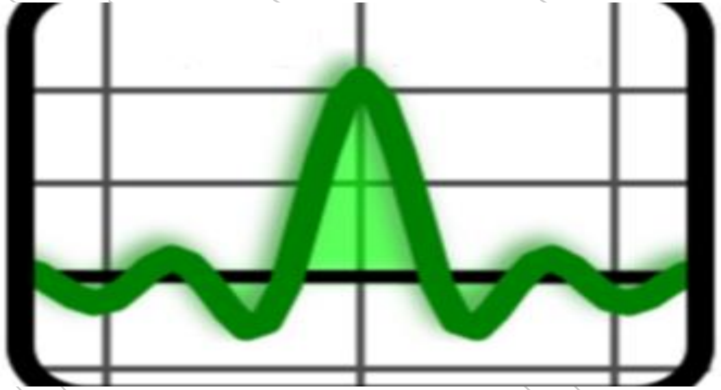

Signals & Systems for Computer Engineering

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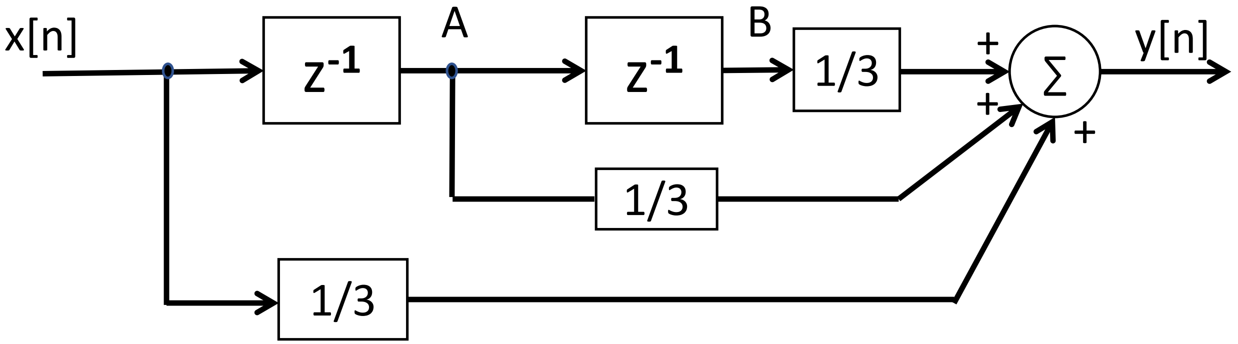
BLG354E / CRN: 21560
3rd Week Lecture

Example: Block diagram of a 2nd order moving average FIR filter is given below.

a) Find its step response

b) Write the transfer function in terms of unit delays (z^{-1})

c) Find the output if the input signal x[n] is discretized from x(t)=10sin(100πt) by samping at 200Hz

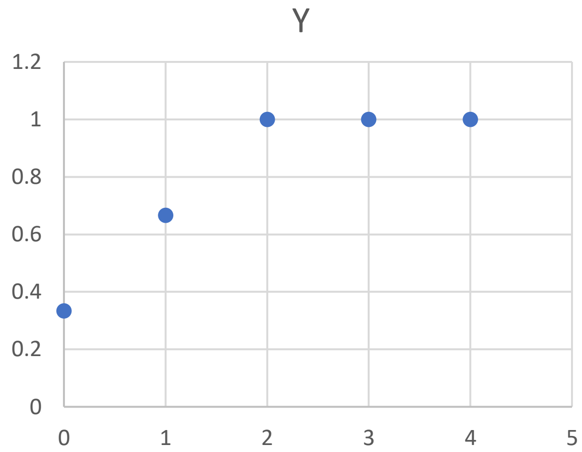


a)

n	X	A	B	Y
0	1	0	0	0.3333
1	1	1	0	0.6667
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

Y=B/3+A/3+X/3

$z^{-1} \rightarrow$ B=A , A=X



b) y[n]=(x[n]+x[n-1]+x[n-2])/3

y[n]=(x[n]+z^{-1}x[n]+z-^{2}x[n])/3

$$T(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1} + z^{-2}}{3}$$

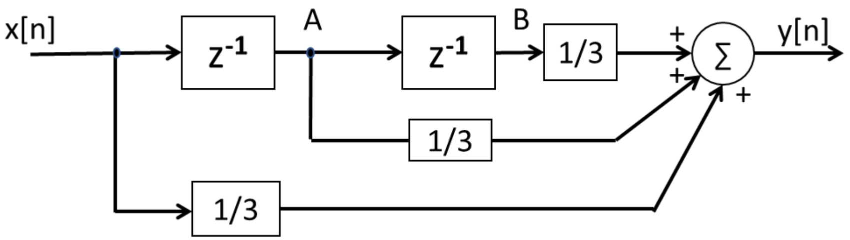
$$T(z) = \frac{z^2 + z + 1}{3z^2}$$

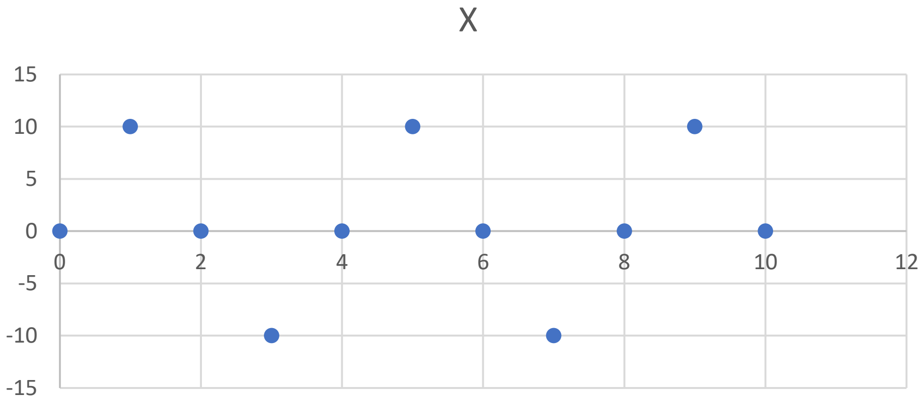
Why is that called FIR Filter ?

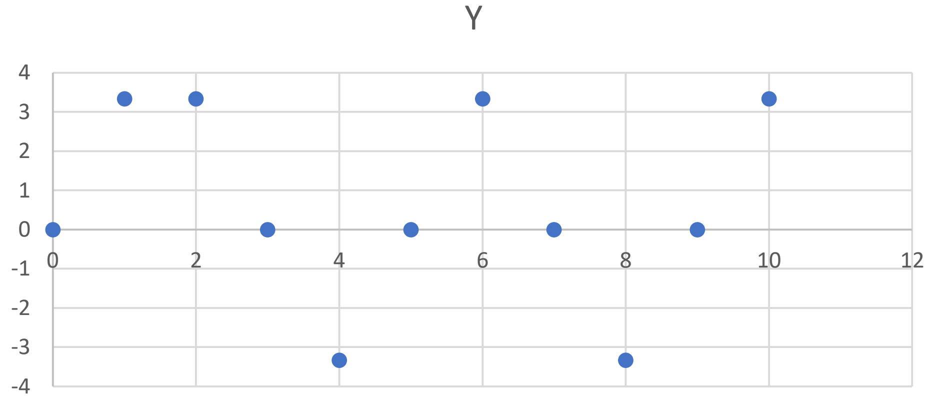
What should be the ideal coefficients for a desired frequency response ?

c) $T_s = 1/200$ $t = n\,T_s = n/200$, $n = 0,1,2,\ldots$

$$x[n] = 10\sin\left(\frac{100\pi}{200}n\right) = 10\sin\left(\frac{\pi}{2}n\right)$$







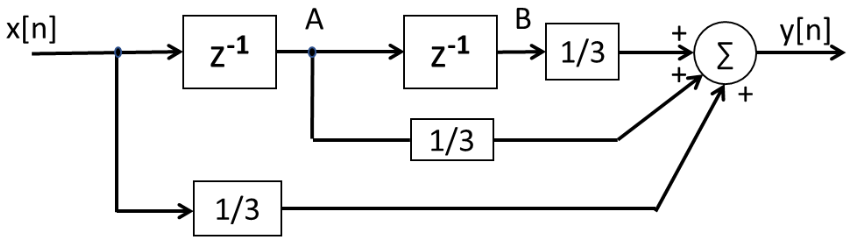
n	X	A	B	Y
0	0.000	0	0	0
1	10.000	0.000	0.000	3.333
2	0.000	10.000	0.000	3.333
3	-10.000	0.000	10.000	0.000
4	0.000	-10.000	0.000	-3.333
5	10.000	0.000	-10.000	0.000
6	0.000	10.000	0.000	3.333
7	-10.000	0.000	10.000	0.000
8	0.000	-10.000	0.000	-3.333
9	10.000	0.000	-10.000	0.000
10	0.000	10.000	0.000	3.333

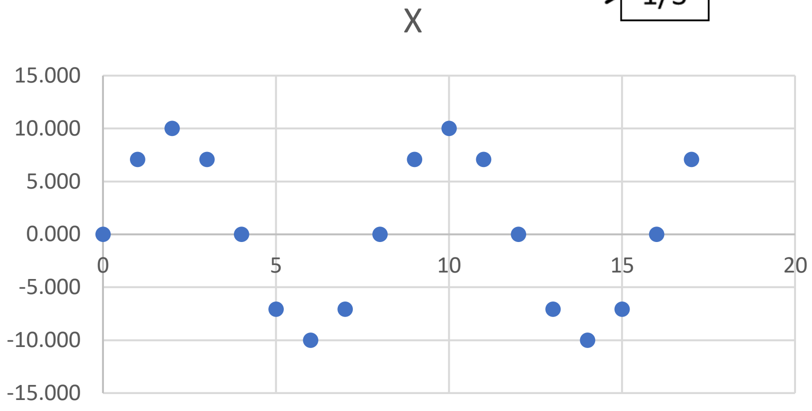
Y=B/3+A/3+X/3

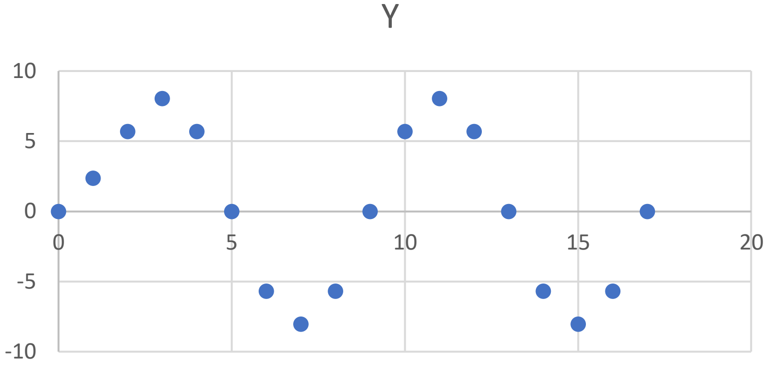
$z^{-1} \rightarrow$ B=A , A=X

$$x[n] = 10\sin(\frac{\pi}{4}n)$$

n	X	A	B	Y
0	0.000	0	0	0
1	7.071	0.000	0.000	2.357
2	10.000	7.071	0.000	5.690
3	7.071	10.000	7.071	8.047
4	0.000	7.071	10.000	5.690
5	-7.071	0.000	7.071	0.000
6	-10.000	-7.071	0.000	-5.690
7	-7.071	-10.000	-7.071	-8.047
8	0.000	-7.071	-10.000	-5.690
9	7.071	0.000	-7.071	0.000
10	10.000	7.071	0.000	5.690
11	7.071	10.000	7.071	8.047
12	0.000	7.071	10.000	5.690
13	-7.071	0.000	7.071	0.000
14	-10.000	-7.071	0.000	-5.690
15	-7.071	-10.000	-7.071	-8.047
16	0.000	-7.071	-10.000	-5.690
17	7.071	0.000	-7.071	0.000

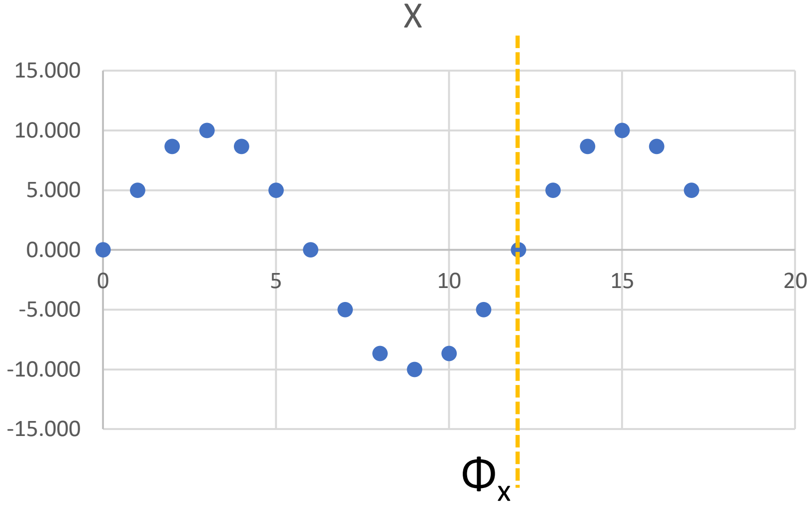


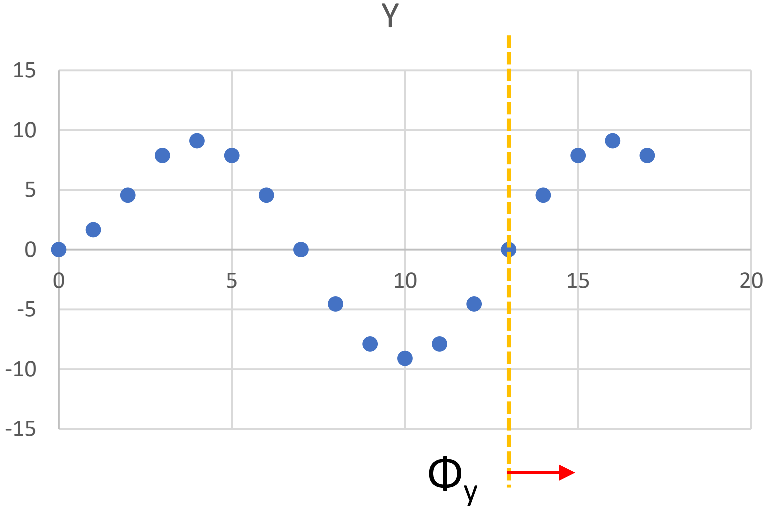




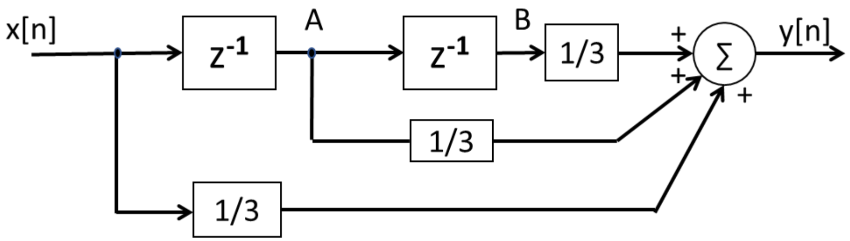
$$x[n] = 10\sin(\frac{\pi}{6}n)$$

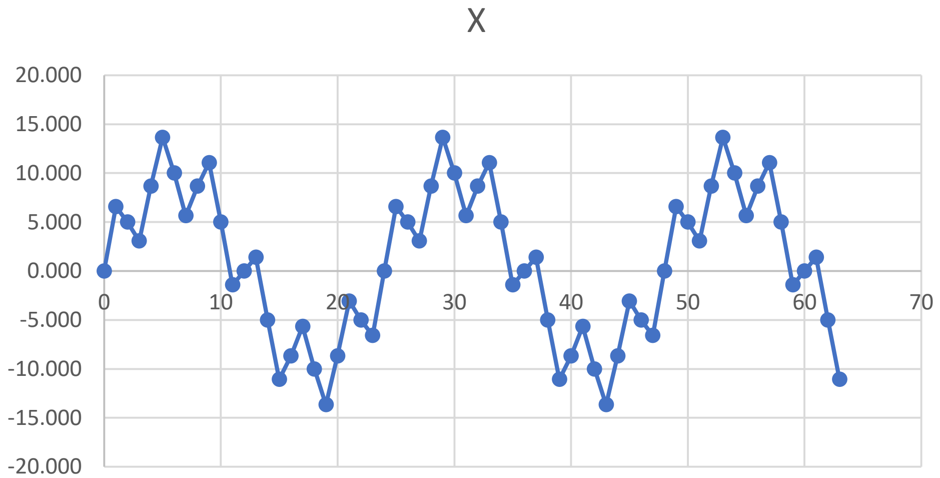
n	X	A	B	Y
0	0.000	0	0	0
1	5.000	0.000	0.000	1.667
2	8.660	5.000	0.000	4.553
3	10.000	8.660	5.000	7.887
4	8.660	10.000	8.660	9.107
5	5.000	8.660	10.000	7.887
6	0.000	5.000	8.660	4.553
7	-5.000	0.000	5.000	0.000
8	-8.660	-5.000	0.000	-4.553
9	-10.000	-8.660	-5.000	-7.887
10	-8.660	-10.000	-8.660	-9.107
11	-5.000	-8.660	-10.000	-7.887
12	0.000	-5.000	-8.660	-4.553
13	5.000	0.000	-5.000	0.000
14	8.660	5.000	0.000	4.553
15	10.000	8.660	5.000	7.887
16	8.660	10.000	8.660	9.107
17	5.000	8.660	10.000	7.887





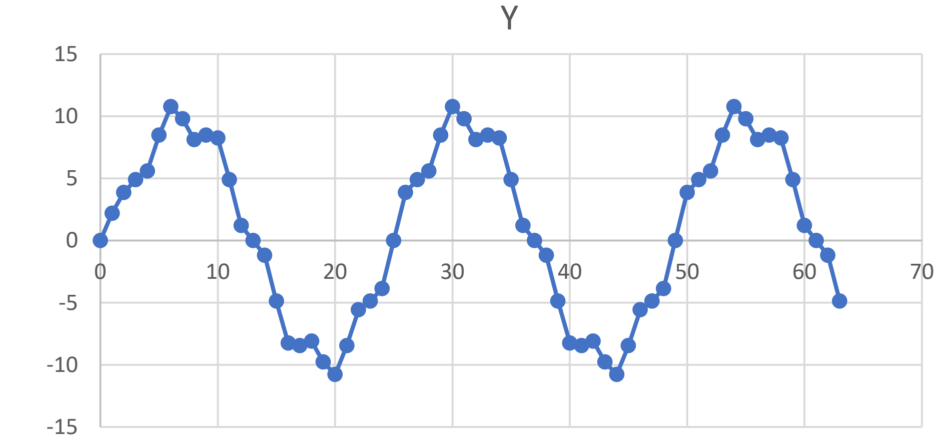
$$\Delta\Phi(\omega) = \Phi_x - \Phi_y$$





Input signal x[n]

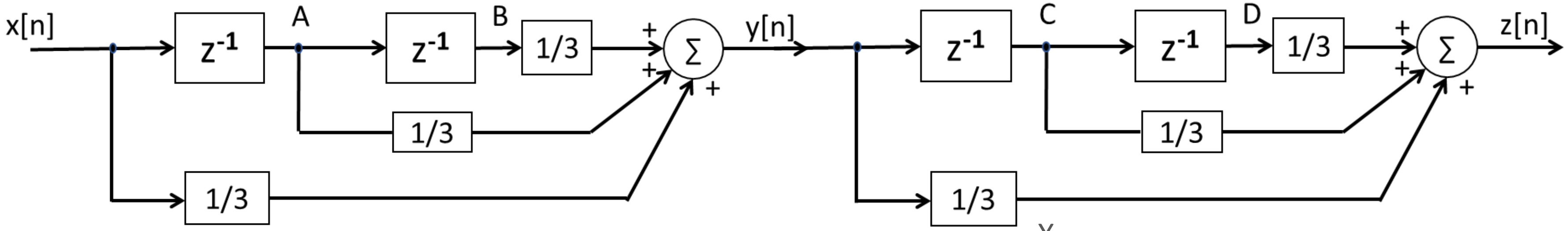
$$x[n] = 10\sin\left(\frac{\pi}{12}n\right) + 4\sin\left(\frac{\pi}{12}n\right)$$



Output signal y[n]

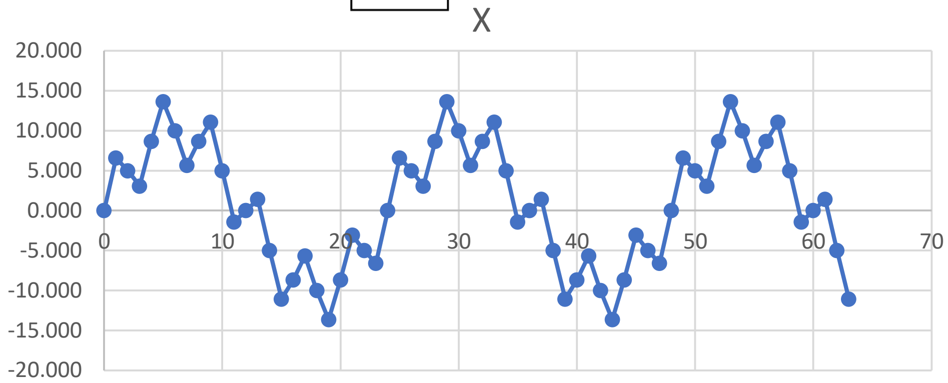
n	X	A	B	Y
0	0.000	0	0	0
1	6.588	0.000	0.000	2.196
2	5.000	6.588	0.000	3.863
3	3.071	5.000	6.588	4.886
4	8.660	3.071	5.000	5.577
5	13.659	8.660	3.071	8.464
6	10.000	13.659	8.660	10.773
7	5.659	10.000	13.659	9.773
8	8.660	5.659	10.000	8.107
9	11.071	8.660	5.659	8.464
10	5.000	11.071	8.660	8.244
11	-1.412	5.000	11.071	4.886
12	0.000	-1.412	5.000	1.196
13	1.412	0.000	-1.412	0.000
14	-5.000	1.412	0.000	-1.196
15	-11.071	-5.000	1.412	-4.886
16	-8.660	-11.071	-5.000	-8.244
17	-5.659	-8.660	-11.071	-8.464
18	-10.000	-5.659	-8.660	-8.106
19	-13.659	-10.000	-5.659	-9.773
20	-8.660	-13.659	-10.000	-10.773
21	-3.071	-8.660	-13.659	-8.464
22	-5.000	-3.071	-8.660	-5.577
23	-6.588	-5.000	-3.071	-4.886
24	0.000	-6.588	-5.000	-3.863
25	6.588	0.000	-6.588	0.000
26	5.000	6.588	0.000	3.863
27	3.071	5.000	6.588	4.886
28	8.660	3.071	5.000	5.577
29	13.659	8.660	3.071	8.463
30	10.000	13.659	8.660	10.773
31	5.659	10.000	13.659	9.773
32	8.660	5.659	10.000	8.107
33	11.071	8.660	5.659	8.464
34	5.000	11.071	8.660	8.244
35	-1.412	5.000	11.071	4.887
36	0.000	-1.412	5.000	1.196
37	1.412	0.000	-1.412	0.000
38	-5.000	1.412	0.000	-1.196
39	-11.071	-5.000	1.412	-4.886
40	-8.660	-11.071	-5.000	-8.244
41	-5.659	-8.660	-11.071	-8.464
42	-10.000	-5.659	-8.660	-8.106
43	-13.659	-10.000	-5.659	-9.773
44	-8.661	-13.659	-10.000	-10.773
45	-3.071	-8.661	-13.659	-8.464
46	-5.000	-3.071	-8.661	-5.577
47	-6.588	-5.000	-3.071	-4.886
48	0.000	-6.588	-5.000	-3.863
49	6.588	0.000	-6.588	0.000
50	5.000	6.588	0.000	3.863
51	3.071	5.000	6.588	4.886
52	8.660	3.071	5.000	5.577
53	13.659	8.660	3.071	8.463
54	10.000	13.659	8.660	10.773
55	5.659	10.000	13.659	9.773
56	8.660	5.659	10.000	8.107
57	11.071	8.660	5.659	8.463
58	5.000	11.071	8.660	8.244
59	-1.412	5.000	11.071	4.887
60	0.000	-1.412	5.000	1.196
61	1.412	0.000	-1.412	0.000
62	-5.000	1.412	0.000	-1.196
63	-11.071	-5.000	1.412	-4.886

n	X	A	B	Y	C	D	Z
0	0.000	0	0	0	0	0	0
1	6.588	0.000	0.000	2.196	0.000	0.000	0.732
2	5.000	6.588	0.000	3.863	2.196	0.000	2.020
3	3.071	5.000	6.588	4.886	3.863	2.196	3.648
4	8.660	3.071	5.000	5.577	4.886	3.863	4.775
5	13.659	8.660	3.071	8.464	5.577	4.886	6.309
6	10.000	13.659	8.660	10.773	8.464	5.577	8.271
7	5.659	10.000	13.659	9.773	10.773	8.464	9.670
8	8.660	5.659	10.000	8.107	9.773	10.773	9.551
9	11.071	8.660	5.659	8.464	8.107	9.773	8.781
10	5.000	11.071	8.660	8.244	8.464	8.107	8.271
11	-1.412	5.000	11.071	4.886	8.244	8.464	7.198
12	0.000	-1.412	5.000	1.196	4.886	8.244	4.775
13	1.412	0.000	-1.412	0.000	1.196	4.886	2.028
14	-5.000	1.412	0.000	-1.196	0.000	1.196	0.000
15	-11.071	-5.000	1.412	-4.886	-1.196	0.000	-2.027
16	-8.660	-11.071	-5.000	-8.244	-4.886	-1.196	-4.775
17	-5.659	-8.660	-11.071	-8.464	-8.244	-4.886	-7.198
18	-10.000	-5.659	-8.660	-8.106	-8.464	-8.244	-8.271
19	-13.659	-10.000	-5.659	-9.773	-8.106	-8.464	-8.781
20	-8.660	-13.659	-10.000	-10.773	-9.773	-8.106	-9.551
21	-3.071	-8.660	-13.659	-8.464	-10.773	-9.773	-9.670
22	-5.000	-3.071	-8.660	-5.577	-8.464	-10.773	-8.271
23	-6.588	-5.000	-3.071	-4.886	-5.577	-8.464	-6.309
24	0.000	-6.588	-5.000	-3.863	-4.886	-5.577	-4.775
25	6.588	0.000	-6.588	0.000	-3.863	-4.886	-2.916
26	5.000	6.588	0.000	3.863	0.000	-3.863	0.000
27	3.071	5.000	6.588	4.886	3.863	0.000	2.916
28	8.660	3.071	5.000	5.577	4.886	3.863	4.775
29	13.659	8.660	3.071	8.463	5.577	4.886	6.309
30	10.000	13.659	8.660	10.773	8.463	5.577	8.271
31	5.659	10.000	13.659	9.773	10.773	8.463	9.670
32	8.660	5.659	10.000	8.107	9.773	10.773	9.551
33	11.071	8.660	5.659	8.464	8.107	9.773	8.781
34	5.000	11.071	8.660	8.244	8.464	8.107	8.271
35	-1.412	5.000	11.071	4.887	8.244	8.464	7.198
36	0.000	-1.412	5.000	1.196	4.887	8.244	4.776
37	1.412	0.000	-1.412	0.000	1.196	4.887	2.028
38	-5.000	1.412	0.000	-1.196	0.000	1.196	0.000
39	-11.071	-5.000	1.412	-4.886	-1.196	0.000	-2.027
40	-8.660	-11.071	-5.000	-8.244	-4.886	-1.196	-4.775
41	-5.659	-8.660	-11.071	-8.464	-8.244	-4.886	-7.198
42	-10.000	-5.659	-8.660	-8.106	-8.464	-8.244	-8.271
43	-13.659	-10.000	-5.659	-9.773	-8.106	-8.464	-8.781
44	-8.661	-13.659	-10.000	-10.773	-9.773	-8.106	-9.551
45	-3.071	-8.661	-13.659	-8.464	-10.773	-9.773	-9.670
46	-5.000	-3.071	-8.661	-5.577	-8.464	-10.773	-8.271
47	-6.588	-5.000	-3.071	-4.886	-5.577	-8.464	-6.309
48	0.000	-6.588	-5.000	-3.863	-4.886	-5.577	-4.775
49	6.588	0.000	-6.588	0.000	-3.863	-4.886	-2.916
50	5.000	6.588	0.000	3.863	0.000	-3.863	0.000
51	3.071	5.000	6.588	4.886	3.863	0.000	2.916
52	8.660	3.071	5.000	5.577	4.886	3.863	4.775
53	13.659	8.660	3.071	8.463	5.577	4.886	6.309
54	10.000	13.659	8.660	10.773	8.463	5.577	8.271
55	5.659	10.000	13.659	9.773	10.773	8.463	9.670
56	8.660	5.659	10.000	8.107	9.773	10.773	9.551
57	11.071	8.660	5.659	8.463	8.107	9.773	8.781
58	5.000	11.071	8.660	8.244	8.463	8.107	8.271
59	-1.412	5.000	11.071	4.887	8.244	8.463	7.198
60	0.000	-1.412	5.000	1.196	4.887	8.244	4.776
61	1.412	0.000	-1.412	0.000	1.196	4.887	2.028
62	-5.000	1.412	0.000	-1.196	0.000	1.196	0.000
63	-11.071	-5.000	1.412	-4.886	-1.196	0.000	-2.027

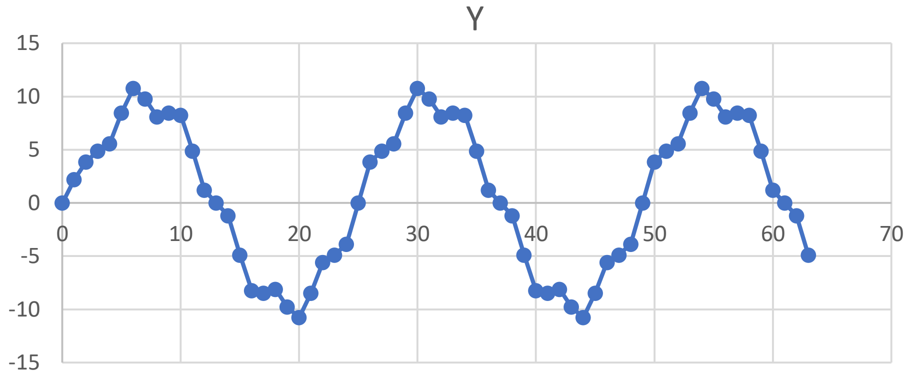


Input signal:

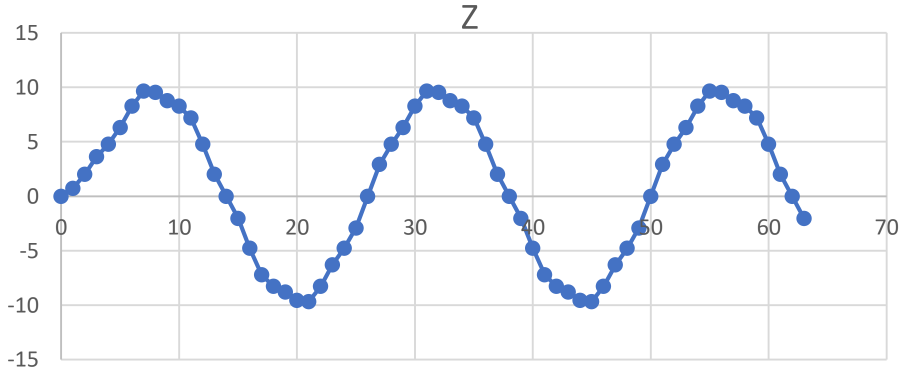
$$x[n] = 10\sin\left(\frac{\pi}{12}n\right) + 4\sin\left(\frac{\pi}{12}n\right)$$



Output of the first stage MOV FIR:



Output of the second stage MOV FIR:



Questions:

How is the dependency between phase response of the system and the applied signal properties ?

How is the dependency between frequency response of the system and the applied signal properties ?

How can we determine the system transfer function which performs the desired frequency response ?

How can we define the discrete time system that performs the equivalent transfer function designed for CT signals ?

How can we implement software that performs the desired frequency response on an embedded digital device?

Answers: Chapter 7+

Linear Time-Invariant (LTI) Systems

Linearity and time-invariance are two most important properties in classification of the systems.

Input-output relationship of an LTI systems can be described in terms of convolution operation.

Impulse Response:

Response of the continuous-time LTI system when the impulse δ(t) signal is applied to its input, called the impulse response and denoted by "h(t)":

$$h(t) = \mathbf{T}\{\delta(t)\}$$

Both the system and integration operator are linear

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$y(t) = \mathbf{T}\{x(t)\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\} = \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau$$

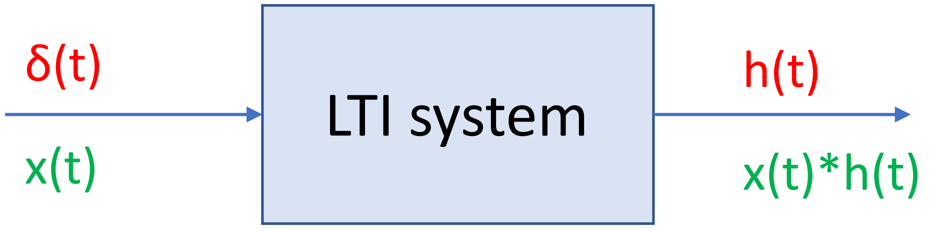
The system is time invariant → $h(t-\tau) = \mathbf{T}\{\delta(t-\tau)\}$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

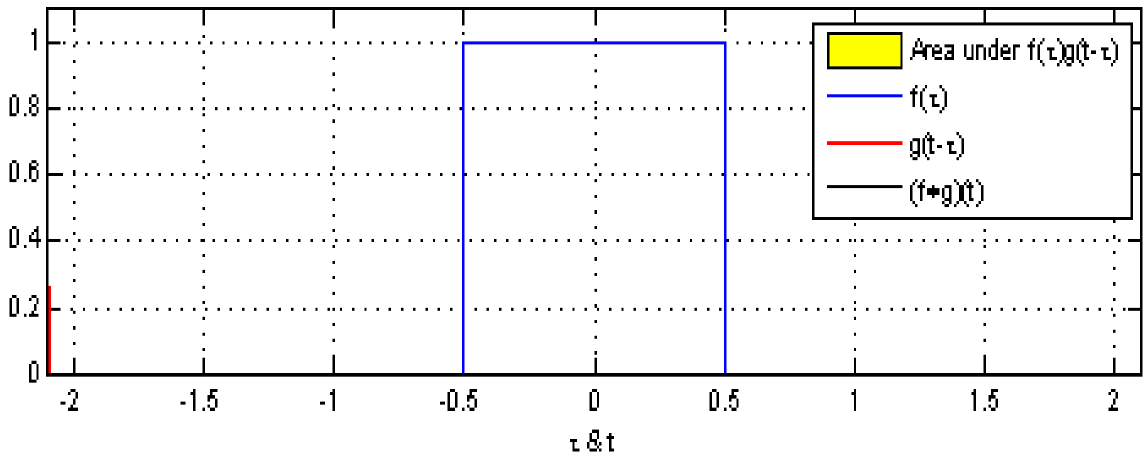
Continuous-time LTI systems can be characterized by their impulse response h(t)

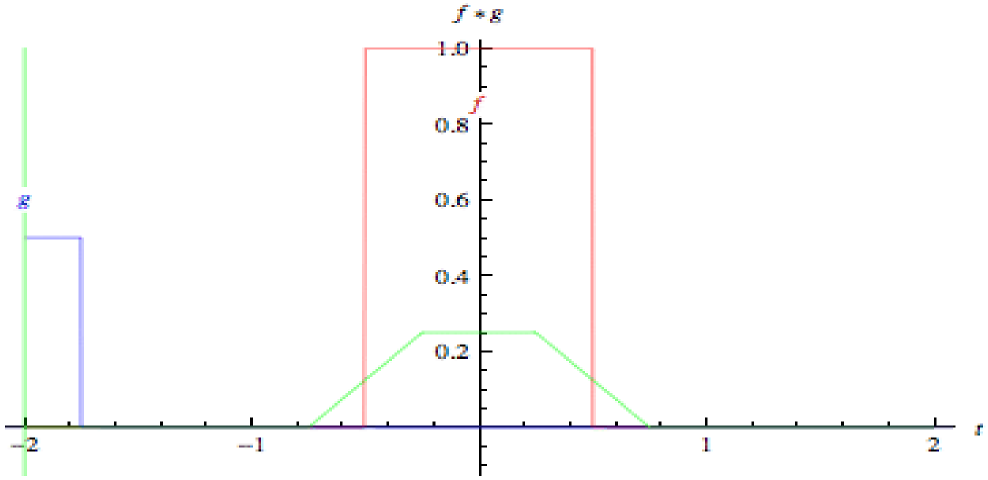
Convolution Integral:

Convolution of two continuous-time signals x(t) and h(t) is stated as $y(t) = x(t) * h(t) = \underbrace{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}_{\text{Convolution integral}}$



<u>If the impulse response of an LTI system is known then its output can be found for any other input signal through the convolution</u>





Convolution Algorithm:

By applying the commutative property of convolution to $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

we get → $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$

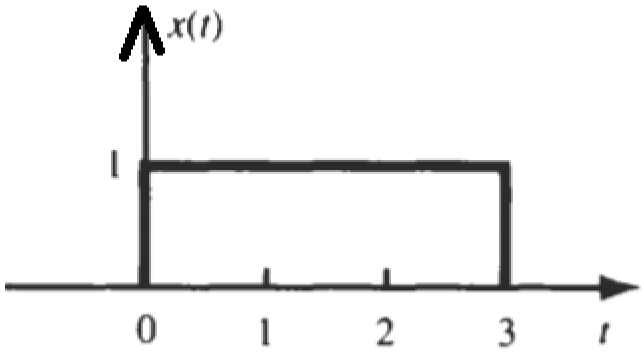
Step 1: Impulse response h(τ) is reflected about the origin (time-reversed) to obtain h(-τ) and then shifted by t to form h(t-τ) = h[-(τ-t)] which is a function of T.

Step 2: h(t-τ) and the signal x(τ) and are multiplied together for all values of τ with fixed t.

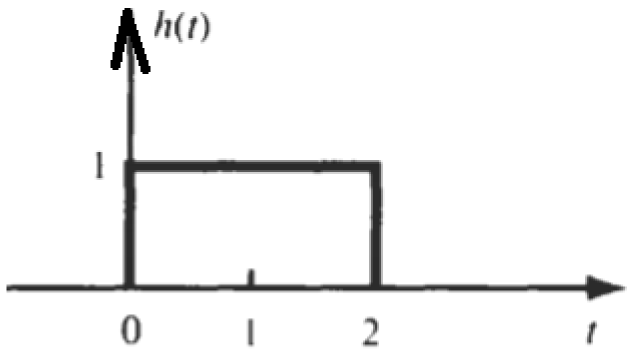
Step 3: "x(τ)h(t-τ)" is integrated over all T and that yields a single output value y(t)

Step 4: Steps 1 to 3 are repeated as t varies over -∞ to +∞ to calculate the total convolution as y(t).

Example: Find y(t)=x(t)*h(t) for x(t) and h(t) shown below



x(t)=u(t)-u(t-3)



h(t)=u(t)-u(t-2)

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau=\int_{-\infty}^{\infty}[u(\tau)-u(\tau-3)][u(t-\tau)-u(t-\tau-2)]\,d\tau$$

$$=\int_{-\infty}^{\infty}u(\tau)u(t-\tau)\,d\tau-\int_{-\infty}^{\infty}u(\tau)u(t-2-\tau)\,d\tau-\int_{-\infty}^{\infty}u(\tau-3)u(t-\tau)\,d\tau+\int_{-\infty}^{\infty}u(\tau-3)u(t-2-\tau)\,d\tau$$

$$u(\tau)u(t-\tau)=\begin{cases}1 & 0<\tau<t,\ t>0\\ 0 & \text{otherwise}\end{cases}$$

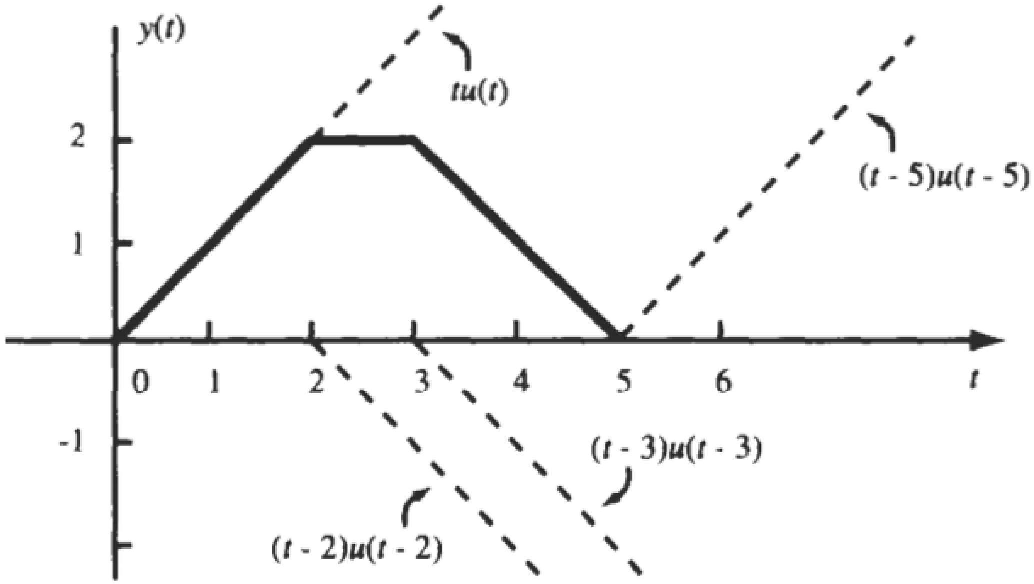
$$u(\tau-3)u(t-\tau)=\begin{cases}1 & 3<\tau<t,\ t>3\\ 0 & \text{otherwise}\end{cases}$$

$$u(\tau)u(t-2-\tau)=\begin{cases}1 & 0<\tau<t-2,\ t>2\\ 0 & \text{otherwise}\end{cases}$$

$$u(\tau-3)u(t-2-\tau)=\begin{cases}1 & 3<\tau<t-2,\ t>5\\ 0 & \text{otherwise}\end{cases}$$

$$y(t) = \left(\int_0^t d\tau\right)u(t) - \left(\int_0^{t-2} d\tau\right)u(t-2) - \left(\int_3^t d\tau\right)u(t-3) + \left(\int_3^{t-2} d\tau\right)u(t-5)$$

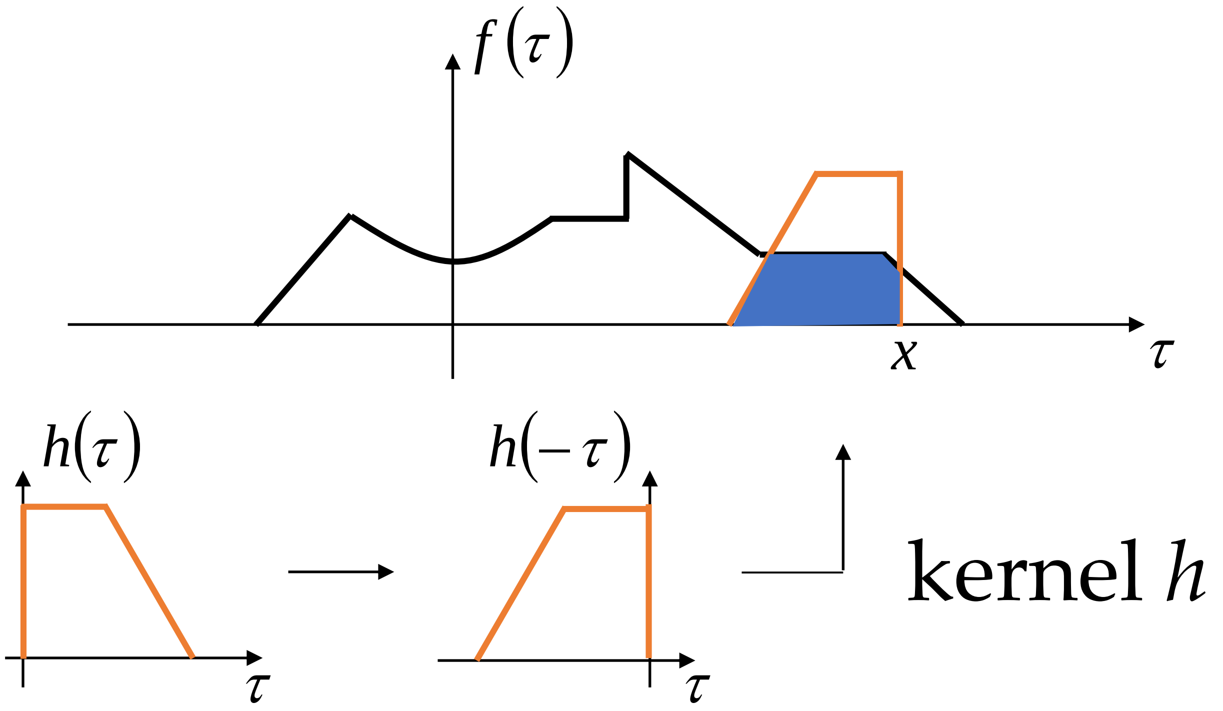
$$= tu(t) - (t-2)u(t-2) - (t-3)u(t-3) + (t-5)u(t-5)$$



Kernel in convolution

Convolution is linear and shift invariant

$$g(x) = \int_{-\infty}^{\infty} f(\tau)h(x-\tau)d\tau \qquad g = f * h$$

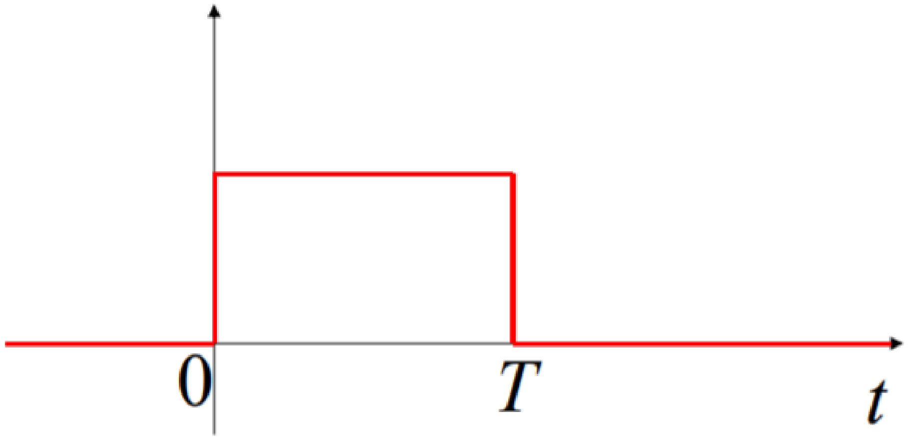


Output has a duration longer than the input indicates that convolution often acts like a low pass filter and smooths the signal.

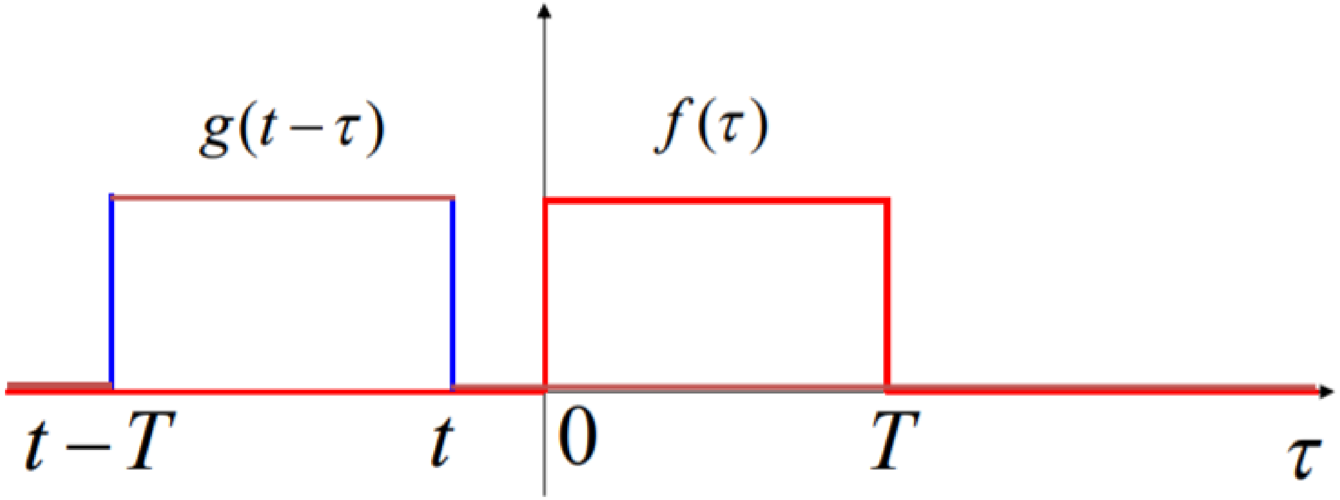
Graphical Convolution:

Suppose that f(t) = g(t) where f(t) is the rectangular pulse depicted in figure, of height 1.

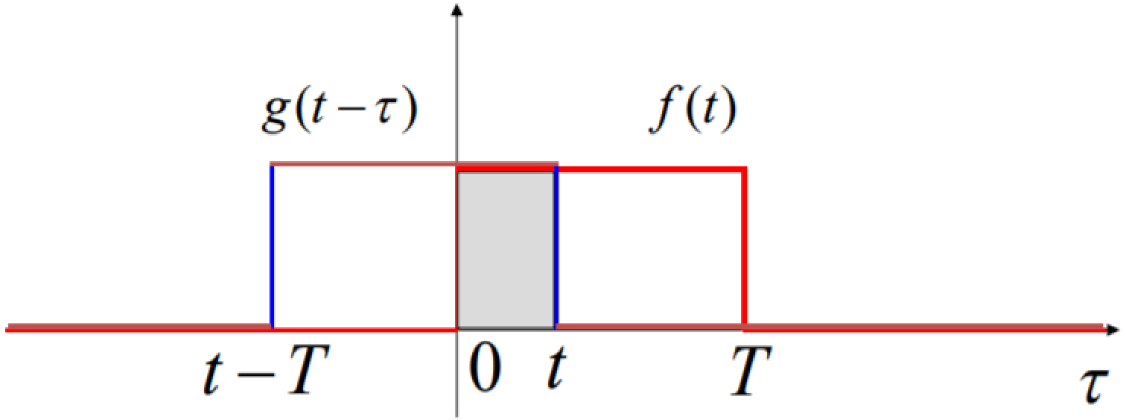
$$f(t) = g(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$



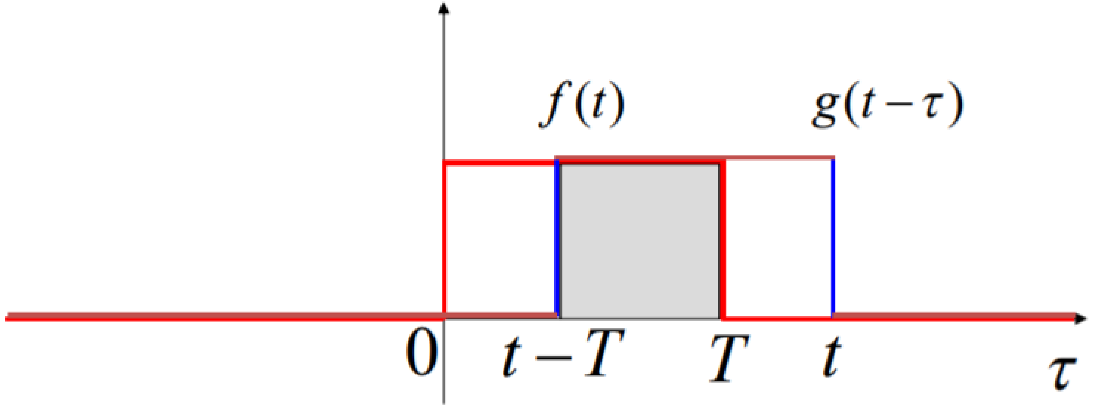
For t<0 :



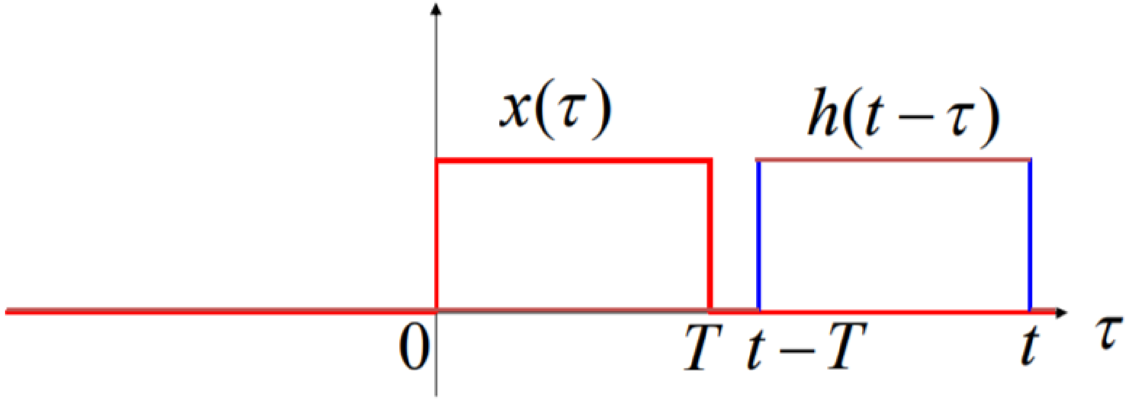
For 0≤t<T :



For T≤t≤ 2T :



For 2T<t :

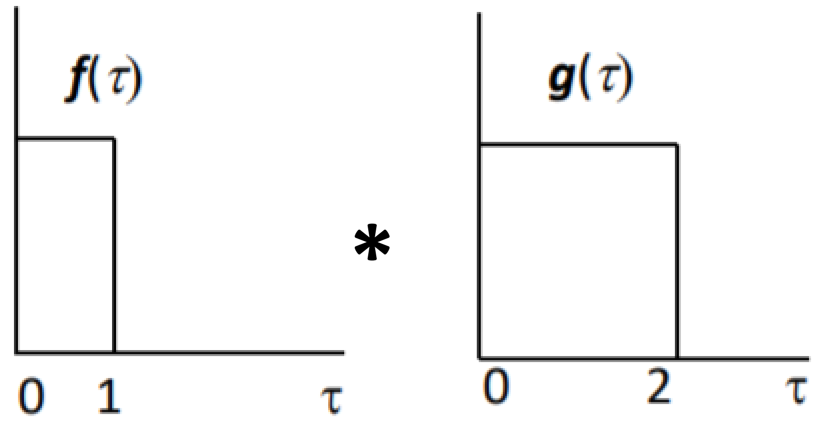


$$\int_0^t 1.1 d\tau = t$$

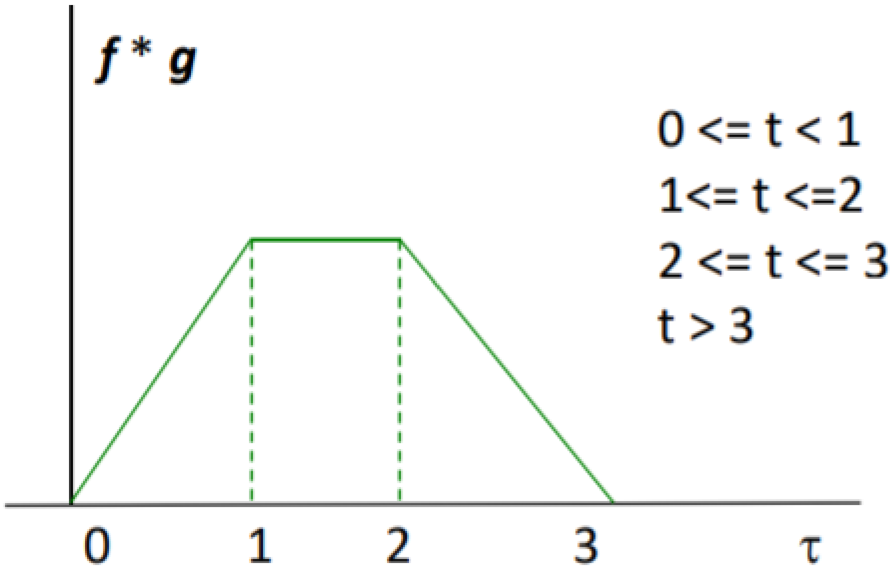
$$\int_{t-T}^{T} 1.1 d\tau = 2T - t$$

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < T \\ 2T - t, & T \le t \le 2T \\ 0, & t > 2T \end{cases}$$

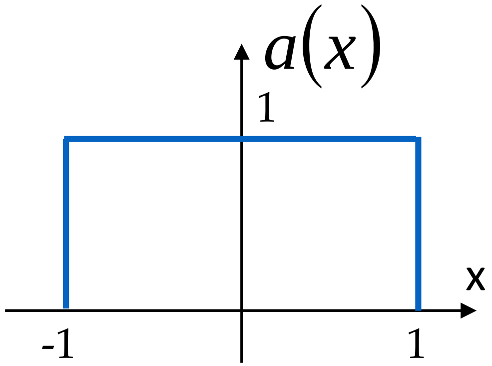
Example:

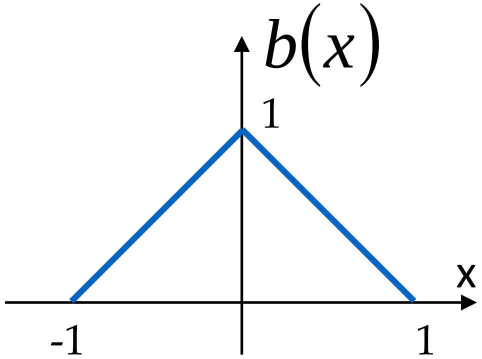


$$y = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

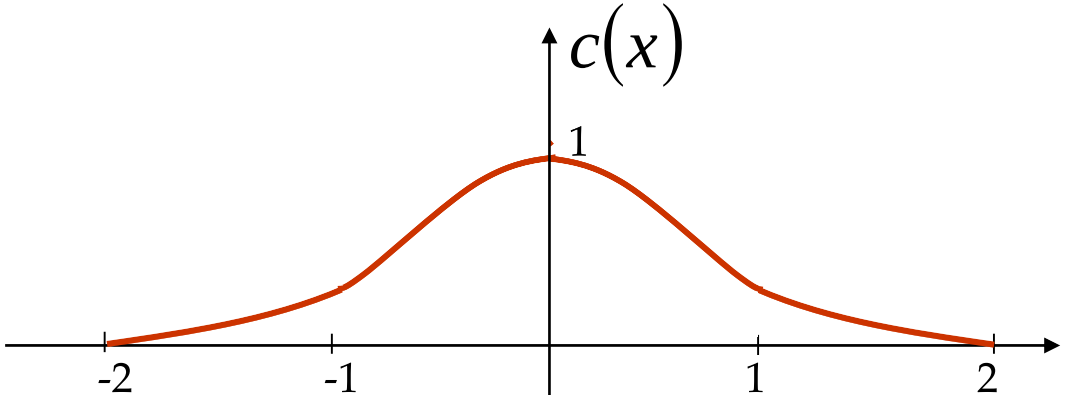


Example: Find the convolution between a(x) and b(x)



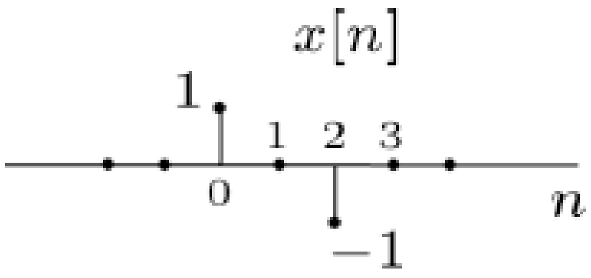


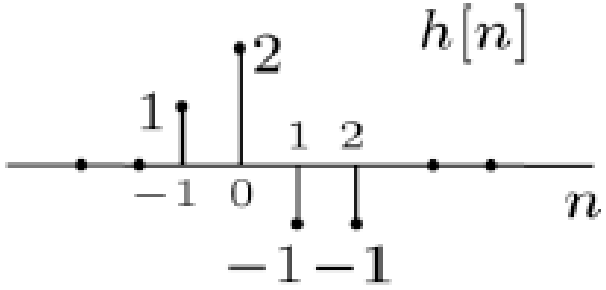
$$c = a * b$$



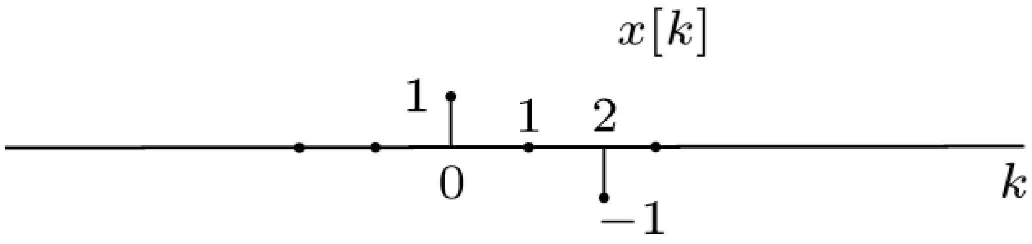
Example:

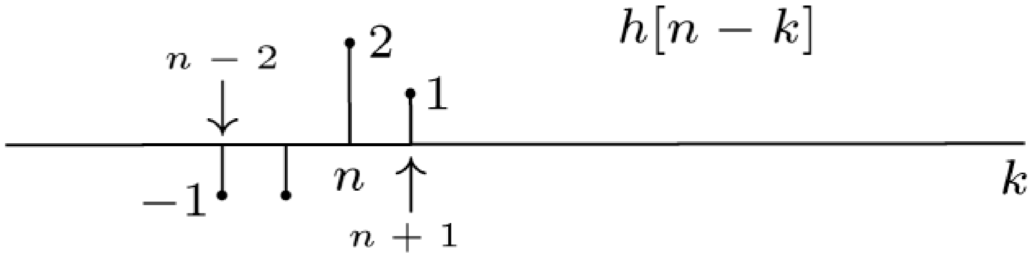
Find convolution of x[n] and h[n] by graphical shifting the h[n] over x[n]





$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

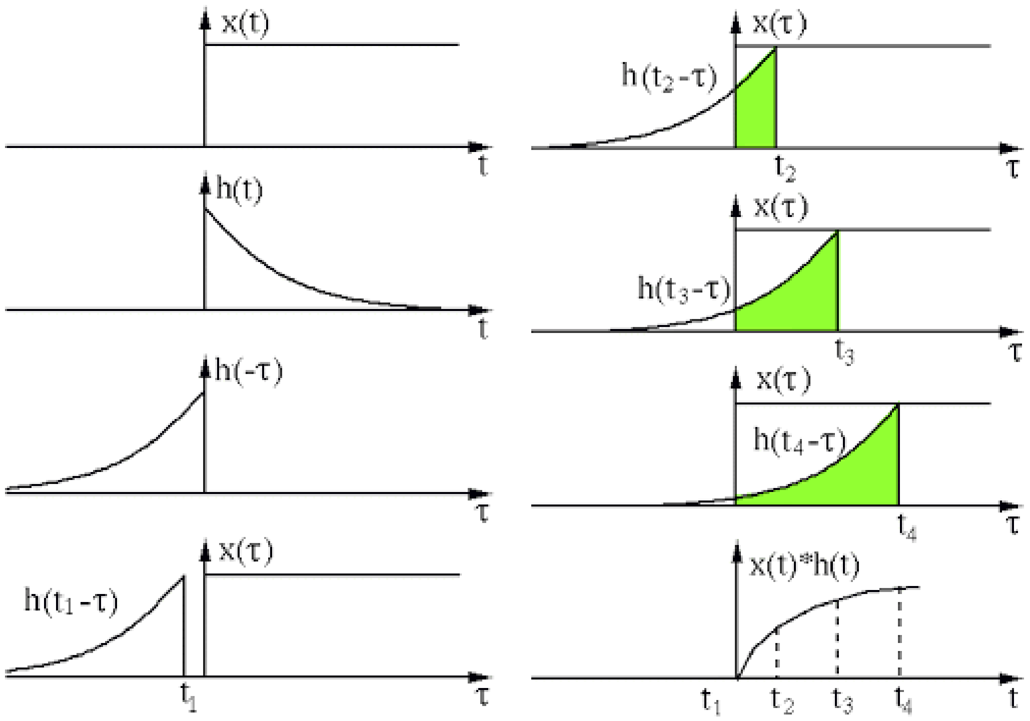




Example:

Impulse response h(t) of a continuous time LTI system is given by $h(t) = e^{-\alpha t}u(t)$ α>

Find output of the system y(t) for x(t)=u(t)



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$y(t) = \int_0^t e^{-\alpha\tau}\,d\tau = \frac{1}{\alpha}(1 - e^{-\alpha t})$$

$$y(t) = \frac{1}{\alpha}(1 - e^{-\alpha t})u(t)$$

Ref: http://fourier.eng.hmc.edu/e161/lectures/convolution/index.html

Convolution properties:

Commutativity: $x[n] * h[n] = h[n] * x[n]$

Associativity: $\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$

Distributivity: $x[n] * \{h_1[n]\} + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$

Time-Frequency Domain Transformation: $x_1(t)*x_2(t) \leftrightarrow X_1(\omega)\cdot X_2(\omega)$ This will be proven after introduction of the Fourier Transform

Proof: Prove the commutativity property of convolution for DT signals

$$x[n] * h[n] = h[n] * x[n]$$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Variable change: $n - k = m$

$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n]$$

Implications of the Convolution Properties

- **Commutative:**

$$x[n]*h[n]=h[n]*x[n]$$

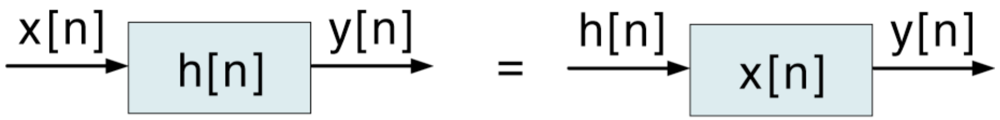
- **Distributive:**

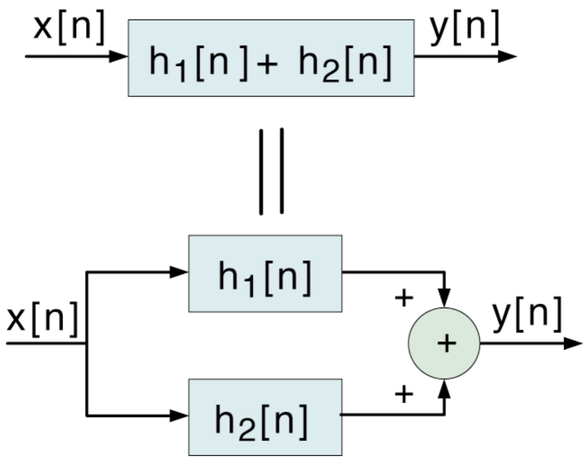
$$x[n]*(h_1[n]+h_2[n]) =$$
$$(x[n]*h_1[n])+(x[n]*h_2[n])$$

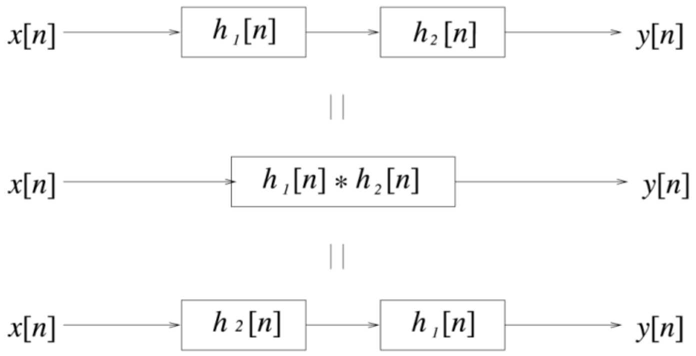
- **Associative:**

$$x[n]*h_1[n]*h_2[n]=$$
$$(x[n]*h_1[n])*h_2[n]=$$
$$(x[n]*h_2[n])*h_1[n]$$

- **Implications:**

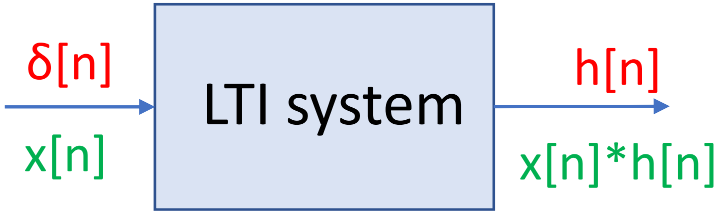






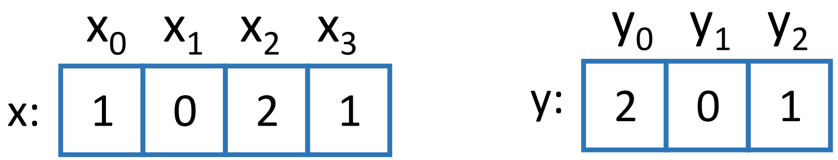
Convolution of discrete time signals:

Output of any discrete-time LTI system is the convolution of the input x[n] with the impulse response h[n] of the system



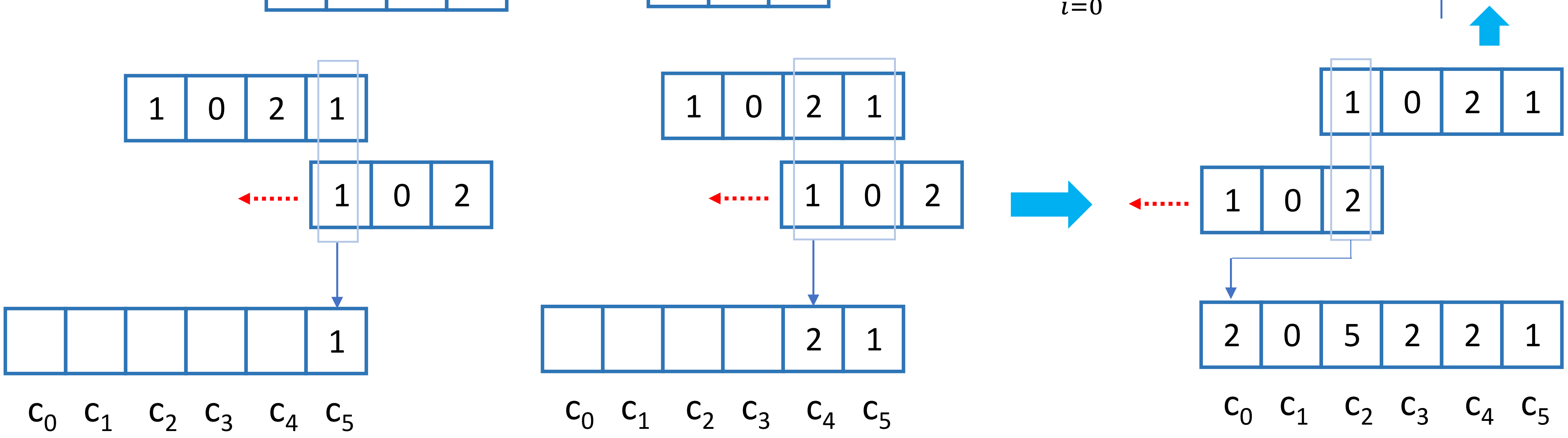
$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$ Convolution sum

Example: x[n]={1, 0, 2, 1} and y[n]={2, 0, 1} are two DT signals. Find their convolution c[n]=x[n]*y[n]



$$c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

c[n]={2, 0, 5, 2, 2, 1}

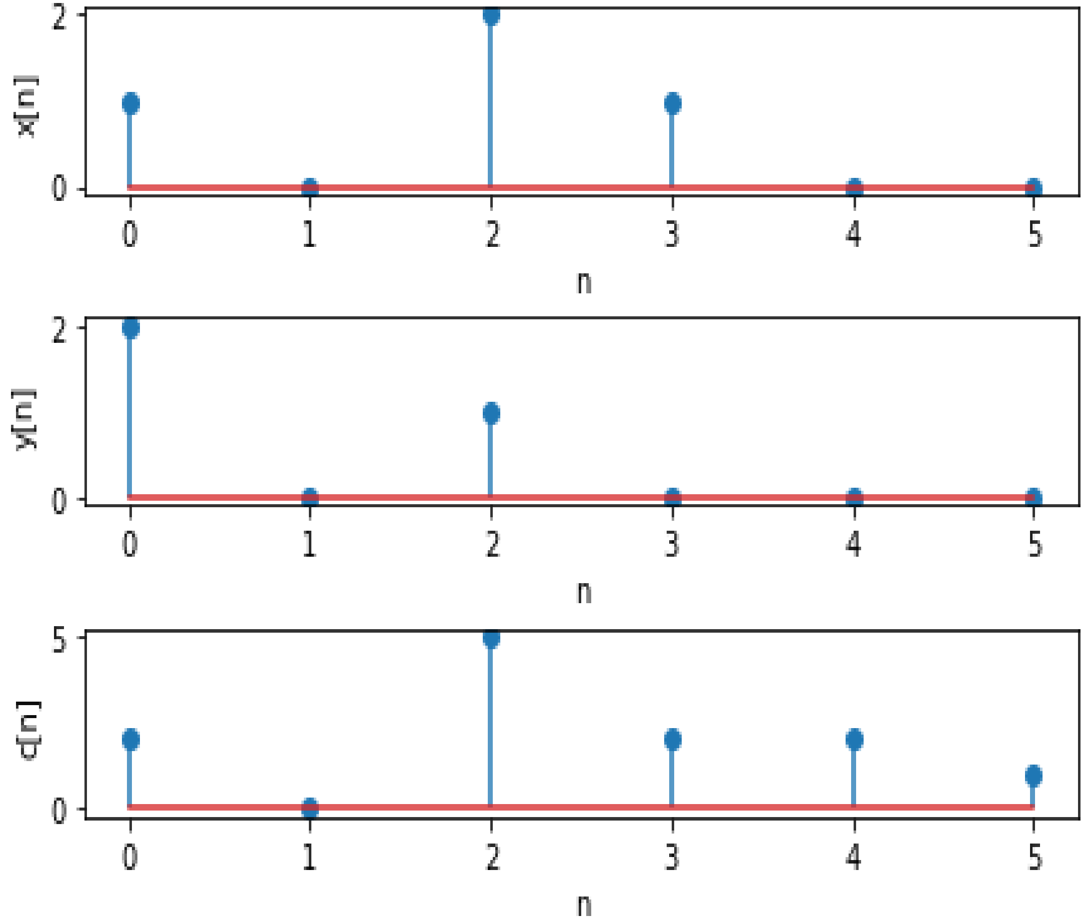


Convolution in Python :

```
import numpy as np
import matplotlib.pyplot as plt

x,y=[1,0, 2, 1], [2,0, 1]
c=np.convolve(x, y)
n=np.arange(0,(len(c)))
print("convolution:  x[n]*y[n]=",c)

plt.subplot(3,1,1);
plt.xlabel('n');
plt.ylabel('x[n]');
# zero padding for plot of x[n]
xp=np.pad(x, (0,(len(c)-len(x))), 'constant')
plt.stem(n, xp);

plt.subplot(3,1,2);
plt.xlabel('n');
plt.ylabel('y[n]');
# zero padding for the plot of y[n]
yp=np.pad(y, (0,(len(c)-len(y))), 'constant')
plt.stem(n, yp);

plt.subplot(3,1,3);
plt.xlabel('n');
plt.ylabel('c[n]');
plt.stem(n, c);

plt.tight_layout(pad=0.5)
plt.show()
```

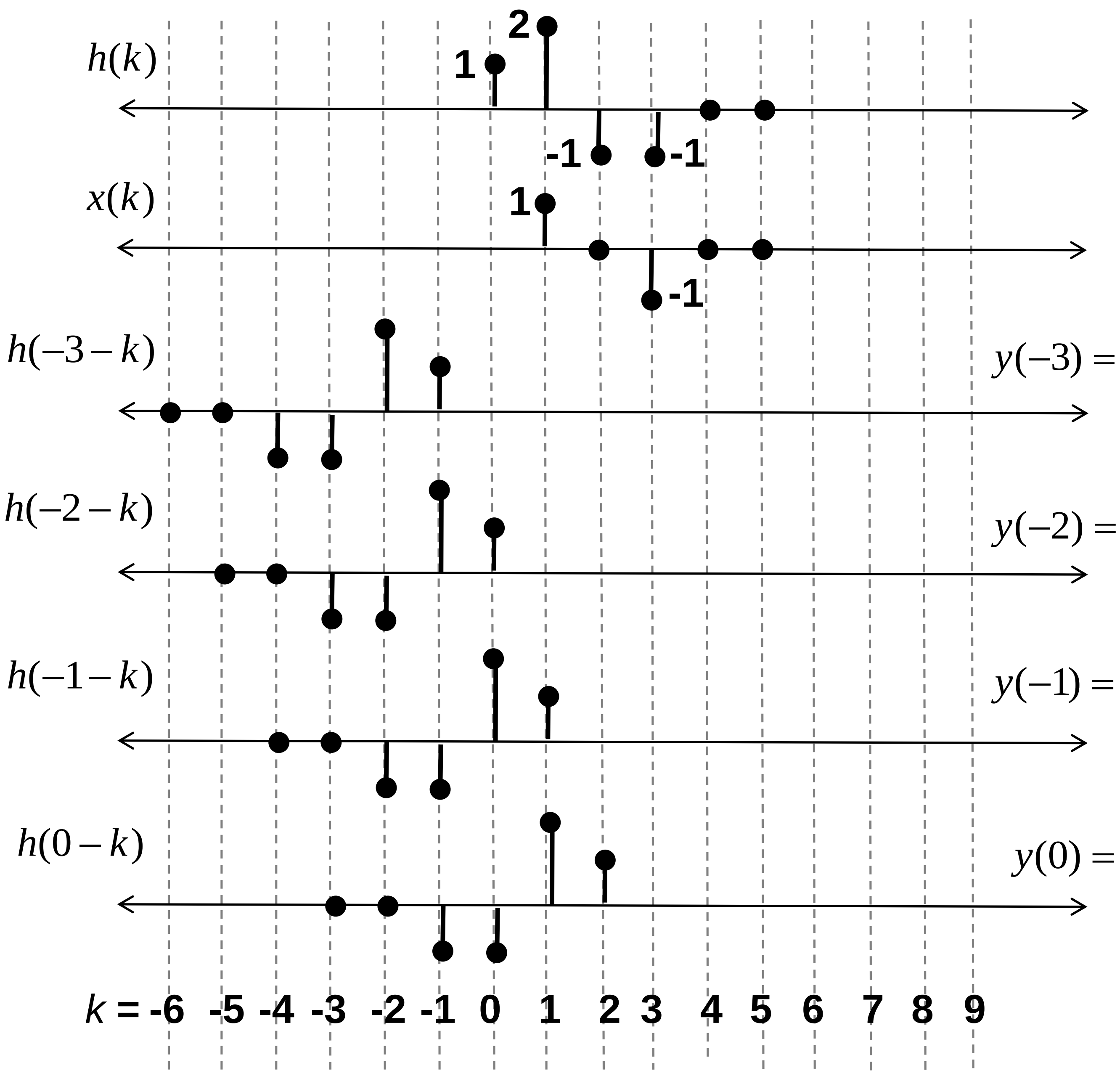
c=np.convolve (x, y, mode)

mode is optional: {'full', 'same', 'valid'}



Size of c: len(c)=(len(x)+len(y)-1)

Graphical Convolution Example (1/2)



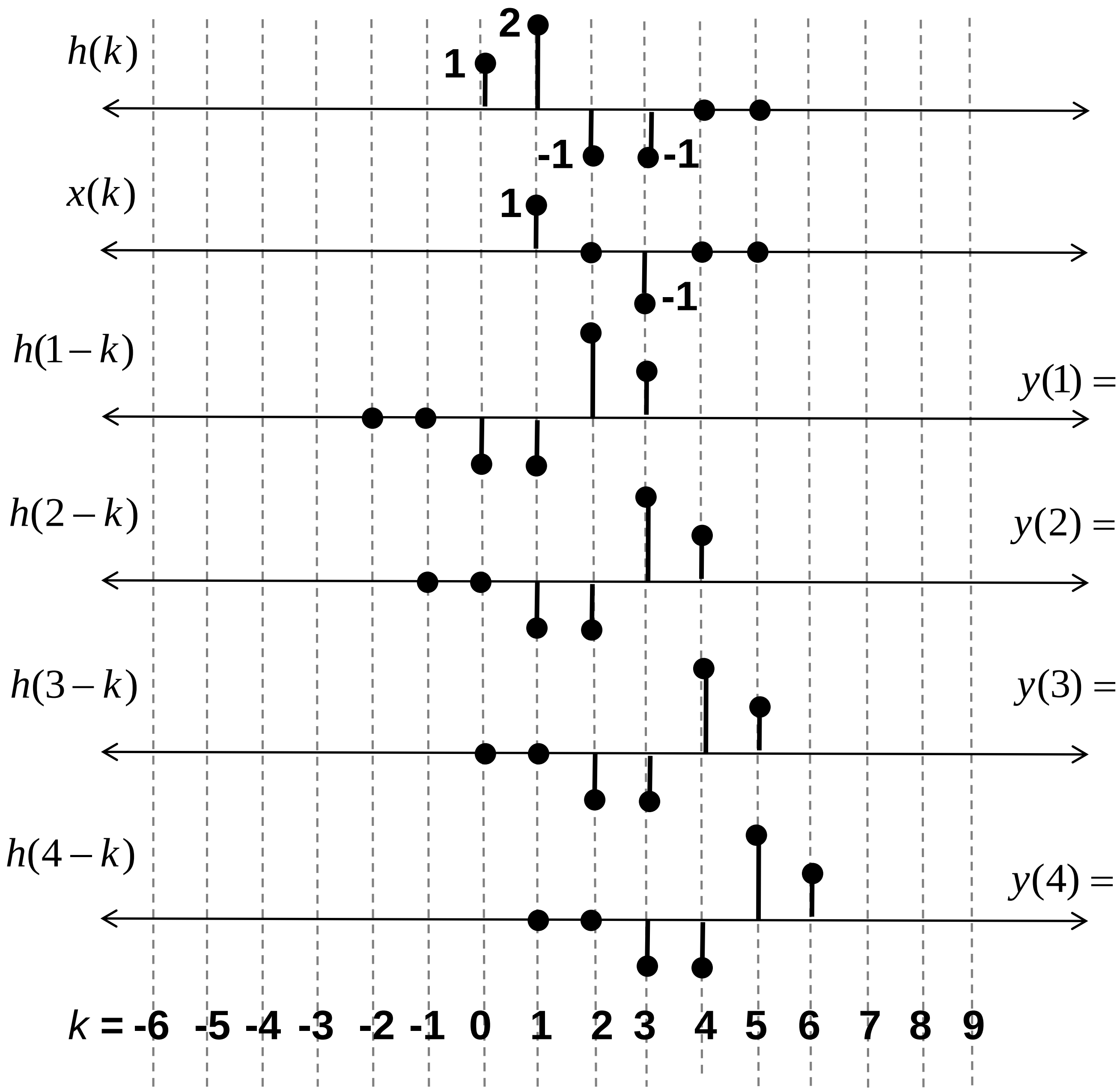
$$y(-3) = \sum_{k=-\infty}^{-\infty} x(k)h(-3-k) = 0$$

$$y(-2) = \sum_{k=-\infty}^{-\infty} x(k)h(-2-k) = 0$$

$$y(-1) = (1)(1) = 1$$

$$y(0) = (1)(0) + (2)(1) = 2$$

Graphical Convolution Example (2/2)



$$y(1) = (-1)(1) + (2)(0) + (-1)(1) = -2$$

$$y(2) = (1)(0) + (2)(-1) + (-1)(0) + (-1)(1) = -2$$

$$y(3) = (1)(0) + (2)(0) + (-1)(-1) + (-1)(0) = 1$$

$$y(4) = (-1)(-1) = 1$$

* Duration of x[n] : L_x = 3 samples

* Duration of h[n] : L_h=4 samples

* *Duration of* $y[n]$: $L_y = L_x + L_h - 1$

Examples of Discrete Time Convolution

- **Example: unit step**

$$h[n] = u[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, u[n-k] = \sum_{k=-\infty}^{n} x[k]$$

- **Example: unit-pulse**

$$h[n] = \delta[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k] = x[n]$$

- **Example: delayed unit-pulse**

$$h[n] = \delta[n - n_0]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, \delta[n - n_0 - k] = x[n - n_0]$$

- **Example: integration**

$$x[n] = u[n]$$

$$h[n] = a^n u[n] \quad |a| < 1$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} u[n] a^n u[n]$$

$$= (1)\delta[n] + (1+a)\delta[n-1] + \ldots$$

$$= \begin{cases} 1 & n = 0 \\ \dfrac{1-a^{n+1}}{1-a} & n > 0 \end{cases}$$