Quantifiers (First Order Logic)

Propositional logic (i.e., assertions without quantifiers) is not enough to symbolize some commonly used statements in Math. Consider for instance the following deduction, which is valid.

"Ayşe is a mathematician. All mathematicians study calculus. Thus, Ayşe studies calculus"

let us try to symbolize it in Proposition Logic. We should represent each component assertions by some capital letters. Let us call

A= Ayşe is a mathematician, B= All mathematicians study calculus, C= Ayşe studies calculus

So we symbolize the given deduction as

$$\underbrace{A, B}_{\text{hypotheses}},\ \therefore \underbrace{C}_{\text{conclusion.}}$$

We easily see that this symbolized deduction is invalid (A = true, B = true, C = false is a counter example). Although the sentences represented by the letters A, B, C are related to eachothers, these relations can't be seen when the sentences are replaced by the letters A, B, C. This problem occurs when sentences contain quantifiers "all" and "some". Furthermore, there are many definitions and results in Math containing "all" and "some". Consider for instance the limit definition in calculus.

By a <u>set</u> we mean a (well defined) collection of objects, these objects are called the <u>elements</u> of the set (or the <u>members</u> of the set) ; if A denotes a set and if a denotes an element of A, we write $a \in A$ to indicate this ; we use the symbol $a \notin A$ for the negation of $a \in A$. For instance, if $A = \{1, 2, \square, \bigstar\}$ then $1 \in A$, $\square \in A$, $\sqrt{2} \notin A$, $\mathbb{G} \notin A$. We will study Sets in coming lectures. A set without any elements is called the empty set and it is denoted by ϕ.

Open statement, predicate, variable

A declarative sentence is called an <u>open statement</u> if the following three conditions hold:

(1) It contains one or more variables

(2) It is not an assertion

(3) It becomes an assertion when the variables in it are replaced by certain allowable choices.

The allowable choices are usually members of a set, which is called the universe of the variable.

Ex: "The number $x+2$ is an even integer" This is an open statement, x is the variable of it, for x we may allow for instance all integers (so the set of all integers is the universe of the variable). If we put for instance $x=1$, the open statement becomes "The number 3 is an even integer" which is an assertion whose truth value is false. An open statement is also called a predicate.

We usually denote open sentences by the notations $P(x)$, $Q(x)$, $P(x,y)$, $x \mathrel{L} y$, ...

Ex: $P(x)$: The number x is odd	$P(2)$, a false assertion
$Q(x,y)$: The number x is greater than y	$Q(2,1)$, a true assertion
$x \mathrel{L} y$: The number $x-y$ is positive	$2 \mathrel{L} 3$, a false assertion

Universal Quantifier, $\forall$

The symbol "$\forall x$" means "for all x". If the variable x is allowed to be in a set A, we sometimes use the symbol "$\forall x \in A$". The symbol "$\forall x \in A$" means "for all x in A". If the symbols "$\forall x$" and "$\forall x \in A$" are followed by an open statement $P(x)$ with variable x, then they become assertions. That is, $\forall x\, P(x)$ and $(\forall x \in A)\, P(x)$ are both assertions (i.e., they are either true or false but not both). These assertions may also be written as $\forall x, P(x)$ and $\forall x \in A, P(x)$ with commas (The textbook prefers the second)

$\forall x, P(x)$ means any of the following:	$\forall x \in A, P(x)$ means any of the following:
for all x, $P(x)$	for all x in A, $P(x)$
for any x, $P(x)$	for any x in A, $P(x)$
for each x, $P(x)$	for each x in A, $P(x)$
for every x, $P(x)$	for every x in A, $P(x)$

The truth value of $\forall x, P(x)$ is given by

assertion	When is it true	When is it false
$\forall x, P(x)$	For every replacement (of the variable x by) a from the universe, $P(a)$ is true	There is at least one replacement (of the variable x by) a from the universe for which $P(a)$ is false
$\forall x \in A, P(x)$	$P(a)$ is true for all a in A	There is at least one a in A for which $P(a)$ is false

Existential Quantifier, $\exists$

The symbol "$\exists x$" means "there exists some x such that". If the variable x is allowed to be in a set A, we sometimes use the symol "$\exists x \in A$". The symbol "$\exists x \in A$" means "there exists some a in A such that". If the symbols "$\exists x$" and "$\exists x \in A$" are followed by an open statement $P(x)$ with variable x, they become assertions (i.e., sentences which are either true or false but not both). That is, $\exists x\, P(x)$ and $(\exists x \in A)\, P(x)$ are both assertions. We may also use notations $\exists x, P(x)$ and $\exists x \in A, P(x)$ with commas to denote them (The textbook prefers the notation).

$\exists x, P(x)$ means any of the following:

- for some x, $P(x)$
- for at least one x, $P(x)$
- there exists an (some) x such that $P(x)$
- there is an (some) x such that $P(x)$

$\exists x \in A$, $P(x)$ means any of the following:

- for some x in A, $P(x)$
- for at least one x in A, $P(x)$
- there exists an (some) x in A such that $P(x)$
- there is an (some) x in A such that $P(x)$

The truth value of $\exists x, P(x)$ is given by

assertion	When is it true	When is it false
$\exists x, P(x)$	For some (at least one) a in the universe, $P(a)$ is true	For every a in the universe, $P(a)$ is false
$\exists x \in A, P(x)$	There is at least one element a in A for which $P(a)$ is true	For every element a in A, $P(a)$ is false

As in Propositional Logic, two quantified assertions $\mathcal{A}$ and $\mathcal{B}$ are called <u>logically equivalent</u> if they have the same truth values (equivalently, if $\mathcal{A} \longleftrightarrow \mathcal{B}$ is a tautology) We use the notation $\mathcal{A} \equiv \mathcal{B}$ to indicate that $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent.

<u>Fact</u>: (When there are naturally more allowable choices for the variable x than the elements of a set A), we have the following logical equivalences:

$$\forall x \in A,\ P(x) \equiv \forall x\ (x \in A \to P(x))$$

$$\exists x \in A,\ P(x) \equiv \exists x\ (x \in A \wedge P(x))$$

<u>Proof</u>: Consider the first equivalence. We will justify that the LHS and the RHS have the same truth values. Suppose that the LHS is true. Take any a from the universe. We have two cases to consider either $a \in A$ or $a \notin A$. If $a \in A$, then $P(a)$ is true (because the LHS is true) and so "$a \in A \to P(a)$" is true If $a \notin A$, then "$a \in A$" is false and the implication "$a \in A \to P(a)$" is true. So we have justified that "$a \in A \to P(a)$" is true for any element a of the universe. Hence the RHS is true. Suppose now the LHS is false Then there is a $c \in A$ such that $P(c)$ is false. Then "$c \in A \to P(c)$" is false, implying that the RHS is false.

Consider the second equivalence. If the LHS is true, then there is an element a in A such that $P(a)$ is true, and so for this a both "$a \in A$" and "$P(a)$" are true implying that "$a \in A \wedge P(a)$" is true for some element a in A, thus the RHS is true. Consider now the case for which the LHS is false. Choose any element b from the universe of the variable. There are two possibilities: $b \in A$ or $b \notin A$. If $b \in A$, then $P(b)$ is false (because the LHS is false) and so "$b \in A \wedge P(b)$" is false. If $b \notin A$, then "$b \in A$" is false and so "$b \in A \wedge P(b)$" is false. Consequently, we have observed that "$b \in A \wedge P(b)$" is false for every b in the universe. This means that the RHS $\exists x\ (x \in A \wedge P(x))$ is false.

<u>Ex</u>: "There is a book on my table which is red"

"There is a book which is on my table and red"

It is clear that both sentences have the same meaning. Let us write them in the notations of the First Order Logic. Consider the symbolization key

$P(x) =$ x is a red book ; $A =$ the set of all books on my table.

The first sentence becomes "$(\exists x \in A)\ P(x)$", the second sentence becomes "$\exists x\,(x \in A \wedge P(x))$"

<u>Ex</u>: Let A be any set of integers. Consider the following assertions:

"For all x in A, x is an even integer" and "For all integer x, if $x \in A$ then x is even"

It is clear that both sentences have the same meaning. Letting $P(x) =$ x is an even integer, we may symbolize them as "$(\forall x \in A) P(x)$" and "$\forall x\,(x \in A \to P(x))$"

<u>Ex</u>: (1) Write "the square of any real number is nonnegative" in symbolic form:

The given sentence may be written as "$\underbrace{\text{for any real number } x}_{\forall x \in \mathbb{R}}$, $\underbrace{x^2 \text{ is nonnegative}}_{= P(x)}$"

the set of real numbers

(2) Write "$n^2 < 2^n$ for all natural numbers $n \geq 5$" in symbolic form:

$$(\forall n \in \mathbb{N})\ (n \geq 5 \longrightarrow n^2 < 2^n)$$

the set of natural numbers

$$\forall n\left(\left[(n \in \mathbb{N}) \wedge (n \geq 5)\right] \longrightarrow n^2 < 2^n\right)$$

$$\forall n\left(n \in \mathbb{N} \to \left[n \geq 5 \to n^2 < 2^n\right]\right)$$

<u>Ex</u>: Consider the following open statements and suppose that the universe of the variable is the set of all real numbers.

$P(x): x \geq 1$, $Q(x): x^2 \geq 1$, $R(x): x^2 - 2x - 3 = 0$, $S(x): x^2 - 9 > 9$

Translate the following into plain English and determine their truth values:

(1) $\exists x, (P(x) \wedge R(x))$

"There is a real number x such that $x \geq 1$ and $x^2 - 2x - 3 = 0$". This is true because $x = 3$ satisfies the conditions.

(2) $\forall x \left(P(x) \to Q(x)\right)$

"For all real numbers x, if $x \geqslant 1$ then $x^2 \geqslant 1$". This is true

(3) $\forall x \left(Q(x) \to S(x)\right)$

"For all real numbers x, if $x^2 \geqslant 1$ then $x^2 - 9 \geqslant 9$" This is false because for instance the real number $x = 2$ does not satisfied the conditions. If "$\forall x$ (   )" is not true, then any specific choice for x making "(   )" false may be called as a counter example.

(4) $\forall x \left(P(x) \vee Q(x)\right)$

"For any real number x, either $x \geqslant 1$ or $x^2 \geqslant 1$". This is false, for instance $x = 1/2$ is a counterexample.

Ex: Let A and B be some sets of integers. Write each of the following in the notations of First Order Logic:

(1) Every element of A is negative:

$(\forall x \in A)(x < 0)$

(2) Every element of A is smaller than any element of B:

$\forall x \in A, \left(\forall y \in B, (x < y)\right)$

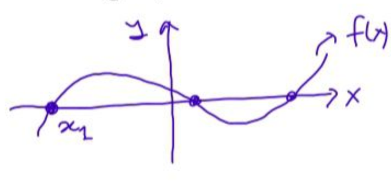
("For all x in A, $x < y$ for all y in B" $\equiv$ "For any $x \in A$ and for any $y \in B$, $x < y$")

(3) There is an element of A which is bigger than every element of B:

$\exists x \in A, \left(\forall y \in B, (x > y)\right)$

"There is an element x of A such that $x > y$ for all y in B"

Ex: Let f be a real valued real function. Symbolize "The graph of f does not intersect the x-axis"



Note that the graph intersects at the x-axis at a point $x = x_1$ iff $f(x_1) = 0$. So the answer is

"$(\forall x \in \mathbb{R})(f(x) \neq 0)$"

<u>Ex</u> Symbolize "175 is not the largest integer"

$(\exists x \in \mathbb{Z})(x > 175)$, "There is an integer x such that $x > 175$"

the set of integers

$\equiv$

$\neg\left((\forall x \in \mathbb{Z})(175 \geqslant x)\right)$ It is not the case that 175 is the largest integer

<u>Ex</u> Symbolize "1/2 is not an integer"

$(\forall x \in \mathbb{Z})(x \neq 1/2)$ "For all integers x, $x \neq 1/2$"

$\equiv$

$\neg\left(\exists x \in \mathbb{Z},\ x = 1/2\right)$ "There is no integer x such that $x = 1/2$"

(i.e., it is not the case that there is an integer x such that $x = 1/2$)

Negation of quantified assertions:

Let A be a set of integers. Consider the following negated quantified assertions:

It is not the case that "every element of A is even" $\equiv$ Some element of A is not even

It is not the case that "for all x in A, $\underbrace{x \text{ is even}}_{\text{"}P(x)\text{"}}$" $\equiv$ There exists an x in A such that x is not even

$$\neg\left((\forall x \in A)\, P(x)\right) \equiv (\exists x \in A)(\neg P(x))$$

The following negation rules for quantified statements are obvious from truth values:

<u>Fact:</u>

$$\neg(\forall x,\ \mathcal{A}) \equiv \exists x,\ \neg\mathcal{A} \qquad \neg\left((\forall x \in A)\,\mathcal{A}\right) \equiv (\exists x \in A)(\neg\mathcal{A})$$

$$\neg(\exists x,\ \mathcal{A}) \equiv \forall x,\ \neg\mathcal{A} \qquad \neg\left((\exists x \in A)\,\mathcal{A}\right) \equiv (\forall x \in A)(\neg\mathcal{A})$$

Note that the negation interchanges the quantifiers $\forall$ and $\exists$. Iterating it we have for instance $\neg\left((\forall x)(\exists y)\, R(x,y)\right) \equiv (\exists x)(\forall y)(\neg R(x,y))$

<u>Ex</u>: Negate and simplify each of the following

(1) $\forall x\,(P(x) \to Q(x))$

$$\neg[\forall x\,(P(x) \to Q(x))] \equiv \exists x\,[\neg(P(x) \to Q(x))] \underset{\substack{\uparrow \\ A\to B \equiv \neg A \vee B}}{\equiv} \exists x\,[\neg(\neg P(x) \vee Q(x))]$$

$$\underset{\substack{\nearrow \\ \text{De Morgan's Laws} \\ \neg(A\vee B) = \neg A \wedge \neg B}}{\equiv} \exists x\,[\neg(\neg P(x)) \wedge \neg Q(x)] \underset{\substack{\uparrow \\ \text{Double Negation} \\ \neg\neg A \equiv A}}{\equiv} \exists x\,(P(x) \wedge \neg Q(x))$$

(2) $\exists x \in \mathbb{R},\ (2 \le x < 5)$

"$2 \le x < 5$" $\equiv$ "$2 \le x$" and "$x < 5$"

$$\neg[\exists x \in \mathbb{R},\ (2 \le x < 5)] \equiv \forall x \in \mathbb{R},\ \neg(2 \le x < 5) \equiv \forall x \in \mathbb{R},\ \neg((2 \le x) \wedge (x < 5))$$

$$\underset{\substack{\nearrow \\ \text{De Morgan's Law}}}{\equiv} \forall x \in \mathbb{R},\ (\neg(2 \le x) \vee \neg(x < 5)) \equiv \forall x \in \mathbb{R},\ ((x < 2) \vee (x \ge 5))$$

(3) $(\forall x \in A)(\exists y \in B)(\forall z \in C)\ (x + y = z)$

$$\neg[(\forall x \in A)(\exists y \in B)(\forall z \in C)\ (x+y=z)] \equiv \exists x \in A,\ (\forall y \in B,\ (\exists z \in C,\ (x+y \ne z)))$$

<u>Ex</u>: Symbolize and negate "There is a man who is the father of every other man" by using the symbolization key $A(x,y) =$ "x is the father of y" where the universe is the set of all men.

"There is a man x such that x is the father of all the other men"

"There is a man x such that, for all man y, if $y \ne x$ then x is the father of y"

$\exists x, (\forall y, ((y \ne x) \to A(x,y)))$. Its negation is

$$\neg[(\exists x)(\forall y)((y \ne x) \to A(x,y))] \overset{\text{"}M\to N \equiv \neg M \vee N\text{"}}{\equiv} \forall x, (\exists y,\ \neg[\neg(y \ne x) \vee A(x,y)])$$

$$\underset{\substack{\swarrow \\ \text{Double Negation} \\ \text{and De Morgan's Laws}}}{\equiv} \forall x, [\exists y,\ (y \ne x \wedge \neg A(x,y))]$$

Its translation to English is

"For all man x, there is a man y such that $y \neq x$ and x is not the father of y"

"For any man there is another man whose father is not him"

Ex Symbolize and negate "All girls are beatiful but some boys are not handsome" by using $A(x)$: "the girl x is beatiful" and $B(y)$: "the boy y is handsome"

$$\forall x\, A(x) \wedge \exists y\, B(y)$$

$$\neg\left[\forall x\, A(x) \wedge \exists y\, B(y)\right] \underset{\substack{\downarrow \\ \text{De Morgan's Law}}}{\equiv} \neg\left(\forall x A(x)\right) \vee \neg\left(\exists y\, B(y)\right) \equiv \exists x(\neg A(x)) \vee \forall y\,(\neg B(y))$$

"Either some girls are not beatiful or all boys are not handsome"

"Either there is a girl who is not beatiful or all boys are not handsome"

"If all girls are beatiful then all boys are not handsome"

$$M \to N \equiv \neg M \vee N$$

Ex (Definition of limit in calculus)

Let f be a real valued real function. Recall from Calculus that we write $\lim_{x \to x_0} f(x) = L$

if "for any $\varepsilon > 0$ there is a $\delta > 0$ such that the following implication holds for all x: $\quad 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$"

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x)\left(0 < |x - x_0| < \delta \to |f(x) - L| < \varepsilon\right)$$

(If, for instance, A is the set of all positive real numbers, then instead of writing $\forall x \in A$ (or $\exists x \in A$) we may write $\forall x > 0$ (or $\exists x > 0$))

Its negation is

$$(\exists \varepsilon > 0)\,(\forall \delta > 0)\,(\exists x)\left(0 < |x - x_0| < \delta \wedge |f(x) - L| \geq \varepsilon\right)$$

where we used the equivalence "$M \to N \equiv \neg M \vee N$". The negation in English is "There is an $\varepsilon > 0$ such that for all $\delta > 0$ there is an x satisfying both $0 < |x - x_0| < \delta$ and $|f(x) - L| \geq \varepsilon$", which is the definition of $\lim_{x \to x_0} f(x) \neq L$.

Ex (The order of $\forall$ and $\exists$ are important)

Consider the open statement $A(x,y)$: "$x - y = 2$", and the assertions

$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\, A(x,y)$ and $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})\, A(x,y)$

$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\, A(x,y) \equiv$ "For all real numbers x, there is a real number y such that $x-y=2$"

This is true because given any real number x we may put $y = x - 2$, which is a real number satisfying $x - y = 2$

$(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})\, A(x,y) \equiv$ "There is a real number y such that $x-y=2$ for all real numbers x"

This is false. If it were true we would see that this some real number y must be -2 by letting $x = 0$ and y must be -1 by letting $x = 1$.

Fact:

1) $\forall x \, \forall y \, P(x,y) \equiv \forall y \, \forall x \, P(x,y)$
2) $\exists x \, \exists y \, P(x,y) \equiv \exists y \, \exists x \, P(x,y)$
3) $\exists x \, (A(x) \wedge B(x)) \not\equiv \exists x \, A(x) \wedge \exists x \, B(x)$
4) $\forall x \, (A(x) \wedge B(x)) \equiv \forall x \, A(x) \wedge \forall x \, B(x)$
5) $\forall x \, (A(x) \vee B(x)) \not\equiv \forall x \, A(x) \vee \forall x \, B(x)$
6) $\exists x \, (A(x) \vee B(x)) \equiv \exists x \, A(x) \vee \exists x \, B(x)$

Proof: Easy. Exercise

Ex: $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\,(x = y) \not\equiv (\exists y \in \mathbb{R})(\forall x \in \mathbb{R})\,(x = y)$

because TRUE but FALSE. Why?

Fact (vacuously true/false assertions)

"$(\forall x \in \underset{\substack{\uparrow \\ \text{the empty set}}}{\emptyset})\ P(x)$" is a tautology (i.e., it is true regardless of what $P(x)$ is)

"$(\exists x \in \emptyset)\ P(x)$" is a contradiction (i.e., it is false regardless of what $P(x)$ is)

Proof:

$$(\forall x \in \emptyset)\ P(x) \equiv \forall x\ (\underbrace{x \in \emptyset}_{\text{false for any } x} \longrightarrow P(x)) = \text{true}$$

$$(\exists x \in \emptyset)\ P(x) \equiv \exists x\ (\underbrace{(x \in \emptyset)}_{\text{false for any } x} \wedge P(x)) = \text{false}$$

Ex: "All prime numbers that are divisible by 6 have 5 digits" is vacuously true, because the set of prime number that are divisible by 6 is empty (denoting this set by A and letting $P(x)$: "x have 5 digits", the given assertion becomes $(\forall x \in A)\ P(x)$, and as we said $A = \emptyset$).

Unique existensial quantifier, $\exists!$

The notation "$\exists!\, x$" means that "there is a unique x such that", and the notation "$\exists!\, x \in A$" means that "there is a unique x in A such that". If they are followed by an open statement (i.e., predicate), they both become assertions. The assertion "$\exists!\, x\ P(x)$" is true precisely when there is exactly one element a of the universe making $P(a)$ true. Similarly, the assertion "$(\exists!\, x \in A)\ P(x)$" is true precisely when there is exactly one element a in A for which $P(a)$ is true. Of course, instead of "there is a unique" we may write "there is only one" or "there is exactly one".

Remark:

$\exists!\, x, P(x)$ is true $\underset{}{\overset{\text{if and only if}}{\overset{\uparrow}{\text{iff}}}}$ there is a unique a such that $P(a)$ is true

iff there is an a such that $P(a)$ is true, and for all y
if $P(y)$ is true then $y = a$

iff $(\exists x)\big((P(x)) \wedge \forall y, (P(y) \to y = x)\big)$

Therefore,

$$(\exists!\, x)\, P(x) \equiv (\exists x)\big[P(x) \wedge (\forall y)\,(P(y) \to y = x)\big]$$

Similarly,

$$(\exists!\, x \in A)\, P(x) \equiv (\exists x \in A)\big[P(x) \wedge (\forall y \in A)\,(P(y) \to y = x)\big]$$

Ex: "There is a unique positive real number whose square is 25"

$(\exists!\, x \in \mathbb{R}^+)\, P(x)$, where $P(x)$: "the square of x is 25"

the set of positive real numbers

$$(\exists x \in \mathbb{R}^+)\big[P(x) \wedge (\forall y \in \mathbb{R}^+)\,(P(y) \to y = x)\big]$$

"There is a positive real number x whose square is 25, and $y = x$ for any positive real number y whose square is 25"