



# Operations Research / Management Science

Operations Research (Management Science) is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources.

# Overview of Quantitative Analysis

## Quantitative Analysis:

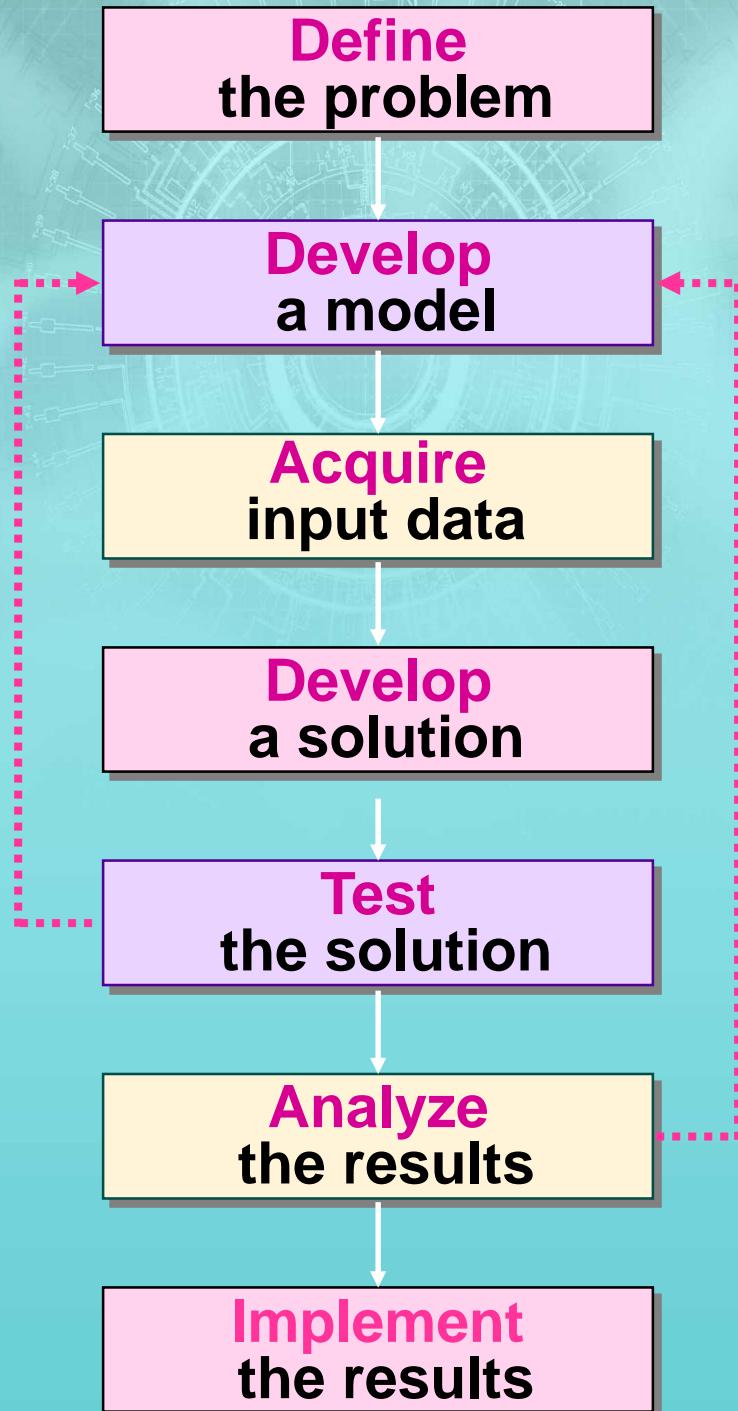
A scientific approach to managerial decision making whereby raw data are processed and manipulated resulting in meaningful information.



## Qualitative Factors:

Information that may be difficult to quantify but can affect the decision-making process such as the weather, state, and federal legislation.

# The QA Approach: Fig 1.1



# Define the Problem

## Problem Definition:

A clear and concise statement that gives direction and meaning to the subsequent QA steps and requires specific, measurable objectives.

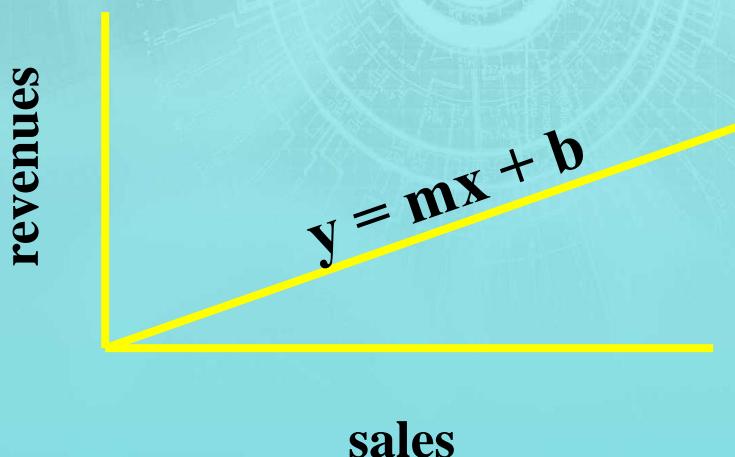
**THIS MAY BE THE MOST DIFFICULT STEP!**

**...because true problem causes must be identified and the relationship of the problem to other organizational processes must be considered.**

# Develop the Model

## Quantitative Analysis Model:

A realistic, solvable, and understandable mathematical statement showing the relationship between variables.



**Models contain both controllable (decision variables) and uncontrollable variables and parameters. Typically, parameters are known quantities (salary of sales force) while variables are unknown (sales quantity).**

# Acquire Data

## Model Data:

Accurate input data that may come from a variety of sources such as company reports, company documents, interviews, on-site direct measurement, or statistical sampling.

**Garbage In**

=

**Garbage Out**

# Develop a Solution

## Model Solution:

- The best model solution is found by manipulating the model variables until a practical and implemental solution is obtained.
- Manipulation can be done by solving the equation(s), trying various approaches (trial and error), trying all possible variables (complete enumeration), and/or implementing an algorithm (repeating a series of steps).

# Test the Solution

## Model Testing:

The collection of data from a different source to validate the accuracy and completeness and sensibility of both the model and model input data ~ consistency of results is key!

# Analyze the Results

## Results Analysis:

Understanding actions implied by the solution and their implications, as well as conducting a sensitivity analysis (a change to input values or the model) to evaluate the impact of a change in model parameters.

**Sensitivity analyses allow the “what-ifs” to be answered.**

# Implement the Results

## Results Implementation:

The incorporation of the solution into the company and the monitoring of the results.

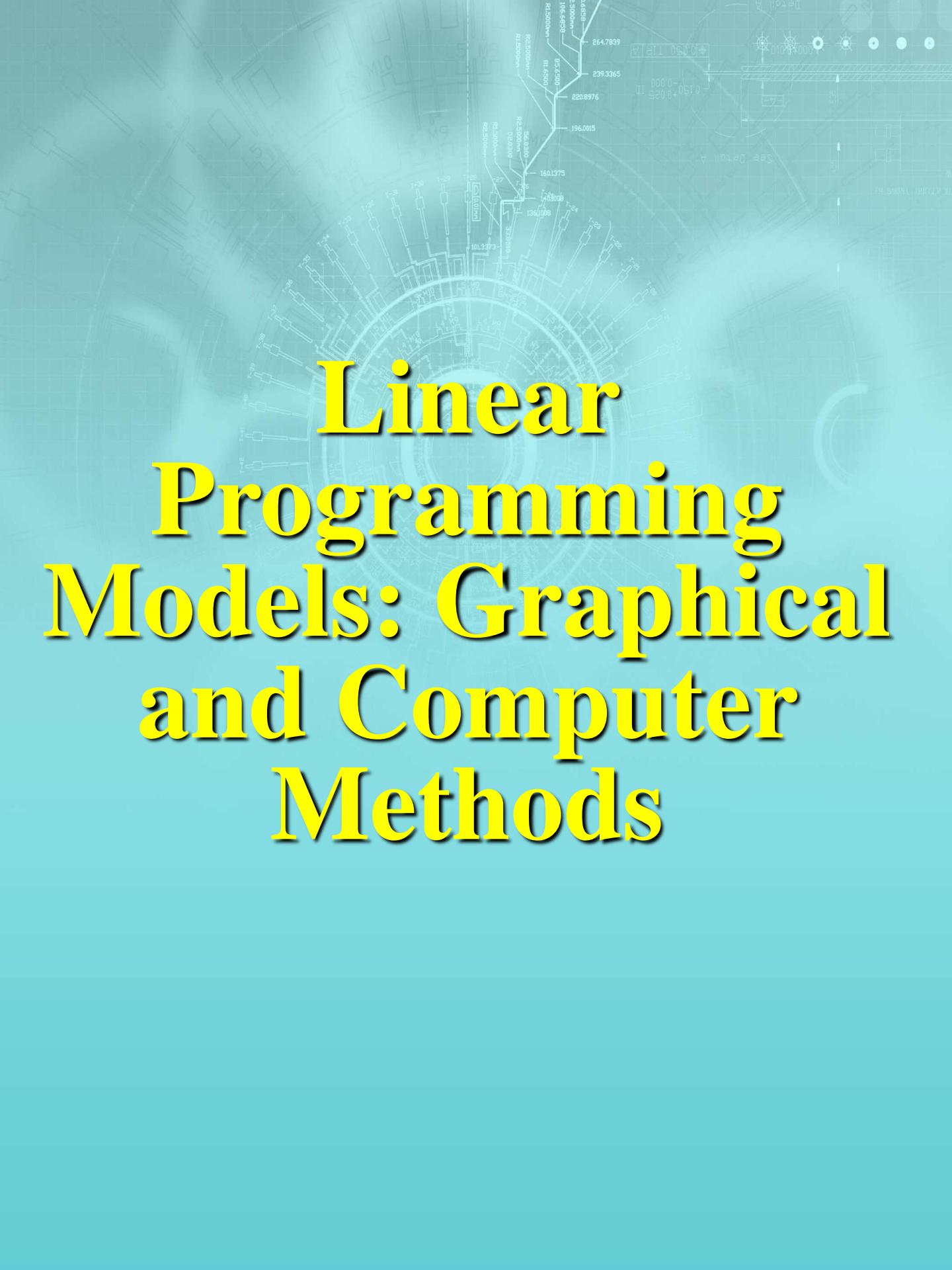
# Modeling in the Real World

Real World Models can be:

- *Complex,*
- *expensive, and*
- *difficult to sell.*

BUT...

*Real world models are used in the real world by real organizations to solve real problems!*



# Linear Programming Models: Graphical and Computer Methods

# Learning Objectives

**Students will be able to:**

1. Understand the basic assumptions and properties of linear programming (LP).
2. Graphically solve any LP problem that has only two variables by both the corner point and isoprofit line methods.
3. Understand special issues in LP such as infeasibility, unboundedness, redundancy, and alternative optimal solutions.
4. Understand the role of sensitivity analysis.
5. Use Excel spreadsheets to solve LP problems.

# Chapter Outline

- 7.1** Introduction
- 7.2** Requirements of a Linear Programming Problem
- 7.3** Formulating LP Problems
- 7.4** Graphical Solution to an LP Problem
- 7.5** Solving Flair Furniture's LP Problem using QM for Windows and Excel
- 7.6** Solving Minimization Problems
- 7.7** Four Special Cases in LP
- 7.8** Sensitivity Analysis

# Introduction

Linear programming (LP) is

- a widely used mathematical modeling technique
- designed to help managers in planning and decision making
- related to resource allocation.
- **LP** is a technique that helps in resource allocation decisions.

Programming refers to

- modeling and solving a problem mathematically.

# Examples of Successful LP Applications

1. Development of a production schedule that will
  - satisfy future demands for a firm's production
  - while *minimizing* total production and inventory costs
2. Selection of product mix in a factory to
  - make best use of machine-hours and labor-hours available
  - while *maximizing* the firm's products

# Examples of Successful LP Applications (*continued*)

3. Determination of grades of petroleum products to yield the *maximum* profit
4. Selection of different blends of raw materials to feed mills to produce finished feed combinations at *minimum* cost
5. Determination of a distribution system that will *minimize* total shipping cost from several warehouses to various market locations

# Requirements of a Linear Programming Problem

*All LP problems have 4 properties in common:*

- All problems seek to maximize or minimize some quantity (the objective function).
- The presence of restrictions or constraints limits the degree to which we can pursue our objective.
- There must be alternative courses of action to choose from.
- The objective and constraints in linear programming problems must be expressed in terms of linear equations or inequalities.

# 5 Basic Assumptions of Linear Programming

## 1. Certainty:

- numbers in the objective and constraints are known with certainty and do not change during the period being studied

## 2. Proportionality:

- exists in the objective and constraints
- constancy between production increases and resource utilization

## 3. Additivity:

- the total of all activities equals the sum of the individual activities

# 5 Basic Assumptions of Linear Programming

*(continued)*

## 4. Divisibility:

- solutions need not be in whole numbers (integers)
- solutions are divisible, and may take any fractional value

## 5. Non-negativity:

- all answers or variables are greater than or equal to ( $\geq$ ) zero
- negative values of physical quantities are impossible

## Linear Programming

$$\text{Max (Min)} Z = \sum c_i x_i$$

subject to

$$\sum a_i x_i \leq b_i$$

$$\begin{matrix} \\ \\ \geq \end{matrix}$$

$$x_i \geq 0, a_i, b_i \in \mathbb{R}$$

$$\text{Max } Z = 3x_1 + 4x_2$$

s.t

$$x_1 + x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

$$\underline{x_1 + x_2 + s_1 = 10 \Rightarrow x_2 = 10 - x_1 - s_1, s_1 \geq 0}$$

$$\text{so Max } Z = 3x_1 + 40 - 4x_1 - 4s_1$$

$$\text{Max } Z = 40 - x_1 - 4s_1 / \underline{\text{Max } Z = 40}, x_1 = s_1 = 0$$

$$x_2 = 10$$

$$\text{Min } Z = 3X_1 + 4X_2$$

s.t

$$X_1 + X_2 \geq 10, \quad X_1, X_2 \geq 0.$$

$$X_1 + X_2 - S_1 = 10 \Rightarrow X_1 = 10 - X_2 + S_1$$

$$\text{Min } Z = 30 - 3X_2 + 3S_1 + 4X_2 \Rightarrow 30 + X_2 + 3S_1$$

$$\text{Min } Z = 30, \quad X_2 = S_1 = 0, \quad X_1 = 10$$

$$\text{Max } Z = 3X_1 + 4X_2$$

s.t

$$X_1 + X_2 \leq 10$$

$$2X_1 + X_2 \leq 15$$

$$X_1, X_2 \geq 0.$$

I  
II

on I;

$$X_2 = 10 - X_1 - S_1$$

submit into (II) and Max Z

$$\text{Max } Z = 3X_1 + 40 - 4X_1 - 4S_1 \Rightarrow \text{Max } Z = 40 - X_1 - 4S_1$$

$$2X_1 + 10 - X_1 - S_1 + S_2 = 15 \Rightarrow X_1 - S_1 + S_2 = 5$$

$$\text{Max } Z = 40 - X_1 - S_1 + S_2 \Rightarrow 40 - 5 = 35$$

$$\text{Max } Z = 40, \quad X_1 = S_1 = 0.$$

$$X_2 = 10, \quad S_2 = 5$$

# Formulating Linear Programming Problems

- Formulating a linear program involves developing a mathematical model to represent the managerial problem.
- Once the managerial problem is understood, begin to develop the mathematical statement of the problem.
- The steps in formulating a linear program follow on the next slide.

# Formulating Linear Programming Problems

*(continued)*

## Steps in LP Formulations

- 1.** Completely understand the managerial problem being faced.
- 2.** Identify the objective and the constraints.
- 3.** Define the decision variables.
- 4.** Use the decision variables to write mathematical expressions for the objective function and the constraints.

# Formulating Linear Programming Problems

*(continued)*

## The *Product Mix Problem*

- Two or more products are usually produced using limited resources such as
  - personnel, machines, raw materials, and so on.
- The profit that the firm seeks to maximize is based on the profit contribution per unit of each product.
- The company would like to determine how many units of each product it should produce so as to maximize overall profit given its limited resources.
- A problem of this type is formulated in the following example on the next slide.

# Flair Furniture Company

## Data - Table 7.1

### Hours Required to Produce One Unit

Department	$T$ Tables	$C$ Chairs	Available Hours This Week
• Carpentry	4	3	240
• Painting & Varnishing	2	1	100
<b>Profit per unit</b>	<b>\$7</b>	<b>\$5</b>	

*Identify the objective and constraints:*

Maximize Profit

Subject to:

Hours of carpentry time used  $\leq$  240 hrs./per week

Hours of paint. & varnishing used  $\leq$  100 hrs./wk.

# Flair Furniture Company

## Data - Table 7.1

*Identify the objective and constraints:*

Maximize Profit

Subject to:

Hours of carpentry time used  $\leq$  240 hrs.per week

Hours of paint. & varnishing used  $\leq$  100 hrs./wk

*Define decision variables:*

Let  $T$  = number of tables produced each week

$C$  = number of chairs produced each week

# Flair Furniture Company

## Data - Table 7.1

### Hours Required to Produce One Unit

Department	T Tables	C Chairs	Available Hours This Week
• Carpentry	4	3	240
• Painting & Varnishing	2	1	100
<b>Profit per unit</b>	<b>\$7</b>	<b>\$5</b>	

### *Mathematical formulation:*

$$\text{Max. profit (z)} = 7T + 5C$$

$$\text{Subject to: } 4T + 3C \leq 240 \text{ (Carpentry)}$$

$$2T + 1C \leq 100 \text{ (Paint & Varnishing)}$$

$$T \geq 0 \quad (1^{\text{st}} \text{ nonnegative cons})$$

$$C \geq 0 \quad (2^{\text{nd}} \text{ nonnegative cons})$$

# Flair Furniture Company Constraints

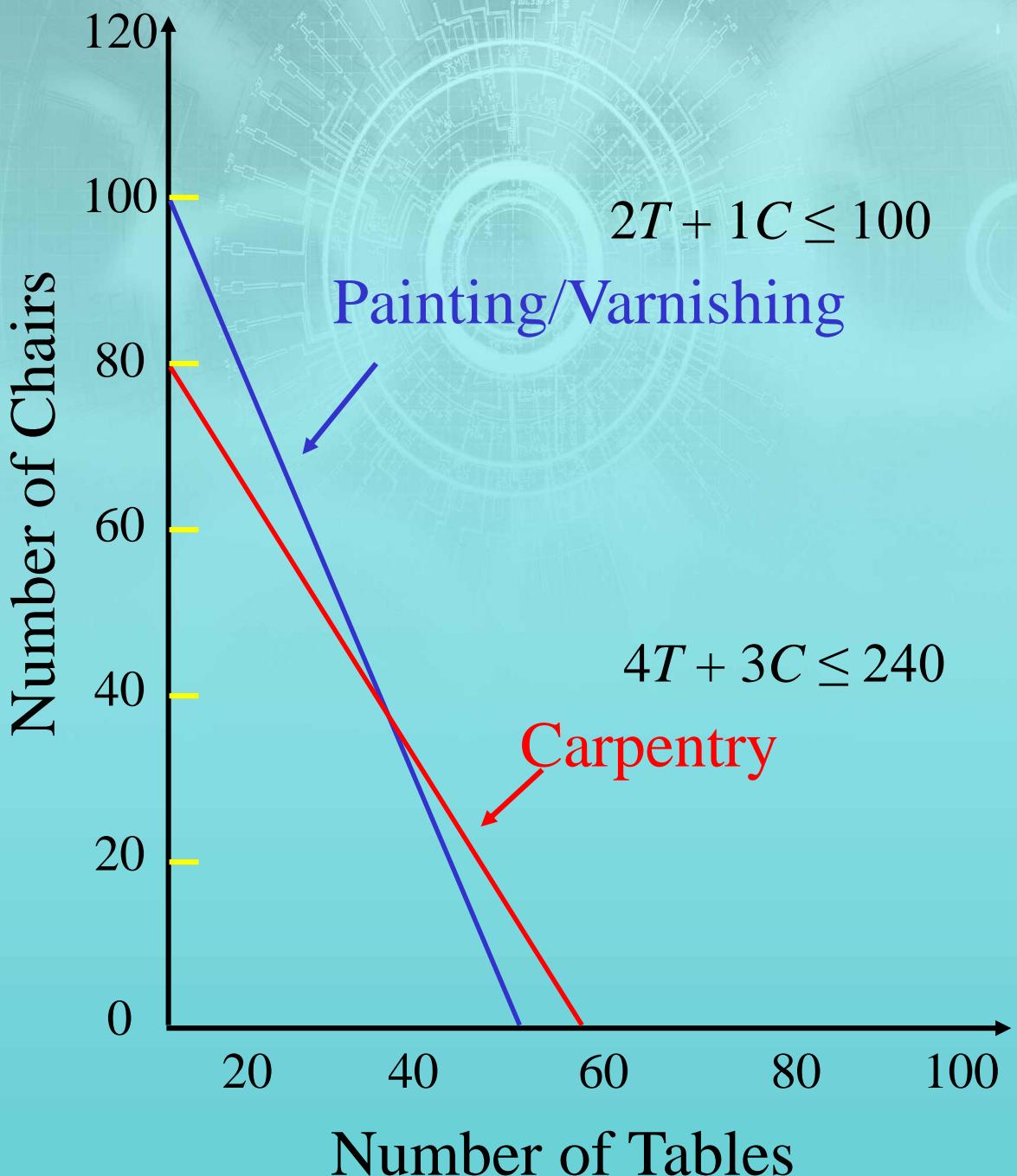
The easiest way to solve a small LP problem, such as that of the Flair Furniture Company, is with the *graphical solution approach*.

The graphical method works only when there are **two** decision variables, but it provides valuable insight into how larger problems are structured.

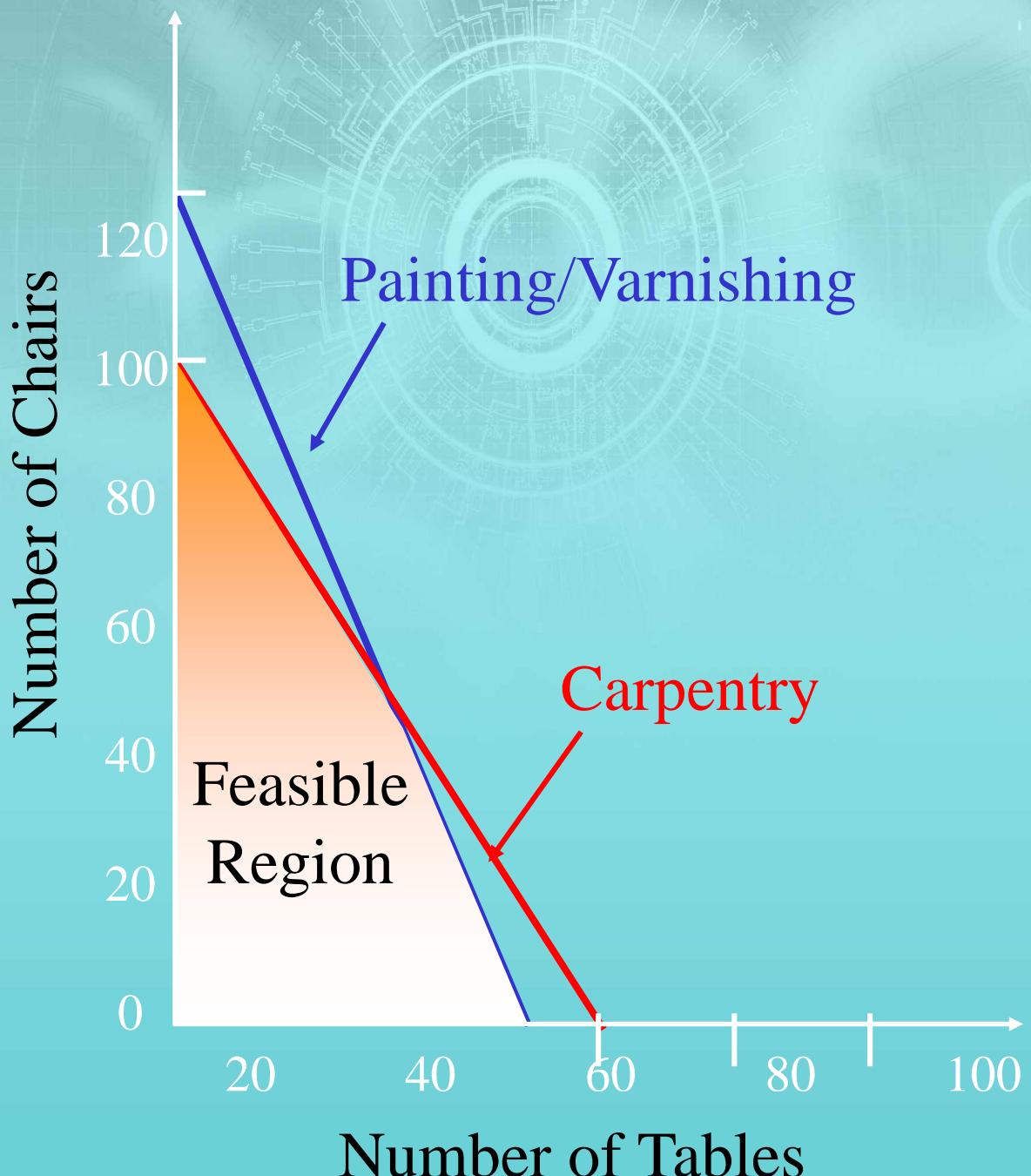
When there are more than two variables, it is not possible to plot the solution on a two-dimensional graph; a more complex approach is needed.

But the graphical method is invaluable in providing us with insights into how other approaches work.

# Flair Furniture Company Constraints



# Flair Furniture Company Feasible Region



# Isoprofit Lines Steps

1. Graph all constraints and find the feasible region.
2. Select a specific profit (or cost) line and graph it to find the slope.
3. Move the objective function line in the direction of increasing profit (or decreasing cost) while maintaining the slope. The last point it touches in the feasible region is the optimal solution.
4. Find the values of the decision variables at this last point and compute the profit (or cost).

# Flair Furniture Company Isoprofit Lines

## Isoprofit Line Solution Method

- Start by letting profits equal some arbitrary but small dollar amount.
- Choose a profit of, say, \$210.
  - This is a profit level that can be obtained easily without violating either of the two constraints.
- The objective function can be written as  $\$210 = 7T + 5C$ .

# Flair Furniture Company Isoprofit Lines

## Isoprofit Line Solution Method

- The objective function is just the equation of a line called an *isoprofit line*.
  - It represents all combinations of  $(T, C)$  that would yield a total profit of \$210.
- To plot the profit line, proceed exactly as done to plot a constraint line:
  - First, let  $T = 0$  and solve for the point at which the line crosses the  $C$  axis.
  - Then, let  $C = 0$  and solve for  $T$ .
  - ✓  $\$210 = \$7(0) + \$5(C)$
  - ✓  $C = 42$  chairs
- Then, let  $C = 0$  and solve for  $T$ .
  - ✓  $\$210 = \$7(T) + \$5(0)$
  - ✓  $T = 30$  tables

# Flair Furniture Company Isoprofit Lines

## Isoprofit Line Solution Method

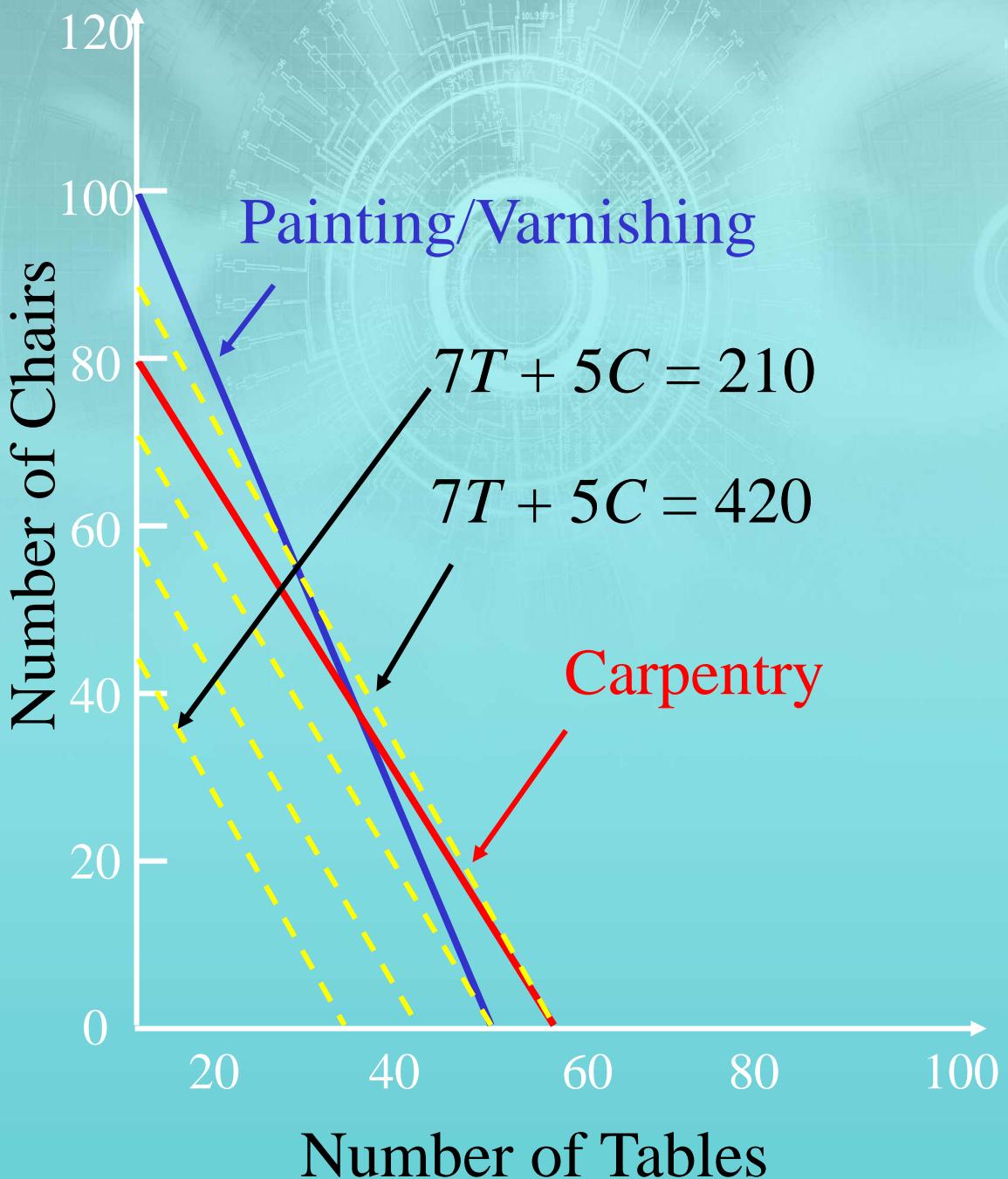
- Next connect these two points with a straight line. This profit line is illustrated in the next slide.
- All points on the line represent feasible solutions that produce an approximate profit of \$210
- Obviously, the isoprofit line for \$210 does not produce the highest possible profit to the firm.
- Try graphing more lines, each yielding a higher profit.
- Another equation,  $\$420 = \$7T + \$5C$ , is plotted in the same fashion as the lower line.

# Flair Furniture Company Isoprofit Lines

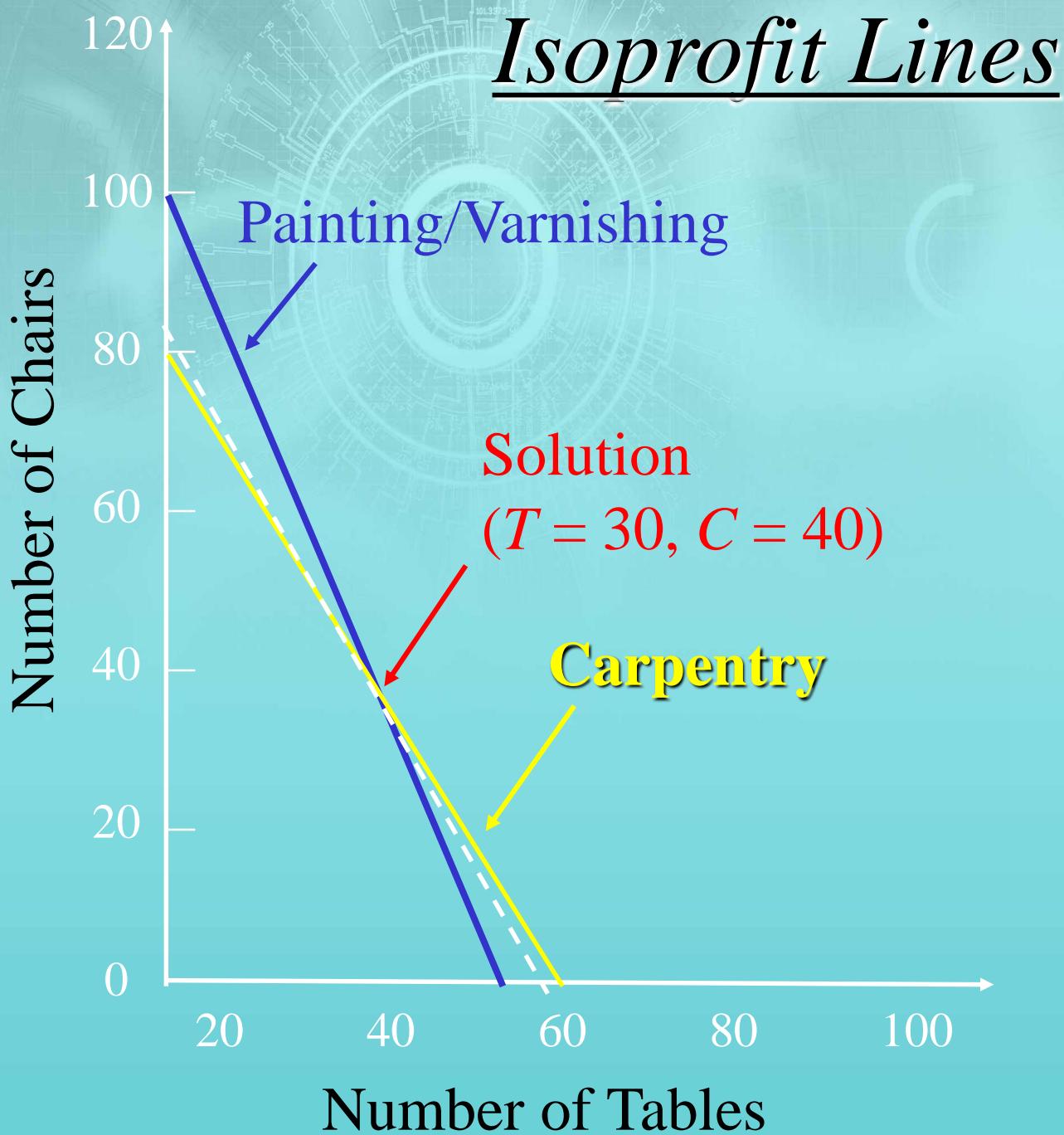
## Isoprofit Line Solution Method

- When  $T = 0$ ,
  - ✓  $\$420 = \$7(0) + 5(C)$
  - ✓  $C = 84$  chairs
- When  $C = 0$ ,
  - ✓  $\$420 = \$7(T) + 5(0)$
  - ✓  $T = 60$  tables
- This line is too high to be considered as it no longer touches the feasible region.
- The highest possible isoprofit line is illustrated in the second following slide. It touches the tip of the feasible region at the corner point ( $T = 30, C = 40$ ) and yields a profit of \$410.

# Flair Furniture Company Isoprofit Lines



# Flair Furniture Company Optimal Solution



# Flair Furniture Company

## Corner Point

### Corner Point Solution Method

- A second approach to solving LP problems
- It involves looking at the profit at every corner point of the feasible region
- The mathematical theory behind LP is that the optimal solution must lie at one of the corner points in the feasible region

# Corner Point

## Corner Point Solution Method, Summary

- 1.** Graph all constraints and find the feasible region.
- 2.** Find the corner points of the feasible region.
- 3.** Compute the profit (or cost) at each of the feasible corner points.
- 4.** Select the corner point with the best value of the objective function found in step 3. This is the optimal solution.

# Flair Furniture Company

## Corner Point

### Corner Point Solution Method

- The feasible region for the Flair Furniture Company problem is a four-sided polygon with four corner, or extreme, points.
- These points are labeled 1 ,2 ,3 , and 4 on the next graph.
- To find the  $(T, C)$  values producing the maximum profit, find the coordinates of each corner point and test their profit levels.

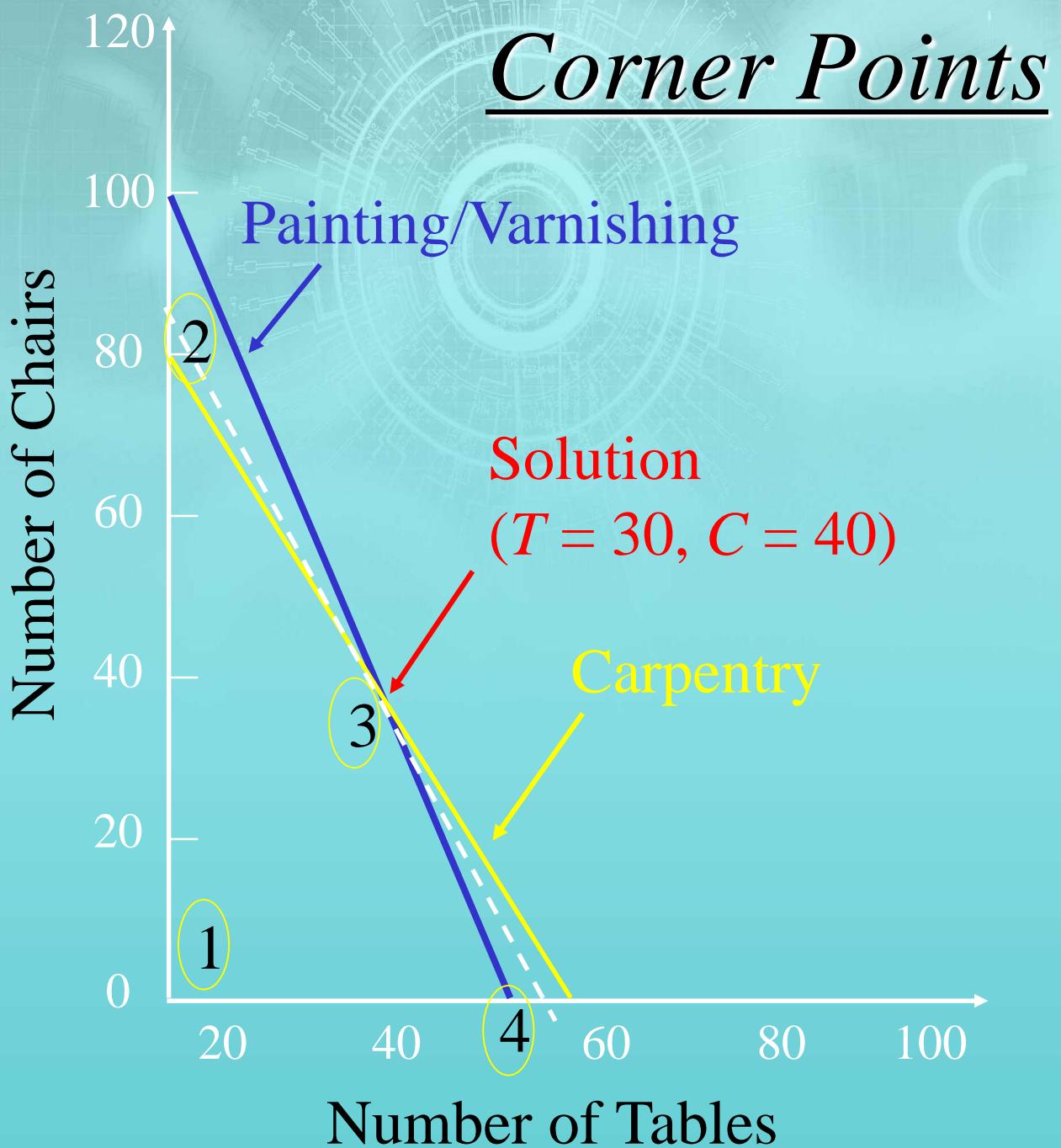
Point 1: $(T = 0, C = 0)$  profit =  $\$7(0) + \$5(0) = \$0$

Point 2: $(T = 0, C = 80)$  profit =  $\$7(0) + \$5(80) = \$400$

Point 3: $(T = 30, C = 40)$  profit =  $\$7(30) + \$5(40) = \$410$

Point 4 :  $(T = 50, C = 0)$  profit =  $\$7(50) + \$5(0) = \$350$

# Flair Furniture Company Optimal Solution



# Flair Furniture - QM for Windows

To use QM for Windows,

1. Select the Linear Programming module.
2. Then specify
  - the number of constraints (other than the non-negativity constraints, as it is assumed that the variables must be nonnegative),
  - the number of variables, and
  - whether the objective is to be maximized or minimized.

For the Flair Furniture Company problem, there are two constraints and two variables.

3. Once these numbers are specified, the input window opens as shown on the next slide.

# Flair Furniture - QM for Windows (*continued*)

The screenshot shows the QM for Windows software interface. The title bar reads "QM for Windows - [Data Table]". The menu bar includes File, Edit, View, Module, Format, Tools, Window, and Help. On the left, there is an "Objective" section with radio buttons for Maximize (selected) and Minimize. To the right, an "Instruction" field says "Enter the name for this variable. Almost any character is permissible." Below this is a data table titled "(untitled)". The table has columns for X1, X2, and RHS. It contains three rows: one for the objective function (Maximize) with values 0, 0, and an empty cell for RHS; one for Constraint 1 with values 0, 0, and <=; and one for Constraint 2 with values 0, 0, and <=.

	X1	X2	RHS
Maximize	0	0	
Constraint 1	0	0	<=
Constraint 2	0	0	<=

4. Next, the coefficients for the objective function and the constraints can be entered.
- Placing the cursor over the X1 or X2 and typing a new name such as Tables and Chairs will change the variable names.
  - The constraint names can be similarly modified.
  - When you select the **Solve** button, you get the output shown in the next slide.

# Flair Furniture - QM for Windows (*continued*)

Linear Programming

File Edit View Module Tables Window Help



Print Screen Step Edit Data

Instruction: There are more results available in additional windows. These may be opened by double clicking or using the WINDOW option in the Main Menu.

Objective

Maximize  
 Minimize

Linear Programming Results

Flair Furniture Company Solution

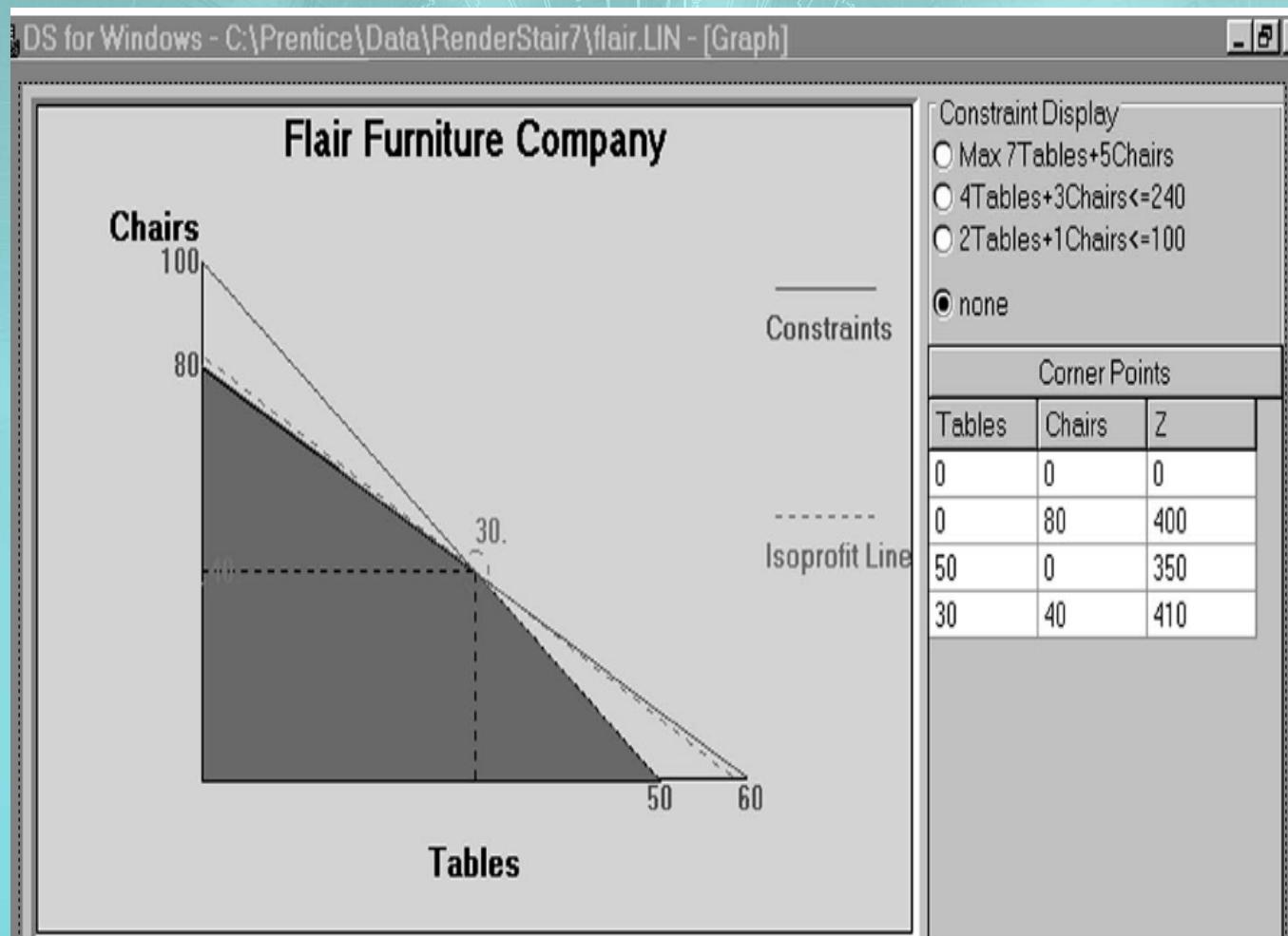
	Tables	Chairs	RHS	Dual	
Maximize	7.	5.			
Carpentry	4.	3.	$\leq$	240.	1.5
Painting	2.	1.	$\leq$	100.	0.5
Solution→	30.	40.		410.	

# Flair Furniture - QM for Windows (*continued*)

5. Modify the problem by selecting the **Edit** button and returning to the input screen to make any desired changes.
6. Once these numbers are specified, the input window opens as shown on the next slide.
7. Once the problem has been solved, a graph may be displayed by selecting Window—Graph from the menu bar in QM for Windows.
8. The next slide shows the output for the graphical solution.

*Notice that in addition to the graph, the corner points and the original problem are also shown.*

# Flair Furniture - QM for Windows (*continued*)



# Flair Furniture – Solver in Excel

To use Solver, open an Excel sheet and:

1. Enter the variable names and the coefficients for the objective function and constraints.
2. Specify cells where the values of the variables will be located. The solution will be put here.
3. Write a formula to compute the value of the objective function. The SUMPRODUCT function is helpful with this.
4. Write formulas to compute the left-hand sides of the constraints. Formulas may be copied and pasted to these cells.
5. Indicate constraint signs ( $\leq$ ,  $=$ , and  $\geq$ ) for display purposes only. The actual signs must be entered into Solver later, but having these displayed in the spreadsheet is helpful.
6. Input the right-hand side values for each constraint.

# Flair Furniture – Solver in Excel (*continued*)

Once the problem is entered in an Excel sheet, follow these steps to use Solver:

1. In Excel, select **Tools—Solver**.
  - If Solver does not appear on the menu under **Tools**, select **Tools—Add-ins** and then check the box next to **Solver Add-in**. Solver then will display in the Tools drop-down menu.
2. Once Solver has been selected, a window will open for the input of the Solver Parameters. Move the cursor to the **Set Target Cell** box and fill in the cell that is used to calculate the value of the objective function.
3. Move the cursor to the **By Changing Cells** box and input the cells that will contain the values for the variables.
4. Move the cursor to the **Subject to the Constraints** box, and then select **Add**.

# Flair Furniture – Solver in Excel (*continued*)

Solver steps, continued:

5. The **Cell Reference** box is for the range of cells that contain the left-hand sides of the constraints.
6. Select the  $\leq$  to change the type of constraint if necessary. Since this setting is the default, no change is necessary.
  - If there had been some  $\geq$  or  $=$  constraints in addition to the  $\leq$  constraints, it would be best to input all of one type [e.g.,  $\leq$ ] first, and then select Add to input the other type of constraint.
7. Move the cursor to the Constraint box to input the right-hand sides of the constraints. Select **Add** to finish this set of constraints and begin inputting another set, or select **OK** if there are no other constraints to add.

# Flair Furniture – Solver in Excel (*continued*)

Solver steps, continued:

8. From the Solver Parameters window, select **Options** and check **Assume Linear Model** and check **Assume Non-negative**; then click **OK**.
9. Review the information in the Solver window to make sure it is correct, and click **Solve**.
10. The Solver Solutions window is displayed and indicates that a solution was found. The values for the variables, the objective function, and the slacks are shown also.
11. Select **Keep Solver Solution** and the values in the spreadsheet will be kept at the optimal solution.
12. You may select what sort of additional information (e.g., **Sensitivity**) is to be presented from the reports Window (discussed later). You may select any of these and select **OK** to have these automatically generated.

# Flair Furniture – Solver in Excel (*continued*)

- On the next slide is an example of the type of worksheet used in Excel for Solver.
- The worksheet demonstrates some of the steps previously discussed.

# Flair Furniture – Example Solver Worksheet

Microsoft Excel - LP.XLS

File Edit View Insert Format Tools Data Window Help

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	A	B	C	D	E	F	G	H	I	J
1	Formulas used in spreadsheet solution to Flair Furniture Company									
2	Objective Function	Variables	Variables/ Solution	Constraints	Constants					
3										
4	=7*C4+5*C5	Tables		=4*C4+3*C5	240					
5		Chairs		=2*C4+1*C5	100					
6				=C4	0					
7				=C5	0					
8										
9	Solver Parameters									
10	Set Target Cell: \$A\$4 <input type="button" value="Solve"/> Equal to: <input checked="" type="radio"/> Max <input type="radio"/> Min <input type="radio"/> Value of: 0 <input type="button" value="Close"/> By Changing Cells: \$C\$4:\$C\$5 <input type="button" value="Guess"/> Subject to the Constraints: \$D\$4 <= \$E\$4 <input type="button" value="Add..."/> \$D\$5 <= \$E\$5 <input type="button" value="Change..."/> \$D\$6 >= 0 <input type="button" value="Delete"/> \$D\$7 >= 0 <input type="button" value="Reset All"/> <input type="button" value="Options..."/> <input type="button" value="Help"/>									
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24	Sheet1	Sheet2	Sheet3	Sheet4	Sheet5	Sheet6	Sheet7	Sheet		

For Help on dialog settings, press F1

# Solving Minimization Problems

Many LP problems minimize an objective, such as cost, instead of maximizing a profit function. For example,

- A restaurant may wish to develop a work schedule to meet staffing needs while minimizing the total number of employees.

**Or,**

- A manufacturer may seek to distribute its products from several factories to its many regional warehouses in such a way as to minimize total shipping costs.

**Or,**

- A hospital may want to provide a daily meal plan for its patients that meets certain nutritional standards while minimizing food purchase costs.

# Solving Minimization Problems

Minimization problems can be solved graphically by

- first setting up the feasible solution region and then using either
- the corner point method or
- an isocost line approach (which is analogous to the isoprofit approach in maximization problems)

to find the values of the decision variables (e.g.,  $X_1$  and  $X_2$ ) that yield the minimum cost.

## Holiday Meal Turkey Ranch example

- The Holiday Meal Turkey Ranch is considering buying two different brands of turkey feed and blending them to provide a good, low-cost diet for its turkeys. Each feed contains, in varying proportions, some or all of the three nutritional ingredients essential for fattening turkeys. Each pound of brand 1 purchased, for example, contains 5 ounces of ingredient A, 4 ounces of ingredient B, and 0.5 ounce of ingredient C. Each pound of brand 2 contains 10 ounces of ingredient A, 3 ounces of ingredient B, but no ingredient C. The minimum monthly requirement per turkey is 90 ounces of ingredient A, 48 ounces of ingredient B and 1.5 ounces of ingredient C. The brand 1 feed costs the ranch 2 cents a pound, while the brand 2 feed costs 3 cents a pound. The owner of the ranch would like to use LP to determine the lowest-cost diet that meets the minimum monthly intake requirement for each nutritional ingredient

	<b>Brand1</b>	<b>Brand2</b>	<b>Requirem net</b>
A	5	10	90 ounce
B	4	3	48 ounce
C	0,5	-	1,5 ounce
Cost of pound	2 cent	3 cent	

$X_1$  = # of pounds of brand 1 feed purchased

$X_2$  = # of pounds of brand 2 feed purchased

Pound=453gr

Ounce=28gr

# Solving Minimization Problems

*Holiday Meal Turkey Ranch example*

**Minimize:**     $2X_1 + 3X_2$

**Subject to:**

$$5X_1 + 10X_2 \geq 90 \text{ oz.} \quad (\text{A})$$

$$4X_1 + 3X_2 \geq 48 \text{ oz.} \quad (\text{B})$$

$$\frac{1}{2}X_1 \geq 1\frac{1}{2} \text{ oz.} \quad (\text{C})$$

$$X_1, X_2 \geq 0 \quad (\text{D})$$

where,

$X_1$  = # of pounds of brand 1 feed purchased

$X_2$  = # of pounds of brand 2 feed purchased

(A) = ingredient A constraint

(B) = ingredient B constraint

(C) = ingredient C constraint

(D) = non-negativity constraints

# Holiday Meal Turkey Ranch

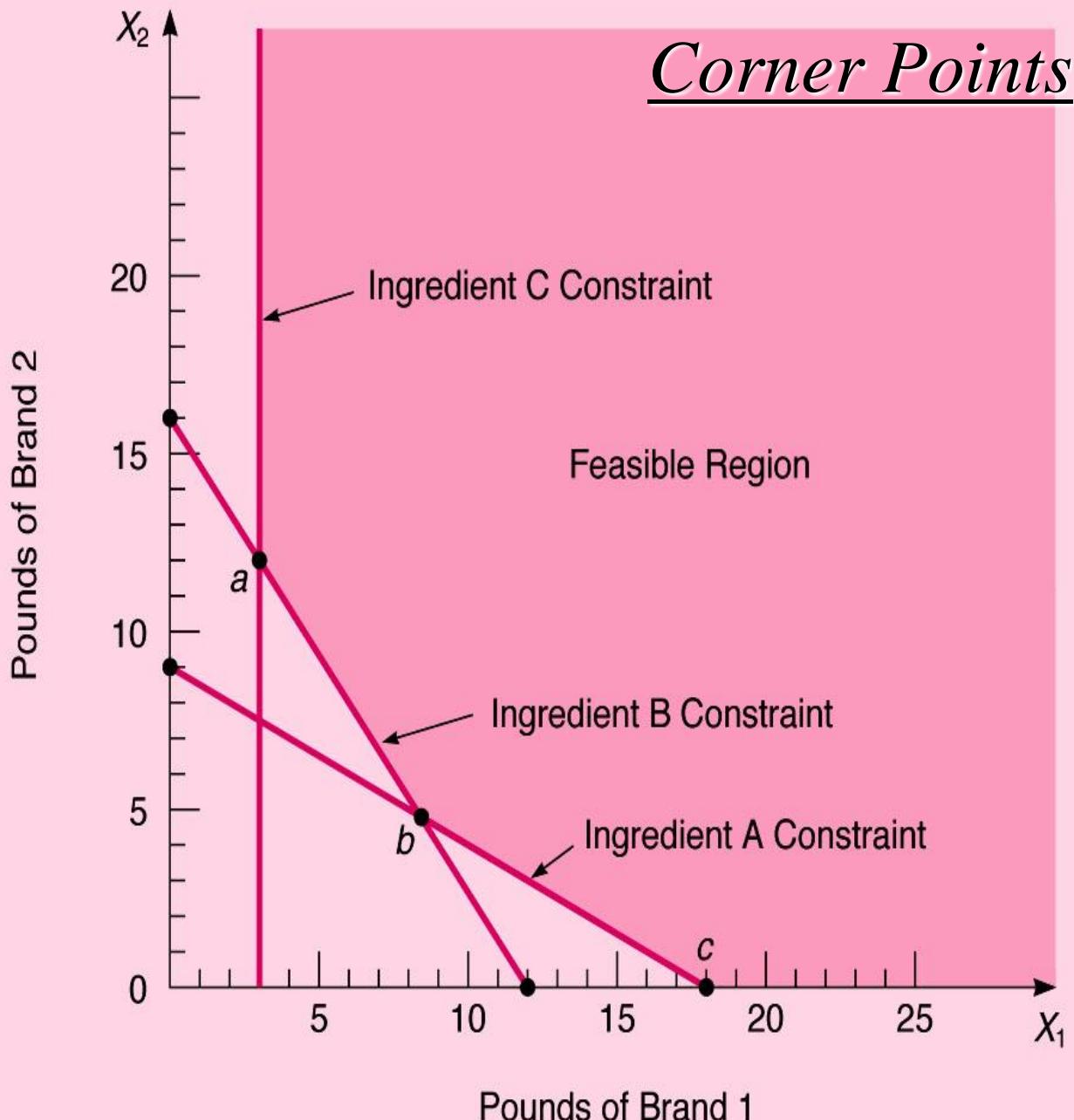
## *Using the Corner Point Method*

To solve this problem:

1. Construct the feasible solution region.
  - This is done by plotting each of the three constraint equations.
2. Find the corner points.
  - This problem has 3 corner points, labeled a, b, and c.
  - Minimization problems are often unbound outward (i.e., to the right and on top), but this causes no difficulty in solving them.
  - As long as they are bounded inward (on the left side and the bottom), corner points may be established.
  - The optimal solution will lie at one of the corners as it would in a maximization problem.

# Holiday Meal Turkey Problem

## Corner Points



# Solving Minimization Problems

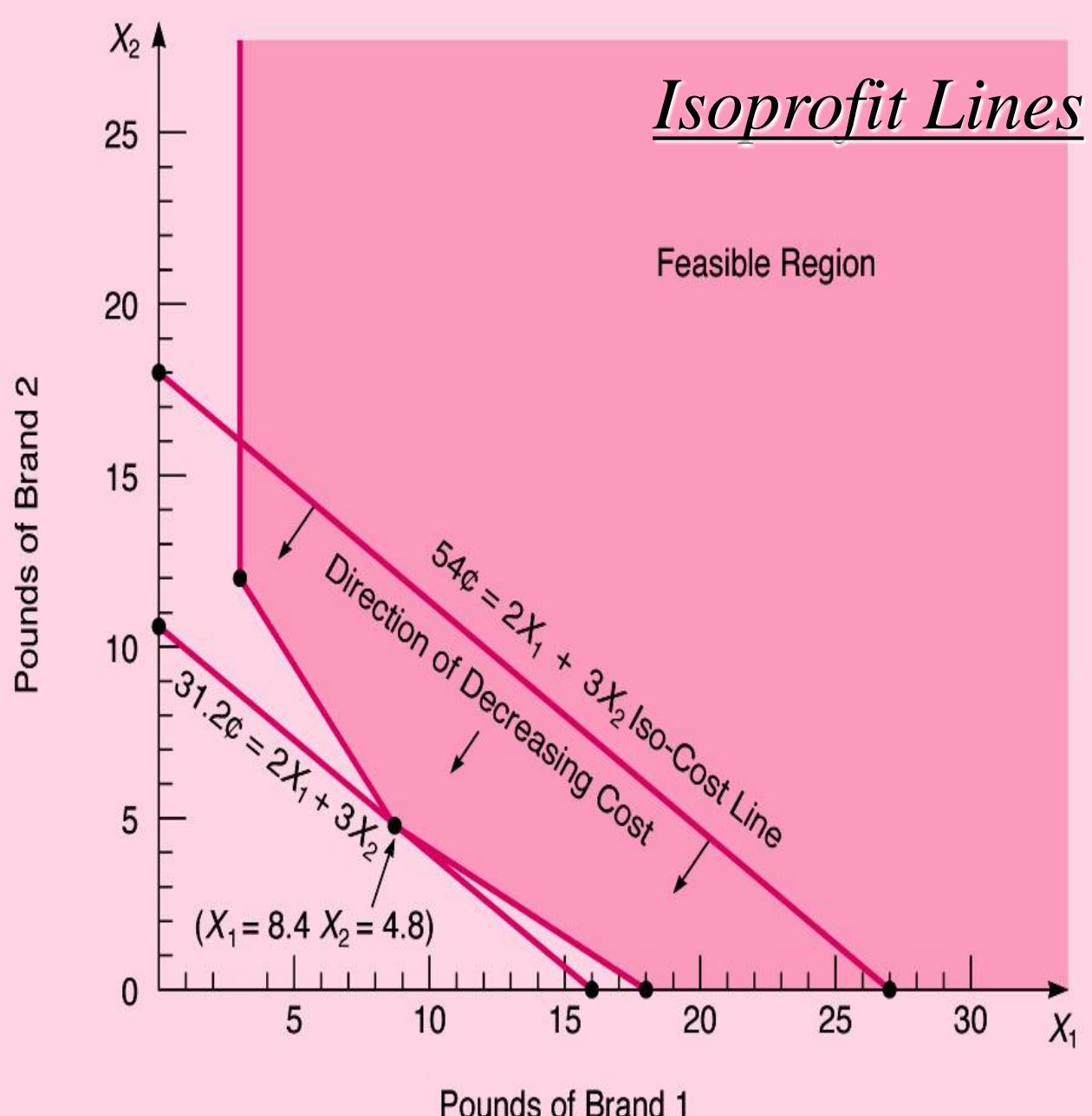
## *Using the Isocost Line Approach*

- As with isoprofit lines, there is no need to compute the cost at each corner point, but instead draw a series of parallel cost lines.
  - The lowest cost line (i.e., the one closest in toward the origin) to touch the feasible region provides the optimal solution corner.
1. Start, for example, by drawing a 54-cent cost line, namely  $54 = 2X_1 + 3X_2$ .
    - Obviously, there are many points in the feasible region that would yield a lower total cost.

# Solving Minimization Problems (*continued*)

2. Proceed to move the isocost line toward the lower left, in a plane parallel to the 54-cent solution line.
3. The last point touched while still in contact with the feasible region is the same as corner point *b* of the Corner Point diagram in the previous slide.
  - It has the coordinates ( $X_1 = 8.4$ ,  $X_2 = 4.8$ ) and an associated cost of 31.2 cents. These are found by solving the two equations for  $X_1$  and  $X_2$ .

# Holiday Meal Turkey Problem



# Special Cases in LP

Four special cases and difficulties arise at times when solving LP problems:

## 1. Infeasibility:

- *lack of a feasible solution region can occur if constraints conflict with one another.*

## 2. Unbounded Solutions:

- *when the objective function in a maximization problem can be infinitely large, the problem is unbounded and is missing one or more constraints.*

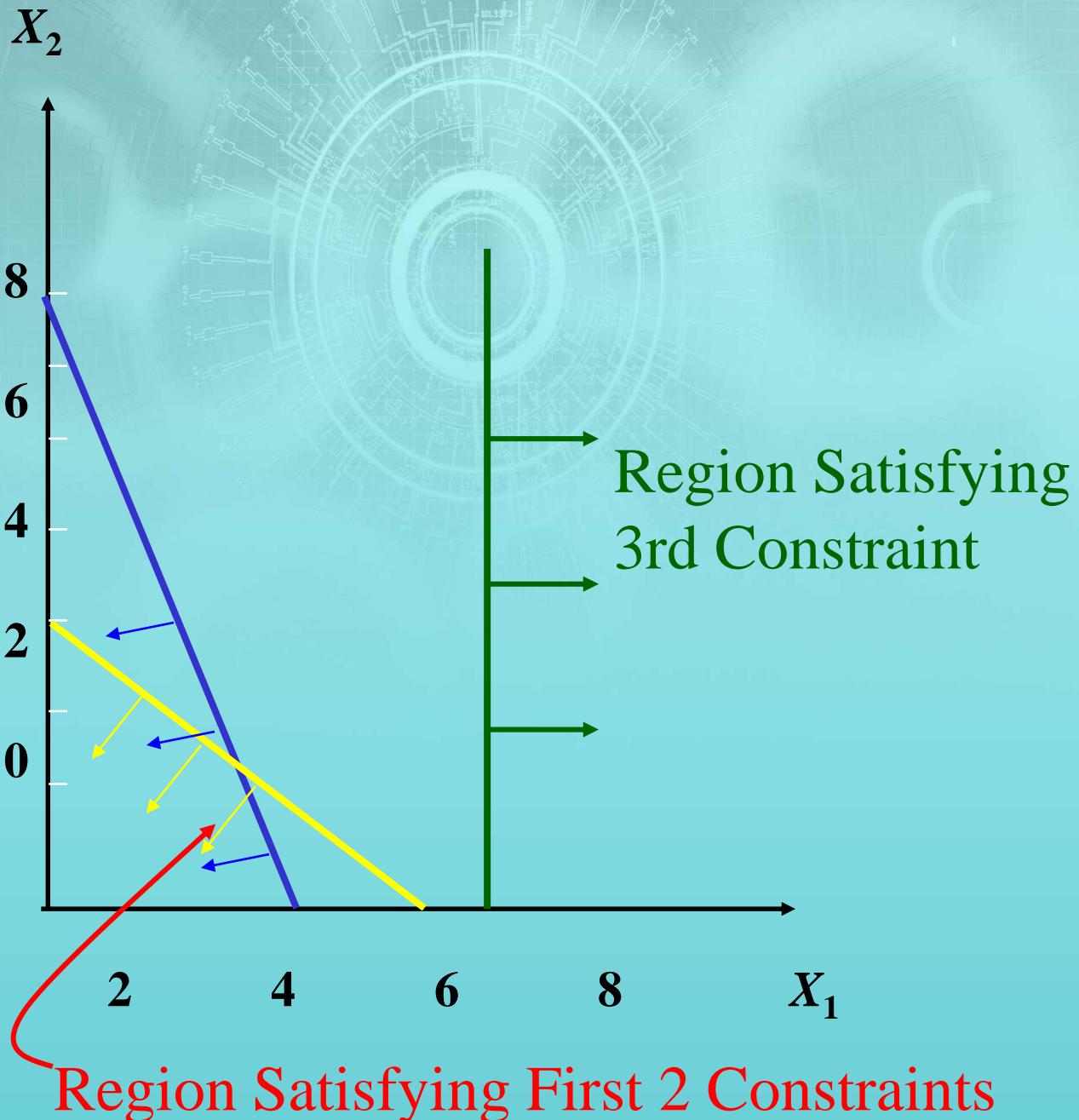
## 3. Redundancy:

- *a redundant constraint is one that does not affect the feasible solution region.*

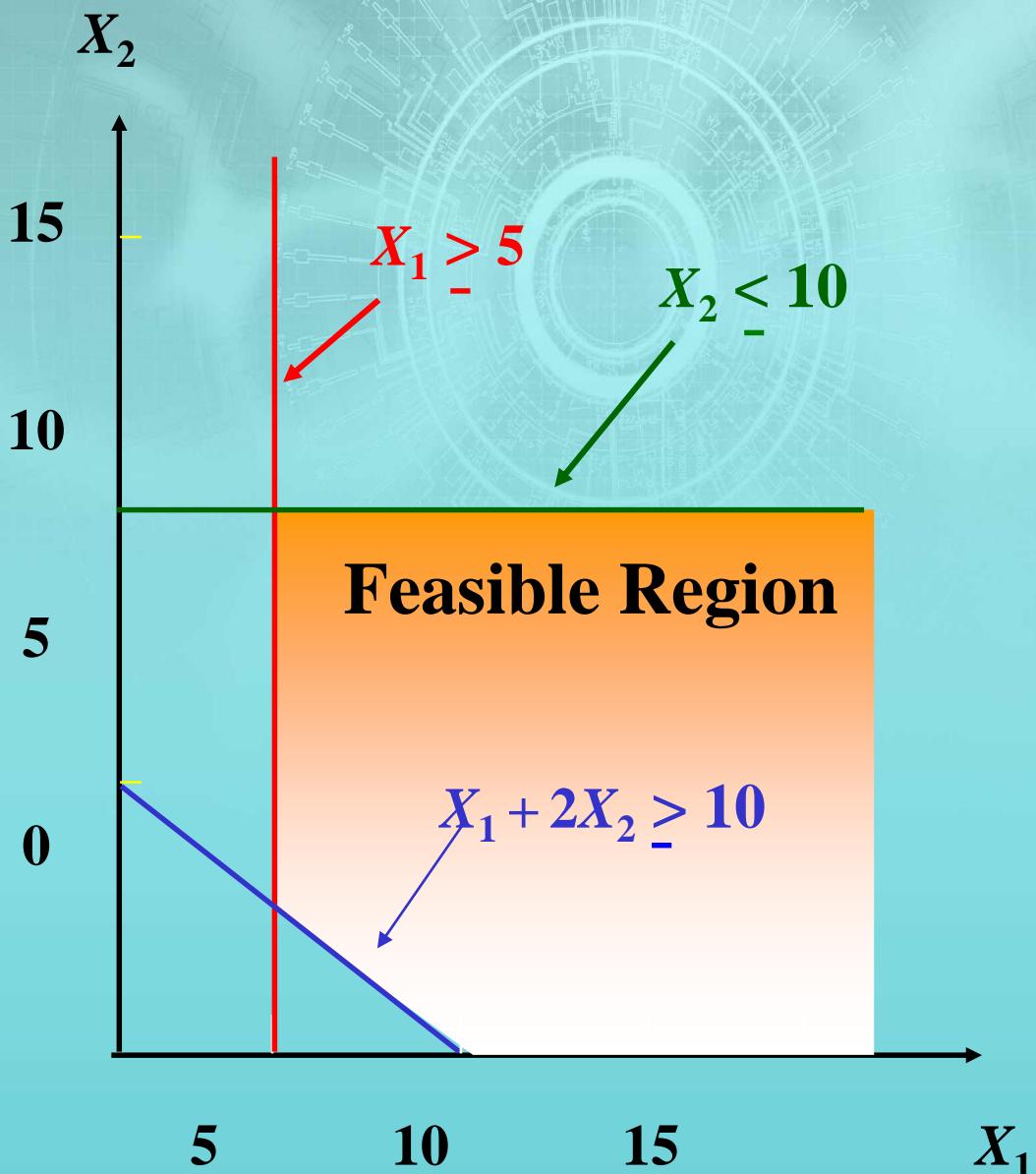
## 4. More than One Optimal Solution:

- *two or more optimal solutions may exist, and this actually allows management great flexibility in deciding which combination to select.*

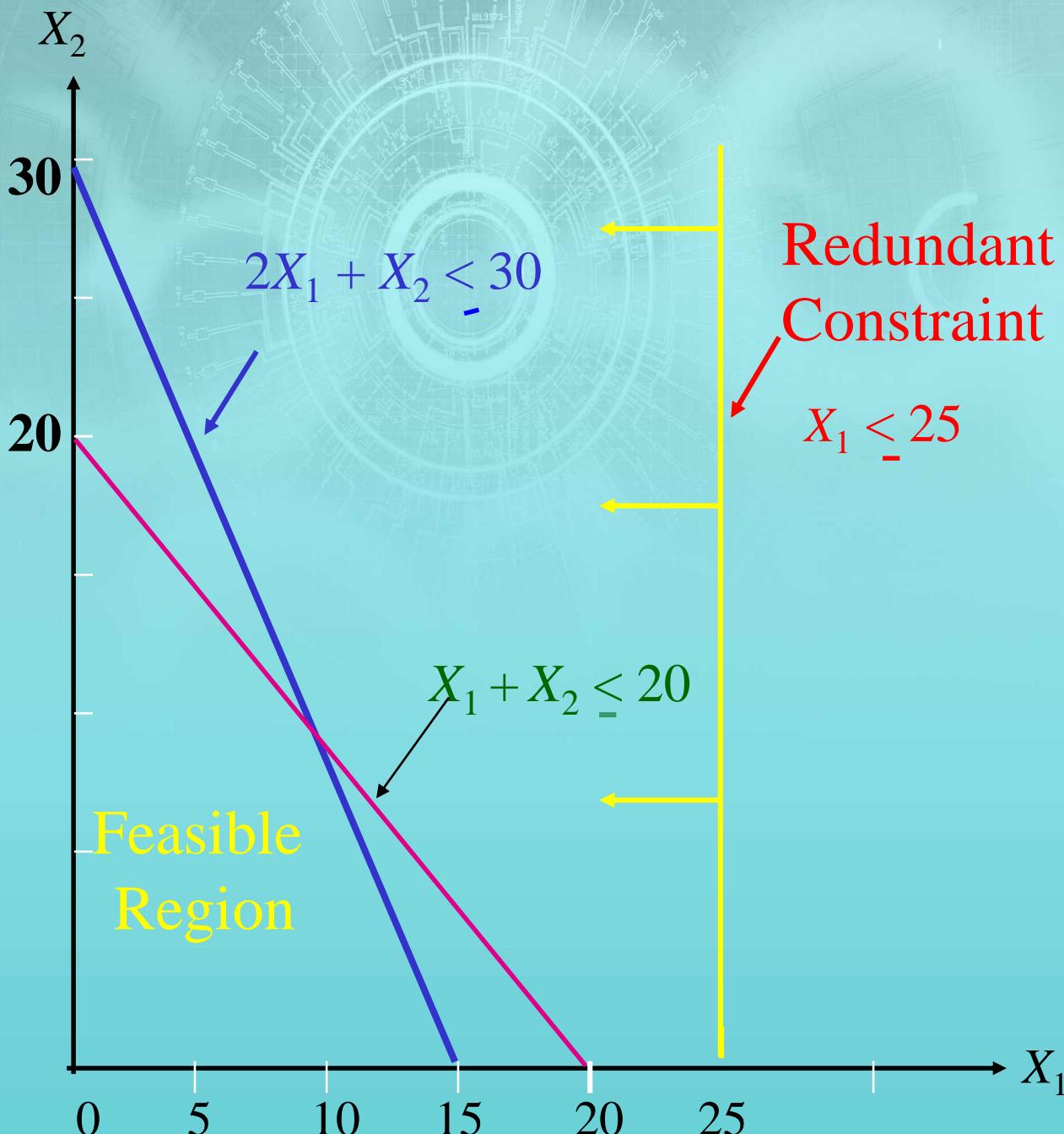
# A Problem with No Feasible Solution



# A Solution Region That is Unbounded to the Right



# A Problem with a Redundant Constraint



# An Example of Alternate Optimal Solutions

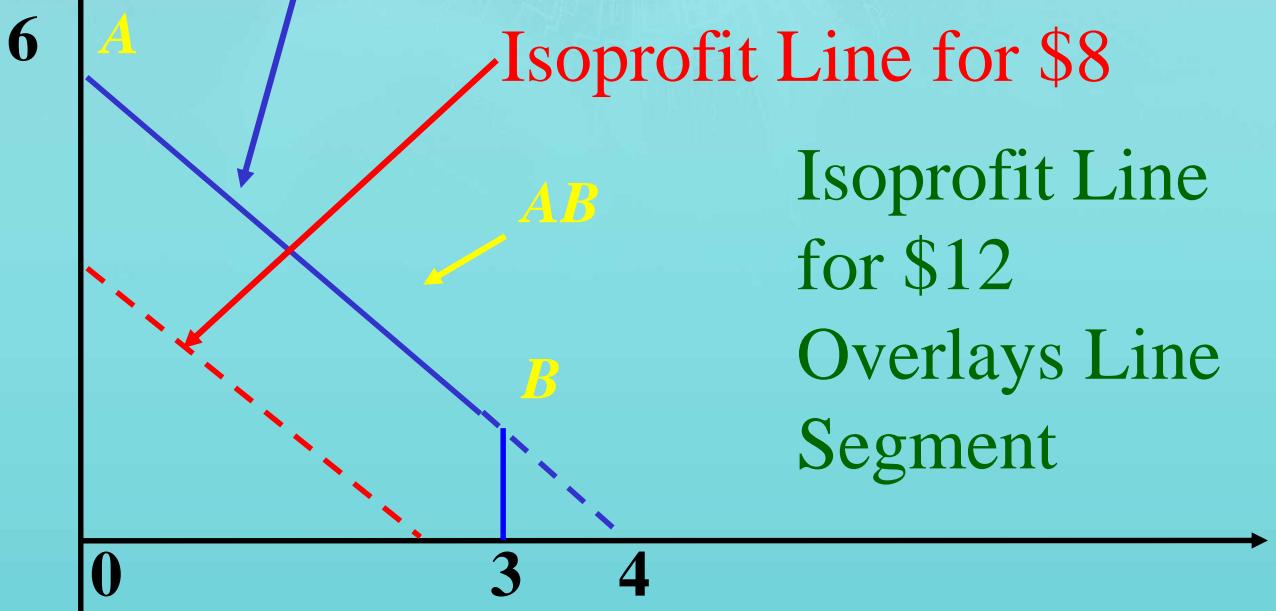
Maximize  $3X_1 + 2X_2$

Subj. To:  $6X_1 + 4X_2 \leq 24$

$X_1 \leq 3$

$X_1, X_2 \geq 0$

Optimal Solution Consists of  
All Combinations of  $X_1$  and  
 $X_2$  Along the AB Segment



# Sensitivity Analysis

- Optimal solutions to LP problems so far have been found under *deterministic assumptions*.
  - This means that we assume complete certainty in the data and relationships of a problem
  - i.e., prices are fixed, resources known, time needed to produce a unit exactly set.
- But in the real world, conditions are dynamic and changing.
- Questions to be addressed include:
  - How sensitive is the optimal solution to changes in profits, resources, or other input parameters?

# Sensitivity Analysis

- One way to reconcile the discrepancy between the deterministic assumptions with dynamic and changing real-world conditions is to determine
  - how *sensitive* the optimal solution is to model assumptions and data.
- An important function of sensitivity analysis is to allow managers to experiment with values of the input parameters.
- Such analyses are used to examine the effects of changes in three areas:
  - contribution rates for each variable,
  - technological coefficients (the numbers in the constraint equations), and
  - available resources (the right-hand-side quantities in each constraint).

# Sensitivity Analysis

- *Sensitivity analysis* is alternatively called
  - *postoptimality analysis*,
  - *parametric programming*, or
  - *optimality analysis*.
- *Sensitivity analysis* also often involves a series of “**what-if?**” questions. *For example:*
  - What if the profit on product 1 increases by 10%?
  - What if less money is available in the advertising budget constraint?
- *Sensitivity analysis* can be used to deal not only with
  - errors in estimating input parameters to the LP model
  - but also with
    - management’s experiments with possible future changes in the firm that may affect profits.

# Sensitivity Analysis

## *Two Means to Conduct Sensitivity Analysis*

1. There are two approaches to determining just how sensitive an optimal solution is to changes.
  - The first is simply a trial-and-error approach.
  - This approach usually involves resolving the entire problem, preferably by computer, each time one input data item or parameter is changed.
  - It can take a long time to test a series of possible changes in this way.

# Sensitivity Analysis

## *Two Means to Conduct Sensitivity Analysis*

2. The preferred approach is the **Analytic Postoptimality Method**.
  - After an LP problem has been solved, attempt to determine a range of changes in problem parameters that
    - will not affect the optimal solution *or*
    - change the variables in the solution.
  - This is done without resolving the whole problem.
  - Postoptimality analysis means examining changes after the optimal solution has been reached.

# Sensitivity Analysis

## Changes in the Objective Function Coefficient

- In real-life problems, contribution rates
  - usually profit or cost
- in the objective functions fluctuate periodically
  - as do most of a firm's expenses.
- Graphically, this means that
  - although the feasible solution region remains exactly the same,
  - the slope of the isoprofit or isocost line will change.
- The objective function coefficient (profit/cost) of any variable can increase or decrease, and
  - the current corner point may remain optimal if the change is not too large.
- However, if this coefficient increases or decreases too much
  - then the optimal solution would be at a different corner point.

# Sensitivity Analysis

## Changes in the Objective Function Coefficient

- Solver in Excel produces a Sensitivity Report (an example is on the next slide).
- Excel provides the allowable increases and decreases for the objective function coefficients.
- By adding the allowable increase to the current value, the upper bound may be obtained.
- For example,
  - The Allowable Increase on the profit (objective coefficient) for CD players is 10, which means that the upper bound on this profit is  $\$50 + \$10 = \$60$ .
  - Similarly, the allowable decrease may be subtracted from the current value to obtain the lower bound.

# Sensitivity Analysis

## Changes in the Objective Function Coefficient

### Excel's Sensitivity Analysis Output for High Note Sound Company

File Edit View Insert Format Tools Data Accounting OM Window Help

	A	B	C	D	E	F	G	H	I
1	Microsoft Excel 8.0 Sensitivity Report								
2	Worksheet: [captures.xls]7.8								
3	Report Created: 8/13/98 10:54:59 AM								
4									
5									
6	Adjustable Cells								
7				Final	Reduced	Objective	Allowable	Allowable	
8	Cell	Name		Value	Cost	Coefficient	Increase	Decrease	
9	\$B\$4	Value CD Players		0	-10	50	10	1E+30	
10	\$C\$4	Value Receivers		20	0	120	1E+30	20	
11									
12	Constraints								
13				Final	Shadow	Constraint	Allowable	Allowable	
14	Cell	Name		Value	Price	R.H. Side	Increase	Decrease	
15	\$D\$9	Electrician hours Used		80	30	80	160	80	
16	\$D\$10	Audio technician hours Used		20	0	60	1E+30	40	
17									
18	We will use 80 hours and 20 hours of electrician and audio technician time, respectively.								
19									
20	If we produce a CD player, our profit will fall by \$10.								
21									
22									
23									

If we produce a CD player, our profit will fall by \$10.

We will use 80 hours and 20 hours of electrician and audio technician time, respectively.

If we use one more electrician hour, our profit will increase by \$30. This is true for up to 160 more hours. The profit will fall by \$30 for each electrician hour less than 80 hours.

# Changes in the Resources or Right-Hand-Side Values

- The right-hand-side values of the constraints often represent resources available to the firm.
- The resources could be labor hours, machine time, or perhaps money or production materials available.
- In the High Note Sound Company example, two resources are
  - hours available of electricians' time and
  - hours of audio technicians' time.
- If additional hours were available, a higher total profit could be realized
- Sensitivity analysis about resources will help answer questions as these:
  - How much should the company be willing to pay for additional hours?
  - Is it profitable to have some electricians work overtime?
  - Should we be willing to pay for more audio technician time?

# Changes in the Resources or Right-Hand-Side Values

- If the right-hand side of a constraint is changed:
  - the feasible region will change (unless the constraint is redundant),
  - and often the optimal solution will change.
- The amount of change in the objective function value that results from a unit change in one of the resources available is called the **dual price** or **dual value**.
- The ***dual price*** for a constraint is the improvement in the objective function value that results from a one-unit increase in the right-hand side of the constraint.

# Changes in the Resources or Right-Hand-Side Values

- The dual price of a resource indicates the amount the objective function will be increased (or decreased) given another unit of the resource.
- However, the amount of possible increase in the right-hand side of a resource is limited.
- If the number of hours increased beyond the upper bound, then the objective function would no longer increase by the dual price.
  - There may be excess (slack) hours of a resource or the objective function may change by an amount different from the dual price.
  - Thus, the dual price is relevant only within limits.
- Both QM for Windows and Excel Solver provide these limits.

# QM For Windows and Changes in Right-Hand-Side Values

- Dual prices will change if the amount of the resource (the right-hand side, RHS, of the constraint) goes
  - above the upper bound or
  - below the lower bound

given in the Ranging section of the QM for Windows output.
- If the dual value of a constraint is zero
  - the slack is positive, indicating unused resource
  - additional amount of resource will simply increase the amount of slack.
- The dual value of zero is relevant as long as the RHS does not go below the lower bound.
- The upper limit of infinity indicates that adding more hours would simply increase the amount of slack.

# Excel Solver and Changes in Right-Hand-Side Values

- Solver gives the **shadow price** instead of the dual price.
- For a maximization problem, the shadow price in the Excel output is equivalent to the dual price in QM for Windows.
- A shadow price is the increase in the objective function value (e.g., profit or cost) that results from a one-unit increase in the right-hand-side of a constraint.
- The **Allowable Increase** and **Allowable Decrease** for the right-hand side of each constraint is provided, and the shadow price is relevant for changes within these limits.
- If a change is made that exceeds these limits, then the problem should be resolved to find the impact of the change.