BLG 336E

Analysis of Algorithms II

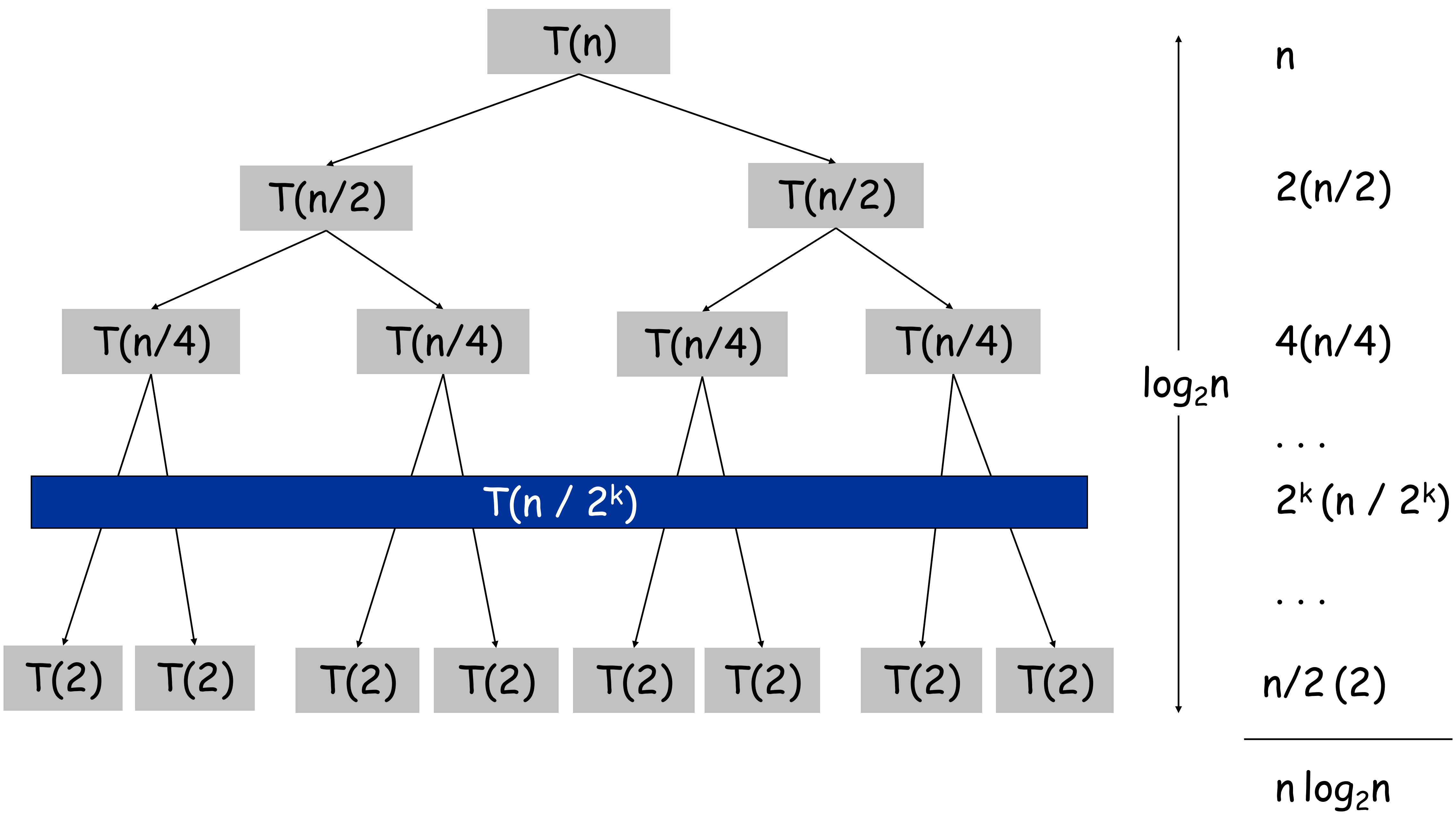
Lecture 8:

Dynamic Programming I

Weighted Interval Scheduling, Segmented Least Squares

Recap- Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



Recap-Why Recurrences?

- The complexity of many interesting algorithms is easily expressed as a recurrence – especially divide and conquer algorithms
- The form of the algorithm often yields the form of the recurrence
- The complexity of recursive algorithms is readily expressed as a recurrence.

Recap-Why solve recurrences?

- To make it easier to compare the complexity of two algorithms
- To make it easier to compare the complexity of the algorithm to standard reference functions.

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d).$$

Recap-Master Theorem

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

- Needlessly recursive integer mult.
  - T(n) = 4 T(n/2) + O(n)
  - T(n) = O(n^2)

  a = 4, b = 2, d = 1 — $a > b^d$ ✓

- Karatsuba integer multiplication
  - T(n) = 3 T(n/2) + O(n)
  - T(n) = O($n^{\log\_2(3)} \approx n^{1.6}$)

  a = 3, b = 2, d = 1 — $a > b^d$ ✓

- MergeSort
  - T(n) = 2T(n/2) + O(n)
  - T(n) = O(nlog(n))

  a = 2, b = 2, d = 1 — $a = b^d$ ✓

- That other one
  - T(n) = T(n/2) + O(n)
  - T(n) = O(n)

  a = 1, b = 2, d = 1 — $a < b^d$ ✓

Recap-Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is sorted. See: 05demo-merge-invert.ppt
- Count inversions where a_i and a_j are in different halves.
- Merge two sorted halves into sorted whole. to maintain sorted invariant

3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2	0	0

13 blue-green inversions: 6 + 3 + 2 + 2 + 0 + 0

Count: O(n)

2	3	7	10	11	14	16	17	18	19	23	25

Merge: O(n)

$$T(n) \le 2T(n/2) + cn \quad \Rightarrow \quad T(n) = O(n \log n)$$

5.5 Integer Multiplication

This was kind of a big deal

$\mathrm{XLIV} \times \mathrm{XCVII} = ?$

$$\begin{array}{r} 44 \\ \times\ 97 \\ \hline \end{array}$$

Integer Multiplication

$$\begin{array}{r} 44 \\ \times\ 97 \\ \hline \end{array}$$

Integer Multiplication

$$
\begin{array}{r}
1234567895931413 \\
\times \quad 4563823520395533 \\
\hline
\end{array}
$$

Integer Multiplication

$$\overbrace{233925720752752384623764283568364918374523856298}^{n}$$
$$4562323582342395285623467235019130750135350013753$$
$$\times$$

???

How fast is the grade-school multiplication algorithm?

(How many one-digit operations?)

About n^2 one-digit operations

At most n^2 multiplications,
and then at most n^2 additions (for carries)
and then I have to add n different 2n-digit numbers…

Big-Oh Notation

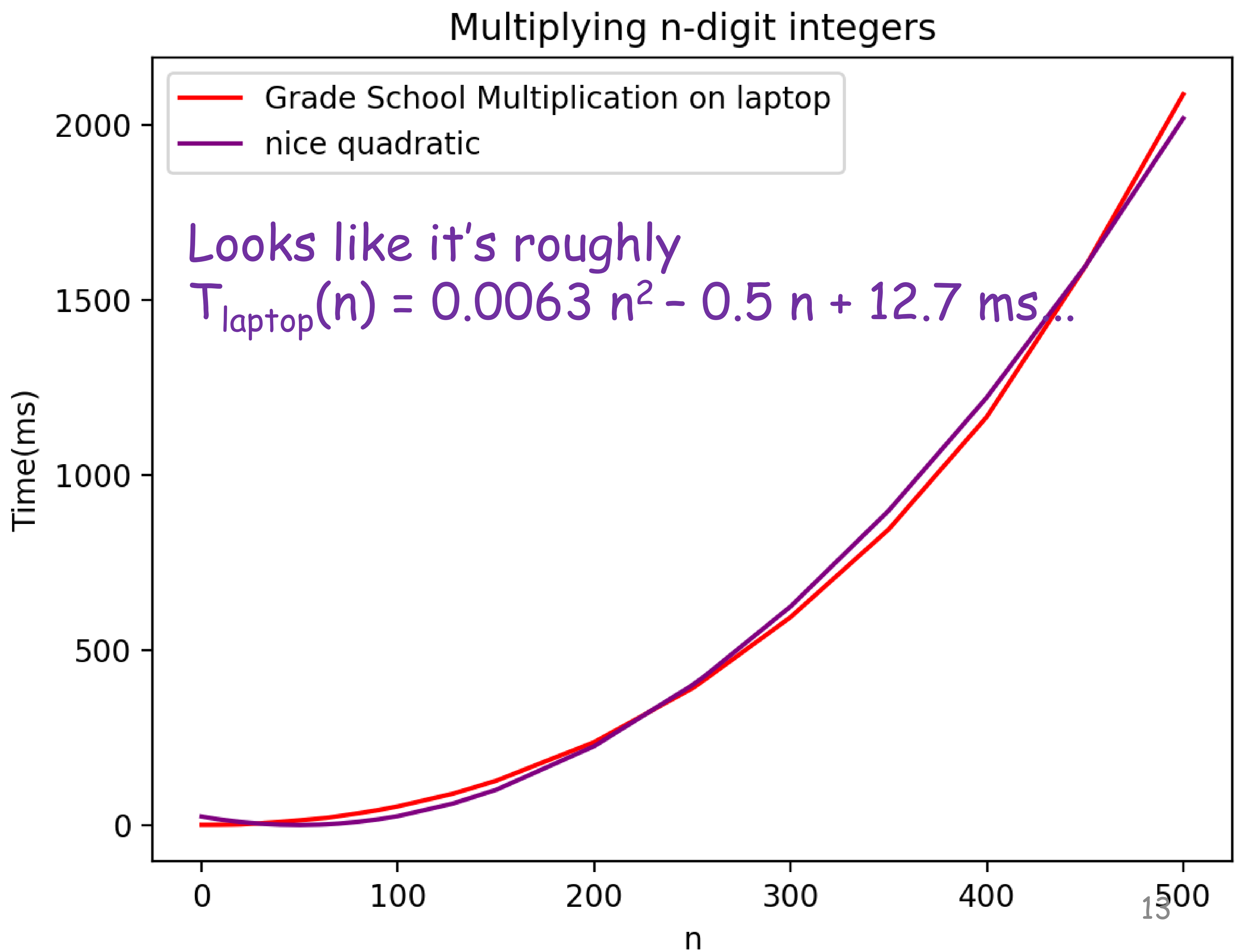
- We say that Grade-School Multiplication
  "runs in time $O(n^2)$"

- Formal definition coming Wednesday!
- Informally, big-Oh notation tells us how the running time scales with the size of the input.

Implemented in Python

highly non-optimized

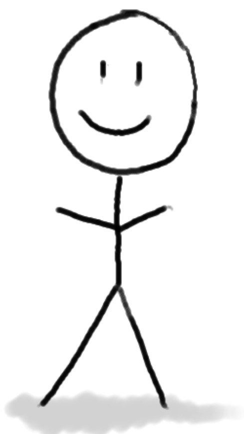
The runtime "scales like" n^2

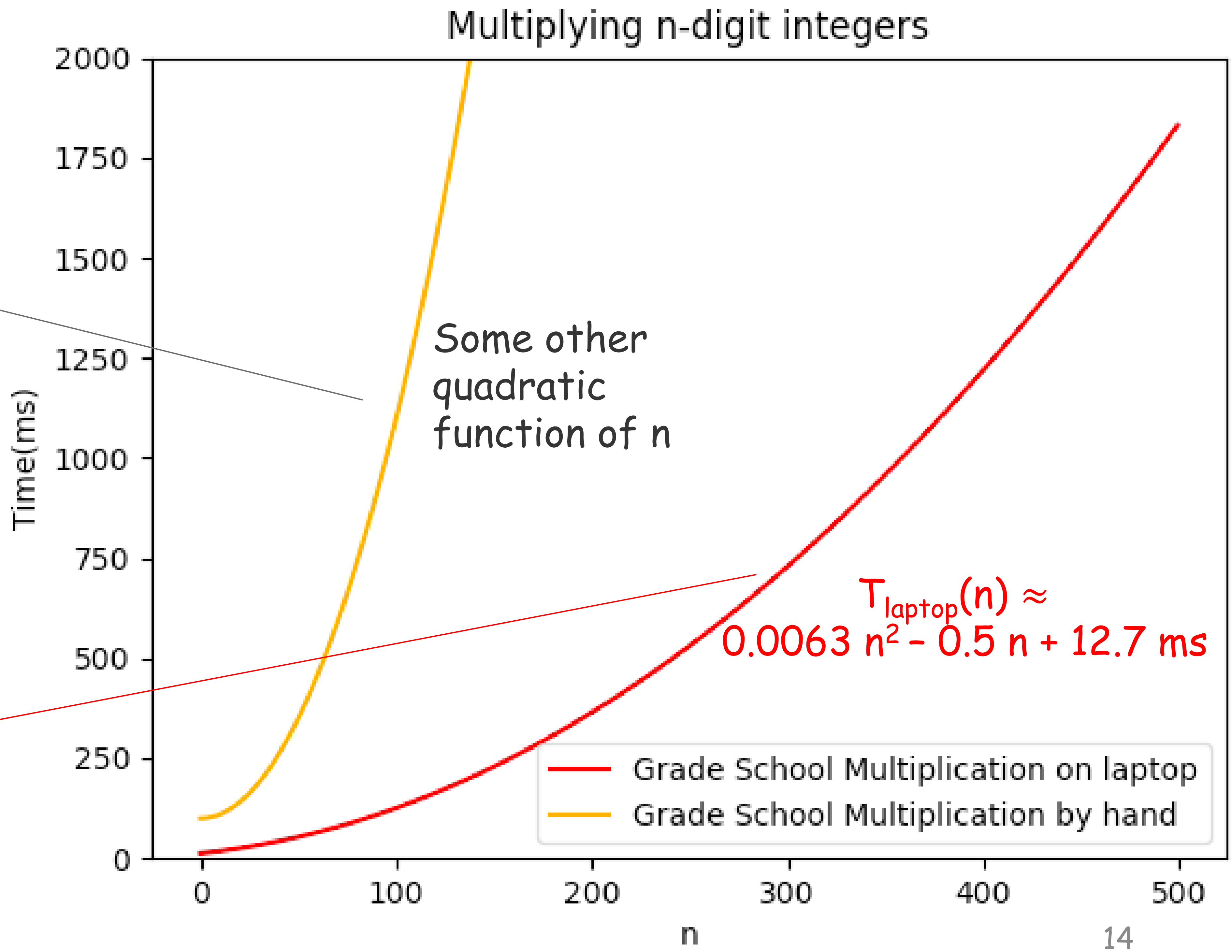


python™

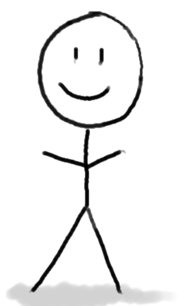
Implemented by hand

The runtime still "scales like" n^2



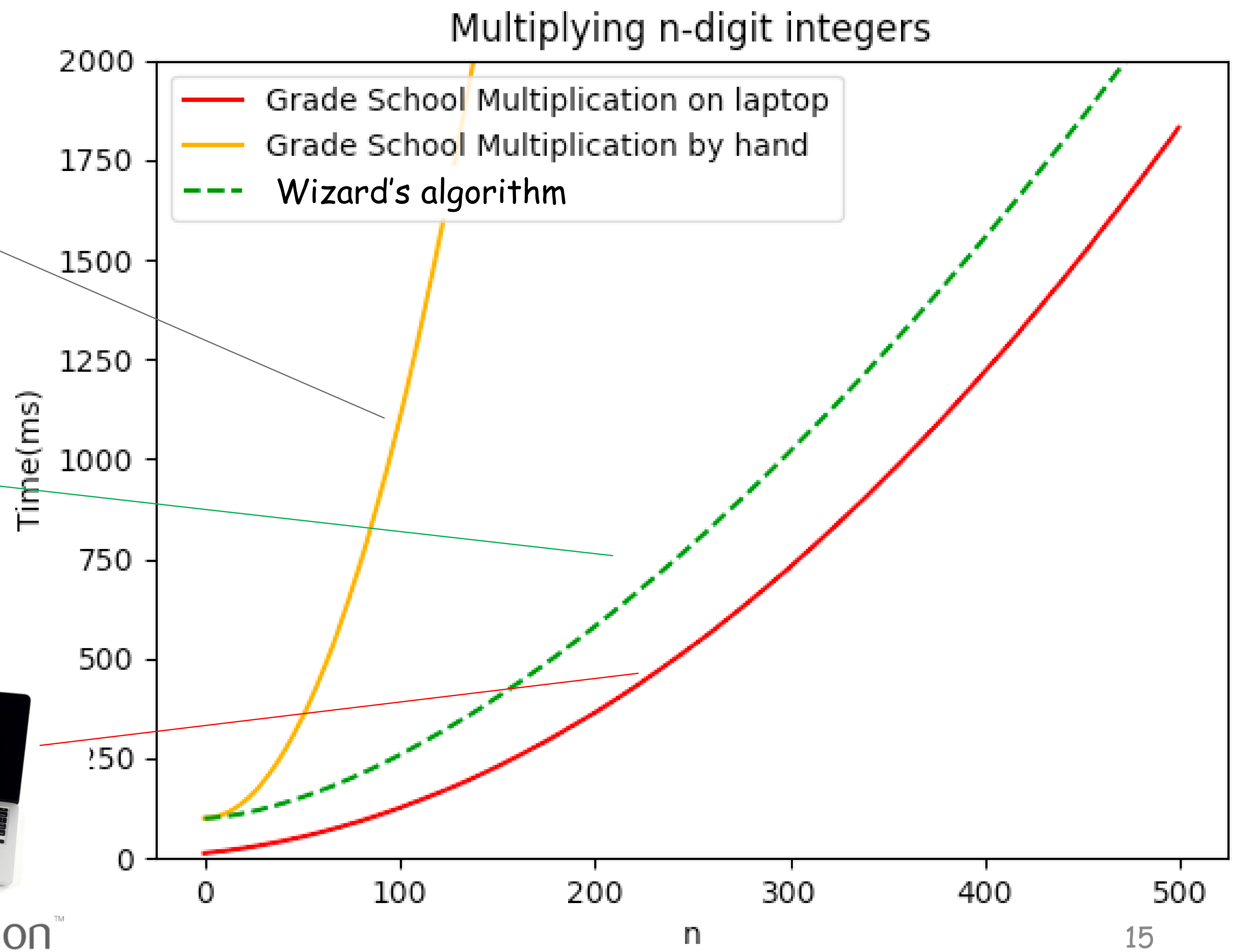


Why is big-Oh notation meaningful?

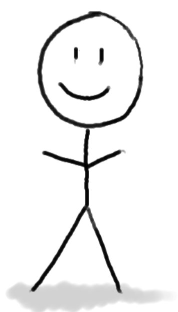


$\approx \frac{n^{1.6}}{10} + 100$

$\approx .0063n^2$

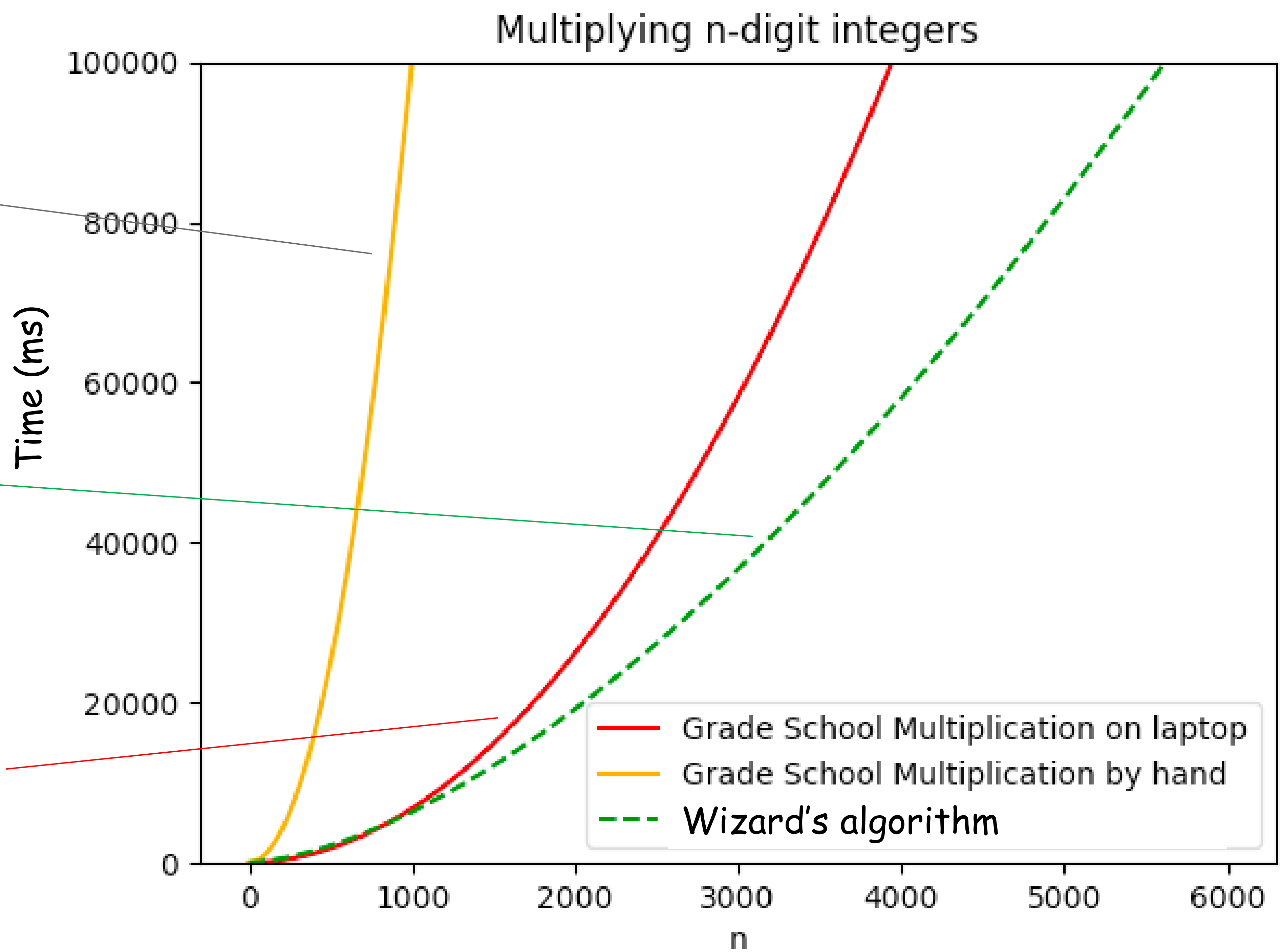


Let n get bigger…



$\approx \frac{n^{1.6}}{10} + 100$

$\approx .0063n^2$



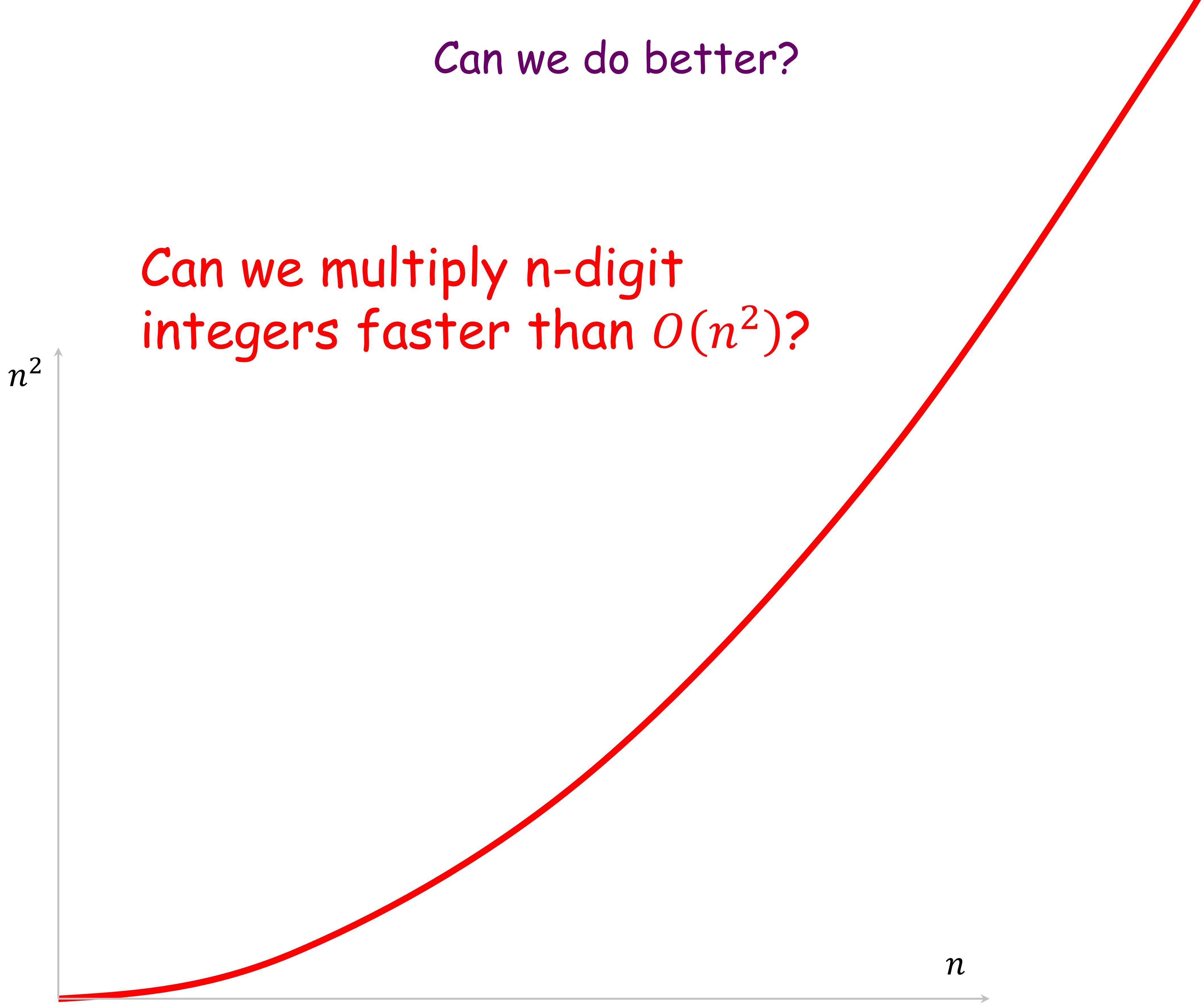
Take-away

- An algorithm that runs in time $O(n^{1.6})$ is "better" than an an algorithm that runs in time $O(n^2)$.

- So the question is...

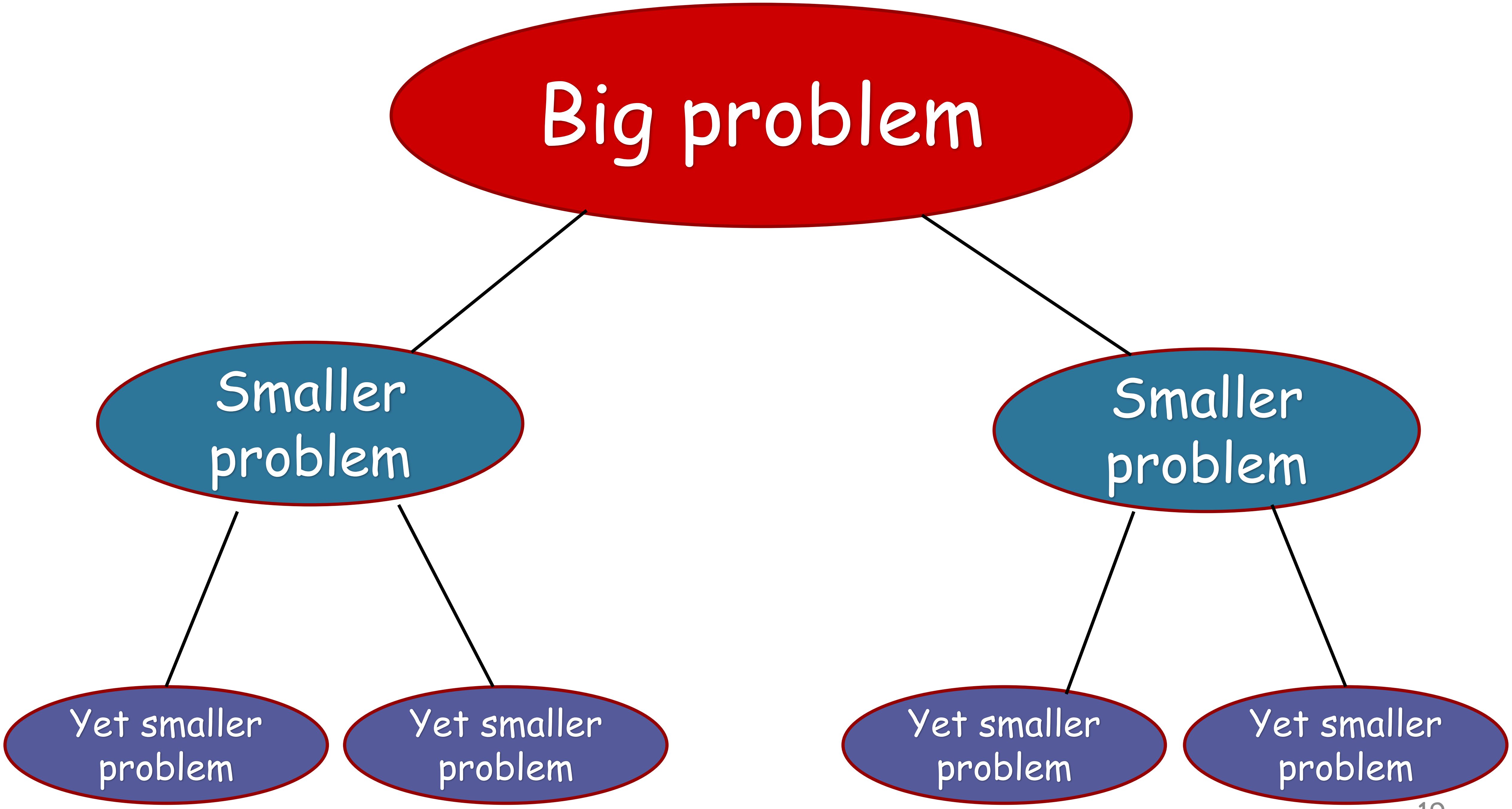
Can we do better?

Can we multiply n-digit integers faster than $O(n^2)$?



Divide and conquer

Break problem up into smaller (easier) sub-problems



Divide and conquer for multiplication

Break up an integer:

$$1234 = 12 \times 100 + 34$$

$$
\begin{aligned}
1234 &\times 5678 \\
&= (12 \times 100 + 34)(56 \times 100 + 78) \\
&= (\underbrace{12 \times 56}_{1})10000 + (\underbrace{34 \times 56}_{2} + \underbrace{12 \times 78}_{3})100 + (\underbrace{34 \times 78}_{4})
\end{aligned}
$$

One 4-digit multiply $\Longrightarrow$ Four 2-digit multiplies

More generally

Suppose n is even

Break up an n-digit integer:

$$[x_1 x_2 \cdots x_n] = [x_1 x_2 \cdots x_{n/2}] \times 10^{n/2} + [x_{n/2+1} x_{n/2+2} \cdots x_n]$$

$$x \times y = (a \times 10^{n/2} + b)(c \times 10^{n/2} + d)$$

$$= \underbrace{(a \times c)}_{1} 10^n + (\underbrace{a \times d}_{2} + \underbrace{c \times b}_{3}) 10^{n/2} + \underbrace{(b \times d)}_{4}$$

One n-digit multiply $\Rightarrow$ Four (n/2)-digit multiplies

Divide and conquer algorithm

not very precisely…

(Assume n is a power of 2…)

x,y are n-digit numbers

Multiply(x, y):

- **If** n=1:
  - **Return** xy

Base case: I've memorized my 1-digit multiplication tables…

- Write $x = a\,10^{\frac{n}{2}} + b$
- Write $y = c\,10^{\frac{n}{2}} + d$

a, b, c, d are n/2-digit numbers

- Recursively compute ac, ad, bc, bd:
  - ac = **Multiply**(a, c), etc..
- Add them up to get xy:
  - xy = ac 10^n + (ad + bc) $10^{n/2}$ + bd

Make this pseudocode more detailed! How should we handle odd n? How should we implement "multiplication by 10^n"?

Think-Pair-Share

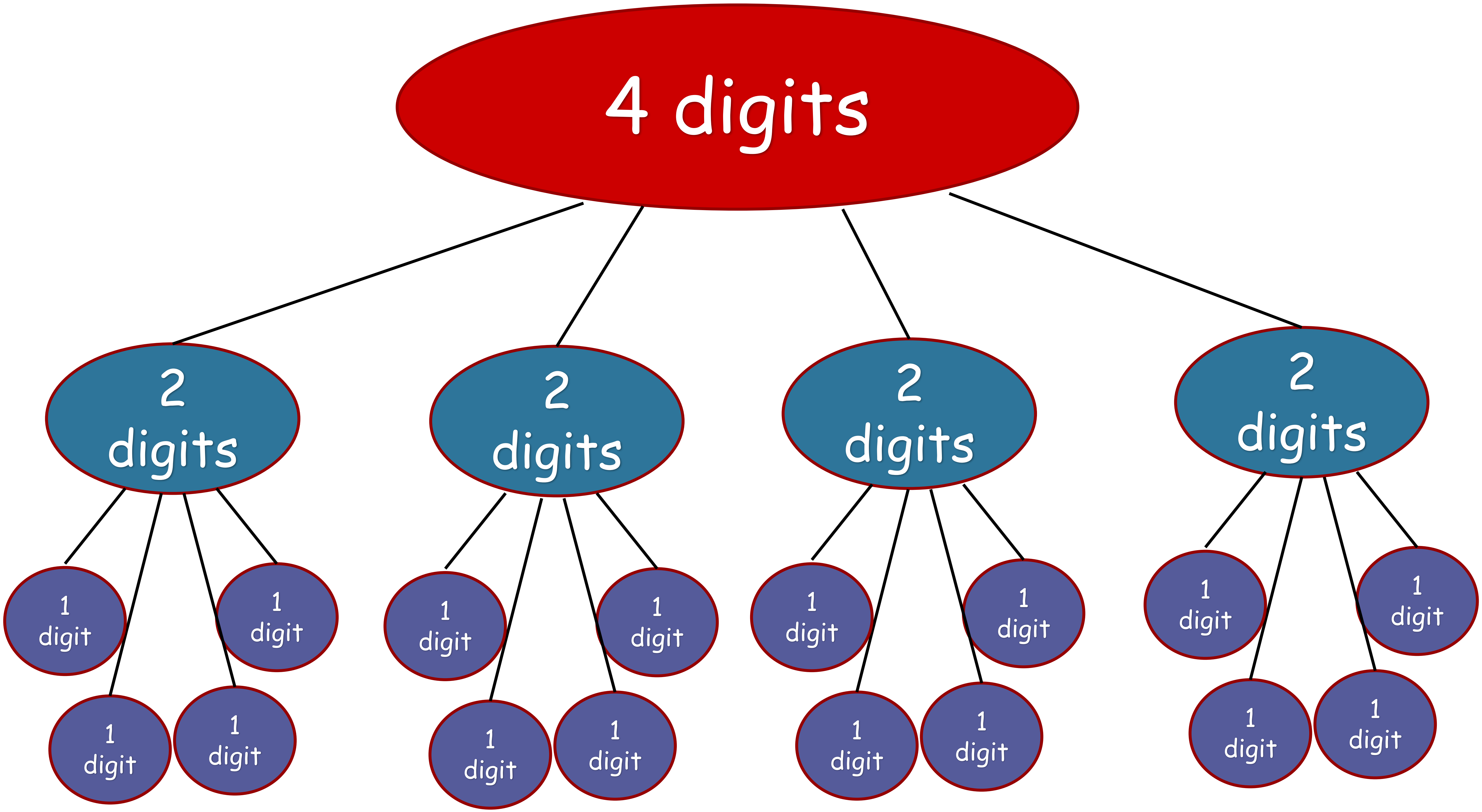
- We saw that this 4-digit multiplication problem broke up into four 2-digit multiplication problems

$$1234 \times 5678$$

- If you recurse on those 2-digit multiplication problems, how many 1-digit multiplications do you end up with total?

Recursion Tree

16 one-digit multiplies!

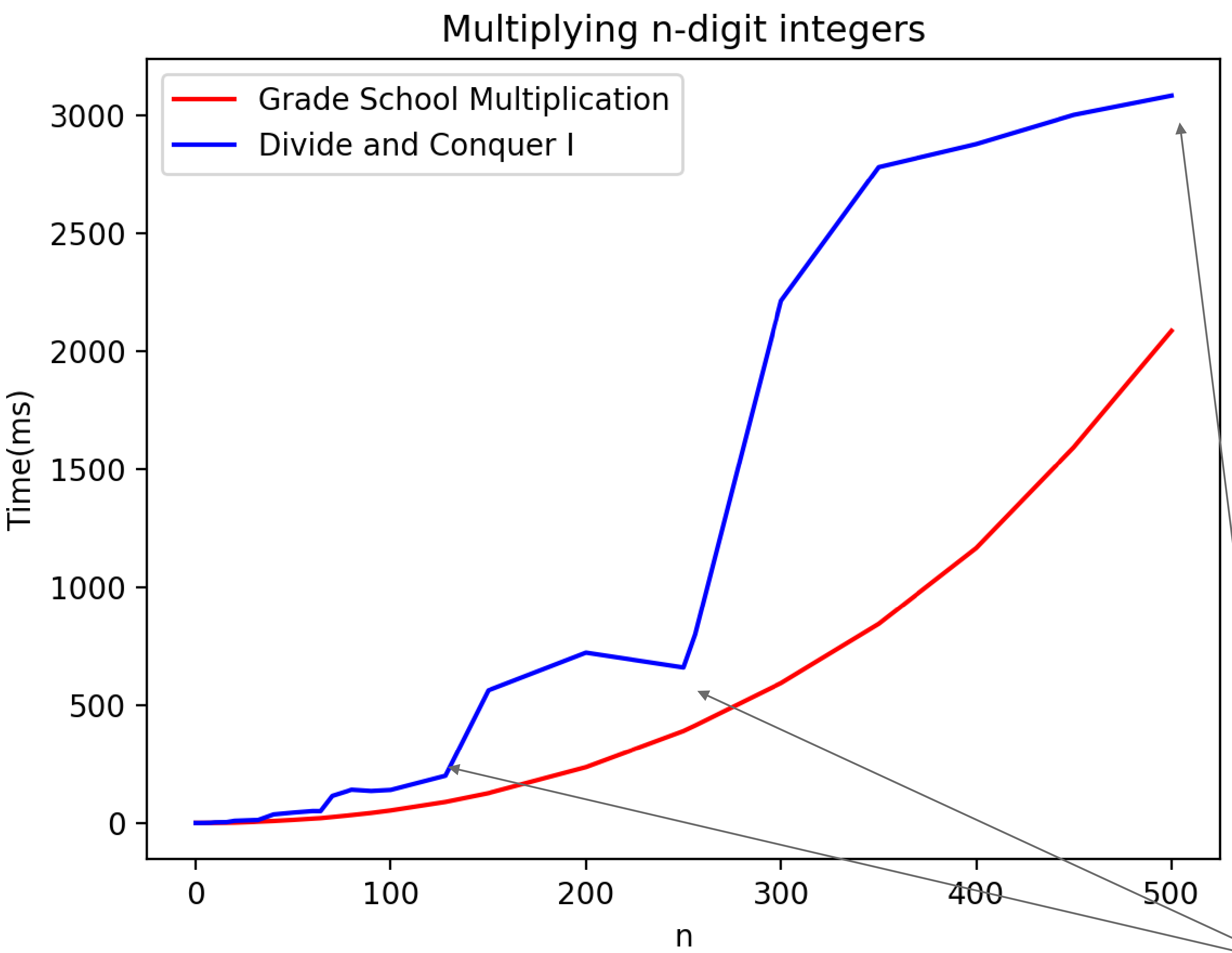


What is the running time?

- Better or worse than the grade school algorithm?
- How do we answer this question?
  1. Try it.
  2. Try to understand it analytically.

1. Try it.

Conjectures about running time?



Doesn't look too good but hard to tell…

Maybe one implementation is slicker than the other?

Maybe if we were to run it to n=10000, things would look different.

Something funny is happening at powers of 2…

2. Try to understand the running time analytically

- ~~Proof by meta-reasoning:~~

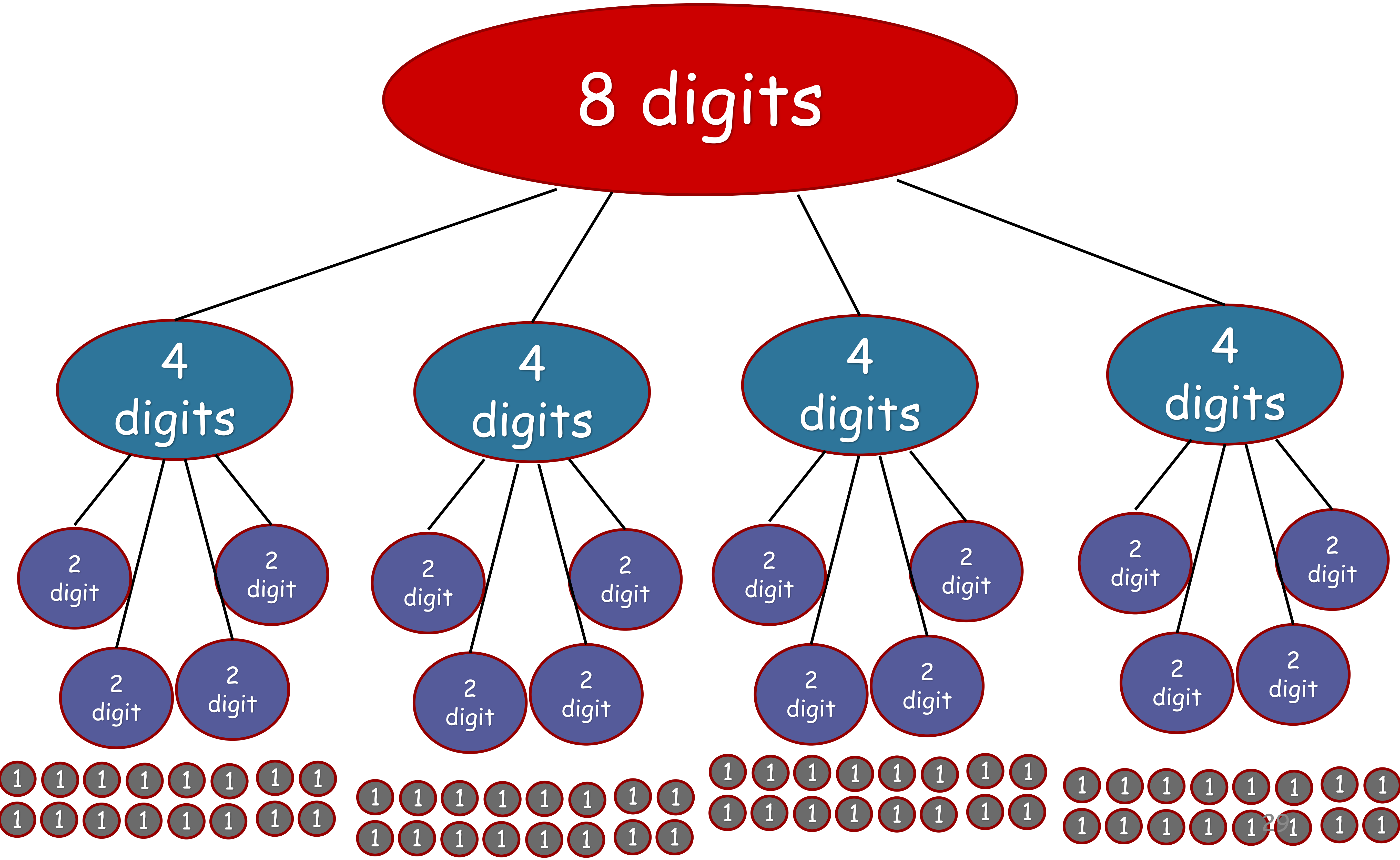
~~It must be faster than the grade school algorithm, because we are learning it in an algorithms class.~~

Think-Pair-Share:

- We saw that multiplying 4-digit numbers resulted in 16 one-digit multiplications.
- How about multiplying 8-digit numbers?
- What do you think about n-digit numbers?

Recursion Tree

64 one-digit multiplies!



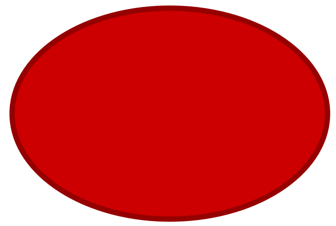
Claim:

The running time of this algorithm

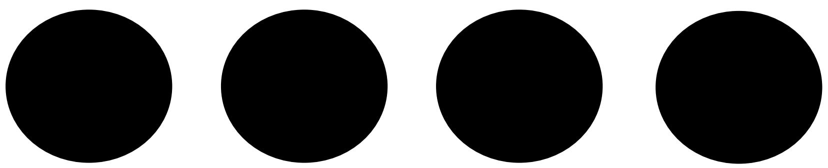
is

AT LEAST n^2 operations.

There are n^2 1-digit problems

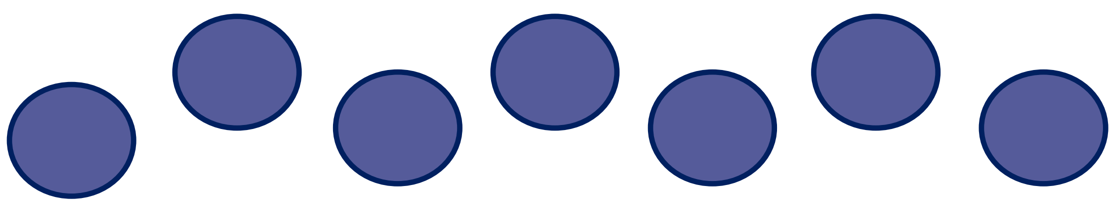


1 problem
of size n



4 problems
of size n/2

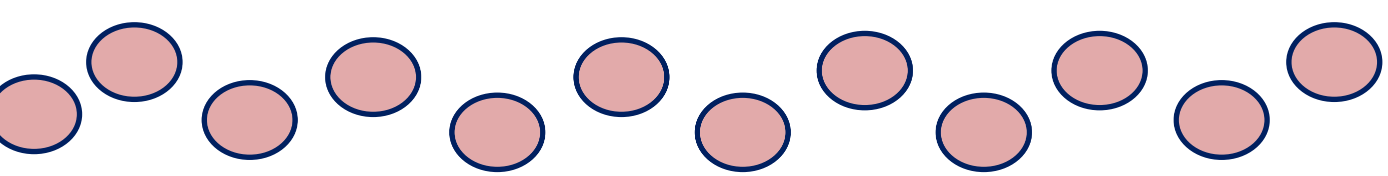
…



4^t problems
of size $n/2^t$

Note: this is just a cartoon – I'm not going to draw all 4^t circles!

…



$\underline{\ n^2\ }$ problems
of size 1

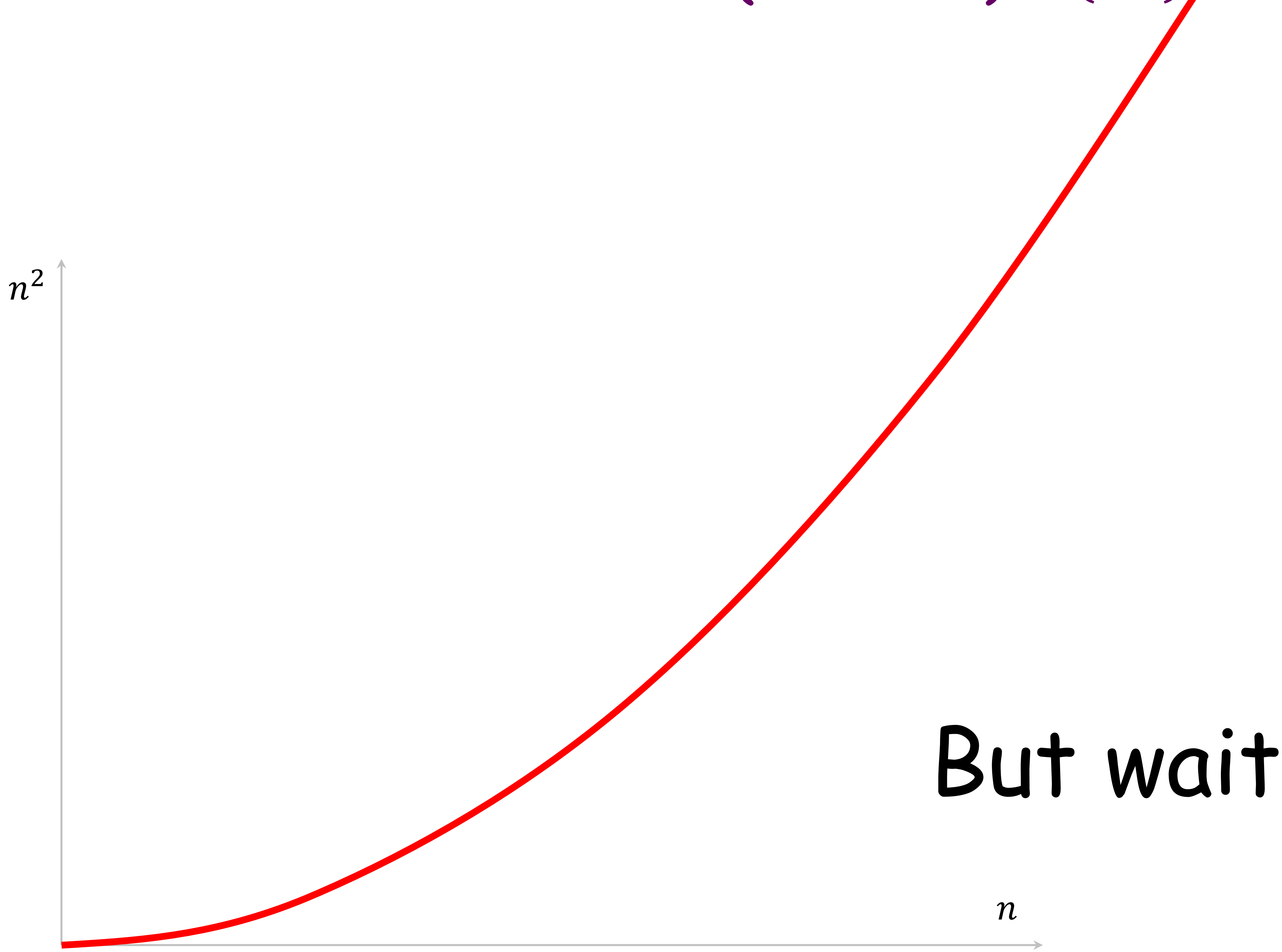
- If you cut n in half $\log_2(n)$ times, you get down to 1.
- So at level
  $t = \log_2(n)$
  we get…

$$4^{\log_2 n} = n^{\log_2 4} = n^2$$

problems of size 1.

That's a bit disappointing

All that work and still (at least) $O(n^2)$...



But wait!!

Divide and conquer can actually make progress

- Karatsuba figured out how to do this better!

$$
\begin{aligned}
xy &= (a \cdot 10^{n/2} + b)(c \cdot 10^{n/2} + d) \\
&= ac \cdot 10^{n} + (ad + bc)10^{n/2} + bd
\end{aligned}
$$

Need these three things

- If only we could recurse on three things instead of four…

Karatsuba integer multiplication

- Recursively compute these THREE things:
  - ac
  - bd
  - (a+b)(c+d)

Subtract these off

get this

(a+b)(c+d) = ac + bd + bc + ad

- Assemble the product:

$$xy = (a \cdot 10^{n/2} + b)(c \cdot 10^{n/2} + d)$$
$$= ac \cdot 10^{n} + (ad + bc)10^{n/2} + bd$$

How would this work?

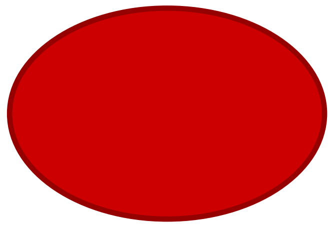
x,y are n-digit numbers

Multiply(x, y):

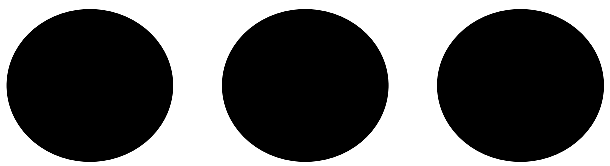
- **If** n=1:
  - **Return** xy
- Write $x = a\,10^{\frac{n}{2}} + b$ and $y = c\,10^{\frac{n}{2}} + d$
- ac = **Multiply**(a, c)
- bd = **Multiply**(b, d)
- z = **Multiply**(a+b, c+d)
- xy = ac 10^n + (z – ac - bd) $10^{n/2}$ + bd
- Return xy

a, b, c, d are n/2-digit numbers

What's the running time?

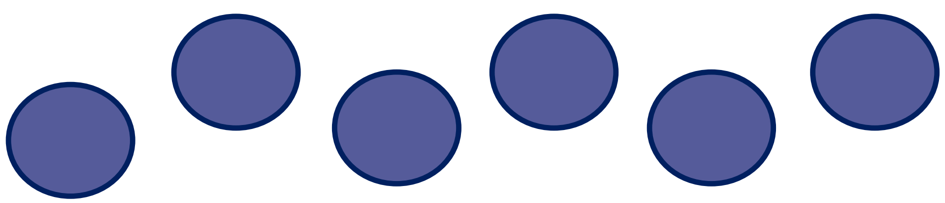


1 problem
of size n



3 problems
of size n/2

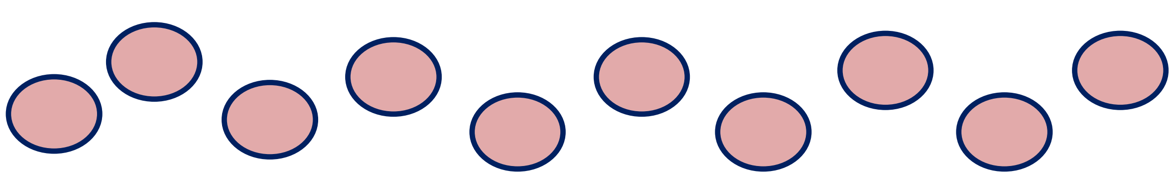
...



3^t problems
of size $n/2^t$

Note: this is just a cartoon – I'm not going to draw all 3^t circles!

...



$\underline{n^{1.6}}$ problems
of size 1

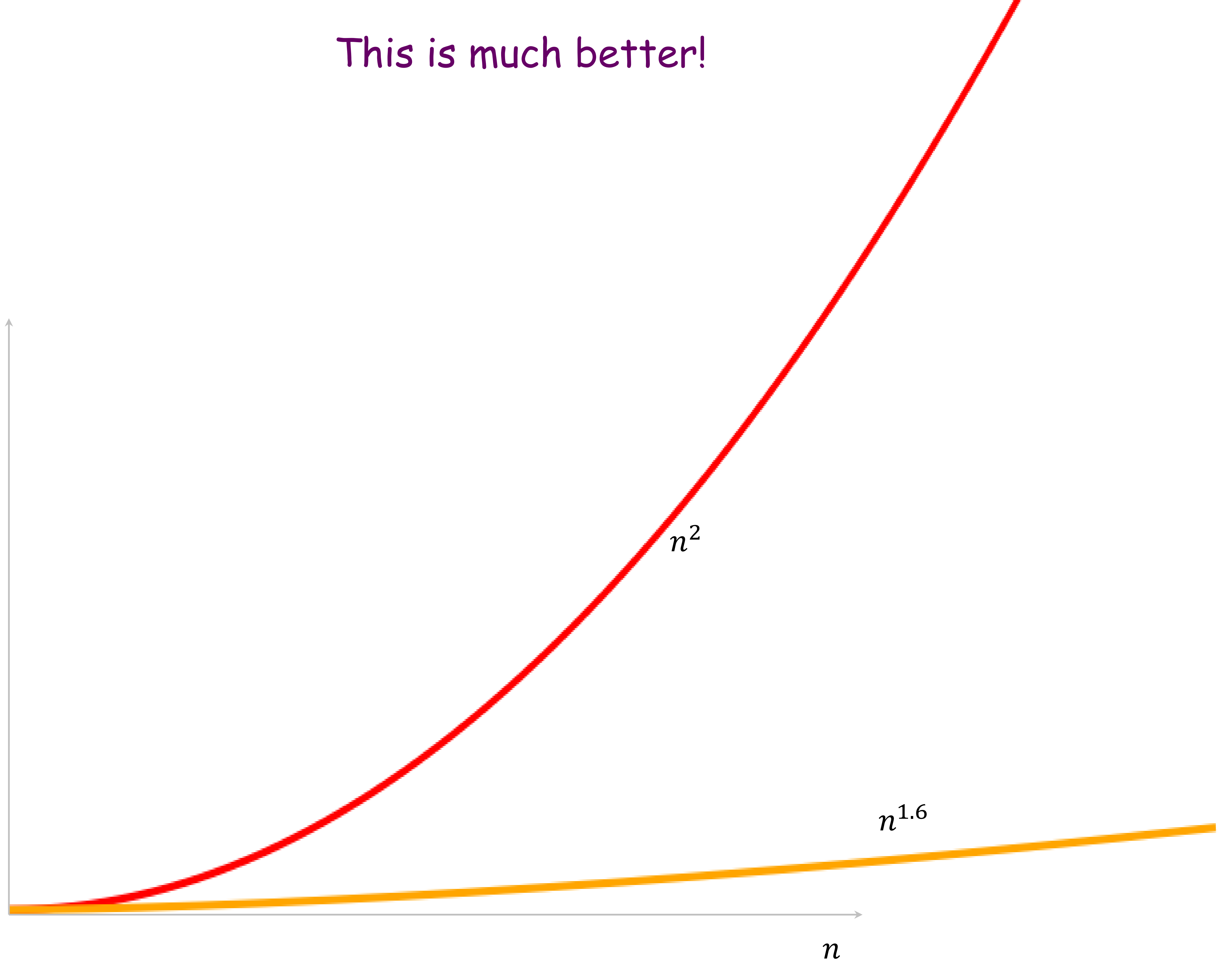
- If you cut n in half $\log_2(n)$ times, you get down to 1.
- So at level
  $t = \log_2(n)$
  we get…

$$3^{\log_2 n} = n^{\log_2 3} \approx n^{1.6}$$

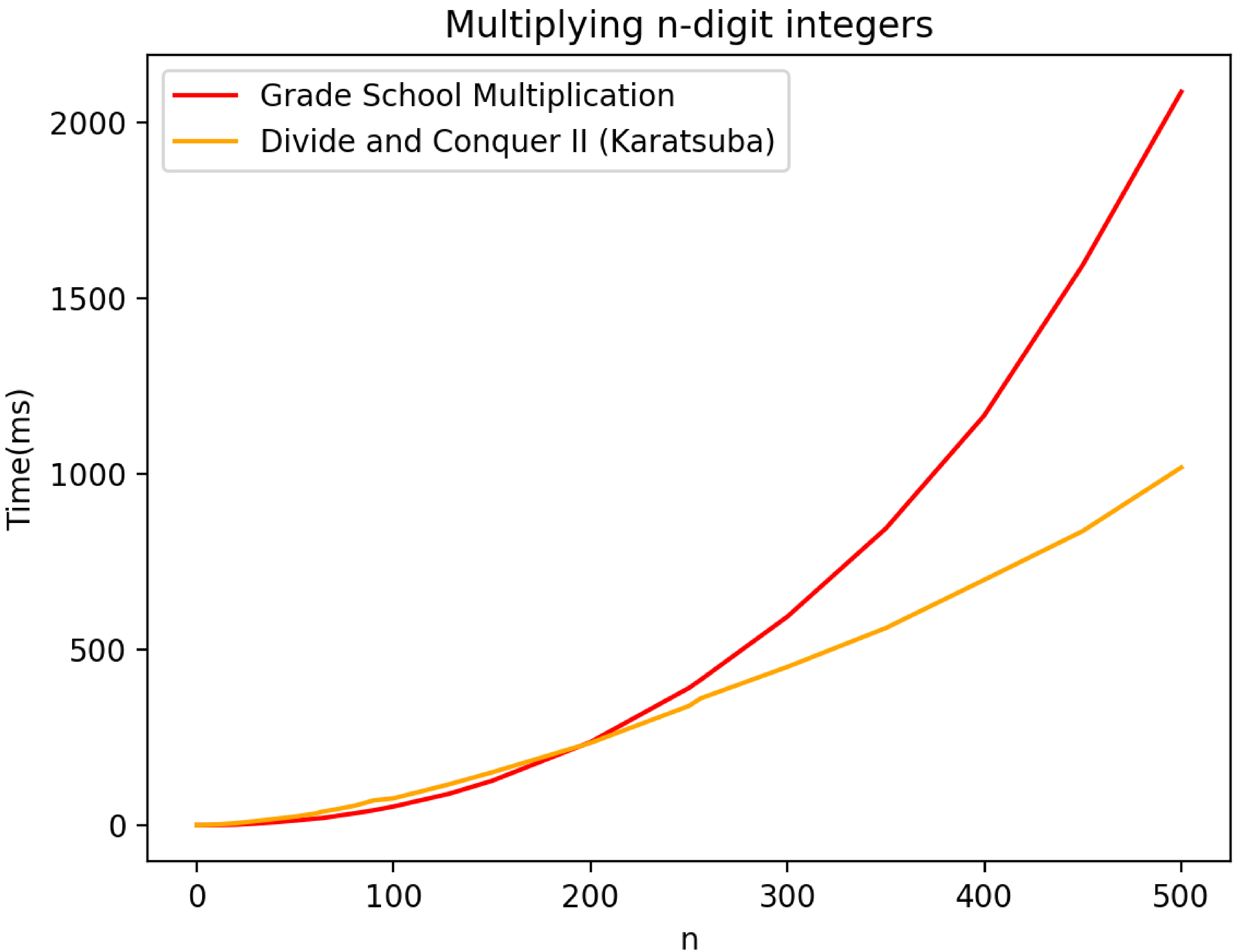
problems of size 1.

We aren't accounting for the work at the higher levels! But we'll see later that this turns out to be okay.

This is much better!



We can even see it in real life!



Integer Arithmetic

Add. Given two n-digit integers a and b, compute a + b.

- O(n) bit operations.

Multiply. Given two n-digit integers a and b, compute a × b.

- Brute force solution: $\Theta(n^2)$ bit operations.

```
    12                      1100
    13                      1101
 X-----                    X-----
    36                      1100
   12                      0000
 +-----                    1100
   156                    1100
                   +-----------
                       10011100
```

Decimal Multiplication   Binary Multiplication

Divide-and-Conquer Multiplication: Warmup

To multiply two n-digit integers:
- Multiply four ½n-digit integers.
- Add two ½n-digit integers, and shift to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) = 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

$$
\mathrm{T}(n) = \underbrace{4T\left(n/2\right)}_{\text{recursive calls}} + \underbrace{cn}_{\text{add, shift}} \Rightarrow \mathrm{T}(n) = \Theta(n^2)
$$

etter

than brute force!

assumes n is a power of 2

Karatsuba Multiplication

To multiply two n-digit integers:

- Add two ½n digit integers.
- Multiply three ½n-digit integers.
- Add, subtract, and shift ½n-digit integers to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0 \\
&= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot ((x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0) + x_0 y_0
\end{aligned}
$$

A B A C C

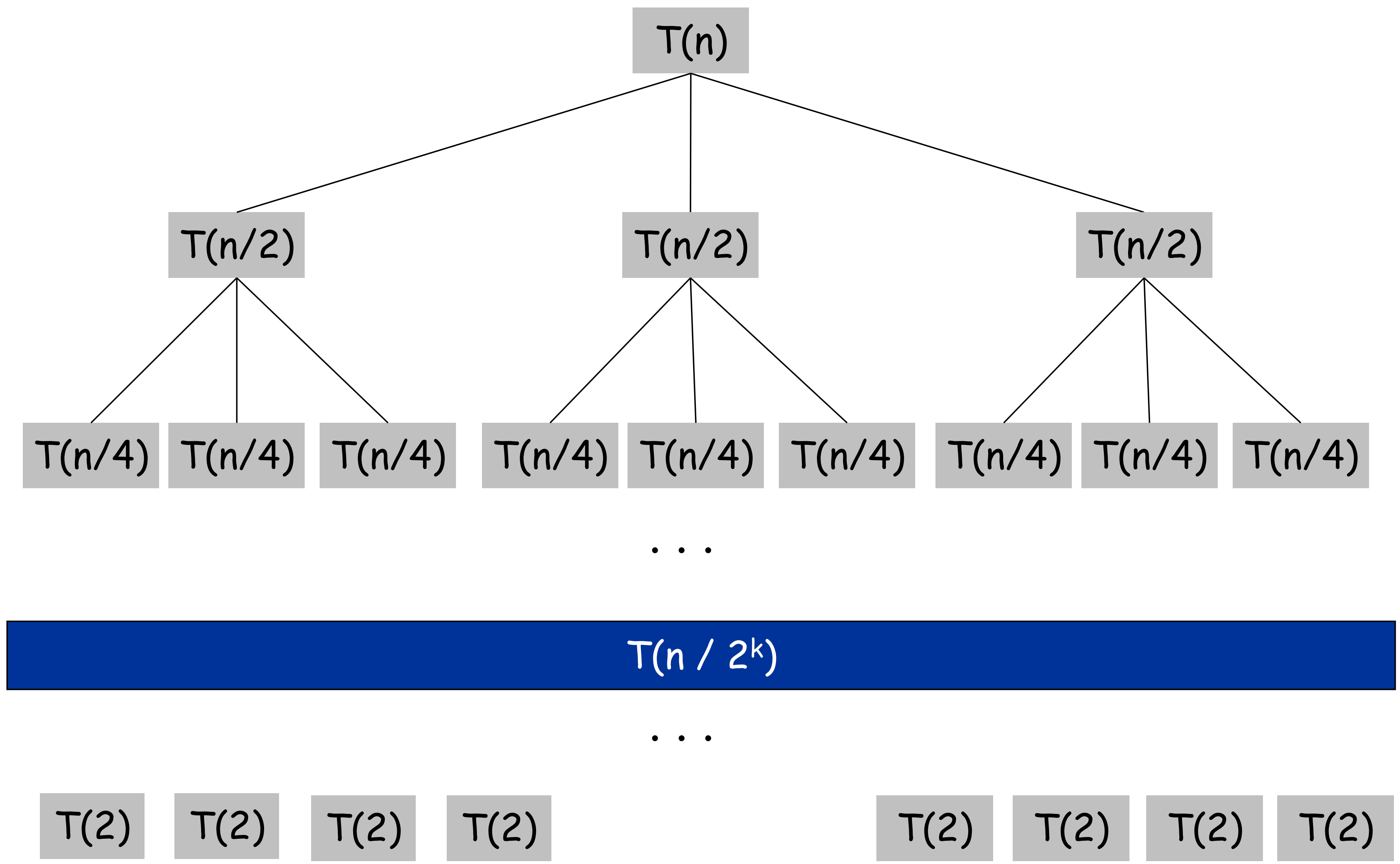
Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$T(n) = \underbrace{3T(n/2)}_{\text{recursive calls}} + \underbrace{cn}_{\text{add, subtract, shift}} \Rightarrow T(n) = O(n^{\log_2 3}) = O(n^{1.59})$$

Karatsuba: Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

$$T(n) = \sum_{k=0}^{\log_2 n} n \left(\tfrac{3}{2}\right)^k = \frac{\left(\tfrac{3}{2}\right)^{1+\log_2 n} - 1}{\tfrac{3}{2} - 1} = 3n^{\log_2 3} - 2$$



n

$3(n/2)$

$9(n/4)$

. . .

$3^k (n / 2^k)$

. . .

$3^{\lg n} (2)$

Matrix Multiplication

Matrix Multiplication

Matrix multiplication. Given two n-by-n matrices A and B, compute C = AB.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Brute force. $\Theta(n^3)$ arithmetic operations.

Fundamental question. Can we improve upon brute force?

Matrix Multiplication: Warmup

Divide-and-conquer.
- Divide: partition A and B into ½n-by-½n blocks.
- Conquer: multiply 8 ½n-by-½n recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= \left(A_{11} \times B_{11}\right) + \left(A_{12} \times B_{21}\right) \\ C_{12} &= \left(A_{11} \times B_{12}\right) + \left(A_{12} \times B_{22}\right) \\ C_{21} &= \left(A_{21} \times B_{11}\right) + \left(A_{22} \times B_{21}\right) \\ C_{22} &= \left(A_{21} \times B_{12}\right) + \left(A_{22} \times B_{22}\right) \end{aligned}$$

$$\mathrm{T}(n) = \underbrace{8T\big(n/2\big)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \quad\Rightarrow\quad \mathrm{T}(n) = \Theta(n^3)$$

Matrix Multiplication: Key Idea

Key idea. multiply 2-by-2 block matrices with only 7 multiplications.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}$$

$$\begin{aligned}
P_1 &= A_{11} \times (B_{12} - B_{22}) \\
P_2 &= (A_{11} + A_{12}) \times B_{22} \\
P_3 &= (A_{21} + A_{22}) \times B_{11} \\
P_4 &= A_{22} \times (B_{21} - B_{11}) \\
P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\
P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\
P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12})
\end{aligned}$$

- 7 multiplications.
- 18 = 10 + 8 additions (or subtractions).

Fast Matrix Multiplication

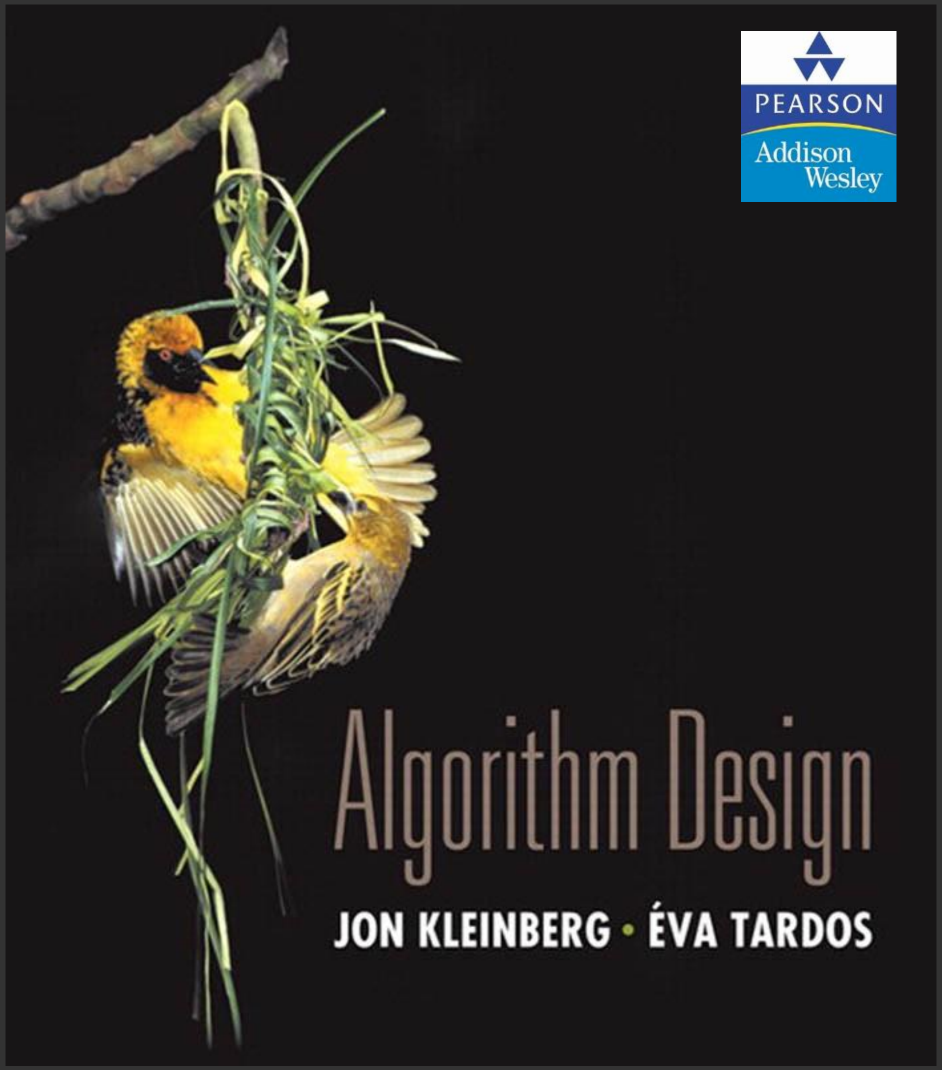
Fast matrix multiplication. (Strassen, 1969)

- Divide: partition A and B into ½n-by-½n blocks.
- Compute: 14 ½n-by-½n matrices via 10 matrix additions.
- Conquer: multiply 7 ½n-by-½n matrices recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

Analysis.

- Assume n is a power of 2.
- T(n) = # arithmetic operations.

$$\mathrm{T}(n) = \underbrace{7T\left(n/2\right)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \quad \Rightarrow \quad \mathrm{T}(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$



6. Dynamic Programming I

- *weighted interval scheduling*
- *segmented least squares*
- *knapsack problem*
- *RNA secondary structure*

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Algorithmic paradigms

Greed. Process the input in some order, myopically making irrevocable decisions.

Divide-and-conquer. Break up a problem into independent subproblems; solve each subproblem; combine solutions to subproblems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping subproblems; combine solutions to smaller subproblems to form solution to large subproblem.

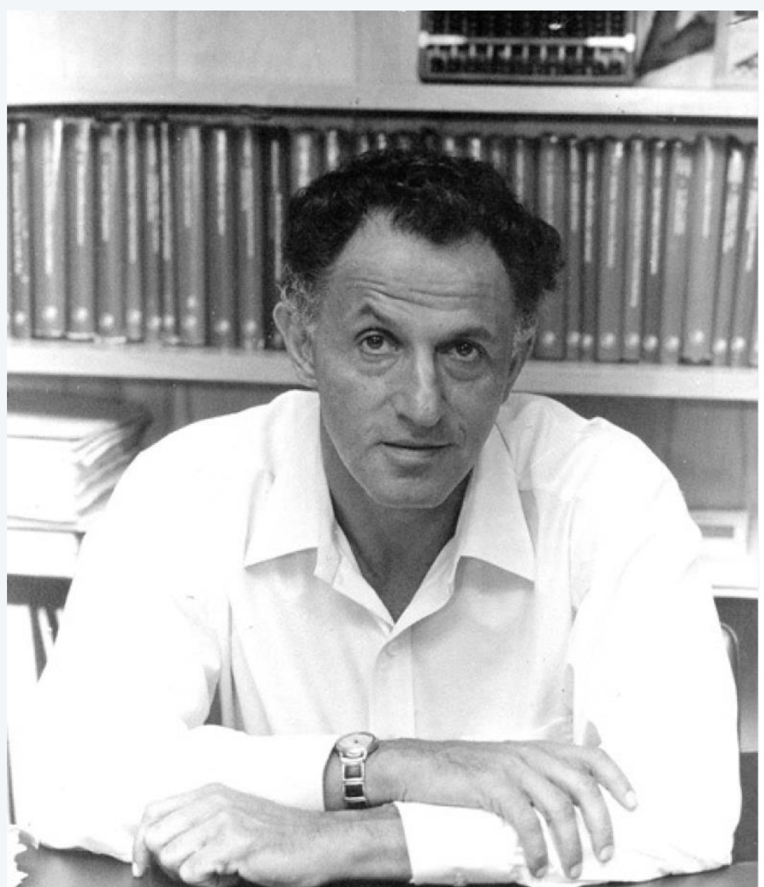
fancy name for
caching intermediate results
in a table for later reuse

Dynamic programming history

Bellman. Pioneered the systematic study of dynamic programming in 1950s.

Etymology.

- Dynamic programming = planning over time.
- Secretary of Defense had pathological fear of mathematical research.
- Bellman sought a “dynamic” adjective to avoid conflict.



THE THEORY OF DYNAMIC PROGRAMMING

RICHARD BELLMAN

1. **Introduction.** Before turning to a discussion of some representative problems which will permit us to exhibit various mathematical features of the theory, let us present a brief survey of the fundamental concepts, hopes, and aspirations of dynamic programming.

To begin with, the theory was created to treat the mathematical problems arising from the study of various multi-stage decision processes, which may roughly be described in the following way: We have a physical system whose state at any time t is determined by a set of quantities which we call state parameters, or state variables. At certain times, which may be prescribed in advance, or which may be determined by the process itself, we are called upon to make decisions which will affect the state of the system. These decisions are equivalent to transformations of the state variables, the choice of a decision being identical with the choice of a transformation. The outcome of the preceding decisions is to be used to guide the choice of future ones, with the purpose of the whole process that of maximizing some function of the parameters describing the final state.

Examples of processes fitting this loose description are furnished by virtually every phase of modern life, from the planning of industrial production lines to the scheduling of patients at a medical clinic; from the determination of long-term investment programs for universities to the determination of a replacement policy for machinery in factories; from the programming of training policies for skilled and unskilled labor to the choice of optimal purchasing and inventory policies for department stores and military establishments.

Dynamic programming applications

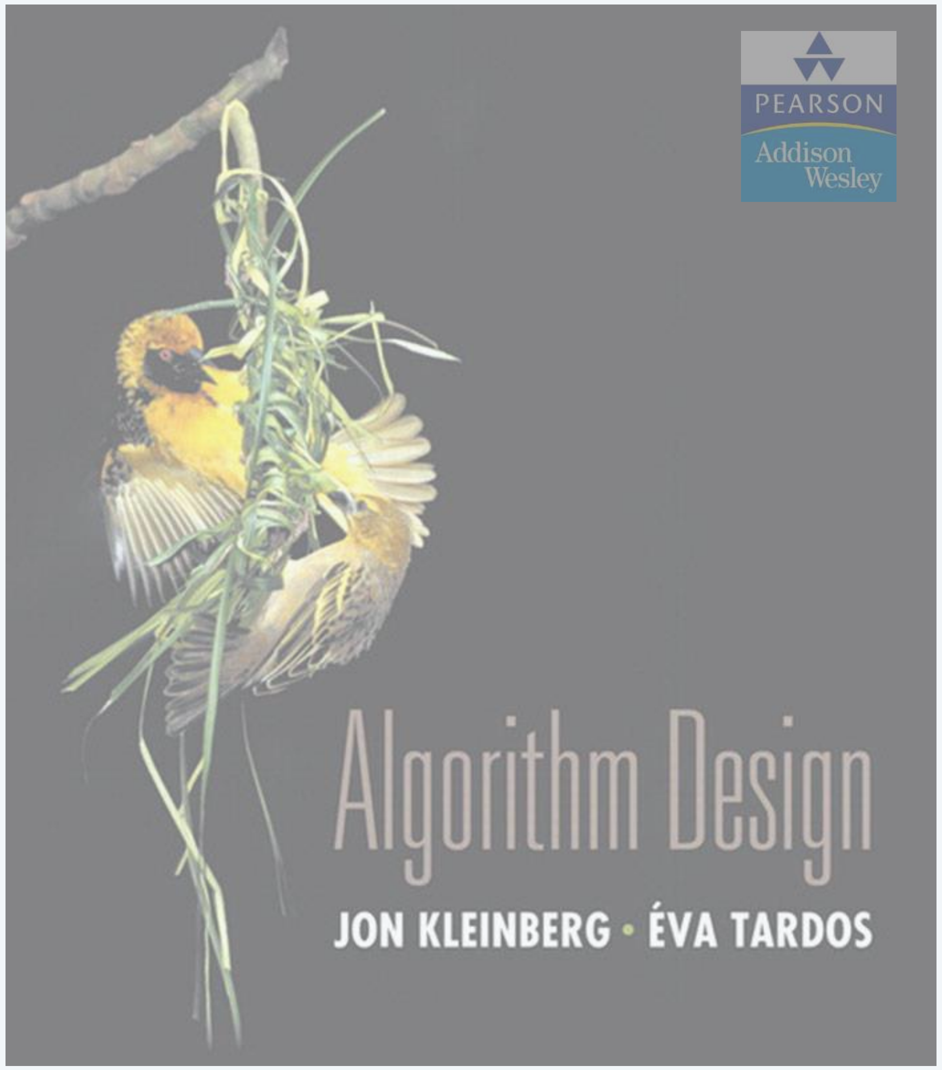
Application areas.

- Computer science: AI, compilers, systems, graphics, theory, ….
- Operations research.
- Information theory.
- Control theory.
- Bioinformatics.

Some famous dynamic programming algorithms.

- Avidan–Shamir for seam carving.
- Unix diff for comparing two files.
- Viterbi for hidden Markov models.
- De Boor for evaluating spline curves.
- Bellman–Ford–Moore for shortest path.
- Knuth–Plass for word wrapping text in T_1X.
- Cocke–Kasami–Younger for parsing context-free grammars.
- Needleman–Wunsch/Smith–Waterman for sequence alignment.

Dynamic programming books



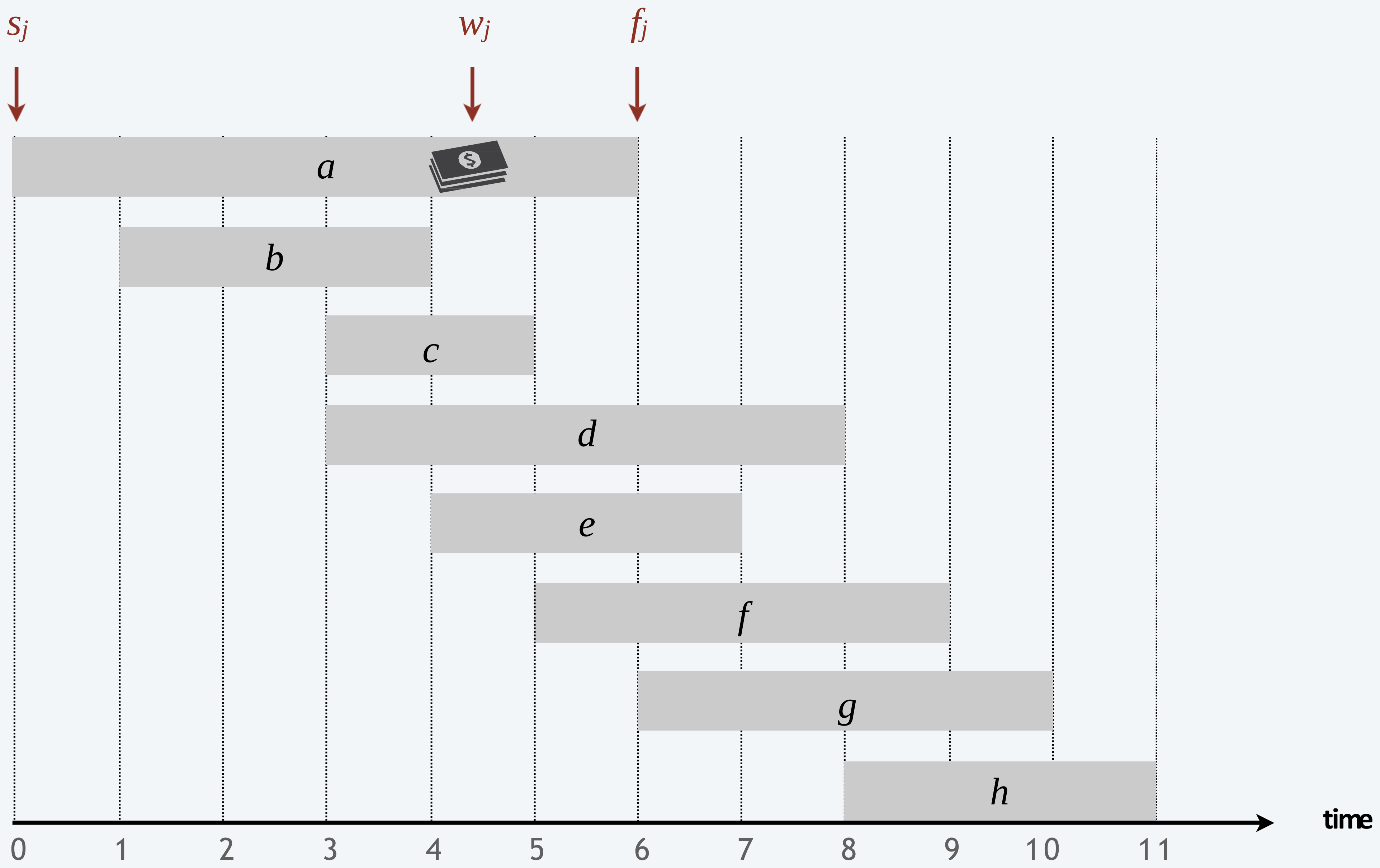
SECTIONS 6.1–6.2

6. DYNAMIC PROGRAMMING I

- *weighted interval scheduling*
- *segmented least squares*
- *knapsack problem*
- *RNA secondary structure*

Weighted interval scheduling

- Job j starts at s_j, finishes at f_j, and has weight $w_j > 0$.
- Two jobs are compatible if they don't overlap.
- Goal: find max-weight subset of mutually compatible jobs.

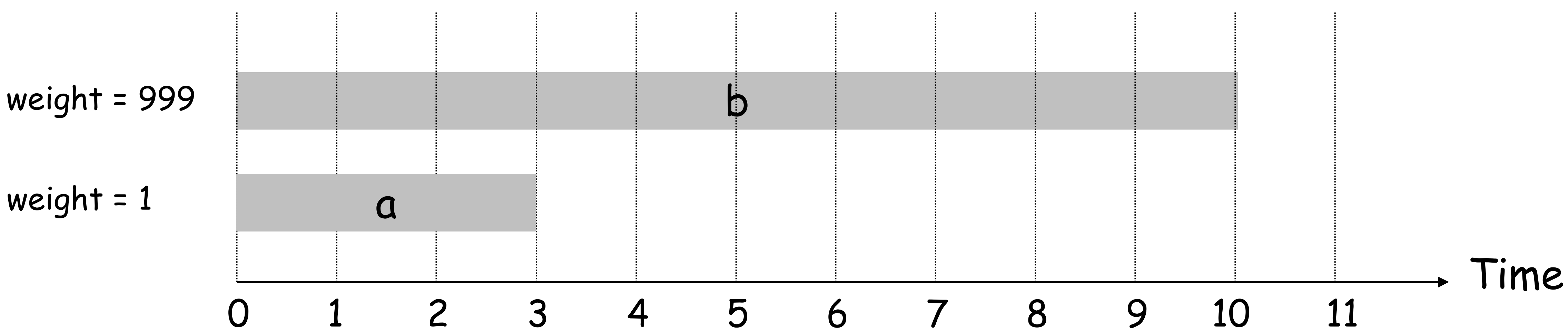


Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.

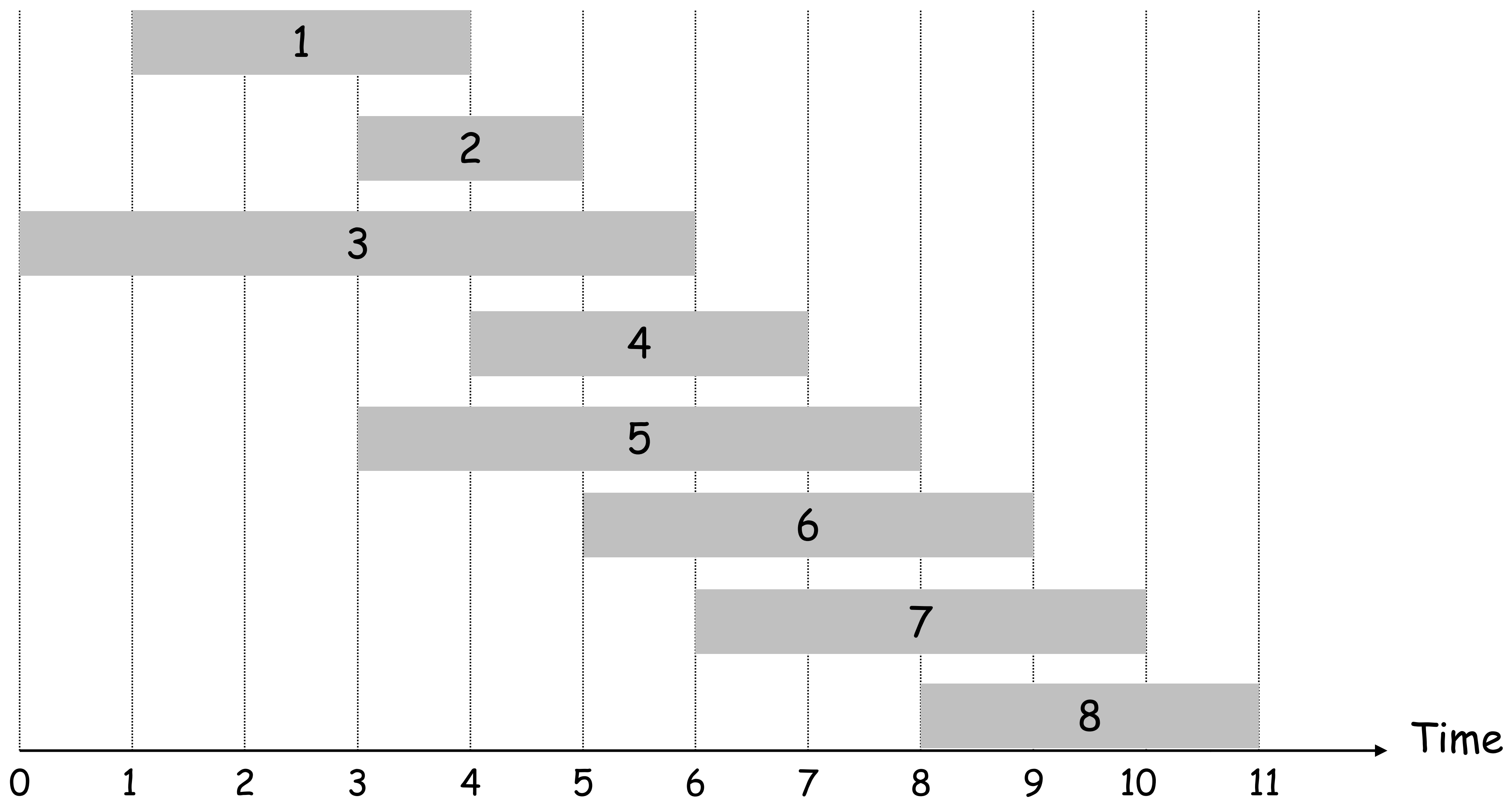


Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$.

Def. p(j) = largest index i < j such that job i is compatible with j.

Ex: p(8) = 5, p(7) = 3, p(2) = 0.



Dynamic Programming: Binary Choice

Notation. OPT(j) = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

- Case 1: OPT selects job j.
  - can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)

optimal substructure

- Case 2: OPT does not select job j.
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left\{ v_j + OPT(p(j)),\ OPT(j-1) \right\} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

Brute force algorithm.

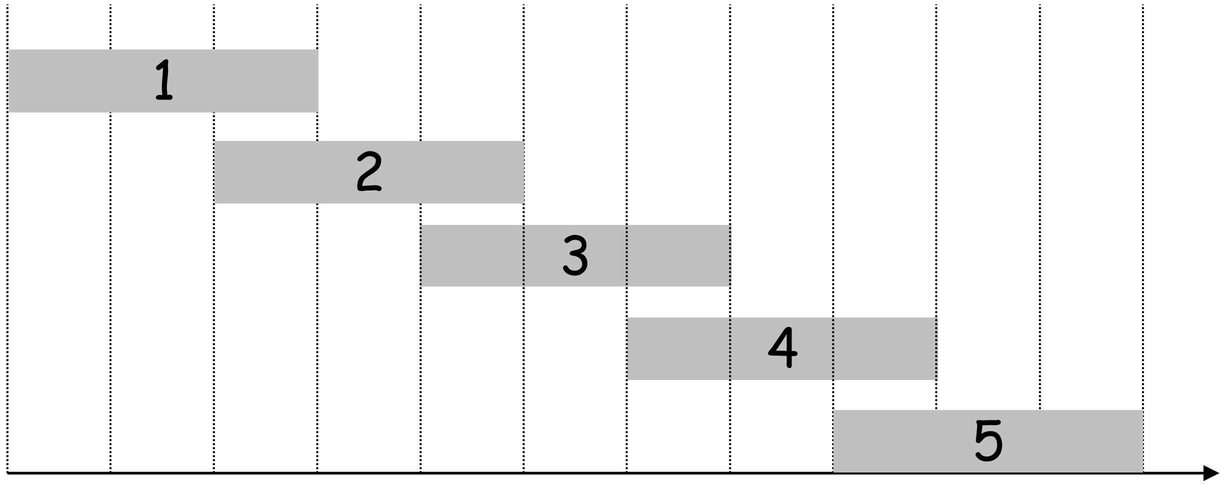
```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max(vj + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

Running time: $T(n) = T(p(n)) + T(n-1) + O(1) = \ldots$

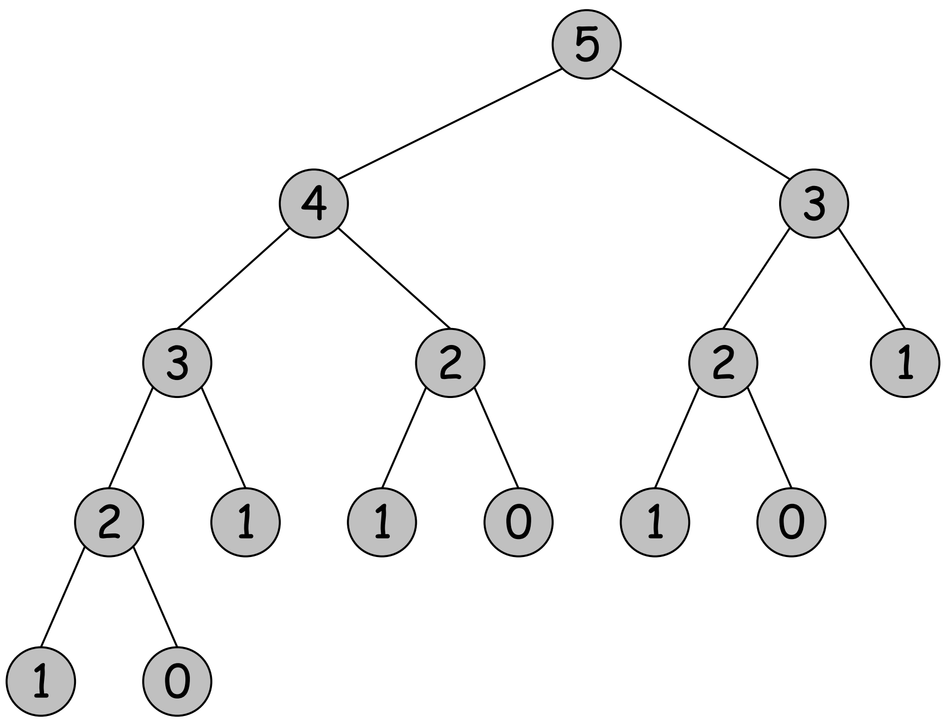
Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems $\Rightarrow$ exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



p(1) = 0, p(j) = j-2



Running time: $T(n) = T(p(n)) + T(n-1) + O(1) = T(n-2) + T(n-1) + O(1) = \Theta(\varphi^n)$
where $\varphi = 1{:}618$ is the golden ratio

Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.
Compute p(1), p(2), …, p(n)

for j = 1 to n
   M[j] = empty   ← global array
M[j] = 0

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max(wj + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
}
```

Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes O(n log n) time.

- Sort by finish time: O(n log n).
- Computing $p(\cdot)$: O(n) after sorting by start time.

- `M-Compute-Opt(j)`: each invocation takes O(1) time and either
  - (i) returns an existing value `M[j]`
  - (ii) fills in one new entry `M[j]` and makes two recursive calls

- Progress measure Φ = # nonempty entries of `M[]`.
  - initially $\Phi = 0$, throughout $\Phi \leq n$.
  - (ii) increases Φ by 1 $\Rightarrow$ at most 2n recursive calls, since there are only n entries to fill.

- Overall running time of `M-Compute-Opt(n)` is O(n). ▪

Remark. O(n) if jobs are pre-sorted by start and finish times.

Automated Memoization

Automated memoization. Many functional programming languages (e.g., Lisp) have built-in support for memoization.

Q. Why not in imperative languages (e.g., Java)?

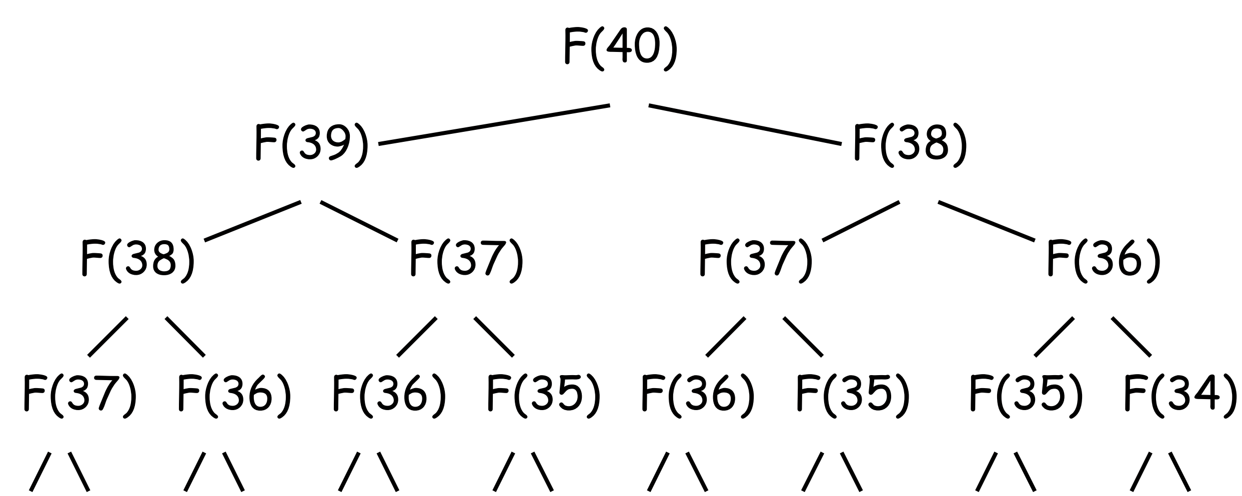
A. compiler's job is easier in pure functional languages since no side effects

```
(defun F (n)
  (if
    (<= n 1)
    n
    (+ (F (- n 1)) (F (- n 2)))))
```

Lisp (efficient)

```
static int F(int n) {
   if (n <= 1) return n;
   else return F(n-1) + F(n-2);
}
```

Java (exponential)



Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?
A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
   if (j = 0)
      output nothing
   else if (v_j + M[p(j)] > M[j-1])
      print j
      Find-Solution(p(j))
   else
      Find-Solution(j-1)
}
```

- # of recursive calls $\leq n \Rightarrow O(n)$.

Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max(vj + M[p(j)], M[j-1])
}
```

Dynamic Programming Overview

Dynamic Programming = Recursion + Memoization

1 Formulate problem recursively in terms of solutions to
  polynomially many sub-problems
2 Solve sub-problems bottom-up, storing optimal solutions

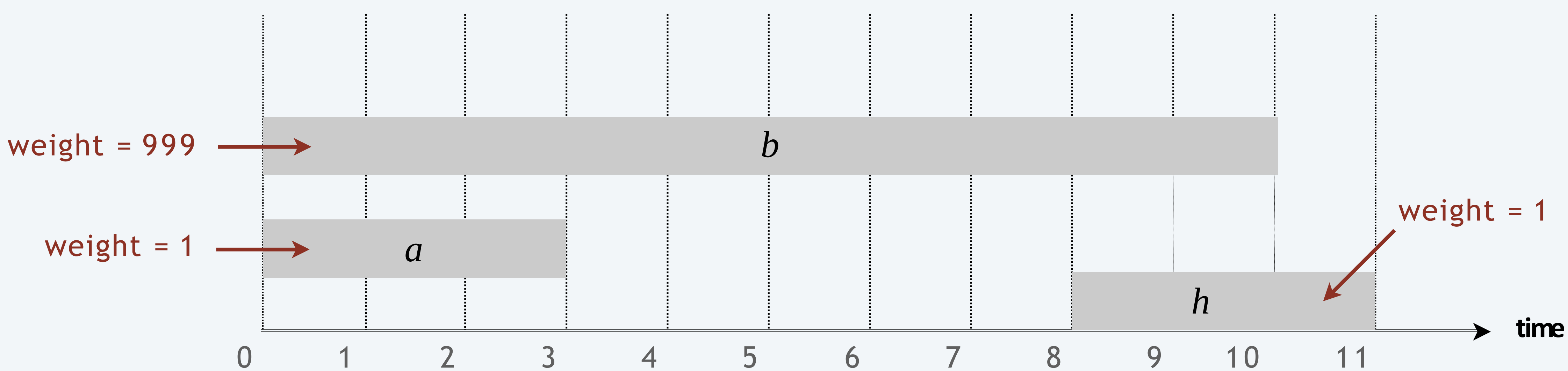
Earliest-finish-time first algorithm

Earliest finish-time first.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Recall. Greedy algorithm is correct if all weights are 1.

Observation. Greedy algorithm fails spectacularly for weighted version.



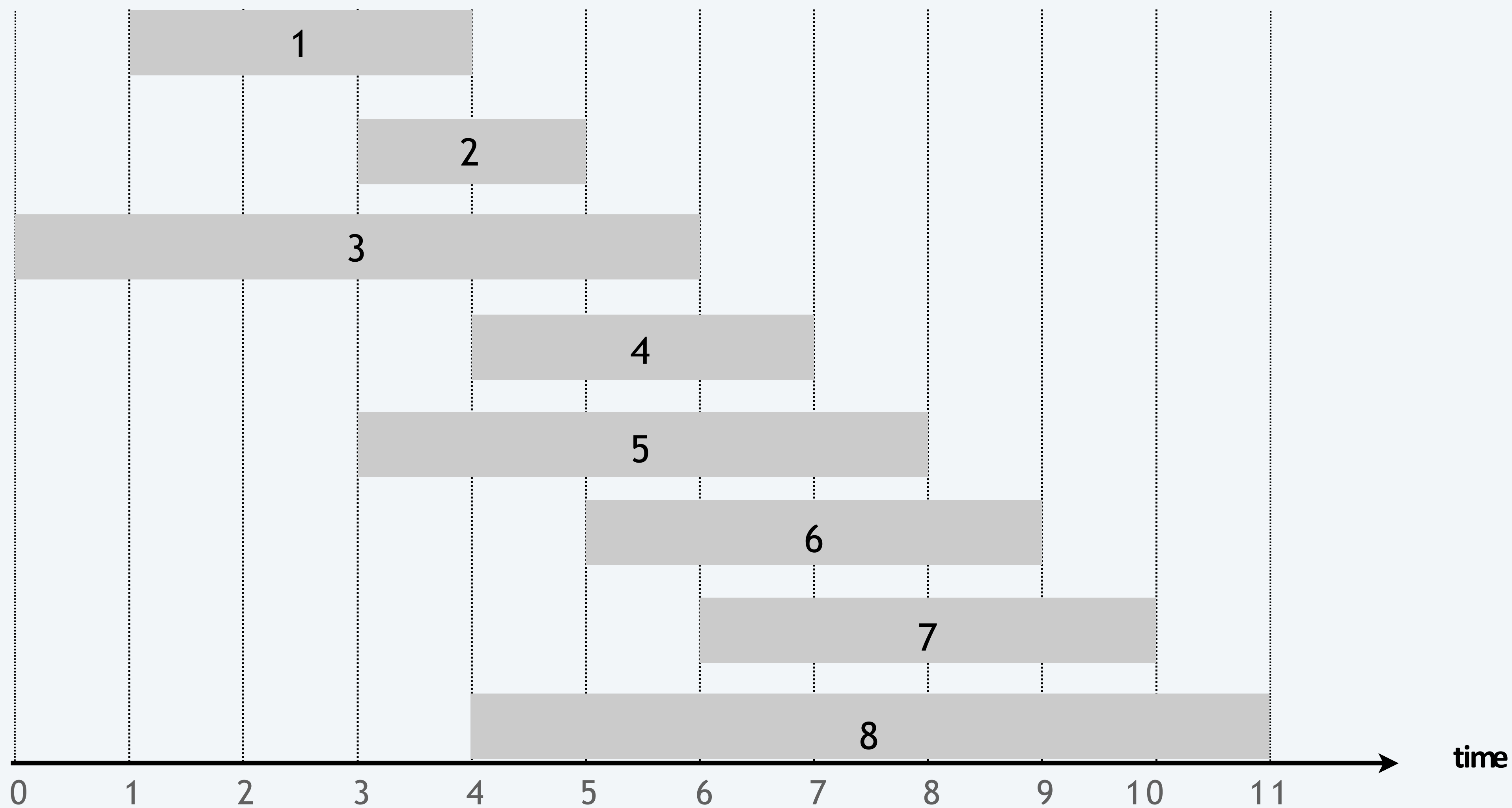
Weighted interval scheduling

Convention. Jobs are in ascending order of finish time: $f_1 \leq f_2 \leq \ldots \leq f_n$.

Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j.

Ex. $p(8) = 1, p(7) = 3, p(2) = 0.$

i is leftmost interval that ends before *j* begins



Dynamic programming: binary choice

Def. $OPT(j)$ = max weight of any subset of mutually compatible jobs for subproblem consisting only of jobs $1, 2, ..., j$.

Goal. $OPT(n)$ = max weight of any subset of mutually compatible jobs.

Case 1. $OPT(j)$ does not select job j.

- Must be an optimal solution to problem consisting of remaining jobs $1, 2, ..., j-1$.

optimal substructure property
(proof via exchange argument)

Case 2. $OPT(j)$ selects job j.

- Collect profit w_j.
- Can't use incompatible jobs $\{ p(j)+1, p(j)+2, ..., j-1 \}$.
- Must include optimal solution to problem consisting of remaining compatible jobs $1, 2, ..., p(j)$.

Bellman equation. $$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ OPT(j-1),\ w_j + OPT(p(j)) \} & \text{if } j > 0 \end{cases}$$

Weighted interval scheduling: brute force

BRUTE-FORCE $(n, s_1, \ldots, s_n, f_1, \ldots, f_n, w_1, \ldots, w_n)$

Sort jobs by finish time and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

Compute $p[1], p[2], \ldots, p[n]$ via binary search.

RETURN COMPUTE-OPT(n).

COMPUTE-OPT(j)

IF ($j = 0$)

RETURN 0.

ELSE

RETURN max {COMPUTE-OPT($j-1$), w_j + COMPUTE-OPT($p[j]$) }.

Dynamic programming: quiz 1

What is running time of COMPUTE-OPT(n) in the worst case?

- **A.** $\Theta(n \log n)$
- **B.** $\Theta(n^2)$
- **C.** $\Theta(1.618^n)$
- **D.** $\Theta(2^n)$ ✓

COMPUTE-OPT(j)

IF (j = 0)

  RETURN 0.

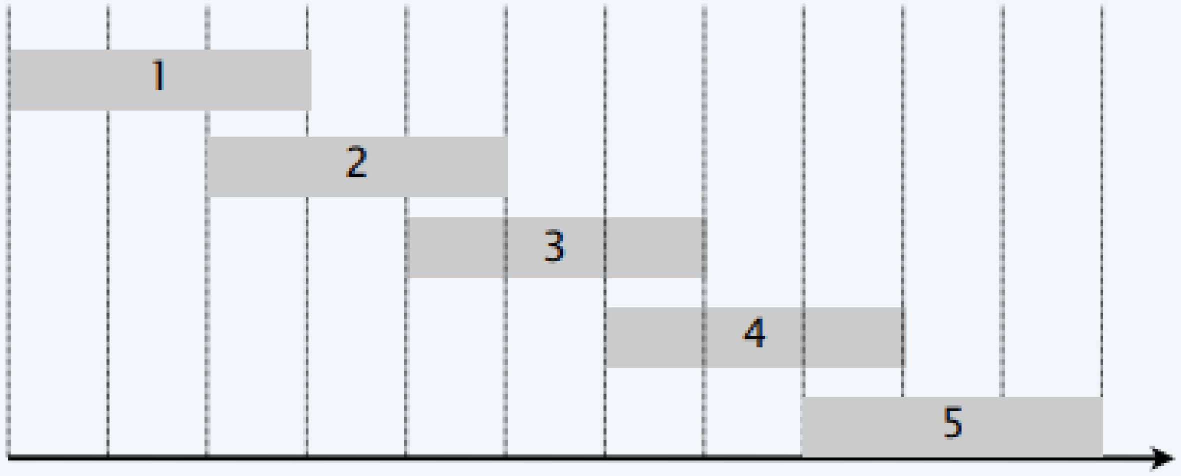
ELSE

  RETURN max {COMPUTE-OPT(j – 1), w_j + COMPUTE-OPT($p[j]$) }.

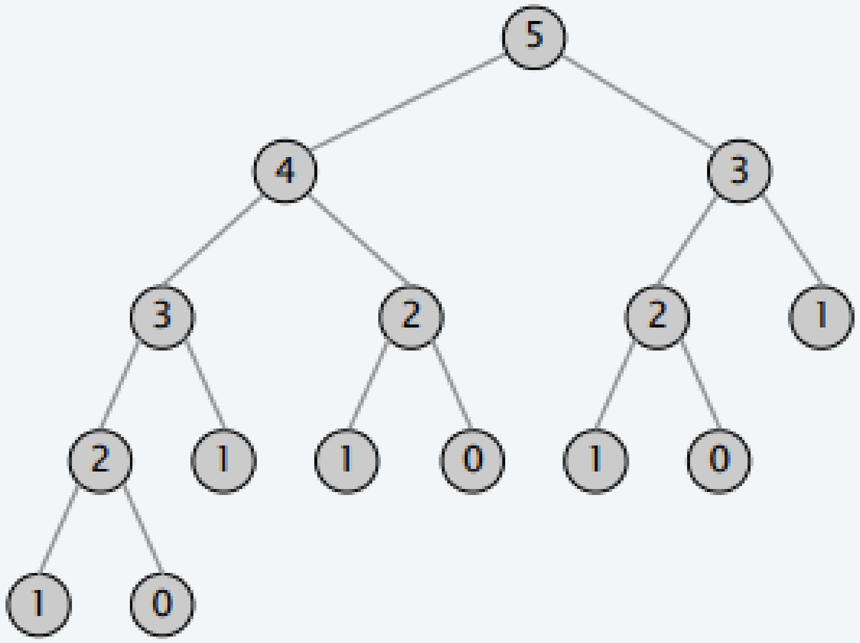
Weighted interval scheduling: brute force

Observation. Recursive algorithm is spectacularly slow because of overlapping subproblems $\Rightarrow$ exponential-time algorithm.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



$p(1) = 0, \ p(j) = j-2$



recursion tree

Weighted interval scheduling: memoization

Top-down dynamic programming (memoization).

- Cache result of subproblem j in $M[j]$.
- Use $M[j]$ to avoid solving subproblem j more than once.

TOP-DOWN($n, s_1, \ldots, s_n, f_1, \ldots, f_n, w_1, \ldots, w_n$)

Sort jobs by finish time and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

Compute $p[1], p[2], \ldots, p[n]$ via binary search.

$M[0] \leftarrow 0.$ ← global array

RETURN M-COMPUTE-OPT(n).

M-COMPUTE-OPT(j)

IF ($M[j]$ is uninitialized)

$M[j] \leftarrow \max \{$ M-COMPUTE-OPT$(j-1)$, w_j + M-COMPUTE-OPT$(p[j])$ $\}$.

RETURN $M[j]$.

Weighted interval scheduling: running time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.

Pf.

- Sort by finish time: $O(n \log n)$ via mergesort.
- Compute $p[j]$ for each j : $O(n \log n)$ via binary search.

- M-COMPUTE-OPT(j): each invocation takes $O(1)$ time and either
  - (1) returns an initialized value $M[j]$
  - (2) initializes $M[j]$ and makes two recursive calls

- Progress measure Φ = # initialized entries among $M[1..n]$.
  - initially $\Phi = 0$; throughout $\Phi \leq n$.
  - (2) increases Φ by $1 \Rightarrow \leq 2n$ recursive calls.

- Overall running time of M-COMPUTE-OPT(n) is $O(n)$. ▪

Those who cannot remember the past are condemned to repeat it.

- Dynamic Programming

Weighted interval scheduling: finding a solution

Q. DP algorithm computes optimal value. How to find optimal solution?

A. Make a second pass by calling FIND-SOLUTION(n).

FIND-SOLUTION(j)

IF ($j = 0$)

RETURN $\varnothing$.

ELSE IF ($w_j + M[p[j]] > M[j-1]$)

RETURN $\{ j \} \cup$ FIND-SOLUTION($p[j]$).

ELSE

RETURN FIND-SOLUTION($j-1$).

$$M[j] = \max \{ M[j-1],\ w_j + M[p[j]] \}.$$

Analysis. # of recursive calls $\leq n \Rightarrow O(n)$.

Weighted interval scheduling: bottom-up dynamic programming

Bottom-up dynamic programming. Unwind recursion.

BOTTOM-UP(n, s_1, …, s_n, f_1, …, f_n, w_1, …, w_n)

Sort jobs by finish time and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

Compute $p[1], p[2], \ldots, p[n]$.

$M[0] \leftarrow 0$.

FOR $j = 1$ TO n

previously computed values

$\quad M[j] \leftarrow \max \{ M[j-1],\ w_j + M[p[j]] \}$.

Running time. The bottom-up version takes $O(n \log n)$ time.

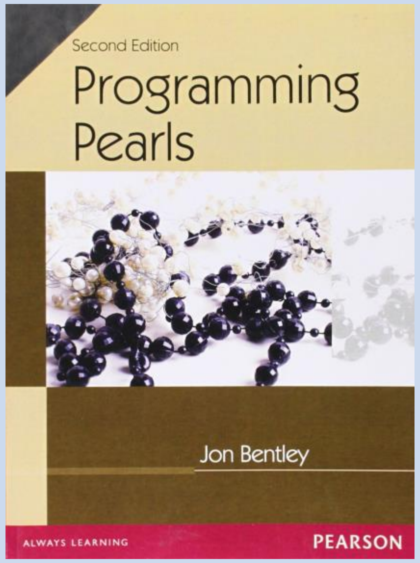
MAXIMUM SUBARRAY PROBLEM

Goal. Given an array x of n integer (positive or negative), find a contiguous subarray whose sum is maximum.

12	5	–1	31	–61	59	26	–53	58	97	–93	–23	84	–15	6

187

Applications. Computer vision, data mining, genomic sequence analysis, technical job interviews, ….



MAXIMUM RECTANGLE PROBLEM

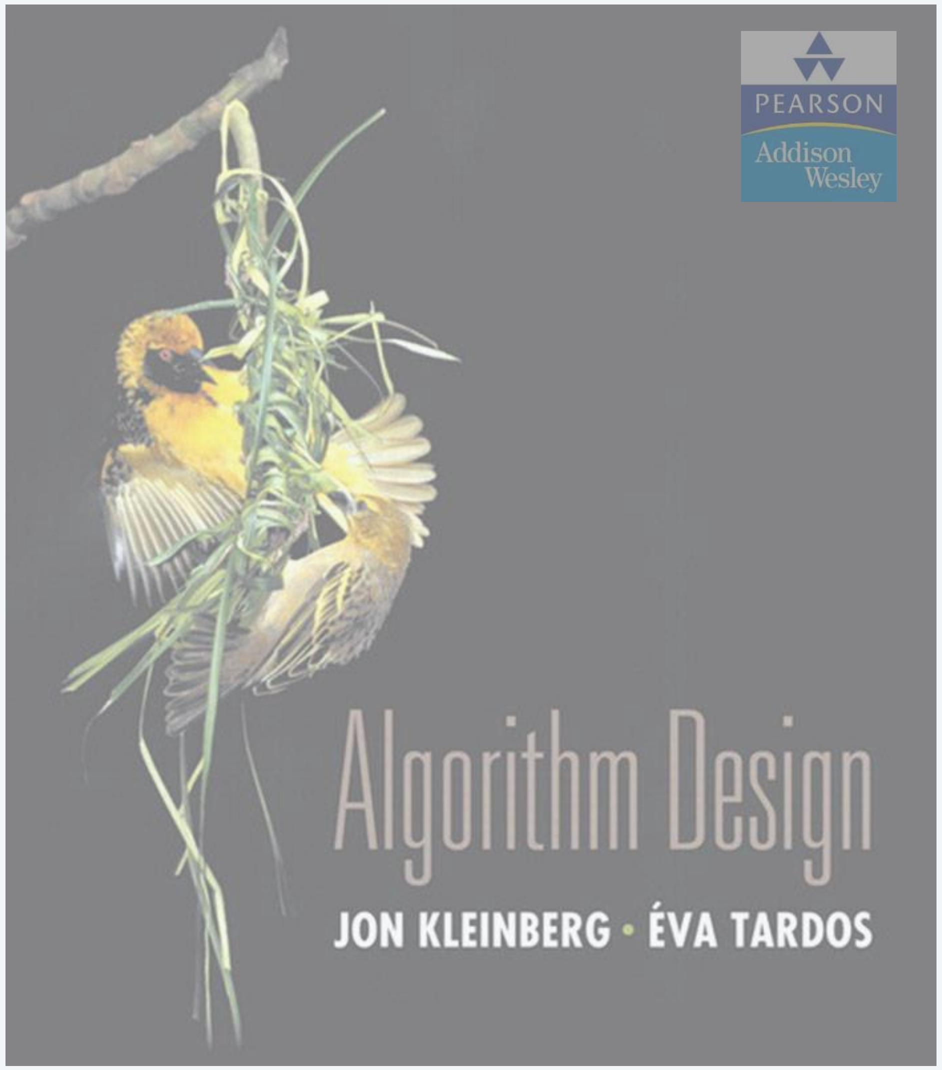
Goal. Given an n-by-n matrix A, find a rectangle whose sum is maximum.

$$A = \begin{bmatrix} -2 & 5 & 0 & -5 & -2 & 2 & -3 \\ 4 & -3 & -1 & 3 & 2 & 1 & -1 \\ -5 & 6 & 3 & -5 & -1 & -4 & -2 \\ -1 & -1 & 3 & -1 & 4 & 1 & 1 \\ 3 & -3 & 2 & 0 & 3 & -3 & -2 \\ -2 & 1 & -2 & 1 & 1 & 3 & -1 \\ 2 & -4 & 0 & 1 & 0 & -3 & -1 \end{bmatrix}$$

13

Applications. Databases, image processing, maximum likelihood estimation, technical job interviews, …

6. DYNAMIC PROGRAMMING



SECTION 6.3

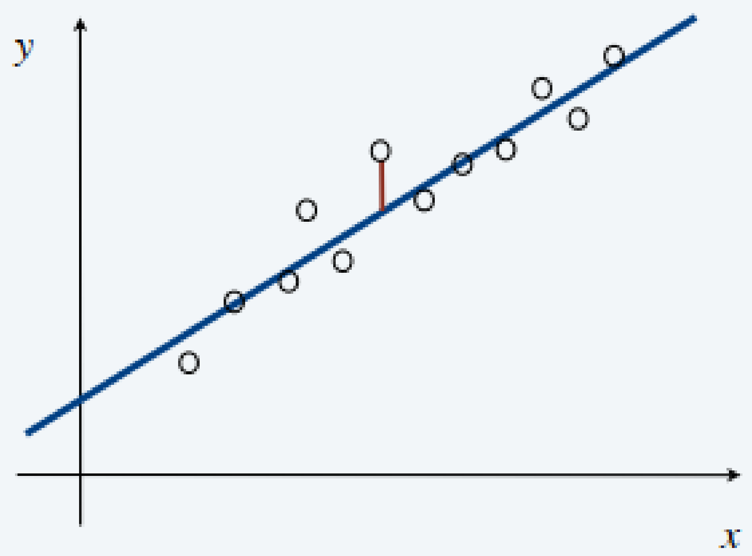
- weighted interval scheduling
- *segmented least squares*
- knapsack problem
- RNA secondary structure

Least squares

Least squares. Foundational problem in statistics.

- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
- Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$



Solution. Calculus $\Rightarrow$ min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

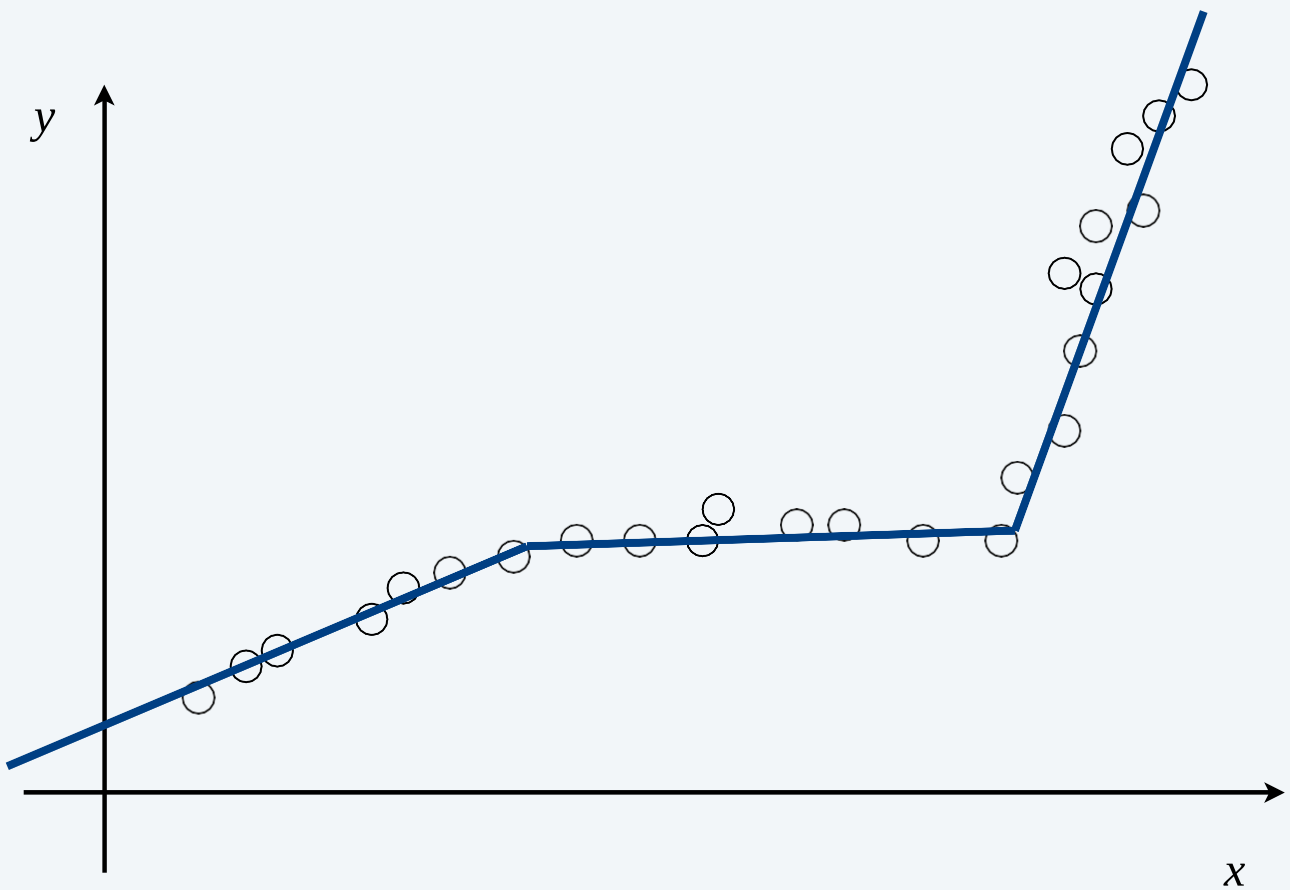
Segmented least squares

Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What is a reasonable choice for $f(x)$ to balance accuracy and parsimony?

goodness of fit — number of lines



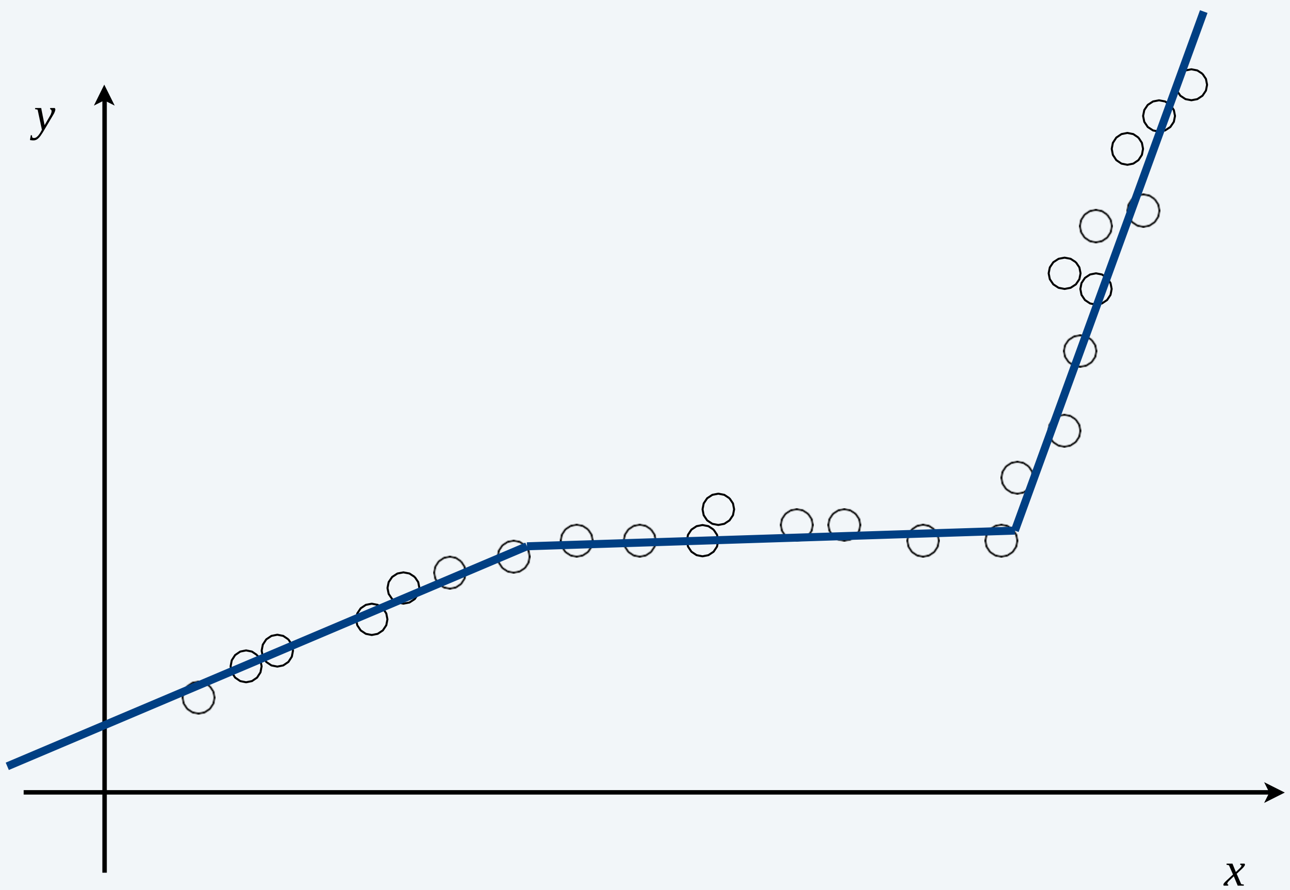
Segmented least squares

Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes $f(x)$.

Goal. Minimize $f(x) = E + c\,L$ for some constant $c > 0$, where

- E = sum of the sums of the squared errors in each segment.
- L = number of lines.



Dynamic programming: multiway choice

Notation.

- $OPT(j)$ = minimum cost for points $p_1, p_2, \ldots, p_j$.
- e_{ij} = SSE for for points $p_i, p_{i+1}, \ldots, p_j$.

To compute $OPT(j)$:

- Last segment uses points $p_i, p_{i+1}, \ldots, p_j$ for some $i \le j$.
- Cost = $e_{ij} + c + OPT(i-1)$. ← optimal substructure property (proof via exchange argument)

Bellman equation.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min\limits_{1 \le i \le j} \{ e_{ij} + c + OPT(i-1) \} & \text{if } j > 0 \end{cases}$$

Segmented least squares algorithm

SEGMENTED-LEAST-SQUARES(n, p_1, …, p_n, c)

FOR j = 1 TO n

  FOR i = 1 TO j

    Compute the SSE e_{ij} for the points p_i, p_{i+1}, …, p_j.

$M[0] \leftarrow 0$.

FOR $j = 1$ TO n

previously computed value

  $M[j] \leftarrow \min_{1 \le i \le j} \{ e_{ij} + c + M[i-1] \}$.

RETURN $M[n]$.

Segmented least squares analysis

Theorem. [Bellman 1961] DP algorithm solves the segmented least squares problem in $O(n^3)$ time and $O(n^2)$ space.

Pf.

- Bottleneck = computing SSE e_{ij} for each i and j.

$$a_{ij} = \frac{n \sum_k x_k y_k - (\sum_k x_k)(\sum_k y_k)}{n \sum_k x_k^2 - (\sum_k x_k)^2}, \quad b_{ij} = \frac{\sum_k y_k - a_{ij} \sum_k x_k}{n}$$

- $O(n)$ to compute e_{ij}. ▪

Remark. Can be improved to $O(n^2)$ time.

- For each i: precompute cumulative sums $\sum_{k=1}^{i} x_k, \quad \sum_{k=1}^{i} y_k, \quad \sum_{k=1}^{i} x_k^2, \quad \sum_{k=1}^{i} x_k y_k$.
- Using cumulative sums, can compute e_{ij} in $O(1)$ time.

Dynamic programming summary

typically, only a polynomial number of subproblems

Outline.

- Define a collection of subproblems.
- Solution to original problem can be computed from subproblems.
- Natural ordering of subproblems from "smallest" to "largest" that enables determining a solution to a subproblem from solutions to smaller subproblems.

Techniques.

- Binary choice: weighted interval scheduling.
- Multiway choice: segmented least squares.
- Adding a new variable: knapsack problem.
- Intervals: RNA secondary structure.

Top-down vs. bottom-up dynamic programming. Opinions differ.

NEXT LECTURE

- Dynamic Programming
- Segmented Least Squares
- Knapsack Problem