MAT281E HW3

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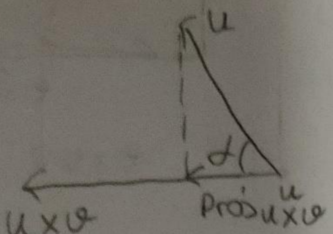
1) 2)

1) $Proj_{u \times v}(u) = x$



$$Proj_{u \times v}^{u} = \frac{\overbrace{(u \times v) \cdot u}^{0}}{\|u \times v\|^2} \cdot u \times v$$

$$Proj_{u \times v}^{u} = 0$$

$$\|Proj_{u \times v}^{u}\| = 0$$

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2) $\|u - v\|^2 = (\sqrt{3})^2$

$$(u-v) \cdot (u-v) = 3$$

$$\underbrace{u \cdot u}_{\underbrace{\|u\|}_{1}} - 2u \cdot v + \underbrace{v \cdot v}_{\underbrace{\|v\|}_{1}} = 3 \rightarrow u \cdot v = \frac{-1}{2}$$

$$\underbrace{u \cdot v}_{\frac{-1}{2}} = \underbrace{\|u\|}_{1} \cdot \underbrace{\|v\|}_{1} \cdot \cos\alpha \qquad \cos\alpha = \frac{-1}{2}$$

$$\|u \times v\| = \underbrace{\|u\|}_{1} \cdot \underbrace{\|v\|}_{1} \cdot \underbrace{\sin\alpha}_{\frac{\sqrt{3}}{2}} \rightarrow (\|u \times v\| \text{ comot be negative.})$$

$$\|u \times v\| = \frac{\sqrt{3}}{2}$$

3)

3)

$$\sum_{n=1}^{\infty} (a^{-n})^2 \to (a^2)^{-n} \to \frac{a^2}{a^2-1}$$

Cauchy - Schwartz Ineq:

$$\sum_{i=1}^{N} a_i b_i \leq \sqrt{\sum_{i=1}^{N} a_i^2} \cdot \sqrt{\sum_{i=1}^{N} b_i^2}$$

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n} \leq \sqrt{\sum_{i=1}^{\infty} e^{-2n}} \cdot \sqrt{\sum_{i=1}^{\infty} \frac{1}{n^2}}$$

$$\leq \frac{e}{\sqrt{e^2-1}} \cdot \frac{\pi}{\sqrt{6}}$$

$$\leq \frac{e\pi}{\sqrt{6e^2-6}}$$

4) 5)

4) $k_1(1,1,2,1) + k_2(1,-2,-2,0) + k_3(0,1,1,-2) = (3,-1,1,4)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & -2 & 1 & -1 \\ 2 & -2 & 1 & 1 \\ 1 & 0 & -2 & 4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to k_1 = 2 \\ \to k_2 = 1 \\ \to k_3 = -1 \\ \\ \end{array}$$

5)

$$\left.\begin{array}{l} w_1 = -x_1 - x_2 + x_3 \\ w_2 = 2x_1 - x_2 - 3x_3 \\ w_3 = -3x_2 - x_3 \end{array}\right) \quad T : \underbrace{R^3}_{\text{domain}} \to \underbrace{R^3}_{\text{codomain}}$$

$$\underbrace{\begin{bmatrix} -1 & -1 & 1 \\ 2 & -1 & -3 \\ 0 & -3 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{w}$$

$A \downarrow$ Standard matrix ($[T]$)

The range of a matrix is column space of its standard matrix.

$$\begin{bmatrix} -1 & -1 & 1 \\ 2 & -1 & -3 \\ 0 & -3 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

c_1 and c_2 contain lead 1's

$$\text{Range}(T_A) = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} \right\}$$

6)

6)

a)
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

b)
$x + z = 0$ plane

$\vec{n} = (1,0,1)$

Standard basis vectors for R^3;

$$\underset{u}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \underset{v}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \underset{z}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$u - \left(u \cdot \frac{\vec{n}}{\|n\|} \cdot \frac{\vec{n}}{\|n\|} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$v - \left(v \cdot \frac{\vec{n}}{\|n\|} \cdot \frac{\vec{n}}{\|n\|} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$z - \left(z \cdot \frac{\vec{n}}{\|n\|} \cdot \frac{\vec{n}}{\|n\|} \right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

6)

c)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

6)

d)

Reflection around $z=y$ plane:

Standard basis:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\quad x \qquad\quad y \qquad\quad z$$

reflection of basis:

$$\left.\begin{aligned} y &\longrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix} \\ z &\rightarrow \begin{bmatrix}0\\1\\0\end{bmatrix} \\ x &\rightarrow \begin{bmatrix}1\\0\\0\end{bmatrix} \end{aligned}\right) \quad [T_1] = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$$

$$[T_2] = \begin{bmatrix}1 & 0 & 0\\0 & \cos 60 & -\sin 60\\0 & \sin 60 & \cos 60\end{bmatrix}$$

$$[T] = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & \cos 60 & -\sin 60\\0 & \sin 60 & \cos 60\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$$

$$[T]\begin{bmatrix}1\\-2\\3\end{bmatrix} = \begin{bmatrix}-2\\3+2\sqrt{3}\\3\sqrt{3}-2\end{bmatrix}$$

6)

e)

$$[T] = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60 & -\sin 60 \\ 0 & \sin 60 & \cos 60 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[T] \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 - 3\sqrt{3}/2 \\ 0 \end{bmatrix}$$

7)

7)

$$[T_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$[T_1][T_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$[T_2][T_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \end{bmatrix}$$

$$[T_1][T_2]x \neq [T_2][T_1]x$$

$$\underline{T_1 \circ T_2 \neq T_2 \circ T_1}$$

8)

8)

a) If a linear transformation is one-to-one, then standard matrix is invertible and vice versa.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & -1 \\ 5 & 6 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & 6/5 \end{bmatrix}$$

→ matrix is invertible, transformation is one-to-one (det $\neq 0$)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 5 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 1/3 & 2/3 \\ 0 & 1 & 0 & -1/2 & -1/6 & 1/6 \\ 0 & 0 & 1 & -9/2 & 1/6 & 5/6 \end{array}\right]$$

$T^{-1}: R^3 \to R^3$ and $[T]^{-1}$

For the range:

Range of T_A is the column space of A:

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & -1 \\ 5 & 6 & -4 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

C_1, C_2 and C_3 contain lead 1's

$$\text{Range}(T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix} \right\}$$

Codomain → R3

The range is a subset of codomain.

8)

b)

$$\begin{bmatrix} 1 & -3 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} = [T] \longrightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \det = 0$$

the Matrix is many-to-one

There is no inverse transformation.

Range:

$$\begin{bmatrix} 1 & -3 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c_1 and c_2 contain lead 1's

$$\text{Range}(T_A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

codomain is R^3

the range is a subset of the codomain.

9)

9) Let u and v in R^2, c and d are scalar.

$$T(cu+dv) = T\left(\begin{bmatrix} cu_1+dv_1 \\ cu_2+dv_2 \end{bmatrix}\right) = \begin{bmatrix} 2(cu_1+dv_1)+cu_2+dv_2 \\ 1-(cu_1+dv_1) \end{bmatrix}$$

$$T(cu)+T(dv) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} dv_1 \\ dv_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2cu_1+cu_2 \\ 1-cu_1 \end{bmatrix} + \begin{bmatrix} 2dv_1+dv_2 \\ 1-dv_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2cu_1+cu_2+2dv_1+dv_2 \\ 2-cu_1-dv_1 \end{bmatrix}$$

$$\begin{bmatrix} 2cu_1+2dv_1+cu_2+dv_2 \\ \underline{1}-cu_1-dv_1 \end{bmatrix} \neq \begin{bmatrix} 2cu_1+cu_2+2dv_1+dv_2 \\ \underline{2}-cu_1-dv_1 \end{bmatrix}$$

$$T(cu+dv) \neq T(cu)+T(dv)$$

<u>Transformation is not linear</u>

10)

10)

$$A(x) = [T]x + t$$

$$A(x) = \begin{bmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

$$A(x) = \begin{bmatrix} T_1 x_1 + T_2 x_2 + T_3 x_3 + t_1 \\ T_4 x_1 + T_5 x_2 + T_6 x_3 + t_2 \\ T_7 x_1 + T_8 x_2 + T_9 x_3 + t_3 \end{bmatrix}$$

$$\begin{bmatrix} A(x) \\ 1 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & T_3 & t_1 \\ T_4 & T_5 & T_6 & t_2 \\ T_7 & T_8 & T_9 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} A(x) \\ 1 \end{bmatrix}}_{Ax} = \underbrace{\begin{bmatrix} [T] & t \\ 0..0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ 1 \end{bmatrix}}_{x}$$

$$A_{4\times4} = \begin{bmatrix} [T] & t \\ 0..0 & 1 \end{bmatrix}$$

11)

11)

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow[\text{onto x-z plane}]{\text{projection}} \begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow[\text{around y axis by } 60^\circ]{\text{Rotation}} \begin{bmatrix}\cos 60 & 0 & \sin 60\\0 & 1 & 0\\-\sin 60 & 0 & \cos 60\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1/2\\0\\-\sqrt{3}/2\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow[\text{onto x-z plane}]{\text{Projection}} \begin{bmatrix}0\\0\\0\end{bmatrix} \xrightarrow[\text{around y axis by } 60^\circ]{\text{Rotation}} \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow[\text{onto x-z plane}]{\text{Projection}} \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow[\text{around y axis by } 60^\circ]{\text{Rotation}} \begin{bmatrix}\cos 60 & 0 & \sin 60\\0 & 1 & 0\\-\sin 60 & 0 & \cos 60\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\sqrt{3}/2\\0\\1/2\end{bmatrix}$$

$$[T] = \begin{bmatrix}1/2 & 0 & \sqrt{3}/2\\0 & 0 & 0\\-\sqrt{3}/2 & 0 & 1/2\end{bmatrix}$$

12)

12) If the standard matrix of the linear transformation is invertible, The transformation is One-to-One. If standard matrix is not invertible, then transformation is many-to-one.

$[T_1]$ is $\underbrace{\text{singular}}_{\det = 0}$. $[T_2]$ has an $\underbrace{\text{inverse}}_{\det \neq 0}$.

$$\det([T_1][T_2]) = \underbrace{\det([T_1])}_{0} . \det([T_2]) = 0$$

$\Rightarrow$ $T_1 \circ T_2$ is not invertible.

$$\det([T_2][T_1]) = \det([T_2]) . \underbrace{\det([T_1])}_{0} = 0$$

$\Rightarrow$ $T_2 \circ T_1$ is not invertible.

13)

13)

a) The set is a vector space. Every plane through the origin is a vector space.

b) The set contains only single object which is zero vector (0,0,0). All of axioms are satisfied. The set is a vector space which is called zero vector space.

c) There is no zero vector. The set cannot be a vector space.

d) The set cannot be a vector space. If k is any scalar and u is in V, ku must be in V.

Ex: $u = (\frac{1}{2}, 0)$ $k = 10$

$v = (5, 0)$ is not in V

14)

14)

a) $\begin{bmatrix} a_1 & b_1 \\ a_1+b_1 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ a_2+b_2 & 0 \end{bmatrix} = \begin{bmatrix} \overset{a}{a_1+a_2} & \overset{b}{b_1+b_2} \\ \underset{a+b}{a_1+b_1+a_2+b_2} & 0 \end{bmatrix}$ ✓

$k \begin{bmatrix} a_1 & b_1 \\ a_1+b_1 & 0 \end{bmatrix} = \begin{bmatrix} \overset{a}{ka_1} & \overset{b}{kb_1} \\ \underset{a+b}{k(a_1+b_1)} & 0 \end{bmatrix}$ ✓

$\begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix}$ is a subset of all 2x2 matrices.

b) $k \begin{bmatrix} a_1 & b_1 \\ a_1a_2 & 0 \end{bmatrix} = \begin{bmatrix} \overset{a}{ka_1} & \overset{b}{kb_1} \\ ka_1b_1 & 0 \end{bmatrix}$ → The matrix is not a subset.

$ka_1b_1 = k^2a_1b_1$ cannot hold except $k=1$, $k=0$

c) $k \begin{bmatrix} a_1 & b_1 \\ a_1+b_1 & 1 \end{bmatrix} = \begin{bmatrix} \overset{a}{ka_1} & \overset{b}{kb_1} \\ \underset{a+b}{ka_1+kb_1} & k \end{bmatrix}$ → The matrix is not a subset.

$k=1$ cannot hold except $k=1$

15)

15)

Let $f(x)$ is discontinuous at the origin,

$g(x) = -f(x)$ which is discontinuous at the origin,

$g(x) + f(x) = 0$ and $y = 0 \notin W$.

The set of functions with discontinuity at the origin <u>cannot</u> form a subspace.

16)

16)

Let A and A' is singular:

$$\det(A) = \det(A') = 0$$

$$A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \det(A) = \begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$A' = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \rightarrow \det(A') = \begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} = 0$$

not singular.

$$A + A' = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \rightarrow \begin{vmatrix} a & 0 \\ 0 & c \end{vmatrix} \neq 0$$

Set of singular matrices with $n \times n$ cannot be a subspace.

Let B and B' is invertible:

not invertible

$$B = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & a \\ x & 0 \end{bmatrix}$$

$$B + B' = \begin{bmatrix} a & a \\ x & x \end{bmatrix} \rightarrow \begin{vmatrix} a & a \\ x & x \end{vmatrix} = 0$$

Set of invertible matrices with $n \times n$ cannot be a subspace.

17)

17)

a)

$$\begin{bmatrix} 1 & 0 & -2 & 2 \\ 2 & 1 & 4 & -4 \\ 1 & -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 2 & 1 & 4 & -4 & 0 \\ 1 & -1 & 3 & 0 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 6/13 & 0 \\ 0 & 1 & 0 & -24/13 & 0 \\ 0 & 0 & 1 & -10/13 & 0 \end{array}\right]$$

$w = t$

$z = 10t/13$

$y = 24t/13$

$x = -6t/13$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} -6/13 \\ 24/13 \\ 10/13 \\ 1 \end{bmatrix}$$

1 free variable,

The solution space is the line through the origin and parallel to the vector $v = \left(\frac{-6}{13}, \frac{24}{13}, \frac{10}{13}, 1\right)$

$$\left.\begin{aligned} x &= -6t/13 \\ y &= 24t/13 \\ z &= 10t/13 \\ w &= t \end{aligned}\right)$$ Parametric equations of the line.

17)

b)

$$A = \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 2 & 1 & 4 & -4 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccccc} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 8 & -8 & 0 \end{array}\right]$$

$w = t$

$z = s$

$y = 8t - 8s$

$x = 2s - 2t$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} 2 \\ -8 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 8 \\ 0 \\ 1 \end{bmatrix}$$

Two free variable,

The solution space is a plane.

$(2s-2t, 8t-8s, s, t)$

17)

c)
$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 2 & 1 & 4 & -4 & 0 \\ 0 & 1 & 6 & -6 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$w = s$
$z = s$
$y = 0$
$x = 0$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

1 free variable,
The solution space is the line through the origin and parallel to the vector $v = (0, 0, 1, 1)$

$$\left.\begin{array}{l} x = 0 \\ y = 0 \\ z = t \\ w = t \end{array}\right) \text{Parametric equation of the line.}$$

17)

d)
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 0 & -4 & 0 \\ 0 & 1 & 6 & 0 \\ -1 & 0 & 8 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = 0$
$y = 0$
$x = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the Solution space is origin $\{0\}$

18)

18) a)

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & x \\ 3 & 0 & 3 & y \\ 2 & -1 & 0 & z \\ 1 & 0 & 1 & w \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc} 1 & 0 & 1 & y/3 \\ 0 & 1 & 2 & (-x/2)+(2y/3) \\ 0 & 0 & 0 & (-x/2)+z \\ 0 & 0 & 0 & w-y/3 \end{array}\right]$$

$$E = \left\{ (x,y,z,w) \in R^4 \mid w - \frac{y}{3} = 0 \; ; \; \frac{-x}{2} + z = 0 \right\}$$

b)

$$\begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{4\times 3} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} k_3 &= t \\ k_2 &= -2t \\ k_1 &= -t \end{aligned}\right)$$

Inf. many solution vectors are linearly dependent.

18)

c) v_1 and v_2 are linearly independent. v_1, v_2 and 2 standard basis vectors can be used as basis for R^4

$$S = \left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

19)

19)

$$W = \begin{vmatrix} \cos 2x & \cos^2 x - \sin^2 x \\ -2\sin 2x & -4\sin x.\cos x \end{vmatrix}$$

$$-4.\sin x.\cos x = -2.\sin 2x$$

$$W = (\cancel{-2.\sin 2x})(\cos 2x) - (\cancel{-2\sin x}).(\cos^2 x - \sin^2 x)$$

$$W = \cos 2x - (\cos^2 x - \sin^2 x)$$

$W = 0 \rightarrow f_1(x)$ and $f_2(x)$ are linearly dependent.

20)

20)

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 4 & -1 & 7 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 & 7/6 \\ 0 & 1 & 0 & 7/6 \\ 0 & 0 & 1 & -7/3 \end{bmatrix}$$

c_1, c_2 and c_3 are linearly independent. (lead 1's)

set of basis $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

$$u_4 = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \frac{7}{6} u_1 + \frac{7}{6} u_2 - \frac{7}{3} u_3$$

$\dim(W)$ = number of vectors in a basis = $\underline{3}$

21) 22)

21) Let u_1, u_2, u_3 are independent vectors and $u_4 = u_1 + u_2 + u_3$

$S = \{u_1, u_2, u_3, u_4\}$ is linearly dependent set since,

$$k_1u_1 + k_2u_2 + k_3u_3 + k_4u_4 = 0$$

$$k_1u_1 + k_2u_2 + k_3u_3 + k_4u_1 + k_4u_2 + k_4u_3 = 0$$

$$\left.\begin{array}{l} k_1 = -s \\ k_2 = -s \\ k_3 = -s \\ k_4 = s \end{array}\right) \text{There are inf many solutions.}$$

If $S - u_4 \rightarrow S' = \{u_1, u_2, u_3\}$ will be a linearly independent set.

S' cannot be a basis, because 3 vectors cannot span R^4

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22) Let $u_1 = \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ y \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $u_3 = \begin{bmatrix} 0 \\ 0 \\ z \\ 0 \\ 0 \end{bmatrix}$ $u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \\ 0 \end{bmatrix}$

$S = \{u_1, u_2, u_3, u_4\}$ is linearly independent set.

If $u_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 9 \end{bmatrix}$ is added to $S \rightarrow S' = \{u_1, u_2, u_3, u_4, u_5\}$

is also a linear independent set. since,

if $\begin{bmatrix} k_1x \\ k_2y \\ k_3z \\ k_4t \\ k_59 \end{bmatrix} = 0$, Only solution is $k_1 = k_2 = k_3 = k_4 = k_5 = 0$

23)

23) $Proj_{u \times v}^{w}$ has the same direction with $u \times v$. Because of $u \times v \perp u$ and v, $Proj_{u \times v}^{w} \perp u, v$ (same direction with $u \times v$)

$u \times v$, u and v are linearly independent then $Proj_{u \times v}^{w}$, u and <u>must</u> be linearly independent.

24)

24) 2x2 matrices can contain maximum 4 independent matrices. Other matrix <u>must</u> be obtained as linear combination of other independed matrices.

Ex:
$$a\begin{bmatrix}1&0\\0&0\end{bmatrix}+b\begin{bmatrix}0&1\\0&0\end{bmatrix}+c\begin{bmatrix}0&0\\1&0\end{bmatrix}+d\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

↳ can be any matrix.

3x3 matrices can contain maximum 9 independent matrices. If there are 4 independent matrices, last matrix may be 5. independed matrix. Therefore <u>every</u> 3x3 matrix may not be obtained as linear combination of other four matrices.

Ex:
$$a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+b\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}+c\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}+d\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\neq\begin{bmatrix}0&0&0\\0&0&0\\0&0&x\end{bmatrix}$$

25)

25)

a) $v_1+v_2+v_3+v_4=0 \rightarrow k_1v_1+k_2v_2+k_3v_3+k_4v_4=0 \rightarrow$ $k_1=0$, $k_2=0$, $k_3=0$, $k_4=0$ Vectors are linearly depended.

$\leftarrow$ the set cannot be basis.

b) $k_1(-x,y)+k_2(x,-y)=0$

$k_1=-1 \quad k_2=1$

$k_1=-2 \quad k_2=2$

$\vdots \qquad \vdots$

inf. many constants. Vectors are linearly dependent. The set cannot be basis

c)

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 2 \\ 3 & 0 & 6 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow$ Vectors are linearly independent. ✓

independent vectors = dimension of vector space ✓

the set can be basis.

25)

d)

the set of vectors that contains zero vector is always a linearly dependent:

If u is zero vector,

$k_1u+k_2v+k_3z \dots = 0 \rightarrow k_1$ can be any real number.

the set cannot be basis.

26)

26) $x_1 \cdot (2,-2,0) + x_2(0,1,-1) + x_3(1,5,4) = (1,1,1)$

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 5 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ -2 & 1 & 5 & 1 \\ 0 & -1 & 4 & 1 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 & 7/20 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 3/10 \end{bmatrix}$$

$x_1 = 7/20 \quad x_2 = 1/5 \quad x_3 = 3/10$

$\underline{W(s) = (7/20,\ 1/5,\ 3/10)}$

27)

27)

a)

$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = 0$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 1 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} k_4 &= s \\ k_3 &= t \\ k_2 &= t/2 - s/2 \\ k_1 &= -t/2 - s/2 \end{aligned}\right\} \text{Inf. many Solution.}$$

$S = \{v_1, v_2, v_3, v_4\}$ is a linearly dependent set.

b)

$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & 3 & -2 & 1 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c_1 and c_2 contains lead 1's.

basis of the space $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$

c)

dimension of the space = number of independent vectors

$= 2$

27)

d)

$$\text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1, v_2\}$$

v_1 and v_2 spans a plane in $R3$

$(2,-1,-1) \times (0,1,3) = (-2,-6,2) \rightarrow$ normal of the plane

$$-2x - 6y + 2z = 0 \rightarrow \text{plane}.$$

e) If vector w in the span, w can be expressible as linear combination of vectors:

$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} k1 \\ k2 \\ k3 \\ k4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 & 1 & 2 \\ -1 & 1 & -1 & 0 & 2 \\ -1 & 3 & -2 & 1 & 0 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

There is no solution.

w is not in span of the given vectors.

28)

28) By the definition, the plane must through the origin to be a vector space. Given plane is not through the origin, the basis cannot be found.

29)

29)

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 2 & -1 & -1 & 2 & 0 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 1 & -4/3 & 0 \end{bmatrix}$$

$$w = t$$
$$z = s$$
$$y = \frac{4t}{3} - s$$
$$x = \frac{-t}{3}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} -1/3 \\ 4/3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Basis} \rightarrow \left\{ \begin{bmatrix} -1/3 \\ 4/3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\dim(W) = \#$ basis vectors

$= 2$

30) 31)

30)

$$t = \frac{x+1}{3}$$

$$y = \frac{x+1}{3} - \frac{1}{3} = \frac{x}{3}$$

$$z = 1 - 3\left(\frac{x+1}{3}\right) = -x$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 1/3 \\ -1 \end{bmatrix}$$

Basis of line $\Rightarrow \begin{bmatrix} 1 \\ 1/3 \\ -1 \end{bmatrix}$

31)

$$\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -2 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 1/2 & 4 \\ 0 & 1 & -1/2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis

set of basis $S = \{1+x^2, 3-2x+x^2\}$

dimension of subspace $= 2$

2 basis cannot span the space of second order polynomials $(\dim(P_2) = 3)$

Any standard basis vector can be added to span the space of the second order polynomials.

basis of the second order polynomials $S = \{1+x^2, 3-2x+x^2, \underbrace{x^2+0x+0}_{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}\}$