Q5 $n^3 < 3^n$ for all $n \geq 4$

Sol: The proof is by induction on n.

For $n=4$, note that $n^3 < 3^n$ becomes $4^3 < 3^4$, that is $64 < 81$, which is true. So the result is true for $n=4$.

Let $k \geq 4$ be a natural number. Suppose that the result is true for $n=k$. That is, suppose that $\boxed{k^3 < 3^k.}$ (We want to prove that the result is true for $n=k+1$) (We want to justify that $\boxed{(k+1)^3 < 3^{k+1}}$)

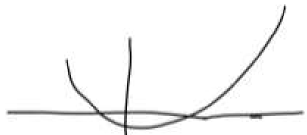
$$\left[(k+1)^3 = \underline{k^3 + 3k^2 + 3k + 1} < 3^k + \underbrace{\boxed{3k^2 + 3k + 1}}_{< 2\,3^k} \overset{?}{<} \underbrace{3^{k+1}}_{3\cdot 3^k} \right]$$

$$3^{k+1} = 3\cdot 3^k \overset{\text{induction hypothesis}}{>} 3k^3 \overset{?}{>} k^3 + 3k^2 + 3k + 1 \quad \Big| \quad 2k^3 > 3k^2 + 3k + 1$$

$k \geq 3$ $\quad k^3 > 3k^2$

$k^3 > 3k+1$

k^2



$k^3 > k^2$ $\quad \boxed{k^2 > 3k+1}$

$k^2 - 3k - 1 = \left(k - \frac{3}{2}\right)^2 - \frac{13}{4}$

$\left(k - \frac{3}{2}\right)^2 = k^2 - 3k + \frac{9}{4} \geq 0$

$k^2 - 3k - 1$

1.5

$k^2 - 3k - 1 = 0$ $\qquad k = \mp \frac{\sqrt{13}}{2} + \frac{3}{2}$ $\qquad k >$

$\log 4$

$\frac{3}{2} + \sqrt{\frac{13}{4}}$

$\boxed{k^2 - 3k - 1 > 0 \quad \text{for } k \geq 4}$

$2k - 3 \qquad k > \frac{3}{2}$

Note that

$$3^{k+1} = 3\cdot 3^k > 3\cdot k^3 = k^3 + k^3 + k^3 > k^3 + 3k^2 + k^3 > k^3 + 3k^2 + k^2$$

By induction hypothesis "$k^3 < 3^k$"

$k^3 > 3k^2$ because $k > 3$

$k^3 > k^2$ becau $k > 1$

$$> k^3 + 3k^2 + 3k + 1 = (k+1)^3$$

$k^2 > 3k+1$ because $k^2 - 3k - 1 > 0$ (by calculus)

Hence we have justified that $(k+1)^3 < 3^{k+1}$, as desired

Consequently, the ~~question~~ result follows by induction

6. $P = \mathbb{Z}[x] = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in \mathbb{Z} \text{ for all } i\}$

$1 + 3x - x^{10}$ $\qquad$ $\frac{1}{2} + \frac{7}{3}x \notin P$

$$\boxed{p(x) \sim q(x) \text{ iff } p(x) - q(x) = c \in \mathbb{Z}}$$

$1 + x + 7x^2 \not\sim 8 + x + 6x^3$ $\qquad$ $1 + x + 7x^2 \sim 8 + x + 7x^2$

(a)

Is $\sim$ reflexive, symmetric, transitive?

reflexive: Let $p(x) \in P$. Is $p(x) \sim p(x)$? Check $p(x) - p(x) = 0 \in \mathbb{Z}$

So $p(x) \sim p(x)$

Hence $\sim$ is reflexive

symmetric: Let $p(x), q(x) \in P$ such that $p(x) \sim q(x)$. Is $q(x) \sim p(x)$?

From $p(x) \sim q(x)$, $p(x) - q(x) \in \mathbb{Z}$. As $p(x) - q(x) \in \mathbb{Z}$, $-(p(x) - q(x)) \in \mathbb{Z}$ too, so $q(x) - p(x) \in \mathbb{Z}$. Hence $q(x) \sim p(x)$. Thus $\sim$ is symmetric.

transitive: Let $p(x) \sim q(x)$ and $q(x) \sim r(x)$ $\exists\, p(x), q(x), r(x) \in P$. Then $p(x) - q(x) \in \mathbb{Z}$ and $q(x) - r(x) \in \mathbb{Z}$. As the sum of two integers is again an integer, $p(x) - q(x) + q(x) - r(x) \in \mathbb{Z}$, so $p(x) - r(x) \in \mathbb{Z}$.

Hence $p(x) \sim r(x)$. Thus $\sim$ is transitive

(b) $\sim$ is an equivalence relation on P. Let $p(x) \in P$. Its equivalence class is

$$[p(x)]_\sim = \{q(x) \in P \mid p(x) \sim q(x)\} = \{q(x) \in P \mid \underbrace{p(x) - q(x)}_{c} \in \mathbb{Z}\}$$

$$p(x) - q(x) \in \mathbb{Z}$$

$P(x)/\sim$ $\qquad$ $\left(p(x) = a_0 + \boxed{a_1 x + \cdots + a_n x^n} \right)$ $\qquad$ $= \{p(x) - c \mid c \in \mathbb{Z}\}$

So $[p(x)]_\sim = \{ p(x) - c \mid c \in \mathbb{Z} \} \longrightarrow \mathbb{Z}$

$p(x) - c \longmapsto c$ is a bijection.

So $|[p(x)]_\sim| = |\mathbb{Z}| = |\mathbb{N}|$ infinite countable

(c) $P/\sim \; = \{ [p(x)]_\sim \mid p(x) \in P \}$ $\quad [1+x]_\sim \quad [7+x]_\sim$

$p(x) \not\sim q(x) \iff [p(x)]_\sim \neq [q(x)]_\sim$

$[x]_\sim, [x^2]_\sim, [x^3]_\sim, [x^4]_\sim, \ldots\ldots [x^n]_\sim, \ldots$ are all distinct.

So in $P/\sim$ there are at least $|\mathbb{N}|$ many elements. So $\underline{\underline{|P/\sim| \geq |\mathbb{N}|}}$

For instance $[x^2+x]_\sim$ is also not equal to any of the above classes

In $P/\sim$ there are at most $|P|$ elements

$P = \mathbb{Z}[x]$ = the set of polynomials

$$\mathbb{Z}[x] - \{0\} = \bigcup_{d=0}^{\infty} P_d$$ P_d = the set of polynomials in P of degree d.

$$P_d \longrightarrow \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{d+1 \text{ times}}$$

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d \longmapsto (a_0, a_1, a_2, \ldots, a_d)$$

This map is bijective

$$|P_d| = |\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{d+1 \text{ times}}| = |\mathbb{N}|$$, P_d = countable.

countable because "the cartesian product of finitely many countable sets is countable"

$$P = \mathbb{Z}[x] = \{0\} \cup \bigcup_{d=0}^{\infty} P_d$$

In the union we have the sets $\{0\}, P_0, P_1, P_2, P_3, \ldots, P_n, \ldots$ which are countably many. Each of these sets is countable.
As "the union of countably many countable sets is countable",
$P = \mathbb{Z}[x]$ is countable. So $|P| = |\mathbb{N}|$

As $|P/\sim| \leq |P|$, $\quad |P/\sim| \leq |\mathbb{N}|$

Consequently, $|P/\sim| = |\mathbb{N}|$

6// $|A| = |\mathbb{R}|$ (It is enough to assume A is infinite)

Recall that
"Any infinite set A contains an infinite countable subset"

Choose an element $a_1 \in A$.

$A - \{a_1\} \neq \emptyset$, choose an element $a_2 \in A - \{a_1\}$

$A - \{a_1, a_2\} \neq \emptyset$ " $a_3 \in A - \{a_1, a_2\}$

$\vdots \qquad \vdots$

$a_{n+1} \in A - \{a_1, a_2, \ldots, a_n\}$

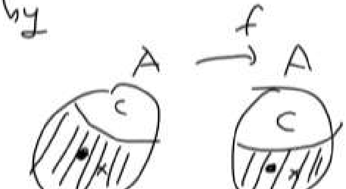
$\vdots$

So $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ is an infinite countable subset of A

Take a countable infinite subset $C = \{a_1, a_2, a_3, a_4, \ldots, a_n, \ldots\} \subseteq A$

Consider the function $f: A \to A$ defined by

$$f(x) = \begin{cases} x & \text{if } x \notin C \\ a_{k-1} & \text{if } x = a_k \text{ and } k > 1 \\ a_1 & \text{if } x = a_1 \end{cases}$$



Note that f is onto but not one to one
$f(a_1) = a_1 = f(a_2)$

$$A = \{a_1, a_2, a_3, a_4, a_5, \dots a_n \dots\} \cup (A-C)$$

$f \downarrow$

$$A = \{a_1, a_2, a_3, a_4, a_5, \dots a_n, \dots\} \cup (A-C)$$

$x \mapsto x$

5 (a) $A \neq \emptyset$, $B \neq \emptyset$. Show that $|A| \leq |A \times B|$

(Recall that $|U| \leq |V|$ iff there is an <u>injective</u> map $U \to V$)

As $B \neq \emptyset$, B has an element. Choose an element $b_0 \in B$. Consider the map $f : A \to A \times B$ given by $f(a) = (a, b_0)$

Note that f is one to one.

5 b $A =$ a countable, $|B| = |\mathbb{R}|$. Show that $|A \cup B| = |\mathbb{R}|$

As the inclusion $\begin{array}{c} B \to A \cup B \\ x \mapsto x \end{array}$ is one to one, $|B| \leq |A \cup B|$.

So $|\mathbb{R}| \leq |A \cup B|$.

We try to find an injective map $A \cup B \to B$

Choose an infinite countable subset $C = \{c_1, c_2, c_3, \dots, c_n, \dots\}$ of B. (because any infinite set contains a countable infinite subset)

Case I: A is infinite

$\Downarrow$

$$A = \{a_1, a_2, a_3, a_4, \dots, a_n, \dots\}$$

Case II: A is finite (exercise)

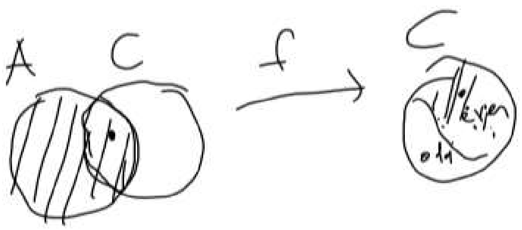
$A \cup C = \overbrace{a_1, a_2, a_3, \cdots, a_n, \cdots}\ \overbrace{c_1, c_2, c_3, \cdots c_n, \cdots}$

$C = c_1, c_2, c_3, \cdots$

Consider the map $f : A \cup C \to C$ given by

$$f(a_k) = c_{2k} \quad \text{and} \quad f(c_k) = c_{2k-1} \text{ if } c_k \notin A$$

$(\text{i.e. if } c_k \in C - A)$



$f(x) \overset{?}{=} f(y)$

$a_1 \to c_2$
$a_2 \to c_4$
$a_3 \to c_6$

$y \notin A$ Why? Exercise

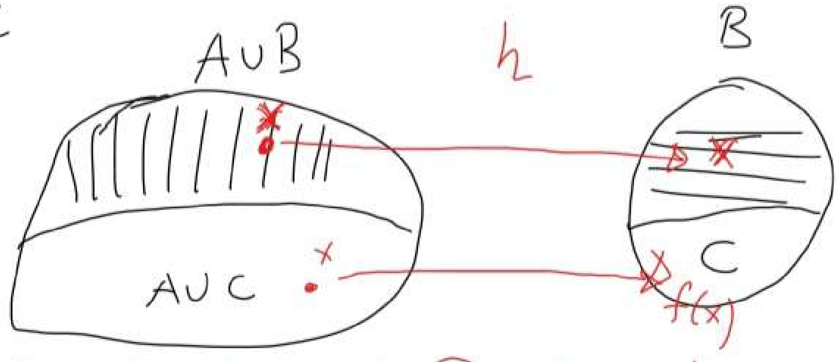
$C \subseteq B$

$h : A \cup B \longrightarrow B$

$A \cup C \longrightarrow C$

Note that

$A \cup B - A \cup C \subseteq B - C$



Note that the map $h : A \cup B \to B$ given by

$$h(x) = \begin{cases} x & \text{if } x \notin A \cup C \\ f(x) & \text{if } x \in A \cup C \end{cases}$$

is injective. So $\underline{\underline{|A \cup B| \leq |B| = |\mathbb{R}|}}$