EHB 211E
Basics of Electrical Circuits

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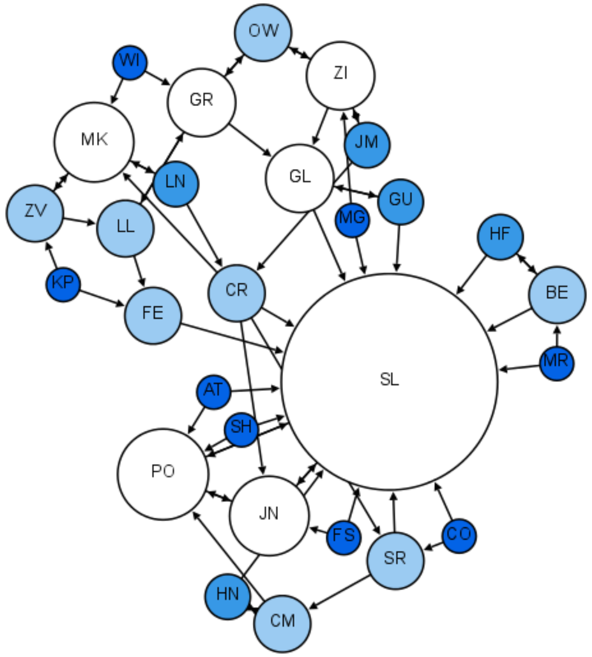
Graph Theory

Linear and Nonlinear Circuit by L.O. Chua, C.A. Desoer, and E.S. Kuh

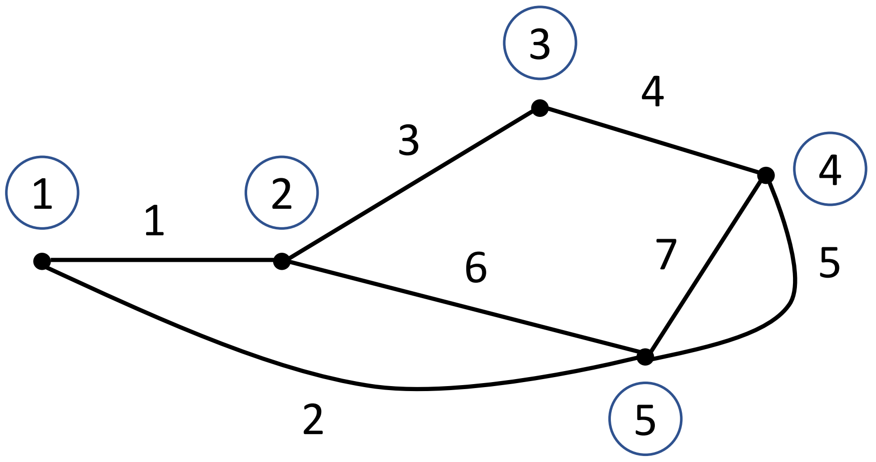
Introduction

- Graph theory (aka network topology): study and model large number of applications in different fields.
  - Computer science: flow of computation, data organization
  - Electrical Engineering: design of circuit connection
  - Physics and chemistry: study molecules (constructing lattice of molecule)
  - Mathematics: minimum cost part, scheduling problem
  - Biology: interaction between species (protein-protein interaction)
  - Linguistics: grammar of language tree
  - Represent routes between cities



Basic Terminology

- What is a graph?
  - Set of nodes connected by branches
- Node (aka vertex): common terminal or point of two or more branches. Node is denoted by n
- Branch (aka edge): line segment that replaces the network element and connects two nodes. Branch is denoted by b.
- Simple example of graph:



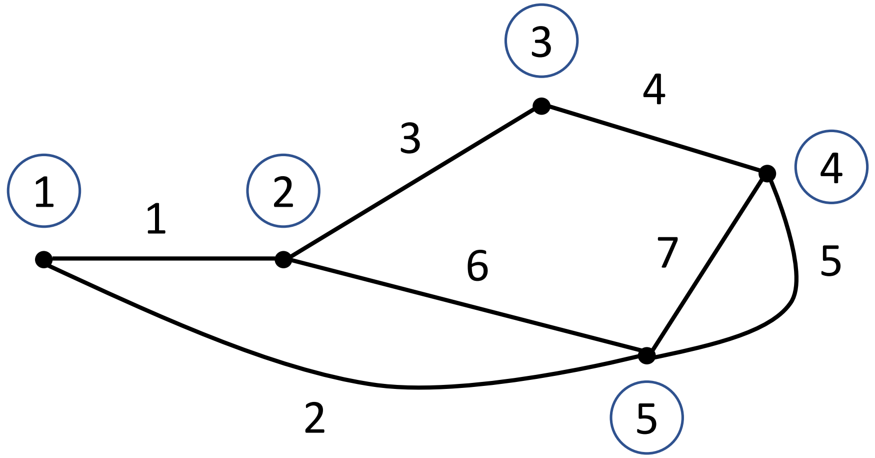
5 nodes and 7 branches

n=5 & b=7

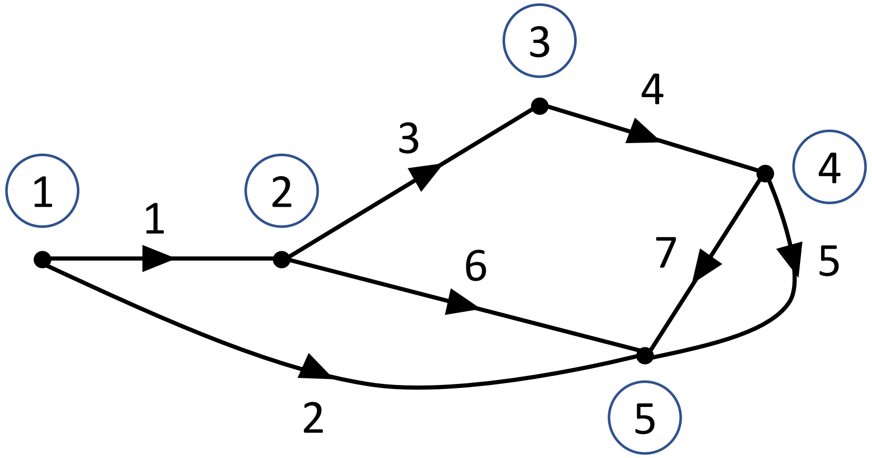
Types of Graphs

- Two types of graphs
  - Undirected graph: direction not specified
  - Directed graph (digraph): all branches are represented with arrows.

Undirected graph



Directed graph

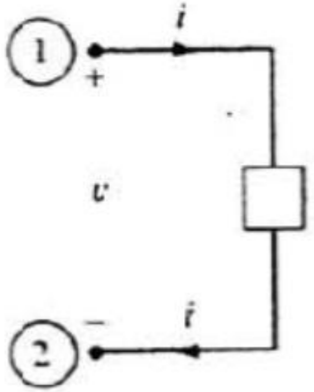


Arrows indicate the direction of current flow in each branch

From Circuits to Graphs

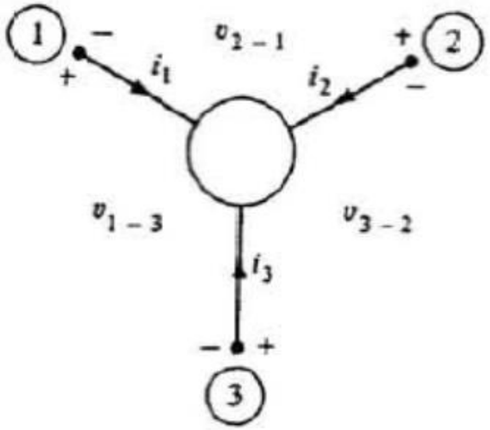
- Graph retains all interconnection properties of the circuit but suppresses the information on the circuit elements.

Two terminal element



digraph

Three terminal element



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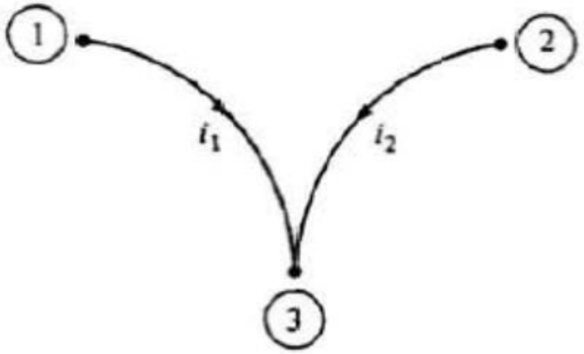


Digraph representation of two terminal element with two nodes and one branch

Voltage in branch: branch voltage
Current in branch: branch current

Power delivered to element: $p(t) = v(t)i(t)$

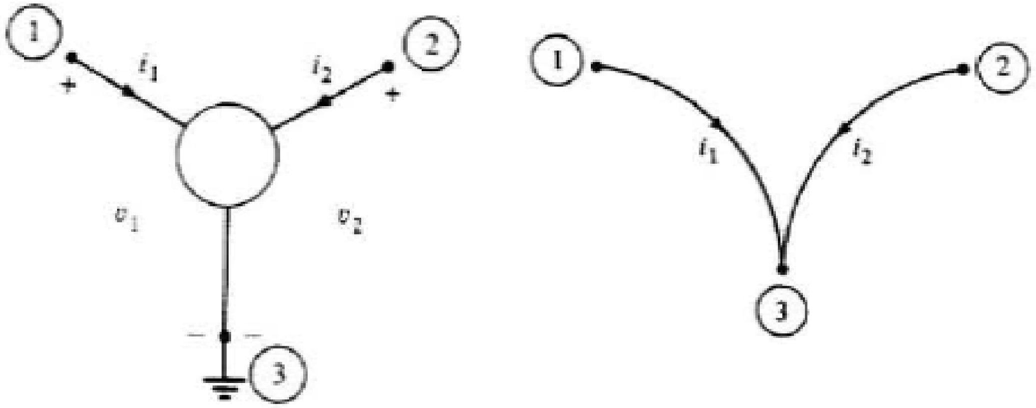
Digraph representation of a three terminal element with node 3 chosen as datum (reference node). Three nodes and two branches.

Apply KVL: Two independent voltages
Apply KCL: Two independent currents

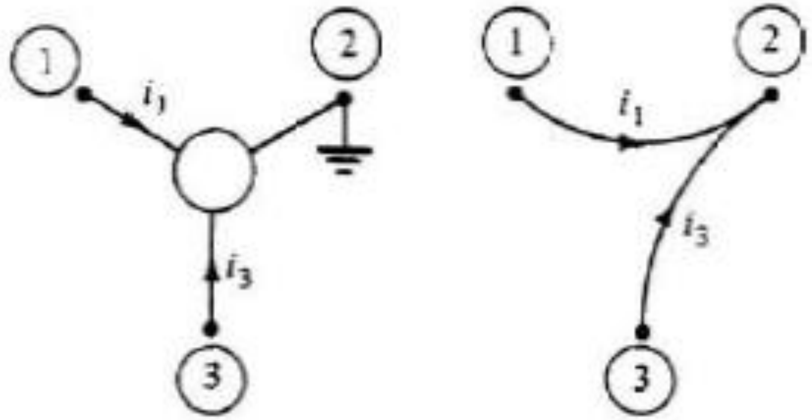
From Circuits to Graphs

- All possibilities for digraph representation of three terminal elements depending on which node chosen as the datum (reference) node.

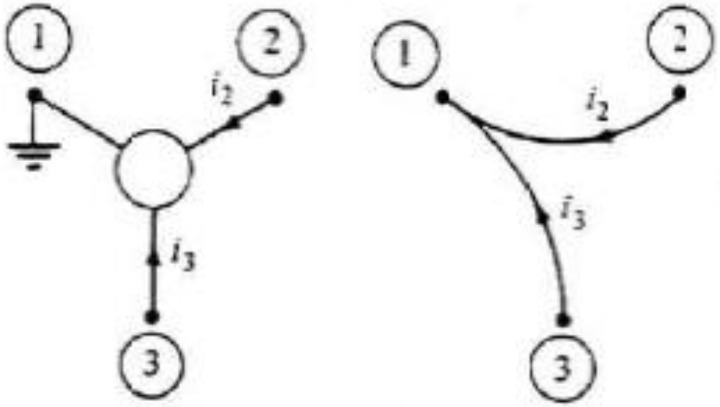
Node 3 (datum node): $v_1 > v_3$ & $v_2 > v_3$



Node 2 (datum node): $v_1 > v_2$ & $v_3 > v_2$



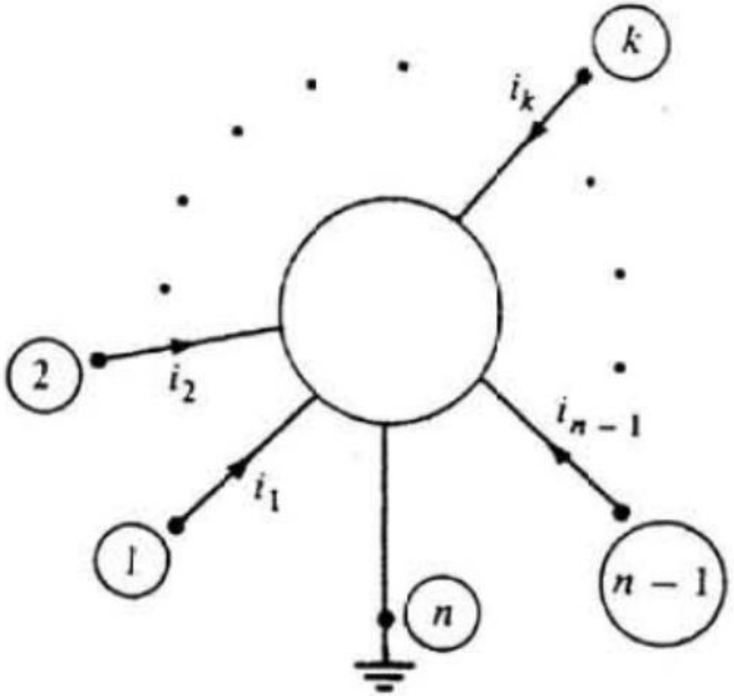
Node 1 (datum node): $v_2 > v_1$ & $v_3 > v_1$

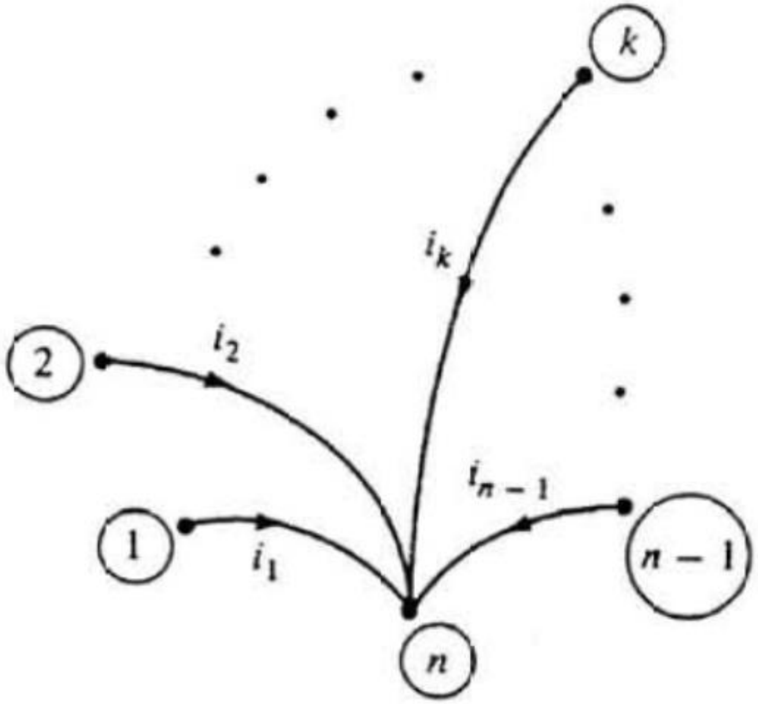


From Circuits to Graphs

- General form of n-terminal elements with digraph representation:

n-terminal element: n nodes and n-1 branches



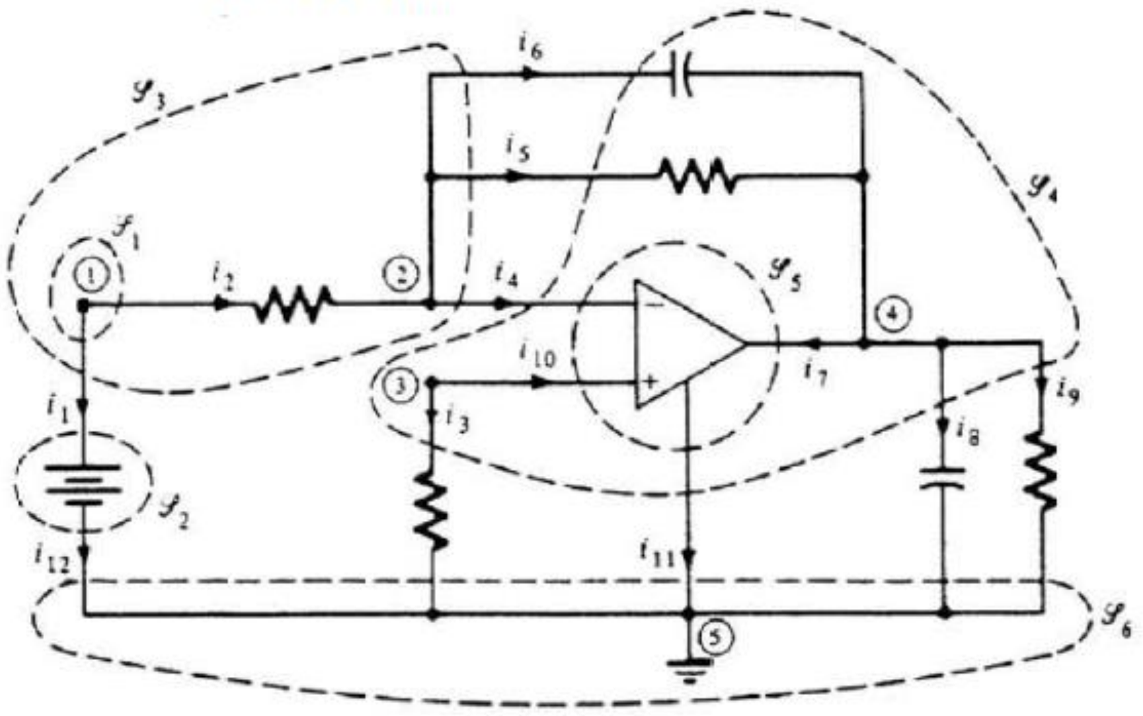


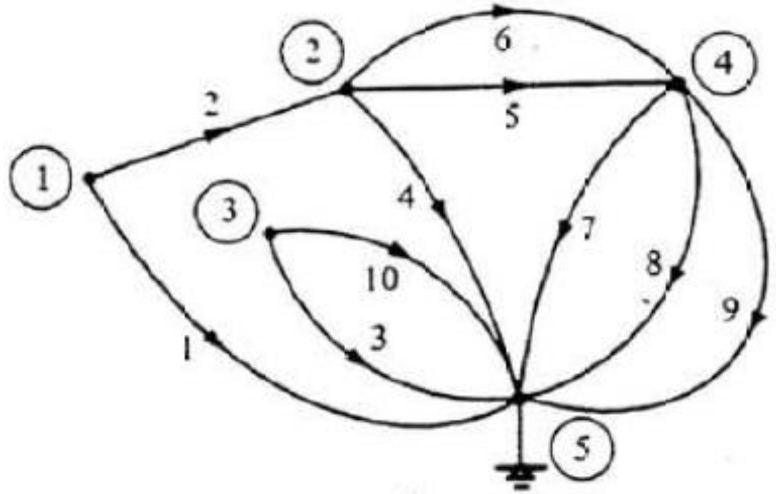
Example

Demonstrate the op amp circuit shown below with the corresponding digraph

Solution:

- Seven two-terminal elements and one four-terminal element
- Recall: n-terminal elements have n-1 branches
- Total number of branches in digraph: 7+4-1=10
- Circuit has five nodes. Thus, digraph has five nodes (node 5: datum node)

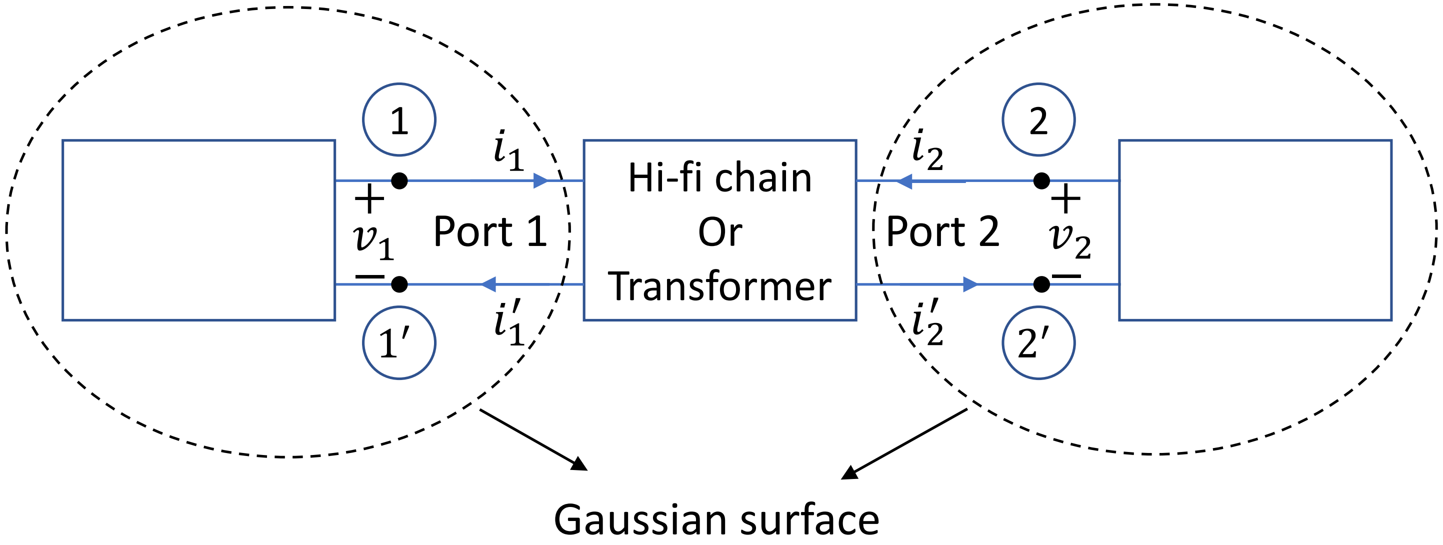




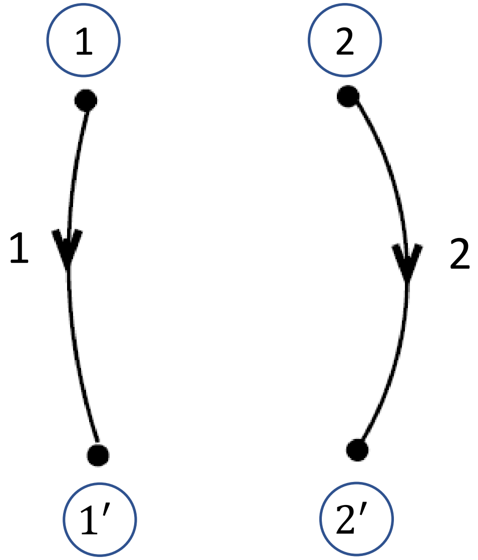
Suppressed circuit elements when drawing the digraph

Two-Ports

- Two-port: circuit element or a circuit with pairs of terminals.
- Example of two-port circuit: Hi-fi chain, two-winding transformer



Digraph representation of two-port circuit



Apply KCL: $i_1 = i_1'$ and $i_2 = i_2'$ (port condition)

Two-port network reduces number of currents from 4 to 2

Power delivered to two-port:

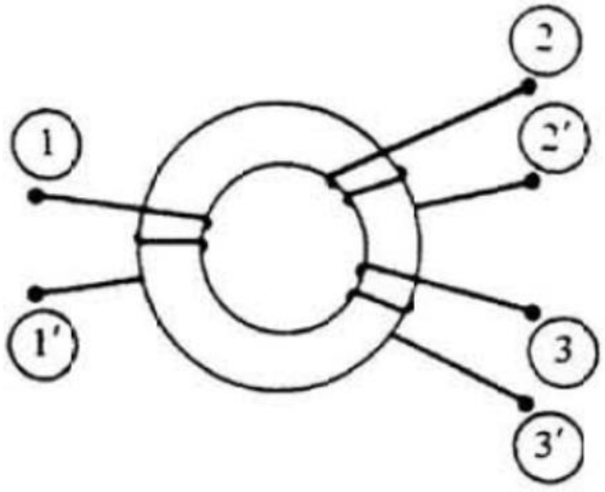
$p(t) = v_1(t)i_1(t) + v_2(t)i_2(t)$

Two-port: 4 nodes and 2 branches

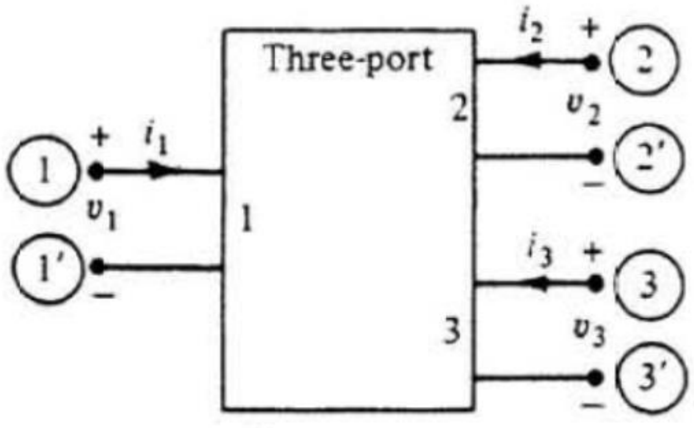
i_1 and i_2: branch currents
v_1 and v_2: branch voltages

Multiport

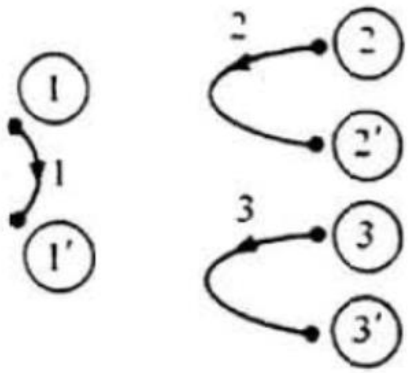
- Generalize the concept of two-port to multiport
- Three-winding transformer: three-port network or circuit.



Three-winding transformer



Three-port



digraph

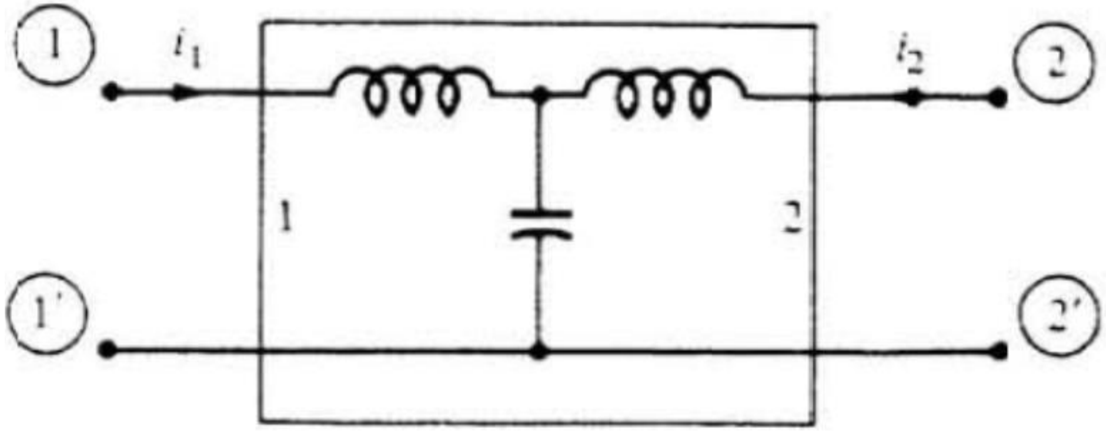
Three-winding transformer: three-port
its digraph: 6 nodes and 3 branches

Recall:
Digraph (directed graph) is a graph that is made up of a set of vertices (nodes) connected by edges (branches)

Grounded Two-Port

- If a common connection exists between node $1'$ and $2'$, it is called grounded two-port. This is equivalent to three terminal element

Grounded two-port



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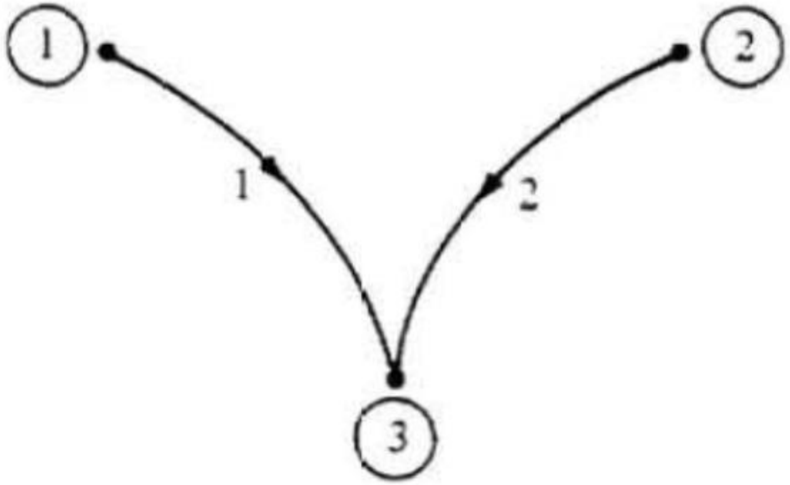
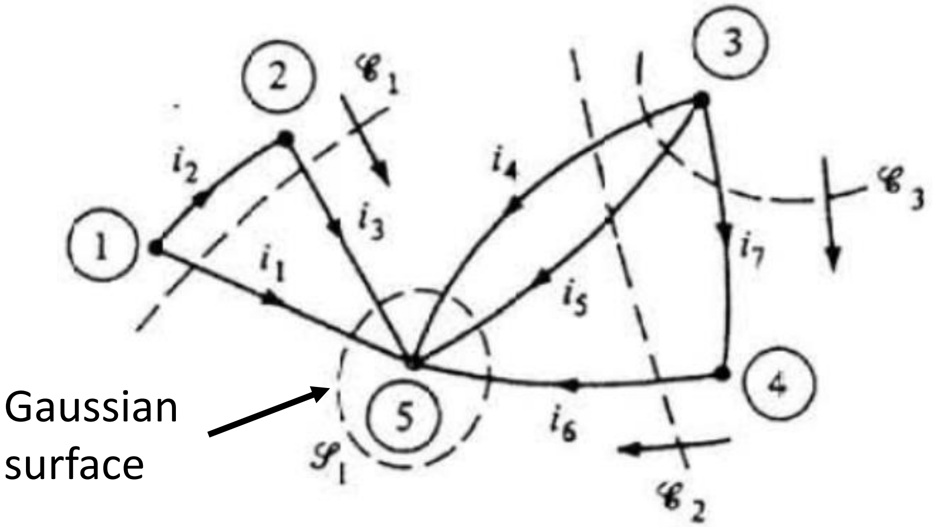


- Digraph of grounded two-port: 3 nodes and 2 branches which are tied together at the common node

Cut Sets

- Cut set is very useful concept in graph theory
- Given a connected digraph G, a set of branches b of G is called a cut set if and only if:
  - The removal of all the branches of the cut set results in a unconnected digraph (resulted digraph is no longer connected).
  - The removal of all but any one branch of G leaves the digraph connected. In other words, if any branch in the set is left intact, the digraph remains connected.



$b_1 = \{1,3\}$ ⟶ b_1 intersects with branch {1,3}

$b_2 = \{4,5,6\}$ ⟶ b_2 intersects with branch {4,5,6}

$b_3 = \{4,5,7\}$ ⟶ b_3 intersects with branch {4,5,7}

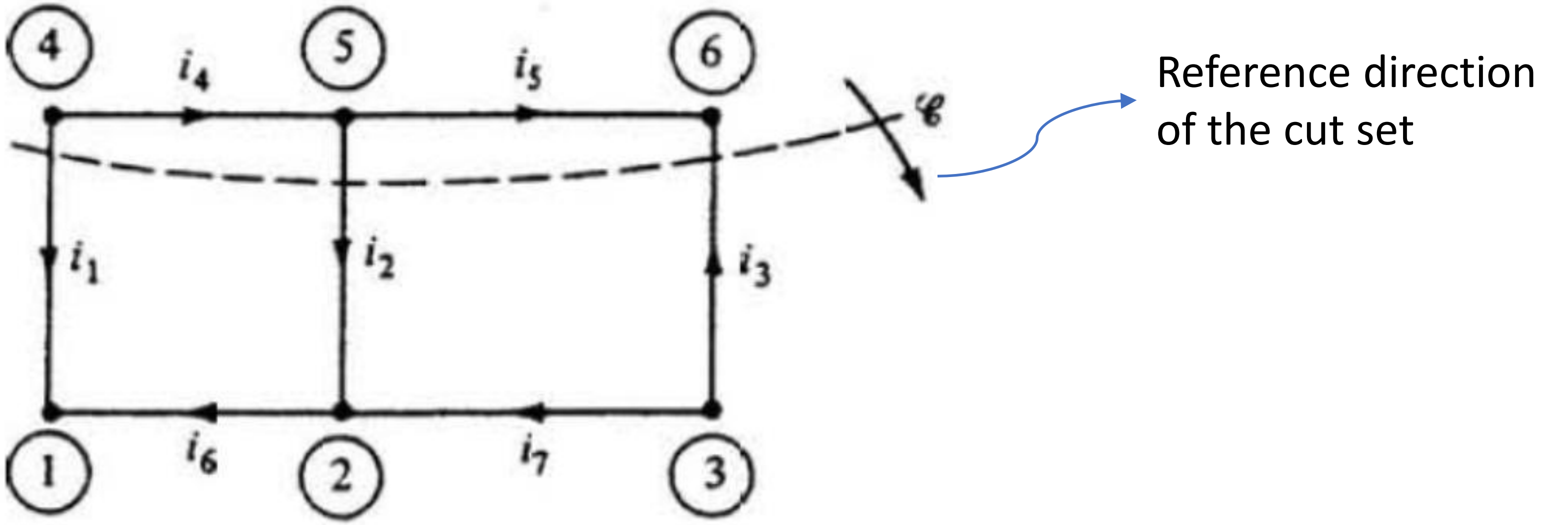
b_1, b_2, b_3 represent the cut set line.

Arrows for b_1, b_2, b_3 define a reference direction for each cut set. A reference direction is chosen arbitrary.

Cut set is a way to isolate node or nodes from the circuit

Cut Sets and KCL

- KCL: algebraic sum of the currents associated with any cut set is equal to zero.

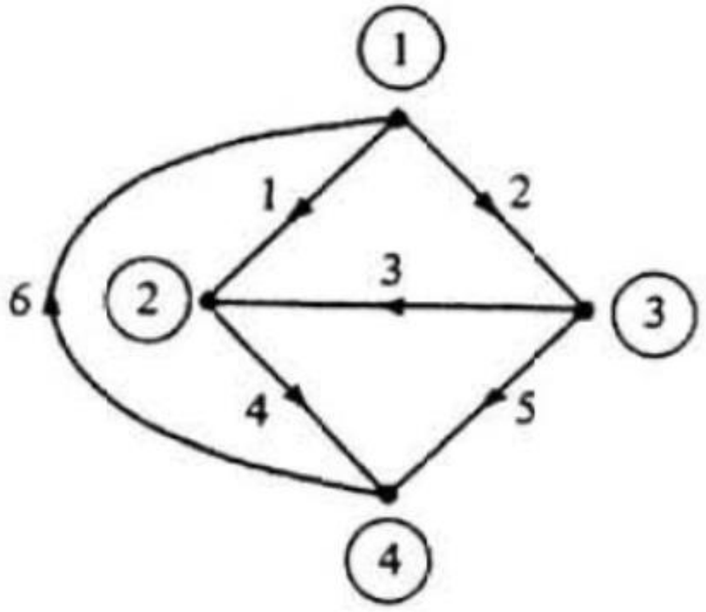


- For the digraph shown in the figure, the cut set $b = \{1,2,3\}$ is indicated by the dashed line cutting through these branches. (it intersects with three branches {1,2,3}).
- Direction of current is positive if it is in the same direction as the reference direction of the cut set.

$$i_1 + i_2 - i_3 = 0$$

Matrix Formation—KCL

- Formation of incidence matrix of the digraph.
- The incidence matrix is denoted by A_a
- If digraph G has n nodes and b branches, then A_a has n rows (one row for each node) and b columns (one column for each branch).



- Digraph: 4 nodes and 6 branches
- By definition: A_a has 4 rows and 6 columns

- Apply KCL
- Before writing KCL equations for each node, assign a reference direction for currents
- The direction of current is positive (+) if branch leaves the node. Otherwise it is negative

Matrix Formation—KCL

- Write the KCL equation for each node:

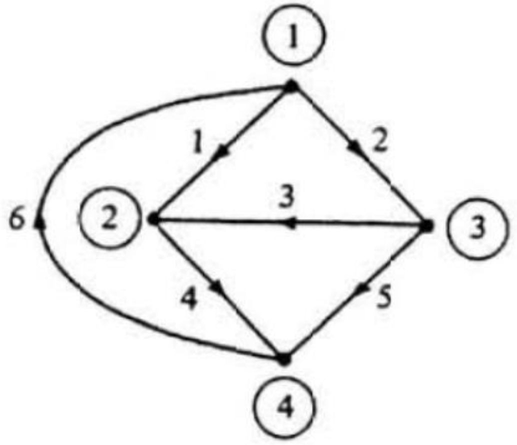
<u>Branches</u>: 1 2 3 4 5 6

$$
\begin{aligned}
&\text{Node 1:} && i_1 + i_2 \qquad\qquad\qquad -i_6 = 0\\
&\text{Node 2:} && -i_1 \qquad - i_3 + i_4 \qquad\qquad = 0\\
&\text{Node 3:} && \qquad -i_2 + i_3 \qquad + i_5 \qquad = 0\\
&\text{Node 4:} && \qquad\qquad\qquad -i_4 - i_5 + i_6 = 0
\end{aligned}
$$

- KCL equations in matrix form as:

$$
\begin{array}{c}
① \rightarrow \\ ② \rightarrow \\ ③ \rightarrow \\ ④ \rightarrow
\end{array}
\underbrace{\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & -1\\
-1 & 0 & -1 & 1 & 0 & 0\\
0 & -1 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & -1 & -1 & 1
\end{bmatrix}}_{\text{branch 1} \qquad \text{branch 6}}
\begin{bmatrix} i_1\\ i_2\\ i_3\\ i_4\\ i_5\\ i_6 \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ 0\\ 0 \end{bmatrix}
$$

4×6 matrix called incidence matrix A_a



- Node equation of digraph can be written as:

$$A_a i = 0$$

A_a: incidence matrix

i: branch current vector (magnitude and direction)

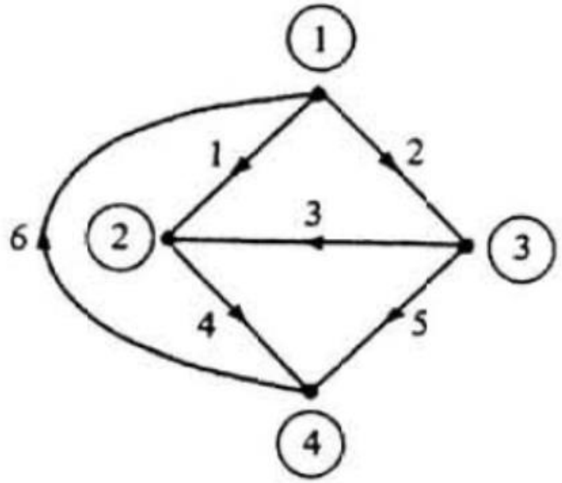
- Rank: number of linear independent equations in a digraph, denoted by r.

$r = n - 1$ $\qquad n$ is number of nodes

- For any connected digraph G with n nodes, if we choose a datum node and throw away the corresponding KCL equation, the remaining n-1 KCL equations are linearly independent

Matrix Formation—KCL

- Assume node 4 is a datum (reference) node
- Delete the last equation (row corresponding datum node).



$$\begin{array}{c} ① \rightarrow \\ ② \rightarrow \\ ③ \rightarrow \end{array} \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \end{bmatrix}}_{\text{Reduced incidence matrix } A} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad Ai = 0$$

- A (reduced incidence matrix) is obtained from A_a (incidence matrix) by deleting the row which corresponds to the chosen datum node
- Addition of all the KCL equation in above matrix will not cancel out each other. Three KCL equations are linearly independent.
- For connected digraph with n nodes, dimension of reduced incidence matrix is $(n-1) \times b$

Matrix Formation—KCL

- Write KCL equations in a matrix form using cut sets
- First, write all distinct cut sets for the given digraph

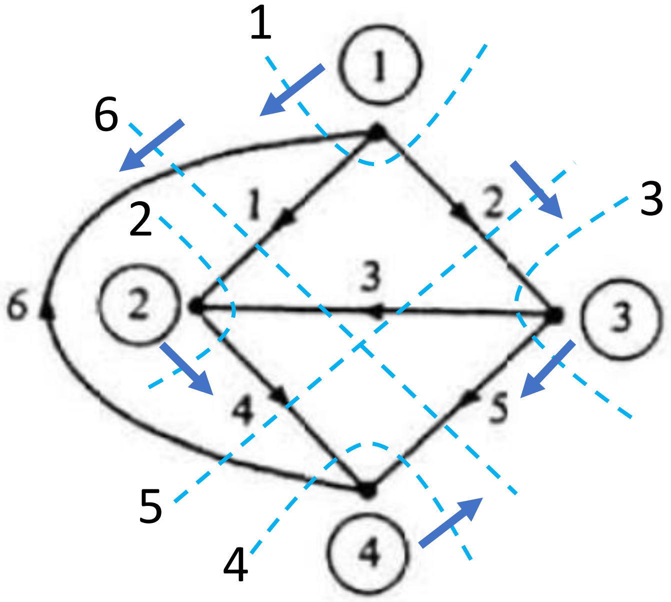
$b_1 = \{1,2,6\}$ $\qquad$ $b_4 = \{4,5,6\}$

$b_2 = \{1,3,4\}$ $\qquad$ $b_5 = \{2,3,4,6\}$

$b_3 = \{2,3,5\}$ $\qquad$ $b_6 = \{1,3,5,6\}$

- Based on the cut sets, the matrix is as follows:

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}}_{\text{Cut-set matrix, } Q_a} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow Q_a i = 0$$



Recall:
Cut set is a partition of the nodes (vertices) of a graph into two disjoint subsets

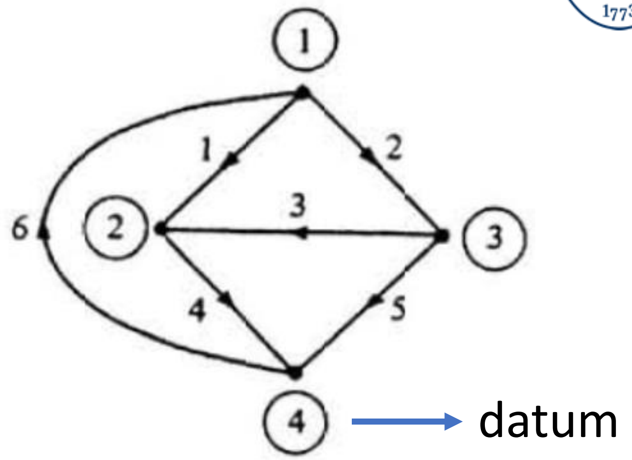
Recall:
Direction of current is positive if it is in the same direction as the reference direction of the cut set.

Rank of cut set matrix: $r = n - 1$

$Q_R i = 0$ (Q_R is called reduced cut-set matrix obtained by deleting redundant equations)

Matrix Formation—KVL

- KVL: another way of obtaining matrix form of the digraph
- Using the associated reference direction, the branch voltages:



$$\left.\begin{aligned} v_1 &= e_1 - e_2 \\ v_2 &= e_1 - e_3 \\ v_3 &= -e_2 + e_3 \\ v_4 &= e_2 - e_4 \\ v_4 &= e_2 \\ v_5 &= e_3 \\ v_6 &= -e_1 \end{aligned}\right\} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{M} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \Rightarrow v = Me$$

$v = Me$

$v = [v_1, v_2, v_3, \dots, v_n]^T$ is the branch voltage vector
$e = [e_1, e_2, e_3, \dots, e_n]^T$ is the node-to-datum voltage vector
M: $b \times (n-1)$ matrix

- Comparing KCL & KVL equations: $M = A^T$ $\longrightarrow$ M matrix equals to transformation of reduced incidence matrix A

$v = A^T e$

Graph Theory—Subgraph

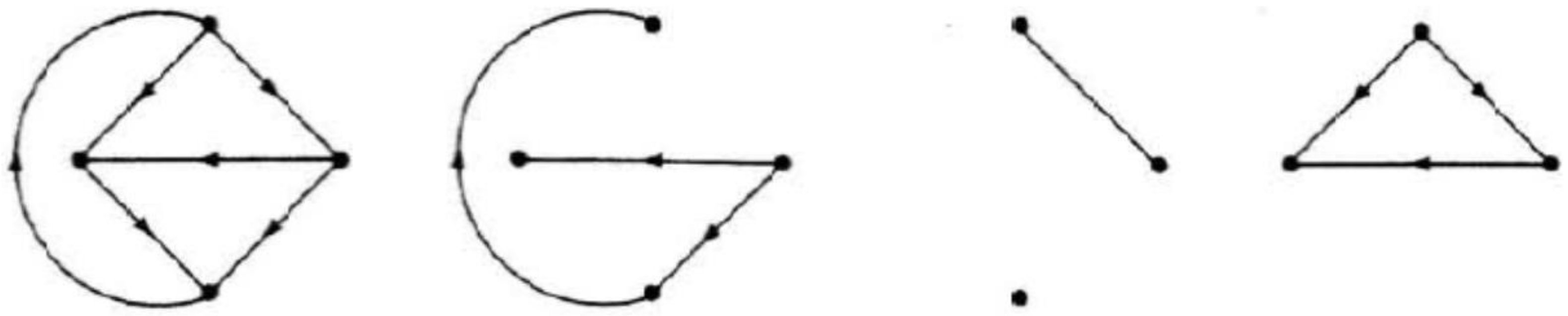
- digraph G is define as a set of vertices (nodes) connected by edges (branches) where the direction of all edges (branches) is specified.

  $G = (v, b)$

- Let G_1 is a subgraph of G if and only if G_1 itself is a graph.

  $G_1 = (v_1, b_1)$ where v_1 is subset of v and b_1 is subset of b

- Example of subgraphs of digraph G:

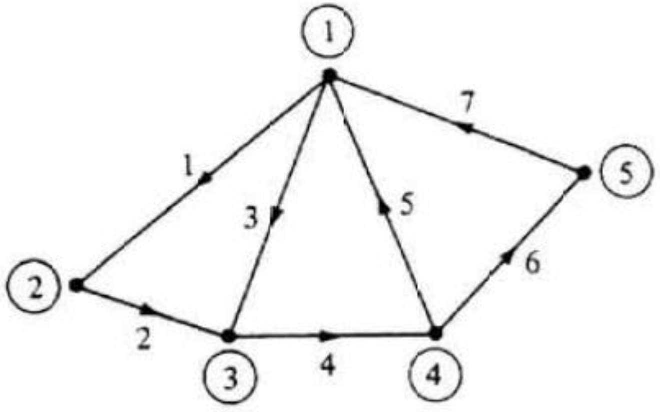


$\mathcal{G}$ $\mathcal{G}_1$ $\mathcal{G}_2$ $\mathcal{G}_3$

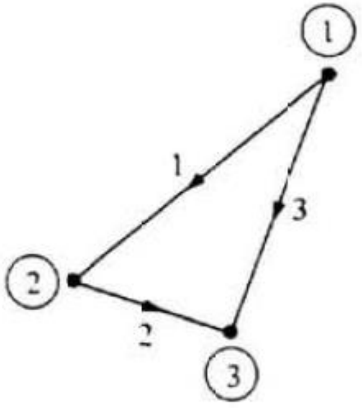
- Graph is connected if there is a path from any point to any other point in the graph.
- Graph G is connected but subgraph does not have to be connected.

Graph Theory—Loop

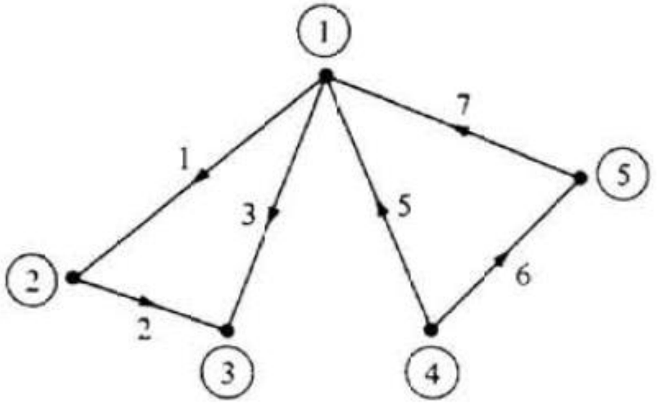
- Loop: connected subgraph of digraph G in which precisely two branches are incident with each other.
- For the loop concept, consider the following digraph G:



Digraph (G)



G_1

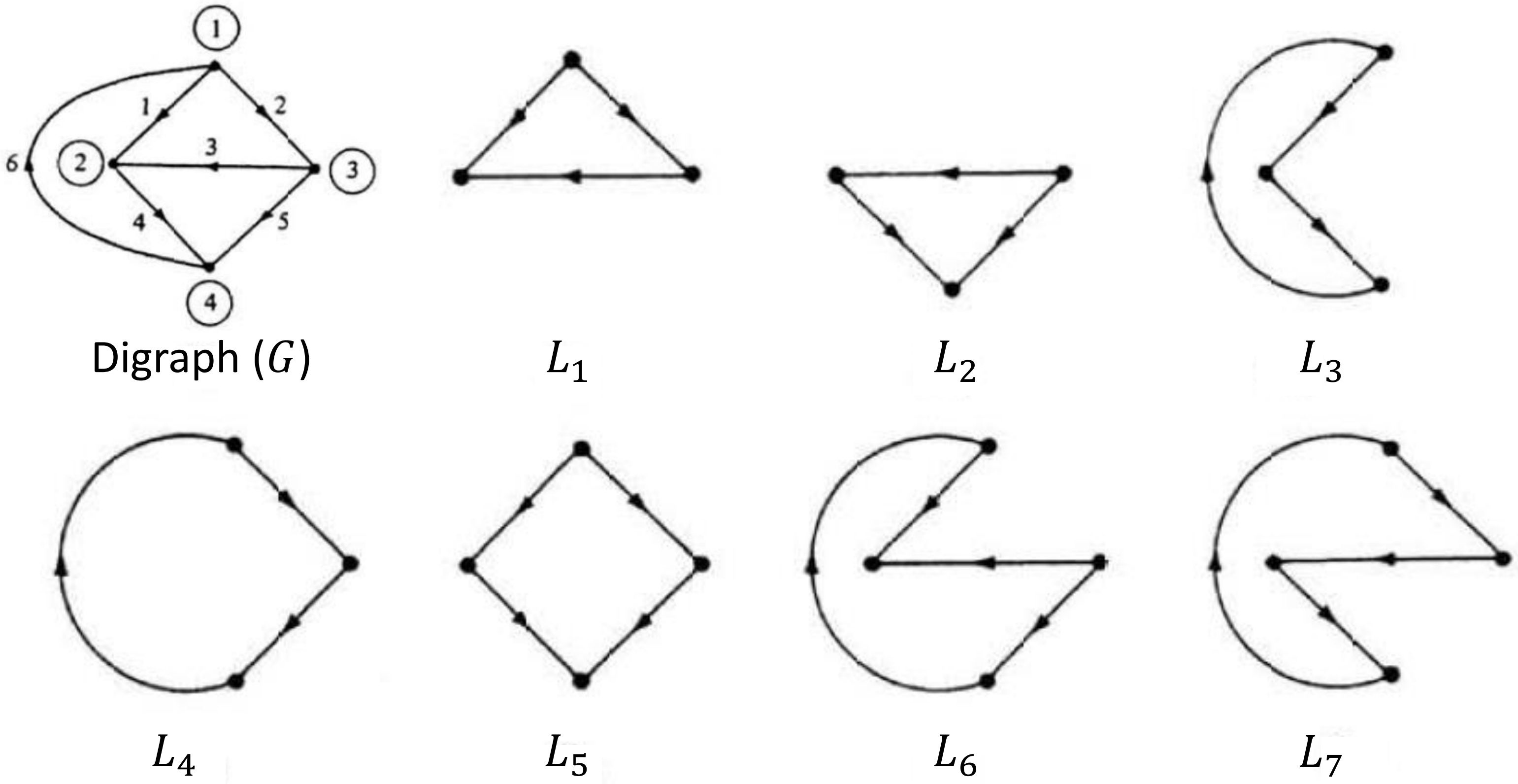


G_2

- G: digraph
- G_1: loop
- G_2: not a loop as it violates the definition of loop at node 1.

Graph Theory—KVL Based on Loops

- KVL equations based on loops
- First, identify all possible distinct loops from the digraph
- Following are all distinct loops:



- Every node has exactly two branches
- 7 distinct loops of digraph G

Graph Theory—KVL Based on Loops

- Apply KVL to each loop:

$L_1: v_1 - v_2 - v_3 = 0$

$L_2: v_3 + v_4 - v_5 = 0$

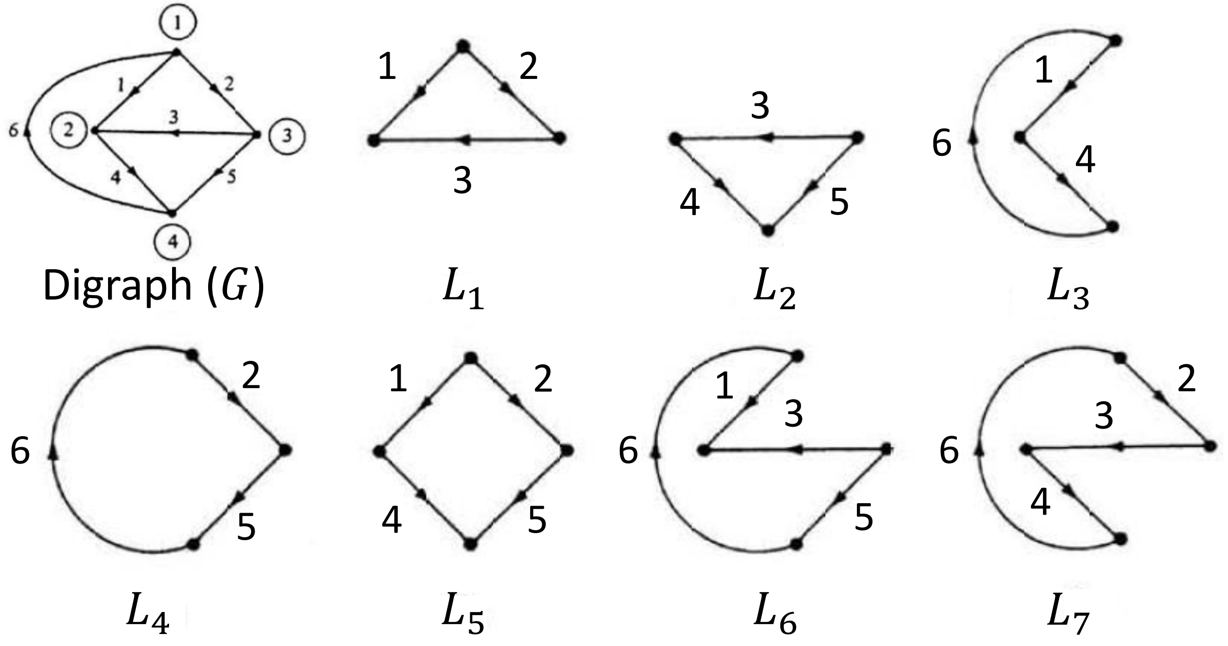
$L_3: v_1 + v_4 + v_6 = 0$

$L_4: v_2 + v_5 + v_6 = 0$

$L_5: v_1 - v_2 + v_4 - v_5 = 0$

$L_6: v_1 - v_3 + v_5 + v_6 = 0$

$L_7: v_2 + v_3 + v_4 + v_6 = 0$



$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}}_{\text{Loop matrix, } B_a} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow B_a v = 0$$

Set of equations in the loop matrix is linearly dependent.

Linear dependency: set of equations are linearly dependent if a linear combination of them is zero.

Graph Theory—KVL Based on Loops

- Minimum set of loops that contains all the information related to KVL equations:

$$l = b - (n - 1)$$

l: minimum number of loops for linearly independent KVL equations
b: number of branches in digraph
n: number of nodes in digraph

- For the digraph, minimum number of linearly independent equations:

$l = 6 - (4 - 1) = 3$ → 3 linearly independent equations contained all the information on the digraph → Choose only 3 loops (or KVL equations)

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}}_{\text{Reduced Loop matrix, } B_R} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

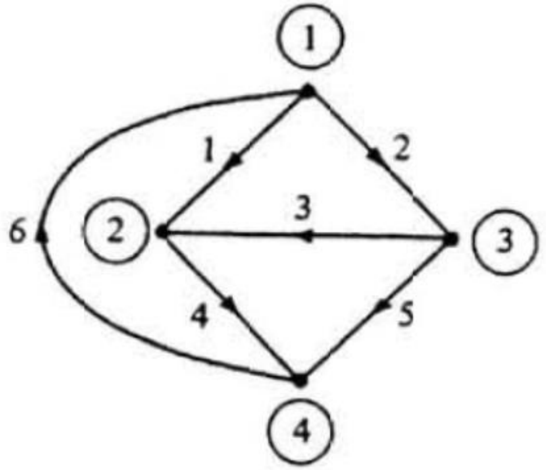
⇒ $L_1, L_2,$ and L_3 are chosen since they contain all the information. Ignore rest of the loops.

$$B_R v = 0$$

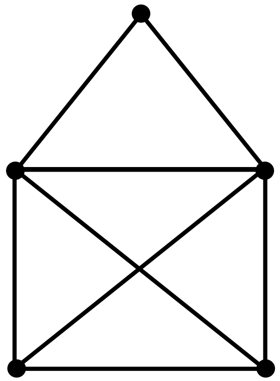
Reduced loop matrix B_R: $b - (n - 1)$ row and b columns

Graph Theory—Planar Graph

- What is a planar graph?
  - A graph which can be drawn on a plane in such a way that no two branches intersect at a point which is not a node.
- Example of planar and non-planar graph:



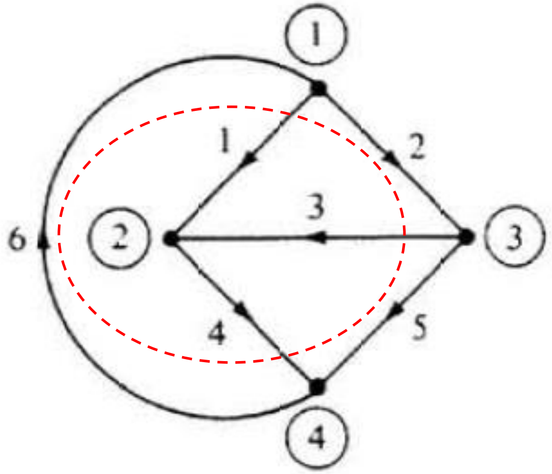
Planar graph (no branches intersect)



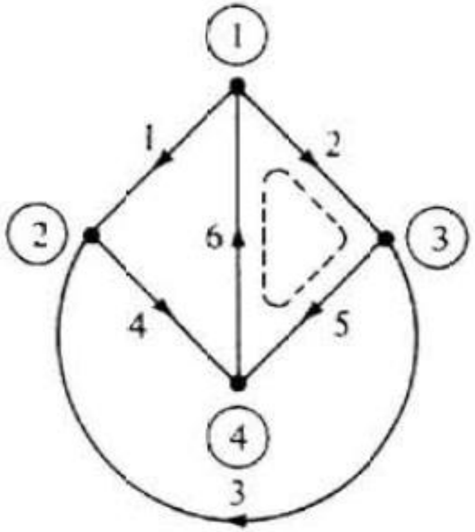
Non-planar graph

Graph Theory—Mesh

- Mesh: a loop consisting of branches encircles nothing in its interior
- Outer mesh: a loop consisting of branches has nothing in its exterior



Graph G

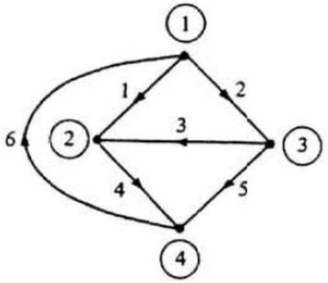


Graph G'

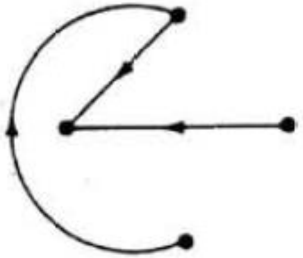
- Graph G': mesh (consisting of branches [2,5,6])
- Graph G: outer mesh (consisting of branches [2,5,6])

Graph Theory—Tree

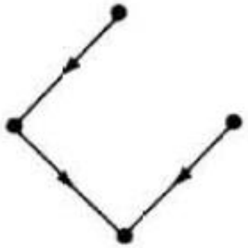
- Tree: one of the most important concept in graph theory
- Tree T of a connected digraph G is a subgraph which satisfy the following fundamental properties:
  - ❑ It is connected (there is a path from any point to any other point)
  - ❑ It contains all the nodes of the digraph
  - ❑ It has no loops
- Consider the following situation:



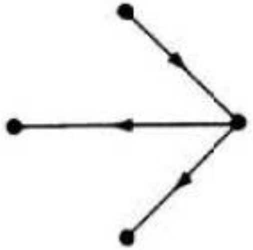
Digraph G



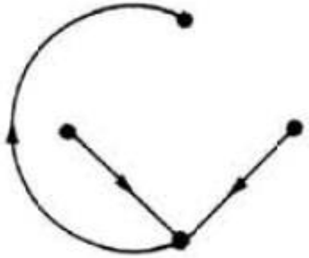
T_1



T_2



T_3



T_4

- T_1, T_2, T_3, and T_4 are four distinct trees of the digraph G

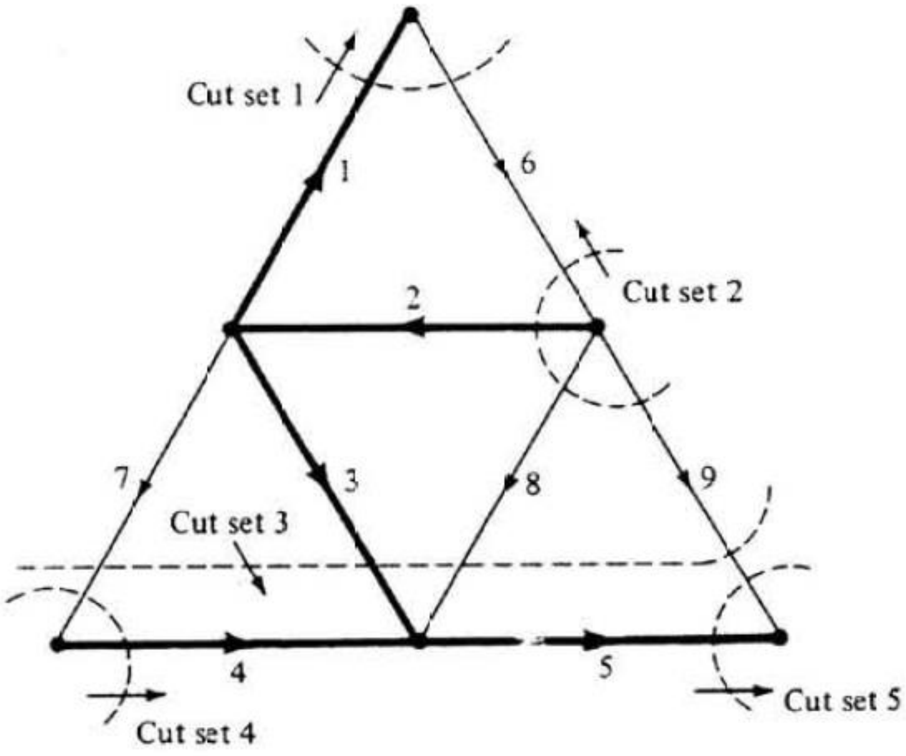
$$\left.\begin{array}{ll} T_1 = \{1,3,6\} & T_3 = \{2,3,5\} \\ T_2 = \{1,4,5\} & T_4 = \{4,5,6\} \end{array}\right\} \text{Each tree consists of three branches}$$

Fundamental Theorem of Graph

- Basic terminology:
  - twig: branches belong to the tree. Also known as tree branches
  - Link: branches that do not belong to the tree. Also known as chord or chord branches
  - Example: $T_1 = \{1,3,6\}$, T_1 has 3 twigs (1,3,6) and 3 links (2,4,5)
- Given a connected digraph G with n nodes, b branches and a tree T;
  - There is a unique path (disregard the branch orientation) along the tree between any pairs of nodes (tree is connected).
  - There are $n-1$ twigs and $l = b-(n-1)$ links.
  - Every twig of T along with some links defines a unique cut set, called the fundamental cut set associated with the twig.
  - Every link of T and the unique path on the tree between its two nodes constitute a unique loop, called the fundamental loop associated with the link.
- Statements mentioned above simply means how to obtain fundamental cut set associated with twig and fundamental loop associated with link from a given digraph.

Fundamental Cut Set Associated with a Tree-KCL

- Consider the following digraph with $b = 9$ and $n = 6$
- First, pick (draw) a tree: solid dark line is a tree
- Second, determine twigs and links
  - twig: $n - 1 = 6 - 1 = 5$ twigs (solid dark line)
  - Link: $l = b - (n - 1) = 9 - 5 = 4$ links (thin solid line)
- Third, determine the fundamental cut set
  - Every twig of tree with some link defines a unique fundamental cut set
- Five fundamental cut sets as:



Cut set 1: $\{1,6\}$

Cut set 2: $\{2,6,8,9\}$

Cut set 3: $\{3,7,8,9\}$

Cut set 4: $\{4,7\}$

Cut set 5: $\{5,9\}$

Recall: Tree

It is connected

It contains all nodes of the digraph

It has no loops

Fundamental Cut Set Associated with a Tree-KCL on Fundamental Cut Set

- KCL equations for five fundamental cut sets:

Cut set 1: $\{1,6\}$ Cut set 2: $\{2,6,8,9\}$ Cut set 3: $\{3,7,8,9\}$ Cut set 4: $\{4,7\}$ Cut set 5: $\{5,9\}$

$$\left.\begin{array}{ll}
\text{Cut set 1:} & i_1 - i_6 = 0 \\
\text{Cut set 2:} & i_2 - i_6 + i_8 + i_9 = 0 \\
\text{Cut set 3:} & i_3 + i_7 + i_8 + i_9 = 0 \\
\text{Cut set 4:} & i_4 - i_7 = 0 \\
\text{Cut set 5:} & i_5 + i_9 = 0
\end{array}\right\}$$

$$\begin{array}{l}
\text{Cut set 1:} \\ \text{Cut set 2:} \\ \text{Cut set 3:} \\ \text{Cut set 4:} \\ \text{Cut set 5:}
\end{array}
\overbrace{\left[\begin{array}{ccccc|cccc}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]}^{Q}
\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ \cdot \\ \cdot \\ \cdot \\ i_9 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$n-1$ twigs: 1_{n-1} (unit matrix) ℓ links: Q_ℓ

$Qi = 0$ where $Q: (\mathrm{n}-1) \times b$
fundamental cut set matrix

$Q = [1_{n-1}, Q_\ell]$

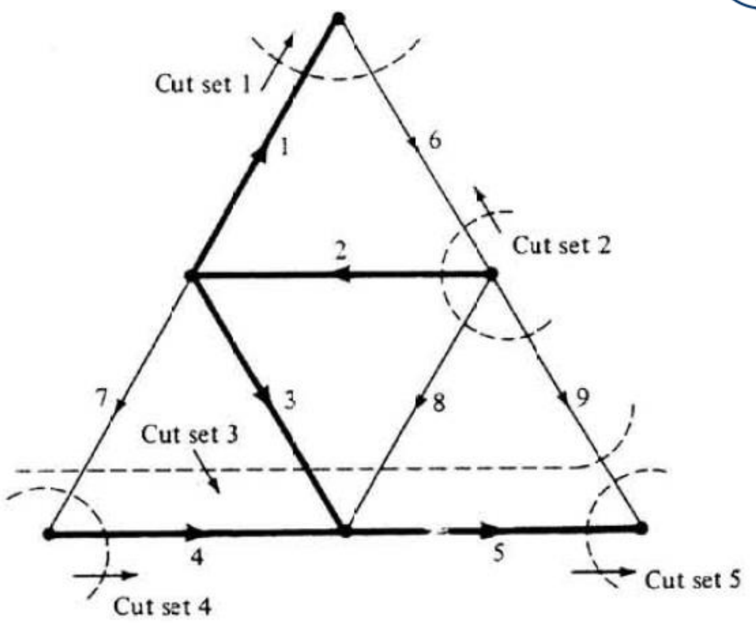
KVL Equations Using Twig Voltages

- Let the twig voltages be:

$v_{t1}, v_{t2}, v_{t3}, v_{t4}, v_{t5}$ (5 twigs).

$v_1 = v_{t1}, v_2 = v_{t2}, v_3 = v_{t3}, v_4 = v_{t4}, v_5 = v_{t5}$

- Four fundamental loops defined by each link
  - Fundamental loop contains only one link and some twigs
  - Apply KVL for fundamental loops defined by four links



$$v_6 + v_2 + v_1 = 0 \Rightarrow v_6 = -v_1 - v_2 = -v_{t1} - v_{t2}$$

$$v_7 + v_4 - v_3 = 0 \Rightarrow v_7 = v_3 - v_4 = v_{t3} - v_{t4}$$

$$v_8 - v_3 - v_2 = 0 \Rightarrow v_8 = v_2 + v_3 = v_{t2} + v_{t3}$$

$$v_9 - v_5 - v_3 - v_2 = 0 \Rightarrow v_9 = v_2 + v_3 + v_5 = v_{t2} + v_{t3} + v_{t5}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ \\ v_9 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}}_{n-1} \begin{bmatrix} v_{t1} \\ v_{t2} \\ v_{t3} \\ v_{t4} \\ v_{t5} \end{bmatrix}$$

- In general, matrix can be written as:

$$v = Q^T v_t$$

v_t: twig voltage vector
Q^T: transpose of fundamental cut-set matrix

Fundamental Loop Matrix Associated with Tree

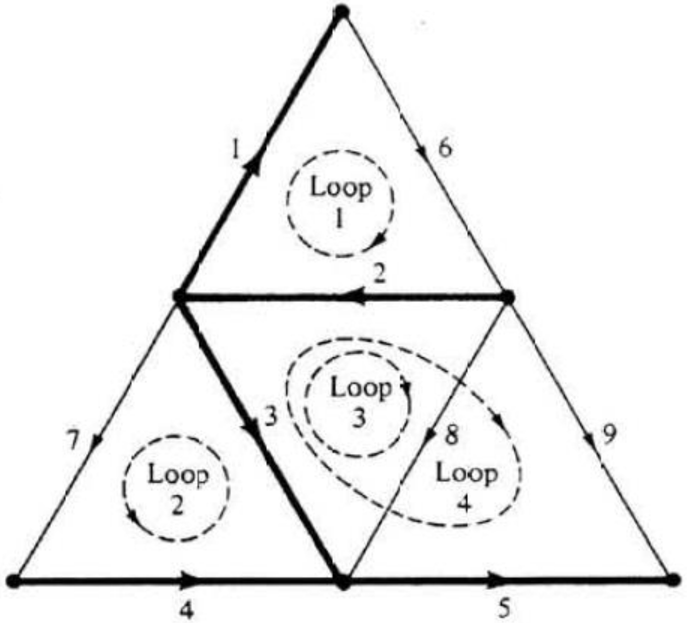
- Determine the KVL equations based on fundamental loops
- Four fundamental loops
- Apply KVL to each fundamental loop:

Loop 1: $v_6 + v_2 + v_1 = 0$

Loop 2: $v_7 + v_4 - v_3 = 0$

Loop 3: $v_8 - v_3 - v_2 = 0$

Loop 4: $v_9 - v_5 - v_3 - v_2 = 0$



Reference direction of each loop is defined by the direction of its associated link

$$\ell \text{ loops} \quad \overbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}}^{B} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow Bv = 0$$

$n-1$ twigs $\qquad$ ℓ links

$$B = [B_t, 1_\ell]$$

1_ℓ: unit matrix

B_t: submatrix of ℓ(loop) rows and n-1 columns

B: $(\ell \times b)$ fundamental loop matrix associated with the tree T