Name: Signature:

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1. (10 points) Prove by induction that

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \in \mathbb{N}$.

For $n = 0$, we see that $\sum_{i=0}^{0} i^2 = 0^2 = 0$ and $\frac{0(0+1)(2\cdot 0+1)}{6} = 0$. So the result is true for $n = 0$. Let k be a natural number. Assume that the result is true for $n = k$. That is, we assume that $\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

Consider now the result for $n = k+1$. Note that

$$\begin{aligned}\sum_{i=0}^{k+1} i^2 &= \left(\sum_{i=0}^{k} i^2\right) + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k+1}{6}\Big(k(2k+1) + 6(k+1)\Big) \\ &= \frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)}{6}\end{aligned}$$

where in the second equlaity we used the induction hypothesis $\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. So the result is true for $n = k+1$

Hence, the result follows from the principle of mathematical induction.

2. (10 points) Given the relation

$$p = \{(x, y) \in \mathbb{Z} \times \mathbb{N} \mid y = \max(0, x)\}$$

from $\mathbb{Z}$ to $\mathbb{N}$, show that:

(a) p is a function.

For any element x of $\mathbb{Z}$, it is clear that the maximum of 0 and x exists and unique (indeed, it is 0 if $x < 0$ and it is x if $x \geq 0$). In other words, for each $x \in \mathbb{Z}$ there is a unique $y \in \mathbb{N}$ such that $(x, y) \in p$. So, p is a function from $\mathbb{Z}$ to $\mathbb{N}$, and it is given by

$$p(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

(b) p is not one to one.

For instance, $-1 \neq -2$ but $p(-1) = 0 = p(-2)$. So p is not one to one.

(c) p is onto.

For any $x \in \mathbb{N}$, note that $x \in \mathbb{Z}$ and $p(x) = x$ because $x \geq 0$. So p is onto.

3. (10 points) Let A be a set and $f : A \to A$ be a function, and let R be the relation on A defined for any elements a and b of A by

$$aRb \text{ if and only if } b = f(a).$$

Prove that:

(a) If $f(f(x)) = x$ for all $x \in A$, then R is symmetric.

Let $(r, s) \in R$ for some $r, s \in A$. $\big($We want to justify that $(s, r) \in R\big)$. Then $s = f(r)$ by the definition of R. From $s = f(r)$, we see that $f(s) = f\big(f(r)\big)$. Using the condition "$f(f(x)) = x$ for all $x \in A$", we see that $f\big(f(r)\big) = r$, and so $f(s) = r$. It now follows from the definition of R that $(s, r) \in R$. Hence, R is symmetric.

(b) If $f(f(x)) = f(x)$ for all $x \in A$, then R is transitive.

Let $(r, s) \in R$ and $(s, t) \in R$ for some $r, s, t \in A$. $\big($We want to justify that $(r, t) \in R\big)$. Then $s = f(r)$ and $t = f(s)$ by the definition of R. From $s = f(r)$, we see that $f(s) = f\big(f(r)\big)$. Using the condition "$f(f(x)) = f(x)$ for all $x \in A$", we see that $f\big(f(r)\big) = f(r)$, and so $f(s) = f(r)$. As $t = f(s)$, we see that $t = f(r)$. It now follows from the definition of R that $(r, t) \in R$. Hence, R is transitive.

4. (10 points) Let $\sim$ be the relation on $(\mathbb{N} - \{0\}) \times \mathbb{N}$ defined for any elements (a, b) and (c, d) of $(\mathbb{N} - \{0\}) \times \mathbb{N}$ by

$$(a, b) \sim (c, d) \text{ if and only if } ad = bc.$$

Prove that:

(a) $\sim$ is an equivalence relation.

Let $(a, b) \in (\mathbb{N} - \{0\}) \times \mathbb{N}$. As $ab = ba$, we see that $(a, b) \sim (a, b)$. So $\sim$ is reflexive.

Let $(x, y), (z, t) \in (\mathbb{N} - \{0\}) \times \mathbb{N}$. Assume that $(x, y) \sim (z, t)$. Then $xt = yz$. So $zy = tx$, implying that $(z, t) \sim (x, y)$. So $\sim$ is symmetric.

Let $(a, b), (c, d), (e, f) \in (\mathbb{N} - \{0\}) \times \mathbb{N}$. Assume that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $ad = bc$ and $cf = de$. From $cf = de$, we see that $acf = ade$. Using $ad = bc$, we see from $acf = ade$ that $acf = bce$. As $c \neq 0$, canceling c we see that $af = be$, implying that $(a, b) \sim (e, f)$. So $\sim$ is transitive.

(b) Find the equivalence classes of $(1, 0)$ and $(1, 1)$.

$$\begin{aligned} [(1,0)] &= \big\{(x, y) \in (\mathbb{N} - \{0\}) \times \mathbb{N} \mid (x, y) \sim (1, 0)\big\} = \big\{(x, y) \in (\mathbb{N} - \{0\}) \times \mathbb{N} \mid x0 = y1\big\} \\ &= \big\{(x, 0) \mid x \in \mathbb{N} - \{0\}\big\} \end{aligned}$$

$$\begin{aligned} [(1,1)] &= \big\{(x, y) \in (\mathbb{N} - \{0\}) \times \mathbb{N} \mid (x, y) \sim (1, 1)\big\} = \big\{(x, y) \in (\mathbb{N} - \{0\}) \times \mathbb{N} \mid x1 = y1\big\} \\ &= \big\{(z, z) \mid z \in \mathbb{N} - \{0\}\big\} \end{aligned}$$

5. (15 points) Let $A = \{a, b, c\}$.

(a) Write as a set of ordered pairs the equivalence relation on A induced by the following partition of A

$$\big\{\{a, b\}, \{c\}\big\}.$$

Recall that $(x, y) \in A \times A$ is in the induced equivalence relation if and only if x and y are both in $\{a, b\}$ or in $\{c\}$. Hence, the induced equivalence relation is

$$\{(a, a), (b, b), (c, c), (a, b), (b, a)\}.$$

(b) How many different equivalence relations can be defined on A?

Recall that there is a bijection between equivalence classes on a set S and the partitions of the set S. So the number of different equivalence relations on A is the number of different partitions of A. As $|A| = 3$, the cardinalities of the sets in a partition of A must be 1,1,1 or 2,1 or 3. That is, the partitions of A are

$$\{\{x\}, \{y\}, \{z\}\}, \quad \{\{x, y\}, \{z\}\}, \quad \{\{x, y, z\}\}$$

where x, y, z are distinct elements of A. So there are $1 + \binom{3}{2} + 1 = 5$ partitions of A. Hence, 5 different equivalence relations can be defined on A.

6. (10 points) Let A be a subset of $\mathbb{Z}$. Consider the relation R on $\mathbb{Z}$ defined for any elements m and n of $\mathbb{Z}$ by

$$(m, n) \in R \text{ if and only if } m - n \in A.$$

Show that if R is an equivalence relation on $\mathbb{Z}$, then A is a subgroup of the group $(\mathbb{Z}, +)$ where $+$ is the usual addition.

As R is an equivalence relation on $\mathbb{Z}$, it is reflexive. In particular, $(1, 1) \in R$. Thus $1 - 1 \in A$. So $0 \in A$. In particular, A is not empty.

Let $a \in A$. Since $a = a - 0$, we see from the definition of R that $(a, 0) \in R$. As R is symmetric, $(0, a) \in R$ and so $0 - a \in A$. Thus $-a \in A$. Consequently, A is closed under taking inverse.

Let $x, y \in A$. As $x = x - 0$ and $y = 0 - (-y)$, it follows from the definition of R that $(x, 0) \in R$ and $(0, -y) \in R$. Using the fact that R is transitive, we get $(x, -y) \in R$. The definition of R implies now that $x + y \in A$. Consequently, A is closed under the addition.

7. (15 points)

(a) Let H be any simple undirected graph with n vertices. For any vertex v, what is the sum of the degree of v in H and the degree of v in the complement of H?

It follows from the definition of the complement of a graph that the sum of the degree of v in H and the degree of v in the complement of H is equal to the degree of v in K_n where K_n is the complete graph on the vertices of H. As in a complete graph with n vertices each vertex is adjacent to every other vertex, the sum we want to determine is $n - 1$.

(b) Let G be a simple undirected graph with vertex set $\{a, b, c, d, e\}$ such that

$$\deg(a) = 3, \quad \deg(b) = 1, \quad \deg(c) = \deg(d) = \deg(e) = 2.$$

Show that G is not isomorphic to its complement.

Using the previous part we see that

$$\deg_{G^c}(a) = 1, \quad \deg_{G^c}(b) = 3, \quad \deg_{G^c}(c) = \deg_{G^c}(d) = \deg_{G^c}(e) = 2$$

where subscript G^c means that the degrees are in the complement G^c of G. As a graph isomorphism preserves the degrees (i.e., the degrees of a vertex and its image under a graph isomorphism are the same), any graph isomorphism $f : G \to G^c$ must satisfy $f(a) = b$ and $f(b) = a$. Recalling the definition of the complement, note that a and b are adjacent in exactly one of G and its complement G^c. As $f(a) = b$ and $f(b) = a$, we see that f does not preserve adjacency. Consequently, f can not be a garph isomorphism.

8. (10 points) Give an example of a bijective function $\mathbb{N} \to \mathbb{Z}$

For instance, we may use even naturals to map onto nonnegative integers and use odd naturals to map onto negative integers. Consider the map

$$f(x) = \begin{cases} x/2, & \text{if } x \text{ is even} \\ -(x+1)/2, & \text{if } x \text{ is odd} \end{cases}$$

$$\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & & 1 & & 2 & & 3 & & 4 & & 5 & \\ & -1 & & -2 & & -3 & & -4 & & -5 & & \end{array}$$

9. (10 points) Let E be an equivalence relation on $\mathbb{R}$ such that the quotient set $\mathbb{R}/E$ (i.e., the set of equivalence classes) is countable. Prove that there is an $r \in \mathbb{R}$ such that the equivalence class $[r]_E$ of r is uncountable.

Recall that if R is an equivalence relation on a set A, then A is the (disjoint) union of the distinct equivalence classes. That is

$$A = \bigcup_{[a] \in A/R} [a].$$

As the union of countably many countable sets is countable, it follows that if A/R is countable and each $[a] \in A/R$ is countable, then A is countable. The result follows because $\mathbb{R}$ is uncountable and it is given that $\mathbb{R}/E$ is countable.