First-Order Differential Equations

Differential Equations and Mathematical Models

An equation relating an unknown function and one or more of its derivatives is called a differential equation .

The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y. ■

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.
2. To find—either exactly or approximately—the appropriate solution of that equation.
3. To interpret the solution that is found.

If C is a constant and

$$y(x) = Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx} = C\left(2xe^{x^2}\right) = (2x)\left(Ce^{x^2}\right) = 2xy.$$

Thus every function $y(x)$ of the form in Eq. (1) *satisfies*—and thus is a solution of—the differential equation

$$\frac{dy}{dx} = 2xy \tag{2}$$

for all x. In particular, Eq. (1) defines an *infinite* family of different solutions of this differential equation, one for each choice of the arbitrary constant C.

This is typical of differential equations. It is also fortunate, because it may allow us to use additional information to select from among all these solutions a particular one that fits the solution under study.

Terminology and Classification

1) Ordinary and partial differential equations:

An ordinary differential equation (ODE) means that the unknown function (dependent variable) depends on only a single independent variables. If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; if they are, the equation is called a partial differential equation (PDE)

$$\frac{dx}{dt} = x^2 + t^2 \quad (ODE)$$

$$\frac{\partial y}{\partial t} = k \cdot \frac{\partial^2 y}{\partial x^2} \quad (PDE)$$

2) System of differential equations:

If there are two or more unknown functions then a system of equations is required

$$\frac{dx}{dt} = x(\alpha - \beta y)$$

$$\frac{dy}{dt} = -y(\gamma - \sigma x).$$

3) Order: The order of a differential equation is the order of the highest derivative that appears in it.

$$\frac{du}{dt} + u = t^2 \quad \text{1st order ODE}$$

$$y^{(4)} + x^2 y^{(3)} + x^2 y = \sin x \quad \text{4th order ODE}$$

$F(x, y, y', y'', \ldots, y^{(n)}) = 0$ (*) is the most general form of an n-th order differential equation with independent variable x and unknown function $y = y(x)$.

4) Solution: We say that the continuous function $u=u(x)$ is a solution of the differential equation (*) on the interval I provided that the derivatives $u', u'', \dots, u^{(n)}$ exist on I and

$$F(x, u, u', u'', \dots, u^{(n)}) = 0 \quad \text{for all } x \text{ in } I.$$

Example: • $y(x) = 2x^{1/2} - x^{1/2}\ln x$ is a solution of the equation $4x^2y'' + y = 0$ for all $x > 0$.

• $y(x) = A\cos 3x + B\sin 3x$ is a two-parameter family of the solutions of the equation $y'' + 9y = 0$.

5) Initial Value Problem (IVP):

IVP consists of a differential equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

together with an initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$, $y''(x_0) = y_0''$, $\dots$, $y^{(n-1)}(x_0) = y_0^{(n-1)}$.

6) Linear and Nonlinear Equations:

The general form of an n-th order linear differential equation is

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + a_2(t)\, y^{(n-2)} + \dots + a_n(t)\, y = g(t).$$

otherwise, it is said to be nonlinear ODE.

$$\frac{dy}{dt} + ty = 0 \qquad \text{linear}$$

$$\frac{d^3y}{dt^3} + t \cdot \frac{dy}{dt} + (\cos^2 t) \cdot y = t^3 \qquad \text{linear}$$

$$y \cdot y' + (\cos t) y = \sin t \qquad \text{non linear}$$

$$y'' + \sin(x+y) = \sin t \qquad \text{nonlinear}$$

In this chapter, we concentrate on first-order differential equations of the form $\dfrac{dy}{dx} = f(x,y)$.

Integrals as General and Particular Solutions

The first-order equation $dy/dx = f(x, y)$ takes an especially simple form if the right-hand-side function f does not actually involve the dependent variable y, so

$$\frac{dy}{dx} = f(x). \tag{1}$$

In this special case we need only integrate both sides of Eq. (1) to obtain

$$y(x) = \int f(x)\,dx + C. \tag{2}$$

This is a **general solution** of Eq. (1), meaning that it involves an arbitrary constant C, and for every choice of C it is a solution of the differential equation in (1). If $G(x)$ is a particular antiderivative of f—that is, if $G'(x) \equiv f(x)$—then

$$y(x) = G(x) + C. \tag{3}$$

To satisfy an initial condition $y(x_0) = y_0$, we need only substitute $x = x_0$ and $y = y_0$ into Eq. (3) to obtain $y_0 = G(x_0) + C$, so that $C = y_0 - G(x_0)$. With this choice of C, we obtain the **particular solution** of Eq. (1) satisfying the initial value problem

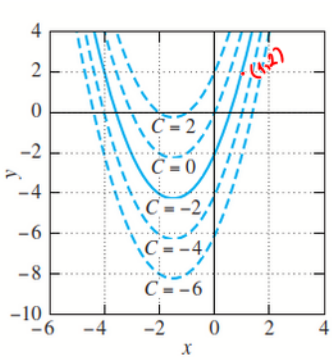
$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0.$$

Example: Solve the initial value problem

$$\frac{dy}{dx} = 2x + 3, \qquad y(1) = 2.$$

$$\frac{dy}{dx} = 2x + 3 \Rightarrow y(x) = \int (2x+3)\,dx = x^2 + 3x + C$$

$$y(1) = 1 + 3 \cdot 1 + C = 2 \Rightarrow C = -2$$



$y(x) = x^2 + 3x - 2.$ is the solution of IVP.

Existence and Uniqueness of Solutions

Example: (Failure of existence) The initial value problem

$$y' = \frac{1}{x} \quad , \quad y(0) = 0$$

has no solution, because no solution $y(x) = \int \frac{1}{x}\,dx = \ln|x| + C$ of the differential equation is defined at $x = 0$.

Example: (Failure of uniqueness) Verify that the initial value problem $y' = 2\sqrt{y}$, $x > 0$, $y(0) = 0$. has two different solutions $y_1(x) = x^2$ and $y_2(x) = 0$.

$$y_1(x) = x^2 \qquad y_1' = 2x = 2.\sqrt{x^2} = 2\sqrt{y} \quad , \quad y(0) = 0.$$

$$y_2(x) = 0 \quad , \quad y_2' = 0 = 2\sqrt{y_2} \qquad y_2(0) = 0.$$

THEOREM 1 Existence and Uniqueness of Solutions

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the xy-plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has one and only one solution that is defined on the interval I.

Example: $\frac{dy}{dx} = -y$ $\qquad$ $\frac{dy}{dx} = f(x,y)$

$f(x,y) = -y$ are $\frac{\partial f}{\partial y} = -1$ continuous everywhere, so Theorem implies the existence of a unique solution for any initial data (a,b).

Example: $\frac{dy}{dx} = 2\sqrt{y}$

$f(x,y) = 2\sqrt{y}$ is continuous wherever $y \geq 0$, but the partial derivative $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}$ is discontinuous when $y = 0$, and hence at the point $(0,0)$. This is why it is possible for there to exist two different solutions $y_1(x) = x^2$ and $y_2(x) = 0$, each of which satisfies the initial condition $y(0) = 0$.

Example: $\dfrac{dy}{dx} = y^2$, $y(0) = 1$.

$f(x,y) = y^2$ $\quad \dfrac{\partial f}{\partial y} = 2y \Rightarrow$ Both of them are continuous everywhere in the xy-plane. Theorem guarantees a unique solution of IVP. Indeed, this is the solution

$$y(x) = \frac{1}{1-x}.$$

But $y(x) = \dfrac{1}{1-x}$ is discontinuous at $x = 1$, so our unique continuous solution does not exist $-2 < x < 2$. Thus the solution interval I of theorem may not be as wide as the rectangle R where f and $\dfrac{\partial f}{\partial y}$ are continuous.

Example: Consider the differential equation

$$x \frac{dy}{dx} = 2y$$

a) Apply the theorem of existence and uniqueness:

$$\frac{dy}{dx} = 2\frac{y}{x} \quad , \quad f(x,y) = 2\frac{y}{x} \quad , \quad \frac{\partial f}{\partial y} = \frac{2}{x}.$$

f and $\frac{\partial f}{\partial y}$ are continuous in the xy-plane where $x \neq 0$.

So, there is a unique solution near any point in the xy-plane where $x \neq 0$.

b) Verify that $y = cx^2$ is a solution of the differential equation given in (a).

$$y(x) = cx^2 \qquad \frac{dy}{dx} = 2cx = 2c\frac{x^2}{x} = 2\frac{y}{x}.$$

c) For what values of a and b does the IVP

$$x\frac{dy}{dx} = 2y, \quad y(a) = b$$

i) a unique solution $(a \neq 0)$

ii) no solution $(a = 0$ and $b \neq 0)$

iii) Infinitely many solutions $(a = 0$ and $b = 0)$

$y(x) = Cx^2$ $y(a) = b$ $\Rightarrow y(a) = Ca^2 = b$

$x = a, \; y = b$ $C = \frac{b}{a^2}$

$y(x) = \frac{b}{a^2}x^2$

$b = Ca^2 \Rightarrow 0 = C \cdot 0$

For all C, $y(x) = Cx^2$ is a solution of the IVP.

Thus, it has infinitely many solutions when $a = b = 0$.

Separable Equations and Applications

The first-order differential equation

$$\frac{dy}{dx} = H(x, y) \tag{1}$$

is called **separable** provided that $H(x, y)$ can be written as the product of a function of x and a function of y:

$$\frac{dy}{dx} = g(x) \cdot h(y) = \frac{g(x)}{f(y)}, \quad \text{where } h(y) = \frac{1}{f(y)}$$

$$\int f(y)\, dy = \int g(x)\, dx$$

$$\Rightarrow F(y) = G(x) + C, \quad \text{where } F'(y) = f(y) \text{ and } G'(x) = g(x).$$

Example:

Solve the initial value problem

$$\frac{dy}{dx} = -6xy, \quad y(0) = 7.$$

$$\frac{dy}{dx} = -6xy \Rightarrow \int \frac{dy}{y} = -6\int x\,dx$$

$$\ln|y| = -3x^2 + C.$$

$$|y| = e^{-3x^2+C} = e^{-3x^2} \cdot e^{C} = e^{-3x^2} \cdot C$$

$$\Rightarrow y = \mp C \cdot e^{-3x^2}$$

$$\Rightarrow y = A \cdot e^{-3x^2}, \quad \text{where } A = \mp C.$$

$$y(0) = 7 \Rightarrow y(0) = A = 7 \Rightarrow A = 7$$

$$y(x) = 7 \cdot e^{-3x^2}.$$

! A solution of a differential equation that contains an 'arbitrary constant" (like the constant C) is commonly called a general solution of the differential equation; any particular choice of a specific value for C yields a single particular solution of the equation.

It is common for a nonlinear first-order differential equation to have both a general solution involving an arbitrary constant C and one or several particular solutions that cannot obtained by selecting a value of C. These exceptional solutions are cannot be obtained by selecting a value for C. These exceptional solutions are frequently called singular solutions.

Example: Find all solutions of the differential equation

$$\frac{dy}{dx} = 6x\,(y-1)^{2/3}$$

$$\int \frac{1}{3(y-1)^{2/3}}\,dy = \int 2x\,dx$$

$$(y-1)^{1/3} = x^2$$

$\Rightarrow$ $(y-1)^{1/3} = x^2 + C$ is a general solution, but $y(x) = 1$ is also a solution. No value of C gives the singular solution $y(x) = 1$. Hence the solution of the given differential equation with the initial condition $y(1) = 1$ is not unique and the function $f(x,y) = 6x\,(y-1)^{2/3}$ is not differentiable.

Example: Solve IVP

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}, \quad y(1) = 3.$$

$$\int (3y^2-5)\,dy = \int (4-2x)\,dx$$

$$\Rightarrow \quad y^3 - 5y = 4x - x^2 + C$$

$$y(1) = 3 \quad \Rightarrow \quad 3^3 - 5.3 = 4.1 - 1^2 + C \quad \Rightarrow \quad C = 9$$

$$y^3 - 5y = 4x - x^2 + 9$$

This equation is not readily solved for y as an explicit function of x.

The equation $K(x, y) = 0$ is commonly called an **implicit solution** of a differential equation if it is satisfied (on some interval) by some solution $y = y(x)$ of the differential equation. But note that a particular solution $y = y(x)$ of $K(x, y) = 0$ may or may not satisfy a given initial condition. For example, differentiation of $x^2 + y^2 = 4$ yields

$$x + y\frac{dy}{dx} = 0,$$

so $x^2 + y^2 = 4$ is an implicit solution of the differential equation $x + yy' = 0$. But only the first of the two explicit solutions

$$y(x) = +\sqrt{4 - x^2} \quad \text{and} \quad y(x) = -\sqrt{4 - x^2}$$

satisfies the initial condition $y(0) = 2$

$x^2 + y^2 = 4$

$2x + 2y\,y' = 0$

$\frac{dy}{dx}$

$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$|y| = \sqrt{4 - x^2}$

$y = \mp\sqrt{4 - x^2}$

Remark 1: You should not assume that every possible algebraic solution $y = y(x)$ of an implicit solution satisfies the same differential equation. For instance, if we multiply the implicit solution $x^2 + y^2 - 4 = 0$ by the factor $(y - 2x)$, then we get the new implicit solution

$$(y-2x)(x^2+y^2-4)=0$$

that yields (or "contains") not only the previously noted explicit solutions $y = +\sqrt{4-x^2}$ and $y = -\sqrt{4-x^2}$ of the differential equation $x + yy' = 0$, but also the additional function $y = 2x$ that does *not* satisfy this differential equation.

$$(y-2x)(x^2+y^2-4)=0$$

$$(y'-2).(x^2+y^2-4) + (y-2x).(2x+2y.y')=0$$

$$(x^2+y^2-4+2y.(y-2x)).y' - 2(x^2+y^2-4)+2x(y-2x)=0$$

$$y' = \frac{2(3x^2+y^2-xy-4)}{x^2+3y^2-4xy-4}$$

$$x+y.y' = \frac{x^3+3xy^2-4x^2y-4x+6x^2y+2y^3-2xy^2-8y}{x^2+3y^2-4xy-4} \neq 0.$$

Remark 2: Similarly, solutions of a given differential equation can be either gained or lost when it is multiplied or divided by an algebraic factor. For instance, consider the differential equation

$$(y-2x)y\frac{dy}{dx} = -x(y-2x) \tag{9}$$

having the obvious solution $y = 2x$. But if we divide both sides by the common factor $(y-2x)$, then we get the previously discussed differential equation

$$y\frac{dy}{dx} = -x, \quad \text{or} \quad x + y\frac{dy}{dx} = 0, \tag{10}$$

of which $y = 2x$ is *not* a solution. Thus we "lose" the solution $y = 2x$ of Eq. (9) upon its division by the factor $(y-2x)$; alternatively, we "gain" this new solution when we multiply Eq. (10) by $(y-2x)$.

Linear First-Order Equations

To solve the equation

$$\frac{dy}{dx} = 2xy \quad (y > 0),$$

we multiply both sides by the factor $1/y$ to get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x; \quad \text{that is,} \quad D_x(\ln y) = D_x\left(x^2\right).$$

$$\Rightarrow \ln y = x^2 + C.$$

For this reason, the function $\rho(x) = \frac{1}{y}$ is called an integrating factor for the original equation.

Definition: An integrating factor for a differential equation is a function $\rho(x,y)$ such that the multiplication of each side of the differential equation by $\rho(x,y)$ yields an equation which each side is recognizable as derivative.

There is a standard technique for solving the linear first-order equation.

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

on an interval on which the coefficient functions $P(x)$ and $Q(x)$ are continuous.

$$e^{\int P(x)dx} \cdot \frac{dy}{dx} + P(x) \cdot e^{\int P(x)dx} \cdot y = Q(x) \cdot e^{\int P(x)dx}$$

$$D_x \left[y(x) \cdot e^{\int P(x)dx} \right] = Q(x)\, e^{\int P(x)dx}$$

because $D_x \left(\int P(x)\, dx \right) = P(x)$

$$y(x) \cdot e^{\int P(x)dx} = \int Q(x) \cdot e^{\int P(x)dx}\, dx + C$$

$$y(x) = e^{-\int P(x)dx} \left[\int \left(Q(x)\, e^{\int P(x)dx} \right) dx + C \right].$$

Thus $\rho(x) = e^{\int P(x)dx}$ is an integrating factor for the the linear first-order differential equation.

Remark! You need not supply explicitly a constant of integration when you found $g(x)$

Let $\int p(x)\,dx = R(x) + K$

$$g(x) = e^{\int p(x)dx} = e^{R(x)+K} = e^{K} \cdot e^{R(x)}$$

e^{K} does not affect materially the result of the differential eqn. multiplying both sides of by $\rho(x)$, so we take $K=0$.

METHOD: SOLUTION OF FIRST-ORDER EQUATIONS

1. Begin by calculating the integrating factor $\rho(x) = e^{\int P(x)\,dx}$.
2. Then multiply both sides of the differential equation by $\rho(x)$.
3. Next, recognize the left-hand side of the resulting equation as the derivative of a product:

$$D_x[\rho(x)y(x)] = \rho(x)Q(x).$$

4. Finally, integrate this equation,

$$\rho(x)y(x) = \int \rho(x)Q(x)\,dx + C,$$

then solve for y to obtain the general solution of the original differential equation.

Example: Solve the initial value problem

$$\frac{dy}{dx} - y = \frac{11}{8} e^{-x/3}, \quad y(0) = -1.$$

1. Begin by calculating the integrating factor $\rho(x) = e^{\int P(x)\,dx}$.

$$\rho(x) = e^{\int P(x)dx} = e^{\int (-1)dx} = e^{-x}$$

2. Then multiply both sides of the differential equation by $\rho(x)$.

$$e^{-x} \cdot \frac{dy}{dx} - e^{-x} y = \frac{11}{8} \cdot e^{-4x/3}$$

3. Next, recognize the left-hand side of the resulting equation as the derivative of a product:

$$D_x[\rho(x)y(x)] = \rho(x)Q(x).$$

$$\frac{d}{dx}(e^{-x} \cdot y) = \frac{11}{8} e^{-4x/3}$$

4. Finally, integrate this equation,

$$\rho(x)y(x) = \int \rho(x)Q(x)\,dx + C,$$

then solve for y to obtain the general solution of the original differential equation.

$$e^{-x} \cdot y = \int \frac{11}{8} \cdot e^{-4x/3}\, dx = -\frac{33}{32} e^{-4x/3} + C.$$

$$y(x) = -\frac{33}{32} \cdot e^{-x/3} + C \cdot e^{x}$$

5. Substitute an initial condition $y(x_0) = y_0$ into the general solution and solve for the value of C.

$$y(0) = -\frac{33}{32} + C = -1 \Rightarrow C = \frac{1}{32}$$

$$\Rightarrow y(x) = -\frac{33}{32} e^{-x/3} + \frac{1}{32} e^{x}.$$

Example: Find a general solution of

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x$$

1) $$\frac{dy}{dx} + \underbrace{\frac{3x}{x^2+1}}_{P(x)} y = \underbrace{\frac{6x}{x^2+1}}_{Q(x)}$$

$$\rho(x) = \exp\left(\int \frac{3x}{x^2+1}\,dx\right) = \exp\left(\frac{3}{2}\int \frac{2x\,dx}{x^2+1}\right)$$

$$= \exp\left(\frac{3}{2}\cdot \ln(x^2+1)\right)$$

$$= \exp\left(\ln(x^2+1)^{3/2}\right)$$

$$= (x^2+1)^{3/2}.$$

2) $(x^2+1)^{3/2}\dfrac{dy}{dx} + 3x\,(x^2+1)^{1/2}\,y = 6x\,(x^2+1)^{3/2}$

3) $D_x\left[(x^2+1)^{3/2}\cdot y\right] = 6x\,(x^2+1)^{3/2}$

4) $(x^2+1)^{3/2}\cdot y = \int 6x\,(x^2+1)^{3/2}\,dx = 2.(x^2+1)^{3/2} + c$

$\Rightarrow \quad y(x) = 2 + C.(x^2+1)^{-3/2}$

THEOREM 1 The Linear First-Order Equation

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0 \tag{11}$$

has a unique solution $y(x)$ on I, given by the formula in Eq. (6) with an appropriate value of C.