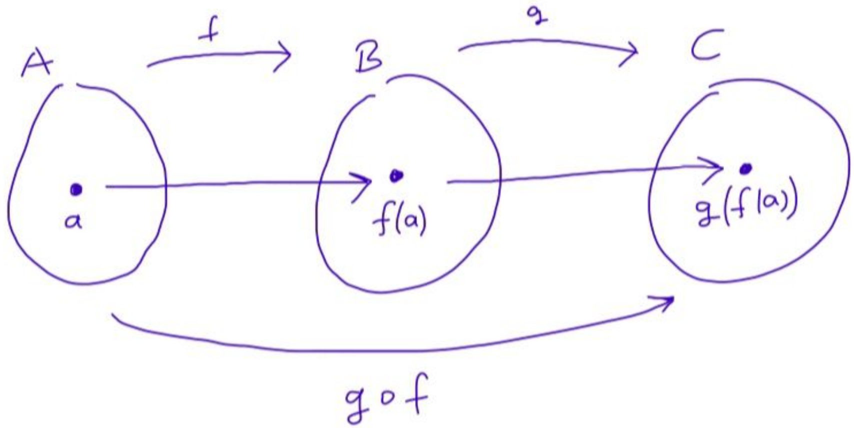
Composition

Definition: Let $f: A \to B$ and $g: B \to C$ be functions such that the codomain of f and the domain of g are the same. The composition of f and g is the function

$$g \circ f : A \to C \text{ defined by } (g \circ f)(a) = g(f(a)) \text{ for all } a \in A.$$



(Note that in $g \circ f$ we first apply f then g)

Ex: P = the set of all people. Consider the following functions

$f: P \to P$, $f(x)$ = the father of x ; $m: P \to P$, $m(x)$ = the mother of x. Then:

$(f \circ f)(x) = f(f(x))$ = the father of $f(x)$ = the father of the father of x

= the grandfather of x

$(f \circ m)(x) = f(m(x))$ = the father of the mother of x

$(m \circ f)(x) = m(f(x))$ = the mother of the father of x

Note that $f \circ m \neq m \circ f$

Ex: $f, g: \mathbb{R} \to \mathbb{R}$ are defined as $f(x) = 2x$ and $g(x) = x^2$. Then

$(f \circ g)(x) = f(g(x)) = 2\, g(x) = 2x^2$ and $(g \circ f)(x) = g(f(x)) = (f(x))^2 = 4x^2$

Ex: Let $f, g: \mathbb{R} \to \mathbb{R}$ be the maps

$$f(x) = \begin{cases} x+7, & \text{if } x \geq 1 \\ -x^2, & \text{if } x < 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 4x, & \text{if } x > 6 \\ 3x-1, & \text{if } x \leq 6 \end{cases}$$

Then

$$(f \circ g)(x) = f(g(x)) = \begin{cases} g(x)+7, & \text{if } g(x) \geq 1 \\ -g(x)^2, & \text{if } g(x) < 1 \end{cases}$$

When $g(x) \geq 1$?

Case I: $x > 6$. Then, $g(x) \geq 1 \iff 4x \geq 1 \iff x \geq \frac{1}{4}$. So $x > 6$

Case II: $x \leq 6$. Then, $g(x) \geq 1 \iff 3x-1 \geq 1 \iff x \geq \frac{2}{3}$. So $\frac{2}{3} \leq x \leq 6$

Consequently, $g(x) \geq 1 \iff x > 6$ or $\frac{2}{3} \leq x \leq 6$.

$$(f \circ g)(x) = \begin{cases} g(x)+7, & \text{if } g(x) \geq 1 \\ -g(x)^2, & \text{if } g(x) < 1 \end{cases} = \begin{cases} 4x+7, & \text{if } x > 6 \\ 3x+6, & \text{if } \frac{2}{3} \leq x \leq 6 \\ -(3x-1)^2, & \text{if } x < \frac{2}{3} \end{cases}$$

Ex: Let $f: A \to B$ and $g: B \to C$ be functions. Prove that

(1) If $g \circ f$ is injective then f is injective

(2) If $g \circ f$ is surjective then g is surjective

(3) If $g \circ f$ is bijective then f is injective and g is surjective

Proof:

(1): Let a_1 and a_2 be any elements of A. Suppose that $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. By the definition of composition, $(g \circ f)(a_1) = (g \circ f)(a_2)$. As $g \circ f$ is injective, $a_1 = a_2$ as desired. Therefore f is injective.

(2) (We want to show that $g: B \to C$ is surjective) Let $c \in C$. As $g \circ f: A \to C$ is surjective, there is an $a \in A$ such that $(g \circ f)(a) = c$. Put $b = f(a)$. Note that $b \in B$ and $g(b) = g(f(a)) = (g \circ f)(a) = c$. Hence g is surjective

(3) Exercise $\square$

Ex: Regarding functions as sets find the subset $g \circ f$ of $A \times C$ in terms of the subsets f of $A \times B$ and g of $B \times C$

Sol: Recall first that, for any function $h: U \to V$, $h(x) = y$ iff $(x,y) \in h \subseteq U \times V$

$(x,z) \in g \circ f$ iff $g \circ f(x) = z$ iff $g(f(x)) = z$ iff $g(y) = z$ and $f(x) = y$ $\exists y \in B$

iff $(y,z) \in g$ and $(x,y) \in f$ for some $y \in B$

Thus, $g \circ f = \{(a,c) \in A \times C \mid (a,b) \in f \text{ and } (b,c) \in g \text{ for some } b \in B\}$

Ex: Let A and B be sets and let $f: A \to B$ be a function. Define a function

$F: P(B) \to P(A)$ by $F(S) = f^{-1}(S) = \{a \in A \mid f(a) \in S\}$ for any $S \in P(B)$

where $P(\,)$ denotes the power set. Show that f is surjective if and only if F is injective.

Proof ($\Rightarrow$): Assume that f is surjective. We want to show that F is injective. Take any two elements U and V of $P(B)$ such that $F(U) = F(V)$. (We want to show that $U = V$) It follows from the definition of F that $f^{-1}(U) = f^{-1}(V)$.

Let $x \in U$. As $f: A \to B$ is surjective and $x \in U \subseteq B$, there is an $a \in A$ such that $x = f(a)$. From the definition of the preimage $f^{-1}(U) = \{z \in A \mid f(z) \in U\}$ we see that $a \in f^{-1}(U)$. So $a \in f^{-1}(V)$, implying that $x = f(a) \in V$. Thus $U \subseteq V$. Similarly, we may show that $V \subseteq U$. Having justified that $U \subseteq V$ and $V \subseteq U$, we conclude that $U = V$. Consequently, F is injective.

($\Leftarrow$): Assume that F is injective. We want to prove that $f:A\to B$ is surjective. Let b be any element of B. (We want to find an element $a\in A$ such that $f(a)=b$) Now $\{b\}$ and ϕ are distinct elements of $P(B)$. As $F:P(B)\to P(A)$ is injective, $F(\{b\})\neq F(\phi)$. Using the definition of F, we see that $f^{-1}(\{b\})\neq f^{-1}(\phi)$. As $f^{-1}(\phi)=\phi$, $f^{-1}(\{b\})\neq\phi$. Thus there is an $a\in f^{-1}(\{b\})$. Thus $f(a)=b$ for some $a\in f^{-1}(\{b\})\subseteq A$. As a result f is surjective. $\square$

Ex: Let A,B,C be sets, and let $f:A\to B$ and $g:B\to C$ be maps. Show that:

(1) If f is surjective and g is not injective, then $g\circ f$ is not injective

(2) If $g\circ f$ is surjective and g is injective, then f is surjective

Proof: (1): As g is not injective, there are elements b_1 and b_2 of B such that $b_1\neq b_2$ and $g(b_1)\neq g(b_2)$. As $f:A\to B$ is surjective and $b_1,b_2\in B$, there are elements a_1 and $a_2\in A$ such that $f(a_1)=b_1$ and $f(a_2)=b_2$. From $b_1\neq b_2$ it follows $a_1\neq a_2$. But, $(g\circ f)(a_1)=g(f(a_1))=g(b_1)\neq g(b_2)=g(f(a_2))=(g\circ f)(a_2)$. Thus $g\circ f$ is not injective.

(2) Exercise. $\square$

Fact: Let A,B,C,D be sets and let $f:A\to B$, $g:B\to C$, $h:C\to D$ be functions. Then:

(1) $(h\circ g)\circ f = h\circ(g\circ f)$ (Function composition is associative)

(2) $f\circ 1_A = f = 1_B\circ f$ where $1_U: U\to U$ denotes the identity map on U for any set U.

Proof: (1): (Recall that two functions F and G are equal if and only if their domains, codomains and rules are the same) Note that $(h\circ g)\circ f$ and $h\circ(g\circ f)$

are both functions from A to D. To finish the proof we need to justify that $((h\circ g)\circ f)(a) = (h\circ(g\circ f))(a)$ for all $a\in A$. Indeed this easily follows from the definition of the composition. Note that for any $a\in A$

$$((h\circ g)\circ f)(a) = (h\circ g)(f(a)) = h\big(g(f(a))\big) = h\big((g\circ f)(a)\big) = (h\circ(g\circ f))(a)$$

Therefore $(h\circ g)\circ f = h\circ(g\circ f)$

(2) Exercise. $\square$

Invertible functions, inverse of a function

<u>Definition</u>: Let A and B be sets. A function $f: A\to B$ is called <u>invertible</u> if there is a function $g: B\to A$ such that $f\circ g = 1_B$ and $g\circ f = 1_A$ where 1_U denotes the identity function on U. Any such function g is called an <u>inverse</u> of f

<u>Proposition</u>: If a function $f: A\to B$ is invertible, then f has a unique inverse
(The unique inverse of f is called <u>the inverse</u> function of f and denoted <u>by f^{-1}</u>)

<u>Proof</u>: As f is invertible, f must have an inverse. (We know that f has at least one inverse. To show that f has exactly one inverse we need to show that f has at most one inverse. For this we may prove that any two inverses of f are equal). Suppose that $g_1: B\to A$ and $g_2: B\to A$ are both inverse of f. Then by the definition $g_1\circ f = 1_A$ and $f\circ g_2 = 1_B$. Now,

$$g_1 = g_1\circ 1_B = g_1\circ(f\circ g_2) \underset{\substack{\uparrow \\ \text{associativity of composition.}}}{=} (g_1\circ f)\circ g_2 = 1_A\circ g_2 = g_2$$

$\square$

Checking whether a given function has an inverse (i.e. whether it is invertible) by using the definition may not be easy. Instead we may use the following:

<u>Theorem</u>: A function $f: A\to B$ is invertible iff f is bijective

<u>Proof</u>: ($\Leftarrow$) Suppose that f is bijective. So f is one to one and onto. As f

is onto, for any $b \in B$ there is an $a \in A$ such that $f(a) = b$. Using the fact that f is injective, we see that if for some $b \in B$ there are elements a_1 and a_2 of A such that $f(a_1) = b$ and $f(a_2) = b$ then $a_1 = a_2$. Consequently, we have proved that "for any $b \in B$ there is a unique $a \in A$ such that $f(a) = b$". Now consider the function $g : B \to A$ defined for any $b \in B$ by $g(b) =$ the unique element $a \in A$ such that $f(a) = b$. Finally, $\forall y \in B$,

$$(f \circ g)(y) = f(g(y)) = f(x) = y = 1_B(y), \text{ so } f \circ g = 1_B, \text{ and } \forall u \in A,$$

the unique element x of A such that $f(x) = y$

$$(g \circ f)(u) = g(f(u)) = g(v) = u = 1_A(u), \text{ so } g \circ f = 1_A$$

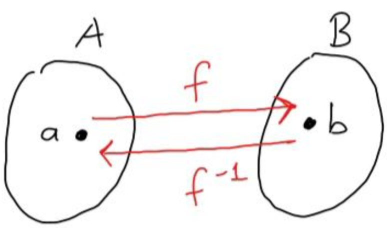
let $v = f(u) \in B$

The unique element ? of A such that $f(?) = v$ is u

($\Leftarrow$) Suppose that f is invertible and h is the inverse of f. Then, $f \circ h = 1_B$ and $h \circ f = 1_A$. (In a previous example we proved that "if $F \circ G$ is injective then G is injective" and "if $F \circ G$ is surjective then F is surjective". As the identity map 1_U is both injective and surjective, it readily follows that f is bijective). Details are left as an exercise. □

The proof of the above theorem explains how to find the inverse of a function.

<u>Remark</u>: Let $f : A \to B$ be a bijective function. So it has the inverse function $f^{-1} : B \to A$. Then:



(1) $f(a) = b$ iff $f^{-1}(b) = a$

(2) Given the rule $f(x)$ of f to find the rule $f^{-1}(x)$ of x, we

first let $y=f(x)$, and then using the equation $y=f(x)$ we solve x in terms of y and get $x=f^{-1}(y)$, finally putting x instead of y in $f^{-1}(y)$ we get the rule $f^{-1}(x)$ of f^{-1}.

<u>Proof</u>: (1) $f(a)=b \Rightarrow \underbrace{f^{-1}(f(a))}_{(\underbrace{f^{-1}\circ f}_{1_A})(a)}=f^{-1}(b) \Rightarrow a=f^{-1}(b)$

$a=f^{-1}(b) \Rightarrow f(a)=\underbrace{f(f^{-1}(b))}_{(\underbrace{f\circ f^{-1}}_{1_B})(b)} \Rightarrow f(a)=b$

(2) $y=f(x) \Rightarrow f^{-1}(y)=x$ by part (1). So to find $f^{-1}(y)$ we need to solve $y=f(x)$ for x and write x in terms of y. $\square$

<u>Ex</u>: Determine whether the given functions are invertible and if possible find their inverses:

(1) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=2x+3$

f is invertible because f is one to one and onto (Exercise).

$y=2x+3 \Rightarrow x=\underbrace{\frac{y-3}{2}}_{f^{-1}(y)} \Rightarrow f^{-1}(x)=\frac{x-3}{2}$

(2) $f:\mathbb{N}\to\mathbb{N}$, $f(n)=n+1$

As $f(n)\geq 1$, there is no $m\in\mathbb{N}$ such that $f(m)=0$. So f is not surjective, implying that f does not have inverse. (Although there is no function $g:\mathbb{N}\to\mathbb{N}$ such that both $f\circ g = 1_{\mathbb{N}}$ and $g\circ f = 1_{\mathbb{N}}$, we may find many functions $g:\mathbb{N}\to\mathbb{N}$ satisfying only $g\circ f = 1_{\mathbb{N}}$. Keep reading)

Ex: Let A and B be sets, and let $f: A\to B$ be a function.

(1) (We say that f has a right inverse if there is a function $g: B\to A$ such that $f\circ g = 1_B$. Any such g is called a right inverse of f). Prove that:

f has a right inverse iff f is surjective

(2) (We say that f has a left inverse if there is a function $g: B\to A$ such that $g\circ f = 1_A$. Any such g is called a left inverse of g). Prove that:

f has a left inverse iff f is injective

(3) Prove that if f has a left inverse g and f has a right inverse h, then $g=h$.

(4) Prove that f has inverse iff f has both a left inverse and a right inverse

(5) Find all left/right inverses of $f:\mathbb{R}\to[0,\infty)$ given by $f(x)=x^2$

(6) Find all left/right inverses of $f:\mathbb{N}\to\mathbb{N}$ given by $f(n)=n+1$

Proof:

(1): ($\Leftarrow$): Suppose that $f:A\to B$ is surjective. We want to show that f has a right inverse. In other words, we want to construct a function $g: B\to A$ such that $f\circ g = 1_B$. (Note that for such a function g we see from $f\circ g = 1_B$ that $f(g(b))=b$. So $g(b)$ must be defined to be an element $z\in A$ such that $f(z)=b$. Such elements z exist because f is onto. We now understand how to construct g and can continue with the proof). As $f:A\to B$ is surjective, $f^{-1}(\{b\}) = \{a\in A \mid f(a)=b\} \neq \emptyset$. So we may choose an element

$x_b \in f^{-1}(\{b\})$ for all $b \in B$. (Indeed, we use the axiom choice to pick the elements x_b). Now consider the function $g: B \to A$ defined by $g(b) = x_b$. Note for any $b \in B$ that

$$(f \circ g)(b) = f(g(b)) = f(x_b) = b = 1_B(b), \text{ so } f \circ g = 1_B. \text{ Thus, } g \text{ is a}$$

$x_b \in f^{-1}(\{b\}) = \{a \in A \mid f(a) = b\}$

right inverse of f.

($\Leftarrow$): Assume that f has a right inverse g. Then $f \circ g = 1_B$. As 1_B is surjective, (by a previous example) we see that f is surjective. (Exercise).

(2) ($\Rightarrow$): Suppose that $f: A \to B$ is injective. We want to construct a function $g: B \to A$ such that $g \circ f = 1_A$. (For such a function g we see from $g \circ f = 1_A$ that $g(f(a)) = a$). Choose an element $a_0 \in A$. Consider the function

$$g: B \to A \text{ defined by } g(b) = \begin{cases} \text{the unique element } a \text{ of } A \text{ such that } f(a) = b, & \text{if } b \in f(A) \\ a_0 & , \text{if } b \notin f(A) \end{cases}$$

Note that as $f: A \to B$ is injective, for any $b \in f(A) = \{f(x) \mid x \in A\}$ there is a unique $a \in A$ such that $f(a) = b$. Note for any $\alpha \in A$ that

$$(g \circ f)(\alpha) = g(f(\alpha)) = g(b) = \alpha = 1_A(\alpha), \text{ so } g \circ f = 1_A. \text{ Thus, } g \text{ is a}$$

put $b = f(\alpha)$ the unique element ? of A such that $f(?) = b$ is α

left inverse of f.

($\Leftarrow$): Exercise

(3): Exercise

(4): Exercise

(5): $f : \mathbb{R} \to [0,\infty)$, $f(x) = x^2$

Left inverses:

$g : [0,\infty) \to \mathbb{R}$ is a left inverse of $f \iff g \circ f = 1_{\mathbb{R}} \iff (g\circ f)(r) = 1_{\mathbb{R}}(r) \ \forall r \in \mathbb{R}$

$$\iff \underbrace{g(r^2) = r \ \forall r \in \mathbb{R}}$$

Putting $r=-1$ and $r=1$, this implies that $g(1) = -1$ and $g(1) = 1$

So g cannot be a function. Hence, g has no left inverse

Right inverses:

$g : [0,\infty) \to \mathbb{R}$ is a right inverse of $f \iff f \circ g = 1_{[0,\infty)} \iff (f\circ g)(r) = 1_{[0,\infty)}(r) \ \forall r \in [0,\infty)$

$$\iff (g(r))^2 = r \quad \forall r \in \mathbb{R} \text{ with } r \geq 0$$

$$\iff g(r) = \mp\sqrt{r} \quad \forall r \in \mathbb{R} \text{ with } r \geq 0$$

So there are infinitely many right inverses of f. For instance

$$g(r) = \begin{cases} \sqrt{r} & \text{if } r \geq 10 \\ -\sqrt{r} & \text{if } r < 10 \end{cases} \qquad \text{is a right inverse of } f.$$

(6) $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n+1$

Left inverses:

$g : \mathbb{N} \to \mathbb{N}$ is a left inverse of $f \iff g \circ f = 1_{\mathbb{N}} \iff (g\circ f)(m) = 1_{\mathbb{N}}(m) \ \forall m \in \mathbb{N}$

$$\iff g(m+1) = m \ \forall m \in \mathbb{N} \iff \underbrace{g(z) = z-1 \ \forall z \in \mathbb{N} \text{ with } z \geq 1.}$$

gives no restriction on $g(0)$

So $g(0)$ is arbitrary

Hence, there are infinitely many left inverses of f; they are

$$g_{n_0}(n) = \begin{cases} n_0, & \text{if } n = 0 \\ n-1, & \text{if } n > 0 \end{cases} \qquad \text{where } n_0 \in \mathbb{N} \text{ is arbitrary}$$

Right inverses: Exercise

Fact: For any bijective functions f and g such that $f\circ g$ is defined, $f\circ g$ is bijective (and so invertible) and $(f\circ g)^{-1} = g^{-1}\circ f^{-1}$

Proof: The proof of "$f\circ g$ is bijective" is left as an exercise. To show that $(f\circ g)^{-1} = g^{-1}\circ f^{-1}$ we should justify that $(f\circ g)\circ(g^{-1}\circ f^{-1}) = id$ and $(g^{-1}\circ f^{-1})\circ(f\circ g) = id$ where id denotes the identity function. Indeed,

$$(f\circ g)\circ(g^{-1}\circ f^{-1}) \underset{\uparrow}{=} \Big(f\circ \underbrace{(g\circ g^{-1})}_{id}\Big)\circ f^{-1} = (f\circ id)\circ f^{-1} = f\circ f^{-1} = id.$$

function composition is associative

$$(g^{-1}\circ f^{-1})\circ(f\circ g) = \cdots\cdots\cdots = id \quad \text{(exercise)}. \quad \square$$

Ex: Let $f: A\to B$ be a function. Show that:

(1) f is injective $\Leftrightarrow$ $f\circ g = f\circ h$ implies $g = h$ for any set C and for any functions $g, h: C\to A$

(2) f is surjective $\Leftrightarrow$ $g\circ f = h\circ f$ implies $g = h$ for any set C and for any functions $g, h: B\to C$

Proof:

(1): ($\Rightarrow$): Assume that f is injective. Let $g, h: C\to A$ be functions such that $f\circ g = f\circ h$. We want to show that $g = h$. As they have the same domain and the same codomain, it is enough to justify that $g(c) = h(c)$ for all $c\in C$.

Let $c\in C$. As $f\circ g = f\circ h$, $(f\circ g)(c) = (f\circ h)(c)$ and so $f(g(c)) = f(h(c))$. Since f is injective, it follows from $f(g(c)) = f(h(c))$ that $g(c) = h(c)$, as desired.

($\Leftarrow$): We want to show that f is injective. The proof will be by contradiction. Suppose for a contradiction that f is not injective. Then there are $a_1, a_2 \in A$

such that $f(a_1)=f(a_2)$ but $a_1 \neq a_2$. (To be able to use the assumption "$f\circ g = f\circ h$ implies $g=h$", somehow we should construct functions $g,h: C \to A$ such that $f\circ g = f\circ h$ but $g \neq h$. To be able to use "$f(a_1)=f(a_2)$ but $a_1 \neq a_2$", somehow we should have $g(c)=a_1$ and $h(c)=a_2$ for some $c\in C$). Consider the set $C=\{1\}$ and the functions $g,h: C\to A$ defined by $g(1)=a_1$ and $h(1)=a_2$.

Now, $f\circ g,\ f\circ h: \underset{\{1\}}{\overset{\shortparallel}{C}} \to B$, and $(f\circ g)(x) = f(a_1) = f(a_2) = (f\circ h)(x)$ for all $x\in C$

Thus $f\circ g = f\circ h$. But $g\neq h$ because $g(1)=a_1 \neq a_2 = h(1)$. This is a contradiction to the assumption.

(2): Exercise.

—— ∘ ——

Set of functions, Characteristic function.

For any sets A and B we let $\mathcal{F}(A,B)=\{f \mid f: A\to B \text{ is a function}\}$ be the set of all functions from A to B.

For any set A and for any subset $B\subseteq A$ we define the <u>characteristic function</u> $\chi_B : A \to \{0,1\}$ of B in A as $\chi_B(a)=\begin{cases} 1 & , \text{ if } a\in B \\ 0 & , \text{ if } a\notin B\end{cases}$

<u>Ex:</u> Let $f: A\to M$ and $g: B\to N$ be two bijective functions. Show that there is a bijective function $\mathcal{F}(A,B)\to\mathcal{F}(M,N)$

<u>Sol:</u> Want to find a bijection $\Psi: \mathcal{F}(A,B) \to \mathcal{F}(M,N)$

$$\sigma: A\to B \mapsto \underbrace{\Psi(\sigma)}_{?}: M\to N$$

The obvious choice for ? is $g\circ\sigma\circ f^{-1}$.

Define Ψ by $\Psi(\sigma) = g\circ\sigma\circ f^{-1}$ for all $\sigma\in\mathcal{F}(A,B)$.

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ M \arrow[r, "?"'] & N \end{tikzcd}$$

f^{-1} exists!

Justify that Ψ is a function from $\mathcal{F}(A,B)$ to $\mathcal{F}(M,N)$ and that Ψ is bijective. (Exercise)

—— ∘ ——

Exercise: Let A be a set. Consider the following functions

$\Psi : \mathcal{F}(A,\{0,1\}) \to \mathcal{P}(A)$ defined by $\Psi(f) = f^{-1}(1) = \{a \in A \mid f(a) = 1\}$ for all $f \in \mathcal{F}(A,\{0,1\})$

$\Phi : \mathcal{P}(A) \to \mathcal{F}(A,\{0,1\})$ defined by $\Phi(B) = \chi_B$ for all $B \in \mathcal{P}(A)$.

Show that:

(1) Ψ is a function from $\mathcal{F}(A,\{0,1\})$ to $\mathcal{P}(A)$, and Φ is a function from $\mathcal{P}(A)$ to $\mathcal{F}(A,\{0,1\})$.

(2) Ψ and Φ are bijections, and Ψ and Φ are inverses of each others

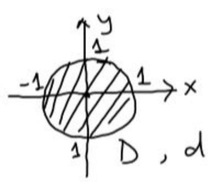
(3) There is a bijection between $\mathcal{F}(A,\{0,1\})$ and $\mathcal{P}(A)$.

Schröder–Bernstein Theorem

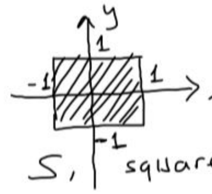
As its proof is long (but not hard to understand), we state the theorem without proof. It is usally used to show that some infinite sets have the same cardinality.

Theorem (Schröder–Bernstein Theorem)
Let A and B be two sets. If there is an injective function $A \to B$ and there is an injective function $B \to A$, then there is a bijective function $A \to B$

Ex: Let $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ and $S = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x,y \leq 1\}$



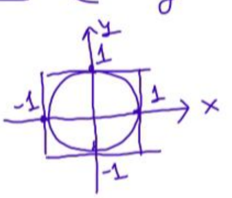
D, disc



S, square

Show that there is a bijection $h : D \to S$

Sol: (Finding an explicit bijective function $h : D \to S$ is not easy! Think for a while)



Note that $D \subseteq S$. So the inclusion map $i : D \to S$ given by $i(d) = d$ for all $d \in D$ is an injective function.

Consider now the function $f : S \to D$ given by $f(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right)$ for all $(x,y) \in S$. Note that

$(x,y) \in S \iff -1 \leq x,y \leq 1 \iff -\frac{1}{2} \leq \frac{x}{2}, \frac{y}{2} \leq \frac{1}{2} \implies \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 \leq \frac{1}{2} \implies \left(\frac{x}{2}, \frac{y}{2}\right) \in D$

Thus f is a function from S to D. It is obvious that f is injective.

As there is an injective function $i : D \to S$ and there is an injective function $f : S \to D$, it follows from the Schröder-Bernstein Theorem that there is a bijective function $h : D \to S$.

—o—

The solution of the previous example suggests

Exercise: Let A be a set and $B \subseteq A$ be a subset. Prove that: there is a bijective function $A \to B$ if and only if there is an injective function $A \to B$

We finish with the following result, not related to Schröder-Bernstein Theorem

Fact: Let A and B be two finite sets and let $f : A \to B$ be a function. Then:

(1) If f is injective then $|A| \leq |B|$

(2) If f is surjective then $|A| \geq |B|$

(3) Assume that $|A| = |B|$. Then,

$$f \text{ is injective} \iff f \text{ is surjective} \iff f \text{ is bijective}$$

Proof: Suppose $|A| = m$ and $|B| = n$.

(1): Let $A = \{a_1, a_2, \dots, a_m\}$. As $f : A \to B$ is injective, $f(a_1), f(a_2), \dots, f(a_m)$ are mutually (i.e., pairwisely) distinct elements of B. So B has at least $m = |A|$ elements. Thus $|B| \geq |A|$

(2): Let $B = \{b_1, b_2, \dots, b_n\}$. As $f : A \to B$ is surjective, for each b_i there is an $a_i \in A$ such that $f(a_i) = b_i$. Note that $a_1, a_2, \dots, a_n$ are mutually distinct elements of A, because $a_r = a_s$ implies that $f(a_r) = f(a_s)$ and so $b_r = b_s$. Therefore, A has at least $n = |B|$ elements. Thus $|A| \geq |B|$.

(3): We prove here only "f is injective $\implies$ f is surjective" as an illustration. The rest is exercise: The function $A \to f(A)$ defined for all $a \in A$ by $a \mapsto f(a)$ is injective (because $f : A \to B$) is injective. Part (1) implies that $|A| \leq |f(A)|$. As $|A| = |B|$, $|B| \leq |f(A)|$. As $f(A) \subseteq B$ and B is finite, $f(A) = B$. So f is surjective. $\square$

EXERCISES 14.2. State the usual name for each composition. (Ignore the fact that **sister**, **daughter**, and many of the other relations are not functions.)

1) husband $\circ$ sister

2) husband $\circ$ mother

3) husband $\circ$ wife

4) husband $\circ$ daughter

5) mother $\circ$ sister

6) daughter $\circ$ sister

7) parent $\circ$ parent

8) child $\circ$ child

9) parent $\circ$ parent $\circ$ parent

10) child $\circ$ brother $\circ$ parent

EXERCISES 14.7.

1) The formulas define functions f and g from $\mathbb{R}$ to $\mathbb{R}$. Find formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

   (a) $f(x) = 3x + 1$ and $g(x) = x^2 + 2$

   (b) $f(x) = 3x + 1$ and $g(x) = (x - 1)/3$

   (c) $f(x) = ax + b$ and $g(x) = cx + d$ (where $a, b, c, d \in \mathbb{R}$)

   (d) $f(x) = |x|$ and $g(x) = x^2$

   (e) $f(x) = |x|$ and $g(x) = -x$

2) Let $A = \{1, 2, 3, 4\}$, $B = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$, and $C = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. The sets of ordered pairs in each part are functions $f \colon A \to B$ and $g \colon B \to C$. Represent $g \circ f$ as a set of ordered pairs.

   (a) $f = \{(1, \mathsf{a}), (2, \mathsf{b}), (3, \mathsf{c}), (4, \mathsf{d})\}$,
   $g = \{(\mathsf{a}, \clubsuit), (\mathsf{b}, \diamondsuit), (\mathsf{c}, \heartsuit), (\mathsf{d}, \spadesuit)\}$

   (b) $f = \{(1, \mathsf{a}), (2, \mathsf{b}), (3, \mathsf{c}), (4, \mathsf{d})\}$,
   $g = \{(\mathsf{a}, \clubsuit), (\mathsf{b}, \clubsuit), (\mathsf{c}, \clubsuit), (\mathsf{d}, \clubsuit)\}$

   (c) $f = \{(1, \mathsf{b}), (2, \mathsf{c}), (3, \mathsf{d}), (4, \mathsf{a})\}$,
   $g = \{(\mathsf{a}, \clubsuit), (\mathsf{b}, \spadesuit), (\mathsf{c}, \heartsuit), (\mathsf{d}, \diamondsuit)\}$

   (d) $f = \{(1, \mathsf{a}), (2, \mathsf{b}), (3, \mathsf{c}), (4, \mathsf{d})\}$,
   $g = \{(\mathsf{a}, \clubsuit), (\mathsf{b}, \clubsuit), (\mathsf{c}, \heartsuit), (\mathsf{d}, \spadesuit)\}$

   (e) $f = \{(1, \mathsf{a}), (2, \mathsf{b}), (3, \mathsf{a}), (4, \mathsf{b})\}$,
   $g = \{(\mathsf{a}, \clubsuit), (\mathsf{b}, \clubsuit), (\mathsf{c}, \heartsuit), (\mathsf{d}, \spadesuit)\}$

3) Suppose $f \colon A \to B$ and $g \colon B \to C$. Show that if

$$(g \circ f)(a) = a, \text{ for every } a \in A,$$

   then f is one-to-one.

EXERCISES 14.8.

1) Suppose $f: A \to B$ and $g: B \to C$. Show that if f and g are one-to-one, then $g \circ f$ is one-to-one.
2) Suppose $f: A \to B$ and $g: B \to C$. Show that if f and g are onto, then $g \circ f$ is onto.
3) Suppose $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is one-to-one, then f is one-to-one.
4) Suppose $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is onto, then g is onto.
5) Give an example of functions $f: A \to B$ and $g: B \to C$, such that $g \circ f$ is onto, but f is *not* onto. [*Hint:* Let $A = B = \mathbb{R}$, $C = [0, \infty)$, $f(x) = x^2$, and $g(x) = x^2$.]
6) Suppose $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is onto, and g is one-to-one, then f is onto.
7) Suppose $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is one-to-one, and the range of f is B, then g is one-to-one.
8) Define $f: [0, \infty) \to \mathbb{R}$ by $f(x) = x$ and $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = |x|$. Show that $g \circ f$ is one-to-one, but g is *not* one-to-one.

9) Suppose f and g are functions from A to A. If $f(a) = a$ for every $a \in A$, then what are $f \circ g$ and $g \circ f$?
10) (*harder*) Suppose $f: A \to B$ and $g: B \to C$. Write a definition of $g \circ f$ purely in terms of sets of ordered pairs. That is, find a predicate $P(x, y)$, such that

$$g \circ f = \{\, (a, c) \in A \times C \mid P(a, c) \,\}.$$

The predicate cannot use the notation $f(x)$ or $g(x)$. Instead, it should refer to the ordered pairs that are elements of f and g.