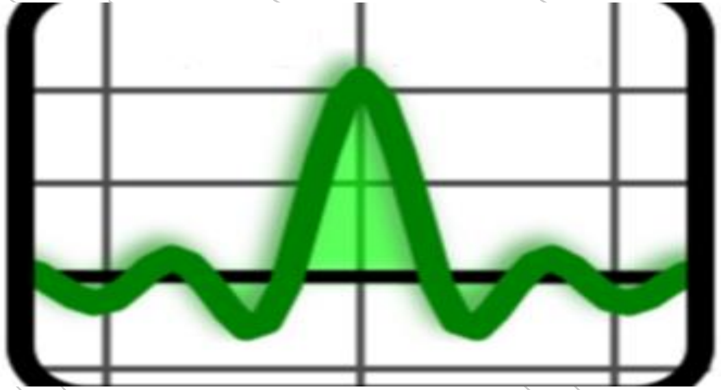

Signals & Systems For Computer Engineering

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BLG354E

10th Week Lecture

Common Fourier Transform Pairs

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \mathscr{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

x(t) and X(ω) form a Fourier transform pair denoted by

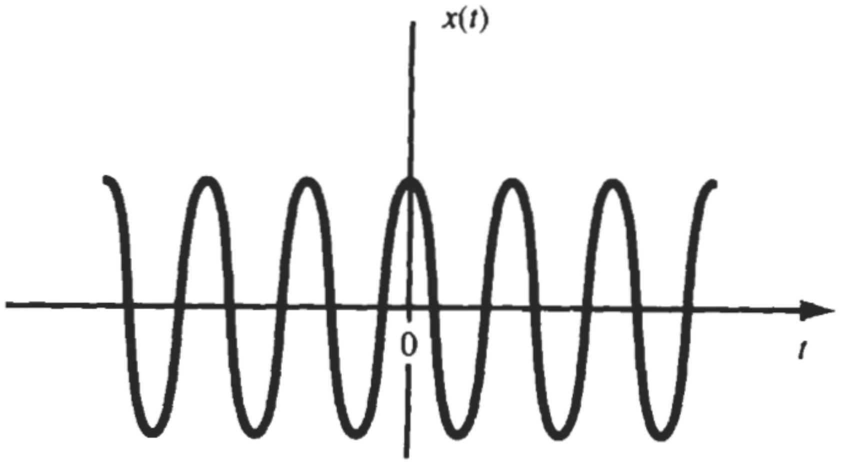
x(t) ↔ X(ω)

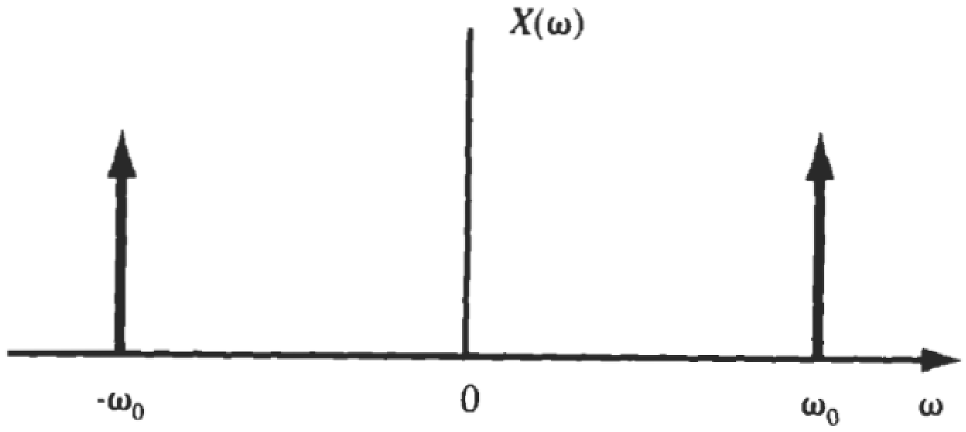
$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\dfrac{1}{j\omega}$
$e^{-at}u(t), a>0$	$\dfrac{1}{j\omega+a}$
$te^{-at}u(t), a>0$	$\dfrac{1}{(j\omega+a)^2}$
$e^{-a\|t\|}, a>0$	$\dfrac{2a}{a^2+\omega^2}$
$\dfrac{1}{a^2+t^2}$	$e^{-a\|\omega\|}$
$e^{-at^2}, a>0$	$\sqrt{\dfrac{\pi}{a}}\, e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \|t\|<a\\ 0 & \|t\|>a\end{cases}$	$2a\dfrac{\sin\omega a}{\omega a}$
$\dfrac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \|\omega\|<a\\ 0 & \|\omega\|>a\end{cases}$
$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0), \omega_0=\dfrac{2\pi}{T}$

Find the Fourier Transform of the signal $x(t)=\cos(\omega_0 t)$

$$\cos \omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

By using linearity property, $$\cos \omega_0 t \longleftrightarrow \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$





This is why sinusoidal modulation signals are used in bandwidth limited communication systems

$$\mathscr{F}\left[x(t)\cos \omega_0 t\right] = \mathscr{F}\left[\frac{1}{2}x(t)\,e^{j\omega_0 t} + \frac{1}{2}x(t)\,e^{-j\omega_0 t}\right]$$
$$= \frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)$$

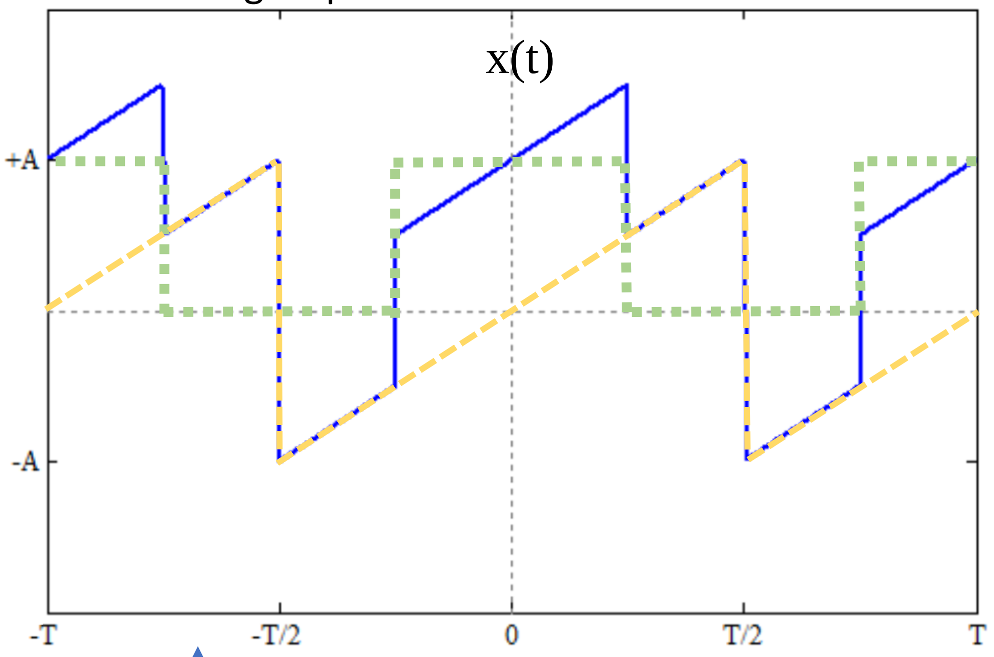
$$x(t)\cos \omega_0 t \longleftrightarrow \frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)$$

Application Fields of the Fourier Transform in Engineering

1) **Communication systems and devices :** Analog and digital filters, spectral Analysis, modulation, Software Defined Radio, BW management, cognitive radio, signal quality
2) **Machine Learning :** Spectral feature extraction, abstraction
3) **Data Processing :** Data compression (derivatives Cosine transform, DFT, FFT)
4) **Image processing :** Transformation , representation , encoding , smoothing and sharping images, Image analysis. 2 or higher dimensional Fourier Transform are being used in image processing
5) **Data analytics :** Time series prediction (DFT, FFT, STFT, ...) , high – pass , low-pass , and band-pass filters. Signal and noise (SNR) estimation by encoding the time series, .
6) **Energy systems and electronic devices:** Harmonic analysis, filtering, stability analysis, energy and signal quality management
7) **Mechanical systems :** Vibration analysis and management, structural diagnostics, design methods
8) **Control systems :** Frequency response, regulators stability analysis, frequency domain control
9) **………**

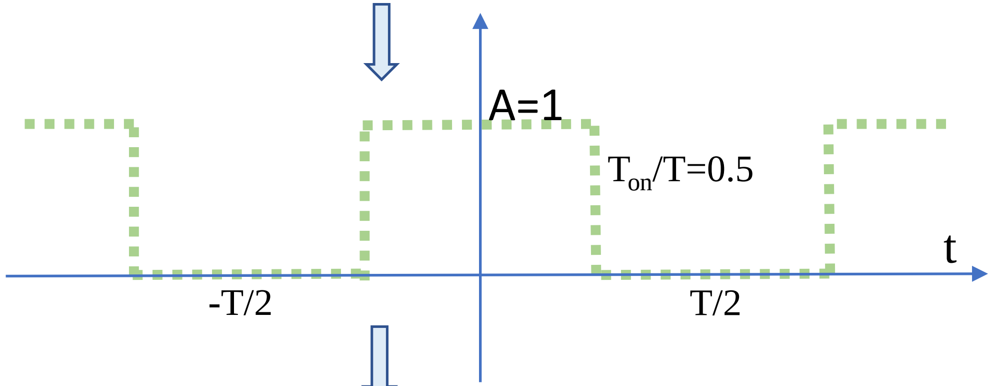
Example: Find the Fourier Series representation of the signal that is sum of sawtooth and square waves (50% duty cycle)

This signal pattern is neither even nor odd



$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t \right)$$

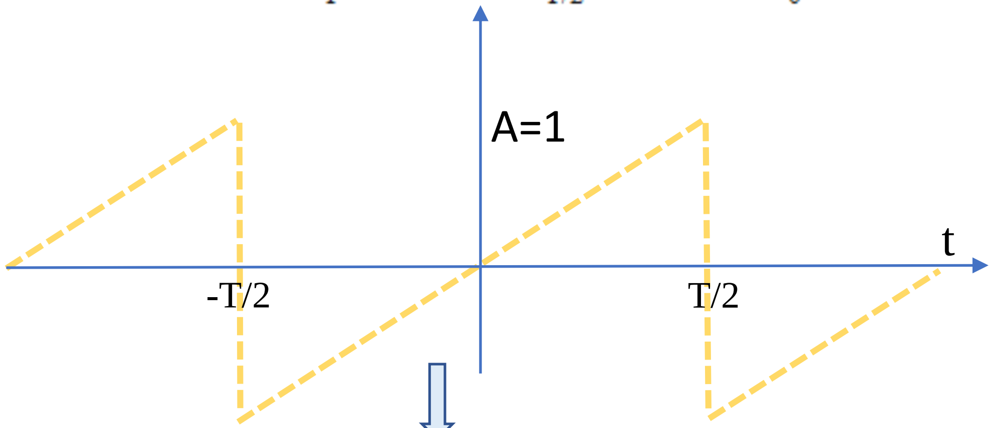
$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos k\omega_0 t \, dt \qquad b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t \, dt$$



$$a_n = 2\frac{A}{n\pi} \sin\left(n\pi \frac{T_{on}}{T}\right) = 2\frac{A}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$a_n = \begin{cases} 2\frac{A}{n\pi}\left(-1^{\frac{n-1}{2}}\right), & n \; odd \\ 0, & n \; even, \; n \neq 0 \end{cases}$$

$a_0 = 0.5$



$$b_n = \frac{2}{T} \frac{AT\left(\sin(\pi n) - \pi n \cos(\pi n)\right)}{\pi^2 n^2}$$

$$b_n = -\frac{2A}{\pi n}(-1)^n$$

n=k	0	1	2	3	4	5	6	7
a_n	0.5	0.6366	0	-0.2122	0	0.1273	0	-0.0909
b_n	----	0.6366	-0.3183	0.2122	-0.1592	0.1273	-0.1061	0.0909

$$c_0 = a_0 \; and \; c_n = \frac{a_n}{2} - j\frac{b_n}{2} \; for n \neq 0, \qquad c_{-n} = c_n^*$$

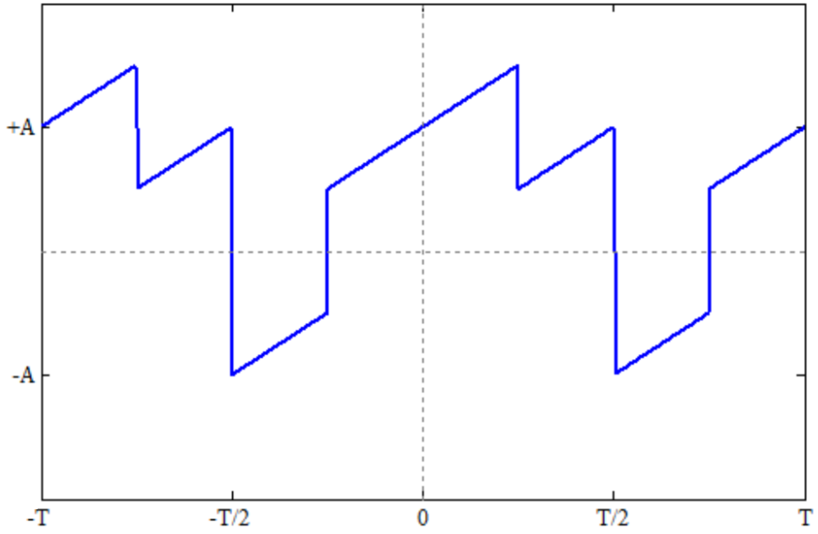
n=k	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
c_n	-0.0455 + 0.0455j	-0.0531j	0.0637 + 0.0637j	-0.0796j	-0.1061 + 0.1061j	-0.1592j	0.3183 + 0.3183j	0.5	0.3183 - 0.3183j	0.1592j	-0.1061 - 0.1061j	0.0796j	0.0637 - 0.0637j	0.0531j	-0.091

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

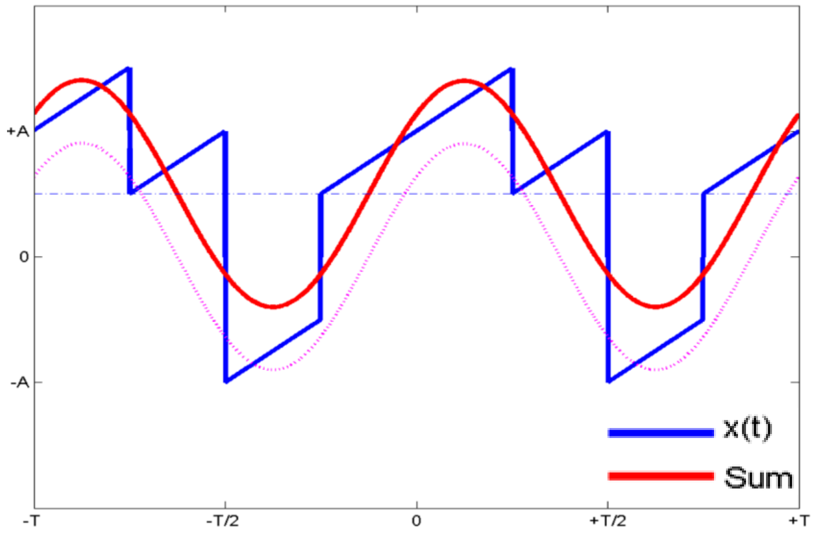
where c_k are complex Fourier coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

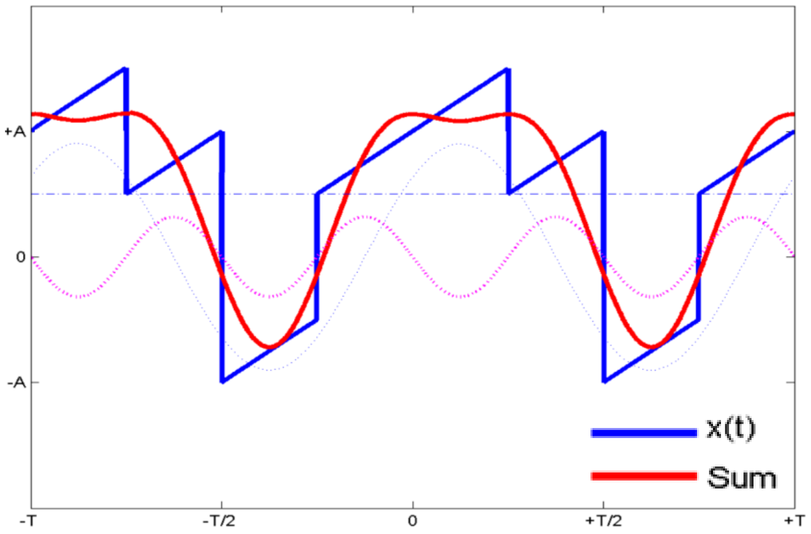
signal x(t)



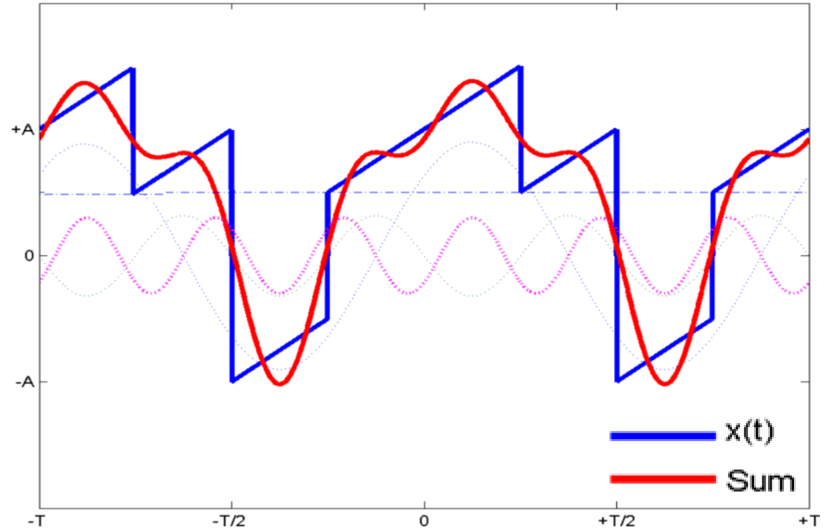
signal x(t)+1st harmonic



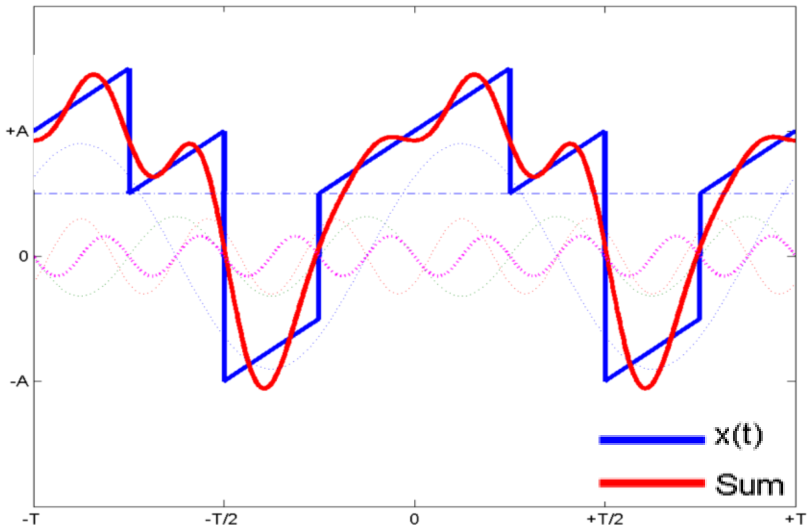
signal x(t)+1st+2nd harmonics



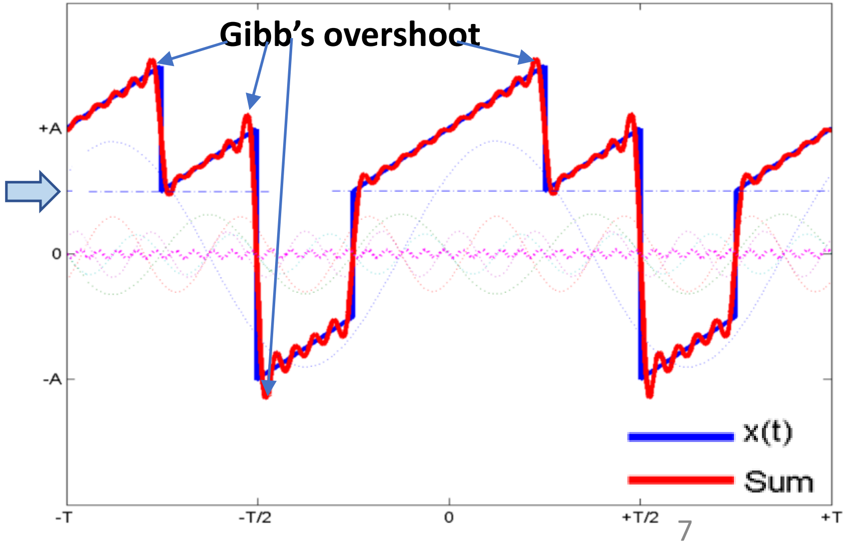
signal x(t)+1st+2nd+3rd harmonics



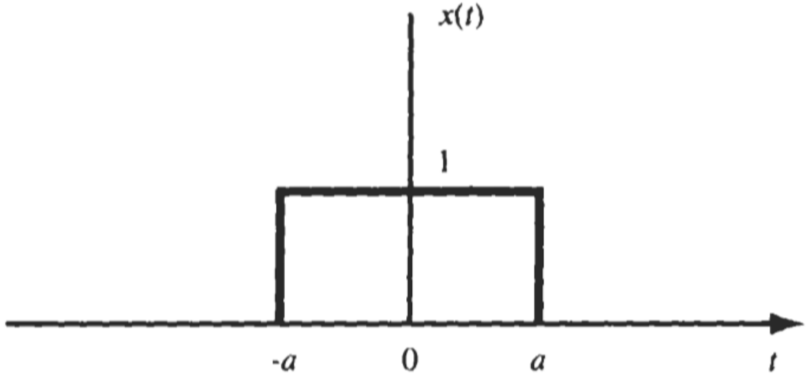
signal x(t)+1st+2nd+3rd+4th harmonics



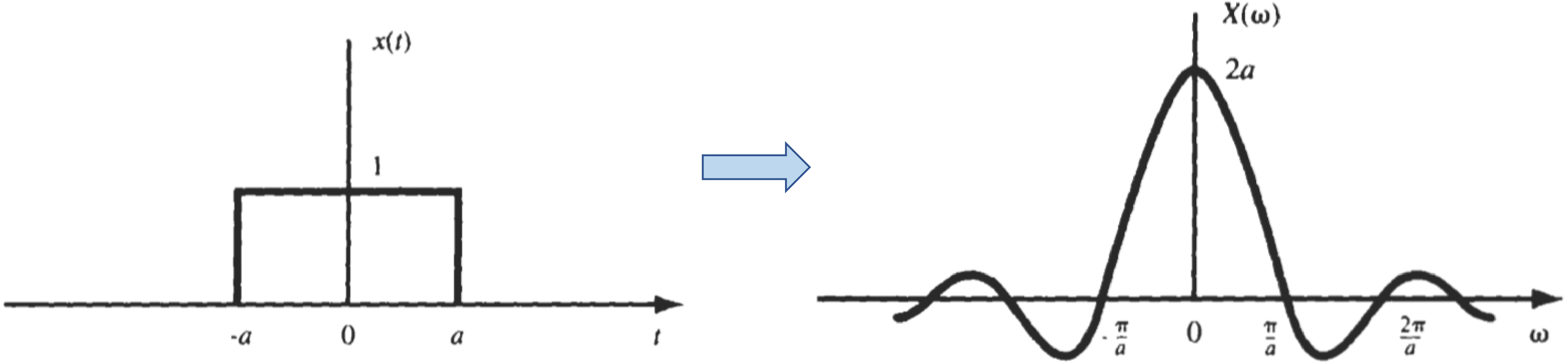
signal x(t)+1st+2nd+3rd+...+20th harmonics



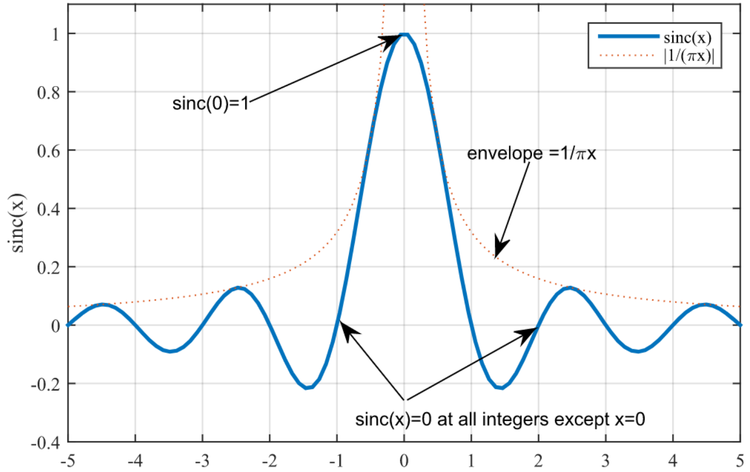
Let us find the Fourier transform of the rectangular pulse signal $x(t) = p_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$



$$X(\omega) = \int_{-\infty}^{\infty} p_a(t)\, e^{-j\omega t}\, dt = \int_{-a}^{a} e^{-j\omega t}\, dt = \frac{1}{j\omega}\left(e^{j\omega a} - e^{-j\omega a}\right) = 2\frac{\sin \omega a}{\omega} = 2a\frac{\sin \omega a}{\omega a}$$



Let us find the Fourier transform of the sinc(x) function where $\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$



$$x(t) = \frac{\sin at}{\pi t} \qquad p_a(t) \longleftrightarrow 2\frac{\sin \omega a}{\omega}$$

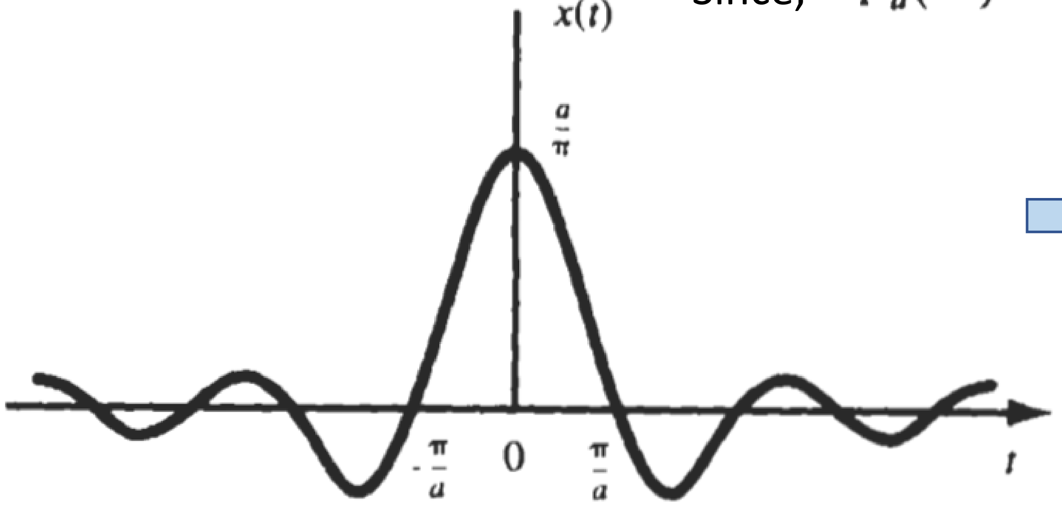
By the duality property,

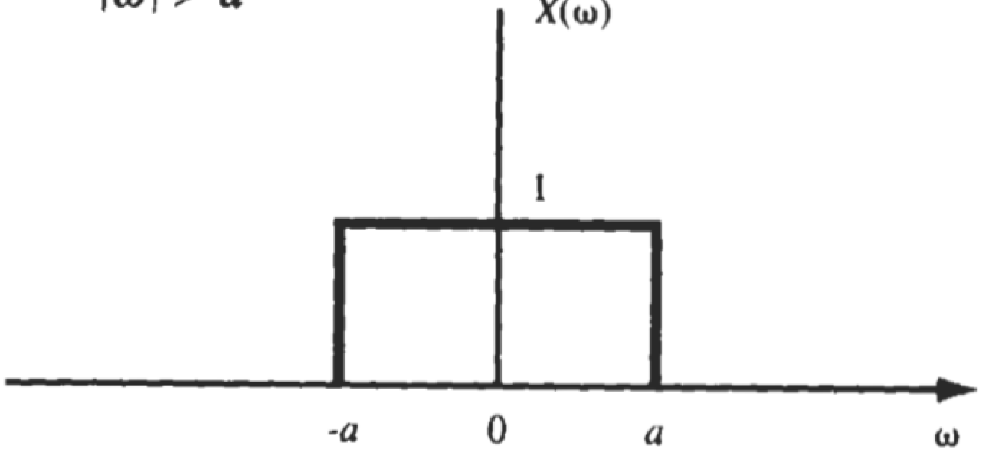
$$\boxed{\begin{array}{l} x(t) \leftrightarrow X(\omega) \\ X(t) \leftrightarrow 2\pi\, x(-\omega) \end{array}}$$

$$2\frac{\sin at}{t} \longleftrightarrow 2\pi p_a(-\omega)$$

Divide both sides by 2π $\rightarrow$ $\dfrac{\sin at}{\pi t} \longleftrightarrow p_a(-\omega) = p_a(\omega)$

Since, $p_a(\omega) = \begin{cases} 1 & |\omega| < a \\ 0 & |\omega| > a \end{cases}$





Prove the convolution theorem by using Fourier Transform properties

$$x_1(t)*x_2(t) \leftrightarrow X_1(\omega)\cdot X_2(\omega)$$

By the definition, $$\mathscr{F}[x_1(t)*x_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau\right]e^{-j\omega t}\,dt$$

By changing the order of integrations we get, $$\mathscr{F}[x_1(t)*x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}\,dt\right]d\tau$$

By the time shifting property → $$\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}\,dt = X_2(\omega)e^{-j\omega\tau}$$

$$\mathscr{F}[x_1(t)*x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)X_2(\omega)e^{-j\omega\tau}\,d\tau = \left[\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}\,d\tau\right]X_2(\omega) = X_1(\omega)X_2(\omega)$$

Therefore $$x_1(t)*x_2(t) \leftrightarrow X_1(\omega)\cdot X_2(\omega)$$

Example: $\frac{dy(t)}{dt} + 2y(t) = x(t)$ describes the input x(t) – output y(t) relationship of an LTI system

a) Find the transfer function of this system in frequency domain
b) Find the output of this system y(t), in time domain, for x(t)=u(t)
c) Write pseudo code that emulates this system on a computer for the sampling period T_s (**Exercise**)

Fourier transform of the system $j\omega Y(\omega) + 2Y(\omega) = X(\omega)$ → $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2 + j\omega}$

Fourier transform of the input signal x(t)=u(t) → $X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$

Output of the system for the specified input signal: $Y(\omega) = X(\omega)H(\omega) = \left[\pi\delta(\omega) + \frac{1}{j\omega}\right]\frac{1}{2 + j\omega}$

By using partial fraction expansion method: Y(ω) $= \pi\delta(\omega)\frac{1}{2 + j\omega} + \frac{1}{j\omega(2 + j\omega)} = \frac{\pi}{2}\delta(\omega) + \frac{1}{2}\frac{1}{j\omega} - \frac{1}{2}\frac{1}{2 + j\omega}$

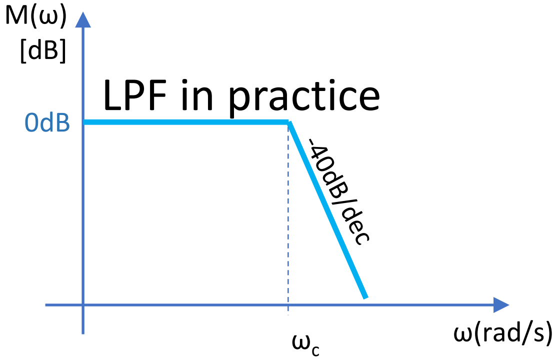
Since $f(\omega)\delta(\omega) = f(0)\delta(\omega)$ we get Y(ω) $= \frac{1}{2}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - \frac{1}{2}\frac{1}{2 + j\omega}$

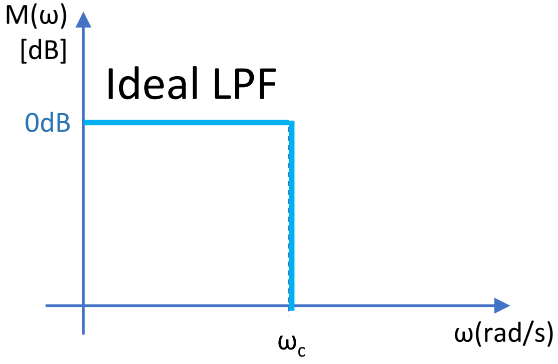
Hence by using the table: $y(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t) = \frac{1}{2}(1 - e^{-2t})u(t)$

(Remark: using the Laplace transform could be easier due to transform of u(t))

Ideal Low Pass Filter (LPF) is characterized by $H(\omega)=\begin{cases}1 & |\omega|<\omega_c \\ 0 & |\omega|>\omega_c\end{cases}$

Find the output of ideal LPF for the input x(t)=sinc(t)





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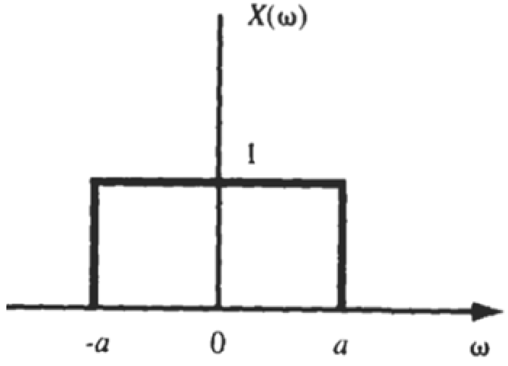
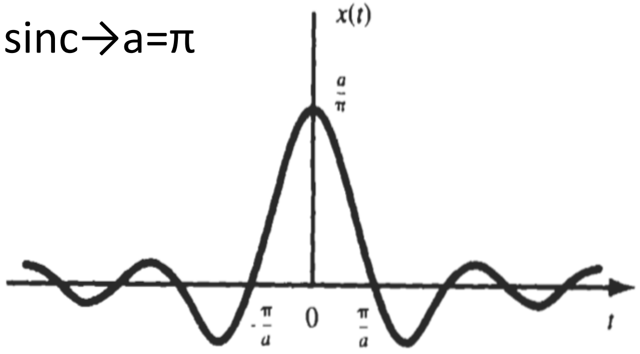


$$x(t)=\frac{\sin at}{\pi t}$$

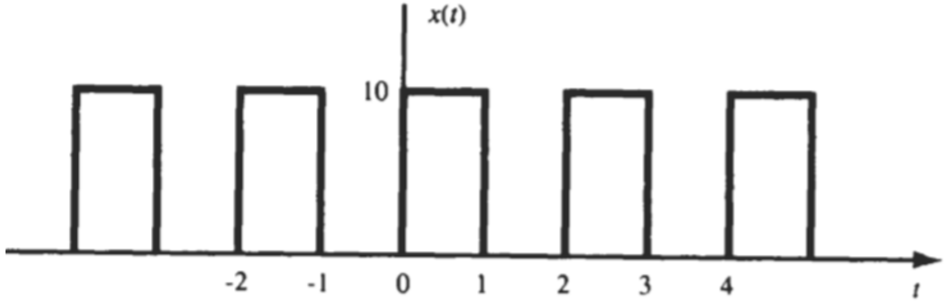
sinc→a=π



When $\omega_c>a$ → $Y(\omega)=X(\omega)H(\omega)=X(\omega)$ → $y(t)=x(t)=\dfrac{\sin at}{\pi t}$

When $\omega_c<a$ → $Y(\omega)=X(\omega)H(\omega)=H(\omega)$ → $y(t)=h(t)=\dfrac{\sin \omega_c t}{\pi t}$

Find the output of ideal LPF H(ω) for the below given periodic input signal where $H(\omega) = \begin{cases} 1 & |\omega| < 4\pi \\ 0 & |\omega| > 4\pi \end{cases}$



Trigonometric Fourier Series is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}$$

As previously shown $x(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right)$

Where A=10 and $\omega_0=\pi$ here for the given x(t) → $x(t) = 5 + \frac{20}{\pi}\left(\sin \pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + \cdots\right)$

Since angular frequency ω>4π will be rejected by the given ideal LPF,
output y(t) will be consisting of harmonics having angular frequency less than 4π.

$$y(t) = 5 + \frac{20}{\pi}\sin \pi t + \frac{20}{3\pi}\sin 3\pi t$$