Chp 22: Gauss's Law

Goals for Chapter 22

- To use the electric field at a surface to determine the charge within the surface
- To learn the meaning of electric flux and how to calculate it
- To learn the relationship between the electric flux through a surface and the charge within the surface
- To use Gauss's law to calculate electric fields
- To learn where the charge on a conductor is located

Charge and electric flux

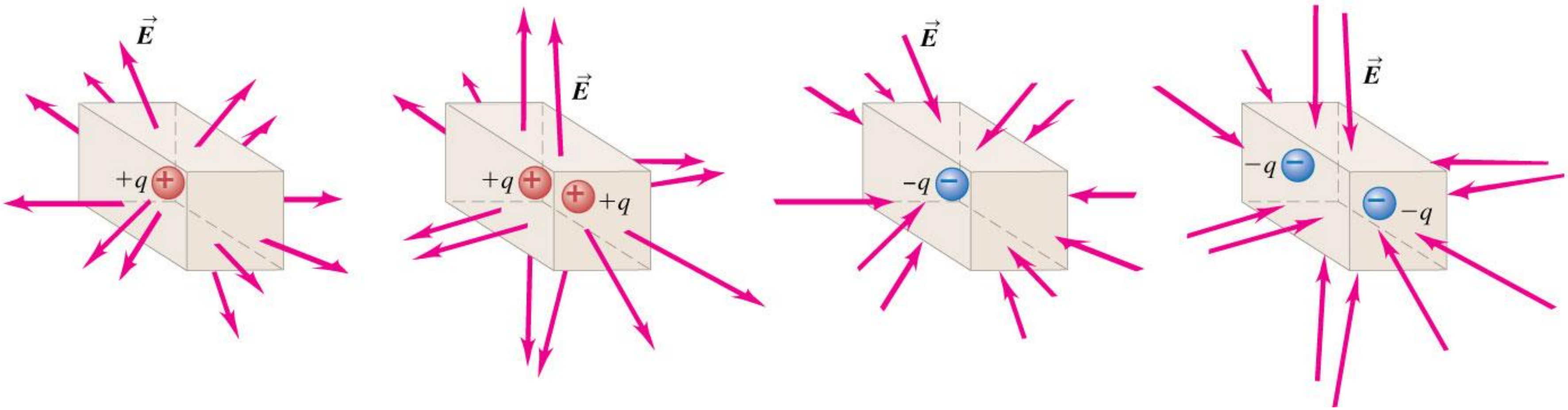
Positive charge within the box produces outward *electric flux* through the surface of the box, and negative charge produces inward flux.

(a) Positive charge inside box, outward flux

(b) Positive charges inside box, outward flux

(c) Negative charge inside box, inward flux

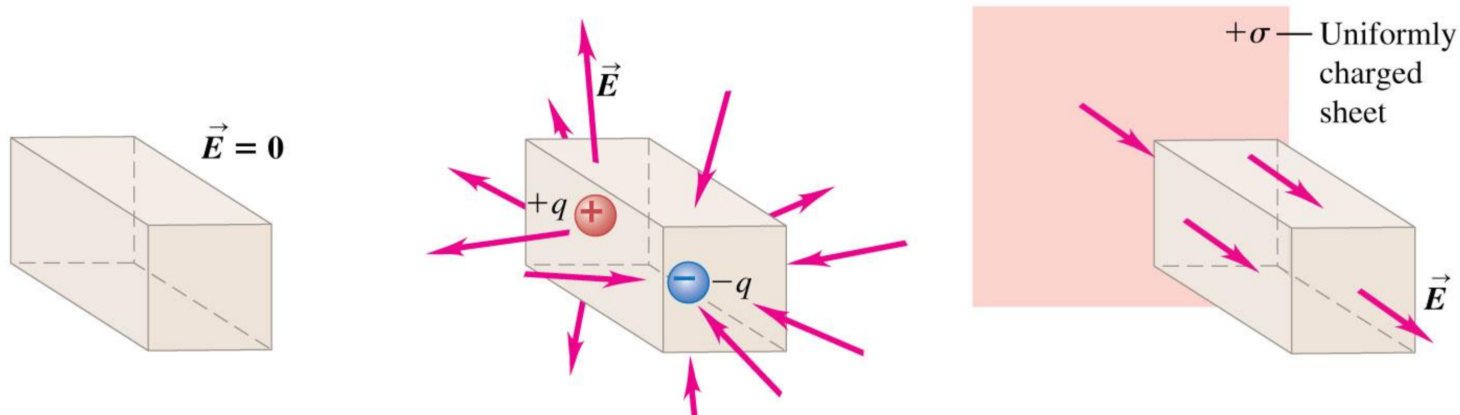
(d) Negative charges inside box, inward flux



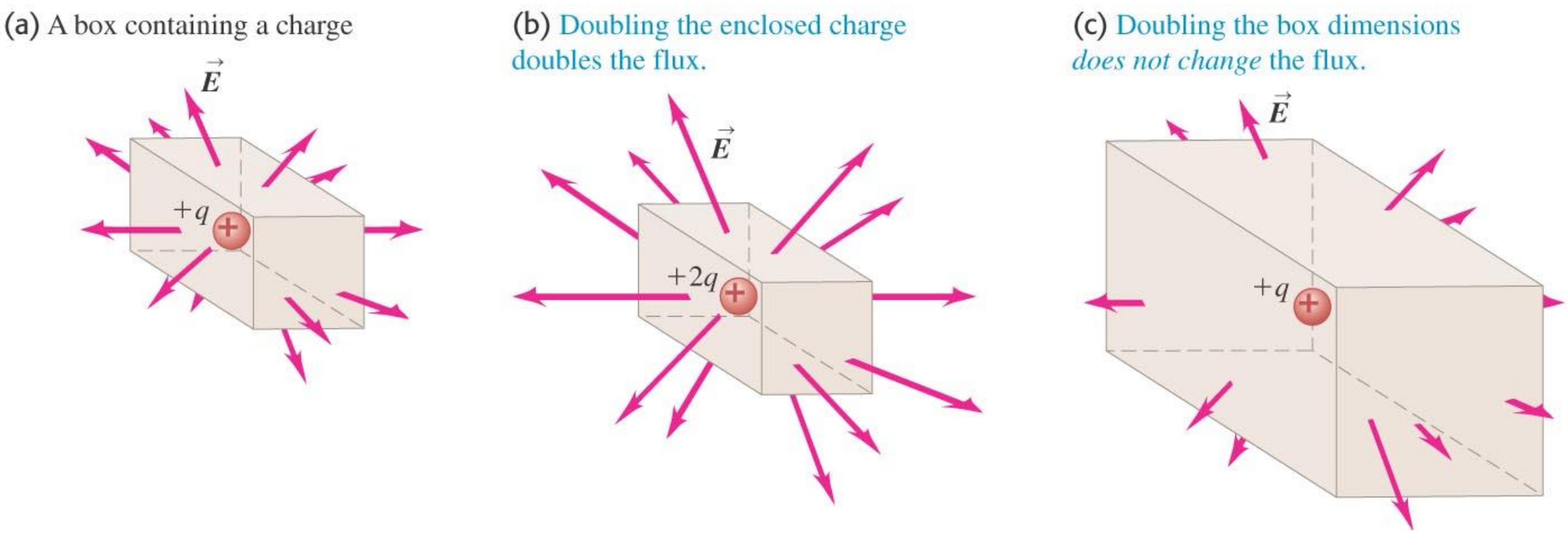
(a) No charge inside box, zero flux

(b) Zero *net* charge inside box, inward flux cancels outward flux.

(c) No charge inside box, inward flux cancels outward flux.

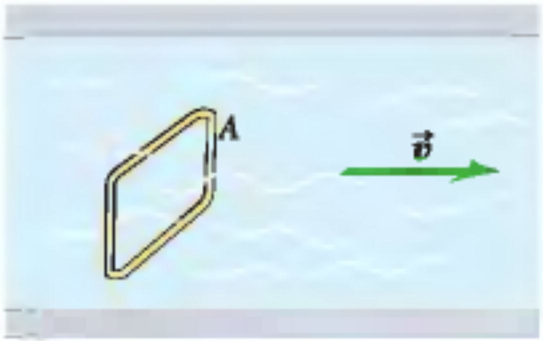


doubling the charge within the box doubles the flux, but doubling the size of the box does not change the flux.

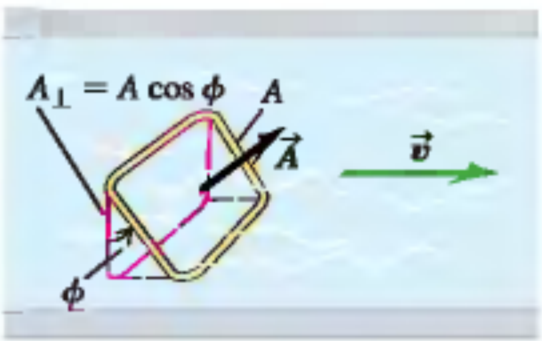


Flux: Fluid-Flow Analogy

(a) A wire rectangle in a fluid



(b) The wire rectangle tilted by an angle ϕ



In the Figure a fluid flows steadily from left to right. Let's examine the volume flow rate dV/ dt (in. say, cubic meters per second) through the wire rectangle with area A. When the area is perpendicular to the flow velocity (a) and the flow velocity is the same at all points in the fluid, the volume flow rate dV/dt is the area A multiplied by the flow speed v:

$$\frac{dV}{dt} = vA$$

When the rectangle is tilted at an angle ϕ (b) the area that counts is the silhouette area that we see when we look in the direction of v. This area, which is outlined in red is ACos ϕ. It is the projection of the area A onto a surface perpendicular to v. Then the volume flow rate through A is

$$\frac{dV}{dt} = vA\cos\phi$$

$$\rightarrow \quad \frac{dV}{dt} = \vec{v}\cdot\vec{A}$$

Flux of a Uniform Electric Field

Using the analogy between electric field and fluid flow. we now define electric flux in the same way as we have just defined the volume flow rate of a fluid; we simply replace the fluid velocity v by the electric field E. We define electric flux as:

$$\Phi_E = \vec{E} \cdot \vec{A} \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)}$$

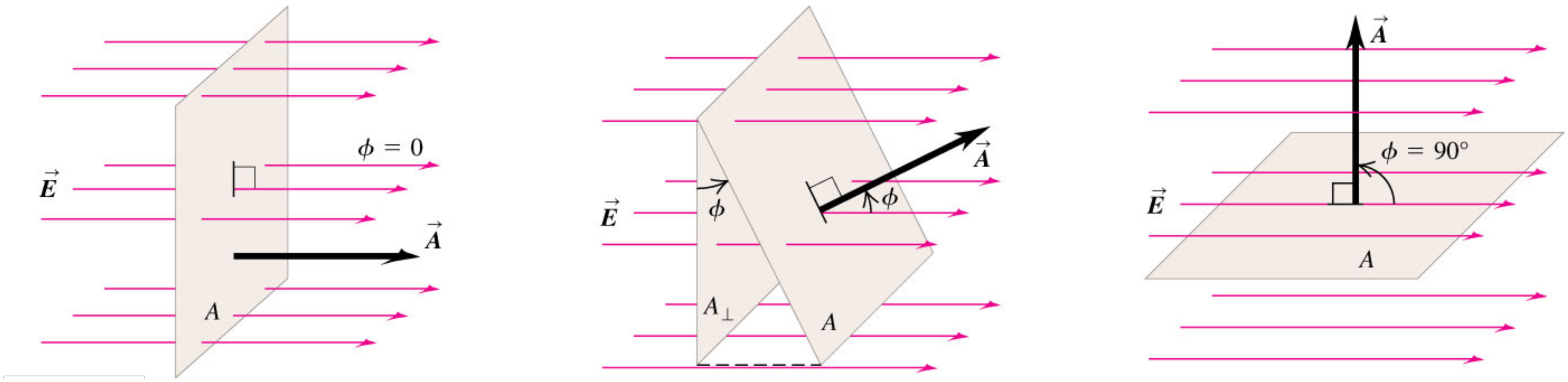
$$\vec{A} = A\hat{n}$$

unit normal vector

(a) Surface is face-on to electric field:
- $\vec{E}$ and $\vec{A}$ are parallel (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 0$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA$.

(b) Surface is tilted from a face-on orientation by an angle ϕ:
- The angle between $\vec{E}$ and $\vec{A}$ is ϕ.
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos \phi$.

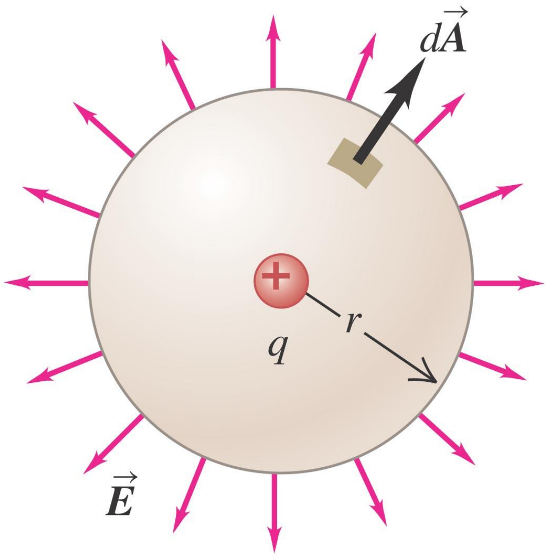
(c) Surface is edge-on to electric field:
- $\vec{E}$ and $\vec{A}$ are perpendicular (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 90°$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos 90° = 0$.



Flux of a Nonuniform Electric Field

$$\Phi_E = \int E\cos\phi \, dA = \int E_\perp \, dA = \int \vec{E} \cdot d\vec{A} \qquad \text{(general definition of electric flux)}$$

Example: Electric flux through a sphere



A positive point charge q = 3.0 micro Coulomb is surrounded by a sphere with radius 0.20 m centered on the charge. Find the electric flux through the sphere due to this charge.

$$E = \frac{q}{4\pi\epsilon_0 r^2} = (9.0 \times 10^9\,\mathrm{N\cdot m^2/C^2})\frac{3.0 \times 10^{-6}\,\mathrm{C}}{(0.20\,\mathrm{m})^2}$$
$$= 6.75 \times 10^5\,\mathrm{N/C}$$

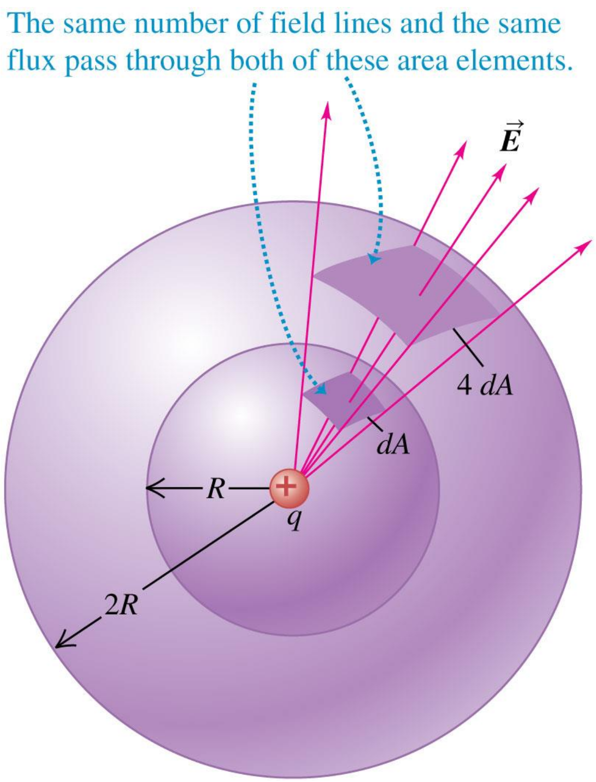
$$\vec{E}\cdot d\vec{A} = E\,dA,$$

➔ $$\Phi_E = EA = (6.75 \times 10^5\,\mathrm{N/C})(4\pi)(0.20\,\mathrm{m})^2$$
$$= 3.4 \times 10^5\,\mathrm{N\cdot m^2/C}$$

Gauss's Law

- Gauss's law is an alternative to Coulomb's law and is completely equivalent to it.
- Carl Friedrich Gauss (1777-1855) formulated this law.

Assume a point charge inside a Spherical Surface → The flux through the sphere is independent of the size of the sphere and depends only on the charge inside. Figure at the left shows why this is so. The magnitude E of the electric field at every point on the surface is given by



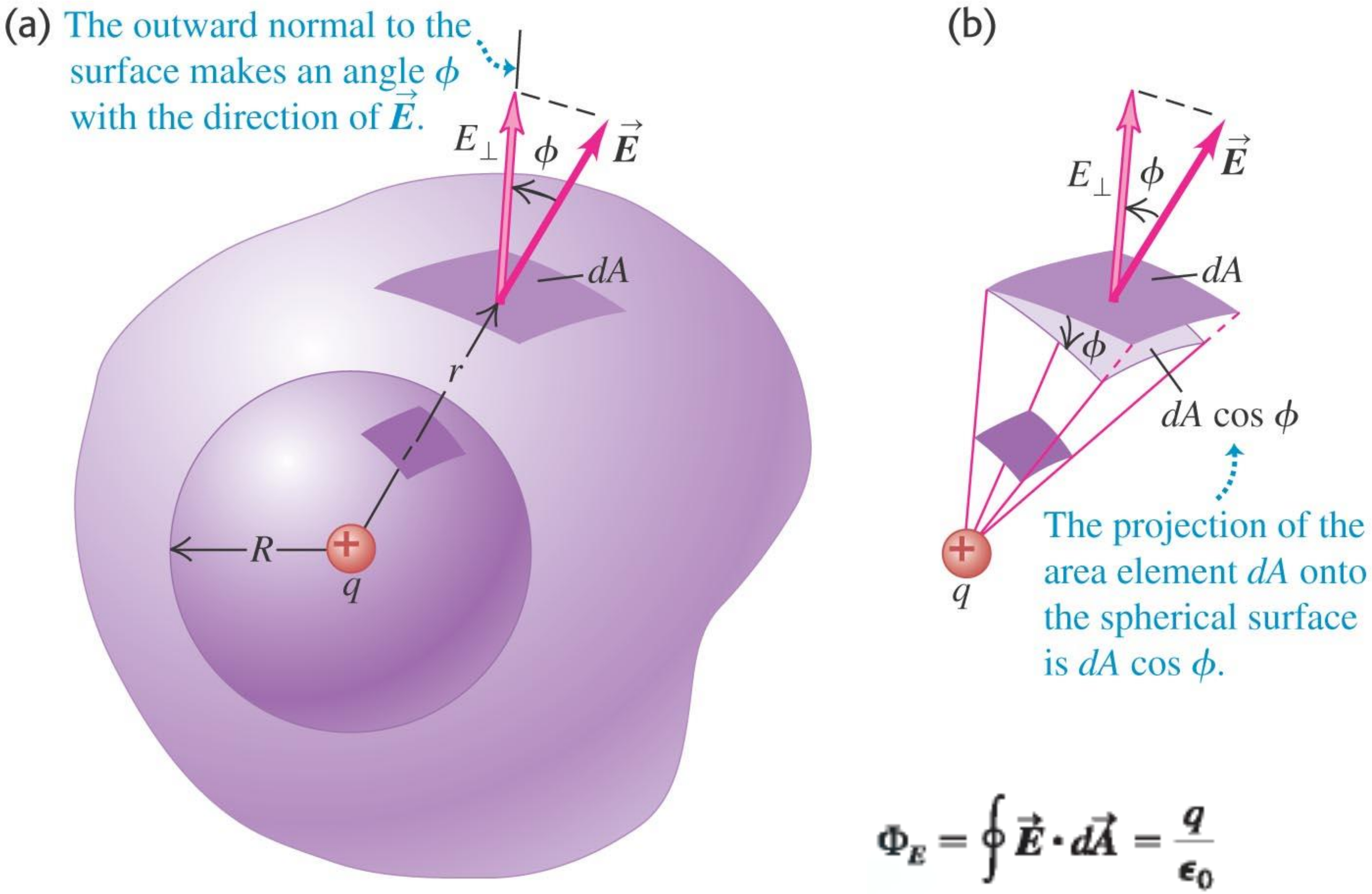
$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$

At each point on the surface, E is perpendicular to the surface, and its magnitude is the same at every point. The total electric flux is the product of the field magnitude E and the total area of the sphere:

$$\rightarrow \quad \Phi_E = EA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}(4\pi R^2) = \frac{q}{\epsilon_0}$$

Point charge inside a nonspherical surface

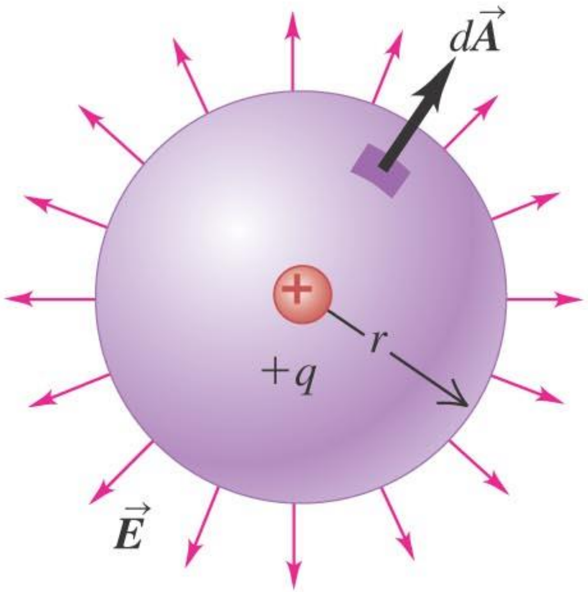
the flux is independent of the surface and depends only on the charge inside.



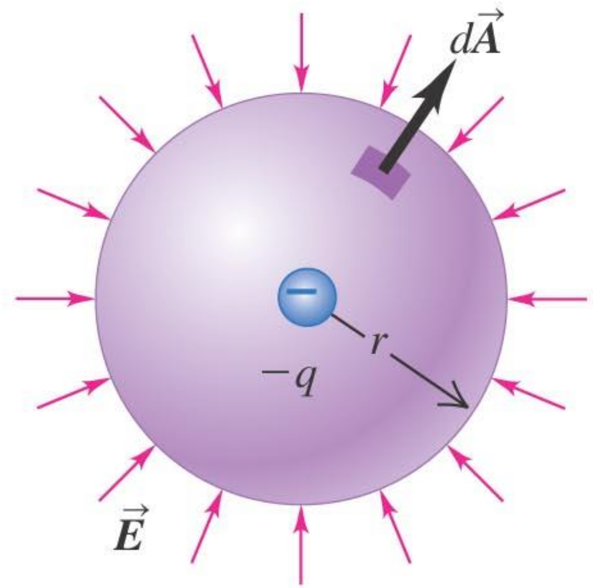
$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Positive and negative flux

(a) Gaussian surface around positive charge: positive (outward) flux



(b) Gaussian surface around negative charge: negative (inward) flux

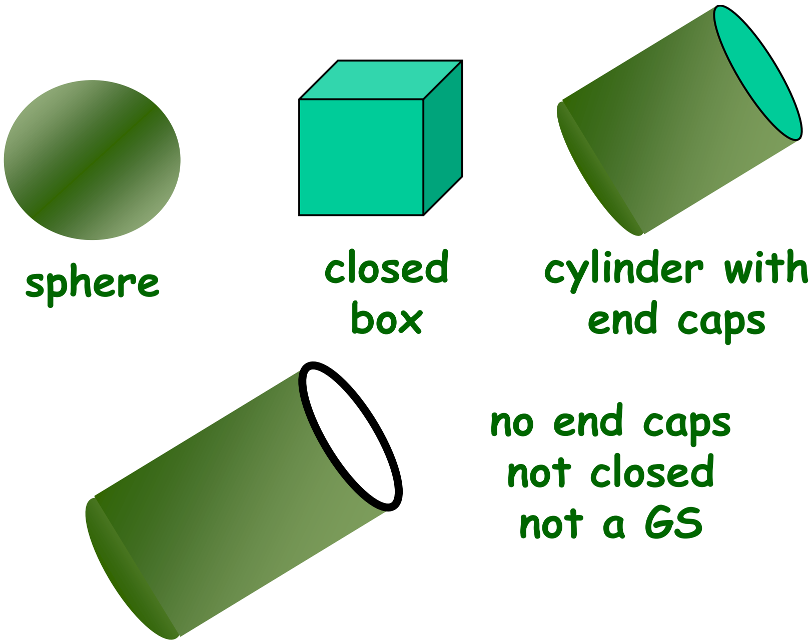


Flux through a closed or open surface S: Integrate on S

$$\Phi_E \equiv \oint_S d\Phi_E = \oint_S \vec{E} \circ \hat{n}\, dA$$

To do this: evaluate integrand at all points on surface S

Gaussian Surfaces:



Field lines cross a closed surface:

- Once (or an odd number of times) for charges that are inside
- Twice (or an even number of times) for charges that are outside
- Choose surface to match the field's symmetry where possible

The flux of electric field crossing a closed surface equals the net charge inside the surface (times a constant). Simple example: charge at center of a spherical "Gaussian surface"

$$\text{Flux} \equiv \Phi = \text{Field x Surface Area} = E \times S = \frac{Q}{4\pi\varepsilon_0 R^2} \times 4\pi R^2 = Q/\varepsilon_0$$

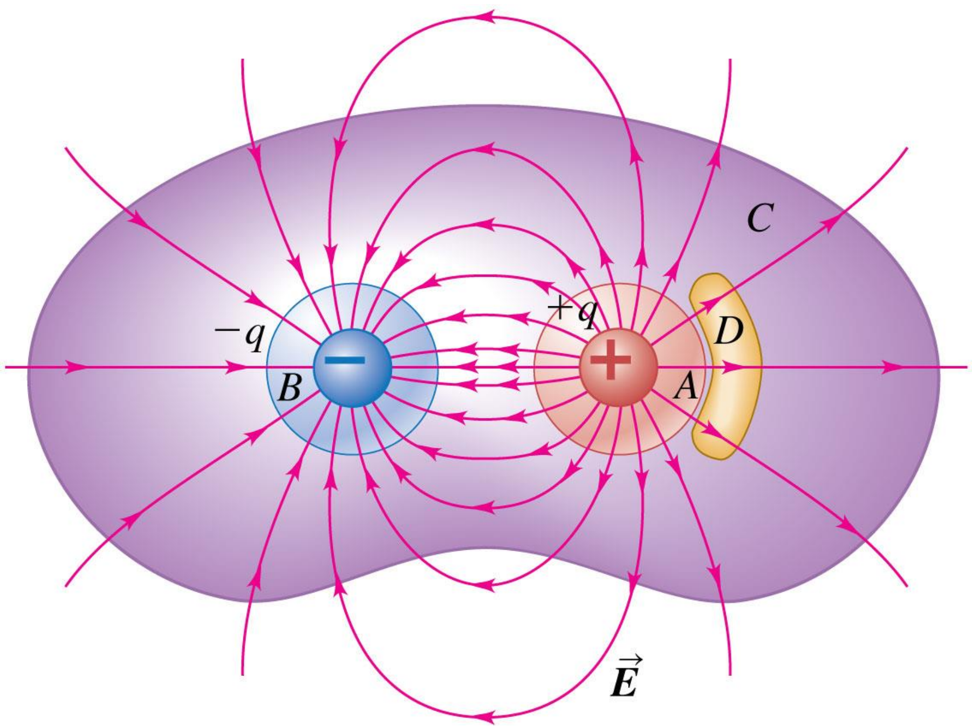
Units: Nm^2/C.

General Form of Gauss's Law

The total electric flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ϵ_0.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0} \qquad \text{(Gauss's law)}$$

$$\Phi_E = \oint E\cos\phi \, dA = \oint E_\perp \, dA = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0} \qquad \text{(various forms of Gauss's law)}$$



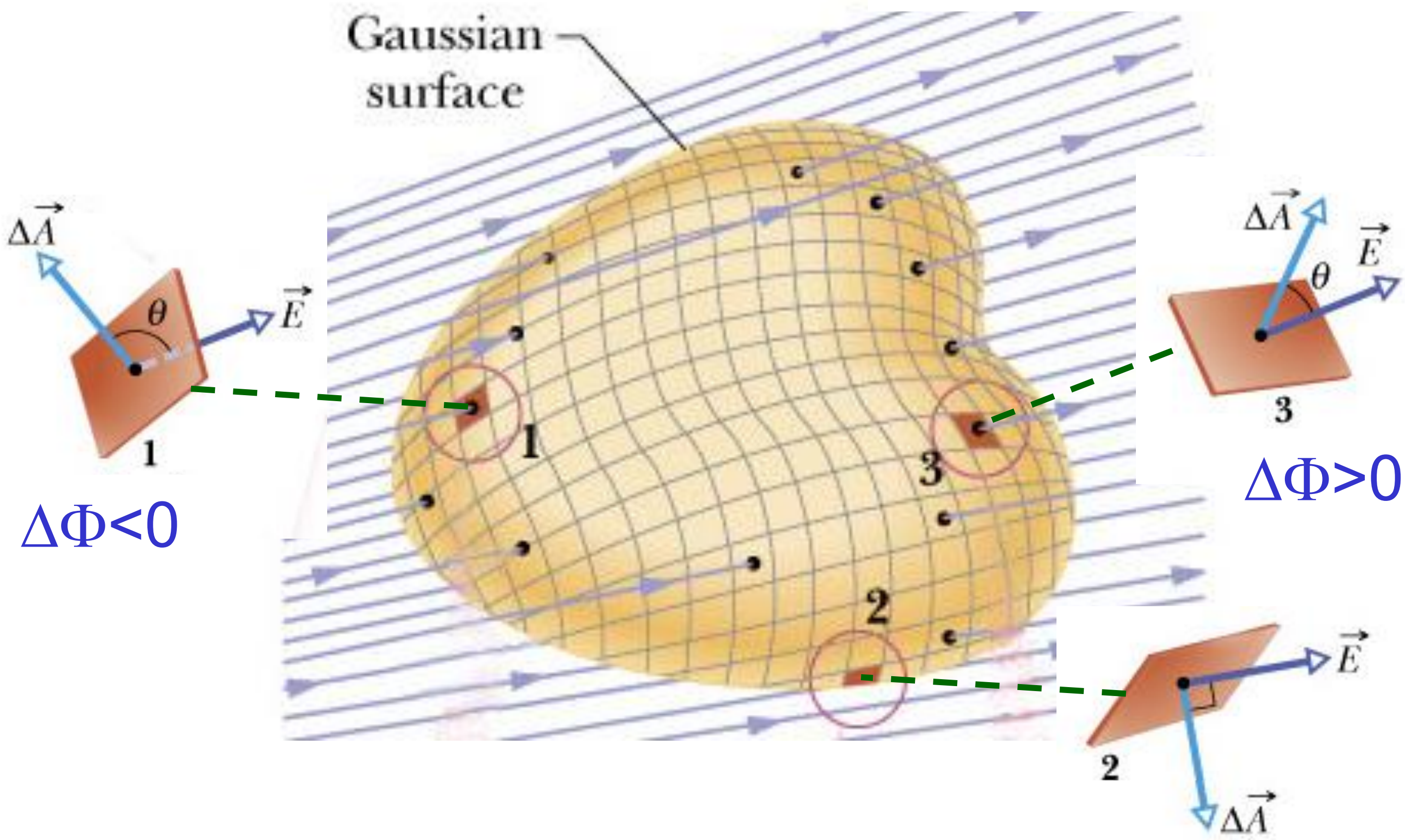
- Total flux from
  A is q/ε_0
  B is $-q/\varepsilon_0$
  C is zero
  D is zero

ΔΦ depends on the angle between the field and chunks of area

$$\Delta\Phi = \vec{E} \circ \hat{n}\Delta A = EA\cos(\theta)$$

$$\Delta\vec{A} = \hat{n}\Delta A$$

$\hat{n} \equiv$ outward unit vector



$\Delta\Phi<0$

$\Delta\Phi>0$

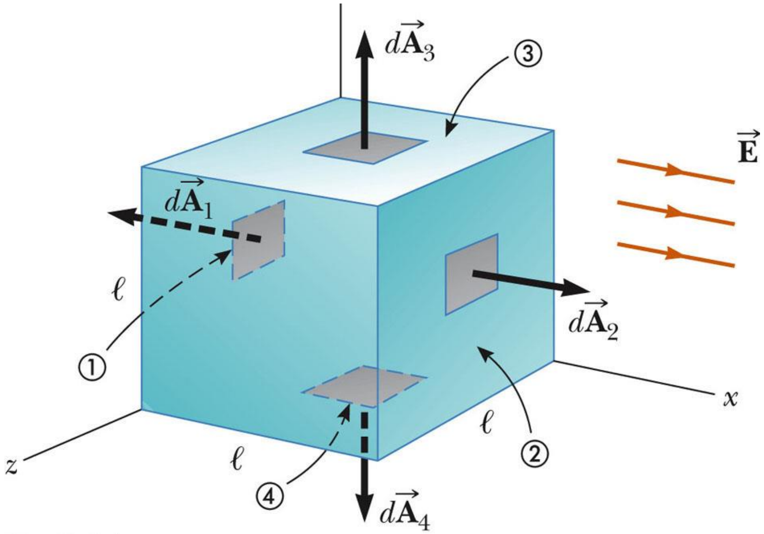
$\Delta\Phi=0$

Evaluating flux through closed or open surfaces

Special case: field is constant across pieces of the surface

EXAMPLE: Flux through a cube

Assume:



Uniform E field everywhere
Directed along x-axis
Cube faces normal to axes
Each side has area l^2
Field lines cut through two surface areas and are tangent to the other four surface areas
For side 1, flux $\phi = -E\,l^2$
For side 2, flux $\phi = +E\,l^2$
For the other four sides, flux $\phi = 0$
Therefore, *total* flux $\phi = 0$

$$\Phi_E = \sum_{i=1,6} \Delta\Phi_i = 0$$

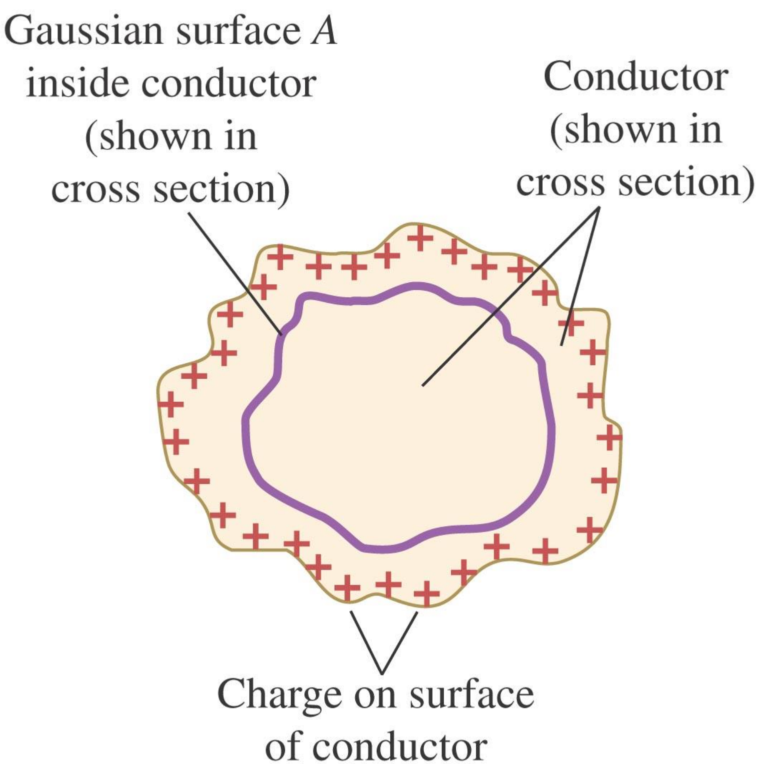
$$\Delta\Phi_E \equiv \vec{E} \circ \hat{n}\Delta A \quad \text{(a scalar)}$$

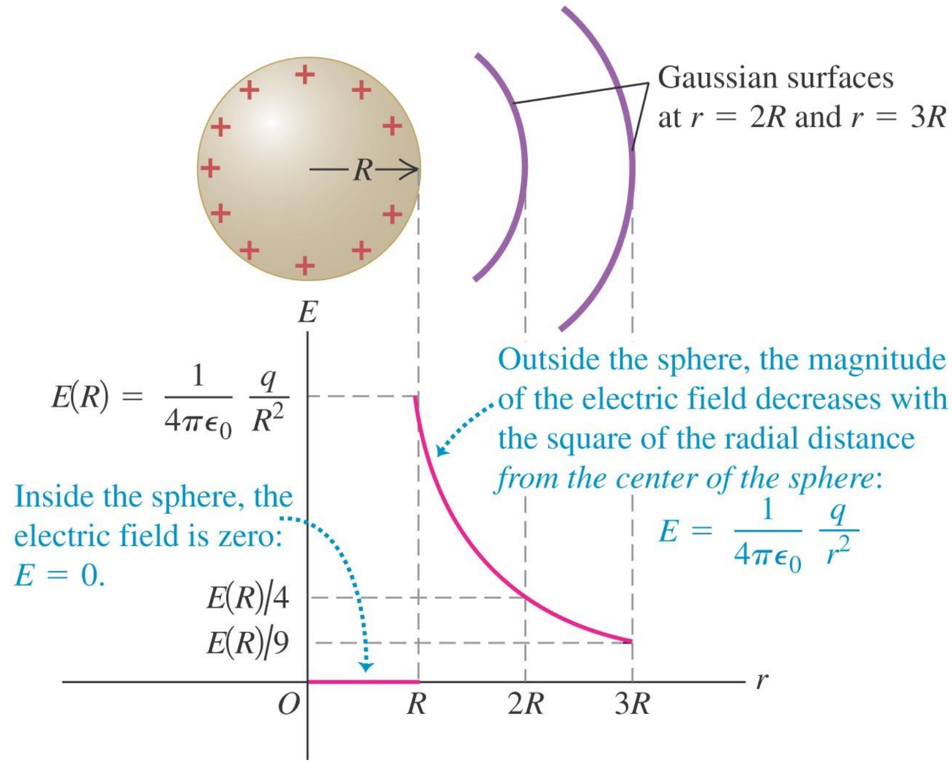
Units of Φ are : Nm^2/C

$$\Phi_E \equiv \sum_{\text{small areas}} \Delta\Phi_E = \sum_{\text{small areas}} \vec{E} \circ \hat{n}\Delta A$$

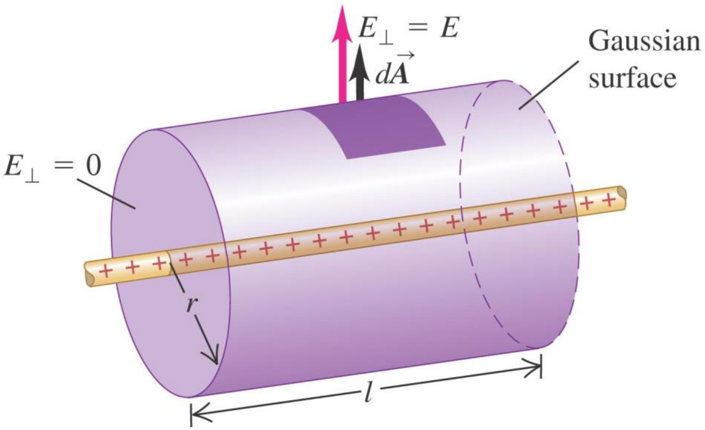
Applications of Gauss's law

Under electrostatic conditions, any excess charge on a conductor resides entirely on its surface.





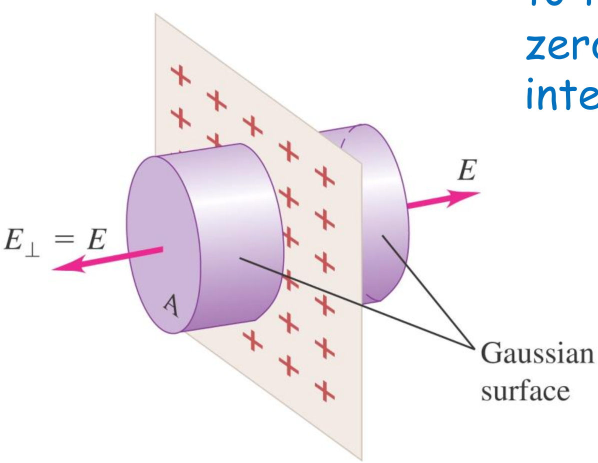
Field of a line charge



$$\Phi_E = (E)(2\pi r l) = \frac{\lambda l}{\epsilon_0}$$

$$\rightarrow \quad E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$$

Field of a sheet of charge



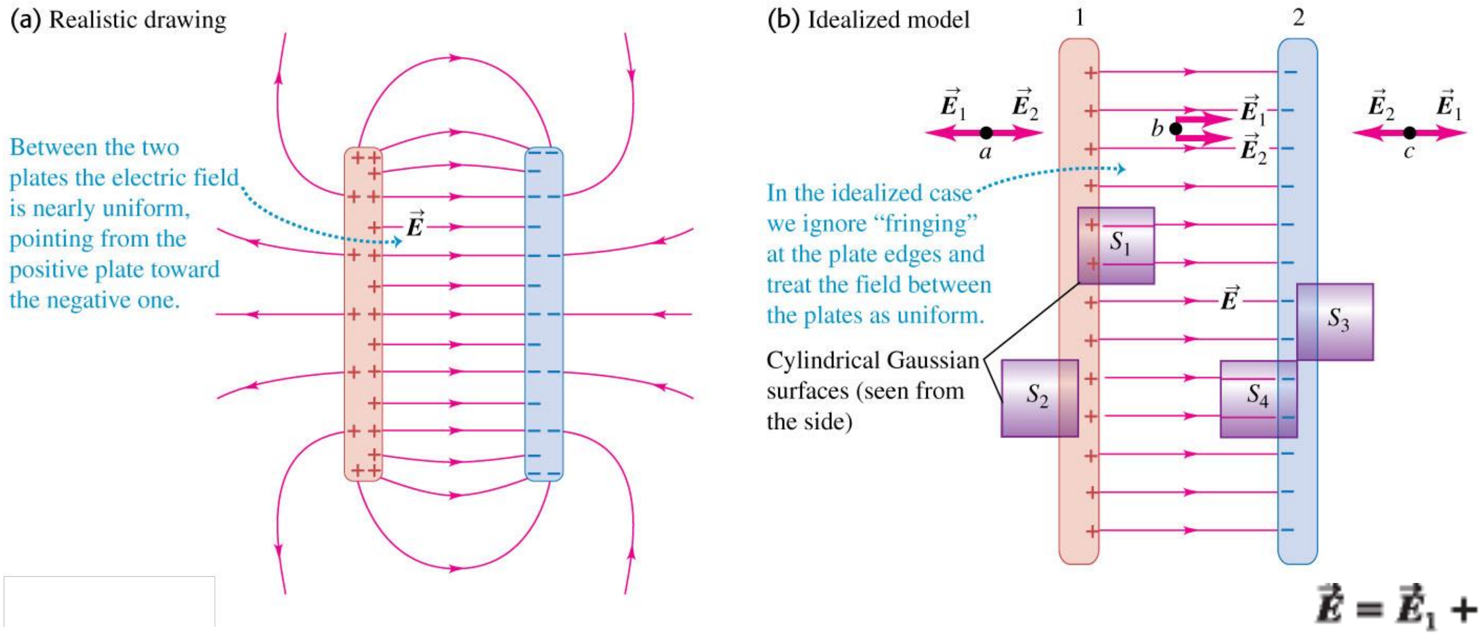
Because E is perpendicular to the charged sheet, it is parallel to the curved side walls of the cylinder, so $E_\perp$ at these walls is zero and there is no flux through these walls. The total flux integral in Gauss's law is 2EA. The net charge is

$$Q_{encl} = \sigma A.$$

$$\rightarrow \quad 2EA = \frac{\sigma A}{\epsilon_0}$$

$$\rightarrow \quad E = \frac{\sigma}{2\epsilon_0}$$

Field between two parallel conducting plates



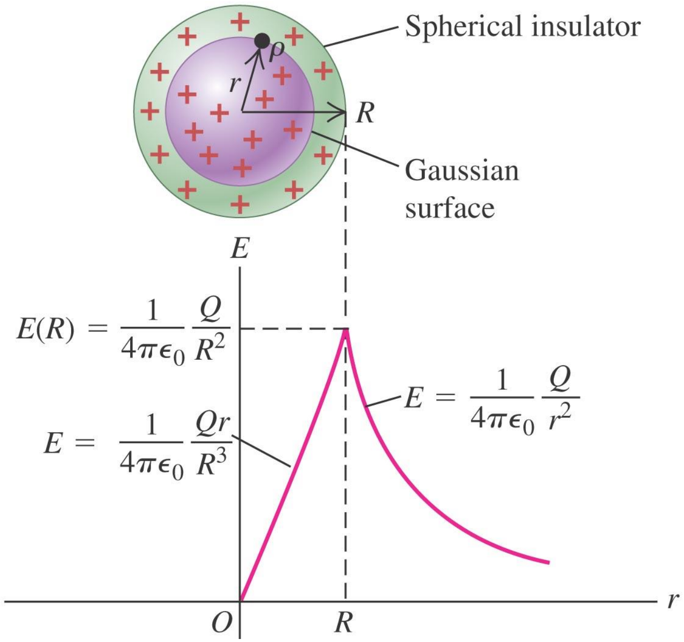
$$\vec{E} = \vec{E}_1 + \vec{E}_2.$$

There is no flux through the side walls of the cylinder, since these walls are parallel to E. So the total flux integral in Gauss's law is EA.
The net charge enclosed by the cylinder is EA

$\rightarrow$ $$EA = \frac{\sigma A}{\epsilon_0}$$

$\rightarrow$ $$E = \frac{\sigma}{\epsilon_0}$$

A uniformly charged sphere



Positive electric charge Q is distributed uniformly throughout the volume of an insulating sphere with radius R. Find the magnitude of the electric field at a point P a distance r from the center of the sphere.
The total electric flux through the Gaussian surface is the product of E and the total area of the surface A.

$$A = 4\pi r^2,$$

$$\rightarrow \quad \Phi_E = 4\pi r^2 E.$$

$r < R$ ➔ The volume charge density is the charge Q divided by the volume of the entire charged sphere of radius R

$$\rightarrow \quad \rho = \frac{Q}{4\pi R^3/3}$$

The volume V_{encl} enclosed by the Gaussian swface is 4/3 π r^3, so the total charge Q_{encl} enclosed by that surface is

$$Q_{encl} = \rho V_{encl} = \left(\frac{Q}{4\pi R^3/3}\right)\left(\frac{4}{3}\pi r^3\right) = Q\frac{r^3}{R^3} \quad \rightarrow \quad 4\pi r^2 E = \frac{Q}{\epsilon_0}\frac{r^3}{R^3} \quad \rightarrow \quad E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}$$

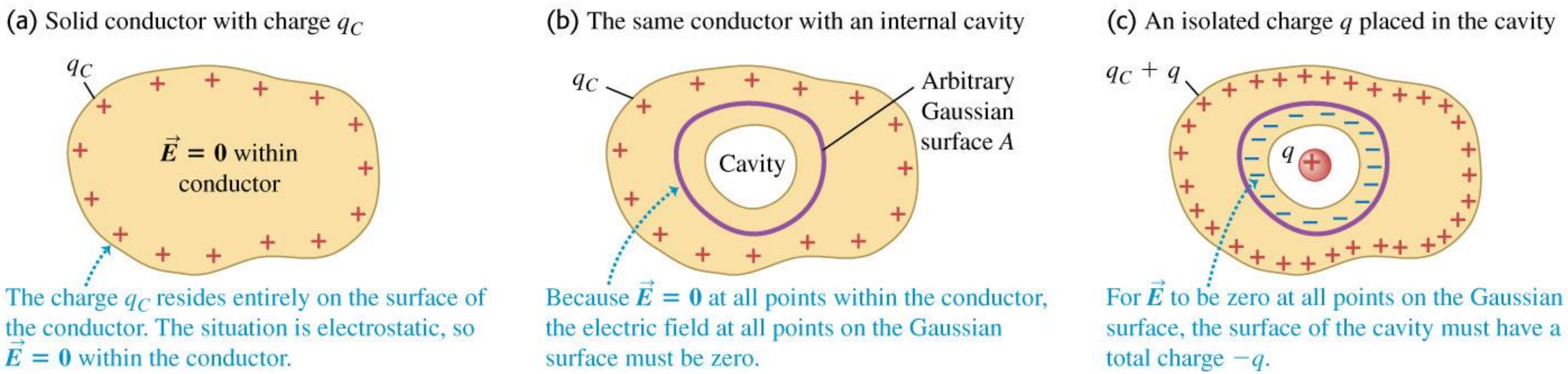
$r > R$ ➔ $Q_{encl} = Q$ $\quad \rightarrow \quad 4\pi r^2 E = \frac{Q}{\epsilon_0} \quad \rightarrow \quad E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$

Charges on conductors

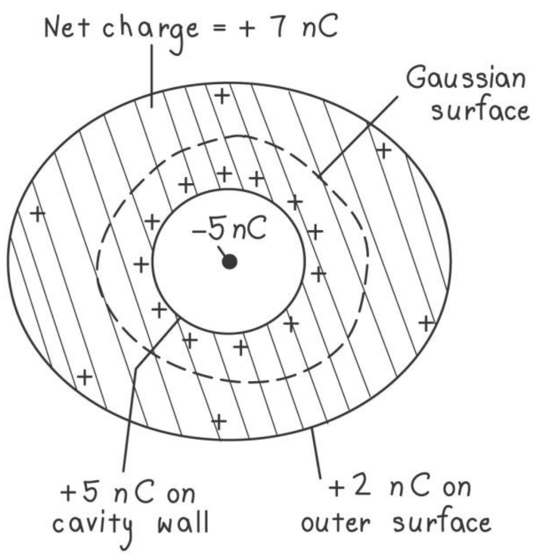
- Suppose we place a small body with a charge q inside a cavity within a conductor.
- The conductor is uncharged and is insulated from the charge q. Again E = 0 everywhere on surface A, so according to Gauss's law the total charge inside this surface must be zero.
- Therefore there must be a charge -q distributed on the surface of the cavity, drawn there by the charge q inside the cavity. The total charge on the conductor must remain zero, so a charge +q must appear either on its outer surface or inside the material.
- But we showed that in an electrostatic situation there can't be any excess charge

within the material of a conductor. So we conclude that the charge + q must

- appear on the outer surface.
- By the same reasoning, if the conductor originally had a charge q_c, then the total charge on the outer surface must be q_c + q after the charge q is inserted into the cavity.



Example: A conductor with a cavity

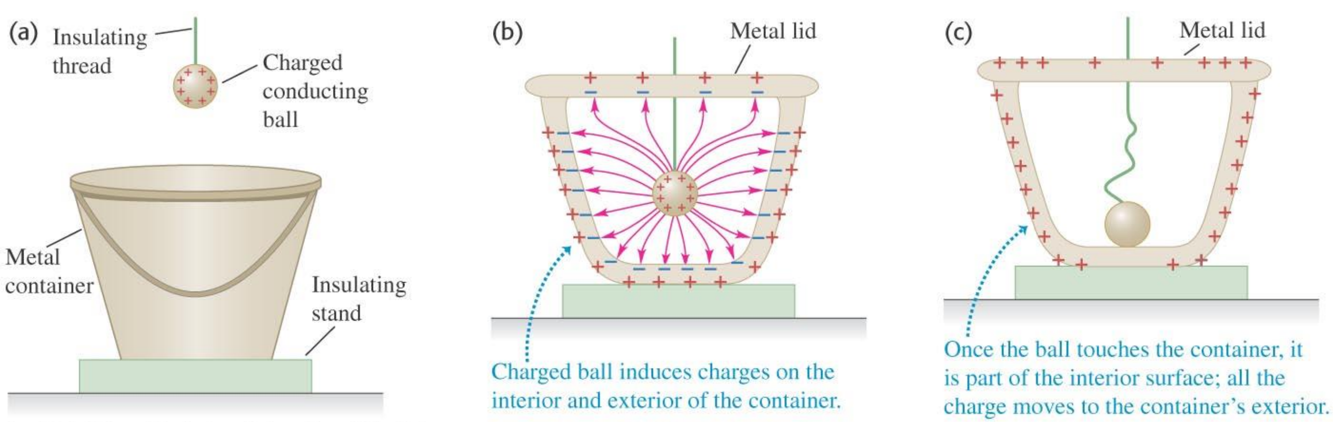


A solid conductor with a cavity carries a total charge of + 7 nC.
Within the cavity, insulated from the conductor, is a point charge of -5 nC. How much charge is on each surface (inner and outer) of the conductor?

If the charge in the cavity is q = -5 nC, the charge on the inner cavity surface must be $-q$ =- (-5 nC) = +5 nC. The conductor carries a total charge of + 7 nC, none of which is in the interior of the material. If + 5 nC is on the inner surface of the cavity, then there must be (+7 nC) - (+5 nC) = +2nC on the outer surface of the conductor.

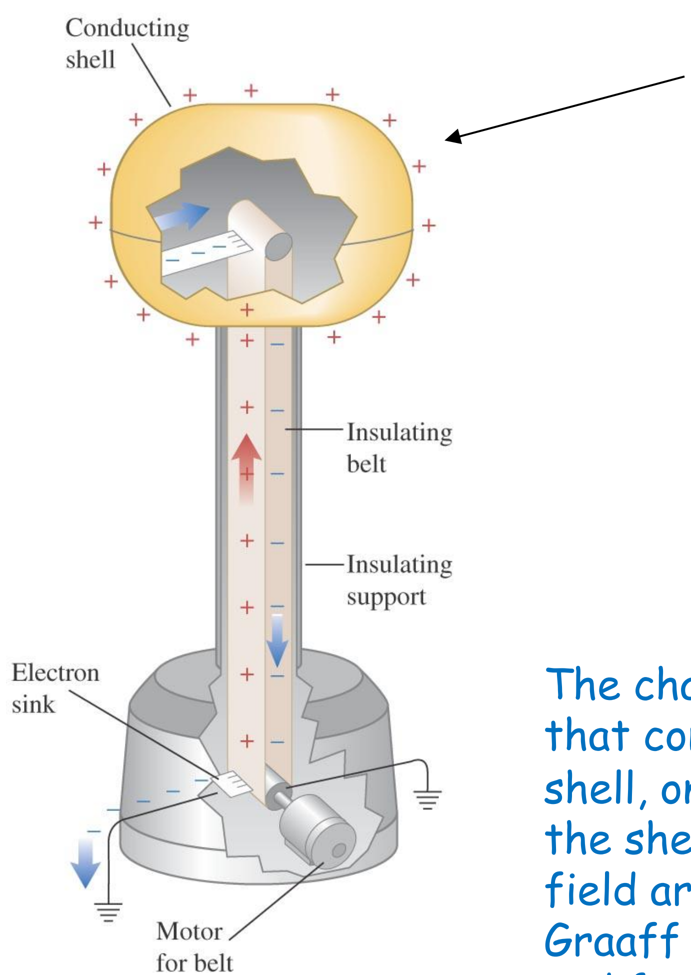
Testing Gauss's Law Experimentally

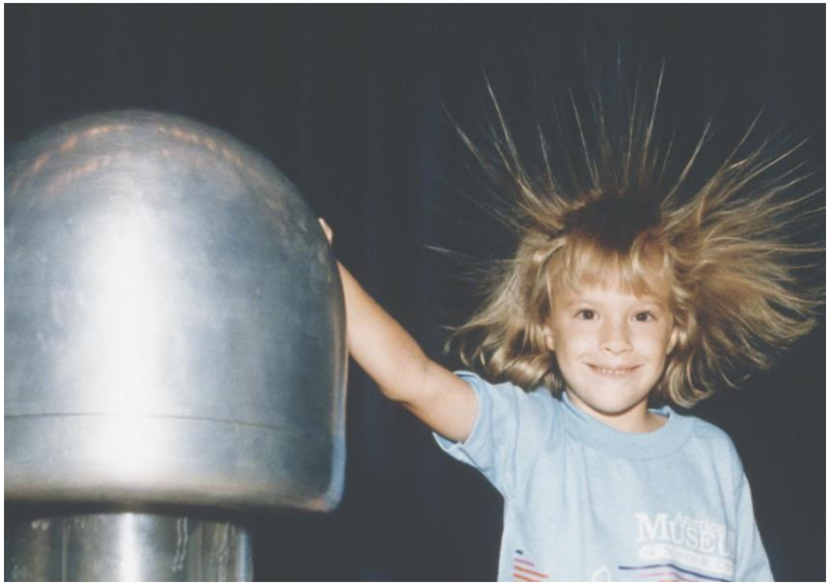
This experiment was performed in the 19th century by the English scientist Michael Faraday, using a metal icepail with a lid, and it is called Faraday's icepail experiment.



The Van de Graaff generator

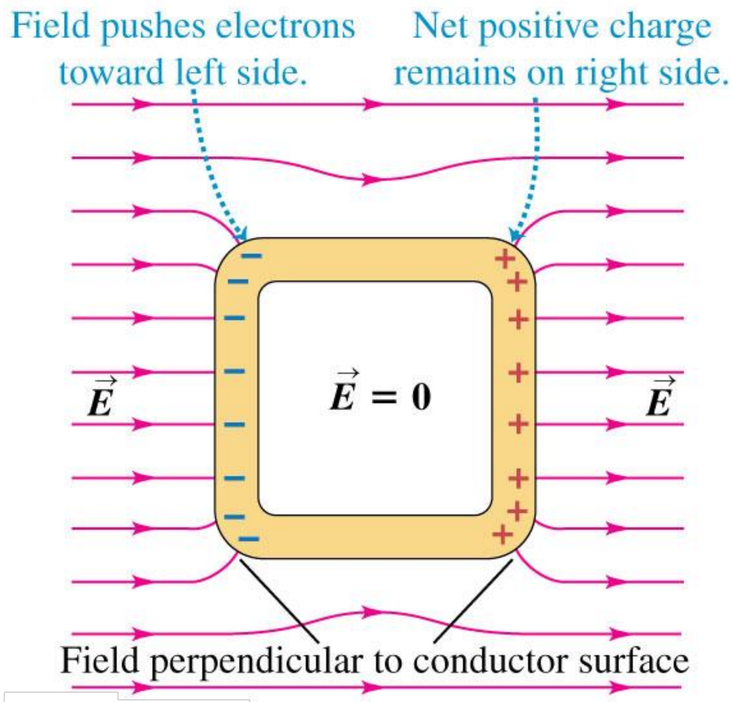
The Van de Graaff generator, shown in Figure below, operates on the same principle as in Faraday's icepail experiment.





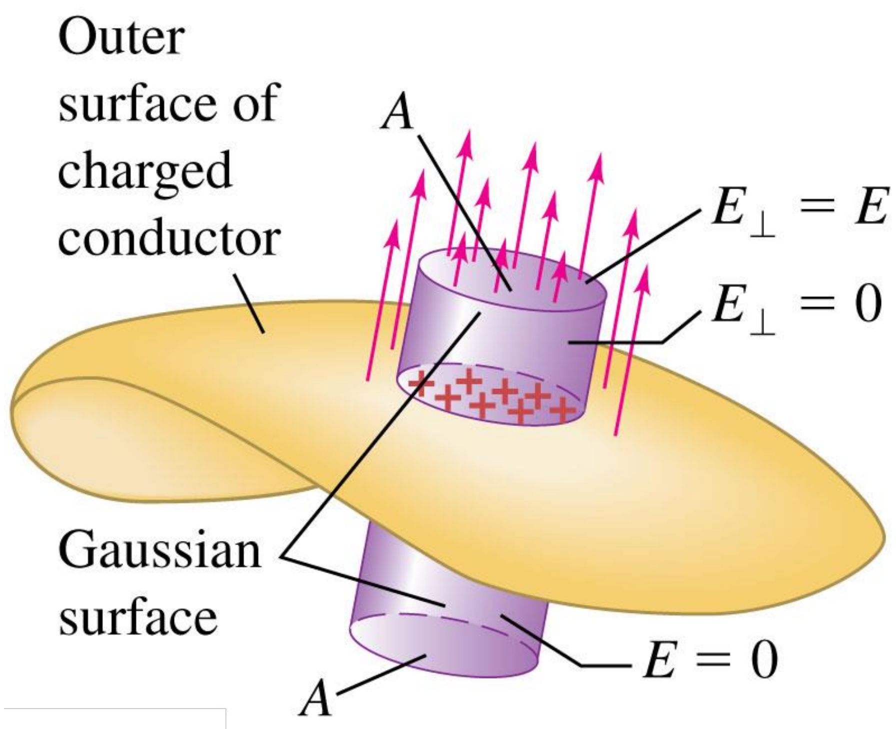
The charged conducting sphere is replaced by a charged belt that continuously carries charge to the inside of a conducting shell, only to have it carried away to the outside surface of the shell. As a result, the charge on the shell and the electric field around it can become very large very rapidly. The Van de Graaff generator is used as an accelerator of charged particles and for physics demonstrations.

Electrostatic shielding: Faraday's Cage



Suppose we have a very sensitive electronic instrument that we want to protect from stray electric fields that might cause erroneous measurements. We surround the instrument with a conducting box, or we line the walls, floor, and ceiling of the room with a conducting material such as sheet copper. The external electric field redistributes the free electrons in the conductor, leaving a net positive charge on the outer surface in some regions and a net negative charge in others. This charge distribution causes an additional electric field such that the total field at every point inside the box is zero, as Gauss's law says it must be.

Field at the surface of a conductor



The electric field at the surface of any conductor is always perpendicular to the surface and has magnitude

$$\sigma/\varepsilon_0$$

$$E_\perp A = \frac{\sigma A}{\epsilon_0} \quad \text{and} \quad E_\perp = \frac{\sigma}{\epsilon_0} \quad \text{(field at the surface of a conductor)}$$