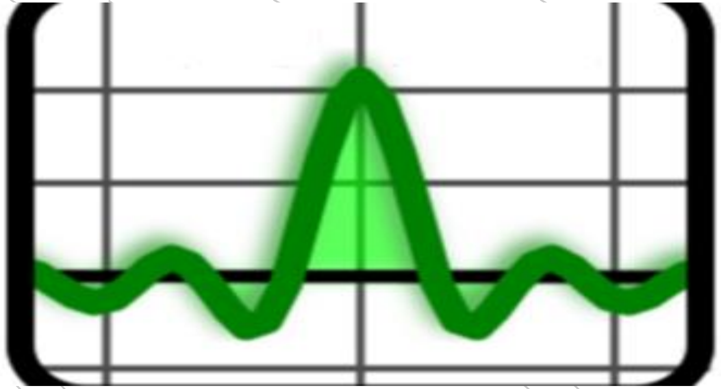

Signals & Systems for Computer Engineering

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BLG354E

11th Week Lecture

$\mathcal{Z}$-Transform Applications in Discrete Time LTI Systems

- $\mathcal{Z}$-transform is the discrete-time counterpart of the Laplace transform $z = e^{Ts}$
- The Laplace transform converts integrodifferential equations into algebraic equations. In a similar manner, the z-transform converts difference equations into algebraic equations, thereby simplifying the analysis of discrete-time systems.

Discrete-time LTI system with impulse response h[n], the output y[n] of the system to the complex exponential input of the form z^n is

$$y[n] = \mathbf{T}\{z^n\} = H(z)z^n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

discrete-time signal x[n], the z-transform X(z) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

z is generally complex-valued and is expressed in polar form as $z = re^{j\Omega}$, r is the magnitude of z and Ω is the angle of z

if x[n] = 0 for n<0 then unilateral Z transform can be stated as $X_I(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$

Region of convergence (ROC):

Find the z transform and the ROC of the signal sequence $x[n]=a^n u[n]$ where a is a real number

By the definition,

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

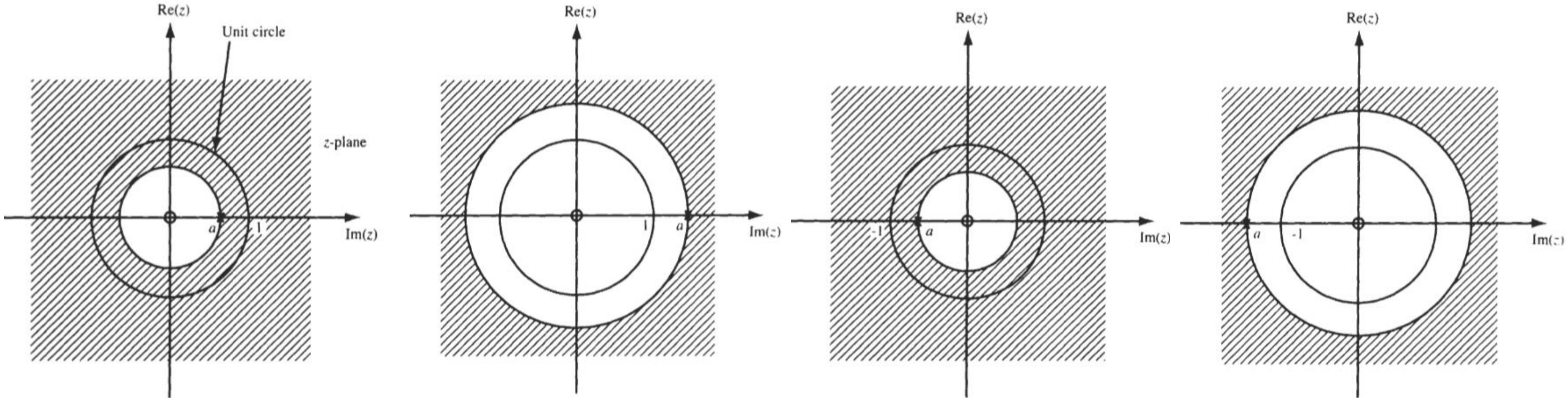
To satisfy ROC,

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$X(z) = \frac{z}{z-a} \qquad |z| > |a|$$

$$X(z) = \frac{z}{z-a} \qquad |z| > |a|$$ → ROC:



0<a<1 | 1<a | -1<a<0 | a<-1

Example:

Find the $\mathcal{Z}$ transform and ROC of the signal sequence x[n] defined as $x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-2}^{3} x[n]z^{-n} = x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

$$X(z) = 5z^{2} + 3z - 2 + 4z^{-2} - 3z^{-3}$$

Every term in $X(z)$ will be finite and consequently $X(z)$ will converge for the z not equal to zero or infinity

ROC of $X(z)$ is $0 < |z| < \infty$

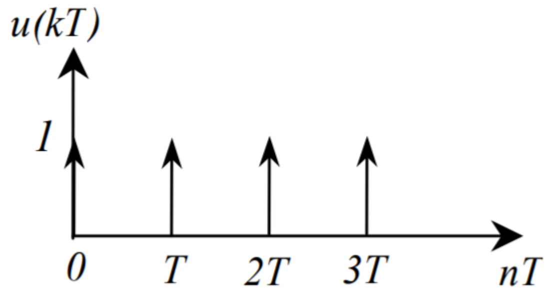
z-TRANSFORM OF SOME COMMON SEQUENCES

x[n]=δ[n] $\rightarrow$ $$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^{-0} = 1 \qquad \text{all } z$$

x[n]=u[n]

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\rightarrow$ $$u(kT) = 1; \quad k = 0,1,2,3,\ldots$$



$$X(z) = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots \quad \rightarrow \quad u[n] \leftrightarrow \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \qquad |z| > 1$$

$$x(t) = e^{-at} \qquad t \geq 0$$

If x(t) is sampled at T=1/f $$X(z) = 1 + \frac{e^{-aT}}{z} + \frac{e^{-a2T}}{z^2} + \ldots$$

$$X(z) = 1 + \frac{e^{-aT}}{z} + \left(\frac{e^{-aT}}{z}\right)^2 + \left(\frac{e^{-aT}}{z}\right)^3 + \ldots = \frac{z}{z - e^{-aT}} \quad \leftrightarrow \quad \mathcal{L}\,[e^{-at}] = \frac{1}{s + a}$$

Example: Find the $\mathcal{Z}$ transform of x[n]=-a^{n}u[-n-1]

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$\sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1-a^{-1}z} \qquad \text{if } |a^{-1}z| < 1 \text{ or } |z| < |a|$$

Hence,

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a} = \frac{1}{1-az^{-1}} \qquad \text{for} \quad |z| < |a|$$

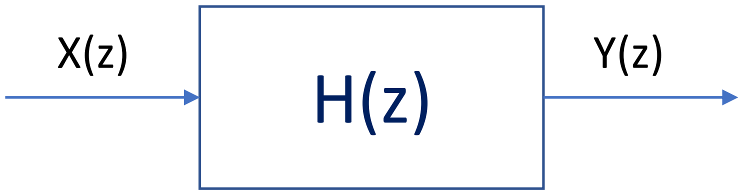
Z Transform of Finite Sequence

consider the sequence of sampled numbers, x[k]

$$x[k] = \{1, -2, 4, 5, 3, -1, 0, 0, 0, 0 \ldots\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = 1 - 2z^{-1} + 4z^{-2} + 5z^{-3} - z^{-4}$$



If the system transfer function H(z) is known then the output signal can be found via inverse transform of Y(z) where $Y(z)=X(z)\cdot H(z)$

Properties of the $\mathcal{Z}$ Transform

Property	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(z)+a_2X_2(z)$	$R' \supset R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	$R' \supset R \cap \{0<\lvert z\rvert<\infty\}$
Multiplication by z_0^n	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$R' = \lvert z_0\rvert R$
Multiplication by $e^{j\Omega_0 n}$	$e^{j\Omega_0 n}x[n]$	$X(e^{-j\Omega_0}z)$	$R' = R$
Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$R' = \frac{1}{R}$
Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$	$R' = R$
Accumulation	$\sum_{k=-\infty}^{n} x[n]$	$\frac{1}{1-z^{-1}}X(z)$	$R' \supset R \cap \{\lvert z\rvert>1\}$
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	$R' \supset R_1 \cap R_2$

Common $\mathcal{Z}$ Transform Pairs

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\dfrac{1}{1-z^{-1}}, \dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\dfrac{1}{1-z^{-1}}, \dfrac{z}{z-1}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 if $(m>0)$ or ∞ if $(m<0)$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}, \dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}, \dfrac{z}{z-a}$	$\lvert z < \lvert a\rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}, \dfrac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}, \dfrac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
$(n+1)a^n u[n]$	$\dfrac{1}{(1-az^{-1})^2}, \left[\dfrac{z}{z-a}\right]^2$	$\lvert z\rvert > \lvert a\rvert$
$(\cos \Omega_0 n)u[n]$	$\dfrac{z^2-(\cos \Omega_0)z}{z^2-(2\cos \Omega_0)z+1}$	$\lvert z\rvert > 1$
$(\sin \Omega_0 n)u[n]$	$\dfrac{(\sin \Omega_0)z}{z^2-(2\cos \Omega_0)z+1}$	$\lvert z\rvert > 1$
$(r^n \cos \Omega_0 n)u[n]$	$\dfrac{z^2-(r\cos \Omega_0)z}{z^2-(2r\cos \Omega_0)z+r^2}$	$\lvert z\rvert > r$
$(r^n \sin \Omega_0 n)u[n]$	$\dfrac{(r\sin \Omega_0)z}{z^2-(2r\cos \Omega_0)z+r^2}$	$\lvert z\rvert > r$
$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Inverse Z-Transform

Finding the sequence x[n] from its z-transform X(z) is called the inverse z-transform, symbolically denoted as

$$x[n]=\mathcal{Z}^{-1}\{X(z)\}$$

Method 1: Inversion Formula

$$x[n]=\frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz$$

where C is a counterclockwise contour of integration enclosing the origin

Method 2: Use of Tables of z-Transform Pairs:

X(z) is expressed as a sum $X(z)=X_1(z)+\cdots+X_n(z)$

$$x[n]=x_1[n]+\cdots+x_n[n]$$

where $X_1(z)$, $X_2(z)$, . . . , $X_n(z)$ are functions with known inverse transforms $x_1[n]$, $x_2[n]$…, $x_n[n]$

Method 3: Power Series Expansion:

$$X[z]=\sum_{n=-\infty}^{\infty}x[n]z^{-n}=\cdots+x[-2]z^{2}+x[-1]z+x[0]+x[1]z^{-1}+x[2]z^{-2}+\cdots$$

This method is useful for a finite-length sequence where X(z) may have no simpler form than a polynomial in z^{-1}

Method 4: Partial-Fraction Expansion:

Partial-fraction expansion method provides inverse z-transform, especially when X(z) is a rational function of z.

$$X(z) = \frac{N(z)}{D(z)} = k\frac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)}$$

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots + \frac{c_n}{z-p_n} = \frac{c_0}{z} + \sum_{k=1}^{n}\frac{c_k}{z-p_k}$$

where $\quad c_0 = X(z)|_{z=0} \qquad c_k = (z-p_k)\left.\frac{X(z)}{z}\right|_{z=p_k}$

$$X(z) = c_0 + c_1\frac{z}{z-p_1} + \cdots + c_n\frac{z}{z-p_n} = c_0 + \sum_{k=1}^{n} c_k\frac{z}{z-p_k}$$

Hence we can inverse transform from each term separately by using the table

If m>n then a polynomial of z must be added to the right-hand side of the equation so that for m>n, the complete partial-fraction expansion would have the form

$$X(z) = \sum_{q=0}^{m-n} b_q z^q + \sum_{k=1}^{n} c_k \frac{z}{z - p_k}$$

If X(z) has multiple-order poles

$$\frac{\lambda_1}{z - p_i} + \frac{\lambda_2}{(z - p_i)^2} + \cdots + \frac{\lambda_r}{(z - p_i)^r}$$

where p_i is the multiple pole with multiplicity r. The expansion of $X(z)/z$ will consist of terms of the form

$$\lambda_{r-k} = \frac{1}{k!} \frac{d^k}{dz^k} \left[(z - p_i)^r \frac{X(z)}{z} \right] \Bigg|_{z = p_i}$$

Example: Find the inverse z-transform of the following X(z) by using the power series expansion method

$$X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| > 1$$

ROC is lzl> 1 and x[n] is a right-sided sequence. Therefore, we must divide so as to obtain a series in power of z^{-1} as follows

$$\begin{array}{r|l}
 & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots \\ \hline
2z^2 - 3z + 1 & z \\
 & \underline{z - \frac{3}{2} + \frac{1}{2}z^{-1}} \\
 & \quad \frac{3}{2} - \frac{1}{2}z^{-1} \\
 & \quad \underline{\frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}} \\
 & \qquad \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \\
 & \qquad \vdots
\end{array}$$

$$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots$$

$$x[n] = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\}$$

Example: Find the inverse $\mathcal{Z}$ transform of $F(z)=\dfrac{2z^2+z}{z^2-1.5z+0.5}$ by using the table and power series methods

$$\frac{F(z)}{z}=\frac{2z+1}{z^2-1.5z+0.5}=\frac{2z+1}{(z-1)(z-0.5)}=\frac{A}{z-1}+\frac{B}{z-0.5}$$

$$=\frac{6}{z-1}+\frac{-4}{z-0.5}=6\frac{z}{z-1}-4\frac{z}{z-0.5}$$

$\frac{z}{z-1} \rightarrow u[k]$, $\frac{z}{z-0.5} \rightarrow 0.5^k$

$$f[k]=6u[k]-4\cdot 0.5^k$$

$$f=\{2,4,5,5.5,\cdots\}$$

$$
\begin{array}{r|l}
 & 2+4z^{-1}+5z^{-2}+\cdots \\
\hline
z^2-1.5z+0.5 & 2z^2+z \\
 & 2z^2-3z+1 \\
\hline
 & 4z-1 \\
 & 4z-6+2z^{-1} \\
\hline
 & 5-2z^{-1} \\
 & 5-7.5z^{-1}+2.5z^{-2} \\
\hline
 & \vdots
\end{array}
\longrightarrow F(z)=2+4z^{-1}+5z^{-2}+\cdots
$$

$$f[k]=\{2,4,5,\cdots\}$$

Verify results in previous example by using Scipy library in Python:

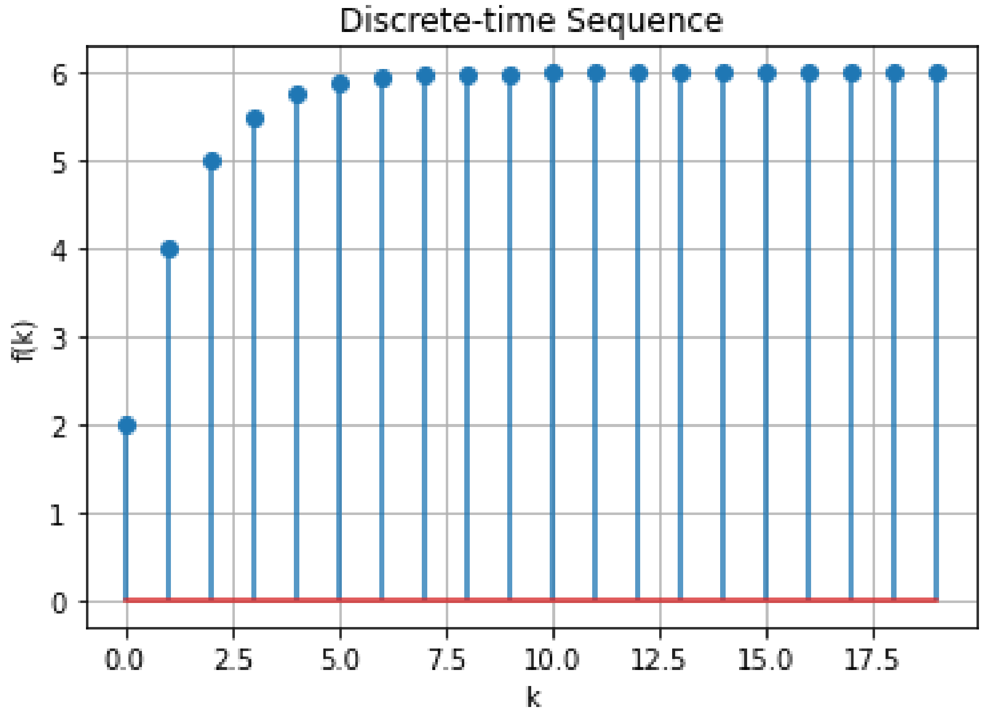
```
import numpy as np
import matplotlib.pyplot as plt
from scipy import signal

# Define the numerator and denominator coefficients of the transfer
function F(z)
numerator = [2, 1, 0]  # Coefficients of z^2 + z
denominator = [1, -1.5, 0.5]  # Coefficients of z^2 - 1.5z + 0.5

# Compute the inverse Z-transform
_, f = signal.dimpulse((numerator, denominator, 1), n=20)
f = np.squeeze(f)  # Remove unnecessary dimensions

# Extract time indices
k = np.arange(len(f))

# Plot the discrete-time sequence
plt.stem(k, f)
plt.xlabel('k')
plt.ylabel('f(k)')
plt.title('Discrete-time Sequence')
plt.grid(True)
plt.show()
```



$$f[k] = \{2, 4, 5, \cdots\} \checkmark \quad \leftrightarrow \quad F(z) = \frac{2z^2 + z}{z^2 - 1.5z + 0.5}$$

Example:

Find the inverse z transform of $X(z) = \dfrac{z}{2z^2 - 3z + 1} \qquad |z| > 1$

$$X(z) = \frac{z}{2z^2 - 3z + 1} = \frac{z}{2(z-1)(z-\frac{1}{2})}$$

By using the partial fraction expansion method: $\dfrac{X(z)}{z} = \dfrac{1}{2z^2 - 3z + 1} = \dfrac{1}{2(z-1)(z-\frac{1}{2})} = \dfrac{c_1}{z-1} + \dfrac{c_2}{z-\frac{1}{2}}$

$$c_1 = \left.\frac{1}{2(z-\frac{1}{2})}\right|_{z=1} = 1 \qquad c_2 = \left.\frac{1}{2(z-1)}\right|_{z=1/2} = -1$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-\frac{1}{2}} \qquad |z| > 1$$

Since the ROC of X(z) is |z|> 1, x[n] is a right-sided sequence: $x[n] = u[n] - (\frac{1}{2})^n u[n] = \left[1 - (\frac{1}{2})^n\right] u[n]$

$$x[n] = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\}$$

Example:

Find the inverse z transform of $X(z)=\dfrac{z}{z(z-1)(z-2)^2} \qquad |z|>2$

By using the partial fraction expansion method: $$\frac{X(z)}{z}=\frac{1}{(z-1)(z-2)^2}=\frac{c_1}{z-1}+\frac{\lambda_1}{z-2}+\frac{\lambda_2}{(z-2)^2}$$

where $c_1=\left.\dfrac{1}{(z-2)^2}\right|_{z=1}=1 \qquad \lambda_2=\left.\dfrac{1}{z-1}\right|_{z=2}=1$

$$\frac{1}{(z-1)(z-2)^2}=\frac{1}{z-1}+\frac{\lambda_1}{z-2}+\frac{1}{(z-2)^2}$$

By setting z=0 we get $-\dfrac{1}{4}=-1-\dfrac{\lambda_1}{2}+\dfrac{1}{4}\longrightarrow\lambda_1=-1$

$$X(z)=\frac{z}{z-1}-\frac{z}{z-2}+\frac{z}{(z-2)^2} \qquad |z|>2$$

Since the ROC of X(z) is lzl> 2, x[n] is a right-sided sequence: $x[n]=(1-2^n+n2^{n-1})u[n]$

Convolution with z-transform

Z Transform has the convolution property in the same way as it was for the Laplace Transform. Convolution of two signal sequences x[k] and h[k] can be stated as y[k] where,

$$y[k] = \sum_{\ell=0}^{\infty} x[\ell]\,h[k-\ell]$$

$$= \sum_{\ell=0}^{\infty} h[\ell]\,x[k-\ell]$$

$$= x[k] * h[k] \quad \leftrightarrow \quad Y(z) = X(z)H(z)$$

Example:

Calculate the convolution y[n] of the sequences $v[n]=\{v_n\} = \{a^n\}$, $w[n]=\{w_n\}= \{b^n\}$, $a \neq b$

$$y_n = \sum_{k=0}^{n} v_k w_{n-k} = \sum_{k=0}^{n} a^k b^{n-k} = b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k = b^n \left(1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \ldots \left(\frac{a}{b}\right)^n\right)$$

The bracketed sum involves $n+1$ terms of a geometric series of common ratio $\frac{a}{b}$ → $y_n = b^n \frac{\left(1 - \left(\frac{a}{b}\right)^{n+1}\right)}{1 - \frac{a}{b}}$

Z transforms of v[n] and and w[n] are: $V(z) = \frac{z}{z-a}$ $W(z) = \frac{z}{z-b}$

$$y_n = \mathbb{Z}^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = \frac{b^{n+1} - a^{n+1}}{(b-a)}$$

using partial fractions or residues

Example:

A discrete time signal sequence is given as $x[n]=2^n$ (n>0). Find the convolution $y[n]=x[n]*x[n]$

a) by using the definition of the convolution
b) by using the Z transform

a) $x[n]=\{1,2,4,8,\ldots\}$ $\quad y[n]=\{2^n\}*\{2^n\}=\sum_{k=0}^{n} 2^k 2^{n-k}=2^n\sum_{k=0}^{n} 1=(n+1)2^n$

b) $\mathcal{Z}$ transform of x[n] : $\quad X(z)=\mathcal{Z}\{2^n\}=\frac{z}{z-2}$ $\quad \rightarrow y[n]=\mathcal{Z}^{-1}\{\frac{z^2}{(z-2)^2}\}$

Y(z) has a second order pole at z=2. If we use the residue method then we get,

$$y[n]=\operatorname{Res}\left(\frac{z^{n+1}}{(z-2)^2},\ 2\right)=\left[\frac{d}{dz}z^{n+1}\right]_2=(n+1)2^n$$

Example: Z transform can also be used for solving the difference equations

For example, $8x[k+2]-6x[k+1]+x[k]=0$

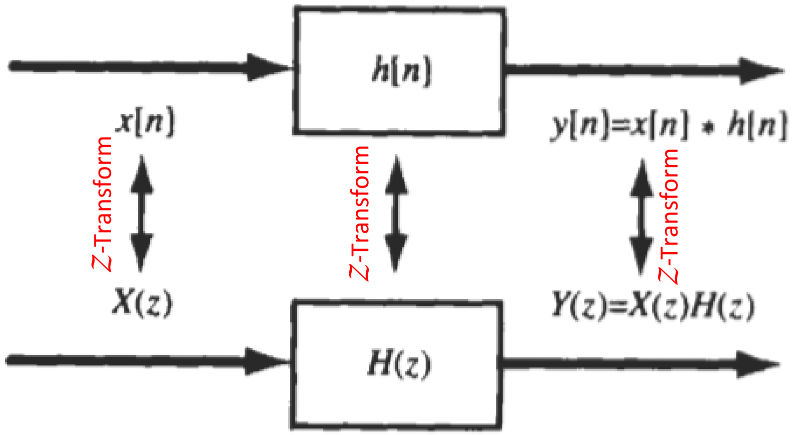
$$\overbrace{8z^2X[z]-8zx[1]}-6zX[z]+X[z]=0 \qquad x[1]=1, x[0]=0$$

$$\frac{X[z]}{z}=\frac{8}{8z^2-6z+1}=\frac{8}{2z-1}-\frac{16}{4z-1}$$

$$X[z]=\frac{4z}{z-\frac{1}{2}}-\frac{4z}{z-\frac{1}{4}}$$

$$x[k]=4\{(\frac{1}{2})^k+(\frac{1}{4})^k\}$$

TRANSFER FUNCTION OF DISCRETE-TIME LTI SYSTEMS



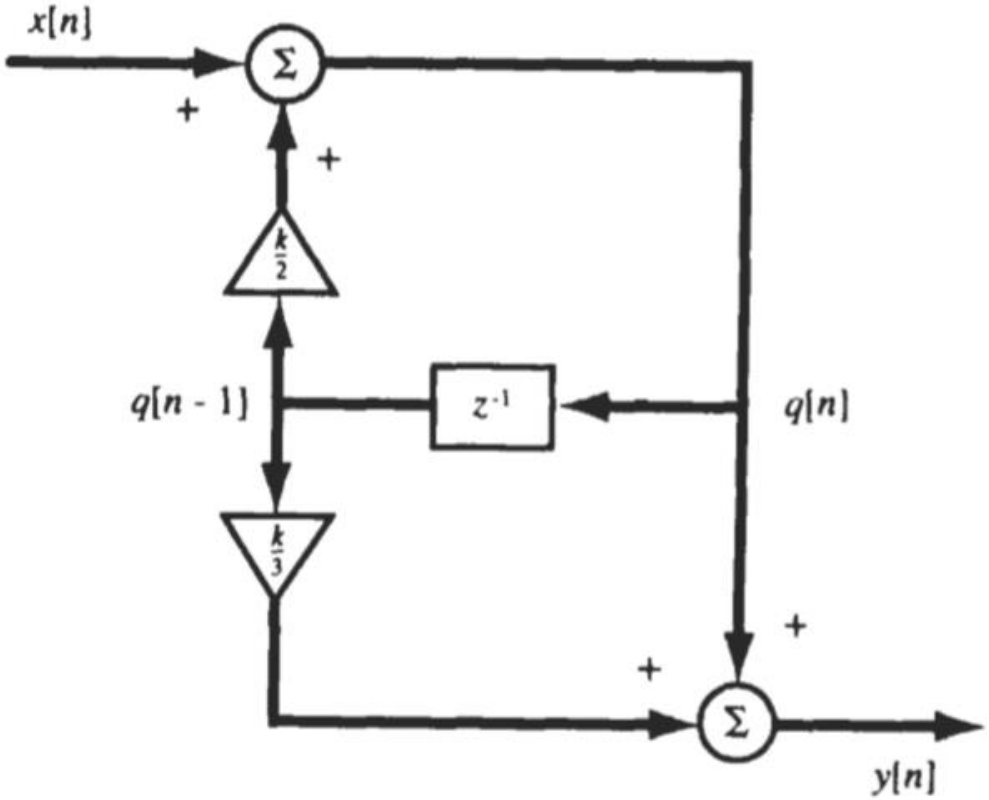
Output y[n] of a discrete-time LTI system equals the convolution of the input x[n] with the impulse response h[n]

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

Example: Find the k values that makes the system shown in the figure BIBO stable.



$$q[n] = x[n] + \frac{k}{2}q[n-1] \qquad \rightarrow \qquad Q(z) = X(z) + \frac{k}{2}z^{-1}Q(z)$$

$$y[n] = q[n] + \frac{k}{3}q[n-1] \qquad \qquad Y(z) = Q(z) + \frac{k}{3}z^{-1}Q(z)$$

$$\left(1 - \frac{k}{2} z^{-1}\right) Q(z) = X(z)$$

$$\left(1 + \frac{k}{3} z^{-1}\right) Q(z) = Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + (k/3) z^{-1}}{1 - (k/2) z^{-1}}$$

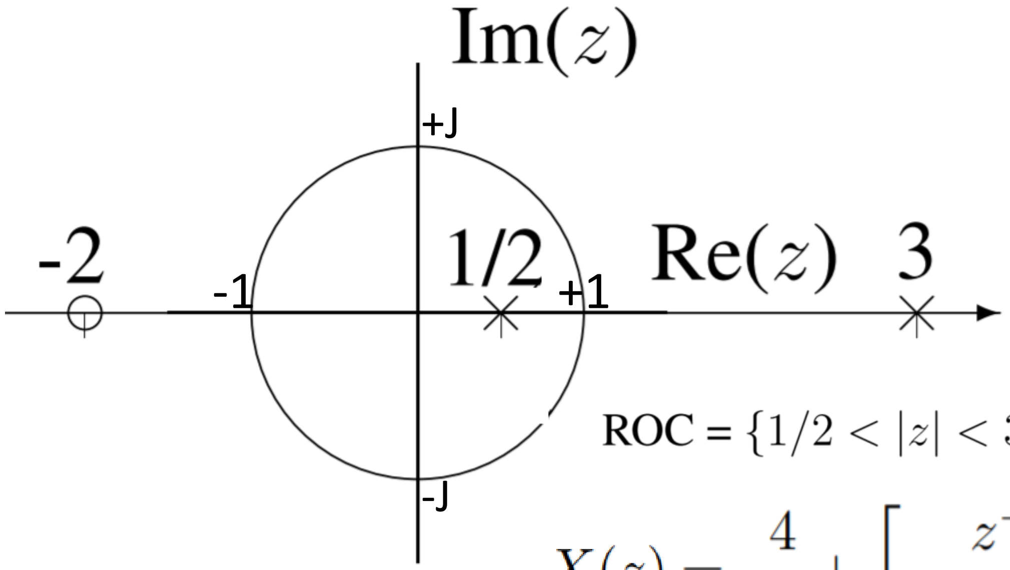
$$= \frac{z + k/3}{z - k/2} \qquad |z| > \left|\frac{k}{2}\right|$$

The system has one zero at z = -k/3 and one pole at z = k/2 → ROC is lzl> lk/2l

Therefore system will be BIB0 stable if the ROC contains the unit circle, lzl= 1 → the system is stable only if IkI<2

Example:

z-transform of the signal X(z) has the pole-zero plot shown below. Find the DT signal x[n]



ROC = $\{1/2 < |z| < 3\}$

$$\rightarrow X(z) = \frac{z+2}{(z-3)(z-1/2)} = \frac{z^{-1}+2z^{-2}}{(1-3z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$X(z) = \frac{4}{3} + \left[\frac{z^{-1}+2z^{-2}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}} - \frac{2}{3/2}\right] = \frac{4}{3} + \frac{z^{-1}+2z^{-2}-\frac{4}{3}\left[1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}\right]}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}}$$

$$X(z) = \frac{4}{3} + \frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{(1-3z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{4}{3} + \frac{r_1}{1-3z^{-1}} + \frac{r_2}{1-\frac{1}{2}z^{-1}}$$

(Partial Fraction Expansion)

$$r_1 = \left.\frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z=3} = \frac{2}{3} \qquad r_2 = \left.\frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{1-3z^{-1}}\right|_{z=1/2} = -2 \rightarrow X(z) = \frac{4}{3} + \frac{\frac{2}{3}}{1-3z^{-1}} + \frac{-2}{1-\frac{1}{2}z^{-1}}$$

anti-casual

$$x[n] = \frac{4}{3}\delta[n] - \frac{2}{3}3^n \boxed{u[-n-1]} - 2\left(\frac{1}{2}\right)^n u[n]$$

Example

The convolution of $f_1(n) = \{2, 1, -3\}$ for $n = 0, 1,$ and 2, and $f_2(n) = \{1, 1, 1, 1\}$ for $n = 0, 1, 2,$ and 3 is

$$G(z) = F_1(z)\,F_2(z) = (2 + z^{-1} - 3z^{-2})(1 + z^{-1} + z^{-2} + z^{-3}) = 2 + 3z^{-1} - 2z^{-4} - 3z^{-5}$$

which indicates that the output is $g(n) = \{2, 3, 0, 0, -2, -3\}$ which can easily be found by simply convoluting $f_1(n)$ and $f_2(n)$.