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Solutions of Question 3 in Homework 1

Only Question 3 will be solved and graded! Each part is worth approximately 9 points.

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(3) (a) Suppose that the universes for the variables x and y are the set of all cars. Let a denote my car. Translate the following quantified assertions into sentences in plain English where the open statements $L(x, y), M(x, y)$, and $N(x, y)$ are given as follows:

$L(x, y) = x$ is faster than y; $M(x, y) = x$ is more expensive than y; $N(x, y) = x$ is older than y.

(i) $(\exists x)(\forall y)L(x, y)$
There is a car x such that, for all cars y, x is faster than y.
There is a car x such that x is faster than y for all cars y.

$\boxed{\text{"There is a car which is faster than every car"}}$

(ii) $(\forall x)(\exists y)M(x, y)$
For all cars x, there exists a car y such that x is more expensive than y.
For any car x, there is a car y such that y is cheaper than x.

$\boxed{\text{"For each car there is a car which is cheaper"}}$

(iii) $(\exists y)(\forall x)\big(L(x, y) \vee N(x, y)\big)$
There is a car y such that, for all cars x, either x is faster than y or x is older than y.
There is a car y such that, either x is faster than y or x is older than y for all cars x.
There is a car y such that, either y is slower than x or y is newer than x for all cars x.

$\boxed{\text{"There is a car which is either slower or newer than any car"}}$

(iv) $(\forall y)(\exists x)\big(\neg M(x, y) \rightarrow L(x, y)\big)$
For all cars y, there is a car x such that, if x is not more expensive than y then x is faster than y.
(To write the above sentence without variables x, y we may use equivalence $P \rightarrow Q \equiv \neg P \vee Q$)
For any car y, there is a car x such that, either x is more expensive than y or x is faster than y.
For any car y, there is a car x such that, either y is cheaper than x or y is slower than x.

$\boxed{\text{"For any car there is car which is either cheaper or slower"}}$

(v) $\forall x L(a, x)$
For all cars x, a is faster than x.
a is faster than x for all cars x.

$\boxed{\text{"My car is faster than every car"}}$

(vi) $\forall x\Big(\big(M(x, a) \wedge N(x, a)\big) \rightarrow \neg L(x, a)\Big)$
For all cars x, if x is more expensive than a and x is older than a, then x is not faster than a.
(To write the above sentence without variables x, y we may argue as follows: Let

$$A = \big\{x \mid M(x, a) \wedge N(x, a)\big\}$$

be the set of all cars x such that x is more expensive than a and x is older than a. Recalling the logical equivalence

$$(\forall z \in S)P(z) \equiv \forall z\big(z \in S \rightarrow P(z)\big)$$

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where S is a set and $P(z)$ is an open statement, we see that

$$\forall x\Big(\big(M(x,a) \wedge N(x,a)\big) \rightarrow \neg L(x,a)\Big) \equiv (\forall x \in A)\neg L(x,a). \quad)$$

$\boxed{\text{“Any car which is more expensive and older than my car is not faster than my car”}}$

(b) Write a negation in plain English for each of the following quantified assertions.

(i) There is a place in the world such that everyone who enter the place goes blind.
Given sentence may be symbolized as

$$(\exists p)(\forall x)\big(xEp \rightarrow B(x)\big)$$

where “$xEp = x$ enters p”, “$B(x) = x$ goes blind”, the universe for p is the all places in the world and the universe for x is the all people. Its negation is

$$\neg\left((\exists p)(\forall x)\big(xEp \rightarrow B(x)\big)\right) \equiv (\forall p)(\exists x)\neg\big(xEp \rightarrow B(x)\big) \equiv (\forall p)(\exists x)\big(xEp \wedge \neg B(x)\big)$$

where we used the equivalence $P \rightarrow Q \equiv \neg P \vee Q$. The negation can be translated as

$\boxed{\text{“For each place in the world there is a person who enters the place and does not go blind”}}$

(ii) At least one person in Istanbul owns every newspapers published yesterday.
Given sentence may be symbolized as

$$(\exists x)(\forall y)xOy$$

where “$xOy = x$ owns y”, the universe for x is the all people in Istanbul and the universe for y is the all published newspapers yesterday. Its negation is

$$\neg\left((\exists x)(\forall y)xOy\right) \equiv (\forall x)(\exists y)\neg xOy$$

The negation can be translated as

$\boxed{\text{“For each person in Istanbul there is a newspaper published yesterday which is not owned by the person”}}$

(iii) Everyone has a secret that is not known by more than two people.
Given sentence may be symbolized as

$$(\forall x)(\exists s)\big(xHs \wedge \neg K(s)\big)$$

where “$xHs = x$ has s”, “$K(s) = s$ is known by more than two people”, the universe for x is the all people and the universe for s is the all secrets. Its negation is

$$\neg\left((\forall x)(\exists s)\big(xHs \wedge \neg K(s)\big)\right) \equiv (\exists x)(\forall s)\big(\neg xHs \vee K(s)\big)$$

The negation can be translated as
“There is a person x such that, for all secrets s, either x does not have s or s is known by more than two people” To write the negation without variables x, s we may use the equivalences

$$P \rightarrow Q \equiv \neg P \vee Q \qquad \text{and} \qquad (\forall z \in A)R(z) \equiv \forall z\big(z \in A \rightarrow R(z)\big)$$

where A is a set. So

$$(\exists x)(\forall s)\big(\neg xHs \vee K(s)\big) \equiv (\exists x)(\forall s)\big(xHs \rightarrow K(s)\big) \equiv (\exists x)(\forall s)\big(s \in A_x \rightarrow K(s)\big) \equiv (\exists x)(\forall s \in A_x)K(s)$$

where for any person x the set $A_x = \big\{t \mid xHt\big\}$ is the set of all secrets t such that x has t. Note that $t \in A_x$ if and only if xHt. Now the negation $(\exists x)(\forall s \in A_x)K(s)$ can be translated as
“There is a person x such that, for all secrets s that x has, s is known by more than two people”

$\boxed{\text{“There is a person each of whose secrets is known by more than two people”}}$

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(iv) Every integer is either odd or even, but not both.
Given sentence may be symbolized as

$$(\forall z \in \mathbb{Z}) \big[\big(z \in O \vee z \in E \big) \wedge \neg (z \in O \wedge z \in E) \big]$$

where O is the set of odd integers and E is the set of even integers. Its negation is

$$\neg \Big((\forall z \in \mathbb{Z}) \big[\big(z \in O \vee z \in E \big) \wedge \neg (z \in O \wedge z \in E) \big] \Big) \equiv (\exists z \in \mathbb{Z}) \big[\big(z \notin O \wedge z \notin E \big) \vee (z \in O \wedge z \in E) \big]$$

The negation can be translated as
"There is an integer z such that, either z is not odd and not even, or z is both odd and even"
To rewrite in a simpler form we may use the equivalence $P \to Q \equiv \neg P \vee Q$ to see that

$$(\exists z \in \mathbb{Z}) \big[\big(z \notin O \wedge z \notin E \big) \vee (z \in O \wedge z \in E) \big] \equiv (\exists z \in \mathbb{Z}) \big[\big(z \in O \vee z \in E \big) \to (z \in O \wedge z \in E) \big]$$

Now the negation can be translated as

$\boxed{\text{"There is an integer such that if it is either odd or even, then it is both odd and even"}}$

(Some symbols above may be simplified if we remember the definitions of the union and the intersection of the sets because

$$\big(z \in O \vee z \in E \big) \equiv z \in (O \cup E) \quad \text{and} \quad \big(z \in O \wedge z \in E \big) \equiv z \in (O \cap E) \;\;)$$

(v) There are real numbers a and b such that $a < s < b$ for any element s of the set S.
Given sentence may be symbolized as

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})(\forall s \in S)(a < s < b)$$

Its negation is

$$\neg \left((\exists a \in \mathbb{R})(\exists b \in \mathbb{R})(\forall s \in S)(a < s < b) \right) \equiv (\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\exists s \in S) \neg (a < s < b)$$

The negation can be translated as

$\boxed{\text{"For all real numbers } a \text{ and } b \text{ there is an element } s \text{ of the set } S \text{ such that it is not the case that } a < s < b\text{"}}$

To rewrite it in a simpler form we may argue as follows: For this we should assume that S is a nonempty set! Note that for the condition $a < s < b$ to be satisfied for all $s \in S$, the condition $a < b$ must already be satisfied. So the given sentence may be written and symbolized as follows:
"There are real numbers a and b satisfying $a < b$ such that $a < s < b$ for any element s of the nonempty set S"

$$(\exists a \in \mathbb{R})(\exists b \in \mathbb{R}) \Big((a < b) \wedge (\forall s \in S)(a < s < b) \Big)$$

Its negation is

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R}) \Big(\neg (a < b) \vee \neg \left[(\forall s \in S)(a < s < b) \right] \Big) \equiv (\forall a \in \mathbb{R})(\forall b \in \mathbb{R}) \Big((a < b) \to \neg \left[(\forall s \in S)(a < s < b) \right] \Big)$$

The negation can be translated as
"For all real numbers a and b if $a < b$ then S is not a subset of the interval (a, b)"

$\boxed{\text{"} S \text{ is not a subset of the interval } (a, b) \text{ for any real numbers } a \text{ and } b \text{ satisfying } a < b\text{"}}$