Inverses and the Adjoint Matrix

The inverse $\mathbf{A}^{-1}$ of the invertible $n \times n$ matrix $\mathbf{A}$ can be found by solving the matrix equation

$$\mathbf{AX} = \mathbf{I}.$$

If we write the coefficient matrix $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, the unknown matrix $\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$, and the identity matrix $\mathbf{I} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$ in terms of their columns, then

$$\mathbf{A}\mathbf{x}_j = \mathbf{e}_j$$

for $j = 1, 2, \ldots, n$. Each of these n equations can then be solved explicitly using Cramer's rule.

$$x_{ij} = \frac{\begin{vmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{i-1} & \mathbf{e}_j & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n \end{vmatrix}}{|\mathbf{A}|}$$

for $i,\ j = 1, 2, \ldots, n$.

THEOREM 5 The Inverse Matrix

The inverse of the invertible matrix $\mathbf{A}$ is given by the formula

$$\mathbf{A}^{-1} = \frac{\left[A_{ij} \right]^T}{|\mathbf{A}|}, \tag{23}$$

where, as usual, A_{ij} denotes the *ij*th cofactor of $\mathbf{A}$; that is, A_{ij} is the product of $(-1)^{i+j}$ and the *ij*th minor determinant of $\mathbf{A}$.

We see in (23) the *transpose* of the **cofactor matrix** $\left[A_{ij} \right]$ of the $n \times n$ matrix $\mathbf{A}$. This transposed cofactor matrix is called the **adjoint matrix** of $\mathbf{A}$ and is denoted by

$$\operatorname{adj} \mathbf{A} = \left[A_{ij} \right]^T = \left[A_{ji} \right].$$

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{bmatrix}$$

$\Rightarrow |A| = 29 \neq 0 \quad A^{-1} = ?$

$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} = -17$

$A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 4 & 5 \\ -3 & -1 \end{vmatrix} = -11$

$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 4 & 2 \\ -3 & 3 \end{vmatrix} = 18$

$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 4 & 5 \\ 3 & -1 \end{vmatrix} = 19$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 5 \\ -3 & -1 \end{vmatrix} = 14$

$A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix} = -15$

$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 4 & 5 \\ 2 & 5 \end{vmatrix} = 10$

$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 5 \\ 4 & 5 \end{vmatrix} = 15$

$A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = -14$

$$\operatorname{Adj} A = [A_{ij}]^T = \begin{bmatrix} -17 & -11 & 18 \\ 19 & 14 & -15 \\ 10 & 15 & -14 \end{bmatrix}^T = \begin{bmatrix} -17 & 19 & 10 \\ -11 & 14 & 15 \\ 18 & -15 & -14 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{29} \cdot \begin{bmatrix} -17 & 19 & 10 \\ -11 & 14 & 15 \\ 18 & -15 & -14 \end{bmatrix}.$$

Vector Spaces

The Vector Space $\mathbf{R}^n$ and Subspaces

An ***n*-tuple** of real numbers is an (ordered) list $(x_1, x_2, x_3, \ldots, x_n)$ of n real numbers. Thus $(1, 3, 4, 2)$ is a 4-tuple and $(0, -3, 7, 5, 2, -1)$ is a 6-tuple. A 2-tuple is an ordered pair and a 3-tuple is an ordered triple of real numbers

DEFINITION ***n*-Space $\mathbf{R}^n$**

The ***n*-dimensional space $\mathbf{R}^n$** is the set of all n-tuples $(x_1, x_2, x_3, \ldots, x_n)$ of real numbers.

The elements of n-space $\mathbf{R}^n$ are called **points** or **vectors**, and we ordinarily use boldface letters to denote vectors. The ith entry of the vector $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$ is called its ith **coordinate** or its ith **component**. For consistency with matrix operations, we agree that

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

so that the $n \times 1$ matrix, or *column* vector, is simply another notation for the same ordered list of n real numbers.

If $n > 3$, we cannot visualize vectors in $\mathbf{R}^n$ in the concrete way that we can "see" vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$.

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbf{R}^n$, then their **sum** is the vector $\mathbf{u} + \mathbf{v}$ given by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n). \tag{1}$$

If c is a scalar—a real number—then the scalar multiple cu is also defined in componentwise fashion:

$$c\mathbf{u} = (cu_1, cu_2, \ldots, cu_n).$$

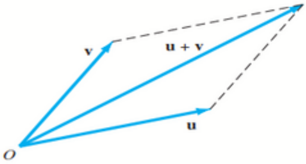

FIGURE 4.2.2. The parallelogram law for addition of vectors, in $\mathbf{R}^n$ just as in $\mathbf{R}^2$ or $\mathbf{R}^3$.

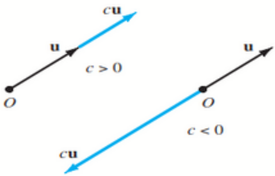

FIGURE 4.2.3. Multiplication of the vector $\mathbf{u}$ by the scalar c, in $\mathbf{R}^n$ just as in $\mathbf{R}^2$ or $\mathbf{R}^3$.

Definition of a Vector Space

Let V be a set of elements called *vectors*, in which the operations of addition of vectors and multiplication of vectors by scalars are defined. That is, given vectors $\mathbf{u}$ and $\mathbf{v}$ in V and a scalar c, the vectors $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ are also in V (so that V is *closed* under vector addition and multiplication by scalars). Then, with these operations, V is called a **vector space** provided that—given any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and any scalars a and b—the following properties hold true:

(a)	$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	(commutativity)
(b)	$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$	(associativity)
(c)	$\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$	(zero element)
(d)	$\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$	(additive inverse)
(e)	$a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$	(distributivity)
(f)	$(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$	
(g)	$a(b\mathbf{u})=(ab)\mathbf{u}$	
(h)	$(1)\mathbf{u}=\mathbf{u}$	

In property (c), it is meant that there exists a **zero vector** $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$. The zero vector in $\mathbf{R}^n$ is

$$\mathbf{0} = (0, 0, \ldots, 0).$$

Similarly, property (d) actually means that, given the vector $\mathbf{u}$ in V, there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. In $\mathbf{R}^n$, we clearly have

$$-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n).$$

The fact that $\mathbf{R} = \mathbf{R}^1$ satisfies properties (a)–(h), with the real numbers playing the dual roles of scalars *and* vectors, means that the real line may be regarded as a vector space. If $n > 1$, then each of properties (a)–(h) may be readily verified for $\mathbf{R}^n$ by working with components and applying the corresponding properties of real numbers. For example,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)\\ &= (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n)\\ &= \mathbf{v} + \mathbf{u}.\end{aligned}$$

Thus n-space $\mathbf{R}^n$ *is a vector space* with the operations of addition of vectors and multiplication of vectors by scalars.

Let $\mathscr{F}$ be the set of all real-valued functions defined on the real number line $\mathbf{R}$. Then each vector in $\mathscr{F}$ is a function $\mathbf{f}$ such that the real number $\mathbf{f}(x)$ is defined for all x in $\mathbf{R}$. Given $\mathbf{f}$ and $\mathbf{g}$ in $\mathscr{F}$ and a real number c, the functions $\mathbf{f}+\mathbf{g}$ and $c\mathbf{f}$ are defined in the natural way,

$$(\mathbf{f}+\mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x)$$

and

$$(c\mathbf{f})(x) = c(\mathbf{f}(x)).$$

Then each of properties (a)–(h) of a vector space follows readily from the corresponding property of the real numbers. For instance, if a is a scalar, then

Subspaces

Let $W = \{\mathbf{0}\}$ be the subset of $\mathbf{R}^n$. Then W satisfies properties (a)–(h) of a vector space. Thus W is itself a vector space. A subset of a vector space V that is itself a vector space is called a *subspace* of V.

DEFINITION Subspace

Let W be a nonempty subset of the vector space V. Then W is a **subspace** of V provided that W itself is a vector space with the operations of addition and multiplication by scalars as defined in V.

THEOREM 1 Conditions for a Subspace

The nonempty subset W of the vector space V is a subspace of V if and only if it satisfies the following two conditions:

(i) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is also in W.

(ii) If $\mathbf{u}$ is in W and c is a scalar, then the vector $c\mathbf{u}$ is also in W.

Example:

Let W be the subset of $\mathbf{R}^n$ consisting of all those vectors $(x_1, x_2, \ldots, x_n)$ whose coordinates satisfy the single homogeneous linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0,$$

where the given coefficients $a_1, a_2, \ldots, a_n$ are not all zero.

show that W *is* a subspace of $\mathbf{R}^n$.

* If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are vectors in W, $u+v$

$$a_1(u_1+v_1) + a_2(u_2+v_2) \cdots + a_n(u_n+v_2)$$
$$= (a_1u_1 + a_2u_2 + \cdots + a_nu_n) + (a_1v_1 + a_2v_2 + \cdots + a_nv_n)$$
$$= 0 + 0 = 0 \quad \Rightarrow \quad u+v \in W.$$

+ If c is scalar, then

$$a_1(cu_1) + a_2(cu_2) + \cdots + a_n(cu_n) = c\,(a_1u_1 + a_2u_2 + \cdots + a_nu_n)$$
$$= c \cdot 0 = 0 \Rightarrow cu \in W.$$

Example:

Let W be the set of all those vectors (x_1, x_2, x_3, x_4) in $\mathbf{R}^4$ whose four coordinates are all nonnegative: $x_i \geq 0$ for $i = 1, 2, 3, 4$. Then it should be clear that the sum of two vectors in W is also a vector in W, because the sum of two nonnegative numbers is nonnegative. Thus W satisfies condition (i) of Theorem 1. But if we take $\mathbf{u} = (1, 1, 1, 1)$ in W and $c = -1$, then we find that the scalar multiple

$$c\mathbf{u} = (-1)(1, 1, 1, 1) = (-1, -1, -1, -1)$$

is *not* in W. Thus W fails to satisfy condition (ii) and therefore is *not* a subspace of $\mathbf{R}^4$. ■

Example:

Let W be the set of all those vectors (x_1, x_2, x_3, x_4) in $\mathbf{R}^4$ such that $x_1x_4 = 0$. Now W satisfies condition (ii) of Theorem 1, because $x_1x_4 = 0$ implies that $(cx_1)(cx_4) = 0$ for any scalar c. But if we take the vectors $\mathbf{u} = (1, 1, 0, 0)$ and $\mathbf{v} = (0, 0, 1, 1)$ in W, we see that their sum $\mathbf{u} + \mathbf{v} = (1, 1, 1, 1)$ is *not* in W. Thus W does not satisfy condition (i) and therefore is *not* a subspace of $\mathbf{R}^4$. ■

$u = (x_1, x_2, x_3, x_4) \in W \quad : \quad x_1 \cdot x_4 = 0.$

$u, v \in W \qquad u+v \overset{?}{\in} W, \qquad c\,u \overset{?}{\in} W$

$x_1 \cdot x_4 = 0$

$c \cdot u = (c \cdot x_1, cx_2, cx_3, cx_4)$

$(cx_1)(cx_4) = c^2 \cdot \underbrace{x_1 \cdot x_4}_{=0} = 0$

$c \cdot u \in W.$

$u = (1,1,0,0) \in W$

$v = (0,0,1,1) \in W$

$u+v\,(1,1,1,1) \notin W$

THEOREM 2 Solution Subspaces

If $\mathbf{A}$ is a (constant) $m \times n$ matrix, then the solution set of the homogeneous linear system

$$\mathbf{Ax} = \mathbf{0} \tag{3}$$

is a subspace of $\mathbf{R}^n$.

***Proof*:** Let W denote the solution set of Eq. (3). If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{Au} = \mathbf{Av} = \mathbf{0}$. Hence

$$\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus the sum $\mathbf{u} + \mathbf{v}$ is also in W, and hence W satisfies condition (i) of Theorem 1. If c is a scalar, then

$$\mathbf{A}(c\mathbf{u}) = c(\mathbf{Au}) = c\mathbf{0} = \mathbf{0},$$

so $c\mathbf{u}$ is in W if $\mathbf{u}$ is in W. Thus W also satisfies condition (ii) of Theorem 1. It therefore follows that W is a subspace of $\mathbf{R}^n$. ■

Note that the solution set of a non-homogeneous linear system

$$\mathbf{Ax} = \mathbf{b} \qquad c.u \qquad \underline{\underline{Au=b}}$$

with $\mathbf{b} \neq \mathbf{0}$ is *never* a subspace. For if $\mathbf{u}$ were a solution vector of the system then

$$\mathbf{A}(2\mathbf{u}) = 2(\mathbf{Au}) = 2\mathbf{b} \neq \mathbf{b} \qquad A.(cu) \neq b$$

because $\mathbf{b} \neq \mathbf{0}$. Thus $2\mathbf{u}$ is *not* a solution vector.

subspaces of $\mathbf{R}^n$ lie the **zero subspace** $\{\mathbf{0}\}$ and $\mathbf{R}^n$ itself. Every other subspace of $\mathbf{R}^n$, each one that is neither $\{\mathbf{0}\}$ nor $\mathbf{R}^n$, is called a **proper subspace** of $\mathbf{R}^n$.

$$
\begin{aligned}
x_1 + 3x_2 - 15x_3 + \ \ 7x_4 &= 0\\
x_1 + 4x_2 - 19x_3 + 10x_4 &= 0\\
2x_1 + 5x_2 - 26x_3 + 11x_4 &= 0.
\end{aligned}
$$

Find a solution space.

$$
\left[\begin{array}{cccc|c} 1 & 3 & -15 & 7 & 0\\ 1 & 4 & -19 & 10 & 0\\ 2 & 5 & -26 & 11 & 0 \end{array}\right]
\xrightarrow[R_3 \to (-2)R_1 + R_3]{R_2 \to R_2 - R_1}
\left[\begin{array}{cccc|c} 1 & 3 & -15 & 7 & 0\\ 0 & 1 & -4 & 3 & 0\\ 0 & -1 & 4 & -3 & 0 \end{array}\right]
$$

$$
\xrightarrow[R_3 \to R_2 + R_3]{R_1 \to (-3)R_2 + R_1}
\begin{array}{c}
\left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 0\\ 0 & 1 & -4 & 3 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\\
\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ x_1 & x_2 & x_3 & x_4 \end{array}
\end{array}
$$

$$
\begin{aligned}
x_1 - 3x_3 - 2x_4 &= 0\\
x_2 - 4x_3 + 3x_4 &=
\end{aligned}
$$

$$
x_3 = s,\quad x_4 = t \Rightarrow x_2 = 4s - 3t,\quad x_1 = 3s + 2t
$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s + 2t \\ 4s - 3t \\ s \\ t \end{bmatrix} = s \cdot \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{u} + t \underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{v} = s \cdot u + t v$$

$$W = \left\{ su + tv \;\middle|\; s, t \in \mathbb{R} \right\}.$$

Linear Combinations and Independence of Vectors

The vector $\mathbf{w}$ is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ provided that there exist scalars $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

Example:

Determine whether the vector $\mathbf{w} = (2, -6, 3)$ in $\mathbf{R}^3$ is a linear combination of the vectors $\mathbf{v}_1 = (1, -2, -1)$ and $\mathbf{v}_2 = (3, -5, 4)$ or not.

$$c_1 v_1 + c_2 v_2 \stackrel{?}{=} w$$

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix} \Rightarrow \begin{aligned} c_1 + 3c_2 &= 2 \\ -2c_1 - 5c_2 &= -6 \\ -c_1 + 4c_2 &= 3 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ -2 & -5 & -6 \\ -1 & 4 & 3 \end{array}\right] \xrightarrow[R_3 \to R_1 + R_3]{R_2 \to 2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 7 & 5 \end{array}\right] \xrightarrow{R_3 \to (-7)R_1 + R_3} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 19 \end{array}\right]$$

The system is inconsistent, so the scalars c_1 and c_2 do not exists. Thus, w is not a linear combination of v_1 and v_2.

Suppose that $v_1, v_2, \dots, v_k$ are vectors in a vector space V. Then we say that the vectors $v_1, v_2, \dots, v_k$ span the vector space V provided that every vector in V is a linear combination of these k vectors. We may also say that the set $S = \{v_1, v_2, \dots, v_k\}$ of vectors is a spanning set for V.

Example: $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$ span $\mathbb{R}^3$

because, for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$x = x_1 \cdot i + x_2 \cdot j + x_3 \cdot k. \rightarrow x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

$$\underline{\underline{(3,2,1)}} = 3.(1,0,0) + 2.(0,1,0) + 1.(0,0,1)$$

$$= 3.i + 2.j + 1.k$$

$\mathbb{R}^3 = \text{span}\{i, j, k\}.$

THEOREM 1 The Span of a Set of Vectors

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in the vector space V. Then the set W of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a subspace of V.

***Proof*:** We must show that W is closed under addition of vectors and multiplication by scalars. If

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k$$

are vectors in W, then

$$\mathbf{u} + \mathbf{v} = (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2 + \cdots + (a_k + b_k)\mathbf{v}_k$$

and

$$c\mathbf{u} = (ca_1)\mathbf{v}_1 + (ca_2)\mathbf{v}_2 + \cdots + (ca_k)\mathbf{v}_k$$

for any scalar c. Thus $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and therefore are vectors in W. It is clear that W is nonempty, and hence W is a subspace of V. ■

Example:

Determine whether the vector $\mathbf{w} = (-7, 7, 11)$ in $\mathbf{R}^3$ is a linear combination of the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (-4, -1, 2)$, and $\mathbf{v}_3 = (-3, 1, 3)$, or not.

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \\ 11 \end{bmatrix}$$

$$c_1 - 4c_2 - 3c_3 = -7$$
$$2c_1 - c_2 + c_3 = 7$$
$$c_1 + 2c_2 + 3c_3 = 11$$

$$\left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{array}\right] \xrightarrow[R_3 \to (-1)R_1 + R_3]{R_2 \to (-2)R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 6 & 6 & 18 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{6} R_3]{R_2 \to \frac{1}{7} \cdot R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow[R_3 \to (-1)R_2 + R_3]{R_1 \to (4)R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_1 + c_3 = 5$ $\qquad c_3 = t \Rightarrow c_1 = 5-t, \quad c_2 = 3-t$

$c_2 + c_3 = 3$ $\qquad w = (5-t)v_1 + (3-t)v_2 + tv_3 \quad t \in \mathbb{R}.$

Let $t=1 \Rightarrow c_1 = 4 \quad c_2 = 2 \quad c_3 = 1$

$$w = 4v_1 + 2v_2 + v_3$$

Let $t=-2 \Rightarrow c_1 = 7 \quad c_2 = 5 \quad c_3 = -2$

$$w = 7v_1 + 5v_2 - 2v_3.$$

DEFINITION Linear Independence

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to be **linearly independent** provided that the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \tag{4}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$. That is, the only linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ that represents the zero vector $\mathbf{0}$ is the trivial combination $0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k$.

Example: The standard unit vectors

$$\begin{aligned} e_1 &= (1,0,\cdots,0) \\ e_2 &= (0,1,0,\cdots,0) \\ &\vdots \\ e_n &= (0,0,\cdots,0,1) \end{aligned}$$

in $\mathbb{R}^n$ are linearly independent.

$c_1e_1 + c_2e_2 + \cdots + c_ne_n = 0 \Rightarrow (c_1, c_2, \cdots, c_n) = (0,0,\cdots,0)$

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0.$

Example: Determine whether the vectors $v_1=(1,2,2,1)$, $v_2=(2,3,4,1)$ and $v_3=(3,8,7,5)$ in $\mathbb{R}^4$ are linearly independent or not.

$$c_1v_1+c_2v_2+c_3v_3=0$$

$$c_1+2c_2+3c_3=0$$
$$2c_1+3c_2+8c_3=0$$
$$2c_1+4c_2+7c_3=0$$
$$c_1+c_2+5c_3=0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 8 & 0 \\ 2 & 4 & 7 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right] \xrightarrow[\substack{R_3\to(-2)R_1+R_2 \\ R_4\to(-1)R_1+R_4}]{R_2\to(-2)R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_2\to(-1)R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_4\to R_2+R_4} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$c_1+2c_2+3c_3=0$$
$$c_2-2c_3=0$$
$$c_3=0$$

$$\Rightarrow c_1=0,\ c_2=0,\ c_3=0.$$

If $v_1, v_2, \dots, v_k$ are linearly independent, the set $S=\{v_1, v_2, \dots, v_k\}$ is linearly independent.

Now we show that *the coefficients in a linear combination of the linearly independent vectors* $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ *are unique.* If both

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k \tag{5}$$

and

$$\mathbf{w} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k, \tag{6}$$

then

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k,$$

so it follows that

$a_2 - b_2 = 0$ $a_2 = b_2$ $a_k - b_k = 0$ $a_k = b_k$

$a_1 - b_1 = 0$ $a_1 = b_1$

$$(\underbrace{a_1 - b_1}_{=0})\mathbf{v}_1 + (\underbrace{a_2 - b_2}_{=0})\mathbf{v}_2 + \cdots + (\underbrace{a_k - b_k}_{0})\mathbf{v}_k = \mathbf{0}. \tag{7}$$

Because the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent, each of the coefficients in (7) must vanish. Therefore, $a_1 = b_1, a_2 = b_2, \dots, a_k = b_k$, so we have shown that the linear combinations in (5) and (6) actually are identical. Hence, if a vector $\mathbf{w}$ is in the set span$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, then it can be expressed in only one way as a linear combination of these linearly independent vectors.

A set of vectors is called **linearly dependent** provided it is not linearly independent. Hence the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent if and only if there exist scalars $c_1, c_2, \ldots, c_k$ *not all zero* such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}. \tag{8}$$

In short, a (finite) set of vectors is linearly dependent provided that some *nontrivial* linear combination of them equals the zero vector.

Example: Let $v_1 = (2,1,3)$, $v_2 = (5,-2,4)$, $v_3 = (3,8,-6)$, $v_4 = (2,7,-4)$

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$$

$$\begin{aligned} 2c_1 + 5c_2 + 3c_3 + 2c_4 &= 0 \\ c_1 - 2c_2 + 8c_3 + 7c_4 &= 0 \\ 3c_1 + 4c_2 - 6c_3 - 4c_4 &= 0 \end{aligned}$$

has a nontrivial solution because it has more unknowns than equations. Thus $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

THEOREM 2 Independence of n Vectors in R^n

The n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbf{R}^n$ are linearly independent if and only if the $n \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

having them as its column vectors has nonzero determinant.

THEOREM 3 Independence of Fewer Than n Vectors in R^n

Consider k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbf{R}^n$, with $k < n$. Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix}$$

be the $n \times k$ matrix having them as its column vectors. Then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if and only if some $k \times k$ submatrix of $\mathbf{A}$ has nonzero determinant.

Example: $e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$, $e_n = (0, 0, \cdots, 1)$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I. \qquad |A| = |I| = 1 \neq 0.$$

$\Rightarrow \{e_1, e_2, \cdots, e_n\}$ is linearly independent.

Example: $v_1 = (2,1,3)$ and $v_2 = (1,0,1)$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow v_1, v_2 \text{ are linearly independent.}$$

$v_1 = (1,2,3)$ and $v_2 = (2,4,6)$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \qquad \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 0 \quad \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$$

Bases and Dimension for Vector Spaces

DEFINITION Basis

A finite set S of vectors in a vector space V is called a **basis** for V provided that

(a) the vectors in S are linearly independent, and

(b) the vectors in S span V.

The **standard basis** for $\mathbf{R}^n$ consists of the unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \{e_1, e_2, \ldots, e_n\}$ span $\mathbb{R}^n$

$$\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \{e_1, e_2, \ldots, e_n\} \text{ linearly independent.}$$

Example: Show that any set of n linearly independent vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

Let $v_1, v_2, \ldots, v_n$ be n linearly independent vectors in $\mathbb{R}^n$. It is known that any set of more than n vectors in $\mathbb{R}^n$ is linearly dependent. Hence given a vector w in $\mathbb{R}^n$ there exists scalars $c, c_1, c_2, \ldots, c_n$ not all zero such that

$$w = \left(-\frac{c_1}{c}\right)v_1 + \left(-\frac{c_2}{c}\right)v_2 + \cdots + \left(-\frac{c_n}{c}\right)v_n$$

$$cw = -c_1v_1 - c_2v_2 - \cdots - c_nv_n$$

$$cw + c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0 \qquad (*)$$

If c were zero, then the vectors $v_1, v_2, \ldots, v_n$ are linearly dependent. Hence $c \neq 0$, so $(*)$ can be solved for w as a linear combination of $v_1, v_2, \ldots, v_n$. Thus the linearly independent vectors $v_1, v_2, \ldots, v_n$ also span $\mathbb{R}^n$ and therefore constitute a basis for $\mathbb{R}^n$.

Example: Is $\{(1,-1,-2,-3),(1,-1,2,3),(1,-1,-3,-2),(0,3,-1,2)\}$ a basis for $\mathbb{R}^4$.

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 3 \\ -2 & 2 & -3 & -1 \\ -3 & 3 & -2 & 2 \end{vmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ -2 & 2 & -3 & -1 \\ -3 & 3 & -2 & 2 \end{vmatrix}$$

$$= 3 \cdot (-1)^{2+4} \cdot \begin{vmatrix} 1 & 1 & 1 \\ -2 & 2 & -3 \\ -3 & 3 & -2 \end{vmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 + (2)R_1} 3 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & 6 & 1 \end{vmatrix}$$

$$= 3 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 4 & -1 \\ 6 & 1 \end{vmatrix} = 3 \cdot (4+6) = 30 \neq 0$$

$\Rightarrow$ The set is a basis for $\mathbb{R}^4$

THEOREM 1 Bases as Maximal Linearly Independent Sets

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for the vector space V. Then any set of more than n vectors in V is linearly dependent.

THEOREM 2 The Dimension of a Vector Space

Any two bases for a vector space consist of the same number of vectors.

A nonzero vector space V is called **finite dimensional** provided that there exists a basis for V consisting of a finite number of vectors from V. In this case the number n of vectors in each basis for V is called the **dimension** of V, denoted by $n = \dim V$. Then V is an *n-dimensional* vector space.

Note that the zero vector space $\{\mathbf{0}\}$ has no basis because it contains *no* linearly independent set of vectors. (Sometimes it is convenient to adopt the convention that the null set is a basis for $\{\mathbf{0}\}$.) Here we define $\dim\{\mathbf{0}\}$ to be zero. A nonzero vector space that has no finite basis is called **infinite dimensional**.

Let $\mathcal{P}$ be the set of all polynomials of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where the largest exponent $n \geq 0$ that appears is the *degree* of the polynomial $p(x)$, and the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real numbers. We add polynomials in $\mathcal{P}$ and multiply them by scalars in the usual way—that is, by collecting coefficients of like powers of x. For instance, if