THOMAS' CALCULUS

MEDIA UPGRADE

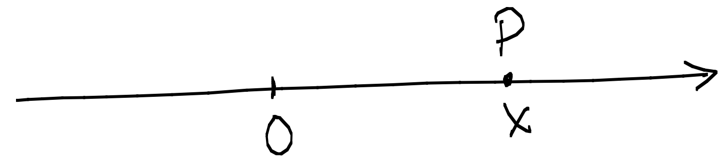
Chapter 12

Vectors and The Geometry of Space

12.1

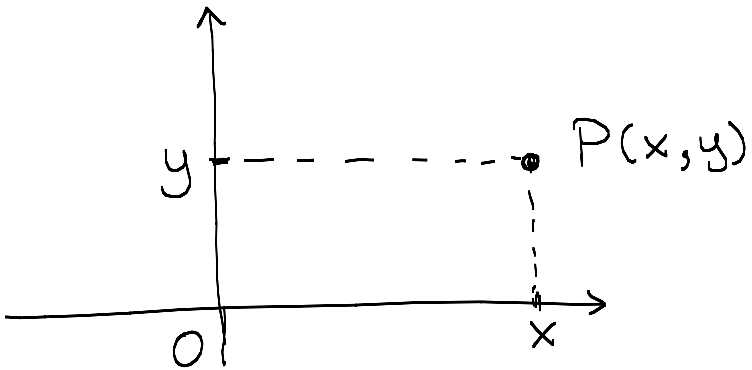
Three-Dimensional Coordinate Systems

- $\mathbb{R} = \{x : x \in \mathbb{R}\}$: 1-dimensional Real Line



x : directed distance from O

- $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$: 2-dimensional Real Plane



- $\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$: 3-dimensional Real Space (Euclidean Space)

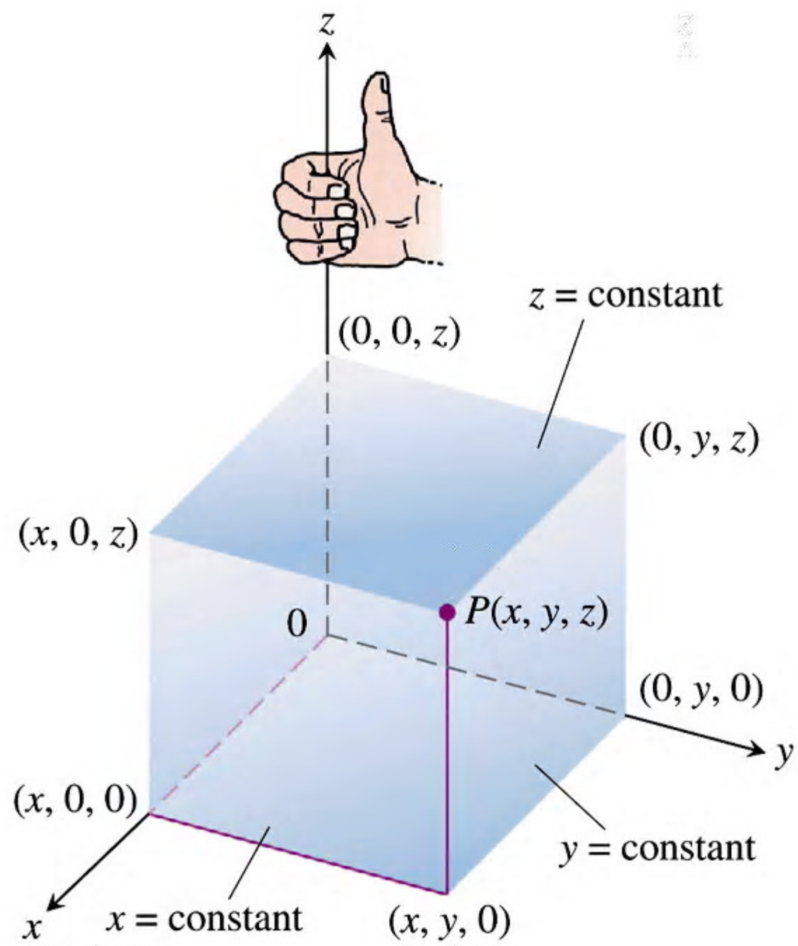


FIGURE 12.1 The Cartesian coordinate system is right-handed.

Coordinate Planes

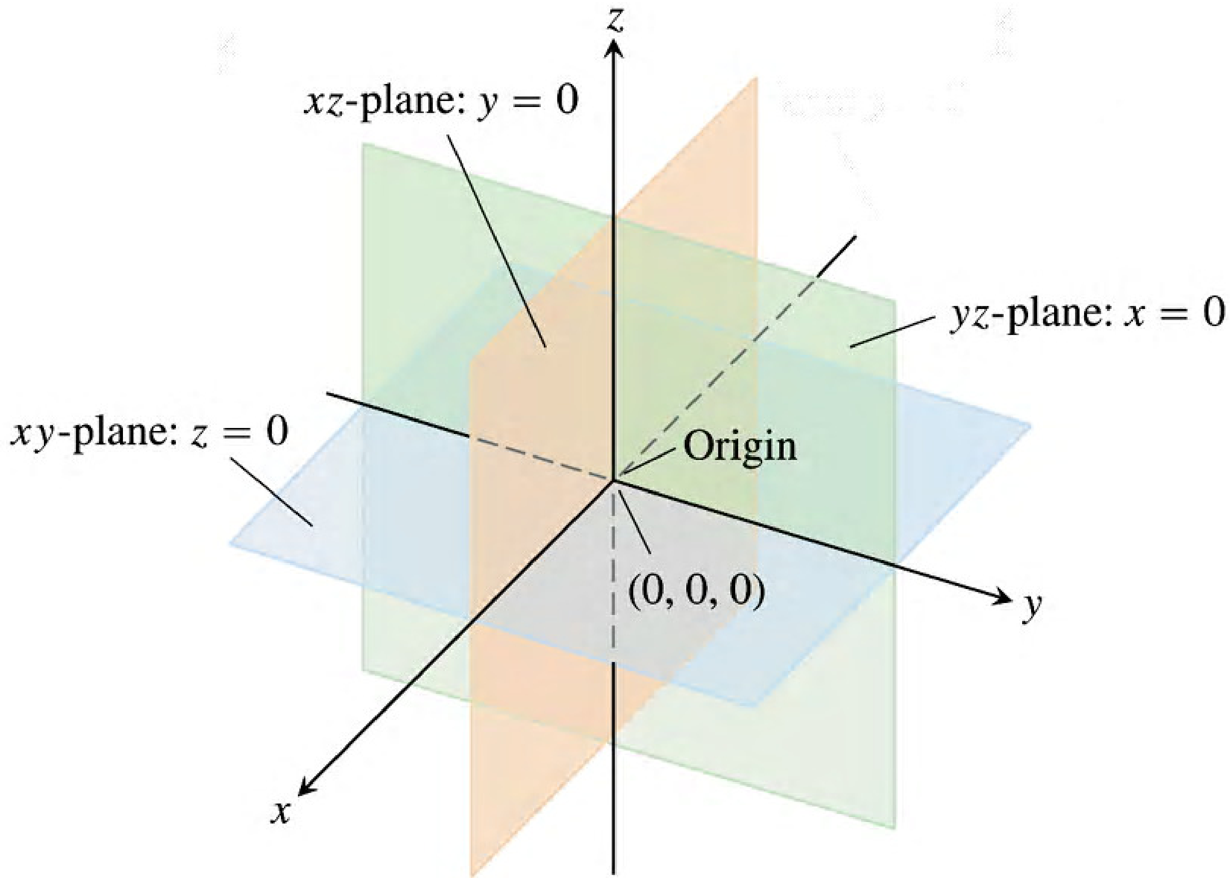


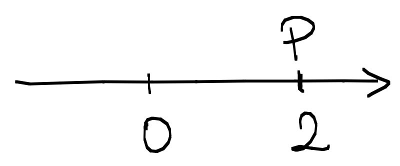
FIGURE 12.2 The planes $x = 0, y = 0$, and $z = 0$ divide space into eight octants.

1st Octant: $\{P(x,y,z) : x \geq 0, y \geq 0, z \geq 0\}$

No conventional numbering for the other octants

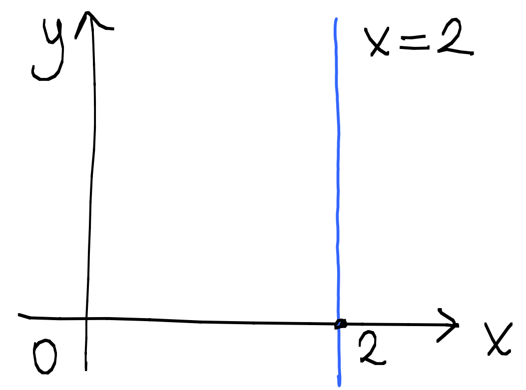
Consider the equation $x=2$:

- On $\mathbb{R}$



$x=2$ defines a point P

- On $\mathbb{R}^2$



$x=2$ defines a line ℓ

$\ell : \{(x,y) : x=2,\ y \in \mathbb{R}\}$

- On $\mathbb{R}^3$, $x=2$ defines a plane which is the shift of yz-plane $(x=0)$ 2 units to the front.

$\{(x,y,z) : x=2,\ y,z \in \mathbb{R}\}$

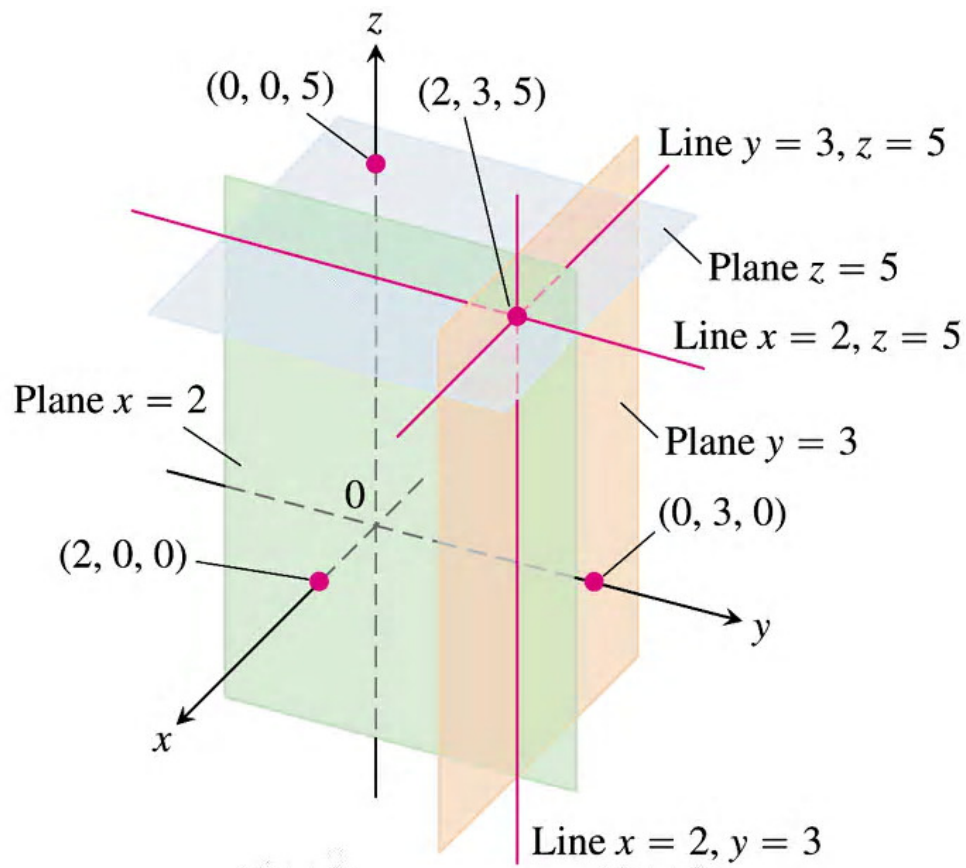
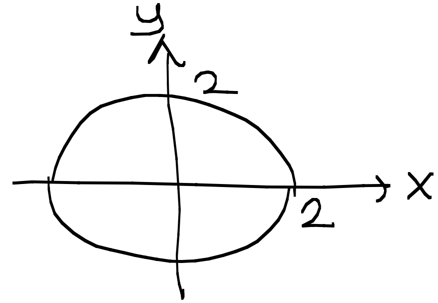


FIGURE 12.3 The planes $x = 2$, $y = 3$, and $z = 5$ determine three lines through the point (2, 3, 5).

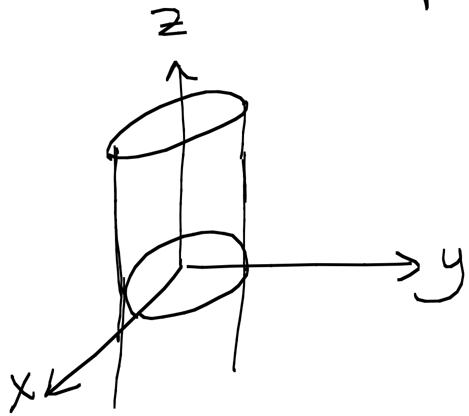
EXAMPLE 1 Interpreting Equations and Inequalities Geometrically

(a) $z \geq 0$	The half-space consisting of the points on and above the xy-plane.
(b) $x = -3$	The plane perpendicular to the x-axis at $x = -3$. This plane lies parallel to the yz-plane and 3 units behind it.
(c) $z = 0, x \leq 0, y \geq 0$	The second quadrant of the xy-plane.
(d) $x \geq 0, y \geq 0, z \geq 0$	The first octant.
(e) $-1 \leq y \leq 1$	The slab between the planes $y = -1$ and $y = 1$ (planes included).
(f) $y = -2, z = 2$	The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x-axis. ■

Consider the equation $x^2+y^2=4$.



- On $\mathbb{R}^2$, it defines a circle
  $C=\{(x,y): x^2+y^2=4\}$



- On $\mathbb{R}^3$, it defines a right circular (infinite) cylinder
  $C=\{(x,y,z): x^2+y^2=4, z\in\mathbb{R}\}$

EXAMPLE 2 Graphing Equations

What points $P(x, y, z)$ satisfy the equations

$$x^2+y^2=4 \quad \text{and} \quad z=3?$$

Solution The points lie in the horizontal plane $z=3$ and, in this plane, make up the circle $x^2+y^2=4$. We call this set of points "the circle $x^2+y^2=4$ in the plane $z=3$" or, more simply, "the circle $x^2+y^2=4$, $z=3$" (Figure 12.4). ■

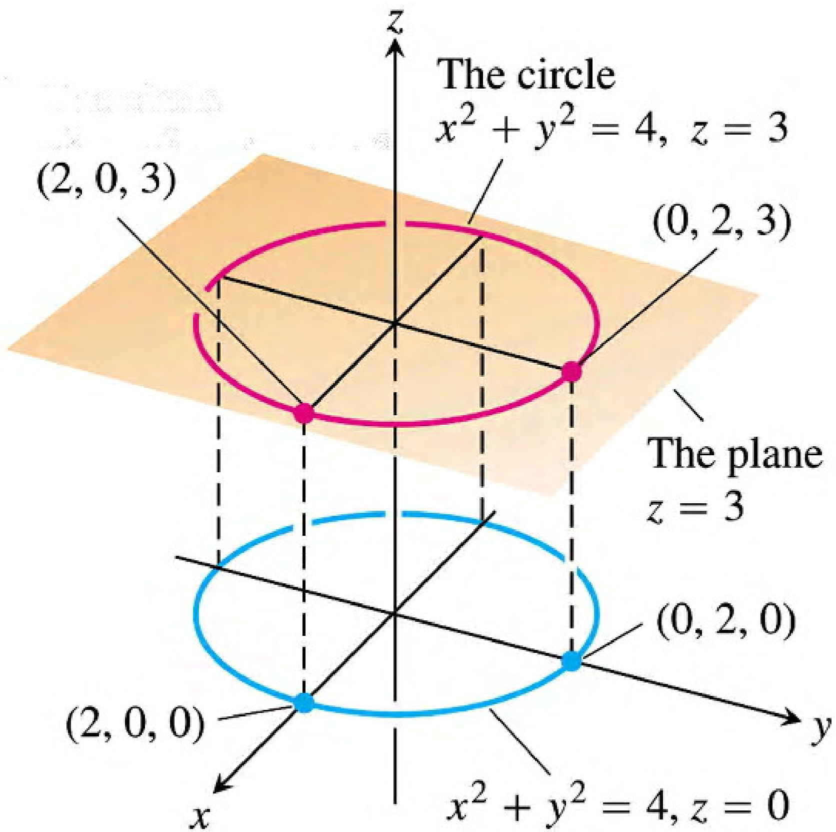


FIGURE 12.4 The circle $x^2 + y^2 = 4$ in the plane $z = 3$ (Example 2).

- The distance between $P_1(x_1)$ and $P_2(x_2)$ on $\mathbb{R}$ is $|P_1P_2| = \sqrt{(x_2-x_1)^2} = |x_2-x_1|$
- The distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on $\mathbb{R}^2$ is $|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

on $\mathbb{R}^3$

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

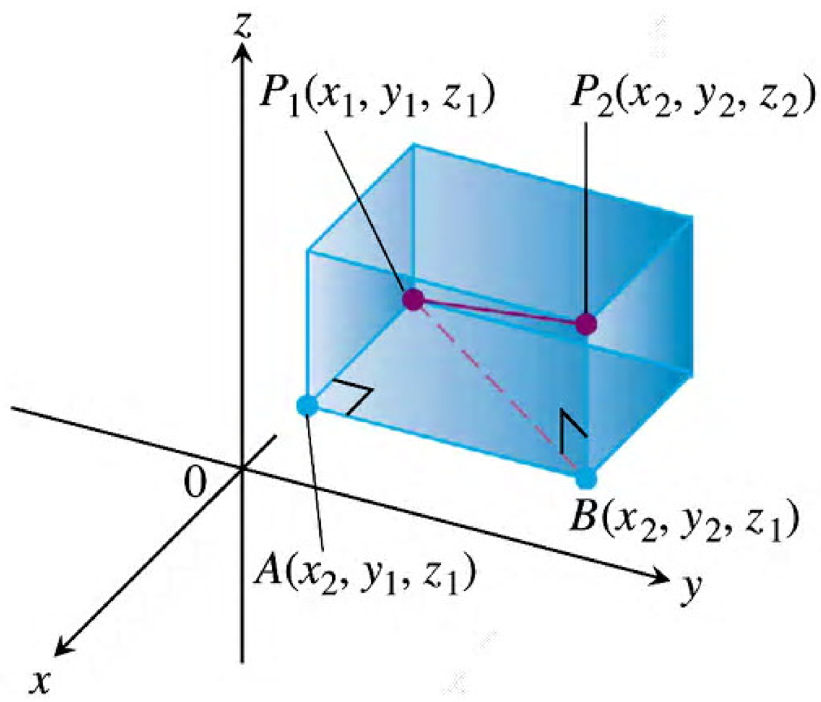


FIGURE 12.5 We find the distance between P_1 and P_2 by applying the Pythagorean theorem to the right triangles P_1AB and P_1BP_2.

EXAMPLE 3 Finding the Distance Between Two Points

The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$\begin{aligned}
|P_1P_2| &= \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2} \\
&= \sqrt{16 + 4 + 25} \\
&= \sqrt{45} \approx 6.708.
\end{aligned}$$

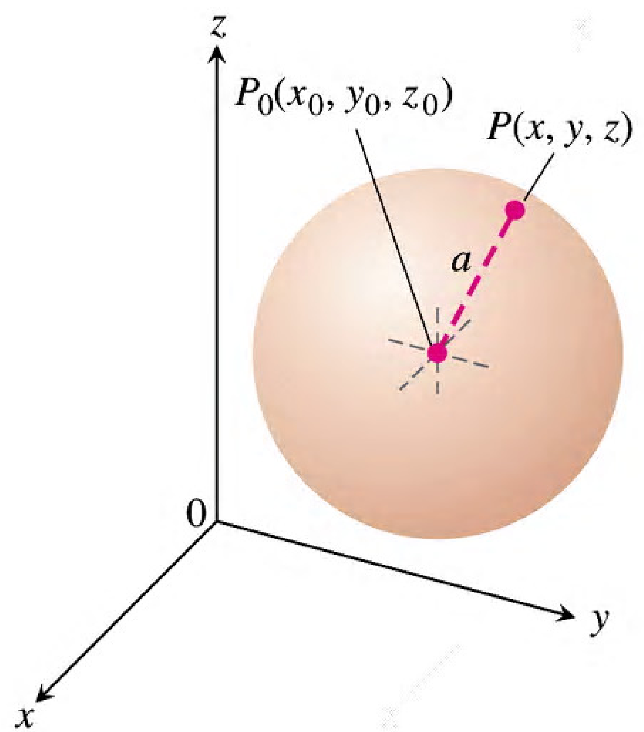


FIGURE 12.6 The standard equation of the sphere of radius a centered at the point (x_0, y_0, z_0) is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

EXAMPLE 4 Finding the Center and Radius of a Sphere

Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Solution We find the center and radius of a sphere the way we find the center and radius of a circle: Complete the squares on the x-, y-, and z-terms as necessary and write each quadratic as a squared linear expression. Then, from the equation in standard form, read off the center and radius. For the sphere here, we have

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = -1 + \frac{9}{4} + 4 = \frac{21}{4}.$$

From this standard form, we read that $x_0 = -3/2$, $y_0 = 0$, $z_0 = 2$, and $a = \sqrt{21}/2$. The center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$.

EXAMPLE 5 Interpreting Equations and Inequalities

(a) $x^2 + y^2 + z^2 < 4$	The interior of the sphere $x^2 + y^2 + z^2 = 4$.
(b) $x^2 + y^2 + z^2 \leq 4$	The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.
(c) $x^2 + y^2 + z^2 > 4$	The exterior of the sphere $x^2 + y^2 + z^2 = 4$.
(d) $x^2 + y^2 + z^2 = 4, z \leq 0$	The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy-plane (the plane $z = 0$).

■

12.2

Vectors

Some of the things we measure are determined by their magnitudes and directions.

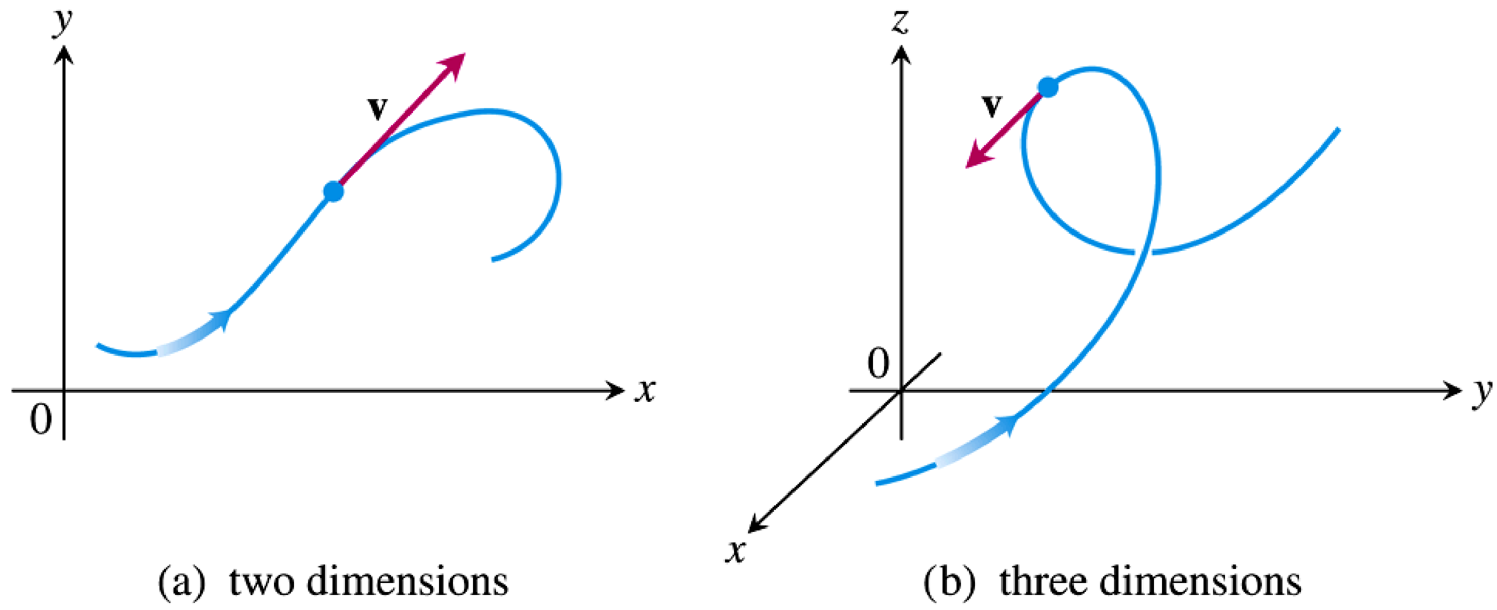


FIGURE 12.8 The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

DEFINITIONS **Vector, Initial and Terminal Point, Length**

A **vector** in the plane is a directed line segment. The directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B; its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

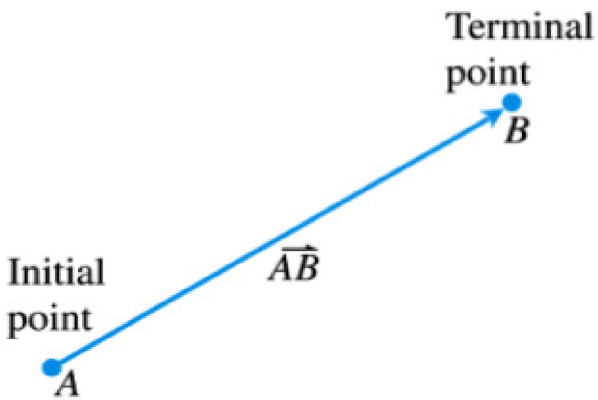


FIGURE 12.7 The directed line segment $\overrightarrow{AB}$.

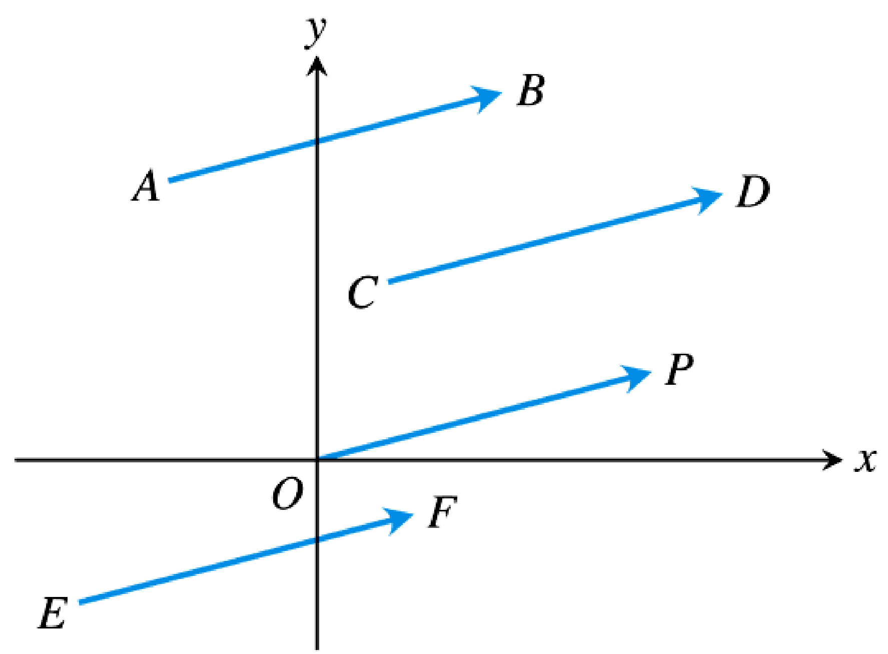


FIGURE 12.9 The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \underline{\overrightarrow{OP}} = \overrightarrow{EF}$.

standard position

DEFINITION **Component Form**

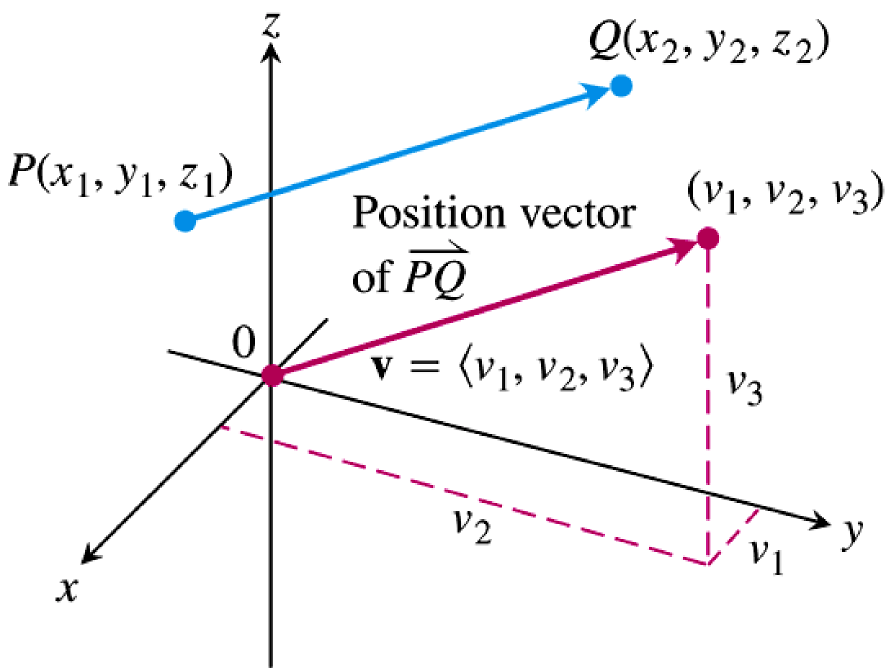
If $\mathbf{v}$ is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If $\mathbf{v}$ is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

v_1, v_2 and v_3 are called the components of $\mathbf{v}$.



$V_1 = X_2 - X_1$

$V_2 = y_2 - y_1$

$V_3 = z_2 - z_1$

FIGURE 12.10 A vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and $\mathbf{v}$ are parallel and have the same length.

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(See Figure 12.10.)

The only vector with length 0 is the **zero vector** $\mathbf{0} = \langle 0, 0 \rangle$ or $\mathbf{0} = \langle 0, 0, 0 \rangle$. This vector is also the only vector with no specific direction.

EXAMPLE 1 Component Form and Length of a Vector

Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution

(a) The standard position vector $\mathbf{v}$ representing $\overrightarrow{PQ}$ has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \qquad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

and

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

(b) The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

■

DEFINITIONS **Vector Addition and Multiplication of a Vector by a Scalar**

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

A scalar is simply a real number which can be positive, negative, or zero.

The definitions apply to planar vectors (but just two components)

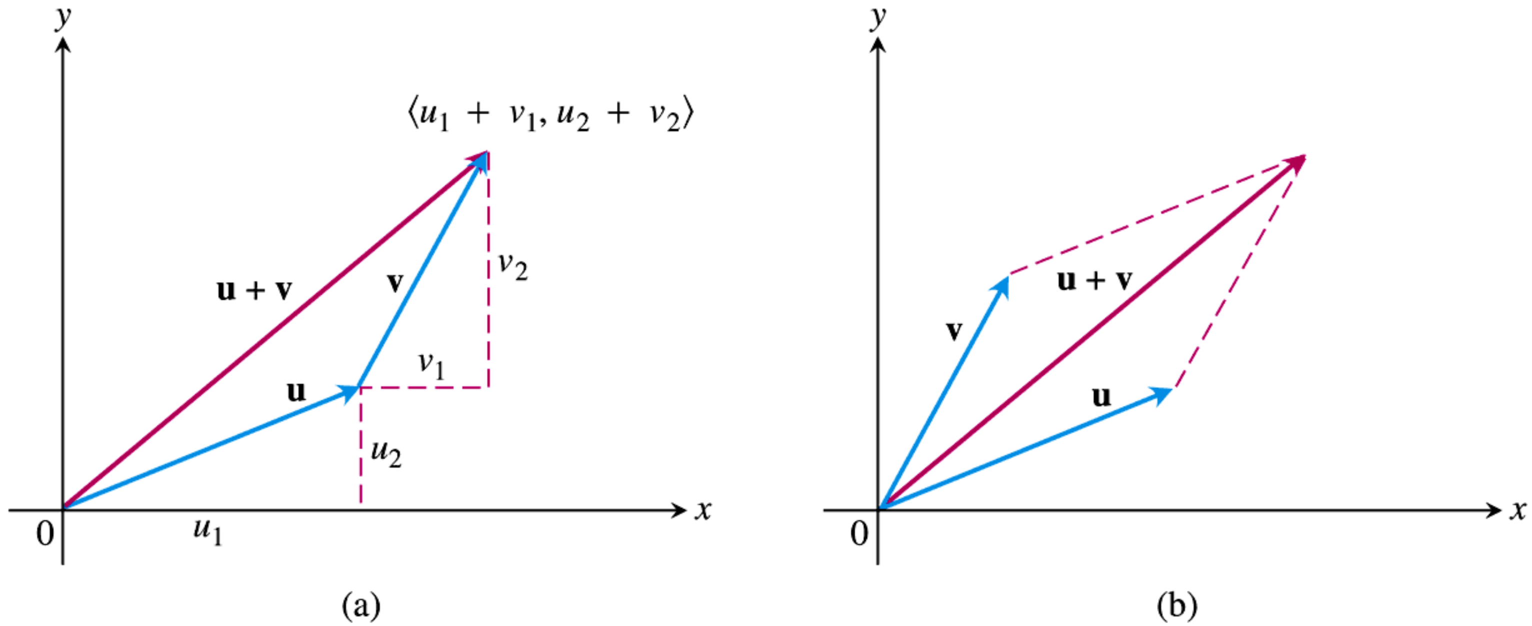


FIGURE 12.12 (a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition.

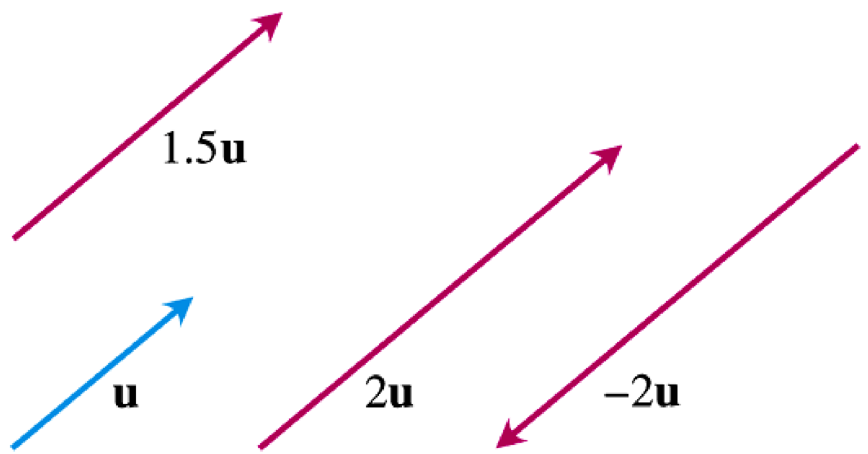
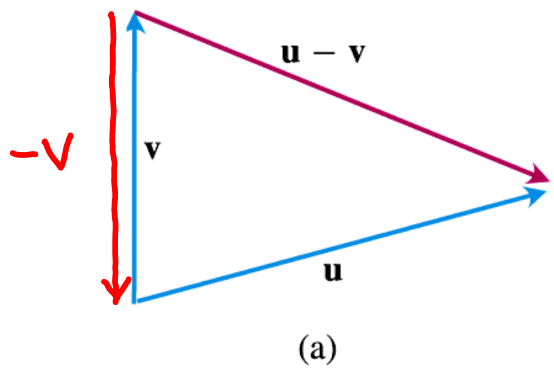


FIGURE 12.13 Scalar multiples of **u**.

- If $k>0$ then $\mathbf{u}$ and $k\mathbf{u}$ have the same direction.
- If $k<0$ then $\mathbf{u}$ and $k\mathbf{u}$ have the opposite directions.
- $|k\mathbf{u}| = |k|\,|\mathbf{u}|$.
- $-\mathbf{u} = (-1)\mathbf{u}$ and $|-\mathbf{u}| = |\mathbf{u}|$.



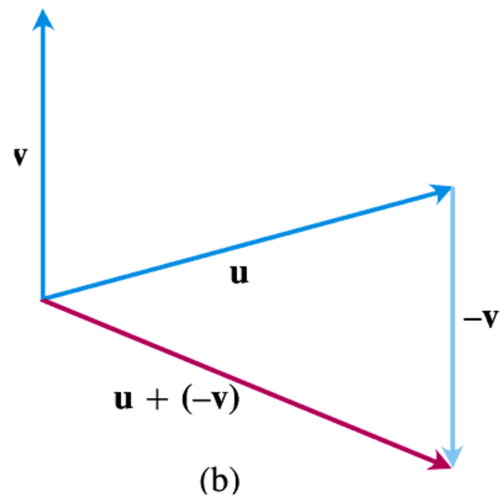


FIGURE 12.14 (a) The vector $\mathbf{u} - \mathbf{v}$, when added to $\mathbf{v}$, gives $\mathbf{u}$. (b) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$. $\Rightarrow (\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u}$

By the difference $\mathbf{u} - \mathbf{v}$ of two vectors, we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \mathbf{u} + (-1)\mathbf{v}$$

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

EXAMPLE 3 Performing Operations on Vectors

Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find

(a) $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left|\frac{1}{2}\mathbf{u}\right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $$\left|\frac{1}{2}\mathbf{u}\right| = \left|\left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$$

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit Vectors

A vector $\mathbf{v}$ of length 1 is called a **unit vector**. The **standard unit vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \qquad \mathbf{j} = \langle 0, 1, 0 \rangle, \qquad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as follows:

$$\begin{aligned} \mathbf{v} = \langle v_1, v_2, v_3 \rangle &= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}. \end{aligned}$$

We call the scalar (or number) v_1 the **i-component** of the vector $\mathbf{v}$, v_2 the **j-component**, and v_3 the **k-component**. In component form, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

(Figure 12.16).

Whenever $\mathbf{v} \neq \mathbf{0}$, its length $|\mathbf{v}|$ is not zero and

$$\left| \frac{1}{|\mathbf{v}|}\mathbf{v} \right| = \frac{1}{|\mathbf{v}|}|\mathbf{v}| = 1.$$

That is, $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of $\mathbf{v}$, called **the direction** of the nonzero vector $\mathbf{v}$.

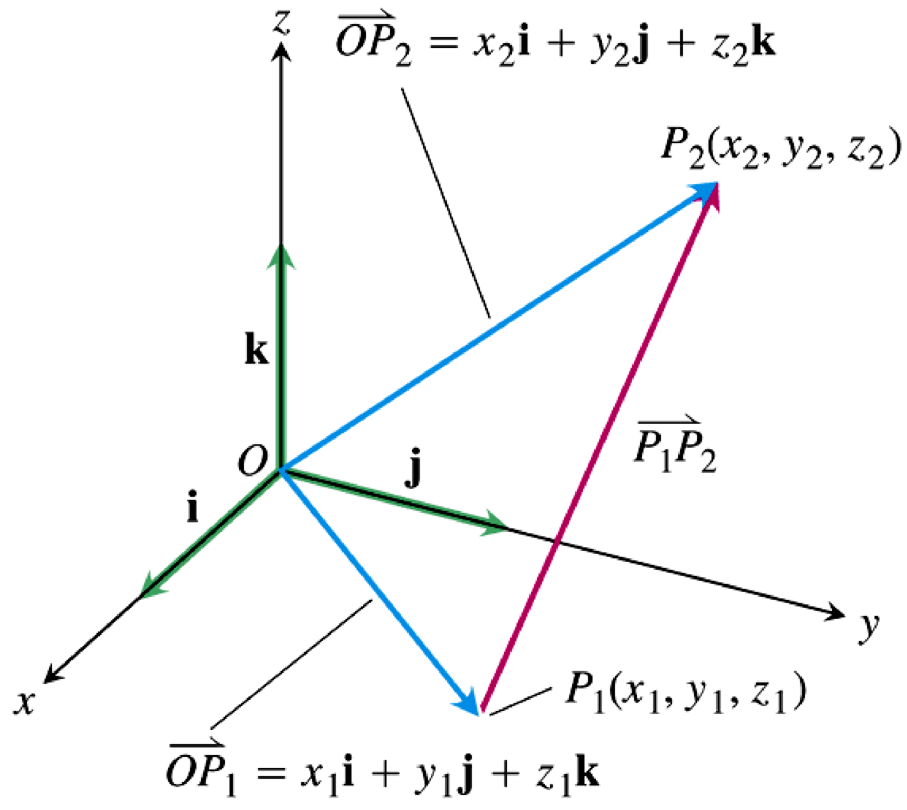


FIGURE 12.16 The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

The vector $\frac{1}{|\mathbf{v}|}\mathbf{v}$ is shortly denoted by $\frac{\mathbf{v}}{|\mathbf{v}|}$

scalar

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of $\mathbf{v}$;

2. the equation $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ in terms of its length and direction.

EXAMPLE 5 Finding a Vector's Direction

Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

The unit vector $\mathbf{u}$ is the direction of $\overrightarrow{P_1P_2}$. ■

* On $\mathbb{R}^2$, the standard unit vectors are $i = \langle 1, 0 \rangle$ and $j = \langle 0, 1 \rangle$

EXAMPLE 6 Expressing Velocity as Speed Times Direction

If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times a unit vector in the direction of motion.

Solution Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ has the same direction as $\mathbf{v}$:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underbrace{5}_{\substack{\text{Length} \\ \text{(speed)}}}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$

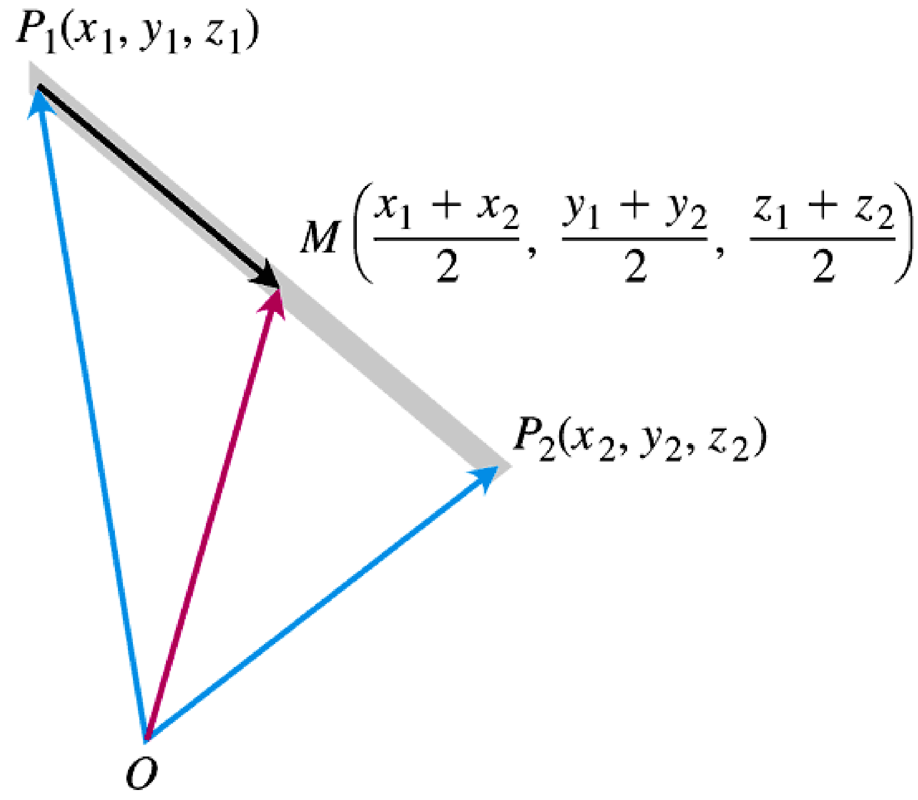


FIGURE 12.17 The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

To see why, observe (Figure 12.17) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) \\ &= \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

12.3

The Dot Product

In this section, we show how to calculate easily the angle between two vectors directly from their components. A key part of the calculation is an expression called the *dot product*. Dot products are also called *inner* or *scalar* products because the product results in a scalar, not a vector. After investigating the dot product, we apply it to finding the projection of one vector onto another.

The angle θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. $0 \leq \theta \leq \pi$. If $\mathbf{u}$ and $\mathbf{v}$ have the same direction then $\theta = 0$.

If $\mathbf{u}$ and $\mathbf{v}$ have the opposite directions then $\theta = \pi$.

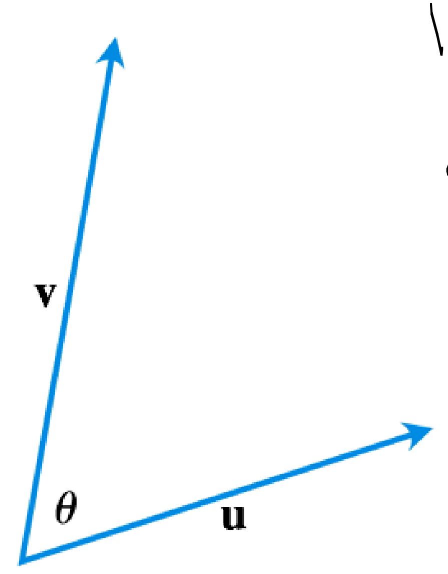


FIGURE 12.19 The angle between **u** and **v**.

THEOREM 1 Angle Between Two Vectors

The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

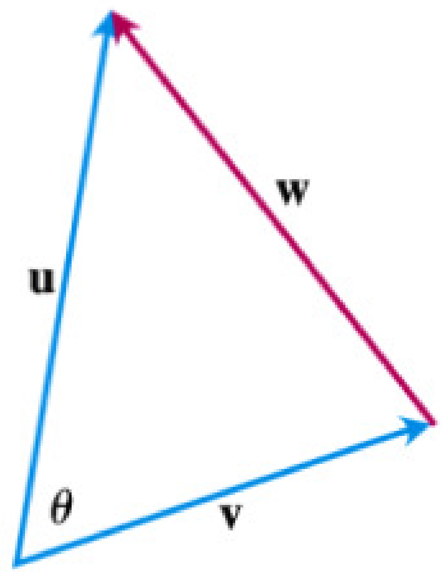


FIGURE 12.20 The parallelogram law of addition of vectors gives $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

Proof of Theorem 1 Applying the law of cosines (Equation (6), Section 1.6) to the triangle in Figure 12.20, we find that

$$|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta \qquad \text{Law of cosines}$$

$$2|\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2.$$

Because $\mathbf{w} = \mathbf{u} - \mathbf{v}$, the component form of $\mathbf{w}$ is $\langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$. So

$$\begin{aligned}
|\mathbf{u}|^2 &= \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = u_1^2 + u_2^2 + u_3^2 \\
|\mathbf{v}|^2 &= \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 = v_1^2 + v_2^2 + v_3^2 \\
|\mathbf{w}|^2 &= \left(\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}\right)^2 \\
&= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\
&= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2
\end{aligned}$$

and

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Therefore,

$$\begin{aligned}
2|\mathbf{u}||\mathbf{v}| \cos\theta &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3) \\
|\mathbf{u}||\mathbf{v}| \cos\theta &= u_1v_1 + u_2v_2 + u_3v_3 \\
\cos\theta &= \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}
\end{aligned}$$

So

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right) \qquad \blacksquare$$

DEFINITION **Dot Product**

The **dot product** $\mathbf{u} \cdot \mathbf{v}$ ("$\mathbf{u}$ dot $\mathbf{v}$") of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

With the notation of the dot product, the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ can be written as

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

Equivalently, $\mathbf{u} . \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$

EXAMPLE 1 Finding Dot Products

(a) $$\begin{aligned}\langle 1, -2, -1\rangle \cdot \langle -6, 2, -3\rangle &= (1)(-6) + (-2)(2) + (-1)(-3)\\ &= -6 - 4 + 3 = -7\end{aligned}$$

(b) $$\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right)\cdot(4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$$

EXAMPLE 2 Finding the Angle Between Two Vectors in Space

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution We use the formula above:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right)$$

$$= \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians}.$$

The angle formula applies to two-dimensional vectors as well.

The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (or perpendicular) if the angle between them is $\frac{\pi}{2}$. For such vectors $\mathbf{u} \cdot \mathbf{v} = 0$.

DEFINITION **Orthogonal Vectors**

Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

EXAMPLE 4 Applying the Definition of Orthogonality

(a) $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

(b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.

(c) $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ since

$$\begin{aligned} \mathbf{0} \cdot \mathbf{u} &= \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) \\ &= 0. \end{aligned}$$

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Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ $= (v + w) \cdot u$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0$.

Proofs of Properties 1 and 3 The properties are easy to prove using the definition. For instance, here are the proofs of Properties 1 and 3.

1. $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \mathbf{v} \cdot \mathbf{u}$

3. $$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \\ &= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) \\ &= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 + u_3 v_3 + u_3 w_3 \\ &= (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3) \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \end{aligned}$$

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The **vector projection** of $\mathbf{u} = \overrightarrow{PQ}$ onto a nonzero vector $\mathbf{v} = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.

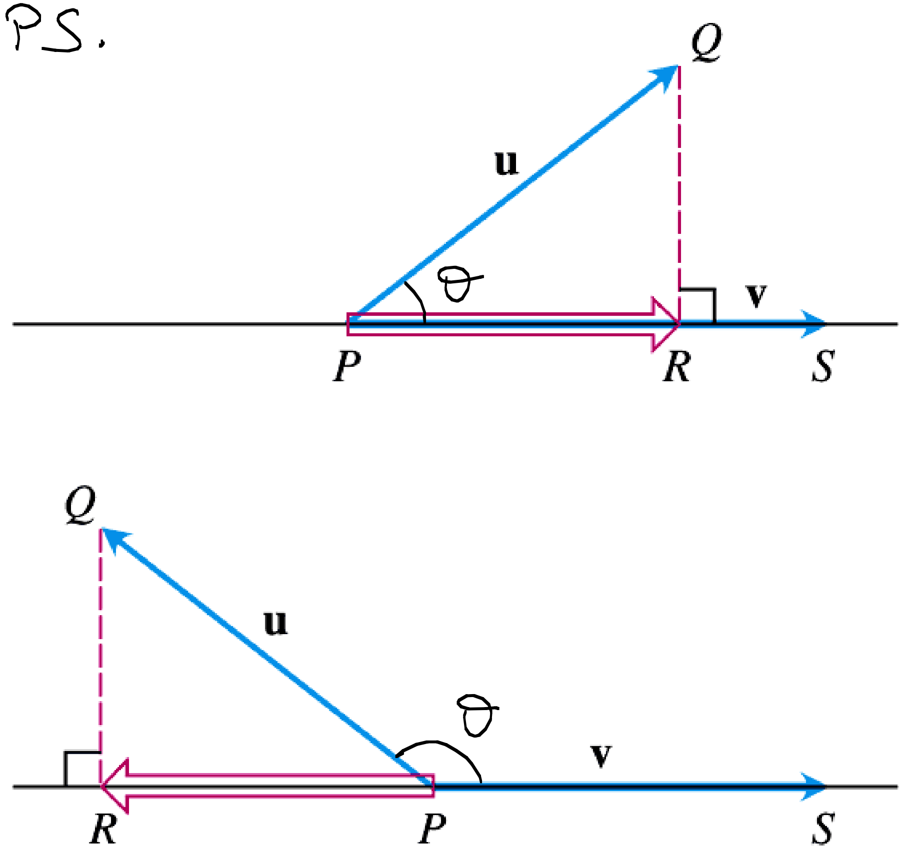


FIGURE 12.22 The vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

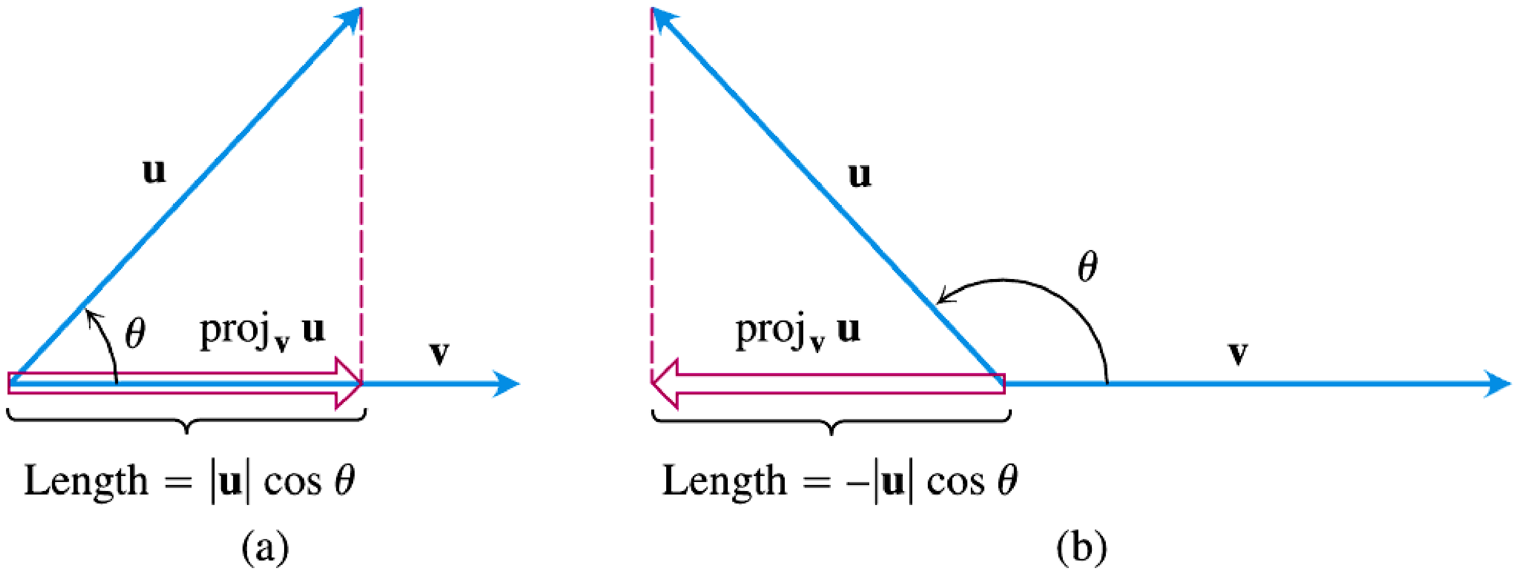


FIGURE 12.24 The length of $\text{proj}_{\mathbf{v}}\mathbf{u}$ is (a) $|\mathbf{u}|\cos\theta$ if $\cos\theta \geq 0$ and (b) $-|\mathbf{u}|\cos\theta$ if $\cos\theta < 0$. The direction of $\text{proj}_{\mathbf{v}}\mathbf{u}$ is (a) $\frac{\mathbf{v}}{|\mathbf{v}|}$ if $\cos\theta \geq 0$ and (b) $-\frac{\mathbf{v}}{|\mathbf{v}|}$ if $\cos\theta < 0$.

In both cases, $\text{proj}_{\mathbf{v}}\mathbf{u} = (|\mathbf{u}|\cos\theta)\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{|\mathbf{u}||\mathbf{v}|\cos\theta}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}$

$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

Vector projection of **u** onto **v**:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \tag{1}$$

Scalar component of **u** in the direction of **v**:

$$|\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} \tag{2}$$

Equations (1) and (2) also apply to two-dimensional vectors.

Application in Physics

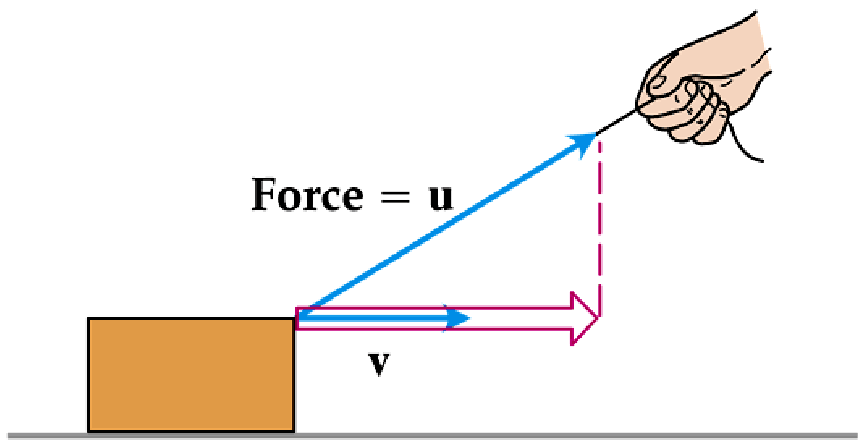


FIGURE 12.23 If we pull on the box with force **u**, the effective force moving the box forward in the direction **v** is the projection of **u** onto **v**.

EXAMPLE 5 Finding the Vector Projection

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$.

Solution We find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ from Equation (1):

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6-6-4}{1+4+4}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})\\ &= -\frac{4}{9}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

We find the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ from Equation (2):

$$\begin{aligned}|\mathbf{u}|\cos\theta &= \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)\\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.\end{aligned}$$

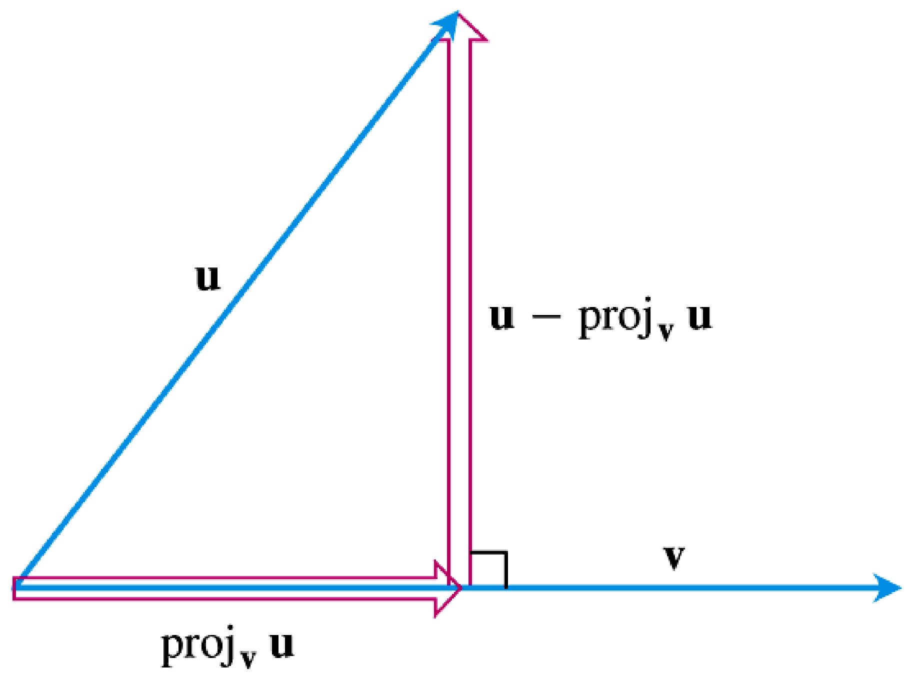


FIGURE 12.26 Writing **u** as the sum of vectors parallel and orthogonal to **v**.

How to Write **u** as a Vector Parallel to **v** Plus a Vector Orthogonal to **v**

$$
\begin{aligned}
\mathbf{u} &= \operatorname{proj}_{\mathbf{v}} \mathbf{u} + (\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) \\
&= \underbrace{\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}}_{\text{Parallel to } \mathbf{v}} + \underbrace{\left(\mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\right)}_{\text{Orthogonal to } \mathbf{v}}
\end{aligned}
$$

EXAMPLE 8 Force on a Spacecraft

A force $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is applied to a spacecraft with velocity vector $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$. Express $\mathbf{F}$ as a sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

Solution

$$\begin{aligned}
\mathbf{F} &= \operatorname{proj}_{\mathbf{v}} \mathbf{F} + (\mathbf{F} - \operatorname{proj}_{\mathbf{v}} \mathbf{F}) \\
&= \frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \left(\mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right) \\
&= \left(\frac{6-1}{9+1}\right)\mathbf{v} + \left(\mathbf{F} - \left(\frac{6-1}{9+1}\right)\mathbf{v}\right) \\
&= \frac{5}{10}(3\mathbf{i} - \mathbf{j}) + \left(2\mathbf{i} + \mathbf{j} - 3\mathbf{k} - \frac{5}{10}(3\mathbf{i} - \mathbf{j})\right) \\
&= \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right) + \left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right).
\end{aligned}$$

The force $(3/2)\mathbf{i} - (1/2)\mathbf{j}$ is the effective force parallel to the velocity $\mathbf{v}$. The force $(1/2)\mathbf{i} + (3/2)\mathbf{j} - 3\mathbf{k}$ is orthogonal to $\mathbf{v}$. To check that this vector is orthogonal to $\mathbf{v}$, we find the dot product:

$$\left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right)\cdot(3\mathbf{i} - \mathbf{j}) = \frac{3}{2} - \frac{3}{2} = 0.$$

■

12.4

The Cross Product

Cross products are widely used to describe the effects of forces in studies of electricity, magnetism, fluid flows, and orbital mechanics. This section presents the mathematical properties that account for the use of cross products in these fields.

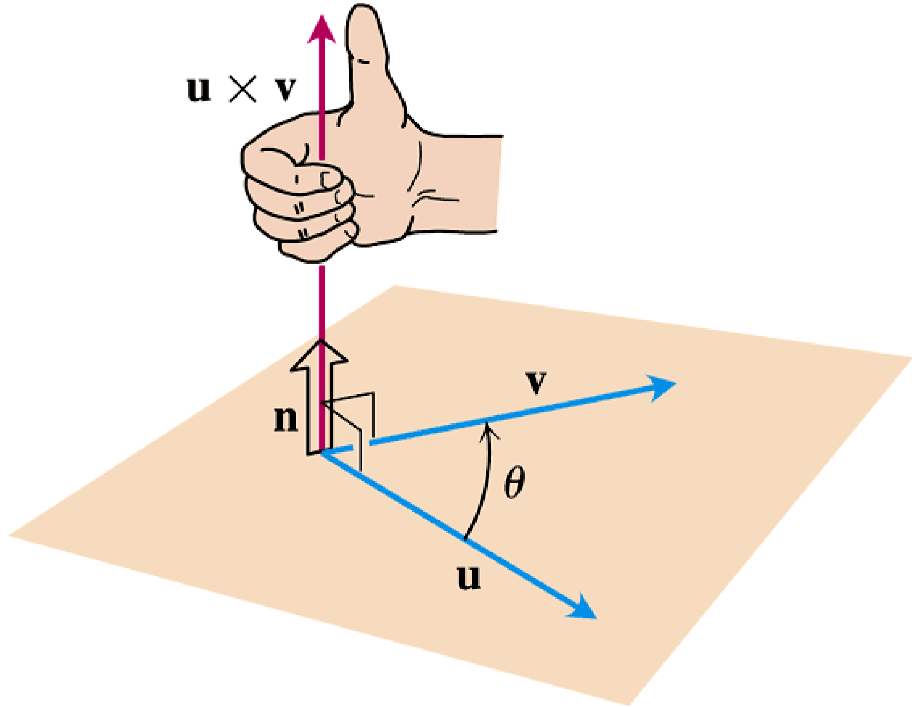


FIGURE 12.27 The construction of $\mathbf{u} \times \mathbf{v}$.

We start with two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in space. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, they determine a plane. We select a unit vector $\mathbf{n}$ perpendicular to the plane by the **right-hand rule**. This means that we choose $\mathbf{n}$ to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from $\mathbf{u}$ to $\mathbf{v}$ (Figure 12.27). Then the **cross product** $\mathbf{u} \times \mathbf{v}$ ("$\mathbf{u}$ cross $\mathbf{v}$") is the *vector* defined as follows.

DEFINITION **Cross Product**

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\, \mathbf{n}$$

Unlike the dot product, the cross product is a vector. For this reason it's also called the **vector product** of $\mathbf{u}$ and $\mathbf{v}$, and applies *only* to vectors in space. The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ because it is a scalar multiple of $\mathbf{n}$.

Since the sines of 0 and π are both zero, it makes sense to define the cross product of two parallel nonzero vectors to be $\mathbf{0}$. If one or both of $\mathbf{u}$ and $\mathbf{v}$ are zero, we also define $\mathbf{u} \times \mathbf{v}$ to be zero. This way, the cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is zero if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel or one or both of them are zero.

Parallel Vectors

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

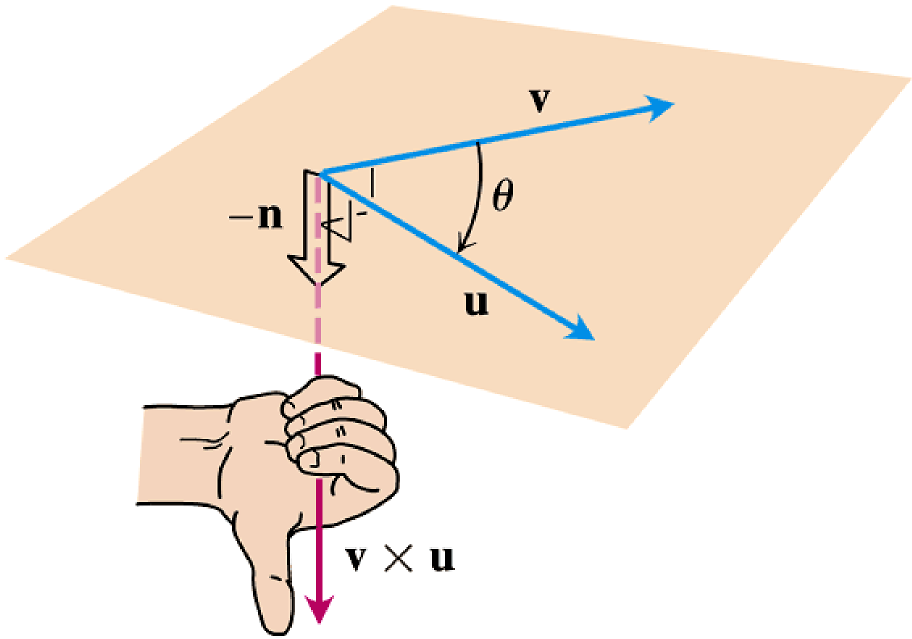


FIGURE 12.28 The construction of $\mathbf{v} \times \mathbf{u}$.

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|\,|\mathbf{n}| = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|.$$

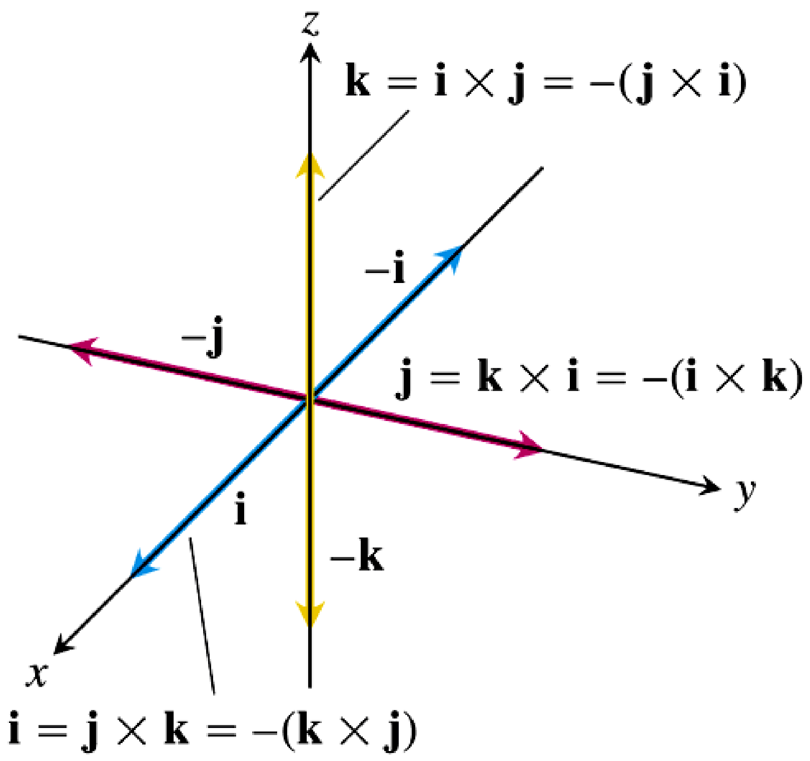


FIGURE 12.29 The pairwise cross products of **i**, **j**, and **k**.

$$i \times i = j \times j = k \times k = 0$$

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin\theta||\mathbf{n}| = |\mathbf{u}||\mathbf{v}||\sin\theta|$$

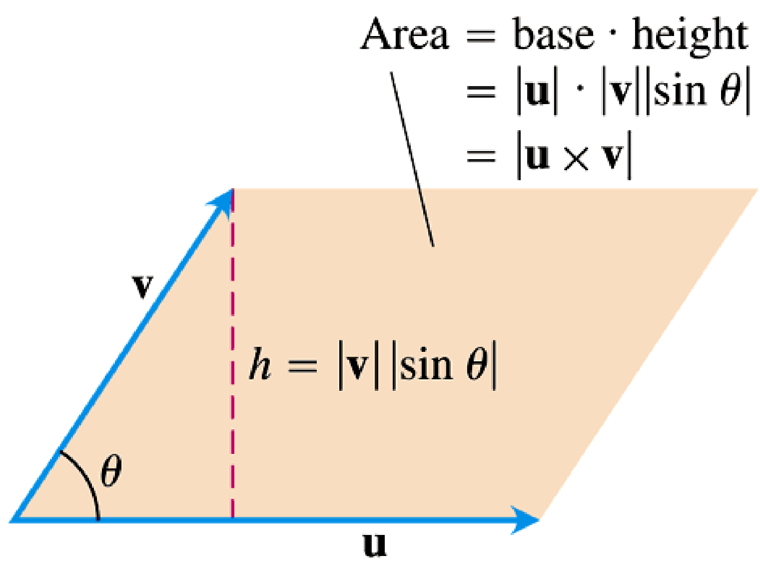


FIGURE 12.30 The parallelogram determined by **u** and **v**.

Determinants

2×2 and 3×3 determinants are evaluated as follows:

$\circledast$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$\circledast$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

EXAMPLE

$$\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} = (2)(3) - (1)(-4) = 6 + 4 = 10$$

EXAMPLE

$$\begin{vmatrix} -5 & 3 & 1 \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = (-5) \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - (3) \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + (1) \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$$

$$= -5(1 - 3) - 3(2 + 4) + 1(6 + 4)$$

$$= 10 - 18 + 10 = 2$$

Determinant Formula for $\mathbf{u} \times \mathbf{v}$

Our next objective is to calculate $\mathbf{u} \times \mathbf{v}$ from the components of $\mathbf{u}$ and $\mathbf{v}$ relative to a Cartesian coordinate system.

Suppose that

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \qquad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Then the distributive laws and the rules for multiplying $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ tell us that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} \\
&\quad + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} + u_2v_3\mathbf{j} \times \mathbf{k} \\
&\quad + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

The terms in the last line are the same as the terms in the expansion of the symbolic determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Calculating Cross Products Using Determinants

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

EXAMPLE 1 Calculating Cross Products with Determinants

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

EXAMPLE 2 Finding Vectors Perpendicular to a Plane

Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

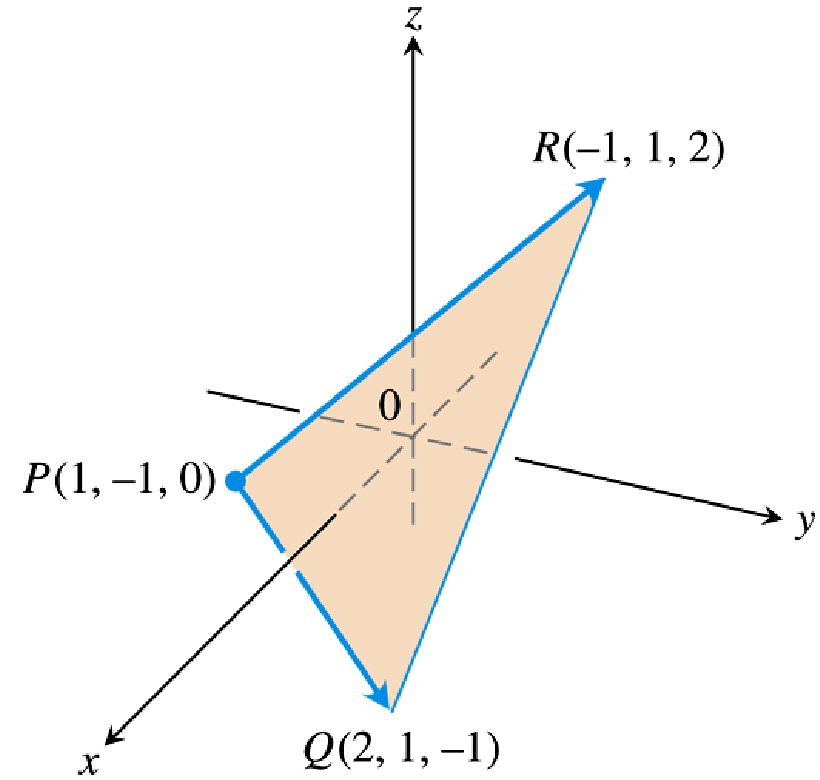


FIGURE 12.31 The area of triangle PQR is half of $|\overrightarrow{PQ} \times \overrightarrow{PR}|$ (Example 2).

Solution The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$

EXAMPLE 3 Finding the Area of a Triangle

Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

Solution The area of the parallelogram determined by P, Q, and R is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}| \qquad \text{Values from Example 2.}$$

$$= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.$$

The triangle's area is half of this, or $3\sqrt{2}$.

Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (in that order).

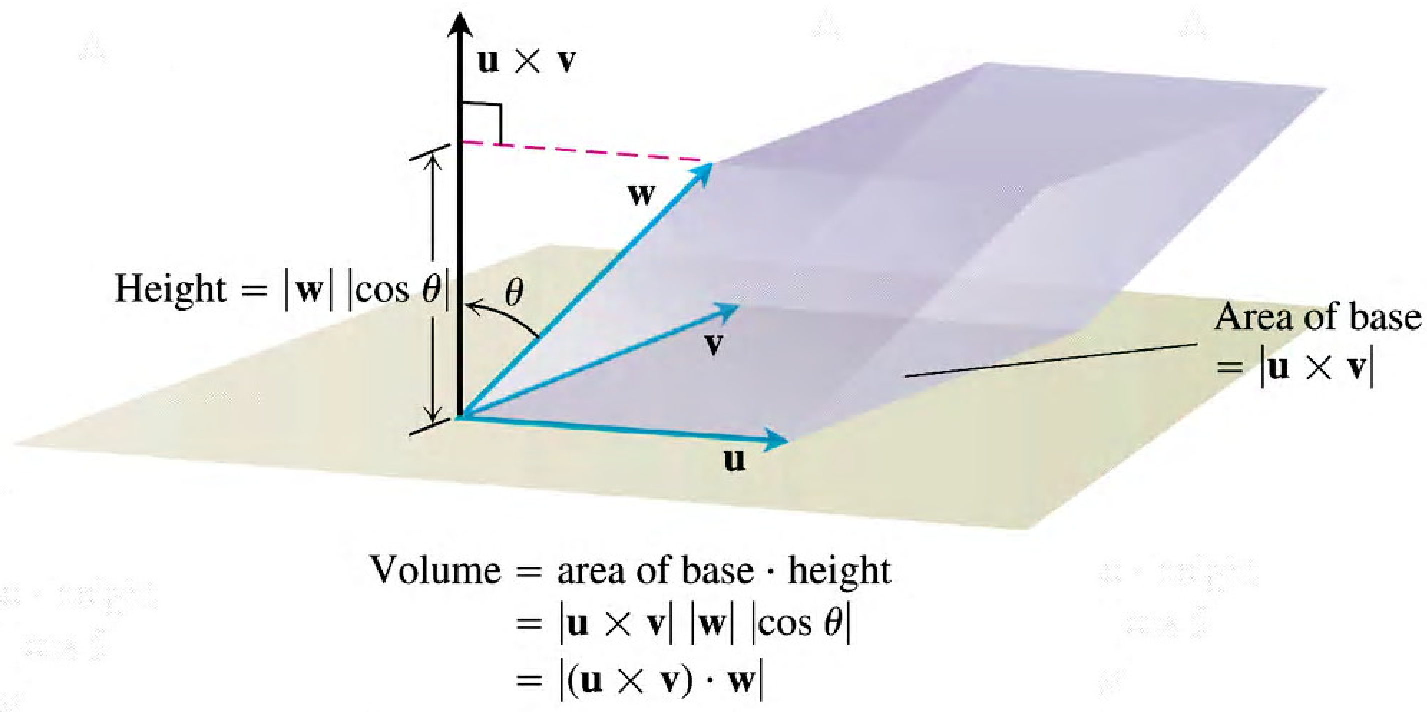


FIGURE 12.34 The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

Calculating the Triple Scalar Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Proof:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right] \cdot \mathbf{w}$$

$$= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

EXAMPLE 6 Finding the Volume of a Parallelepiped

Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution Using the rule for calculating determinants, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed.

12.5

Lines and Planes in Space

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space.

In the plane, a line is determined by a point and a number giving the slope of the line. In space a line is determined by a point and a *vector* giving the direction of the line.

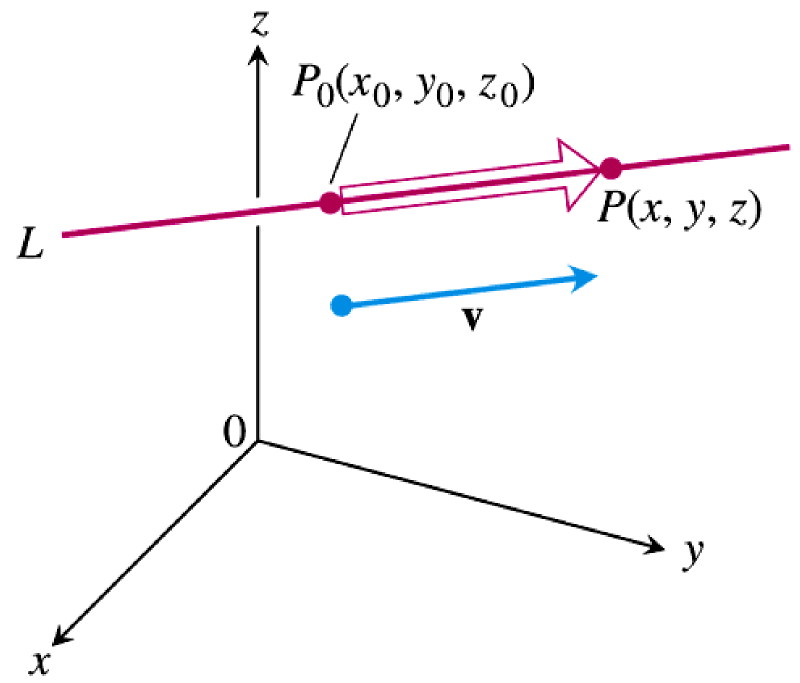
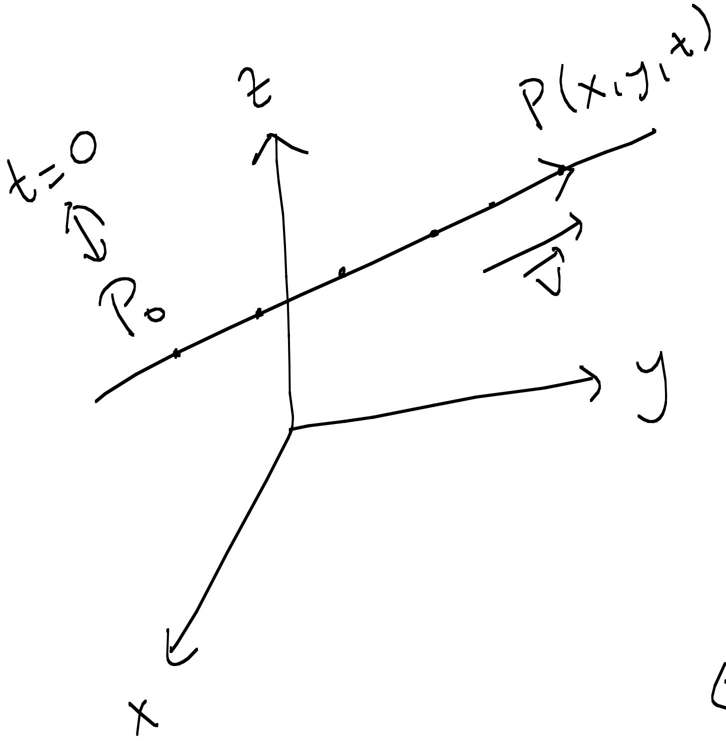


FIGURE 12.35 A point P lies on L through P_0 parallel to $\mathbf{v}$ if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$.



$$\overrightarrow{P_0P} \;//\; \vec{v}$$

$$\Leftrightarrow \quad \overrightarrow{P_0P} = t \cdot \vec{v} \quad (t \in \mathbb{R})$$

$$\Leftrightarrow \quad \langle x - x_0, y - y_0, z - z_0 \rangle = t \langle v_1, v_2, v_3 \rangle$$

$$x - x_0 = t v_1 \quad \Rightarrow \quad \begin{cases} x = t \cdot v_1 + x_0 \\ y = t \cdot v_2 + y_0 \\ z = t \cdot v_3 + z_0 \end{cases}$$

$$t \in \mathbb{R}$$

scalar parametric
line eqn.

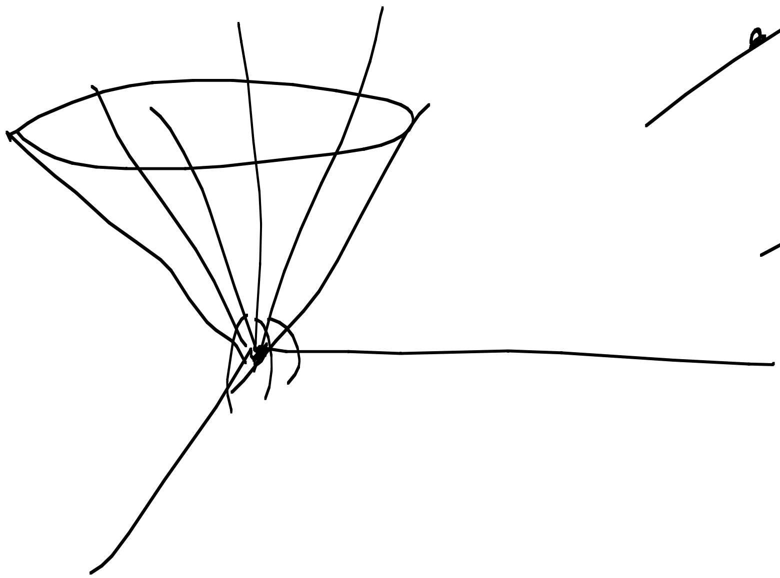
vector
parametric
line eqn.

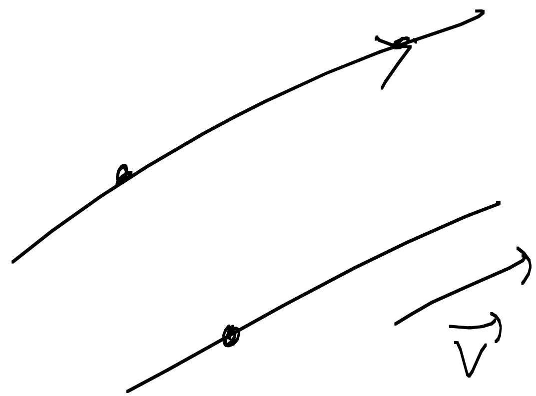
$$\vec{r}(t) = (tv_1 + x_0)\,i + (tv_2 + y_0)\,j + (tv_3 + z_0)\,k$$

$t \in \mathbb{R}$

standard
form

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$





Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \tag{3}$$

EXAMPLE 1 Parametrizing a Line Through a Point Parallel to a Vector

Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (Figure 12.36).

Solution With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

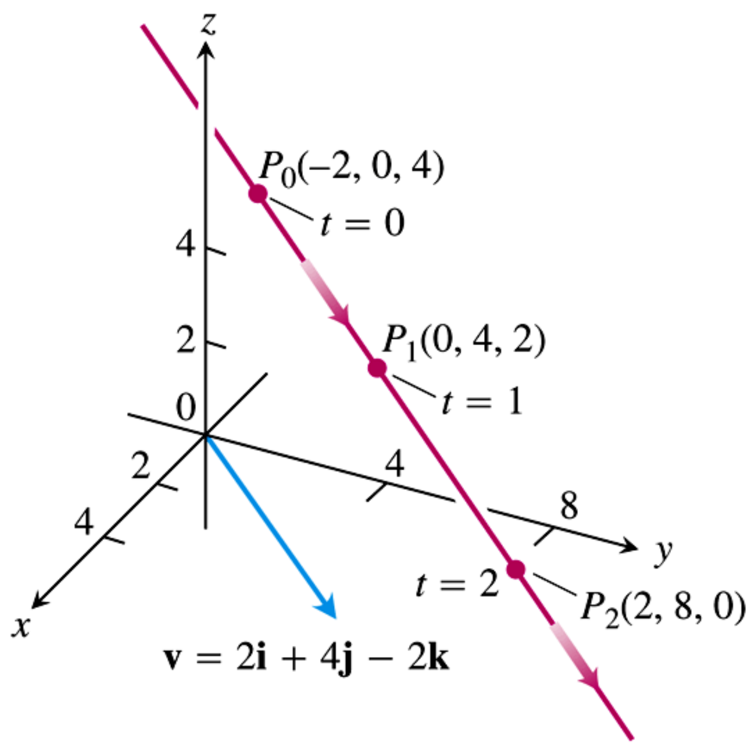


FIGURE 12.36 Selected points and parameter values on the line $x = -2 + 2t, y = 4t, z = 4 - 2t$. The arrows show the direction of increasing t (Example 1).

EXAMPLE 2 Parametrizing a Line Through Two Points

Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution The vector

$$\begin{aligned}\overrightarrow{PQ} &= (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} \\ &= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}\end{aligned}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t.

EXAMPLE 3 Parametrizing a Line Segment

Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Figure 12.37).

Solution We begin with equations for the line through P and Q, taking them, in this case, from Example 2:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \leq t \leq 1$ to parametrize the segment:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \leq t \leq 1.$$

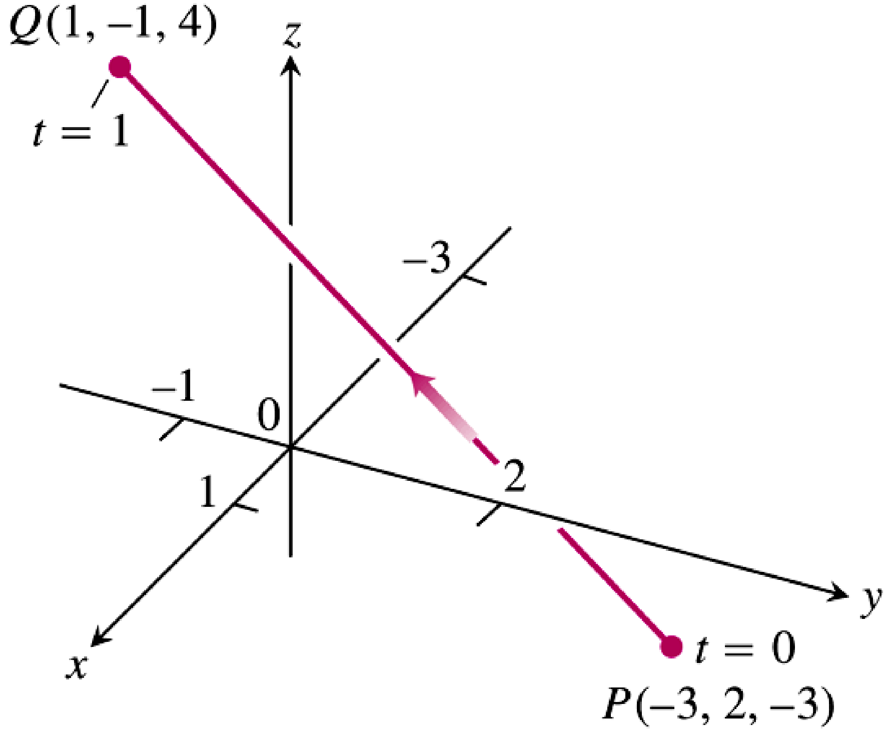


FIGURE 12.37 Example 3 derives a parametrization of line segment PQ. The arrow shows the direction of increasing t.

The Distance from a Point to a Line in Space

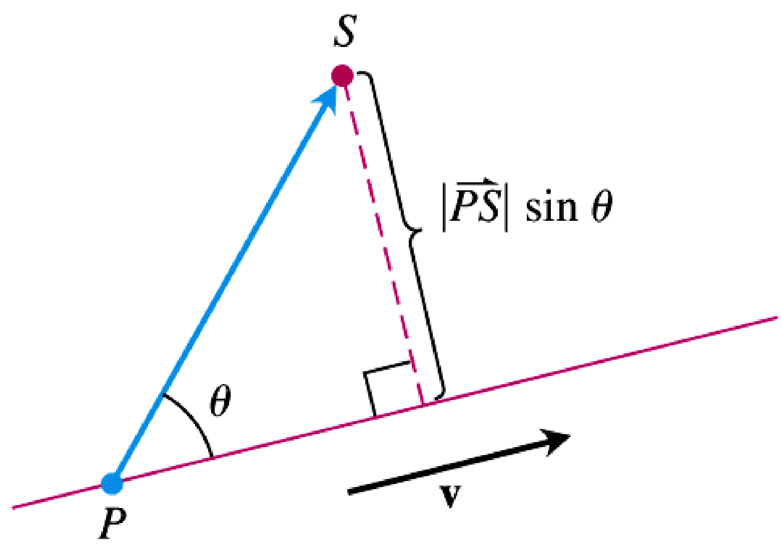
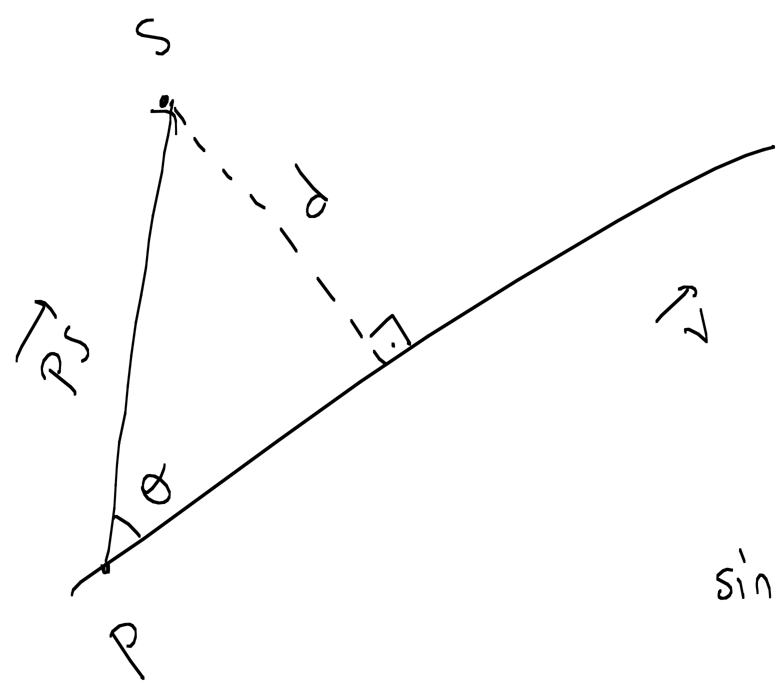
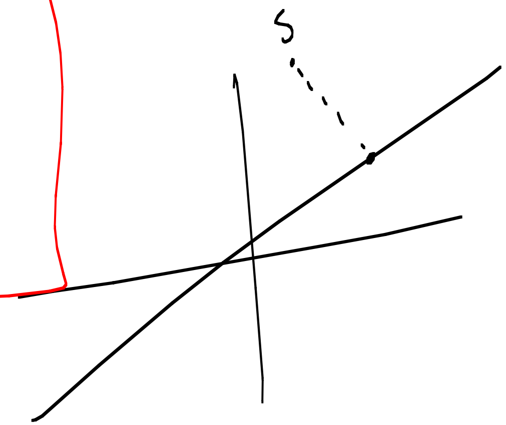


FIGURE 12.38 The distance from S to the line through P parallel to $\mathbf{v}$ is $|\overrightarrow{PS}| \sin \theta$, where θ is the angle between $\overrightarrow{PS}$ and $\mathbf{v}$.

$$\text{distance} = |\overrightarrow{PS}| . \sin\theta = \frac{|\overrightarrow{PS}|.|v|.\sin\theta}{|v|} = \frac{|\overrightarrow{PS} \times v|}{|v|}$$



$$|u \times v| = |u|\,|v| \sin\alpha$$
$$\sin\alpha = \frac{|u \times v|}{|u|\,|v|}$$



$$\sin\theta = \frac{d}{|PS|}$$

$$\Rightarrow d = |\vec{PS}| \cdot \sin\theta = \cancel{|\vec{PS}|} \frac{|PS \times \vec{v}|}{\cancel{|\vec{PS}|}\,|\vec{v}|}$$

$$= \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

Distance from a Point S to a Line Through P Parallel to $\mathbf{v}$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \tag{5}$$

EXAMPLE 5 Finding Distance from a Point to a Line

Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \qquad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its "tilt" or orientation. This "tilt" is defined by specifying a vector that is perpendicular or normal to the plane.

Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ (Figure 12.39). Thus, the dot product $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

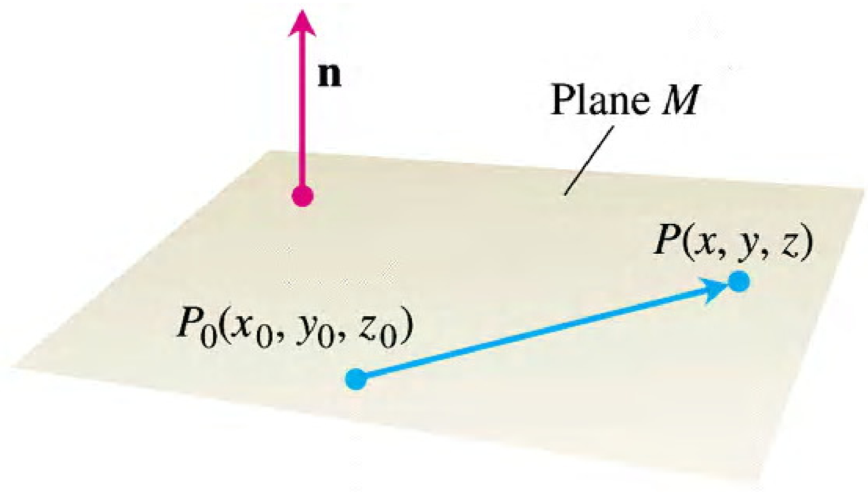
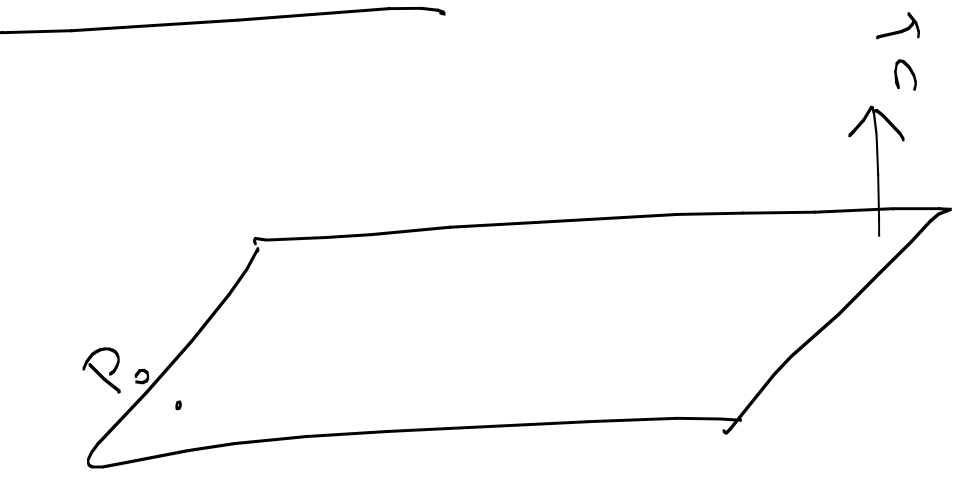
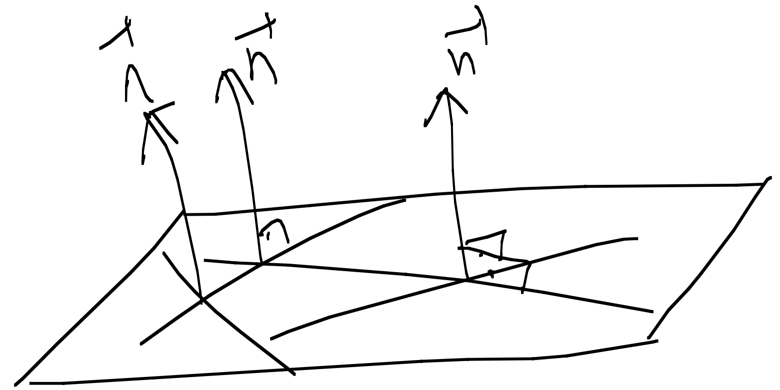


FIGURE 12.39 The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

Plane Eqn.

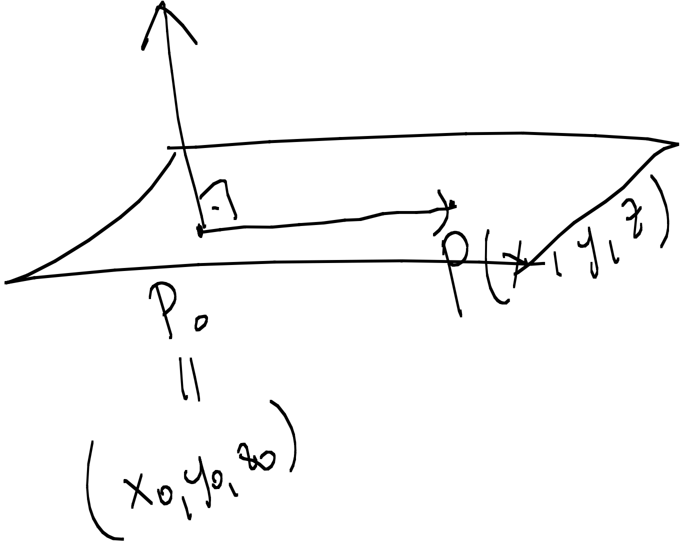


$\vec{n}$: normal vector to a plane if it is perpendicular to any line inside that plane



Aim: Find the eqn. of the plane through P_0 with normal $\vec{n}$

$\vec{n} = Ai + Bj + Ck$



$\overrightarrow{P_0P} \perp \vec{n}$

$\overrightarrow{P_0P} \cdot \vec{n} = 0$

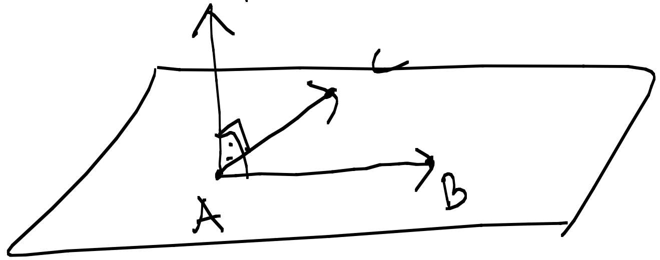
$\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle A, B, C \rangle = 0$

$$\boxed{A(x-x_0) + B(y-y_0) + C(z-z_0) = 0}$$

<u>Rk</u> We can also write the plane eqn. if we are given 3 points on this plane provided that these points are not <u>colinear</u>.

on the same line

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$



Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation:	$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$
Component equation:	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
Component equation simplified:	$Ax + By + Cz = D,$ where
	$D = Ax_0 + By_0 + Cz_0$

EXAMPLE 6 Finding an Equation for a Plane

Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22. \qquad \blacksquare$$

Notice in Example 6 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of x, y, and z in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

EXAMPLE 7 Finding an Equation for a Plane Through Three Points

Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of A(0, 0, 1) into the component form of the equation to obtain

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

$$3x + 2y + 6z = 6.$$

■

Lines of Intersection

Just as lines are parallel if and only if they have the same direction, two planes are **parallel** if and only if their normals are parallel, or $\mathbf{n}_1 = k\mathbf{n}_2$ for some scalar k. Two planes that are not parallel intersect in a line.

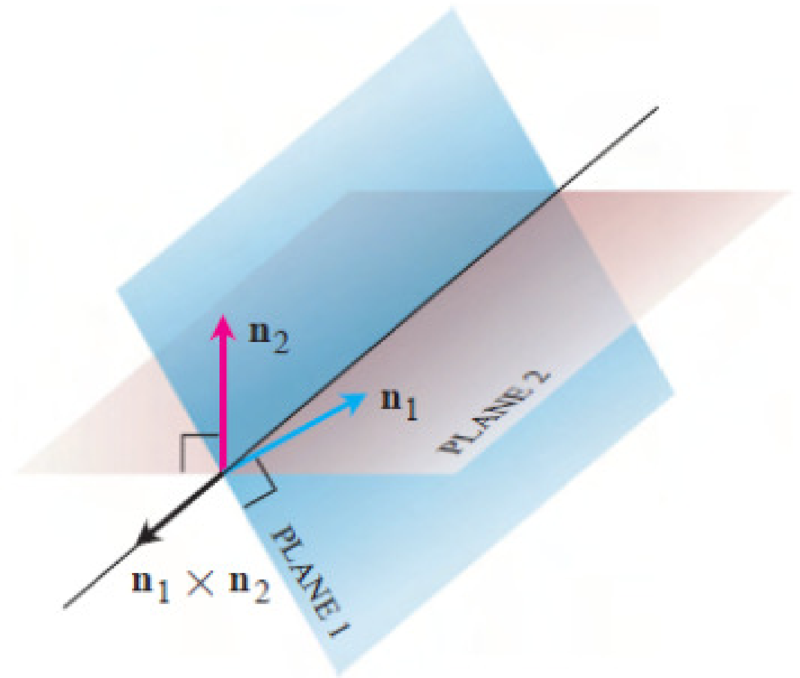


FIGURE 12.40 How the line of intersection of two planes is related to the planes' normal vectors (Example 8).

EXAMPLE 8 Finding a Vector Parallel to the Line of Intersection of Two Planes

Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The line of intersection of two planes is perpendicular to both planes' normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ (Figure 12.40) and therefore parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Any nonzero scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$ will do as well. ■

EXAMPLE 9 Parametrizing the Line of Intersection of Two Planes

Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Solution We find a vector parallel to the line and a point on the line and use Equations (3).

Example 8 identifies $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \qquad y = -1 + 2t, \qquad z = 15t.$$

The choice $z = 0$ is arbitrary and we could have chosen $z = 1$ or $z = -1$ just as well. Or we could have let $x = 0$ and solved for y and z. The different choices would simply give different parametrizations of the same line. ■

2nd way (Ex. 9) Make a common solution

$$\begin{array}{r} 3x - 6y - 2z = 15 \\ -\,(2x + y - 2z = 5) \\ \hline x - 7y = 10 \end{array} \Rightarrow x = 10 + 7y$$

$$2z = 3x - 6y - 15 = 3(10 + 7y) - 6y - 15$$

$$= 15 + 15y$$

$$\begin{cases} x = 10 + 7t \\ y = t \\ z = \frac{15}{2}t + \frac{15}{2} \end{cases} \quad t \in \mathbb{R}$$

1st way

$\vec{v} = \langle 14, 2, 15 \rangle$

$\vec{v_1} = \langle 7, 1, \frac{15}{2} \rangle$

2nd way

$(3, -1, 0)$

$t = -1$

EXAMPLE 10 Finding the Intersection of a Line and a Plane

Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Solution The point

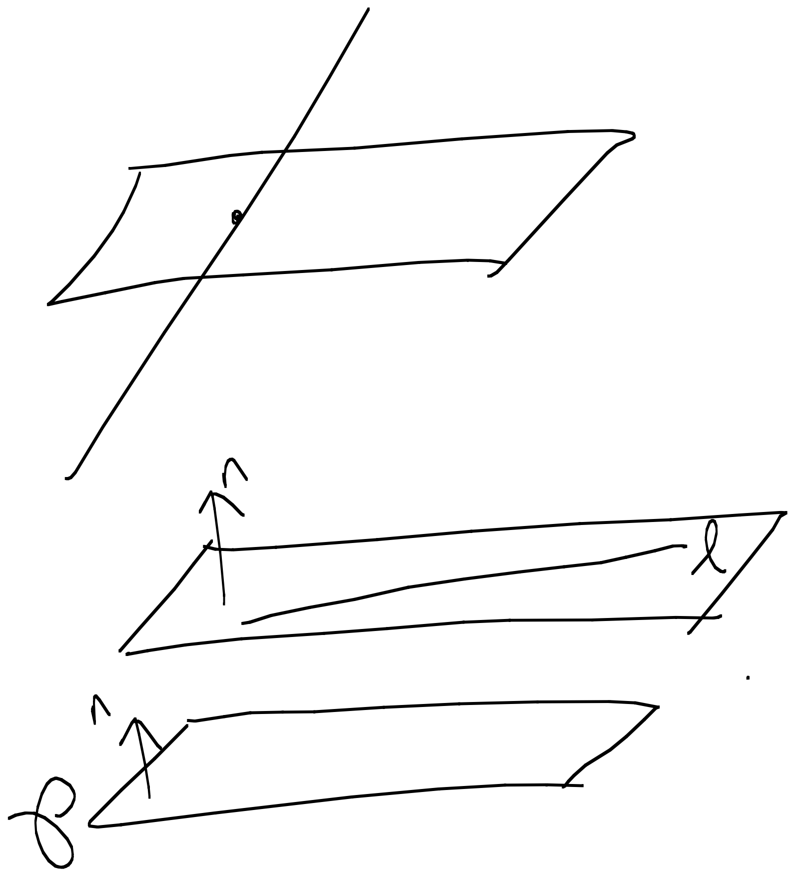
$$\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$\begin{aligned} 3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right).$$



Remark: Given a plane $\mathcal{P}$ some lines may not necessarily intersect $\mathcal{P}$.

if the line ℓ is in a plane parallel to $\mathcal{P}$ then they will not intersect

This is the case if $\vec{n} \perp \ell$

Ex

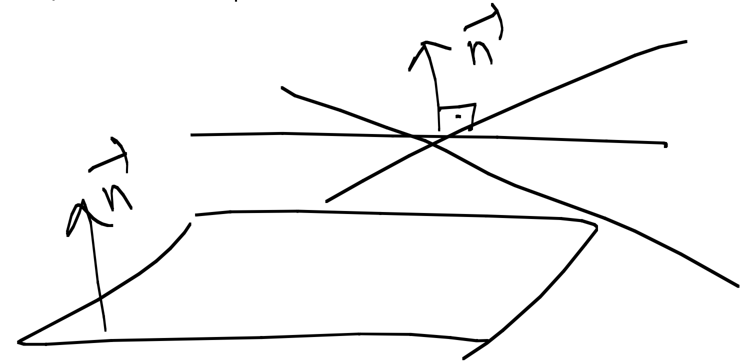
Do the plane $x - y = 5$ and the line $x = 1 - t \quad y = 3 - t \quad z = 2t - 1$ intersect?

$$(1-t) - (3-t) \stackrel{?}{=} 5$$

$$-2 = 5 \quad \text{contradiction} \quad \Rightarrow \quad \text{there's no intersection}$$

OR

$$\langle -1, -1, 2 \rangle \cdot \langle 1, -1, 0 \rangle = 0$$



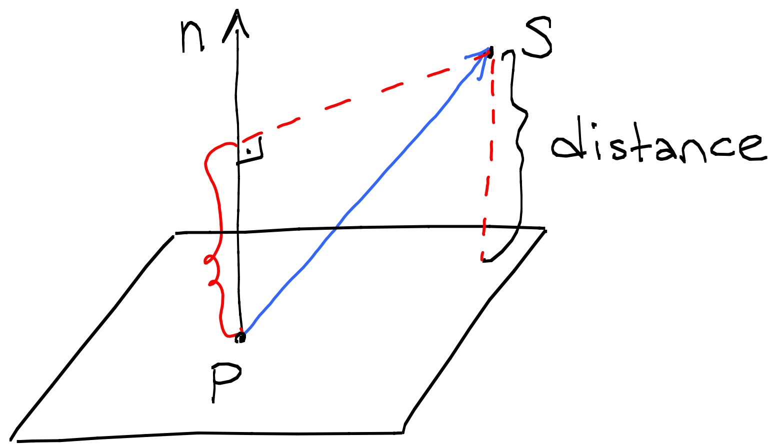
no intersection

The Distance from a Point to a Plane

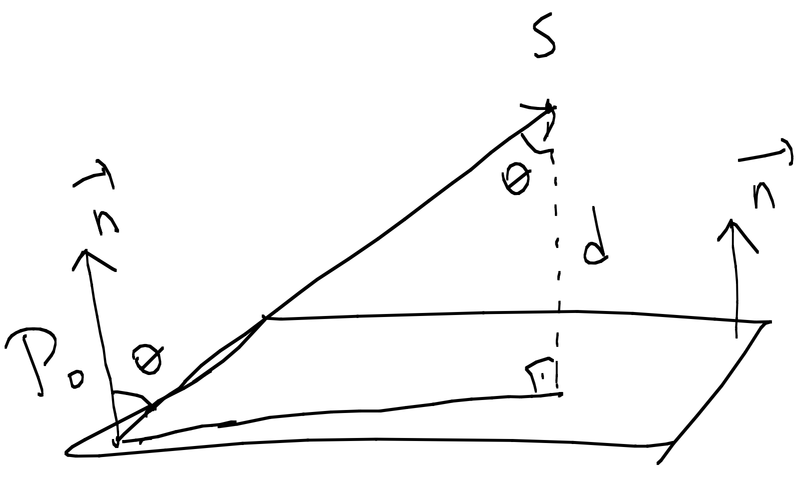
If P is a point on a plane with normal $\mathbf{n}$, then the distance from any point S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$. That is, the distance from S to the plane is

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \tag{6}$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane.



Distance from a Pt. to a Plane



$$\cos\theta = \frac{d}{|\overrightarrow{P_0S}|}$$

$$\frac{\overrightarrow{P_0S}\cdot\vec{n}}{|\overrightarrow{P_0S}|\,|\vec{n}|} = \frac{d}{|\overrightarrow{P_0S}|}$$

$$\Rightarrow \quad d = \left|\frac{\overrightarrow{P_0S}\cdot n}{|n|}\right|$$

EXAMPLE 11 **Finding the Distance from a Point to a Plane**

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Find a point P from the plane and apply the formula:

Letting $x = y = 0$ we get $z = 1 \Rightarrow (0, 0, 1)$

or $x = z = 0$ we get $y = 3 \Rightarrow (0, 3, 0)$

Let's use $P(0, 3, 0)$:

$\langle \vec{PS} \rangle = \langle 1, -2, 3 \rangle$ $\qquad \vec{n} = \langle 3, 2, 6 \rangle$

$$\Rightarrow \quad d = \frac{|\vec{PS} \cdot \vec{n}|}{|\vec{n}|} = \frac{|3 - 4 + 18|}{\sqrt{9 + 4 + 36}} = \frac{17}{7}$$

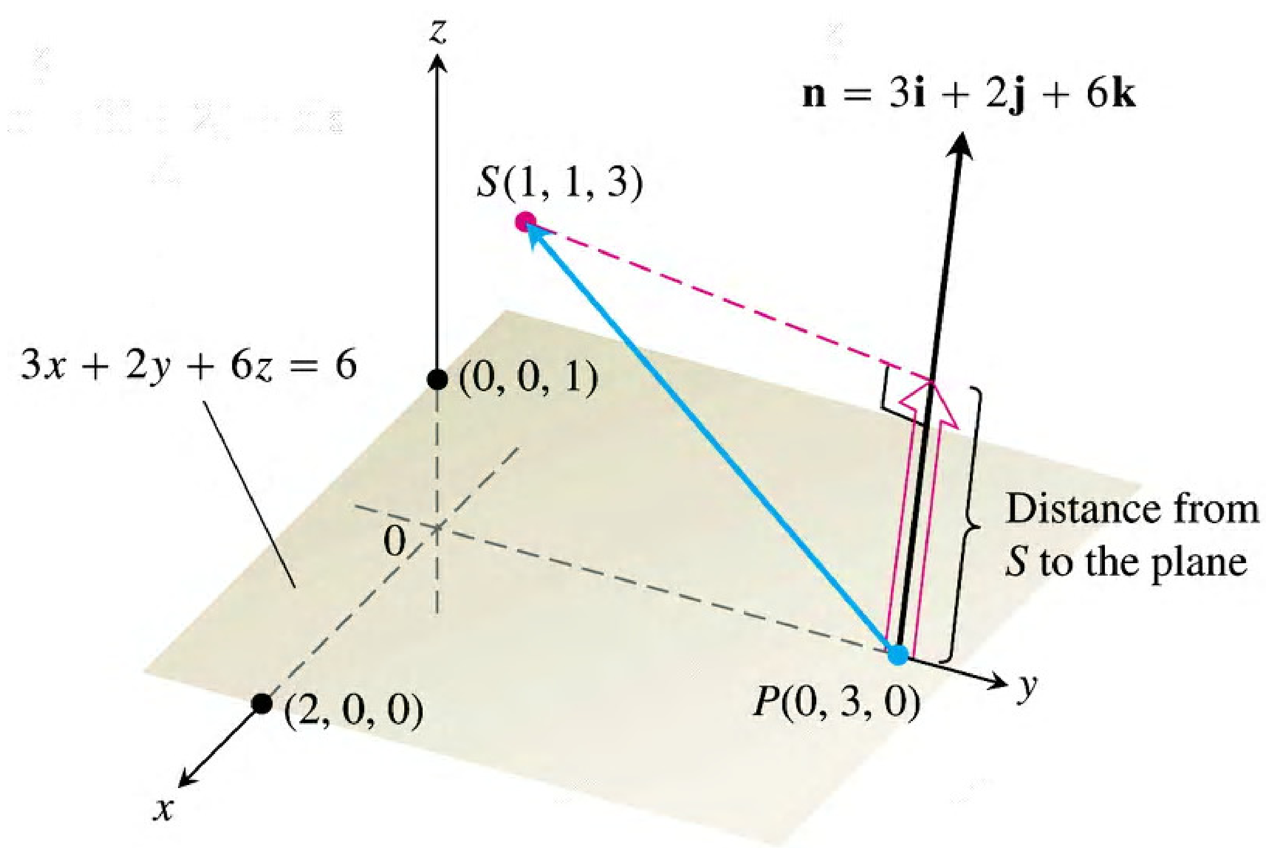


FIGURE 12.41 The distance from S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$ (Example 11).

Remark -

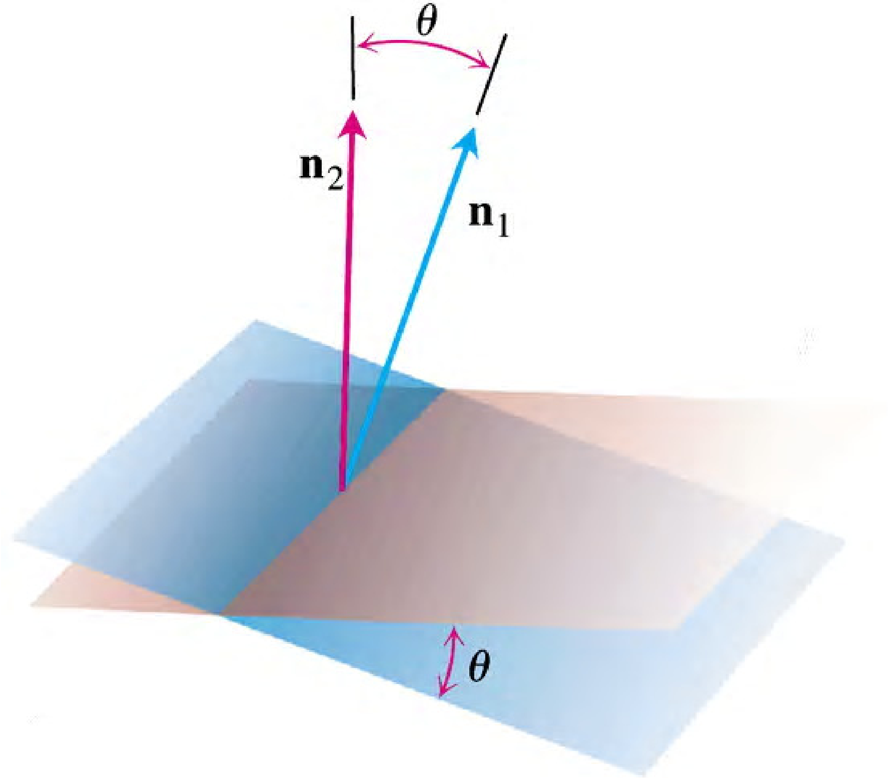


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.

12.6

Cylinders and Quadric Surfaces

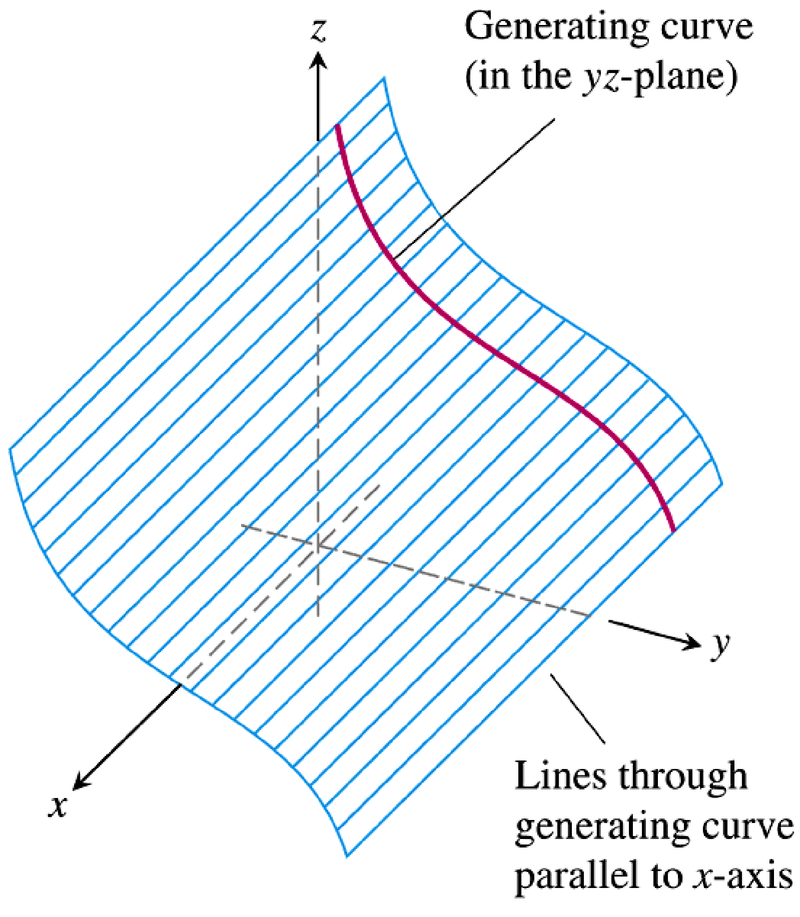


FIGURE 12.43 A cylinder and generating curve.

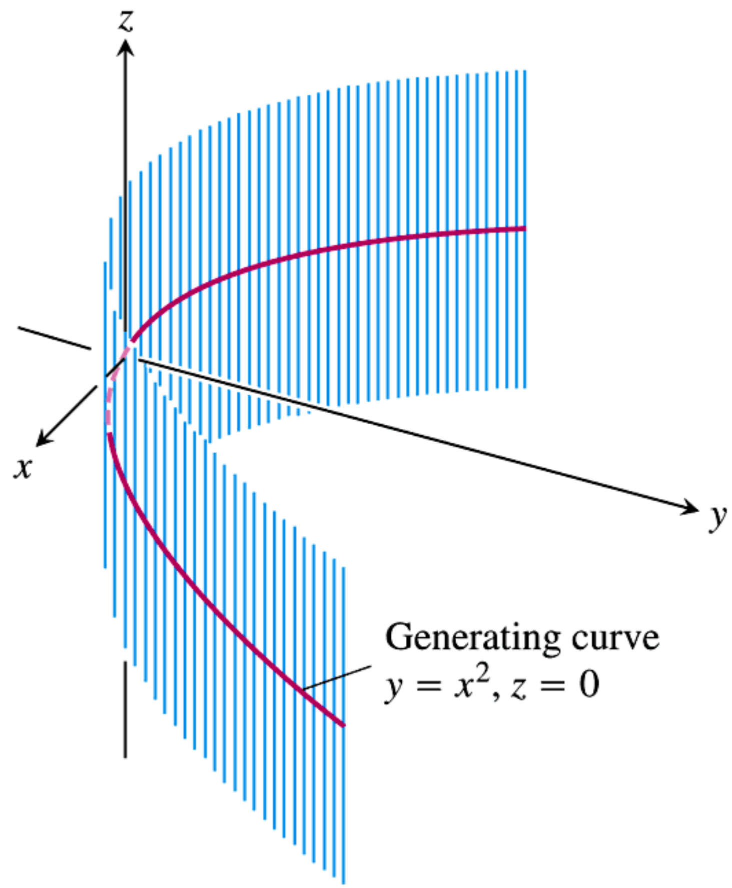


FIGURE 12.44 The cylinder of lines passing through the parabola $y = x^2$ in the xy-plane parallel to the z-axis (Example 1).

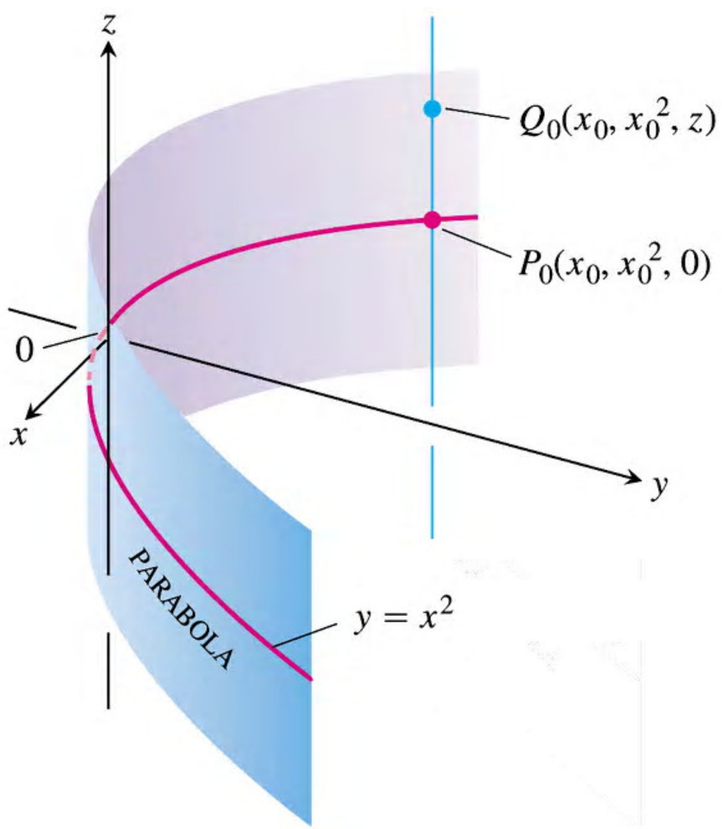


FIGURE 12.45 Every point of the cylinder in Figure 12.44 has coordinates of the form (x_0, x_0^2, z). We call it "the cylinder $y = x^2$."

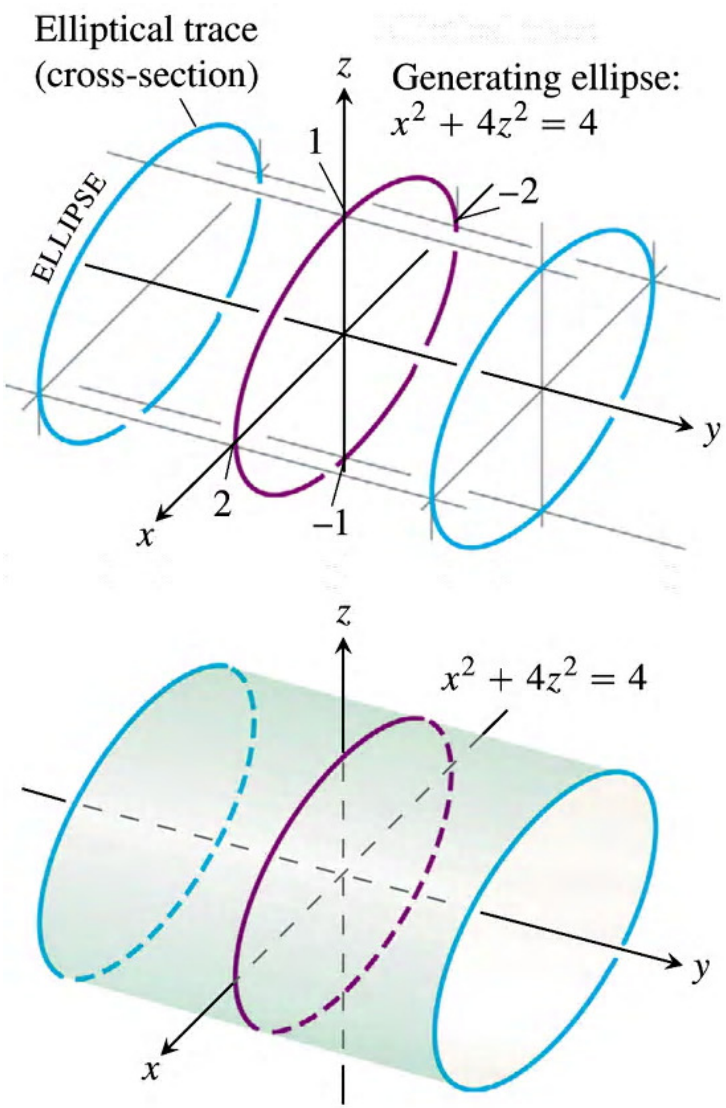


FIGURE 12.46 The elliptical cylinder $x^2 + 4z^2 = 4$ is made of lines parallel to the y-axis and passing through the ellipse $x^2 + 4z^2 = 4$ in the xz-plane. The cross-sections or "traces" of the cylinder in planes perpendicular to the y-axis are ellipses congruent to the generating ellipse. The cylinder extends along the entire y-axis.

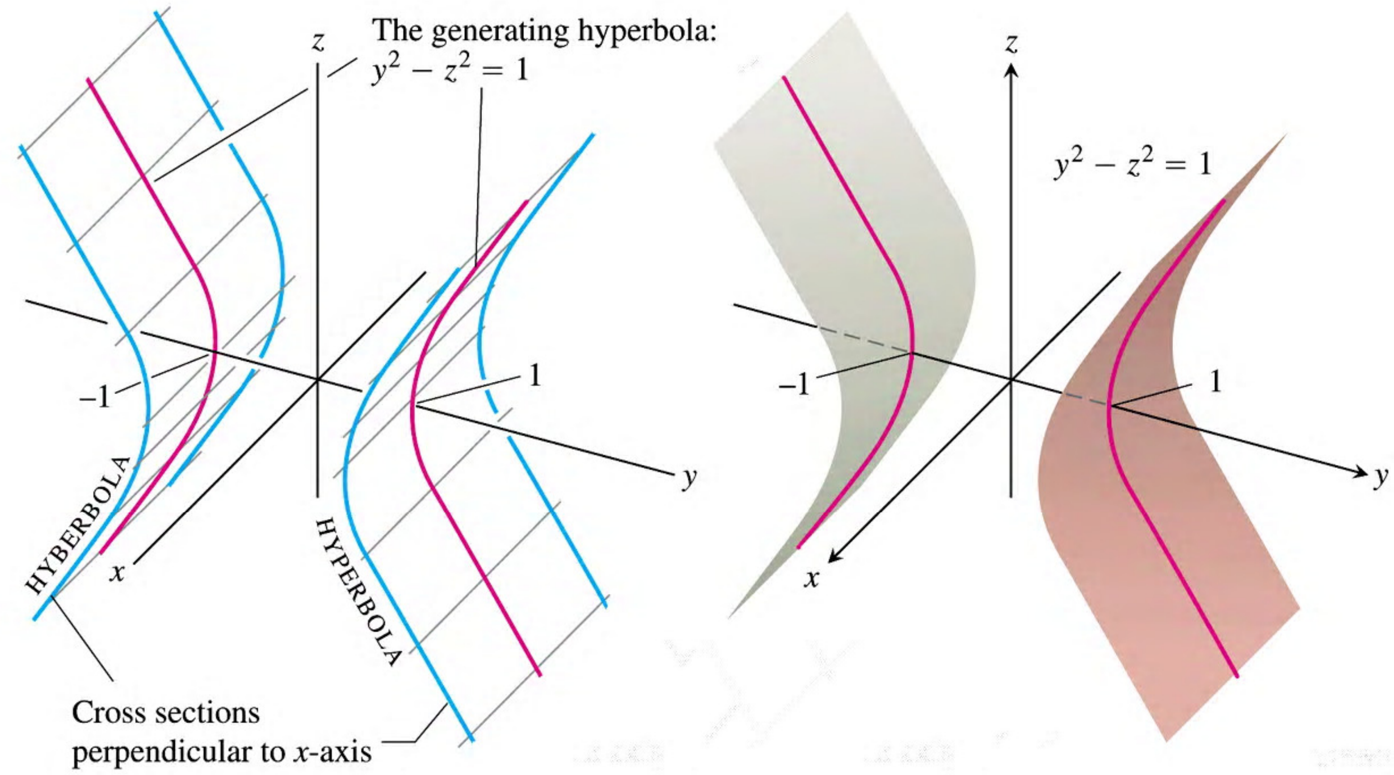


FIGURE 12.47 The hyperbolic cylinder $y^2 - z^2 = 1$ is made of lines parallel to the x-axis and passing through the hyperbola $y^2 - z^2 = 1$ in the yz-plane. The cross-sections of the cylinder in planes perpendicular to the x-axis are hyperbolas congruent to the generating hyperbola.

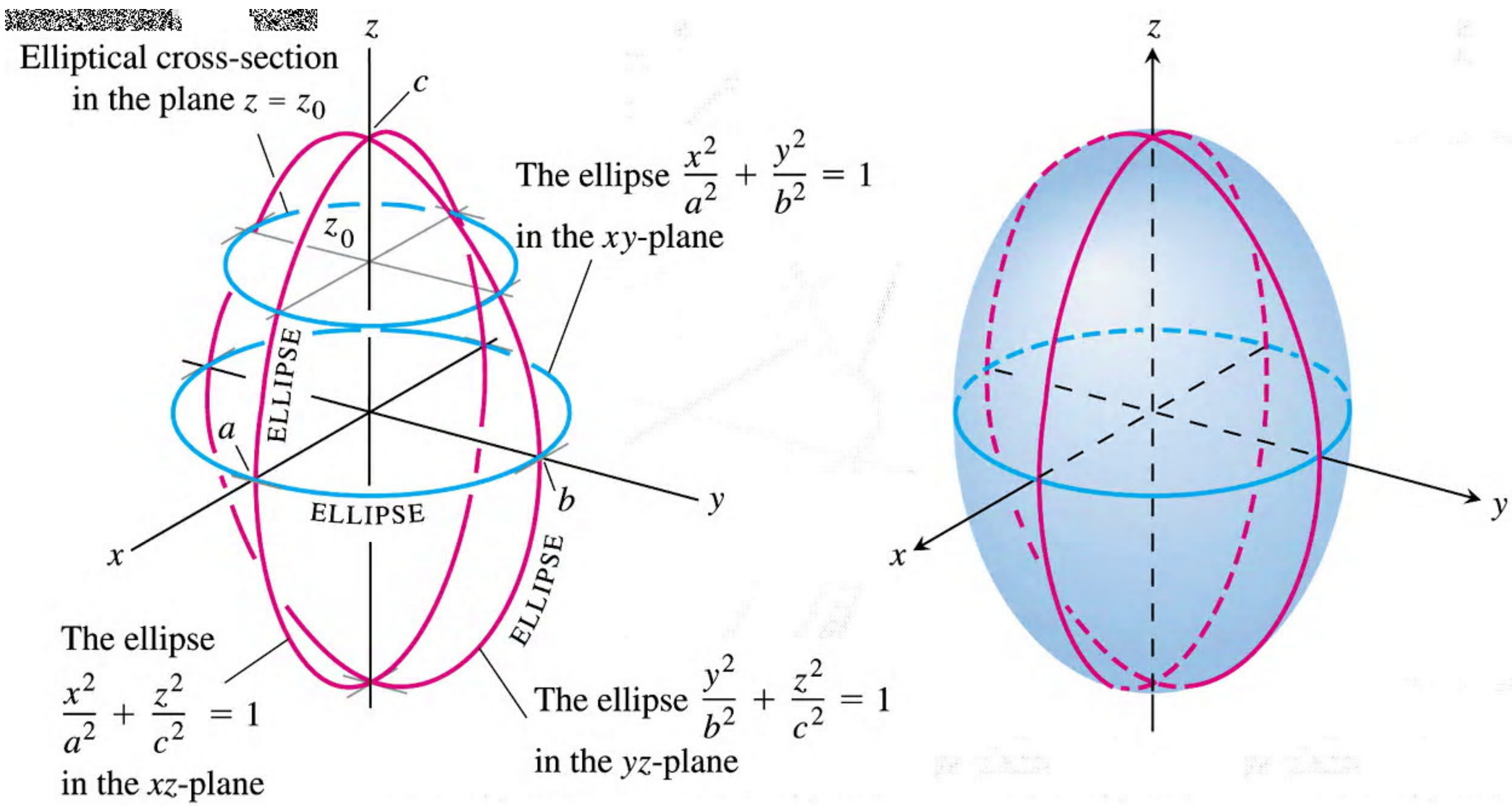


FIGURE 12.48 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in Example 2 has elliptical cross-sections in each of the three coordinate planes.

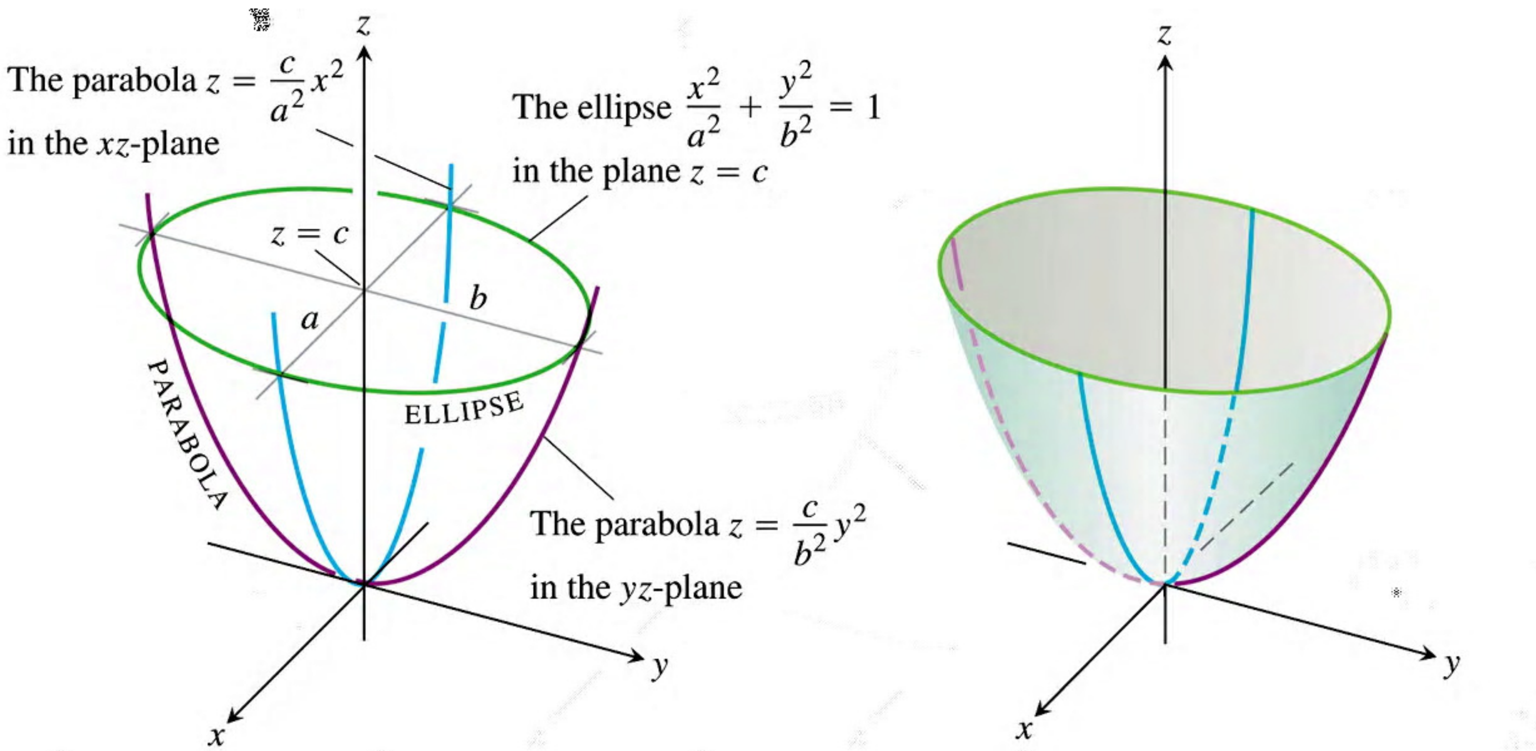


FIGURE 12.49 The elliptical paraboloid $(x^2/a^2) + (y^2/b^2) = z/c$ in Example 3, shown for $c > 0$. The cross-sections perpendicular to the z-axis above the xy-plane are ellipses. The cross-sections in the planes that contain the z-axis are parabolas.

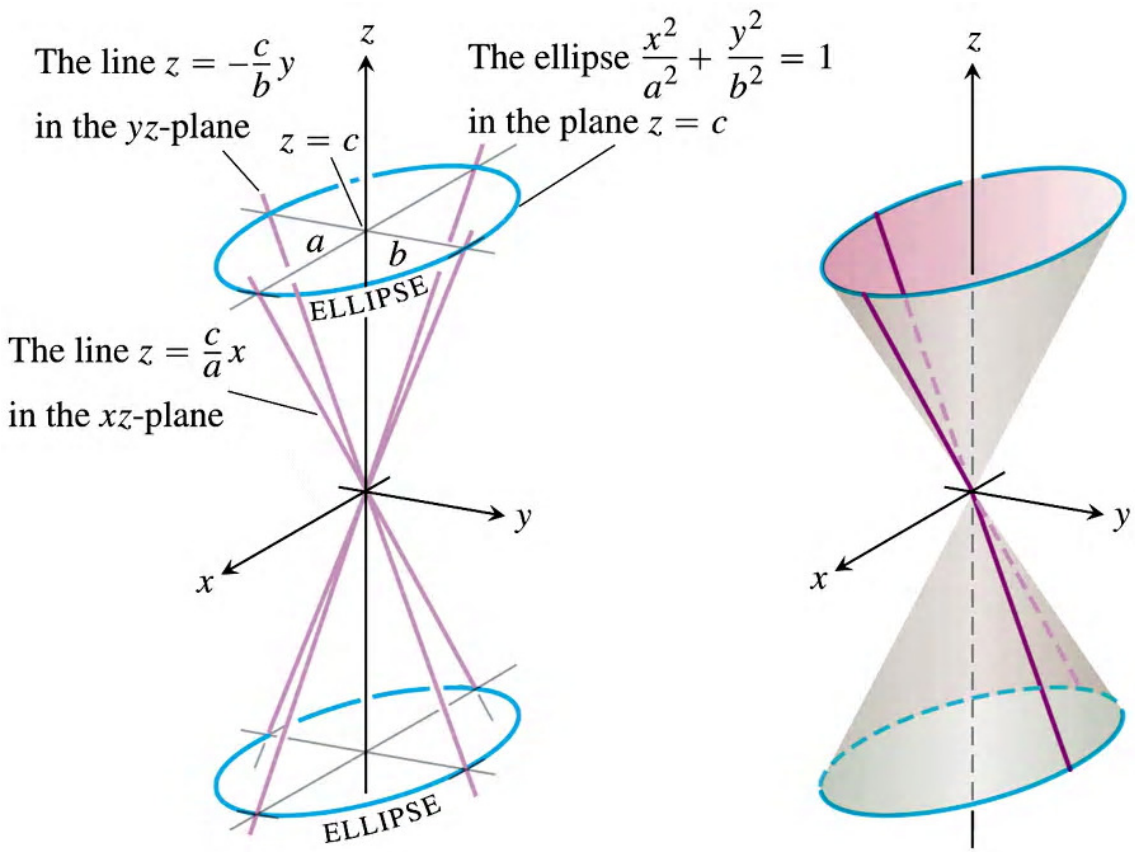


FIGURE 12.50 The elliptical cone $(x^2/a^2) + (y^2/b^2) = (z^2/c^2)$ in Example 4. Planes perpendicular to the z-axis cut the cone in ellipses above and below the xy-plane. Vertical planes that contain the z-axis cut it in pairs of intersecting lines.

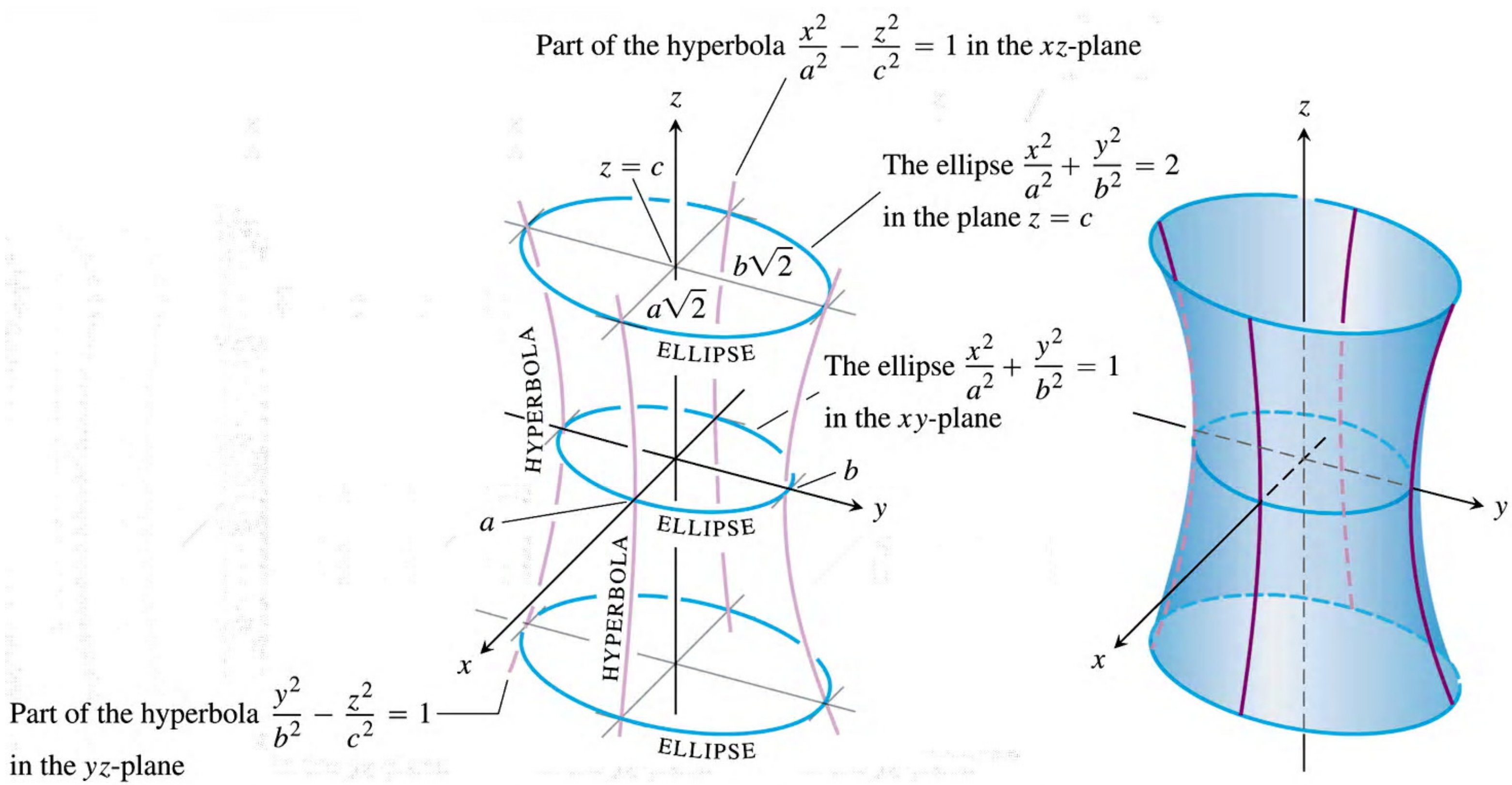


FIGURE 12.51 The hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ in Example 5. Planes perpendicular to the z-axis cut it in ellipses. Vertical planes containing the z-axis cut it in hyperbolas.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

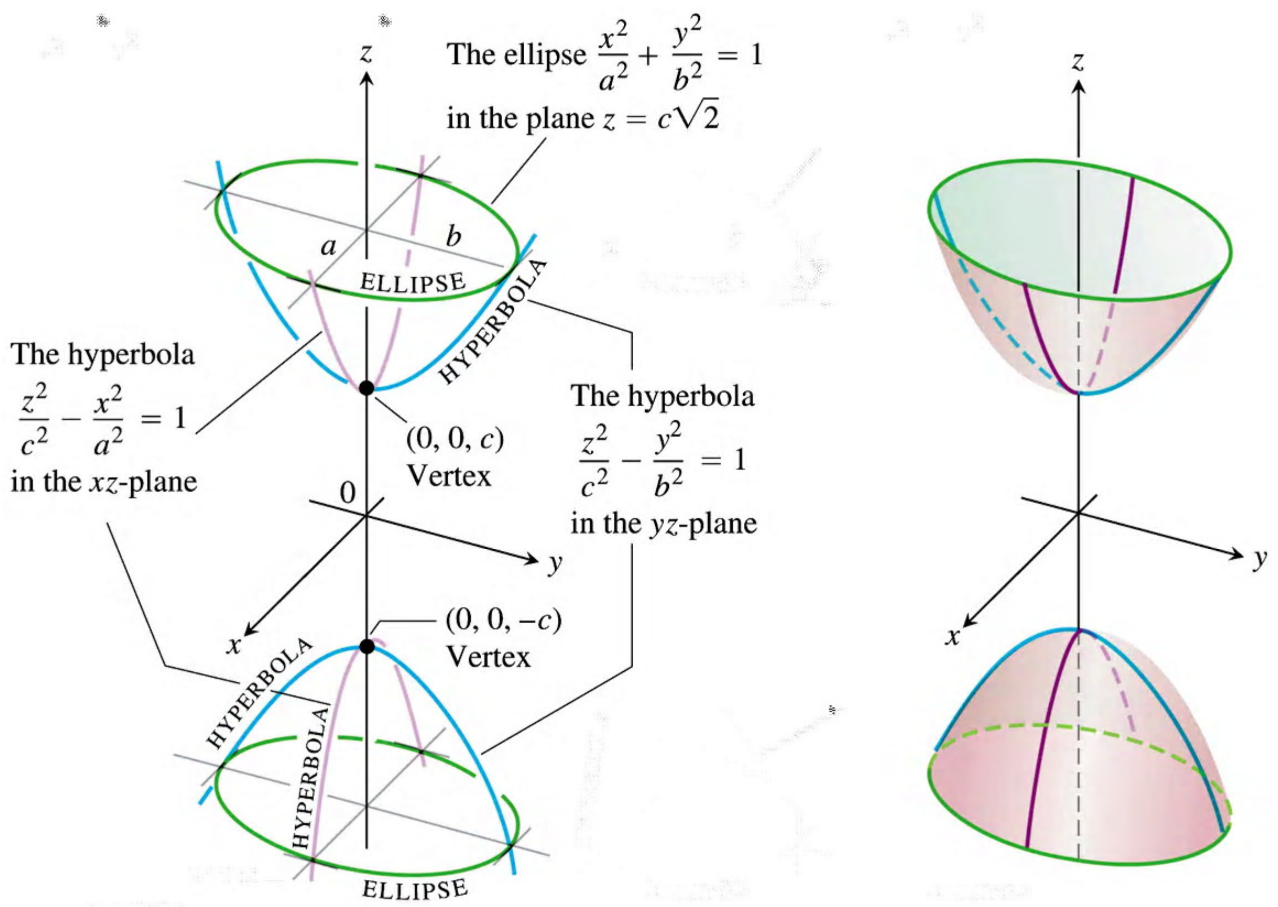


FIGURE 12.52 The hyperboloid $(z^2/c^2) - (x^2/a^2) - (y^2/b^2) = 1$ in Example 6. Planes perpendicular to the z-axis above and below the vertices cut it in ellipses. Vertical planes containing the z-axis cut it in hyperbolas.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

origin is a saddle point

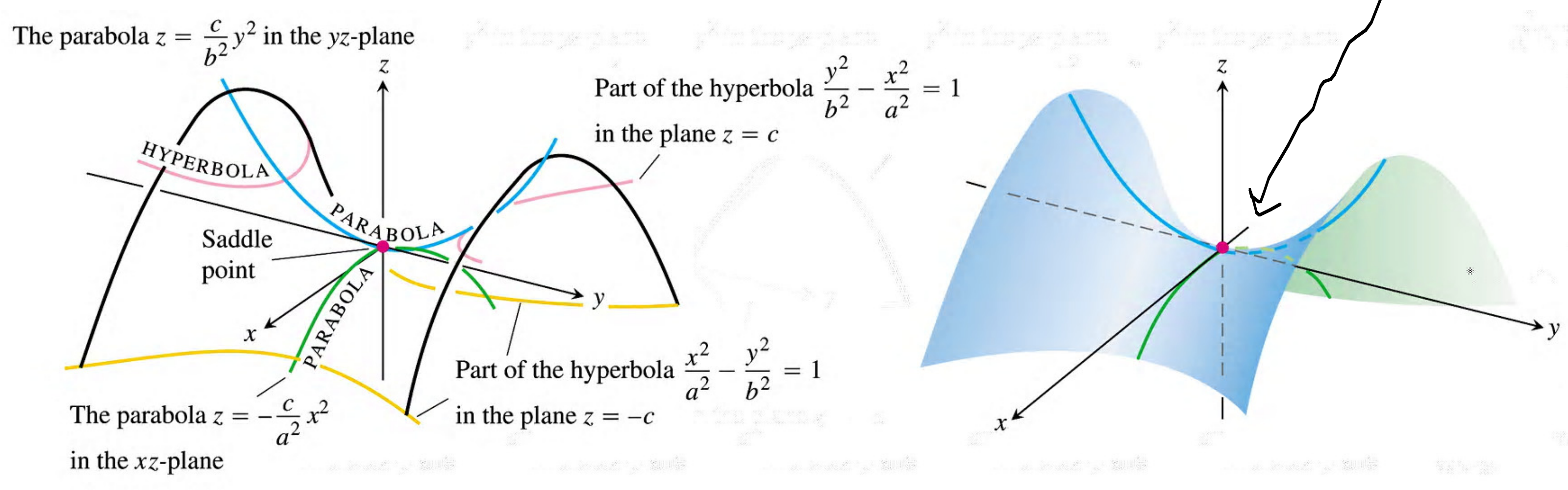


FIGURE 12.54 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$, $c > 0$. The cross-sections in planes perpendicular to the z-axis above and below the xy-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$$