SOME IMPORTANTES CONTINUOUS RANDOM VARIABLES

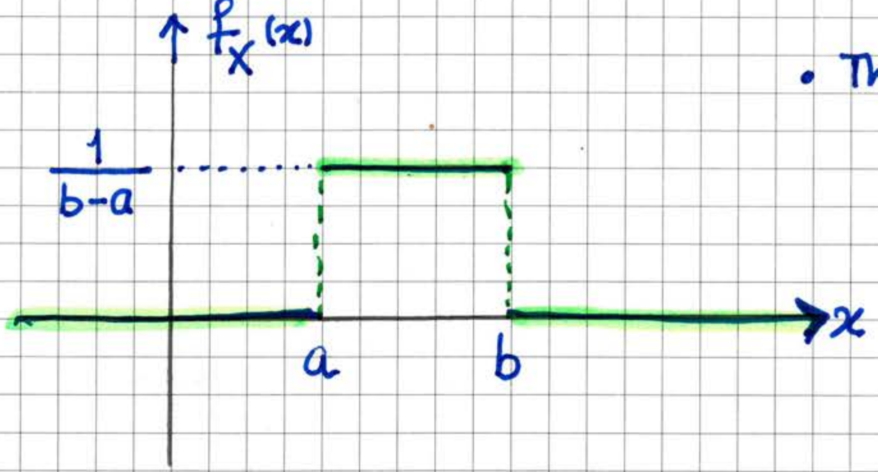
- Most physical measurements, do not produce a discrete set of values but rather a continuum of values such as the maximum temperature measured during the day.
- Other physical quantities such as time, length, area, pressure, load, intensity, etc., when they need to be described probabilistically, are modeled by continuous random variables.
- Now, we are going to discuss some important continuous random variables. These are as follows:
  - Uniform distribution
  - Exponential distribution
  - Gaussian or Normal distribution.

UNIFORM DISTRIBUTION

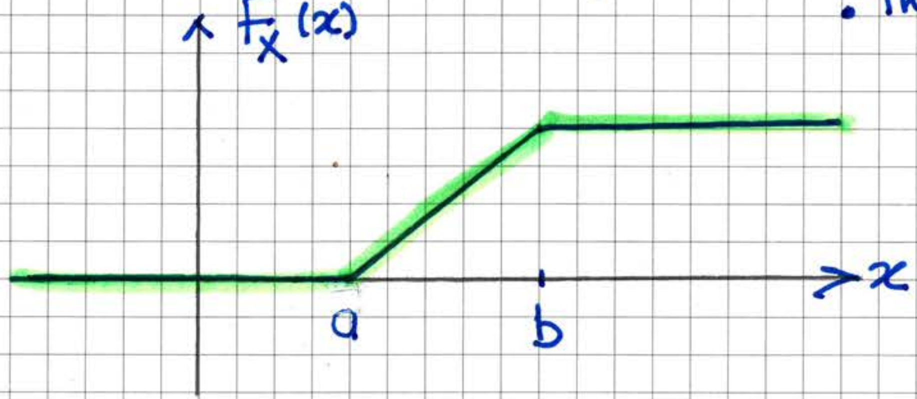
- A continuous random variable X has a uniform distribution over an interval a to b $(b > a)$ if it is equally likely to take on any value in this interval.

- The probability density function of X is constant over intercal (a, b) and has the form

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} , & \text{for } a \le x \le b \\ 0 , & \text{otherwise.} \end{cases}$$



- The pdf, $f_X(x)$.



- The cdf, $F_X(x)$.

- It is given the shorthand notation

$$X \sim \mathcal{U}(a, b).$$

- The cdf of X can be found by integrating the pdf of the random variable:

$$F_X(x) = P_r\{X \le x\}$$

$$= \int_a^x \frac{1}{b-a}\,dx$$

$$= \begin{cases} 0 & , \quad x < a \\ \dfrac{x-a}{b-a} & , \quad a \le x < b \\ 1 & , \quad x \geq b. \end{cases}$$

- The expected value of X is

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$= \int_a^b x\left(\frac{1}{b-a}\right)dx = \frac{a+b}{2}.$$

- The variance of X is

$$VAR[X] = \sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx$$

$$= \int_a^b \left[x - \left(\frac{a+b}{2}\right)\right]^2 \left(\frac{1}{b-a}\right)dx = \frac{(b-a)^2}{12}.$$

- The Characteristic Function is

$$\Phi_X(t) = E\left[e^{jtX}\right]$$

$$= \int_{-\infty}^{\infty} e^{jtx} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{b-a} e^{jtx}\,dx = \frac{e^{jtb} - e^{jta}}{jt(b-a)}$$

for $t \neq 0$.

- If the interval is $[-a, a]$ then, we can write

$$\Phi_X(t) = \frac{1}{at}\left(\frac{e^{jta} - e^{-jta}}{2j}\right)$$

$$= \frac{\sin at}{at}, \quad \text{for } t \neq 0.$$

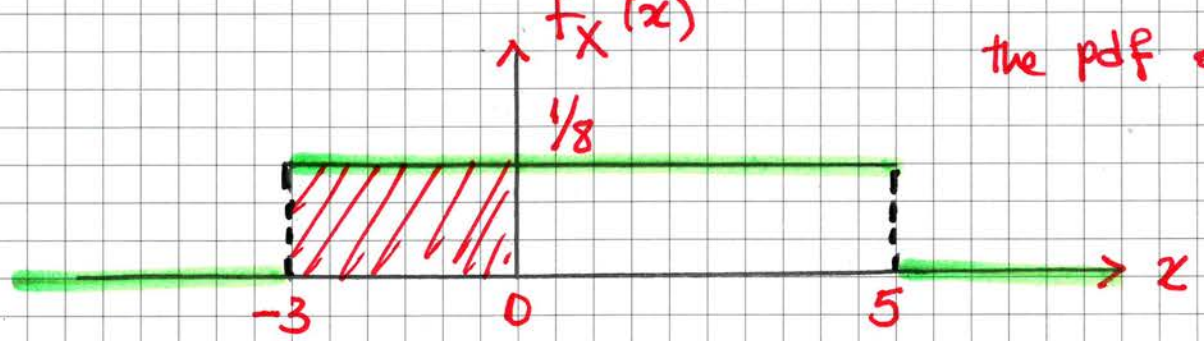
Example: [T.T. Soong]

- Due to the unpredictable traffic, the time required by a certain student to travel from his home to his morning class is uniformly distributed between 22 and 30 minutes.
- If he leaves home at precisely 7:35 a.m, what is the probability that he will not be late for class, which begins promptly at 8 a.m.?

Answer:

Let X be the class arrivel time of the student in minutes after 8:00 a.m. It than has a uniform distribution by

$$f_X(x) = \begin{cases} 1/8 , & \text{for } -3 \le x \le 5 \\ 0 , & \text{otherwise} \end{cases}$$



the pdf of X.

- We are interested in the probability

$$\Pr\{-3 \le X \le 0\} = 3\left(\frac{1}{8}\right) = \frac{3}{8}.$$

- This probability is equal to the ratio of the shaded area.

Remark:

- If a random variable X is uniformly distributed over an interval A, then the probability of X taking values in a subinterval B is given by.

$$\Pr\{X \text{ in } B\} = \frac{\text{length of } B}{\text{length of } A}.$$

EXPONENTIAL DISTRIBUTION

- The exponential random variable is closely related to the Poisson distribution. Indeed, if

$$X \sim \text{Pois}(\lambda),$$

λ is the number of occurrence in the unit time interval. we can write the random variable X in the interval $[0,t]$;

- The pmf of the Poisson random variable can be expressed as follows;

$$p_X(x) = \begin{cases} \dfrac{(\lambda t)^x}{x!} e^{-\lambda t} & ; \quad x = 0,1,2,\dots \\ 0 & ; \quad \text{otherwise} \end{cases}$$

- $p_X(0)$ is considered that no occurrences on $[0,t]$. This is given by.

$$p_X(0) = \frac{(\lambda t)^0}{0!} e^{-\lambda t}$$

- Another interpretation of $p_X(0) = e^{-\lambda t}$ is that this is the probability that the time occurrence is greater than t.
- Consider this time as a random variable T, we note that

$$p_X(0) = Pr\{T > t\} = e^{-\lambda t}, \quad t \geq 0.$$

- If we let the time vary and consider the random variable T as the ~~time~~ occurrence, then

$$F_T(t) = Pr\{T \leq t\}$$
$$= 1 - e^{-\lambda t}, \quad \text{for } t \geq 0.$$

- Since

$$f_T(t) = \frac{d}{dt} F_T(t),$$

we obtain the pdf as

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0. \end{cases}$$

- λ is the parameter of the distribution. Its mean and variance are obtained as follows:

$$E[T] = \mu_T = \int_{-\infty}^{\infty} t\, f_T(t)\, dt$$
$$= \int_0^{\infty} t \lambda e^{-\lambda t} dt = \frac{1}{\lambda}.$$

Remark:

- We can establish this result that the interarrival time between Poisson arrivals has an exponential distribution.
- the parameter λ in the distribution of T is the mean arrival rate associated with Poisson arrivals.

Example:

The probability that a telephone call lasts no more than t minutes is often modeled as an exponential cdf

$$F_T(t) = \begin{cases} 1 - e^{-t/3}, & t \geq 0 \\ 0, & \text{otherwise}. \end{cases}$$

(a) What is the probability that a conversation will last between 2 and 4 minutes.

$$\Pr\{2 \leq T \leq 4\} = F_T(4) - F_T(2)$$
$$= 1 - e^{-4/3} - (1 - e^{-2/3})$$
$$= e^{-2/3} - e^{-4/3} = 0.25.$$

(b) What is the duration in minutes of a telephone conversation?

$$f_T(t) = \frac{d}{dt} F_T(t)$$

$$= \begin{cases} \frac{1}{3} e^{-t/3} & , \quad t \geq 0 \\ 0 & , \quad t < 0 . \end{cases}$$

Example: [Montgomery and Runger]

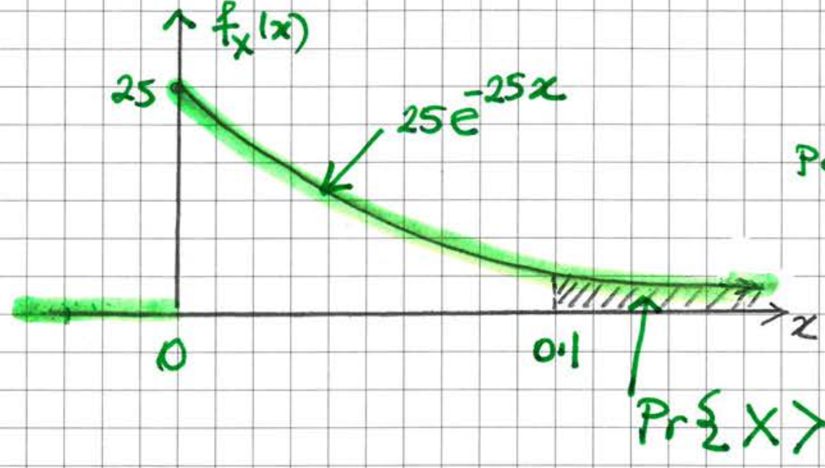
- In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour.
- What is the probability that there are no log-ons in an interval of 6 minutes?

Let X denote the time in hours from the start of the interval until the first log-on. Then X has an exponential distribution with $\lambda = 25$ log-ons per hour.

Its pdf is

$$f_X(x) = \begin{cases} 25\, e^{-25x} & , \quad x \geq 0 \\ 0 & , \quad x < 0 . \end{cases}$$

We are interested in the probability that X exceeds 6 minutes. Because λ is given in log-ons per hour, we express all time units in hours. That is 6 minutes = 0.1 hour.



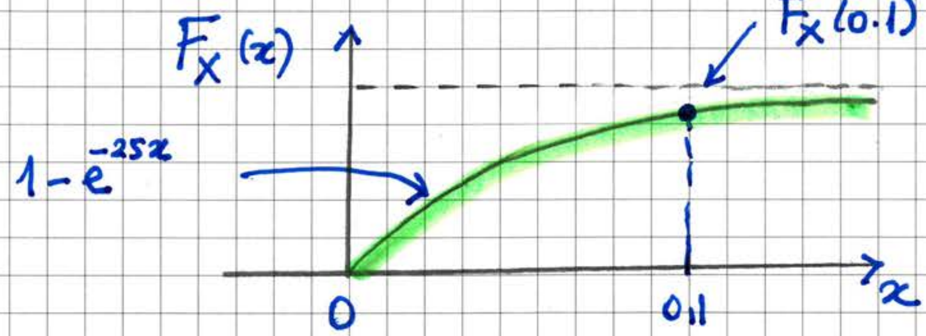
Pdf for this example.

- The probability requested is shown as the shaded area under the probablity density function

$$Pr\{X > 0.1\} = \int_{0.1}^{\infty} 25\, e^{-25x} dx$$

$$= e^{-25(0.1)} = 0.082.$$

- We can obtain the same result from the cdf as follows



$$Pr\{X > 0.1\} = 1 - F_X(0.1)$$

$$= e^{-25(0.1)} = 0.082.$$

- What is the probability that the time until the next log-on is between 2 and 3 minutes?

We convert all units to hours,

$$Pr\{0.033 < X < 0.05\} = \int_{0.033}^{0.05} 25\, e^{-25x}\, dx$$

$$= -e^{-25x}\Big|_{0.033}^{0.05} = 0.152$$

An alternative solution is

$$Pr\{0.033 < X < 0.05\} = F_X(0.05) - F_X(0.033)$$

$$= 0.152.$$

- Determine the interval of time such that the probability that no log-on occurs in the interval is 0.90.

The question asks for the length of time x such that $Pr\{X > x\} = 0.90$. Now

$$Pr\{X > x\} = e^{-25x} = 0.90$$

We take the natural log of both sides,

to obtain

$$-25x = \ln(0.90) = -0.1054.$$

Hence

$$x = 0.00421 \text{ hour} = 0.25 \text{ minute}.$$

Find the mean time until the next log-on,

$$\mu_X = \frac{1}{25} = 0.04 \text{ hour}$$
$$= 2.4 \text{ minutes}.$$

The standard deviation of the time until the next log-on is

$$\sigma_X = \frac{1}{25} \text{ hours} = 2.4 \text{ minutes}.$$

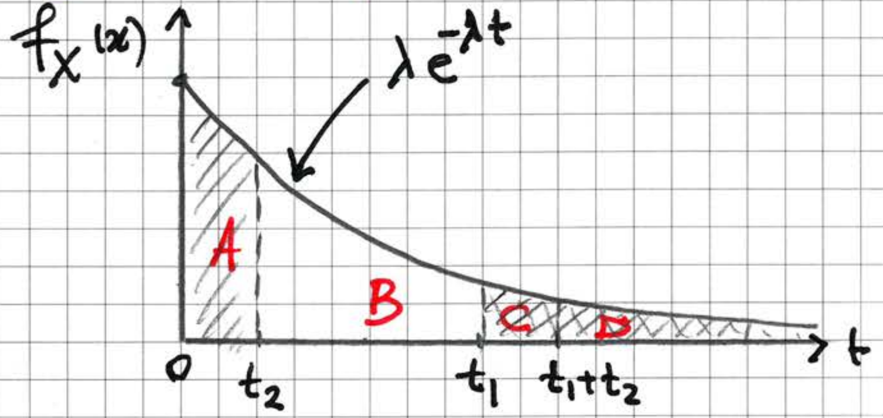
Remark:

- Our starting point for observing the system does not matter. However, if there are high-use periods during the day, such as right after 8:00 A.M, followed by a period of low use, a Poisson process is not appropriate model for logs-ons and the distribution is not appropriate for computing probabilities. It might be reasonable to model each of the high- and

and low-use periods by a separate Poisson process, employing a larger value for λ during the high-use periods and a smaller value otherwise.

LACK OF MEMORY PROPERTY

(of an exponential random variable)



- The exponential distribution is the only continuous distribution with this property.

For an exponential random variable X,

$$Pr\{X < t_1+t_2 \mid X > t_1\} = Pr\{X < t_2\}.$$

$$Pr\{X < t_1+t_2 \mid X > t_2\} = \frac{Pr\{t_1 < X < t_1+t_2\}}{Pr\{X > t_2\}}$$

$$= \frac{\int_{t_1}^{t_1+t_2} \lambda e^{-\lambda x} dx}{\int_{t_1}^{\infty} \lambda e^{-\lambda x} dx} = 1 - e^{-\lambda t_2} = Pr\{X < t_2\}.$$

Indeed,

$$\Pr\{X < t_2\} = \int_0^{t_2} \lambda e^{-\lambda x} dx = 1 - e^{-\lambda t_2}.$$

- We can graphically illustrate the lack of memory property.

- The area of region A divided by the total area under the probability density function

$$A + B + C + D = 1$$

equals $\Pr\{X < t_2\}$. $\left(= \frac{\text{Area } A}{\underbrace{A+B+C+D}_{1}} \right)$

- The area of region C divided by the area C+D equals

$$\Pr\{X < t_1 + t_2 \mid X > t_1\} = \frac{\text{Area } C}{\text{Area } C+D}.$$

Remark:

An exponential random variable is the continuous analog of a geometric random variable, and they share a similar lack of memory property.

Example: [Montgomory and Runger]

- Let X denote the time between detections of a particle with a Geiger counter and assume that X has an exponential distribution with

$$E[X] = 1.4 \text{ minutes.}$$

Therefore, the parameter of the distribution is $\lambda = 1/E[X] = 1/1.4$ detection per minute.

$$f_X(x) = \left(\frac{1}{1.4}\right) e^{-\frac{x}{1.4}}, \quad \text{for } x \geq 0 \qquad \text{(pdf)}$$

and

$$F_X(x) = 1 - e^{-\frac{x}{1.4}}, \quad \text{for } x \geq 0 \qquad \text{(cdf)}$$

- The probability that we detect a particale within 30 seconds of starting the counter is.

$$\Pr\{X < 0.5\} = F_X(0.5)$$

$$= 1 - e^{-\frac{0.5}{1.4}} = 0.30.$$

- In this calculation, all units are converted to minutes.

- Now, suppose we turn on the Geiger counter and wait 3 minutes without detecting a particle.
- What is the probability that a particale is detected in the next 30 seconds?

Warning!

Because we have already been waiting for 3 minutes, we feel that we are "due". That is, the probability of a detection in the next 30 seconds should be greater than 0.3.

However, for an exponential distribution, this is not true. The requested probability can be expressed as the conditional probability that

$$\Pr\{X < 3.5 \mid X > 3\}.$$

From the definition of conditional probability, we have

$$\Pr\{X < 3.5 \mid X > 3\} = \frac{\Pr\{3 < X < 3.5\}}{\Pr\{X > 3\}}.$$

From the cdf of X, $F_X(x) = 1 - e^{-\frac{x}{1.4}}$,
we can write,

$$\Pr\{3 < X < 3.5\} = F_X(3.5) - F_X(3)$$

$$= \left[1 - e^{-\frac{3.5}{1.4}}\right] - \left[1 - e^{-\frac{3}{1.4}}\right]$$

$$= 0.035$$

and

$$\Pr\{X > 3\} = 1 - F_X(3)$$

$$= e^{-\frac{3}{1.4}} = 0.117$$

Therefore,

$$\Pr\{3 < X < 3.5\} = \frac{0.035}{0.117}$$

$$= 0.30 \quad .$$

We observe that waiting 3 minutes without a detection does not change the probability of a detection in the next 30 seconds.

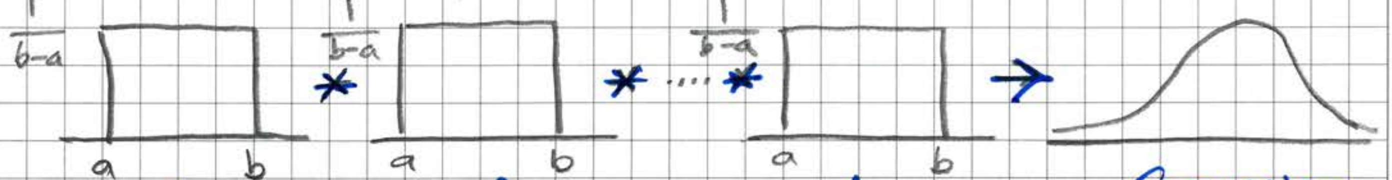
GAUSSIAN (NORMAL) DISTRIBUTION
(Random Variable)

- This distribution is very important in both the theory and applications. Indeed, this random variable appears so often in problems involving randomness. Therefore, it is known as the "normal" random variable.

- According to the central limit theorem Gaussian random variable X consists of the sum of a large number of "small" random variables.

If $X_1, X_2, \dots, X_n$ independent and identically distributed (iid) random variables, $\mu_i = \mu < 0$ and $\sigma_i^2 = \sigma^2 < 0$, Then

$$X_1 + X_2 + \cdots + X_n \longrightarrow X$$ Always Gaussian?



uniform uniform ... uniform Gaussian

$Exp(\lambda) * Exp(\lambda) * \cdots * Exp(\lambda) \to$ Gaussian

$Bern(p) * Bern(p) * \cdots * Exp(\lambda) \to$ Gaussian

- The normal distribution is in many respects the cornerstone of statistics.

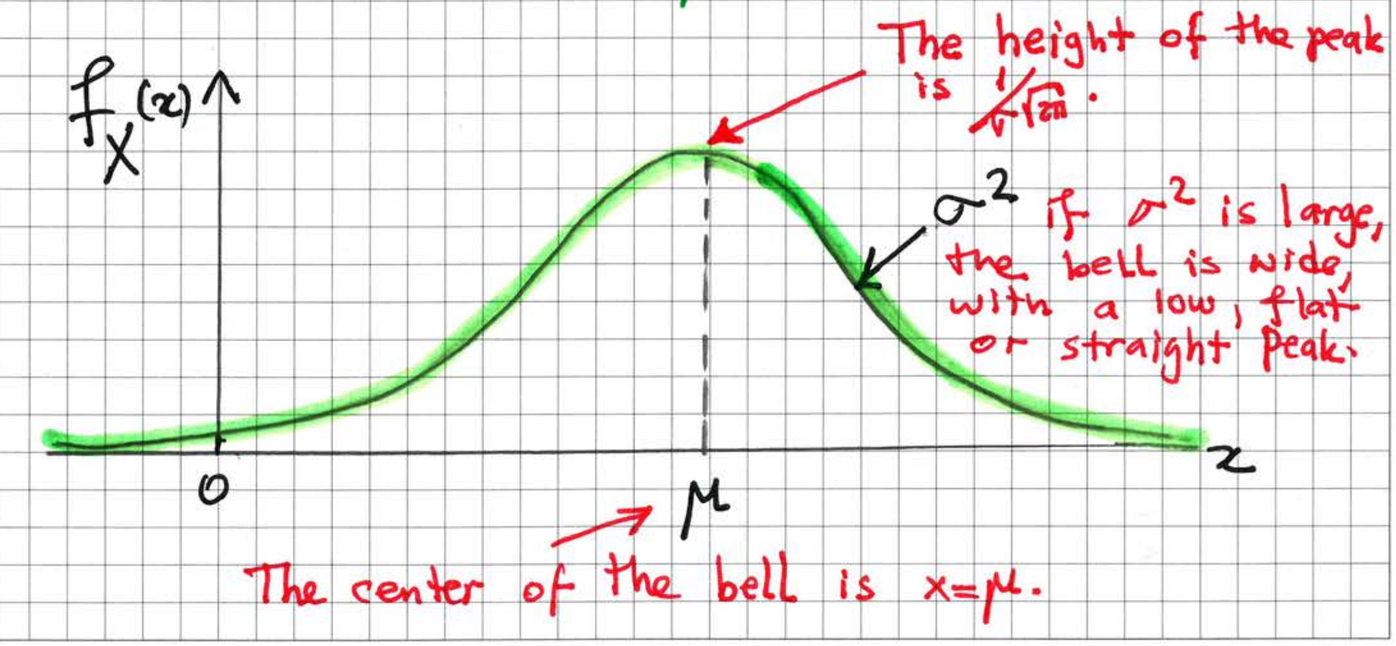
Definition:

A random variable X is said to have a normal distribution with mean μ, $(-\infty < \mu < \infty)$ and variance σ^2, $(\sigma^2 > 0)$, if it has the density function,

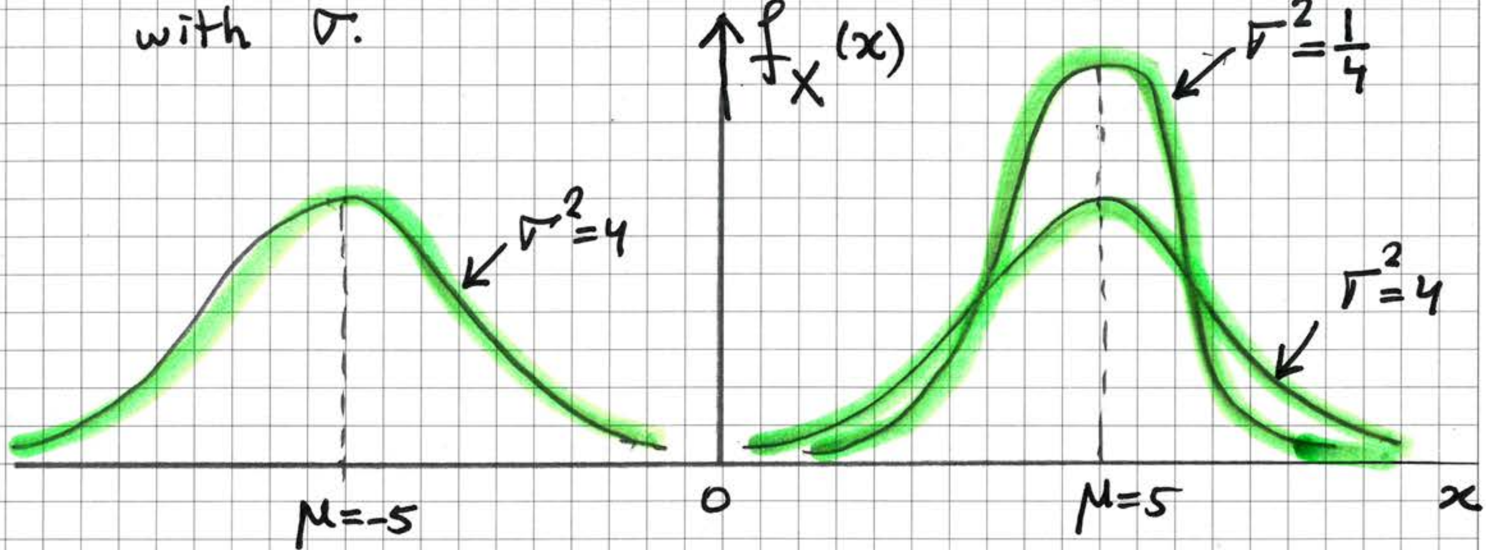
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

for $-\infty < x < \infty$.

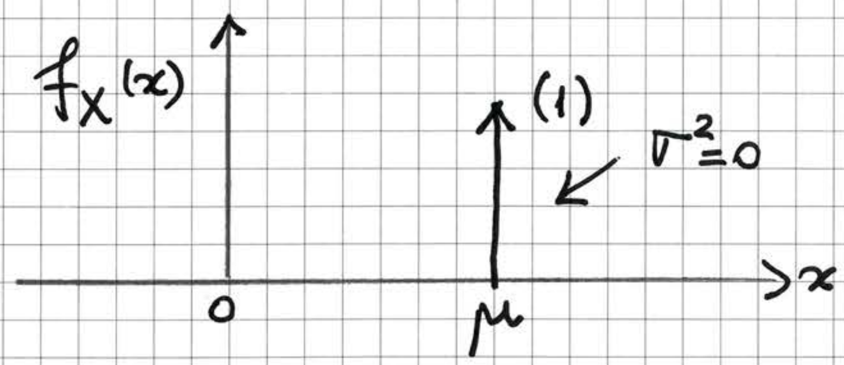
The shorthand notation is $X \sim \mathcal{N}(\mu, \sigma^2)$. This indicates that the random variable X normally distributed with mean μ and variance σ^2.



- Figure shows that the normal pdf is a "bell-shaped" curve centered and symmetric about μ and whose "width" increases with σ.



- Note that at the limit case, $\sigma^2 \to 0$, and the random variable is a deterministic number, $X=\mu$. There is no randomness.



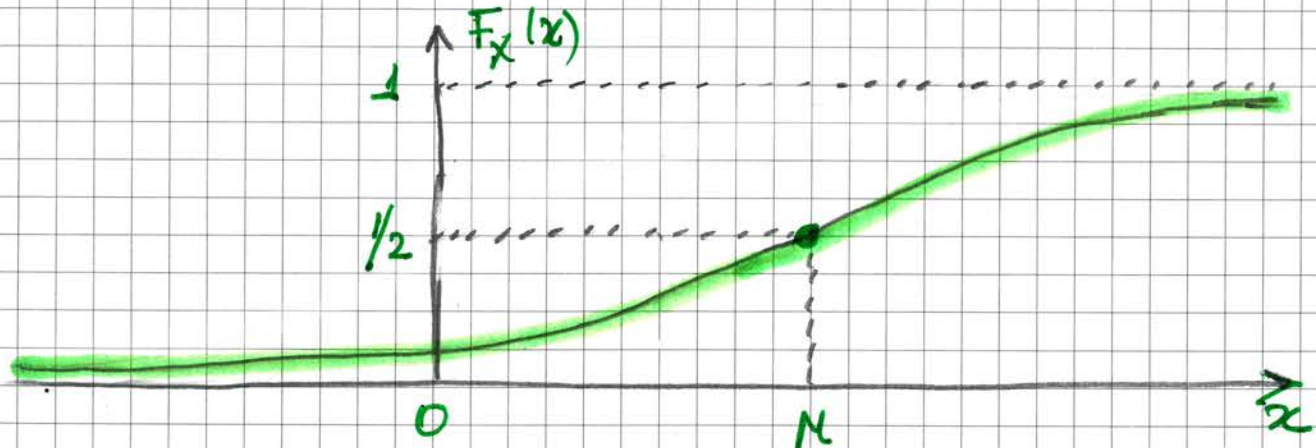
$$f_X(x) = \delta(x-\mu).$$

The cdf (probability distribution function) of the Gaussian random is given by

$$F_X(x) = P_r\{X \leq x\}$$

$$= \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(t-\mu)} dt$$

$$-\infty < x < \infty$$



For $\mu=0$ and $\sigma^2=1$, we call the normal random variable as standart normal random variable, and the random variable is shown as Z.

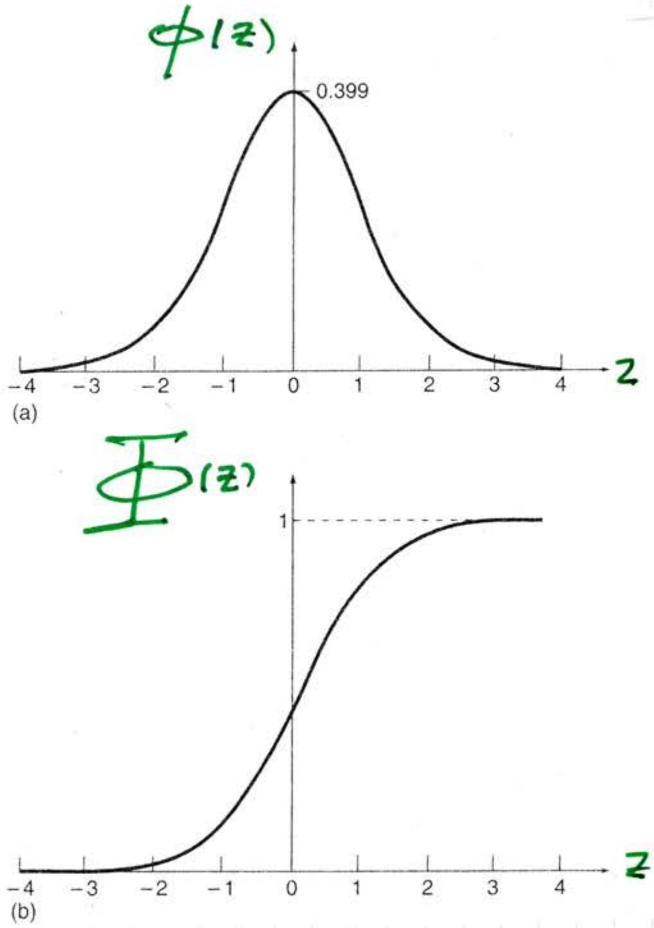
Normal R.V.	Standard Normal R.V
$X \sim N(\mu, \sigma^2)$	$Z \sim N(0,1)$

$X \longrightarrow Z$

$f_X(x) \longrightarrow \phi(z)$

$F_Z(x) \longrightarrow \Phi(z)$.

Subject
Name
Probability and Statistics
(3)



pdf:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$-\infty < z < \infty$$

cdf:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

Theorem 1:

If X is a $N(\mu, \sigma^2)$ random variable, then

$$E[X] = \mu$$

$$VAR[X] = \sigma^2$$

and the area under a Gaussian pdf is

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = 1.$$

Theorem 2:

If X is $\mathcal{N}(\mu, \sigma^2)$,
$Y = aX + b$ random variable
is also normal,

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2).$$

Remark:

From theorem 1, we obtain that the standart normal random variable, $Z \sim \mathcal{N}(0,1)$ has

$$E[Z] = 0 \quad \text{and} \quad VAR[Z] = 1.$$

Theorem 3:

If X is a normal random variable, $X \sim \mathcal{N}(\mu, \sigma^2)$, the cdf of X is

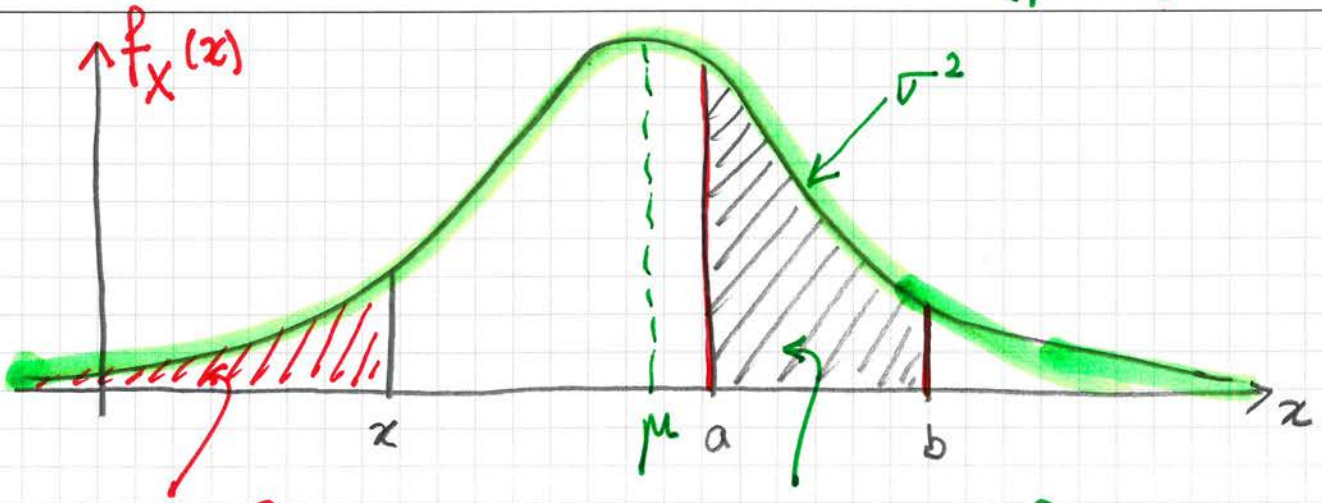
$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

and the probability that X is in the interval $[a, b]$ is

$$\Pr\{a \leq X \leq b\} = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

ibis

$X \sim N(\mu, \sigma^2)$



$\Pr\{X \le x\}$

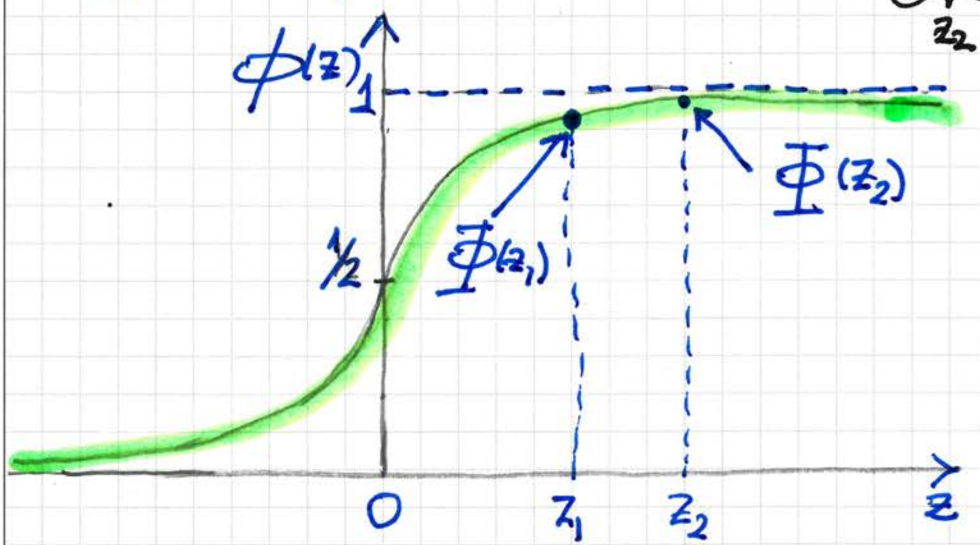
$\Downarrow$

$$\Phi\left(\frac{x-\mu}{\sigma}\right)$$

$\Pr\{a \le X \le b\}$

$\Downarrow$

$$\Phi\underbrace{\left(\frac{b-\mu}{\sigma}\right)}_{z_2} - \Phi\underbrace{\left(\frac{a-\mu}{\sigma}\right)}_{z_1}$$



* $\Phi(z)$ given with a table for $0 \le z \le 3$.
(Page 349, textbook)

$\Phi(0) = \frac{1}{2}$

$\Phi(3) \cong 1$

$\Phi(-z) = 1 - \Phi(z)$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du = \Pr\{Z \le z\}.$$

Proof:

Remember $X \sim N(\mu, \sigma^2)$ is the normal random variable

$Z \sim N(0,1)$ is the standard normal random variable

We have

$$F_X(x) = P_r\{X \le x\}$$
$$= \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt .$$

For a sample value x of the normal random variable X, the corresponding sample value of z of the standard normal random variable of Z is

$$z = \frac{x-\mu}{\sigma}$$

Therefore, after the change of variable

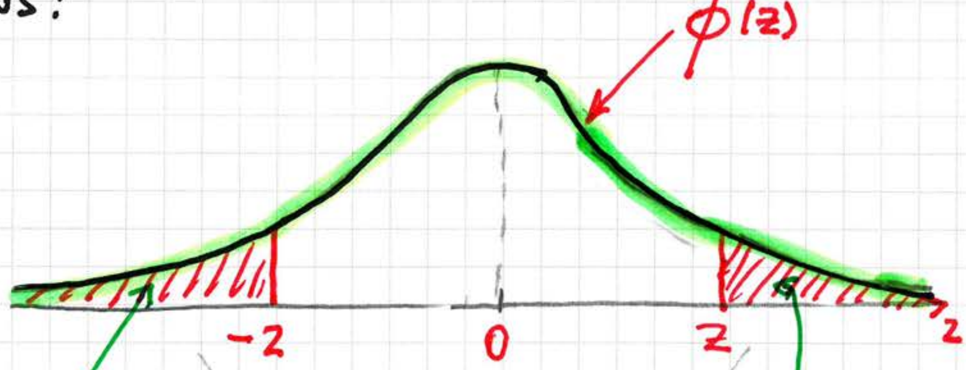
$$F_X(x) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}_{\phi(z) \text{ is the pdf of the standard normal r.v.}} dz$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \phi(z)\, dz$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right).$$ ← The cdf of the standard normal r.v.

Remark:

There are only positive values of z for $\Phi(z)$ at our table. Therefore, the nagative values will be calculated as follows:



$\Pr\{Z \leq -z\} = \Phi(-z)$

$\Pr\{Z > z\} = 1 - \Pr\{Z \leq z\}$
$= 1 - \Phi(z).$

Due to the symmetry

$$\phi(-z) = 1 - \phi(z).$$

Example: [T.T. Soong]

Owing to many independent error sources, the length of a manufactured machine part is normally distributed with $\mu = 11$ cm and $\sigma = 0.2$ cm, namely, $X \sim N(11, 0.04)$.

If specifications require that the length be between 10.6 cm and 11.2 cm, what proportion of the manufactured parts will be rejected on average?

Answer:

If X is used to denote the part length in centimeters, it is reasonable to assume that it is distributed as

$$X \sim N(11, 0.04).$$

Therefore, on average, the proportion of acceptable parts is

$$Pr\{10.6 \leq X \leq 11.2\}.$$

Then, we can write,

$$Pr\{10.6 \leq X \leq 11.2\} = Pr\left\{\frac{10.6-11}{0.2} \leq Z \leq \frac{11.2-11}{0.2}\right\}$$

$$= \Phi\{-2 \leq Z \leq 1\}$$

$$= \Phi(1) - \Phi(-2)$$

$$= \Phi(1) - [1 - \Phi(2)]$$

$$= \Phi(1) + \Phi(2) - 1$$

$$= 0.8413 + 0.9772 - 1 = 0.8185.$$

The desired answer, the probability of rejection is

$$Pr\{\text{rejection}\} = 1 - 0.8185 = 0.1815.$$

Example:

Let us compute

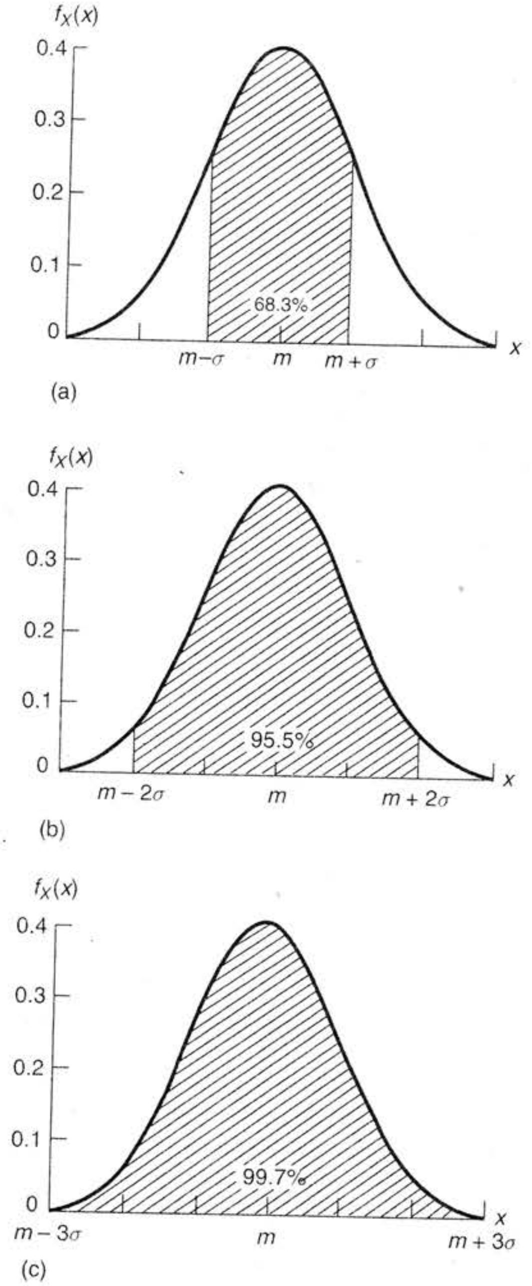
$Pr\{\mu - k\sigma \leq X \leq \mu + k\sigma\}$

where $X \sim \mathcal{N}(\mu, \sigma^2)$.

We can write,

$$Pr\{\mu - k\sigma \leq X \leq \mu + k\sigma\}$$
$$= Pr\{-k \leq Z \leq k\}$$
$$= \Phi(k) - \Phi(-k)$$
$$= \Phi(k) - [1 - \Phi(k)]$$
$$= 2\Phi(k) - 1.$$

From Table: $\Phi(1) = 0.8413$, $\Phi(2) = 0.9772$ and $\Phi(3) = 0.9987$,



- We get 68.3%, 95.5% and 99.7% of the area under a normal pdf, respectively, in the ranges $\mu \mp \sigma$, $\mu \mp 2\sigma$, and $\mu \mp 3\sigma$.
- For example, the chances are about 99.7% that a randomly selected sample from a normal distribution is within the rang of $\mu \mp 3\sigma$ [see above figure(c)]

FUNCTIONS OF RANDOM VARIABLES

(one random variable)

We want to determine the relationship between probability distributions of two random variables X and Y when they are related by

$$Y = g(X).$$

Namely, given the probability distribution (cdf) of X, $F_X(x)$, probability mass function (pmf), $P_X(x)$ or probability density function (pdf), we are interested in corresponding distribution for Y, $F_Y(y)$, $P_Y(y)$, or $f_Y(y)$.

(A) Discrete Random Variables

Suppose that possible values of X can be enumerated as $x_1, x_2 \dots$. $Y = g(X)$ shows that the corresponding possible values of Y may be enumerated as $y = g(x_1)$, $y_2 = g(x_2), \dots$.

Let the pmf of X be given by.

$$P_X(x_i) = p_i \quad , \text{ for } i = 1, 2, \dots$$

Then the pmf of y is simply determined as

$$P_Y(y_i) = P_Y\left[g(x_i)\right] = P_i \quad , i=1,2,\ldots.$$

Example: [Soong]

The pmf of a random variable X is given as

$$P_X(x) = \begin{cases} 1/2 & , \text{ for } x=-1 \\ 1/4 & , \text{ for } x=0 \\ 1/8 & , \text{ for } x=1 \\ 1/8 & , \text{ for } x=2. \end{cases}$$

The random variable Y is related to X by

$$Y = 2X+1.$$

Determine the pmf of Y.

Answer: The corresponding values of Y are:

$$g(-1) = 2\cdot(-1)+1 = -1$$
$$g(0) = 2\cdot(0)+1 = 1$$
$$g(1) = 2\cdot(1)+1 = 3$$
$$g(2) = 2\cdot(2)+1 = 5.$$

Hence, the pmf of Y is given by

$$P_Y(y) = \begin{cases} 1/2 & , \text{ for } y = -1 \\ 1/4 & , \text{ for } y = 1 \\ 1/8 & , \text{ for } y = 3 \\ 1/8 & , \text{ for } y = 5 \end{cases}.$$

Example:

The same X as given in the above example, determine the pmf Y if

$$Y = 2X^2 + 1.$$

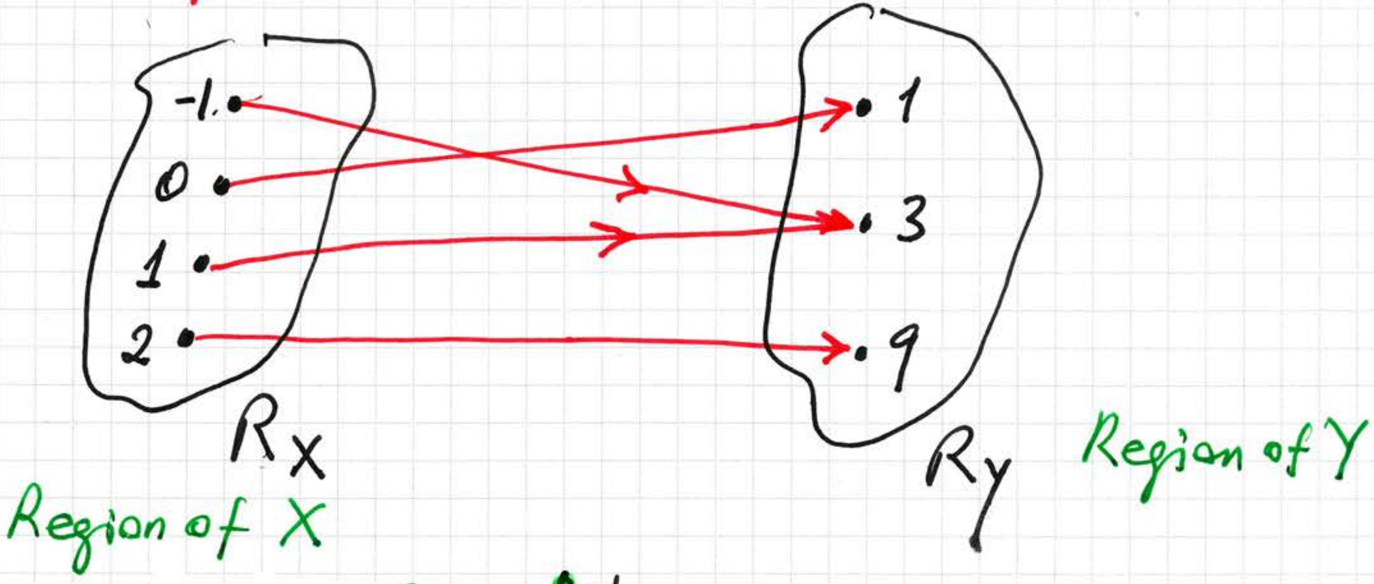
In this case, the corresponding values of Y are:

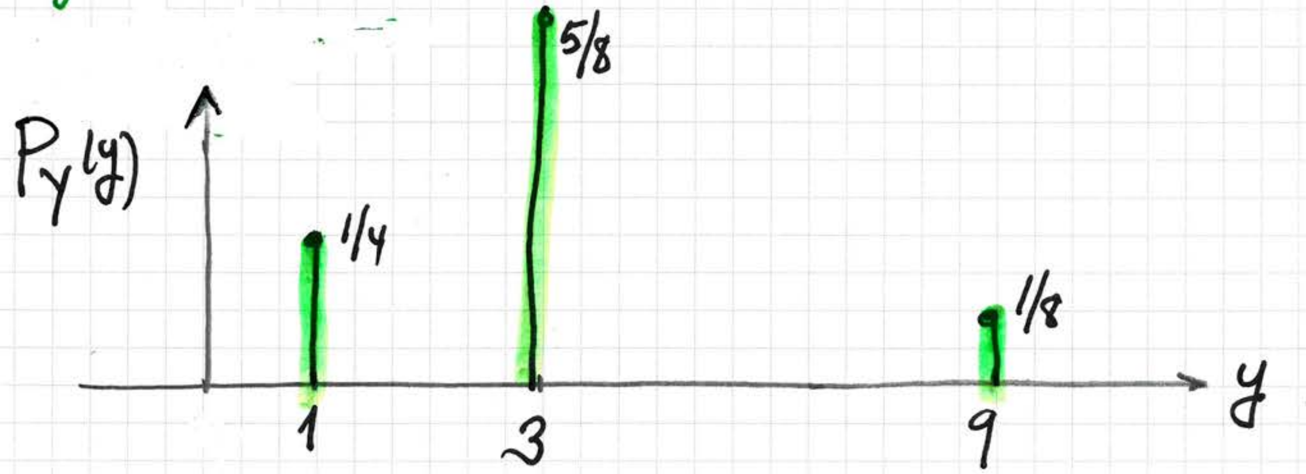
$$g(-1) = 2(-1)^2 + 1 = 3$$
$$g(0) = 2(0)^2 + 1 = 1$$
$$g(1) = 2(1)^2 + 1 = 3$$
$$g(2) = 2(2)^2 + 1 = 9.$$

Hence, the pmf of Y is given by

$$P_Y(y) = \begin{cases} 1/4 & , \text{ for } y=1 \\ 5/8 \left(= \frac{1}{2}+\frac{1}{8}\right) & , \text{ for } Y=3 \\ 1/8 & , \text{ for } Y=9 . \end{cases}$$

- Note that many-to-one transformations are possible.





(B) Continuous Random Variables

We have a continuous random variable with known cdf, $F_X(x)$, or pdf, $f_X(x)$. If the transformation $Y = g(X)$ is continuous in X and strictly monotonic (increasing or decreasing) function, then

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{d\, g^{-1}(y)}{dy} \right|.$$

- The proof of this theorem is given in page 141. [kayran].

Remark:

- For continuous random variables, we follow a two-step procedure:

  (1) Find the cdf

  $$F_Y(y) = \Pr\{Y \leq y\}.$$

  - This is obtained from $F_X(x)$.

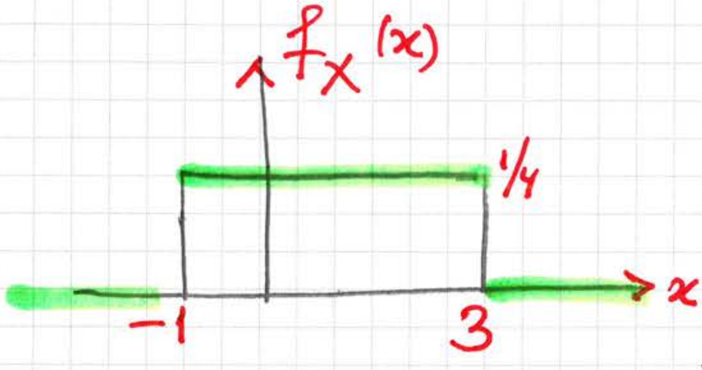
  (2) Compute the pdf of Y, $f_Y(y)$, by calculating the derivative

  $$f_Y(y) = \frac{d}{dy} F_Y(y).$$

This procedure always work and it easy to remomber: Read the textbook.

Example:

- The continuous Random variable X is is uniformly distributed as shown in Figure.

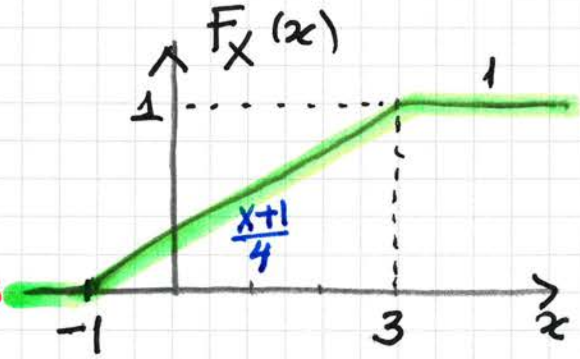


- The transformation or the relationship between X and Y random variables is

$$Y = 10X.$$

- First we calculate the cdf of X,

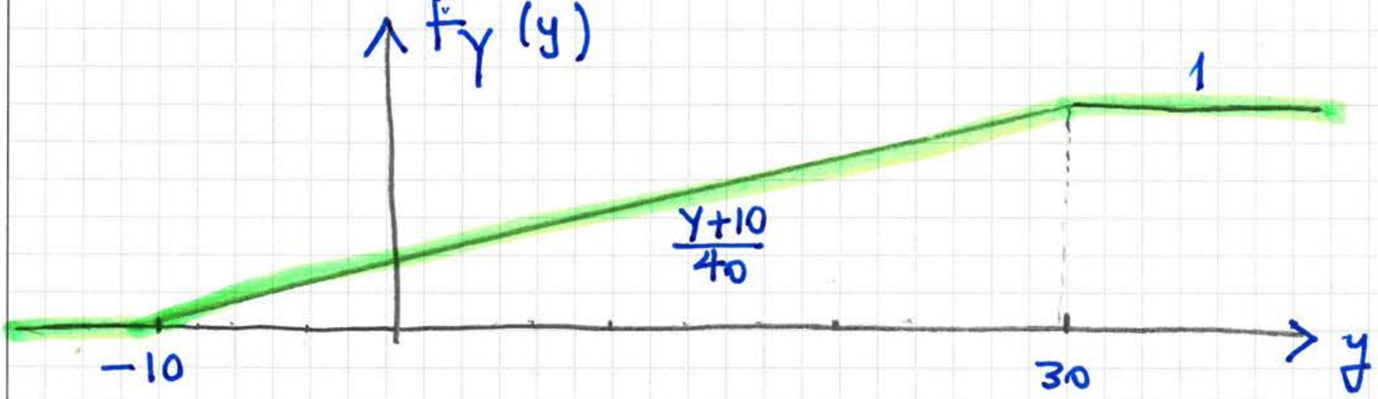
$$F_X(x) = \begin{cases} 0 & , \quad x < -1 \\ \dfrac{x+1}{4} & , \quad -1 \le x \le 3 \\ 1 & , \quad x > 3 \end{cases}$$



- We compute $F_Y(y)$ as follows:

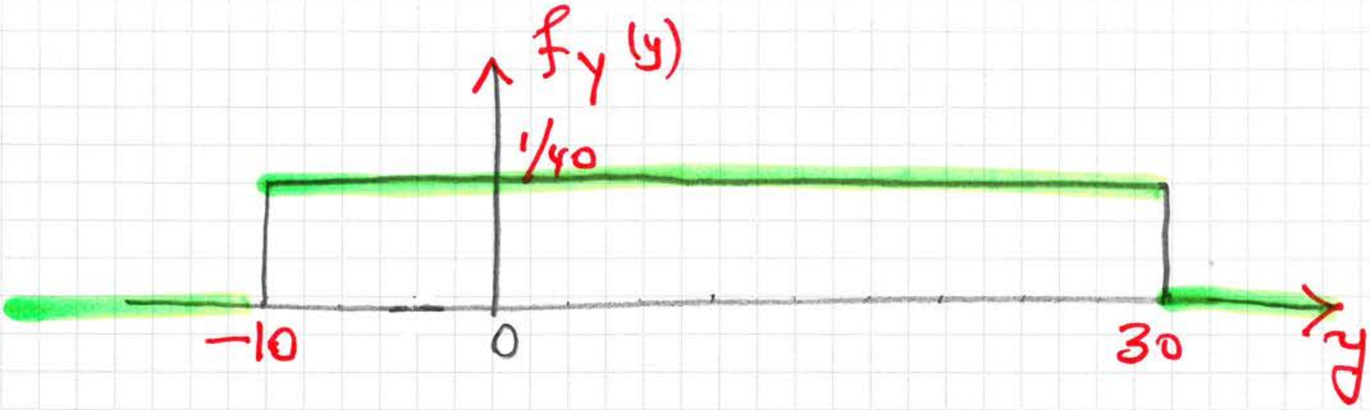
$$F_Y(y) = P_r\{Y \le y\} = P_r\{10X \le y\} = P_r\{X \le y/10\}$$

$$= F_X(y/10).$$



- the pdf of Y is obtained by differentiating $F_Y(y)$ with respect to y:

$$f_Y(y) = \frac{d}{dy} F_Y = \begin{cases} 0 & x < -10, \\ 1/40 & -10 \le x \le 30, \\ 0 & x > 30. \end{cases}$$



Chebyshev Inequality:

- The knowledge of mean and variance of a random variable, although very useful, it is not sufficient to determine its distribution and therefore does not permit us to give answers to such questions; "What is $P_r\{X \geq 5\}$?

- However, it is possible to establish some probability bounds knowing only the mean and variance, without having its probability function $f_X(x)$ or $F_X(x)$.

Theorem:

The Chebyshev inequality states that

$$\Pr\{ |X-\mu_X| \geq k\sigma_X \} \leq \frac{1}{k^2}$$

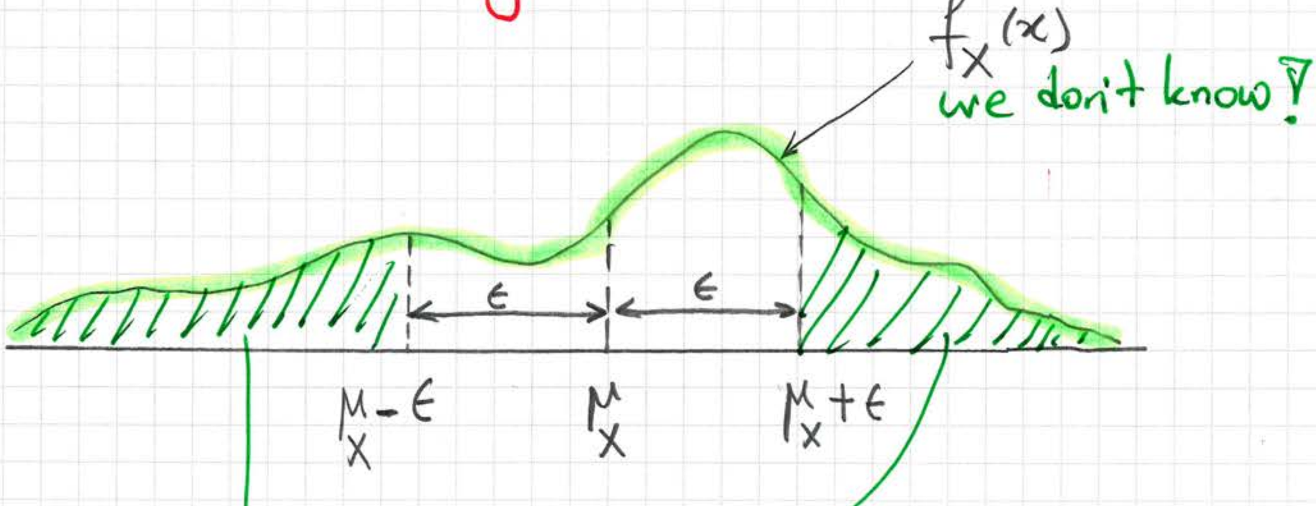
for any $k>0$.

OR $$\Pr\{ |X-\mu_X| \geq \epsilon \} \leq \frac{\sigma_X^2}{\epsilon^2}$$

for any $\epsilon>0$.

OR $$\Pr\{ |X-\mu_X| < k\sigma_X \} \geq 1-\frac{1}{k^2}$$

for any $k>0$.



- There is an example in the textbook.

The area = the Probability $\leq \frac{\sigma_X^2}{\epsilon^2}$.

Remarks:-

- A measure of concentration of a random variable near its mean μ_X and its variance σ_X^2.
- The Chebyshev inequality shows that X is outside an arbitrary interval $(\mu-\epsilon, \mu+\epsilon)$ is negligible if the ratio σ/ϵ is sufficiently small.
- $Pr\{|X-\mu| \geq \epsilon\} \leq \frac{\sigma_X^2}{\epsilon^2}$ tells that, If $\sigma=0$, then the probability that X is outside the interval $(\mu-\epsilon, \mu+\epsilon)$ equals 0 for any ϵ. Hence

$$X=\mu \quad \text{with probability } 1.$$

- For specific densities, the bound is too high. Suppose, for example, that X is normal. In this case, $Pr\{|X-\mu| \geq 3\sigma\} = 2-2\Phi(3) = 0.0027$ However, Chebyshev inequality, yields

$$Pr\{|X-\mu| \geq 3\sigma\} \leq 1/9.$$

- The significance of Chebyshev's inequality is the fact that it holds for any $f(x)$ and it can be used even if $f_X(x)$ is not known.

Example [T.T. Soong]

The angle Φ of a pendulum as measured from the vertical is a random variable uniformly distributed over the interval $(-\pi/2 < \Phi < \pi/2)$.

Pendulum.

Determine the pdf of Y, the horizontal distance, as show in this Figure.

The transformation equation in this case is

$$Y = \tan \Phi$$

where

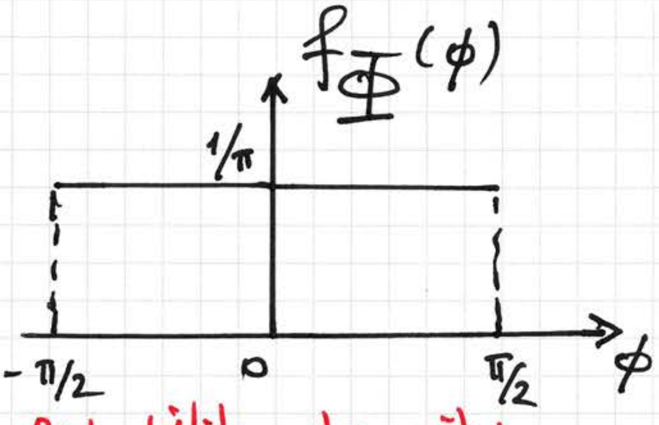
$$f_{\Phi}(\phi) = \begin{cases} 1/\pi, & \text{for } -\frac{\pi}{2} < \phi < \frac{\pi}{2} \\ 0, & \text{otherwise}. \end{cases}$$

As shown in the figure, $Y = \tan \Phi$ equation is monotono within the range $-\pi/2 < \phi < \pi/2$. Therefor, we have

$$g^{-1}(y) = \tan^{-1} y.$$

and

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{1+y^2}.$$



Probability density function, $f_{\Phi}(\phi)$.

The pdf of Y is given by

$$f_Y(y) = f_\Phi\left[g^{-1}(y)\right]\left|\frac{dg^{-1}(y)}{dy}\right|$$

$$= \frac{f_\Phi(\tan^{-1} y)}{1+y^2}$$

$$= \frac{1}{\pi}\frac{1}{(1+y^2)},$$

for $-\infty < y < \infty$.

$Y = \tan\Phi$ transformation

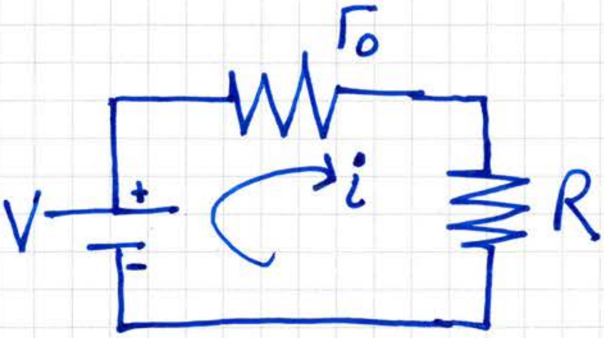
The range space R_Y corresponding to $-\pi/2 < \phi < \pi/2$ is $-\infty < y < \infty$. The pdf $f_Y(y)$ is thus valid for the whole range of y.

The random variable Y has the so-called Cauchy distribution and is plotted in the following figure:

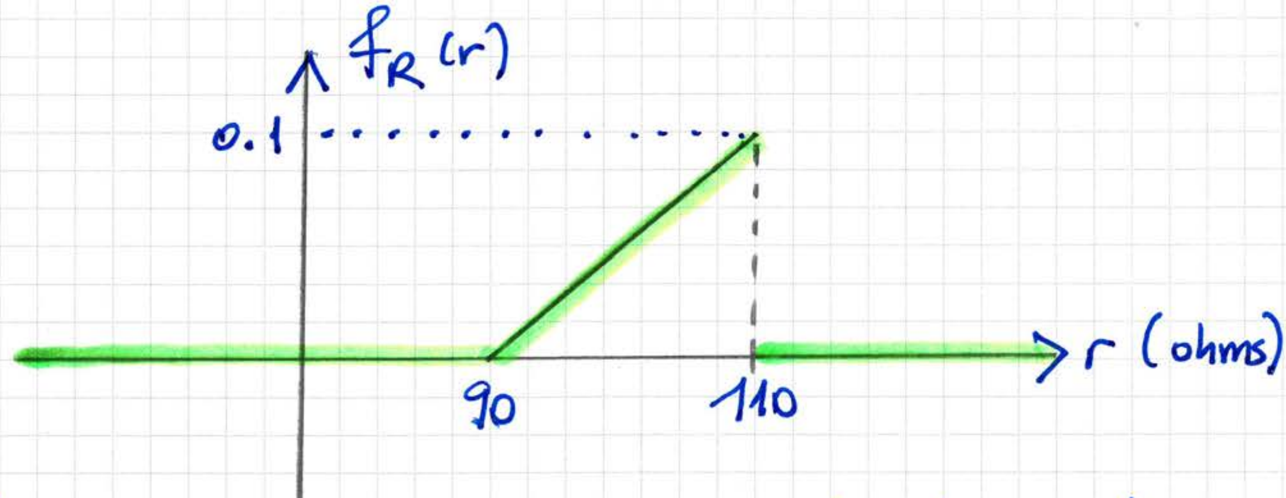
Probability density function, $f_Y(y)$.

Example:

The resistance R in this circuit is random and has a triangular distribution, as follows



$$f_R(r) = \begin{cases} 0.005(r-90) & , \text{ for } 90 \le r \le 110; \\ 0 & , \text{ otherwise} \end{cases}$$



The circuit has a constant $i = 0.1$ A and a constant resistance $r_0 = 100\,\Omega$; we can determine the pdf of the voltage, V.

The realionship between V and R is

$$V = i\,(R + r_0)$$
$$= 0.1\,(R + 100) \quad ; \quad V = g(R)$$

$f_V(v)$ we will find!

$f_R(r)$ we know

• The range $R_R = \{ r : 90 \leq r \leq 110 \}$ corresponds to $R_V = \{ v : 19 \leq v \leq 21 \}$. It is clear that $f_V(v)$ is zero outside the interval $19 \leq v \leq 21$.

• In this interval, since $V = 0.1(R+100)$ equation represent a strictly monotonic function, $f_V(v)$ is obtained as follows,

$$f_V(v) = f_R(g^{-1}(v)) \left| \frac{d}{dv} g^{-1}(v) \right| , \quad 19 \leq v < 21,$$

where

$$v = g(r) \qquad \Rightarrow \qquad r = g^{-1}(v)$$
$$v = 0.1(r+100) \qquad\qquad r = -100 + 10v$$

and

$$\frac{d}{dv} g^{-1}(v) = 10 ,$$

Then, we have,

$$f_V(v) = 0.005(-100 + 10v - 90)(10)$$
$$= 0.5(v-19) \text{ , for } 19 < v < 21 .$$

and

$$f_V(v) = 0, \quad \text{otherwise.}$$