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Solutions of Question 3 in Homework 3

Only Question 3 will be solved and graded! Each part is worth 50 points.

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(3) (a) Let R be an equivalence relation on a finite set A. Prove that $|R| - |A|$ is even.

As $R \subseteq A \times A$, we have two possibilities for any element of R. It is of the form (x, x) or of the form (x, y) where $x, y \in A$ such that $x \neq y$. So, we may write R as a union of two disjoint sets S and T :

$$R = S \cup T \qquad \text{where} \qquad S = \{(x, x) \in R \mid x \in A\} \qquad \text{and} \qquad T = \{(x, y) \in R \mid x, y \in A, x \neq y\}.$$

As R is reflexive, $(a, a) \in R$ for any $a \in A$ and so $S = \{(a, a) \mid a \in A\}$. Thus $|A| = |S|$. As $S \cap T = \emptyset$,

$$|R| - |A| = |R| - |S| = |T|.$$

As R is symmetric, for any $(x, y) \in T$ it follows that $(y, x) \in T$. Since $(x, y) \neq (y, x)$ for any $(x, y) \in T$ (because $x \neq y$), the elements of T can be grouped into distinct pairs of the form $\{(x, y), (y, x)\}$. Note also that for any two such pairs $\{(x, y), (y, x)\}, \{(u, v), (v, u)\}$ we have either

$$\{(x, y), (y, x)\} \cap \{(u, v), (v, u)\} = \emptyset \qquad \text{or} \qquad \{(x, y), (y, x)\} = \{(u, v), (v, u)\}.$$

Consequently, $|T|$ is even. As $|R| - |A| = |T|$, we conclude that $|R| - |A|$ is even .

Note that we did not use the transitivity of the relation. So, we have just proved that "if R is a reflexive and symmetric relation on a finite set A then $|R| - |A|$ is even".

(b) Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4\}$. Let E be the equivalence relation on $\mathcal{P}(A)$ defined for any elements X and Y of $\mathcal{P}(A)$ by

$$(X, Y) \in E \text{ if and only if } X \cap B = Y \cap B.$$

Find the equivalence classes and write the quotient set A/E.

Note first that

$$\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}, \{1,2,3\}, \{1,2,3,4\}\Big\}.$$

Take any element of $\mathcal{P}(A)$. For instance take $\emptyset$. Then

$$[\emptyset] = \Big\{X \in \mathcal{P}(A) \mid X \cap \{3,4\} = \emptyset \cap \{3,4\}\Big\} = \Big\{X \in \mathcal{P}(A) \mid X \cap \{3,4\} = \emptyset\Big\} = \Big\{\emptyset, \{1\}, \{2\}, \{1,2\}\Big\}.$$

Take any element of $\mathcal{P}(A) - [\emptyset]$. For instance take $\{3\}$. Then

$$\big[\{3\}\big] = \Big\{X \in \mathcal{P}(A) \mid X \cap \{3,4\} = \{3\} \cap \{3,4\}\Big\} = \Big\{X \in \mathcal{P}(A) \mid X \cap \{3,4\} = \{3\}\Big\} = \Big\{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Big\}$$

Take any element of $\mathcal{P}(A) - \big([\emptyset] \cup [\{3\}]\big)$. For instance take $\{4\}$. Then

$$\big[\{4\}\big] = \Big\{X \in \mathcal{P}(A) \mid X \cap \{3,4\} = \{4\} \cap \{3,4\}\Big\} = \Big\{X \in \mathcal{P}(A) \mid X \cap \{3,4\} = \{4\}\Big\} = \Big\{\{4\}, \{1,4\}, \{2,4\}, \{1,2,4\}\Big\}$$

Take any element of $\mathcal{P}(A) - \big([\emptyset] \cup [\{3\}] \cup [\{4\}]\big)$. For instance take $\{3,4\}$. Then

$$\begin{aligned} \big[\{3,4\}\big] &= \Big\{X \in \mathcal{P}(A) \mid X \cap \{3,4\} = \{3,4\} \cap \{3,4\}\Big\} = \Big\{X \in \mathcal{P}(A) \mid X \cap \{3,4\} = \{3,4\}\Big\} \\ &= \Big\{\{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\Big\} \end{aligned}$$

As $[\emptyset] \cup [\{3\}] \cup [\{4\}] \cup [\{3,4\}] = \mathcal{P}(A)$, there is no other distinct equivalence class and so the are four distinct equivalence classes which are

$$[\emptyset], [\{3\}], [\{4\}], [\{3,4\}].$$

So

$$A/R = \Big\{[\emptyset], [\{3\}], [\{4\}], [\{3,4\}]\Big\}$$

(Note also that

$$[\emptyset] = [\{1\}] = [\{2\}] = [\{1,2\}], \qquad [\{3\}] = [\{1,3\}] = [\{2,3\}] = [\{1,2,3\}],$$
$$[\{4\}] = [\{1,4\}] = [\{2,4\}] = [\{1,2,4\}], \qquad [\{3,4\}] = [\{1,3,4\}] = [\{2,3,4\}] = [\{1,2,3,4\}] \;)$$