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Solutions of Question 1 in Homework 2

Only Question 1 will be solved and graded! Each part is worth approximately 17 points.

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(1) Prove or disprove each of the following:

(a) There are sets A, B and C satisfying the conditions: $A \cap B \neq \emptyset$ and $A \cap C = \emptyset$ and $(A \cap B) - C = \emptyset$.
FALSE! Suppose for a contradiction that there are such sets A, B, C. As $A \cap B \neq \emptyset$, there is an x such that $x \in A \cap B$. Then $x \in A$ and $x \in B$. As $A \cap C = \emptyset$, it follows that $x \notin C$. But then from $x \in A \cap B$ and $x \notin C$, we see that $x \in (A \cap B) - C$. In particular, $(A \cap B) - C \neq \emptyset$. A contradiction to the third condition.

(b) $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$ for any sets A and B where $\mathcal{P}(\ )$ denotes the power set of its argument.
FALSE! Indeed

$$\mathcal{P}(A - B) \not\subseteq \mathcal{P}(A) - \mathcal{P}(B)$$

for any sets A and B : As the empty set is a subset of any set, $\emptyset \in \mathcal{P}(S)$ for any set S. So

$$\emptyset \in \mathcal{P}(A - B) \quad \text{but} \quad \emptyset \notin \mathcal{P}(A) - \mathcal{P}(B).$$

Therefore, $\mathcal{P}(A - B)$ is not a subset of $\mathcal{P}(A) - \mathcal{P}(B)$.

(Exercise: Show that $\mathcal{P}(A - B) - \{\emptyset\} \subseteq \mathcal{P}(A) - \mathcal{P}(B)$ for any sets A and B.)

(c) For any sets A, B, C, if $A \cup C = B \cup C$ and $A - C = B - C$, then $A = B$.
FALSE! Many counterexamples may be given. For instance, if $A = \emptyset$ and $B = C \neq \emptyset$, then

$$A \cup C = B = B \cup C \quad \text{and} \quad A - C = \emptyset = B - C \quad \text{but} \quad A \neq B.$$

For another counterexample, let $A = \{1, 2, 3\}$, $B = \{1, 2\}, C = \{3\}$.

(d) For any sets A and B, every subset of $A \times B$ is of the form $U \times V$ for some subset U of A and for some subset V of B.
FALSE! Consider, for instance, the sets $A = \{1, 2\}$ and $B = \{a, b\}$ and the subset $\{(1, a), (2, b)\}$ of $A \times B$. Suppose for a contradiction that

$$\{(1, a), (2, b)\} = U \times V$$

for some subset U of A and for some subset V of B. Then, $1 \in U, 2 \in U$ and $a \in V, b \in V$. These imply that $(1, b) \in U \times V$. Hence,

$$(1, b) \in \{(1, a), (2, b)\},$$

which is a contradiction.

(Exercise: Let A and B be sets. Show that every subset of $A \times B$ is of the form $U \times V$ for some subset U of A and for some subset V of B if and only if $|A| \leq 1$ or $|B| \leq 1$.)

(e) For any sets A, B, P, Q, if $A \times B \subseteq P \times Q$ then $A \subseteq P$.
FALSE! Indeed, $B = \emptyset$ and A, P are any sets such that $A \not\subseteq P$ form a counterexample, because $A \times \emptyset = \emptyset$ for any set A and the empty set is a subset of every set.

(Exercise: Show that, for any sets A, B, P, Q, if $A \times B \subseteq P \times Q$ and $B \neq \emptyset$ then $A \subseteq P$.)

(f) $A \cap \mathcal{P}(A) = \emptyset$ for any nonempty set A.
FALSE! For instance, $A = \{\emptyset\}$ forms a counterexample, because

$$\mathcal{P}(A) = \{\emptyset, \{\emptyset\}\} \quad \text{and} \quad A \cap \mathcal{P}(A) = A = \{\emptyset\} \neq \emptyset.$$