Chapter 9

Linear Programming: The Simplex Method

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Learning Objectives

Students will be able to:

1. Convert LP constraints to equalities with slack, surplus, and artificial variables.
2. Set up and solve LP problems with simplex tableaus.
3. Interpret the meaning of every number in a simplex tableau.
4. Recognize special cases such as infeasibility, unboundedness, and degeneracy.
5. Use the simplex tables to conduct sensitivity analysis.
6. Construct the dual problem from the primal problem.

Chapter Outline

Chapter Outline
(continued)

Introduction

- Graphical methods are fine for 2 variables.
- But most LP problems are too complex for simple graphical procedures.
- The Simplex Method:
  - examines corner points, like in graphing;
  - systematically examines corner points, using algebra, until an optimal solution is found;
  - does its searching iteratively.

Introduction

Why study the Simplex Method?

- The Simplex Method:
  - Provides the optimal solution to the X_i variables and the maximum profit (or minimum cost).
  - Provides important economic information.
- Understanding how the Simplex Method works is important because
  - it allows the use of computer generated solutions to be run successfully, and
  - it allows for understanding how to interpret LP computer printouts.

Setting Up the Simplex Tableau

- The first example examined is to solve a maximization problem.
- The Flair Furniture Company from Chapter 7 is examined.
- First, start by reviewing the original equations.

Flair Furniture Company

Hours Required to Produce One Unit

Department	T Tables	C Chairs	Available Hours This Week
Painting/Varnishing	2	1	100
Carpentry	4	3	240
Profit/unit	$7	$5	

Objective: *Maximize:* $7T + 5C$

Constraints:

$$2T + 1C \leq 100 \quad \text{(painting \& varnishing)}$$

$$4T + 3C \leq 240 \quad \text{(carpentry)}$$

$$T, C \geq 0 \quad \text{(nonnegativity constraints)}$$

Making Constraints into Equations

1st Step

- Convert each inequality constraint into an equation.
- Less-than-or-equal-to constraints ($\leq$) are converted to equations by adding a *slack variable* to each.
  - Slack variables represent unused resources.

- For the Flair Furniture problem, define the slacks as:
  - S_1 = unused hours in the painting dept
  - S_2 = unused hours in the carpentry dept

- The constraints are now written as:
  - $2T + 1C + S_1 = 100$
  - $4T + 3C + S_2 = 240$

Making Constraints into Equations

1st Step, continued

- Slack variables not appearing in an equation are added with a coefficient of 0.
  - This allows all the variables to be monitored at all times.
- The final Simplex Method equations appear as:
  - $2T + 1C + 1S_1 + 0S_2 = 100$
  - $4T + 3C + 0S_1 + 1S_2 = 240$
  - $T, C, S_1, S_2 \geq 0$
- The slacks are added to the objective coefficient with 0 profit coefficients.
- The objective function, then, is:
  - maximize profit:
    $\$7T + \$5C + \$0S_1 + \$0S_2$

An Algebraic Initial Solution

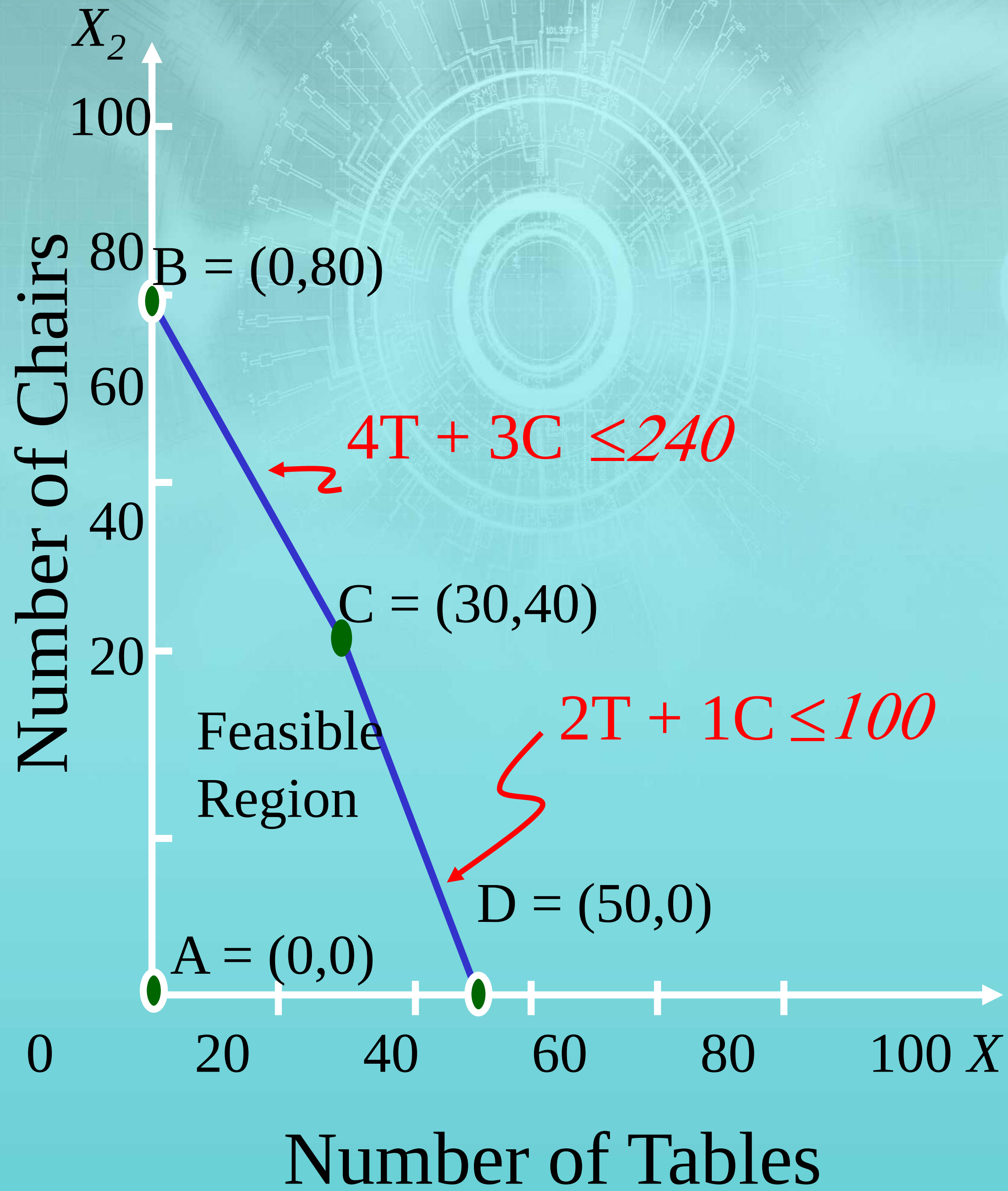
- There are now 2 (*n*) equations and 4 variables.
- When the number of variables is > than the number of equations, the system can only be solved by setting some of the variables equal to 0.
- A basic feasible solution to a system of *n* equations is found by setting all but *n* variables equal to 0 and solving for the other variables.
- Solutions found in this manner are called *basic feasible solutions*.

An Algebraic Initial Solution

Using the figure on the next slide, the following information can be observed:

- The Simplex Method starts with an initial feasible solution with all real variables (T and C) set to 0 [Point A on the graph].
- The Simplex Method will always start at this point and then move up or over to the corner point that provides the most improved profit [Points B or D].
- The method will move to a new corner point [C], which is the optimal point in this example.
- The method considers only feasible solutions and will only touch the corner points of the feasible region.

Flair Furniture Company's Feasible Region & Corner Points



Flair Furniture's First Simplex Tableau

The Next Step

- All the coefficients of all the equations and objective function need to be tabular form.
- The two constraints are written below.
  - o The numbers (2,1,1,0) in the first row are the coefficients of the first equation,
  - o The numbers (4,3,0,1) in the second row are the coefficients of the second equation.
- The first *simplex tableau* is on the next slide.

Constraints in tabular form:

Solution Mix	T	C	S_1	S_2	*Quantity*
S_1	2	1	1	0	100
S_2	4	3	0	1	240

Constraint equation rows

Flair Furniture's Initial Simplex Tableau

Profit per Unit Column
Prod. Mix Column
Real Variables Columns
Slack Variables Columns
Constant Column

C_j	Solution Mix	\$7	\$5	\$0	\$0		Profit per unit row
		T	C	S_1	S_2	Quantity	
\$0	S_1	2	1	1	0	100	Constraint equation rows
\$0	S_2	4	3	0	1	240	
	Z_j	\$0	\$0	\$0	\$0	\$0	Gross Profit row
	$C_j - Z_j$	\$7	\$5	\$0	\$0	\$0	Net Profit row

Flair Furniture's Initial Simplex Tableau

Profit per Unit Column — *Prod. Mix Column* — *Real Variables Columns* — *Slack Variables Columns* — *Constant Column*

C_j		\$7	\$5	\$0	\$0		*Profit per unit row*
	Solution Mix	T	C	S_1	S_2	*Quantity*	
\$0	S_1	2	1	1	0	100	*Constraint equation rows*
\$0	S_2	4	3	0	1	240	
	Z_j	\$0	\$0	\$0	\$0	\$0	*Gross Profit row*
	C_j - Z_j	\$7	\$5	\$0	\$0	\$0	*Net Profit row*

Flair Furniture's First Simplex Tableau

- The initial solution is a called a *basic feasible solution* and can be written as a vector:

$$\begin{bmatrix} T \\ C \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 240 \end{bmatrix}$$

- The solution mix is referred to as the *basis* and all variables in the basis are called *basic*.
- *Nonbasic variables* are those set equal to zero in the basis.

Flair Furniture's First Simplex Tableau

Substitution Rates

- Substitution rates are numbers in the body of the simplex table.
- The entries in the table in the columns under each variable are the coefficients of that variable.

Solution Mix	T	C	S_1	S_2	*Quantity*
S_1	2	1	1	0	100
S_2	4	3	0	1	240

Under T are the coefficients $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Under C are the coefficients $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Under S_1 are the coefficients $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Under S_2 are the coefficients $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Flair Furniture's First Simplex Tableau

Substitution Rates, continued

- The numbers in the body of the simplex tableau can be thought of as substitution rates.
- *For example,*
  - o To make T larger than 0, every unit increase of T will require 2 units of S_1 and 4 units of S_2 must be removed.
  - o The substitution rates for each unit of C is 1 unit of S_1 and 3 units of S_2.
- For any variable ever to appear in the solution mix column, it must have the number 1 someplace in its column and 0's in every other place in that column.

Column S_1 has $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so it is in the solution.

Column S_2 has $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so it is in the solution.

Flair Furniture's First Simplex Tableau

Adding the Objective Function

- Next step is to add a row to reflect the objective function values for each variable.
- These values are called contribution rates and are labeled as C_j. They appear above each variable, as in the following tableau:

$C_j \rightarrow$		\$7	\$5	\$0	\$0	
$\downarrow$	*Solution Mix*	T	C	S_1	S_2	*Quantity*
\$0	S_1	2	1	1	0	100
\$0	S_2	4	3	0	1	240

- The unit profit rates are found in the top row and the leftmost column.
    - C_j indicates the unit profit for each variable currently in the solution mix.

Flair Furniture's First Simplex Tableau

The Z_j and $C_j - Z_j$ Rows

- The *Zj* value for the *Quantity* column provides the total contribution (gross profit in this case) of the given solution.
- The *Zj* values for the other columns (variables *T*, *C*, S_1, and S_2) represent the gross profit *given up* by adding one unit of this variable into the current solution.
- In this example, there is no profit *lost* by adding one unit of either *T* (tables), *C* (chairs), S_1, or S_2.

Flair Furniture's First Simplex Tableau

The Z_j and $C_j - Z_j$ Rows

- The $Cj - Zj$ number in each column represents the net profit that will result from introducing 1 unit of each product or variable into the solution,
  - i.e., the profit gained minus the profit given up.
  - It is not calculated for the quantity column.
- To compute these numbers, simply subtract the Zj total for each column from the Cj value at the very top of that variable's column.
- The calculations for the net profit per unit (the $Cj - Zj$ row) in this example follow.

Flair Furniture's First Simplex Tableau

The Z_j and $C_j - Z_j$ Rows

calculations for the net profit per unit

	COLUMN			
	T	C	S_1	S_2
C_j	$7	$5	$0	$0
Z_j	$0	$0	$0	$0
$C_j - Z_j$	$7	$5	$0	$0

- When a profit of $0 is computed, the initial solution is not optimal.
- Examining the numbers in the *Cj - Zj* row shows that the total profit can be increased by $7 for each unit of *T* (tables) and by $5 for each unit of *C* (chairs) added to the solution mix.
  - A negative number in the *Cj - Zj* row would tell us that profits would *decrease* if the corresponding variable were added to the solution mix.
- An optimal solution is reached in the simplex method when the *Cj - Zj* row contains no positive numbers.

Simplex Steps for Maximization

- After the initial tableau is completed, proceed through a series of five steps to compute all the numbers needed in the next tableau.
  - The calculations are not difficult, but they are complex; the smallest arithmetic error can produce a wrong answer.

The 5 Steps

1. Choose the variable with the greatest positive $C_j - Z_j$ to enter the solution.
2. Determine the row to be replaced by selecting that one with the smallest (non-negative) quantity-to-pivot-column ratio.
3. Calculate the new values for the pivot row.
4. Calculate the new values for the other row(s).
5. Calculate the C_j and $C_j - Z_j$ values for this tableau.
   - If there are any $C_j - Z_j$ values greater than zero, return to step 1.

Pivot Row, Pivot Number Identified in the Initial Simplex Tableau

	$C_j \rightarrow$	\$7	\$5	\$0	\$0	
$\downarrow$	Solution Mix	T	C	S_1	S_2	Quantity
\$0	S_1	2	1	1	0	100
\$0	S_2	4	3	0	1	240
	Z_j	\$0	\$0	\$0	\$0	\$0
	$C_j - Z_j$	\$7	\$5	\$0	\$0	

Pivot row

Pivot number

Pivot column

Largest (C_j - Z_j) value

Pivot Row Changed

Row divided by 2

C_j →		\$7	\$5	\$0	\$0	
↓	*Solution Mix*	T	C	S_1	S_2	*Quantity*
\$7	T	1	1/2	1/2	0	50
\$0	S_2	4	3	0	1	240
	Z_j					
	$C_j - Z_j$					

This row will be changed next to get a zero in the pivot column.

Calculating the New S_2 Row for Flair's Second Tableau

Equation 9-1

$$\begin{pmatrix}\text{New}\\ \text{Row}\\ \text{Numbers}\end{pmatrix} = \begin{pmatrix}\text{Numbers}\\ \text{in old}\\ \text{row}\end{pmatrix} - \left[\begin{pmatrix}\text{Number}\\ \text{above}\\ \text{or below}\\ \text{pivot}\\ \text{number}\end{pmatrix} \times \begin{pmatrix}\text{Corresponding}\\ \text{number}\\ \text{in the}\\ \text{new row}\end{pmatrix}\right]$$

0	=	4	-	(4)	x	(1)
1	=	3	-	(4)	x	(1/2)
-2	=	0	-	(4)	x	(1/2)
1	=	1	-	(4)	x	(0)
40	=	240	-	(4)	x	(50)

Completed Second Simplex Tableau for Flair Furniture

C_j →		$7	$5	$0	$0	
↓	*Solution Mix*	*T*	*C*	S_1	S_2	*Quantity*
$7	*T*	1	1/2	1/2	0	50
$0	S_2	0	1	-2	1	40
	Z_j	$7	$7/2	$7/2	$0	$350
	$C_j - Z_j$	$0	$3/2	-$7/2	$0	

Row divided by 2

Row calculated using Equation 9-1 as shown in previous slide

Pivot Row, Column, and Number Identified in Second Simplex Tableau

C_j		\$7	\$5	\$0	\$0	
	Solution Mix	T	C	S_1	S_2	*Quantity*
\$7	T	1	1/2	1/2	0	50
\$0	S_2	0	1	-2	1	40
	Z_j	\$7	\$7/2	\$7/2	\$0	\$350 (Total Profit)
	$C_j - Z_j$	\$0	\$3/2	-\$7/2	\$0	

Pivot row

Pivot number

Pivot column

Calculating the New *T* Row for Flair's Third Tableau

$$\begin{pmatrix}\text{New}\\\text{Row}\\\text{Numbers}\end{pmatrix} = \begin{pmatrix}\text{Numbers}\\\text{in old}\\\text{row}\end{pmatrix} - \left[\begin{pmatrix}\text{Number}\\\text{above}\\\text{or below}\\\text{pivot}\\\text{number}\end{pmatrix} \times \begin{pmatrix}\text{Corresponding}\\\text{number}\\\text{in the}\\\text{new row}\end{pmatrix}\right]$$

1	=	1	-	(1/2)	x	(0)
0	=	1/2	-	(1/2)	x	(1)
3/2	=	1/2	-	(1/2)	x	(-2)
-1/2	=	0	-	(1/2)	x	(1)
30	=	50	-	(1/2)	x	(40)

Final Simplex Tableau for the Flair Furniture Problem

C_j →		$7	$5	$0	$0	
↓	*Solution Mix*	*T*	*C*	S_1	S_2	*Quantity*
$7	*T*	1	0	3/2	-1/2	30
$5	*C*	0	1	-2	1	40
	Z_j	$7	5	$1/2	$3/2	$410
	$C_j - Z_j$	$0	$0	-$1/2	-$3/2	

Since every number in the last row is 0 or negative, an optimal solution has been found. **The solution is:**

T = 30 tables
C = 40 chairs
S_1 = 0 slack hours in painting
S_2 = 0 slack hours in carpentry
profit = $410 for the optimal solution

Example

Max Z = 50 x +20 y

s.t

2 x + 4 y <= 400

100 x +50 y <= 8000

x <= 60

x, y are positive R

$$\text{Max } Z = 50x + 20Y + 0S1 + 0S2 + 0S3$$

s.t

$$2x + 4y + S1 = 400$$

$$100x + 50y + S2 = 8000$$

$$x + S3 = 60$$

		x	y	S1	S2	S3	
	Cj	50	20	0	0	0	
S1	0	2	4	1	0	0	400
S2	0	100	50	0	1	0	8000
S3	0	1	0	0	0	1	60
zj		0	0	0	0	0	0
cj-zj		50	20	0	0	0	
S1	0	0	4	1	0	-2	280
S2	0	0	50	0	1	-100	2000
x	50	1	0	0	0	1	60
zj		50	0	0	0	50	3000
cj-zj		0	20	0	0	-50	
S1	0	0	0	1	-0,08	6	120
y	20	0	1	0	0,02	-2	40
x	50	1	0	0	0	1	60
zj		50	20	0	0,4	10	3800
cj-zj		0	0	0	-0,4	-10	

Surplus and Artificial Variables

- To use the simplex method with greater-than-or-equal-to (≥) constraints and equalities, each of these must be converted to a special form similar to that made for the less-than-or-equal-to (≤) constraints.
- If they are not, the simplex technique is unable to set up an initial solution in the first tableau.
- Consider the following two constraints to see how to convert typical constraints greater-than-or-equal-to (≥) constraints and equalities:

Constraint 1: $5X_1 + 10X_2 + 8X_3 \geq 210$

Constraint 2: $25X_1 + 30X_2 = 900$

First constraint;

$5X_1 + 10X_2 + 8X_3 - S1 = 210$

$S1 \geq 0$

While constructing the initial table, we start from the origin where X1=X2=0

In the constraint, when X1=X2=0 then S1 becomes -210 but it was positive. Contradiction!

Second constraint;

$25X_1 + 30X_2 \qquad = 900$

Again at the initial point, X1=X2=0

İt becomes 0=900

Surplus and Artificial Variables

- Greater-than-or-equal-to (≥) constraints (e.g., constraint #1) require a different approach than do the less-than-or-equal-to (≤) constraints seen in the Flair Furniture problem.
- They involve the subtraction of a *surplus variable* rather than the addition of a slack variable.
- The surplus variable tells us how much the solution exceeds the constraint resource.
- Because of its analogy to a slack variable, surplus is sometimes simply called *negative slack*.
- To convert the first constraint, begin by subtracting a surplus variable, S_1, to create an equality.

Surplus and Artificial Variables

- There is one more step in preparing a ≥ constraint for the simplex method.
- Artificial variables are needed in ≥ and = constraints.
- Simply add the artificial variable, A_1, to the constraint as follows:

Constraint 1 *completed*:

$$5X_1 + 10X_2 + 8X_3 - S_1 + A_1 = 210$$

- Now, not only the X_1, X_2, and X_3 variables may be set to 0 in the initial simplex solution, but the S_1 surplus variable as well.
- This leaves $A_1 = 210$.

Surplus and Artificial Variables

- Constraint 2 is already an equality.

So why worry about it?

- To be included in the initial simplex solution even
  an equality must have an artificial variable added to it.

Constraint 2 *rewritten*:

$$25X_1 + 30X_2 + A_2 = 900$$

- The reason for artificial variables in an equality constraint concerns finding an initial LP solution.
- Depending on the complexity of the constraint, it can be *extremely* difficult to visualize a set of initial solutions.
- By adding artificial variables, such as A_2, an automatic initial solution can be found.
- In this case, when X_1 and X_2 are set equal to 0, $A_2 = 900$.

Surplus and Artificial Variables

- Artificial variables have no meaning in a physical sense and are nothing more than computational tools for generating initial LP solutions.
- Before the final simplex solution has been reached, all artificial variables must be gone from the solution mix.
- Artificial variables have no physical meaning and drop out of the solution mix before the final tableau.
- This matter is handled through the problem's objective function.

Surplus and Artificial Variables

- Whenever an artificial or surplus variable is added to one of the constraints, it must also be included
  - in the other equations and
  - in the problem's objective function.

  just as was done for slack variables.
- Since artificial variables must be forced out of the solution, assign a very high cost to each.
- Simply use the letter ***M*** to represent a very large number.
- Surplus variables, like slack variables, carry a zero cost.
- In maximization problems, use negative ***M***.

Surplus and Artificial Variables

- If a problem had an objective function that read:

  minimize cost = $\$5X_1 + \$9X_2 + \$7X_3$

- and constraints such as the two mentioned previously, the completed objective function and constraints would appear as follows:

minimize cost =

$$\$5X_1 + \$9X_2 + \$7X_3 + \$0S_1 + \$MA_1 + \$MA_2$$

subject to:

$$5X_1 + 10X_2 + 8X_3 - 1S_1 + 1A_1 + 0A_2 = 210$$

$$25X_1 + 30X_2 + 0X_3 + 0S_1 + 0A_1 + 1A_2 = 900$$

Procedures LP Minimization

I. Formulate the LP problem's objective function and constraints.

II. Include:

* slack variables in each less-than-or-equal-to constraint,
* artificial variables in each equality constraint, and
* both surplus and artificial variables in each greater-than-or-equal-to constraint.

Then add all of these variables to the problem's objective function.

III. Develop an initial simplex tableau with artificial and slack variables in the basis and their variables (the X_i's) set equal to 0. Compute the Z_j and C_j - Z_j values for this tableau.

Simplex Steps for Minimization

IV. Follow these five steps until an optimal solution has been reached:

1. Choose the variable with the greatest negative $C_j - Z_j$ to enter the solution.
2. Determine the row to be replaced by selecting that one with the smallest (non-negative) quantity-to-pivot-column ratio.
3. Calculate the new values for the pivot row.
4. Calculate the new values for the other row(s).
5. Calculate the C_j and $C_j - Z_j$ values for this tableau. If there are any $C_j - Z_j$ values less than zero, return to step 1.

		X1	X2	X3	A1	S1	A2	
Cj		5	9	7	M	0	M	Quantity
Iteration 1								
M	A1	5	10	8	1	-1	0	210
M	A2	25	30	0	0	0	1	900
	zj	30M	40M	8M	M	-M	M	1110M
	cj-zj	5-30M	9-40M	7-8M	0	M	0	
Iteration 2								
9	X2	0,5	1	0,8	0,1	-0,1	0	21
M	A2	10	0	-24	-3	3	1	270
	zj	10M+4,5	9	7,2-24M	0,9-3M	3M-0,9	M	270M
	cj-zj	0,5-10M	0	24M-0,2	M-4	-3	0	
Iteration 3								
9	X2	0	1	2	0,25	-0,25	-0,05	7,5
5	X1	1	0	-2,4	-0,3	0,3	0,1	27
	zj	5	9	6	0,75	-0,75	0,05	202,5
	cj-zj	0	0	1	M-0,75	0,75	M-0,05	

Special Cases

- In Chapter 7 some special cases that may arise when solving LP problems graphically were addressed (Section 8 of Chapter 7).
- These cases are discussed again, this time as they refer to the simplex method.
- These four cases are:
  1. Infeasibility
  2. Unbounded Solutions
  3. Degeneracy
  4. Multiple Optimal Solutions

Special Cases Infeasibility

- *Infeasibility* exists when there is no solution that satisfies all of the problem's constraints.
- In the simplex method, an *infeasible* solution is indicated by looking at the final tableau.
- All C_j - Z_j row entries will be of the proper sign to imply optimality, but an artificial variable will still be in the solution mix.
- A situation with no feasible solution may exist if the problem was formulated improperly.
  - Possibly conflicting constraints.

Indication of *infeasibility*:

- even though all C_j - Z_j
  - are positive or 0 (the criterion for an optimal solution in a *minimization* case) an artificial variable remains in the solution mix, or
  - are negative or 0 (the criterion for an optimal solution in a *maximization* case) all non-slack variables are already in the basis.

Max prob

		x	y	S1	S2	A1	
	Cj	2	1	0	0	-M	
A1	-M	0	0	0	-1	0	5
x	2	1	1	1	0	1	15
zj		2	2	2	M	2	30-5M
cj-zj		0	-1	-2	-M	-M-2	

Special Cases Unbounded Solutions

- *Unboundedness* describes linear programs that do not have finite solutions.
- It occurs in maximization problems, for example, when a solution variable can be made infinitely large without violating a constraint.
- In the simplex method, the condition of *unboundedness* will be discovered prior to reaching the final tableau.
- The problem will be manifested when trying to decide which variable to remove from the solution mix.
- As seen earlier, the procedure is to divide each quantity column number by the corresponding pivot column number.
  - * The row with the smallest positive ratio is replaced.
  - * But if all the ratios turn out to be negative or undefined, it indicates that the problem is unbounded.

Max prob

		x	y	S1	S2		
	Cj	2	1	0	0		
S1	0	-2	0	0	-1		5
y	1	-1	1	1	0		15
zj		-1	1	1	0		15
cj-zj		3	0	-1	0		

Special Cases Degeneracy

- *Degeneracy* is another situation that can occur when solving an LP problem using the simplex method.
- It develops when three constraints pass through a single point.
- **For example,**
  - suppose a problem has only these three constraints: $X_1 \leq 10$, $X_2 \leq 10$, and $X_1 + X_2 \leq 20$
  - All three constraint lines will pass through the point (10, 10) .
- *Degeneracy* is first recognized when the ratio calculations are made.
- If there is a ***tie*** for the smallest ratio, this is a signal that *degeneracy* exists.
- As a result of this, when the next tableau is developed, one of the variables in the solution mix will have a value of zero.

Special Cases Degeneracy

- Degeneracy could lead to a situation known as *cycling*, in which the simplex algorithm alternates back and forth between the same nonoptimal solutions;
  - that is, it puts a new variable in, then takes it out in the next tableau, puts it back in, and so on.
- One simple way of dealing with the issue is to select either row in question arbitrarily.
- If unlucky and cycling does occur, simply go back and select a competing row.

Max problem

		x	y	S1	S2	S3	
	Cj	5	8	0	0	0	
y	8	0,25	1	2	0	0	10
S2	0	4	0	2	2	3	20
S1	0	2	0	1	0	1	10
zj		2	8	16	0	0	80
cj-zj		3	0	-16	0	0	

Special Cases
Multiple Optimal Solutions

- Multiple, or alternate, optimal solutions can be spotted when the simplex method is being used by looking at the final tableau.
- If the C_j-Z_j value is equal to 0 for a variable that is not in the solution mix, more than one optimal solution exists.

Max prob

		x	y	S1	S2		
	Cj	2	1	0	0		
S2	0	0	1	0	1		10
x	2	1	0,5	1	0		30
zj		2	1	2	0		60
cj-zj		0	0	0	0		

Sensitivity Analysis

- *Sensitivity Analysis* shows how the optimal solution and the value of its objective function change, given changes in various inputs to the problem.
- Computer programs handling LP problems of all sizes provide *sensitivity analysis* as an important output feature.
- Those programs use the information provided in the final simplex tableau to compute ranges for the objective function coefficients and ranges for the RHS values.
- They also provide "*shadow prices*," a concept that will be discussed.

Sensitivity Analysis
High Note Sound Company

Example from Chapter 7

maximize profit $= \$50X_1 + \$120X_2$

subject to:

$2X_1 + 4X_2 \leq 80$ (hours of electrician time)

$3X_1 + 1X_2 \leq 60$ (hours of technician time)

Optimal solution:

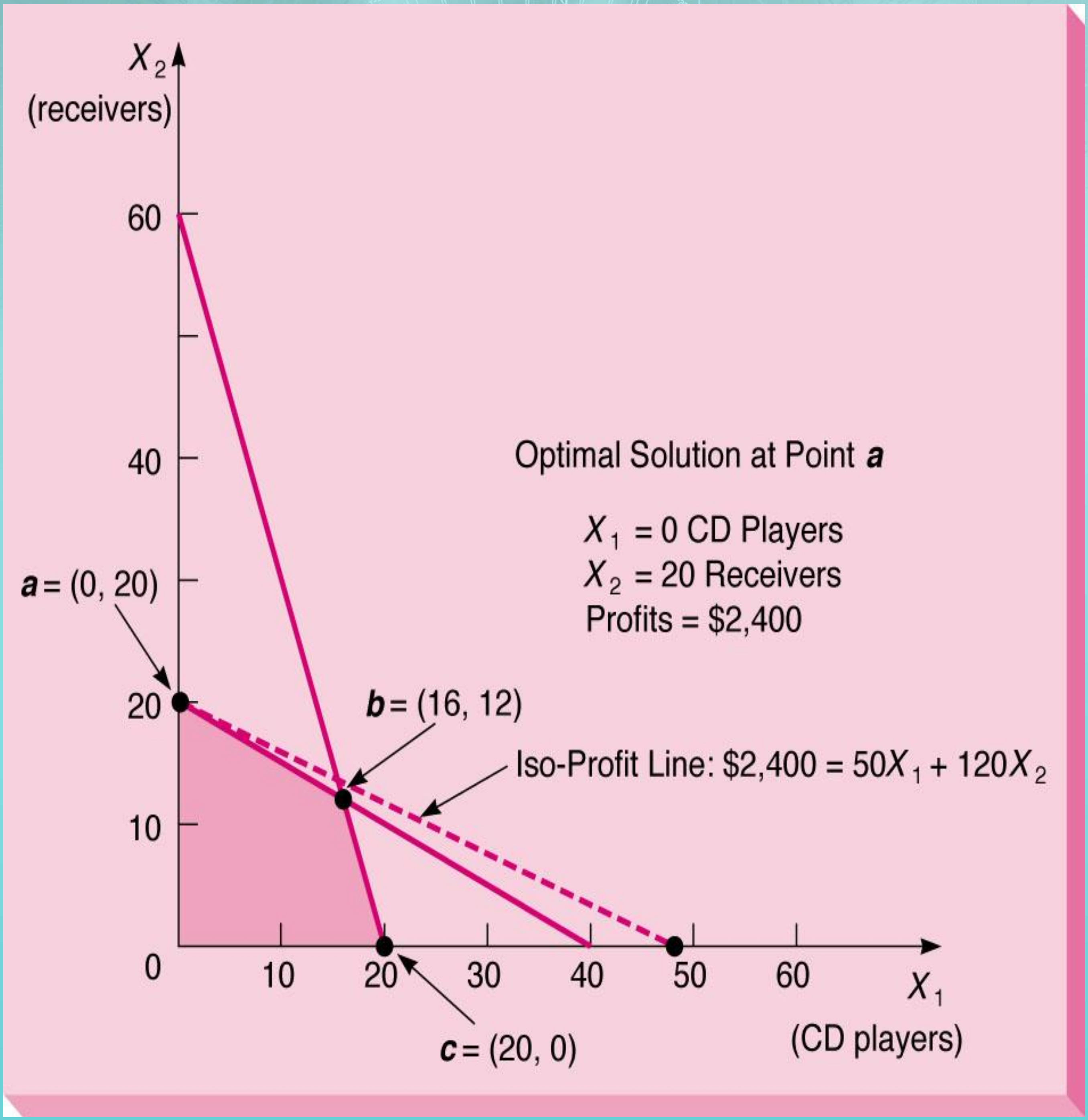
$X_2 = 20$ receivers
$S_2 = 40$ hours slack in technician time

Basic variables

$X_1 = 0$ CD players
$S_1 = 0$ hours slack in electrician time

Nonbasic variables

Sensitivity Analysis
High Note Sound Company



Simplex Solution
High Note Sound Company

C_j		50	120	0	0	
	Solution Mix	X_1	X_2	S_1	S_2	Quantity
120	X_2	1/2	1	1/4	0	20
0	S_2	5/2	0	-1/4	1	40
	Z_j	60	120	30	0	2400
	$C_j - Z_j$	-10	0	-30	0	

The solution is optimal as long as all $C_j – Z_j \leq 0$

Nonbasic Objective Function Coefficients

Goal: How much would the Objective function coefficients have to change before one of the nonbasic variables would enter the solution mix and replace one of the basic variables?

C_j		50	120	0	0	
	Solution Mix	X_1	X_2	S_1	S_2	Quantity
120	X_2	1/2	1	1/4	0	20
0	S_2	5/2	0	-1/4	1	40
	Z_j	60	120	30	0	2400
	$C_j - Z_j$	-10	0	-30	0	

The answer lies in the $C_j - Z_j$ row of the final simplex tableau.

Nonbasic Objective Function Coefficients

- In a maximization problem, the basis will not change unless the $C_j - Z_j$ value of one of the nonbasic variables becomes positive.
  - o The current solution will be optimal as long as all numbers in the bottom row are 0.
  - o The solution is NOT optimal if X_1's $C_j - Z_j$ value is > 0, or if S_1's $C_j - Z_j$ value is < 0.

- The values of C_j for X_1 that do not bring about any change in the optimal solution are given by $C_j - Z_j \leq 0$, or $C_j \leq Z_j$.

Nonbasic Objective Function Coefficients

- In this example:
  - For X_1, $C_1 = \$50$ and $Z_1 = \$60$ meaning current solution is optimal given the profit per CD increases less than \$10 and is < \$60

- In both cases, when maximizing an objective, the value of C_j may be increased up to the value of Z_j.
- The value of C_j for nonbasic variables can be decreased to negative infinity without affecting the solution.
  - The range over which C_j rates for nonbasic variables can vary without causing a change in the optimal solution mix is called the range of insignificance:
  - ✓ $-\infty \le C_j$ (for X_1) $\le \$60$

Simplex Solution High Note Sound Company

C_j		50	120	0	0	
	Sol Mix	X_1	X_2	S_1	S_2	Qty
120	X_1	½	1	1/4	0	20
0	S_2	5/2	0	-1/4	1	40
	Z_j	60	120	30	0	40
	$C_j - Z_j$	0	0	-30	0	2400

Solution mix changes if X_1 profit is greater than 60.

Basic Objective Function Coefficients

- A change in the profit or cost of a basic variable can affect the C_j - Z_j values of *all* nonbasic variables because this C_j is not only in the C_j row but also in the C_j column.
  - This impacts the Z_j row.
- Consider changing the profit contribution of stereo receivers in this problem.
- Currently, the objective function coefficient is $120.
- The change in this value can be denoted by the Greek capital letter delta (Δ).
- The final simplex tableau is reworked on the following slide.

Basic Objective Function Coefficients

C_j		50	120	0	0	
	Solution Mix	X_1	X_2	S_1	S_2	Quantity
120 + Δ	X_1	1/2	1	1/4	0	20
0	S_2	5/2	0	-1/4	1	40
	Z_j	60+ Δ/2	120+ Δ	30+ Δ/4	0	2400+20 Δ
	C_j - Z_j	-10- Δ/2	0	-30- Δ/4	0	

- The new C_j-Z_j values for nonbasic variables X_1 and S_1 were determined in exactly the same way as earlier.
- But wherever the C_j value for X_2 of \$120 was seen before, a new value of \$120 + Δ is used.

Changes in Resources or RHS Values

- Making changes in the RHS values of constraints (in this example, the resources of electricians' and audio technicians' time) result in changes in the feasible region and often the optimal solution.
- The *shadow price* is the value of one additional unit of a scarce resource.
- *Shadow pricing* provides an important piece of economic information.

Changes in Resources or RHS Values

- The worth of additional resources is valuable information to management.
- For example, questions as the following can be useful to management:
  - Exactly how much should a firm be willing to pay to make additional resources available?
  - Is one more hour of machine time worth $1 or $5 or $20?
  - Is it worthwhile to pay workers an overtime rate to stay one extra hour each night to increase production output?

Changes in Resources or RHS Values

- This information is available by looking at the final simplex tableau of an LP problem.
- An important property of the C_j - Z_j row is that the negatives of the numbers in its slack variable (S_i) columns provide us with what we call shadow prices.
- A *shadow price* is the change in value of the objective function from an increase of one unit of a scarce resource.
  - (e.g., by making one more hour of machine time or labor time or other resource available).

Simplex Solution
High Note Sound Company

C_j		50	120	0	0	
	Sol Mix	X_1	X_2	S_1	S_2	Qty
120	X_1	½	1	1/4	0	20
0	S_2	5/2	0	-1/4	1	40
	Z_j	60	120	30	0	40
	$C_j - Z_j$	0	0	-30	0	2400

Objective increases by 30 if 1 additional hour of electrician time is available.

Changes in Resources or RHS Values

Interpreting the Results

- The final tableau indicates that the optimal solution is $X_1 = 0$, $X_2 = 20$, $S_1 = 0$, and $S_2 = 40$ and that profit = \$2,400.
  - S_1 represents slack availability of the electricians' resource and S_2 the unused time in the audio technicians' department.
- The firm is considering hiring an extra electrician on a part-time basis.
  - Assume it costs \$22 per hour in wages and benefits to bring the part-timer on board.

Should the firm do this?

- The answer is ***yes***;
  - the shadow price of the electrician time resource is \$30.
- The firm will *net* \$8 (= \$30 - \$22) for every hour the new worker helps in the production process.

Changes in Resources or RHS Values

Interpreting the Results

Should High Note also hire a part-time audio technician at a rate of $14 per hour?

- The answer is ***no*;**
  - o the shadow price is $0, implying no increase in the objective function by making more of this second resource available.
- Why?
- Because not all of the resource is currently being used—40 hours are still available.
- It would hardly pay to buy more of the resource.

Right-Hand-Side Ranging

- The range over which shadow prices remain valid is called *right-hand-side ranging*.
- *Ranging* employs the simplex process to find the minimum ratio for a new variable.
- Looking at S_1 for ranging, ratios are calculated from the S_1 column and quantity:

  Smallest positive ratio = 20/(1/4) = 80

  Smallest negative ratio = 40/(-1/4) = -160

C_j		50	120	0	0	
	Solution Mix	X_1	X_2	S_1	S_2	Quantity
120	X_2	1/2	1	1/4	0	20
0	S_2	5/2	0	-1/4	1	40
	Z_j	60	120	30	0	2400
	$C_j - Z_j$	-10	0	-30	0	

Right-Hand-Side Ranging

- The *smallest positive ratio* (80 in this example) tells by how many hours the electricians' time resource can be *reduced* without altering the current solution mix.
- The RHS resource can be reduced by as much as 80 hours,
  - ✓ basically from the current 80 hours all the way down to 0 hours,

  without causing a basic variable to be pivoted out of the solution.
- The *smallest (in absolute value) negative ratio* (160) tells us the number of hours that can be added to the resource before the solution mix changes.
- Here, electricians' time can be increased by 160 hours, up to 240 (= 80 currently + 160 may be added) hours.
- The range of electricians' time over which the shadow price of $30 is proved valid.
  - ✓ That range is from 0 to 240 hours.

Right-Hand-Side Ranging

- The audio technician resource is slightly different in that all 60 hours of time originally available have not been used.
  - o S_2 = 40 hours in final tableau.
- Applying the ratio test, the number of audio technicians' hours can be reduced by only 40 (the smallest positive ratio = 40/1) before a shortage occurs.
- Since not all the hours currently available are being used, they can be increased indefinitely without altering the problem's solution.
  - o There are no negative substitution rates in the S_2 column, so there are no negative ratios.
- So, the valid range for *this* shadow price would be from 20 (= 60 40) hours to an unbounded upper limit.

Sensitivity Analysis by Computer

- Solver in Excel has the capability of producing sensitivity analysis that includes the shadow prices of resources.
- The ranging is performed by Excel and highlighted.
- No need for calculating smallest positive or negative ratios.
- Sensitivity is an option once the Solver program is run for a problem.
  - It is a Report option.

Steps to Form the Dual

- Every LP primal has a dual which provides useful economic information.

To form the dual, first express the primal as a maximization problem with all $\leq$ constraints.

- o The dual will be a minimization problem.
- o The right-hand-side values of the primal constraints become the objective coefficients of the dual.
- o The primal objective function coefficients become the right-hand-side of the dual constraints.
- o The transpose of the primal constraint coefficients become the dual constraint coefficients.
- o Constraint $\leq$ signs become $\geq$ signs.

Primal and Dual

$$\begin{rcases} \text{Max } Z = 3X_1 + 4X_2 \\ \text{s.t} \\ \text{I/} \quad X_1 + 2X_2 \leq 50 \\ \text{II/} \quad 3X_1 + X_2 \leq 30 \\ X_1, X_2 \geq 0 \end{rcases} \text{Primal}$$

$3 \times \text{I} + 0 \times \text{II}$, $\quad 3X_1 + 6X_2 \leq 150$

$Z = 3X_1 + 4X_2 \leq 3X_1 + 6X_2 \leq 150$, So, $Z \leq 150$

$0 \times \text{I} + 4 \times \text{II}$, $\quad 12X_1 + 4X_2 \leq 120$

$Z = 3X_1 + 4X_2 \leq 12X_1 + 4X_2 \leq 120$, So, $Z \leq 120$

$\text{I} + 2 \times \text{II}$, $\quad 7X_1 + 4X_2 \leq 110$

$Z = 3X_1 + 4X_2 \leq 7X_1 + 4X_2 \leq 110$, So, $Z \leq 110$

$y_1 \times \text{I} + y_2 \times \text{II}$

$$\begin{array}{r|cc} y_1 & 1 & 2 \\ + \; y_2 & 3 & 1 \\ \hline & \multicolumn{2}{c}{y_1 + 3y_2 \geq 3} \\ & \multicolumn{2}{c}{\dots y_2 \geq 4} \end{array}$$

$Z = 3X_1 + 4X_2$

looking for the min Value of RHS value

Primal and Dual

Primal

$$Max Z = \sum_j c_j x_j$$

$$\sum_j a_{ij} x_j \le b_i \quad \forall i$$

$$x_i \ge 0$$

Dual

$$Min Z = \sum_j b_i y_i$$

$$\sum_j a_{ij} y_i \ge c_j \quad \forall j$$

$$y_i \ge 0$$

Primal and Dual

Primal:

Max: $50X_1 + 120X_2$

Subject to:

$$2X_1 + 4X_2 \le 80$$

$$3X_1 + 1X_2 \le 60$$

Dual

Min: $80U_1 + 60U_2$

Subject to:

$$2U_1 + 3U_2 \ge 50$$

$$4U_1 + 1U_2 \ge 120$$

- **Primal**

Min $Z = 3\,x_1 + 2{,}5\,x_2$

$2\,x_1 + 4\,x_2 \geq 40$

$3\,x_1 + 2\,x_2 \geq 50$

$x_1, x_2 \geq 0$

- **Dual**

Max $Z = 40\,y_1 + 50\,y_2$

$2\,y_1 + 3\,y_2 \leq 3$

$4\,y_1 + 2y_2 \leq 2{,}5$

$y_1, y_2 \geq 0$

Comparison of the Primal and Dual Optimal Tableaus

Primal's Optimal Solution

C_j	Solution Mix	Quantity	$50	$120	$0	$0
			X_1	X_2	S_1	S_2
$7	X_2	20	1/2	1	1/4	0
$5	S_2	40	5/2	0	-1/4	1
	Z_j	$2,400	60	120	30	0
	$C_j - Z_j$		-10	0	-30	0

Dual's Optimal Solution

C_j	Solution Mix	Quantity	80	60	$0	$0	M	M
			X_1	X_2	S_1	S_2	A_1	A_2
$7	U_1	30	1	1/4	0	-1/4	0	1/2
$5	S_1	10	0	-5/2	1	-1/2	-1	1/2
	Z_j	$2,400	80	20	0	-20	0	40
	$C_j - Z_j$		$0	40	0	20	M	M-40

A maximum value of Z in a primal problem equals the minimum value of W in the dual problem.

Any pair of primal and dual problems can be converted to each other.

The dual of a dual problem always is the primal problem.

$$\begin{array}{ll} \text{Min} & 0.4x_1 + 0.5x_2 \\ \text{s.t.} & 0.3x_1 + 0.1x_2 \le 2.7 \\ & 0.5x_1 + 0.5x_2 = 6 \\ & 0.6x_1 + 0.4x_2 \ge 6 \\ & x_1 \ge 0, \quad x_2 \ge 0 \end{array}$$

$$\Downarrow$$

$$\begin{array}{lll} \text{Min} & 0.4x_1 + 0.5x_2 & \\ \text{s.t.} & -0.3x_1 - 0.1x_2 \ge -2.7 & [\mathrm{y}_1] \\ & 0.5x_1 + 0.5x_2 \ge 6 & [\mathrm{y}_2^+] \\ & -0.5x_1 - 0.5x_2 \ge -6 & [\mathrm{y}_2^-] \\ & 0.6x_1 + 0.4x_2 \ge 6 & [\mathrm{y}_3] \\ & x_1 \ge 0, \quad x_2 \ge 0 & \end{array}$$

Max $\quad -2.7y_1 + 6(y_2^+ - y_2^-) + 6y_3$

s.t.

$$-0.3y_1 + 0.5(y_2^+ - y_2^-) + 0.6y_3 \le 0.4$$

$$-0.1y_1 + 0.5(y_2^+ - y_2^-) + 0.4y_3 \le 0.5$$

$$y_1 \ge 0,\ y_2^+ \ge 0,\ y_2^- \ge 0,\ y_3 \ge 0.$$

$$\Downarrow$$

Max $\quad -2.7y_1 + 6y_2 + 6y_3$

s.t.

$$-0.3y_1 + 0.5y_2 + 0.6y_3 \le 0.4$$

$$-0.1y_1 + 0.5y_2 + 0.4y_3 \le 0.5$$

$$y_1 \ge 0,\ y_2 : \text{URS},\ y_3 \ge 0.$$

- **Primal**

Min Z = $3x_1 + 2{,}5x_2$

$2x_1 + 4x_2 \geq 40$

$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0$

		x_1	x_2	A1	S1	A2	S2	
Cj		3	2,5	M	0	M	0	
x_2	2,5	0	1	0,375	-0,375	-0,25	0,25	2,5
x_1	3	1	0	-0,25	0,25	0,5	-0,5	15
zj		3	2,5	-0,1875	0,1875	-0,875	0,875	51,25
cj-zj		0	0	0,1875	-0,1875	0,875	-0,875	

x_1=15, x_2=2,5, Z=51,25

shadow prices; S1=0,1875, S2=0,875

- **Dual**

Max Z = $40\,y_1 + 50\,y_2$

$2\,y_1 + 3\,y_2 \le 3$

$4\,y_1 + 2y_2 \le 2{,}5$

$y_1, y_2 \ge 0$

		x_1	x_2	S1	S2	
	Cj	40	50	0	0	
x_2	50	0	1	0,5	-0,25	0,875
x_1	40	1	0	-0,25	0,375	0,1875
zj		40	50	15	2,5	51,25
cj-zj		0	0	-15	-2,5	

x_1=0,1875, x_2=0,875, Z=51,25

shadow prices; S1=15, S2=2,5

The value of obj func in both is 51,25

Solution of primal x_1=15, x_2=2,5 are shadow prices in dual. Shadow prices in primal are solutions in dual

Primal vs Dual

- **If the solution of one the problem is infinity then there is no solution of the other problem**
- **Both problems may not have solutions**
- **The solutions of both problems can not be infinity at the same time**
- **If one of the problem has an optimal solution then the other one also has.**