
# THOMAS' CALCULUS

MEDIA UPGRADE

# Chapter 15

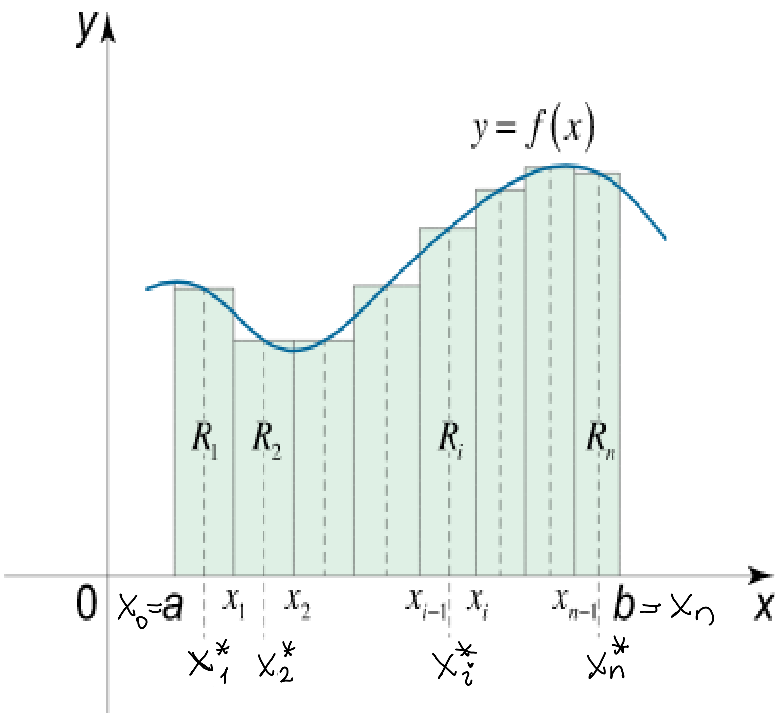
Multiple Integrals

# 15.1

Double Integrals

**Reminding:** Let $f(x)$ be a continuous nonnegative funct. defined on $[a,b]$. Let $\mathcal{P}=\{a=x_0, x_1, x_2, \dots, x_n=b\}$ be a partition of $[a,b]$ where $x_i=a+i.\Delta x$ and $\Delta x=\frac{b-a}{n}$.

Let $x_i^* \in [x_{i-1}, x_i]$ be an arbitrary point.




$$S=\{(x,y): a\le x\le b,\ 0\le y\le f(x)\}$$

Area of $S$ can be approximated by a Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$

(the sum of the areas of rectangles).

$$\text{Area of } S=\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right)$$

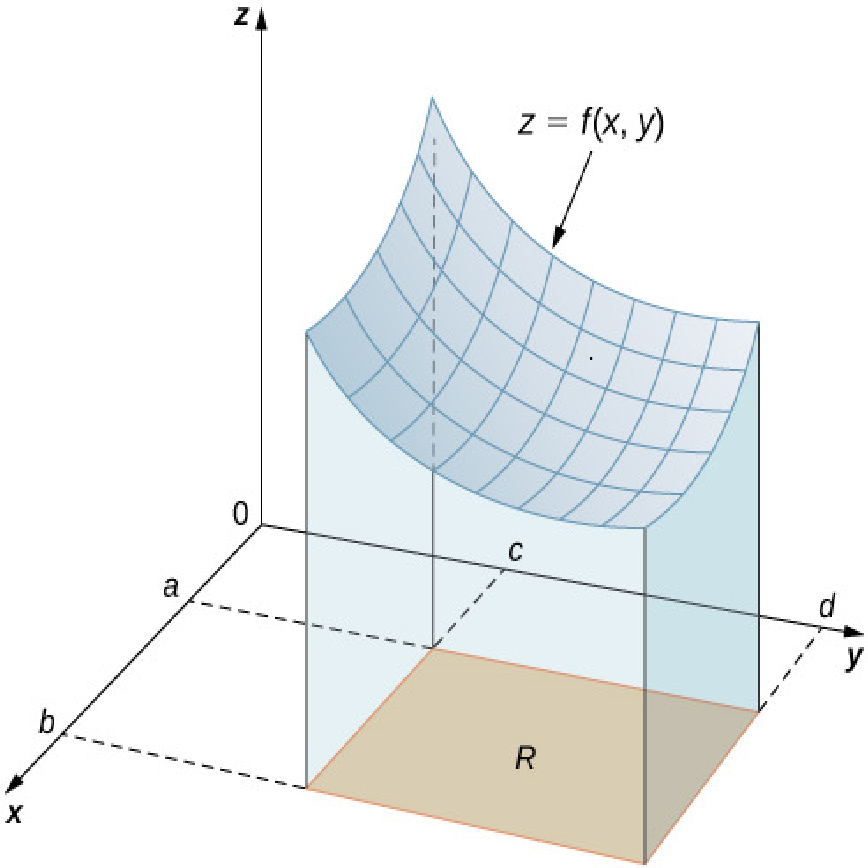
For <u>any</u> function $f(x)$ defined on $[a,b]$, the <u>definite integral</u> of $f(x)$ over $[a,b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\,\Delta x\right)$$

provided that the limit exists.

If $f(x)\geq 0$ then $\int_a^b f(x)\,dx$ is the area of the region between $y=f(x)$ and the $x$-axis $(y=0)$ from $x=a$ to $x=b$.

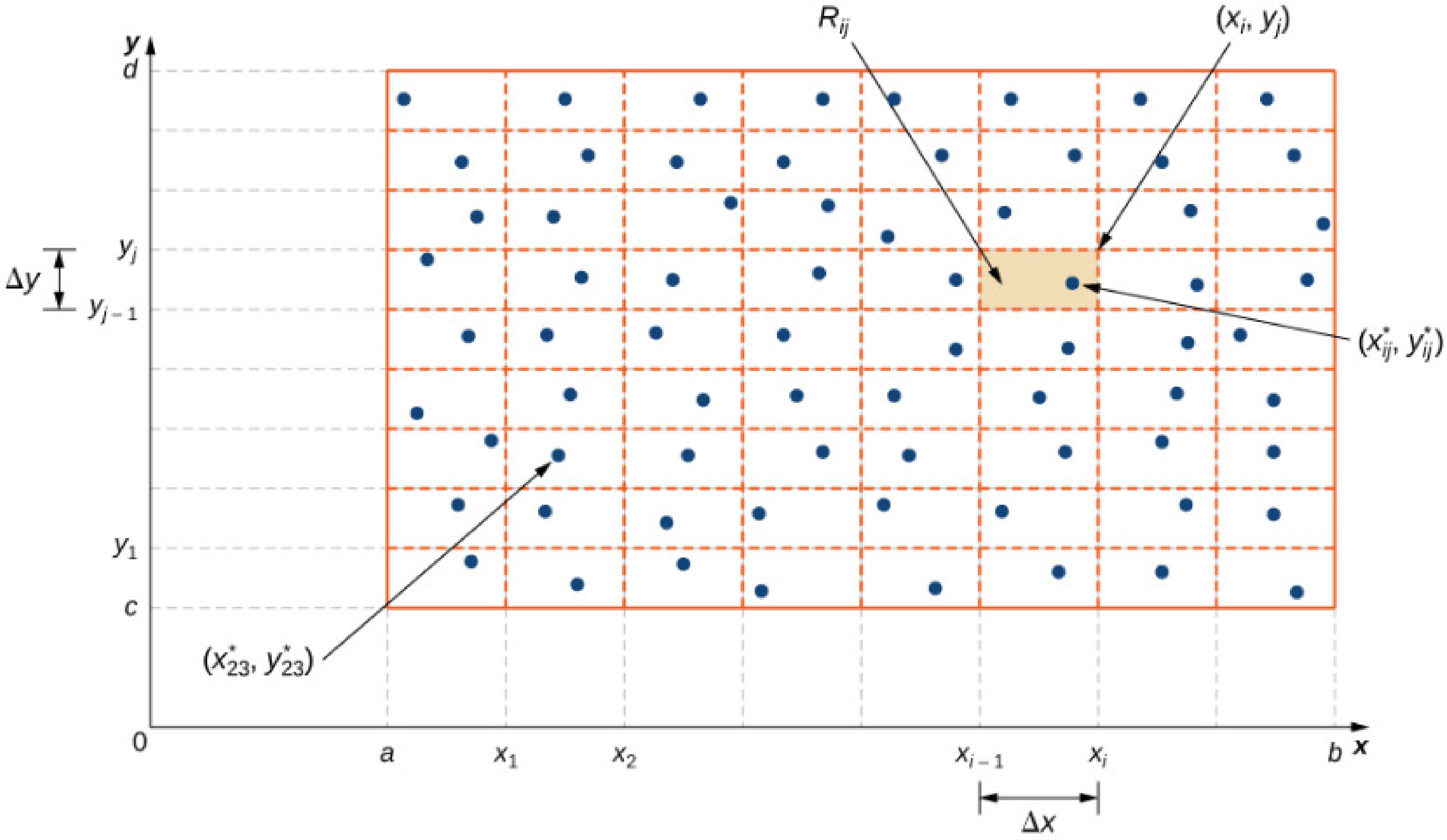
Let $f(x,y)$ be a continuous, nonnegative function defined on the closed region $R = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\}$. Let $S$ be the solid bounded below by the region $R$ and above by the surface $z = f(x,y)$ (the graph of $f(x,y)$).



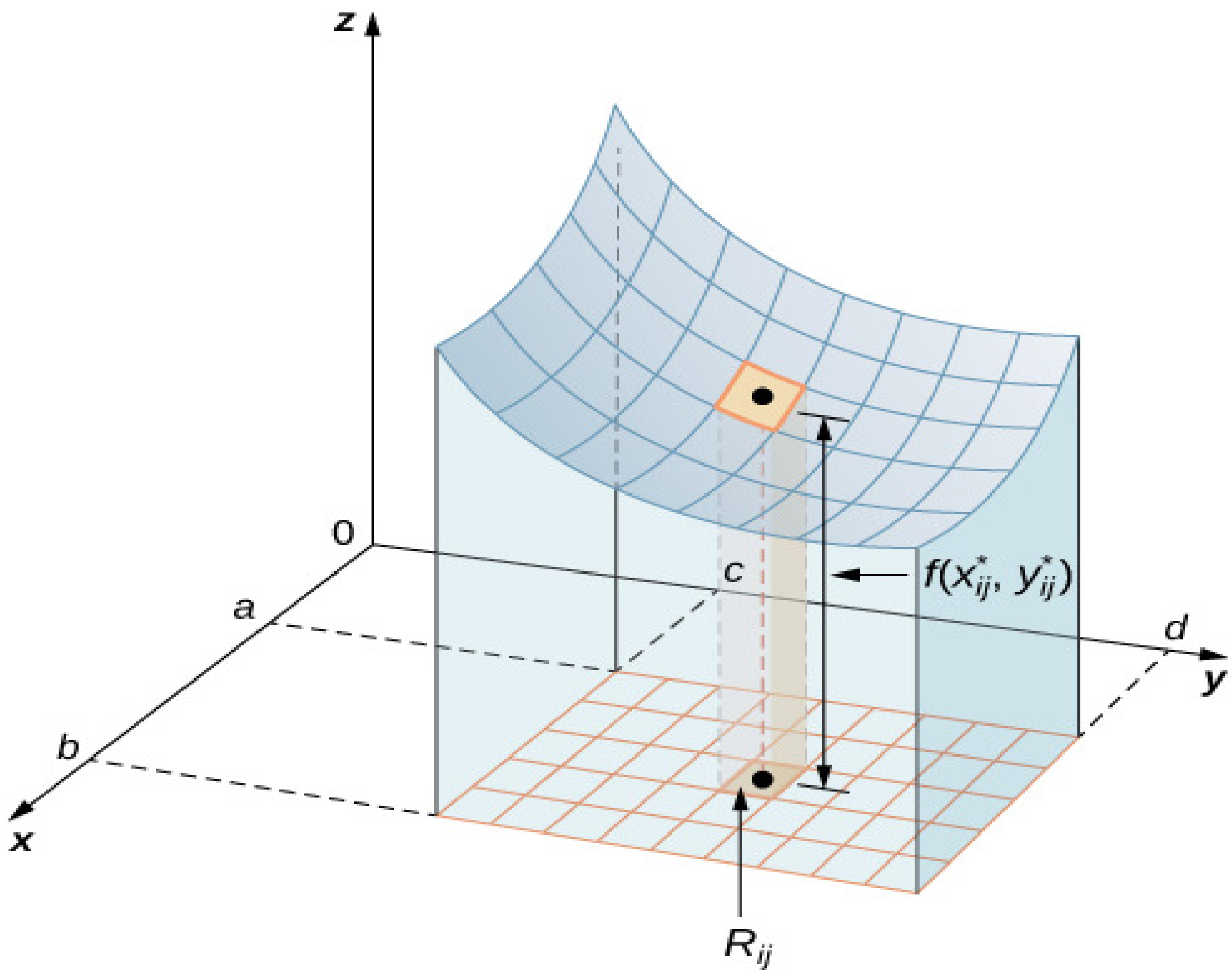

We want to find the volume $V$ of the solid $S$.

Divide the intervals $[a,b]$ and $[c,d]$ into $m$ and $n$ subintervals of equal lengths $\Delta x = \frac{b-a}{m}$ and $\Delta y = \frac{d-c}{n}$, respectively.

This gives a partition of the region R into subrectangles $R_{ij}$ with sides $\Delta x$ and $\Delta y$, and area $\Delta A = \Delta x \cdot \Delta y$



Let $(x_{ij}^*, y_{ij}^*) \in R_{ij}$ be an arbitrary point for each $i \in \{1,2,\dots,m\}$, $j \in \{1,2,\dots,n\}$. Then the volume of a thin rectangular box above the rectangle $R_{ij}$ is $f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$



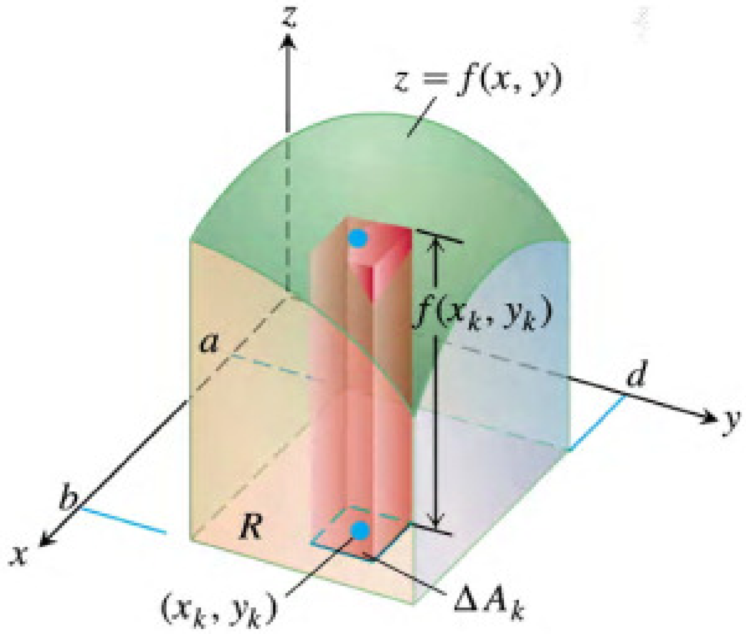
The volume $V$ of the solid $S$ can be approximated by the sum of the volumes of the rectangular boxes:

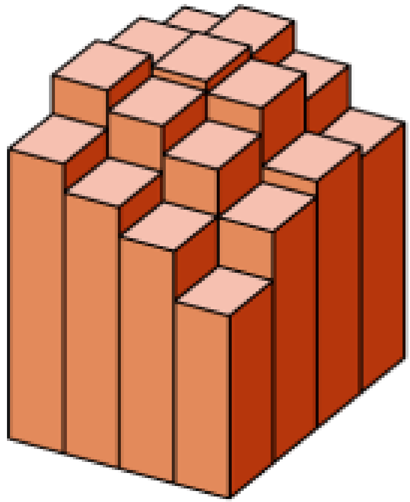
$$V \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^{*}, y_{ij}^{*}) . \Delta A \qquad \text{(double Riemann sum)}$$

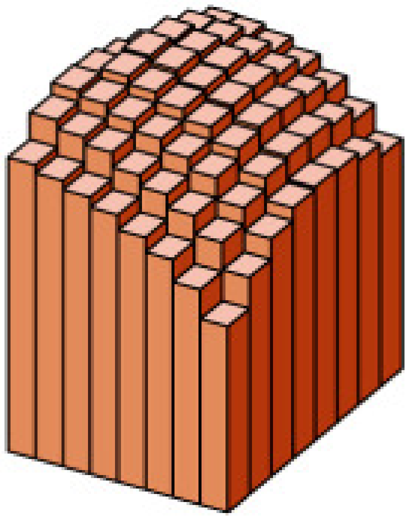
We obtain a better approximation to the volume if $m$ and $n$ become larger, see next page.

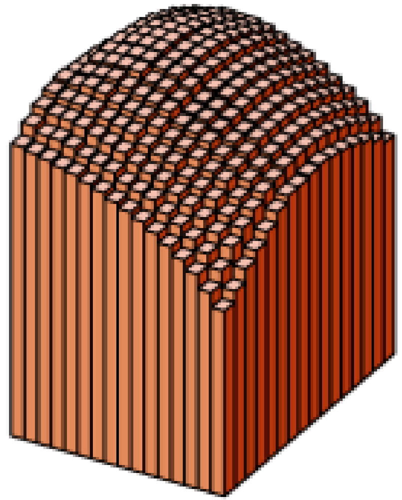
$$V = \lim_{m,n \to \infty} \left( \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^{*}, y_{ij}^{*}) . \Delta A \right)$$

Note that, this limit exists since $f(x,y)$ is continuous and $R$ is bounded.


z
z = f(x, y)
f(x_k, y_k)
a
d
y
b
x
R
(x_k, y_k)
ΔA_k







# Definition

Let $f(x,y)$ be a function defined on a rectangular region $R=\{(x,y): a\le x\le b\ ,\ c\le y\le d\}$, $\{R_{ij}\}$ be a partition of the region $R$ as above.

Let $(x_{ij}^*, y_{ij}^*)\in R_{ij}$ be an arbitrary point for each $i\in\{1,2,\dots,m\}$, $j\in\{1,2,\dots,n\}$. Then the double integral of $f(x,y)$ over $R$ is defined as:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\left(\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*).\Delta A\right)$$

provided that the limit exists.

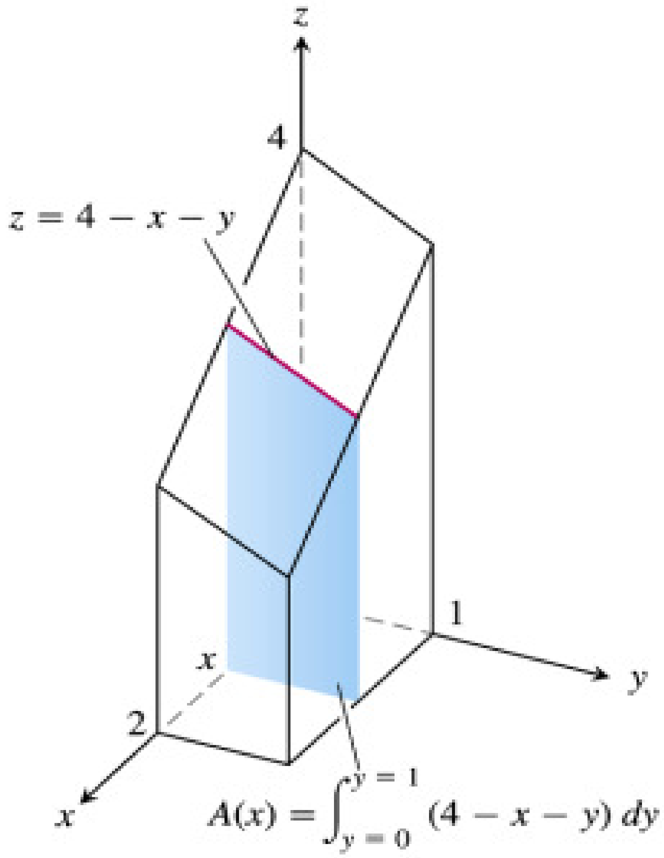
If the limit exists then $f(x,y)$ is said to be integrable over $R$.

If $f(x,y) \geq 0$ over $R$ then $\iint_R f(x,y)\, dA$ is the volume of the solid $S$ bounded by $R$ and the surface $z = f(x,y)$.

**Example:** Evaluate the volume under the plane $z = 4 - x - y = f(x,y)$ over the rectangular region $R = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2 \ , \ 0 \leq y \leq 1\}$ in the xy-plane.

We apply the method of slicing from MAT103,

with slices perpendicular to the x-axis:




Then the volume is:

$$V = \int_{x=0}^{x=2} A(x)\,dx,$$

where $A(x)$ is the cross-sectional area at $x$.

Since $A(x)$ is the area under the curve $z = 4 - x - y$ in the plane of cross-section at $x$, it can be calculated as the integral $A(x) = \int_{y=0}^{y=1} (4 - x - y)\, dy$.

In evaluation of $A(x)$, $x$ is held fixed since the integration is with respect to $y$.

$$\text{Volume} = \int_{x=0}^{x=2} A(x)\, dx = \int_{x=0}^{x=2} \left( \int_{y=0}^{y=1} (4 - x - y)\, dy \right) dx$$

$$\text{Volume} = \int_{x=0}^{x=2} \left[ 4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx = \int_{x=0}^{x=2} \left( \frac{7}{2} - x \right) dx$$

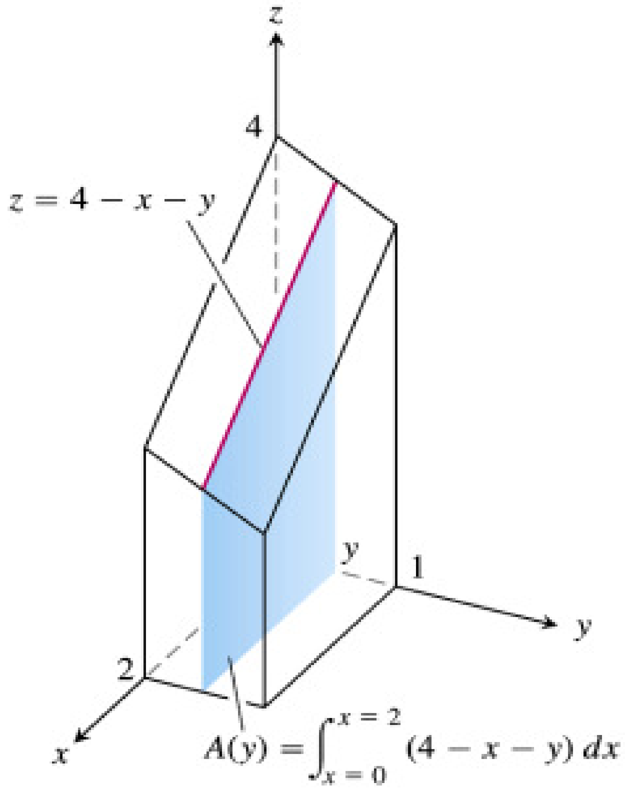
$$= \left[ \frac{7}{2} x - \frac{x^2}{2} \right]_0^2 = 5.$$

What would have happened if we had calculated the volume by slicing with planes perpendicular to the y-axis?

The typical cross-section area is:

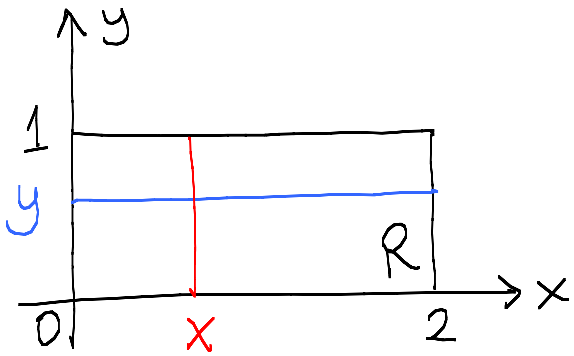
$$A(y) = \int_{x=0}^{x=2} (4-x-y)\,dx = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2}$$

$$= 6 - 2y.$$




$$\text{Volume} = \int_{y=0}^{y=1} A(y)\,dy = \int_{y=0}^{y=1} (6-2y)\,dy$$

$$= \left[6y - y^2\right]_0^1 = 5$$



**THEOREM 1 Fubini's Theorem (First Form)**

If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

For rectangular regions, one can change the order of integration by simply interchanging the limits.

$\int_c^d \int_a^b f(x,y)\,dx\,dy$ and $\int_a^b \int_c^d f(x,y)\,dy\,dx$ are called iterated integrals of $f(x,y)$ over $R$.

**Notation :**

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left( \int_a^b f(x,y)\,dx \right) dy = \int_c^d dy \int_a^b f(x,y)\,dx .$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left( \int_c^d f(x,y)\,dy \right) dx = \int_a^b dx \int_c^d f(x,y)\,dy .$$

## EXAMPLE 1 Evaluating a Double Integral

Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 1 - 6x^2 y \qquad \text{and} \qquad R: \quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

**Solution** By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_{0}^{2} (1 - 6x^2 y)\, dx\, dy = \int_{-1}^{1} \Big[x - 2x^3 y\Big]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (2 - 16y)\, dy = \Big[2y - 8y^2\Big]_{-1}^{1} = 4.$$

Reversing the order of integration gives the same answer:

$$\int_{0}^{2}\int_{-1}^{1} (1 - 6x^2 y)\, dy\, dx = \int_{0}^{2} \Big[y - 3x^2 y^2\Big]_{y=-1}^{y=1} dx$$

$$= \int_{0}^{2} [(1 - 3x^2) - (-1 - 3x^2)]\, dx$$

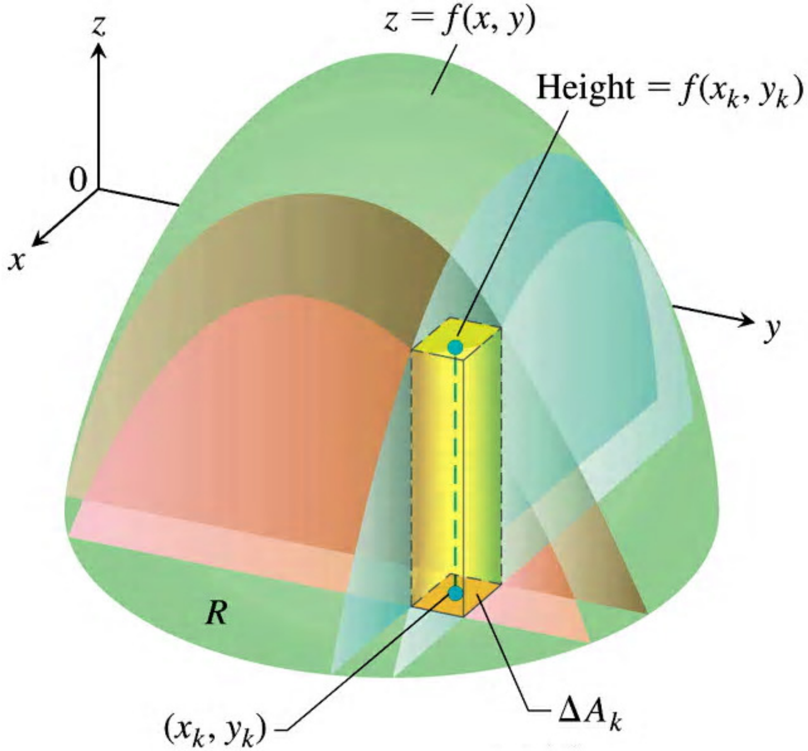
$$= \int_{0}^{2} 2\, dx = 4.$$

# Double Integrals over Bounded Nonrectangular Regions

Let $R$ be a nonrectangular region (bounded by curves) and let $f(x,y)$ be a function defined on $R$. Then using a partition of $R$ (similar to the rectangular regions) we define the double integral of $f(x,y)$ over $R$.

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\left(\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*).\Delta A\right)$$

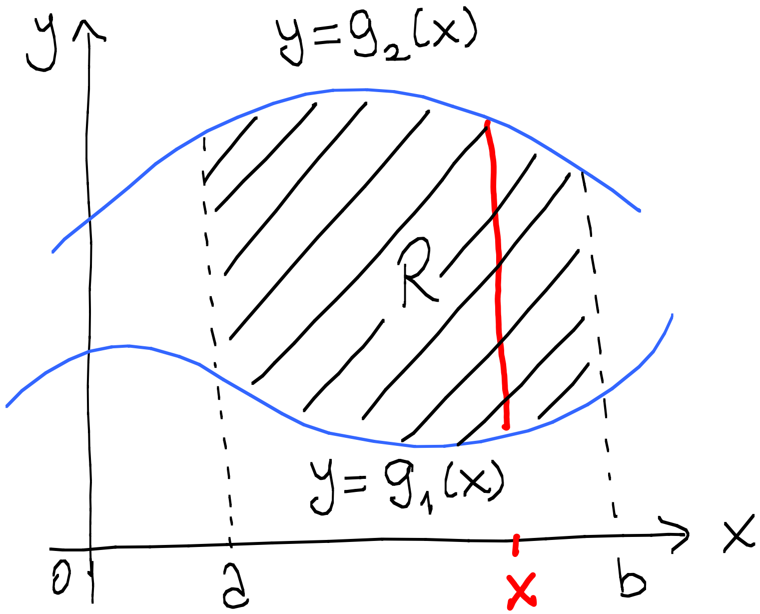
provided that the limit exists.




$$\text{Volume} = \lim \sum f(x_k, y_k)\,\Delta A_k = \iint_R f(x, y)\,dA$$

**FIGURE 15.8** We define the volumes of solids with curved bases the same way we define the volumes of solids with rectangular bases.

Let $R$ be a region defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where $g_1(x)$ and $g_2(x)$ are continuous functions on $[a,b]$ and let $f(x,y)$ be a positive continuous function defined on $R$.




Then $\iint_R f(x,y)\,dA$ is the volume of the solid bounded by $R$ and $z=f(x,y)$ which can be calculated by slicing with planes perpendicular to x-axis.

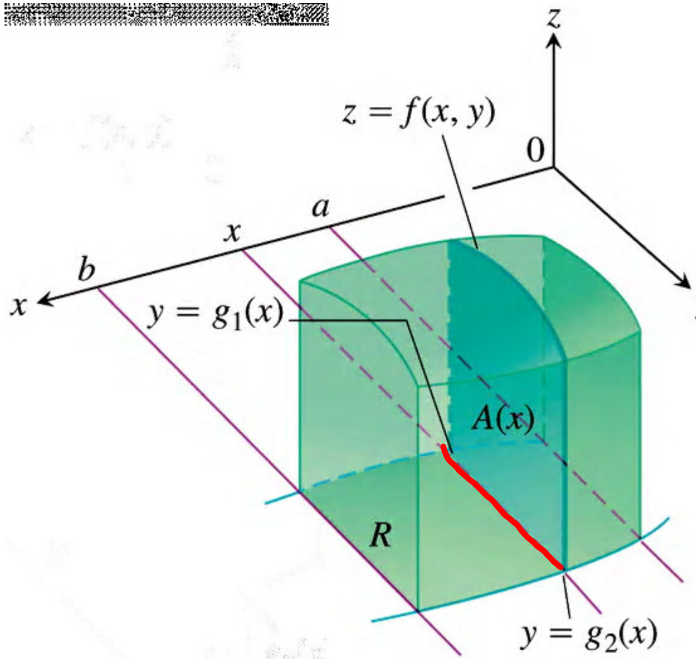



**FIGURE 15.9** The area of the vertical slice shown here is

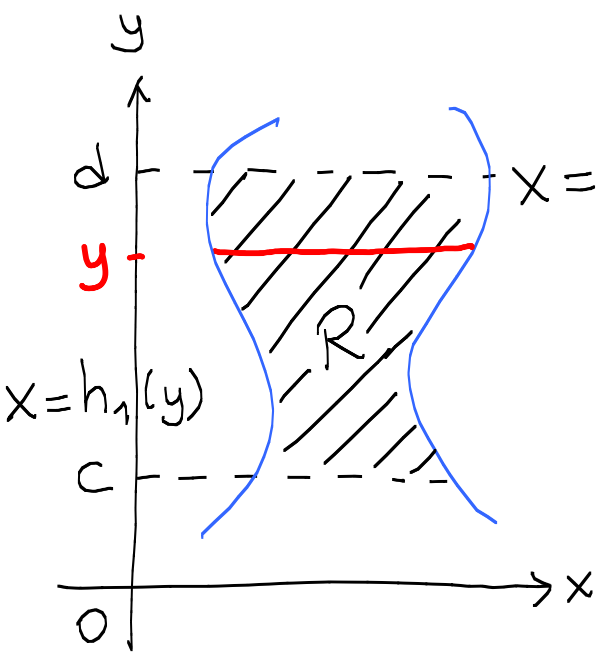
$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$.

Then the volume is:

$$V = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Similarly, if $R$ is a region defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where $h_1(y)$ and $h_2(y)$ are continuous functions on $[c,d]$ and $f(x,y)$ is a positive continuous function defined on $R$. Then the volume of the solid bounded by $R$ and $z=f(x,y)$ can be calculated by slicing with planes perpendicular to $y$-axis.



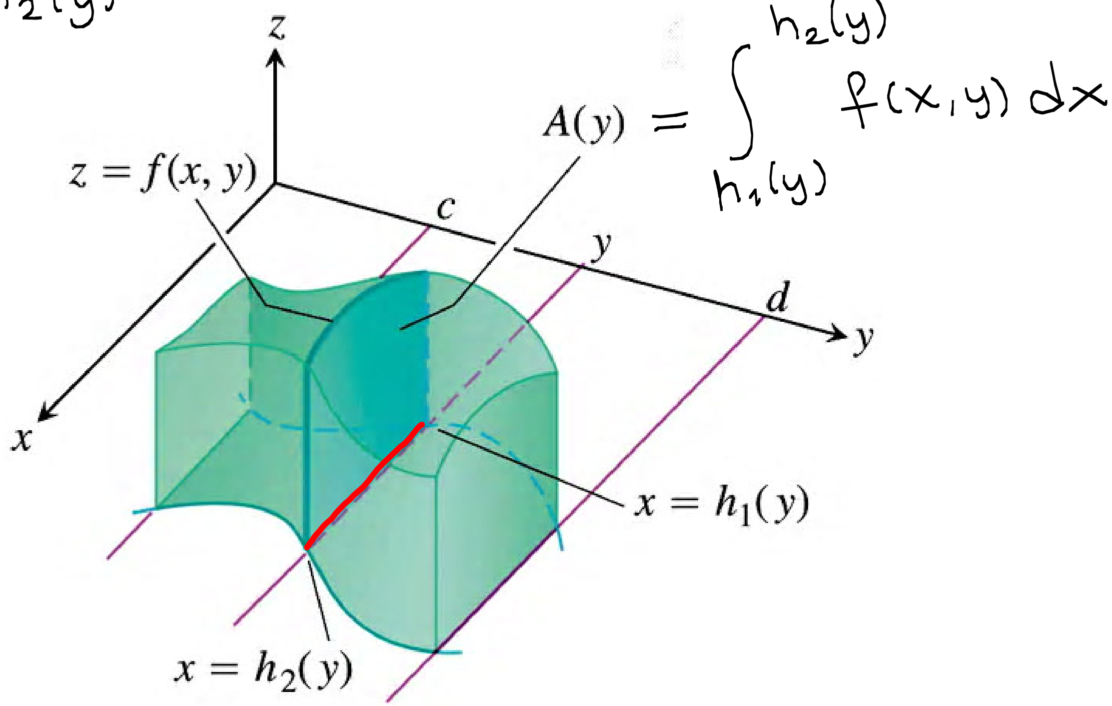




**FIGURE 15.10** The volume of the solid shown here is

$$\int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy = \iint_R f(x,y)\,dA.$$

**THEOREM 2 Fubini's Theorem (Stronger Form)**

Let $f(x, y)$ be continuous on a region $R$.

1. If $R$ is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with $g_1$ and $g_2$ continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

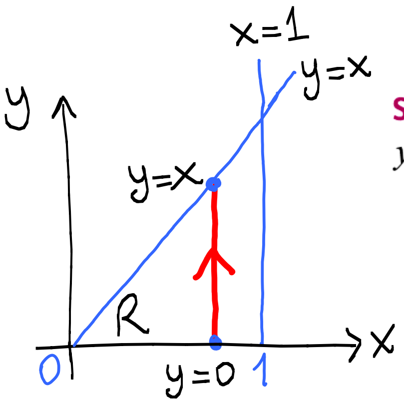
2. If $R$ is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with $h_1$ and $h_2$ continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

**EXAMPLE 2** Finding Volume

Find the volume of the prism whose base is the triangle in the $xy$-plane bounded by the $x$-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

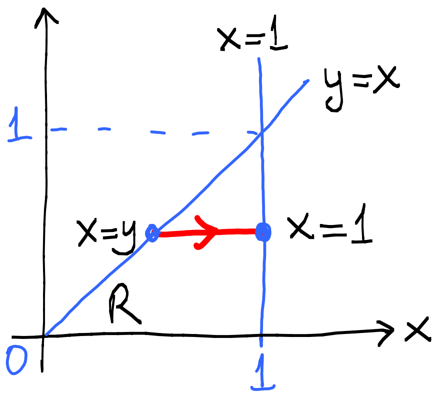



**Solution** See Figure 15.11 on page 1075. For any $x$ between 0 and 1, $y$ may vary from $y = 0$ to $y = x$ (Figure 15.11b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed (Figure 15.11c), the integral for the volume is




$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be.

$R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\} = \{(x,y) : 0 \le y \le 1,\ y \le x \le 1\}$

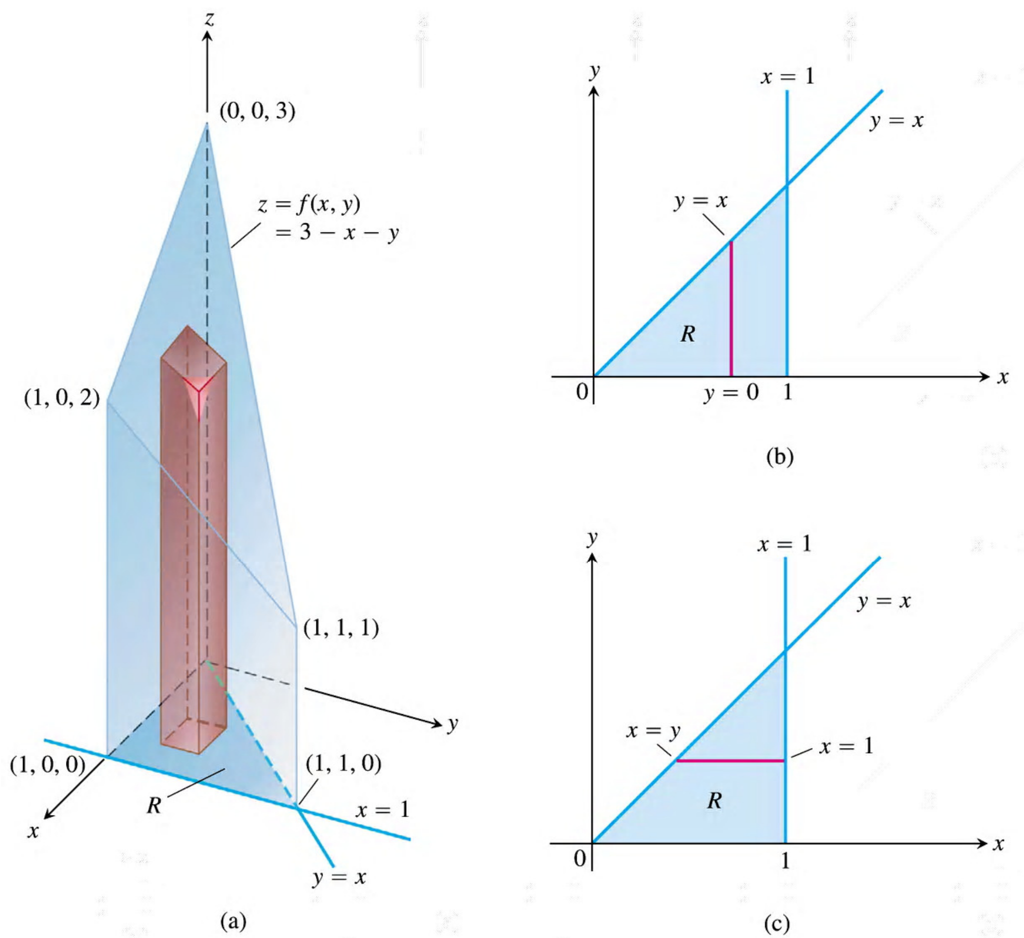
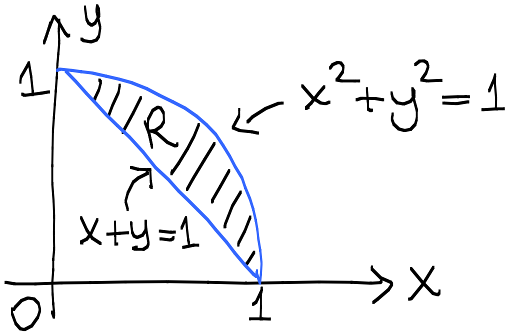
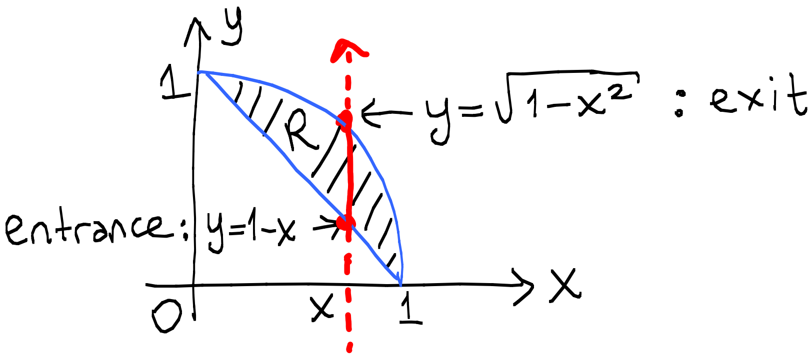



**FIGURE 15.11** (a) Prism with a triangular base in the $xy$-plane. The volume of this prism is defined as a double integral over $R$. To evaluate it as an iterated integral, we may integrate first with respect to $y$ and then with respect to $x$, or the other way around (Example 2).
(b) Integration limits of

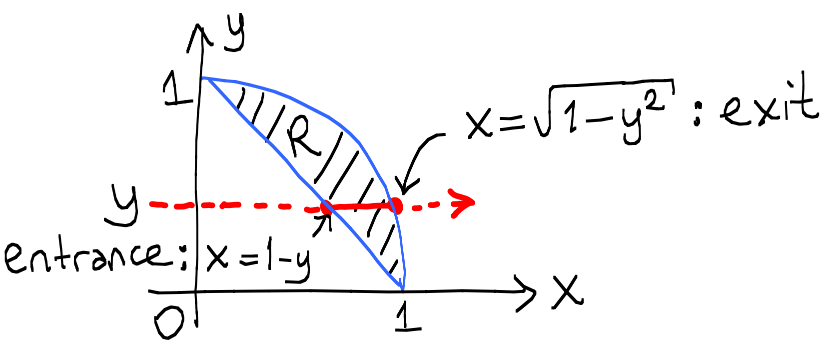
# Finding Limits of Integration




$$\iint_R f(x,y)\,dA = \int_{x=?}^{x=?}\int_{y=?}^{y=?} f(x,y)\,dy\,dx = \int_{y=?}^{y=?}\int_{x=?}^{x=?} f(x,y)\,dx\,dy$$




$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$$




$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=1}\int_{x=1-y}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

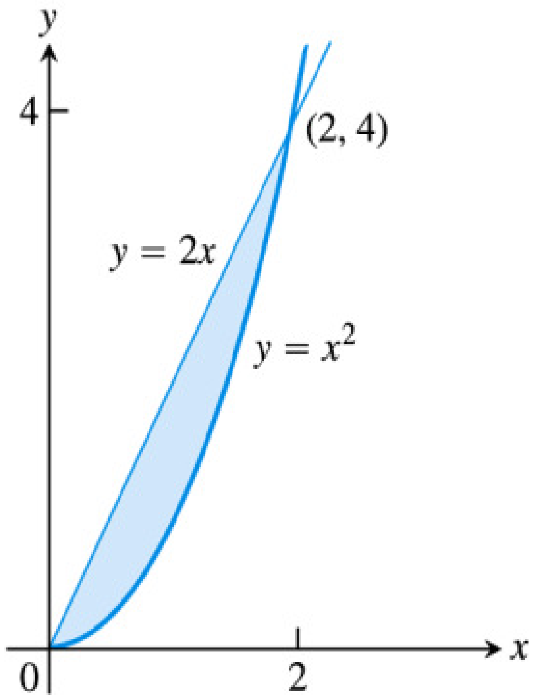
**Example:** Sketch the region of integration for the integral

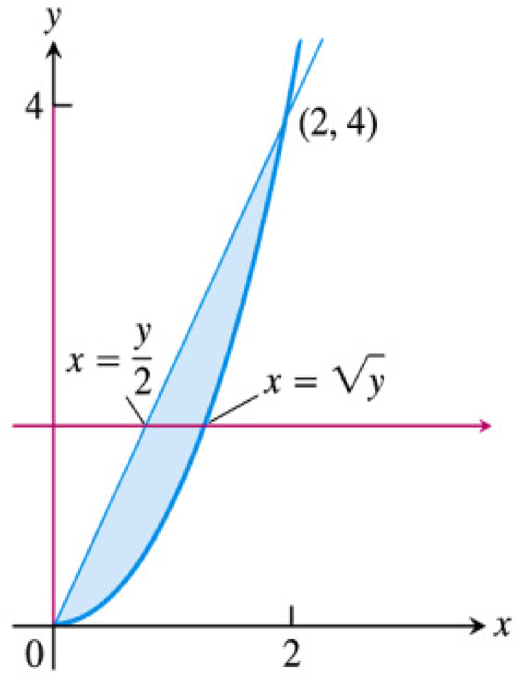
$$\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx$$

and write an equivalent integral with the order of integration reversed.

<u>Solution:</u> $R$: $x^2 \le y \le 2x$ & $0 \le x \le 2$

$R$ is bounded by $y = x^2$ and $y = 2x$ from $x = 0$ to $x = 2$.



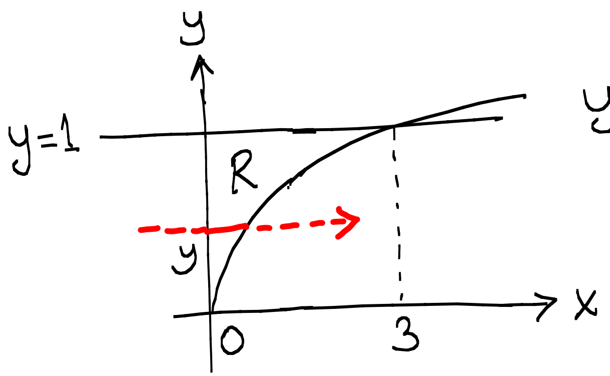

$$I = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\, dx\, dy$$

**Example:** Evaluate the integral $I = \int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx$.

There is no direct way to evaluate this integral in this order, we need to reverse the order of integration.

$R: 0 \le x \le 3,\ \sqrt{x/3} \le y \le 1$. R is bounded by the curves $y=\sqrt{x/3}$ and $y=1$ from $x=0$ to $x=3$.

$R: 0 \le y \le 1,\quad 0 \le x \le 3y^2$




$y=\sqrt{x/3}$
$\Updownarrow$
$x=3y^2,\ y \ge 0$

$$I = \int_0^1 \int_0^{3y^2} e^{y^3}\, dx\, dy = \int_0^1 \left(x e^{y^3}\right)\Big|_{x=0}^{x=3y^2} dy$$

$$= \int_0^1 3y^2 e^{y^3}\, dy = e^{y^3}\Big|_0^1 = e^1 - e^0 = e-1$$

**Properties of Double Integrals**

If $f(x, y)$ and $g(x, y)$ are continuous, then

**1.** *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA$ (any number $c$)

**2.** *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\,dA = \iint_R f(x, y)\,dA \pm \iint_R g(x, y)\,dA$$

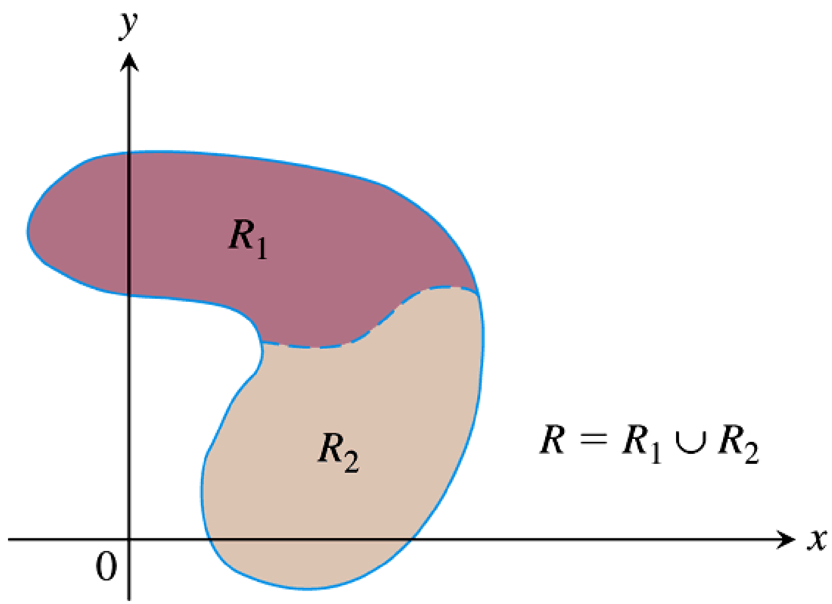
**3.** *Domination:*

**(a)** $\displaystyle\iint_R f(x, y)\,dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \text{ on } R$

**(b)** $\displaystyle\iint_R f(x, y)\,dA \geq \iint_R g(x, y)\,dA \quad \text{if} \quad f(x, y) \geq g(x, y) \text{ on } R$

**4.** *Additivity:* $\displaystyle\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$

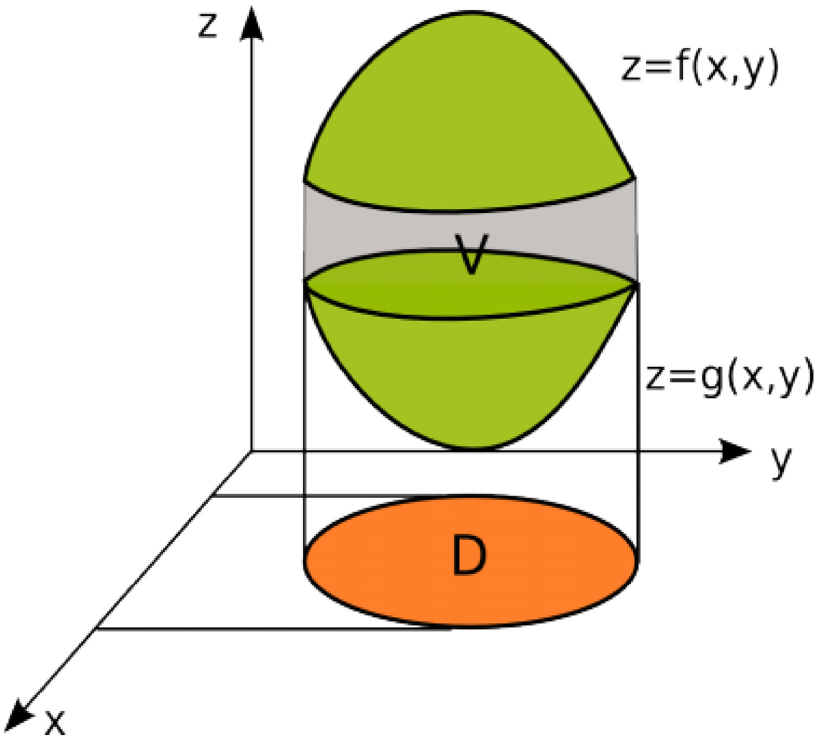
if $R$ is the union of two nonoverlapping regions $R_1$ and $R_2$ (Figure 15.7).




$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

**FIGURE 15.7** The Additivity Property for rectangular regions holds for regions bounded by continuous curves.

Assume that two surfaces $z=f(x,y)$ and $z=g(x,y)$ are such that $f(x,y) \geq g(x,y)$ over a region D in xy-plane. The volume of the solid bounded above by $z=f(x,y)$ and below by $z=g(x,y)$ over the region D can be evaluated as the double integral of the difference $f(x,y)-g(x,y)$ over the region D.



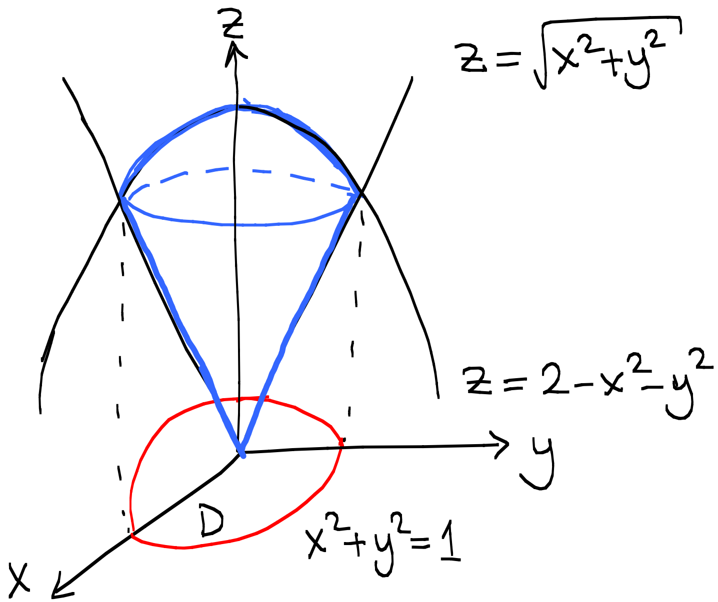

$$V=\iint_D (f(x,y)-g(x,y))\,dA$$

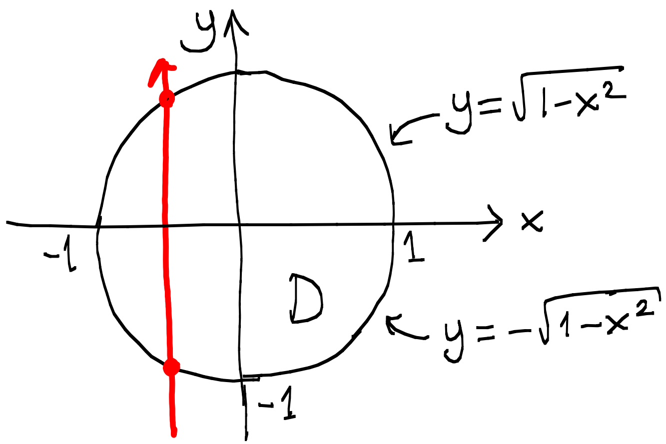
**Example:** Find the volume of the solid bounded above by the paraboloid $z = 2-x^2-y^2$ and below by the cone $z=\sqrt{x^2+y^2}$.

Solution: $V = \iint_D \left[ (2-x^2-y^2) - \sqrt{x^2+y^2} \right] dy\,dx$

where D is the projection of the solid onto the xy-plane whose boundary is the projection of the intersection of the paraboloid and the cone:

$$z = 2-x^2-y^2 = \sqrt{x^2+y^2} \iff x^2+y^2=1$$



$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left( (2-x^2-y^2) - \sqrt{x^2+y^2} \right) dy\, dx$$

In polar coordinates:

$$V = \int_{0}^{2\pi} \int_{0}^{1} \left( 2 - r^2 - r \right) r\, dr\, d\theta \underset{\uparrow}{=} \frac{5\pi}{6}$$

Exercise!

# 15.2

## Area, Moments, and Centers of Mass

If we take $f(x,y)=1$ in the definition of the double integral then the Riemann sum approximates the area of the region $R$, and its limit gives the exact area of $R$.

**DEFINITION Area**

The **area** of a closed, bounded plane region $R$ is

$$A = \iint_R dA.$$

## EXAMPLE 1 Finding Area

Find the area of the region $R$ bounded by $y = x$ and $y = x^2$ in the first quadrant.

**Solution** We sketch the region (Figure 15.15), noting where the two curves intersect, and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[ y \Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

Notice that the single integral $\int_0^1 (x - x^2)\, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.5.

$$A = \int_0^1 \int_y^{\sqrt{y}} dx\, dy = \int_0^1 \Big[ x \Big]_y^{\sqrt{y}} dy = \int_0^1 (\sqrt{y} - y)\, dy = \left[ \frac{2}{3} y^{3/2} - \frac{y^2}{2} \right]_0^1 = \frac{1}{6}.$$

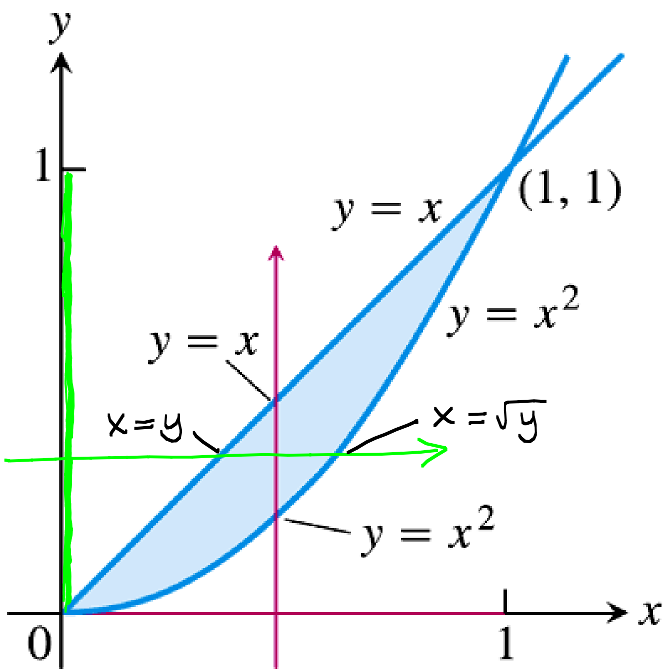
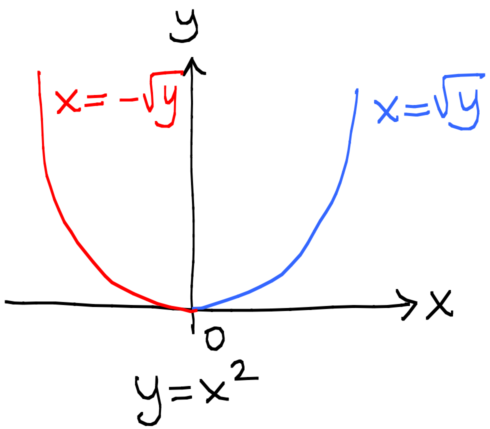



**FIGURE 15.15** The region in Example 1.

**Example:** Find the area of the region $R$ enclosed by the parabola $y=x^2$ and the line $y=x+2$.



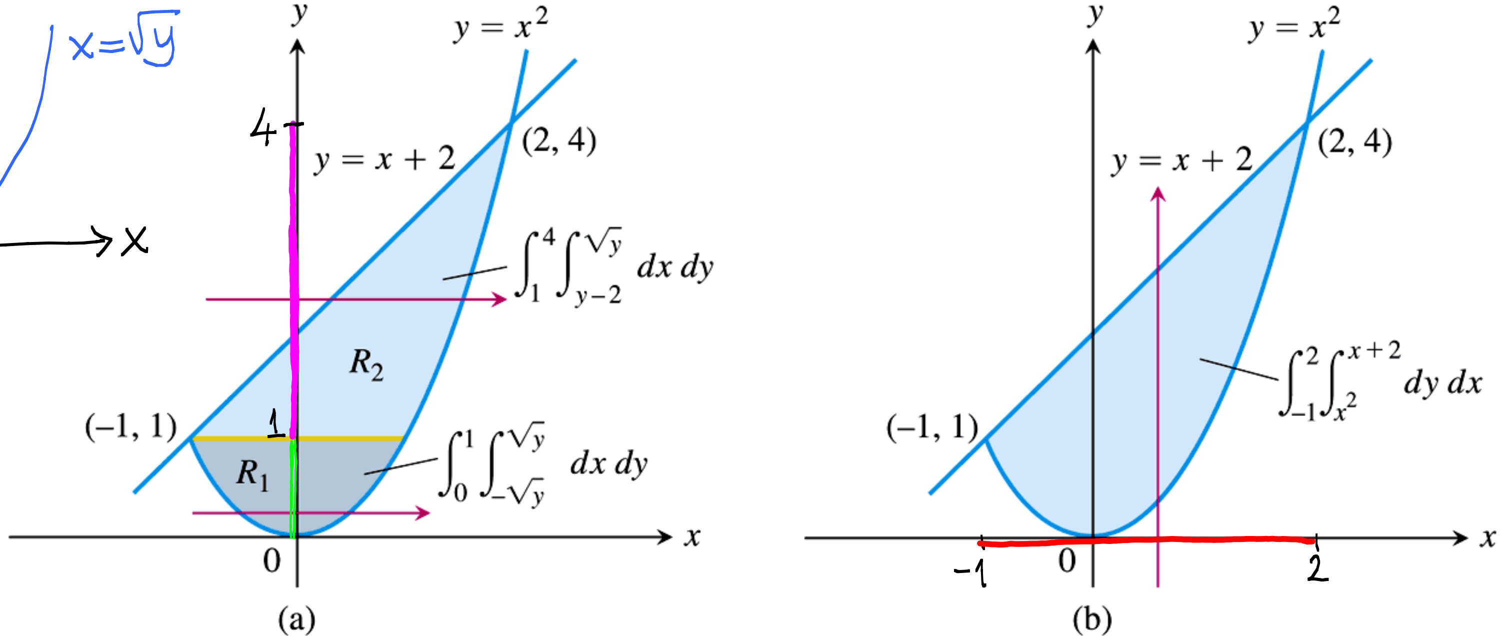




**FIGURE 15.16** Calculating this area takes (a) two double integrals if the first integration is with respect to $x$, but (b) only one if the first integration is with respect to $y$ (Example 2).

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx = \frac{9}{2}$$

Exercise!

$$\textbf{Average value} \text{ of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f\, dA. \tag{3}$$

**EXAMPLE 3** Finding Average Value

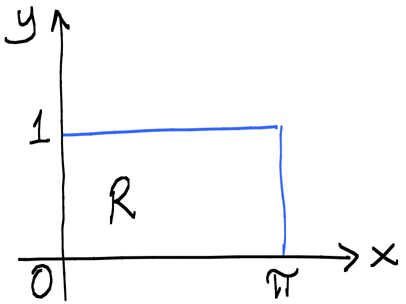
Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R$: $0 \leq x \leq \pi$, $0 \leq y \leq 1$.

**Solution** The value of the integral of $f$ over $R$ is

$$\int_0^{\pi} \int_0^1 x \cos xy \, dy \, dx = \int_0^{\pi} \left[ \sin xy \right]_{y=0}^{y=1} dx \qquad \int x \cos xy \, dy = \sin xy + C$$

$$= \int_0^{\pi} (\sin x - 0) \, dx = -\cos x \Big]_0^{\pi} = 1 + 1 = 2.$$

The area of $R$ is $\pi$. The average value of $f$ over $R$ is $2/\pi$.



# 15.3

## Double Integrals in Polar Form

Integrals are sometimes easier to evaluate if we change to polar coordinates.

Suppose that a function $f(r, \theta)$ is defined over a region $R$ that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \leq g_1(\theta) \leq g_2(\theta) \leq a$ for every value $\theta \in [\alpha, \beta]$.

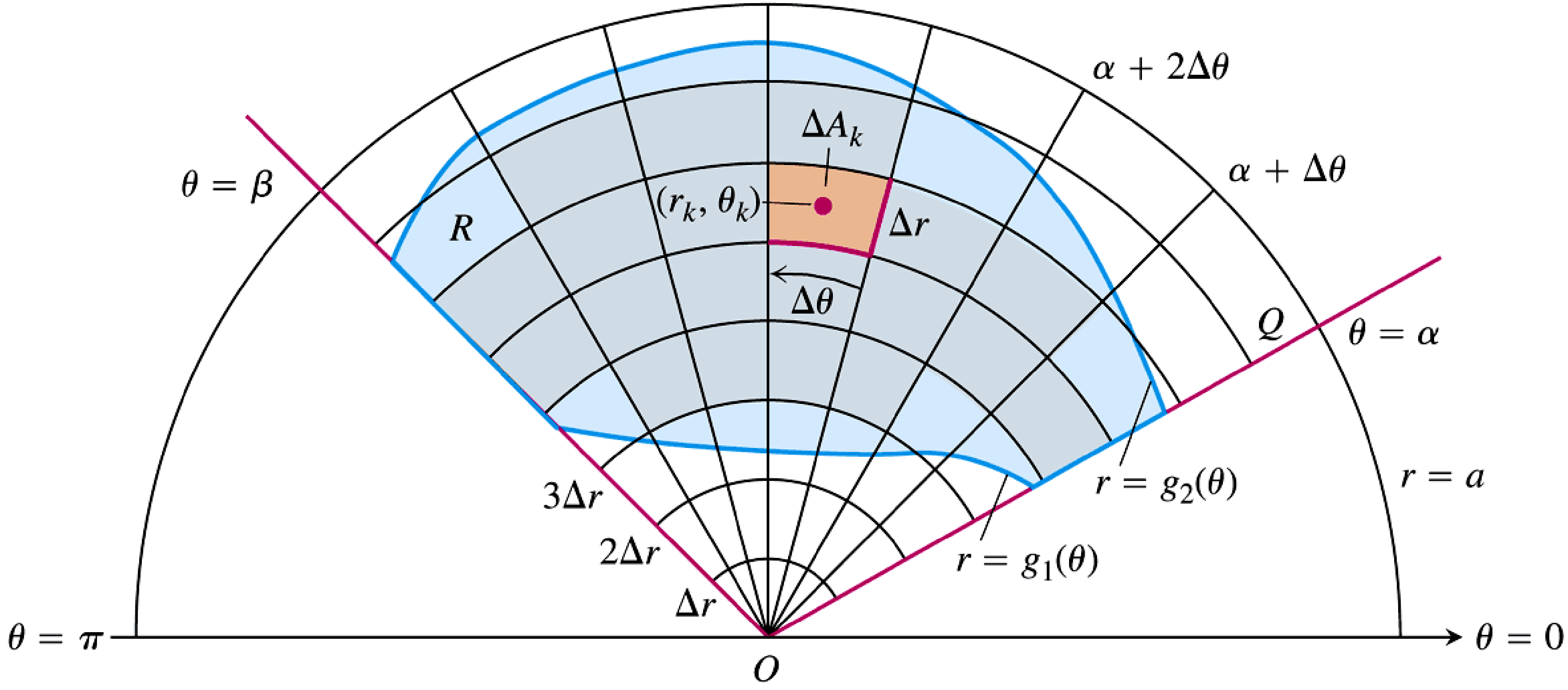



**FIGURE 15.21** The region $R$: $g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, is contained in the fan-shaped region $Q$: $0 \le r \le a$, $\alpha \le \theta \le \beta$. The partition of $Q$ by circular arcs and rays induces a partition of $R$.

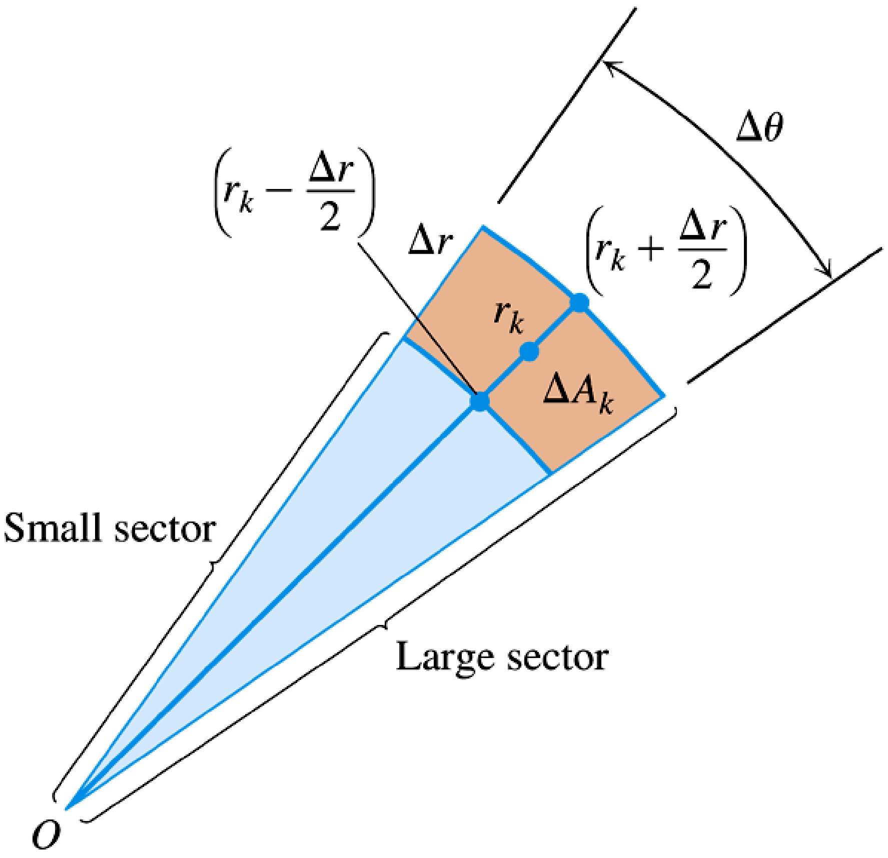



**FIGURE 15.22** The observation that

$$\Delta A_k = \begin{pmatrix} \text{area of} \\ \text{large sector} \end{pmatrix} - \begin{pmatrix} \text{area of} \\ \text{small sector} \end{pmatrix} = \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right]$$

leads to the formula $\Delta A_k = r_k\,\Delta r\,\Delta\theta$.

$$\iint_R f(r,\theta)\, dA = \lim_{n\to\infty} \left( \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\, \Delta r\, \Delta\theta \right)$$

A version of Fubini's Theorem:

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

## EXAMPLE 1 Finding Limits of Integration

Find the limits of integration for integrating $f(r, \theta)$ over the region $R$ that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

### Solution

1. We first sketch the region and label the bounding curves (Figure 15.23).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters $R$ where $r = 1$ and leaves where $r = 1 + \cos\theta$.
3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect $R$ run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$

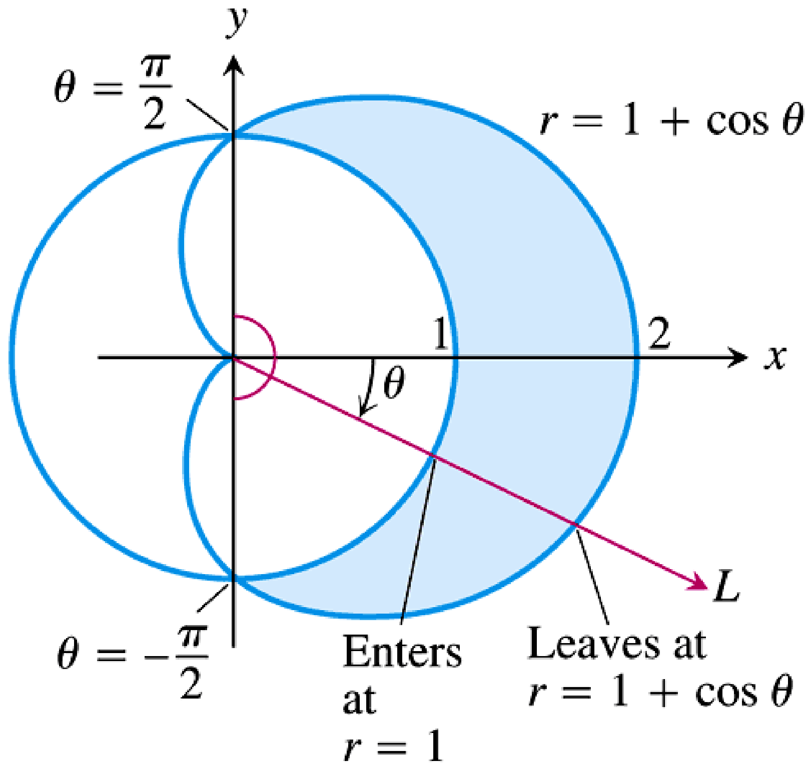
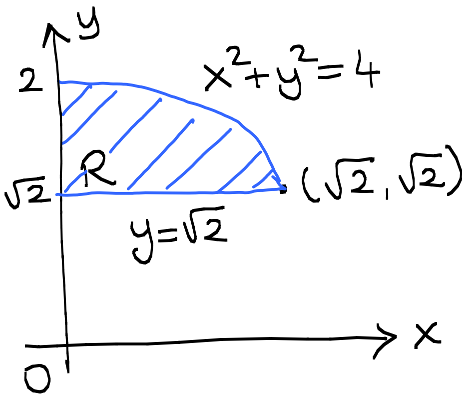
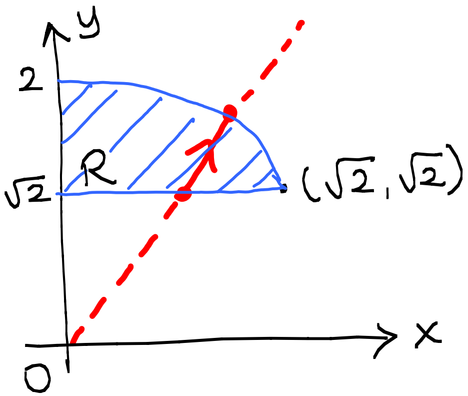



**FIGURE 15.23** Finding the limits of integration in polar coordinates for the region in Example 1.

## Example:

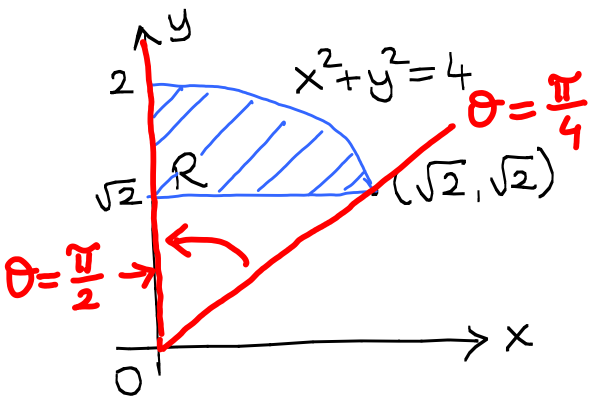



$$I = \iint_R f(r,\theta)\, dA = \int_{\theta=?}^{\theta=?} \int_{r=?}^{r=?} f(r,\theta)\, r.dr.d\theta$$




$$y = r.\sin\theta = \sqrt{2} \Rightarrow r = \frac{\sqrt{2}}{\sin\theta} = \sqrt{2}.\csc\theta \ \text{: entrance}$$

$$x^2+y^2=r^2=4 \Rightarrow r=2 \ \text{: exit}$$




$$\left.\begin{array}{l} \theta = \frac{\pi}{4} \text{ : lower limit} \\ \theta = \frac{\pi}{2} \text{ : upper limit} \end{array}\right\} \quad I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\sqrt{2}\csc\theta}^{2} f(r,\theta)\, r.dr.d\theta$$

To evaluate the area of the region $R$ substitute $f(r, \theta) = 1$:

**Area in Polar Coordinates**

The area of a closed and bounded region $R$ in the polar coordinate plane is

$$A = \iint_R r \, dr \, d\theta.$$

**Example:** Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$

Solution:

$$A = 4 \int_0^{\pi/4} \int_0^{\sqrt{4\cos 2\theta}} r \, dr \, d\theta = 4 \int_0^{\pi/4} \left[ \frac{r^2}{2} \right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4 \int_0^{\pi/4} 2\cos 2\theta \, d\theta = 4 \sin(2\theta) \Big|_0^{\pi/4} = 4$$

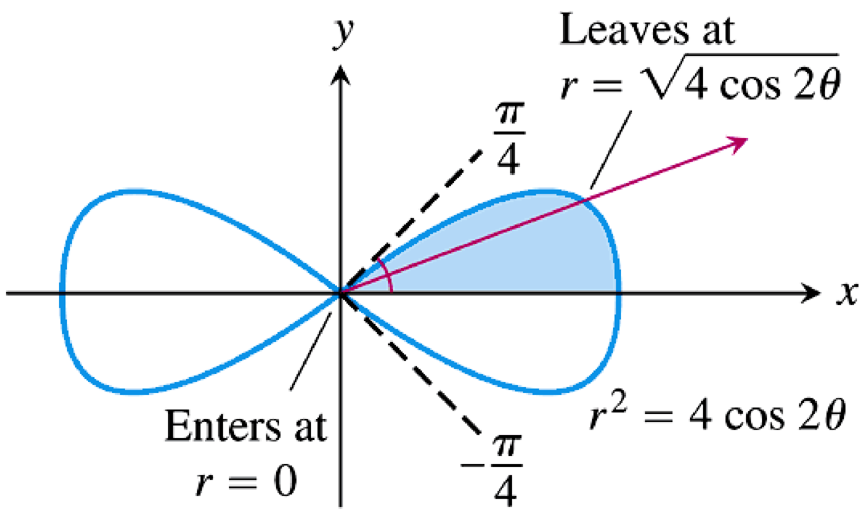



**FIGURE 15.24** To integrate over the shaded region, we run $r$ from 0 to $\sqrt{4 \cos 2\theta}$ and $\theta$ from 0 to $\pi/4$ (Example 2).

**Example:** Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$ where $R$ is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$.

<u>Solution:</u> There is no direct way to integrate $e^{x^2+y^2}$ with respect to $x$ or $y$.

Substitute $x=r\cos\theta$, $y=r\sin\theta$ and $dy\,dx=r\,dr\,d\theta$

$$\iint_R e^{x^2+y^2}\,dy\,dx=\int_0^{\pi}\int_0^{1} e^{r^2}\,r\,dr\,d\theta=\int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta=\int_0^{\pi}\frac{e-1}{2}\,d\theta$$

$$=\frac{e-1}{2}\,\theta\Big|_0^{\pi}=\frac{\pi}{2}(e-1)$$

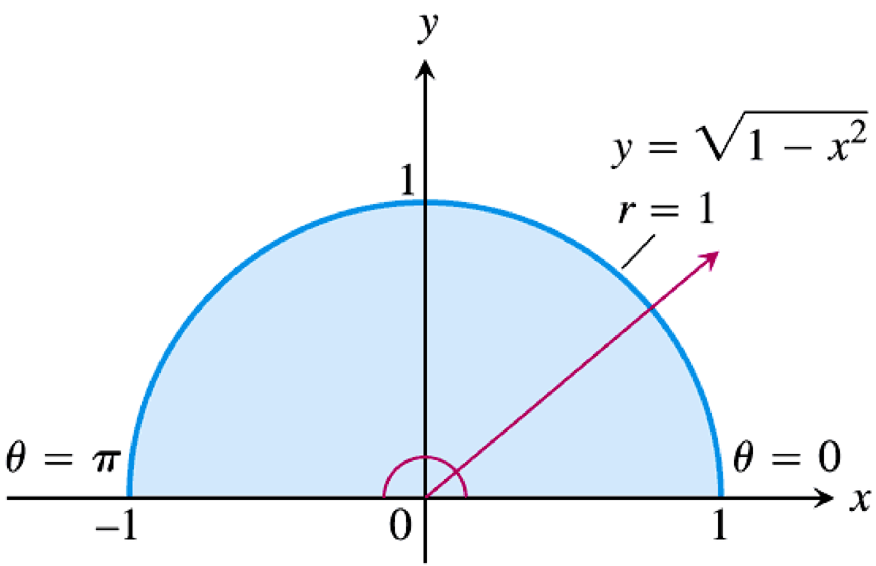



**FIGURE 15.26** The semicircular region in Example 4 is the region

$$0 \le r \le 1, \qquad 0 \le \theta \le \pi.$$

# 15.4

## Triple Integrals in Rectangular Coordinates

Let $F(x,y,z)$ be a function defined on a closed bounded region D

Form a partition of D into rectangular boxes with dimensions $\Delta x_k, \Delta y_k, \Delta z_k$,

and volume

$$\Delta V_k = \Delta x_k \cdot \Delta y_k \cdot \Delta z_k$$

$$\iiint_D F(x,y,z)\, dV$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} F(x_k,y_k,z_k)\cdot \Delta V_k\right)$$

$$= \iiint_D F(x,y,z)\, dx\, dy\, dz$$

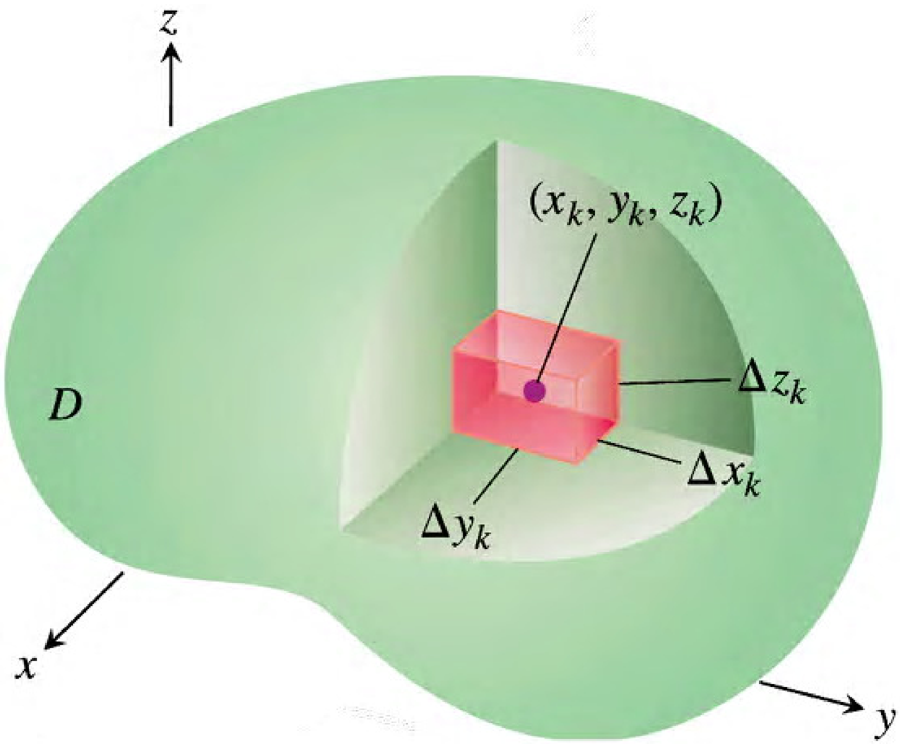



**FIGURE 15.27** Partitioning a solid with rectangular cells of volume $\Delta V_k$.

If $F(x,y,z)=1$ then the integral over $D$ gives the volume of $D$.

**DEFINITION** **Volume**

The **volume** of a closed, bounded region $D$ in space is

$$V = \iiint_D dV.$$

## Finding Limits of Integration

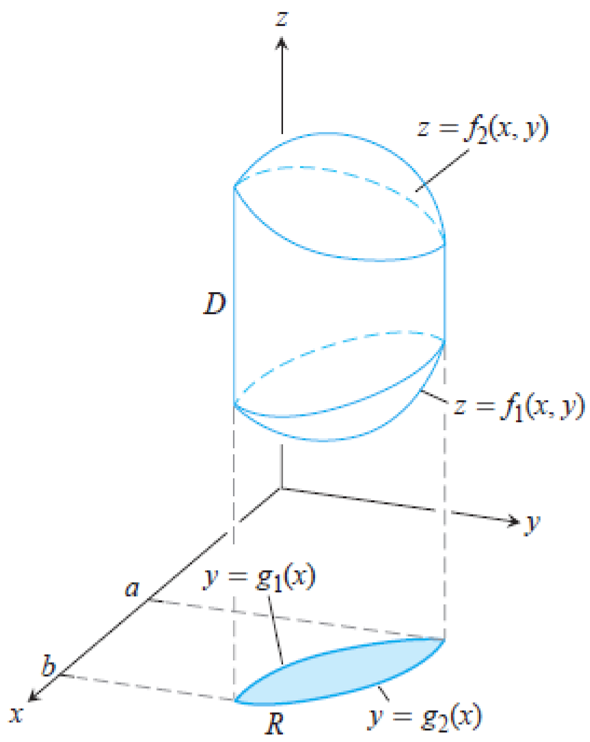
We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem (Section 15.1) to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these single integrals.

To evaluate

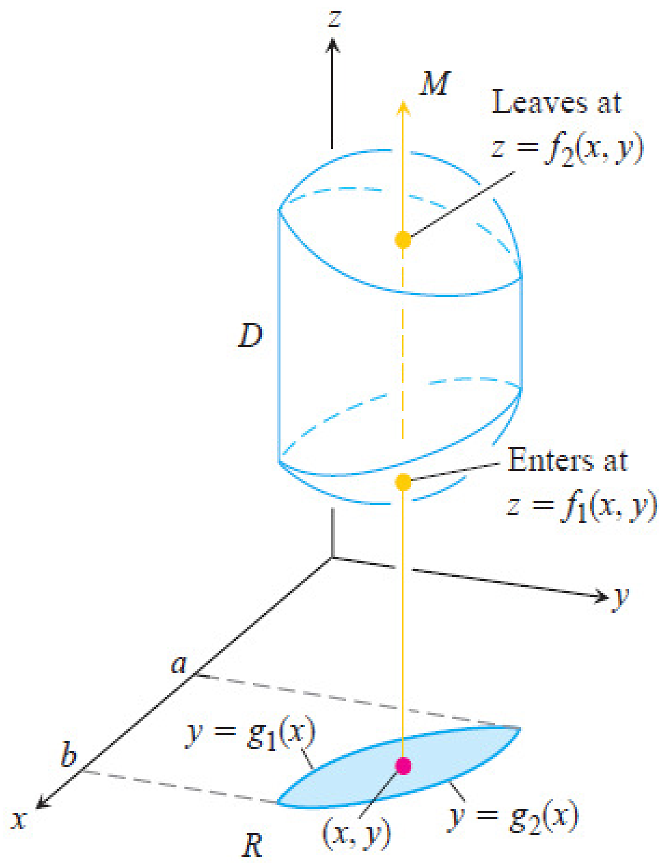
$$\iiint_D F(x, y, z)\, dV$$

over a region $D$, integrate first with respect to $z$, then with respect to $y$, finally with $x$.

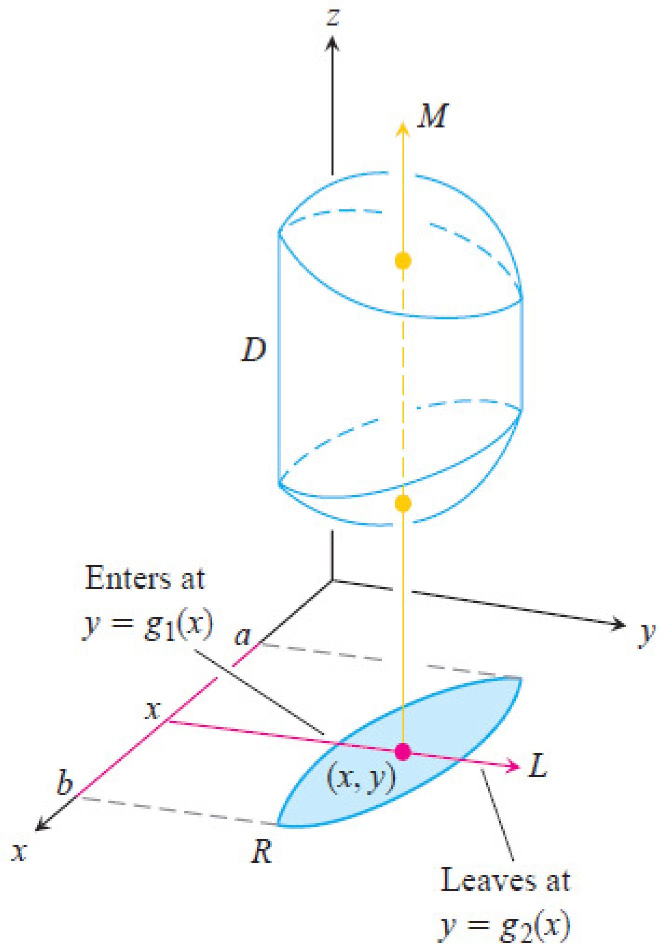
1. *Sketch:* Sketch the region $D$ along with its "shadow" $R$ (vertical projection) in the $xy$-plane. Label the upper and lower bounding surfaces of $D$ and the upper and lower bounding curves of $R$.

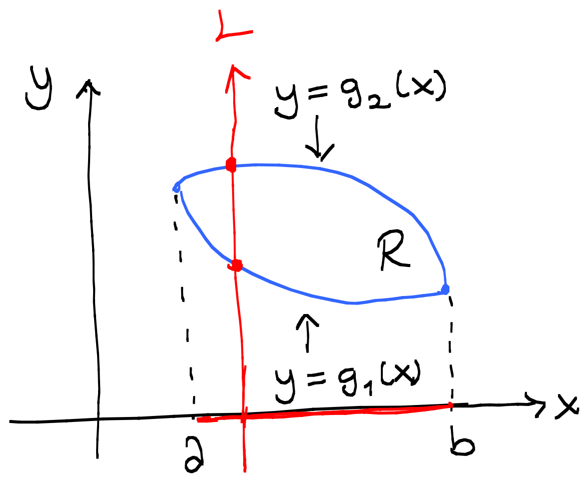


2. *Find the z-limits of integration:* Draw a line $M$ passing through a typical point $(x, y)$ in $R$ parallel to the $z$-axis. As $z$ increases, $M$ enters $D$ at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the $z$-limits of integration.



**3.** *Find the y-limits of integration:* Draw a line $L$ through $(x, y)$ parallel to the $y$-axis. As $y$ increases, $L$ enters $R$ at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the $y$-limits of integration.



4. *Find the x-limits of integration:* Choose $x$-limits that include all lines through $R$ parallel to the $y$-axis ($x = a$ and $x = b$ in the preceding figure). These are the $x$-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

Follow similar procedures if you change the order of integration. The “shadow” of region $D$ lies in the plane of the last two variables with respect to which the iterated integration takes place.

The above procedure applies whenever a solid region $D$ is bounded above and below by a surface, and when the “shadow” region $R$ is bounded by a lower and upper curve. It does not apply to regions with complicated holes through them, although sometimes such regions can be subdivided into simpler regions for which the procedure does apply.

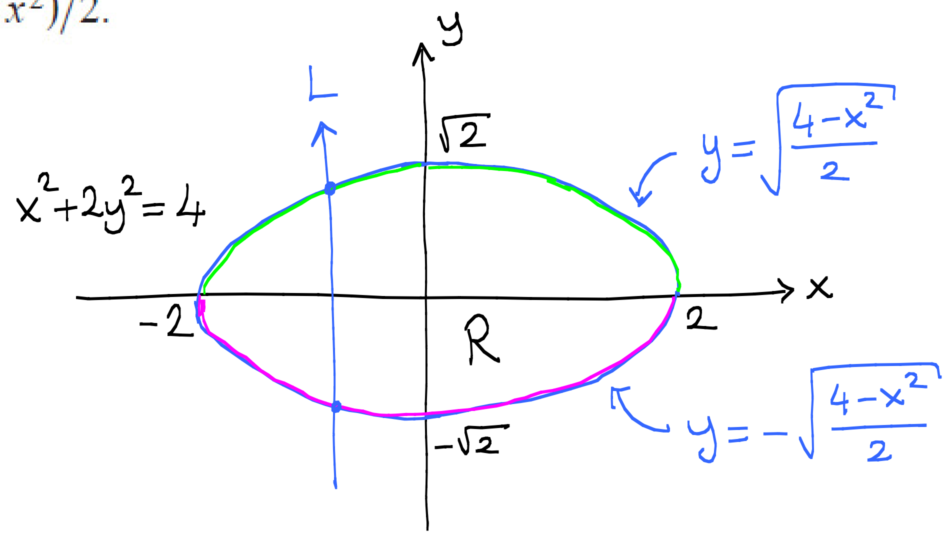
## EXAMPLE 1 Finding a Volume

Find the volume of the region $D$ enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

**Solution** The volume is

$$V = \iiint_D dz\, dy\, dx,$$

the integral of $F(x, y, z) = 1$ over $D$. To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure 15.28) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region $R$, the projection of $D$ onto the $xy$-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of $R$ is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.



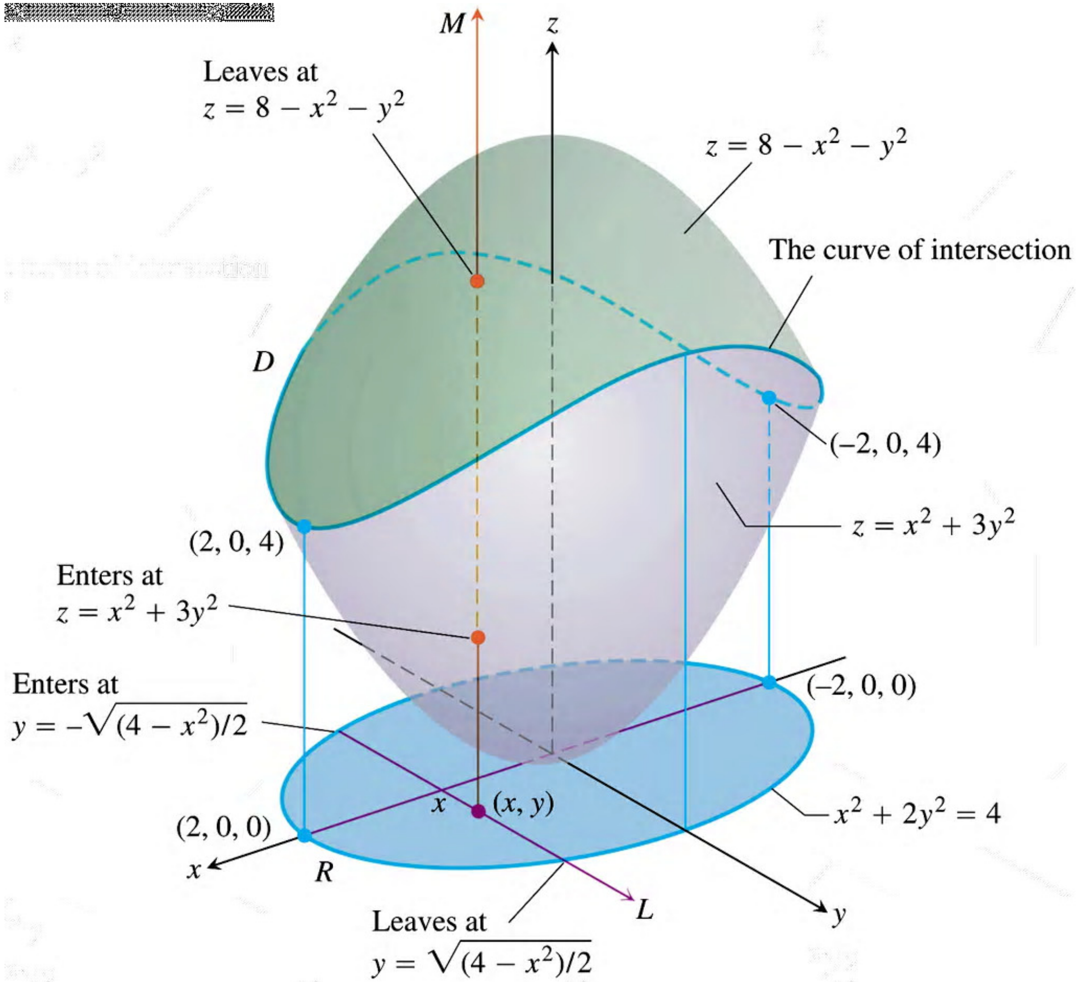



**FIGURE 15.28** The volume of the region enclosed by two paraboloids, calculated in Example 1.

Now we find the $z$-limits of integration. The line $M$ passing through a typical point $(x, y)$ in $R$ parallel to the $z$-axis enters $D$ at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

Next we find the $y$-limits of integration. The line $L$ through $(x, y)$ parallel to the $y$-axis enters $R$ at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Finally we find the $x$-limits of integration. As $L$ sweeps across $R$, the value of $x$ varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of $D$ is

$$V = \iiint_D dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$\left(\begin{aligned} z &= f(x,y) = 8 - x^2 - y^2 \\ z &= g(x,y) = x^2 + 3y^2 \end{aligned}\right)$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\,dy\,dx = \iint_R (f(x,y) - g(x,y))\,dy\,dx$$

$$= \int_{-2}^{2}\left[(8 - 2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2}\left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right) dx$$

$$= \int_{-2}^{2}\left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4 - x^2)^{3/2}\,dx$$

$$= 8\pi\sqrt{2}.$$ After integration with the substitution $x = 2\sin u$. ■

## EXAMPLE 2 Finding the Limits of Integration in the Order *dy dz dx*

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron $D$ with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$.

**Solution** We sketch $D$ along with its "shadow" $R$ in the $xz$-plane (Figure 15.29). The upper (right-hand) bounding surface of $D$ lies in the plane $y = 1$. The lower (left-hand) bounding surface lies in the plane $y = x + z$. The upper boundary of $R$ is the line $z = 1 - x$. The lower boundary is the line $z = 0$.

First we find the $y$-limits of integration. The line through a typical point $(x, z)$ in $R$ parallel to the $y$-axis enters $D$ at $y = x + z$ and leaves at $y = 1$.

Next we find the $z$-limits of integration. The line $L$ through $(x, z)$ parallel to the $z$-axis enters $R$ at $z = 0$ and leaves at $z = 1 - x$.

Finally we find the $x$-limits of integration. As $L$ sweeps across $R$, the value of $x$ varies from $x = 0$ to $x = 1$. The integral is

$$\int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x, y, z)\, dy\, dz\, dx. \qquad \blacksquare$$

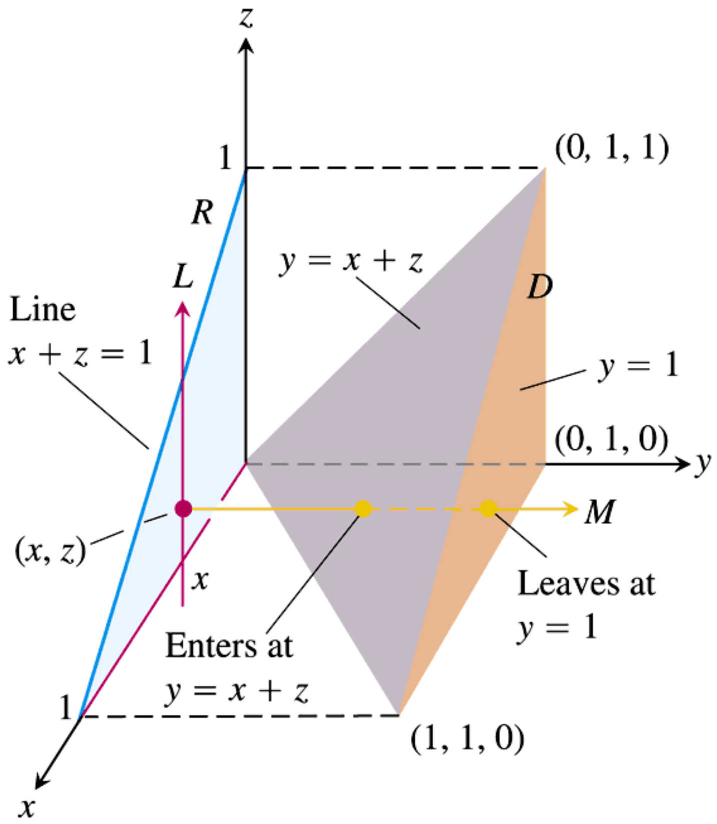



**FIGURE 15.29** Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron $D$ (Example 2).

## EXAMPLE 3 Revisiting Example 2 Using the Order *dz dy dx*

To integrate $F(x, y, z)$ over the tetrahedron $D$ in the order $dz\,dy\,dx$, we perform the steps in the following way.

First we find the $z$-limits of integration. A line parallel to the $z$-axis through a typical point $(x, y)$ in the $xy$-plane "shadow" enters the tetrahedron at $z = 0$ and exits through the upper plane where $z = y - x$ (Figure 15.29).

Next we find the $y$-limits of integration. On the $xy$-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line through $(x, y)$ parallel to the $y$-axis enters the shadow in the $xy$-plane at $y = x$ and exits at $y = 1$.

Finally we find the $x$-limits of integration. As the line parallel to the $y$-axis in the previous step sweeps out the shadow, the value of $x$ varies from $x = 0$ to $x = 1$ at the point $(1, 1, 0)$. The integral is

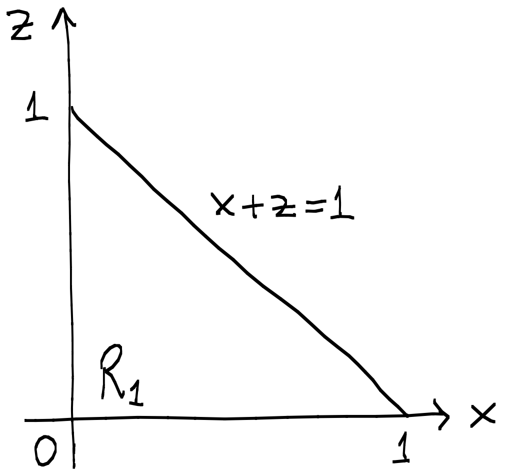
$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\, dz\, dy\, dx.$$

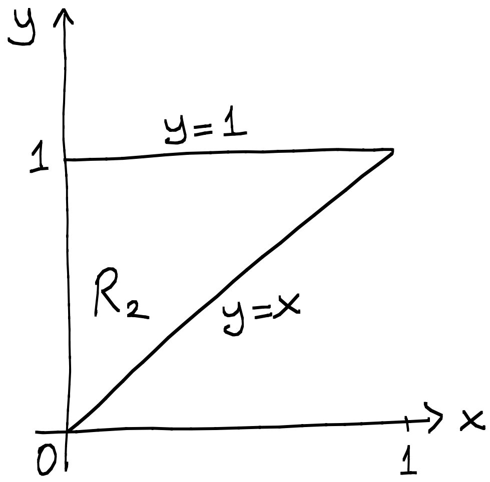
For example, if $F(x, y, z) = 1$, we would find the volume of the tetrahedron to be

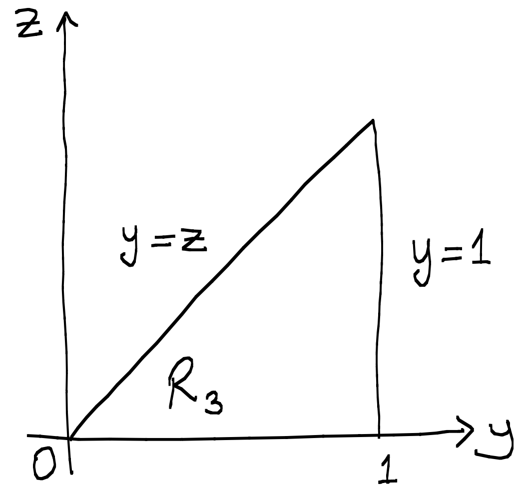
$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\, dy\, dx = \int_0^1 \int_x^1 (y-x)\, dy\, dx = \int_0^1 \left[\frac{y^2}{2} - xy\right]_{y=x}^{y=1} dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx = \left[\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1 = \frac{1}{6}$$

Projections of D onto the coordinate planes:




$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^{1} dy\,dz\,dx = \frac{1}{6}$$

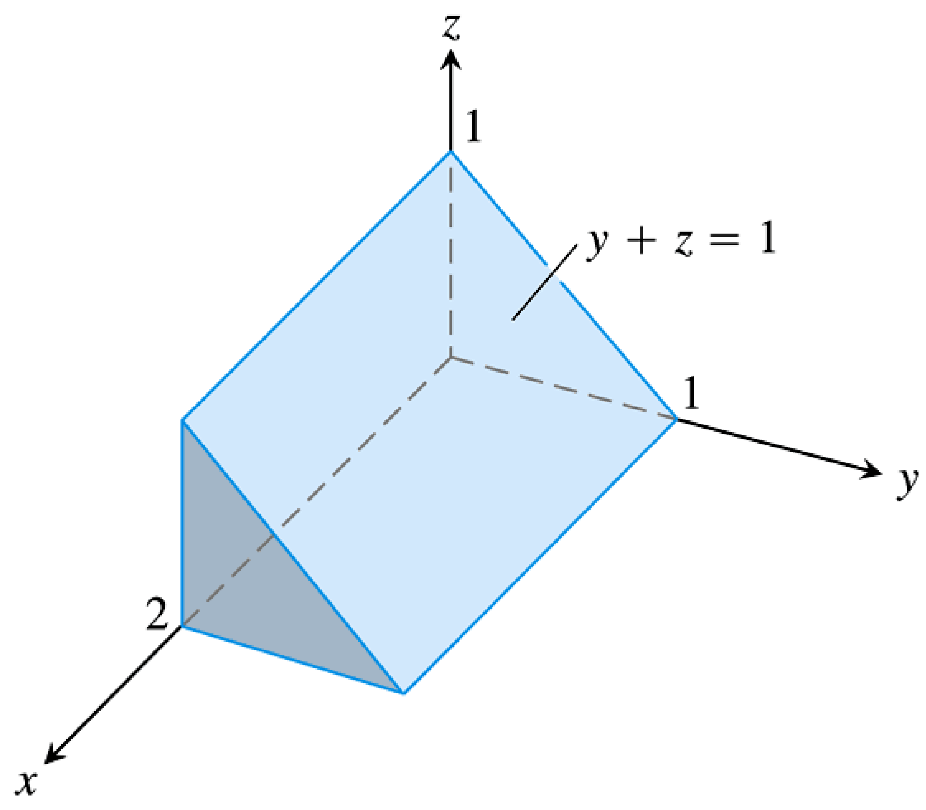
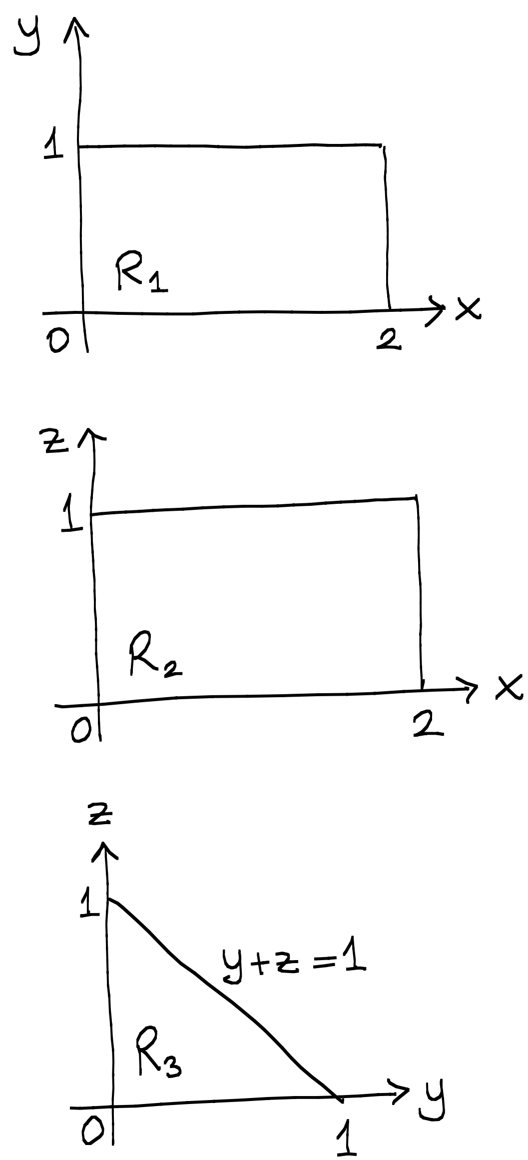



**FIGURE 15.30** Example 4 gives six different iterated triple integrals for the volume of this prism.



## EXAMPLE 4 Using Different Orders of Integration

Each of the following integrals gives the volume of the solid shown in Figure 15.30.

**(a)** $\displaystyle\int_0^1\int_0^{1-z}\int_0^2 dx\,dy\,dz$ **(b)** $\displaystyle\int_0^1\int_0^{1-y}\int_0^2 dx\,dz\,dy$

**(c)** $\displaystyle\int_0^1\int_0^2\int_0^{1-z} dy\,dx\,dz$ **(d)** $\displaystyle\int_0^2\int_0^1\int_0^{1-z} dy\,dz\,dx$

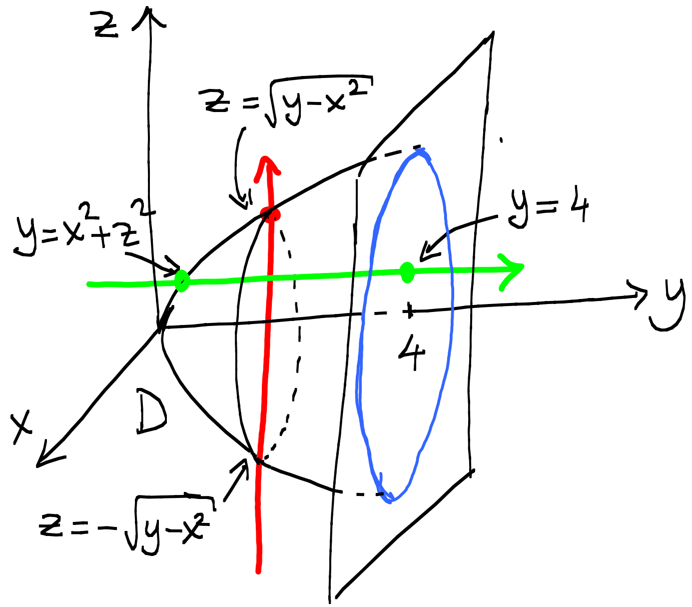
**(e)** $\displaystyle\int_0^1\int_0^2\int_0^{1-y} dz\,dx\,dy$ **(f)** $\displaystyle\int_0^2\int_0^1\int_0^{1-y} dz\,dy\,dx$

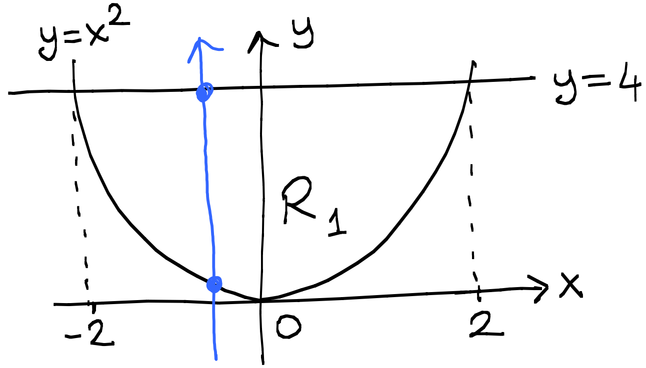
$$V=\int_0^1\int_0^{1-y}\int_0^2 dx\,dz\,dy=\int_0^1\int_0^{1-y}2\,dz\,dy=\int_0^1 2(1-y)\,dy=1$$

$$V=\int_0^1\int_0^2\int_0^{1-z} dy\,dx\,dz=\int_0^1\int_0^2(1-z)\,dx\,dz=\int_0^1 2(1-z)\,dz=1$$

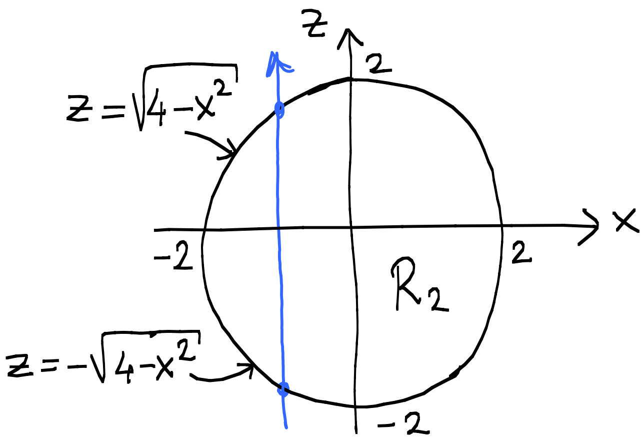
$$V=\int_0^2\int_0^1\int_0^{1-y} dz\,dy\,dx=\int_0^2\int_0^1(1-y)\,dy\,dx=\int_0^2\frac{1}{2}\,dx=1$$

**Example:** Express $I = \iiint_D \sqrt{x^2+z^2}\, dV$ where $D$ is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y=4$, and $dV = dz\,dy\,dx$ and $dV = dy\,dz\,dx$.




$$I = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\; dz\, dy\, dx$$




$$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\; dy\, dz\, dx$$

## Properties of Triple Integrals

If $F = F(x, y, z)$ and $G = G(x, y, z)$ are continuous, then

1. *Constant Multiple:* $\displaystyle\iiint_D kF\,dV = k\iiint_D F\,dV$ (any number $k$)

2. *Sum and Difference:* $\displaystyle\iiint_D (F \pm G)\,dV = \iiint_D F\,dV \pm \iiint_D G\,dV$

3. *Domination:*

   **(a)** $\displaystyle\iiint_D F\,dV \geq 0$ if $F \geq 0$ on $D$

   **(b)** $\displaystyle\iiint_D F\,dV \geq \iiint_D G\,dV$ if $F \geq G$ on $D$

4. *Additivity:* $\displaystyle\iiint_D F\,dV = \iiint_{D_1} F\,dV + \iiint_{D_2} F\,dV$

   if $D$ is the union of two nonoverlapping regions $D_1$ and $D_2$.

# 15.6

## Triple Integrals in Cylindrical and Spherical Coordinates

When evaluating $\iiint_D f(x,y,z)\,dV$, if D is bounded by a cylinder, cone or sphere, using cylindrical or spherical coordinates simplifies the calculations.

### DEFINITION Cylindrical Coordinates

**Cylindrical coordinates** represent a point $P$ in space by ordered triples $(r, \theta, z)$ in which

1. $r$ and $\theta$ are polar coordinates for the vertical projection of $P$ on the $xy$-plane
2. $z$ is the rectangular vertical coordinate.

r is the (shortest) distance from z-axis.

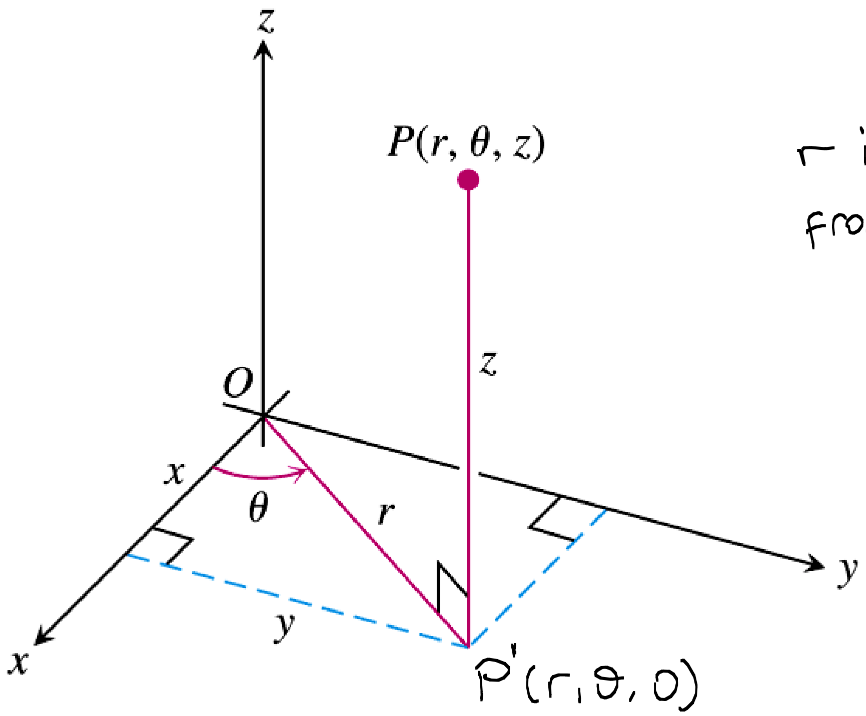



**FIGURE 15.36** The cylindrical coordinates of a point in space are $r$, $\theta$, and $z$.

**Equations Relating Rectangular $(x, y, z)$ and Cylindrical $(r, \theta, z)$ Coordinates**

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

- The equation $r=0$ describes the $z$-axis.
- The equation $r=a$, $a>0$ describes the cylinder about $z$-axis with radius $a$.
- The equation $\theta=\theta_0$ describes the plane that contains the $z$-axis and makes an angle $\theta_0$ with the positive $x$-axis.
- The equation $z=z_0$ describes the plane through $(0,0,z_0)$ perpendicular to the $z$-axis (parallel to the $xy$-plane).

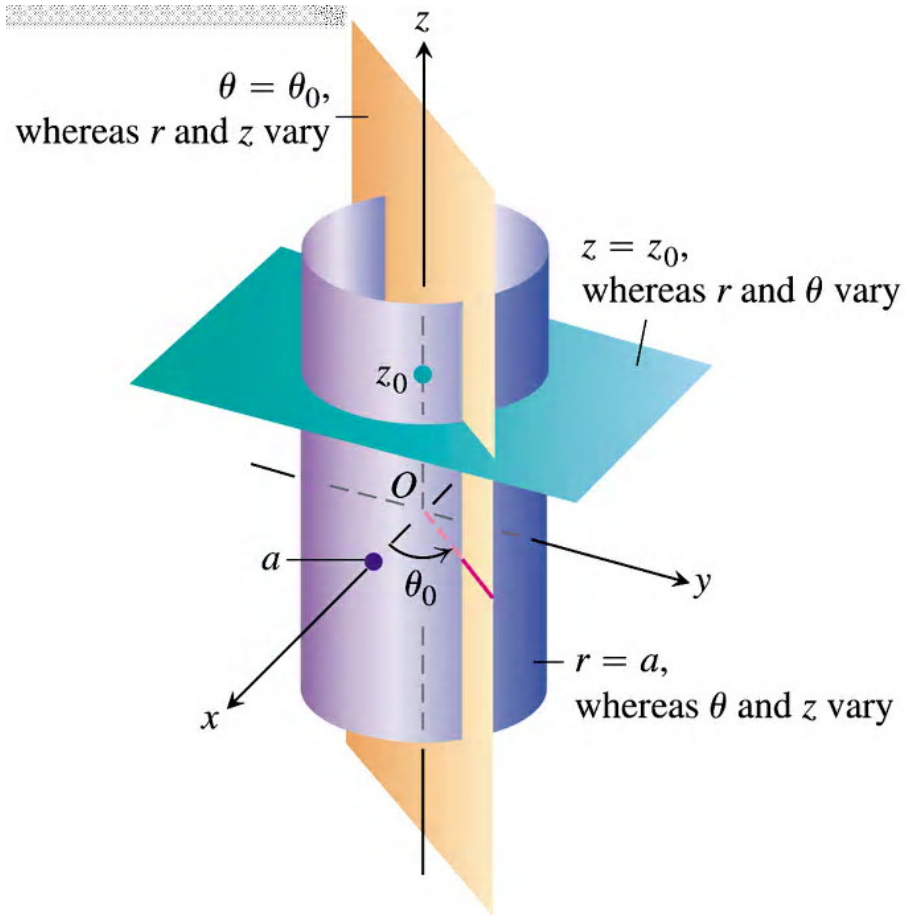



**FIGURE 15.37** Constant-coordinate equations in cylindrical coordinates yield cylinders and planes.

When computing triple integrals over a region D in cylindrical coordinates, we partition the region into cylindrical wedges rather than rectangular boxes.

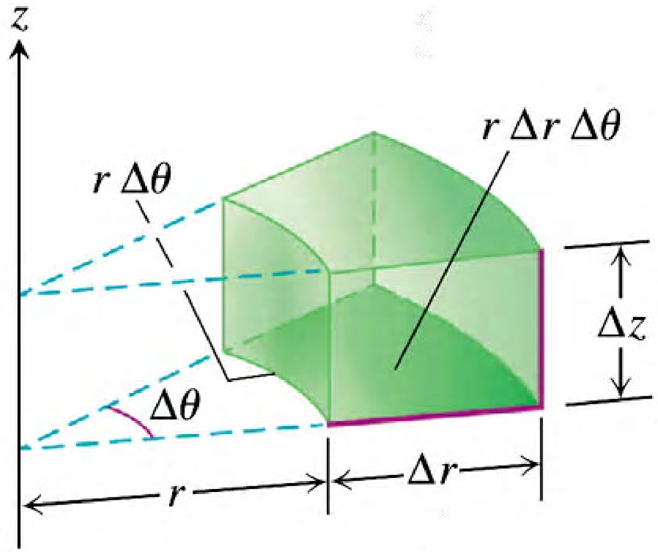



**FIGURE 15.38** In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$.

$$\iiint_D f(x,y,z)\,dV = \iiint_D f(r\cos\theta, r\sin\theta, z)\,dz\,r\,dr\,d\theta$$

**Example 1:** Let $D$ be a region bounded below by the plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$, and above by the paraboloid $z=x^2+y^2$. Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over $D$.

<u>Solution:</u> Let $R$ be the base of $D$ (projection of $D$ on the $xy$-plane). The boundary of $R$ is the circle $x^2+(y-1)^2=1$ (intersection of $z=0$ and $x^2+(y-1)^2=1$). Its polar equation is:

$$x^2+(y-1)^2=1 \Leftrightarrow x^2+y^2-2y=0 \Leftrightarrow r^2-2r\sin\theta=0 \Leftrightarrow r=2\sin\theta \quad (r\neq 0)$$

The polar equation for $z=0$ is $z=0$ and for $z=x^2+y^2$ is $z=r^2$.

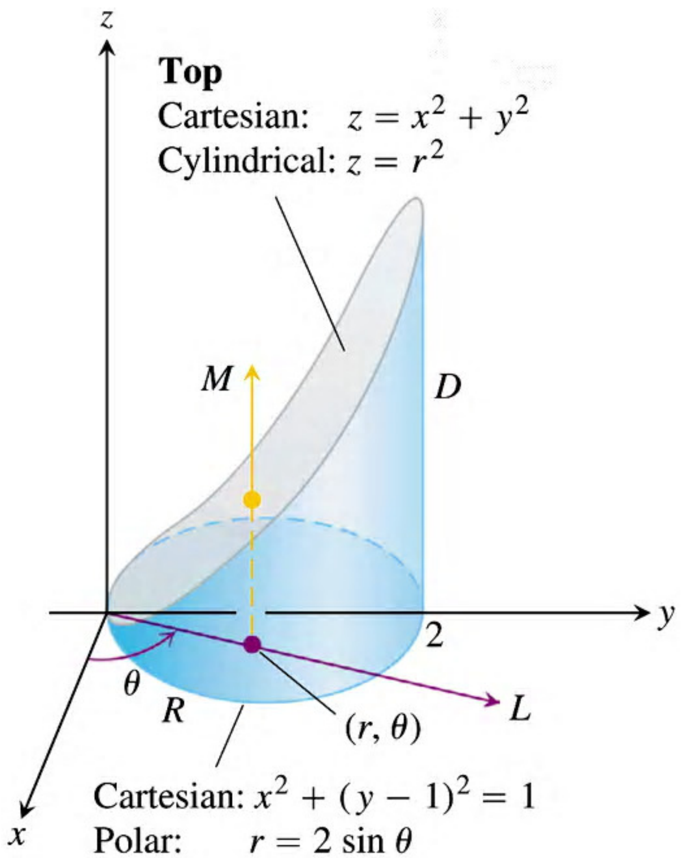
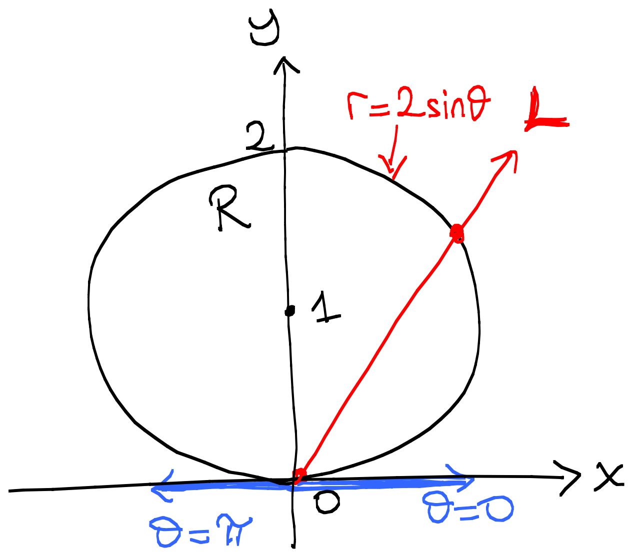



**FIGURE 15.39** Finding the limits of integration for evaluating an integral in l coordinates (Example 1).



For the z-limits of integration, draw a line M parallel to the z-axis. M enters the region D at $z=0$ and exits at $z=r^2$.

For the r-limit of integration, draw a ray L to the region R in the xy-plane. L enters the region R from the origin $(r=0)$ and exits at $r = 2\sin\theta$.

For the $\theta$-limits of integration, sweep the ray L across the region R. The angle $\theta$ between L and the positive x-axis runs from $\theta=0$ to $\theta=\pi$.

$$\iiint_D f(r,\theta,z)\, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r,\theta,z)\, dz\, r\, dr\, d\theta$$

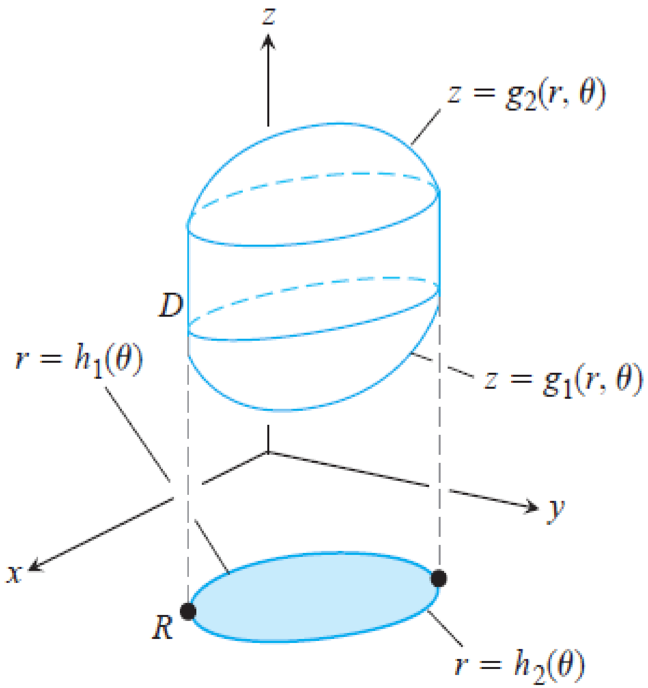
## How to Integrate in Cylindrical Coordinates

To evaluate

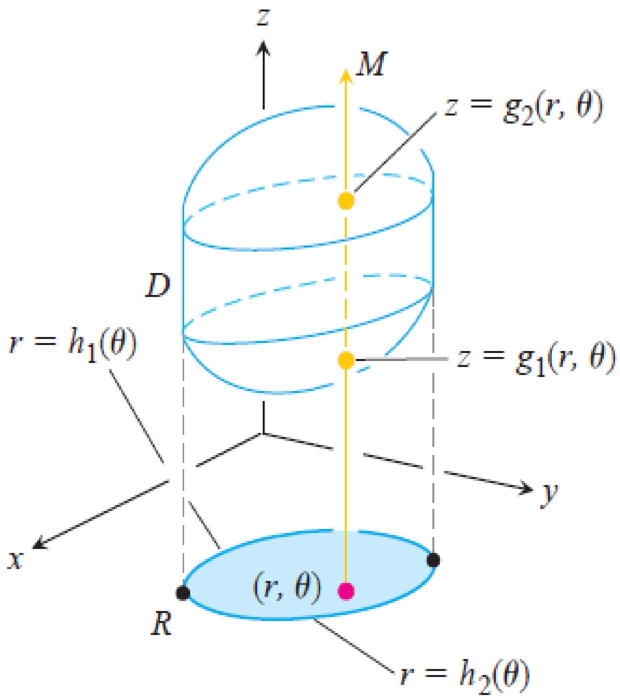
$$\iiint_D f(r, \theta, z)\, dV$$

over a region $D$ in space in cylindrical coordinates, integrating first with respect to $z$, then with respect to $r$, and finally with respect to $\theta$, take the following steps.

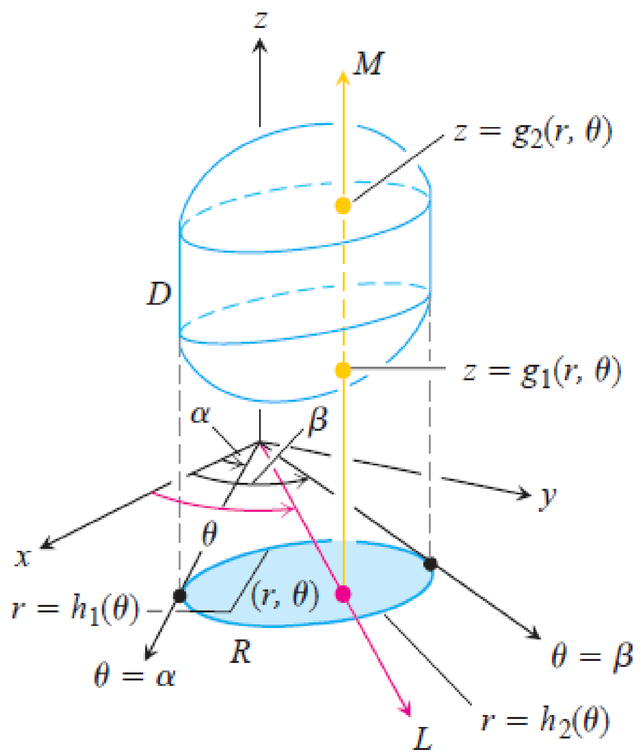
1. *Sketch*. Sketch the region $D$ along with its projection $R$ on the $xy$-plane. Label the surfaces and curves that bound $D$ and $R$.

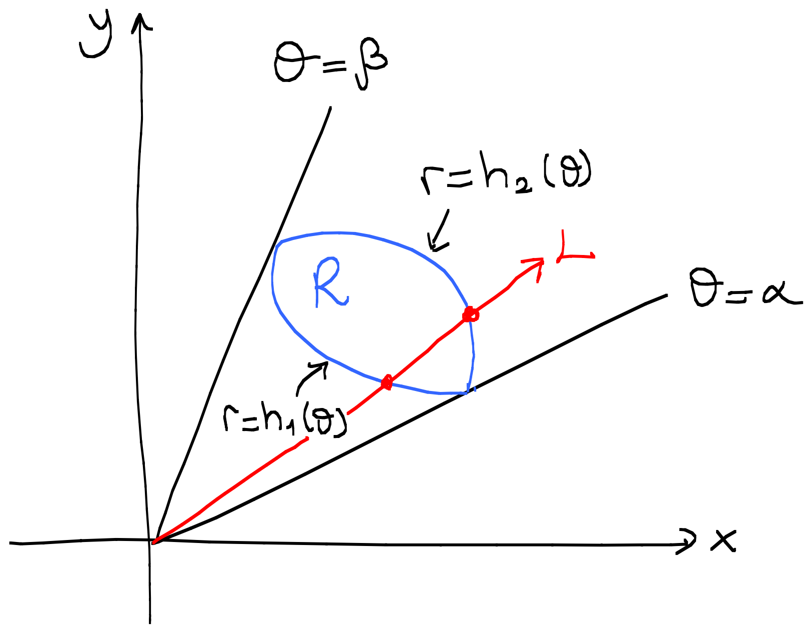


**2.** *Find the z-limits of integration*. Draw a line $M$ through a typical point $(r, \theta)$ of $R$ parallel to the $z$-axis. As $z$ increases, $M$ enters $D$ at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the $z$-limits of integration.



**3.** *Find the r-limits of integration.* Draw a ray $L$ through $(r, \theta)$ from the origin. The ray enters $R$ at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the $r$-limits of integration.

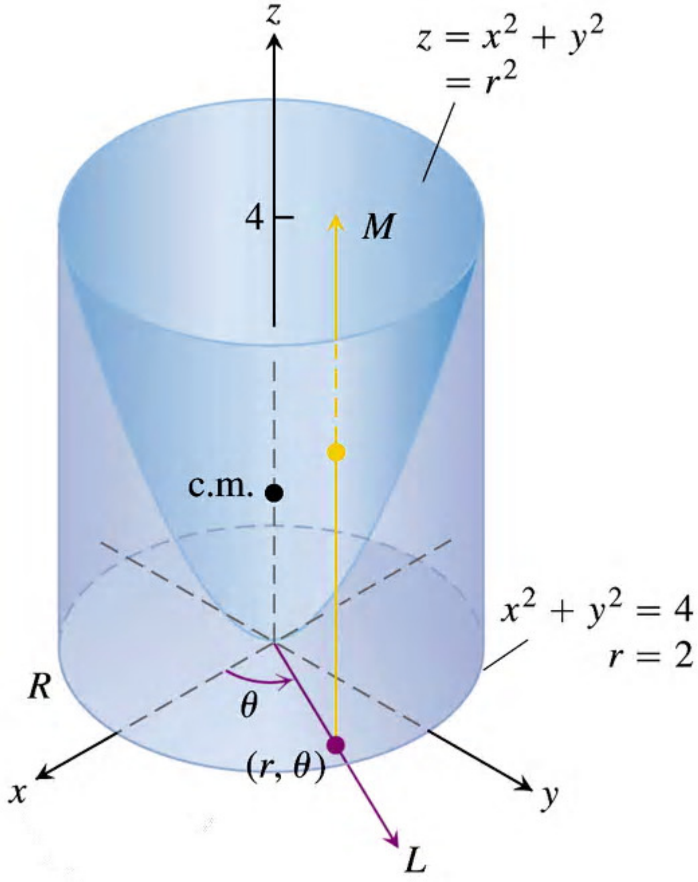


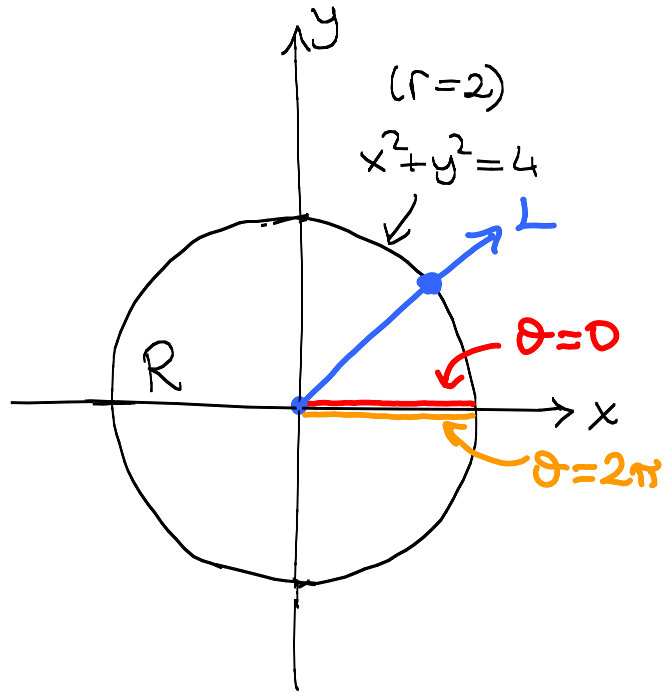




**4.** *Find the θ-limits of integration.* As $L$ sweeps across $R$, the angle $\theta$ it makes with the positive $x$-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the $\theta$-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

**Example 2:** Find the volume of the solid enclosed by the cylinder $x^2+y^2=4$, bounded above by the paraboloid $z=x^2+y^2$, and below by the xy-plane.




$$\text{Volume} = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} 1 \cdot dz\, r\, dr\, d\theta = 8\pi \text{ (Exercise)}$$

# Spherical Coordinates and Integration

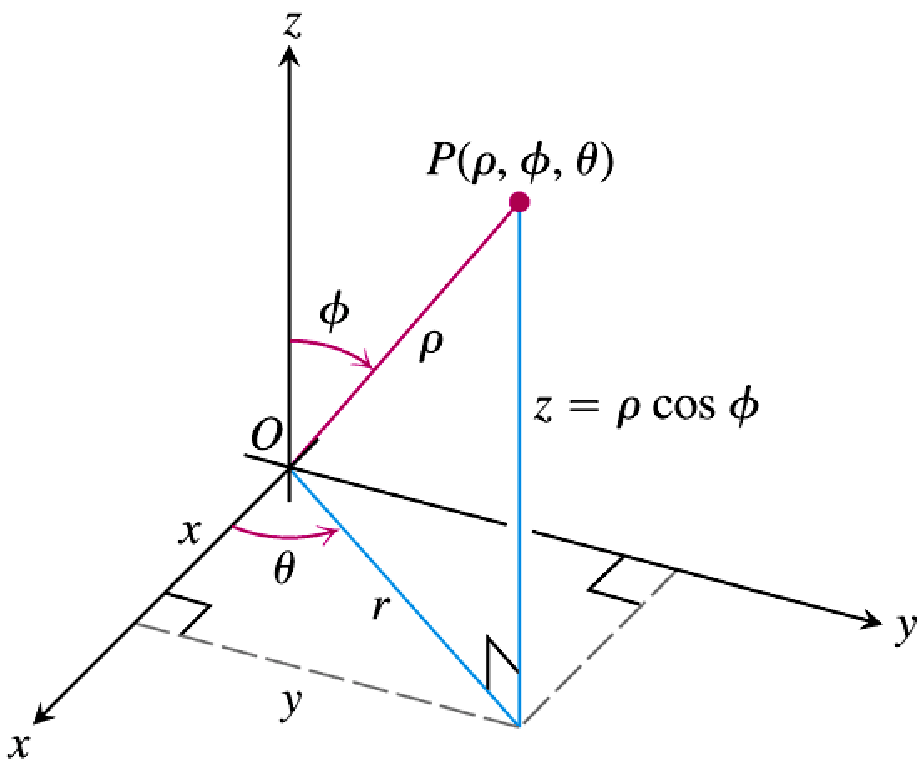



**FIGURE 15.41** The spherical coordinates $\rho$, $\phi$, and $\theta$ and their relation to $x$, $y$, $z$, and $r$.

### DEFINITION Spherical Coordinates

**Spherical coordinates** represent a point $P$ in space by ordered triples $(\rho, \phi, \theta)$ in which

1. $\rho$ is the distance from $P$ to the origin.
2. $\phi$ is the angle $\overrightarrow{OP}$ makes with the positive $z$-axis $(0 \leq \phi \leq \pi)$.
3. $\theta$ is the angle from cylindrical coordinates.

Unlike $r$, the variable $\rho$ is never negative.

- The equation $\rho = 0$ describes the origin
- The equation $\rho = a$, $a > 0$ describes the sphere of radius $a$ centered at origin.
- The equation $\phi = 0$ describes the positive $z$-axis.
- The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the $z$-axis. (xy-plane is interpreted as the cone $\phi = \frac{\pi}{2}$). If $\phi_0 > \frac{\pi}{2}$ then the cone $\phi = \phi_0$ opens downward.
- The equation $\phi = \pi$ describes the negative $z$-axis.

- The equation $\theta = \theta_0$ describes the half plane that contains the $z$-axis and makes an angle $\theta_0$ with the positive $x$-axis. (it is the half plane since $\rho \geq 0$ and $0 \leq \phi \leq \pi$).

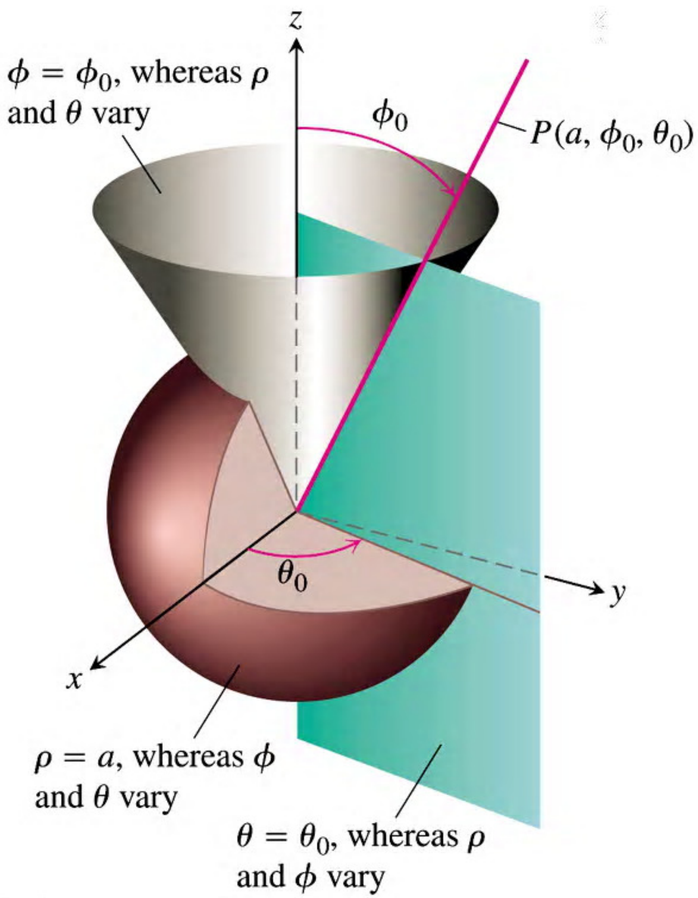



**FIGURE 15.42** Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

**Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates**

$$
\begin{aligned}
r &= \rho \sin \phi, \qquad & x &= r \cos \theta = \rho \sin \phi \cos \theta, \\
z &= \rho \cos \phi, \qquad & y &= r \sin \theta = \rho \sin \phi \sin \theta, \\
\rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}. & &
\end{aligned}
\tag{1}
$$

**EXAMPLE 3** Converting Cartesian to Spherical

Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$.

**Solution** We use Equations (1) to substitute for $x$, $y$, and $z$:

$$
\begin{aligned}
x^2 + y^2 + (z-1)^2 &= 1 \\
\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 &= 1 \qquad \text{Equations (1)} \\
\rho^2 \sin^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 &= 1 \\
\rho^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} &= 2\rho\cos\phi \\
\rho^2 &= 2\rho\cos\phi \\
\rho &= 2\cos\phi .
\end{aligned}
$$

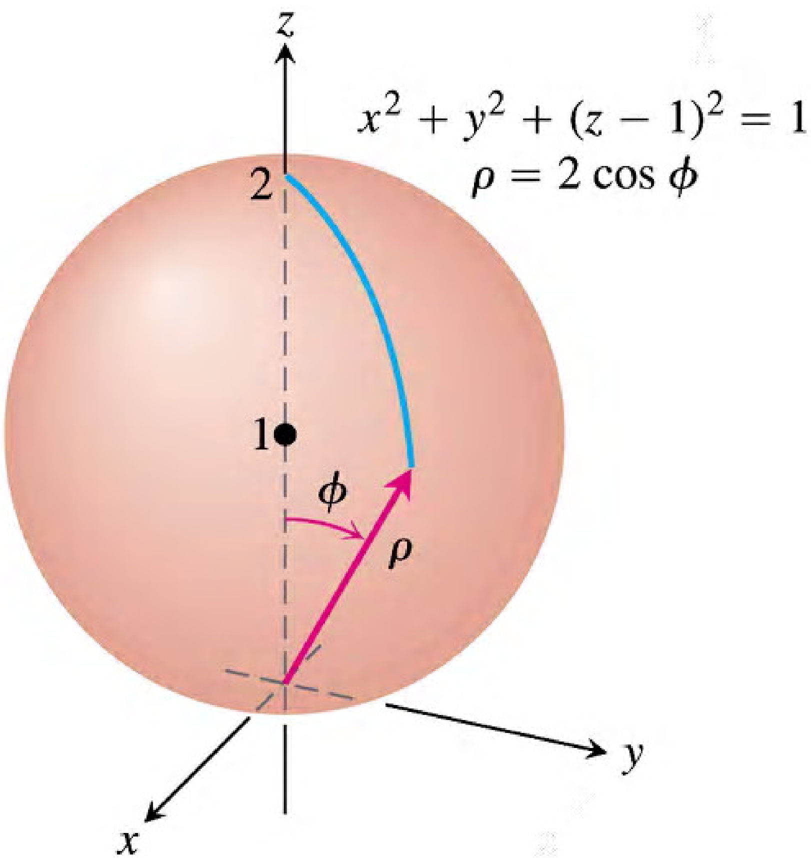



**FIGURE 15.43** The sphere in Example 3.

## EXAMPLE 4 Converting Cartesian to Spherical

Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$ (Figure 15.44).

**Solution 1** *Use geometry*. The cone is symmetric with respect to the $z$-axis and cuts the first quadrant of the $yz$-plane along the line $z = y$. The angle between the cone and the positive $z$-axis is therefore $\pi/4$ radians. The cone consists of the points whose spherical coordinates have $\phi$ equal to $\pi/4$, so its equation is $\phi = \pi/4$.

**Solution 2** *Use algebra*. If we use Equations (1) to substitute for $x$, $y$, and $z$ we obtain the same result:

$$
\begin{aligned}
z &= \sqrt{x^2 + y^2} \\
\rho \cos \phi &= \sqrt{\rho^2 \sin^2 \phi} && \text{Example 3} \\
\rho \cos \phi &= \rho \sin \phi && \rho \geq 0,\ \sin \phi \geq 0 \\
\cos \phi &= \sin \phi \\
\phi &= \frac{\pi}{4}. && 0 \leq \phi \leq \pi
\end{aligned}
$$

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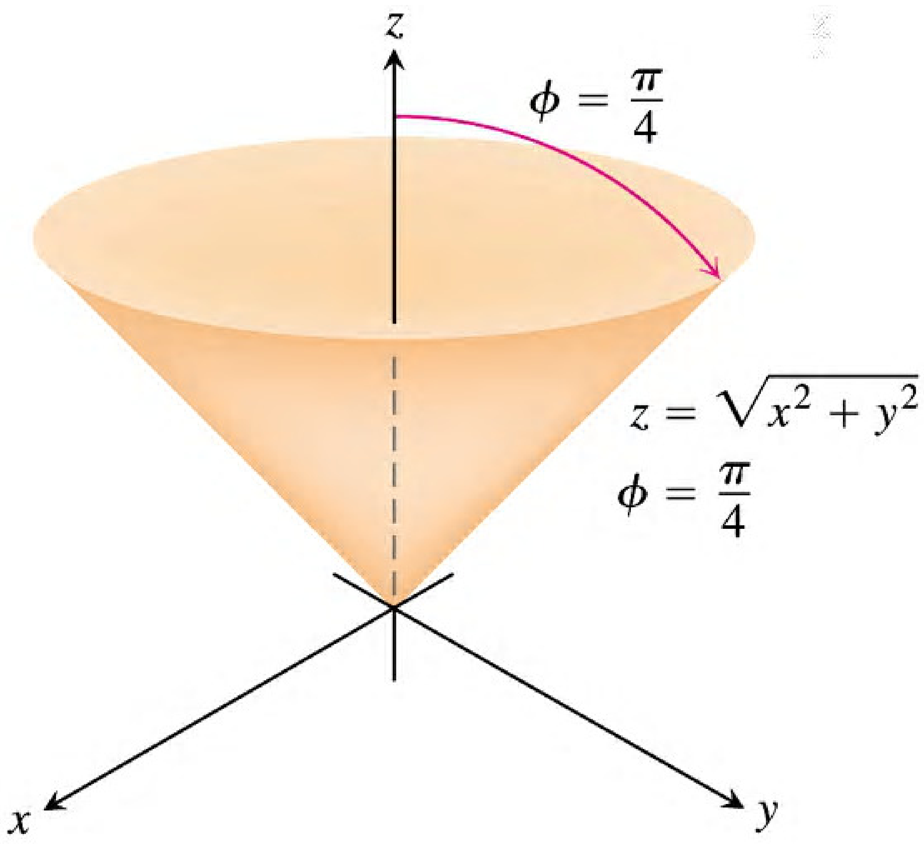



**FIGURE 15.44** The cone in Example 4.

When computing triple integrals over a region D in spherical coordinates, we partition the region into spherical wedges.

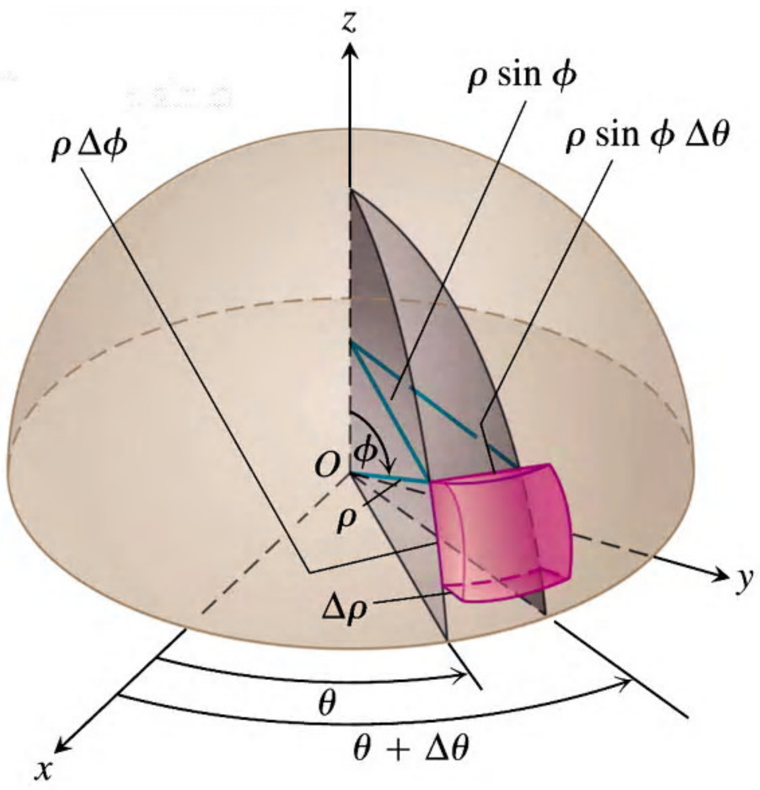



**FIGURE 15.45** In spherical coordinates

$$dV = d\rho \cdot \rho\, d\phi \cdot \rho \sin\phi\, d\theta = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

$$\iiint_D F(\rho, \phi, \theta)\, dV = \iiint_D F(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

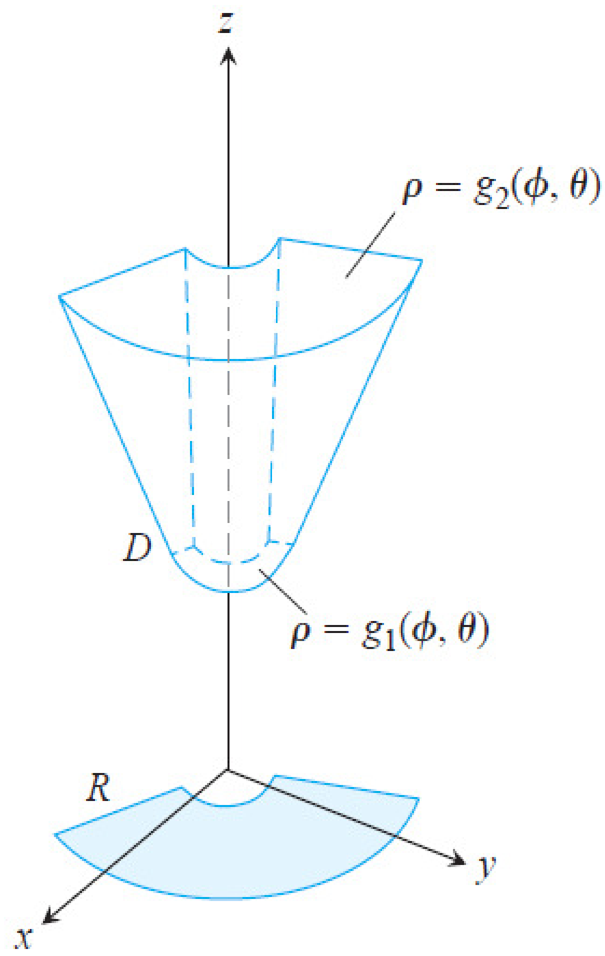
## How to Integrate in Spherical Coordinates

To evaluate

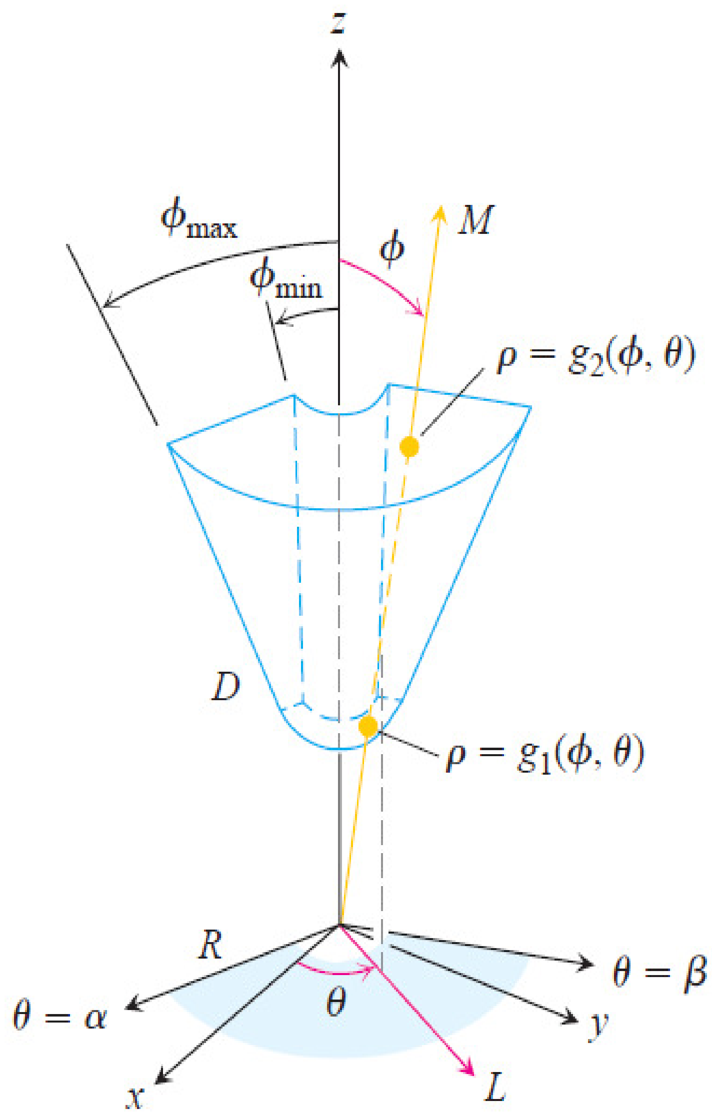
$$\iiint_D f(\rho, \phi, \theta)\, dV$$

over a region $D$ in space in spherical coordinates, integrating first with respect to $\rho$, then with respect to $\phi$, and finally with respect to $\theta$, take the following steps.

**1.** *Sketch*. Sketch the region $D$ along with its projection $R$ on the $xy$-plane. Label the surfaces that bound $D$.



**2.** *Find the ρ-limits of integration*. Draw a ray $M$ from the origin through $D$ making an angle $\phi$ with the positive $z$-axis. Also draw the projection of $M$ on the $xy$-plane (call the projection $L$). The ray $L$ makes an angle $\theta$ with the positive $x$-axis. As $\rho$ increases, $M$ enters $D$ at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the $\rho$-limits of integration.



3. *Find the $\phi$-limits of integration*. For any given $\theta$, the angle $\phi$ that $M$ makes with the $z$-axis runs from $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$. These are the $\phi$ -limits of integration.

4. *Find the $\theta$-limits of integration*. The ray $L$ sweeps over $R$ as $\theta$ runs from $\alpha$ to $\beta$. These are the $\theta$-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi, \theta)}^{\rho=g_2(\phi, \theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

**EXAMPLE 5** Finding a Volume in Spherical Coordinates

Find the volume of the "ice cream cone" $D$ cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

**Solution** The volume is $V = \iiint_D \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over $D$.

To find the limits of integration for evaluating the integral, we begin by sketching $D$ and its projection $R$ on the $xy$-plane (Figure 15.46).

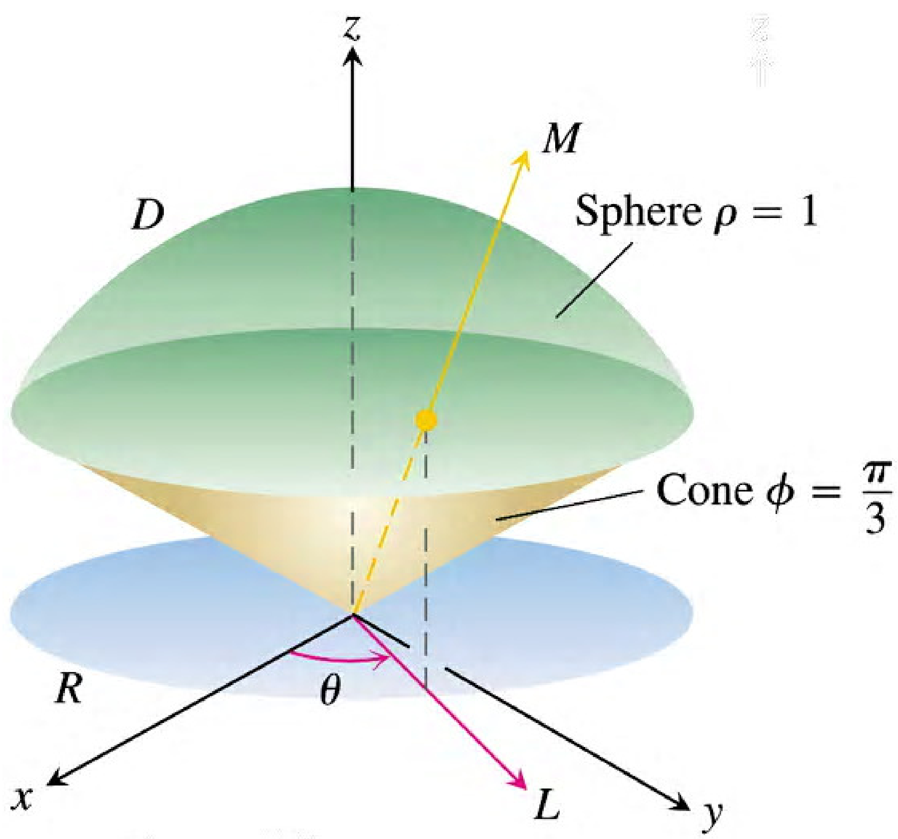



**FIGURE 15.46** The ice cream cone in Example 5.

*The $\rho$-limits of integration.* We draw a ray $M$ from the origin through $D$ making an angle $\phi$ with the positive $z$-axis. We also draw $L$, the projection of $M$ on the $xy$-plane, along with the angle $\theta$ that $L$ makes with the positive $x$-axis. Ray $M$ enters $D$ at $\rho = 0$ and leaves at $\rho = 1$.

*The $\phi$-limits of integration.* The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive $z$-axis. For any given $\theta$, the angle $\phi$ can run from $\phi = 0$ to $\phi = \pi/3$.

*The $\theta$-limits of integration.* The ray $L$ sweeps over $R$ as $\theta$ runs from 0 to $2\pi$. The volume is

$$
\begin{aligned}
V &= \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/3} \left[ \frac{\rho^3}{3} \right]_0^1 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta \\
&= \int_0^{2\pi} \left[ -\frac{1}{3} \cos\phi \right]_0^{\pi/3} d\theta = \int_0^{2\pi} \left( -\frac{1}{6} + \frac{1}{3} \right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.
\end{aligned}
$$

■

## Coordinate Conversion Formulas

| Cylindrical to Rectangular | Spherical to Rectangular | Spherical to Cylindrical |
|---|---|---|
| $x = r\cos\theta$ | $x = \rho\sin\phi\cos\theta$ | $r = \rho\sin\phi$ |
| $y = r\sin\theta$ | $y = \rho\sin\phi\sin\theta$ | $z = \rho\cos\phi$ |
| $z = z$ | $z = \rho\cos\phi$ | $\theta = \theta$ |

Corresponding formulas for $dV$ in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

# 15.7

## Substitutions in Multiple Integrals

Suppose that a region $G$ in the $uv$-plane is transformed one-to-one into the region $R$ in the $xy$-plane by equations of the form

$$x = g(u, v), \qquad y = h(u, v),$$

as suggested in Figure 15.47. We call $R$ the **image** of $G$ under the transformation, and $G$ the **preimage** of $R$. Any function $f(x, y)$ defined on $R$ can be thought of as a function

$f(g(u, v), h(u, v))$ defined on $G$ as well. How is the integral of $f(x, y)$ over $R$ related to the integral of $f(g(u, v), h(u, v))$ over $G$?

The answer is: If $g$, $h$, and $f$ have continuous partial derivatives and $J(u, v)$ (to be discussed in a moment) is zero only at isolated points, if at all, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v))\,|J(u, v)|\, du\, dv. \tag{1}$$

The factor $J(u,v)$ measures how much the transformation is expanding or contracting the area around a point in $G$ as $G$ is transformed into $R$.

## Definition Jacobian

The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v), y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \tag{2}$$

The Jacobian is also denoted by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

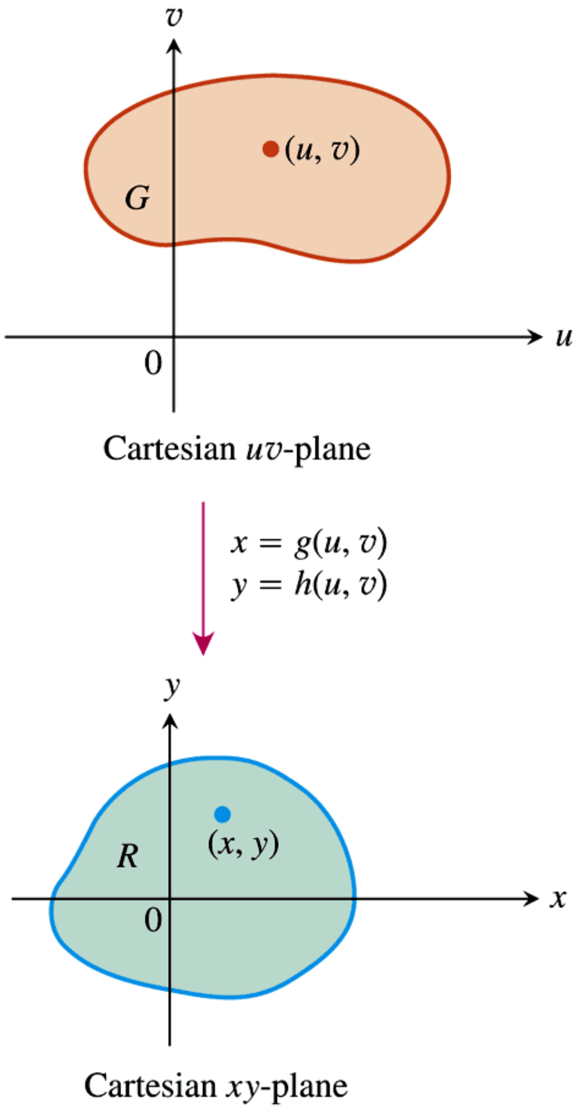



**FIGURE 15.47** The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to change an integral over a region $R$ in the $xy$-plane into an integral over a region $G$ in the $uv$-plane.

For polar coordinates, we have $r$ and $\theta$ in place of $u$ and $v$. With $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Hence, Equation (1) becomes

$$\begin{aligned} \iint_R f(x,y)\,dx\,dy &= \iint_G f(r\cos\theta, r\sin\theta)\,|r|\,dr\,d\theta \\ &= \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta, \qquad \text{If } r \geq 0 \qquad (3) \end{aligned}$$

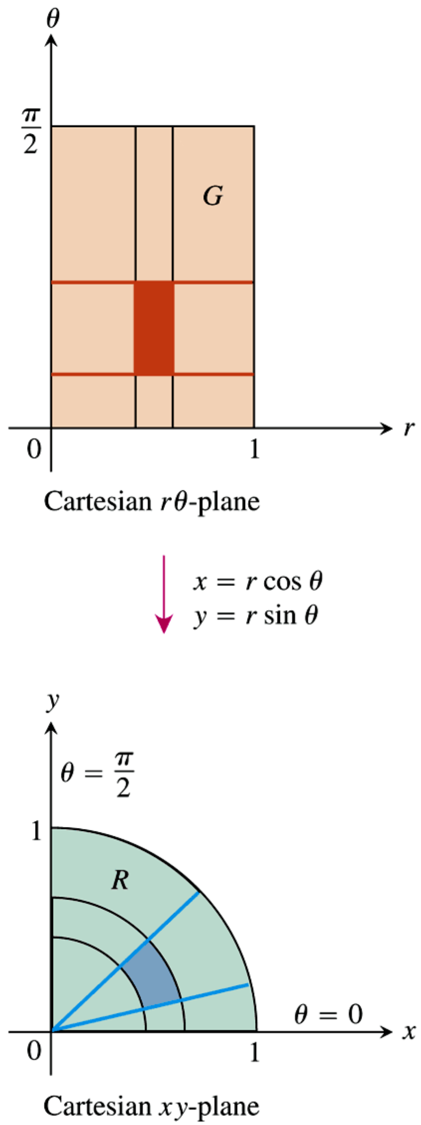



**FIGURE 15.48** The equations $x = r\cos\theta$, $y = r\sin\theta$ transform $G$ into $R$.

**EXAMPLE 1** Applying a Transformation to Integrate

Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\, dx\, dy$$

by applying the transformation

$$u = \frac{2x - y}{2}, \qquad v = \frac{y}{2} \tag{4}$$

and integrating over an appropriate region in the $uv$-plane.

**Solution** We sketch the region $R$ of integration in the $xy$-plane and identify its boundaries (Figure 15.49).

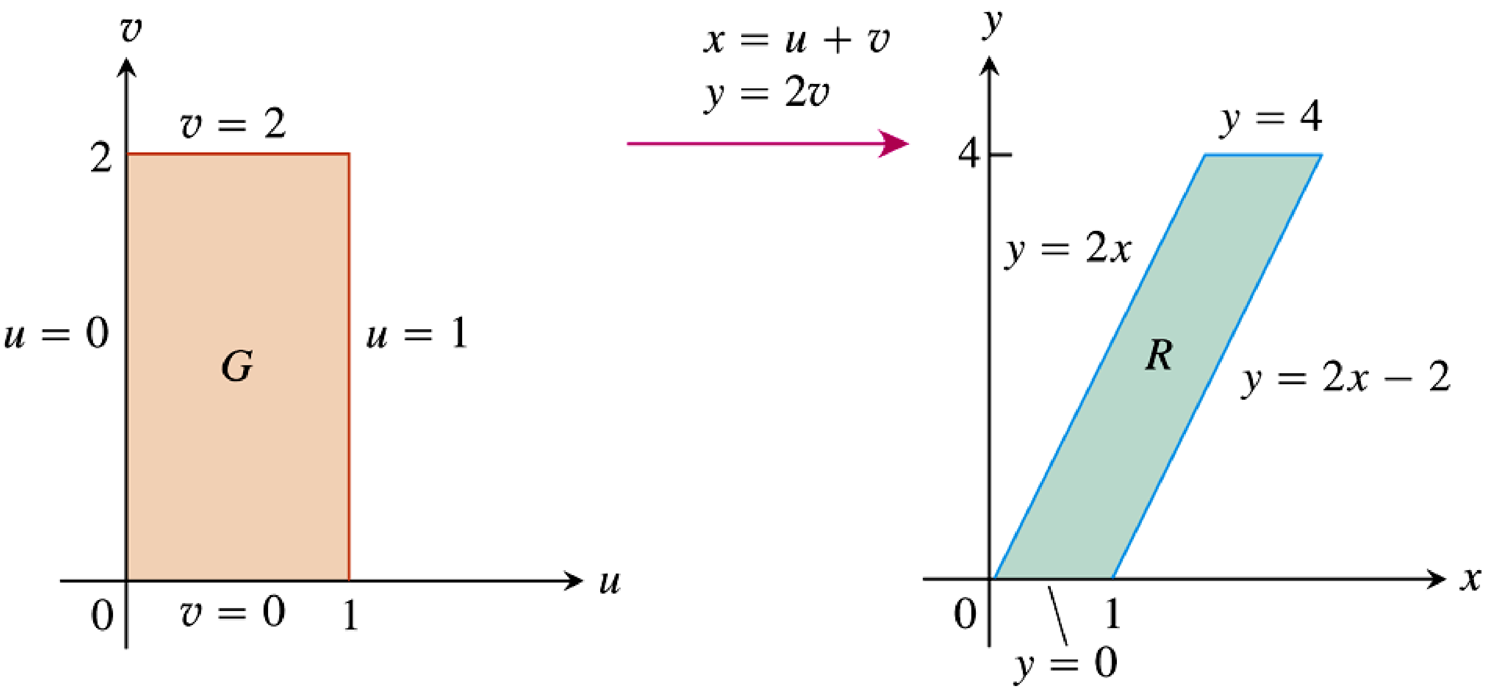



**FIGURE 15.49** The equations $x = u + v$ and $y = 2v$ transform $G$ into $R$. Reversing the transformation by the equations $u = (2x - y)/2$ and $v = y/2$ transforms $R$ into $G$ (Example 1).

To apply Equation (1), we need to find the corresponding $uv$-region $G$ and the Jacobian of the transformation. To find them, we first solve Equations (4) for $x$ and $y$ in terms of $u$ and $v$. Routine algebra gives

$$x = u + v \qquad y = 2v. \tag{5}$$

We then find the boundaries of $G$ by substituting these expressions into the equations for the boundaries of $R$ (Figure 15.49).

| ***xy*-equations for the boundary of *R*** | **Corresponding *uv*-equations for the boundary of *G*** | **Simplified *uv*-equations** |
|---|---|---|
| $x = y/2$ | $u + v = 2v/2 = v$ | $u = 0$ |
| $x = (y/2) + 1$ | $u + v = (2v/2) + 1 = v + 1$ | $u = 1$ |
| $y = 0$ | $2v = 0$ | $v = 0$ |
| $y = 4$ | $2v = 4$ | $v = 2$ |

The Jacobian of the transformation (again from Equations (5)) is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u + v) & \dfrac{\partial}{\partial v}(u + v) \\[2ex] \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

We now have everything we need to apply Equation (1):

$$\begin{aligned} \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\, dx\, dy &= \int_{v=0}^{v=2} \int_{u=0}^{u=1} u \left| J(u, v) \right| du\, dv \\ &= \int_0^2 \int_0^1 (u)(2)\, du\, dv = \int_0^2 \Big[ u^2 \Big]_0^1 dv = \int_0^2 dv = 2. \end{aligned}$$

■

## EXAMPLE 2 Applying a Transformation to Integrate

Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx.$$

**Solution** We sketch the region $R$ of integration in the $xy$-plane and identify its boundaries (Figure 15.50). The integrand suggests the transformation $u = x + y$ and $v = y - 2x$. Routine algebra produces $x$ and $y$ as functions of $u$ and $v$:

$$x = \frac{u}{3} - \frac{v}{3}, \qquad y = \frac{2u}{3} + \frac{v}{3}. \tag{6}$$

From Equations (6), we can find the boundaries of the $uv$-region $G$ (Figure 15.50).

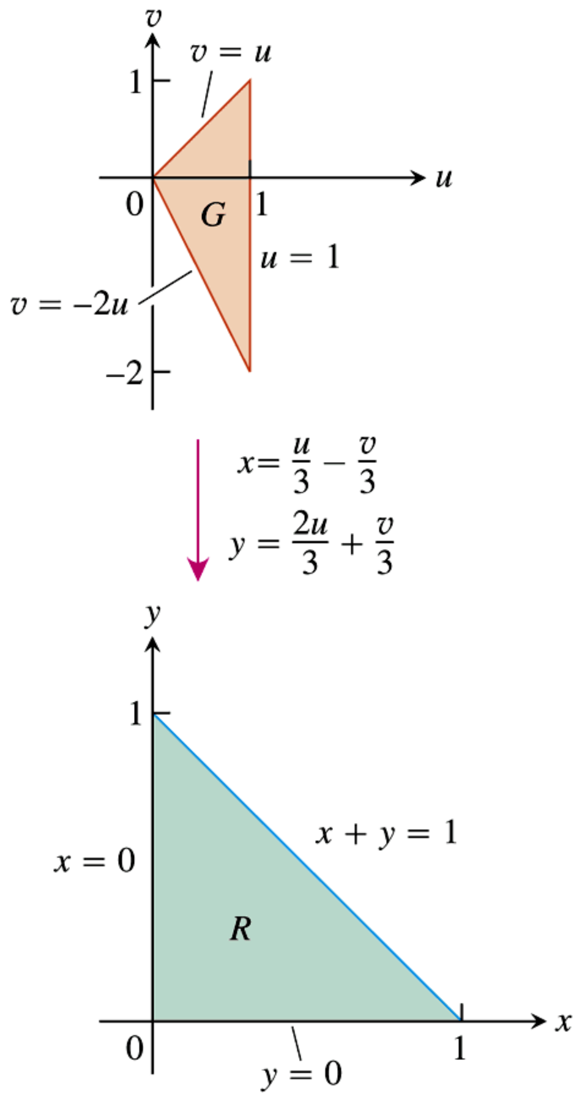



**FIGURE 15.50** The equations $x = (u/3) - (v/3)$ and $y = (2u/3) + (v/3)$ transform $G$ into $R$. Reversing the transformation by the equations $u = x + y$ and $v = y - 2x$ transforms $R$ into $G$ (Example 2).

| $xy$-equations for the boundary of $R$ | Corresponding $uv$-equations for the boundary of $G$ | Simplified $uv$-equations |
|---|---|---|
| $x + y = 1$ | $\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$ | $u = 1$ |
| $x = 0$ | $\frac{u}{3} - \frac{v}{3} = 0$ | $v = u$ |
| $y = 0$ | $\frac{2u}{3} + \frac{v}{3} = 0$ | $v = -2u$ |

The Jacobian of the transformation in Equations (6) is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{3} & -\dfrac{1}{3} \\[2ex] \dfrac{2}{3} & \dfrac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

Applying Equation (1), we evaluate the integral:

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2} v^2 \,|J(u,v)|\,dv\,du$$

$$= \int_0^1 \int_{-2u}^{u} u^{1/2} v^2 \left(\frac{1}{3}\right) dv\,du = \frac{1}{3}\int_0^1 u^{1/2} \left[\frac{1}{3} v^3\right]_{v=-2u}^{v=u} du$$

$$= \frac{1}{9}\int_0^1 u^{1/2}(u^3 + 8u^3)\,du = \int_0^1 u^{7/2}\,du = \frac{2}{9} u^{9/2}\Big]_0^1 = \frac{2}{9}. \qquad \blacksquare$$

## Substitutions in Triple Integrals

Suppose that a region $G$ in $uvw$-space is transformed one-to-one into the region $D$ in $xyz$-space by differentiable equations of the form

$$x = g(u, v, w), \qquad y = h(u, v, w), \qquad z = k(u, v, w),$$

as suggested in Figure 15.51. Then any function $F(x, y, z)$ defined on $D$ can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

defined on $G$. If $g$, $h$, and $k$ have continuous first partial derivatives, then the integral of $F(x, y, z)$ over $D$ is related to the integral of $H(u, v, w)$ over $G$ by the equation

$$\iiint\limits_D F(x, y, z)\, dx\, dy\, dz = \iiint\limits_G H(u, v, w)\,|J(u, v, w)|\, du\, dv\, dw. \tag{7}$$

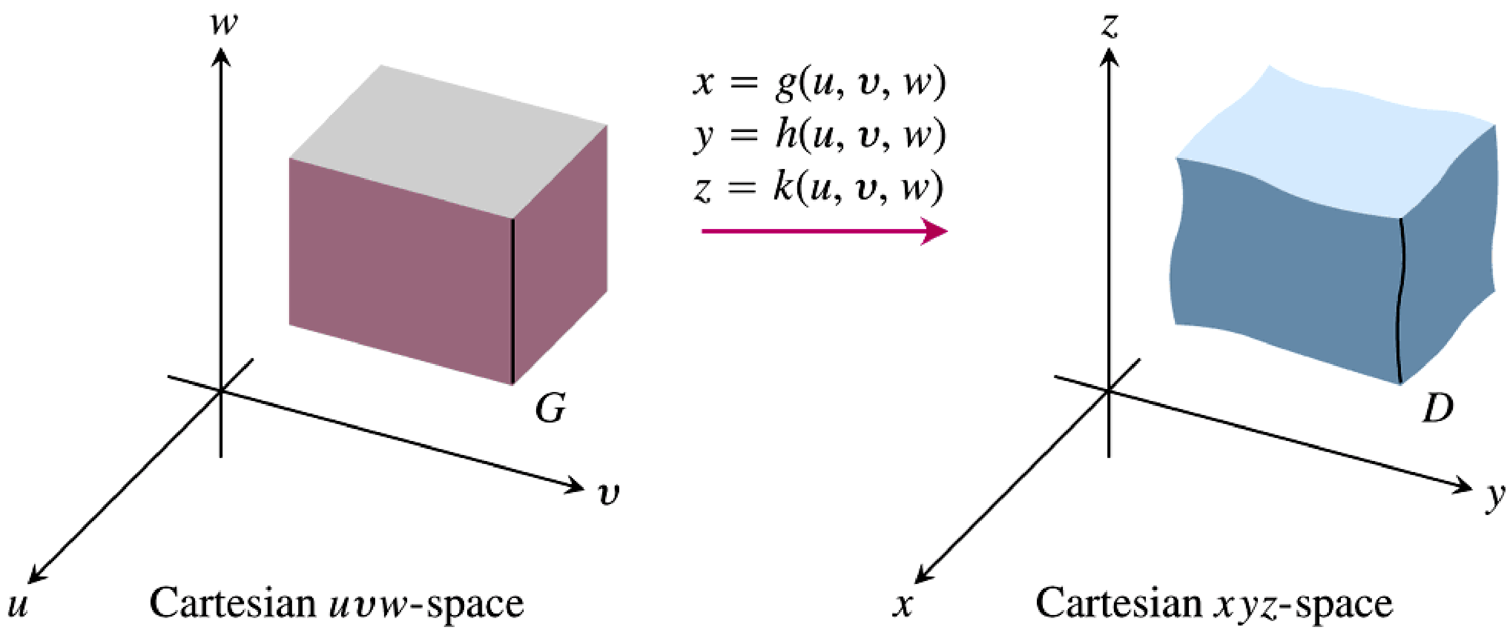



**FIGURE 15.51** The equations $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$ allow us to change an integral over a region $D$ in Cartesian $xyz$-space into an integral over a region $G$ in Cartesian $uvw$-space.

The factor $J(u, v, w)$, whose absolute value appears in this equation, is the **Jacobian determinant**

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

This determinant measures how much the volume near a point in $G$ is being expanded or contracted by the transformation from $(u, v, w)$ to $(x, y, z)$ coordinates.

For cylindrical coordinates, $r$, $\theta$, and $z$ take the place of $u$, $v$, and $w$. The transformation from *Cartesian* $r\theta z$-space to Cartesian $xyz$-space is given by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

(Figure 15.52). The Jacobian of the transformation is

$$J(r, \theta, z) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\[2ex] \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(r, \theta, z)\,|r|\, dr\, d\theta\, dz.$$

We can drop the absolute value signs whenever $r \geq 0$.

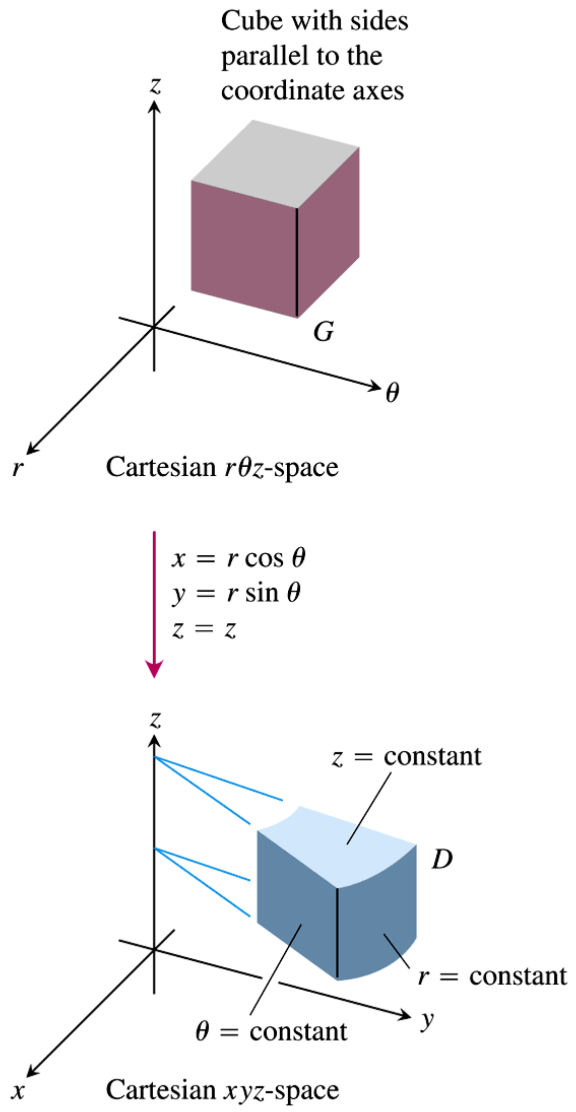



**FIGURE 15.52** The equations $x = r\cos\theta, y = r\sin\theta$, and $z = z$ transform the cube $G$ into a cylindrical wedge $D$.

For spherical coordinates, $\rho$, $\phi$, and $\theta$ take the place of $u$, $v$, and $w$. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian $xyz$-space is given by

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi$$

(Figure 15.53). The Jacobian of the transformation is

$$J(\rho, \phi, \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin\phi$$

The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(\rho, \phi, \theta)\,|p^2 \sin\phi|\, d\rho\, d\phi\, d\theta.$$

We can drop the absolute value signs because $\sin\phi$ is never negative for $0 \leq \phi \leq \pi$.

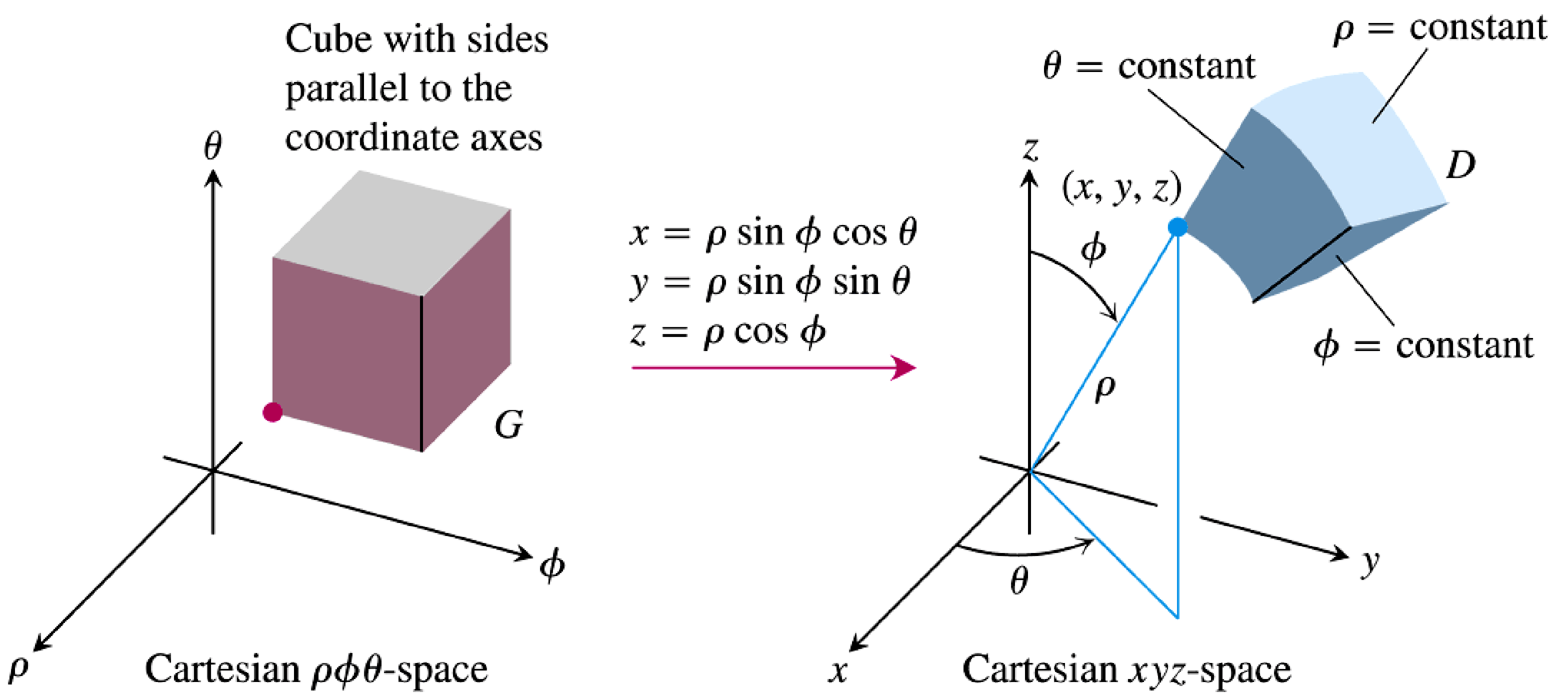



**FIGURE 15.53** The equations $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$ transform the cube $G$ into the spherical wedge $D$.

## EXAMPLE 3 Applying a Transformation to Integrate

Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left( \frac{2x - y}{2} + \frac{z}{3} \right) dx\, dy\, dz$$

by applying the transformation

$$u = (2x - y)/2, \qquad v = y/2, \qquad w = z/3 \tag{8}$$

and integrating over an appropriate region in $uvw$-space.

**Solution** We sketch the region $D$ of integration in $xyz$-space and identify its boundaries (Figure 15.54). In this case, the bounding surfaces are planes.

To apply Equation (7), we need to find the corresponding $uvw$-region $G$ and the Jacobian of the transformation. To find them, we first solve Equations (8) for $x$, $y$, and $z$ in terms of $u$, $v$, and $w$. Routine algebra gives

$$x = u + v, \qquad y = 2v, \qquad z = 3w. \tag{9}$$

We then find the boundaries of $G$ by substituting these expressions into the equations for the boundaries of $D$:

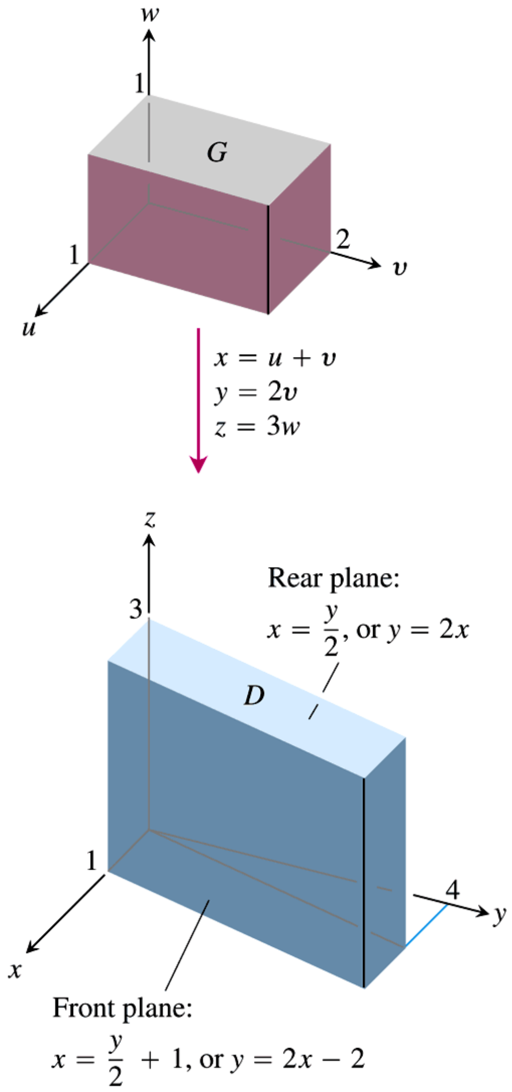



**FIGURE 15.54** The equations $x = u + v$, $y = 2v$, and $z = 3w$ transform $G$ into $D$. Reversing the transformation by the equations $u = (2x - y)/2$, $v = y/2$, and $w = z/3$ transforms $D$ into $G$ (Example 3).

| *xyz*-equations for the boundary of ***D*** | Corresponding *uvw*-equations for the boundary of ***G*** | Simplified *uvw*-equations |
|---|---|---|
| $x = y/2$ | $u + v = 2v/2 = v$ | $u = 0$ |
| $x = (y/2) + 1$ | $u + v = (2v/2) + 1 = v + 1$ | $u = 1$ |
| $y = 0$ | $2v = 0$ | $v = 0$ |
| $y = 4$ | $2v = 4$ | $v = 2$ |
| $z = 0$ | $3w = 0$ | $w = 0$ |
| $z = 3$ | $3w = 3$ | $w = 1$ |

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Equation (7):

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left( \frac{2x - y}{2} + \frac{z}{3} \right) dx\, dy\, dz$$

$$= \int_0^1 \int_0^2 \int_0^1 (u + w) \left| J(u, v, w) \right| du\, dv\, dw$$

$$= \int_0^1 \int_0^2 \int_0^1 (u + w)(6)\, du\, dv\, dw = 6 \int_0^1 \int_0^2 \left[ \frac{u^2}{2} + uw \right]_0^1 dv\, dw$$

$$= 6 \int_0^1 \int_0^2 \left( \frac{1}{2} + w \right) dv\, dw = 6 \int_0^1 \left[ \frac{v}{2} + vw \right]_0^2 dw = 6 \int_0^1 (1 + 2w)\, dw$$

$$= 6\left[ w + w^2 \right]_0^1 = 6(2) = 12. \quad \blacksquare$$