THEOREM 1 The Linear First-Order Equation

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0 \tag{11}$$

has a unique solution $y(x)$ on I, given by the formula in Eq. (6) with an appropriate value of C.

Remark[1]: This theorem gives a solution on the entire interval I for a linear diff. equation, in contrast with previous existence and uniqueness theorem, which guarantees only a solution on a possibly smaller interval.

Remark[2]: A linear first-order differential equation has no singular solutions.

Remark[3]: The appropriate value of the constant C can be selected "automatically" by writing

$$\rho(x) = \exp\left(\int_{x_0}^{x} P(t)\,dt\right).$$

$$y(x) = \frac{1}{\rho(x)}\left[y_0 + \int_{x_0}^{x} \rho(x)\,Q(t)\,dt\right].$$

$$\rho(x)\frac{dy}{dx} + \rho(x)P(x)\,y = \rho(x)\,Q(x)$$

$$\frac{d}{dx}\left(\rho(x)\cdot y(x)\right) = \rho(x)\,Q(x)$$

$$\int_{x_0}^{x}\frac{d}{dx}\left(\rho(t)\cdot y(t)\right)dt = \int_{x_0}^{x}\rho(x)\,Q(x)\,dx$$

$$\rho(t)\,y(t)\Big|_{x_0}^{x} = \int_{x_0}^{x}\rho(t)\,Q(t)\,dt$$

$$\rho(x)\,y(x) - \underbrace{\rho(x_0)}_{e^{\int_{x_0}^{x_0}p(t)dt}=1}\cdot y(x_0) = \int_{x_0}^{x}\rho(t)\,Q(t)\,dt$$

$$y(x) = \frac{1}{\rho(x)}\cdot\left[\,y(x_0) + \int_{x_0}^{x}\rho(t)\,Q(t)\,dt\,\right].$$

Example:

$y' + p(x)y = Q(x)$

Solve the initial value problem

$$x^2\frac{dy}{dx} + xy = \sin x, \quad y(\underset{x_0}{\underset{\downarrow}{1}}) = y_0.$$

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = \frac{\sin x}{x^2}$$

$P(x) = \frac{1}{x}$, $Q(x) = \frac{\sin x}{x^2}$ are continuous on $(-\infty, 0) \cup (0, \infty)$.

Since $x_0 = 1 \in (0, \infty)$. I.V.P has a unique solution on $(0, \infty)$.

$$\rho(x) = e^{\int_{x_0}^{x} p(t)\,dt} = \exp\left[\int_1^x \frac{1}{t}\,dt\right] = \exp\left(\ln t \Big|_1^x\right)$$

$$= \exp(\ln x - \underbrace{\ln 1}_{=0})$$

$$= e^{\ln x} = x.$$

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = \frac{\sin x}{x^2}$$

$$x\frac{dy}{dx} + y = \frac{\sin x}{x}$$

$$\frac{d}{dx}(x \cdot y) = \frac{\sin x}{x}$$

$$\int_1^x \frac{d}{dt}(t \cdot y)\, dt = \int_1^x \frac{\sin t}{t}\, dt$$

$$t \cdot y(t) \Big|_1^x = \int_1^x \frac{\sin t}{t}\, dt$$

$$x \cdot y(x) - y(1) = \int_1^x \frac{\sin t}{t}\, dt \Rightarrow y(x) = \frac{1}{x} \cdot \left[y_0 + \int_1^x \frac{\sin t}{t}\, dt \right]$$

Properties of Natural Logs

Product property	$\ln ab = \ln a + \ln b$
Quotient property	$\ln \frac{a}{b} = \ln a - \ln b$
Power property	$\ln m^p = p \ln m$
Exponential & Logarithmic	$e^{\ln x} = x$
Inverse property	$\ln e^x = x$ for $x > 0$
One-to-one property of exponents	if $e^x = e^y$, then $x = y$
One-to-one property of logarithms	if $\ln x = \ln y$, then $x = y$

Example: Solve the differential equation

$$(x + ye^{y})\frac{dy}{dx} = 1$$

by regarding y as the independent variable rather than x.

$$(x+ye^{y})\frac{dy}{dx}\,dx = (1)\,dx \Rightarrow (x+ye^{y})\,dy = dx$$

$$\Rightarrow \frac{dx}{dy} = (x+ye^{y}) \qquad (x(y))$$

$$\frac{dx}{dy} = x + ye^{y} \Rightarrow \frac{dx}{dy} \underset{P(y)}{\underbrace{-1}} x = \underset{Q(y)}{\underbrace{y.e^{y}}}$$

$$* \quad \rho(y) = \exp\left(\int(-1)\,dy\right) = e^{-y}$$

$$e^{-y}\frac{dx}{dy} - xe^{-y} = y \Rightarrow \frac{d}{dy}(xe^{-y}) = y \Rightarrow xe^{-y} = \int y + c$$

Substitution Methods and Exact Equations

Solve the differential equation

$$\frac{dy}{dx} = (x + y + 3)^2.$$

Define $v = x+y+3 \Rightarrow y = v - x - 3 \qquad \frac{dy}{dx} = \frac{dv}{dx} - 1$

$(x,y) \leftrightarrow (x,v)$

$$\frac{dv}{dx} - 1 = v^2 \Rightarrow \frac{dv}{dx} = v^2+1 \Rightarrow \frac{dv}{v^2+1} = dx$$

$$\Rightarrow \quad x = \int \frac{dv}{v^2+1} = \tan^{-1} v + C = \tan^{-1}(x+y+3) + C$$

$$\tan^{-1}(x+y+3) = x - C \Rightarrow \tan(x-C) = x+y+3$$

$$\rightsquigarrow y(x) = -x-3+\tan(x-C).$$

Substitution Method:

$$\frac{dy}{dx} = f(x,y)$$

Define $v = \alpha(x,y) \Rightarrow y = \beta(x,v)$

$$\frac{dy}{dx} = \frac{\partial \beta}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial \beta}{\partial v} \cdot \frac{dv}{dx} = f\left(x, \beta(x,v)\right)$$

$$\Rightarrow \quad \frac{dv}{dx} = g(x,v).$$

If this new equation is either seperable or linear, then we can apply the methods of preceding sections to solve it.

Example: Solve the differential equation

$$2x\, e^{2y} \frac{dy}{dx} = 3x^4 + e^{2y}$$

by substituting $v = e^{2y}$.

$\rightarrow \left(e^{2y}\right)' = (2y)' \cdot e^{2y}$
$= 2 \cdot y' \cdot e^{2y}$

$$\frac{dv}{dx} = 2 \cdot e^{2y} \cdot \frac{dy}{dx}$$

$$x \cdot \frac{dv}{dx} = 2x\, e^{2y} \frac{dy}{dx} = 3x^4 + e^{2y} = 3x^4 + v$$

$$x \cdot \frac{dv}{dx} = 3x^4 + v \Rightarrow \frac{dv}{dx} \underbrace{- \frac{1}{x}}_{P(x)} \cdot v = \underbrace{3x^3}_{Q(x)}$$

$$\rho(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$\frac{d}{dx}\left(v \cdot \frac{1}{x}\right) = 3x^2 \Rightarrow v \cdot \frac{1}{x} = x^3 + C \Rightarrow v(x) = x^4 + Cx$$

$$\Rightarrow e^{2y} = x^4 + Cx$$

$$\rightsquigarrow y(x) = \frac{1}{2} \ln |x^4 + Cx|$$

Example: Show that the substitution $v = ax + by + c$ transforms the differential equation

$$\frac{dy}{dx} = F(ax + by + c)$$

into a seperable equation.

$$v = ax + by + c \Rightarrow \frac{dv}{dx} = a + b \cdot \frac{dy}{dx}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b} \cdot \left(\frac{dv}{dx} - a\right) \qquad \frac{dy}{dx} = F(\underbrace{ax + by + c}_{v})$$

$$\frac{1}{b} \cdot \left(\frac{dv}{dx} - a\right) = F(v) \Rightarrow \frac{dv}{dx} - a = bF(v).$$

$$\Rightarrow \frac{dv}{bF(v) + a} = dx.$$

Homogeneous Equations

A **homogeneous** first-order differential equation is one that can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right). \tag{7}$$

If we make the substitutions

$$v = \frac{y}{x}, \quad y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}, \tag{8}$$

then Eq. (7) is transformed into the *separable* equation

$$x\frac{dv}{dx} = F(v) - v.$$

$$\Rightarrow \frac{dv}{F(v) - v} = \frac{dx}{x}.$$

Example:

Solve the differential equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2.$$

$$\frac{dy}{dx} = 2\cdot\left(\frac{x}{y}\right) + \frac{3}{2}\left(\frac{y}{x}\right)$$

$y = vx \Rightarrow \dfrac{dy}{dx} = v + x\cdot\dfrac{dv}{dx}$, $\quad v = \dfrac{y}{x} \rightarrow \dfrac{1}{v} = \dfrac{x}{y}$

$$v + x\cdot\frac{dv}{dx} = \frac{2}{v} + \frac{3}{2}v \Rightarrow x\frac{dv}{dx} = \frac{2}{v} + \frac{v}{2} = \frac{v^2+4}{2v}$$

$$\Rightarrow \frac{dx}{x} = \frac{2v\,dv}{v^2+4} \Rightarrow \int\frac{dx}{x} = \int\frac{2v}{v^2+4}\,dv$$

$v^2+4 = u$

$2v\,dv = du$

$\int\frac{du}{u}$

$$\Rightarrow \ln|x| = \ln(v^2+4) - \ln C.$$

$$\Rightarrow \ln|x| = \ln(v^2+4) - \ln c \Rightarrow v^2+4 = c.|x|.$$

$$\ln c = \ln(v^2+4) - \ln|x|$$

$$\ln c = \ln \frac{v^2+4}{|x|}$$

$$C = \frac{v^2+4}{|x|} \Rightarrow v^2+4 = c.|x|$$

$$k = \begin{cases} \text{positive}, & x>0 \\ 0, & x=0 \\ \text{negative}, & x<0 \end{cases}$$

$$\Rightarrow \frac{y^2}{x^2} + 4 = c.|x| \Rightarrow y^2 + 4x^2 + kx^3$$

$$y(x) = \mp\sqrt{kx^3 - 4x^2} = \mp\sqrt{(kx-4)x^2}$$

$$kx-4 > 0 \Rightarrow x > \frac{4}{k}$$

$$kx-4 < 0 \Rightarrow x < \frac{4}{k}$$

Thus $y(x) = \pm\sqrt{kx^3 - 4x^2}$ are defined if $x > 4/k$ if $k>0$ and for $x<4/k$ if $k<0$.

Example:

Solve the initial value problem

$$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad y(x_0) = 0,$$

where $x_0 > 0$.

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$$

$$\frac{y}{x} = v \quad , \quad y = v.x \qquad \frac{dy}{dx} = v + x.\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v + \sqrt{1-v^2} \Rightarrow x\frac{dv}{dx} = \sqrt{1-v^2}$$

$$\int \frac{dv}{\sqrt{1-v^2}} = \int \frac{dx}{x} \Rightarrow \sin^{-1} v = \ln x + C.$$

We don't need to write $\ln|x|$ because $x>0$ near $x = x_0 > 0$.

$$\sin^{-1} v = \ln x + c$$

$$\sin^{-1}\frac{y}{x} = \ln x + c \Rightarrow y(x) = x . \sin(\ln x + c)$$

$$y(x_0) = 0 \Rightarrow y(x_0) = \underbrace{x_0}_{\neq 0} . \underbrace{\sin(\ln(x_0) + c)}_{=0} = 0$$

$$\ln(x_0) + c = 0 \Rightarrow \ln(x_0) = -c$$

$$c = -\ln x_0$$

$$\Rightarrow y(x) = x \sin x \left(\ln x - \ln(x_0) \right)$$

$$\rightsquigarrow y(x) = x \sin x \left(\ln \frac{x}{x_0} \right).$$

Example:

Solve the differential equation

$$\frac{dy}{dx} = \frac{x - y - 1}{x + y + 3}$$

by finding h and k so that the substitutions $x = u + h$, $y = v + k$ transform it into the homogeneous equation

$$\frac{dv}{du} = \frac{u - v}{u + v}.$$

$x = u + h, \Rightarrow dx = du$

$y = v + k \Rightarrow dy = dv$

$$\underbrace{\frac{dy}{dx} \cdot dx}_{dy} = \frac{x-y-1}{x+y+3} dx \quad \Rightarrow dy = \frac{x-y-1}{x+y+3} dx$$

$$\Rightarrow dv = \frac{u+h-v-k-1}{u+h+v+h+3} du$$

$$\Rightarrow \frac{dv}{du} = \frac{u-v+\overbrace{h-k-1}^{0}}{u+v+\underbrace{h+k+3}_{0}}$$

$$\left.\begin{array}{l} h-k-1=0 \\ h+k+3=0 \end{array}\right\} \quad \left.\begin{array}{l} h-k=1 \\ h+k=-3 \end{array}\right\} h=-1, \; k=-2$$

$x = u-1, \; y = v-2.$

$$\frac{dv}{du} = \frac{u-v}{u+v} \Rightarrow \frac{dv}{du} = \frac{1-\frac{v}{u}}{1+\frac{v}{u}}$$

$$\frac{v}{u} = w$$

$$v = u \cdot w$$

$$\frac{dv}{du} = w + u \cdot \frac{dw}{du}$$

$$w + u \cdot \frac{dw}{du} = \frac{1-w}{1+w} \Rightarrow u \cdot \frac{dw}{du} = \frac{1-w}{1+w} - w$$

$$\Rightarrow u \cdot \frac{dw}{du} = \frac{1-w-w-w^2}{1+w} \Rightarrow u \cdot \frac{dw}{du} = \frac{1-2w-w^2}{1+w}$$

$$\Rightarrow \frac{du}{u} = -\frac{1+w}{w^2+2w-1} dw.$$

$$w^2+2w-1 = \tilde{u}$$

$$(2w+2)\,dw = d\tilde{u}$$

$$2(w+1)\,dw = d\tilde{u}$$

$$\frac{1}{2}\int \frac{d\tilde{u}}{\tilde{u}}$$

$$\Rightarrow \ln u = -\frac{1}{2}\left[\ln(w^2+2w-1) - \ln c\right]$$

$$\Rightarrow \underline{\underline{2}} \ln u + \ln(w^2+2w-1) = \ln c \Rightarrow u^2 \cdot (w^2+2w-1) = C$$

$$\ln(u^2) \cdot (w^2+2w-1) = \ln c$$

$x = u-1, \quad y = v-h$

$$u^2(w^2+2w-1)=C$$

$$u^2\left(\frac{v^2}{u^2}+2\cdot\frac{v}{u}-1\right)=C \Rightarrow v^2+2v.u-u^2=C \quad , \quad u = x+1, \quad v = y+2$$

$$\Rightarrow (y+2)^2+2.(x+1).(y+2)-(x+1)^2=C.$$

$$\Rightarrow y^2+2xy-x^2+6y+2x=\underbrace{C-7}_{D}.$$

Bernoulli Equations

A first-order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{9}$$

is called a **Bernoulli equation.** If either $n = 0$ or $n = 1$, then Eq. (9) is linear.
, the substitution

$$v = y^{1-n} \quad , \quad \frac{dv}{dx} = (1-n) \cdot y^{-n} \frac{dy}{dx} \tag{10}$$

transforms Eq. (9) into the linear equation

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

Example:

The equation

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}$$

is neither separable nor linear nor homogeneous, but it is a Bernoulli equation with $n = \frac{4}{3}$, $1 - n = -\frac{1}{3}$. The substitutions

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}$$

$$\nearrow \quad x \cdot y^{-4/3}\frac{dy}{dx} + 6 \cdot y^{-1/3} = 3x$$

$v = y^{1-n}$, $\quad v = y^{-1/3}$, $\quad y = v^{-3}$, $\quad \frac{dy}{dx} = -3 \cdot v^{-4} \cdot \frac{dv}{dx}$

$$x \cdot (v^{-3})^{-4/3} \cdot \left(-3 \cdot v^{-4}\frac{dv}{dx}\right) + 6 \cdot v = 3x$$

$$\Rightarrow -3x\, v^{4} \cdot v^{-4}\frac{dv}{dx} + 6v = 3x \quad \Rightarrow \quad \frac{dv}{dx} - \frac{2}{x}v = -1.$$

$$\frac{dv}{dx} - \underset{\substack{\downarrow \\ P(x)}}{\frac{2}{x}} v = \underset{\substack{\downarrow \\ Q(x)}}{-1}.$$

$$\rho(x) = \exp\left(\int -\frac{2}{x} dx \right) = e^{-2\int \frac{dx}{x}}$$

$$= e^{-2\ln x} = e^{\ln x^{-2}}$$

$$= \frac{1}{x^2}.$$

$$\frac{1}{x^2} \cdot \frac{dv}{dx} - \frac{2}{x^3} \cdot v = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(v \cdot \frac{1}{x^2} \right) = -\frac{1}{x^2} \Rightarrow v \cdot \frac{1}{x^2} = \frac{1}{x} + C \Rightarrow v(x) = x + Cx^2$$

$$\Rightarrow y^{-1/3} = x + Cx^2$$

$$\left(y^{-1/3} \right)^{-3} = \left(x + Cx^2 \right)^{-3}$$

$$\Rightarrow \underline{\underline{y(x)}} = \frac{1}{(x+Cx^2)^3}.$$

Example!

$$y^2(xy'+y)\cdot(1+x^4)^{1/2}=x$$

$$xy'+y=\frac{x}{y^2(1+x^4)^{1/2}}, \quad y\neq 0.$$

$$y'+\frac{1}{x}y=\frac{1}{(1+x^4)^{1/2}}\cdot y^{-2} \qquad (\text{Bernoulli}).$$

$n=-2, \quad v=y^{1-n}$

$$\Rightarrow \quad y^2\cdot y'+\frac{1}{x}y^3=\frac{1}{(1+x^4)^{1/2}}$$

$v=y^3$

$* \; y=v^{1/3}$

$* \; \frac{dy}{dx}=\frac{1}{3}v^{-2/3}\cdot\frac{dv}{dx}$

$$\Rightarrow \quad \frac{dv}{dx}+\underbrace{\frac{3}{x}}_{p(x)}v=\frac{3}{(1+x^4)^{1/2}} \qquad (\text{Linear})$$

$$\rho(x)=\exp\left(\int\frac{3}{x}\,dx\right)=e^{3\ln x}=x^3$$

$$x^3 \frac{dv}{dx} + 3x^2 v = \frac{3x^3}{(1+x^4)^{1/2}}$$

$$\frac{d}{dx}\left(x^3.v\right) = \frac{3x^3}{(1+x^4)^{1/2}} \Rightarrow x^3.v = \int \frac{3x^3}{(1+x^4)^{1/2}}\,dx$$

$1+x^4 = u$

$4x^3\,dx = du$

$x^3\,dx = \frac{du}{4}$

$3x^3\,dx = \frac{3}{4}du$

$\frac{3}{4}\int \frac{du}{u^{1/2}}$

$$\Rightarrow x^3.v = \frac{3}{4}\,2.\sqrt{1+x^4} + C.$$

$$v(x) = \frac{3}{2x^3}.\sqrt{1+x^4} + \frac{C}{x^3}$$

$v = y^3$

$$y^3 = \frac{3\sqrt{1+x^4} + \overbrace{2C}^{D}}{2x^3} \qquad (D = 2C)$$

$$y(x) = \frac{\sqrt[3]{D + 3\sqrt{1+x^4}}}{\sqrt[3]{2}\,x}.$$

Exact Differential Equations

The general first-order differential equation

$$y' = f(x,y)$$

can be written in its differential form

$$M(x,y)\,dx + N(x,y)\,dy \quad (*),$$

where $M = f(x,y)$ and $N(x,y) = -1$.

If there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N,$$

then the equation

$$F(x,y) = C$$

implicitly defines a general solution of $(*)$ and $(*)$ is called an exact differential equation.

$M(x,y)\,dx + N(x,y)\,dy = 0$

$\frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} dy = 0 \qquad dF = 0$

$F(x, y(x)) = C.$

THEOREM 1 Criterion for Exactness

Suppose that the functions $M(x, y)$ and $N(x, y)$ are continuous and have continuous first-order partial derivatives in the open rectangle R: $a < x < b$, $c < y < d$. Then the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{23}$$

is exact in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{24}$$

at each point of R. That is, there exists a function $F(x, y)$ defined on R with $\partial F/\partial x = M$ and $\partial F/\partial y = N$ if and only if Eq. (24) holds on R.

proof: "$\Rightarrow$": Suppose that $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact.
Then there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N.$$

Since $M(x,y)$ and $N(x,y)$ have continuous partial derivatives

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

"$\Leftarrow$": Suppose that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. We need to construct a function such that

$$① \ \frac{\partial F}{\partial x} = M \quad \text{and} \quad ② \ \frac{\partial F}{\partial y} = N$$

$$① \ \frac{\partial F}{\partial x} = M \Rightarrow \int \frac{\partial F}{\partial x}\,dx = \int M\,dx$$

$$\Rightarrow F(x,y) = \int M(x,y)\,dx + g(y)$$

(2) $N = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + g'(y)$

$\Rightarrow g'(y) = N - \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right)$

It sufficies to show that $N - \frac{\partial}{\partial y}\int M(x,y)dy$ is a function of y only.

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M(x,y)\,dx\right) = \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int M(x,y)\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

Thus $g(y) = \int \left[N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)dy\right)\right] dy$

$$F(x,y) = \int M(x,y)\,dx + \int \left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx\right) dy$$

Example: Solve $y^3 dx + 3xy^2 dy = 0$

$$y^3 dx + x\, d(y^3) = 0 \rightarrow d(xy^3) = 0$$

OR

$$y^3 + 3xy^2 \frac{dy}{dx} = 0$$

$$\rightarrow \frac{d}{dx}(x \cdot y^3) = 0$$

$$\Rightarrow F(x,y) = xy^3 = C.$$

Notice that, $M = y^3$, $N = 3xy^2$

$$M_y = 3y^2 = N_x = 3y^2$$

Example:

Solve the differential equation

$$\underbrace{(6xy - y^3)}_{M(x,y)}\, dx + \underbrace{(4y + 3x^2 - 3xy^2)}_{N(x,y)}\, dy = 0.$$

$M(x,y) = 6xy - y^3$ $\quad M_y = 6x - 3y^2$

$N(x,y) = 4y + 3x^2 - 3xy^2$ $\quad N_x = 6x - 3y^2$

$\left.\begin{matrix} M_y = 6x - 3y^2 \\ N_x = 6x - 3y^2 \end{matrix}\right\} M_y = N_x \rightsquigarrow$ It's exact.

Find $F(x,y)$ such that ① $M = F_x$ and ② $N = F_y$

① $F_x = 6xy - y^3 \Rightarrow F(x,y) = \int (6xy - y^3)\, dx$

$= 3x^2y - y^3x + g(y)$

② $F_y = 3x^2 - 3y^2x + g'(y) = N = 4y + 3x^2 - 3xy^2$

$\Rightarrow \quad g'(y) = 4y \quad \Rightarrow \quad g(y) = 2y^2 + C$

$$g(y) = 2y^2 + C$$

$$F(x,y) = 3x^2y - y^3x + g(y)$$

$$= 3x^2y - y^3x + 2y^2 + C.$$

Thus, $3x^2y - y^3x + 2y^2 = C$ (general solution)