MAT281E Linear Algebra and Applications Homework 4 – Due January 11, 2022

Late submissions will not be graded.

1. What is the dimension of the columnspace of $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$? Is there a solution to the general equation $A\mathbf{x} = \mathbf{b}$ for any given $\mathbf{b}$?

2. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & -2 & 2 \end{bmatrix}$ and $b = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

   a) Is b in the columnspace of the matrix A. If so write b as a linear combination of the column vectors of this matrix.

   b) What is the rank of A?

   c) What is the dimension of the nullspace (nullity)of the matrix?

   d) Write the homogeneous solution and the general solution?

3. Let $A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix}$

   a) Find a basis for the nullspace of A.

   b) What is nullity of A?

   c) What is the rank of A?

   d) Determine the homogeneous solution and the general solution of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$.

   e) Can you determine solution(s) of $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$. Explain.

4. Reduce A to a row-echelon form. Appropriately select the rows of the reduced row echelon matrix and the columns of A to determine bases for the rowspace and columnspace of A, respectively.

$$A = \begin{bmatrix} 1 & 0 & 1 & -4 \\ 2 & 0 & 3 & -1 \\ 2 & 0 & 4 & 6 \end{bmatrix}$$

5. By making use of row-reduction, appropriately select columns of A to determine basis for the columnspace of A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix}$$

6. If E is an elementary matrix, do A and EA have the same columnspace? Explain. (Think of $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$)

7. Determine a basis for the rowspace of $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -4 & 6 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ by choosing from the row vectors of A. (Hint: Need to apply row reduction to $\underline{\underline{A}}^T$ to identify the lin. independent columns of A^T)

8. Find all 3x3 matrices for which the homogeneous system has a solution space as the line $x = 2t$, $y = t$, $z = t/2$. (Hint: Write the row reduced augmented matrix from given information.)What is the rank in this case?

9. What conditions must be satisfied for the overdetermined system below to be consistent? Do not try to solve for the solution naively. Apply row reduction. If the system is consistent does a unique solution exist?

$$x_1 + x_2 - x_3 = b_1$$

$$2x_1 - x_2 + 3x_3 = b_2$$

$$-x_1 + 3x_2 + x_3 = b_3$$

$$2x_2 - x_3 = b_4$$

10. We have a 3x6 matrix A with rank 3. What is the number of free variables in the solution to the system $A\mathbf{x} = 0$? For a given **b,** are we guaranteed to have a solution to $A\mathbf{x} = \mathbf{b}$? If we have a solution, what is the dimension of the solution space? Explain without using an example.

11. For a 4x3 matrix can the nullspace, the column space and row space all be a line through the origin? For a 2x4 matrix can the nullspace, the column space and row space all be a plane through the origin?

12. Consider matrix $A = \begin{bmatrix} 4 & 2 \\ t & 1 \\ 3 & t \end{bmatrix}$. Is the column space of A the same for all t? Is the row space of A the same for all t? Is the solution space of $A\mathbf{x} = \mathbf{b}$ for a given **b** in the span of the column vectors the same for all t? (Hint: Apply row reduction)

13. Find the characteristic equation, eigenvalues, eigenvectors and the bases for the eigenspaces of the following matrices. For each eigenvalue also find the rank of $\lambda I - A$:

a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$

14. Find the eigenvalues and eigenvectors of A^5 for matrix in part b) of problem above. How many eigenspaces does it have? What is the dimension of each eigenspace and what are the bases vectors for it?

15. Can you always diagonalize a matrix with fewer eigenvalues than the matrix dimension? Explain.

16. Let the characteristic polynomial of a matrix be $p(\lambda) = \lambda^2 - 9$

a) What are the dimensions of the matrix?

b) What is the determinant of the matrix?

c) Suppose the matrix is invertible. What is the characteristic polynomial for the inverse matrix? Relate eigenvalues of a matrix to the eigenvalues of its inverse.

d) What is the characteristic polynomial of the transpose of the matrix? Explain your reasoning.

17. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

Is the matrix A diagonalizable. If so find a matrix P that diagonalizes A. Can you write A as a linear combination of rank 1 matrices formed from its eigenvectors. Determine the eigendecomposition $A = P\Lambda P^{-1}$

18. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Is the matrix A diagonalizable. If so find a matrix P that orthogonally diagonalizes A. Can you write A in diagonal form as a linear combination of rank 1 matrices formed from its eigenvectors. Note that A is real symmetric so that you do not have to compute the inverse of P.

19. Does $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have a zero eigenvalue based on your observation of its rows or columns? Do not compute the characteristic polynomial. Explain.

20. Determine an orthonormal basis that spans the rows of $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ using the Gram Schmidt orthogonalization process. Determine a decomposition of A into the product of a lower triangular matrix and an orthogonal matrix. (Hint: How should you apply QR decomposition here?)

21. Find the least squares solution for the linear system $A\mathbf{x} = \mathbf{b}$ and the best approximation for $\mathbf{b}$ if $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$.