 4 & 1 \end{bmatrix}$$ does not exist.

18. Describe all possible r.r.e.f.s (reduced row echelon forms) of i)2x4 matrix ii) 4 × 3 matrix.

19. Compute the determinant of the following matrix using a combination of any suitable methods with low overall complexity :

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

21. Find A^{120} if $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$