MAT281E Homework 3. Due December 29, 2021. Late submissions will not be graded.

1. What is $\left\| proj_{\underline{u}\times\underline{v}}(\underline{u}) \right\|$?

2. Find $\|\underline{u}\times\underline{v}\|$ if $\|\underline{u}-\underline{v}\|=\sqrt{3}$ and $\|\underline{u}\|=\|\underline{v}\|=1$

3. Given that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty}a^{-n}=\frac{a}{1-a}$, $|a|<1$ determine an reasonable upper bound to $\sum_{n=1}^{\infty}\frac{e^{-n}}{n}$ by applyin g the Cauchy Schwartz inequality. Treat the sequences as infinite dimensional vectors.

4. Can you represent $(3\ \text{-}1\ 1\ 4)$ as a linear combination of . $(1\ 1\ 2\ 1)$, $(1\ \text{-}2\ \text{-}2\ 0)$, $(0\ 1\ 1\ \text{-}2)$? If so, find the coefficients(weights).

5. Find the domain, codomain, range and the standard matrix for the linear transformation defined by the equations
$$w_1=-x_1-x_2+x_3$$
$$w_2=2x_1-x_2-3x_3$$
$$w_3=-3x_2-x_3$$
(Hint: For range, apply row reduction and read the constraint (if any) on w_i in the row echelon form)

6. Use matrix multiplication to find
   a. The reflection of $(1\ 1\ 0)$ about the z-axis
   b. The orthogonal projection of $(1\ \text{-}3\ 1)$ onto the $x=-z$ plane
   c. The orthogonal projection of $(1\ \text{-}1\ 5)$ onto the x axis followed by reflection about the x axis
   d. The reflection around $z=y$ plane of $(1\ \text{-}2\ 3)$, followed by rotation around the x-axis by 60°, followed by dilation with factor 2. What is the Standard matrix for the stated composition?
   e. The dilation with factor of 2 of $(1\ \text{-}2\ 3)$ followed by rotation around x axis by 60°, followed by orthogonal projection onto the y-axis followed by contraction with factor ½. What is the Standard matrix for the stated composition?

7. Let T_1 be the orthogonal projection onto the z axis, T_2 be the rotation with respect to x axis by θ. Is it true that $T_1\circ T_2=T_2\circ T_1$? Explain.

8. For each part below:Is the linear transformation given below one-to-one or many-to-one? What is the inverse transformation? Is the range a subset of the codomain?

a) $w_1=x_1+x_2-x_3$, $w_2=2x_1-3x_2-x_3$, $w_3=5x_1+6x_2-4x_3$

b) $w_1=x_1-3x_2-x_3$, $w_2=x_1+x_2-x_3$, $w_3=x_1-x_3$

9. Let $T(x_1,x_2)=(2x_1+x_2,1-x_1)$. Is the transformation linear? Explain.

10. Let matrix $\underline{\underline{A}}_{4x4}$ govern the affine transformation $A:\Re^3\rightarrow\Re^3$. Show that $\underline{\underline{A}}_{4x4}=\begin{bmatrix}\underline{\underline{L}} & \underline{t}\\ 0...0 & 1\end{bmatrix}$

where $\underline{\underline{L}}$ is the the standard matrix of a linear transformation and $\underline{t}$ is a translation vector.

11. Let a linear transformation $T:\Re^3 \rightarrow \Re^3$ orthogonally project a vector onto the x-z plane followed by rotation around the y-axis by 60°. Determine the standard matrix for this transformation from the images of the standard basis vectors.

12. Let T_1 and T_2 be two linear transformations where the domain and the codomain are $\Re^3$. If T_1 is many-to-one and T_2 is one-to-one, are the standard matrices for the compositions $T_1 \circ T_2$, $T_2 \circ T_1$ invertible? Explain.

13. Is the set of all pairs of real numbers of the form (x, y, z) a vector space if

    a. $x+z=2y$

    b. $x^2+y^2+z^2=0$

    c. $x-y=1$

    d. $x+y<1$

    Explain.

14. Is the set of all 2x2 matrices of the form $\begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix}$ a subspace of the set of all 2x2 matrices?

    Explain. Repeat for $\begin{bmatrix} a & b \\ ab & 0 \end{bmatrix}$, and $\begin{bmatrix} a & b \\ a+b & 1 \end{bmatrix}$.

15. Do the set of functions with a discontinuity at the origin form a subspace?

16. Do the set of invertible matrices with dimension nxn form a subspace of the set of matrices with dimension nxn? Do the set of singular matrices with dimension nxn form a subspace of the set of matrices with dimension nxn? Explain.

17. For each part below describe the solution space (subspace) of the homogeneous system $\underline{\underline{A}}\underline{x}=\underline{0}$

    a) $\underline{\underline{A}}=\begin{bmatrix} 1 & 0 & -2 & 2 \\ 2 & 1 & 4 & -4 \\ 1 & -1 & 3 & 0 \end{bmatrix}$ b) $\underline{\underline{A}}=\begin{bmatrix} 1 & 0 & -2 & 2 \\ 2 & 1 & 4 & -4 \end{bmatrix}$ c) $\underline{\underline{A}}=\begin{bmatrix} 1 & 0 & -2 & 2 \\ 2 & 1 & 4 & -4 \\ 0 & 1 & 6 & -6 \end{bmatrix}$ d) $\underline{\underline{A}}=\begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & -4 \\ 0 & 1 & 6 \\ -1 & 0 & 8 \end{bmatrix}$

    Specifically find equation(s) that describe the subspace

18. Let $\underline{v}_1=(43\ 21), \underline{v}_2=(-2\ 0\ -10), \underline{v}_3=(0\ 3\ 01)$

    a. Find the subspace of $\Re^4$ spanned by these vectors.

    b. Are the vectors linearly independent?

    c. Determine a basis for $\Re^4$ that contains as many of the above three vectors as possible.

19. Determine whether the following functions are linearly dependent or not by applying the Wronskian.

    $f_1(x)=\cos 2x \quad f_2(x)=\cos^2 x-\sin^2 x$,

20. Determine a basis set, express the other vectors in terms of the basis vectors. What is the dimension of the subspace spanned by these vectors?

    $\underline{u}_1=(1\ -2\ 0)\ \underline{u}_2=(1\ 2\ 4)\ \underline{u}_3=(1\ 0\ -1)\ \underline{u}_4=(0\ 0\ 7)$

21. Suppose we have a set of 4 linearly dependent vectors in $\Re^4$. Is it possible to get a smaller linearly independent set by removing one or more vectors to from the set? Is this a basis for $\Re^4$?.

22. Suppose we have a set of 4 linearly independent vectors in $\Re^5$. Is it possible to get a larger linearly independent set by adding one or more vectors to this set? Explain.

23. If $\underline{u}$, $\underline{v}$ and $\underline{w}$ are linearly independent vectors other than zero vector, are the vectors $\underline{u}, \underline{v}, proj_{\underline{u}\times\underline{v}}\,\underline{w}$ necessarily linearly independent? Explain. Note that any set containing the zero vector is linearly dependent.

24. Consider a set of five arbitrary 2x2 matrices. Can you always write one as a linear combination of the others? Explain. Repeat for five arbitrary 3x3 matrices.

25. For each of the following sets explain whether or not the set is/could be a basis for the space mentioned.

a) Four vectors in $\Re^6$ that form a loop when end-to-end connected.

b) $(-x, y)$ and $(x, -y)$ for any $x \neq 0 \;\; y \neq 0$ in $\Re^2$.

c) $(1\;\;2\;\;3)$, $(1\;\;2\;\;0)$, $(-1\;\;2\;\;6)$ in $\Re^3$

d) Any set of vectors that includes the zero vector in $\Re^n$.

26. Find the coordinates of $\underline{w}$ relative to the basis $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ by solving a linear system.

$\underline{w} = (1\;\;1\;\;1)$, $\underline{v}_1 = (2\;\;-2\;\;0)$, $\underline{v}_2 = (0\;\;1\;\;-1)$, $\underline{v}_3 = (1\;\;5\;\;4)$

27. Let $\underline{w} = (2\;\;2\;\;0)$, $\underline{v}_1 = (2\;\;-1\;\;-1)$, $\underline{v}_2 = (0\;\;1\;\;3)$, $\underline{v}_3 = (1\;\;-1\;\;-2)$, $\underline{v}_4 = (1\;\;0\;\;1)$,

a) Do the vectors $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ form a linearly independent set?

b) Determine a suitable subset of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ as a basis for the space spanned by $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$.

c) What is the dimension of the space spanned by $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$?

d) Describe the span of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ based on your answer to b) and c).

e) Is vector $\underline{w}$ in the span of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$?

28. Can you determine a basis for the plane $x - y + z = 2$ that spans only the plane?

29. Let $(w, x, y, z) \in \Re^4$. Determine a basis for the subspace of $\Re^4$ formed by the intersection of the plane $x + y - w + z = 0$ with the plane $2x - y + 2w - z = 0$? What is the dimension of this subspace?

30. Can you determine a basis for the line described as $x = 3t - 1, y = t - \dfrac{1}{3}, z = 1 - 3t$ that spans only the line?

31. Determine a basis from the following set of second order polynomials. What is the dimension of the (sub)space that it spans? Does this basis span the space of the second order polynomials? If no, then augment the basis by adding suitably chosen polynomial(s) so that the enlarged set spans the space of the second order polynomials.

$$p_1(x) = x^2 + 1,\; p_2(x) = x^2 - 2x + 3,\; p_3(x) = x - 1,\; p_4(x) = 3x^2 + 2x + 1$$

(Hint: Use the standard basis for the space of second order polynomials. Work with coordinate vectors written relative to this basis)