BLG 336E

Analysis of Algorithms II

Lecture 9:

Dynamic Programming II

DNA Sequencing, Knapsack Problem

Last time

Dynamic Programming!

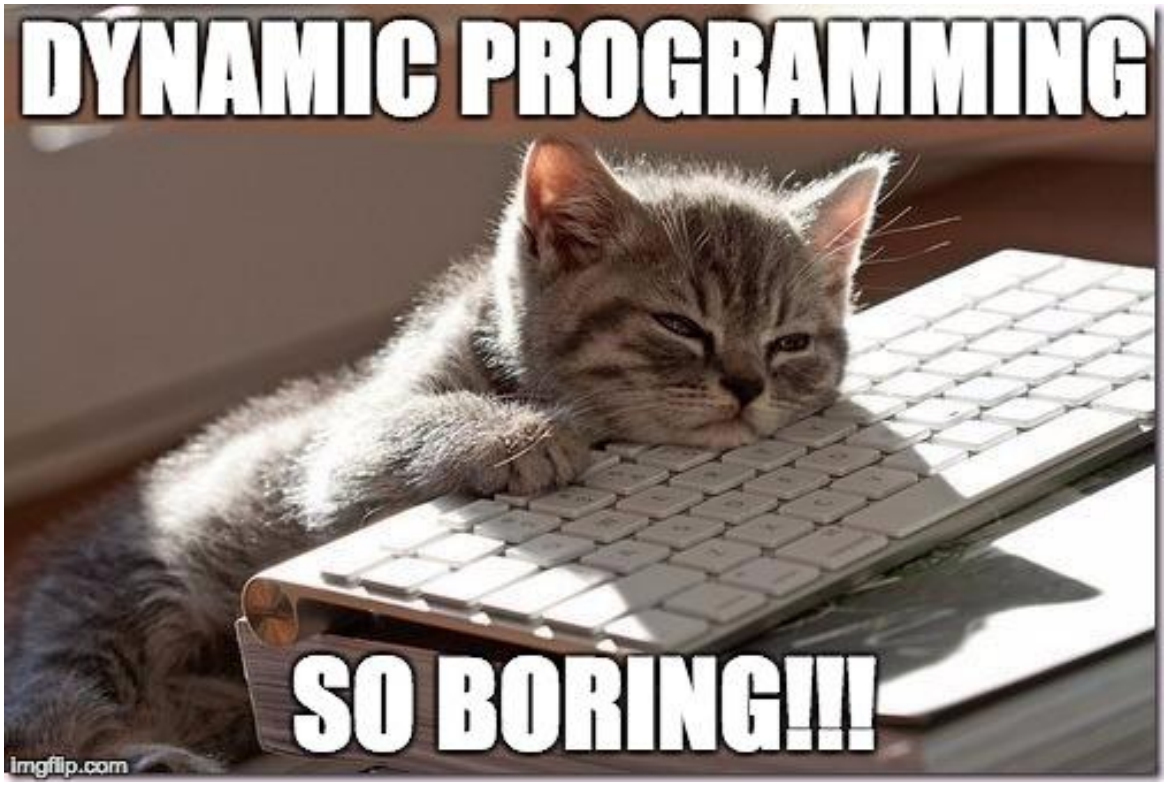
- Dynamic programming is an algorithm design paradigm.
- Basic idea:
  - Identify **optimal sub-structure**
    - Optimum to the big problem is built out of optima of small sub-problems
  - Take advantage of **overlapping sub-problems**
    - Only solve each sub-problem once, then use it again and again
  - Keep track of the solutions to sub-problems in a table as you build to the final solution.

Today

- Examples of dynamic programming:
  1. Longest common subsequence
  2. Knapsack problem
     - Two versions!
  3. Independent sets in trees
     - If we have time…
     - (If not the slides will be there as a reference)

The goal of this lecture

- For you to get really bored of dynamic programming



Longest Common Subsequence

- How similar are these two species?

DNA:

AGCCCTAAGGGCTACCTAGCTT

DNA:

GACAGCCTACAAGCGTTAGCTTG

Longest Common Subsequence

- How similar are these two species?

DNA:

AGCCCTAAGGGCTACCTAGCTT

DNA:

GACAGCCTACAAGCGTTAGCTTG

- Pretty similar, their DNA has a long common subsequence:

AGCCTAAGCTTAGCTT

Longest Common Subsequence

- Subsequence:
  - BDFH is a **subsequence** of ABCDEFGH
- If X and Y are sequences, a **common subsequence** is a sequence which is a subsequence of both.
  - BDFH is a **common subsequence** of ABCDEFGH and of ABDFGHI
- A **longest common subsequence**…
  - …is a common subsequence that is longest.
  - The **longest common subsequence** of ABCDEFGH and ABDFGHI is ABDFGH.

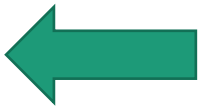
We sometimes want to find these

- Applications in bioinformatics

- The unix command `diff`
- Merging in version control
  - svn, git, etc…

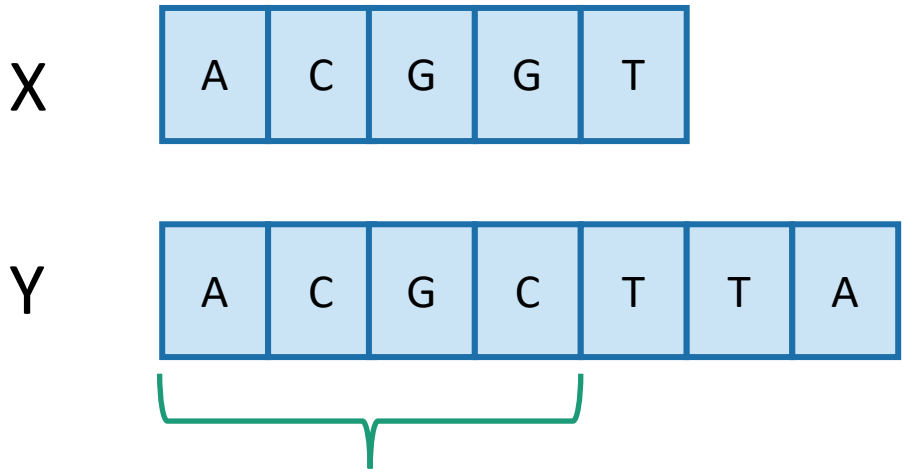
```
[DN0a22a660:~ mary$ cat file1
A
B
C
D
E
F
G
H
[DN0a22a660:~ mary$ cat file2
A
B
D
F
G
H
I
[DN0a22a660:~ mary$ diff file1 file2
3d2
< C
5d3
< E
8a7
> I
DN0a22a660:~ mary$
```

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure. 
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.
- **Step 3:** Use dynamic programming to find the length of the longest common subsequence.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual LCS.
- **Step 5:** If needed, code this up like a reasonable person.

Step 1: Optimal substructure

Prefixes:



Notation: denote this prefix **ACGC** by Y_4

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)

Optimal substructure ctd.

- Subproblem:
  - finding LCS's of prefixes of X and Y.

- Why is this a good choice?
  - There's some relationship between LCS's of prefixes and LCS's of the whole things.
  - These subproblems overlap a lot.

To see this formally, on to…

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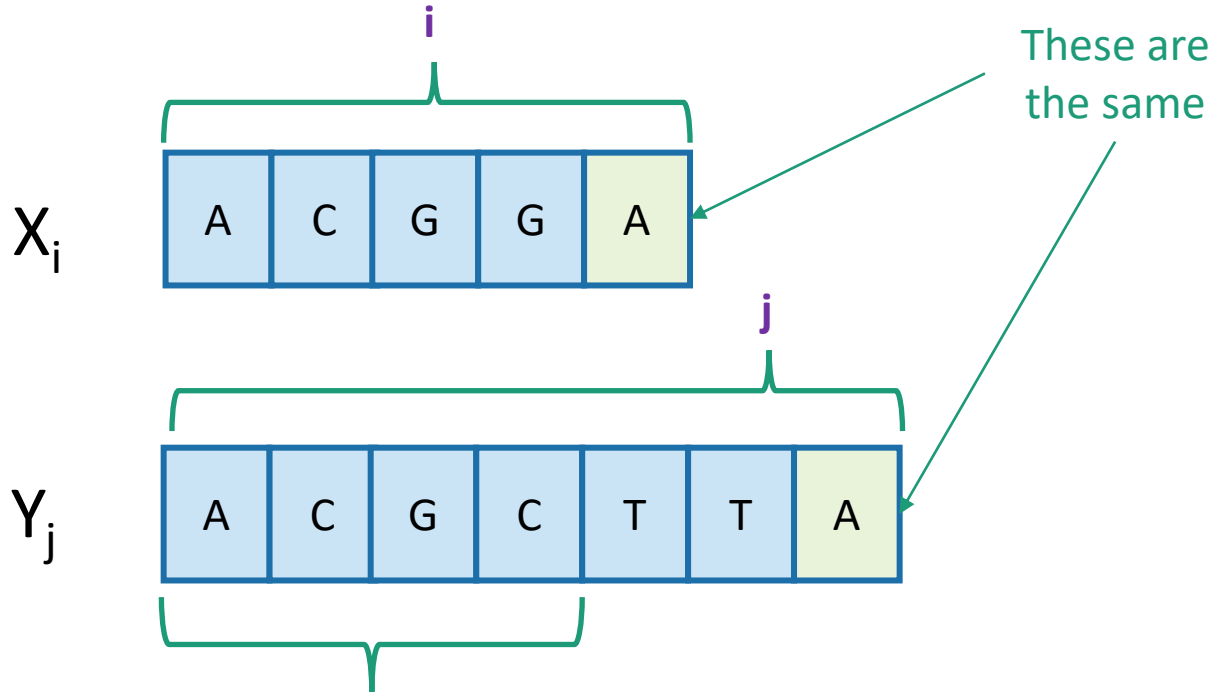
Two cases

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)

Case 1: X[i] = Y[j]



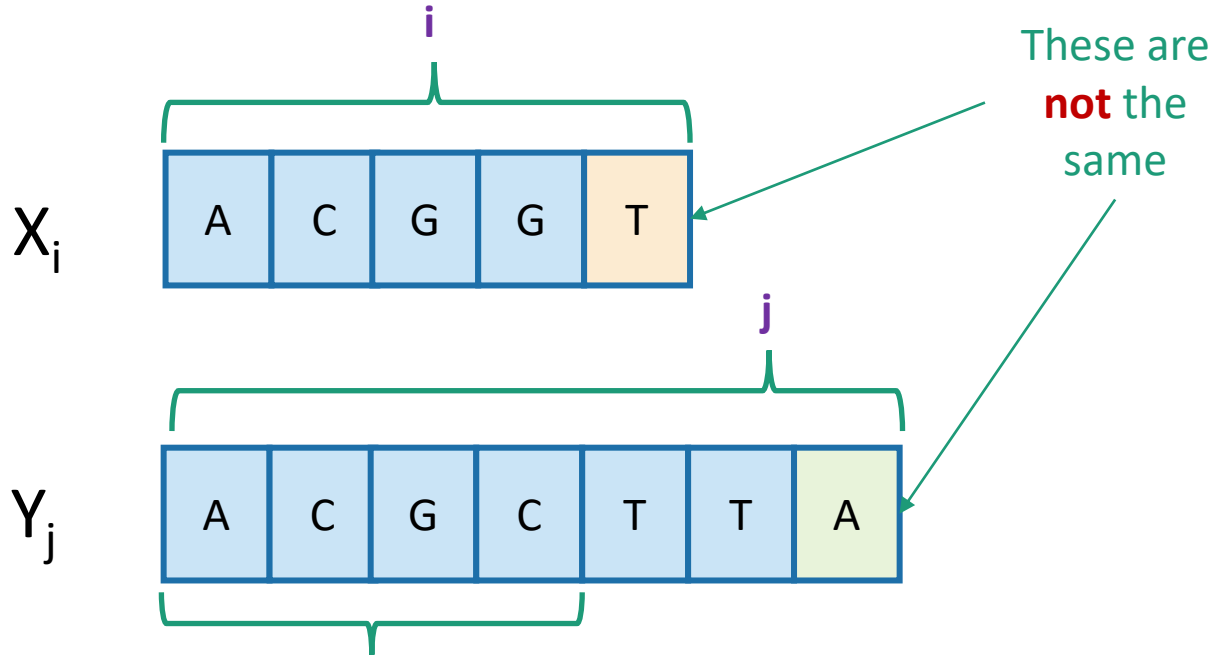
Notation: denote this prefix **ACGC** by Y_4

- Then C[i,j] = 1 + C[i-1,j-1].
  - because $LCS(X_i,Y_j) = LCS(X_{i-1},Y_{j-1})$ followed by A

Two cases

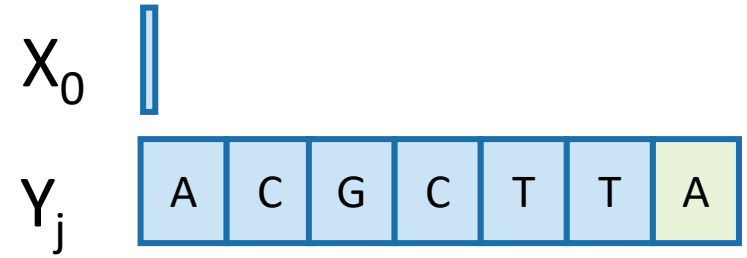
- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)

Case 2: X[i] != Y[j]



- Then C[i,j] = max{ C[i-1,j], C[i,j-1] }.
  - either $LCS(X_i,Y_j) = LCS(X_{i-1},Y_j)$ and T is not involved,
  - or $LCS(X_i,Y_j) = LCS(X_i,Y_{j-1})$ and A is not involved,

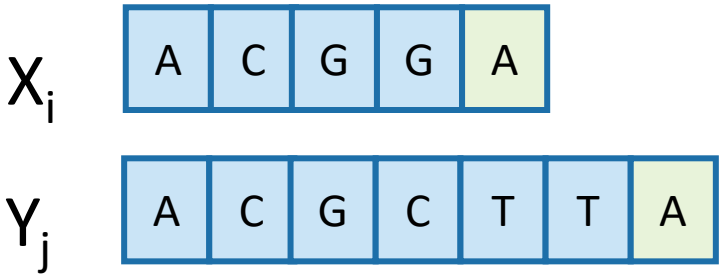
Recursive formulation
of the optimal solution



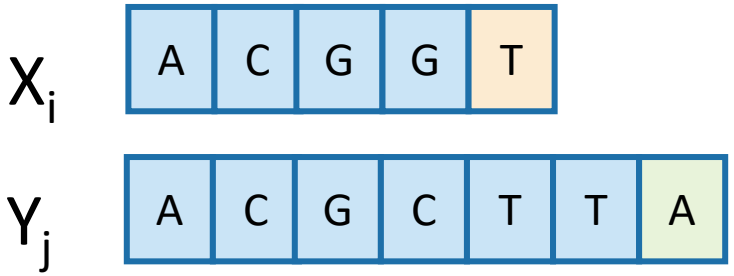
Case 0

- $C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$

Case 1



Case 2



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LCS DP OMG BBQ

- **LCS**(X, Y):
  - C[i,0] = C[0,j] = 0 for all i = 1,…,m, j=1,…n.
  - **For** i = 1,…,m and j = 1,…,n:
    - **If** X[i] = Y[j]:
      - C[i,j] = C[i-1,j-1] + 1
    - **Else:**
      - C[i,j] = max{ C[i,j-1], C[i-1,j] }

Running time: O(nm)

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

A	C	T	G

X

A	0	0	0	0	0
C	0				
G	0				
G	0				
A	0				
	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

X \ Y		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

So the LCM of X and Y has length 3.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

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Example

X

A	C	G	G	A

Y

A	C	T	G

Y

A	C	T	G

X

A	0	0	0	0	0
C	0				
G	0				
G	0				
A	0				
	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

A	C	T	G

X

A	0	0	0	0	0
C	0	1	1	1	1
G	0	1	2	2	2
G	0	1	2	2	3
A	0	1	2	2	3
	0	1	2	2	3

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

A	C	T	G

X

A	0	0	0	0	0
C	0	1	1	1	1
G	0	1	2	2	2
G	0	1	2	2	3
A	0	1	2	2	3
	0	1	2	2	3

- Once we've filled this in, we can work backwards.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

A	C	T	G

X

A	0	0	0	0	0
C	0	1	1	1	1
G	0	1	2	2	2
G	0	1	2	2	3
A	0	1	2	2	3
	0	1	2	2	3

- Once we've filled this in, we can work backwards.

That 3 must have come from the 3 above it.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

	A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
X G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

This 3 came from that 2 – we found a match!

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

A	C	T	G

X

A
C
G
G
A

0	0	0	0	0
0	1	1	1	1
0	1	2	2	2
0	1	2	2	3
0	1	2	2	3
0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

That 2 may as well have come from this other 2.

G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

A	C	T	G

X

A	0	0	0	0	0
C	0	1	1	1	1
G	0	1	2	2	2
G	0	1	2	2	3
A	0	1	2	2	3
	0	1	2	2	3

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G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

A	C	T	G

X

A				
C				
G				
G				
A				

0	0	0	0	0
0	1	1	1	1
0	1	2	2	2
0	1	2	2	3
0	1	2	2	3
0	1	2	2	3

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C	G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X

A	C	G	G	A

Y

A	C	T	G

Y

A	C	T	G

X

A	0	0	0	0	0
C	0	1	1	1	1
G	0	1	2	2	2
G	0	1	2	2	3
A	0	1	2	2	3
	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

A	C	G

This is the LCS!

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

This gives an algorithm to recover the actual LCS not just its length

- It runs in time O(n + m)
  - We walk up and left in an n-by-m array
  - We can only do that for n + m steps.
- So actually recovering the LCS from the table is much faster than building the table was.
- We can find LCS(X,Y) in time O(mn).

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This pseudocode actually isn't so bad

- If we are only interested in the length of the LCS:
  - Since we go across the table one-row-at-a-time, we can only keep two rows if we want.
  - If we want to recover the LCS, we need to keep the whole table.

- Can we do better than O(mn) time?
  - A bit better.
    - By a log factor or so.
  - But doing much better (polynomially better) is an open problem!
    - If you can do it let me know :D

What have we learned?

- We can find LCS(X,Y) in time O(nm)
  - if |Y|=n, |X|=m
- We went through the steps of coming up with a dynamic programming algorithm.
  - We kept a 2-dimensional table, breaking down the problem by decrementing the length of X and Y.

Example 2: Knapsack Problem

- We have n items with weights and values:

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- And we have a knapsack:
  - it can only carry so much weight:

Capacity: 10

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

🌮 🌮 💡 💡

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

💡 🍉 🌮

Total weight: 9
Total value: 35

Some notation

Item:					
Weight:	w_1	w_2	w_3	...	w_n
Value:	v_1	v_2	v_3		v_n

Capacity: W

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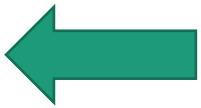
- **Step 1:** Identify optimal substructure. 
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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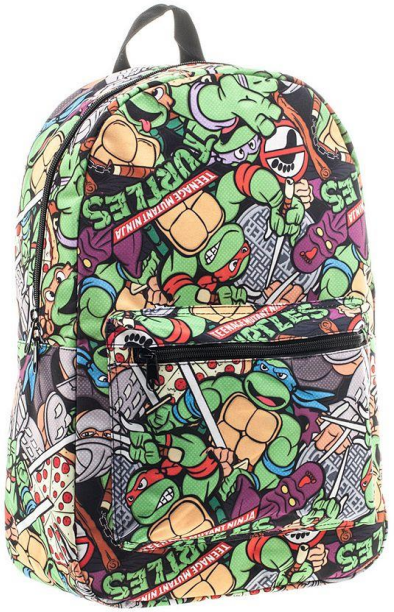
Optimal substructure

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks

Then larger knapsacks



Then larger knapsacks

Optimal substructure

item i

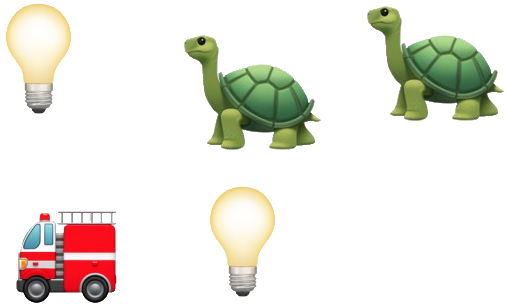
- Suppose this is an optimal solution for capacity x:

Say that the optimal solution contains at least one copy of item i.

Weight w_i
Value v_i

Capacity x
Value V

- Then this optimal for capacity x - w_i:



If I could do better than the second solution, then adding a turtle to that improvement would improve the first solution.

Capacity x – w_i
Value V - v_i

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Recursive relationship

- Let K[x] be the optimal value for capacity x.

$$K[x] = \max_i \{ \quad + \quad \}$$

The maximum is over all i so that $w_i \leq x$.

Optimal way to fill the smaller knapsack

The value of item i.

$$K[x] = \max_i \{ K[x - w_i] + v_i \}$$

- (And K[x] = 0 if the maximum is empty).
  - That is, there are no i so that $w_i \leq x$

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Let's write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
  - **return** K[W]

Running time: O(nW)

K[x] = $\max_i$ {  +  }

= $\max_i$ { K[x – w_i] + v_i }

Why does this work?

Because our recursive relationship makes sense.

Can we do better?

- We only need log(W) bits to write down the input W and to write down all the weights.
- Maybe we could have an algorithm that runs in time O(nlog(W)) instead of O(nW)?
- Or even O($n^{1000000}$ $\log^{1000000}(W)$)?

- Open problem!
  - (But probably the answer is **no**…otherwise P = NP)

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Let's write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
  - **return** K[W]

$K[x] = \max_i \{$  +  $\}$

$= \max_i \{ K[x - w_i] + v_i \}$

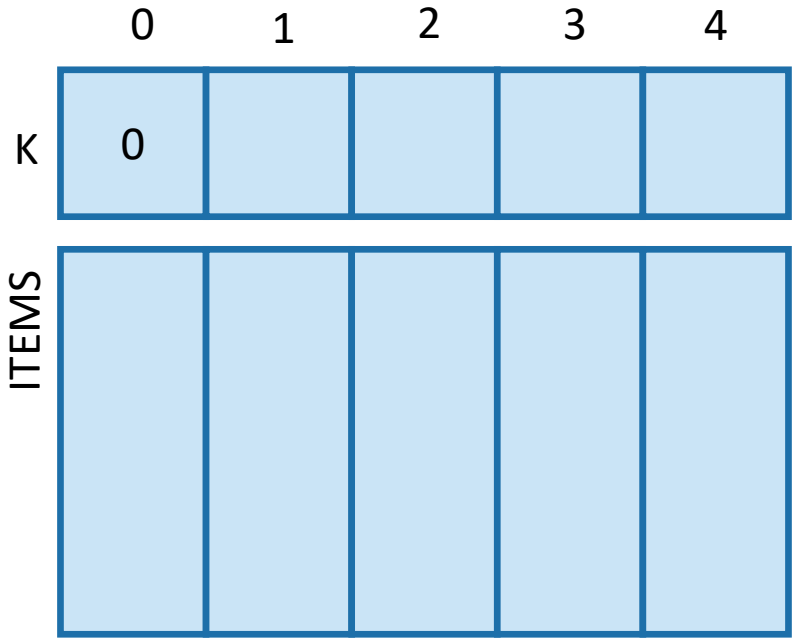
Let's write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

K[x] = $\max_i$ {  +  }

= $\max_i$ { K[x – w_i] + v_i }

Example



- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item
  - **return** ITEMS[W]

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1			
ITEMS					

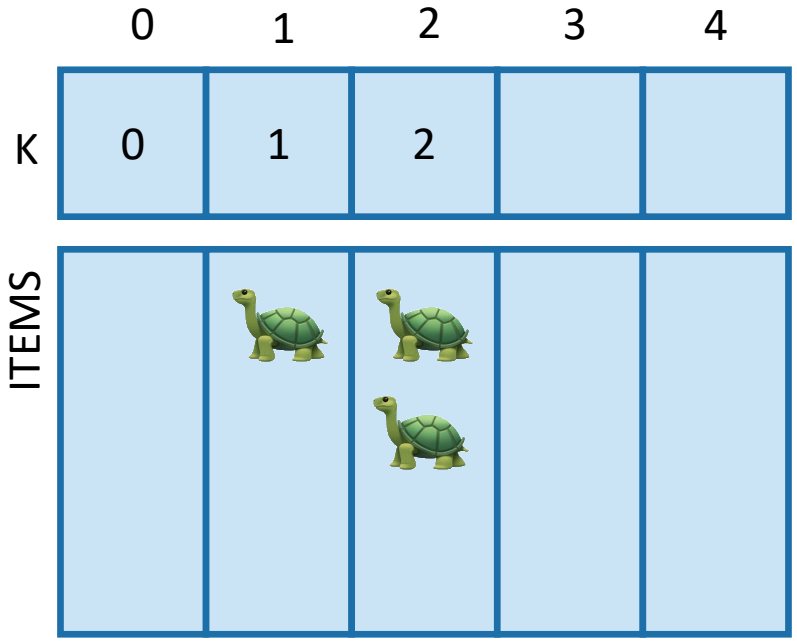
ITEMS[1] = ITEMS[0] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example



ITEMS[2] = ITEMS[1] + 🐢

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item
  - **return** ITEMS[W]

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4		
ITEMS		🐢	💡		

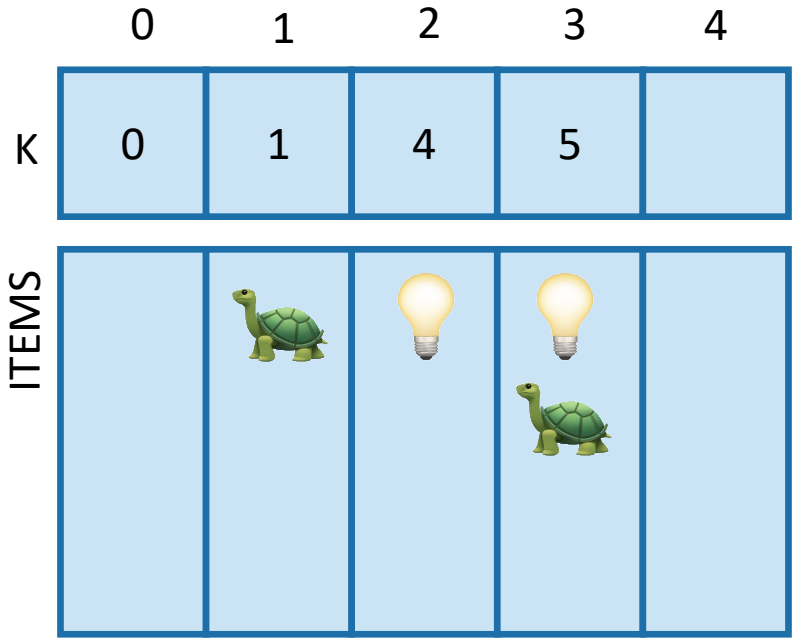
ITEMS[2] = ITEMS[0] + 💡

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item
  - **return** ITEMS[W]

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example



ITEMS[3] = ITEMS[2] + 🐢

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item
  - **return** ITEMS[W]

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	
ITEMS		🐢	💡	🍉	

ITEMS[3] = ITEMS[0] + 🍉

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item
  - **return** ITEMS[W]

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	7
ITEMS					 

ITEMS[4] = ITEMS[3] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	8
ITEMS		🐢	💡	🍉	💡 💡

ITEMS[4] = ITEMS[2] + 💡

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item
  - **return** ITEMS[W]

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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- **Step 5:** If needed, code this up like a reasonable person.

(Pass)

What have we learned?

- We can solve unbounded knapsack in time O(nW).
  - If there are n items and our knapsack has capacity W.
- We again went through the steps to create DP solution:
  - We kept a one-dimensional table, creating smaller problems by making the knapsack smaller.

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

🌮 🌮 💡 💡

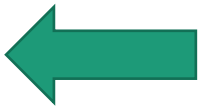
Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

💡 🍉 🌮

Total weight: 9
Total value: 35

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure. 
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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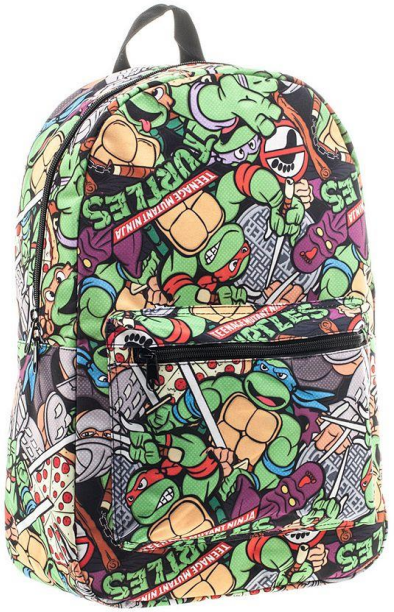
- **Step 5:** If needed, code this up like a reasonable person.

Optimal substructure: try 1

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks

Then larger knapsacks



Then larger knapsacks

This won’t quite work…

- We are only allowed **one copy of each item**.
- The sub-problem needs to “know” what items we’ve used and what we haven’t.

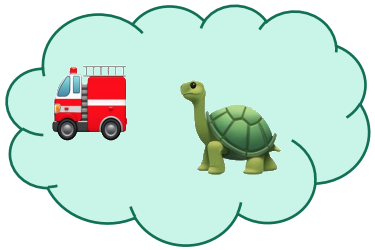
Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

First solve the problem with few items

We'll still increase the size of the knapsacks.

Then more items

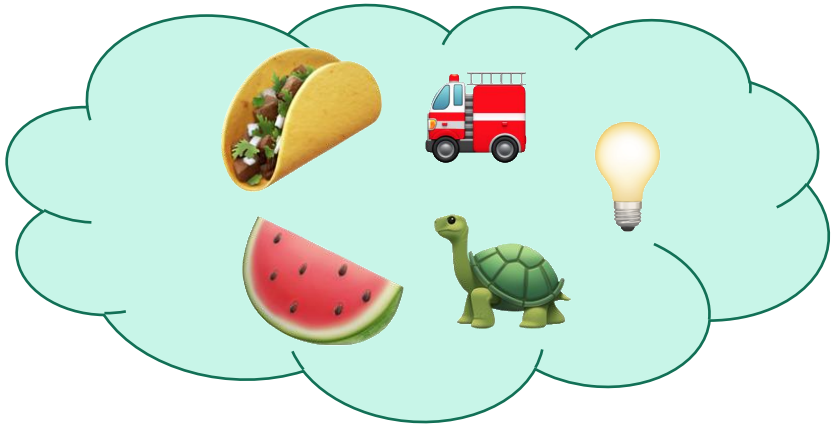


Then yet more items

(We'll keep a two-dimensional table).

Our sub-problems:

- Indexed by x and j



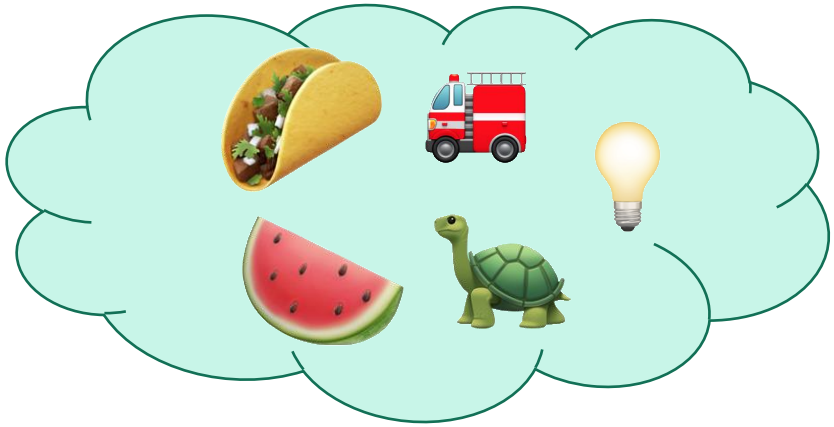
First j items

Capacity x

Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.
- **Case 2**: Optimal solution for j items does use item j.



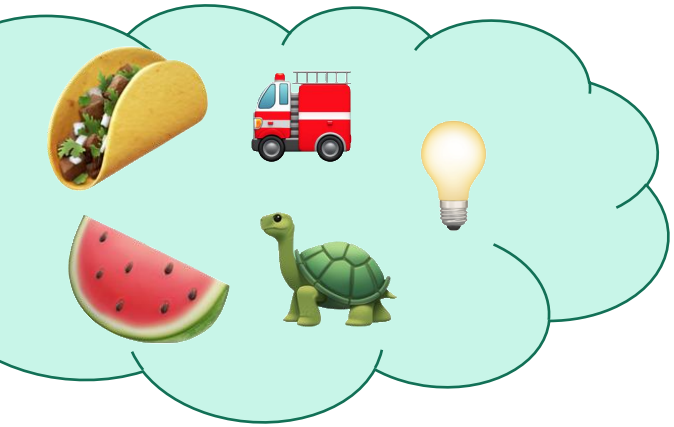
First j items

Capacity x

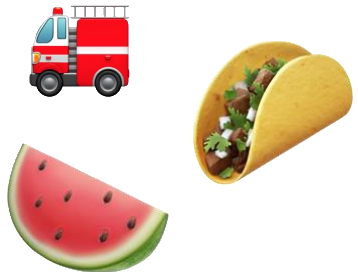
Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.

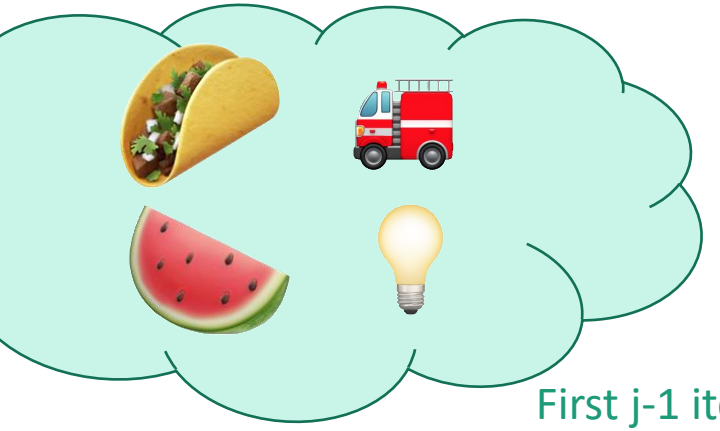


First j items

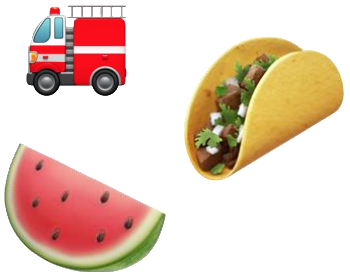


Capacity x
Value V
Use only the first j items

- Then this is an optimal solution for j-1 items:



First j-1 items

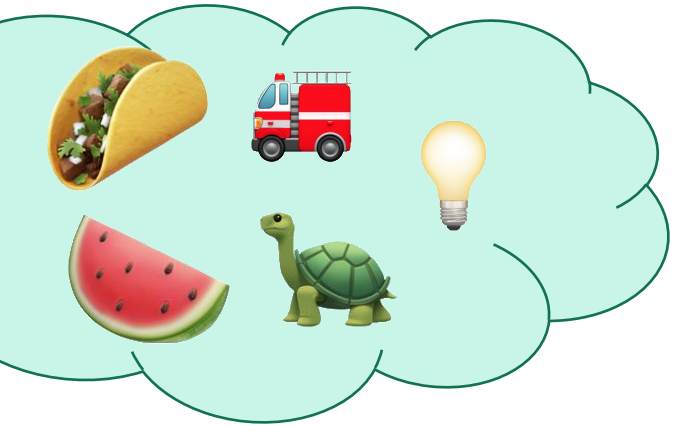


Capacity x
Value V
Use only the first j-1 items.

Two cases

item j

- **Case 2**: Optimal solution for j items uses item j.



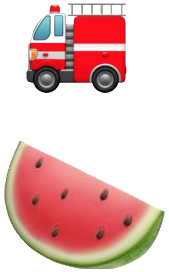
First j items

Weight w_j
Value v_j

Capacity x
Value V
Use only the first j items

- Then this is an optimal solution for j-1 items and a smaller knapsack:

First j-1 items



Capacity $x - w_i$
Value $V - v_i$
Use only the first j-1 items.

Recipe for applying Dynamic Programming

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- **Step 5:** If needed, code this up like a reasonable person.

Recursive relationship

- Let K[x,j] be the optimal value for:
  - capacity x,
  - with j items.

$$K[x,j] = \max\{ K[x, j-1] , K[x - w_j, j-1] + v_j \}$$

Case 1 Case 2

- (And K[x,0] = 0 and K[0,j] = 0).

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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- **Step 3:** Use dynamic programming to find the value of the optimal solution.
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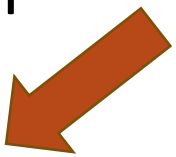


Bottom-up DP algorithm

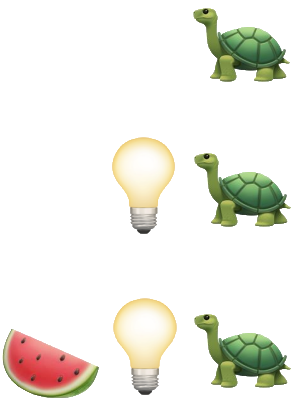
- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1] Case 1
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ } Case 2
  - **return** K[W,n]

Running time O(nW)

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0			
j=2	0			
j=3	0			



current entry

relevant previous entry

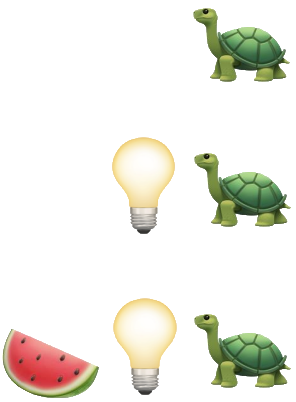
Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	0		
j=2	0			
j=3	0			



current entry

relevant previous entry

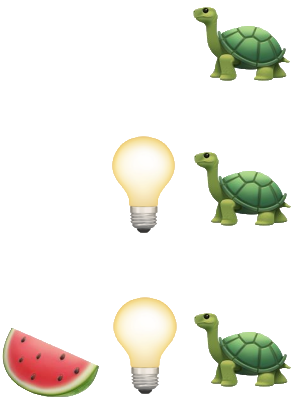
Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0			
j=3	0			



current entry

relevant previous entry

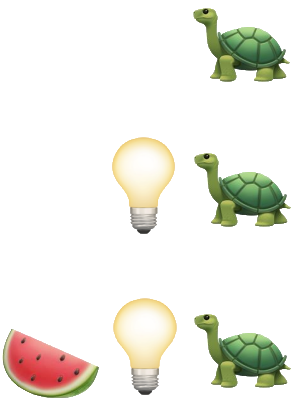
Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0	1		
j=3	0			



current entry

relevant previous entry

Item:			
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Example

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  - K[x,0] = 0 for all x = 0,…,W
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  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	0	
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:			
Weight:	1	2	3
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Capacity: 3

Example

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      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
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      - **if** $w_j \leq x$:
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  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	1	
j=3	0	1		

current entry

relevant previous entry

Item:			
Weight:	1	2	3
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Example

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  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	4	
j=3	0	1		






current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
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  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	4	
j=3	0	1	4	

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
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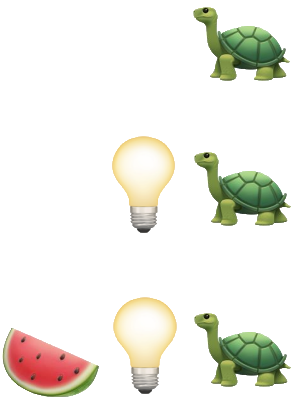
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	0
j=2	0	1	4	
j=3	0	1	4	



current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

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  - K[x,0] = 0 for all x = 0,…,W
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        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	
j=3	0	1	4	

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
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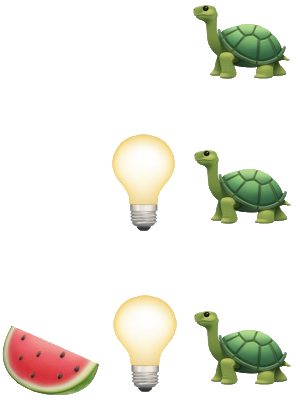
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	1
j=3	0	1	4	



current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
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  - **for** x = 1,…,W:
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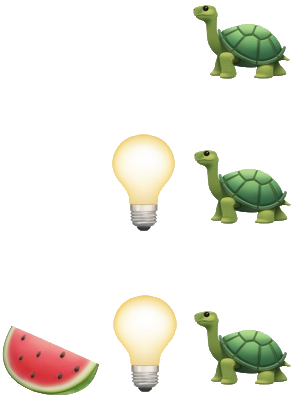
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	




current entry


relevant previous entry

Item:			
			
Weight:	1	2	3
Value:	1	4	6


Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
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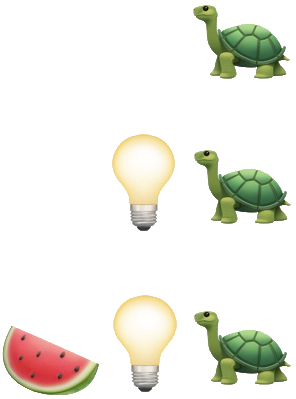
	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	5



current entry

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Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

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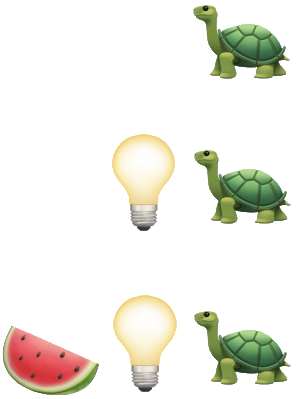
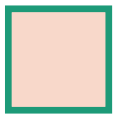
	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	6




current entry


relevant previous entry

Item:			
			
Weight:	1	2	3
Value:	1	4	6


Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
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  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	6

So the optimal solution is to put one watermelon in your knapsack!

current entry

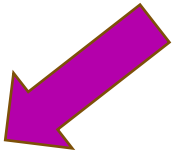
relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.



What have we learned?

- We can solve 0/1 knapsack in time O(nW).
  - If there are n items and our knapsack has capacity W.

- We again went through the steps to create DP solution:
  - We kept a two-dimensional table, creating smaller problems by restricting the set of allowable items.

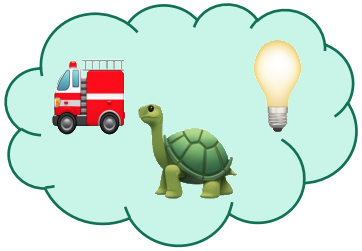
Question

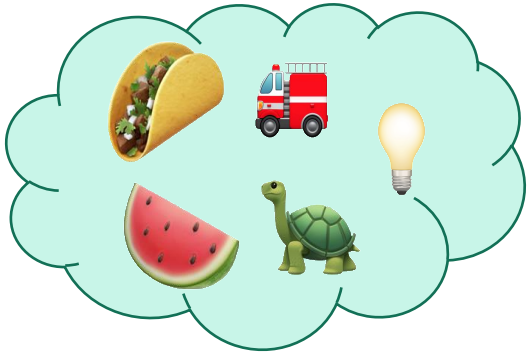
- How did we know which substructure to use in which variant of knapsack?

Answer in retrospect:

This one made sense for unbounded knapsack because it doesn't have any memory of what items have been used.

vs.



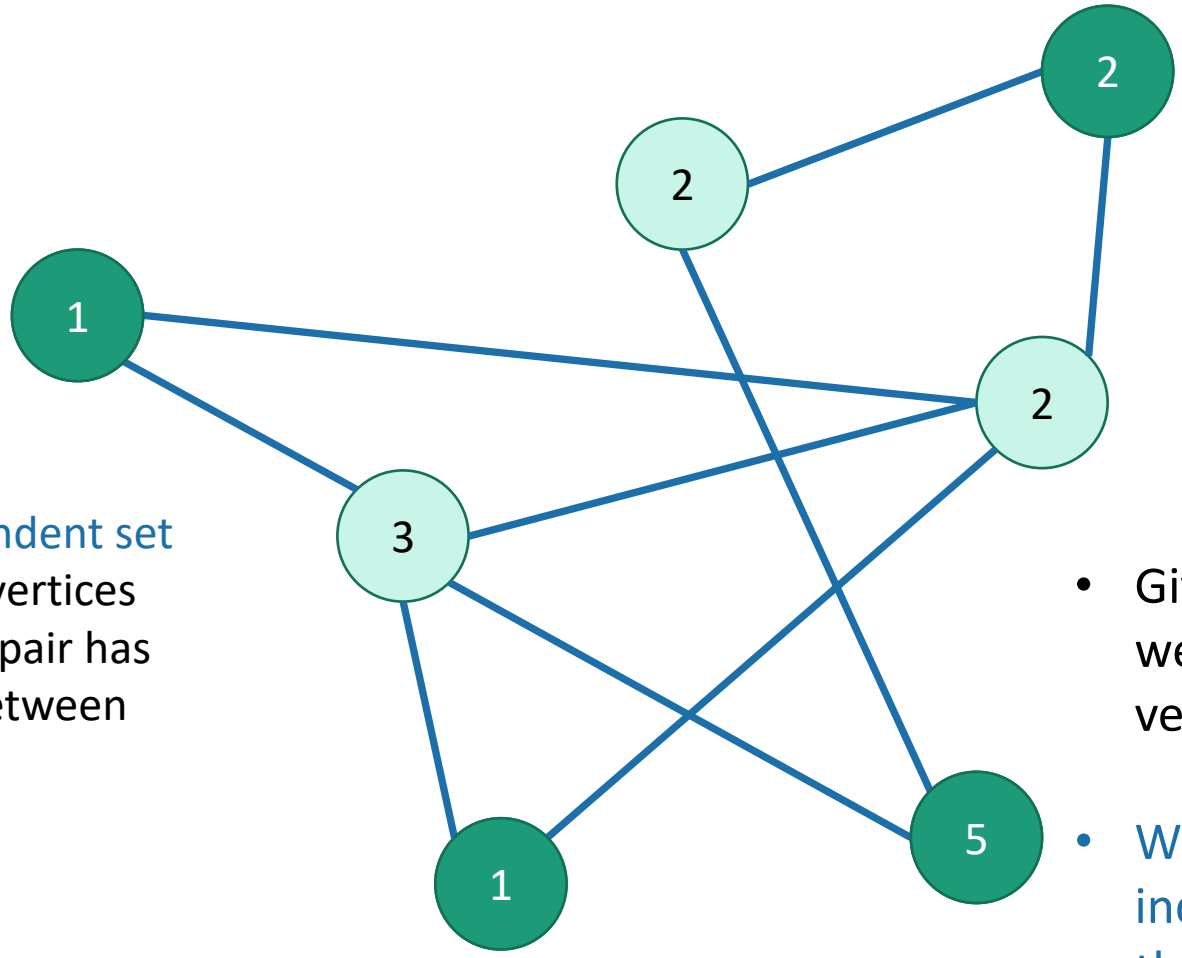


In 0/1 knapsack, we can only use each item once, so it makes sense to leave out one item at a time.

Operational Answer: try some stuff, see what works!

Example 3: Independent Set

if we still have time



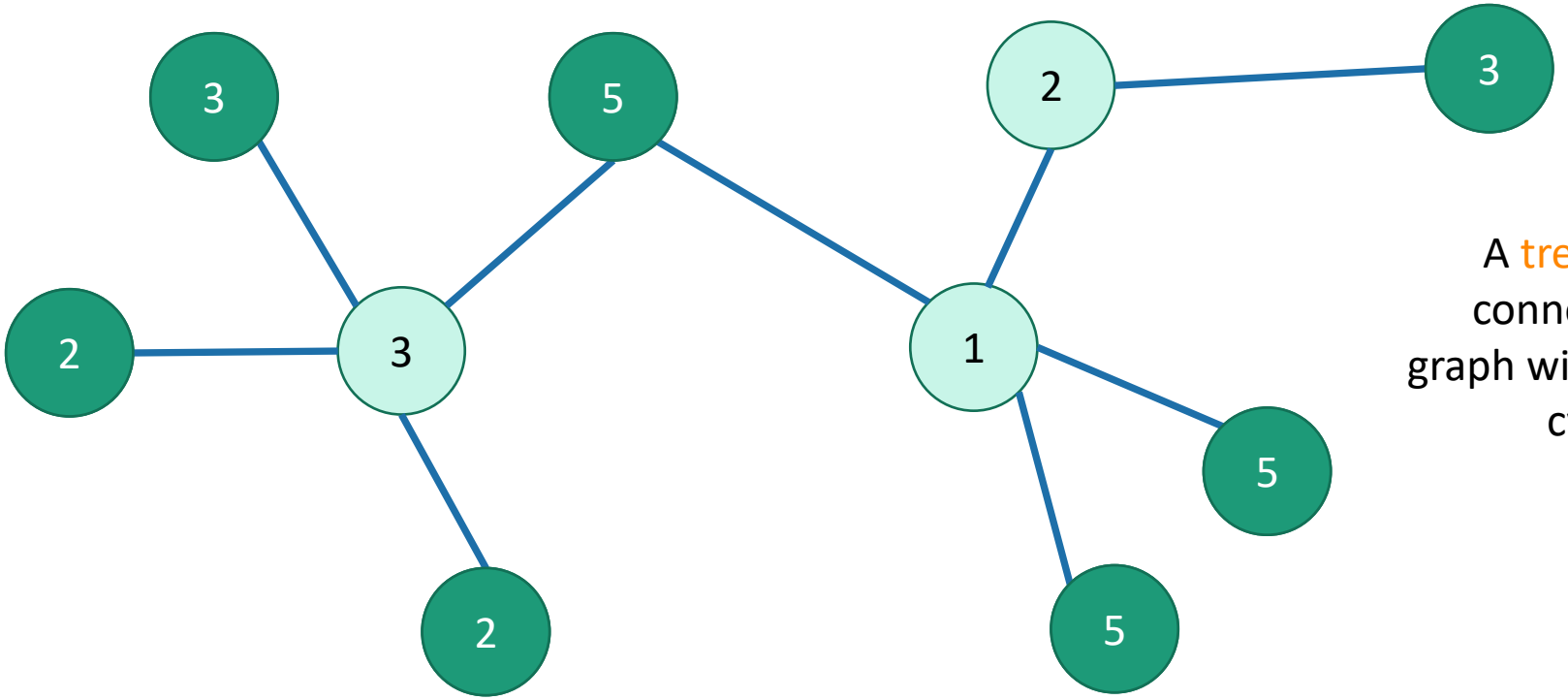
An independent set is a set of vertices so that no pair has an edge between them.

- Given a graph with weights on the vertices…
- What is the independent set with the largest weight?

Actually this problem is NP-complete.

So we are unlikely to find an efficient algorithm

- But if we also assume that the graph is a tree…

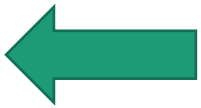


A tree is a connected graph with no cycles.

Problem:

find a maximal independent set in a tree (with vertex weights).

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure. 
- **Step 2:** Find a recursive formulation for the value of the optimal solution
- **Step 3:** Use dynamic programming to find the value of the optimal solution
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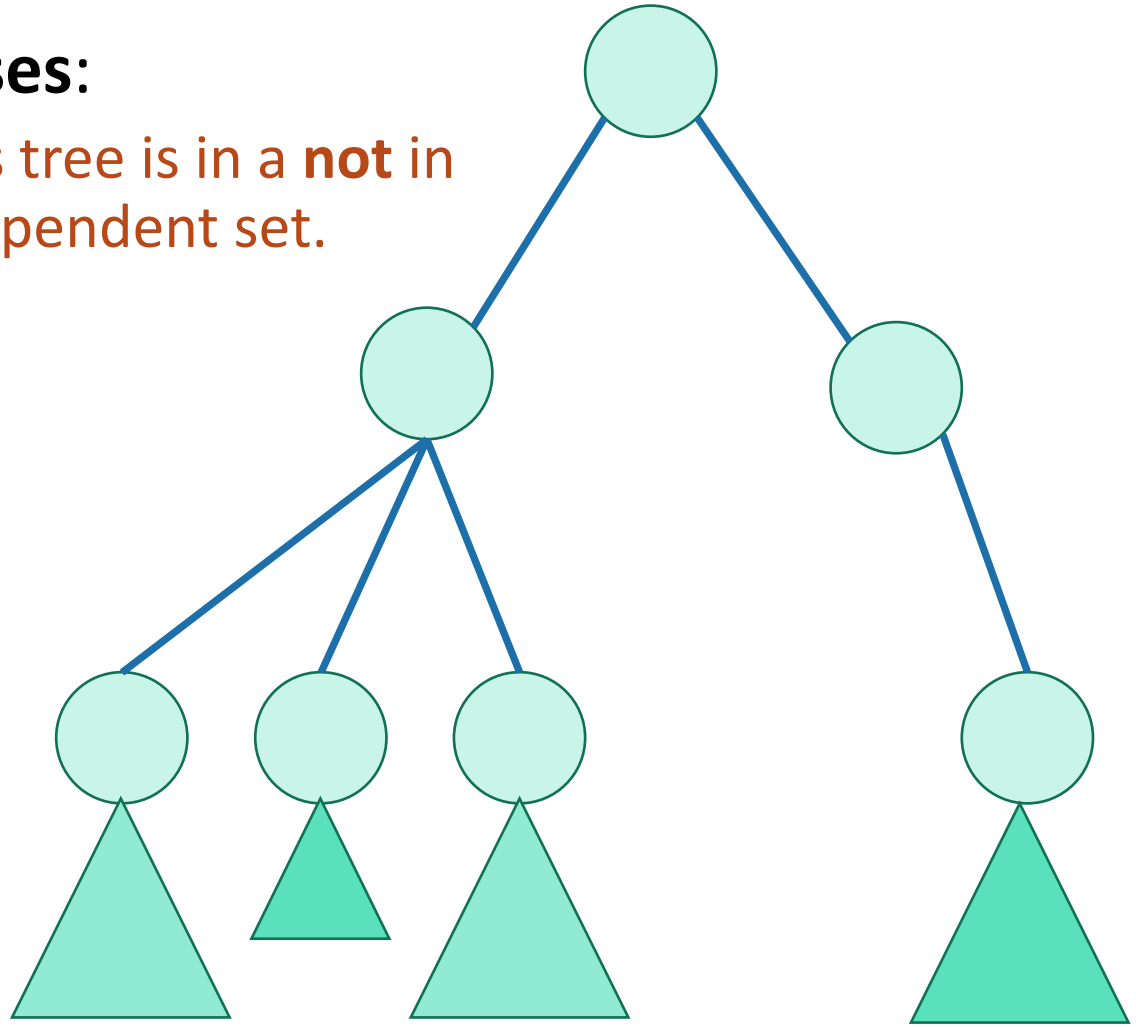
- **Step 5:** If needed, code this up like a reasonable person.

Optimal substructure

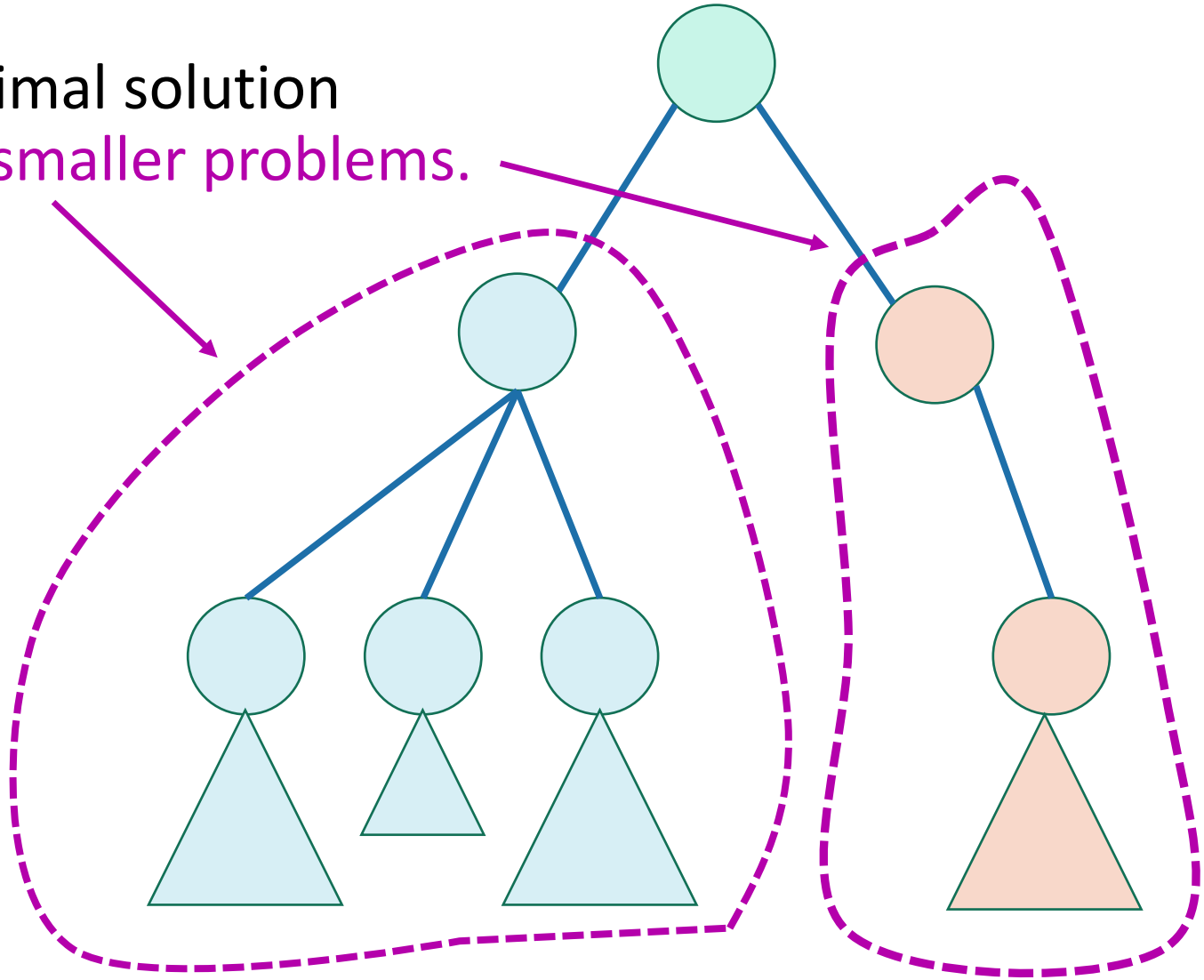
- **Subtrees** are a natural candidate.
- There are **two cases**:
  1. The root of this tree is in a **not** in a maximal independent set.
  2. Or it is.



Case 1:

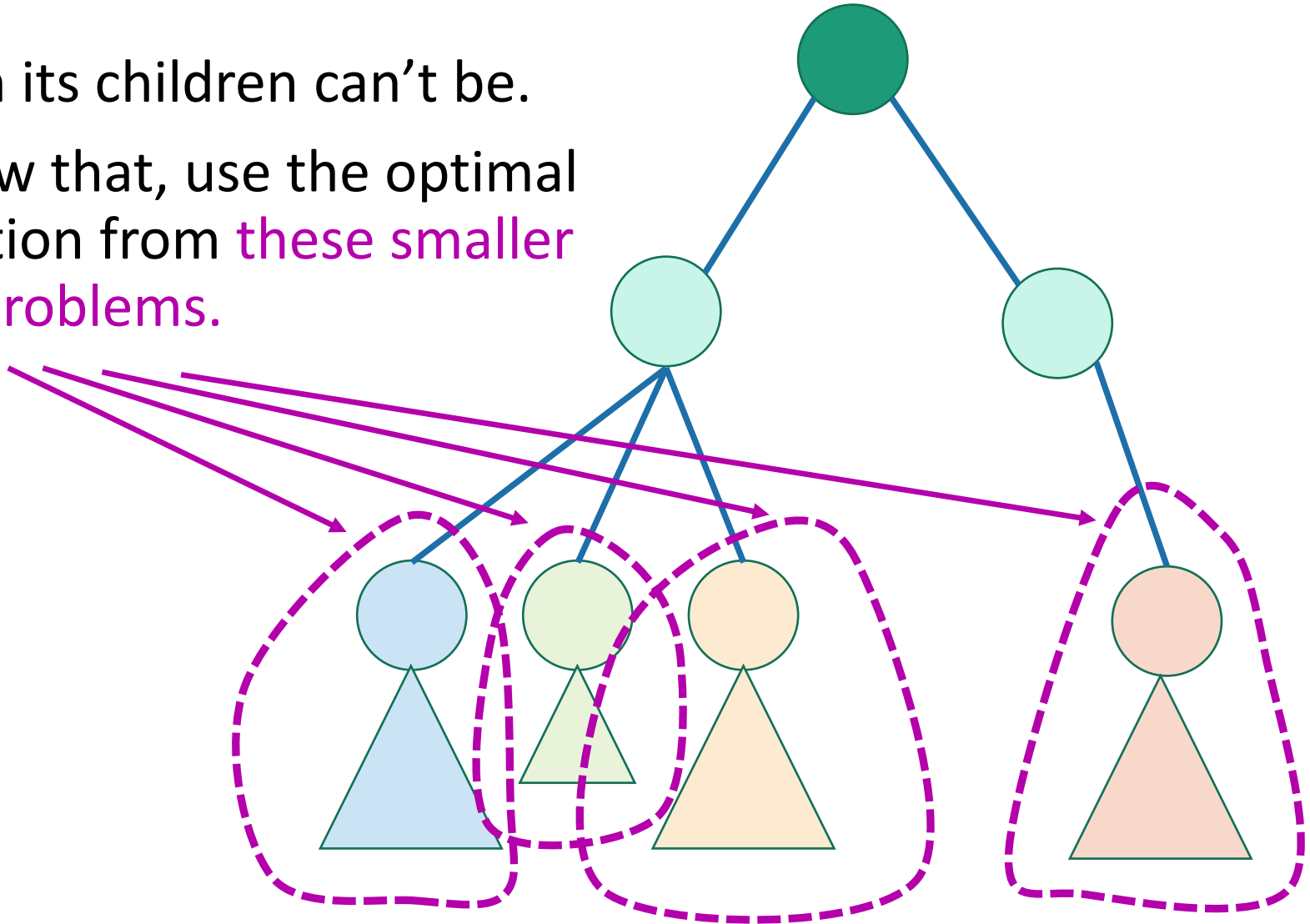
the root is **not** in an maximal independent set

- Use the optimal solution from these smaller problems.



Case 2:
the root is in an maximal independent set

- Then its children can't be.
- Below that, use the optimal solution from these smaller subproblems.



Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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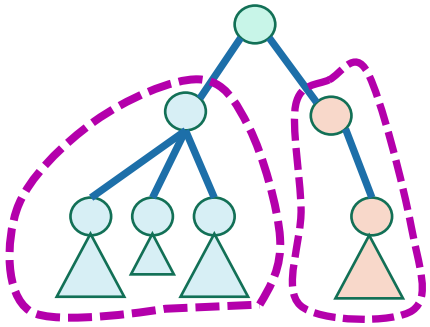
- **Step 5:** If needed, code this up like a reasonable person.

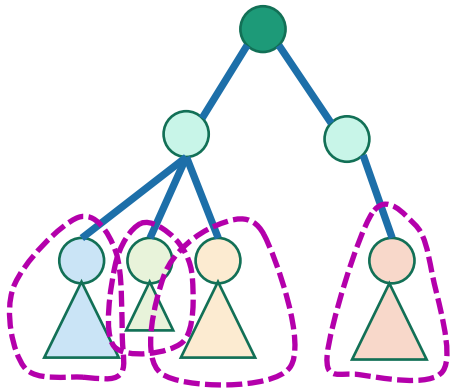
Recursive formulation: try 1

- Let A[u] be the weight of a maximal independent set in the tree rooted at u.



- $A[u] = \max \begin{cases} \sum_{v \in u.\text{children}} A[v] \\ \text{weight}(u) + \sum_{v \in u.\text{grandchildren}} A[v] \end{cases}$

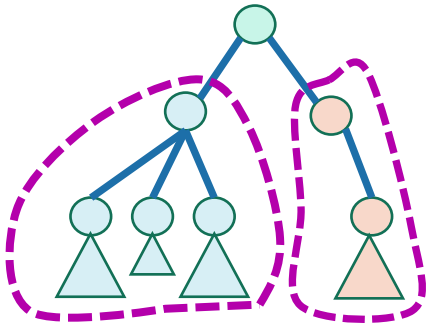
When we implement this, how do we keep track of this term?



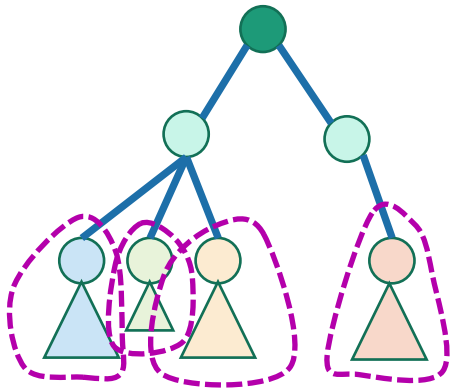
Recursive formulation: try 2

Keep two arrays!

- Let A[u] be the weight of a maximal independent set in the tree rooted at u.
- Let B[u] = $\sum_{v \in u.\text{children}} A[v]$



- $A[u] = \max \begin{cases} \sum_{v \in u.\text{children}} A[v] \\ \text{weight}(u) + \sum_{v \in u.\text{children}} B[v] \end{cases}$



Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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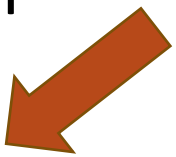


A top-down DP algorithm

- MIS_subtree(u):
  - **if** u is a leaf:
    - A[u] = weight(u)
    - B[u] = 0
  - **else**:
    - **for** v in u.children:
      - MIS_subtree(v)
    - $A[u] = \max\{ \sum_{v \in u.\text{children}} A[v] , \text{weight}(u) + \sum_{v \in u.\text{children}} B[v] \}$
    - $B[u] = \sum_{v \in u.\text{children}} A[v]$
- MIS(T):
  - MIS_subtree(T.root)
  - **return** A[T.root]

Initialize global arrays A, B that we will use in all of the recursive calls.

Running time?

- We visit each vertex once, and at every vertex we do O(1) work:
  - Make a recursive call
  - look stuff up in tables
- Running time is O(|V|)

Why is this different from divide-and-conquer?

That's always worked for us with tree problems before…

- MIS_subtree(u):
  - **if** u is a leaf:
    - **return** weight(u)
  - **else**:
    - **for** v in u.children:
      - MIS_subtree(v)
    - **return** $\max\{ \sum_{v \in u.\text{children}} \text{MIS\_subtree}(v) ,$

$$\text{weight}(u) + \sum_{v \in u.\text{grandchildren}} \text{MIS\_subtree}(v) \}$$

This is exactly the same pseudocode, except we've ditched the table and are just calling MIS_subtree(v) instead of looking up A[v] or B[v].

- MIS(T):
  - **return** MIS_subtree(T.root)

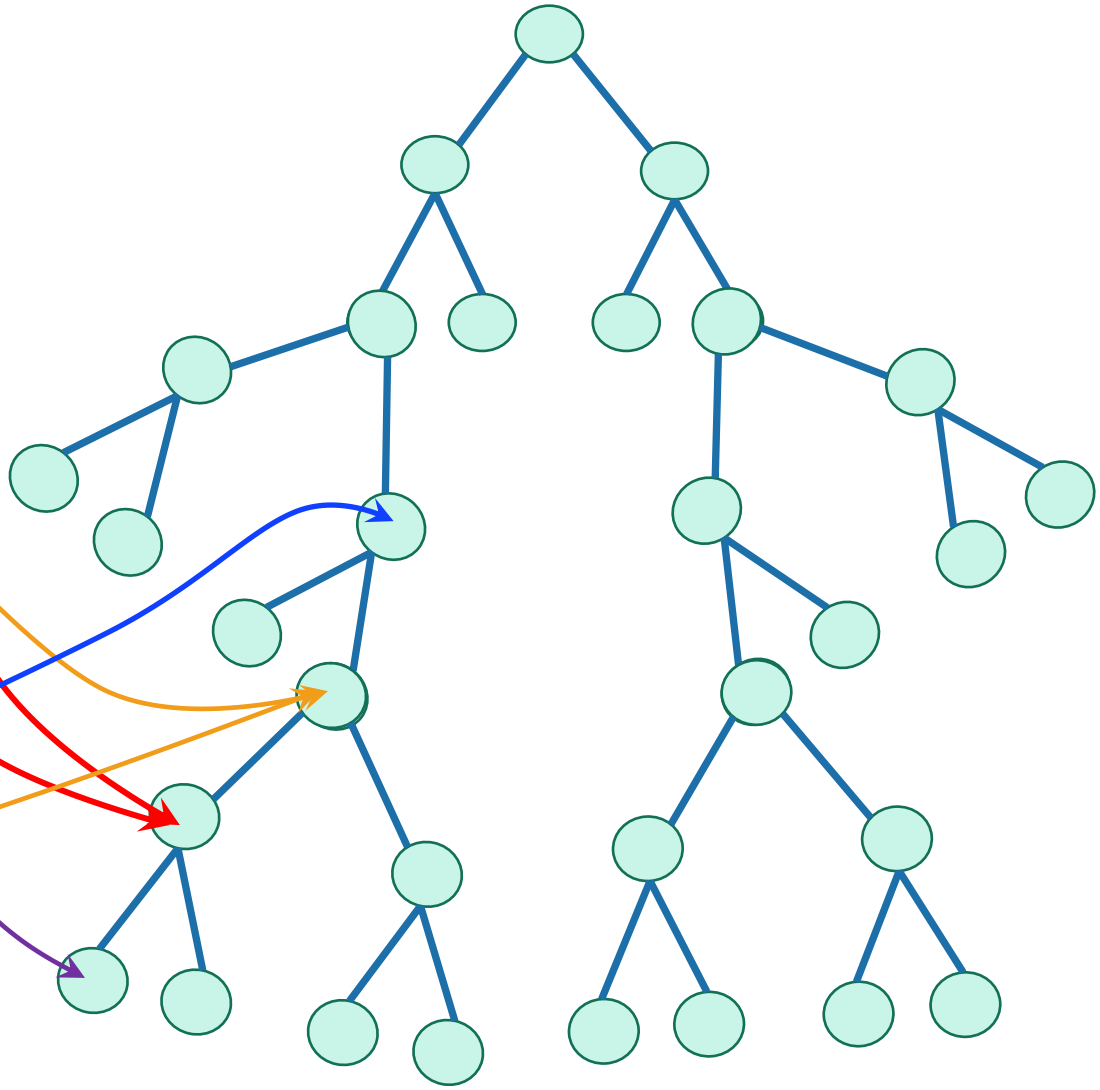
Why is this different from divide-and-conquer?

That's always worked for us with tree problems before…

How often would we ask about the subtree rooted **here**?

Once for **this node** and once for **this one**.

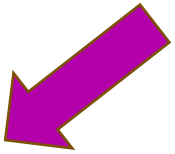
But we then ask about **this node** twice, **here** and **here**.



This will blow up exponentially without using dynamic programming to take advantage of **overlapping subproblems**.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.



What have we learned?

- We can find maximal independent sets in trees in time O(|V|) using dynamic programming!

- For this example, it was natural to implement our DP algorithm in a top-down way.

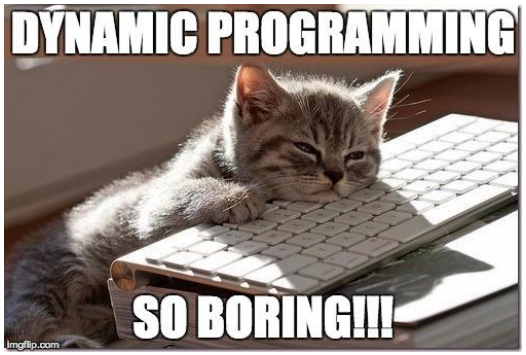
Recap

- Today we saw examples of how to come up with dynamic programming algorithms.
  - Longest Common Subsequence
  - Knapsack two ways
  - (If time) maximal independent set in trees.
- There is a **recipe** for dynamic programming algorithms.

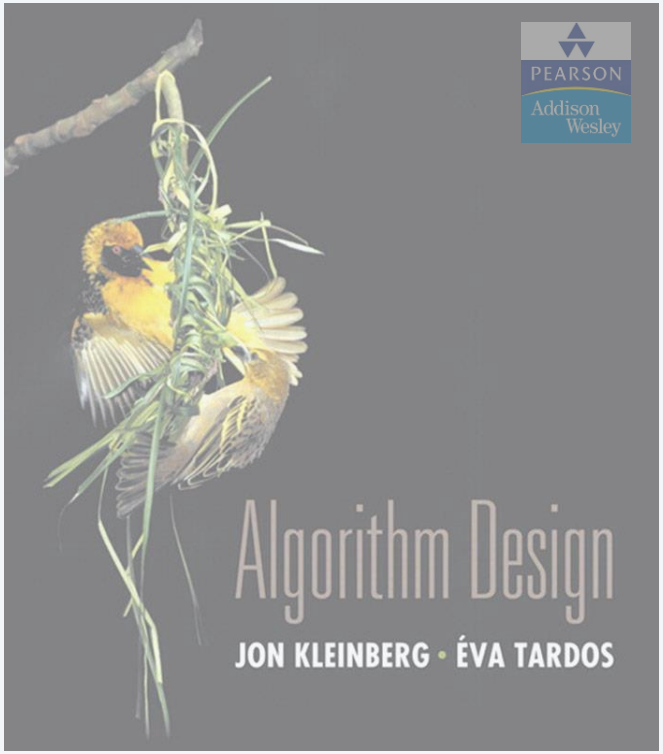
Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Recap



- Today we saw examples of how to come up with dynamic programming algorithms.
  - Longest Common Subsequence
  - Knapsack two ways
  - (If time) maximal independent set in trees.
- There is a **recipe** for dynamic programming algorithms.
- Sometimes coming up with the right substructure takes some creativity



SECTION 6.4

6. DYNAMIC PROGRAMMING

- *weighted interval scheduling*
- *segmented least squares*
- *knapsack problem*
- *RNA secondary structure*

Knapsack problem

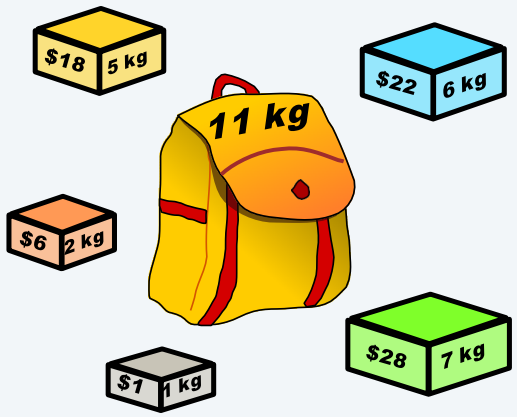
Goal. Pack knapsack so as to maximize total value of items taken.

- There are n items: item i provides value $v_i > 0$ and weighs $w_i > 0$.
- Value of a subset of items = sum of values of individual items.
- Knapsack has weight limit of W.

Ex. The subset $\{ 1, 2, 5 \}$ has value \$35 (and weight 10).

Ex. The subset $\{ 3, 4 \}$ has value \$40 (and weight 11).

Assumption. All values and weights are integral.



i	v_i	w_i
1	\$1	1 kg
2	\$6	2 kg
3	\$18	5 kg
4	\$22	6 kg
5	\$28	7 kg

weights and values can be arbitrary positive integers

knapsack instance (weight limit W = 11)

Dynamic programming: quiz 2

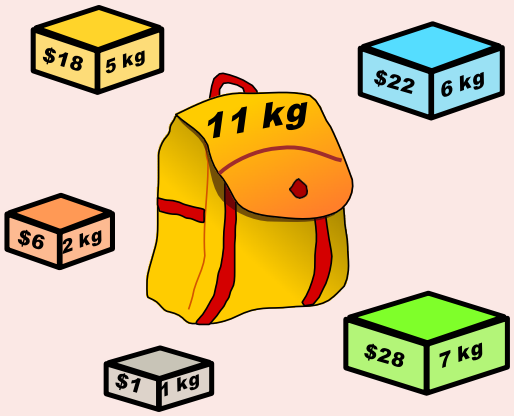
Which algorithm solves knapsack problem?

A. Greedy-by-value: repeatedly add item with maximum v_i.

B. Greedy-by-weight: repeatedly add item with minimum w_i.

C. Greedy-by-ratio: repeatedly add item with maximum ratio v_i / w_i.

D. None of the above.



i	v_i	w_i
1	$1	1 kg
2	$6	2 kg
3	$18	5 kg
4	$22	6 kg
5	$28	7 kg

knapsack instance
(weight limit W = 11)

Dynamic programming: quiz 3

Which subproblems?

A. $OPT(w)$ = optimal value of knapsack problem with weight limit w.

B. $OPT(i)$ = optimal value of knapsack problem with items $1, \ldots, i$.

C. $OPT(i, w)$ = optimal value of knapsack problem with items $1, \ldots, i$ subject to weight limit w. 

D. Any of the above.

Dynamic programming: two variables

Def. $OPT(i, w)$ = optimal value of knapsack problem with items $1, \ldots, i$, subject to weight limit w.

Goal. $OPT(n, W)$.

Case 1. $OPT(i, w)$ does not select item i. ← possibly because $w_i > w$

- $OPT(i, w)$ selects best of $\{ 1, 2, \ldots, i-1 \}$ subject to weight limit w.

Case 2. $OPT(i, w)$ selects item i. ← optimal substructure property (proof via exchange argument)

- Collect value v_i.
- New weight limit $= w - w_i$.
- $OPT(i, w)$ selects best of $\{ 1, 2, \ldots, i-1 \}$ subject to new weight limit.

Bellman equation.

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max \{ OPT(i-1, w),\ v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$

Knapsack problem: bottom-up dynamic programming

KNAPSACK(n, W, w_1, …, w_n, v_1, …, v_n)

FOR $w = 0$ TO W

  $M[0, w] \leftarrow 0$.

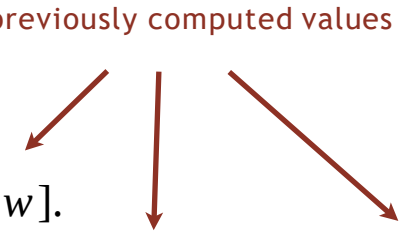
FOR $i = 1$ TO n

  FOR $w = 0$ TO W

    IF $(w_i > w)$ $M[i, w] \leftarrow M[i-1, w]$.

    ELSE $M[i, w] \leftarrow \max \{ M[i-1, w],\ v_i + M[i-1, w-w_i] \}$.

RETURN $M[n, W]$.



$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\ v_i + OPT(i-1,w-w_i)\,\} & \text{otherwise} \end{cases}$$

Knapsack problem: bottom-up dynamic programming demo

i	v_i	w_i
1	\$1	1 kg
2	\$6	2 kg
3	\$18	5 kg
4	\$22	6 kg
5	\$28	7 kg

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{OPT(i-1,w),\ v_i + OPT(i-1,w-w_i)\} & \text{otherwise} \end{cases}$$

weight limit w

subset of items 1, …, i	0	1	2	3	4	5	6	7	8	9	10	11
{ }	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	35	40

OPT(i, w) = optimal value of knapsack problem with items 1, …, i, subject to weight limit w

Knapsack Problem

Truck – 10t capacity

Optimum cargo combination:

- Item 1: $5 (3t)
- Item 2: $7 (4t)
- Item 3: $8 (5t)

Knapsack Problem

Output function f(i,w)

⇔

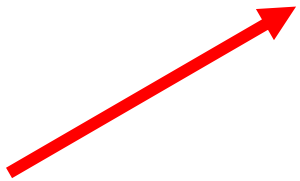
Optimum output of a combination of items 1 to i with a cumulated weight of w or less.

- Item 1: x1=$5 ; w1=3t
- Item 2: x2=$7 ; w2=4t
- Item 3: x3=$8 ; w3=5t

Knapsack Problem

Output function f(i,w)

f(i,w)=Max[xi + f(i,w-wi) ; f(i-1,w)]



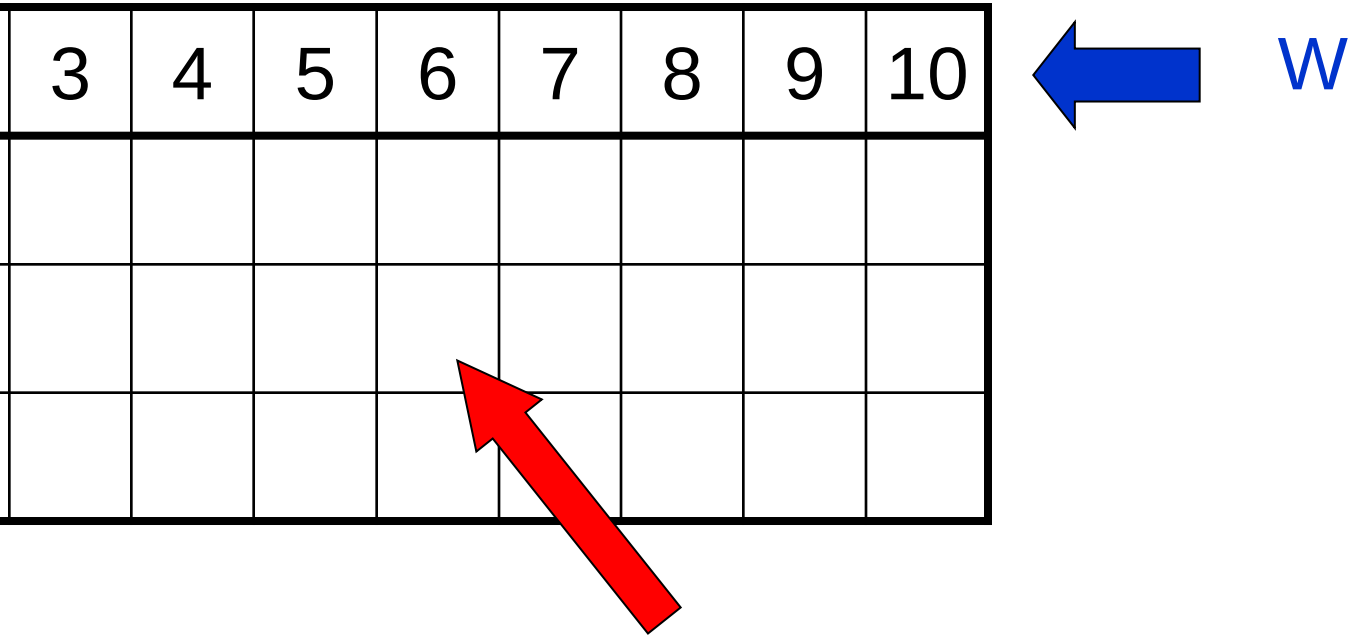
ONE Item i + optimum combination of weight w-wi

NO Item i + optimum combination items 1 to i-1

Knapsack Problem

Table

	1	2	3	4	5	6	7	8	9	10
1										
2										
3										



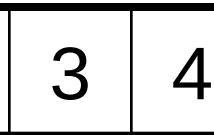
- Item 1: x1=$5 ; w1=3t
- Item 2: x2=$7 ; w2=4t
- Item 3: x3=$8 ; w3=5t

- Item 1: x1=$5 ; w1=3t
- Item 2: x2=$7 ; w2=4t
- Item 3: x3=$8 ; w3=5t

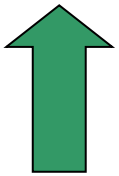
Knapsack Problem

Table

	1	2	3	4	5	6	7	8	9	10
1			**Using only item 1**							
2										
3										



W



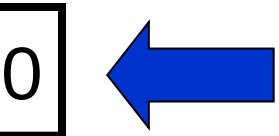
i

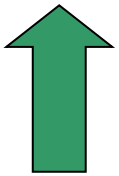
- Item 1: x1=$5 ; w1=3t
- Item 2: x2=$7 ; w2=4t
- Item 3: x3=$8 ; w3=5t

Knapsack Problem

Table

	1	2	3	4	5	6	7	8	9	10
1										
2					**Using only item 1 & 2**					
3										



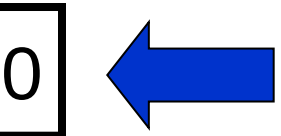


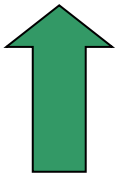
- Item 1: x1=$5 ; w1=3t
- Item 2: x2=$7 ; w2=4t
- Item 3: x3=$8 ; w3=5t

Knapsack Problem

Table

	1	2	3	4	5	6	7	8	9	10
1										
2										
3			**Using items 1, 2 & 3**							





- Item 1: x1=$5 ; w1=3t
- Item 2: x2=$7 ; w2=4t
- Item 3: x3=$8 ; w3=5t

Knapsack Problem

Table

	1	2	3	4	5	6	7	8	9	10
1	0	0	5	5	5	10				
2										
3										

W

0 items n° 1

1 items n° 1
w1 = 3

2 items n° 1
2 w1 = 6

•Item 1: x1=$5 ; w1=3t
•Item 2: x2=$7 ; w2=4t
•Item 3: x3=$8 ; w3=5t

Knapsack Problem

Table

	1	2	3	4	5	6	7	8	9	10
1	0	0	5	5	5	10	10	10	15	15
2	0	0	5	7						
3										

w – w2 =
5 – 4 = 1

+ x2 (= 7)

$$f(i,w)=\text{Max}[\ x_i + f(i,w-w_i)\ ;\ f(i-1,w)\ ]$$

•Item 1: x1=$5 ; w1=3t
•Item 2: x2=$7 ; w2=4t
•Item 3: x3=$8 ; w3=5t

Knapsack Problem

Table

	1	2	3	4	5	6	7	8	9	10
1	0	0	5	5	5	10	10	10	15	15
2	0	0	5	7	7					
3										

+ x2 (= 7)

$$f(i,w)=\text{Max}[\ x_i + f(i,w-w_i)\ ;\ f(i-1,w)\ ]$$

•Item 1: x1=$5 ; w1=3t
•Item 2: x2=$7 ; w2=4t
•Item 3: x3=$8 ; w3=5t

Knapsack Problem

Table

	1	2	3	4	5	6	7	8	9	10
1	0	0	5	5	5	10	10	10	15	15
2	0	0	5	7	7					
3										

w – w2 =
6 – 4 = 2

+ x2 (= 7)

f(i,w)=Max[xi + f(i,w-wi) ; f(i-1,w)]

•Item 1: x1=$5 ; w1=3t
•Item 2: x2=$7 ; w2=4t
•Item 3: x3=$8 ; w3=5t

Knapsack Problem

Table

	1	2	3	4	5	6	7	8	9	10
1	0	0	5	5	5	10	10	10	15	15
2	0	0	5	7	7	10				
3										

+ x2 (= 7)

f(i,w)=Max[xi + f(i,w-wi) ; f(i-1,w)]

- Item 1: x1=$5 ; w1=3t
- Item 2: x2=$7 ; w2=4t
- Item 3: x3=$8 ; w3=5t

Knapsack Problem

COMPLETED TABLE

	1	2	3	4	5	6	7	8	9	10
1	0	0	5	5	5	10	10	10	15	15
2	0	0	5	7	7	10	12	14	15	17
3	0	0	5	7	8	10	12	14	15	17

- Item 1: x1=$5 ; w1=3t
- Item 2: x2=$7 ; w2=4t
- Item 3: x3=$8 ; w3=5t

Knapsack Problem

Path

	1	2	3	4	5	6	7	8	9	10
1	0	0	5	5	5	10	10	10	15	15
2	0	0	5	7	7	10	12	14	15	17
3	0	0	5	7	8	10	12	14	15	17

Optimal: 2 x Item 1 + 1 x Item 2

Knapsack problem: running time

Theorem. The DP algorithm solves the knapsack problem with n items and maximum weight W in $\Theta(n\ W)$ time and $\Theta(n\ W)$ space.

Pf.

weights are integers between 1 and W

- Takes $O(1)$ time per table entry.
- There are $\Theta(n\ W)$ table entries.
- After computing optimal values, can trace back to find solution:
  $OPT(i, w)$ takes item i iff $M[i, w] > M[i-1, w]$. ▪

Remarks.

- Algorithm depends critically on assumption that weights are integral.
- Assumption that values are integral was not used.

Knapsack Problem: Bottom-Up

Knapsack. Fill up an n-by-W array.

```
Input: n, w_1,…,w_N, v_1,…,v_N

for w = 0 to W
   M[0, w] = 0

for i = 1 to n
   for w = 1 to W
      if (w_i > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], v_i + M[i-1, w-w_i ]}

return M[n, W]
```

Knapsack Algorithm

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Dynamic programming: quiz 4

Does there exist a poly-time algorithm for the knapsack problem?

A. Yes, because the DP algorithm takes $\Theta(n\ W)$ time.

B. No, because $\Theta(n\ W)$ is not a polynomial function of the input size.

C. No, because the problem is **NP**-hard.

D. Unknown.

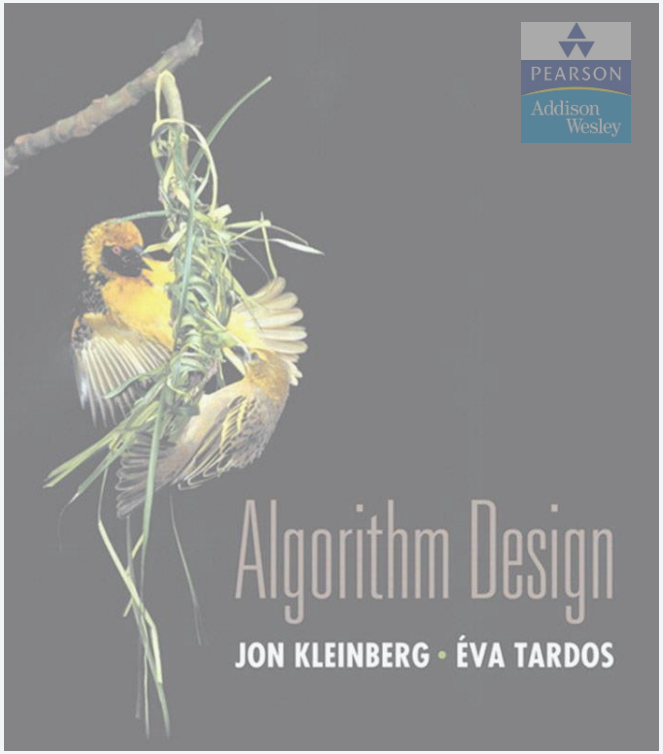
COIN CHANGING

Problem. Given n coin denominations $\{ c_1, c_2, \ldots, c_n \}$ and a target value V, find the fewest coins needed to make change for V (or report impossible).

Recall. Greedy cashier's algorithm is optimal for U.S. coin denominations, but not for arbitrary coin denominations.

Ex. { 1, 10, 21, 34, 70, 100, 350, 1295, 1500 }.

Optimal. 140¢ = 70 + 70.



SECTION 6.5

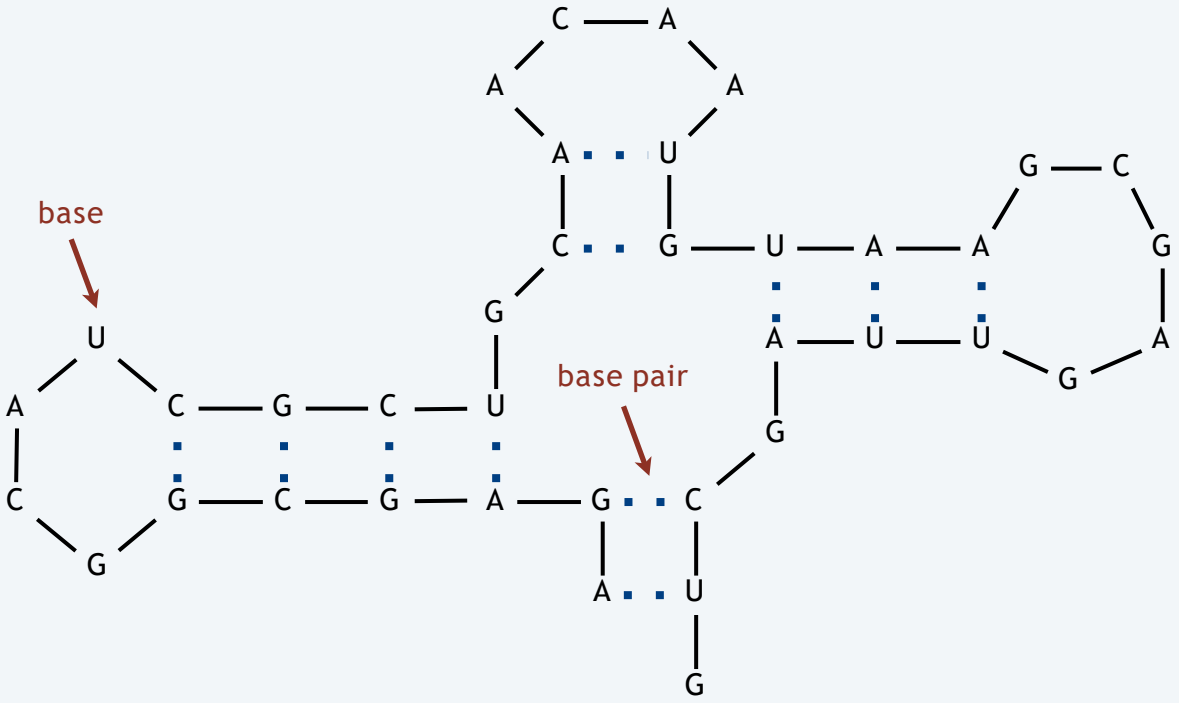
6. DYNAMIC PROGRAMMING I

- *weighted interval scheduling*
- *segmented least squares*
- *knapsack problem*
- *RNA secondary structure*

RNA secondary structure

RNA. String $B = b_1 b_2 \ldots b_n$ over alphabet $\{ \mathrm{A}, \mathrm{C}, \mathrm{G}, \mathrm{U} \}$.

Secondary structure. RNA is single-stranded so it tends to loop back and form base pairs with itself. This structure is essential for understanding behavior of molecule.

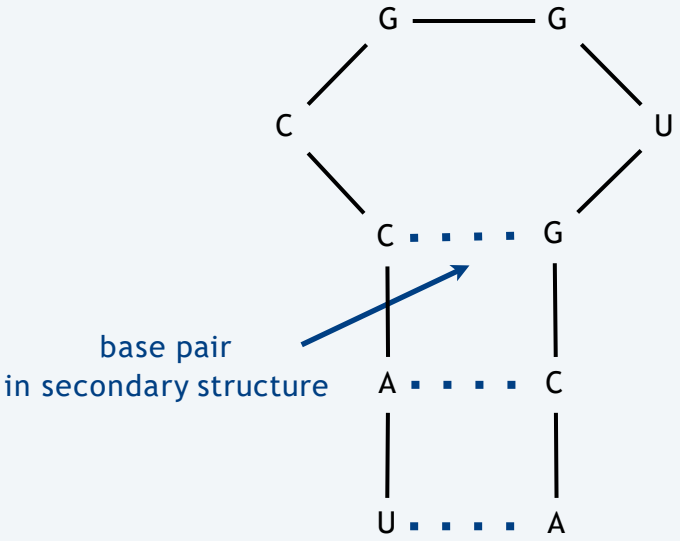


RNA secondary structure for GUCGAUUGAGCGAAUGUAACAACGUGGCUACGGCGAGA

RNA secondary structure

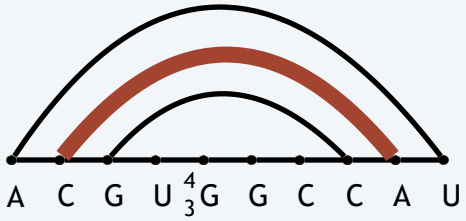
Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

- [Watson-Crick] S is a matching and each pair in S is a Watson-Crick complement: A–U, U–A, C–G, or G–C.



B = ACGUGGCCCAU

$S = \{ (b_1, b_{10}), (b_2, b_9), (b_3, b_8) \}$



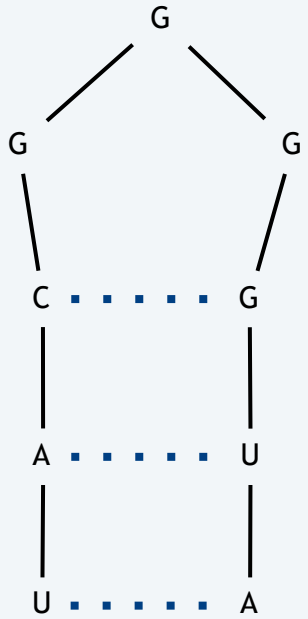
S is not a secondary structure

(C-A is not a valid Watson-Crick pair)

RNA secondary structure

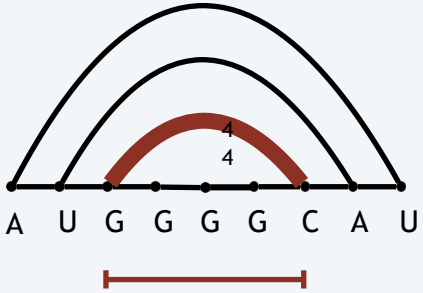
Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

- [Watson-Crick] S is a matching and each pair in S is a Watson-Crick complement: A–U, U–A, C–G, or G–C.
- [No sharp turns] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.



B = AUGGGGCAU

$S = \{ (b_1, b_{10}), (b_2, b_9), (b_3, b_8) \}$



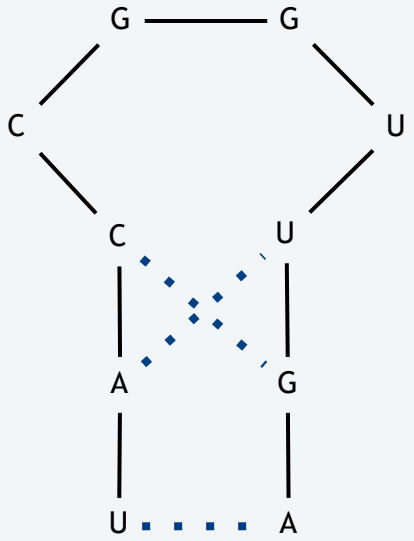
S is not a secondary structure

(≤4 intervening bases between G and C)

RNA secondary structure

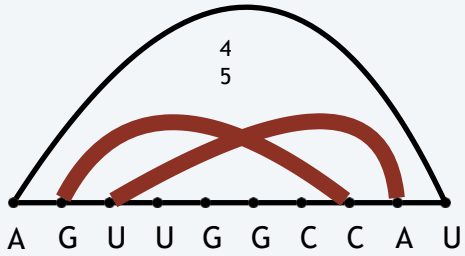
Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

- [Watson-Crick] S is a matching and each pair in S is a Watson-Crick complement: A–U, U–A, C–G, or G–C.
- [No sharp turns] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.
- [Non-crossing] If (b_i, b_j) and (b_k, b_ℓ) are two pairs in S, then we cannot have $i < k < j < \ell$.



B = ACUUGGCCAU

$S = \{ (b_1, b_{10}), (b_2, b_8), (b_3, b_9) \}$

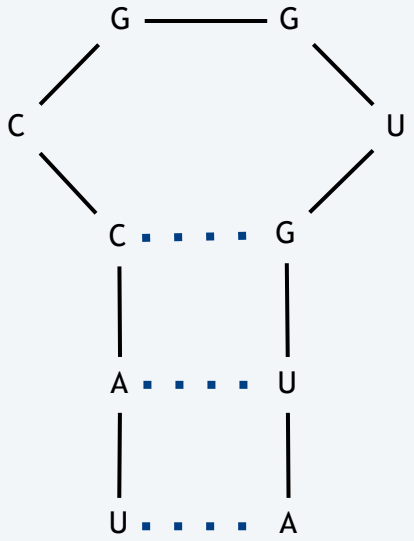


S is not a secondary structure (G-C and U-A cross)

RNA secondary structure

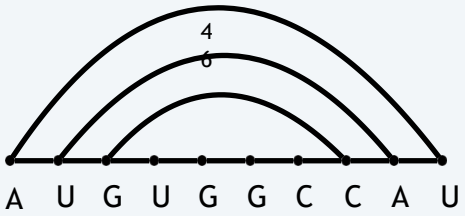
Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

- [Watson-Crick] S is a matching and each pair in S is a Watson-Crick complement: A–U, U–A, C–G, or G–C.
- [No sharp turns] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.
- [Non-crossing] If (b_i, b_j) and (b_k, b_ℓ) are two pairs in S, then we cannot have $i < k < j < \ell$.



B = AUGUGGCCAU

$S = \{ (b_1, b_{10}), (b_2, b_9), (b_3, b_8) \}$



S is a secondary structure (with 3 base pairs)

RNA secondary structure

Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

- [Watson–Crick] S is a matching and each pair in S is a Watson–Crick complement: A–U, U–A, C–G, or G–C.
- [No sharp turns] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.
- [Non-crossing] If (b_i, b_j) and (b_k, b_ℓ) are two pairs in S, then we cannot have $i < k < j < \ell$.

Free-energy hypothesis. RNA molecule will form the secondary structure with the minimum total free energy.

approximate by number of base pairs
(more base pairs $\Rightarrow$ lower free energy)

Goal. Given an RNA molecule $B = b_1 b_2 \ldots b_n$, find a secondary structure S that maximizes the number of base pairs.

Dynamic programming: quiz 5

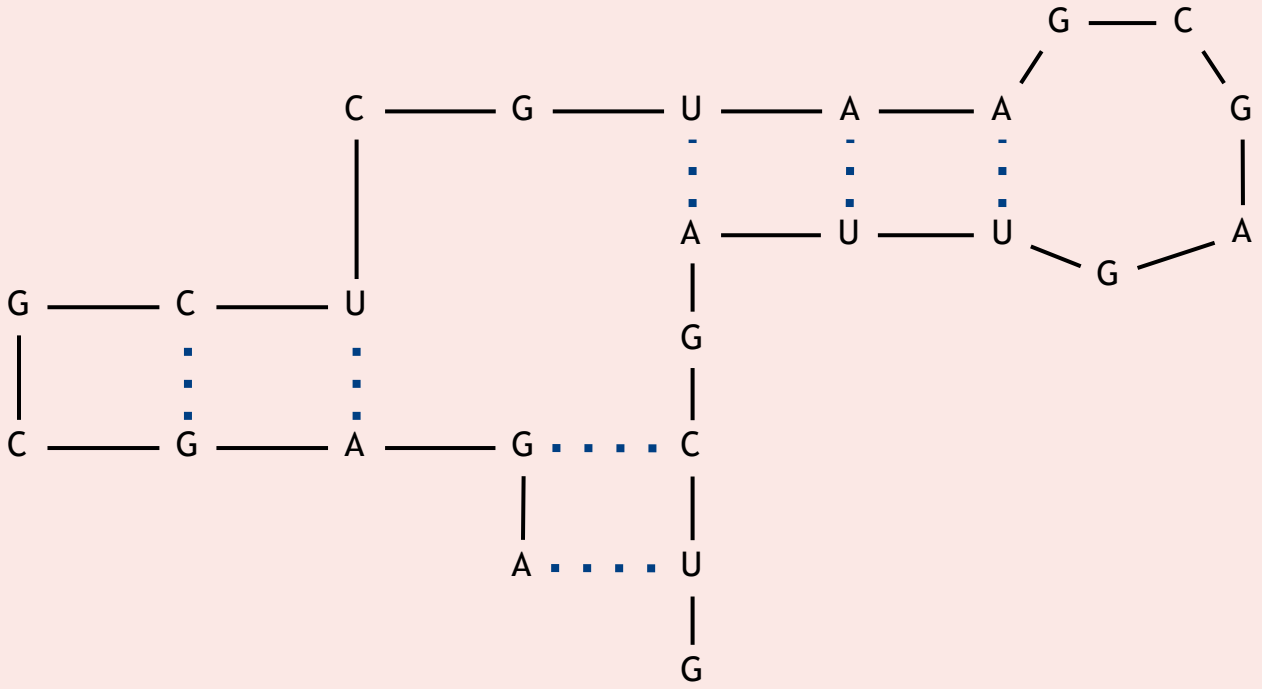
Is the following a secondary structure?

A. Yes.

B. No, violates Watson–Crick condition.

C. No, violates no-sharp-turns condition.

D. No, violates no-crossing condition.



Dynamic programming: quiz 6

Which subproblems?

A. $OPT(j)$ = max number of base pairs in secondary structure of the substring $b_1 b_2 \ldots b_j$.

B. $OPT(j)$ = max number of base pairs in secondary structure of the substring $b_j b_{j+1} \ldots b_n$.

C. Either A or B.

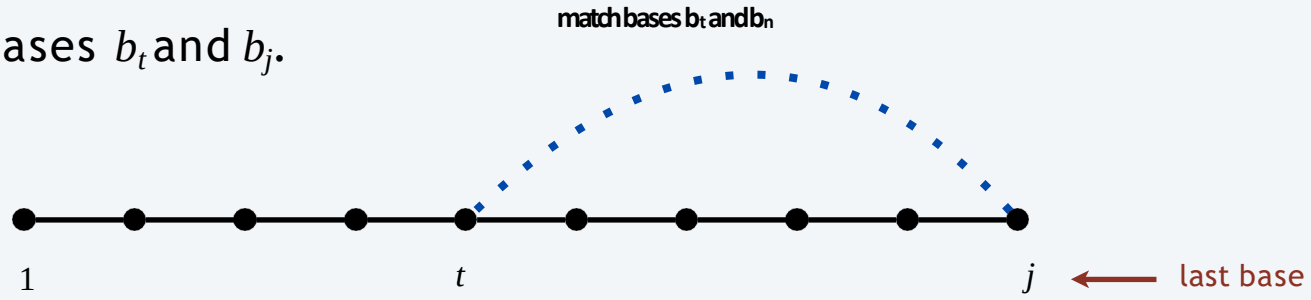
D. Neither A nor B.

RNA secondary structure: subproblems

First attempt. $OPT(j)$ = maximum number of base pairs in a secondary structure of the substring $b_1 b_2 \ldots b_j$.

Goal. $OPT(n)$.

Choice. Match bases b_t and b_j.



Difficulty. Results in two subproblems (but one of wrong form).

- Find secondary structure in $b_1 b_2 \ldots b_{t-1}$. ← $OPT(t-1)$
- Find secondary structure in $b_{t+1} b_{t+2} \ldots b_{j-1}$. ← need more subproblems (first base no longer b_1)

Dynamic programming over intervals

Def. $OPT(i, j)$ = maximum number of base pairs in a secondary structure of the substring $b_i b_{i+1} \ldots b_j$.

Case 1. If $i \geq j - 4$.

- $OPT(i, j) = 0$ by no-sharp-turns condition.

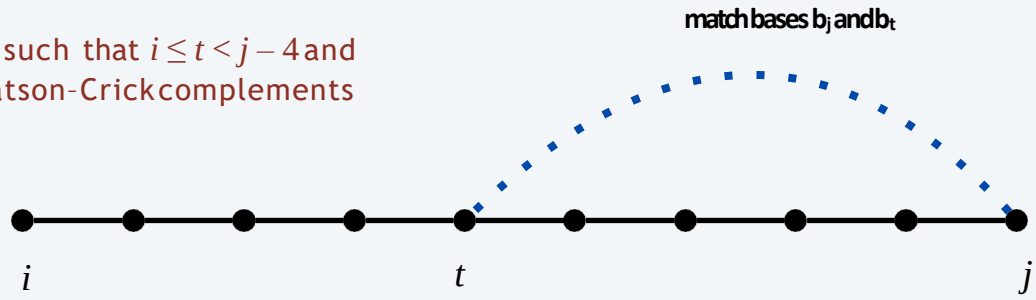
Case 2. Base b_j is not involved in a pair.

- $OPT(i, j) = \mathrm{OPT}(i, j-1)$.

Case 3. Base b_j pairs with b_t for some $i \leq t < j - 4$.

- Non-crossing condition decouples resulting two subproblems.
- $OPT(i, j) = 1 + \max_t \{ OPT(i, t-1) + OPT(t+1, j-1) \}$.

take max over t such that $i \leq t < j - 4$ and b_t and b_j are Watson-Crick complements



Dynamic programming: quiz 7

In which order to compute $OPT(i, j)$?

A. Increasing i, then j.

B. Increasing j, then i. 

C. Either A or B.

D. Neither A nor B.

Bottom-up dynamic programming over intervals

Q. In which order to solve the subproblems?

A. Do shortest intervals first—increasing order of $j - i$.

RNA-SECONDARY-STRUCTURE$(n, b_1, \ldots, b_n)$

FOR $k = 5$ TO $n - 1$

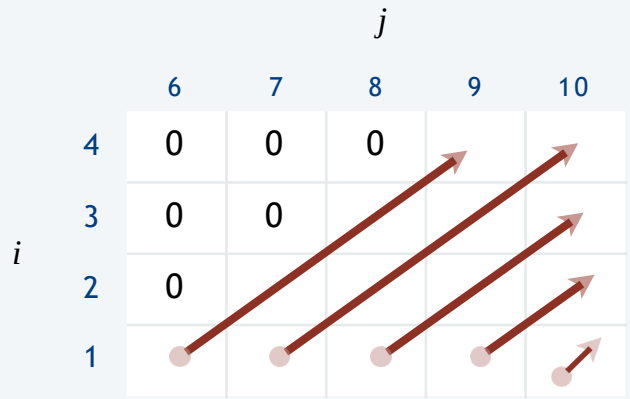
  FOR $i = 1$ TO $n - k$

    $j \leftarrow i + k$.

    Compute $M[i, j]$ using formula.

RETURN $M[1, n]$.

all needed values are already computed



order in which to solve subproblems

5

3

Theorem. The DP algorithm solves the RNA secondary structure problem in $O(n^3)$ time and $O(n^2)$ space.

Dynamic programming summary

Outline.

typically, only a polynomial number of subproblems

- Define a collection of subproblems.
- Solution to original problem can be computed from subproblems.
- Natural ordering of subproblems from "smallest" to "largest" that enables determining a solution to a subproblem from solutions to smaller subproblems.

Techniques.

- Binary choice: weighted interval scheduling.
- Multiway choice: segmented least squares.
- Adding a new variable: knapsack problem.
- Intervals: RNA secondary structure.

Top-down vs. bottom-up dynamic programming. Opinions differ.

NEXT LECTURE

- Min-cut
- Karger's Algorithm