THOMAS' CALCULUS

MEDIA UPGRADE

Chapter 13

Vector-Valued Functions and Motion in Space

13.1

Vector Functions

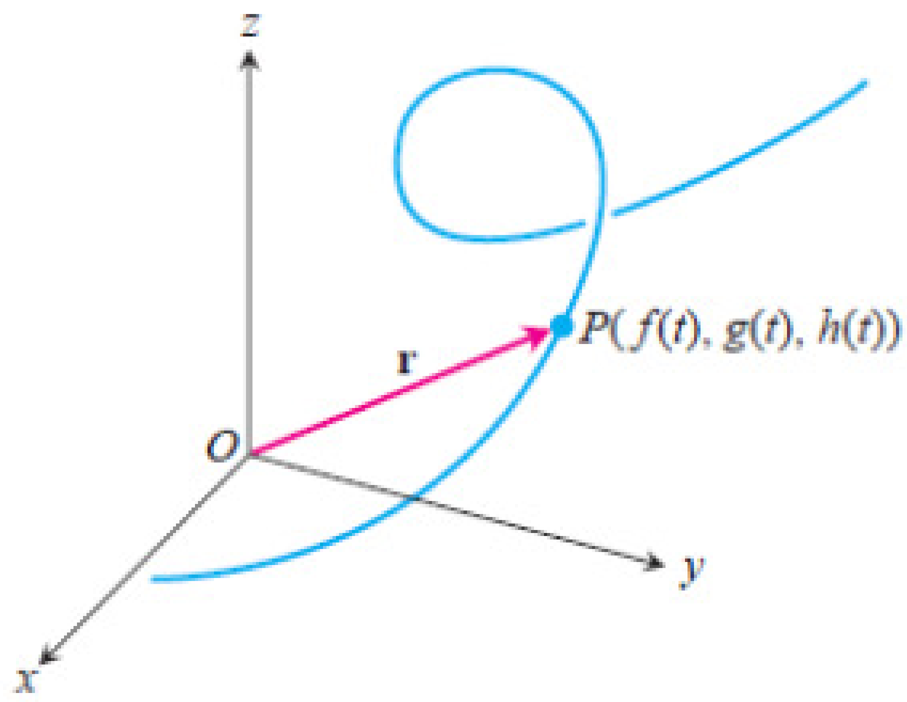


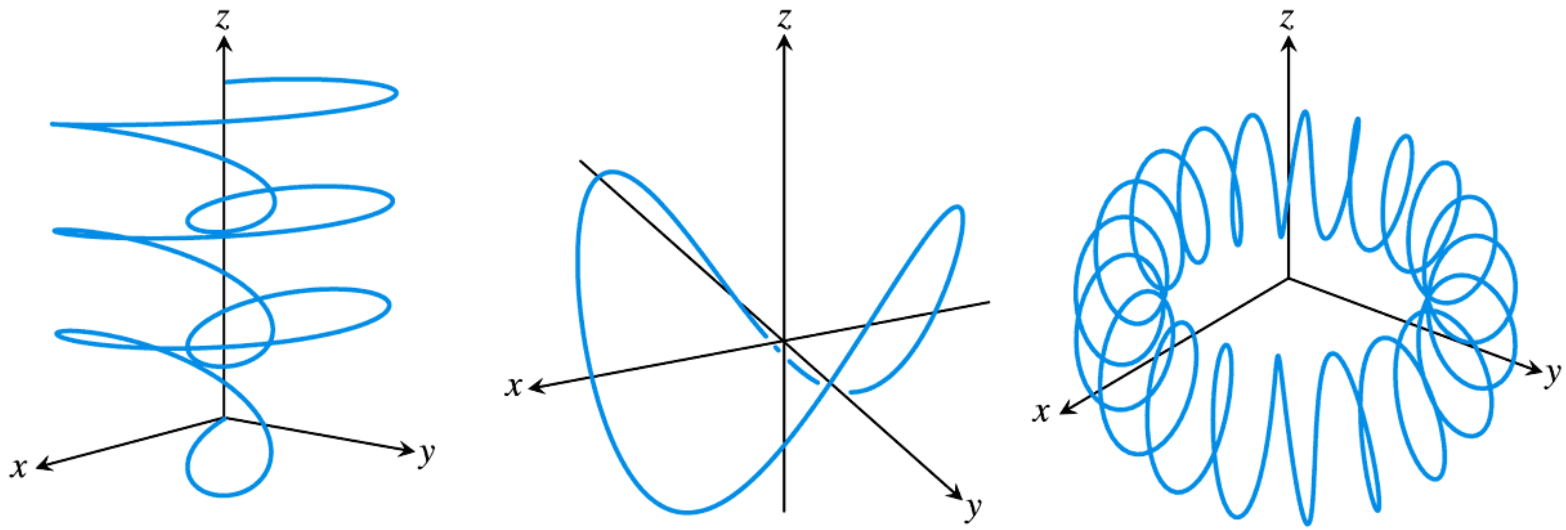
FIGURE 13.1 The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

A particle is moving in $\mathbb{R}^3$ and at time t it is at the point $P(f(t), g(t), h(t))$

i.e. its position is parametrized by t with coordinate functions

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

Then $\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$, $t \longmapsto \langle f(t), g(t), h(t) \rangle = \vec{r}(t)$ is a vector function.



$\mathbf{r}(t) = (\sin 3t)(\cos t)\mathbf{i} + (\sin 3t)(\sin t)\mathbf{j} + t\mathbf{k}$

(a)

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$

(b)

$\mathbf{r}(t) = (4 + \sin 20t)(\cos t)\mathbf{i} + (4 + \sin 20t)(\sin t)\mathbf{j} + (\cos 20t)\mathbf{k}$

(c)

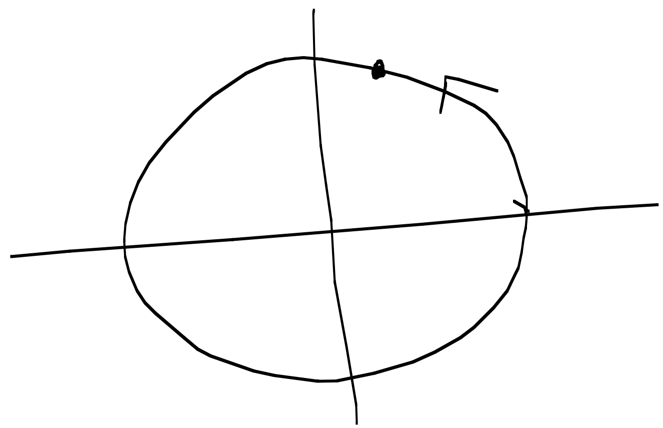
FIGURE 13.2 Computer-generated space curves are defined by the position vectors $\mathbf{r}(t)$.

<u>Ex</u> Consider the unit circle $x^2+y^2=1$

$$\vec{r}(t) = \langle \cos t, \sin t \rangle \qquad t \in [0, 2\pi]$$

$$= (\cos t)\,\vec{i} + (\sin t)\,\vec{j}$$

gives the position vector of a particle moving on the unit circle in counterclockwise direction.



Ex $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ $\mathbb{R} \longrightarrow \mathbb{R}^3$

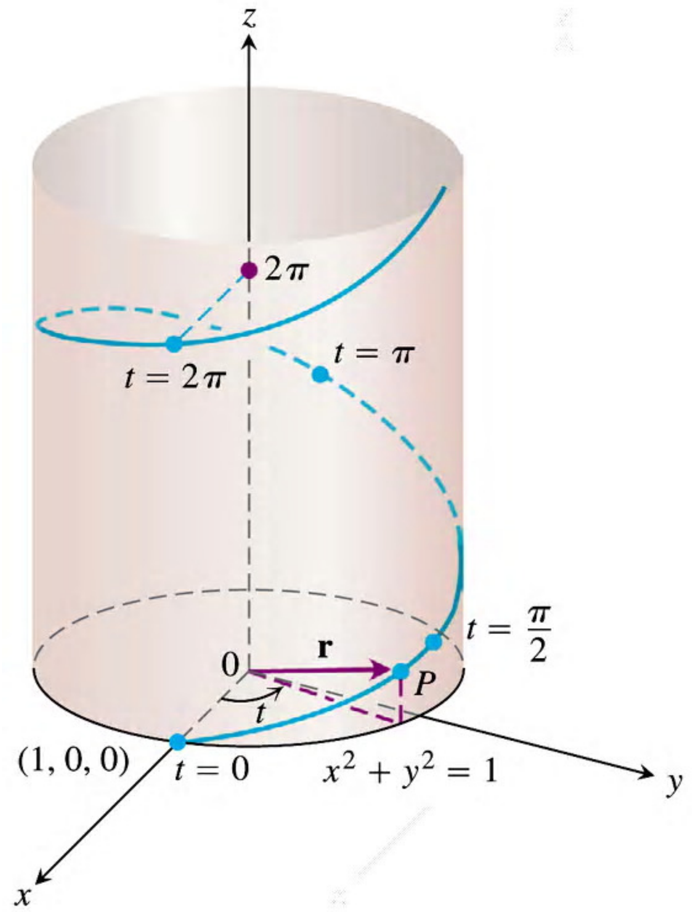
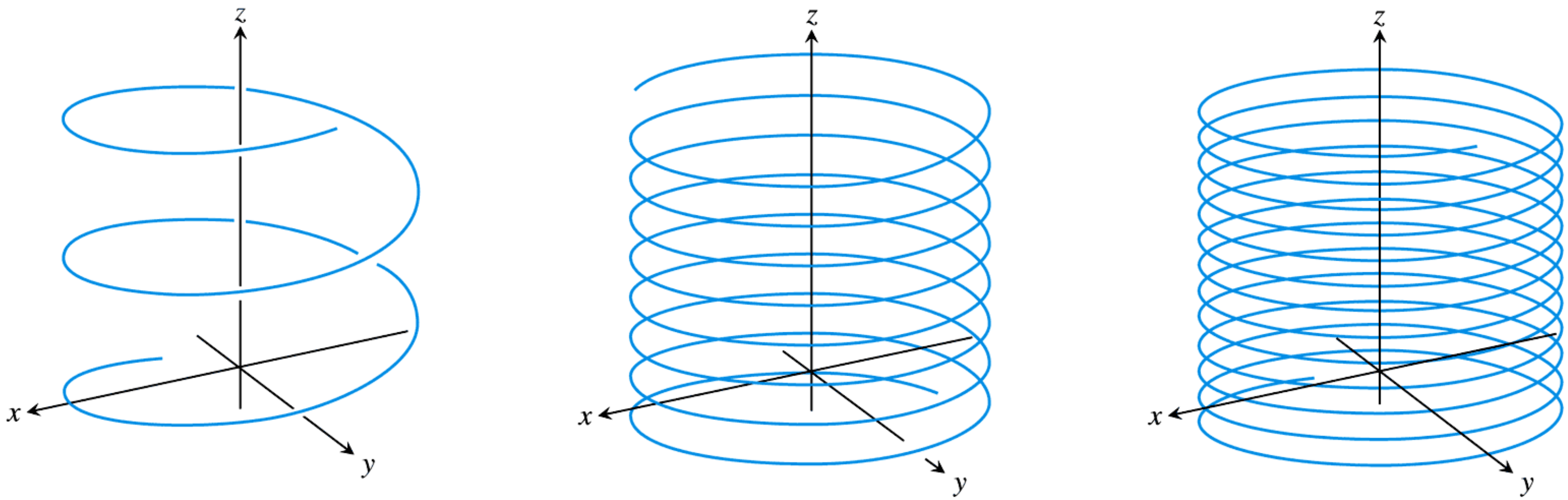


FIGURE 13.3 The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ (Example 1).



$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$ $\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$

FIGURE 13.4 Helices drawn by computer.

DEFINITION **Limit of Vector Functions**

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has **limit** $\mathbf{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all t

$$0 < |t - t_0| < \delta \qquad \Rightarrow \qquad |\mathbf{r}(t) - \mathbf{L}| < \epsilon .$$

EXAMPLE 2 Finding Limits of Vector Functions

If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\begin{aligned}\lim_{t\to\pi/4} \mathbf{r}(t) &= \left(\lim_{t\to\pi/4} \cos t\right)\mathbf{i} + \left(\lim_{t\to\pi/4} \sin t\right)\mathbf{j} + \left(\lim_{t\to\pi/4} t\right)\mathbf{k} \\ &= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}.\end{aligned}$$

DEFINITION **Continuous at a Point**

A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

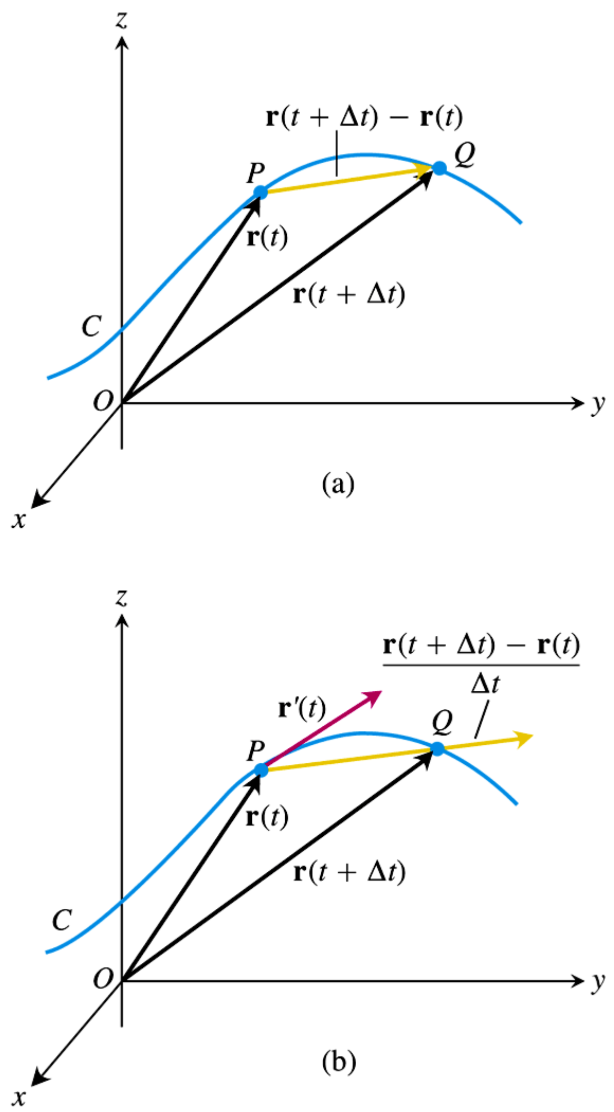


FIGURE 13.5 As $\Delta t \to 0$, the point Q approaches the point P along the curve C. In the limit, the vector $\overrightarrow{PQ}/\Delta t$ becomes the tangent vector $\mathbf{r}'(t)$.

DEFINITION Derivative

The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at *t*** if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

DEFINITIONS **Velocity, Direction, Speed, Acceleration**

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

<u>Ex</u> $\vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ t^2 \rangle$

velocity: $\vec{v} = \dfrac{d\vec{r}}{dt} = \langle -3\sin t,\ 3\cos t,\ 2t \rangle$

speed: $|\vec{v}(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$

$= \sqrt{9 + 4t^2}$

acceleration $\vec{a}(t) = \dfrac{d\vec{v}}{dt} = \langle -3\cos t,\ -3\sin t,\ 2 \rangle$

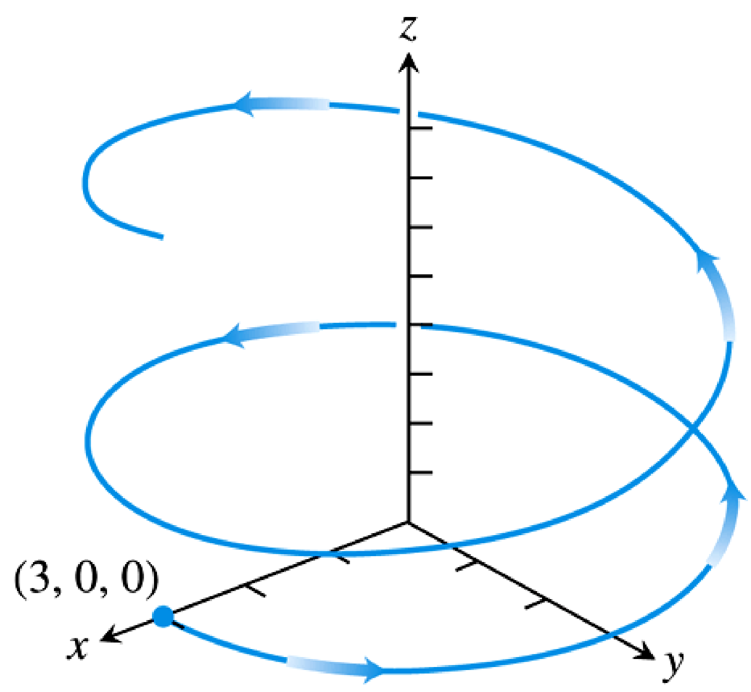


FIGURE 13.7 The path of a hang glider with position vector $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$ (Example 4).

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$

2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$

5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

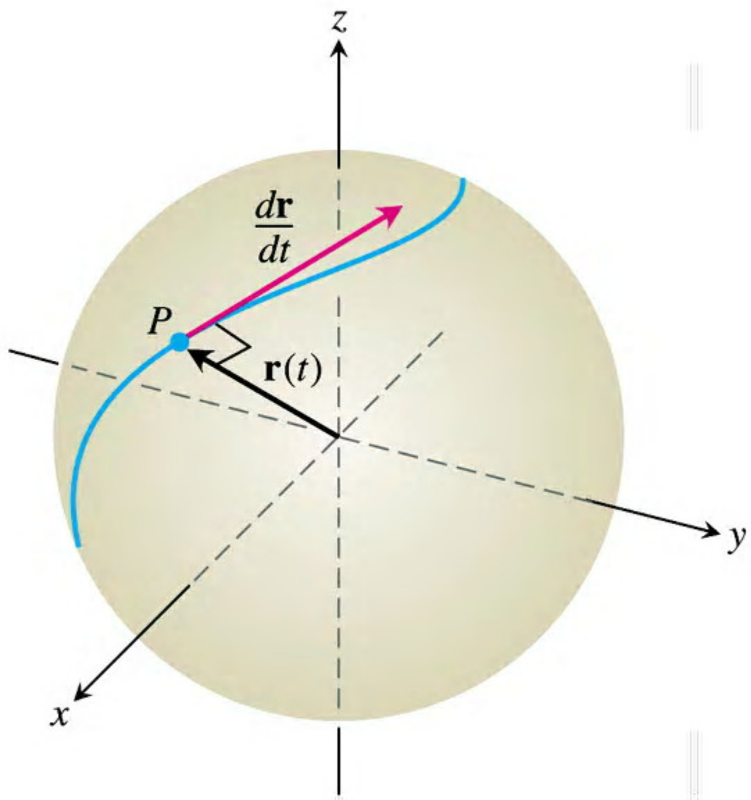


FIGURE 13.8 If a particle moves on a sphere in such a way that its position $\mathbf{r}$ is a differentiable function of time, then $\mathbf{r} \cdot (d\mathbf{r}/dt) = 0$.

If $\mathbf{r}$ is a differentiable vector function of t of constant length, then

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0. \tag{4}$$

(Pf) $|r(t)| = C$: constant

$\Rightarrow |r|^2 = r \cdot r = c^2$ is also constant

Diff. both sides $\vec{r} \cdot \frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt} \cdot r = 0$

$\Rightarrow 2\left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) = 0$

$\square$

EXAMPLE 5 Supporting Equation (4)

Show that $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \sqrt{3}\mathbf{k}$ has constant length and is orthogonal to its derivative.

Solution

$$\begin{aligned}
\mathbf{r}(t) &= (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \sqrt{3}\mathbf{k} \\
|\mathbf{r}(t)| &= \sqrt{(\sin t)^2 + (\cos t)^2 + \left(\sqrt{3}\right)^2} = \sqrt{1 + 3} = 2 \\
\frac{d\mathbf{r}}{dt} &= (\cos t)\mathbf{i} - (\sin t)\mathbf{j} \\
\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} &= \sin t \cos t - \sin t \cos t = 0
\end{aligned}$$

■

DEFINITION Indefinite Integral

The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$

EXAMPLE 6 Finding Indefinite Integrals

$$\int ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\, dt = \left(\int \cos t\, dt\right)\mathbf{i} + \left(\int dt\right)\mathbf{j} - \left(\int 2t\, dt\right)\mathbf{k} \tag{5}$$

$$= (\sin t + C_1)\mathbf{i} + (t + C_2)\mathbf{j} - (t^2 + C_3)\mathbf{k} \tag{6}$$

$$= (\sin t)\mathbf{i} + t\mathbf{j} - t^2\mathbf{k} + \mathbf{C} \qquad \mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} - C_3\mathbf{k}$$

As in the integration of scalar functions, we recommend that you skip the steps in Equations (5) and (6) and go directly to the final form. Find an antiderivative for each component and add a constant vector at the end.

DEFINITION **Definite Integral**

If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

EXAMPLE 7 Evaluating Definite Integrals

$$\begin{aligned}\int_0^{\pi}((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt &= \left(\int_0^{\pi}\cos t\,dt\right)\mathbf{i} + \left(\int_0^{\pi}dt\right)\mathbf{j} - \left(\int_0^{\pi}2t\,dt\right)\mathbf{k}\\ &= \Big[\sin t\Big]_0^{\pi}\,\mathbf{i} + \Big[t\Big]_0^{\pi}\,\mathbf{j} - \Big[t^2\Big]_0^{\pi}\,\mathbf{k}\\ &= [0 - 0]\mathbf{i} + [\pi - 0]\mathbf{j} - [\pi^2 - 0^2]\mathbf{k}\\ &= \pi\mathbf{j} - \pi^2\mathbf{k}\end{aligned}$$

13.3

Arc Length and the Unit Tangent Vector **T**

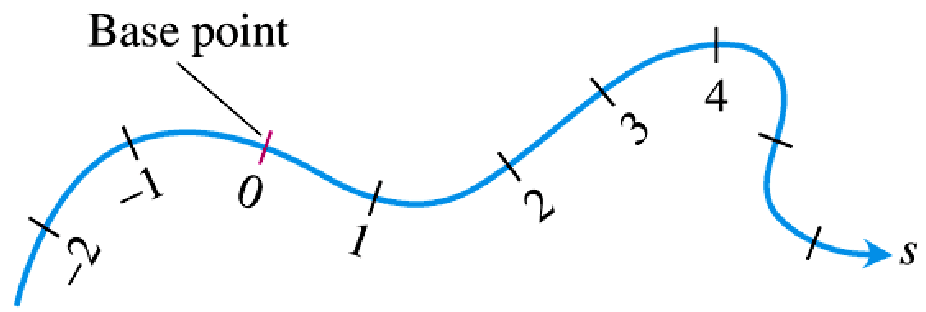


FIGURE 13.14 Smooth curves can be scaled like number lines, the coordinate of each point being its directed distance along the curve from a preselected base point.

DEFINITION **Length of a Smooth Curve**

The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \tag{1}$$

Arc Length Formula

$$L = \int_a^b |\mathbf{v}|\, dt \tag{2}$$

EXAMPLE 1 Distance Traveled by a Glider

A glider is soaring upward along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. How far does the glider travel along its path from $t = 0$ to $t = 2\pi \approx 6.28$ sec?

Solution The path segment during this time corresponds to one full turn of the helix (Figure 13.15). The length of this portion of the curve is

$$\begin{aligned} L &= \int_a^b |\mathbf{v}|\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2} \text{ units of length.} \end{aligned}$$

This is $\sqrt{2}$ times the length of the circle in the xy-plane over which the helix stands.

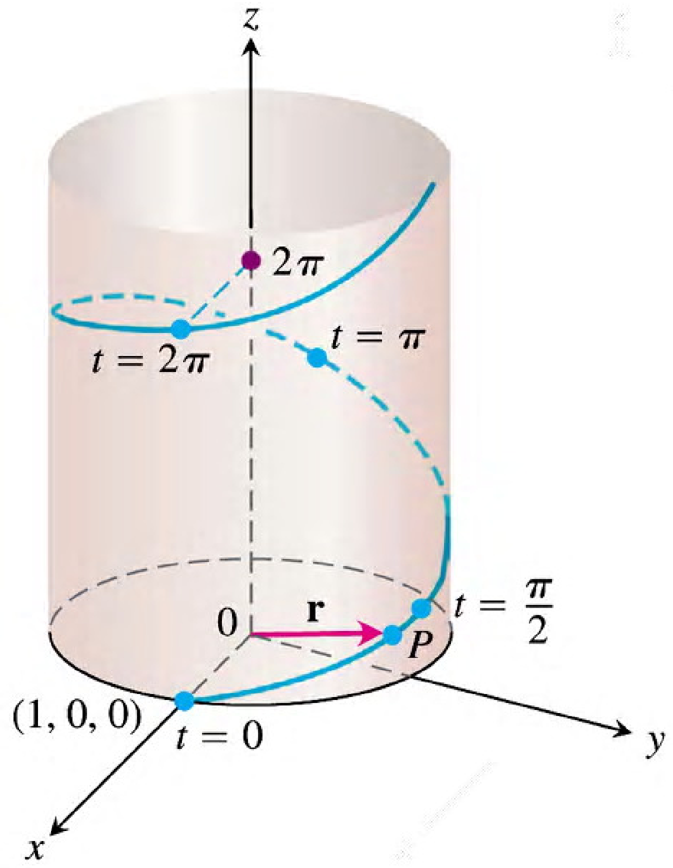
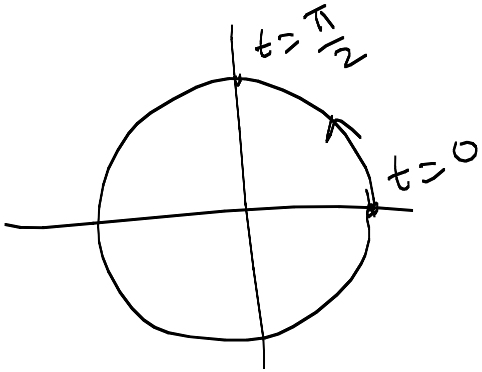


FIGURE 13.15 The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ in Example 1.

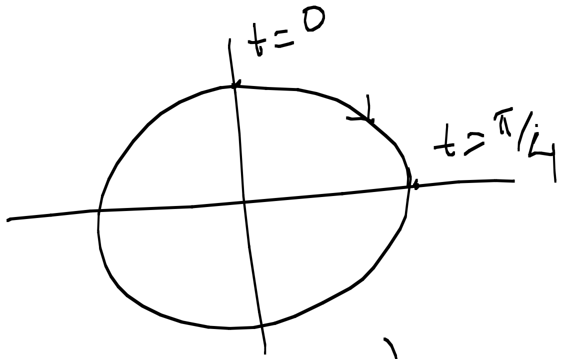
Remark A curve may have different param.

Consider $x^2+y^2=1$



$x=\cos t$
$y=\sin t$
$t\in[0,2\pi]$

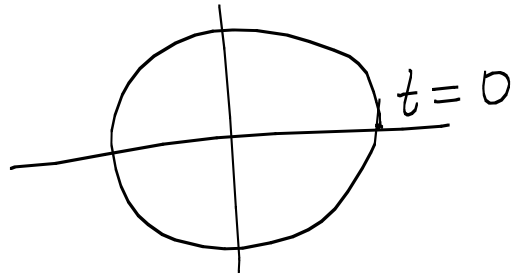
$|v|=1$



$x=\sin(2t)$
$y=\cos(2t)$
$t\in[0,2\pi]$

(2 turns in clockwise dir.)

$|v|=2$



$x=\cos(5t)$
$y=\sin(5t)$
$t\in[0,2\pi]$

(5 turns in count. clock)

$|v|=5$

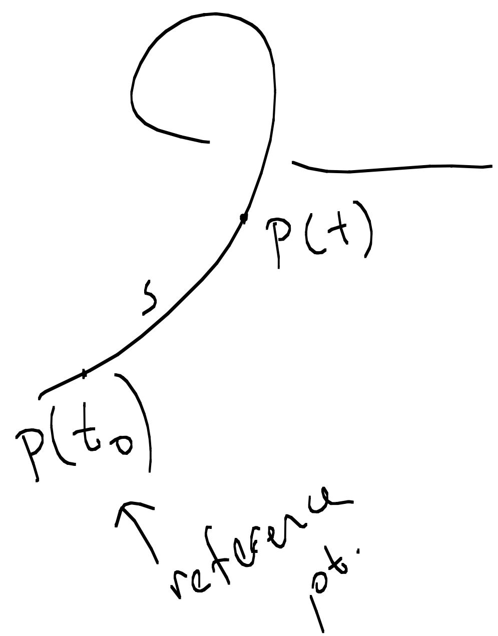
__Aim:__ Find a parametrization of $\vec{r}(t)$ so that $|v| = 1$

In this case $\left(\begin{array}{c}\text{distance taken} \\ \text{by the particle}\end{array}\right) = \left(\begin{array}{c}\text{time} \\ \text{passed}\end{array}\right)$

We must have the distance as our parameter

$$s(t) = \int_{t_0}^{t} |v(\tau)| \, d\tau$$

is the arclength parameter



τ : tau

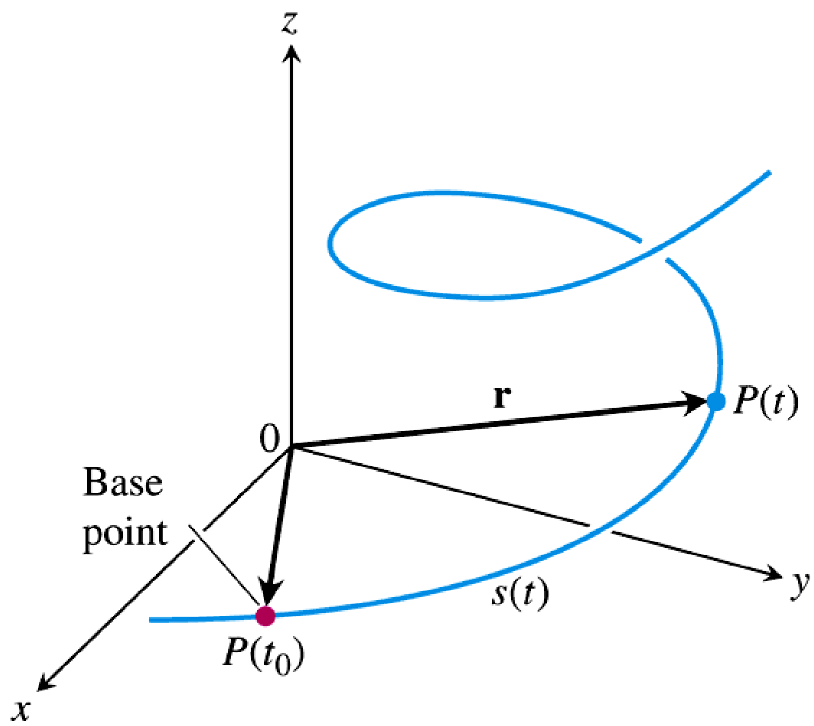


FIGURE 13.16 The directed distance along the curve from $P(t_0)$ to any point $P(t)$ is

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau .$$

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau \qquad (3)$$

EXAMPLE 3 Distance Along a Line

Show that if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the arc length parameter along the line

$$\mathbf{r}(t) = (x_0 + tu_1)\mathbf{i} + (y_0 + tu_2)\mathbf{j} + (z_0 + tu_3)\mathbf{k}$$

from the point $P_0(x_0, y_0, z_0)$ where $t = 0$ is t itself.

Solution

$$\mathbf{v} = \frac{d}{dt}(x_0 + tu_1)\mathbf{i} + \frac{d}{dt}(y_0 + tu_2)\mathbf{j} + \frac{d}{dt}(z_0 + tu_3)\mathbf{k} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \mathbf{u},$$

so

$$s(t) = \int_0^t |\mathbf{v}|\, d\tau = \int_0^t |\mathbf{u}|\, d\tau = \int_0^t 1\, d\tau = t. \qquad \blacksquare$$

With arclength parameter $|v| = 1$

$$|v(s)| = \left|\frac{dr}{ds}\right| = \left|\frac{dr/dt}{ds/dt}\right| = \frac{|v(t)|}{|v(t)|} = 1$$

$\Rightarrow$ $\dfrac{d\vec{r}}{ds}$ is a unit vector called the unit tangent vector

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|} \quad : \text{ unit tangent vector}$$

DEFINITION Unit Tangent Vector

The **unit tangent vector** of a smooth curve $\mathbf{r}(t)$ is

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{v}}{|\mathbf{v}|}. \tag{5}$$

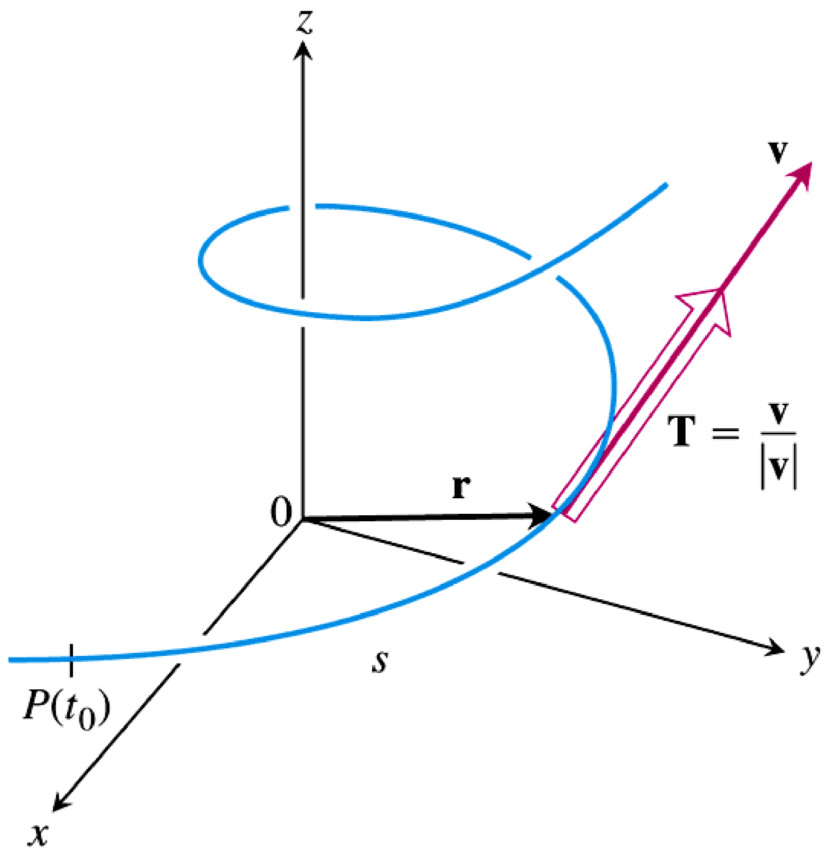


FIGURE 13.17 We find the unit tangent vector $\mathbf{T}$ by dividing $\mathbf{v}$ by $|\mathbf{v}|$.

EXAMPLE 4 Finding the Unit Tangent Vector $\mathbf{T}$

Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$$

representing the path of the glider in Example 4, Section 13.1.

Solution In that example, we found

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k}$$

and

$$|\mathbf{v}| = \sqrt{9 + 4t^2}.$$

Thus,

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3\sin t}{\sqrt{9+4t^2}}\mathbf{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9+4t^2}}\mathbf{k}.$$

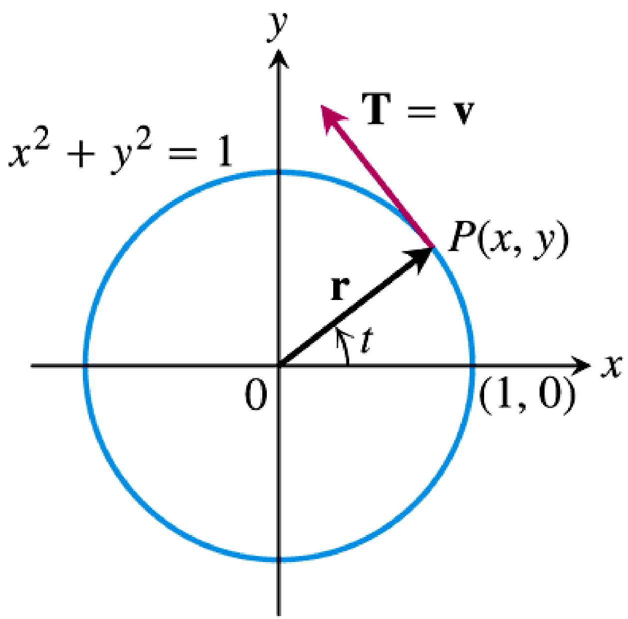


FIGURE 13.18 The motion $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ (Example 5).

Some Past Exam Questions ↓ Ch 12 & Ch 13

b) Find the length of the curve defined by the position vector

$$\vec{r}(t) = (e^t \cos t)\,\vec{i} + (e^t \sin t)\,\vec{j} + e^t\,\vec{k}\,, \quad -\ln 4 \le t \le 0\,.$$

[10p] b) $\vec{r}(t) = (1+t, -t^2, 1+t^3)$ eğrisinin $t = 1$ noktasındaki teğet doğrusunun **xy**-düzlemiyle kesiştiği noktayı bulunuz.

[10p] **c)** Find the length of the curve defined by the position vector
$\vec{r}(t) = (\sin t - t\cos t)\,\vec{i} + (\cos t + t\sin t)\,\vec{j} + t^2\,\vec{k}$ for $0 \leqslant t \leqslant 2$.

2

[15p] a) $L_1 : x = 2 - t,\quad y = -1 + 2t,\quad z = 1 + t,\quad -\infty < t < \infty$
$L_2 : x = -1 + 3s,\quad y = -2 + s,\quad z = -1 + 2s,\quad -\infty < s < \infty$
are given. Find the point of intersection of L_1 and L_2. Then, find the equation of the plane determined by these lines.

[10p] **b)** Find the distance from the point $(1, -2, 4)$ to the plane $2x + y - 3z = 12$.

b) Given two lines

$$\mathbf{L_1} : x = 1 + t\ ,\ y = 1 - t\ ,\ z = 2t\ ,\ -\infty < t < \infty$$

$$\mathbf{L_2} : x = 2 - s\ ,\ y = s\ ,\ z = 2\ ,\ -\infty < s < \infty$$

i) Find the intersection point of $\mathbf{L_1}$ and $\mathbf{L_2}$.

ii) Find the equation of the plane that contains these lines.

a) Show that the line $\mathbf{L_1 : x = 1 + 2t,\ y = 1 - t,\ z = 2t;\ -\infty < t < \infty}$ is perpendicular to the line $\mathbf{L_2 : x = 1 + 3s, y = 4s,\ z = 1 - s;\ -\infty < s < \infty}$.

b) Find the length of $\vec{\mathbf{r}}(\mathbf{t}) = \cos(\mathbf{t^2})\,\vec{\mathbf{i}} + \sin(\mathbf{t^2})\,\vec{\mathbf{j}} + \frac{\mathbf{1}}{\mathbf{6}}\,(\mathbf{4t+1})^{\mathbf{3/2}}\vec{\mathbf{k}}$ for $\mathbf{0 \le t \le 1}$.

[**12p**] **a)** The curve $\vec{\mathbf{r}}(\mathbf{t}) = (-\mathbf{t^2} + \mathbf{1})\,\vec{\mathbf{i}} + (\mathbf{t^3} + \mathbf{2})\,\vec{\mathbf{j}} + \mathbf{4\,t}\,\vec{\mathbf{k}}$ is given. Find the equation of the plane intersecting the curve at $\mathbf{t = 2}$ and perpendicular to the line that is tangent to the curve at $\mathbf{t = 1}$.

[**13p**] **b)** Find the distance from the point $\mathbf{P\,(1, -3, 6)}$ to the plane $\mathbf{2x - 3y + 6z = 12}$.

[**08p**] **b)** Find the distance from the point $\mathbf{A(2, -2, 1)}$ to the plane $\mathbf{3x + y - 2z = 4}$.

[**10p**] **c)** Find the length of the curve $\vec{\mathbf{r}}(\mathbf{t}) = \ln(\cos(\mathbf{t}))\,\vec{\mathbf{i}} + (\tan(\mathbf{t}) - \mathbf{t})\,\vec{\mathbf{k}}$ for $\mathbf{0 \le t \le \frac{\pi}{3}}$.