EHB 211E
Basics of Electrical Circuits

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State-Space Representation and Nonlinear Circuit Elements

Linear and Nonlinear Circuit by L.O. Chua, C.A. Desoer, and E.S. Kuh

Introduction

- What is state-space representation?
  - Mathematical model of a physical system as a set of input, output, and state variable related by 1st order differential equations.
  - In other words, a system by a series of 1st order differential equations. Highest order of derivative is 1st derivative.
  - It is also known as state-space model.
- Why do we use state-space representation?
  - State-space representation allows us to understand complex systems.
  - As systems become more complex, representing them with transfer function becomes harder.
  - More useful way to solve complex systems as it can handle multiple inputs and outputs as opposed to transfer function.
- State-space representation of a system is given by

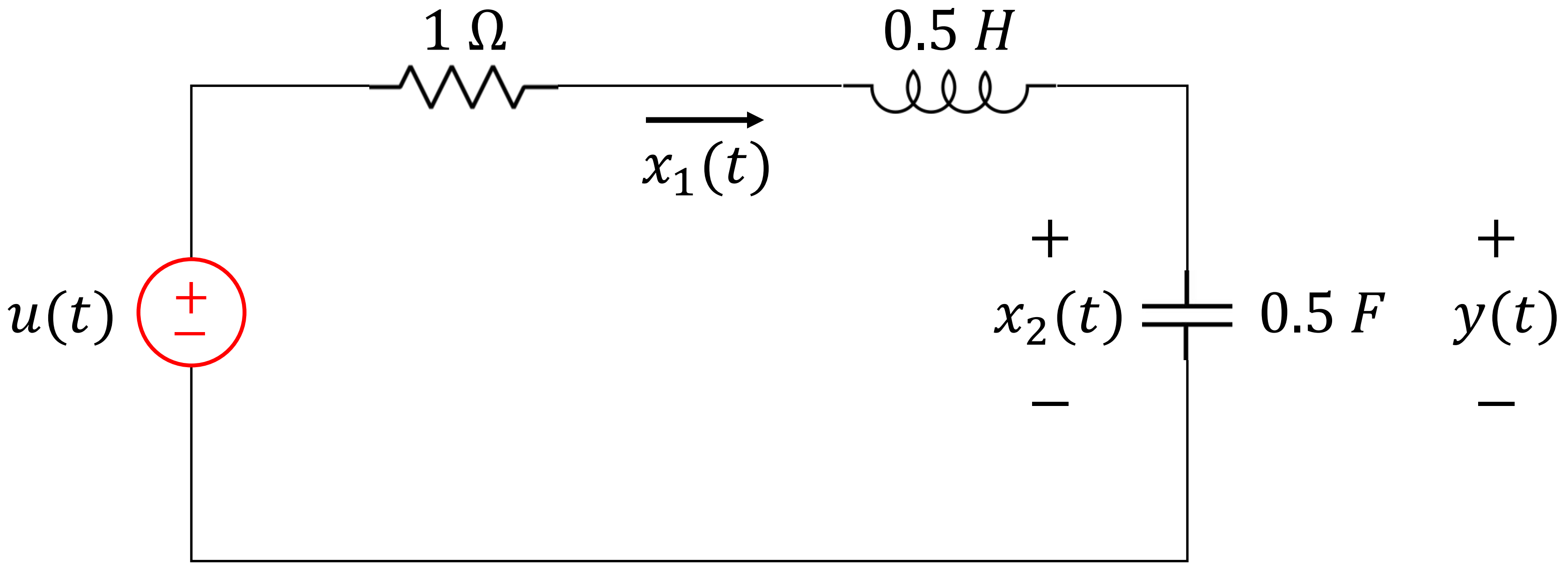
$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$$

x: state vector
$\dot{x}$: derivative of state vector
u: input vector
y: output vector
A, B, C, D: matrixes

Example 1

Consider the circuit shown below. Represent the system in a state-space form (adapted from Assoc. Prof. Onur Ferhanoğlu).



Solution:

Apply KVL: $-u + V_R + V_L + V_C = 0$ → Input, state variable, and output as function of time. Drop (t) for simplicity.

By definition, inductor: $V_L = L\dfrac{di}{dt}$

By definition, capacitor: $i_C = C\dfrac{dv}{dt}$

$x_1(t)$: current and $x_2(t)$: voltage across capacitor.

Solution

$$V_R = x_1 \times 1 = x_1$$

$$V_L = L\frac{di}{dt} \Rightarrow V_L = 0.5\frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{d}{dt}(x_1) \Rightarrow \frac{di}{dt} = \dot{x}_1 \Rightarrow V_L = 0.5\,\dot{x}_1$$

$$i_C = C\frac{dv}{dt} \Rightarrow i_C = 0.5\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = \frac{d}{dt}(x_2) = \dot{x}_2$$

$$i_C = x_1$$

$$V_C = x_2$$

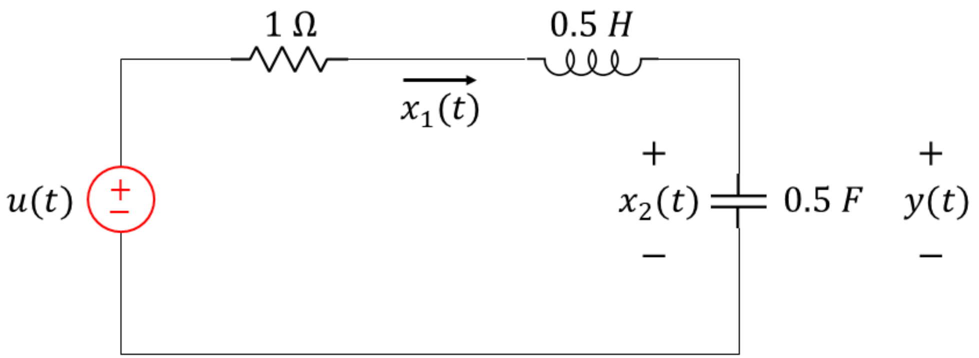
$$x_1 = 0.5\,\dot{x}_2 \Rightarrow \dot{x}_2 = 2\,x_1$$

$$-u + x_1 + 0.5\,\dot{x}_1 + x_2 = 0 \Rightarrow \dot{x}_1 = -2x_2 - 2x_1 + 2u$$

$$y = x_2$$

$$\left.\begin{aligned} \dot{x}_1 &= -2x_1 - 2x_2 + 2u \\ \dot{x}_2 &= 2\,x_1 \end{aligned}\right\} \text{State equations}$$

$$y = x_2 \;\} \text{ Output equations}$$



Solution

$$\left.\begin{aligned}\dot{x}_1 &= -2x_1 - 2x_2 + 2u \\ \dot{x}_2 &= 2\,x_1\end{aligned}\right\}$$ State equations

$$y = x_2 \}$$ Output equations

General equations

$$\vec{\dot{x}} = A\vec{x} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

Equations can be written as:

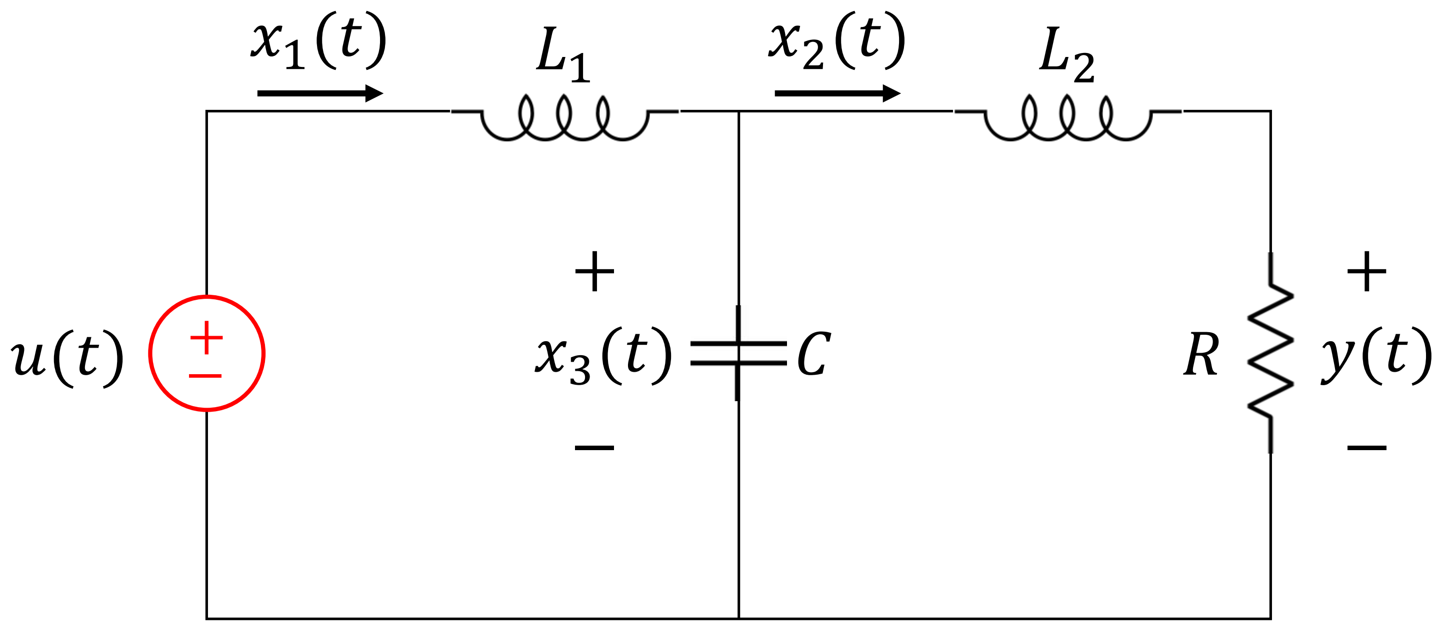
$$\underbrace{\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}-2 & -2 \\ 2 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1 \\ x_2\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}2 \\ 0\end{bmatrix}}_{B}\underbrace{u(t)}_{u} \longleftarrow \text{Single output}$$

State-space form of the output equation:

$$\underbrace{[y]}_{y} = \underbrace{[0\ 1]}_{C}\underbrace{\begin{bmatrix}x_1 \\ x_2\end{bmatrix}}_{x} + \underbrace{[0]}_{D}\underbrace{u(t)}_{u}$$

Example 2

Determine the state-space equations for the circuit shown below (adapted from Assoc. Prof. Onur Ferhanoğlu).



Solution:

Apply KVL to the left loop: $-u(t) + V_{L1} + V_C = 0$

$$V_L = L\frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{d}{dt}(x_1) \Rightarrow \frac{di}{dt} = \dot{x}_1 \Rightarrow V_{L1} = L_1\dot{x}_1(t) \qquad V_C = x_3(t)$$

$$-u(t) + L_1\dot{x}_1(t) + x_3(t) = 0 \Rightarrow \dot{x}_1(t) = -\frac{1}{L_1}x_3(t) + \frac{1}{L_1}u(t)$$

Solution

Apply KVL to the right loop: $-V_C + V_{L2} + V_R = 0$ $\qquad V_C = x_3(t)$

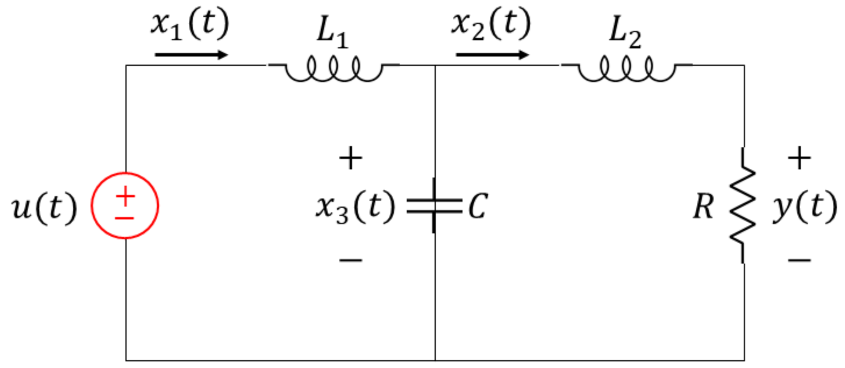
$$V_L = L\frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{d}{dt}(x_2) \Rightarrow \frac{di}{dt} = \dot{x}_2 \Rightarrow V_{L2} = L_2\dot{x}_2(t) \qquad V_R = x_2(t)R$$

$$-x_3(t) + L_2\dot{x}_2(t) + x_2(t)R = 0 \Rightarrow \dot{x}_2(t) = -\frac{R}{L_2}x_2(t) + \frac{1}{L_2}x_3(t)$$

KCL: $x_1(t) = x_2(t) + i_C$

$$i_C = C\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = \frac{d}{dt}(x_3) = \dot{x}_3 \Rightarrow i_C = C\dot{x}_3(\mathrm{t})$$

$$x_1(t) = x_2(t) + C\dot{x}_3(\mathrm{t}) \Rightarrow \dot{x}_3(\mathrm{t}) = \frac{1}{C}x_1(t) - \frac{1}{C}x_2(t)$$



Equation for output: $y(t) = x_2(t)R$

Solution

All equations:

$$\dot{x}_1(t) = -\frac{1}{L_1}x_3(t) + \frac{1}{L_1}u(t)$$

$$\dot{x}_2(t) = -\frac{R}{L_2}x_2(t) + \frac{1}{L_2}x_3(t)$$

$$\dot{x}_3(\mathrm{t}) = \frac{1}{C}x_1(t) - \frac{1}{C}x_2(t)$$

State equations

$$y(t) = x_2(t)R$$

Output equations

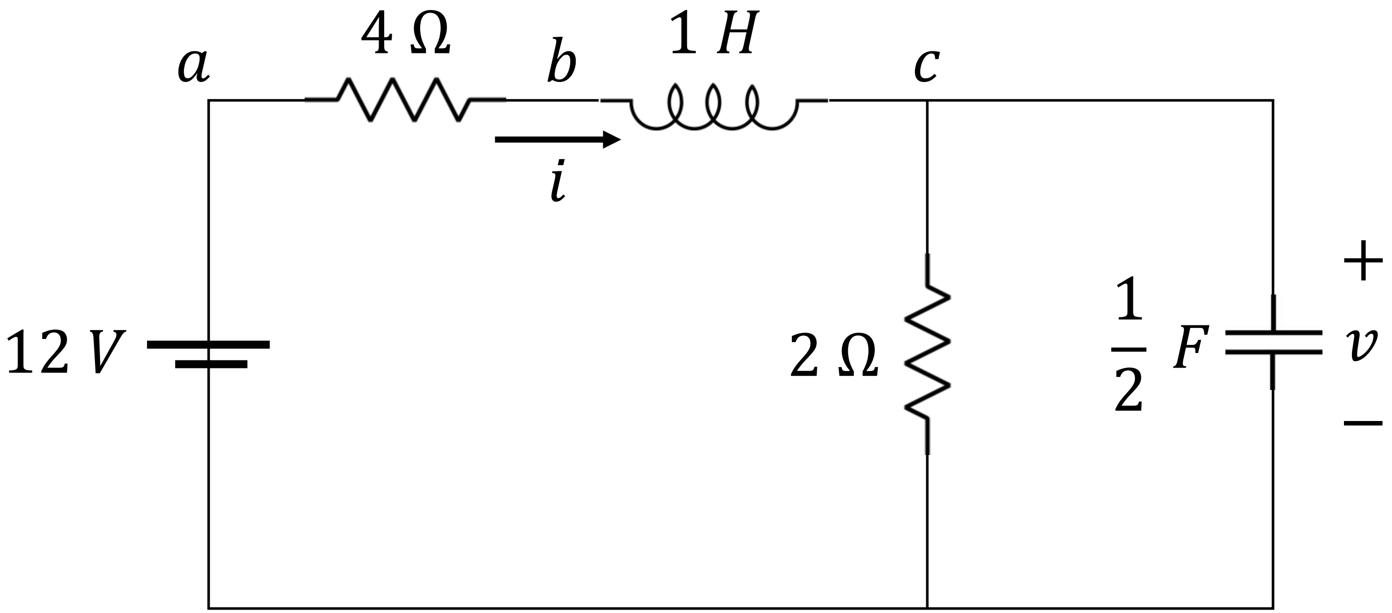
Matrix form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad R \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$

Example 3

Determine the state equations of the system shown below (adapted from C.T. Pan's note).



Solution:

Represent the system in terms of voltage v and current i since these are the state variables.

Apply KCL at node c: $i = \frac{v}{2} + C\frac{dv}{dt} \Rightarrow C\frac{dv}{dt} = i - \frac{v}{2} \Rightarrow \underbrace{\frac{dv}{dt} = -\frac{1}{2C}v + \frac{1}{C}i}_{\text{State equation}}$

Solution

Apply KVL to the left loop: $-12 + 4i + V_L + v = 0$

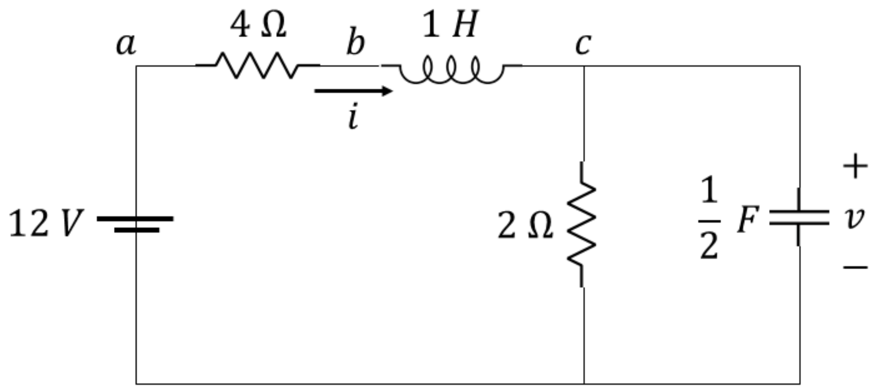
$$V_L = L\frac{di}{dt} \quad \Rightarrow \quad \frac{di}{dt} = \frac{d}{dt}(i) \Rightarrow \frac{di}{dt} = \dot{i}$$

$$-12 + 4i + L\dot{i} + v = 0$$

Represent current as $\frac{d}{dt}$

$$-12 + 4i + L\frac{di}{dt} + v = 0 \quad \Rightarrow \quad \underbrace{\frac{di}{dt} = -\frac{1}{L}v - \frac{4}{L}i + \frac{1}{L}12}_{\text{State equation}}$$

Note that no output equation since it is not stated in the circuit



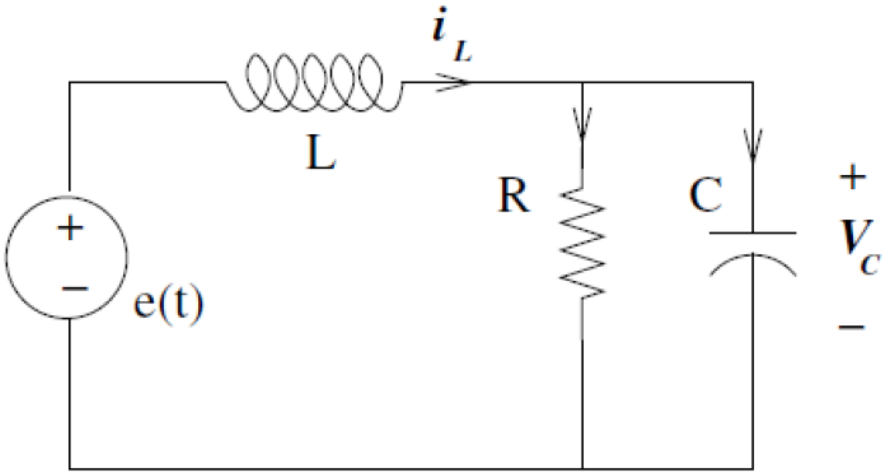
$$\frac{d}{dt}\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} -1/2C & 1/C \\ -1/L & -4/L \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} 12$$

or

$$\begin{bmatrix} \frac{dv}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{4}{L} \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} 12$$

Example 4

Write the state variable equations in the circuit shown below (adapted from Müştak E. Yalçın’s note).



Solution:

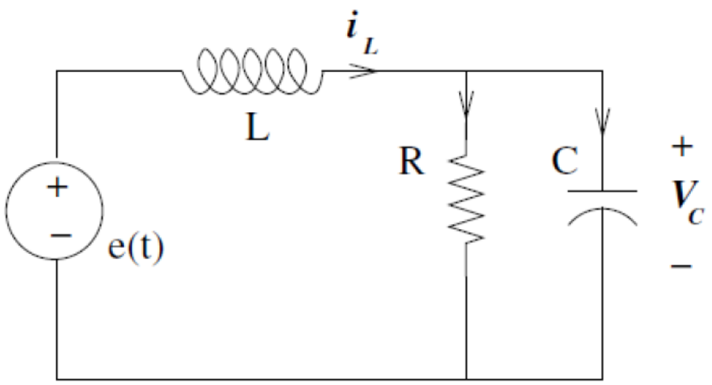
Apply KCL: $i_L = i_R + i_C$ $\quad$ $i_R = V_C / R$ $\quad$ $G = \dfrac{1}{R}$ $\quad$ $i_R = G V_C$

$$i_L = G V_C + C\frac{dV_C}{dt} \Rightarrow C\frac{dV_C}{dt} = -G V_C + i_L \Rightarrow \underbrace{\frac{dV_C}{dt} = -\frac{G}{C} V_C + \frac{1}{C} i_L}_{\text{State equation}}$$

Solution

Apply KVL to the left loop: $-e + V_L + V_R = 0$

$V_R = V_C$ (since R||C) $\quad V_L = L\dfrac{di_L}{dt}$



$$-e + L\frac{di_L}{dt} + V_C = 0 \Rightarrow \underbrace{\frac{di_L}{dt} = -\frac{1}{L}V_C + \frac{1}{L}e}_{\text{State equation}}$$

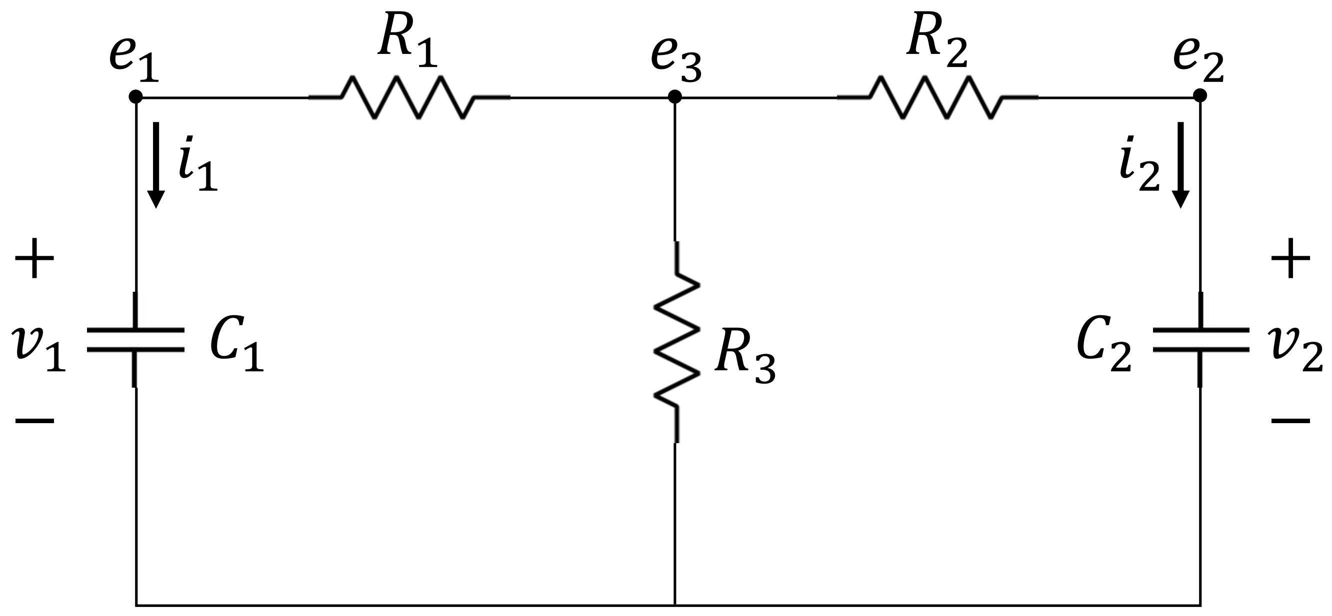
Output is not specified in the circuit

$$\begin{bmatrix} \frac{dV_C}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -G/C & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e$$

Example 5

Represent the system shown below using state-space model. $R_1 = R_2 = R_3 = 1\,\Omega$ and $C_1 = C_2 = 1\,F$ (adapted from Eytan Modiano's note-MIT).

Solution:



Apply KCL (each node):

$$e_1:\ i_1 = \frac{e_3 - e_1}{R_1} = \frac{e_3 - e_1}{1}$$

$$e_2:\ i_2 = \frac{e_3 - e_2}{R_2} = \frac{e_3 - e_2}{1}$$

$$e_3:\ \frac{e_3}{R_3} = \frac{e_3}{1} \qquad i_1 + i_2 + i_3 = 0$$

$$\frac{e_3 - e_1}{1} + \frac{e_3 - e_2}{1} + \frac{e_3}{1} = 0 \Rightarrow 3e_3 - e_1 - e_2 = 0 \Rightarrow e_3 = \frac{e_1 + e_2}{3}$$

e_1, e_2, and e_3 are node voltages and can be replaced by V_1, V_2, and V_3, respectively.

Solution

Substitute e_3 into i_1 and i_2 equations

$$i_1 = \frac{e_3 - e_1}{1} = \frac{\frac{e_1 + e_2}{3} - e_1}{1} \Rightarrow i_1 = \frac{-2V_1 + V_2}{3}$$

$$i_2 = \frac{e_3 - e_2}{1} = \frac{\frac{e_1 + e_2}{3} - e_2}{1} \Rightarrow i_2 = \frac{V_1 - 2V_2}{3}$$

$$i_1 = C\frac{dV_1}{dt} = 1\frac{dV_1}{dt} = \frac{dV_1}{dt}$$

$$i_2 = C\frac{dV_2}{dt} = 1\frac{dV_2}{dt} = \frac{dV_2}{dt}$$

Solution

$$\left.\begin{aligned}\frac{dV_1}{dt} &= -\frac{2}{3}V_1 + \frac{1}{3}V_2 \\ \frac{dV_2}{dt} &= \frac{1}{3}V_1 - \frac{2}{3}V_2\end{aligned}\right\}\text{ State equations}$$

$$\begin{bmatrix}\frac{dV_1}{dt}\\ \frac{dV_2}{dt}\end{bmatrix} = \begin{bmatrix}-\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{2}{3}\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}u(t)$$

or

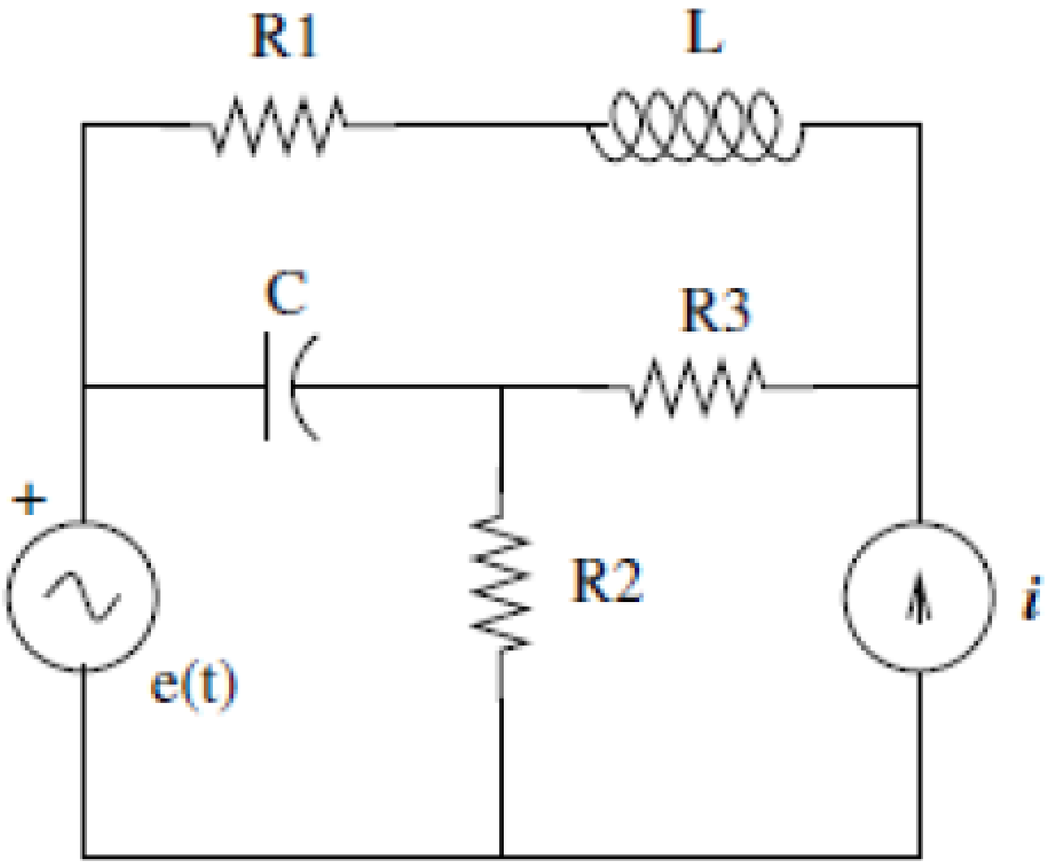
$$\begin{bmatrix}\dot{V}_1\\ \dot{V}_2\end{bmatrix} = \begin{bmatrix}-\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{2}{3}\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix}$$

or

$$\frac{d}{dt}\begin{bmatrix}V_1\\ V_2\end{bmatrix} = \begin{bmatrix}-\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{2}{3}\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix}$$

Example 6

Write the state equations for the circuit given below (adapted from Müştak E. Yalçın’s note).



Solution

Apply KCL

at node a: $i + i_L + i_3 = 0$

$i_c = i_3 + i_2 \Rightarrow i_3 = i_c - i_2$

$i + i_L + i_C - i_2 = 0$

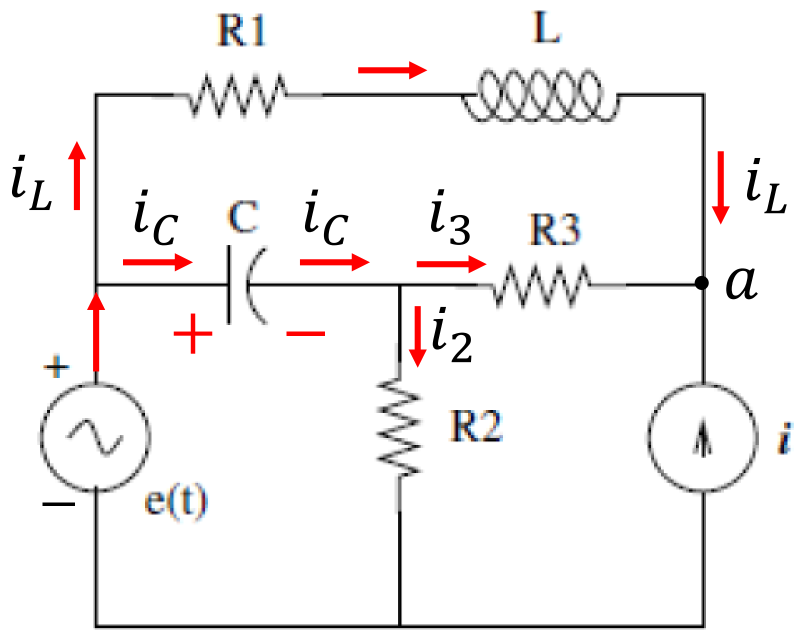
Apply KVL to the upper loop:

$$-V_C + V_1 + V_L - V_3 = 0$$

$$i_C = C\frac{dV_C}{dt} \qquad V_L = L\frac{di_L}{dt}$$

$$i + i_L + C\frac{dV_C}{dt} - i_2 = 0 \Rightarrow C\frac{dV_C}{dt} = -i_L + i_2 - i$$

$$-V_C + V_1 + L\frac{di_L}{dt} - V_3 = 0 \Rightarrow L\frac{di_L}{dt} = V_3 + V_C - V_1$$



Circuit has two inputs, voltage and current source.

Since variable are V_C and i_L, write equations in terms of V_C and i_L.

Solution

$$i_2 = \frac{-V_C - (-e)}{R_2} = \frac{e - V_C}{R_2}$$

$$V_1 = i_L R_1$$

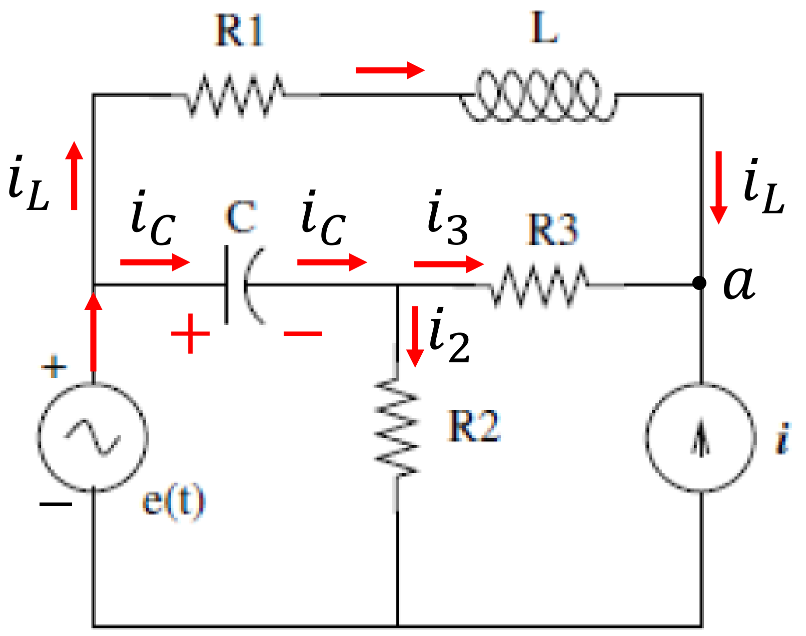
$$V_3 = R_3 i_3 \text{ where } i_3 = i_c - i_2$$

$$V_3 = R_3(i_c - i_2) \qquad \text{where } \; i_C = -i_L + i_2 - i$$

$$V_3 = R_3(-i_L + i_2 - i - i_2) \Rightarrow V_3 = R_3(-i_L - i)$$

$$C\frac{dV_C}{dt} = -i_L + i_2 - i \Rightarrow \frac{dV_C}{dt} = -\frac{1}{C}i_L + \frac{e - V_C}{R_2 C} - \frac{1}{C}i \qquad \text{or}$$

$$\frac{dV_C}{dt} = -\frac{1}{R_2 C}V_C - \frac{1}{C}i_L + \frac{1}{R_2 C}e - \frac{1}{C}i \quad \Big\} \text{ State equations}$$



Solution

$$L\frac{di_L}{dt} = V_3 + V_C - V_1 \;\Rightarrow\; L\frac{di_L}{dt} = R_3(-i_L - i) + V_C - i_L R_1$$

$$\frac{di_L}{dt} = -\frac{R_3}{L}i_L - \frac{R_3}{L}i + \frac{1}{\mathrm{L}}V_C - \frac{R_1}{L}i_L \qquad \text{or}$$

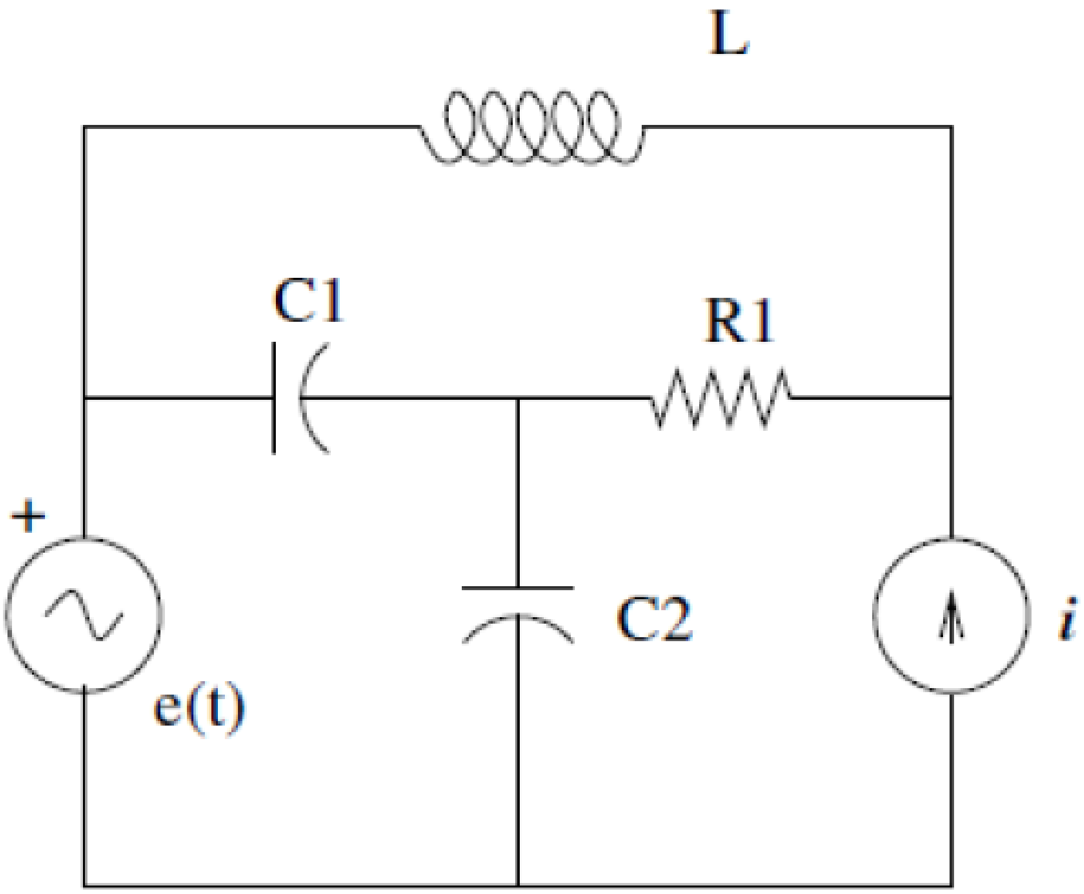
$$\frac{di_L}{dt} = \frac{1}{\mathrm{L}}V_C + \left(-\left(\frac{R_1 + R_3}{L}\right)\right)i_L - \frac{R_3}{L}i$$

State equations

$$\frac{d}{dt}\begin{bmatrix} V_C \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_2 C} & \frac{-1}{C} \\ \frac{1}{L} & \frac{-(R_3+R_1)}{L} \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{R_2 C} \\ 0 \end{bmatrix} e(t) + \begin{bmatrix} \frac{-1}{C} \\ -\frac{R_3}{L} \end{bmatrix} i$$

Example 7

Write the state equations for the circuit shown below (adapted from Müştak E. Yalçın's note).



Solution

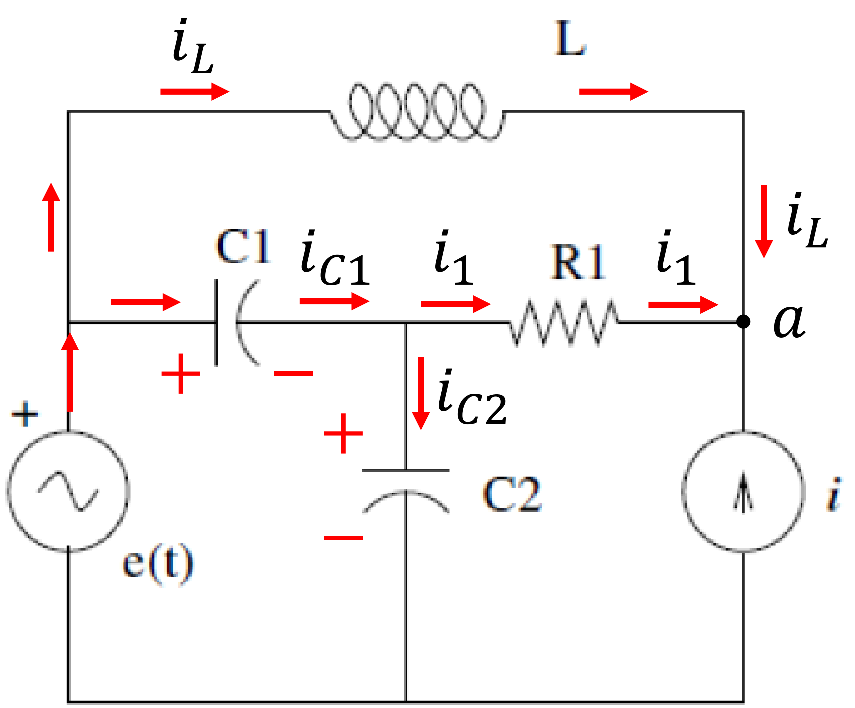
Apply KCL at node a: $i_1 + i_L + i = 0$

$$i_{C1} = i_1 + i_{C2} \Rightarrow i_1 = i_{C1} - i_{C2}$$

$$i_{C1} - i_{C2} + i_L + i = 0$$

Apply KVL to the upper loop:

$$-V_{C1} + V_L - V_1 = 0$$



$$i_{C1} = C_1 \frac{dV_{C1}}{dt} \qquad V_L = L\frac{di_L}{dt}$$

$$C_1 \frac{dV_{C1}}{dt} - i_{C2} + i_L + i = 0 \Rightarrow C_1 \frac{dV_{C1}}{dt} = -i_L - i + i_{C2} \longrightarrow$$ 1st equation

$$-V_{C1} + L\frac{di_L}{dt} - V_1 = 0 \Rightarrow L\frac{di_L}{dt} = V_1 + V_{C1} \longrightarrow$$ 2nd equation

Solution

$$i_{C2} = C_2 \frac{dV_{C2}}{dt}$$

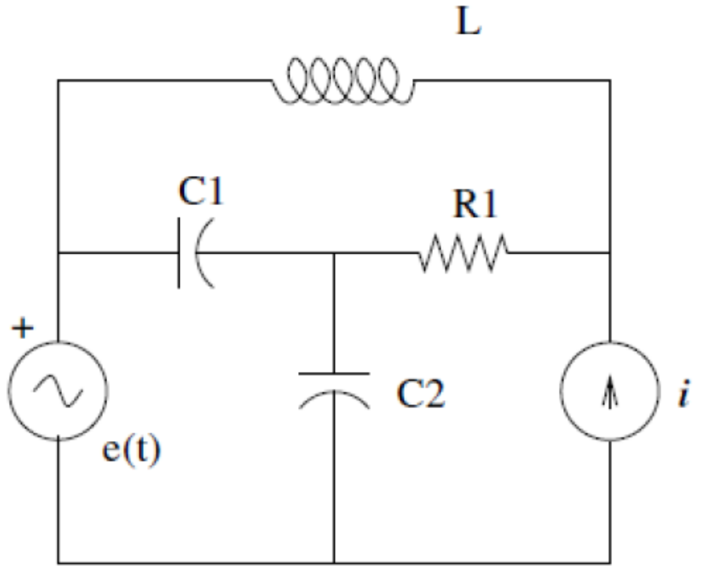
Apply KVL to the lower left loop:

$V_{C2} = e - V_{C1}$ ➡ Take the derivative of both sides

$\frac{dV_{C2}}{dt} = \frac{de}{dt} - \frac{dV_{C1}}{dt}$ ➡ Multiply both sides with C_2

$\underbrace{C_2 \frac{dV_{C2}}{dt}}_{i_{C2}} = C_2 \frac{de}{dt} - C_2 \frac{dV_{C1}}{dt}$ ➡ Substitute into equation 1

$$C_1 \frac{dV_{C1}}{dt} = -i_L - i + C_2 \frac{de}{dt} - C_2 \frac{dV_{C1}}{dt} \Rightarrow C_1 \frac{dV_{C1}}{dt} + C_2 \frac{dV_{C1}}{dt} = -i_L - i + C_2 \frac{de}{dt}$$



Solution

$$C_1 \frac{dV_{C1}}{dt} + C_2 \frac{dV_{C1}}{dt} = -i_L - i + C_2 \frac{de}{dt} \Rightarrow \frac{dV_{C1}}{dt}(C_1 + C_2) = -i_L - i + C_2 \frac{de}{dt}$$

$$\frac{dV_{C1}}{dt} = -\frac{1}{C_1 + C_2} i_L - \frac{1}{C_1 + C_2} i + \frac{C_2}{C_1 + C_2} \frac{de}{dt} \longrightarrow$$ 1st state equ.

$$L \frac{di_L}{dt} = V_1 + V_{C1} \qquad V_1 = i_1 R_1 \qquad i_1 + i_L + i = 0 \Rightarrow i_1 = -(i_L + i)$$

$$V_1 = -R_1(i_L + i)$$

$$L \frac{di_L}{dt} = -R_1(i_L + i) + V_{C1} \Rightarrow \frac{di_L}{dt} = -\frac{R_1}{L} i_L - \frac{R_1}{L} i + \frac{1}{L} V_{C1} \longrightarrow$$ 2nd state equ.

Solution

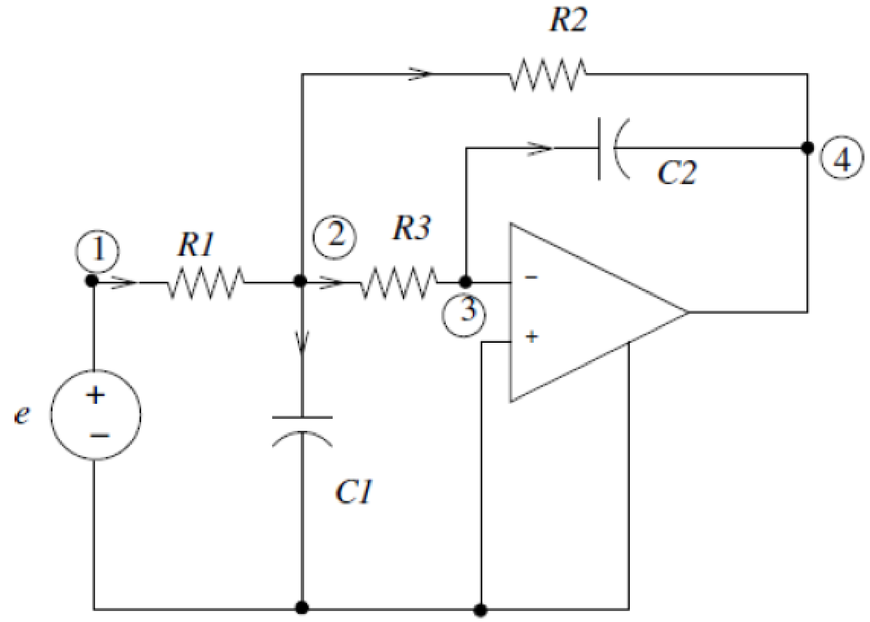
$$\left.\begin{aligned}\frac{dV_{C1}}{dt} &= -\frac{1}{C_1 + C_2} i_L + \frac{C_2}{C_1 + C_2}\frac{de}{dt} - \frac{1}{C_1 + C_2} i \\ \frac{di_L}{dt} &= \frac{1}{L} V_{C1} - \frac{R_1}{L} i_L - \frac{R_1}{L} i\end{aligned}\right\} \text{State equations}$$

- In matrix form:

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{1}{C_1 + C_2} \\ \dfrac{1}{L} & -\dfrac{R_1}{L} \end{bmatrix}\begin{bmatrix} V_{C1} \\ i_L \end{bmatrix} + \begin{bmatrix} \dfrac{C_2}{C_1 + C_2} \\ 0 \end{bmatrix}\frac{de}{dt} + \begin{bmatrix} -\dfrac{1}{C_1 + C_2} \\ -\dfrac{R_1}{L} \end{bmatrix} i$$

Example 8

For the circuit given below, write state equations (adapted from Müştak E. Yalçın's note).



Solution:

Apply KCL at node 2: $i_1 = i_2 + i_3 + i_{C1}$

Let voltages at each node as follows:

Node 1: V_{d1}

Node 2: V_{d2}

Node 3: V_{d3}

Node 4: V_{d4}

$$i_1 = \frac{V_{d1} - V_{d2}}{R_1} \qquad i_2 = \frac{V_{d2} - V_{d4}}{R_2}$$

$$i_3 = \frac{V_{d2} - V_{d3}}{R_3} \qquad i_{C1} = C1\frac{dV_{C1}}{dt}$$

$$\frac{V_{d1} - V_{d2}}{R_1} = \frac{V_{d2} - V_{d4}}{R_2} + \frac{V_{d2} - V_{d3}}{R_3} + C1\frac{dV_{C1}}{dt}$$

$$C1\frac{dV_{C1}}{dt} = \frac{V_{d1} - V_{d2}}{R_1} - \frac{V_{d2} - V_{d4}}{R_2} - \frac{V_{d2} - V_{d3}}{R_3}$$

Solution

Apply KCL at node 3: $i_3 = i_{C2}$ $\quad i_3 = \dfrac{V_{d2} - V_{d3}}{R_3}$ $\quad i_{C2} = C2\dfrac{dV_{C2}}{dt}$

$$C2\frac{dV_{C2}}{dt} = \frac{V_{d2} - V_{d3}}{R_3}$$

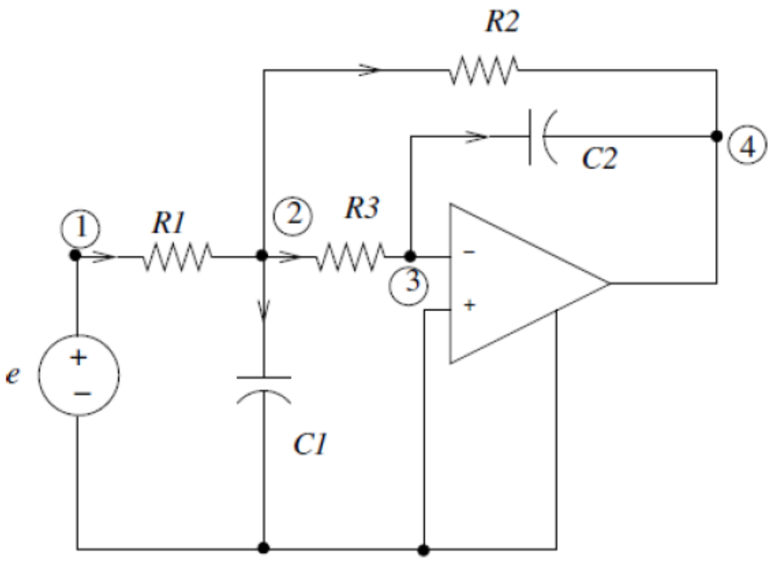
Redefine node voltages as: $V_{d1} = e, V_{d2} = V_{C1}, V_{d3} = 0$, and $V_{d4} = -V_{C2}$

Recall: $G = \dfrac{1}{R}$

$$C1\frac{dV_{C1}}{dt} = G_1(e - V_{C1}) - G_2(V_{C1} - V_{C2}) - G_3(V_{C1} - 0)$$

$$\frac{dV_{C1}}{dt} = \frac{G_1}{C1}(e - V_{C1}) - \frac{G_2}{C1}(V_{C1} - V_{C2}) - \frac{G_3}{C1}V_{C1}$$

$$\frac{dV_{C1}}{dt} = -\left(\frac{G_1 + G_2 + G_3}{C1}\right)V_{C1} - \frac{G_2}{C1}V_{C2} - \frac{G_1}{C1}e$$ $\longrightarrow$ 1st eq.



Solution

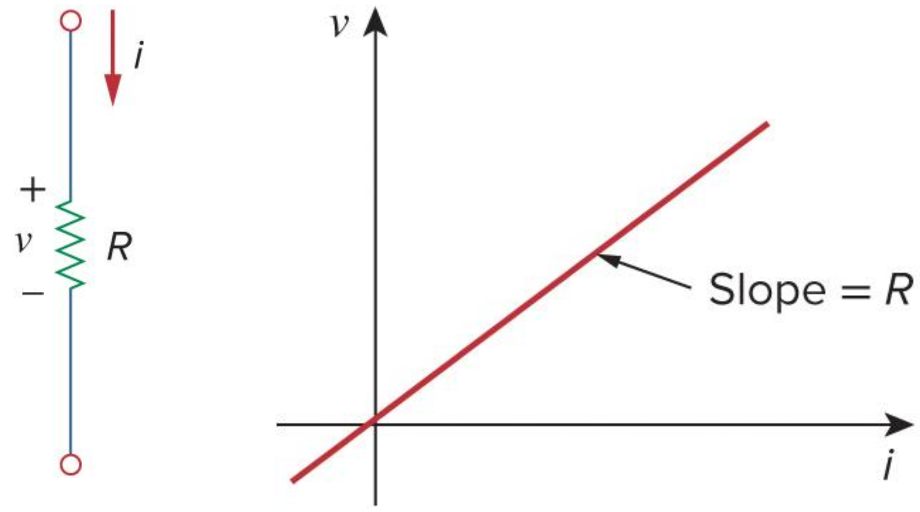
$$C2\frac{dV_{C2}}{dt} = G_3\,(V_{C1} - 0) \Rightarrow \frac{dV_{C2}}{dt} = \frac{G_3}{C2}V_{C1} \longrightarrow \text{2}^{\text{nd}}\ \text{eq.}$$

$$\left.\begin{aligned} \frac{dV_{C1}}{dt} &= -\left(\frac{G_1 + G_2 + G_3}{C1}\right)V_{C1} - \frac{G_2}{C1}V_{C2} - \frac{G_1}{C1}e \\ \frac{dV_{C2}}{dt} &= \frac{G_3}{C2}V_{C1} \end{aligned}\right\} \text{State equations}$$

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} = \begin{bmatrix} -\frac{G_1+G_2+G_3}{C_1} & -\frac{G_2}{C_1} \\ -\frac{G3}{C_2} & 0 \end{bmatrix}\begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} + \begin{bmatrix} \frac{G_1}{C_1}0 \end{bmatrix} e$$

Two terminal Linear and Nonlinear Circuit Elements

- Two types of resistors:
  - Linear resistor
  - Nonlinear resistor
- **Linear resistor:**
  - Two terminal circuit element whose resistance value does not change or vary with the flow of current through it.
  - Current through the resistance is always proportional to the voltage applied across it.



- Linear resistance: Obey Ohm's law

$$v(t) = i(t)R \text{ or } i(t) = Gv(t)$$

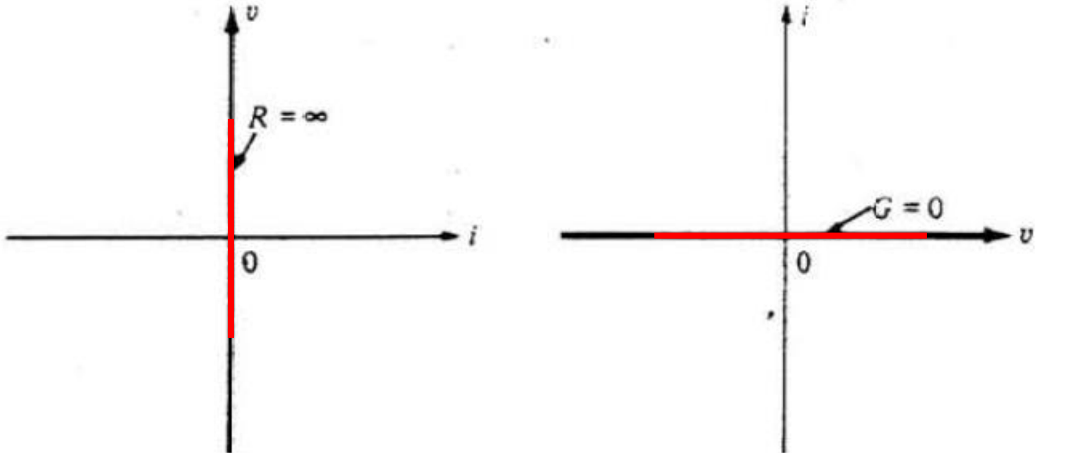
- It has a constant resistance (slope)
- Linearity: Its i-v graph is a straight line passing through the origin

Linear Resistor

- Resistance value for open and short circuit:

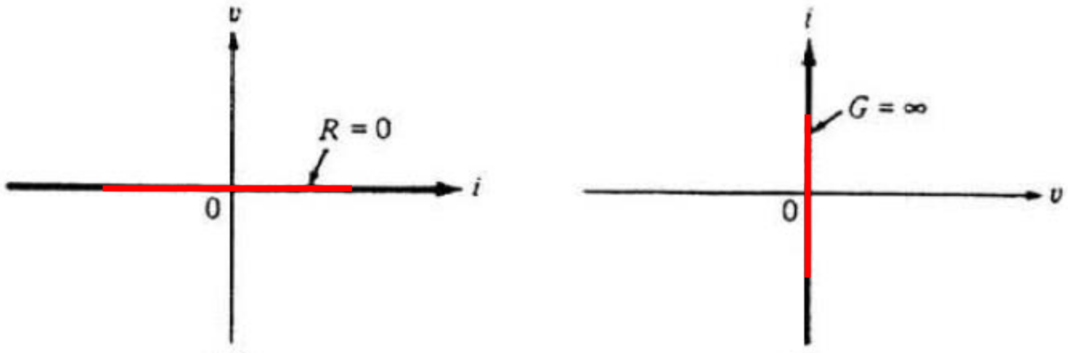
$$v = iR \text{ or } i = Gv$$

- When $R = \infty$, $i = 0$



Characteristics of open circuit

- When $R = 0$, $v = 0$

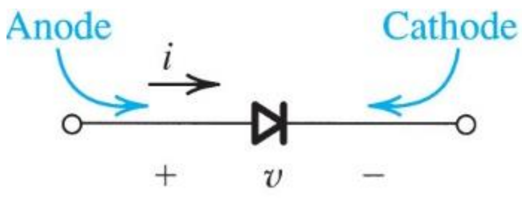


Characteristics of short circuit

Nonlinear Resistor

- What is nonlinear resistor?
  - Circuit element whose voltage and current relation vary nonlinearly as opposed to linear resistor.
  - Current through nonlinear resistor is not proportional to the voltage applied across it.
- Diode: nonlinear resistor.

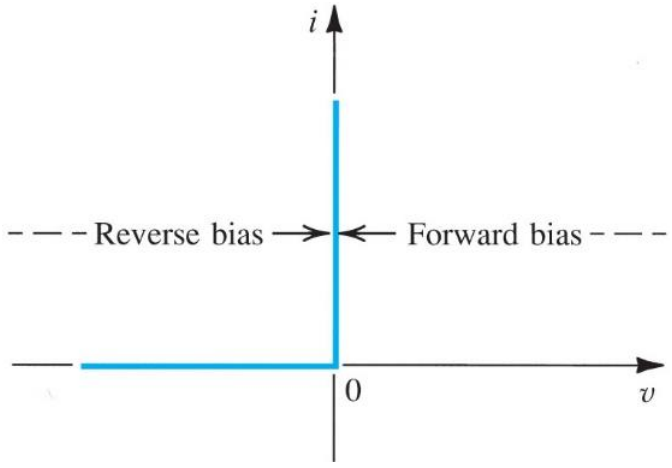
Circuit symbol of diode



Two terminals:
Anode (+) and Cathode (-)

Current flows in the direction of arrowhead

$i - v$ characteristics of ideal diode

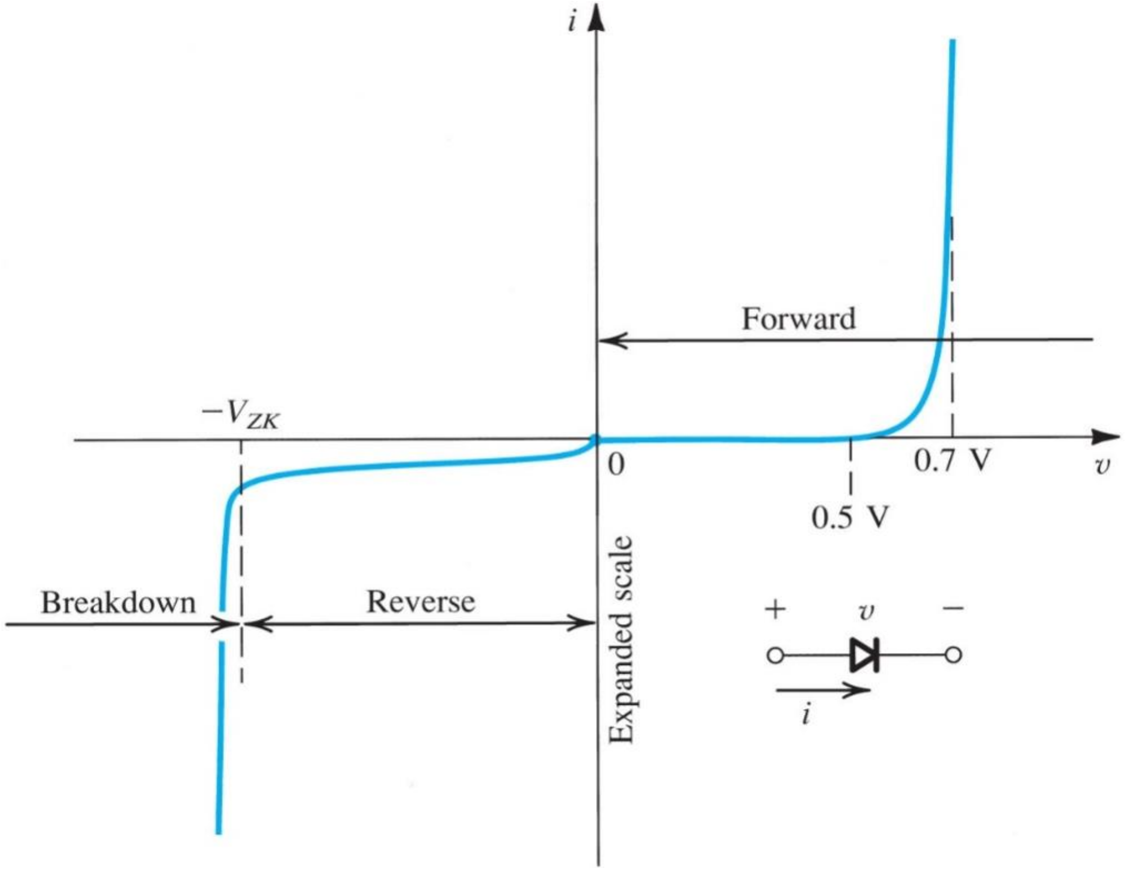


Highly nonlinear

Nonlinear Resistor: pn-Junction Diode

- pn-junction diode is a two terminal semiconductor devices
- Current increases exponentially in the forward bias region.
- pn-Junction diode: nonlinear elements as $i - v$ relationship is not linear.

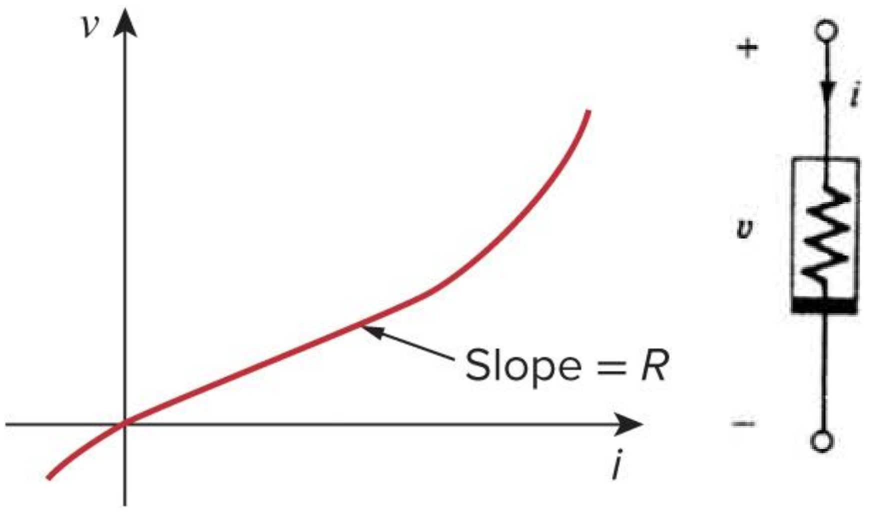
$$i = I_S\left(e^{\frac{v}{V_T}} - 1\right)$$



Analysis of Nonlinear Resistive Circuit

- Reason to analyze nonlinear circuits:
  - Electrical devices such as computer or amplifier are constructed based upon mostly nonlinear circuit.
  - Understanding nonlinear circuit: design superior devices.

Nonlinear resistance: Does not Obey Ohm's law

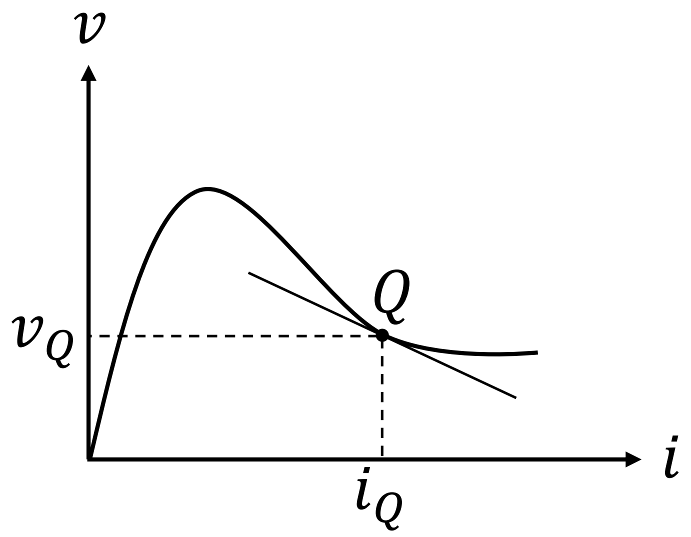


- Its resistance (slope) varies with current

Analysis of Nonlinear Resistive Circuit

- Nonlinear resistor in a circuit can be analyzed using linear approximation.
- Linear approximation is called linearization.
- Nonlinear graph: i and v are not directly proportional
- Point Q: operation point
- Linearized based on the slope at the operation point Q. Resistance is obtained from slope.

$$R_Q = \left.\frac{dv_Q}{dI_Q}\right|_Q$$



- DC operating point is obtained from the dc component of the input signal when ac components are set to zero.
- For ac analysis, determine the slope (R_Q) at operating point Q when dc components are set to zero.

$$R_Q = \left.\frac{df(i)}{di_N}\right|_Q$$

Example 9

For the circuit shown below, $R = 3.5\ \Omega, e_S = 9\ V, e_t(t) = 0.1\sin(10t)$. The nonlinear resistance is characterized by

$$v_R = i_R^3 - 6i_R^2 + 9i_R$$

Determine the solution for v_R (adapted from Müştak E. Yalçın's note).

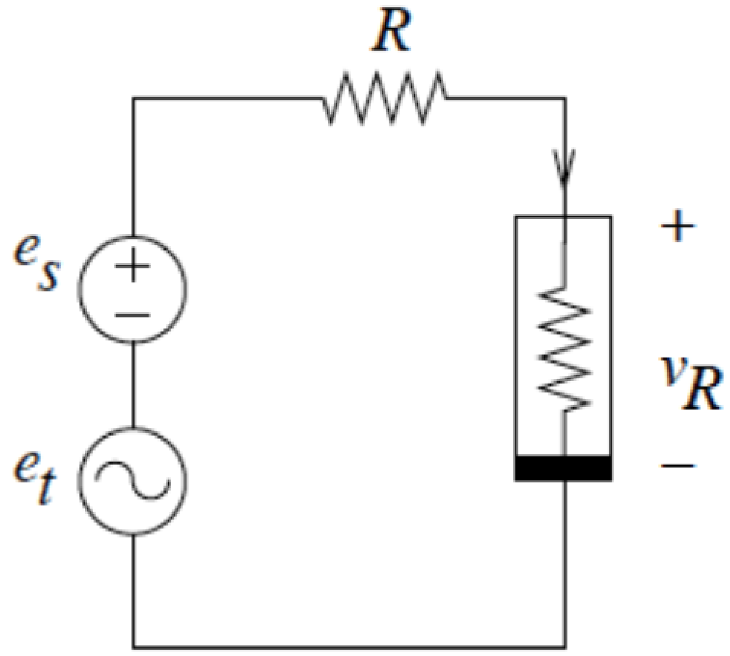
Solution:

Circuit has both ac and dc components:

$e_s = 9\ V \longrightarrow$ dc

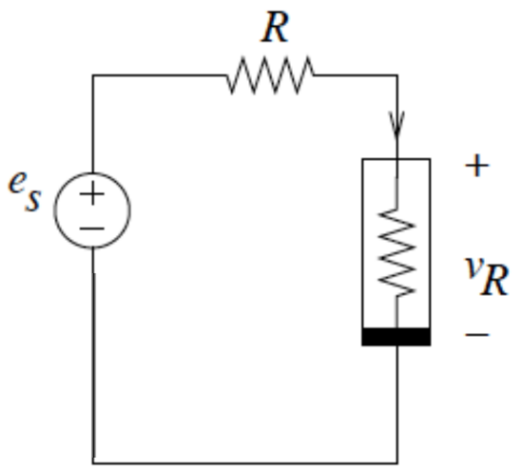
$e_t(t) = 0.1\sin(10t) \longrightarrow$ ac

In order to find complete solution for v_R, determine both ac & dc components of v_R.



Solution

DC analysis: Ignore ac component and find dc operating point based on dc component of the input signal.



Apply KVL: $-e_S + i_R R + V_R = 0 \Rightarrow e_S = i_R R + V_R$

$v_R = i_R^3 - 6i_R^2 + 9i_R \qquad e_s = 9\ V \qquad R = 3.5\ \Omega$

$9 = 3.5i_R + i_R^3 - 6i_R^2 + 9i_R$

$3.5i_R + i_R^3 - 6i_R^2 + 9i_R - 9 = 0 \Rightarrow i_R^3 - 6i_R^2 + 12.5i_R - 9 = 0$

Solving the cubic equation: $i_R = 2\ A$

$v_R = i_R^3 - 6i_R^2 + 9i_R \Rightarrow v_R = 2^3 - 6(2)^2 + 9(2) \Rightarrow v_R = 2\ V$

$$\left.\begin{aligned} I_R &= 2\ A \\ V_R &= 2\ V \end{aligned}\right\} \text{DC operating point}$$

Solution

AC analysis: Ignore dc component and determine R_Q at operating point Q.

$$R_Q = \left.\frac{dv_Q}{dI_Q}\right|_Q \Rightarrow$$ Linearizing the nonlinear resistor around $i_R = 2\ A$

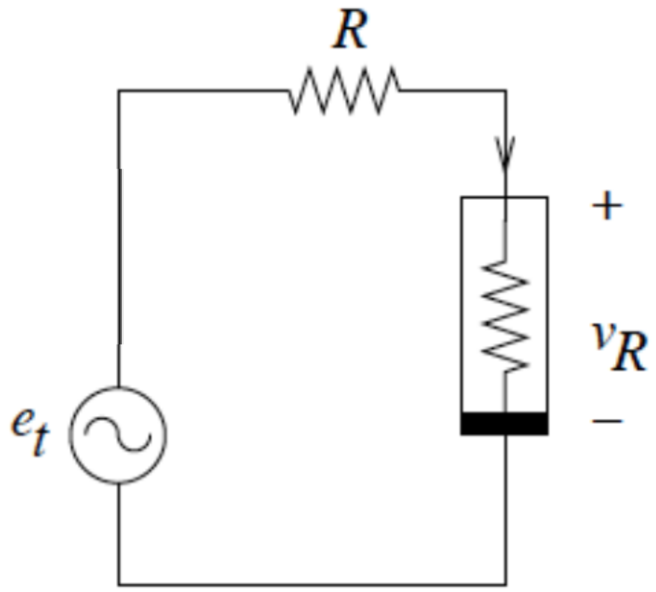
$$R_Q = \frac{d}{dI_Q}(i_R^3 - 6i_R^2 + 9i_R) \Rightarrow R_Q = 3i_R^2 - 12i_R + 9\Big|_{i_R = 2\ A}$$

$$R_Q = 3(2)^2 - 12(2) + 9 \Rightarrow R_Q = -3\ \Omega$$

Find the ac component of v_R

Voltage division: $v_R = \dfrac{R_Q}{R_Q + R} e_t(t)$

$$v_R = \frac{-3}{-3 + 3.5} 0.1 \sin(10t) \Rightarrow v_R = -0.6\sin(10t)$$

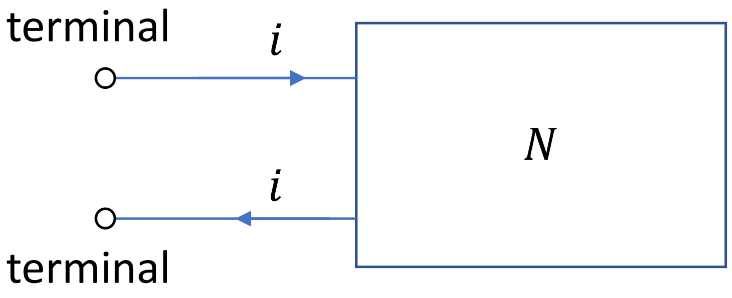


Complete solution (superposition): $v_R = 2 - 0.6\sin(10t)$

Resistive One-Port Network

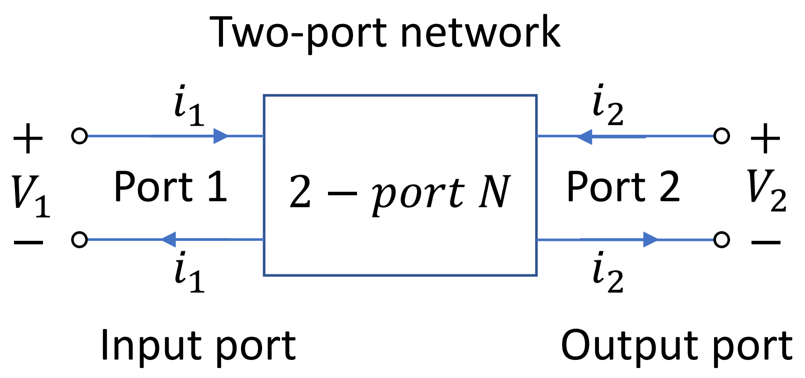
- What is port?
  - A pair of terminals connecting an electrical network or circuit to an external circuit.
- Port network: useful to analyze large and complex circuit.
- Port condition:
  - Current enters through one terminal of the port is equal to the current leaving through second terminal of the port
- Example of one-port network: Resistor, capacitor, inductor.

One-port network



Resistive Two-Port Network

- Electrical circuit or network has two pairs of terminals called two port network.
- Example of two-port network: transformer, filter, transmission line



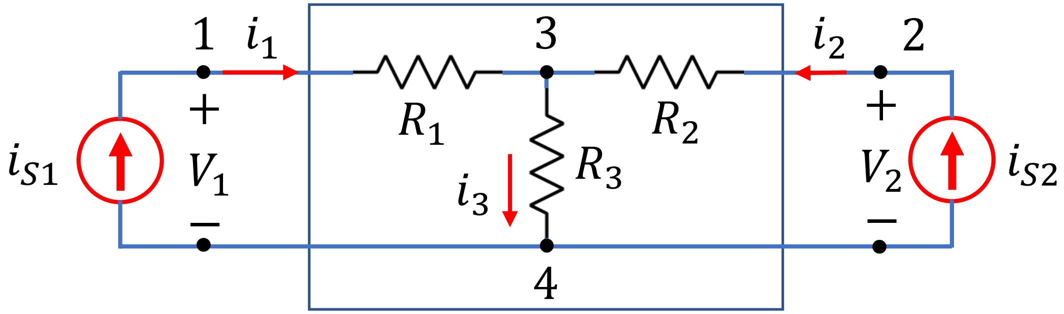
- Two port network has four variables: V_1, V_2, i_1, i_2
- Port voltages and port currents can be represented as:

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad i = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

- Note that more than two-port network is called multiport network.

Example 10

Figure below shows a T-circuit which is placed into a black box to create a network. Two independent current sources are connected to the input and output ports. Represent input and output voltages in matrix from. Network variables are V_1, i_1, V_2, i_2



Solution:

Apply KCL:

$i_{S1} = i_1 \qquad i_{S2} = i_2 \qquad i_3 = i_1 + i_2$

Apply KVL: $-V_1 + R_1 i_1 + R_3 i_3 = 0$

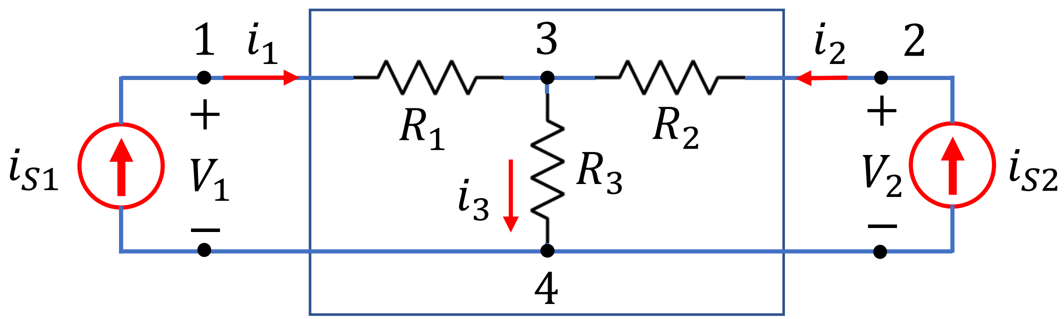
Write i_3 in terms of i_1 and i_2 as they are circuit variables.

$-V_1 + R_1 i_1 + R_3(i_1 + i_2) = 0 \Rightarrow V_1 = R_1 i_1 + R_3 i_1 + R_3 i_2$

$V_1 = (R_1 + R_3) i_1 + R_3 i_2 \longrightarrow$ 1st equation

Solution

Apply KVL: $-V_2 + R_2 i_2 + R_3 i_3 = 0$



$$V_2 = R_2 i_2 + R_3(i_1 + i_2) = 0$$

$$V_2 = R_3 i_1 + (R_2 + R_3) i_2 = 0 \longrightarrow 2^{nd} \text{ equ.}$$

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = Ri = \begin{bmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Current controlled representation

$$R = \begin{bmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{bmatrix} \longrightarrow$$

Resistance matrix of the linear resistive two port

Two currents are sources and two voltages are responses. Thus, i_1 and i_2 are independent variables and V_1 and V_2 are dependent variables (voltages are function of currents).

Solution

- i_1 and i_2 can be solved in terms of V_1 and V_2

$$G = \frac{1}{R} \qquad\qquad i = GV$$

- G is conductance which is inverse of resistance matrix R

$$G = R^{-1} = \frac{1}{R_1R_2 + R_1R_3 + R_2R_3}\begin{bmatrix} R_2 + R_3 & -R_3 \\ -R_3 & R_1 + R_3 \end{bmatrix}$$

→ Conductance matrix of the linear resistive two port

- Current equations:

$$i_1 = \frac{R_2 + R_3}{R_1R_2 + R_1R_3 + R_2R_3}V_1 - \frac{R_3}{R_1R_2 + R_1R_3 + R_2R_3}V_2$$

$$i_2 = \frac{-R_3}{R_1R_2 + R_1R_3 + R_2R_3}V_1 + \frac{R_1 + R_2}{R_1R_2 + R_1R_3 + R_2R_3}V_2$$

$$i = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = GV = [G]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Voltage controlled representation

Six Representations

- For a resistive two-port network, there exists six different representations
- All possible representations are shown in the table below:

Representations	Independent variables	Dependent variables
Current-controlled	i_1, i_2	v_1, v_2
Voltage-controlled	v_1, v_2	i_1, i_2
Hybrid 1	i_1, v_2	v_1, i_2
Hybrid 2	v_1, i_2	i_1, v_2
Transmission 1	v_2, i_2	v_1, i_1
Transmission 2	v_1, i_1	v_2, i_2

Six Representations

- Equations for six different representations of a resistive two-port:

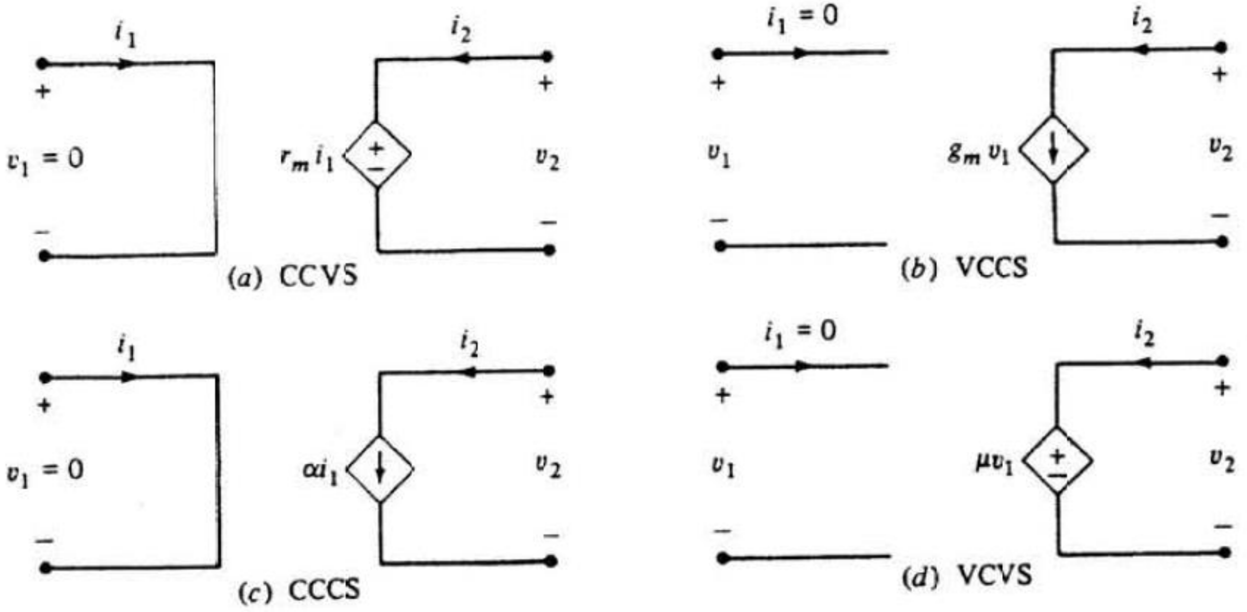
Representations	Scalar equations	Vector equations
Current-controlled	$v_1 = r_{11}i_1 + r_{12}i_2$ $v_2 = r_{21}i_1 + r_{22}i_2$	$\mathbf{v} = \mathbf{R}\mathbf{i}$
Voltage-controlled	$i_1 = g_{11}v_1 + g_{12}v_2$ $i_2 = g_{21}v_1 + g_{22}v_2$	$\mathbf{i} = \mathbf{G}\mathbf{v}$
Hybrid 1	$v_1 = h_{11}i_1 + h_{12}v_2$ $i_2 = h_{21}i_1 + h_{22}v_2$	$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \mathbf{H}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$
Hybrid 2	$i_1 = h'_{11}v_1 + h'_{12}i_2$ $v_2 = h'_{21}v_1 + h'_{22}i_2$	$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \mathbf{H}'\begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$
Transmission 1†	$v_1 = t_{11}v_2 - t_{12}i_2$ $i_1 = t_{21}v_2 - t_{22}i_2$	$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \mathbf{T}\begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$
Transmission 2†	$v_2 = t'_{11}v_1 + t'_{12}i_1$ $-i_2 = t'_{21}v_1 + t'_{22}i_1$	$\begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} = \mathbf{T}'\begin{bmatrix} v_1 \\ i_1 \end{bmatrix}$

Linear Controlled Sources

- Controlled sources are resistive two-port elements consisting of two branches:
  - Primary branch: open circuit or short circuit
  - Secondary branch: voltage source or current source

Constant:

- r_m: transresistance
- g_m: transconductance
- α: current transfer ratio
- μ: voltage transfer ratio



- Each linear controlled source is characterized by two linear equations

CCVS:	$v_1 = 0$	$v_2 = r_m i_1$
VCCS:	$i_1 = 0$	$i_2 = g_m v_1$
CCCS:	$v_1 = 0$	$i_2 = \alpha i_1$
VCVS:	$i_1 = 0$	$v_2 = \mu v_1$

Linear Controlled Sources

- Linear equation for four controlled sources can also be represented in matrix form as:

CCVS: $v_1 = 0 \qquad v_2 = r_m i_1$

VCCS: $i_1 = 0 \qquad i_2 = g_m v_1$

CCCS: $v_1 = 0 \qquad i_2 = \alpha i_1$

VCVS: $i_1 = 0 \qquad v_2 = \mu v_1$

CCVS: $$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ r_m & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

VCCS: $$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ g_m & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

CCCS: $$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

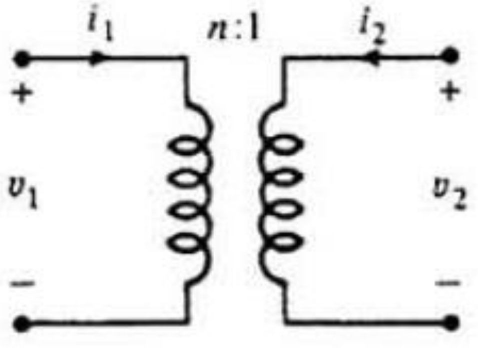
VCVS: $$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

Linear Controlled Sources

- Ideal transformer: ideal two-port resistive circuit element characterizing by

$v_1 = nv_2 \qquad i_2 = -ni_1$ n: real number called turn ratio

Symbol of transformer



$$\frac{v_1}{v_2} = n = \frac{-i_2}{i_1}$$

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = H \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

- Ideal transformer neither dissipate nor stores energy (non-energetic elements)

$p = v_1 i_1 + v_2 i_2 = 0$

Gyrator

- Gyrator: an ideal two-port element defined by the following equations:

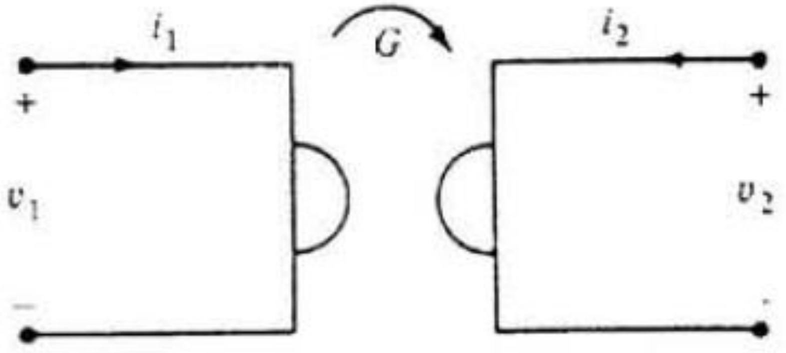
$$i_1 = Gv_2 \qquad i_2 = -Gv_1$$

Constant G is called the gyration conductance

- In vector form, the voltage controlled representation:

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad i = \begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix} v$$

Symbol of gyrator



- Just like Ideal transformer, gyrator is non-energetic elements (power delivered to the two-port is zero).