EHB 211E
Basics of Electrical Circuits

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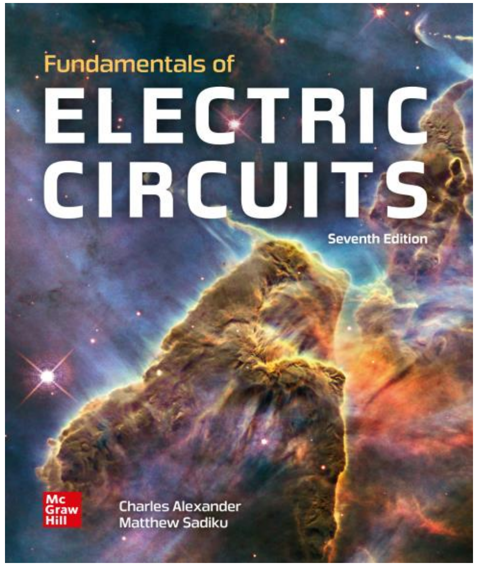
First-Order Circuits



Fundamentals of Electric Circuits by C. Alexander and M. Sadiku, Seventh Edition (McGraw-Hill Education)

Introduction

- Study two types of simple circuits:
  - ❑ Circuit: a resistor and capacitor $\longrightarrow$ RC circuit
  - ❑ Circuit: a resistor and inductor $\longrightarrow$ RL circuit
- Analysis of RC and RL circuits by applying Kirchhoff's law (KCL & KVL)
- Applying Kirchhoff's law to pure resistive circuits: Algebraic equations
- Applying Kirchhoff's law to RC & RL circuits: Differential equations
- Differential equations from RC & RL circuits are of the first-order circuit. Hence, the circuits are known as first-order circuits.
- First-order circuit is characterized by a first-order differential equation.
- Two ways to excite these circuits:
  - ❑ Initial conditions of the storage elements (source-free circuit). No independent element and energy initially stored in the capacitive and inductive element. They may have dependent sources.
  - ❑ Exciting first-order circuits by independent sources.

The Source-Free RC Circuit

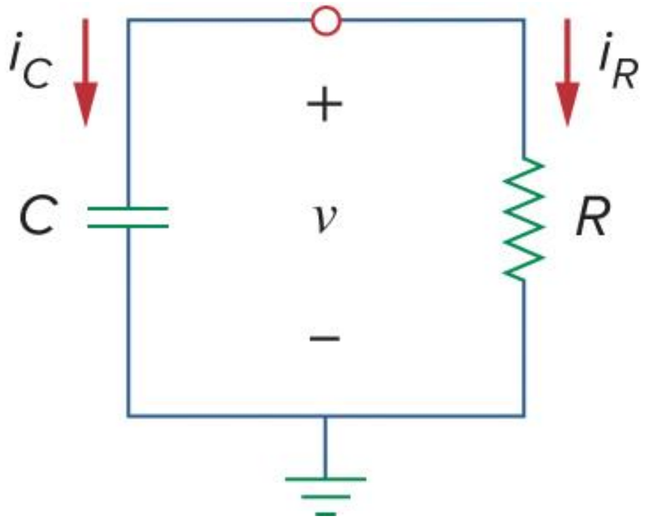
- Source-free RC circuit: dc source is suddenly disconnected.
- Energy already stored in the capacitor and released to the resistors.
- Objective: determine the circuit response
- Voltage across capacitor $v(t)$
- Capacitor is initially charged. Assume $t = 0$, the initial voltage is $v(0) = V_0$
- Energy stored in the capacitor:

$$w(0) = \frac{1}{2}C(V_0)^2 \ at \ t = 0 \longrightarrow \text{Initial energy stored}$$

- Apply KCL to the top node: $i_C + i_R = 0$
- By definition: $i_C = C\frac{dv}{dt}$ and $i_R = \frac{v}{R}$

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \Rightarrow \frac{dv}{dt} + \frac{v}{RC} = 0 \longrightarrow \text{First-order differential equation}$$

A source-free *RC* circuit



The Source-Free RC Circuit

$$\frac{dv}{dt} + \frac{v}{RC} = 0 \quad \Longrightarrow \quad \frac{dv}{dt} = -\frac{v}{RC} \Rightarrow \frac{dv}{v} = -\frac{1}{RC}dt \quad \Longrightarrow$$

Integrate both sides of the equation

$$\int \frac{dv}{v} = -\frac{1}{RC}\int dt \quad \Longrightarrow \quad lnv = -\frac{t}{RC} + A$$

A is the constant of the integral part

$$log_e v = -\frac{t}{RC} + A \quad \Longrightarrow \quad v(t) = e^{-\frac{t}{RC}+A} \quad \Longrightarrow \quad v(t) = e^{-\frac{t}{RC}} \underbrace{\boxed{e^A}}_{\text{Constant}}$$

- Let's call $e^A = V_0$ because at $t = 0$, $v(0) = e^0 e^A = V_0$

$$v(t) = V_0 e^{-\frac{t}{RC}}$$

The Source-Free RC Circuit

$$v(t) = V_0 e^{-\frac{t}{RC}} \longrightarrow$$ Voltage response of RC circuit is exponential decay of the initial voltage

- When $t = 0$, $v(t) = V_0$ (voltage across capacitor is initial voltage).
- When $t = \infty$, $e^{-\frac{\infty}{RC}}$ approaches zero and $v(t) = 0$ (discharging over a certain period of time).
- The response is due to initial voltage not due to some external voltage or current source. Hence, it is called natural response of the circuit.
- Natural response of a circuit refers to the behavior of the circuit itself, with no external sources of excitation.

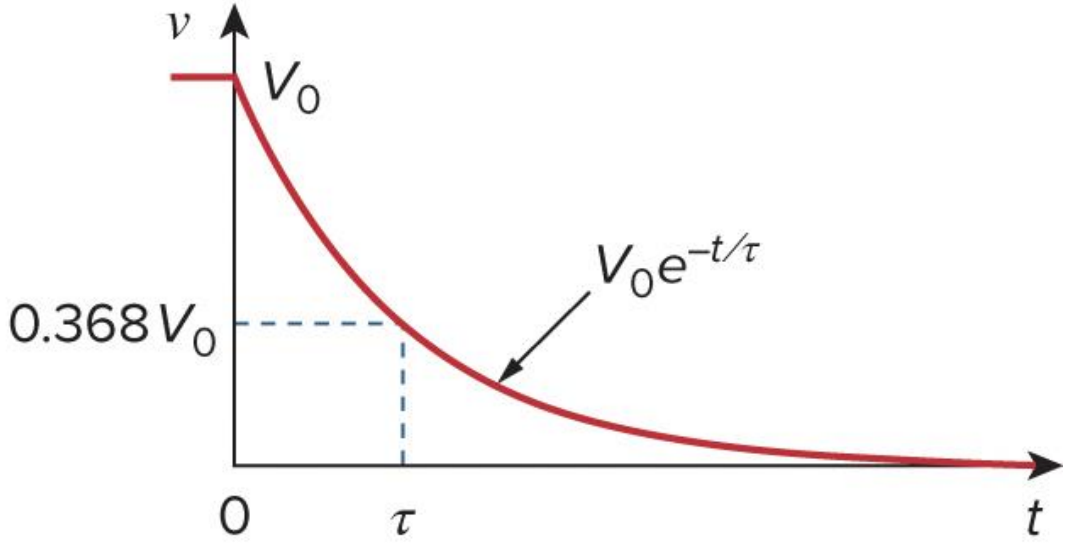
The Source-Free RC Circuit

- Rapid decrease of voltage can be expressed in terms of the time constant τ, unit is second.
- Time constant τ of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its initial value.

At $t = \tau$, $\quad v(t) = V_0 e^{-\frac{\tau}{RC}} \Longrightarrow V_0 e^{-\frac{\tau}{RC}} = V_0 e^{-1} \Rightarrow \frac{\tau}{RC} = 1 \Rightarrow \tau = RC$

$\boxed{\tau = RC}$ Time constant of capacitor

$\boxed{v(t) = V_0 e^{-\frac{t}{\tau}}}$ Voltage response of the RC circuit



The Source-Free RC Circuit

- Table shows how the value of $v(t)/V_0$ changes when time t increases
- After 5τ, $v(t)$ is less than 1% of V_0. It takes approximately 5τ for the circuit reach its final state or steady state (Capacitor is fully discharged after 5τ).
- Smaller the time constant (τ), more rapidly voltage decreases, i.e., faster response.

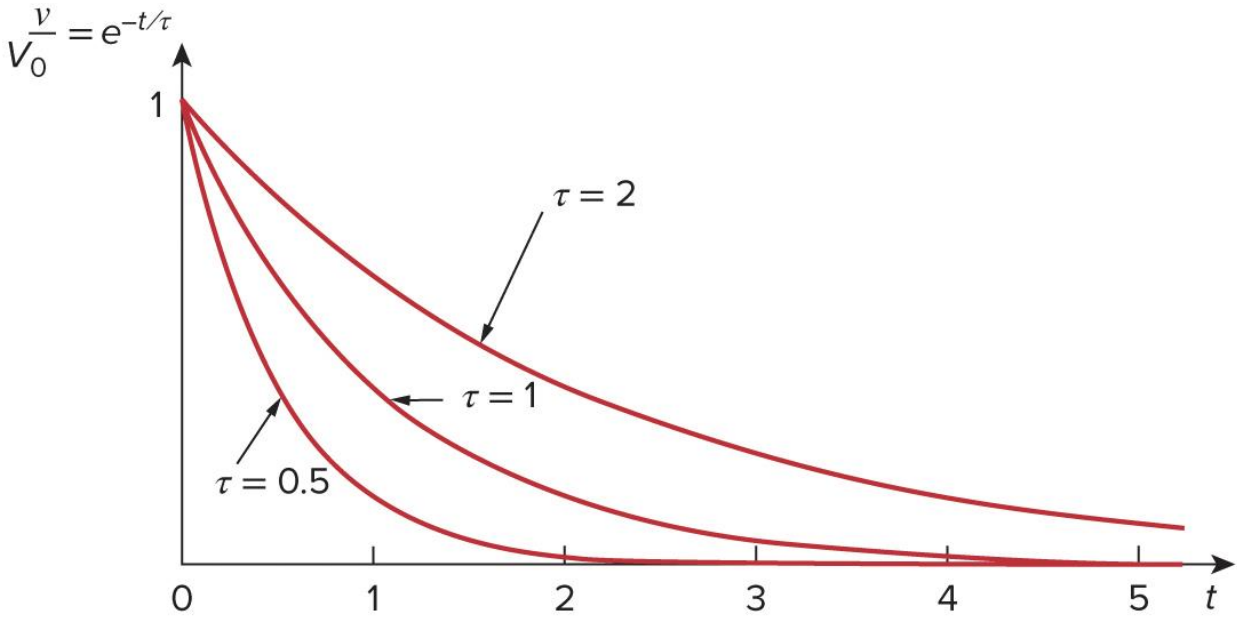


TABLE 7.1

Values of $v(t)/V_o = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

The Source-Free RC Circuit

- Using Ohm's law, the current $i_R(t)$ is expressed as:

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{\tau}}$$

- Power dissipated in the resistor:

$$p(t) = v i_R \Rightarrow p(t) = \left(V_0 e^{-\frac{t}{\tau}}\right)\left(\frac{V_0}{R} e^{-\frac{t}{\tau}}\right) \Rightarrow p(t) = \frac{(V_0)^2}{R} e^{-\frac{2t}{\tau}}$$

- Energy absorbed by the resistor:

$p = \frac{w}{t} \Rightarrow w = pt$ ⇒ Take derivative of both sides of the equation ⇒ $dw = pdt$ ⇒ Integrate both sides

$$\int dw = \int pdt \Rightarrow w_R(t) = \int_0^t p(t)dt \Rightarrow w_R(t) = \int_0^t \frac{(V_0)^2}{R} e^{-\frac{2t}{\tau}} dt = -\frac{\tau (V_0)^2}{2R} e^{-\frac{2t}{\tau}} \Big|_0^t, \quad \tau = RC$$

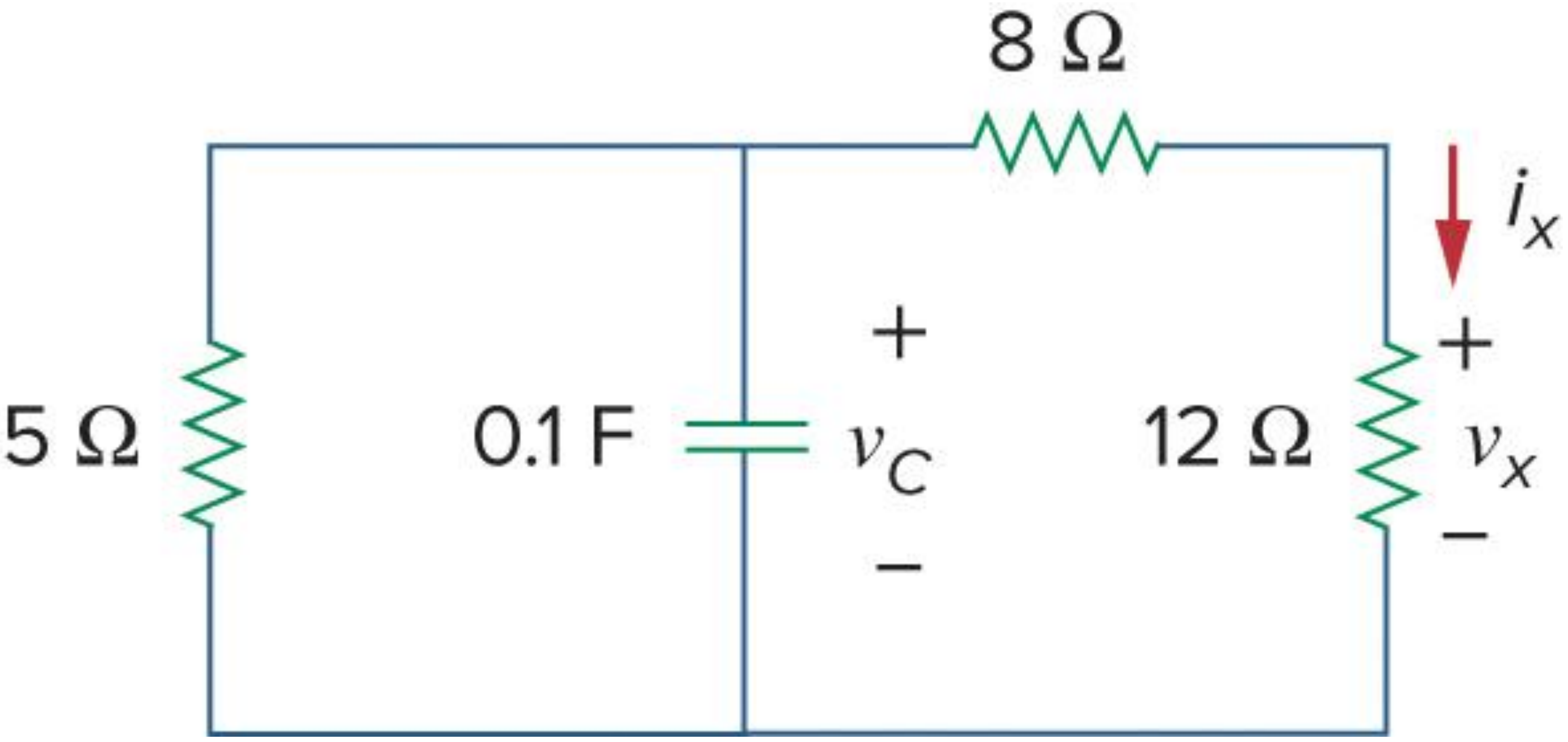
$$= -\frac{RC(V_0)^2}{2R} e^{-\frac{2t}{\tau}} \Big|_0^t = \frac{1}{2} C (V_0)^2 \left(1 - e^{-\frac{2t}{\tau}}\right)$$

- When $t = \infty$, $e^{-\frac{2t}{\tau}}$ approaches 0

$$w_C = \frac{1}{2} C (V_0)^2$$

Example 1

For the circuit shown below, let $v_C(0) = 15\ V$. Find v_C, v_x, and i_x for $t > 0$.



Solution

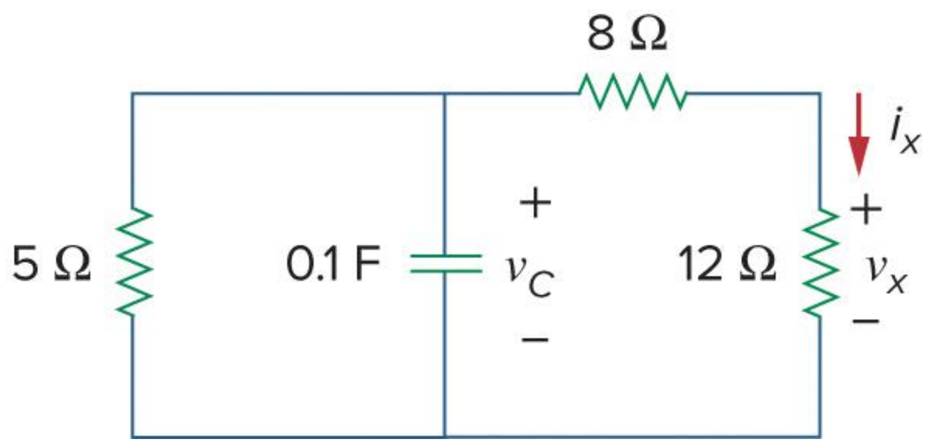
$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

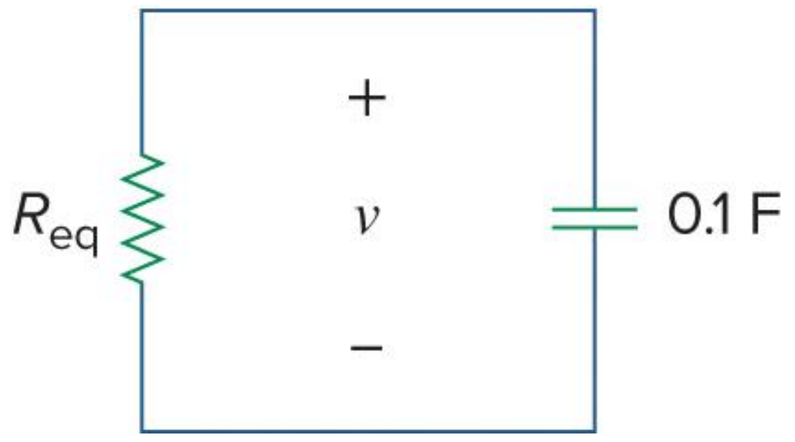
$$\tau = R_{eq}C = 4(0.1) = 0.4\ \text{s}$$

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V}, \qquad v_C = v = 15e^{-2.5t}\ \text{V}$$

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\ \text{V}$$

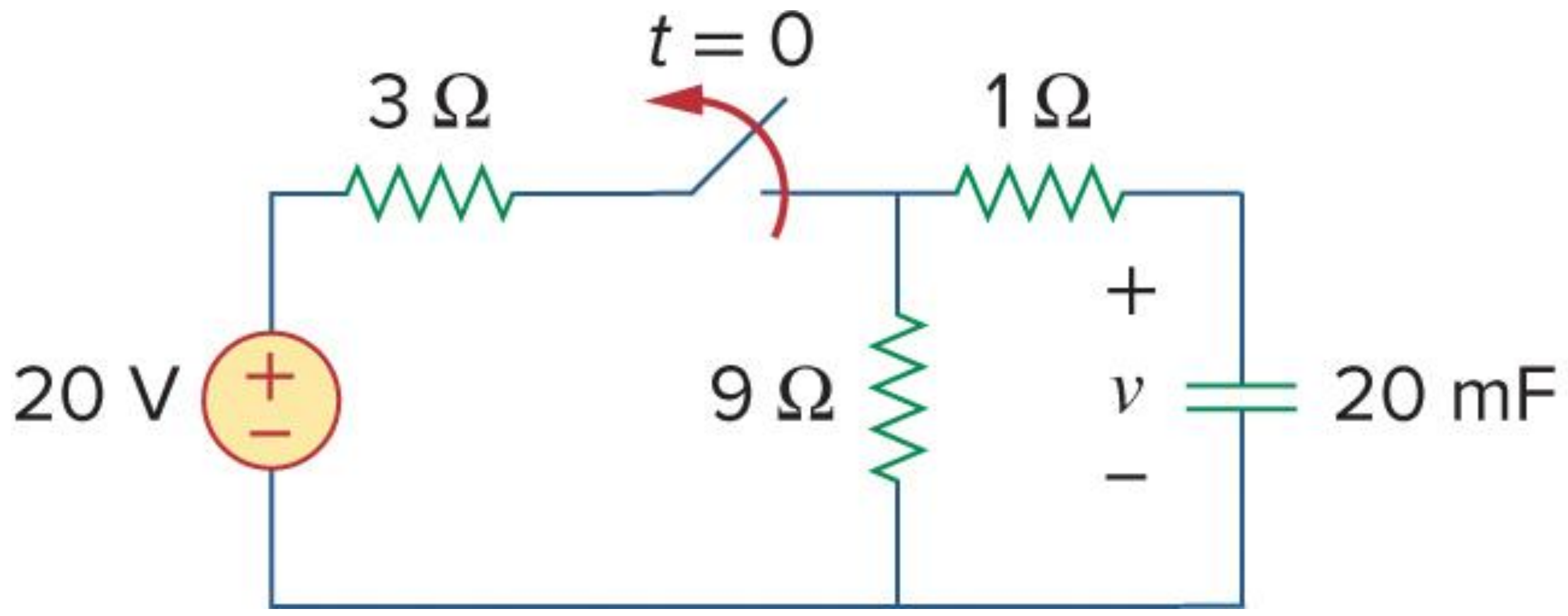
$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$





Example 2

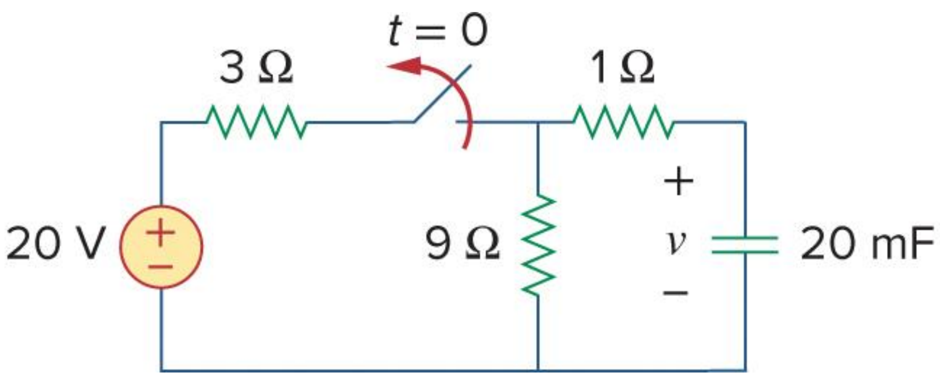
The switch in the circuit shown below has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t > 0$. Calculate the initial energy stored in the capacitor.



Solution

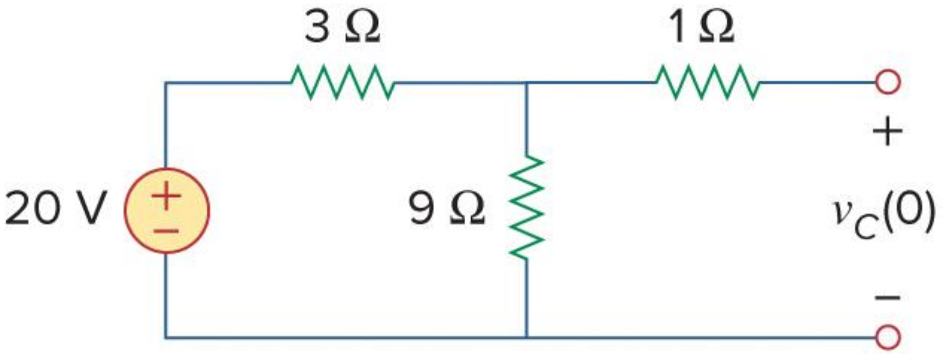
For $t < 0$ the switch is closed; the capacitor is an open circuit to dc

$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

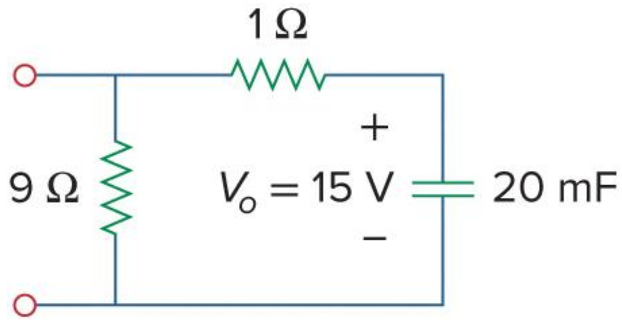


Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$ or

$$v_c(0) = V_0 = 15$$



For $t > 0$ the switch is opened: $R_{eq} = 1 + 9 = 10\ \Omega$



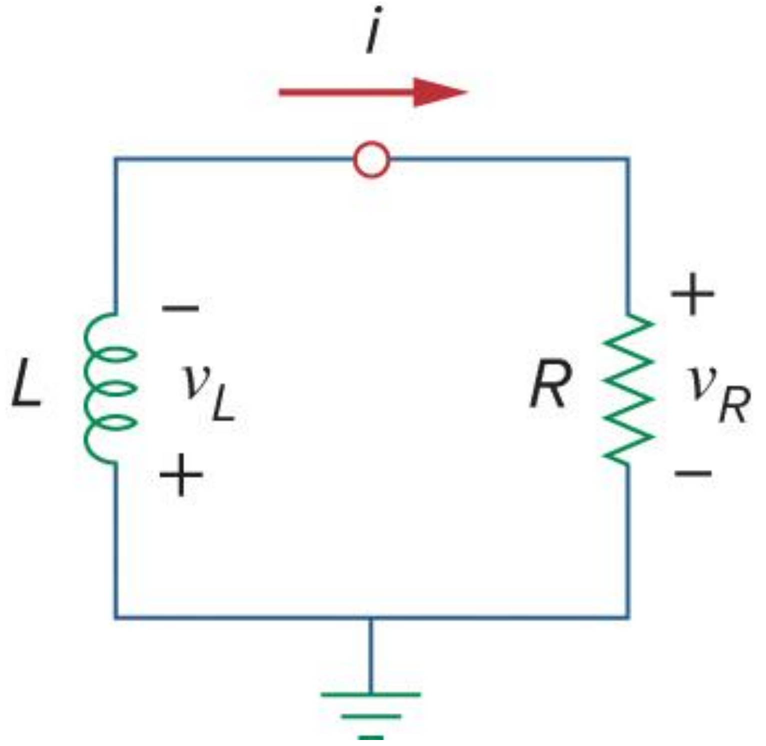
$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2 \text{ s} \qquad v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} \text{ V}$$

$$v(t) = 15e^{-5t} \text{ V} \qquad w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 \text{ J}$$

The Source-Free RL Circuit

- Goal: determine circuit response in RL circuit.
- When $t = 0$, there must be initial current in RL circuit (current excites the circuit)

$i(0) = I_0$ ⟶ When $t = 0$, inductor current equal to initial current.



- Initial energy in the inductor:

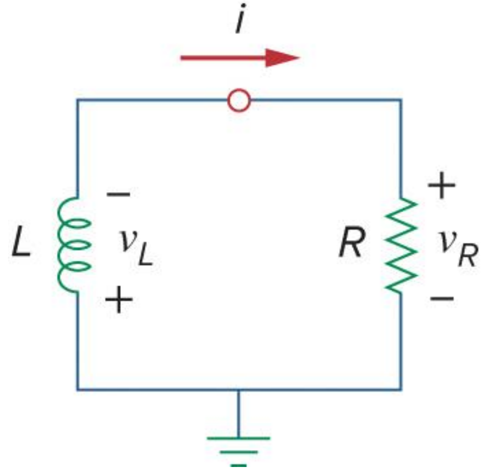
$$w(0) = \frac{1}{2} L (I_0)^2$$

The Source-Free RL Circuit

- Apply KVL to the circuit: $v_L + v_R = 0$, $v_L = L\dfrac{di}{dt}$

$$L\frac{di}{dt} + iR = 0 \Rightarrow \frac{di}{dt} + \frac{R}{L}i = 0 \Rightarrow \frac{di}{dt} = -\frac{R}{L}i \Rightarrow \frac{di}{i} = -\frac{R}{L}dt \Rightarrow$$ Integrate both sides

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt \Rightarrow lni\Big|_{I_0}^{i(t)} = -\frac{R}{L}t\Big|_0^t$$



$$lni - lnI_0 = -\frac{R}{L}t + 0 \Rightarrow \ln\left(\frac{i}{I_0}\right) = -\frac{Rt}{L}$$

$$log_e\left(\frac{i}{I_0}\right) = -\frac{Rt}{L} \Rightarrow \frac{i}{I_0} = e^{-\frac{Rt}{L}} \Rightarrow i(t) = I_0 e^{-\frac{Rt}{L}} \longrightarrow$$ Natural response of RL circuit is exponential decay of the initial current

$\boxed{\tau = \dfrac{L}{R}}$ Time constant of inductor

$$\boxed{i(t) = I_0 e^{-\frac{t}{\tau}}}$$

The Source-Free RL Circuit

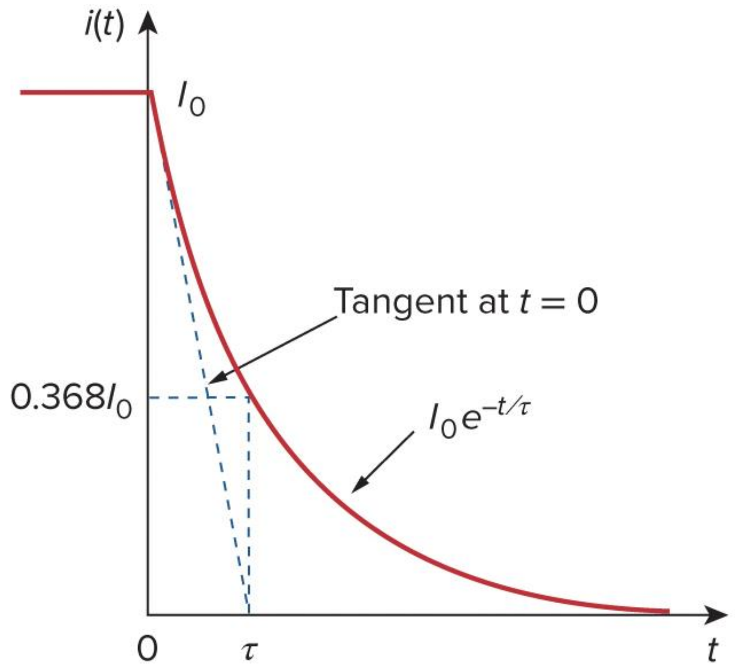
- When $t < 0$, inductor is charged.
- When $t = 0$, inductor excites the circuit by releasing the stored energy.
- Inductor has an exponential decay due to $e^{-\frac{\infty}{\tau}}$ term.
- Similar to capacitor, at time $t = \tau$, current through the circuit is 36.8% smaller than initial value.

$$i(t) = I_0 e^{-\frac{t}{\tau}} \Rightarrow i(\tau) = I_0 e^{-\frac{\tau}{\tau}} = \frac{I_0}{\mathrm{e}^1} = 0.368 I_0$$

$$v_R(t) = iR \Rightarrow v_R(t) = I_0 R e^{-\frac{t}{\tau}}$$

- Power dissipated in the resistor:

$$p = v_R i \Rightarrow p = (I_0)^2 R e^{-\frac{2t}{\tau}}$$



The Source-Free RL Circuit

- Energy absorbed by the resistor:

$p = \dfrac{w}{t} \Rightarrow w = pt$ → Take derivate of both sides of this equation → $dw = pdt$ → Integrate both sides

$\int dw = \int pdt$ → $w_R(t) = \int_0^t p(t)dt \Rightarrow w_R(t) = \int_0^t (I_0)^2 R e^{-\frac{2t}{\tau}} dt = -\frac{\tau}{2}(I_0)^2 R e^{-\frac{2t}{\tau}}$, $\tau = \frac{L}{R}$

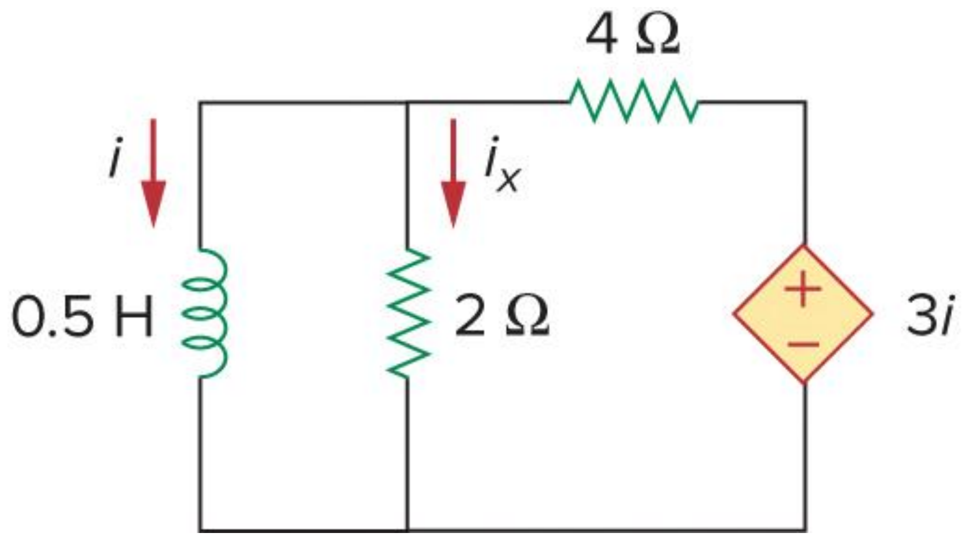
$$w_R(t) = -\frac{1}{2}\frac{L}{R}(I_0)^2 R\left(e^{-\frac{2t}{\tau}}\right)\Big|_0^t \rightarrow w_R(t) = \frac{1}{2}L(I_0)^2\left(1 - e^{-\frac{2t}{\tau}}\right)$$

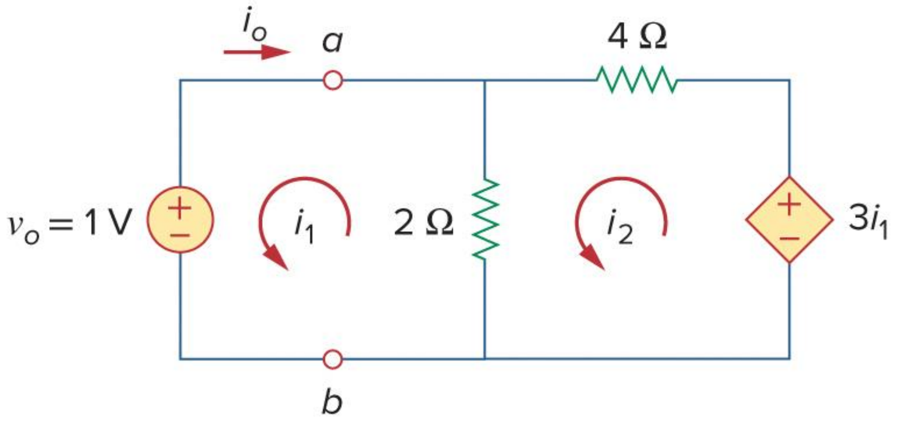
- Energy initially stored in the inductor is eventually dissipated by the resistor.

- When $t \rightarrow \infty, e^{-\frac{2t}{\tau}} \rightarrow 0$ $\quad w_R(t) = \frac{1}{2}L(I_0)^2$ ⟶ Same as $w_L(0)$ which is the initial energy stored in the inductor.

Example 3

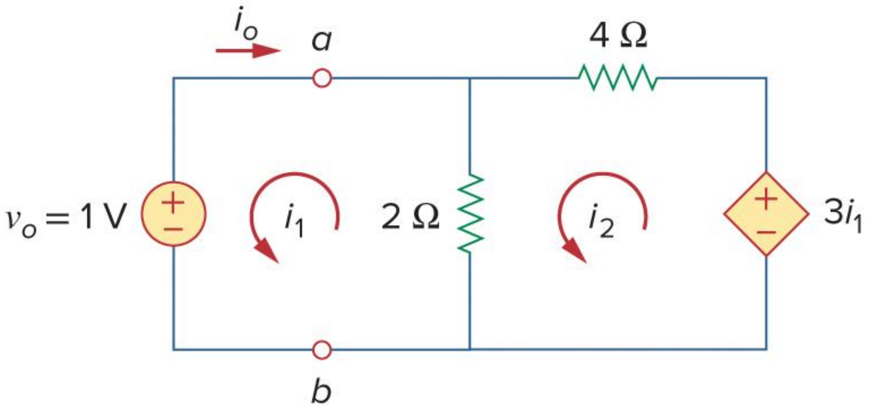
Assuming that $i(0) = 10\text{ A}$, calculate $i(t)$ and $i_x(t)$ in the circuit shown below.





Solution

- Method 1: Find equivalent resistance.
- Since the circuit has only dependent source, insert a test voltage source of 1 V at the inductor terminals a-b and find R_{Th}.
- Keep it in mind that we cannot turn off dependent circuit when applying Thevenin theorem.
- To find R_{Th}, we need to find i_0 as $R_{Th} = \frac{v_0}{i_0} = \frac{1}{i_0}$
- Apply KVL both loops:



$$2(i_1 - i_2) + 1 = 0 \quad \Rightarrow \quad i_1 - i_2 = -\frac{1}{2}$$

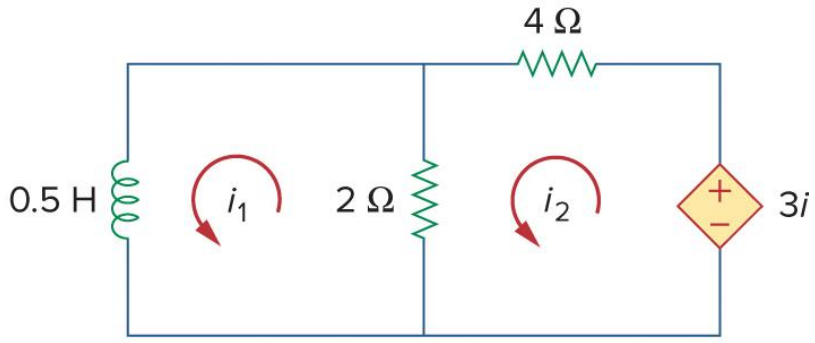
$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1$$

$$i_1 = -3 \text{ A}, \qquad i_o = -i_1 = 3 \text{ A} \qquad\qquad R_{\text{eq}} = R_{\text{Th}} = \frac{v_o}{i_o} = \frac{1}{3}\,\Omega \qquad\qquad \tau = \frac{L}{R_{\text{eq}}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\text{ s}$$

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

Solution

- Method 2: apply KVL



$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0 \quad \text{or} \quad \frac{di_1}{dt} + 4i_1 - 4i_2 = 0$$

For loop 2,

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1 \qquad \frac{di_1}{dt} + \frac{2}{3}i_1 = 0 \qquad \frac{di_1}{i_1} = -\frac{2}{3}dt$$

Since $i_1 = i$, we may replace i_1 with i and integrate: $\ln i \Big|_{i(0)}^{i(t)} = -\frac{2}{3}t \Big|_0^t$ or $\ln \frac{i(t)}{i(0)} = -\frac{2}{3}t$

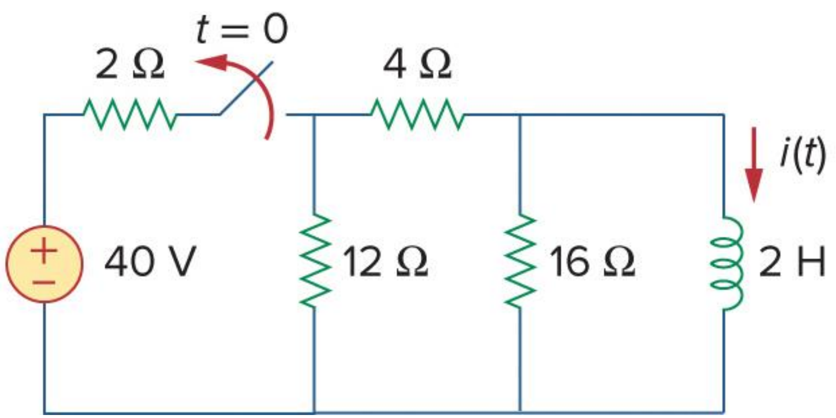
$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

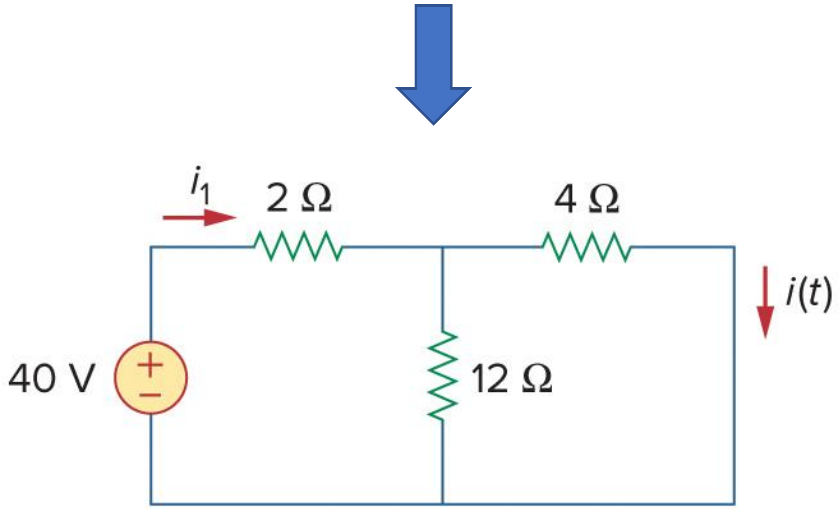
$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t} \text{ V}$$

$$i_x(t) = \frac{v}{2} = -1.6667e^{-(2/3)t} \text{ A}, \qquad t > 0$$

Example 4

The switch in the circuit shown below has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.





Solution

For $t < 0$, the switch is closed. This means that inductor is short circuit to dc condition.

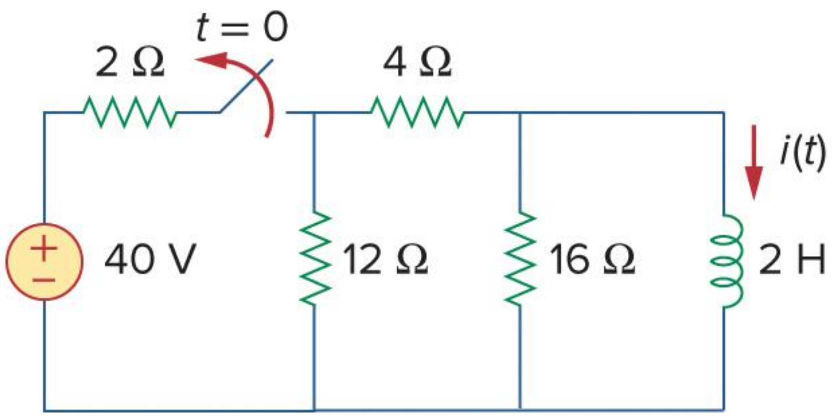
$$\frac{4 \times 12}{4 + 12} = 3\ \Omega \qquad i_1 = \frac{40}{2 + 3} = 8\ \text{A}$$

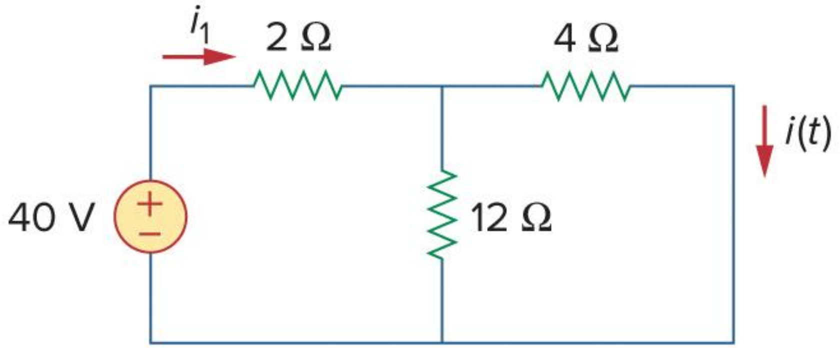
using current division

$$i(t) = \frac{12}{12 + 4} i_1 = 6\ \text{A}, \qquad t < 0$$

Since the current through an inductor cannot change instantaneously,

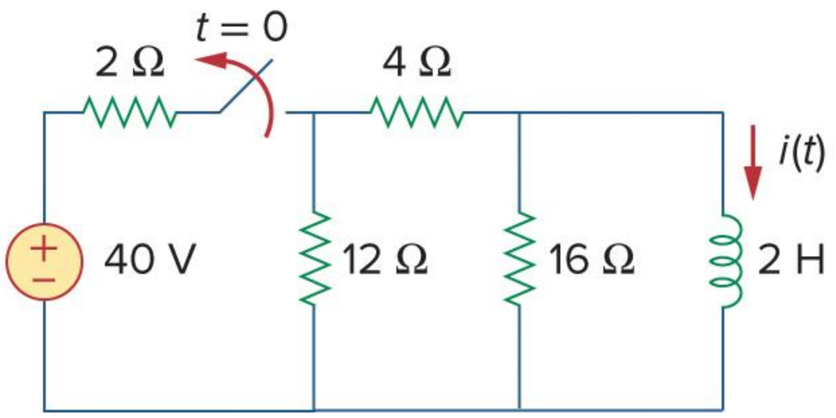
$$i(0) = i(0^-) = 6\ \text{A}$$





Solution

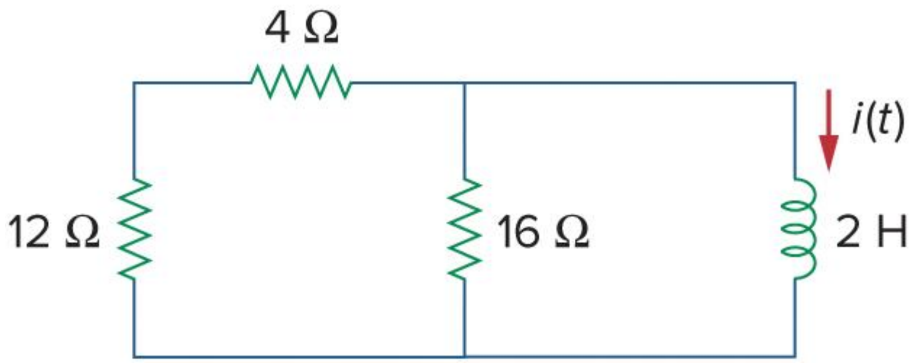
For $t > 0$, the switch is opened and voltage source is disconnected. This means we have a source-free RL circuit as shown below.



To find $i(t)$, we need to use following equation:

$$i(t) = i(0)e^{-\frac{t}{\tau}}$$

To find $i(t)$, determine time constant τ



$$R_{eq} = (12 + 4) \parallel 16 = 8\ \Omega$$

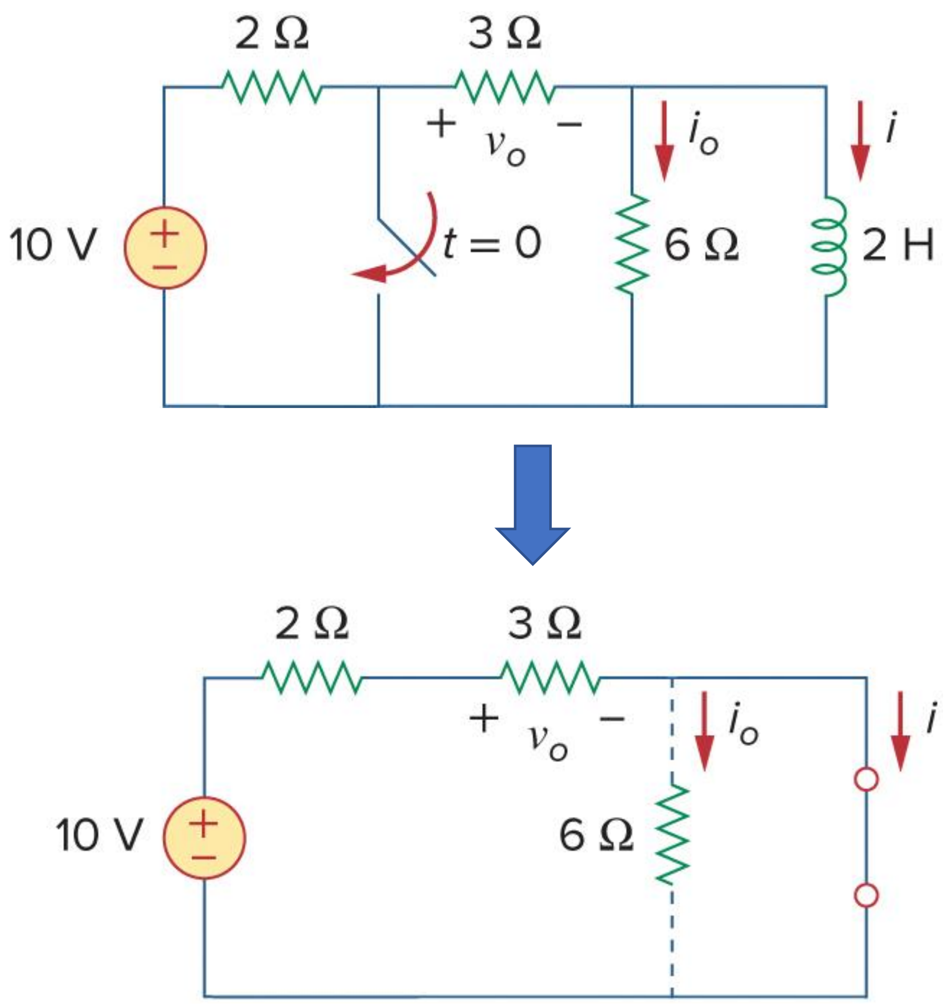
$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4}\ \text{s}$$

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t}\ \text{A}$$

Example 5

In the circuit shown below, find i_0, v_0 and i for all time, assuming that the switch was open for a long time.

Solution:



For $t < 0$, the switch is opened. Since inductor is short circuit to dc, 6 Ω resistor is short circuited. $i_0 = 0$

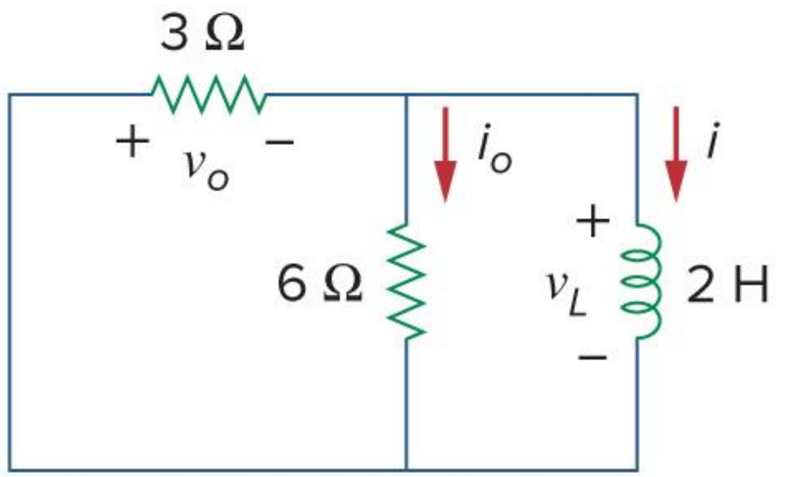
$$i(t) = \frac{10}{2+3} = 2\text{ A}, \qquad t < 0$$

$$v_o(t) = 3i(t) = 6\text{ V}, \qquad t < 0$$

$i(0) = 2\,A$ for $t < 0$ & $t = 0$, $i(0) = i(0^-) = 2\,A$

Solution

For $t > 0$, the switch is closed. The voltage is short circuited.



$$R_{Th} = 3 \parallel 6 = 2\ \Omega \qquad \tau = \frac{L}{R_{Th}} = 1\ \text{s}$$

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \text{A}, \qquad t > 0$$

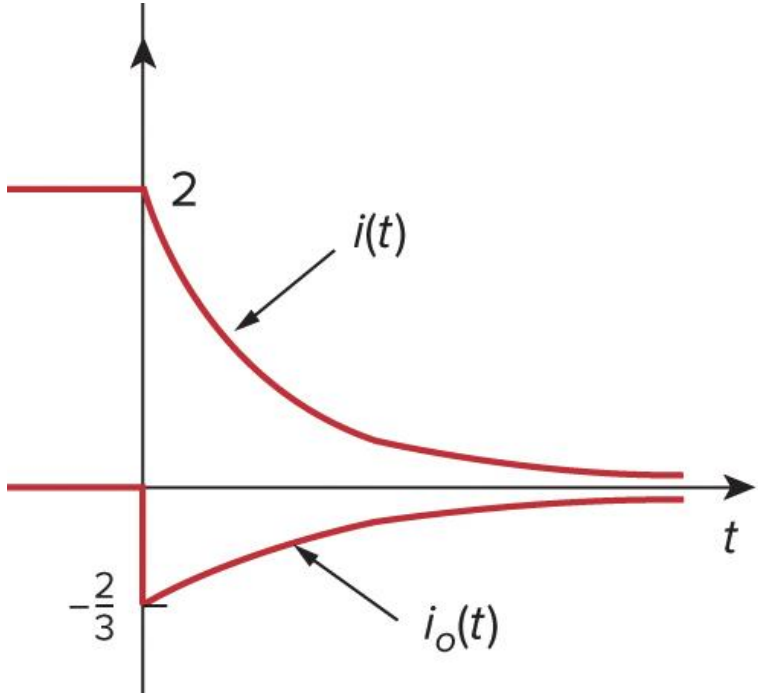
$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\ \text{V}, \qquad t > 0$$

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\ \text{A}, \qquad t > 0$$

Thus, for all time,

$$i_o(t) = \begin{cases} 0\ \text{A}, & t < 0 \\ -\frac{2}{3}e^{-t}\ \text{A}, & t > 0 \end{cases}, \qquad v_o(t) = \begin{cases} 6\ \text{V}, & t < 0 \\ 4e^{-t}\ \text{V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2\ \text{A}, & t < 0 \\ 2e^{-t}\ \text{A}, & t \geq 0 \end{cases}$$



From the Graph and expression: the inductor current is continuous at t=0 (current through inductor cannot change instantly), while resistor current through 6 Ω resistor drop from 0 to -2/3 at t=0, and voltage across 3 Ω resistor drop from 6 V to 4 V at t=0.

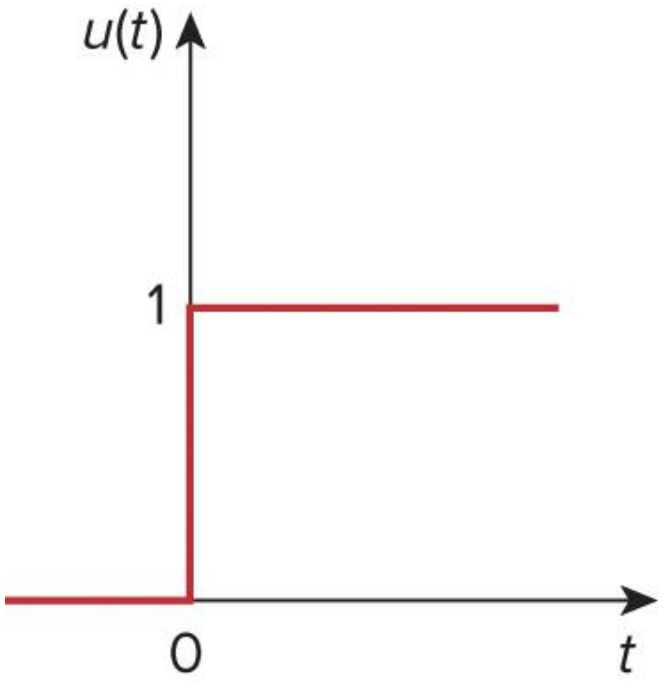
Singularity (Switching) Function

- Singularity (aka switching functions): useful for circuit analysis.
- Good approximations to the switching signals that arises in circuit with switching operation.
- Helpful in the neat, compact description of the step response of RC or RL circuit.
- Singularity function: either discontinuous or have discontinuous derivatives.
- Three most widely used singularity functions:
  - Unit step function
  - Unit impulse function
  - Unit ramp function

Unit Step Function

- Unit step function $u(t)$ is 0 for negative value of t and 1 for positive value of t.
- In mathematically,

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

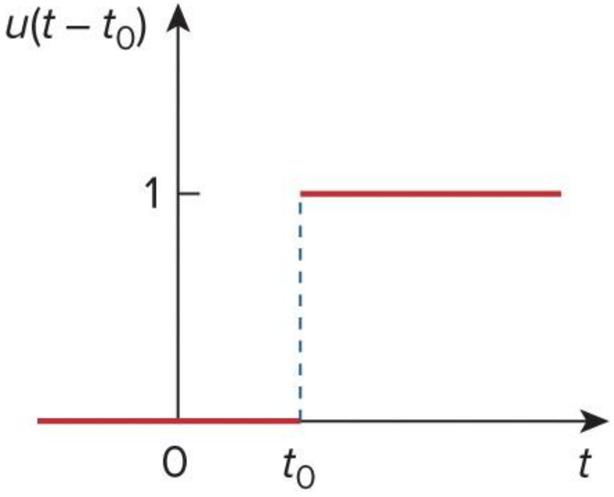


- Unit step function is undefined at $t = 0$ where it changes abruptly from 0 to 1.

Unit Step Function

- Assume that abrupt (sudden) change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$.

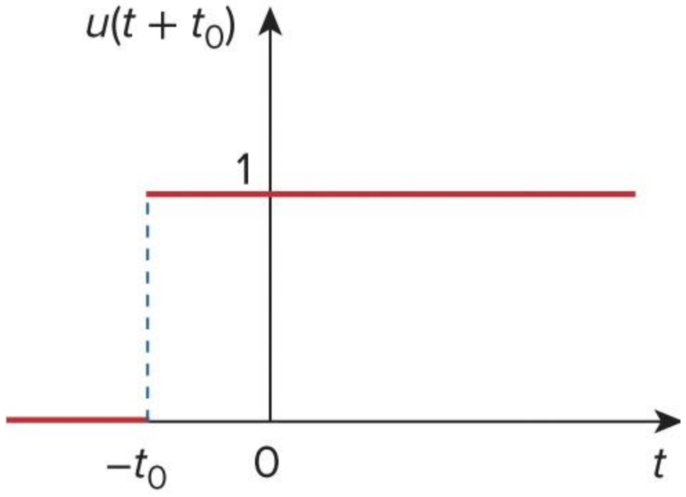
$$u(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$$



$u(t)$ is delayed by t_0 sec

- Assume that abrupt (sudden) change occurs at $t = -t_0$

$$u(t + t_0) = \begin{cases} 0 & t < -t_0 \\ 1 & t > -t_0 \end{cases}$$



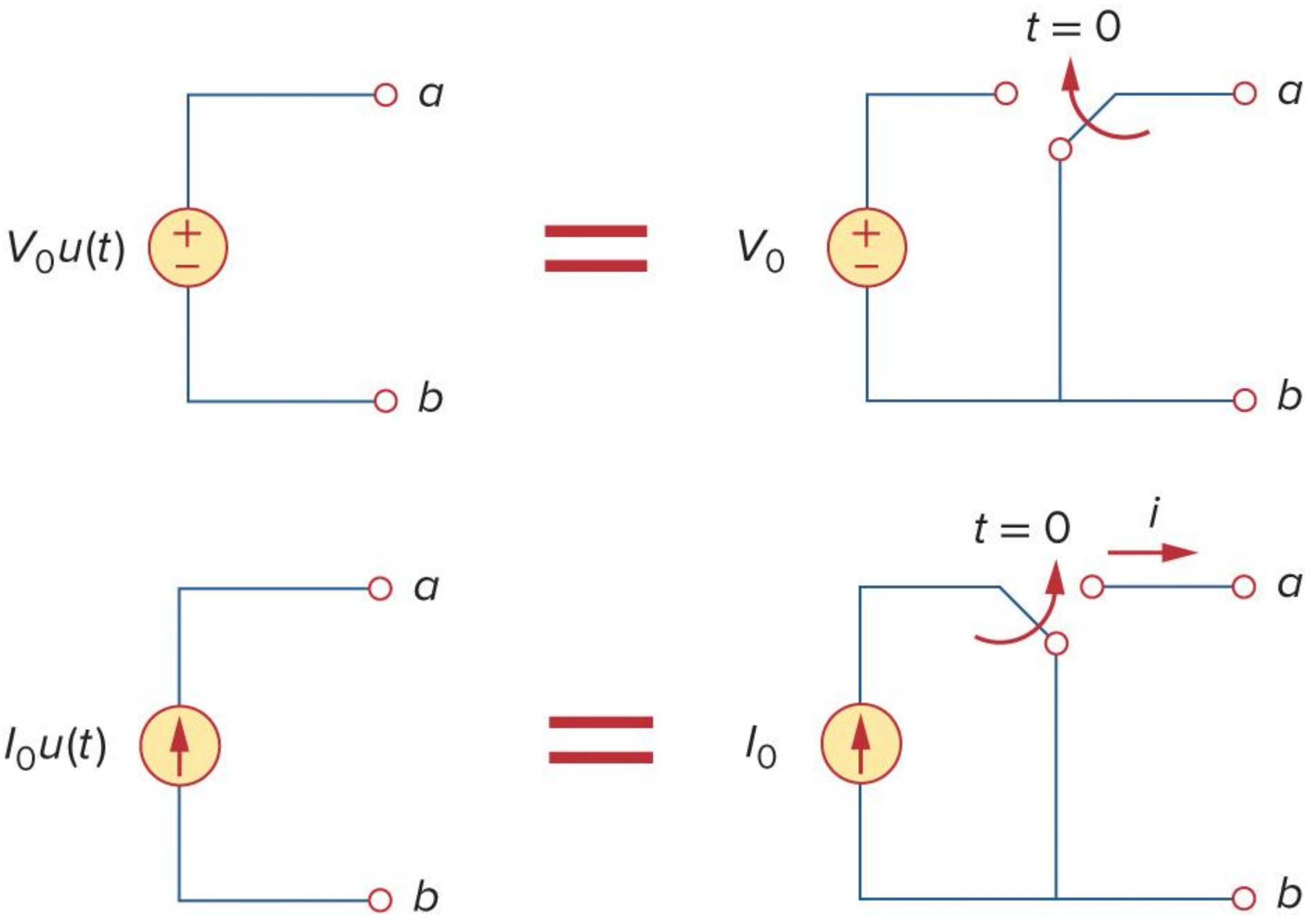
$u(t)$ is advanced by t_0 sec

Unit Step Function

- An abrupt change in voltage and current can be represented by step function.

$$v(t) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases} \Rightarrow v(t) = V_0 u(t - t_0)$$

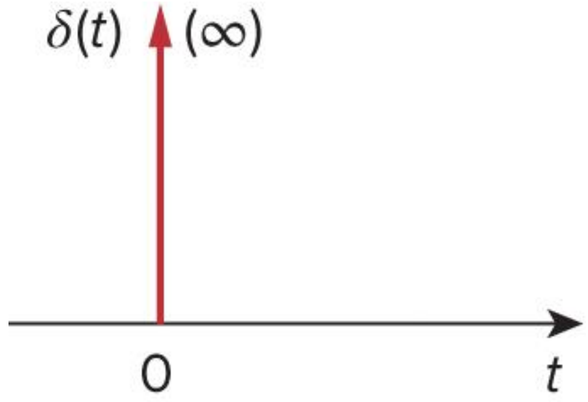
- If $t_0 = 0$, $v(t)$ is simply the step voltage $V_0 u(t)$



Unit Impulse Function

- Derivative of the unit step function $u(t)$ is the unit impulse function (aka delta function) which is represented by $\delta(t)$

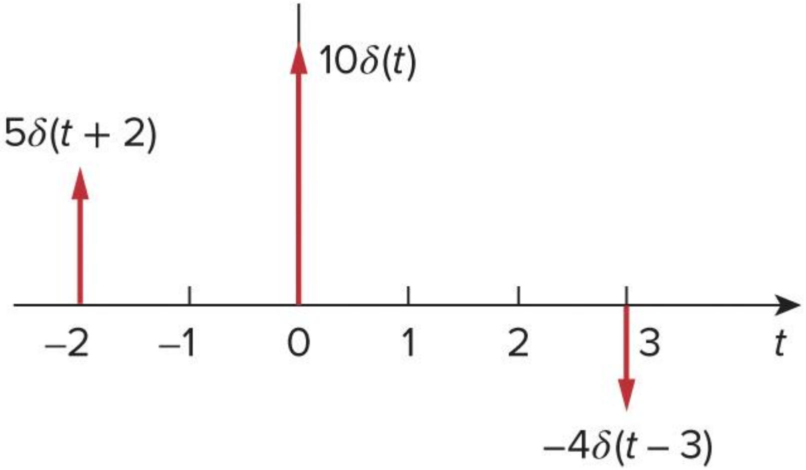
$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0 & t < 0 \\ \text{undefined} & t = 0 \\ 0 & t > 0 \end{cases}$$



- The unit impulse function $\delta(t)$ is zero everywhere except at t=0, where it is undefined.
- The unit impulse may be regarded as an applied or resulting shock. It may be visualized as a very short duration pulse of unit area.
- It can be expressed mathematically as:

$$\int_{0^-}^{0^+} \delta(t)dt = 1$$

The strength of the impulse function.

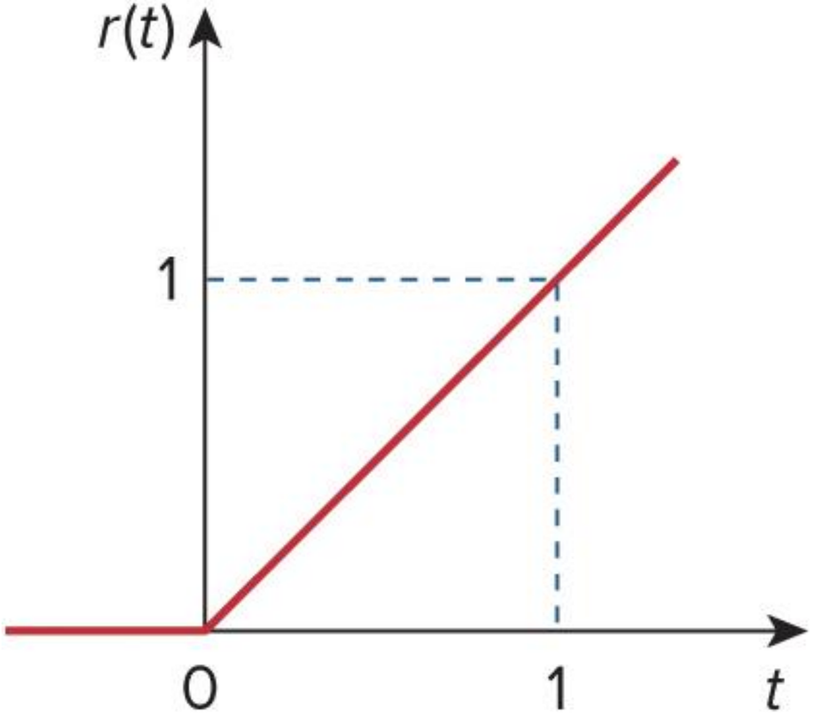


Unit Ramp Function

- Integral of the unit step function $u(t)$ is the unit ramp function $\mathrm{r}(t)$:

$$\mathrm{r}(t) = \int_{-\infty}^{t} u(t)dt = u(t).t$$

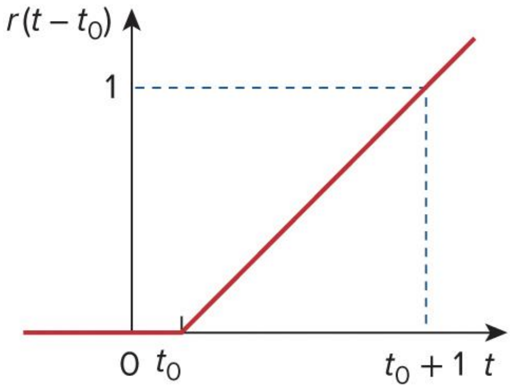
$$r(t) = \begin{cases} 0 & t \leq 0 \\ t & t \geq 0 \end{cases}$$



Unit Ramp Function

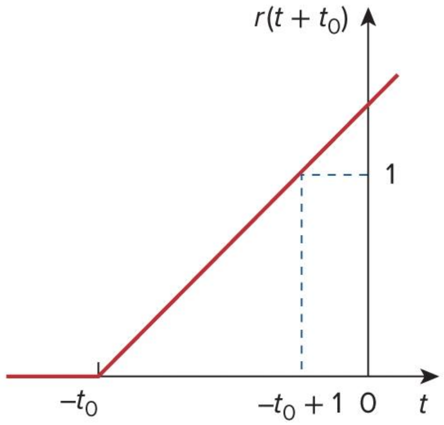
- Delayed unit ramp function

$$r(t-t_0)=\begin{cases}0 & t\le t_0\\ t-t_0 & t\ge t_0\end{cases}$$



- Advanced unit ramp function

$$r(t+t_0)=\begin{cases}0 & t\le -t_0\\ t+t_0 & t\ge -t_0\end{cases}$$

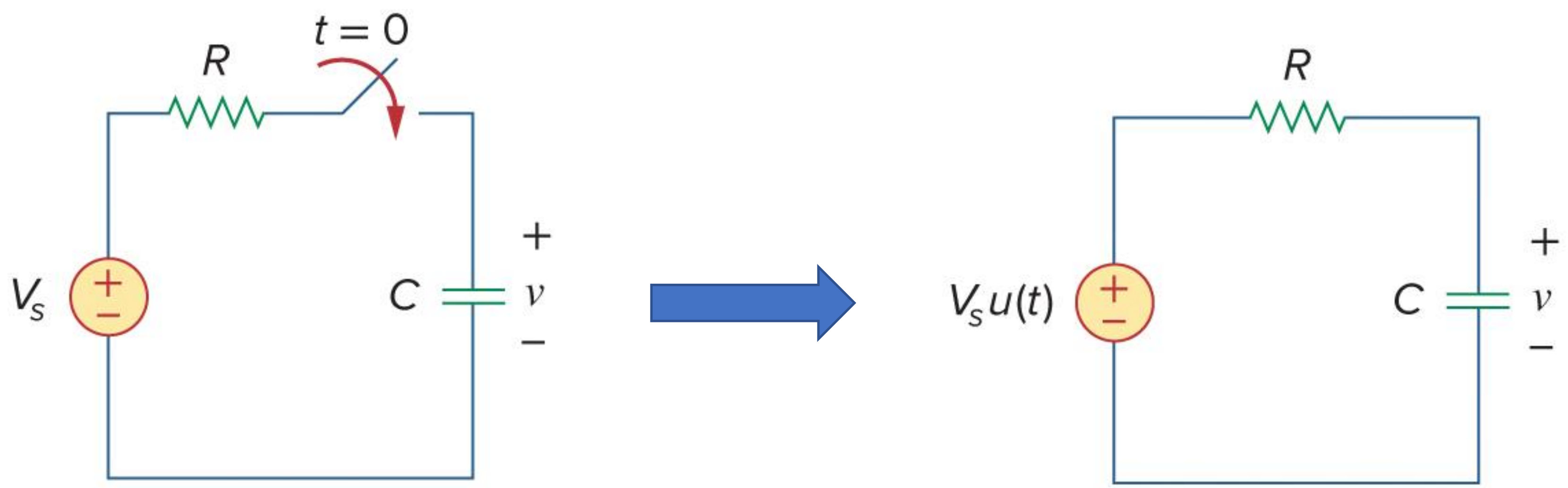


- Three singularity functions are related by differentiation or integration:

$$\delta(t)=\frac{du(t)}{dt}\qquad u(t)=\frac{dr(t)}{dt}\qquad u(t)=\int_{-\infty}^{t}\delta(t).dt\qquad r(t)=\int_{-\infty}^{t}u(t).dt$$

Step Response of an RC Circuit

- When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a step response.
- The step response is the response of the circuit due to a sudden application of a dc voltage or current source.
- Once the switch is closed, there is a sudden application of dc source. Thus, the circuit can be replaced by the following circuit (on the right)



Step Response of an RC Circuit

- Assume V_0 is the initial voltage on the capacitor.
- Voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0$$

0^-: just before switching
0^+: just after switching

- Apply KCL: $C\dfrac{dv}{dt} + \dfrac{v - V_s u(t)}{R} = 0 \Rightarrow C\dfrac{dv}{dt} + \dfrac{v}{R} - \dfrac{V_s u(t)}{R} = 0$

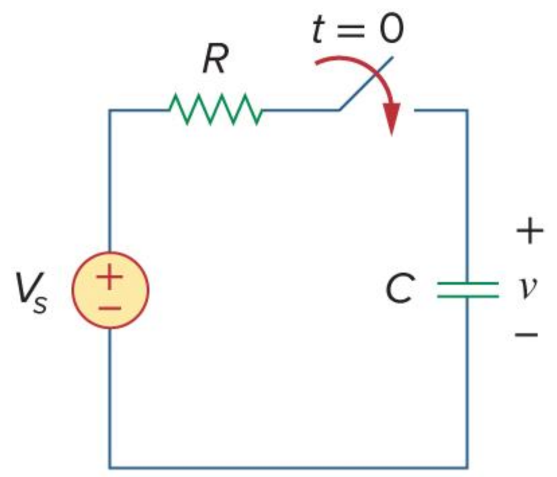
$$C\frac{dv}{dt} + \frac{v}{R} = \frac{V_s u(t)}{R} \Rightarrow \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$$

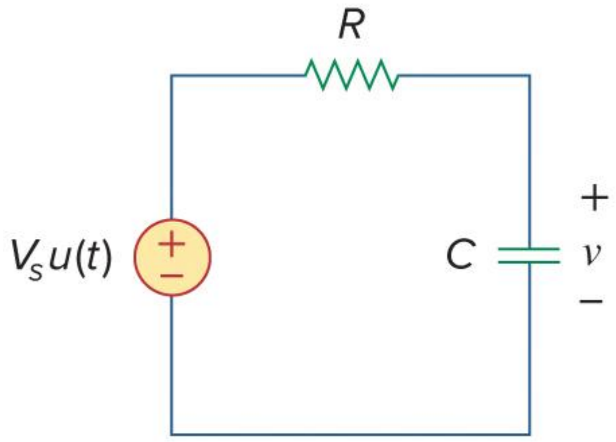
For $t > 0$, unit step function is 1

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \Rightarrow \frac{dv}{dt} = -\frac{v - V_s}{RC} \Rightarrow \frac{dv}{v - V_s} = -\frac{\mathrm{dt}}{RC} \Rightarrow \text{Integrate}$$

$$\int_{v(0)}^{v(t)} \frac{dv}{v - V_s} = -\int_0^t \frac{1}{RC}dt \Rightarrow \ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{\mathrm{t}}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0 \Rightarrow \ln\left(\frac{v(t) - V_s}{V_0 - V_s}\right) = -\frac{t}{RC}$$





Step Response of an RC Circuit

$$\frac{v(t) - V_s}{V_0 - V_s} = e^{-\frac{t}{RC}} \quad \Rightarrow \quad v(t) - V_s = (V_0 - V_s)e^{-\frac{t}{RC}} \qquad \tau = RC$$

$$v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}} \qquad t > 0$$

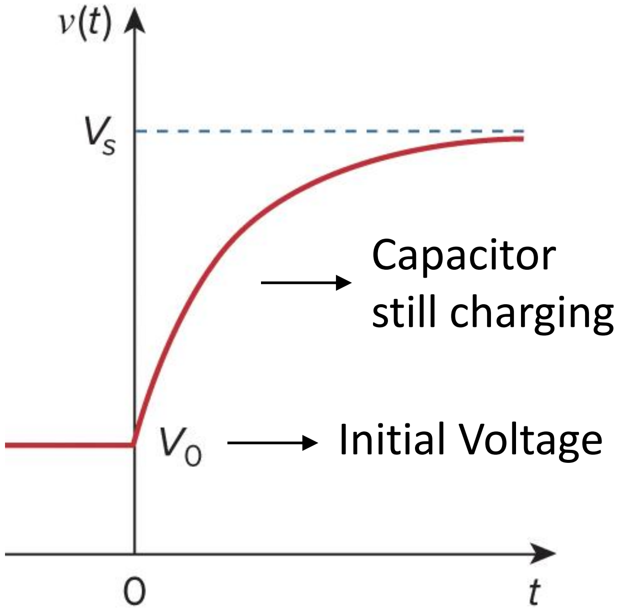
Complete response (or total response) of the RC circuit to a sudden application of dc voltage source

$$\boxed{v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{\tau}} & t > 0 \end{cases}}$$

Assuming capacitor is initially charged

Step Response of an RC Circuit

Assume $V_s > V_0$



$$v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{\tau}} & t > 0 \end{cases}$$

If capacitor is initially uncharged, $V_0 = 0$

$$v(t) = \begin{cases} 0 & t < 0 \\ V_s(1 - e^{-\frac{t}{\tau}}) & t > 0 \end{cases}$$

When $t < 0,\ u(t) = 0$

When $t > 0,\ u(t) = 1$

$$v(t) = V_s\left(1 - e^{-\frac{t}{\tau}}\right)u(t)$$

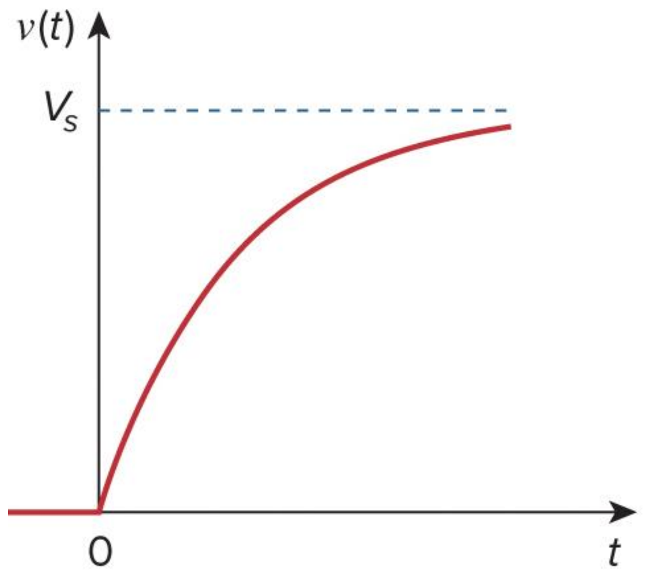
Step Response of an RC Circuit

- Current through the capacitor (capacitor initially uncharged):

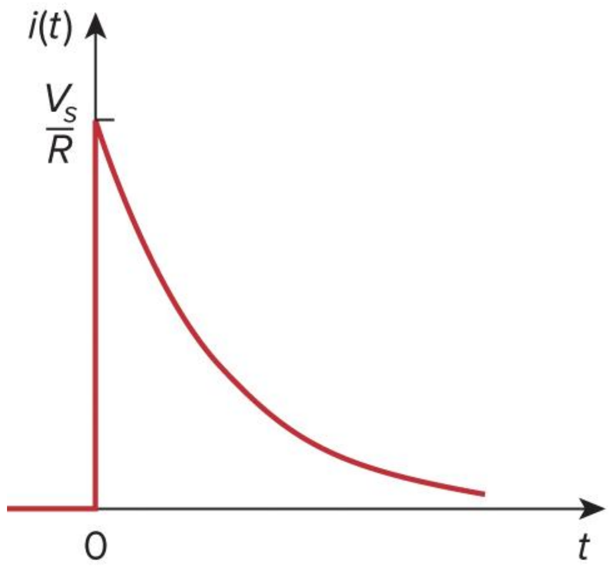
$$i(t) = C\frac{dv}{dt} = C\frac{d}{dt}\left(V_s\left(1 - e^{-\frac{t}{\tau}}\right)\right) \Rightarrow i(t) = C\frac{1}{\tau}V_s e^{-\frac{t}{\tau}} \qquad \tau = RC \qquad t > 0$$

$$i(t) = C\frac{1}{RC}V_s e^{-\frac{t}{\tau}} \Rightarrow i(t) = \frac{V_s}{R}e^{-\frac{t}{\tau}}u(t)$$

Capacitor voltage $\mathrm{v}(t)$



Capacitor current $i(t)$



Step Response of an RC Circuit

- Another way of finding step response of an RC & RL circuits.

$$v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}} \Rightarrow v(t) = V_s + V_0 e^{-\frac{t}{\tau}} - V_s e^{-\frac{t}{\tau}}$$

$$v(t) = V_0 e^{-\frac{t}{\tau}} + V_s(1 - e^{-\frac{t}{\tau}}) \Rightarrow$$ $v(t)$: two components

- Two ways of decomposing into two components:
  - Natural response and forced response
  - Transient response and steady-state response

Step Response of an RC Circuit

- 1st: Natural response and forced response:
  - Total response or complete response can be written as:

Complete response=natural response + forced response
(Storage energy) (Independent source)

$$v = v_n + v_f$$

v_n:natural response

v_f:forced response

$$v_n = V_0 e^{-\frac{t}{\tau}} \Rightarrow \text{Storage energy}$$

$$v_f = V_s(1 - e^{-\frac{t}{\tau}}) \Rightarrow \text{Independent source}$$

- v_n is the natural response that is produced by capacitor.
- v_f is the forced response that is produced by the circuit when external force (a voltage source) is applied.
- Natural response dies out and leaving only the steady-state component of forced response since as time increases, energy stored in the capacitor is reduced.

Step Response of an RC Circuit

- 2nd: Transient response and steady-state response:
  - Total response or complete response can be written as:

Complete response=transient response + steady-state response
(Temporary part) (Permanent part)

$$v = v_t + v_{ss}$$

v_t:transient response

v_{ss}:steady-state response

$$v_t = (V_0 - V_s)e^{-\frac{t}{\tau}}$$ → Temporary part

$$v_{ss} = V_s$$ → Permanent part

- Transient response v_t is temporary response and decay to zero as time approaches infinity.
- The steady-state response v_{ss} remains after the transient response has die out.

Step Response of an RC Circuit

- The complete response of the general equation may be written as:

$$\boxed{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-\frac{t}{\tau}}}$$

- $v(0)$: initial voltage
- $v(\infty)$: final or steady-state value.

- Step response of an RC circuit requires three parameters:
  - The initial capacitor voltage $v(0)$
  - The final capacitor voltage $v(\infty)$
  - The time constant τ

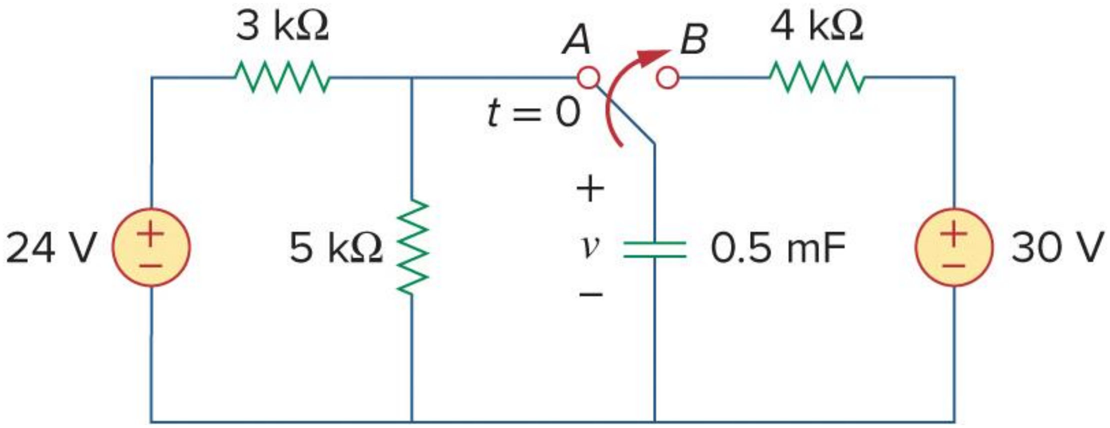
- If the switch changes position at time $t = t_0$ instead of $t = 0$, there is a time delay in the response and the equation becomes:

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-\frac{t-t_0}{\tau}}$$

$v(t_0)$: initial voltage value at $t = t_0$

Example 6

The switch in the circuit shown below has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1\ s$ and $t = 4\ s$.



Solution:

For $t < 0$, the switch is at position *A*. The capacitor acts like an open circuit to dc

$$v(0^-) = \frac{5}{5+3}(24) = 15\ \text{V}$$

Capacitor voltage cannot change instantaneously

$$v(0) = v(0^-) = v(0^+) = 15\ \text{V}$$

For $t > 0$, the switch is at position *B*. Thevenin resistance (equivalent resistance): $R_{Th} = 4\ k\Omega$

$$\tau = R_{\text{Th}}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2\ \text{s}$$

Capacitor acts like an open circuit to dc at steady-state, $v(\infty) = 30\ V$

Complete response:

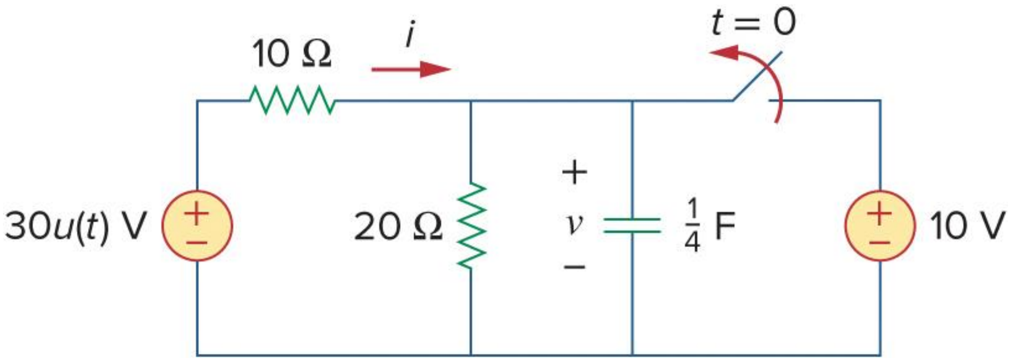
$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t})\ \text{V}$$

At $t = 1$, $\quad v(1) = 30 - 15e^{-0.5} = 20.9\ \text{V}$

At $t = 4$, $\quad v(4) = 30 - 15e^{-2} = 27.97\ \text{V}$

Example 7

In the circuit shown below, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.



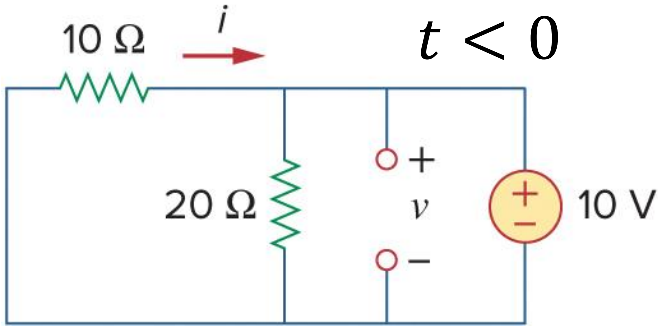
Solution:

By definition of the unit step function:

$$30u(t) = \begin{cases} 0 & t < 0 \\ 30 & t > 0 \end{cases}$$

$$v = 10 \text{ V}$$

$$i = -\frac{v}{10} = -1 \text{ A}$$

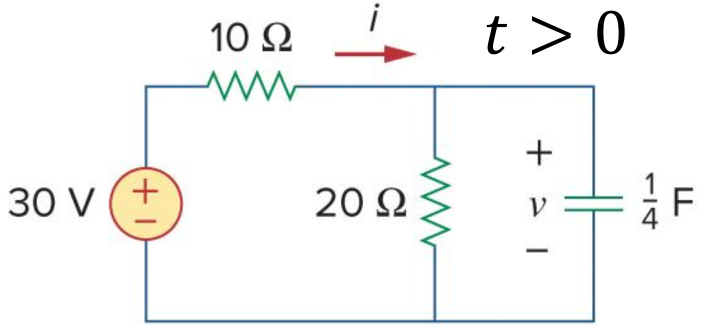


$$v(0) = v(0^-) = 10 \text{ V}$$

$$v(\infty) = \frac{20}{20 + 10}(30) = 20 \text{ V} \qquad R_{\text{Th}} = 10 \,\|\, 20 = \frac{10 \times 20}{30} = \frac{20}{3} \,\Omega$$

$$\tau = R_{\text{Th}} C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3} \text{ s} \qquad v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t}) \text{ V}$$



Apply KCL:

$$i = \frac{v}{20} + C\frac{dv}{dt} = 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t}) \text{ A}$$

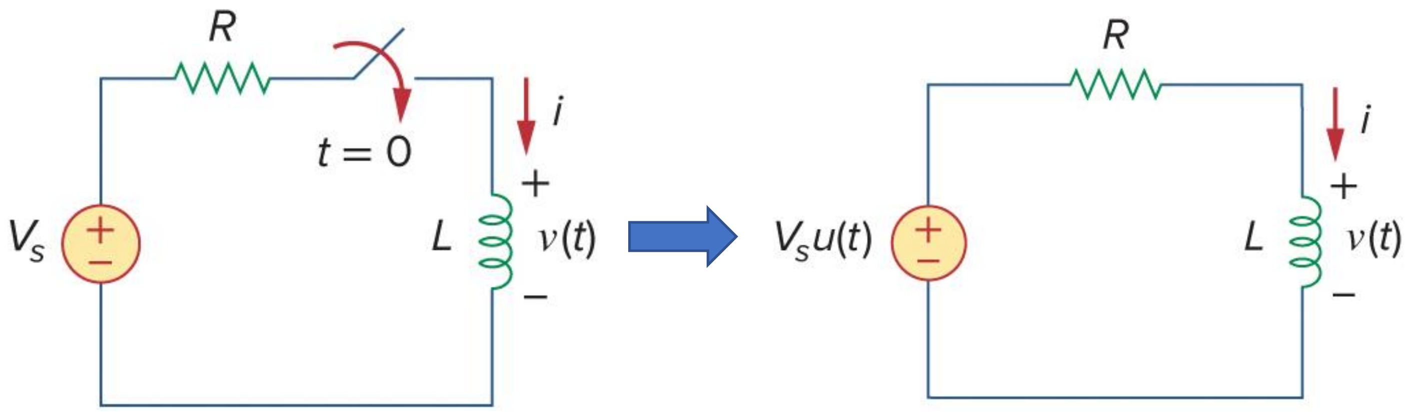
Step Response of an RL circuit

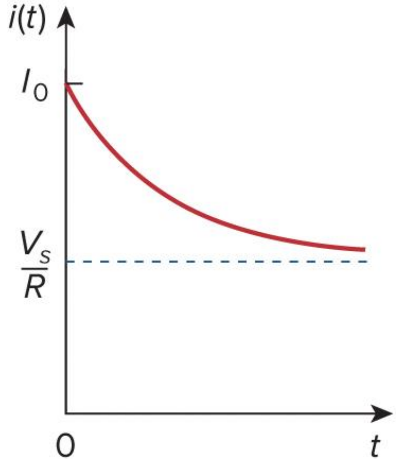
- When the switch is closed, there is a sudden of voltage source is applied and the circuit is replaced by the following circuit (on the right)
- The voltage source (or current source) can be modeled as a step function (aka step response).
- Let the current response is sum of transient and steady-state response

$$i = i_t + i_{ss} \qquad i_t = Ae^{-\frac{t}{\tau}} \qquad \tau = \frac{L}{R} \qquad i_{ss} = \frac{V_s}{R} \qquad i = Ae^{-\frac{t}{\tau}} + \frac{V_s}{R}$$

- Current through inductor cannot change instantaneously: $i(0^+) = i(0^-) = I_0$

At $t = 0$, $I_0 = A + \frac{V_s}{R}$ $\Rightarrow$ $A = I_0 - \frac{V_s}{R}$ $\qquad i(t) = \frac{V_s}{R} + (I_0 - \frac{V_s}{R})e^{-\frac{t}{\tau}}$

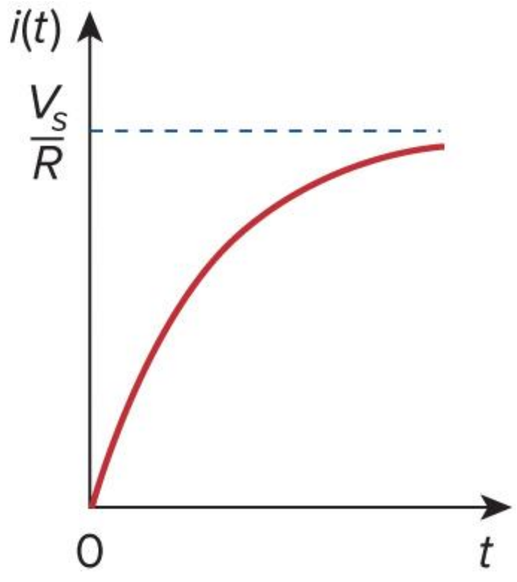




Step Response of an RL circuit

- To find the complete response, RL circuit requires three parameters:
  - The initial inductor current $i(0)$
  - The final inductor current $i(\infty)$
  - The time constant τ

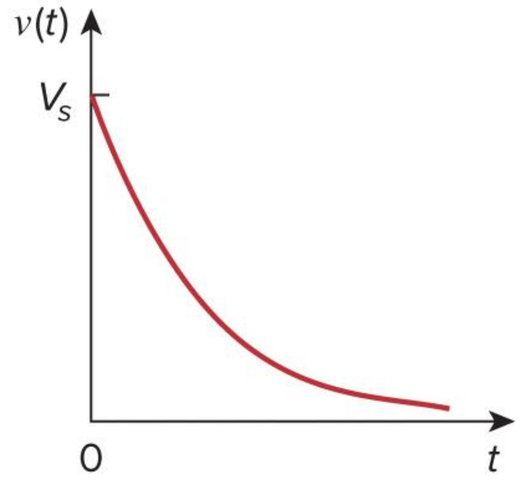
$$\boxed{i(t) = i(\infty) + [i(0) - i(\infty)]e^{-\frac{t}{\tau}}}$$ Complete response



If $I_0 = 0$, $$i(t) = \begin{cases} 0 & t < 0 \\ \frac{V_s}{R}\left(1 - e^{-\frac{t}{\tau}}\right) & t > 0 \end{cases}$$

$$\left.\begin{array}{l} \text{When } t < 0,\ u(t) = 0 \\ \text{When } t > 0,\ u(t) = 1 \end{array}\right\} \quad i(t) = \frac{V_s}{R}\left(1 - e^{-\frac{t}{\tau}}\right)u(t)$$

$$v(t) = L\frac{di}{dt} \Rightarrow v(t) = V_s e^{-\frac{t}{\tau}}$$



Example 8

Find $i(t)$ in the circuit of shown below for $t > 0$. Assume that the switch has been closed for a long time.

Solution:

When $t < 0$, 3 Ω resistor is short-circuited and inductor acts like short circuit (dc condition)

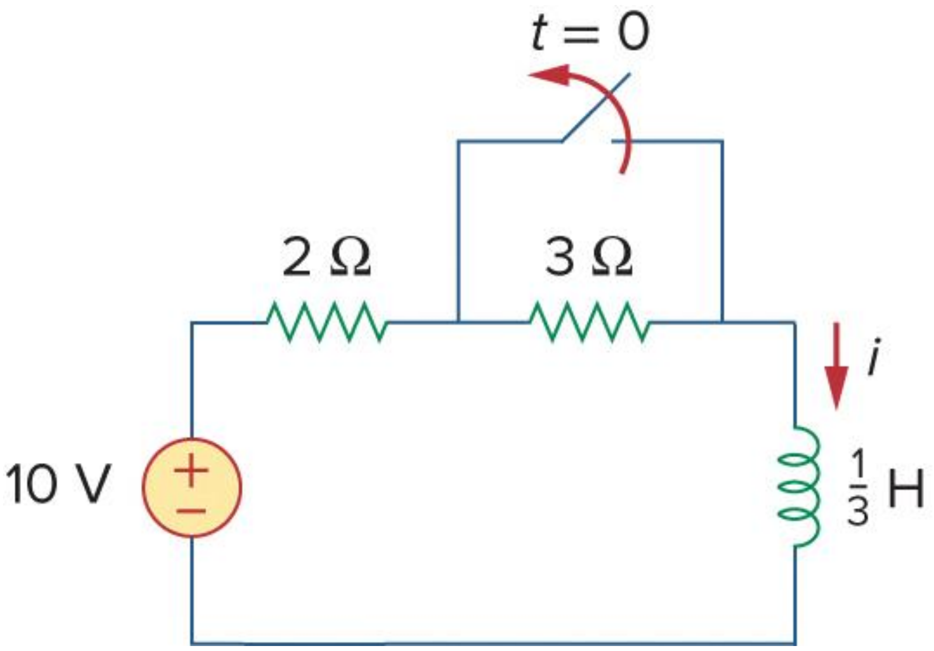
$$i(0^-) = \frac{10}{2} = 5 \text{ A} \qquad i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When $t > 0$, $\quad i(\infty) = \dfrac{10}{2+3} = 2 \text{ A}$

$$R_{\text{Th}} = 2 + 3 = 5\ \Omega \qquad \tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

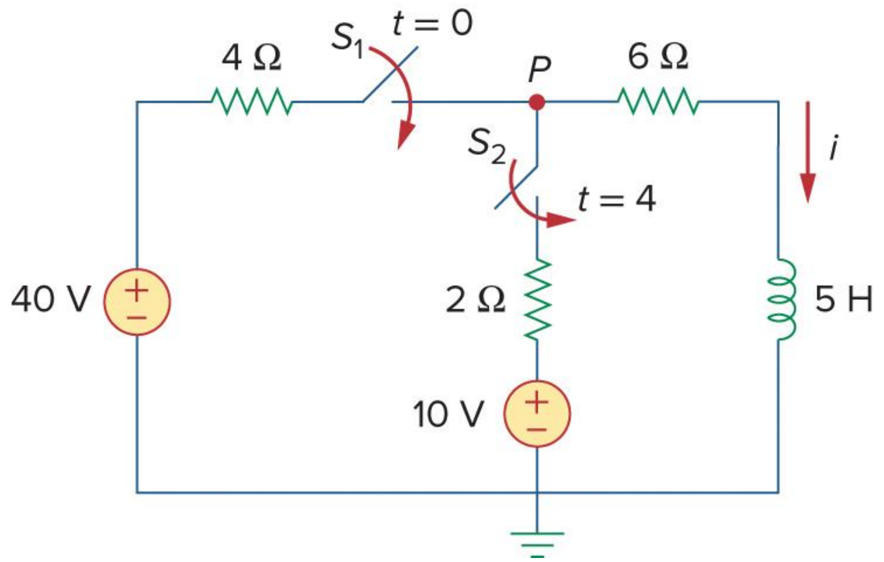
$$= 2 + (5-2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t > 0$$



Example 9

At $t = 0$, switch 1 in the circuit below is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2\ s$ and $t = 5\ s$.

Solution:



For $t < 0$, switches S_1 and S_2 are open so that $i = 0$.

$$i(0^-) = i(0) = i(0^+) = 0$$

For $0 \le t \le 4$, S_1 is closed $\quad i(\infty) = \dfrac{40}{4 + 6} = 4$ A,

$$R_{Th} = 4 + 6 = 10\ \Omega \qquad \tau = \frac{L}{R_{Th}} = \frac{5}{10} = \frac{1}{2}\ \text{s}$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} = 4 + (0 - 4)e^{-2t} = 4(1 - e^{-2t})\ \text{A}, \qquad 0 \le t \le 4$$

For $t \ge 4$, S_2 is closed $\quad i(4) = i(4^-) = 4(1 - e^{-8}) \simeq 4$ A

Using KCL, $\quad \dfrac{40 - v}{4} + \dfrac{10 - v}{2} = \dfrac{v}{6} \Rightarrow v = \dfrac{180}{11}$ V $\qquad i(\infty) = \dfrac{v}{6} = \dfrac{30}{11} = 2.727$ A

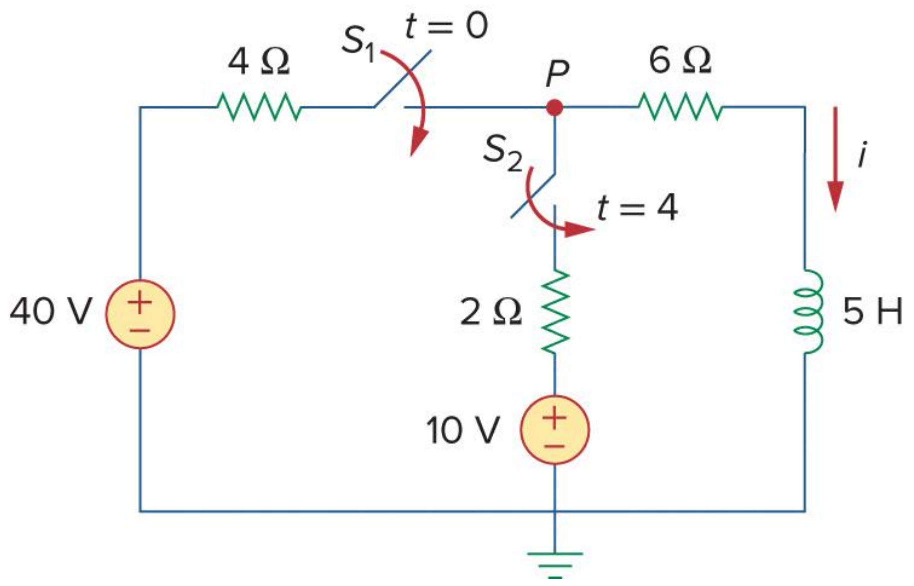
Solution

$$R_{\text{Th}} = 4 \,\|\, 2 + 6 = \frac{4 \times 2}{6} + 6 = \frac{22}{3}\,\Omega \qquad \tau = \frac{L}{R_{\text{Th}}} = \frac{5}{\frac{22}{3}} = \frac{15}{22}\text{ s}$$

$$i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}, \qquad t \geq 4$$

$$i(t) = 2.727 + (4 - 2.727)e^{-(t-4)/\tau}, \qquad \tau = \frac{15}{22}$$

$$= 2.727 + 1.273e^{-1.4667(t-4)}, \qquad t \geq 4$$



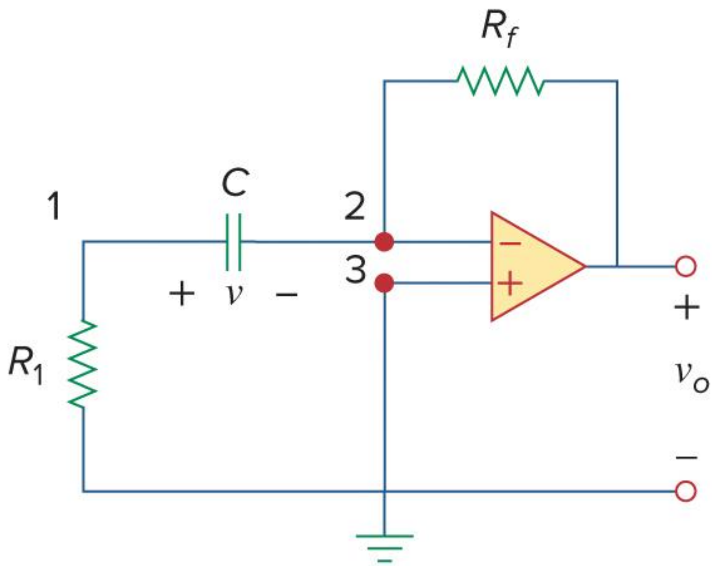
$$i(t) = \begin{cases} 0, & t \leq 0 \\ 4(1 - e^{-2t}), & 0 \leq t \leq 4 \\ 2.727 + 1.273e^{-1.4667(t-4)}, & t \geq 4 \end{cases}$$

At $t = 2$, $\quad i(2) = 4(1 - e^{-4}) = 3.93$ A

At $t = 5$, $\quad i(5) = 2.727 + 1.273e^{-1.4667} = 3.02$ A

First-Order Op Amp Circuit: Example 10

For the op amp circuit shown below, find v_0 for $t > 0$, given that $v(0) = 3\ V$. Let $R_f = 80\ k\Omega$, $R_1 = 20\ k\Omega$, and $C = 5\ \mu F$.



Solution:

Method 1: KCL at node 1: $\dfrac{0 - v_1}{R_1} = C\dfrac{dv}{dt}$

$v_1 = v \qquad \dfrac{dv}{dt} + \dfrac{v}{CR_1} = 0$ Same equation as source free RC

$$v(t) = V_0 e^{-t/\tau}, \qquad \tau = R_1 C \qquad v(0) = 3 = V_0$$

$$\tau = 20 \times 10^3 \times 5 \times 10^{-6} = 0.1 \qquad v(t) = 3e^{-10t}$$

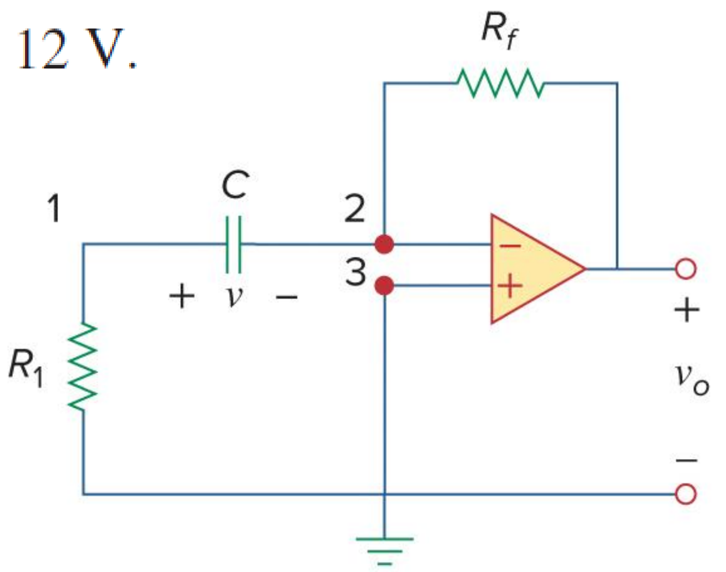
Applying KCL at node 2 gives $C\dfrac{dv}{dt} = \dfrac{0 - v_o}{R_f} \qquad v_o = -R_f C\dfrac{dv}{dt}$

$$v_o = -80 \times 10^3 \times 5 \times 10^{-6}(-30e^{-10t}) = 12e^{-10t}\ \text{V}, \qquad t > 0$$

Solution

Method 2: $v(0^+) = v(0^-) = 3 \text{ V},$

apply KCL at node 2 $\frac{3}{20{,}000} + \frac{0 - v_o(0^+)}{80{,}000} = 0 \quad v_o(0^+) = 12 \text{ V}.$

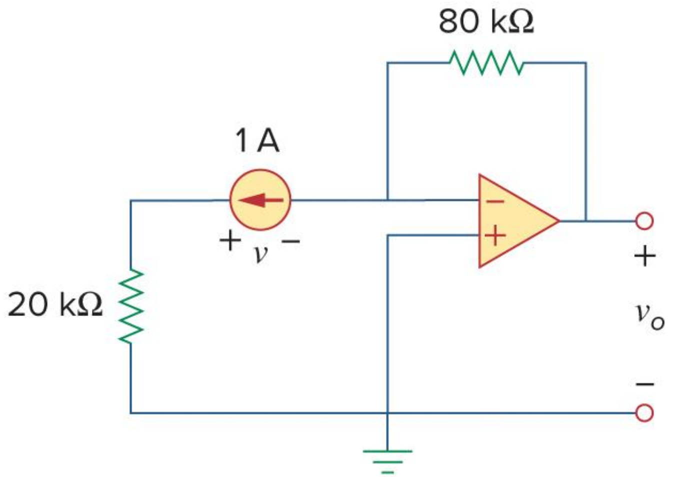


Since the circuit is source free, $v(\infty) = 0$ V.

To find τ, we need to find R_{eq}. Remove capacitor and place 1 A current source (source free circuit). Apply KVL to the input loop:

$$20{,}000(1) - v = 0 \quad \Rightarrow \quad v = 20 \text{ kV}$$

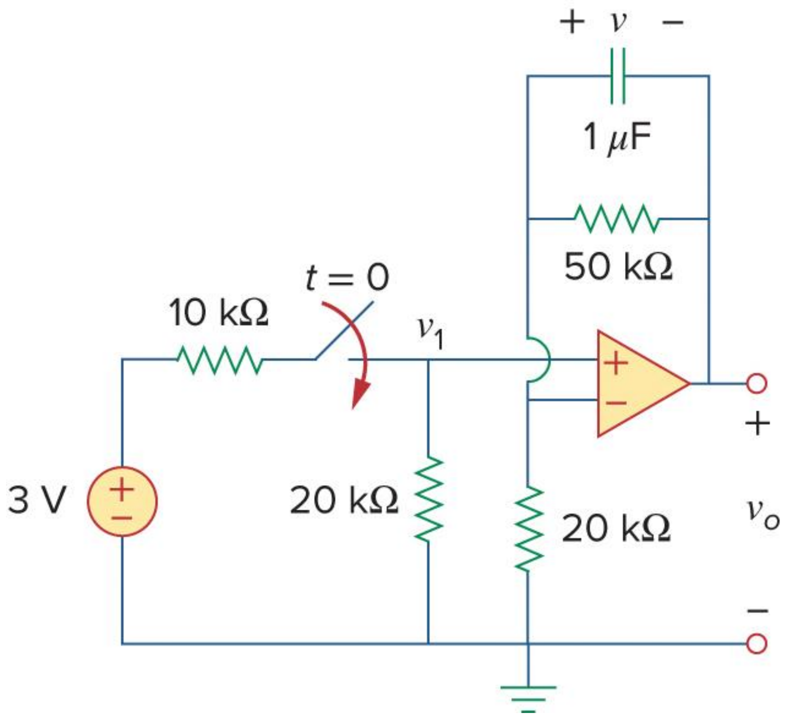
$$R_{\text{eq}} = \frac{v}{1} = 20 \text{ k}\Omega \qquad \tau = R_{\text{eq}}C = 0.1.$$



$$\begin{aligned} v_o(t) &= v_o(\infty) + [v_o(0) - v_o(\infty)]e^{-t/\tau} \\ &= 0 + (12 - 0)e^{-10t} = 12e^{-10t} \text{ V}, \qquad t > 0 \end{aligned}$$

Example 11

Determine $v(t)$ and $v_0(t)$ in the circuit shown below.



Solution:

Since we will find step response, we can write the following equation:

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}, \qquad t > 0$$

$$\tau = RC = 50 \times 10^3 \times 10^{-6} = 0.05$$

$t < 0$, the switch is open and no voltage across capacitor

$$v(0) = 0.$$

$t > 0$, $\quad v_1 = \dfrac{20}{20 + 10} 3 = 2 \text{ V}$

$$\boxed{v_o(\infty) = \left(1 + \frac{50}{20}\right) v_1} = 3.5 \times 2 = 7 \text{ V}$$

Capacitor acts like an open circuit to dc and op amp circuit behaves like an noninverting op amp.

Noninverting op amp:

$$v_0 = (1 + \frac{R_f}{R_1}) v_1$$

$v_1 - v_o = v \qquad v(\infty) = 2 - 7 = -5 \text{ V}$

$$v(t) = -5 + [0 - (-5)]e^{-20t} = 5(e^{-20t} - 1) \text{ V}, \qquad t > 0$$

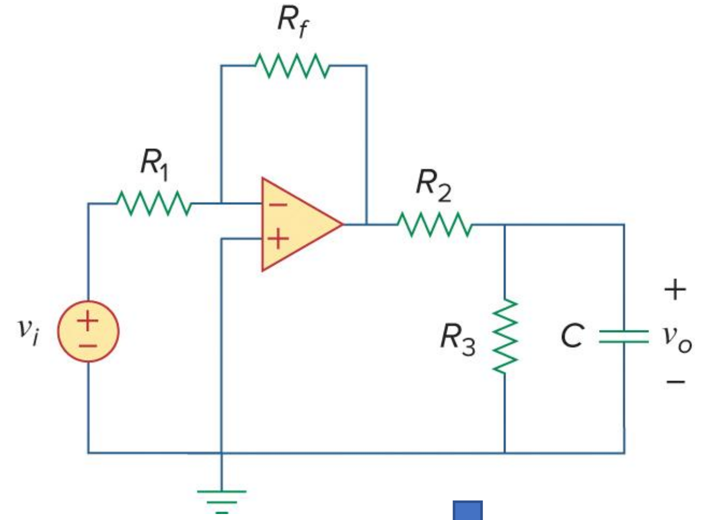
$$v_o(t) = v_1(t) - v(t) = 7 - 5e^{-20t} \text{ V}, \qquad t > 0$$

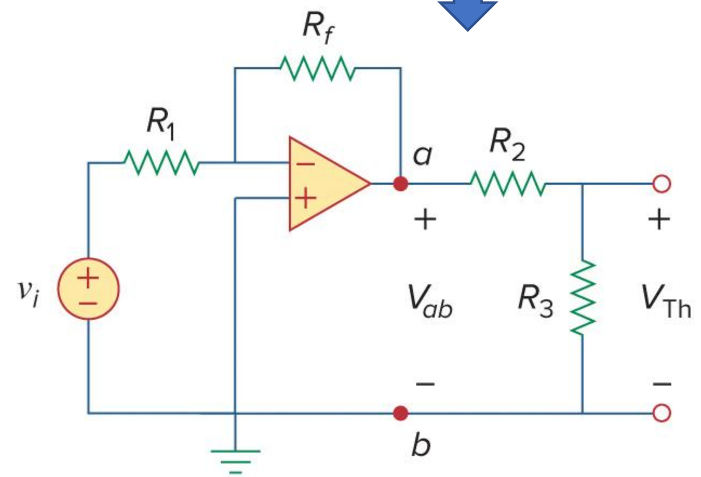
Example 12

Find the step response $v_0(t)$ for $t > 0$ in the op amp circuit shown below. Let $v_i = 2u(t)\ V$, $R_1 = 20\ k\Omega$, $R_f = 50\ k\Omega$, $R_2 = R_3 = 10\ k\Omega$, and $C = 2\ \mu F$.

Solution:

Using Thevenin theorem, we may simplify the circuit. Remove the capacitor and find the Thevenin equivalent



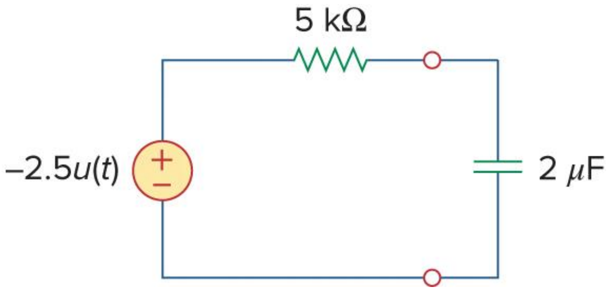


$$V_{ab} = -\frac{R_f}{R_1}v_i \qquad V_{\mathrm{Th}} = \frac{R_3}{R_2 + R_3}V_{ab} = -\frac{R_3}{R_2 + R_3}\frac{R_f}{R_1}v_i$$

$$V_{\mathrm{Th}} = -\frac{R_3}{R_2 + R_3}\frac{R_f}{R_1}v_i = -\frac{10}{20}\frac{50}{20}2u(t) = -2.5u(t)$$

$$R_{\mathrm{Th}} = \frac{R_2 R_3}{R_2 + R_3} = 5\ \mathrm{k}\Omega \longrightarrow$$

R_{Th} can be found by turning off the input voltage v_i. Doing so, v_{ab} will be zero



$$v_o(t) = -2.5(1 - e^{-t/\tau})u(t)$$

$$\tau = R_{\mathrm{Th}}C = 5 \times 10^3 \times 2 \times 10^{-6} = 0.01$$

$$v_o(t) = 2.5(e^{-100t} - 1)u(t)\ \mathrm{V}$$