MAT 281E, Linear Algebra and Applications

Homework 2

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1)

1) Choosing 4th column is more sensible in terms of containing more zeros than other rows or columns (smart choice)

$$A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 4 & 3 & -6 & 0 \\ 0 & 3 & 2 & 1 \\ 3 & 2 & -1 & 0 \end{bmatrix} \quad \det(A) = 1.-1 \underbrace{\begin{vmatrix} 4 & 3 & -6 \\ 0 & 3 & 2 \\ 3 & 2 & -1 \end{vmatrix}}_{\text{Reason: contains zero}} + -1.-1 \underbrace{\begin{vmatrix} 2 & -2 & 3 \\ 4 & 3 & -6 \\ 3 & 2 & 1 \end{vmatrix}}_{\text{Reason: Smaller numbers}} \underbrace{+0 \; +0}_{a_{24} \text{ and } a_{44} \text{ are zero}}$$

$$-1 \left(3 \begin{vmatrix} 4 & -6 \\ 3 & -1 \end{vmatrix} + (-2) \begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} + 0 \right) = -44$$

$$+$$

$$1 \left(3 \begin{vmatrix} -2 & 3 \\ 3 & -6 \end{vmatrix} + -2 \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} + (-1).(1) \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} \right) = 43$$

$\underline{\det(A) = -1}$

2)

2)

1)

$$\det(A)=\begin{vmatrix}2&2&4&6\\1&3&-2&1\\2&8&-4&2\\1&3&6&7\end{vmatrix}\xrightarrow[(r_2\leftrightarrow r_3)]{}-\begin{vmatrix}2&2&4&6\\2&8&-4&2\\1&3&-2&1\\1&3&6&7\end{vmatrix}\xrightarrow[(-2r_3+r_2\rightarrow r_2)]{}-\begin{vmatrix}2&2&4&6\\0&2&0&0\\1&3&-2&1\\1&3&6&7\end{vmatrix}$$

$$\downarrow(-r_4+r_3\rightarrow r_3)$$

$$-\begin{vmatrix}2&2&4&6\\0&2&0&0\\0&0&-8&-6\\1&3&6&7\end{vmatrix}\xleftarrow[(\frac{1}{2}r_1)]{}-2\begin{vmatrix}1&1&2&3\\0&2&0&0\\0&0&-8&-6\\1&3&6&7\end{vmatrix}\xleftarrow[(-r_1+r_4\rightarrow r_4)]{}-2\begin{vmatrix}1&1&2&3\\0&2&0&0\\0&0&-8&-6\\0&2&4&4\end{vmatrix}$$

$$\downarrow(\tfrac{1}{2}r_4)$$

$$-2.2\begin{vmatrix}1&1&2&3\\0&2&0&0\\0&0&-8&-6\\0&1&2&2\end{vmatrix}\xrightarrow[(\frac{1}{2}r_2)]{}-2.2.2\begin{vmatrix}1&1&2&3\\0&1&0&0\\0&0&-8&-6\\0&1&2&2\end{vmatrix}\xrightarrow[(\frac{1}{4}r_3)]{}-8.4\begin{vmatrix}1&1&2&3\\0&1&0&0\\0&0&-2&-3/2\\0&1&2&2\end{vmatrix}$$

$$\downarrow(-r_2+r_4\rightarrow r_4)$$

$$-8.4\begin{vmatrix}1&1&2&3\\0&1&0&0\\0&0&-2&-3/2\\0&0&2&2\end{vmatrix}\xrightarrow[(r_3+r_4\rightarrow r_4)]{}-8.4\begin{vmatrix}1&1&2&3\\0&1&0&0\\0&0&-2&-3/2\\0&0&0&1/2\end{vmatrix}\xrightarrow[2r_4 \text{ and } \frac{-1}{2}r_3]{}-32\begin{vmatrix}1&1&2&3\\0&1&0&0\\0&0&1&3/4\\0&0&0&1\end{vmatrix}$$

32

2)

1)

$\det(A) = 1.1.1.1.32 = 32$

2)

2) As we found with row reduction in (2)(1)

$$\det(A) = \begin{vmatrix} 2 & 2 & 4 & 6 \\ 1 & 3 & -2 & 1 \\ 2 & 8 & -4 & 2 \\ 1 & 3 & 6 & 7 \end{vmatrix} = 32 \begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\det(A) = 32 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{vmatrix} = 32\left(\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\right) = 1.32 = 32$$

3)

3) (a)

$$C_{11} = \begin{bmatrix} \cancel{3} & \cancel{2} & \cancel{7} & \cancel{6} \\ \cancel{-3} & 3 & 4 & 1 \\ \cancel{2} & 1 & -4 & -3 \\ \cancel{-4} & 1 & 2 & -2 \end{bmatrix} \rightarrow \begin{vmatrix} 3 & 4 & 1 \\ 1 & -4 & -3 \\ 1 & 2 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 4 & 1 \\ 1 & -4 & -3 \\ 1 & 2 & -2 \end{vmatrix} = 3(-1)^2 \begin{vmatrix} -4 & -3 \\ 2 & -2 \end{vmatrix} + 1.(-1)^3 \begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix} + 1.(-1)^4 \begin{vmatrix} 4 & 1 \\ -4 & -3 \end{vmatrix}$$

$$= 3(8+6) - 1(-8-2) + 1(-12+4)$$

$$= 42 + 10 - 8 = 44$$

3) (a)

$$C_{22} = \begin{vmatrix} 3 & 7 & 6 \\ 2 & -4 & -3 \\ -4 & 2 & -2 \end{vmatrix} = 3\underbrace{\begin{vmatrix} -4 & -3 \\ 2 & -2 \end{vmatrix}}_{14} - 2\underbrace{\begin{vmatrix} 7 & 6 \\ 2 & -2 \end{vmatrix}}_{-26} - 4\underbrace{\begin{vmatrix} 7 & 6 \\ -4 & -3 \end{vmatrix}}_{3}$$

$$C_{22} = 42 + 52 - 12 = 82$$

$$C_{23} = -\begin{vmatrix} 3 & 2 & 6 \\ 2 & 1 & -3 \\ -4 & 1 & -2 \end{vmatrix} = \left(3\underbrace{\begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix}}_{1} - 2\underbrace{\begin{vmatrix} 2 & 6 \\ 1 & -2 \end{vmatrix}}_{-10} - 4\underbrace{\begin{vmatrix} 2 & 6 \\ 1 & -3 \end{vmatrix}}_{-12}\right) \cdot -1$$

$$C_{23} = 3 + 20 + 48 = 71 \cdot -1 = \underline{-71}$$

$$C_{24} = \begin{vmatrix} 3 & 2 & 7 \\ 2 & 1 & -4 \\ -4 & 1 & 2 \end{vmatrix} = -2\underbrace{\begin{vmatrix} 2 & -4 \\ -4 & 2 \end{vmatrix}}_{-12} + \underbrace{\begin{vmatrix} 3 & 7 \\ -4 & 2 \end{vmatrix}}_{34} - \underbrace{\begin{vmatrix} 3 & 7 \\ 2 & -4 \end{vmatrix}}_{-26}$$

$$C_{24} = 24 + 34 + 26 = 84$$

$$C_{31} = +\begin{vmatrix} 2 & 7 & 6 \\ 3 & 4 & 1 \\ 1 & 2 & -2 \end{vmatrix} = \left(\underbrace{\begin{vmatrix} 7 & 6 \\ 4 & 1 \end{vmatrix}}_{-17} - 2\underbrace{\begin{vmatrix} 2 & 6 \\ 3 & 1 \end{vmatrix}}_{-16} - 2\underbrace{\begin{vmatrix} 2 & 7 \\ 3 & 4 \end{vmatrix}}_{-13}\right) \cdot +1$$

$$C_{31} = -17 + 32 + 26 = 41 \cdot -1 = 41$$

$$C_{32} = -\begin{vmatrix} 3 & 7 & 6 \\ -3 & 4 & 1 \\ -4 & 2 & -2 \end{vmatrix} = \left(3\underbrace{\begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix}}_{-10} + 3\underbrace{\begin{vmatrix} 7 & 6 \\ 2 & -2 \end{vmatrix}}_{-26} - 4\underbrace{\begin{vmatrix} 7 & 6 \\ 4 & 1 \end{vmatrix}}_{-17}\right) \cdot -1$$

$$C_{32} = -30 - 78 + 68 = -40 \cdot -1 = \underline{40}$$

(a)

3)

$$C_{33} = \begin{vmatrix} 3 & 2 & 6 \\ -3 & 3 & 1 \\ -4 & 1 & -2 \end{vmatrix} = 3\underbrace{\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}}_{-7} + 3\underbrace{\begin{vmatrix} 2 & 6 \\ 1 & -2 \end{vmatrix}}_{-10} - 4\underbrace{\begin{vmatrix} 2 & 6 \\ 3 & 1 \end{vmatrix}}_{-16}$$

$$C_{33} = -21 - 30 + 64 = 13$$

$$C_{34} = -\begin{vmatrix} 3 & 2 & 7 \\ -3 & 3 & 4 \\ -4 & 1 & 2 \end{vmatrix} = \left(-2\underbrace{\begin{vmatrix} -3 & 4 \\ -4 & 2 \end{vmatrix}}_{10} + 3\underbrace{\begin{vmatrix} 3 & 7 \\ -4 & 2 \end{vmatrix}}_{34} - \underbrace{\begin{vmatrix} 3 & 7 \\ -3 & 4 \end{vmatrix}}_{33}\right) \cdot -1$$

$$C_{34} = -20 + 102 - 33 = 49 \cdot -1 = \underline{-49}$$

$$C_{41} = -\begin{vmatrix} 2 & 7 & 6 \\ 3 & 4 & 1 \\ 1 & -4 & -3 \end{vmatrix} = \left(2\underbrace{\begin{vmatrix} 4 & 1 \\ -4 & -3 \end{vmatrix}}_{-8} - 3\underbrace{\begin{vmatrix} 7 & 6 \\ -4 & -3 \end{vmatrix}}_{3} + \underbrace{\begin{vmatrix} 7 & 6 \\ 4 & 1 \end{vmatrix}}_{-17}\right) \cdot -1$$

$$C_{41} = -16 - 9 - 17 = 42 \cdot -1 = \underline{-42}$$

$$C_{42} = \begin{vmatrix} 3 & 7 & 6 \\ -3 & 4 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 3\underbrace{\begin{vmatrix} 4 & 1 \\ -4 & -3 \end{vmatrix}}_{-8} + 3\underbrace{\begin{vmatrix} 7 & 6 \\ -4 & -3 \end{vmatrix}}_{3} + 2\underbrace{\begin{vmatrix} 7 & 6 \\ 4 & 1 \end{vmatrix}}_{-17}$$

$$C_{42} = -24 + 9 - 34 = -49$$

$$C_{43} = -\begin{vmatrix} 3 & 2 & 6 \\ -3 & 3 & 1 \\ 2 & 1 & -3 \end{vmatrix} = \left(2\underbrace{\begin{vmatrix} 2 & 6 \\ 3 & 1 \end{vmatrix}}_{-16} - \underbrace{\begin{vmatrix} 3 & 6 \\ -3 & 1 \end{vmatrix}}_{21} - 3\underbrace{\begin{vmatrix} 3 & 2 \\ -3 & 3 \end{vmatrix}}_{15}\right) \cdot -1$$

$$C_{43} = -32 - 21 - 45 = -98 \cdot -1 = \underline{98}$$

(a)

3) $C_{44} = \begin{vmatrix} 3 & 2 & 7 \\ -3 & 3 & 4 \\ 2 & 1 & -4 \end{vmatrix} = 2\underbrace{\begin{vmatrix} 2 & 7 \\ 3 & 4 \end{vmatrix}}_{-13} - \underbrace{\begin{vmatrix} 3 & 7 \\ -3 & 4 \end{vmatrix}}_{33} -4\underbrace{\begin{vmatrix} 3 & 2 \\ -3 & 3 \end{vmatrix}}_{15}$

$C_{44} = -26 - 33 - 60 = \underline{-119}$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} = \begin{bmatrix} 44 & -10 & 51 & -42 \\ -57 & 82 & -71 & 84 \\ 41 & 40 & 13 & -49 \\ 42 & -49 & 98 & -119 \end{bmatrix}$$

adj(A) is transpose of this matrix

$$\text{adj}(A) = \begin{bmatrix} 44 & -57 & 41 & 42 \\ -10 & 82 & 40 & -49 \\ 51 & -71 & 13 & 98 \\ -42 & 84 & -49 & -119 \end{bmatrix}$$

3) (b)

$$A^{-1} = \frac{adj(A)}{det(A)}$$

$$det(A) = \underbrace{3\begin{vmatrix} 3 & 4 & 1 \\ 1 & -4 & -3 \\ 1 & 2 & -2 \end{vmatrix}}_{3 \cdot 44 \text{ (computed in (a))}} - (-3)\begin{vmatrix} 2 & 7 & 6 \\ 1 & -4 & -3 \\ 1 & 2 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & 7 & 6 \\ 3 & 4 & 1 \\ 1 & 2 & -2 \end{vmatrix} - (-4)\begin{vmatrix} 2 & 7 & 6 \\ 3 & 4 & 1 \\ 1 & -4 & -3 \end{vmatrix}$$

$$3\begin{vmatrix} 2 & 7 & 6 \\ 1 & -4 & -3 \\ 1 & 2 & -2 \end{vmatrix} = \left(2\begin{vmatrix} -4 & -3 \\ 2 & -2 \end{vmatrix} - 1\begin{vmatrix} 7 & 6 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 7 & 6 \\ -4 & -3 \end{vmatrix}\right) \cdot 3 = 57 \cdot 3$$

$$2\begin{vmatrix} 2 & 7 & 6 \\ 3 & 4 & 1 \\ 1 & 2 & -2 \end{vmatrix} = \left(2\begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix} - 3\begin{vmatrix} 7 & 6 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 7 & 6 \\ 4 & 1 \end{vmatrix}\right) \cdot 2 = 41 \cdot 2$$

$$4\begin{vmatrix} 2 & 7 & 6 \\ 3 & 4 & 1 \\ 1 & -4 & -3 \end{vmatrix} = \left(2\begin{vmatrix} 4 & 1 \\ -4 & -3 \end{vmatrix} - 3\begin{vmatrix} 7 & 6 \\ -4 & -3 \end{vmatrix} + 1\begin{vmatrix} 7 & 6 \\ 4 & 1 \end{vmatrix}\right) \cdot 4 = -42 \cdot 4$$

$$det(A) = 171 + 82 - 168 + 132 = 217$$

3) (c)

$$A^{-1} = \frac{adj(A)}{det(A)}$$

$$A^{-1} = \begin{bmatrix} 44/217 & -57/217 & 41/217 & 42/217 \\ -10/217 & 82/217 & 40/217 & -49/217 \\ 51/217 & -71/217 & 13/217 & 98/217 \\ -42/217 & 84/217 & -49/217 & -119/217 \end{bmatrix}$$

4)

$$A_2 = \begin{bmatrix} 2 & 3 & 1 & -3 & 4 \\ 0 & 9 & 3 & 0 & 4 \\ 1 & 7 & -4 & -2 & 0 \\ 3 & 5 & 0 & 0 & 2 \\ 0 & -4 & 3 & -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & 1 & -3 & 4 \\ 0 & 0 & 3 & 0 & 4 \\ 1 & 0 & -4 & -2 & 0 \\ 3 & 4 & 0 & 0 & 2 \\ 0 & 2 & 3 & -1 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 2 & 0 & 1 & 3 & 4 \\ 0 & 0 & 3 & 9 & 4 \\ 1 & 0 & -4 & 7 & 0 \\ 3 & 4 & 0 & 5 & 2 \\ 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

(a)

4) According to Cramer's Rule;

$$x_2 = \frac{\det(A_2)}{\det(A)} \qquad x_4 = \frac{\det(A_4)}{\det(A)}$$

$$A_2 = \begin{bmatrix} 2 & 3 & 1 & -3 & 4 \\ 0 & 9 & 3 & 0 & 4 \\ 1 & 7 & -4 & -2 & 0 \\ 3 & 5 & 0 & 0 & 2 \\ 0 & -4 & 3 & -1 & 1 \end{bmatrix}$$

$$\det(A_2) = \begin{vmatrix} 2 & 3 & 1 & -3 & 4 \\ 0 & 9 & 3 & 0 & 4 \\ 1 & 7 & -4 & -2 & 0 \\ 3 & 5 & 0 & 0 & 2 \\ 0 & -4 & 3 & -1 & 1 \end{vmatrix} \xrightarrow{r_3 \leftrightarrow r_1} - \begin{vmatrix} 1 & 7 & -4 & -2 & 0 \\ 0 & 9 & 3 & 0 & 4 \\ 2 & 3 & 1 & -3 & 4 \\ 3 & 5 & 0 & 0 & 2 \\ 0 & -4 & 3 & -1 & 1 \end{vmatrix}$$

$\downarrow -2r_1 + r_3 \to r_3$

$$- \begin{vmatrix} 1 & 7 & -4 & -2 & 0 \\ 0 & 9 & 3 & 0 & 4 \\ 0 & -11 & 9 & 1 & 4 \\ 0 & -16 & 12 & 6 & 2 \\ 0 & -4 & 3 & -1 & 1 \end{vmatrix} \xleftarrow{-3r_1 + r_4 \to r_4} - \begin{vmatrix} 1 & 7 & -4 & -2 & 0 \\ 0 & 9 & 3 & 0 & 4 \\ 0 & -11 & 9 & 1 & 4 \\ 3 & 5 & 0 & 0 & 2 \\ 0 & -4 & 3 & -1 & 1 \end{vmatrix}$$

$\downarrow -4r_5 + r_4 \to r_4$

$$
-\begin{vmatrix} 1 & 7 & -4 & -2 & 0 \\ 0 & 9 & 3 & 0 & 4 \\ 0 & -11 & 9 & 1 & 4 \\ 0 & 0 & 0 & 10 & -2 \\ 0 & -4 & 3 & -1 & 1 \end{vmatrix}
\xrightarrow[r_4 \leftrightarrow r_5]{}
\begin{vmatrix} 1 & 7 & -4 & -2 & 0 \\ 0 & 9 & 3 & 0 & 4 \\ 0 & -11 & 9 & 1 & 4 \\ 0 & -4 & 3 & -1 & 1 \\ 0 & 0 & 0 & 10 & -2 \end{vmatrix}
$$

$$\downarrow \frac{-11}{4} r_4 + r_3 \rightarrow r_3$$

$$
\begin{vmatrix} 1 & 7 & -4 & -2 & 0 \\ 0 & 9 & 3 & 0 & 4 \\ 0 & 0 & 3/4 & 15/4 & 5/4 \\ 0 & 0 & 39/9 & -1 & 25/9 \\ 0 & 0 & 0 & 10 & -2 \end{vmatrix}
\xleftarrow[4/9\, r_2 + r_4 \rightarrow r_4]{}
\begin{vmatrix} 1 & 7 & -4 & -2 & 0 \\ 0 & 9 & 3 & 0 & 4 \\ 0 & 0 & 3/4 & 15/4 & 5/4 \\ 0 & -4 & 3 & -1 & 1 \\ 0 & 0 & 0 & 10 & -2 \end{vmatrix}
$$

$$\downarrow -52/9\, r_3 + r_4 \rightarrow r_4$$

$$
\begin{vmatrix} 1 & 7 & -4 & -2 & 0 \\ 0 & 9 & 3 & 0 & 4 \\ 0 & 0 & 3/4 & 15/4 & 5/4 \\ 0 & 0 & 0 & -68/3 & -40/9 \\ 0 & 0 & 0 & 10 & -2 \end{vmatrix}
\xrightarrow[30/68\, r_4 + r_5 \rightarrow r_5]{}
\begin{vmatrix} 1 & 7 & -4 & -2 & 0 \\ 0 & 9 & 3 & 0 & 4 \\ 0 & 0 & 3/4 & 15/4 & 5/4 \\ 0 & 0 & 0 & -68/3 & -40/9 \\ 0 & 0 & 0 & 0 & \frac{-202}{51} \end{vmatrix}
$$

$$\det(A_2) = 1 \cdot 9 \cdot \frac{3}{4} \cdot \frac{-68}{3} \cdot \frac{-202}{51}$$

$$= 606$$

(a)

4)

$$\det(A) = \begin{vmatrix} 2 & 0 & 1 & -3 & 4 \\ 0 & 0 & 3 & 0 & 4 \\ 1 & 0 & -4 & -2 & 0 \\ 3 & 4 & 0 & 0 & 2 \\ 0 & 2 & 3 & -1 & 1 \end{vmatrix} \xrightarrow{\frac{1}{2}r_1} 2\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 0 & 3 & 0 & 4 \\ 1 & 0 & -4 & -2 & 0 \\ 3 & 4 & 0 & 0 & 2 \\ 0 & 2 & 3 & -1 & 1 \end{vmatrix}$$

$$\downarrow -1r_1 + r_3 \rightarrow r_3$$

$$2\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & -\frac{9}{2} & -\frac{1}{2} & -2 \\ 3 & 4 & 0 & 0 & 2 \\ 0 & 2 & 3 & -1 & 1 \end{vmatrix}$$

$$\downarrow -3r_1 + r_4 \rightarrow r_4$$

$$2\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & -9/2 & -1/2 & -2 \\ 0 & 4 & -3/2 & 9/2 & -4 \\ 0 & 2 & 3 & -1 & 1 \end{vmatrix}$$

$$\downarrow r_2 \leftrightarrow r_4$$

$$-2\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 4 & -3/2 & 9/2 & -4 \\ 0 & 0 & -9/2 & -1/2 & -2 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 2 & 3 & -1 & 1 \end{vmatrix} \xrightarrow{\frac{1}{4}r_2} -8\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 1 & -3/8 & 9/8 & -1 \\ 0 & 0 & -9/2 & -1/2 & -2 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 2 & 3 & -1 & 1 \end{vmatrix}$$

$$\downarrow -2r_2 + r_5 \rightarrow r_5$$

$$-8\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 1 & -3/8 & 9/8 & -1 \\ 0 & 0 & -9/2 & -1/2 & -2 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 15/4 & -13/4 & 3 \end{vmatrix}$$

$$\downarrow r_3 \cdot \frac{-2}{9}$$

$$36\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 4 & -3/2 & 9/2 & -4 \\ 0 & 0 & 1 & 1/9 & 4/9 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 2 & 3 & -1 & 1 \end{vmatrix}$$

$$\downarrow -3r_3 + r_4 \rightarrow r_4$$

4) (a)

$$36\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 1 & -3/8 & 9/8 & -1 \\ 0 & 0 & 1 & 1/9 & \frac{4}{9} \\ 0 & 0 & 0 & -1/3 & 8/3 \\ 0 & 0 & 15/4 & -13/4 & 3 \end{vmatrix} \xrightarrow[\frac{-15}{4} r_3 + r_5 \to r_5]{} 36\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 1 & -3/8 & 9/8 & -1 \\ 0 & 0 & 1 & 1/9 & 4/9 \\ 0 & 0 & 0 & -1/3 & 8/3 \\ 0 & 0 & 0 & -11/3 & 4/3 \end{vmatrix}$$

$$\downarrow -3 r_4$$

$$-12\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 1 & -3/8 & 9/8 & -1 \\ 0 & 0 & 1 & 1/9 & 4/9 \\ 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & -28 \end{vmatrix} \xleftarrow[\frac{11}{3} r_4 + r_5 \to r_5]{} -12\begin{vmatrix} 1 & 0 & 1/2 & -3/2 & 2 \\ 0 & 1 & -3/8 & 9/8 & -1 \\ 0 & 0 & 1 & 1/9 & 4/9 \\ 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & -11/3 & 4/3 \end{vmatrix}$$

$$\det(A) = -12 . 1 . 1 . 1 . 1 . -28$$

$$\det(A) = 336$$

(a)

4)

$$\det(A_4)=\begin{vmatrix}2&0&1&3&4\\0&0&3&9&4\\1&0&-4&7&0\\3&4&0&5&2\\0&2&3&-4&1\end{vmatrix}\xrightarrow{\frac{1}{2}r_1} 2\begin{vmatrix}1&0&1/2&3/2&2\\0&0&3&9&4\\1&0&-4&7&0\\3&4&0&5&2\\0&2&3&-4&1\end{vmatrix}$$

$$\downarrow -1r_1+r_3\to r_3$$

$$2\begin{vmatrix}1&0&1/2&3/2&2\\0&0&3&9&4\\0&0&-9/2&11/2&-2\\0&4&-3/2&1/2&-4\\0&2&3&-4&1\end{vmatrix}\xleftarrow{-3r_1+r_4\to r_4} 2\begin{vmatrix}1&0&1/2&3/2&2\\0&0&3&9&4\\0&0&-9/2&11/2&-2\\3&4&0&5&2\\0&2&3&-4&1\end{vmatrix}$$

$$\downarrow r_2\longleftrightarrow r_4$$

$$-2\begin{vmatrix}1&0&1/2&3/2&2\\0&4&-3/2&1/2&-4\\0&0&-9/2&11/2&-2\\0&0&3&9&4\\0&2&3&-4&1\end{vmatrix}\xrightarrow{\frac{1}{4}r_2} -8\begin{vmatrix}1&0&1/2&3/2&2\\0&1&-3/8&1/8&-1\\0&0&-9/2&11/2&-2\\0&0&3&9&4\\0&2&3&-4&1\end{vmatrix}$$

$$\downarrow -2r_2+r_5\to r_5$$

$$36\begin{vmatrix}1&0&1/2&3/2&2\\0&1&-3/8&1/8&-1\\0&0&1&-11/9&4/9\\0&0&3&9&4\\0&0&15/4&-17/4&3\end{vmatrix}\xleftarrow{-\frac{2}{9}r_3} -8\begin{vmatrix}1&0&1/2&3/2&2\\0&1&-3/8&1/8&-1\\0&0&-9/2&11/2&-2\\0&0&3&9&4\\0&0&15/4&-17/4&3\end{vmatrix}$$

$$\downarrow -3r_3+r_4\to r_4$$

(a)

4)

$$36\begin{vmatrix} 1 & 0 & 1/2 & 3/2 & 2 \\ 0 & 1 & -3/8 & 1/8 & -1 \\ 0 & 0 & 1 & -11/9 & 4/9 \\ 0 & 0 & 0 & 38/3 & 8/3 \\ 0 & 0 & 15/4 & -17/4 & 3 \end{vmatrix} \xrightarrow{\frac{-15}{4}r_3 + r_5 \to r_5} 36\begin{vmatrix} 1 & 0 & 1/2 & 3/2 & 2 \\ 0 & 1 & -3/8 & 1/8 & -1 \\ 0 & 0 & 1 & -11/9 & 4/9 \\ 0 & 0 & 0 & 38/3 & 8/3 \\ 0 & 0 & 0 & 1/3 & 4/3 \end{vmatrix}$$

$$\xrightarrow{\frac{3}{38}r_4} 12{,}38\begin{vmatrix} 1 & 0 & 1/2 & 3/2 & 2 \\ 0 & 1 & -3/8 & 1/8 & -1 \\ 0 & 0 & 1 & -11/9 & 4/9 \\ 0 & 0 & 0 & 1 & 4/19 \\ 0 & 0 & 0 & 1/3 & 4/3 \end{vmatrix} \xrightarrow{\frac{-1}{3}r_4 + r_5 \to r_5} 12.38\begin{vmatrix} 1 & 0 & 1/2 & 3/2 & 2 \\ 0 & 1 & -3/8 & 1/8 & -1 \\ 0 & 0 & 1 & \frac{-11}{9} & \frac{4}{9} \\ 0 & 0 & 0 & 1 & \frac{4}{19} \\ 0 & 0 & 0 & 0 & \frac{24}{19} \end{vmatrix}$$

$$\det(A_4) = 1.1.1.1.\frac{24}{\cancel{19}}.12.\overset{2}{\cancel{38}} = 24.24 = 576$$

$\det(A_2) = 606$

$\det(A) = 336$

$\det(A_4) = 576$

$$x_2 = \frac{606}{336}$$

$$x_4 = \frac{576}{336}$$

5)

5)

$$A = \begin{bmatrix} 2 & 3 & 7 \\ -2 & x & -11 \\ 0 & -3 & x \end{bmatrix} \xrightarrow[r_3 \leftrightarrow r_2]{E_1} \begin{bmatrix} 2 & 3 & 7 \\ 0 & -3 & x \\ -2 & x & -11 \end{bmatrix} \xrightarrow[r_1 + r_3 \to r_3]{E_2} \begin{bmatrix} 2 & 3 & 7 \\ 0 & -3 & x \\ 0 & (x+3) & -4 \end{bmatrix}$$

$$\downarrow E_3 \left(\frac{x+3}{3} r_2 + r_3 \to r_3 \right)$$

$$\begin{bmatrix} 2 & 3 & 7 \\ 0 & -3 & x \\ 0 & 0 & \left(\frac{x^2+3x}{3} - 4 \right) \end{bmatrix}$$

If the matrix is non-invertible,

$\det(A) = 0$

$$-6 \left(\frac{x^2+3x-12}{3} \right) = 0$$

$$x^2 + 3x - 12 = 0$$

$$\Delta = 9 + 48 = 57$$

$$x_1 = \frac{-3 + \sqrt{57}}{2} \qquad x_2 = \frac{-3 - \sqrt{57}}{2}$$

x_1 and x_2 make matrix non-invertible.

6)

6)

$$\left[\begin{array}{cccc|cccc} 3 & 2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

r_4 is not going to change when row operations are used to find A^{-1}. In addition, a_{21}, a_{31}, a_{32} are not going to change (Inverse of an upper triangular matrix is also an upper triangular matrix.)

6) (1)

$$A = \left[\begin{array}{cccc|cccc} 3 & 2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow 1/2\, r_2$$

$$\left[\begin{array}{cccc|cccc} 3 & 2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow r_2 + r_3 \to r_2$$

$$\left[\begin{array}{cccc|cccc} 3 & 2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow r_4 + r_3 \to r_3$$

$$\left[\begin{array}{cccc|cccc} 3 & 2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow -1/2\, r_3$$

$$\left[\begin{array}{cccc|cccc} 3 & 2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow -2r_2 + r_1 \to r_1$$

$$\left[\begin{array}{cccc|cccc} 3 & 0 & 3 & 5 & 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow -3r_3 + r_1 \to r_1$$

$$\left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 5 & 1 & -1 & -1/2 & 3/2 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow -5r_4 + r_1 \to r_1$$

$$\left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & -1 & -1/2 & -7/2 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}r_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & -1/3 & -1/6 & -7/6 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

7)

7)

$$\begin{vmatrix} 1 & 2 & 7 & 6 \\ -4 & -7 & -17 & -13 \\ 2 & 2 & 11 & 7 \\ -3 & 1 & -16 & -21 \end{vmatrix} \xrightarrow{(4r_1+r_2\to r_2)} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 1 & 11 & 11 \\ 2 & 2 & 11 & 7 \\ -3 & 1 & -16 & -21 \end{vmatrix} \xrightarrow{-2r_1+r_3\to r_3} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 1 & 11 & 11 \\ 0 & -2 & -3 & -5 \\ -3 & 1 & -16 & -21 \end{vmatrix} \xrightarrow{3r_1+r_4\to r_4} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 1 & 11 & 11 \\ 0 & -2 & -3 & -5 \\ 0 & 7 & 5 & -3 \end{vmatrix}$$

$$\downarrow 2r_2+r_3\to r_3$$

$$19\begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 1 & 11 & 11 \\ 0 & 0 & 1 & 17/19 \\ 0 & 0 & 0 & \left(\frac{-296}{19}\right) \end{vmatrix} \xleftarrow{\frac{1}{19}r_3} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 1 & 11 & 11 \\ 0 & 0 & 19 & 17 \\ 0 & 0 & 0 & \left(\frac{-296}{19}\right) \end{vmatrix} \xleftarrow{\frac{72}{19}r_3+r_4\to r_4} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 1 & 11 & 11 \\ 0 & 0 & 19 & 17 \\ 0 & 0 & -72 & -80 \end{vmatrix} \xleftarrow{-7r_2+r_4\to r_4} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 1 & 11 & 11 \\ 0 & 0 & 19 & 17 \\ 0 & 7 & 5 & -3 \end{vmatrix}$$

$$\downarrow \frac{-19}{296}r_4$$

$$-296\begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 1 & 11 & 11 \\ 0 & 0 & 1 & 17/19 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\det(A) = \underline{-296}$$

8)

8)

$$\det(A) = \begin{vmatrix} 5 & -1 & 0 \\ -10 & 2 & 7 \\ 0 & -3 & 4 \end{vmatrix} \xrightarrow[\downarrow r_2]{2r_1+r_2} \begin{vmatrix} 5 & -1 & 0 \\ 0 & 0 & 7 \\ 0 & -3 & 4 \end{vmatrix} \xrightarrow[r_2 \to r_3]{} - \begin{vmatrix} 5 & -1 & 0 \\ 0 & -3 & 4 \\ 0 & 0 & 7 \end{vmatrix}$$

$$\det(A) = 105$$

$$\underbrace{\det(A^{-1})}_{1/105} \cdot \underbrace{\det(A)}_{105} = 1$$

$$\underbrace{\det(A^{-1})}_{\frac{1}{105}} = \underbrace{\det((A^{-1})^T)}_{\frac{1}{105}}$$

$$\det((A^{-1})^T)^3 = \left(\frac{1}{105}\right)^3$$

9)

9) Using row operations,

$$\begin{vmatrix} 1 & 2 & 7 & 6 \\ 2 & 6 & 8 & 20 \\ -5 & 7 & 11 & 7 \\ 0 & 1 & -3 & 4 \end{vmatrix} \xrightarrow{-2r_1+r_2\to r_2} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 2 & -6 & 8 \\ -5 & 7 & 11 & 7 \\ 0 & 1 & -3 & 4 \end{vmatrix} \xrightarrow{5r_1+r_3\to r_3} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 2 & -6 & 8 \\ 0 & 17 & 25 & 19 \\ 0 & 1 & -3 & 4 \end{vmatrix}$$

$$\xrightarrow{17/2\, r_2+r_3\to r_3} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 2 & -6 & 8 \\ 0 & 0 & -26 & 53 \\ 0 & 1 & -3 & 4 \end{vmatrix} \xrightarrow{-1/2\, r_2+r_4\to r_4} \begin{vmatrix} 1 & 2 & 7 & 6 \\ 0 & 2 & -6 & 8 \\ 0 & 0 & -26 & 53 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

the product of diagonal entries is zero, matrix is non-invertible ($\det(A)=0$) (upper triangular)

10)

10)

$$(A^{-2}) = (A^{-1})^2 \qquad \underbrace{\det(A)}_{-2} \cdot \underbrace{\det(A^{-1})}_{-1/2} = 1$$

$$\det(A^{-2}B^3) = \det(A^{-2}) \cdot \det(B^3)$$

$$\det((A^{-1})^2) = \underbrace{\det(A^{-1})}_{-1/2} \cdot \underbrace{\det(A^{-1})}_{-1/2}$$

$$\det(B^3) = \underbrace{\det(B)}_{1} \cdot \underbrace{\det(B)}_{1} \cdot \underbrace{\det(B)}_{1}$$

$$\det(A^{-2}B^3) = \underbrace{\det(A^{-2})}_{\frac{1}{4}} \cdot \underbrace{\det(B^3)}_{1} = \frac{1}{4}$$

7

11)

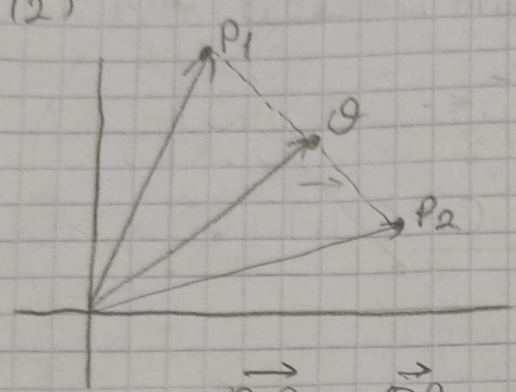
11)

$$\begin{vmatrix} x & y & z & w \\ 2x & y & 2z & 2w \\ x & y & 2z & 2w \\ -x & 0 & 0 & 0 \end{vmatrix} \xrightarrow[r_2]{-r_3+r_2} \begin{vmatrix} x & y & z & w \\ x & 0 & 0 & 0 \\ x & y & 2z & 2w \\ -x & 0 & 0 & 0 \end{vmatrix} \xrightarrow[r_2]{r_2+r_4} \begin{vmatrix} x & y & z & w \\ 0 & 0 & 0 & 0 \\ x & y & 2z & 2w \\ -x & 0 & 0 & 0 \end{vmatrix}$$

the matrix has a row of zeros, so det $=0$

12)

12)



$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$\overrightarrow{O\vartheta} = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{P_1P_2})$$

$$\overrightarrow{O\vartheta} = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1})$$

$$\overrightarrow{O\vartheta} = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2})$$

$$\overrightarrow{O\vartheta}\cdot(\overrightarrow{OP_1}\times\overrightarrow{OP_2}) = \frac{1}{2}\left((\overrightarrow{OP_1} + \overrightarrow{OP_2})\cdot(\overrightarrow{OP_1}\times\overrightarrow{OP_2})\right)$$

$$\underbrace{\overrightarrow{OP_1}\cdot(\overrightarrow{OP_1}\times\overrightarrow{OP_2})}_{0} + \underbrace{\overrightarrow{OP_2}\cdot(\overrightarrow{OP_1}\times\overrightarrow{OP_2})}_{0}$$

$\overrightarrow{O\vartheta}\cdot(\overrightarrow{OP_1}\times\overrightarrow{OP_2}) = 0 \rightarrow \overrightarrow{O\vartheta}$, $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are on the same plane (Volume is zero)

13)

13) $\|(x+y)+(-z)\| \leq \|x+y\| + \|-z\| \rightarrow$ Triangle inequality.

$\|x+y\| \leq \|x\| + \|y\| \rightarrow$ Triangle inequality.

$\|x+y\| + \|-z\| \leq \|x\| + \|y\| + \|-z\|$

Thus,

$\|(x+y)+(-z)\| \leq \|x+y\| + \|-z\| \leq \|x\| + \|y\| + \|-z\|$

$(\|-z\| = \|z\| = (-z \cdot -z)^{1/2} = (z \cdot z)^{1/2})$

$\|(x+y)-z\| \leq \|x\| + \|y\| + \|z\|$

13) Demonstrating in $\mathbb{R}^2$

$x = (x_1, x_2) \quad y = (y_1, y_2) \quad z = (z_1, z_2)$

$\|(x+y)+(-z)\| \leq \|x+y\| + \|z\| \rightarrow \sqrt{(x_1+y_1-z_1)^2 + (x_2+y_2-z_2)^2} \leq \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} + \sqrt{z_1^2+z_2^2}$

$\|z\| + \|x+y\| \leq \|x\| + \|y\| + \|z\| \rightarrow \sqrt{z_1^2+z_2^2} + \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} \leq \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2} + \sqrt{z_1^2+z_2^2}$

$\|(x+y)+(-z)\| \leq \|x+y\| + \|z\| \leq \|x\| + \|y\| + \|z\|$

$\rightarrow \sqrt{(x_1+y_1-z_1)^2 + (x_2+y_2-z_2^2)} \leq \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} + \sqrt{z_1^2+z_2^2} \leq \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2} + \sqrt{z_1^2+z_2^2}$

$\rightarrow \sqrt{(x_1+y_1-z_1)^2 + (x_2+y_2-z_2)^2} \leq \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2} + \sqrt{z_1^2+z_2^2}$

14)

14)

1)

$$a = (a_1, a_2, a_3)$$
$$b = (b_1, b_2, b_3)$$
$$c = (c_1, c_2, c_3)$$

$$b \times c = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = i \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - j \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + k \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{bmatrix} b_2c_3 - c_2b_3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ b_1c_3 - c_1b_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1c_2 - c_1b_2 \end{bmatrix}$$

$$b \times c = \begin{bmatrix} b_2c_3 - c_2b_3 \\ c_1b_3 - b_1c_3 \\ b_1c_2 - c_1b_2 \end{bmatrix}_{3\times1}$$

$$a \cdot (b \times c) = (b \times c)^T a = \begin{bmatrix} b_2c_3 - c_2b_3 & c_1b_3 - b_1c_3 & b_1c_2 - c_1b_2 \end{bmatrix}_{1\times3} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{3\times1}$$

$$= a_1(b_2c_3 - c_2b_3) + a_2(c_1b_3 - b_1c_3) + a_3(b_1c_2 - c_1b_2)$$

14)

1)

$$\det(A) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) + a_2(c_1b_3 - b_1c_3) + a_3(b_1c_2 - c_1b_2) \quad \checkmark$$

det(A) = a.(bxc)

14)

2)

$$a \times c = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$a \times c = \begin{bmatrix} a_2c_3 - c_2a_3 \\ c_1a_3 - a_1c_3 \\ a_1c_2 - c_1a_2 \end{bmatrix}$$

$$-b(a \times c) = (a \times c)^T - b = \begin{bmatrix} a_2c_3 - c_2a_3 & c_1a_3 - a_1c_3 & a_1c_2 - c_1a_2 \end{bmatrix} \begin{bmatrix} -b_1 \\ -b_2 \\ -b_3 \end{bmatrix}$$

$$= b_1(c_2a_3 - a_2c_3) + b_2(a_1c_3 - c_1a_3) + b_3(c_1a_2 - a_1c_2)$$

14)

2)

$$\det(A) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\det(A) = b_1(c_2a_3 - a_2c_3) + b_2(a_1c_3 - c_1a_3) + b_3(c_1a_2 - a_1c_2)$$

$$\det(A) = a.(b \times c)$$

$$\det(A) = -b.(a \times c)$$

$$\underline{a.(b \times c) = -b.(a \times c)}$$

15)

15)

for equality ex.

$x = (1,1)$

$y = (1,1)$

$\sqrt{2} - \sqrt{2} \leq \|0\|$

$0 \leq 0$

$\underline{0 = 0}$

for inequality ex.

$x = (1,3)$

$y = (-2,-3)$

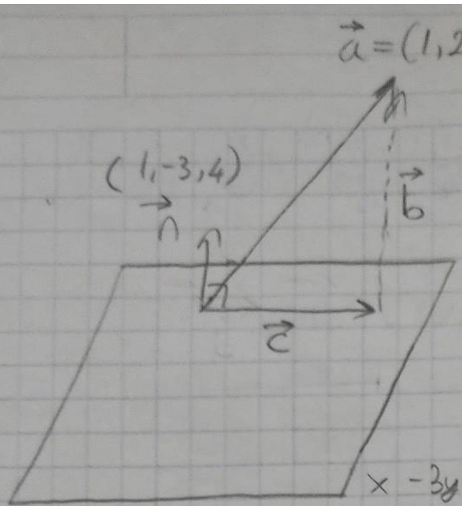
$x - y = (3,6)$

$\big|\|x\| - \|y\|\big| \leq \big|\|x - y\|\big|$

$\underline{\sqrt{13} - \sqrt{10} < \sqrt{45}}$

16)

16)



$x-3y+4z-5=0$

$$\vec{b}=\vec{a}\cdot\frac{\vec{n}}{\|n\|}\cdot\frac{\vec{n}}{\|n\|}$$

$$\frac{(1,-3,4)\cdot(1,2,1)}{\sqrt{1+9+16}=\sqrt{26}}$$

$$\vec{b}=\frac{-1}{\sqrt{26}}\cdot\frac{(1,-3,4)}{\sqrt{26}}$$

$$\vec{c}=\vec{a}-\vec{b}$$

$$(1,2,1)-\left(\frac{-1}{26},\frac{3}{26},\frac{-4}{26}\right)$$

$$\vec{c}=\left(\frac{27}{26},\frac{49}{26},\frac{30}{26}\right)$$

unit norm vector ($\vec{u}$) along $\vec{c}$

$$\vec{u}=\frac{\vec{c}}{\|c\|}=\frac{\left(\frac{27}{26},\frac{49}{26},\frac{30}{26}\right)}{\sqrt{\frac{27^2}{26^2}+\frac{49^2}{26^2}+\frac{30^2}{26^2}}}$$

$$=\left(\frac{27}{\sqrt{4030}},\frac{49}{\sqrt{4030}},\frac{30}{\sqrt{4030}}\right)$$

17)

17)

normal of the first plane : $\vec{n_1} = (1,2,2)$

normal of the second plane : $\vec{n_2} = (1,1,-3)$

$$\vec{n_1} \times \vec{n_2} = \vec{n_3}$$

↳ direction vector of the line.

$$\vec{n_3} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 1 & 1 & -3 \end{vmatrix} = i \begin{vmatrix} 2 & 2 \\ 1 & -3 \end{vmatrix} - j \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= -8i + 5j - k$$

$$= (-8, 5, -1)$$

if the line is orthogonal to any vector, then $\vec{n_3}$ must be orthogonal likewise. Therefore ;

Let asked vector is $\vec{\vartheta} = (a, b, c)$

$(a,b,c) \cdot (1,2,-1) = 0$

$(a,b,c) \cdot (-8,5,-1) = 0$

$\vec{\vartheta} = (a, 3a, 7a)$

$a + 2b - c = 0$

$-/ \ -8a + 5b - c = 0$

$a = a$

$-b = 3a$

$9a - 3b = 0 \quad 3a = b \quad c = 7a$

$a + 6a - c = 0$

$7a = c$

unit vector of $\vec{\vartheta} = \dfrac{\vec{\vartheta}}{\|\vartheta\|}$

$$\frac{\vec{\vartheta}}{\|\vartheta\|} = \frac{(a, 3a, 7a)}{\sqrt{a^2 + 9a^2 + 49a^2}} = \left(\frac{1}{\sqrt{59}}, \frac{3}{\sqrt{59}}, \frac{7}{\sqrt{59}}\right)$$

18)

18) $P = (x, y, z)$

$$\overrightarrow{P_0P} = (x-2, y+1, z-3)$$

$$(x-2, y+1, z-3) \cdot (1, 3, -2) = 0$$
$$(x-2, y+1, z-3) \cdot (-1, 1, 2) = 0$$

$$a1 = x - 2 + 3y + 3 - 2z + 6 = 0$$
$$a2 = 2 - x + y + 1 + 2z - 6 = 0$$

$$a1 = x + 3y - 2z = -7$$
$$a2 = -x + y + 2z = 3$$

$$a1 + a2 = 4y = -4 \qquad y = -1$$

$$x - 3 - 2z = -7$$
$$x - 2z = -4$$
$$x = 2z - 4$$

set of all P's $= \begin{cases} x = t \\ y = -1 \\ z = \dfrac{t+4}{2} \end{cases}$

19)

19) $P=(x,y,z)$ $\qquad\qquad \vec{v}=(2,1,-5)$

$$\vec{P_0P}=(x-1,y-1,z+1)$$

$$(x-1,y-1,z+1)\cdot(2,1,-5)=0$$

$$2x-2+y-1-5z-5=0$$

$2x+y-5z-8=0 \rightarrow$ Equation represents a plane which contains all p points that substantiate $\vec{P_0P}\cdot\vec{v}=0$

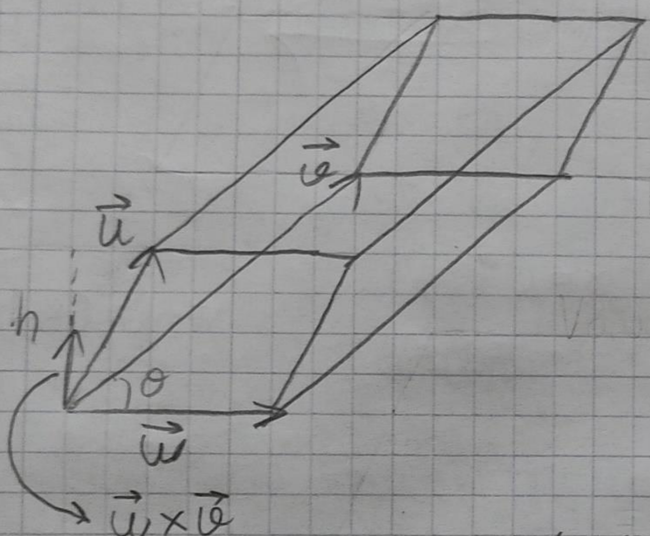
20)

20)

(a) $\vec{u} = (2,0,0)$

$\vec{v} = (2,3,0)$

$\vec{w} = (2,3,2)$



$$h = \frac{|\vec{u} \cdot (\vec{w} \times \vec{v})|}{\|\vec{w} \times \vec{v}\|}$$

$$= \frac{|(2,0,0) \cdot (-6,4,0)|}{\sqrt{52}} = \frac{12}{\sqrt{52}}$$

$$\vec{w} \times \vec{v} = \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ 2, & 3 & 0 \end{vmatrix} = (-6,4,0)$$

$$\text{Base} = (\|\vec{v}\| \cdot \sin(\theta)) \|\vec{w}\| = \|\vec{w} \times \vec{v}\|$$

$$\|\vec{w} \times \vec{v}\| = \sqrt{52}$$

$$\text{Volume} = \underbrace{\text{Base}}_{\sqrt{52}} \times \underbrace{\text{height}}_{\frac{12}{\sqrt{52}}} = 12$$

20) Surface Area:

(b) $2(\|u\| \cdot \sin\alpha \cdot \|\vartheta\| + \|u\| \cdot \sin\beta \cdot \|w\| + \|\vartheta\| \cdot \sin\theta \cdot \|w\|)$

$$2(\underbrace{\|\vec{u} \times \vec{w}\|}_{\sqrt{52}} + \underbrace{\|\vec{u} \times \vec{\vartheta}\|}_{6} + \underbrace{\|\vec{w} \times \vec{\vartheta}\|}_{\sqrt{52}})$$

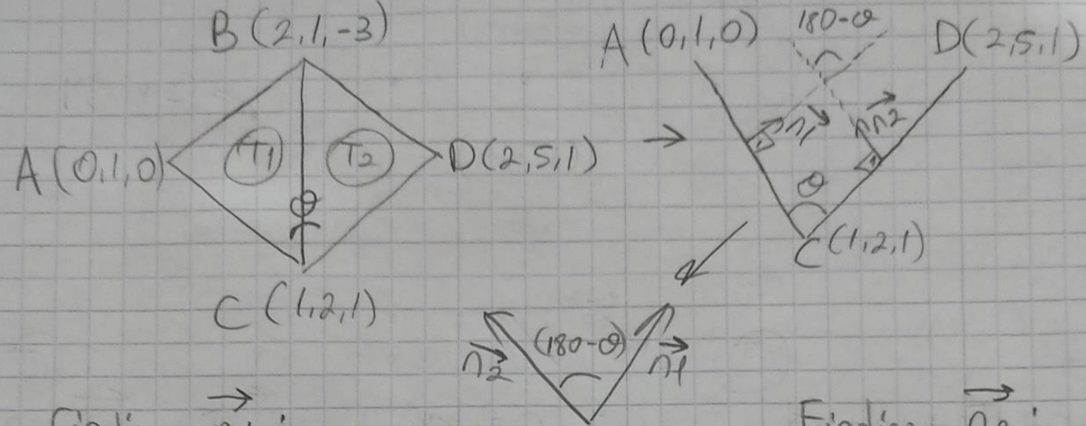
$$\vec{u} \times \vec{w} = \begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 2 & 3 & 2 \end{vmatrix} = (0,-4,6) \qquad \|\vec{u} \times \vec{w}\| = \sqrt{52}$$

$$\vec{u} \times \vec{\vartheta} = \begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 2 & 3 & 0 \end{vmatrix} = (0,0,6) \qquad \|\vec{u} \times \vec{\vartheta}\| = 6$$

$$2(6 + 2(\sqrt{52})) = \underline{12 + 4\sqrt{52}} = \text{Area}$$

21)

21)



Finding $\vec{n_1}$;

$\vec{BA} \times \vec{BC} = \vec{n_1}$

$$\vec{BA} \times \vec{BC} = \begin{vmatrix} i & j & k \\ -2 & 0 & 3 \\ -1 & 1 & 4 \end{vmatrix}$$

$\vec{n_1} = (-3, 5, -2)$

Finding $\vec{n_2}$;

$\vec{BC} \times \vec{BD} = \vec{n_2}$

$$= \begin{vmatrix} i & j & k \\ -1 & 1 & 4 \\ 0 & 4 & 4 \end{vmatrix}$$

$\vec{n_2} = (-12, 4, -4)$

$$\vec{n_1} \cdot \vec{n_2} = \|n_1\| \, \|n_2\| \cdot \cos\theta$$

$$36 + 20 + 8 = \sqrt{9+25+4} \cdot \sqrt{144+16+16} \cdot \cos(180-\theta)$$

$$64 = \sqrt{38} \cdot 4\sqrt{11} \cdot \cos(180-\theta)$$

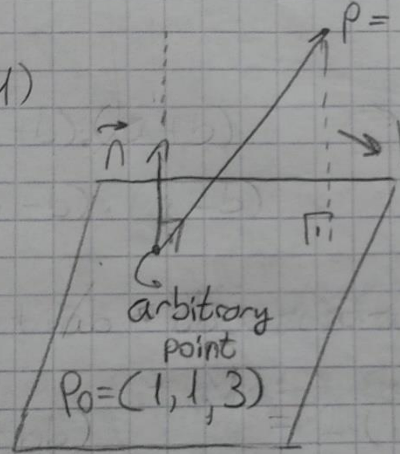
$$\cos(180-\theta) = \frac{64}{4\sqrt{418}} = \frac{16}{\sqrt{418}}$$

$$\cos\theta = \frac{-16}{\sqrt{418}}$$

$\theta \sim 141{,}5$

22)

22) $\vec{n} = (1, -3, 1)$



$$\frac{|\vec{P_0P} \cdot \vec{n}|}{\|n\|} = \frac{|(-1,0,-7)(1,-3,1)|}{\sqrt{1+9+1}}$$

$$|-1-7| = 8$$

$$D = \mathrm{Proj}_{n}^{\vec{P_0P}} = \frac{8}{\sqrt{11}}$$

23)

23) If $(a+b) \perp (a-b)$, then $(a+b) \cdot (a-b) = 0$

$$a(a-b) + b(a-b) = 0$$

$$(a \cdot a) - \cancel{(a \cdot b)} + \cancel{(a \cdot b)} - (b \cdot b) = 0$$

$$\underbrace{a \cdot a}_{\|a\|^2} = \underbrace{b \cdot b}_{\|b\|^2}$$

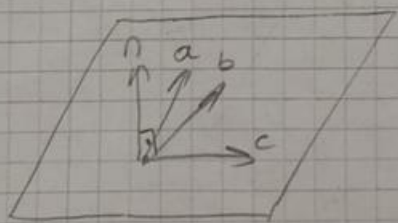
$$\frac{\|a\|^2}{\|b\|^2} = 1$$

$$\frac{\|a\|}{\|b\|} = 1 \quad (\|a\| \text{ and } \|b\| > 0)$$

24)

24)

Let a, b, c are coplanar,



$c \times a = n$

$n \cdot b = 0 \quad (\cos 90 = 0)$

then,

$(c \times a) \cdot b = 0$

$a+b+c=0$

$(c \times (-c-b)) \cdot b = 0$

$= ((c+b) \times c) \cdot b = 0$

$\underbrace{(c \times c)}_{0} + \underbrace{(b \times c)}_{-n}$

$\underbrace{-n \cdot b}_{0} = 0$

$0=0$ ✓

$0=0$ ✓ means that if $a+b+c=0$, then they must be on the same plane.

25)

25)

$$\|a+b\|^2 = ((a+b)\cdot(a+b))^{1/2 \cdot 2}$$

$$\|a-b\|^2 = ((a-b)\cdot(a-b))^{1/2 \cdot 2}$$

$$\begin{array}{l} (a+b)\cdot(a+b) = a\cdot a + 2a\cdot b + b\cdot b \\ (a-b)\cdot(a-b) = a\cdot a - 2a\cdot b + b\cdot b \\ + \\ \hline \end{array}$$

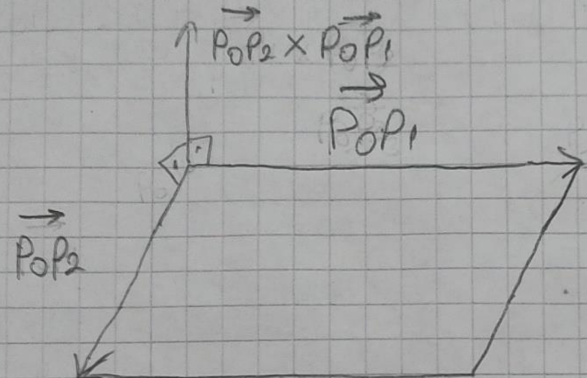
$$2a\cdot a + 2b\cdot b$$

$$2(\underbrace{a\cdot a}_{\|a\|^2} + \underbrace{b\cdot b}_{\|b\|^2}) = 2(\|a\|^2 + \|b\|^2)$$

26)

26) ① If $\vec{P_0P_1} \times \vec{P_2P_3} \neq 0$, then $\vec{P_0P_1}$ and $\vec{P_2P_3}$ cannot be parallel to each other. ($\sin\theta \neq 0$).

Let $\vec{P_0P_1}$ and $\vec{P_0P_2}$ lie in the same plane,



$$\vec{P_0P_2} \cdot (\vec{P_0P_1} \times \vec{P_2P_3}) = (\vec{P_0P_2} \times \vec{P_0P_1}) \cdot \vec{P_2P_3}$$

If $(\vec{P_0P_2} \times \vec{P_0P_1}) \cdot \vec{P_2P_3}$ is zero, then $\vec{P_2P_3}$ is parallel to the plane. This proofs that (in R^3)

② $\vec{P_0P_2}$, $\vec{P_0P_1}$ and $\vec{P_2P_3}$ are on the same plane. (Volume is zero)

According to ① and ② ($\vec{P_0P_1}$ and $\vec{P_2P_3}$ have different directions and they are on the same plane) lines which contain P_0, P_1 and P_2, P_3 must intersect.