MAT 271E: PROBABILITY AND STATISTICS

PROF. DR. CANAN SARICAM

WEEK 3

PROBABILITY

BASIC PROBABILITY CONCEPTS

PROBABILITY

It is the measure of the likeliness that an event will occur.

Random trials

Sample spaces

Events

Random trials

Activities that have uncertain or chance outcomes are common.

EXAMPLES:

- ✓ A farmer plants and eventually harvests a crop. The crop yield is uncertain in advance because of the chance influences of weather and other natural factors. → different crop yields
- ✓ A laboratory centrifuge experiences operational breakdowns from time to time. The cause of the next breakdown is uncertain, there being several possible causes, including different kinds of mechanical and electrical failures. toss dice
- ✓ When we toss the coin, all we want to know is whether we get a " heads" or a "tails".

Random trials

A random trial is an activity in which there are two or more different possible outcomes, and uncertainty exists in advance as to which outcome prevail.

Or;

An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment.**

Sample spaces and basic outcomes

The outcomes of a random trial are usually defined according to the purpose of the study.

The different possible outcomes of a random trial are called the **basic outcomes**, and a set of all basic outcomes is called the **sample space**. The sample space is denoted as S.

Sample spaces and basic outcomes

EXAMPLE:

Assume a cylinder of air is analysed for the presence of a rare molecule. The possible outcomes of this experiment might be summarized simply as {yes} or {no} depending on whether or not the selected cylinder contains the molecule. The sample space contains only two possible outcomes, $S = \{yes, no\}$. Coin

Sample spaces and basic outcomes

EXAMPLE:

Cosider the experiment of selecting two component parts and classifying each as meeting or not meeting the electrical timing requirements of the product.An outcome of this experiment in which the first part selected is acceptable and the second part selected is not acceptable can be denoted as AN. Using this notation, we can represent the sample space of the experiment as the set.

$$S = \{AA, AN, NA, NN\},$$

where the first letter in each pair identifies the classification of the first part and the second letter identifies the classification of the second part.

Sample spaces and basic outcomes

In random experiments that involve selecting items from a batch, we will indicate whether or not a selected item is replaced before the next one is selected. For example, if the batch consists of the three items {a,b,c} and our experiment is to select two items **without replacement**, we can present the sample space as S = {ab, ac, ba, bc, ca, cb }. However, if items are replaced before the next one is selected the sampling is referred to as **with replacement**. Then, the possible outcomes are S = {aa, ab, ac, ba, bb, bc, ca, cb, cc }.

Sample spaces and basic outcomes

SAMPLE SPACE

Discrete

- Finite
- Countably infinite

Continuous

Sample spaces and basic outcomes

The coin-tossing experiment has a discrete finite sample space. Each of the two outcomes may be uniquely identified by a label- heads or tails. The number of outcomes is 2, a finite number.

The outcomes of a countably infinite sample space may be placed in one-to-one correspondence with the set of positive integers. For example, the sample space describing the possible number of heads that can occur in repeated flips of a coin before the first tails occurs is a countably infinite sample space.

Sample spaces and basic outcomes

The second kind of sample space, the continuous sample space, is always uncountably infinite because it is defined on a continuum. Suppose we are measuring the amount of time between the emission of two different particles from a radioactive substance. Our physical environment is a truly continuous phenomenon, the number of possible intervals of time that might occur is infinite. No matter how short an interval you select on a continuum, an infinite number of possibilities still exist on the interval. Obviously, it is impossible to label each and every element of a continuous sample space.

Events

- ✓ Often we are interested in a collection of related outcomes from a random experiment.
- ✓ An **event** is a subset of the basic outcomes of the sample space.
- ✓ An event is said to occur if any one of its basic outcomes is realized in the random trial.

Events

In the example of selecting two component parts, suppose that the set of all outcomes for which at least one part is not acceptable is denoted as E_1. Then,

$$E_1 = \{ AN, NA, NN \}$$

The event that both parts are not acceptable, denoted as E_2 contains only the single outcome $E_2 = \{ NN \}$. Other examples of events are $E_3 = \varnothing$, the **null set**.

Events

Some of the basic set operations are summarized below in terms of events:
The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as

$$E_1 \cup E_2.$$

The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as .

$$E_1 \cap E_2.$$

The **complement** of an event in a sample space is the set of outcomes in the sample
space that are not in the event. We denote the component of the event E as E'.

Events

In the example of selecting two component parts, if E_1={AA,AN,NA} and E_2= { AN, NA, NN }, then

$$E_1 \cup E_2 = \{AA, AN, NA, NN\} = S$$

$$E_1 \cap E_2 = \{ AN, NA \}$$

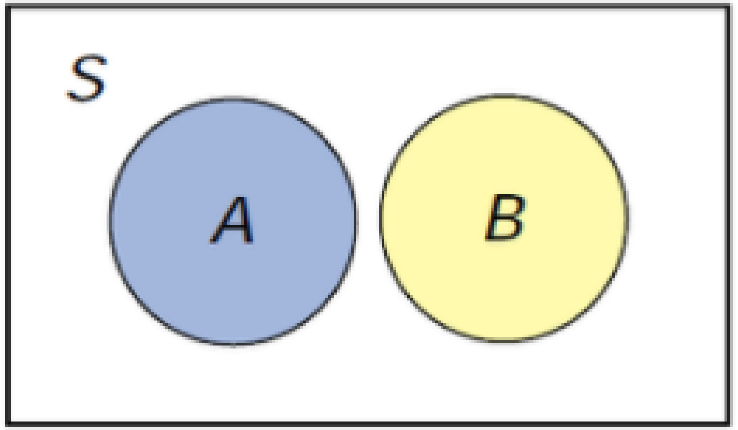
$$E_1' = \{ NN \} \qquad E_2' = \{ AA \}$$

Some events of a sample space have no basic outcomes in common. They are said to be mutually exclusive events. Two events with no outcomes in common have an important relationship.

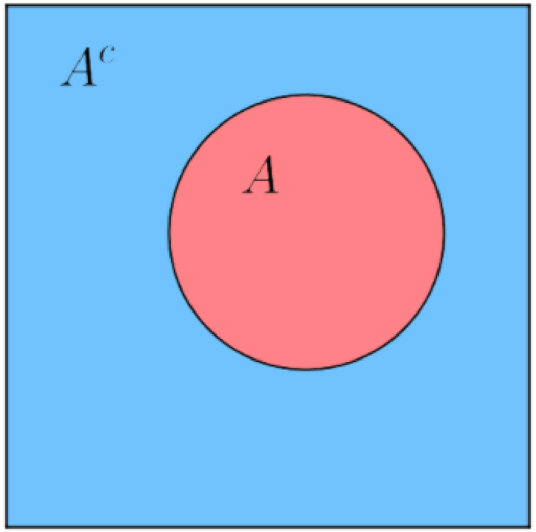
Two events, denoted as E_1 and E_2, such that $E_1 \cap E_2 = \varnothing$, are said to be **mutually exclusive**.

Events

In the example of selecting two component parts, if E_1 and E_2 are not mutually exclusive. However, the two events $E_1 = \{ AA, AN, NA \}$ and $E_3 = \{ NN \}$ are mutually exclusive. An event E and its complement E', are always mutually exclusive.



Venn diagram of mutually exclusive events of A and B



Venn diagram of mutually exclusive events of A and its complement A’

Events

Additional results involving events are summarized below. The definition of the complement of an event implies that

$$(E')' = E$$

The distributive law for set operations implies that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \qquad \text{and} \qquad (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

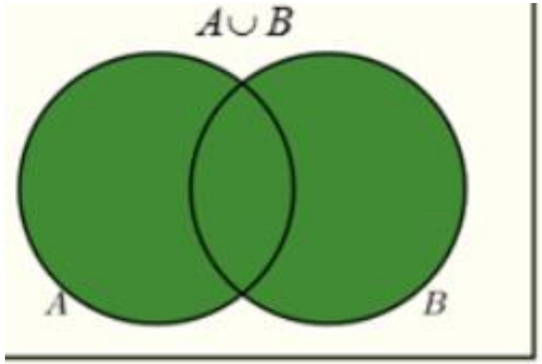
DeMorgan's laws imply that

$$(A \cup B)' = A' \cap B' \qquad \text{and} \qquad (A \cap B)' = A' \cup B'$$

Also, remember that

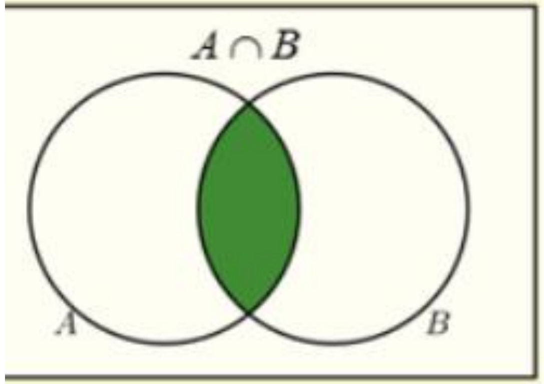
$$A \cap B = B \cap A \qquad \text{and} \qquad A \cup B = B \cup A$$

Venn diagrams



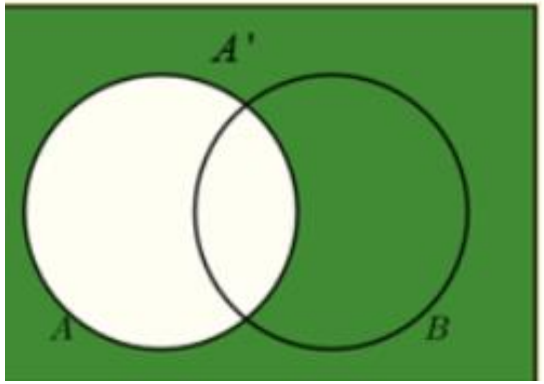
A union B

Elements that belong to either *A* or *B* or both.



A intersect B

Elements that belong to both *A* and *B*.



A complement

Elements that don't belong to *A*.

Example 1

Measurements of the time needed to complete a chemical reaction might be modeled with the sample space $S = R^+$, the set of positive real numbers. Let

$$E_1 = \{x \mid 1 \le x < 10\} \qquad \text{and} \qquad E_2 = \{x \mid 3 < x < 118\}$$

Then,

$$E_1 \cup E_2 = \{x \mid 1 \le x < 118\} \qquad \text{and} \qquad E_1 \cap E_2 = \{x \mid 3 < x < 10\}$$

Also,

$$E_1' = \{x \mid x \ge 10\} \qquad \text{and} \qquad E_1' \cap E_2 = \{x \mid 10 \le x < 118\}$$

Example 2

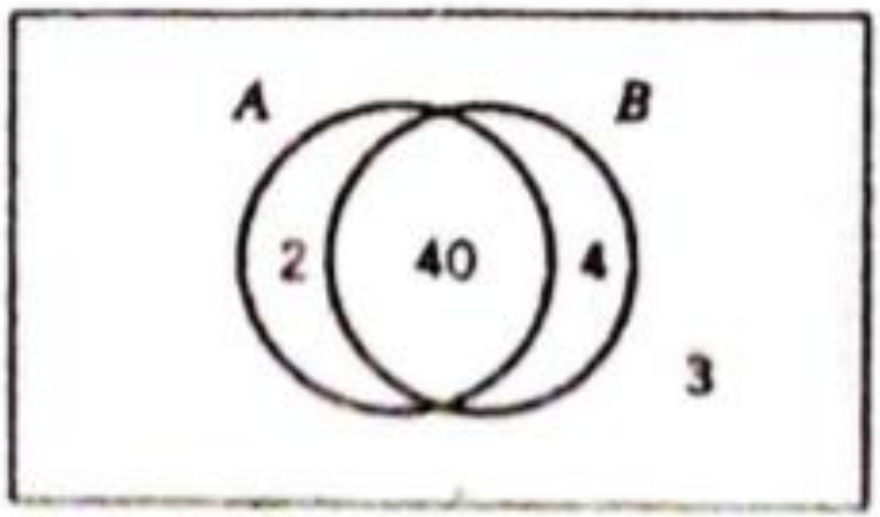
Samples of polycarbonate plastic are analyzed for scratch resistance and shock resistance. The results from 49 samples are summarized as follows.

		shock resistance	
		high	low
scratch resistance	high	40	4
	low	2	3

Let A denote the event that a sample has high shock resistance, and let B denote the event that a sample has high scratch resistance.

Determine the number of samples in $A \cap B$, A', and $A \cup B$.

Example 2



A = high-shock resistance
B = high-scratch resistance

Example 3

Example 4.4 A newly manufactured switching gear may be used in any one of the three modes, in any two of three modes, or in all three modes: manual (M), semiautomatic (S), and automatic (A). In tests on 100 gears, the following uses were found.

Number	Use
15	$M \cap S \cap A$
20	$M \cap S$
10	A-only
20	$S \cap A$
Remainder	M-only

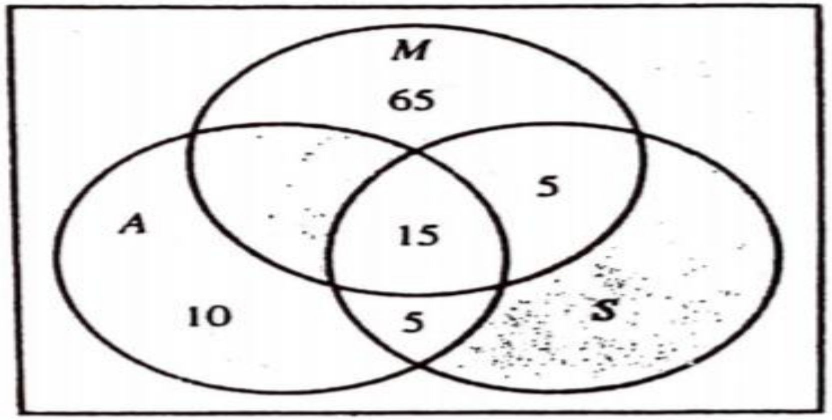
(a) Define M, S, and A as sets of gears used. Draw a Venn diagram, and show the number of gears in each use.

(b) How many gears are in M? in S? in $(M \cup S)$? in $(M \cup S)'$? in $M \cup S'$?

Example 3

Example 4.4 A newly manufactured switching gear may be used in any one of the three modes, in any two of three modes, or in all three modes: manual (M), semiautomatic (S), and automatic (A). In tests on 100 gears, the following uses were found.

Number	Use
15	$M \cap S \cap A$
20	$M \cap S$
10	A-only
20	$S \cap A$
Remainder	M-only



(a) Define M, S, and A as sets of gears used. Draw a Venn diagram, and show the number of gears in each use.

(b) How many gears are in M? in S? in (M ∪ S)? in (M ∪ S)′? in M ∪ S′?

85-25-90-10-95

PROBABILITY AND OPERATIONAL RULES

Probability and operational rules

- **Probability** is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur. “The chance of rain today is 30%” is a statement that quantifies our feeling about the possibility of rain.
- The likelihood of an outcome is quantified by assigning a number from the interval [0, 1] to the outcome (or a percentage from 0 to 100%).
- Higher numbers indicate that the outcome is more likely than lower numbers.
- A probability of 0 indicates an outcome will not occur.
- A probability of 1 indicates an outcome will occur with certainty.

Probability and operational rules

- Whenever a sample space consists of **N** possible outcomes that are equally likely, the probability of each outcome is **1/N**.

- For a discrete sample space, the probability of an event E, denoted as P(E), equals the sum of the probabilities of the outcomes in E.

Probability and operational rules

If S is the sample space and E is any event in a random experiment, the probability of an event E, expressed as P(E), is used to denote the probability that event E will occur, and equals to the sum of the probabilities of the outcomes in E.

(1) The sum of the probabilities of all basic outcomes in the sample space is 1.

$$P(S) = 1$$

(2) Every event has a probability value between 0 and 1.

$$0 \leq P(E) \leq 1$$

(3) The probability of a null set is zero.

$$P(\varnothing) = 0$$

(4) **The complementation rule.** If the complement of the event E is E',

$$P(E) + P(E') = 1$$

Probability and operational rules

(5) If event A is contained in event B, then we should have $P(A) \leq P(B)$.

(6) **Addition Rule.** We often wish to obtain the probability for occurrence of one event or another.

6.1. For any events A and B which are not mutually exclusive:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Subtracting $P(A \cap B)$ is required because $P(A) + P(B)$ adds $P(A \cap B)$ into the sum two times.

Probability and operational rules

6.2. Similar relations apply in the case of three or more events, although the formulas become very complex. For the case of three events;

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

6.3. If A, B, and C are mutually exclusive events, all intersections have a probability of zero, and this is a special case of the general addition rule.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Probability and operational rules

In general, a collection of events, $E_1, E_2, \ldots, E_k$, is said to be mutually exclusive if for all pairs,

$$E_i \cap E_j = \varnothing$$

For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k)$$

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Example 1

A random experiment can result in one of the outcomes $\{a, b, c, d\}$ with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event $\{a, b\}$, B the event $\{b, c, d\}$, and C the event $\{d\}$. Then,

$$P(A) = 0.1 + 0.3 = 0.4$$

$$P(B) = 0.3 + 0.5 + 0.1 = 0.9$$

$$P(C) = 0.1$$

Also, $P(A') = 0.6$, $P(B') = 0.1$, and $P(C') = 0.9$. Furthermore, because $A \cap B = \{b\}$, $P(A \cap B) = 0.3$. Because $A \cup B = \{a, b, c, d\}$, $P(A \cup B) = 0.1 + 0.3 + 0.5 + 0.1 = 1$. Because $A \cap C$ is the null set, $P(A \cap C) = 0$.

Example 2

A visual inspection of a location on wafers from a semiconductor manufacturing process resulted in the following table:

Number of Contamination Particles	Proportion of Wafers
0	0.40
1	0.20
2	0.15
3	0.10
4	0.05
5 or more	0.10

a) If one wafer is selected randomly from this process and the location is inspected, what is the probability that it contains no particles?
b) What is the probability that a wafer contains three or more particles in the inspected location?
c) What is the probability that a wafer contains either 0 or more than three particles in the inspected location?

Example 2

A visual inspection of a location on wafers from a semiconductor manufacturing process resulted in the following table:

Number of Contamination Particles	Proportion of Wafers
0	0.40
1	0.20
2	0.15
3	0.10
4	0.05
5 or more	0.10

a) If one wafer is selected randomly from this process and the location is inspected, what is the probability that it contains no particles? 0.4

b) What is the probability that a wafer contains three or more particles in the inspected location?

$0.1+0.05+0.1=0.25$

c) What is the probability that a wafer contains either 0 or more than three particles in the inspected location?

$0.4+0.05+0.1=0.55$

Example 3

A manufacturer requires that all new product designs be evaluated by potential customers in order to benefit from customer inputs early in the design cycle. Based on historical data, if two customers evaluate the product, and they decide independently whether they like the product, the sample space and probabilities can be modeled as follows.

Customer 1	Customer 2	Probability
(approve,	approve)	0.04
(approve,	change)	0.16
(change,	approve)	0.16
(change,	change)	0.64

a) Let E denote the event that both customers approve the design. Find P(E).

b) Let G denote the event that at least one customer approves the design. Find P(G).

c) Let H denote the event that the second customer approves the design. Find P(H).

Example 3

A manufacturer requires that all new product designs be evaluated by potential customers in order to benefit from customer inputs early in the design cycle. Based on historical data, if two customers evaluate the product, and they decide independently whether they like the product, the sample space and probabilities can be modeled as follows.

Customer 1	Customer 2	Probability
(approve,	approve)	0.04
(approve,	change)	0.16
(change,	approve)	0.16
(change,	change)	0.64

a) Let E denote the event that both customers approve the design. Find P(E). 0.04

b) Let G denote the event that at least one customer approves the design. Find P(G). 0.16+0.16+0.04=0.36

c) Let H denote the event that the second customer approves the design. Find P(H). 0.04+0.16=0.20

Example 4

Table 4-1 Wafers in Semiconductor Manufacturing

		Center of Sputtering Tool	
		no	yes
High	no	514	68
Contamination	yes	112	246

a) Let A denote the event that the wafer contains high level of contamination. Find P(A).
b) Let B denote the event that the wafer is in the center of a sputtering tool. Find P(B).
c) Find P(A∩B).
d) Find P(AUB).
e) Let E be the event that the wafer is neither from the center of the sputtering tool nor contains high levels of contamination. Find P(E).

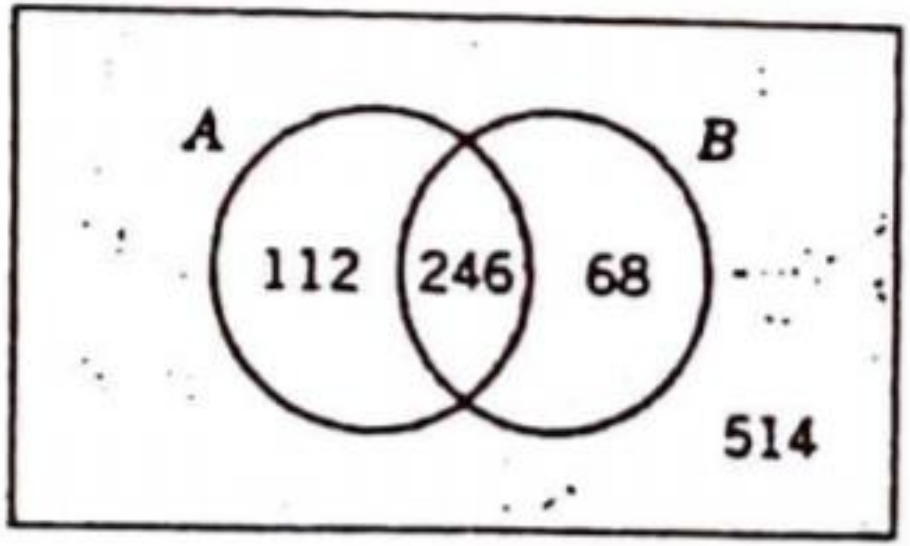
Example 4

Table 4-1 Wafers in Semiconductor Manufacturing

		Center of Sputtering Tool	
		no	yes
High Contamination	no	514	68
	yes	112	246

a) Let A denote the event that the wafer contains high level of contamination. Find P(A). 358/940
b) Let B denote the event that the wafer is in the center of a sputtering tool. Find P(B). 314/940
c) Find P(A∩B). 246/940
d) Find P(AUB). 358/940+314/940-246/940=426/940
e) Let E be the event that the wafer is neither from the center of the sputtering tool nor contains high levels of contamination. Find P(E). 514/940

Example 4



S = 940 wafers

A = high contamination
B = center of sputtering tool

Venn diagram for Example

Example 5

Example 4.11. A vendor's experience has shown that, in units of a particular product, 4 out of 100 electronic components have fabrication errors and 3 out of 100 have both fabrication errors and the presence of impurities. If a particular bin contains components of which 8% have impurities, what is the probability of finding, in a unit from that bin, (a) fabrication errors or impurities, (b) neither fabrication errors nor impurities, and (c) only impurities?

Example 5

Example 4.11. A vendor's experience has shown that, in units of a particular product, 4 out of 100 electronic components have fabrication errors and 3 out of 100 have both fabrication errors and the presence of impurities. If a particular bin contains components of which 8% have impurities, what is the probability of finding, in a unit from that bin, (a) fabrication errors or impurities, (b) neither fabrication errors nor impurities, and (c) only impurities?

Let I be the process of impurities and E be the presence of fabrication errors

a)P(IUE)=0.08+0.04-0.03=0.09

b)P(IUE)'=1-0.09=0.91

c)P(I-only)=0.08-0.03=0.05