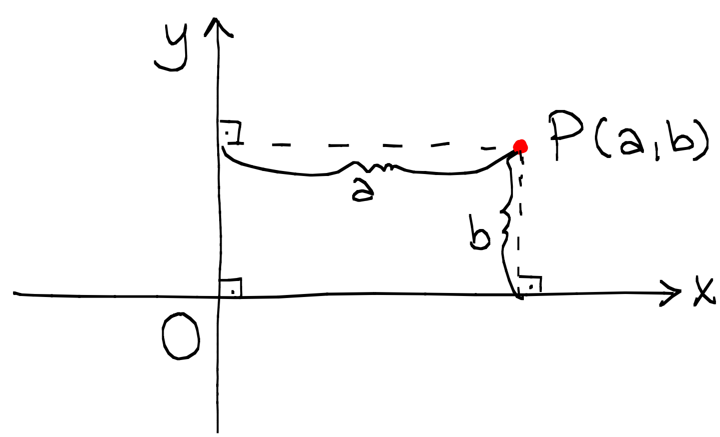
10.5

Polar Coordinates

Reminding: Cartesian Coordinates

There are two perpendicular coordinate axes (x-axis & y-axis) which intersect at a point O, called the origin. Any point P in the plane ($\mathbb{R}^2$) can be represented, in terms of Cartesian coordinates, by an ordered pair of numbers (a,b), which are directed distances from the axes:



Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O (Figure 10.35). Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.

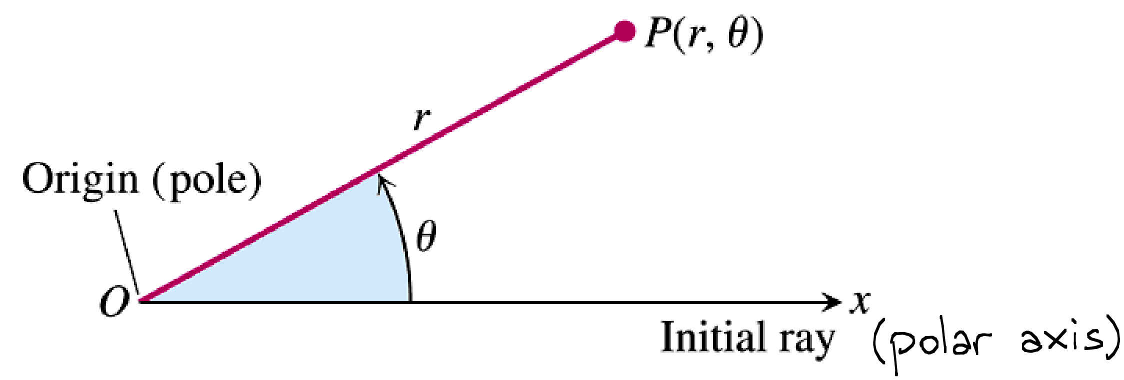


FIGURE 10.35 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray, called the polar axis.

Polar axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

Polar Coordinates

$$P(r, \theta)$$

Directed distance from O to P

Directed angle from initial ray to OP

"Directed" means r and θ can take negative values. θ is positive when measured in the counter-clockwise direction from the polar axis and negative in the clockwise direction.

If $P = O$ (the pole), then $r = 0$, so $(0, \theta)$ represents the pole for any value of θ.

The point $P(r, \theta)$ can be reached by:

1) turning $|\theta|$ radians counterclockwise if $\theta > 0$ and clockwise if $\theta < 0$ from the polar axis and
2) going forward $|r|$ units if $r > 0$ and backward $|r|$ units if $r < 0$.

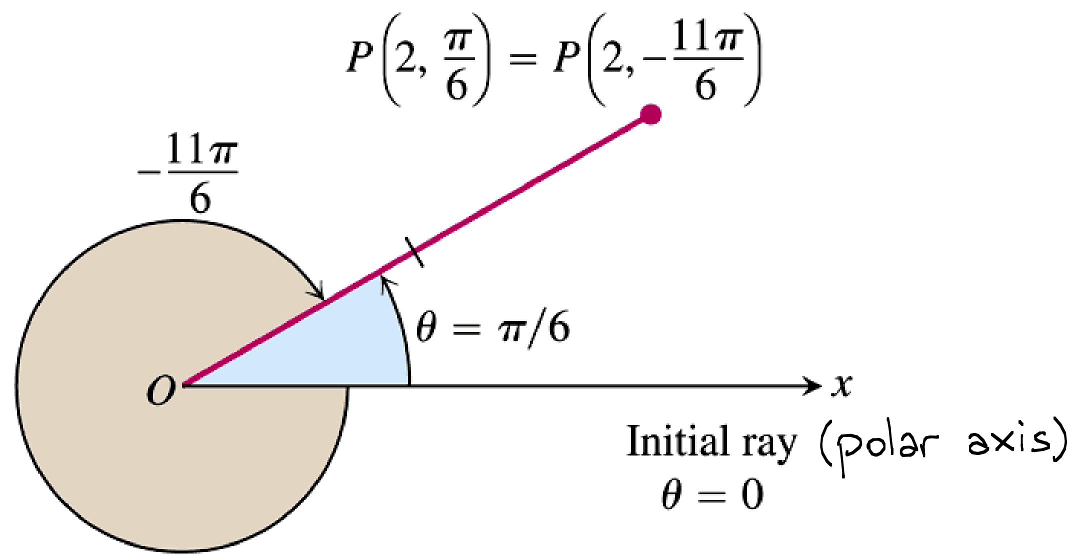


FIGURE 10.36 Polar coordinates are not unique.

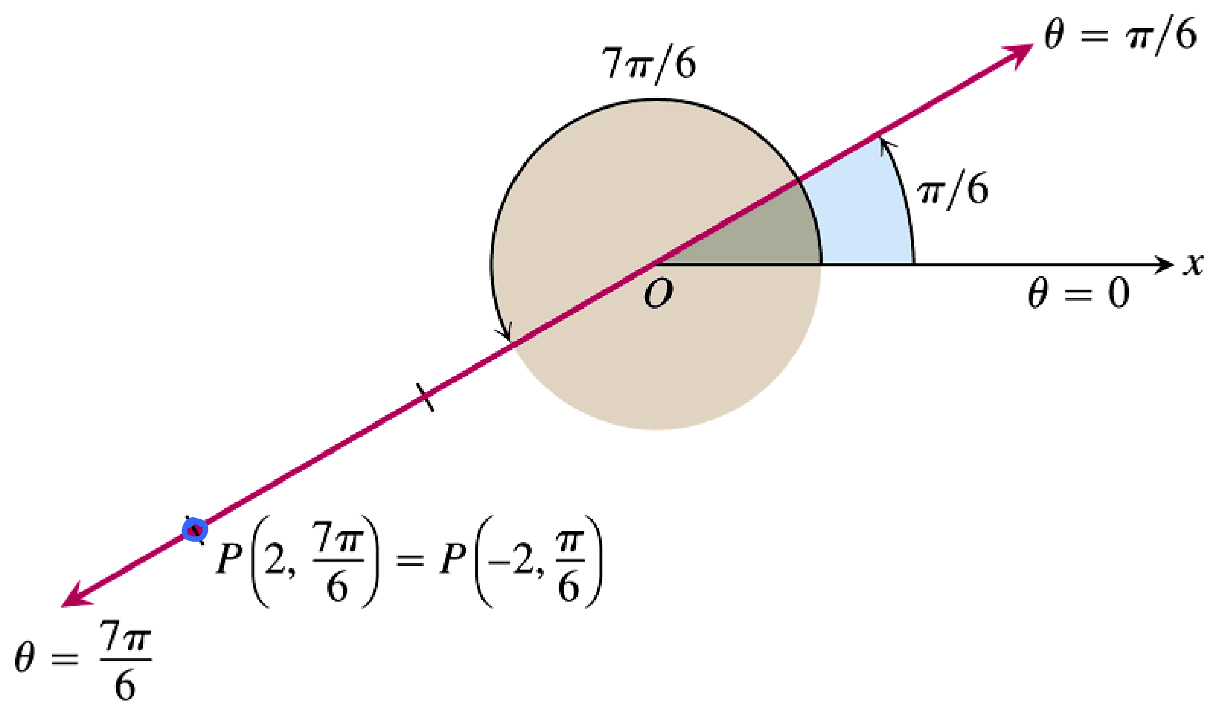


FIGURE 10.37 Polar coordinates can have negative r-values.

Example 1 Find all the polar coordinates of the point $P(2, \frac{\pi}{6})$.

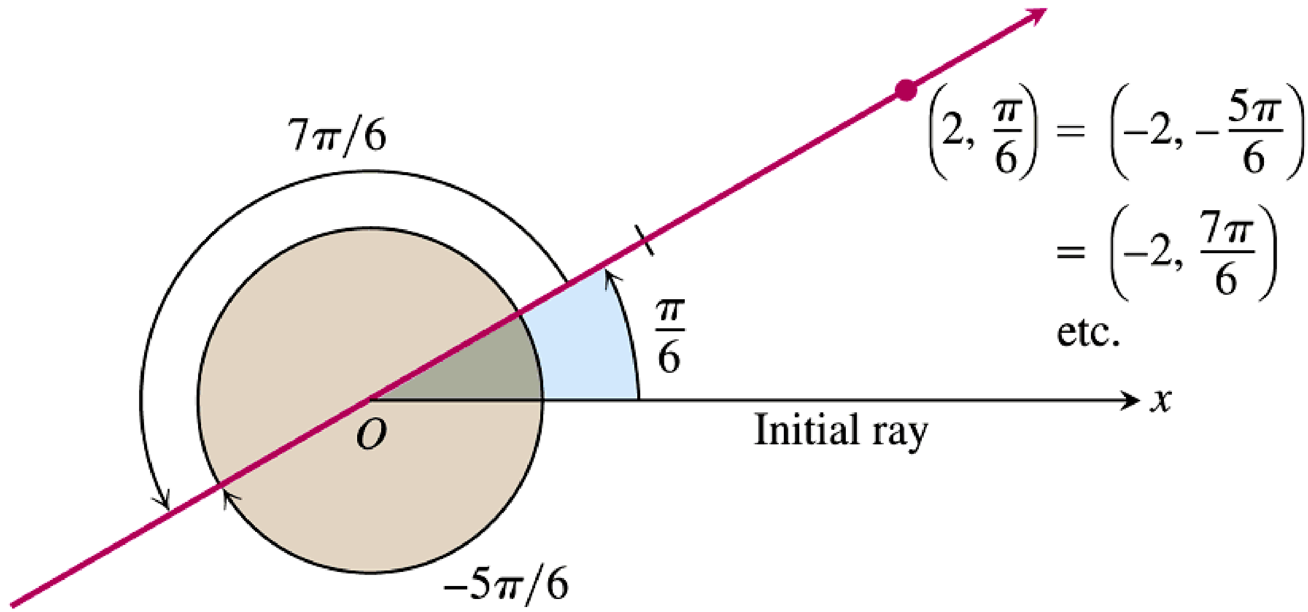
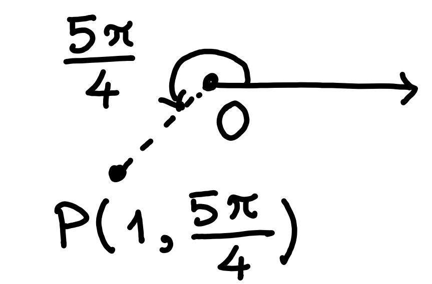
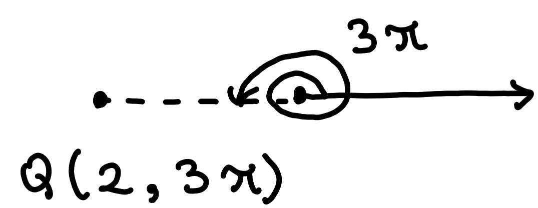


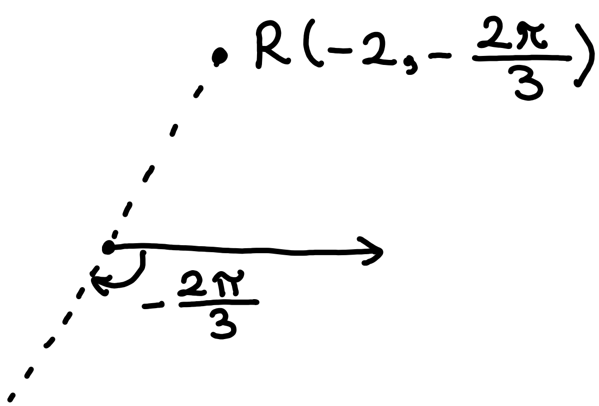
FIGURE 10.38 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

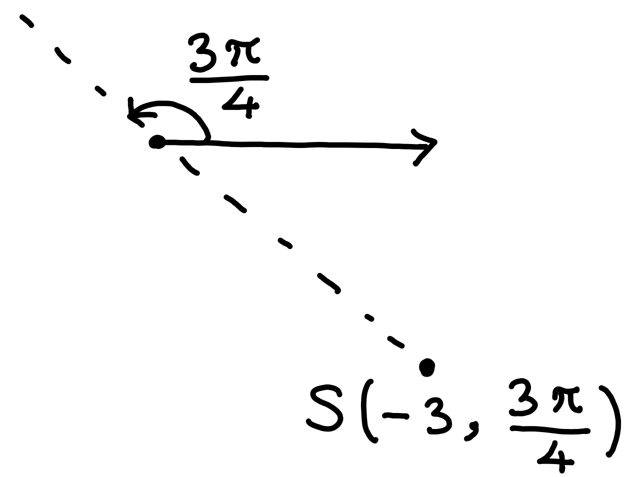
$P(2, \frac{\pi}{6}) = P\left(2, \frac{\pi}{6} + 2k\pi\right) = P\left(-2, \frac{7\pi}{6} + 2k\pi\right)$ for any integer k.

Example 2 Plot the points with polar coordinates $P(1, \frac{5\pi}{4})$, $Q(2, 3\pi)$, $R(-2, -\frac{2\pi}{3})$ and $S(-3, \frac{3\pi}{4})$.









Polar Equations and Graphs

If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O (Figure 10.39).

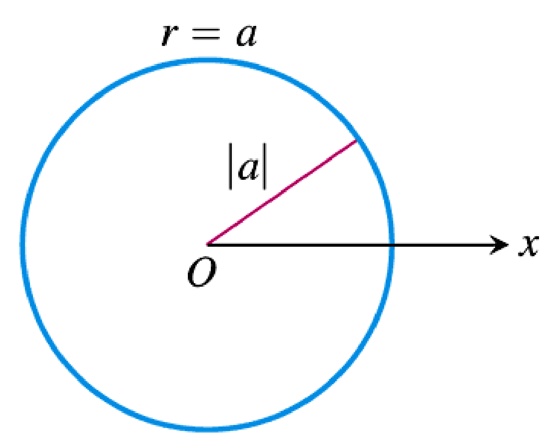


FIGURE 10.39 The polar equation for a circle is $r = a$.

If we hold θ fixed at a constant value $\theta = \theta_0$ and let r vary between $-\infty$ and ∞, the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray.

Equation	Graph
$r = a$	Circle radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

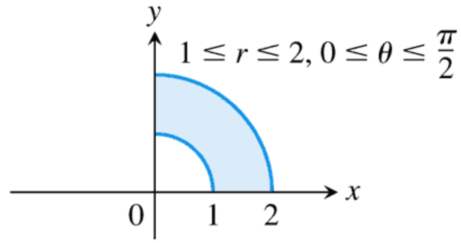
Example 3

(a) The equations $r=1$ and $r=-1$ both represent the circle of radius 1 centered at O (unit circle).

(b) The equations $\theta = \frac{\pi}{6}$, $\theta = \frac{7\pi}{6}$ and $\theta = -\frac{5\pi}{6}$ all represent the same line through O. (see Example 1)

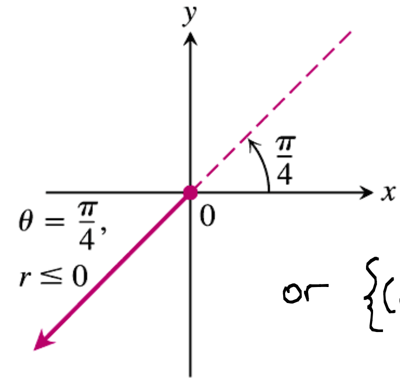
Example 4

(a)



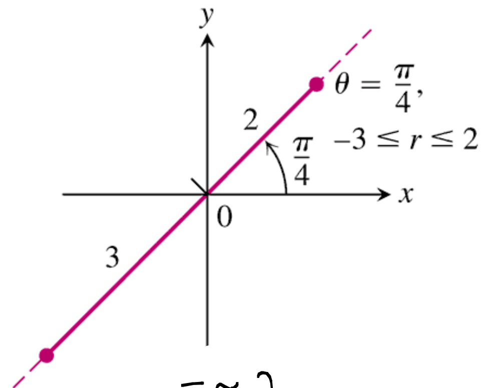
or $\{(r,\theta) : -2 \le r \le -1,\ \pi \le \theta \le \frac{3\pi}{2}\}$

(c)



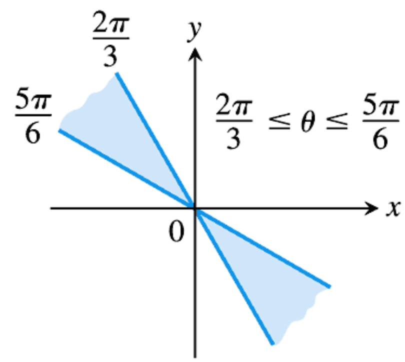
or $\{(r,\theta) : r \ge 0,\ \theta = \frac{5\pi}{4}\}$

(b)



or $\{(r,\theta) : -2 \le r \le 3,\ \theta = \frac{5\pi}{4}\}$

(d)



or $\{(r,\theta) : r \in \mathbb{R},\ -\frac{\pi}{3} \le \theta \le -\frac{\pi}{6}\}$

FIGURE 10.40 The graphs of typical inequalities in r and θ

Relating Polar and Cartesian Coordinates

Identify pole with the origin and initial ray (polar axis) with the positive x-axis.

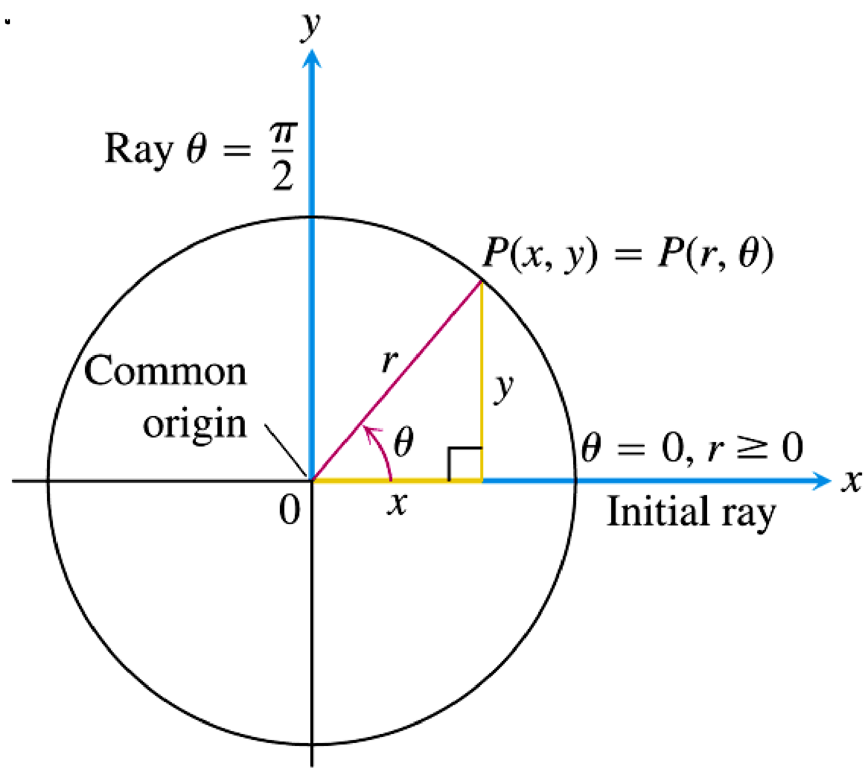


FIGURE 10.41 The usual way to relate polar and Cartesian coordinates.

Equations Relating Polar and Cartesian Coordinates

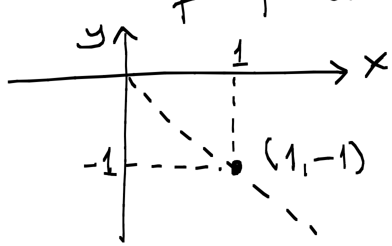
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2$$

Example 5

(a) Represent the point with polar coordinates $(r,\theta) = (2, \frac{\pi}{3})$ in terms of Cartesian coordinates.

$x = 2.\cos(\frac{\pi}{3}) = 1$, $y = 2.\sin(\frac{\pi}{3}) = \sqrt{3}$ $\Rightarrow P(1, \sqrt{3})$

(b) Find a representation of the point with Cartesian coordinates $(x,y) = (1,-1)$ in terms of polar coordinates.



$$\left.\begin{array}{l} x^2+y^2 = 2 = r^2 \\ \Rightarrow r = \mp\sqrt{2} \end{array}\right\}$$

- If $r = \sqrt{2}$ choose $\theta = -\frac{\pi}{4}$
- If $r = -\sqrt{2}$ choose $\theta = \frac{3\pi}{4}$

(not unique!)

EXAMPLE Equivalent Equations

Polar equation	Cartesian equivalent
$r \cos \theta = 2$	$x = 2$
$r^2 \cos \theta \sin \theta = 4$	$xy = 4$
$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r \cos \theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos \theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

With some curves, we are better off with polar coordinates; with others, we aren't.

Example 6 Find a polar equation for the circle $x^2+(y-3)^2=9$.

$x^2+y^2-6y+9=9 \iff r^2-6r\sin\theta=0 \iff r(r-6\sin\theta)=0$

$\iff r=0$ or $r=6\sin\theta$ (the whole circle)

(just origin)

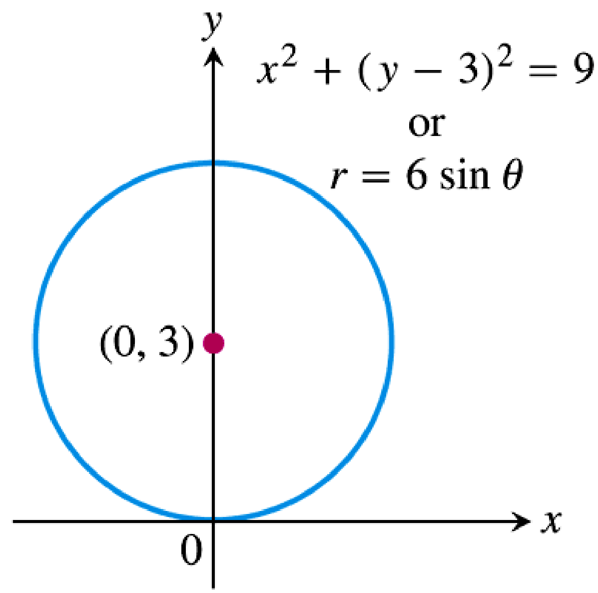


FIGURE 10.42 The circle in Example

EXAMPLE 7 Converting Polar to Cartesian

Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

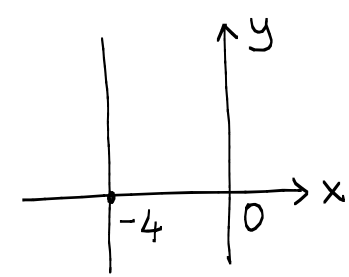
Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation:

$$r\cos\theta = -4$$
$$x = -4$$

The graph: Vertical line through $x = -4$ on the x-axis

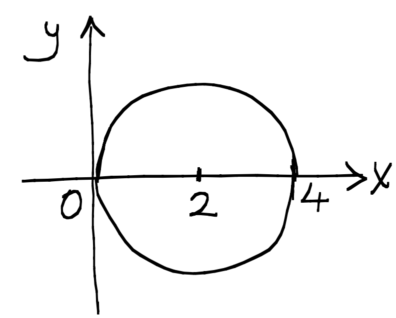


(b) $r^2 = 4r\cos\theta$

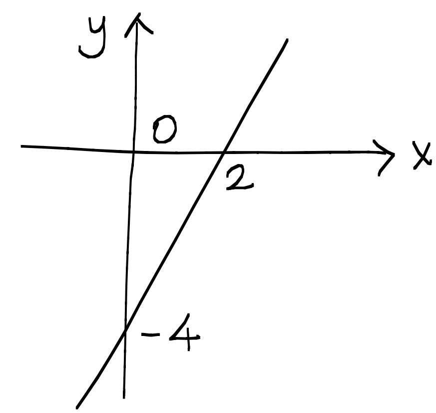
The Cartesian equation:

$$r^2 = 4r\cos\theta$$
$$x^2 + y^2 = 4x$$
$$x^2 - 4x + y^2 = 0$$
$$x^2 - 4x + 4 + y^2 = 4 \quad \text{Completing the square}$$
$$(x-2)^2 + y^2 = 4$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$



(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$



The Cartesian equation:

$$r(2\cos\theta - \sin\theta) = 4$$
$$2r\cos\theta - r\sin\theta = 4$$
$$2x - y = 4$$
$$y = 2x - 4$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$

In the above example, we first convert the polar equation into an equivalent Cartesian equation, then sketch its graph. But this method is not always practical. For example, the equations $r = 1 - \cos\theta$ and $x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$ represent the same curve but the Cartesian version is not easy to sketch. So we should find a way of sketching the polar equation directly.

10.6

Graphing in Polar Coordinates

To sketch a polar curve, we first check if the curve has any symmetries by using the following:

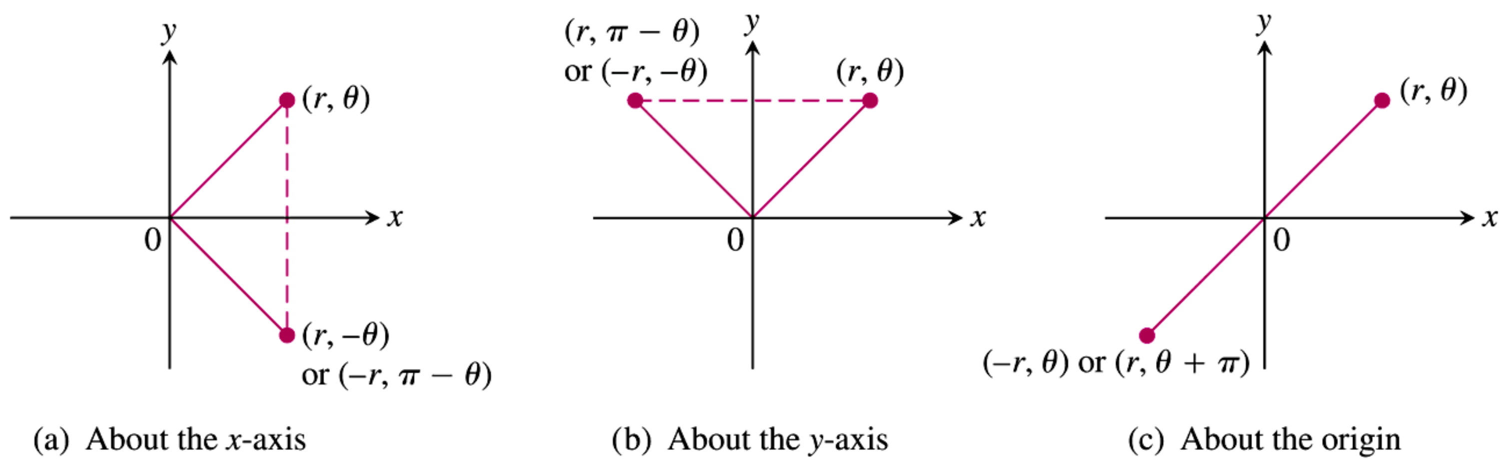


FIGURE 10.43 Three tests for symmetry in polar coordinates.

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 10.43a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.43b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 10.43c).

Example a) The circle $r = 6\sin\theta$ $(x^2 + (y-3)^2 = 9)$ is symmetric about y-axis since $r = 6\sin\theta = 6\sin(\pi - \theta)$
((r,θ) and $(r, \pi-\theta)$ lie on the graph)

b) The circle $r = 2\cos\theta$ $((x-1)^2 + y^2 = 1)$ is symmetric about x-axis since $r = 2\cos\theta = 2\cos(-\theta)$ $[(r,\theta)$ & $(r,-\theta)]$

To sketch the graph of a polar curve, we make a table of (sufficiently many) (r,θ)-values in a suitable domain, and connect them in order of increasing θ.

EXAMPLE A Cardioid

Graph the curve $r = 1 - \cos\theta$.

Solution The curve is symmetric about the x-axis because

$$
\begin{aligned}
(r, \theta) \text{ on the graph} &\Rightarrow r = 1 - \cos\theta \\
&\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\
&\Rightarrow (r, -\theta) \text{ on the graph.}
\end{aligned}
$$

Cosine is periodic with period 2π and the curve is symmetric about the x-axis, so it is enough to consider only when $\theta \in [0, \pi]$. The rest will follow from symmetry.

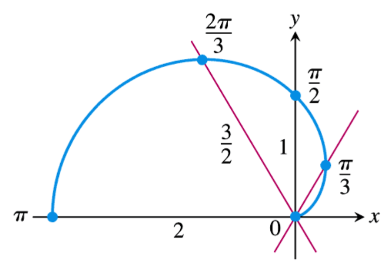
Make the table

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)

As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a min. value of 0 to a max. value of 2.

Plot the points and connect them

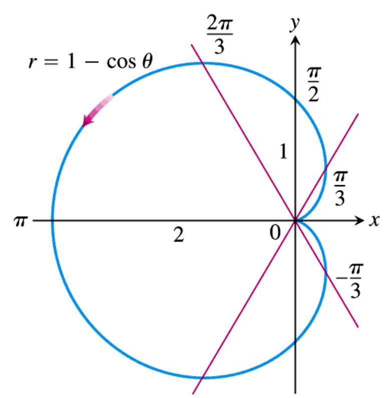


(b)

Recall that the Cartesian equation for the cardioid $r = 1 - \cos\theta$ is:

$$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$$

Take the symmetry



(c)

FIGURE 10.44 The steps in graphing the cardioid $r = 1 - \cos\theta$ (Example). The arrow shows the direction of increasing θ.

EXAMPLE Graph the Curve $r^2 = 4\cos\theta$. (A Lemniscate)

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the x-axis because

$$\begin{aligned} (r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.} \end{aligned}$$

The curve is also symmetric about the origin because

$$\begin{aligned} (r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow (-r)^2 = 4\cos\theta \\ &\Rightarrow (-r, \theta) \text{ on the graph.} \end{aligned}$$

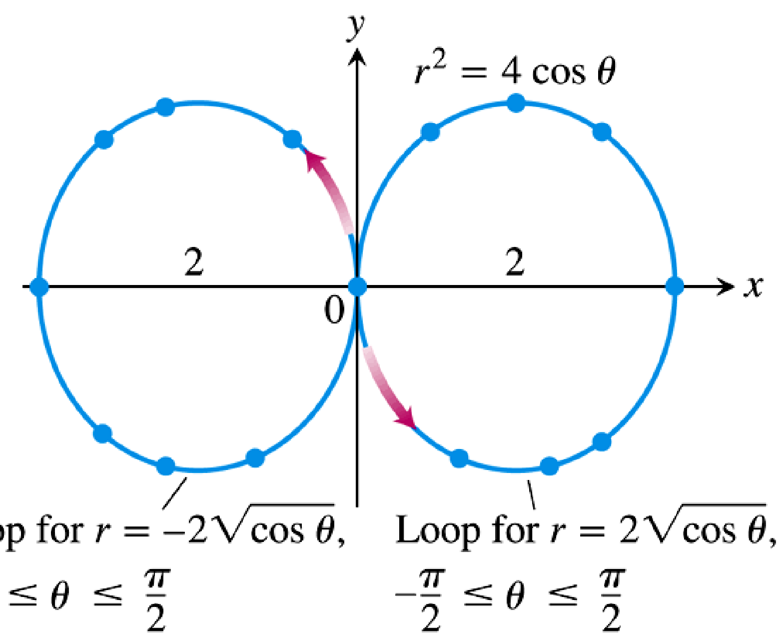
Together, these two symmetries imply symmetry about the y-axis.

For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}.$$

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

FIGURE 10.45 The graph of $r^2 = 4\cos\theta$. The arrows show the direction of increasing θ. The values of r in the table are rounded (Example).

Example Graph the curve $r^2 = \sin(2\theta)$. **(A Lemniscate)**

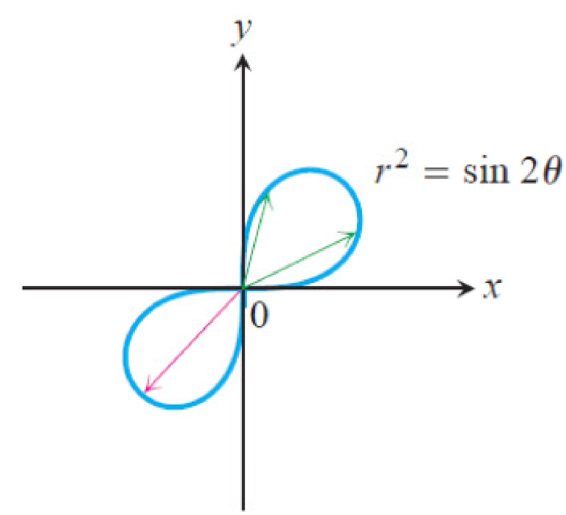
$r^2 = \sin(2\theta) \geq 0 \Rightarrow$ we get the whole graph by running θ from 0 to $\frac{\pi}{2}$. The curve is symmetric about the origin since

$$(r,\theta) \text{ on the graph} \Rightarrow r^2 = \sin(2\theta) \Rightarrow (-r)^2 = \sin(2\theta)$$
$$\Rightarrow (-r,\theta) \text{ on the graph.}$$

For each value of $\theta \in [0, \frac{\pi}{2}]$, we have two values of r:

$$r = \mp\sqrt{\sin(2\theta)}$$

θ	r
0	0
$\pi/12$	$\mp 1/\sqrt{2}$
$\pi/6$	$\mp\sqrt{\sqrt{3}/2}$
$\pi/4$	∓ 1
$\pi/3$	$\mp\sqrt{\sqrt{3}/2}$
$\pi/2$	0



Finding Points Where Polar Graphs Intersect

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. The problem is that a point of intersection may satisfy the equation of one curve with polar coordinates that are different from the ones with which it satisfies the equation of another curve. Thus, solving the equations of two curves simultaneously may not identify all their points of intersection. One sure way to identify all the points of intersection is to graph the equations.

EXAMPLE Deceptive Polar Coordinates

Show that the point $(2, \pi/2)$ lies on the curve $r = 2\cos 2\theta$.

Solution It may seem at first that the point $(2, \pi/2)$ does not lie on the curve because substituting the given coordinates into the equation gives

$$2 = 2\cos 2\left(\frac{\pi}{2}\right) = 2\cos\pi = -2,$$

which is not a true equality. The magnitude is right, but the sign is wrong. This suggests looking for a pair of coordinates for the same given point in which r is negative, for example, $(-2, -(\pi/2))$. If we try these in the equation $r = 2\cos 2\theta$, we find

$$-2 = 2\cos 2\left(-\frac{\pi}{2}\right) = 2(-1) = -2,$$

and the equation is satisfied. The point $(2, \pi/2)$ does lie on the curve.

EXAMPLE Elusive Intersection Points

Find the points of intersection of the curves

$$r^2 = 4\cos\theta \qquad \text{and} \qquad r = 1 - \cos\theta.$$

Solution In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others. In this example, simultaneous solution reveals only two of the four intersection points. The others are found by graphing. (Also, see Exercise 49.)

If we substitute $\cos\theta = r^2/4$ in the equation $r = 1 - \cos\theta$, we get

$$\begin{aligned} r &= 1 - \cos\theta = 1 - \frac{r^2}{4} \\ 4r &= 4 - r^2 \\ r^2 + 4r - 4 &= 0 \\ r &= -2 \pm 2\sqrt{2}. && \text{Quadratic formula} \end{aligned}$$

The value $r = -2 - 2\sqrt{2}$ has too large an absolute value to belong to either curve. The values of θ corresponding to $r = -2 + 2\sqrt{2}$ are

$$\begin{aligned}
\theta &= \cos^{-1}(1 - r) && \text{From } r = 1 - \cos\theta \\
&= \cos^{-1}\left(1 - \left(2\sqrt{2} - 2\right)\right) && \text{Set } r = 2\sqrt{2} - 2. \\
&= \cos^{-1}\left(3 - 2\sqrt{2}\right) \\
&= \pm 80^\circ. && \text{Rounded to the nearest degree}
\end{aligned}$$

We have thus identified two intersection points: $(r, \theta) = (2\sqrt{2} - 2, \pm 80^\circ)$.

If we graph the equations $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$ together (Figure 10.47), as we can now do by combining the graphs in Figures 10.44 and 10.45, we see that the curves also intersect at the point $(2, \pi)$ and the origin. Why weren't the r-values of these points revealed by the simultaneous solution? The answer is that the points $(0, 0)$ and $(2, \pi)$ are not on the curves "simultaneously." They are not reached at the same value of θ. On the curve $r = 1 - \cos\theta$, the point $(2, \pi)$ is reached when $\theta = \pi$. On the curve $r^2 = 4\cos\theta$, it is reached when $\theta = 0$, where it is identified not by the coordinates $(2, \pi)$, which do not satisfy the equation, but by the coordinates $(-2, 0)$, which do. Similarly, the cardioid reaches the origin when $\theta = 0$, but the curve $r^2 = 4\cos\theta$ reaches the origin when $\theta = \pi/2$.

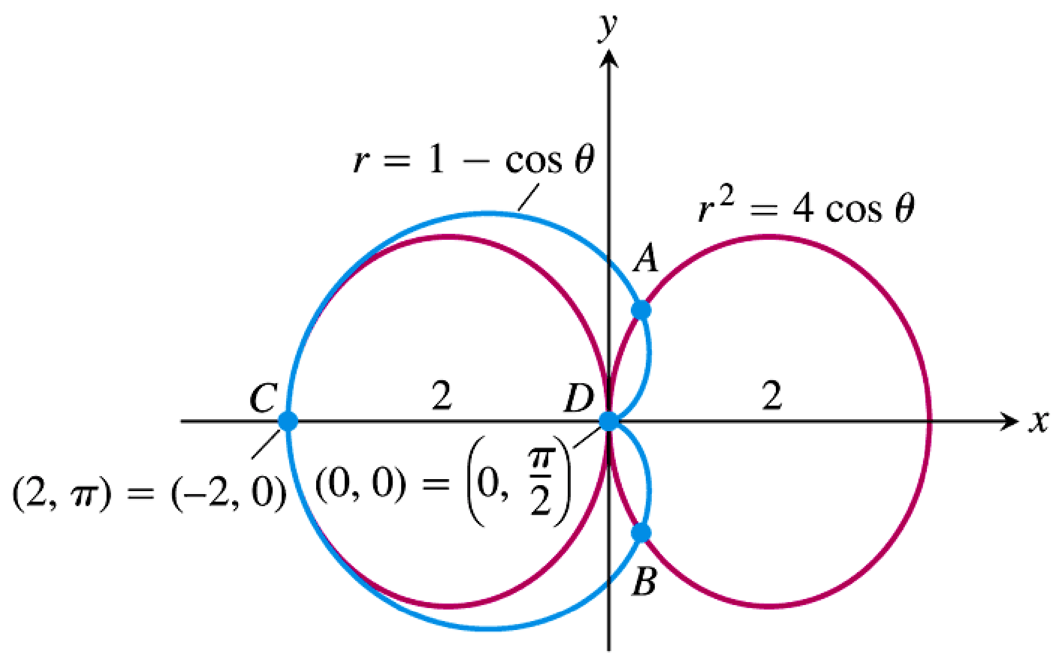


FIGURE 10.47 The four points of intersection of the curves $r = 1 - \cos\theta$ and $r^2 = 4\cos\theta$ (Example). Only A and B were found by simultaneous solution. The other two were disclosed by graphing.

TANGENTS TO POLAR CURVES

Let $r=f(\theta)$ be a polar curve, then $x=r.\cos\theta=f(\theta).\cos\theta$ and $y=r.\sin\theta=f(\theta).\sin\theta$ is a parametrization of the polar curve. Using the slope formula for parametric curves:

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{f'(\theta).\sin\theta+f(\theta).\cos\theta}{f'(\theta).\cos\theta-f(\theta).\sin\theta}\quad \text{provided } \frac{dx}{d\theta}\neq 0 \text{ at } (r,\theta)$$

To locate the horizontal tangents, solve $\frac{dy}{d\theta}=0$ (where $\frac{dx}{d\theta}\neq 0$).

To locate the vertical tangents, solve $\frac{dx}{d\theta}=0$ (where $\frac{dy}{d\theta}\neq 0$).

Example (a) For the cardioid $r=1-\cos\theta$, find the slope of the tangent line when $\theta=\frac{\pi}{6}$.

(b) Find the points on the cardioid $r=1-\cos\theta$ where the tangent line is horizontal or vertical.

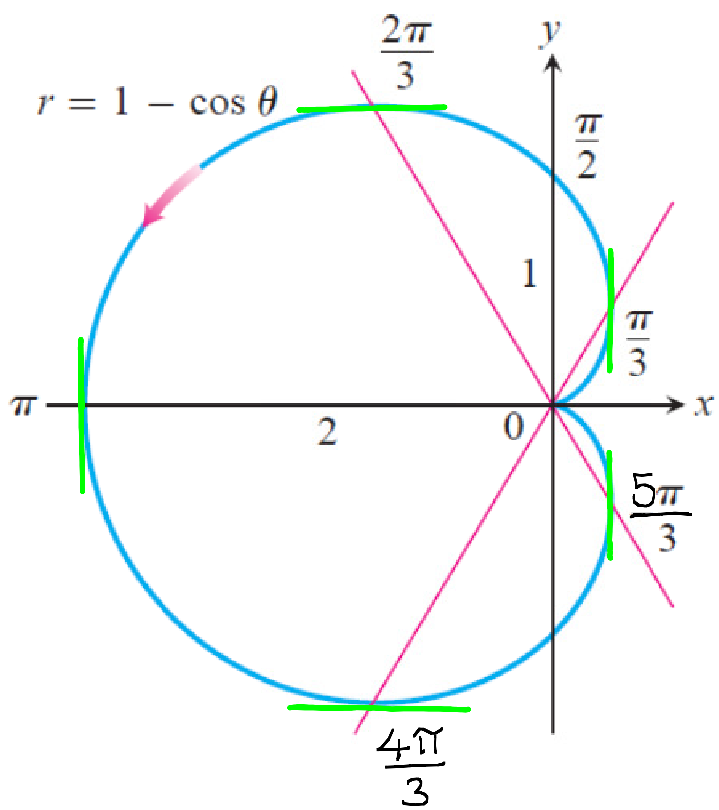
$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}\left((1-\cos\theta).\sin\theta\right)}{\frac{d}{d\theta}\left((1-\cos\theta).\cos\theta\right)} = \frac{\sin\theta.\sin\theta + (1-\cos\theta).\cos\theta}{\sin\theta.\cos\theta + (1-\cos\theta).(-\sin\theta)}$$

$$= \frac{\sin^2\theta - \cos^2\theta + \cos\theta}{2\sin\theta.\cos\theta - \sin\theta} = \frac{1+\cos\theta - 2\cos^2\theta}{\sin\theta(2\cos\theta-1)} = \frac{(1+2\cos\theta)(1-\cos\theta)}{\sin\theta(2\cos\theta-1)}$$

(a) $$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{\left(1+2\frac{\sqrt{3}}{2}\right)\left(1-\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}\left(2.\frac{\sqrt{3}}{2}-1\right)} = \frac{(1+\sqrt{3})(2-\sqrt{3})}{(\sqrt{3}-1)} = \frac{\sqrt{3}-1}{\sqrt{3}-1} = 1$$

(b) $$\frac{dy}{d\theta} = (1+2\cos\theta)(1-\cos\theta) = 0 \iff \theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0$$

$$\frac{dx}{d\theta} = \sin\theta(2\cos\theta-1) = 0 \iff \theta = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$$



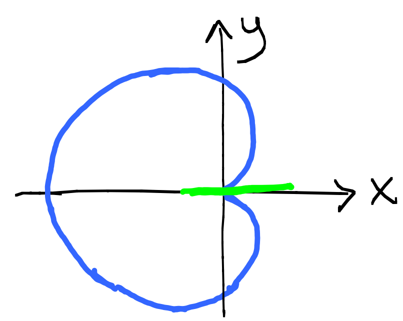
There are horizontal tangents at the points $\left(\frac{3}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{3}{2}, \frac{4\pi}{3}\right)$ and vertical tangents at $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $(2, \pi)$ and $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$.

When $\theta = 0$, both $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ are 0. We should evaluate the limit:

$$\lim_{\theta \to 0} \frac{dy}{dx} = \lim_{\theta \to 0} \frac{(1+2\cos\theta)(1-\cos\theta)}{\sin\theta(2\cos\theta - 1)} = \lim_{\theta \to 0} \frac{1+2\cos\theta}{2\cos\theta - 1} \cdot \lim_{\theta \to 0} \frac{1-\cos\theta}{\sin\theta}$$

$$\lim_{\theta \to 0} \frac{1+2\cos\theta}{2\cos\theta - 1} = 3 \quad \text{and} \quad \lim_{\theta \to 0} \frac{1-\cos\theta}{\sin\theta} \underset{\text{L'H.Rule}}{\overset{\left(\frac{0}{0}\right)}{=}} \lim_{\theta \to 0} \frac{\sin\theta}{\cos\theta} = 0$$

$$\Rightarrow \lim_{\theta \to 0} \frac{dy}{dx} = 0 \quad \Rightarrow \text{At } (0,0) \text{ there is a horizontal tangent}$$



Exercises: 1) Graph the following polar curves:

a) $r = 1 + \sin\theta$ b) $r^2 = -\sin(2\theta)$

2) Find the slope of the tangent line to $r = 1 + \sin\theta$ when $\theta = \frac{\pi}{3}$.

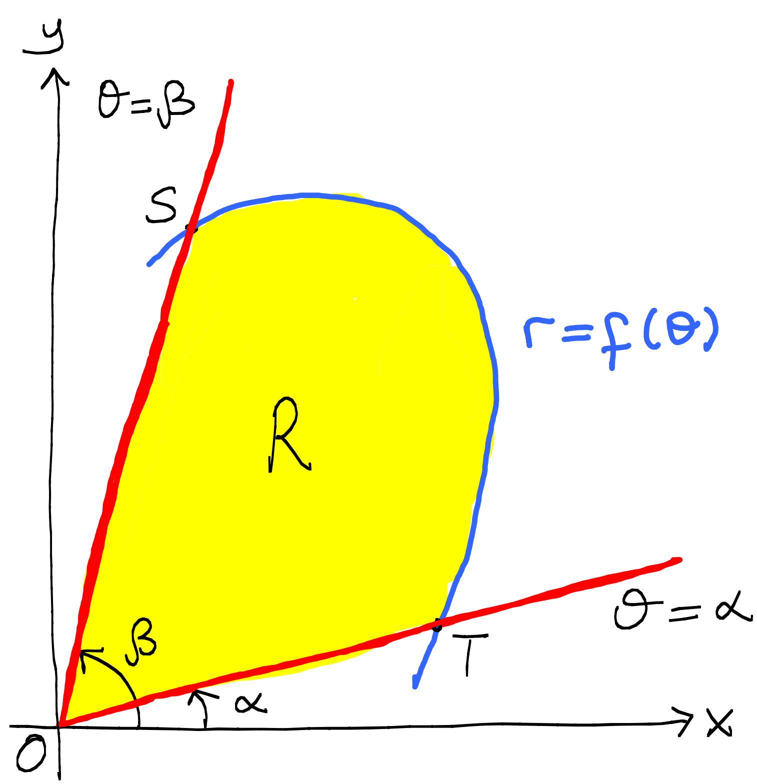
3) Find the points on $r = 1 + \sin\theta$ at which the tangent line is horizontal or vertical.

10.7

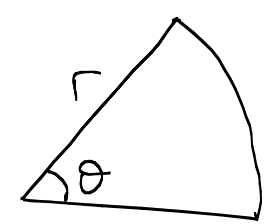
Areas and Lengths in Polar Coordinates

Let R be the region bounded by the polar curve $r=f(\theta)$ and by the rays $\theta=\alpha$ and $\theta=\beta$, where f is a positive continuous function and $0<\beta-\alpha\leq 2\pi$.

We want to develop a formula for the area of R.



We will use the formula for the area of a sector of a circle:



$$A=\frac{1}{2}r^2\theta$$

We approximate the region with circular sectors as follows:
The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is

$$A_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \cdot \Delta\theta_k$$

The area of OTS is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \cdot \Delta\theta_k$$

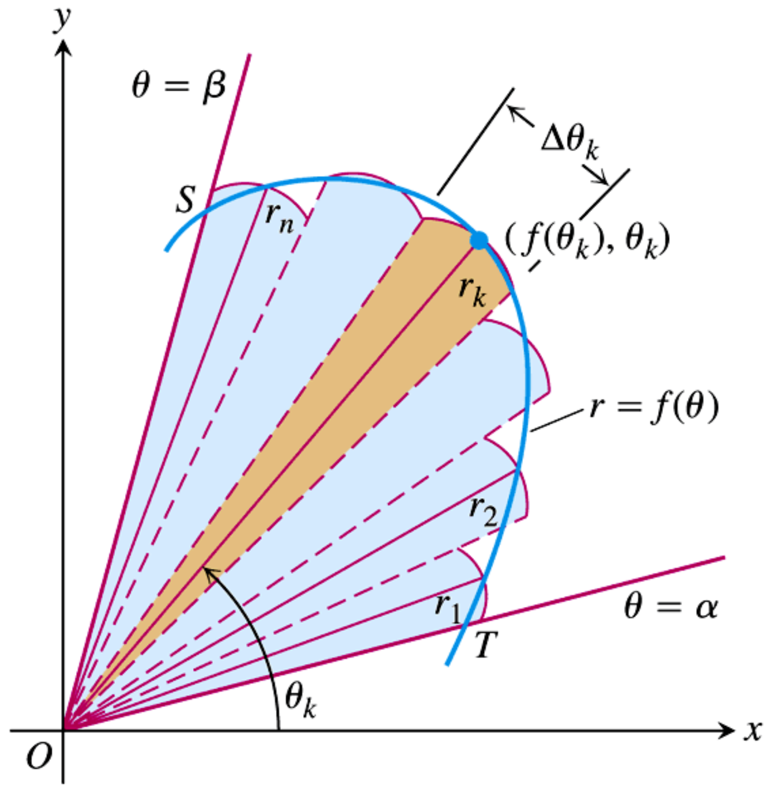


FIGURE 10.48 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

As $n \to \infty$ we obtain the following formula:

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta), \alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

This is the integral of the **area differential** (Figure 10.49)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} (f(\theta))^2 \, d\theta.$$

The choice of α and β is not unique, as long as the criteria $0 < \beta - \alpha \leq 2\pi$ is satisfied.

EXAMPLE 1 Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid (Figure 10.50) and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2\, d\theta &= \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2\, d\theta \\
&= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta)\, d\theta \\
&= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta \\
&= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.
\end{aligned}$$

■

$$R = \{ (r, \theta) : 0 \leq \theta \leq 2\pi \text{ and } 0 \leq r \leq 2(1+\cos\theta) \}$$

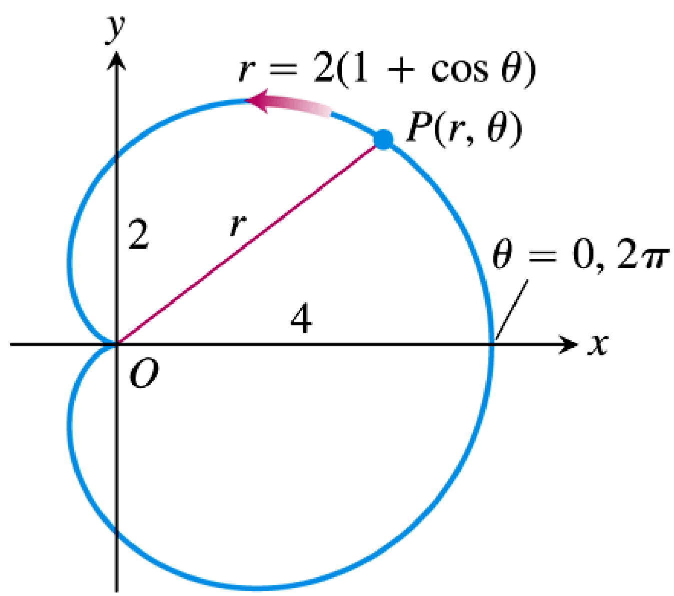


FIGURE 10.50 The cardioid in Example 1.

- $0 < \beta - \alpha \leq 2\pi$
- $0 \leq r_1(\theta) \leq r_2(\theta)$ for all $\theta \in [\alpha, \beta]$

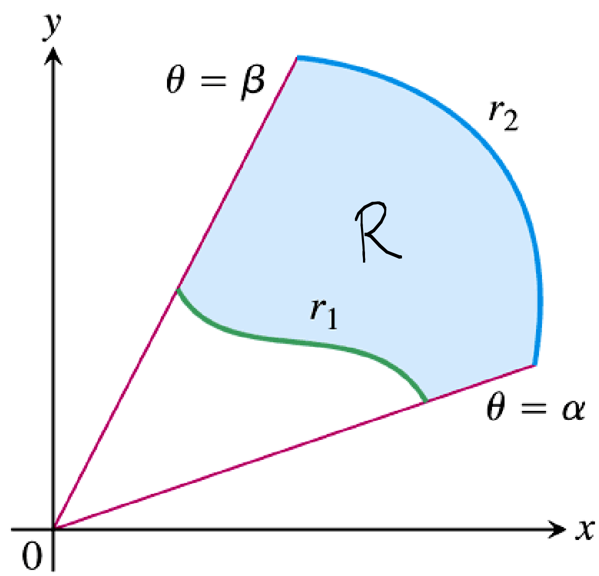
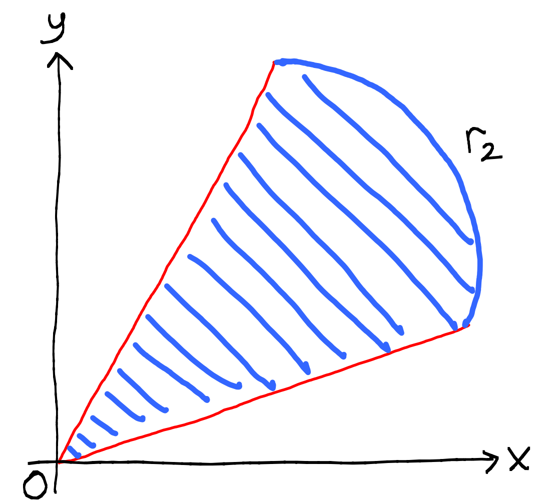
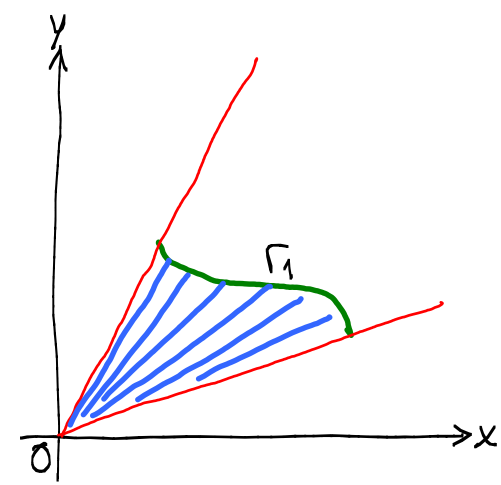


FIGURE 10.52 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

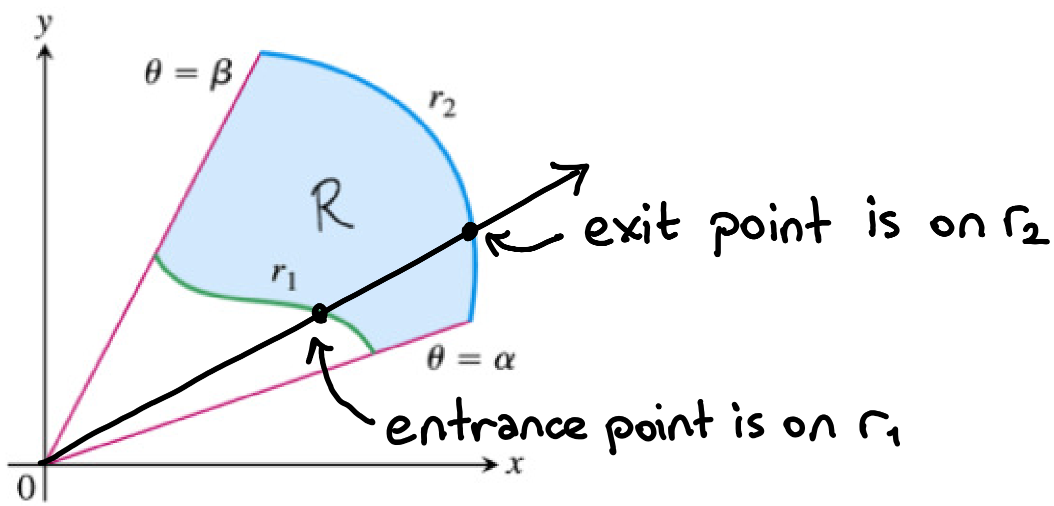




$$R = \{ (r, \theta) : \alpha \leq \theta \leq \beta \text{ and } r_1(\theta) \leq r \leq r_2(\theta) \}$$

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$, $0 < \beta - \alpha \le 2\pi$

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$



EXAMPLE 2 Finding Area Between Polar Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution We sketch the region to determine its boundaries and find the limits of integration (Figure 10.53). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is

$$
\begin{aligned}
A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^{2} - r_1^{2}\right) d\theta \\
&= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^{2} - r_1^{2}\right) d\theta \qquad \text{Symmetry} \\
&= \int_{0}^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \\
&= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_{0}^{\pi/2} \left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.
\end{aligned}
$$

points of intersection:

$r_1 = 1 - \cos\theta = 1 = r_2$

$\Leftrightarrow \cos\theta = 0$

$\Rightarrow \theta = -\frac{\pi}{2}$ or $\frac{\pi}{2}$

You can also choose

$\alpha = \frac{3\pi}{2}$ and $\beta = \frac{5\pi}{2}$

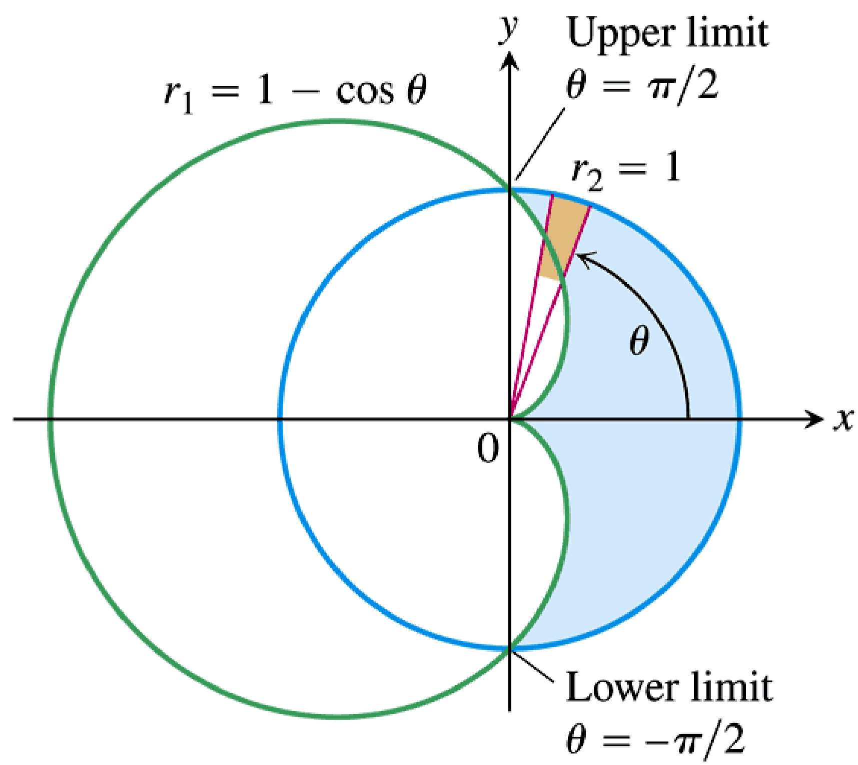


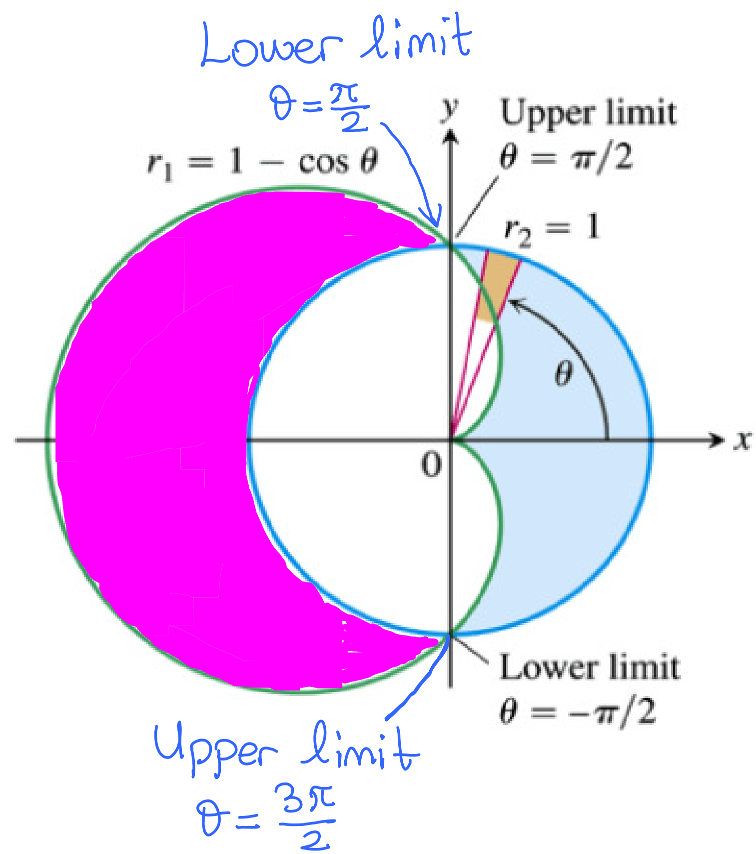
FIGURE 10.53 The region and limits of integration in Example 2.

so that $A = \int_{3\pi/2}^{5\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$ which will have the same result.

Area of the region outside the circle $r=1$ and inside the cardioid $r=1-\cos\theta$ is:

$$A=\int_{\pi/2}^{3\pi/2} \frac{1}{2}\left[(1-\cos\theta)^2-(1)^2\right]d\theta$$

(Area of the pink shaded region)

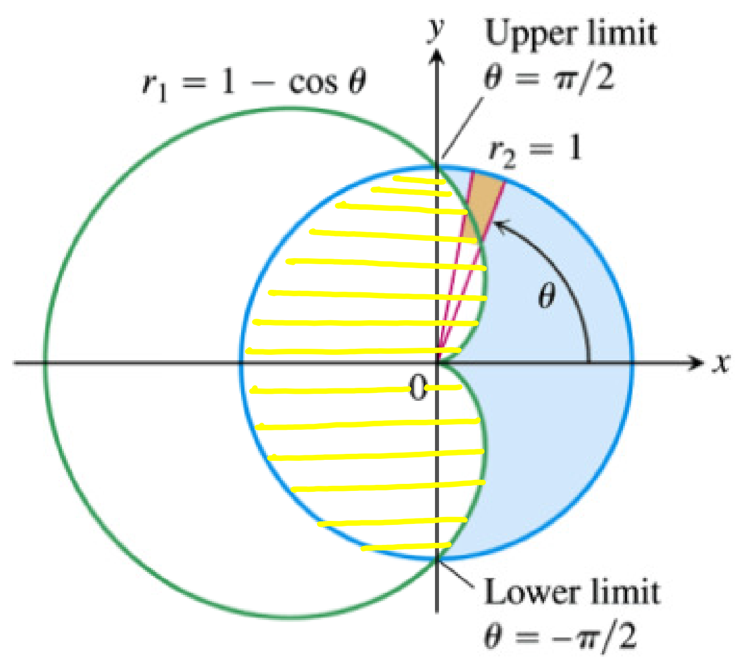


Area of the region shared by the circle $r=1$ and the cardioid $r=1-\cos\theta$ is: (Area of the yellow shaded region)

$$A=\int_0^{\pi/2}\frac{1}{2}(1-\cos\theta)^2\,d\theta+\int_{\pi/2}^{\pi}\frac{1}{2}(1)^2\,d\theta$$

$$+\int_{\pi}^{3\pi/2}\frac{1}{2}(1)^2\,d\theta+\int_{3\pi/2}^{2\pi}\frac{1}{2}(1-\cos\theta)^2\,d\theta$$

$$=2\left(\int_0^{\pi/2}\frac{1}{2}(1-\cos\theta)^2\,d\theta+\int_{\pi/2}^{\pi}\frac{1}{2}(1)^2\,d\theta\right)$$



(You can use the symmetry of the region with respect to x-axis)

EXAMPLE 3 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid and the circle are sketched in Figure 5 and the desired region is shaded. The values of a and b in Formula 4 are determined by finding the points of intersection of the two curves. They intersect when $3\sin\theta = 1 + \sin\theta$, which gives $\sin\theta = \frac{1}{2}$, so $\theta = \pi/6, 5\pi/6$. The desired area can be found by subtracting the area inside the cardioid between $\theta = \pi/6$ and $\theta = 5\pi/6$ from the area inside the circle from $\pi/6$ to $5\pi/6$. Thus

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 + \sin\theta)^2\,d\theta$$

Since the region is symmetric about the vertical axis $\theta = \pi/2$, we can write

$$A = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta)\,d\theta\right]$$

$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta)\,d\theta$$

$$= \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\,d\theta \qquad \left[\text{because } \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)\right]$$

$$= 3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\pi/6}^{\pi/2} = \pi$$

□

(Origin is the 3rd point of intersection which couldn't be found algebraically)

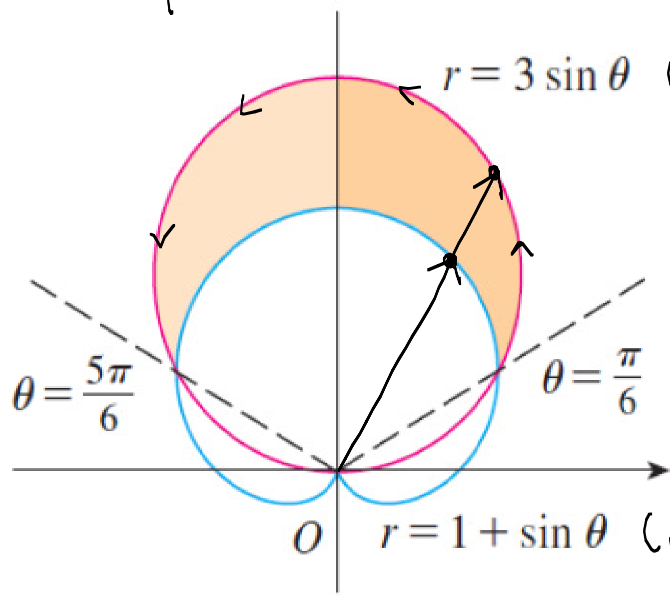


FIGURE 5

Area of the region outside the cirle $r = 3\sin\theta$ and inside the cardioid $r = 1 + \sin\theta$ is :

$$A = 2 \cdot \left(\int_{-\pi/2}^{0} \frac{1}{2}(1+\sin\theta)^2 \, d\theta + \int_{0}^{\pi/6} \frac{1}{2}\left[(1+\sin\theta)^2 - (3\sin\theta)^2\right] d\theta \right)$$

(yellow shaded region below)

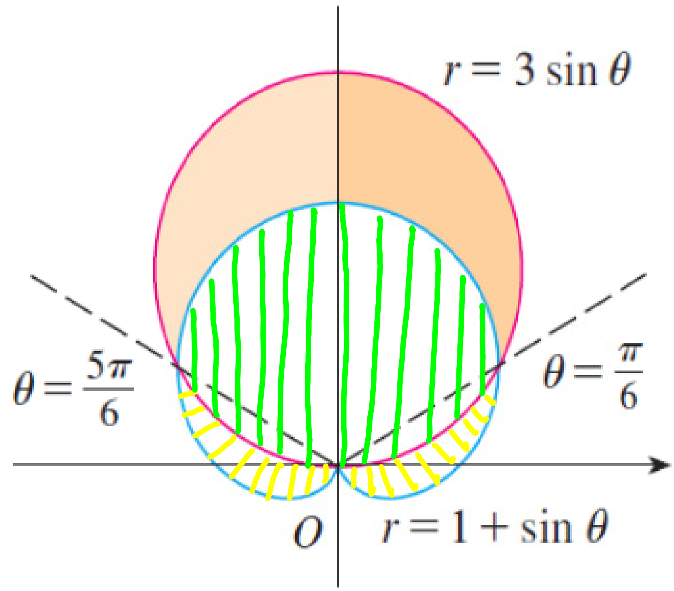


FIGURE 5

Area of the region shared by the circle $r = 3\sin\theta$ and the cardioid $r = 1 + \sin\theta$ is:

$$A = 2\left[\int_0^{\pi/6} \frac{1}{2}(3\sin\theta)^2\,d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(1+\sin\theta)^2\,d\theta\right]$$

(green shaded region)

__Question:__ Express the area of the region enclosed by the circle $r = 3\sin\theta$ (see previous example) as an integral.

$$\text{Area} = \int_0^{\pi} \frac{1}{2}(3\sin\theta)^2\, d\theta = 2\int_0^{\pi/2} \frac{1}{2}(3\sin\theta)^2\, d\theta = \pi \cdot \left(\frac{3}{2}\right)^2$$

Note that the circle is traced once for $\theta \in [0, \pi]$.

When $\theta \in [\pi, 2\pi]$, $r = 3\sin\theta < 0$, so the points are in the backward direction, tracing the circle for the second time.

STRATEGY

To find the area of a region bounded by the polar curves $r=f_1(\theta)$ and $r=f_2(\theta)$:

- Sketch the curves and find the intersection points $\theta=\theta_0$ and $\theta=\theta_1$ (because of periodicity choose θ_0 and θ_1 such that $0<\theta_1-\theta_0\leq 2\pi$)
- Draw rays from origin to the region and specify the entrance curve r_1 and the exit curve r_2.
- Construct the area as an integral

$$A=\int_{\theta_0}^{\theta_1}\frac{1}{2}\left(r_2^2-r_1^2\right)d\theta$$ (use any symmetry if exists)

ARC LENGTH FOR POLAR CURVES

To find the length of a polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$ we will use its parametrization and the length formula for parametric curves.

$$x = r.\cos\theta = f(\theta).\cos\theta \,, \quad y = r.\sin\theta = f(\theta).\sin\theta \,, \quad \alpha \le \theta \le \beta .$$

$$\text{Length } L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = f'(\theta).\cos\theta - f(\theta).\sin\theta \,, \quad \frac{dy}{d\theta} = f'(\theta).\sin\theta + f(\theta).\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (f(\theta))^2 + (f'(\theta))^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

By trigonometric identities

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

EXAMPLE 4 Finding the Length of a Cardioid

Find the length of the cardioid $r = 1 - \cos\theta$.

Solution We sketch the cardioid to determine the limits of integration (Figure 10.54). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.

With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 - \cos\theta)^2 + (\sin\theta)^2 \\ &= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta \end{aligned}$$

and

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \end{aligned}$$

$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \geq 0 \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$

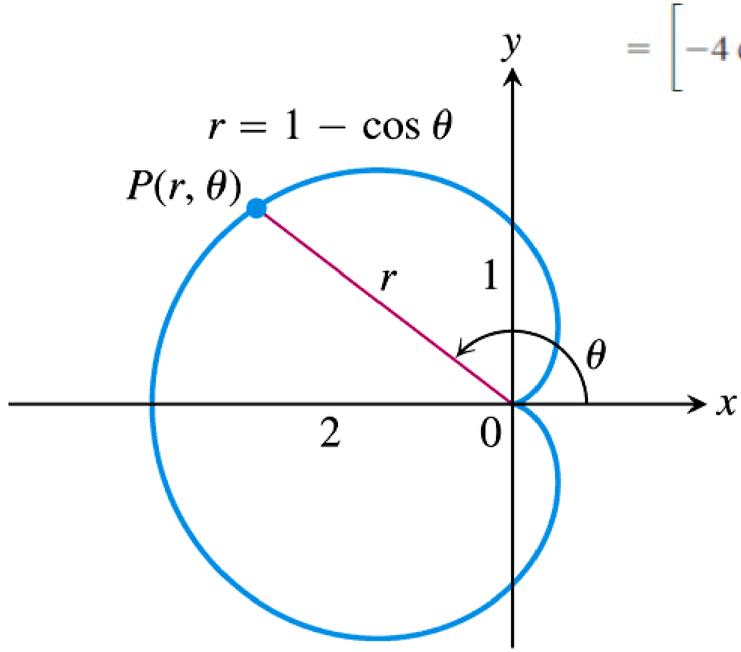


FIGURE 10.54 Calculating the length of a cardioid (Example 4).

Exercise

Find the length of the following curves:

a) $r = \sqrt{1 + \sin 2\theta}\,, \quad 0 \le \theta \le \pi\sqrt{2}.$

b) $r = \cos^3\left(\frac{\theta}{3}\right), \quad 0 \le \theta \le \frac{\pi}{4}$

c) $r = e^{\frac{\theta}{2}}, \quad 0 \le \theta \le \pi$