Inverses of Matrices

Recall that the $n \times n$ **identity matrix** is the diagonal matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

having ones on its main diagonal and zeros elsewhere. It is not difficult to deduce directly from the definition of the matrix product that $\mathbf{I}$ acts like an identity for matrix multiplication:

$$\mathbf{AI} = \mathbf{A} \quad \text{and} \quad \mathbf{IB} = \mathbf{B}$$

The Inverse Matrix $\mathbf{A}^{-1}$

If $a \neq 0$, then there is a number $b = a^{-1} = 1/a$ such that $ab = ba = 1$. Given a nonzero matrix $\mathbf{A}$, we therefore wonder whether there is a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. The following two examples show that the answer to this question depends upon the particular matrix $\mathbf{A}$.

DEFINITION Invertible Matrix

The square matrix $\mathbf{A}$ is called **invertible** if there exists a matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

Example:

$$\mathbf{A} = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix},$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad , \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ A is invertible

Example:

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $AB = BA = I \Rightarrow$

$$AB = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-3c & b-3d \\ -2a+6c & -2b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} a - 3c &= 1 \\ -2a + 6c &= 0 \end{aligned} \qquad \begin{aligned} b - 3d &= 0 \\ -2b + 6d &= 1 \end{aligned} \Rightarrow$$ These equations are inconsistent.

Thus there can exist no 2×2 matrix B such that $AB = I$. So A is not invertible.

A matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ is called an **inverse matrix** of the matrix $\mathbf{A}$. The following theorem says that no matrix can have two different inverse matrices.

THEOREM 1 Uniqueness of Inverse Matrices

If the matrix $\mathbf{A}$ is invertible, then there exists precisely one matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

***Proof*:** If $\mathbf{C}$ is a (possibly different) matrix such that $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$ as well, then the associative law of multiplication gives

$$\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}.$$

Thus $\mathbf{C}$ is in fact the same matrix as $\mathbf{B}$. ■

The unique inverse of an invertible matrix $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$.

Arbitrary integral powers of a square matrix $\mathbf{A}$ are defined as follows, though in the case of a negative exponent we must assume that $\mathbf{A}$ is also invertible. If n is a positive integer, we define

$$\mathbf{A}^0 = \mathbf{I} \qquad \text{and} \qquad \mathbf{A}^1 = \mathbf{A};$$

$$\mathbf{A}^{n+1} = \mathbf{A}^n\mathbf{A} \quad \text{for } n \geq 1;$$

$$\mathbf{A}^{-n} = (\mathbf{A}^{-1})^n.$$

THEOREM 3 Algebra of Inverse Matrices

If the matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size are invertible, then

(a) $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$;

(b) If n is a nonnegative integer, then $\mathbf{A}^n$ is invertible and $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$;

(c) The product $\mathbf{AB}$ is invertible and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}. \tag{11}$$

***Proof of* (c)**

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I};$$

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}.$$

Thus we get $\mathbf{I}$ when we multiply $\mathbf{AB}$ *on either side* by $\mathbf{B}^{-1}\mathbf{A}^{-1}$. Because the inverse of the matrix $\mathbf{AB}$ is unique, this proves that $\mathbf{AB}$ is invertible and that its inverse matrix is $\mathbf{B}^{-1}\mathbf{A}^{-1}$. ■

How to Find $\mathbf{A}^{-1}$

DEFINITION Elementary Matrix

The $n \times n$ matrix $\mathbf{E}$ is called an **elementary matrix** if it can be obtained by performing a *single* elementary row operation on the $n \times n$ identity matrix $\mathbf{I}$.

We obtain some typical elementary matrices as follows.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{(3)R_1} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{E}_1 \xrightarrow{R_2 \to R_1 + R_2} \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$$

is not elementary matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(2)R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \mathbf{E}_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{SWAP}(R_1,\, R_2)]{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}_3$$

The three elementary matrices $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ correspond to three typical elementary row operations. ■

THEOREM 5 Elementary Matrices and Row Operations

If an elementary row operation is performed on the $m \times n$ matrix $\mathbf{A}$, then the result is the product matrix $\mathbf{EA}$, where $\mathbf{E}$ is the elementary matrix obtained by performing the same row operation on the $m \times m$ identity matrix.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \qquad E_1 . A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -1 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \xrightarrow{R_1 \to 3R_1} \begin{bmatrix} 3 & 6 \\ -1 & 4 \end{bmatrix} = E_1 A .$$

Not: A is row equivalent to B $\Leftrightarrow$ There are the elementary matrices $E_1, E_2, \dots, E_k$ such that $E_1 E_2 \cdots E_k A = B$.

Elementary row operations are reversible:

Elementary Row Operation	Inverse Operation
$(c)R_i$	$\frac{1}{c}R_i$
$R_i \leftrightarrow R_j$ SWAP(R_i, R_j)	$R_j \leftrightarrow R_i$ SWAP(R_i, R_j)
$(c)R_i + R_j$	$(-c)R_i + R_j$

FIGURE 3.5.1. Inverse elementary row operations.

Every elementary matrix is inversible.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to 3R_1} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = E_1$$

$$E_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = E_2$$

$$E_1 \cdot E_2 = E_2 \cdot E_1 = I.$$

THEOREM 6 Invertible Matrices and Row Operations

The $n \times n$ matrix $\mathbf{A}$ is invertible if and only if it is row equivalent to the $n \times n$ identity matrix $\mathbf{I}$.

proof: " $\Rightarrow$ " let A be invertiable

$$Ax = 0. \Rightarrow (A^{-1}A)x = 0 \Rightarrow Ix = 0 \Rightarrow x = 0$$

The system $Ax = 0$ has only the trivial solution $x = 0$

Thus A is row equivalent to I.

" $\Leftarrow$ " let A be row equivalent to I, that is there are elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_1 E_2 \ldots E_k A = I$$

Since each elementary matrix E_i is invertible

$$A = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1}$$

$\Rightarrow$ A is invertible.

ALGORITHM Finding $\mathbf{A}^{-1}$

To find the inverse $\mathbf{A}^{-1}$ of the invertible $n \times n$ matrix $\mathbf{A}$, find a sequence of elementary row operations that reduces $\mathbf{A}$ to the $n \times n$ identity matrix $\mathbf{I}$. Then apply the same sequence of operations in the same order to $\mathbf{I}$ to transform it into $\mathbf{A}^{-1}$.

Note: $[A \mid I] \rightarrow [I \mid ?]$

$$A \to I \quad \Rightarrow \quad I = E_1 E_2 \cdots E_k A$$

$$A = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1}$$

$$A^{-1} = (E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1})^{-1}$$

$$= (E_1^{-1})^{-1} (E_2^{-1})^{-1} \cdots (E_k^{-1})^{-1}$$

$$= E_1 E_2 \cdots E_k \cdot I$$

$$E_1 E_2 \cdots E_k [A \mid I] = [E_1 E_2 \cdots E_k A \mid E_1 E_2 \cdots E_k I]$$

$$[I \mid A^{-1}] \Rightarrow ? = A^{-1}$$

Example: Find the inverse of the 3×3 matrix

$$A = \begin{bmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 7 & 3 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 7 & 3 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to (-3)R_1 + R_3]{R_2 \to (-2)R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & -2 & 3 & 0 & -3 & 1 \end{array}\right] \xrightarrow[R_3 \to (2)R_2 + R_3]{R_1 \to R_1 + (-3)R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -3 & 7 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{array}\right] \xrightarrow[R_2 \to (-1)R_3 + R_2]{R_1 \to (-5)R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 42 & -5 \\ 0 & 1 & 0 & 3 & -9 & 1 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

Example: Show that $A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$ is not invertible

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to (2)R_1 + R_2} \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & -3 \\ 0 & 0 \end{matrix}}_{\neq I}$$

Matrix Equations

THEOREM 4 Inverse Matrix Solution of $\mathbf{Ax} = \mathbf{b}$

If the $n \times n$ matrix $\mathbf{A}$ is invertible, then for any n-vector $\mathbf{b}$ the system

$$\mathbf{Ax} = \mathbf{b} \tag{12}$$

has the unique solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{13}$$

that is obtained by multiplying both sides in (12) on the left by the matrix $\mathbf{A}^{-1}$.

***Proof*:** We must show that $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is a solution and that it is the only solution of Eq. (12). First, the computation

$$\mathbf{A}(\mathbf{A}^{-1}\mathbf{b}) = (\mathbf{A}\mathbf{A}^{-1})\mathbf{b} = \mathbf{I}\mathbf{b} = \mathbf{b}$$

shows that $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ *is* a solution. Second, if $\mathbf{x}_1$ is any (possibly different) solution, we observe that multiplication of each side of the equation $\mathbf{A}\mathbf{x}_1 = \mathbf{b}$ on the left by $\mathbf{A}^{-1}$ yields $\mathbf{x}_1 = \mathbf{A}^{-1}\mathbf{b}$, and hence $\mathbf{x}_1$ is the same solution as $\mathbf{x}$ after all. ■

Example: Use the inverse of the coefficent matrix to solve the system

$$2x_1 + 7x_2 + 3x_3 = 1$$

$$x_1 + 3x_2 + 2x_3 = 0 \qquad AX = b$$

$$3x_1 + 7x_2 + 9x_3 = -1$$

$$A = \begin{bmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

$$X = A^{-1} \cdot b = \begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ 1 \end{bmatrix}$$

Example: Find a 3×4 matrix X such that

$$\underbrace{\begin{bmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{bmatrix}}_{A} X = \underbrace{\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & -1 & -1 \end{bmatrix}}_{B} \qquad AX = B$$

$$X = A^{-1} \cdot B = \underset{3\times 3}{\begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix}} \underset{3\times 4}{\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & \underset{b}{-1} & -1 \end{bmatrix}}$$

$$= \begin{bmatrix} -23 & -10 & -8 & 60 \\ 5 & 2 & 2 & -13 \\ 4 & 2 & 1 & -10 \end{bmatrix}.$$

Note that $AX = B$ is a generalization of $Ax = b$.

Nonsingular Matrices

(Invertible)

A square matrix having these equivalent properties is sometimes called a nonsingular matrix .

THEOREM 7 Properties of Nonsingular Matrices

The following properties of an $n \times n$ matrix $\mathbf{A}$ are equivalent.

(a) $\mathbf{A}$ is invertible.

(b) $\mathbf{A}$ is row equivalent to the $n \times n$ identity matrix $\mathbf{I}$.

(c) $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

(d) For every n-vector $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

(e) For every n-vector $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ is consistent.

(e) $\Rightarrow$ (a) Let $AX = b$ be consistent

If $b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th row} = e_j$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_1} + x_2 \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_2} + \cdots + x_n \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{e_n}$$

$$= \underbrace{x_1 e_1}_{u_1} + \underbrace{x_2 e_2}_{u_2} + \cdots + \underbrace{x_n e_n}_{u_n}$$

then

$A \cdot u_j = e_j$. Let $B = [u_1 \ u_2 \ \cdots \ u_n]$

$$AB = A \cdot [u_1 \ u_2 \cdots u_n] = [Au_1 \ Au_2 \ \cdots \ Au_n]$$

$$= [e_1 \ e_2 \ \cdots \ e_n]$$

$$= I.$$

$$AB = I$$

$\Rightarrow$ A is invertible.

Example: Show that the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad-bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{a \neq 0} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to (-c)R_1 + R_2} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \xrightarrow{d-\frac{bc}{a} \neq 0}$$

$\left(d - \frac{bc}{a} = 0 \Rightarrow A^{-1} \text{ doesn't exist}\right)$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{c}{a}\cdot\frac{a}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{R_1 \to \left(-\frac{b}{a}\right)R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-cb} & -\frac{b}{ad-cb} \\ 0 & 1 & \frac{-c}{ad-cb} & \frac{a}{ad-cb} \end{array}\right]$$

$\underbrace{\qquad}_{A^{-1}}$

$$A^{-1} = \frac{1}{ad-cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad-bc \neq 0$ and $a \neq 0$.

Determinants

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$. The number $ad - bc$ is called the **determinant** of the 2×2 matrix $\mathbf{A}$. There are several common notations for determinants:

$$\det \mathbf{A} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{1}$$

$- \qquad +$

The determinant $\det \mathbf{A} = |a_{ij}|$ of a 3×3 matrix $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$ is defined as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \qquad (5$$

DEFINITION Minors and Cofactors

Let $\mathbf{A} = \left[\, a_{ij} \,\right]$ be an $n \times n$ matrix. The ijth **minor** of $\mathbf{A}$ (also called the **minor** of a_{ij}) is the determinant M_{ij} of the $(n-1) \times (n-1)$ submatrix that remains after deleting the ith row and the jth column of $\mathbf{A}$. The ijth **cofactor** A_{ij} of $\mathbf{A}$ (or the **cofactor** of a_{ij}) is defined to be

$$A_{ij} = (-1)^{i+j} M_{ij}. \tag{6}$$

Example:

$$M_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

The cofactor A_{ij} is obtained by attaching the $(-1)^{i+j}$ to M_{ij}

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$A_{11} = + \quad M_{11} \qquad A_{41} = - \quad M_{41}$$

$$\det A = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$
$$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

The last formula is the cofactor expansion of $\det A$ along the first row of A.

DEFINITION ***n* x *n* Determinants**

The **determinant** det $\mathbf{A} = |a_{ij}|$ of an $n \times n$ matrix $\mathbf{A} = [\, a_{ij} \,]$ is defined as

$$\det \mathbf{A} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}. \tag{8}$$

Thus we multiply each element of the first row of $\mathbf{A}$ by its cofactor and then add these n products to get det $\mathbf{A}$.

Example: Evaluate the determinant of

$$A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 7 & 4 & 3 & 5 \\ -6 & 2 & 2 & 4 \end{bmatrix}$$ by using cofactor expansion.

$$\det A = 2 \cdot (-1)^{1+1} \cdot \begin{vmatrix} -1 & 0 & 0 \\ 4 & 3 & 5 \\ 2 & 2 & 4 \end{vmatrix} + 0 \cdot (-1)^3 \cdot \begin{vmatrix} 0 & 0 & 0 \\ 7 & 3 & 5 \\ -6 & 2 & 4 \end{vmatrix} + 0 \cdot (-1)^4 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 7 & 4 & 5 \\ -6 & 2 & 4 \end{vmatrix}$$

$$+ (-3) \cdot (-1)^5 \cdot \begin{vmatrix} 0 & -1 & 0 \\ 7 & 4 & 3 \\ -6 & 2 & 2 \end{vmatrix}$$

$$\det A = 2.(-1)^{1+1} \cdot \begin{vmatrix} -1 & 0 & 0 \\ 4 & 3 & 5 \\ 2 & 2 & 4 \end{vmatrix} + \cdot 3 \cdot \begin{vmatrix} 0 & -1 & 0 \\ 7 & 4 & 3 \\ -6 & 2 & 2 \end{vmatrix}$$

$$= 2.(-1). \begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} + 3.(-1).(-1)^{3}. \begin{vmatrix} 7 & 3 \\ -6 & 2 \end{vmatrix}$$

$$= -2.(12-10) + 3.(14+18) = -4+96 = 92.$$

$$\det \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 7 & 4 & 3 & 5 \\ -6 & 2 & 2 & 4 \end{bmatrix} = (-1).(-1)^{4} \cdot \begin{vmatrix} 2 & 0 & -3 \\ 7 & 3 & 5 \\ -6 & 2 & 4 \end{vmatrix}$$

$$= (-1) \cdot \left\{ 3.(-1)^{4}. \begin{vmatrix} 2 & -3 \\ -6 & 4 \end{vmatrix} + 2.(-1)^{5}. \begin{vmatrix} 2 & -3 \\ 7 & 5 \end{vmatrix} \right\}$$

$$= -\left\{ 3.(8-18) - 2.(10+21) \right\} = -(-30-62) = 92.$$

THEOREM 1 Cofactor Expansions of Determinants

The determinant of an $n \times n$ matrix $\mathbf{A} = [\, a_{ij} \,]$ can be obtained by expansion along any row or column. The cofactor expansion along the ith row is

$$\det \mathbf{A} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}. \tag{9}$$

The cofactor expansion along the jth column is

$$\det \mathbf{A} = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}. \tag{10}$$

Row and Column Properties

1), if the square matrix A has either an all-zero row or an all-zero column , then det A=0.

$$\begin{vmatrix} 0 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 3 & 6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

2) If the n × n matrix B is obtained from A by multiplying a single row (or a column) of A by the constant k , then det B = k det A .

$$A \xrightarrow{R_i \to kR_i} B \qquad |B| = k|A|$$

$$(\text{or } C_i \to kC_i)$$

$$\begin{vmatrix} 7 & 15 & -17 \\ -2 & 9 & 6 \\ 5 & -12 & 10 \end{vmatrix} = 3 \cdot \begin{vmatrix} 7 & 5 & -17 \\ -2 & 9 & 6 \\ 5 & -4 & 10 \end{vmatrix}$$

3) If the $n \times n$ matrix B is obtained from A by interchanging two rows (or two columns), then

$$\det B = -\det A$$

$$A \xrightarrow[\text{(or } C_i \leftrightarrow C_j\text{)}]{R_i \leftrightarrow R_j} B \qquad |B| = -|A|$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \qquad \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 6-4 = 2$$

4) If two rows (or two columns) of the n × n matrix A are identical, then det A = 0.

proof: Let B denote the matrix obtained by interchanging the two identical rows of A

$$\Rightarrow B = A \Rightarrow |B| = |A|$$

By Property 3, $|B| = -|A| \Rightarrow |A| = -|A| \Rightarrow 2 \cdot |A| = 0$

$$\Rightarrow |A| = 0.$$

5) Suppose that the nxn matrices A_1, A_2, and B are identical except for their i th rows- that is, the other n −1 rows of the three matrices are identical- and that the i th row of B is the sum of the i th rows of A_1 and A_2. Then

$$\det B = \det A_1 + \det A_2.$$

This result also holds if columns are involved instead of rows.

$$A_1 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A_1| = 1.(-1).\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2 \qquad |A_2| = 1.(-1).\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$$|B| = 1.(-1).\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$\Rightarrow |B| = |A_1| + |A_2|$$

6) If the n × n matrix B is obtained by adding a constant multiple of one row (or column) of A to another row (or column) of A , then det B = det A.

$$A \xrightarrow[(\text{or } C_i \to C_i + kC_j)]{R_i \to R_i + kR_j} B \quad \Rightarrow \quad |A| = |B|$$

Evaluate the determinant after first simplifying the computation by adding an appropriate multiple of some row and column to other

$$\det A = \begin{vmatrix} 2 & -3 & 4 \\ -1 & 4 & 2 \\ 3 & 10 & 1 \end{vmatrix} \xrightarrow{C_3 \to C_1 + 2C_2} \begin{vmatrix} 2 & -3 & 0 \\ -1 & 4 & 0 \\ 3 & 10 & 7 \end{vmatrix}$$

$$= 7 \cdot \begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix} = 7 \cdot (8-3) = 35.$$

An **upper triangular matrix** is a square matrix having only zeros *below* its main diagonal. A **lower triangular matrix** is a square matrix having only zeros *above* its main diagonal. A **triangular matrix** is one that is either upper triangular or lower triangular, and thus looks like

$$\begin{bmatrix} 1 & 6 & 10 \\ 0 & 5 & 8 \\ 0 & 0 & 7 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & 6 & 5 \end{bmatrix}.$$

upper triangular   lower triangular

7) The determinant of a triangular matrix is equal to the product of its diagonal elements.

$$\begin{vmatrix} 3 & 11 & 9 & 2 \\ 0 & -2 & 8 & -6 \\ 0 & 0 & 5 & 17 \\ 0 & 0 & 0 & -4 \end{vmatrix} = 3 \cdot \begin{vmatrix} -2 & 8 & -6 \\ 0 & 5 & 17 \\ 0 & 0 & -4 \end{vmatrix} = 3.(-2) \cdot \begin{vmatrix} 5 & 17 \\ 0 & -4 \end{vmatrix}$$

$$= 3.(-2) \cdot 5.(-4) = 120$$

Example: Evaluate the determinant

$$\begin{vmatrix} 2 & -1 & 3 \\ -2 & 1 & 5 \\ 4 & -2 & 10 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -1 & 3 \\ -2 & 1 & 5 \\ 4 & -2 & 10 \end{vmatrix} \xrightarrow[R_3 \to R_2 - 2R_1]{R_2 \to R_2 + R_1} \begin{vmatrix} 2 & -1 & 3 \\ 0 & 0 & 8 \\ 0 & 6 & 4 \end{vmatrix} = (2).(0).(4) = 0.$$

The Transpose of a Matrix

The transpose $\mathbf{A}^T$ of a 2×2 matrix $\mathbf{A}$ is obtained by interchanging its off-diagonal elements:

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \tag{12}$$

More generally, the **transpose** of the $m \times n$ matrix $\mathbf{A} = [\, a_{ij} \,]$ is the $n \times m$ matrix $\mathbf{A}^T$ defined by

$$\mathbf{A}^T = [\, a_{ji} \,]. \tag{13}$$

$$\begin{bmatrix} 2 & 0 & 5 \\ 4 & -1 & 7 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 \\ 0 & -1 \\ 5 & 7 \end{bmatrix}$$

If A and B are matrices of appropriate sizes and c is a number, then

(i) $(\mathbf{A}^T)^T = \mathbf{A}$;

(ii) $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$;

(iii) $(c\mathbf{A})^T = c\mathbf{A}^T$;

(iv) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

A^T

8) If A is a square matrix, then det ([redacted]) = det A

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad |A| = 4-6 = -2$

$\Rightarrow |A| = |A^T|$

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad |A^T| = 4-6 = -2$

9) The n × n matrix A is invertible if and only if det A ≠ 0.

Example: Show that the matrix

$$\mathbf{V} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$ is invertiable iff a, b and c are distinct

$$|V| = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-a).(c-a). \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} \xrightarrow{R_3 \to R_3 - R_2} = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)$$

$$|V| \neq 0 \Rightarrow b \neq a, \ c \neq a, \ c \neq b$$

10) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, then

$$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|.$$

11) $|A + B|$ is generally not equal to $|A|+|B|$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$|A| = -6, \quad |B| = 3 \quad |A| + |B| = -6 + 3 = -3$$

$$A + B = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix} \quad |A+B| = 6 - 12 = -6$$

$$\Rightarrow |A+B| \neq |A| + |B|.$$

12) We can calculate the determinant of the inverse of an invertible matrix A :

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I},$$

so

$$|\mathbf{A}|\,|\mathbf{A}^{-1}| = |\mathbf{A}\mathbf{A}^{-1}| = |\mathbf{I}| = 1,$$

and therefore

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}.$$

proof : A is invertible matrix $\Rightarrow$ $AA^{-1} = I$ and $|A| \neq 0$

$|A||A^{-1}| = |I|$ $\Rightarrow$ $\underbrace{|A|}_{\neq 0} . |A^{-1}| = 1$

$\Rightarrow$ $|A^{-1}| = \frac{1}{|A|}$

THEOREM 4 Cramer's Rule

Consider the $n \times n$ linear system $\mathbf{Ax} = \mathbf{b}$, with

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}.$$

If $|\mathbf{A}| \neq 0$, then the ith entry of the unique solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is given by

$$\begin{aligned} x_i &= \frac{\begin{vmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{vmatrix}}{|\mathbf{A}|} \\ &= \frac{1}{|\mathbf{A}|} \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}, \end{aligned} \tag{17}$$

where in the last expression the constant vector $\mathbf{b}$ replaces the ith column vector $\mathbf{a}_i$ of $\mathbf{A}$.

Use Cramer's rule to solve the system

$$\begin{aligned} x_1 + 4x_2 + 5x_3 &= 2 \\ 4x_1 + 2x_2 + 5x_3 &= 3 \\ -3x_1 + 3x_2 - x_3 &= 1. \end{aligned}$$

$$b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 5 \\ 0 & -14 & -15 \\ 0 & 15 & 14 \end{vmatrix} = 1 \cdot \begin{vmatrix} -14 & -15 \\ 15 & 14 \end{vmatrix} = 29.$$

$$x_1 = \frac{1}{29} \cdot \begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 5 \\ 1 & 3 & -1 \end{vmatrix} = \frac{1}{29} \cdot \begin{vmatrix} 0 & -2 & 7 \\ 0 & -7 & 8 \\ 1 & 3 & -1 \end{vmatrix} = \frac{1}{29} \cdot 1 \cdot \begin{vmatrix} -2 & 7 \\ -7 & 8 \end{vmatrix} = \frac{33}{29}$$

$$x_2 = \frac{1}{29} \begin{vmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \\ -3 & 1 & -1 \end{vmatrix} = \frac{1}{29} \cdot \begin{vmatrix} 1 & 2 & 5 \\ 0 & -5 & -15 \\ 0 & 7 & 14 \end{vmatrix} = \frac{1}{29} \cdot 1 \cdot \begin{vmatrix} -5 & -15 \\ 7 & 14 \end{vmatrix} = \frac{35}{29}$$

$$x_3 = \frac{1}{29} \cdot \begin{vmatrix} 1 & 4 & 2 \\ 4 & 2 & -5 \\ -3 & 3 & 7 \end{vmatrix} = \frac{1}{29} \cdot \begin{vmatrix} 1 & 4 & 2 \\ 0 & -14 & -5 \\ 0 & 15 & 7 \end{vmatrix} = \frac{-23}{29}.$$