# MAT281E HW4

Mustafa Can Çalışkan
150200097

1)

1)

$$\begin{array}{c} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix} \\ c_1' \quad c_2' \quad\quad \end{array} \xrightarrow[\text{RREF}]{} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\ c_1 \quad c_2 \quad\quad \end{array}$$

dimension of the column space = # basis of the column space.

$c_1$ and $c_2$ form a basis for RREF → $c_1'$ and $c_2'$ form a basis for A

$$\text{Basis col}(A) = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$\dim(\text{col}(A)) = \underline{2}$

$c_1'$ and $c_2'$ cannot span $R^3$ (dim. of basis = 2). Because of that, All b's may not be expressed as a linear combination of the basis.

Shortly, $c_1'$, $c_2'$ and b may be linearly independent and the system may not be consistent.

2)

2)

a)

$$A = \begin{array}{c} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & -2 & 2 \end{bmatrix} \\ \begin{matrix} c_1' & c_2' & c_3' \end{matrix} \end{array} \xrightarrow[\text{RREF}]{} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} \end{array}$$

Basis of $col(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$

Column vectors of A span R3 $(\dim(col(A)) = 3)$. b is in the column space of A and it can be expressed as a linear combination of the column vectors of A.

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hookrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 2 & 2 & 1 \\ 2 & -2 & 2 & 1 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$6\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = b = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & -2 & 2 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There are three nonzero rows (3 lead 1's) ✓

Row space and column space are both three dimensional ✓

$\text{rank}(A) = 3$

c)

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$x_3 = 0$
$x_2 = 0$
$x_1 = 0$

The space containing only zero vector is considered to be zero dimensional.

$\text{Nullity} = \dim(N(A)) = 0$

2) d)

$$A = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} x1 = 0 \\ x2 = 0 \\ x3 = 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 2 & 2 & 1 \\ 2 & -2 & 2 & 1 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -4 \end{array}\right] \quad \begin{array}{l} x1 = 6 \\ x2 = 3/2 \\ x3 = -4 \end{array}$$

General solution $= X_p + X_h$

$$= \underbrace{\begin{bmatrix} 6 \\ 3/2 \\ -4 \end{bmatrix}}_{X_p} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{X_h}$$

3)

3)

a)

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & 4 & -2 & 2 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = t$

$x_3 = s$

$x_2 = s + t$

$x_1 = -s - 3t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Basis of $N(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

b) Nullity = dim (N(A)) = # basis of N(A) = 2

c)

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two nonzero rows (2 lead 1's)

Rank(A) = 2

3)
d)

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & 4 & -2 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -s-3t \\ x_2 = s+t \\ x_3 = s \\ x_4 = t \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 5 \\ 1 & 0 & 1 & 3 & 3 \\ 2 & 4 & -2 & 2 & 4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 3 \\ 0 & 1 & -1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = 3-s-3t \\ x_2 = -1/2+s+t \\ x_3 = s \\ x_4 = t \end{array}$$

$$x_g = \underbrace{\begin{bmatrix} 3 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{t\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{x_h}$$

3)

e)

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_1'$ $c_2'$ $\qquad$ $c_1$ $c_2$

$c_1$ and $c_2$ form a basis for RREF $\rightarrow$ $c_1'$ and $c_2'$ form a basis for A.

$$\text{Basis of } \operatorname{col}(A) = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right\}$$

$c_1'$ and $c_2'$ cannot span $\mathbb{R}^3$. All b's may not be expressed as a linear combination of the basis. ($c_1'$, $c_2'$ and b can be linearly independent). The system may not be consistent.

4)

4)

$$A=\begin{bmatrix}1&0&1&-4\\2&0&3&-1\\2&0&4&6\end{bmatrix}\xrightarrow[(-2r_1+r_2\to r_2)]{E_1}\begin{bmatrix}1&0&1&-4\\0&0&1&7\\2&0&4&6\end{bmatrix}\xrightarrow[(-2r_1+r_3\to r_3)]{E_2}\begin{bmatrix}1&0&1&-4\\0&0&1&7\\0&0&2&14\end{bmatrix}$$

$$\xrightarrow[(-2r_2+r_4\to r_4)]{E_3}\underbrace{\begin{bmatrix}1&0&1&-4\\0&0&1&7\\0&0&0&0\end{bmatrix}}_{\text{REF}}\xrightarrow[(-r_2+r_1\to r_1)]{E_4}\begin{array}{c}r_1\\r_2\\\\\end{array}\begin{bmatrix}1&0&0&-11\\0&0&1&7\\0&0&0&0\end{bmatrix}$$

The columns with lead 1's corresponding to the original matrix A will be the basis vectors for the column space.

$$\text{Basis of } col(A)=\left\{\begin{bmatrix}1\\2\\2\end{bmatrix},\begin{bmatrix}1\\3\\4\end{bmatrix}\right\}$$

Row operations do not change the row space.
The matrix at the RREF has the same basis as those of A.

The nonzero rows of a matrix in RREF are linearly independent. ($r_1$ and $r_2$)

$$\text{Basis of } row(A)=\left\{[1,0,0,-11],[0,0,1,7]\right\}$$

5)

5) 1)

$$\begin{array}{c}\begin{bmatrix}1&2&3\\2&4&6\\1&0&-2\\0&2&5\end{bmatrix}\\ c_1' \; c_2'\end{array} \xrightarrow{-2r_1+r_2\to r_2} \begin{bmatrix}1&2&3\\0&0&0\\1&0&-2\\0&2&5\end{bmatrix} \xrightarrow{-r_1+r_3\to r_3} \begin{bmatrix}1&2&3\\0&0&0\\0&-2&-5\\0&2&5\end{bmatrix} \xrightarrow{r_2\leftrightarrow r_3} \begin{bmatrix}1&2&3\\0&-2&-5\\0&0&0\\0&2&5\end{bmatrix}$$

$$\xrightarrow{r_2+r_4\to r_4} \begin{bmatrix}1&2&3\\0&-2&-5\\0&0&0\\0&0&0\end{bmatrix} \xrightarrow{r_2+r_1\to r_1} \begin{bmatrix}1&0&-2\\0&-2&-5\\0&0&0\\0&0&0\end{bmatrix} \xrightarrow{-\frac{1}{2}r_2} \begin{array}{c}\begin{bmatrix}1&0&-2\\0&1&5/2\\0&0&0\\0&0&0\end{bmatrix}\\ c_1 \; c_2\end{array}$$

$c_1$ and $c_2$ are linearly independent $\rightarrow$ $c_1'$ and $c_2'$ are linearly independent.

5)
2)

$$\text{Basis of col}(A) = \left\{ \begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\4\\0\\2\end{bmatrix} \right\}$$

6)

6) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

corresponding to $c_1$

contains lead 1's

Basis of $col(A) = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad EA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Basis of $col(EA) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$col(A)$ and $col(EA)$ are different spaces since any vector in $col(EA)$ cannot be written as a linear combination of the basis of $col(A)$ and vice versa.

$$k\begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq a\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

7)

7)

$$A^T = \begin{array}{c} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -4 & 0 & 1 \\ 4 & 6 & -2 & 3 \end{bmatrix} \\ \begin{array}{cccc} c_1' & c_2' & c_3' & \phantom{0} \end{array} \end{array} \xrightarrow{\text{RREF}} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \\ \begin{array}{cccc} c_1 & c_2 & c_3 & \phantom{-1/2} \end{array} \end{array}$$

$c_1$, $c_2$ and $c_3$ are linearly independent

$\rightarrow$ $c_1'$, $c_2'$ and $c_3'$ are linearly independent

$$\text{Basis of } \mathrm{Col}(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$$

$$\text{Basis of } \mathrm{row}(A) = \left\{ [1,2,4], [2,-4,6], [1,0,-2] \right\}$$

8)

8)

$x = t\begin{bmatrix} 2 \\ 1 \\ 1/2 \end{bmatrix}$ Nullity $= 1$

The rank of matrices must be 2 (Nullity + Rank $= n$)

RREF of matrices is $\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ Basis of the row(A) $= \left\{ \begin{bmatrix} 1 & 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \right\}$

All 3x3 matrices:

$\begin{bmatrix} a & x & -4a-2x \\ b & y & -4b-2y \\ c & z & -4c-2z \end{bmatrix}$ $\rightarrow$ Every row must be written as a linear combination of the basis of the rowspace.

9)

9)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 2 & -1 & 3 & b_2 \\ -1 & 3 & 1 & b_3 \\ 0 & 2 & -1 & b_4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & (5b_1+2b_2-b_3)/10 \\ 0 & 1 & 0 & (b_1+b_3)/4 \\ 0 & 0 & 1 & (-5b_1+4b_2+3b_3)/20 \\ 0 & 0 & 0 & b_4 - 3b_1/4 + b_2/5 - 7b_3/20 \end{array}\right]$$

must be zero

$r_1$, $r_2$ and $r_3$ do not contain $b_4$ ✓

The only condition which makes the system consisted is

$b_4 - 3b_1/4 + b_2/5 - 7b_3/20 = 0.$

$$b_4 = \frac{3b_1}{4} - \frac{b_2}{5} + \frac{7b_3}{20}.$$

If the system is consistent → $x_1 = (5b_1 + 2b_2 - b_3)/10$

$x_2 = (b_1 + b_3)/4$

$x_3 = (-5b_1 + 4b_2 + 3b_3)/20$

Unique solution.

10)

10)

$$\underbrace{\#\text{lead } 1\text{'s}}_{\underbrace{\operatorname{rank}(A)}_{3}} + \underbrace{\#\text{free variables}}_{\underbrace{\operatorname{nullity}(A)}_{x}} = \underbrace{\#\text{columns}}_{6}$$

There are 3 free variable

rank(A) = 3 → there are three independent column vectors
→ three basis of the column space

Any $b_{3\times1}$ can be written as a linear combination of these three independed vectors.
→ Any b is in the column space of A
→ For any given b, we can guaranteed to have a solution to $Ax=b$

Dimension of the solution space → # free variables
→ 3

11)

11)

For 4x3 matrix;

# lead 1's + # free voiables = # columns = 3 → RREF

number of the basis of the column Space (dim (col.space))

number of the basis of the nullspace. (dim (Nullspace))

If the col. space a line through the origin → dim (col. space) = 1

" " nullspace " " " " " → dim (Nullspace) = 1

1 + 1 = 2 ≠ 3 (# columns)

→ Col. space and Nullspace cannot be a line through the origin at the same time.

For 2x4 matrix:

In RREF, If the matrix has 2 lead 1's and 2 free voiables, (# lead 1's + # free var. = # columns equation holds)

dim (col. space) is = 2 (Due to dim (Row space) = dim (col. space), dim (Row space) = 2)

dim (Nullspace) = 2

→ Column space, row space and nullspace can be a plane through the origin at the same time.

12)

12)

$$\begin{bmatrix}4&2\\t&1\\3&t\end{bmatrix} \xrightarrow[\frac{1}{4}r_1]{E_1} \begin{bmatrix}1&1/2\\t&1\\3&t\end{bmatrix} \xrightarrow[(-t)r_1+r_2\to r_2]{E_2} \begin{bmatrix}1&1/2\\0&1-t/2\\3&t\end{bmatrix} \xrightarrow[-3r_1+r_3\to r_3]{E_3} \begin{bmatrix}1&1/2\\0&1-t/2\\0&t-3/2\end{bmatrix}$$

$$(t\neq\tfrac{3}{2}) \to \left(\tfrac{t}{2t-3}\right) r_3+r_2\to r_2 \;\Big\downarrow E_4$$

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} \xleftarrow[(\frac{3}{2}-t)r_2+r_3\to r_3]{E_6} \begin{bmatrix}1&0\\0&1\\0&t-3/2\end{bmatrix} \xleftarrow[-\frac{r_2}{2}+r_1\to r_1]{E_5} \begin{bmatrix}1&1/2\\0&1\\0&t-3/2\end{bmatrix}$$

(If $t=\frac{3}{2}$ after $E_3$ $\to \begin{bmatrix}1&1/2\\0&1/4\\0&0\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ ✓

In any case, RREF of $\begin{bmatrix}4&2\\t&1\\3&t\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ for all $t$ values.

Basis of the row space of $A \to \{[1,0],[0,1]\}$

" " " column space of $A \to \left\{\begin{bmatrix}4\\t\\3\end{bmatrix},\begin{bmatrix}2\\1\\t\end{bmatrix}\right\}$

Basis of the solution space of $A \to \{\ \}$ No free variable.

The solution space and the row space are the same for all $t$ values,

but the column space will be different due to containing $t$ variable in the basis.

13)

13)

a)

$\det(\lambda I - A) = 0$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{vmatrix} = 0 \rightarrow (\lambda-1)^2 - 9 = 0 \rightarrow \text{characteristic equation} ✓$$

$$\lambda^2 - 2\lambda - 8 = 0$$

$$\left.\boxed{\begin{matrix} \lambda = 4 \\ \lambda = -2 \end{matrix}}\right) \text{ eigenvalues } ✓$$

$$\begin{bmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{for } \lambda = 4 \rightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ -3 & 3 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 = t$

for $\lambda = 4$, eigenvector is $\begin{bmatrix} t \\ t \end{bmatrix}$

$$\text{for } \lambda = -2 \longrightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ -3 & -3 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$y_1 = -t$

for $\lambda = -2$, eigen vector is $\begin{bmatrix} -t \\ t \end{bmatrix}$

13)

a)

for $\lambda=4$

$x = t\begin{bmatrix}1\\1\end{bmatrix} \rightarrow \begin{bmatrix}1\\1\end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda=4$

for $\lambda=-2$

$x = t\begin{bmatrix}-1\\1\end{bmatrix} \rightarrow \begin{bmatrix}-1\\1\end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda=-2$

for $\lambda=4$

$$\lambda I - A = \begin{bmatrix}3 & -3\\-3 & 3\end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \rightarrow$$ The rank is 1

for $\lambda=-2$

$$\lambda I - A = \begin{bmatrix}-3 & -3\\-3 & -3\end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} \rightarrow$$ The rank is 1

13)

b)

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda & 0 & -1 \\ 1 & (\lambda-1) & 1 \\ -1 & 1 & \lambda \end{vmatrix} = 0 \rightarrow \lambda \begin{vmatrix} \lambda-1 & 1 \\ 1 & \lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & \lambda-1 \\ -1 & 1 \end{vmatrix}$$

$$\lambda(\lambda^2 - \lambda - 1) - 1(1 + \lambda - 1)$$

$$\lambda^3 - \lambda^2 - \lambda - \lambda = 0$$

$$\lambda^3 - \lambda^2 - 2\lambda = 0 \rightarrow \text{characteristic equation}$$

$$\lambda(\lambda - 2).(\lambda + 1) = 0$$

$\lambda = 0$
$\lambda = 2$
$\lambda = -1$
) Eigenvalues

for $\lambda = 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{Rank} = 2$$

$x_3 = 0$
$x_2 = t$
$x_1 = t$

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \rightarrow \text{eignvector.}$$

$$x = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ forms a basis for eigenspace corresponding to $\lambda = 0$

(3) b) for $\lambda = 2$

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{Rank} = 2$$

$x_3 = t$

$x_2 = -3t/2$

$x_1 = t/2$

$$\begin{bmatrix} t/2 \\ -3t/2 \\ t \end{bmatrix} \rightarrow \text{Eigenvector}$$

$$x = t\begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix}$$ forms a basis for eigenspace corresponding to $\lambda = 2$

for $\lambda = -1$

$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{Rank} = 2$$

$x_2 = 0$

$x_3 = t$

$x_1 = -t$

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \rightarrow \text{Eigenvector}$$

$$x = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$ forms a basis for eigenspace corresponding to $\lambda = -1$

13)
c)

$$\begin{vmatrix} (\lambda-1) & 1 & 2 \\ 1 & (\lambda-2) & -5 \\ 0 & -1 & \lambda-3 \end{vmatrix} = (\lambda-1)\begin{vmatrix} (\lambda-2) & -5 \\ -1 & (\lambda-3) \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ -1 & \lambda-3 \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda^2-5\lambda+1) - (\lambda-1)$$

$$\lambda^3-5\lambda^2+\lambda-\lambda^2+5\lambda-1-\lambda+1=0$$

$$\lambda^3-6\lambda^2+5\lambda=0 \rightarrow \text{characteristic equation}$$ ✓

$$\lambda(\lambda-5).(\lambda-1)=0$$

$$\left.\begin{matrix} \lambda=0 \\ \lambda=5 \\ \lambda=1 \end{matrix}\right) \text{Eigenvalues}$$ ✓

for $\lambda=0$,

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 1 & -2 & -5 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \underline{\text{Rank}=2}$$

$$x_3=t$$
$$x_2=-3t$$
$$x_1=-t$$

$$\begin{bmatrix} -t \\ -3t \\ t \end{bmatrix} \rightarrow \underline{\text{Eigenvector}}$$

$$x=t\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

↳ forms a basis for eigenspace corresponding to $\underline{\lambda=0}$

13)

c) for $\lambda = 5$

$$\left[\begin{array}{ccc|c} 4 & 1 & 2 & 0 \\ 1 & 3 & -5 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{Rank} = 2$$

$x_3 = t$

$x_2 = 2t$

$x_1 = -t$

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} \rightarrow \text{Eigenvector}$$

$$x = t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

→ forms a basis for eigenspace corresponding to $\lambda = 5$

for $\lambda = 1$,

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 1 & -1 & -5 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{Rank} = 2$$

$x_3 = t$

$x_2 = -2t$

$x_1 = 3t$

$$\begin{bmatrix} 3t \\ -2t \\ t \end{bmatrix} \rightarrow \text{Eigenvector}$$

$$x = t\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

→ forms a basis for eigenspace corresponding to $\lambda = 1$

14)

14)

$\lambda^k$ is the eigenvalue of $A^k$ and x is the eigenvector.

Eigenvalues of $A \rightarrow 0, 2, -1$

" " $A^5 \rightarrow 0, 2^5, (-1)^5 \rightarrow 0, 32, -1$

Eigenvectors of $A \rightarrow \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}, \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix}, \begin{bmatrix} p/2 \\ -3p/2 \\ p \end{bmatrix} \rightarrow$ Eigenvectors of $A^5$

for $\lambda = 0$ for $\lambda = -1$ for $\lambda = 32$

for $\lambda = 0$

$x = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda = -1$

$x = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 32$

$x = p\begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix}$

There are three eigenspaces ✓

for $\lambda = 0$,

Eigenspace $= \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\} \rightarrow$ dimension = 1 $\rightarrow$ basis $= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda = -1$

Eigenspace $= \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} \rightarrow$ dimension = 1 $\rightarrow$ basis $= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

for $\lambda = 32$

Eigenspace $= \text{span}\left\{\begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix}\right\} \rightarrow$ dimension = 1 $\rightarrow$ basis $= \begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix}$

15)

15) If a matrix $A_{n \times n}$ does not have n distinct eigenvalues (some of them are the same), the matrix is not diagonalizable.

Let $A_{n\times n}$ is any triangular matrix:

$$A_{n\times n} = \begin{bmatrix} n & a_{12} & a_{13} \cdots a_{1n} & & \\ 0 & n & a_{23} \cdots a_{2n} & & \\ 0 & & m & & \\ \vdots & 0 & & p & \\ 0 & 0 & 0 & & q \end{bmatrix}$$

→ The matrix is not diagonalizable ($a_{11}$ and $a_{22}$ are the same)

dimension of $A_{n\times n} = n \times n$

$\#$ eigenvalues $= n-1$

$A_{n\times n}$ is not diagonalizable.

16)

16)

a) For $n \times n$ matrix $\rightarrow$ degree of the characteristic polynomial $= n$

dimension of a matrix $= n \times n = 2 \times 2$ ✓

b)

$\det(\lambda I - A) = P(\lambda) = \lambda^2 - 9$

$\lambda = 0 \rightarrow \det(-A) = -9$

$\det(-A) = (-1)^n . \det(A)$ $n \rightarrow 2$

$\det(-A) = -9 = 1. \det(A)$

$\det(A) = -9$ ✓

c)

$Ax = \lambda x$

$A^{-1}Ax = \lambda A^{-1}x$

$x = \lambda A^{-1}x$

$\frac{1}{\lambda}x = A^{-1}x$ $\rightarrow$ Eigenvalues of $A \rightarrow 3, -3$

" " $A^{-1} \rightarrow 1/3, -1/3$

$\det(A^{-1}) = -1/9$

Characteristic polynomial $= a\lambda^2 + b\lambda - \frac{1}{9}$

$\frac{-1}{9a} = \frac{-1}{9} \rightarrow a = 1$

$-b = 0 \quad b = 0$

" " $= \lambda^2 - \frac{1}{9} = \det(\lambda I - A^{-1})$

16)

d)

$$(\lambda I - A)^T = (\lambda I - A^T)$$

$\lambda I$ is a symmetric matrix. $(\lambda I = (\lambda I)^T)$

$$\det(\lambda I - A) = \det(\lambda I - A^T) = \lambda^2 - 9$$

17)

17) (1)

$$A=\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow$$ The matrix has distinct eigenvalues (different main diagonal entries) (the matrix is diagonalizable)

$$\begin{bmatrix} \lambda-1 & -1 & -2 \\ 0 & \lambda-3 & 4 \\ 0 & 0 & \lambda+1 \end{bmatrix}$$

Eigenvalues $\rightarrow$ $\lambda=1$, $\lambda=3$, $\lambda=-1$ (triangular matrix property)

for $\lambda=1$,

$$\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x_1&=t \\ x_2&=0 \\ x_3&=0 \end{aligned}$$

$$x=t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda=3$

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x_1&=t/2 \\ x_2&=t \\ x_3&=0 \end{aligned}$$

$$x=t\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda=-1$

$$\left[\begin{array}{ccc|c} -2 & -1 & -2 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x_1&=-3t/2 \\ x_2&=t \\ x_3&=t \end{aligned}$$

$$x=t\begin{bmatrix} -3/2 \\ 1 \\ 1 \end{bmatrix}$$

(7) (2)

$$P = \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \checkmark \qquad P^{-1} = \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = P\Lambda P^{-1}$$

$$P^{-1}A = \Lambda P^{-1}$$

$$P^{-1}AP = \Lambda$$

$$\begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \text{Eigen decomposition}$$

The matrix is not symmetric. (P is not orthogonal)

$A = P\Lambda P^{-1} \neq P\Lambda P^{T} \rightarrow$ A cannot be written as a linear combination of rank1 matrices formed from its eigenvectors.

*18)*

18) (1)

$$\begin{vmatrix} \lambda-3 & -1 & 0 \\ -1 & \lambda-2 & -1 \\ 0 & -1 & \lambda-3 \end{vmatrix} = 0 \qquad (\lambda-3)\begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-3 \end{vmatrix} + 1\begin{vmatrix} -1 & 0 \\ -1 & \lambda-3 \end{vmatrix}$$

$$(\lambda-3)(\lambda^2-5\lambda+5) + 3-\lambda$$

$$\lambda^3 - 5\lambda^2 + 5\lambda - 3\lambda^2 + 15\lambda - 15 + 3 - \lambda$$

$$\lambda^3 - 8\lambda^2 + \underbrace{19\lambda - 12}_{7\lambda + 12\lambda} = 0$$

$$\lambda(\lambda-1).(\lambda-7) + 12(\lambda-1)$$

$$(\lambda-1)(\lambda(\lambda-7)+12)$$

$$(\lambda-1)(\lambda-3).(\lambda-4) = 0$$

$\lambda = 1$, $\lambda = 3$, $\lambda = 4$: The matrix has three distinct eigenvalues, the matrix is diagonalizable.

for $\lambda = 1$

$$\left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = t \\ x_2 = -2t \\ x_3 = t \end{array}$$

$$x = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

18)(2) for $\lambda = 3$

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -t \\ x_2 = 0 \\ x_3 = t \end{array}$$

$$x = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda = 4$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = t \\ x_2 = t \\ x_3 = t \end{array}$$

$$x = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Orthogonally diagonalize:

Eigenvectors: $\underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{u_3}$

$v_1 = u_1$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$v_3 = u_3 - \left( \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} \cdot v_1 + \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} \cdot v_2 \right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left( 0 + 0 \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

18) (3)

$$w_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \qquad w_3 = \frac{v_3}{\|v_3\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$w_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$P^{-1}AP = \Lambda \qquad \underline{\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$$

The matrix is symmetric, that means,

$$A = P\Lambda P^{-1} = P\Lambda P^T$$

$$P = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \qquad P^T = \begin{bmatrix} c_1^T \\ c_2^T \\ c_3^T \end{bmatrix}$$

$$A = \underset{1\times 3}{\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} c_1^T \\ c_2^T \\ c_3^T \end{bmatrix}}$$

$$A = \underline{c_1 c_1^T + 3 c_2 c_2^T + 4 c_3 c_3^T} \rightarrow$$ A can be written as a linear comb. of rank 1 matrices.

19)

19)

A matrix is invertible iff $\lambda = 0$ is not an eigenvalue of the matrix and vice versa.

$$\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} \rightarrow \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$$

the matrix is not invertible.
the matrix has $\lambda = 0$ eigenvalue.

20)

20)

1)

$u_1 = (1, -1, 0)$

$u_2 = (-1, 2, 1)$

$u_3 = (-1, 3, 2)$

$u_4 = (0, 1, 3)$

$$v_1 = u_1$$

$$v_2 = u_2 - \left( \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 \right) = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$$v_3 = u_3 - \left( \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 \right)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \text{skipped}$$

$$v_4 = u_4 - \left( \frac{\langle u_4, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u_4, v_2 \rangle}{\|v_2\|^2} v_2 \right)$$

$$= \begin{bmatrix} -2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \quad q_4 = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Orthonormal basis set for rowspace $= \left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$

20)
(2)

$$A^T = Q.R \rightarrow A = R^T Q^T$$

$$A^T = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Applying Gram-Schmidt to column vectors:

$$c1 = v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$v_2 = c_2 - \left( \frac{\langle c_2, v_1 \rangle}{\|v_1\|^2} v_1 \right) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -3/2 \\ 3/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$$v_3 = c_3 - \left( \frac{\langle c_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle c_3, v_2 \rangle}{\|v_2\|^2} v_2 \right) = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_4 = c_4 - \left( \frac{\langle c_4, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle c_4, v_2 \rangle}{\|v_2\|^2} v_2 \right) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 4/6 \\ 10/6 \\ 14/6 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix} \quad \underbrace{q_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \quad q_4 = \begin{bmatrix} -\sqrt{3}/3 \\ -\sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$$

$q_3$ is not orthonormal, although it have to be used to find ($A = R^T Q^T$)

20)

(3)

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}_{3\times 4}}_{A^T} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & \sqrt{6}/6 & 0 & -\sqrt{3}/3 \\ -1/\sqrt{2} & \sqrt{6}/6 & 0 & -\sqrt{3}/3 \\ 0 & \sqrt{6}/3 & 0 & \sqrt{3}/3 \end{bmatrix}_{3\times 4}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & -3/\sqrt{2} & \frac{-2}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \sqrt{6}/2 & \sqrt{6} & 7/\sqrt{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/\sqrt{3} \end{bmatrix}}_{R}$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ -3/\sqrt{2} & \sqrt{6}/2 & 0 & 0 \\ -2/\sqrt{2} & \sqrt{6} & 0 & 0 \\ -1/\sqrt{2} & 7/\sqrt{6} & 0 & 2/\sqrt{3} \end{bmatrix}}_{R^T} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ \sqrt{6}/6 & \sqrt{6}/6 & \sqrt{6}/3 \\ 0 & 0 & 0 \\ -\sqrt{3}/3 & -\sqrt{3}/3 & \sqrt{3}/3 \end{bmatrix}}_{Q^T}$$

is $Q^T$ orthogonal matrix?

21)

21)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Least Square Sol. : $(A^TA)^{-1}A^Tb = x$

$$\left( \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = x$$

$$= \begin{bmatrix} 2/3 \\ 0 \\ 0 \end{bmatrix}$$

Best Approx. :

$$\underset{4\times3}{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 2/3 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \\ 0 \end{bmatrix}$$