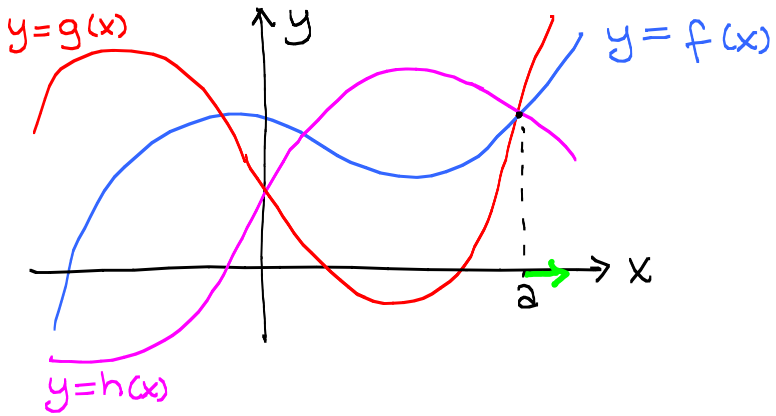
14.5

Directional Derivatives and Gradient Vectors

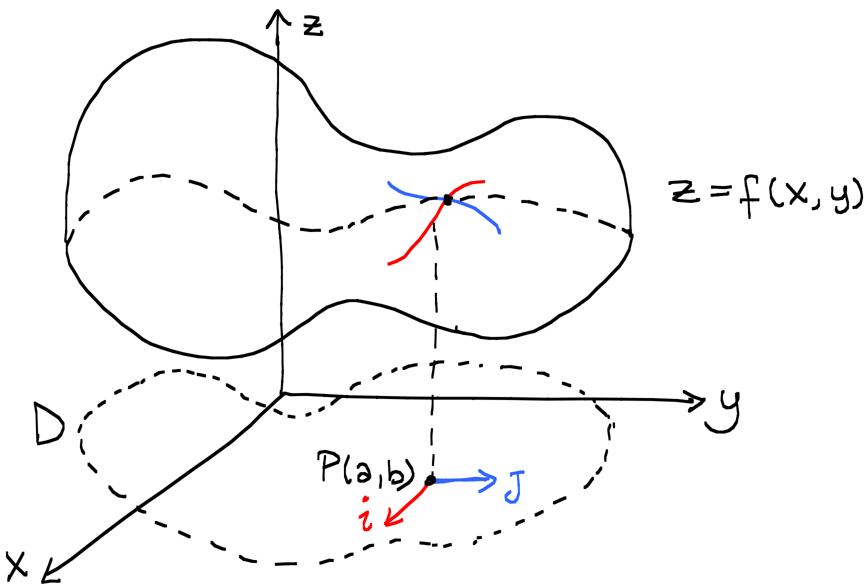
Let $y=f(x)$ be a single variable function on a domain $D \subset \mathbb{R}$ with the following graph. For any $a \in D$:



$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

is the rate of change of f with respect to x (as x increases).

Let $z=f(x,y)$ be a differentiable function of two variables on a domain $D \subset \mathbb{R}^2$. For any point $P(a,b) \in D$:



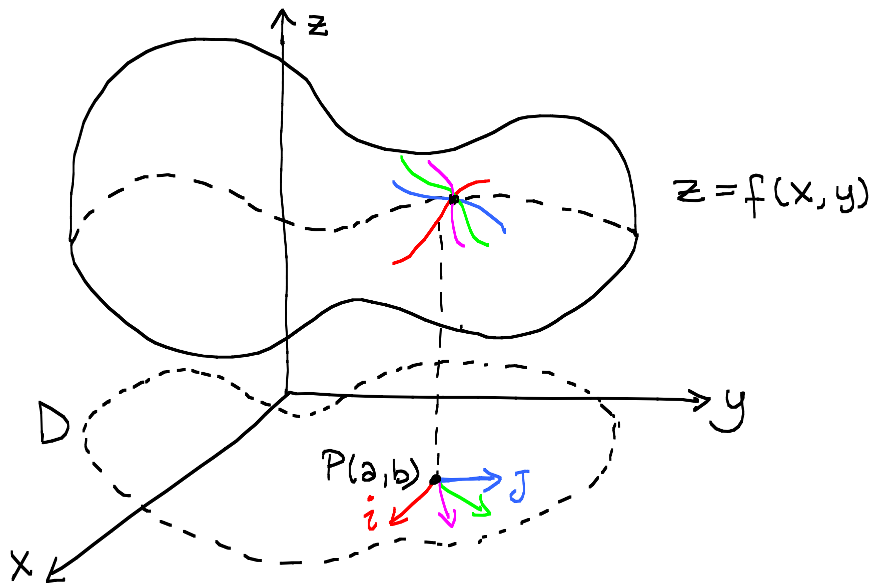
$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

rate of change of f w.r.to x at $P(a,b)$
(in the direction of i)

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

rate of change of f w.r.to y at $P(a,b)$
(in the direction of j)

What about the other directions at $P(a,b)$?



We will generalize the concept of derivative of multivariable functions to any direction.

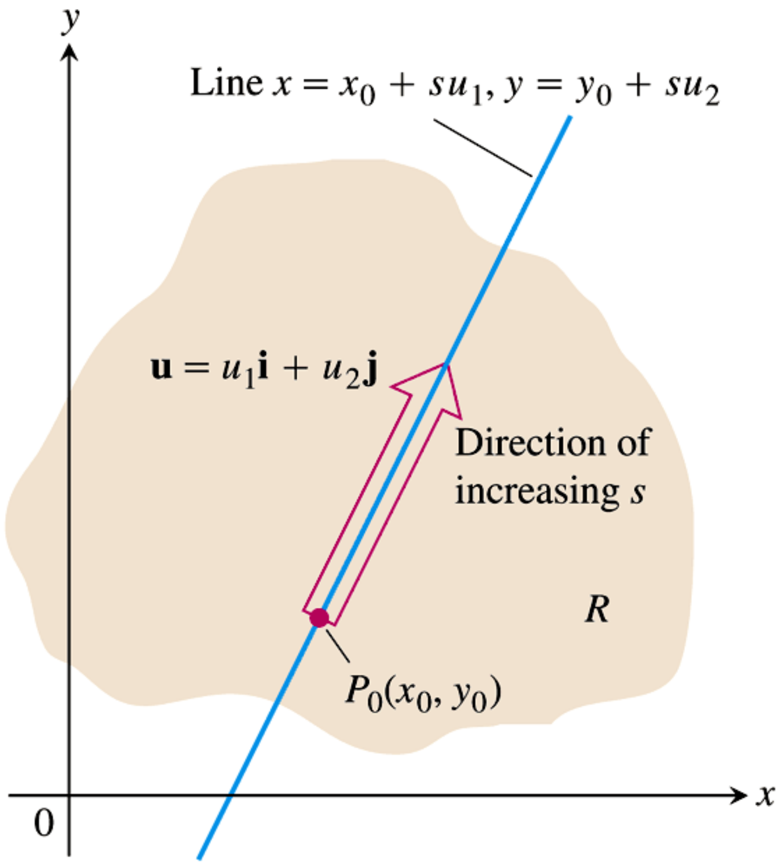


FIGURE 14.24 The rate of change of f in the direction of $\mathbf{u}$ at a point P_0 is the rate at which f changes along this line at P_0.

R: Domain of f,

$P_0(x_0, y_0) \in R$,

$u = u_1 i + u_2 j$ is a unit vector,

s: arclength parameter

DEFINITION Directional Derivative

The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$(D_u f)_{P_0} = \left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \tag{1}$$

provided the limit exists.

$$(D_i f)_{P_0} = f_x(P_0) \quad \& \quad (D_j f)_{P_0} = f_y(P_0)$$

EXAMPLE 1 Finding a Directional Derivative Using the Definition

Find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = \left(1/\sqrt{2}\right)\mathbf{i} + \left(1/\sqrt{2}\right)\mathbf{j}$.

Solution

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \qquad \text{Equation (1)}$$

$$= \lim_{s\to 0} \frac{f\left(1 + s\cdot\frac{1}{\sqrt{2}}, 2 + s\cdot\frac{1}{\sqrt{2}}\right) - f(1, 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1\cdot 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s}$$

$$= \lim_{s\to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s\to 0}\left(\frac{5}{\sqrt{2}} + s\right) = \left(\frac{5}{\sqrt{2}} + 0\right) = \frac{5}{\sqrt{2}}.$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction $\mathbf{u} = \left(1/\sqrt{2}\right)\mathbf{i} + \left(1/\sqrt{2}\right)\mathbf{j}$ is $5/\sqrt{2}$. ■

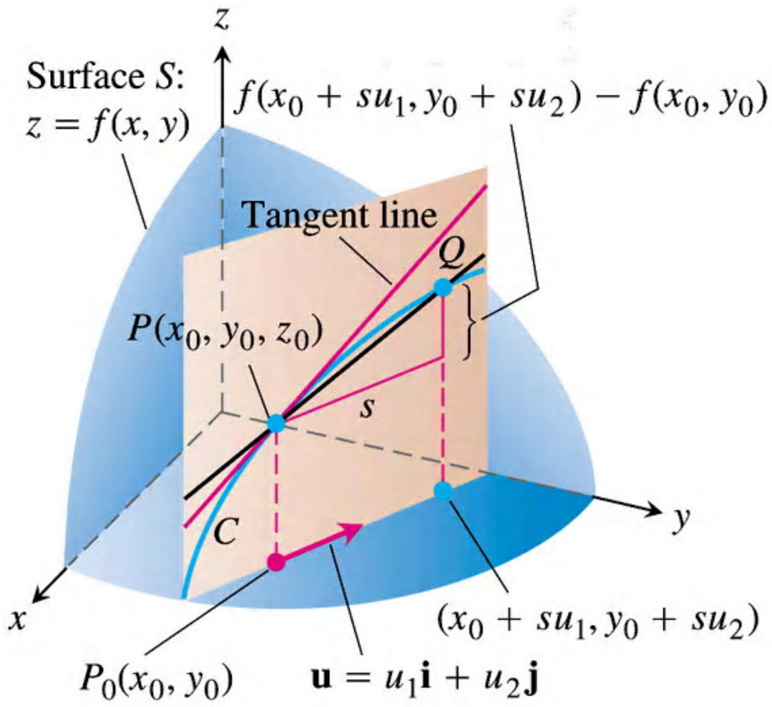


FIGURE 14.25 The slope of curve C at P_0 is $\lim_{Q \to P}$ slope (PQ); this is the directional derivative

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (D_{\mathbf{u}}f)_{P_0}.$$

The curve C is the intersection of the surface $z=f(x,y)$ with the plane through $P_0(x_0,y_0)$, parallel to $u=u_1 i+u_2 j$, perpendicular to the xy-plane.

DEFINITION **Gradient Vector**

The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

Let $x = x_0 + su_1$, $y = y_0 + su_2$ and s be the arclength parameter, then by the Chain Rule:

$$\left(\frac{df}{ds}\right)_{u, P_0} = \frac{d}{ds} f(x_0 + su_1, y_0 + su_2) = \left(\frac{\partial f}{\partial x}\right)_{P_0} \cdot \frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \cdot \frac{dy}{ds} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \cdot u_1 + \left(\frac{\partial f}{\partial y}\right) \cdot u_2$$

$$= \left[\left(\frac{\partial f}{\partial x}\right)_{P_0} i + \left(\frac{\partial f}{\partial y}\right)_{P_0} j\right] \cdot (u_1 i + u_2 j) = (\nabla f)_{P_0} \cdot u$$

So we proved the following theorem:

THEOREM 9 The Directional Derivative Is a Dot Product

If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient f at P_0 and $\mathbf{u}$.

Just like every vector being a linear combination of i and j, every derivative is a linear combination of f_x and f_y:

$$(D_u f)_{P_0} = (\nabla f)_{P_0} \cdot u = \langle f_x(P_0), f_y(P_0) \rangle \cdot \langle u_1, u_2 \rangle = u_1 f_x(P_0) + u_2 f_y(P_0)$$

EXAMPLE 2 Finding the Directional Derivative Using the Gradient

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution The direction of $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.26). The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} \qquad \text{Equation (4)}$$

$$= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1 .$$

■

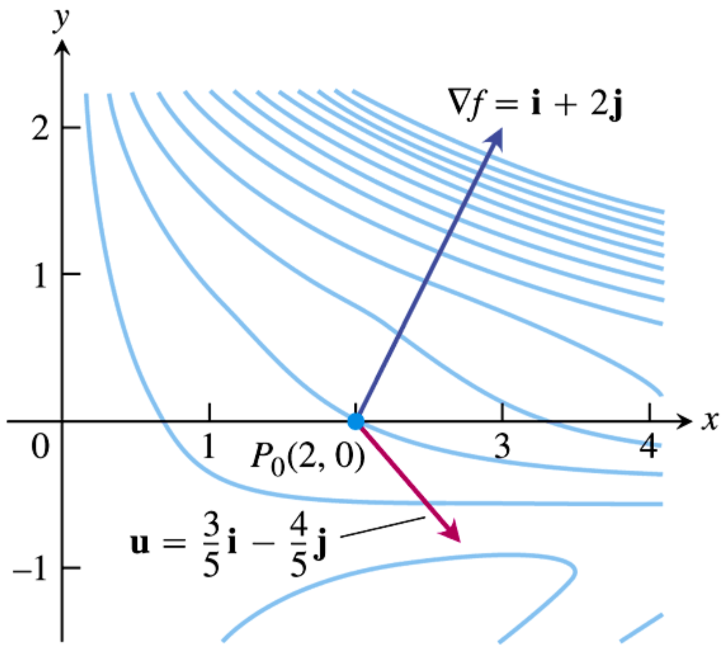


FIGURE 14.26 Picture ∇f as a vector in the domain of f. In the case of $f(x, y) = xe^y + \cos(xy)$, the domain is the entire plane. The rate at which f changes at $(2, 0)$ in the direction $\mathbf{u} = (3/5)\mathbf{i} - (4/5)\mathbf{j}$ is $\nabla f \cdot \mathbf{u} = -1$ (Example 2).

u is a unit vector so $|u| = 1$. θ is the angle between ∇f and u.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos \theta$

1. The function f increases most rapidly when $\cos \theta = 1$ or when $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f| \cos (0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos (\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f| \cos (\pi/2) = |\nabla f| \cdot 0 = 0.$$

Example: Find the directions in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

a) increases most rapidly at $P(1,1)$,

b) decreases most rapidly at $P(1,1)$,

c) has zero change at $P(1,1)$.

Solution: $f_x = x$, $f_y = y$, $\nabla f = x\,i + y\,j \Rightarrow (\nabla f)_{(1,1)} = i + j$

$|(\nabla f)_{(1,1)}| = \sqrt{2} \Rightarrow u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$ is the unit vector in the direction of $(\nabla f)_{(1,1)}$

a) $u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

b) $-u = -\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$

c) $n = -\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$ and $-n = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$ since $\nabla f \cdot n = 0 = \nabla f \cdot (-n)$

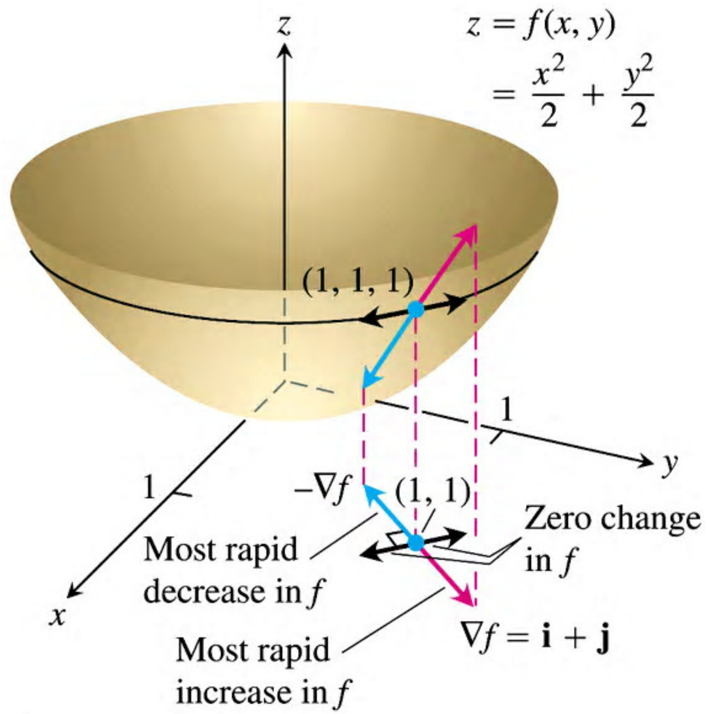


FIGURE 14.27 The direction in which $f(x, y) = (x^2/2) + (y^2/2)$ increases most rapidly at (1, 1) is the direction of $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at (1, 1, 1) (Example 3).

Let $f(x,y)$ be a differentiable function that has a constant value c along a smooth curve $r(t) = g(t)\,i + h(t)\,j$. Then $r(t)$ is a level curve of f at level c:

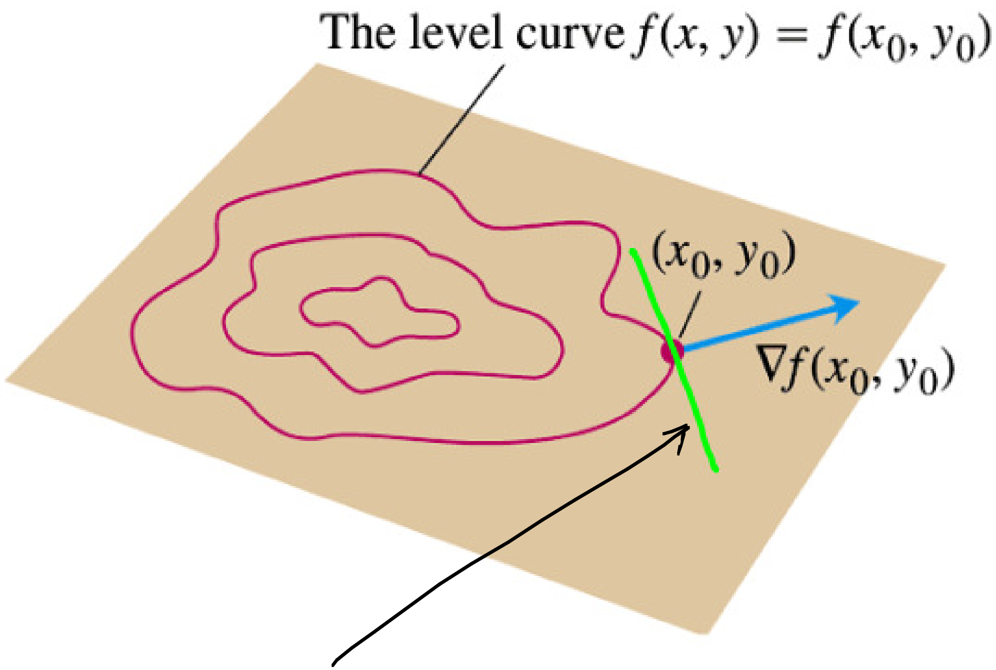
$$f(g(t), h(t)) = c \quad \Rightarrow \frac{d}{dt}\Big(f(g(t), h(t))\Big) = \frac{d}{dt}(c) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} \cdot \frac{dg}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dh}{dt} = 0 \quad \text{(by the Chain Rule)}$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j\right) \cdot \underbrace{\left(\frac{dg}{dt}\, i + \frac{dh}{dt}\, j\right)}_{\frac{dr}{dt}} = 0$$

$\Rightarrow \nabla f \cdot \dfrac{dr}{dt} = 0 \Rightarrow \nabla f$ is normal (orthogonal) to the tangent vector $\dfrac{dr}{dt}$ so it is normal to the level curve.

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure 14.28).



An equation of the tangent line to the curve $f(x,y) = f(x_0, y_0)$ at $P(x_0, y_0)$

$$\left((x-x_0)\,i + (y-y_0)\,j\right) \cdot \left(\nabla f(x_0, y_0)\right) = 0$$

Example: Find an equation for the tangent line to the ellipse $\frac{x^2}{4}+y^2=2$ at the point $P(-2,1)$.

Solution: The ellipse is a level curve of $f(x,y)=\frac{x^2}{4}+y^2$ at the level $f(-2,1)=2$.

$$f_x=\frac{x}{2},\quad f_y=2y,\quad \nabla f=\frac{x}{2}i+2yj \Rightarrow (\nabla f)_{(-2,1)}=-i+2j$$

An equation of the tangent line:

$$(-i+2j)\cdot\left((x-(-2))i+(y-1)j\right)=0$$

$$\Rightarrow (-1)(x+2)+2(y-1)=0 \quad \Rightarrow \quad x-2y=-4$$

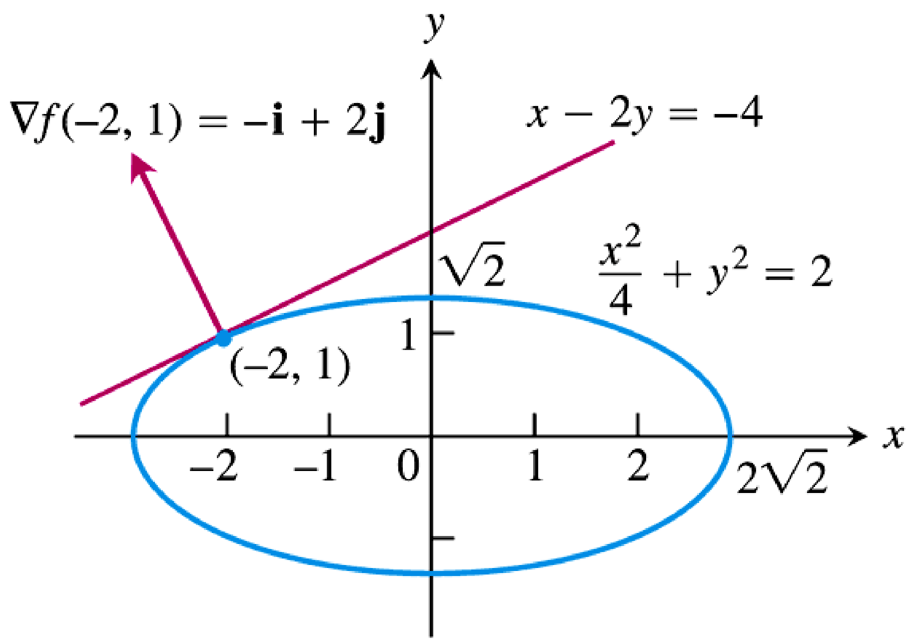


FIGURE 14.29 We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$ (Example 4).

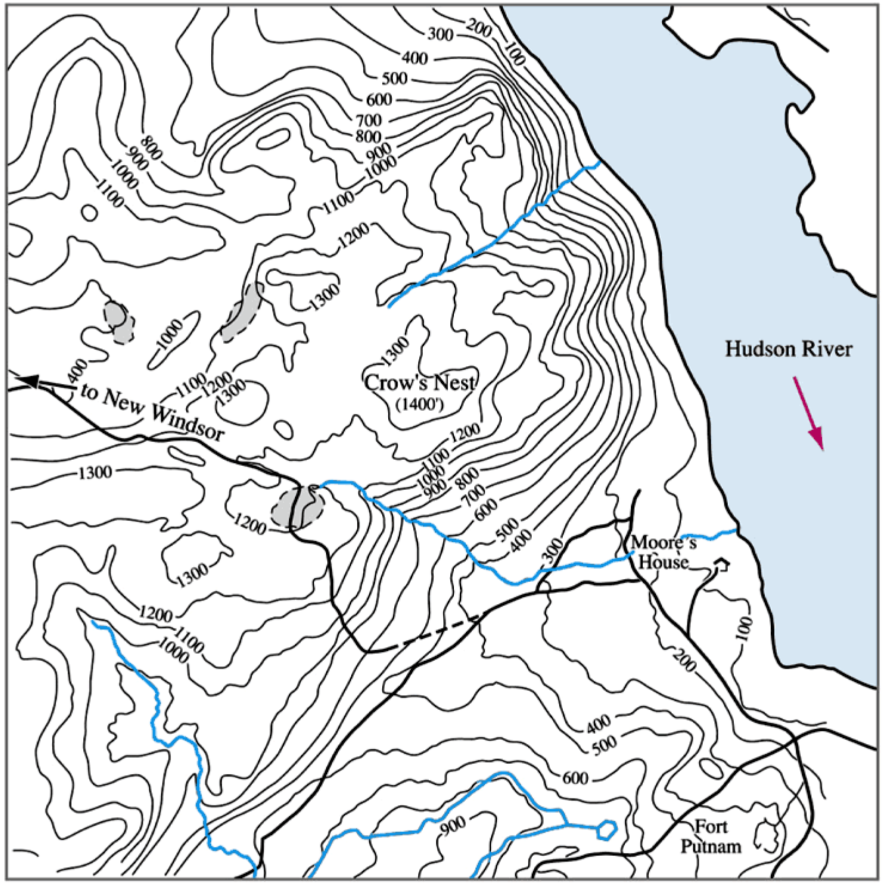


FIGURE 14.23 Contours of the West Point Area in New York show streams, which follow paths of steepest descent, running perpendicular to the contours.

Algebra Rules for Gradients

1. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
2. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
3. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

Functions of Three Variables

For a differentiable function $f(x,y,z)$ and a unit vector $u = u_1 i + u_2 j + u_3 k$ in $\mathbb{R}^3$, we have

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$D_u f = \nabla f \cdot u = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 + \frac{\partial f}{\partial z} u_3$$

$$D_u f = \nabla f \cdot u = |\nabla f|\,|u| \cos\theta = |\nabla f| \cos\theta$$

f increases most rapidly in the direction of ∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f, the derivative (rate of change) is zero.

Example: a) Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P(1,1,0)$ in the direction of $v = 2i - 3j + 6k$. b) In what directions does f change most rapidly at P, and what are the rates of change?

<u>Solution:</u> $|v| = \sqrt{2^2 + (-3)^2 + 6^2} = 7 \Rightarrow u = \frac{v}{|v|} = \frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k$ is the unit vector in the direction of v.

$f_x = 3x^2 - y^2$, $f_y = -2xy$, $f_z = -1 \Rightarrow \nabla f = (3x^2 - y^2)i + (-2xy)j - k$

$(\nabla f)_P = 2i - 2j - k$

$(D_u f)_P = (\nabla f)_P \cdot u = (2i - 2j - k) \cdot \left(\frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k\right) = \frac{4}{7}$

f increases most rapidly in the direction of $(\nabla f)_P$ with the rate:

$|(\nabla f)_P| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

f decreases most rapidly in the direction of $-(\nabla f)_P$ with the rate:

$-|(\nabla f)_P| = -3$

Exercise: 1) Find the directional derivative of the following functions at the given point in the given direction. Find the direction and the rate of maximum increase and decrease.

a) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + \frac{x}{y^2}$, $P(1,1)$, $v = 3i - 2j$

b) $f(x,y,z) = \cos(xy) + e^{yz} + \ln(xz)$, $P(1, 0, \frac{1}{2})$, $v = i + 2j + 2k$

2) If the temperature at $P(x,y,z)$ is given by

$$T(x,y,z) = 85 + \left(1 - \frac{z}{100}\right) e^{-(x^2+y^2)},$$

find the direction from the point $(2,0,99)$ in which the temperature increases most rapidly.

14.6

Tangent Planes and Differentials

Let $r = x(t)i + y(t)j + z(t)k$ be a smooth curve on the level surface $f(x,y,z) = C$ of a differentiable function $f(x,y,z)$.

Then $f(x(t), y(t), z(t)) = C$. Differentiating both sides w.r.to t:

$$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{d}{dt}(C) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = 0 \quad \text{by the Chain Rule}$$

$$\Rightarrow \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \right) \cdot \left(\frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k \right) = \nabla f \cdot \frac{dr}{dt} = 0$$

Thus, at every point along the curve, ∇f is orthogonal to the curve's velocity vector.

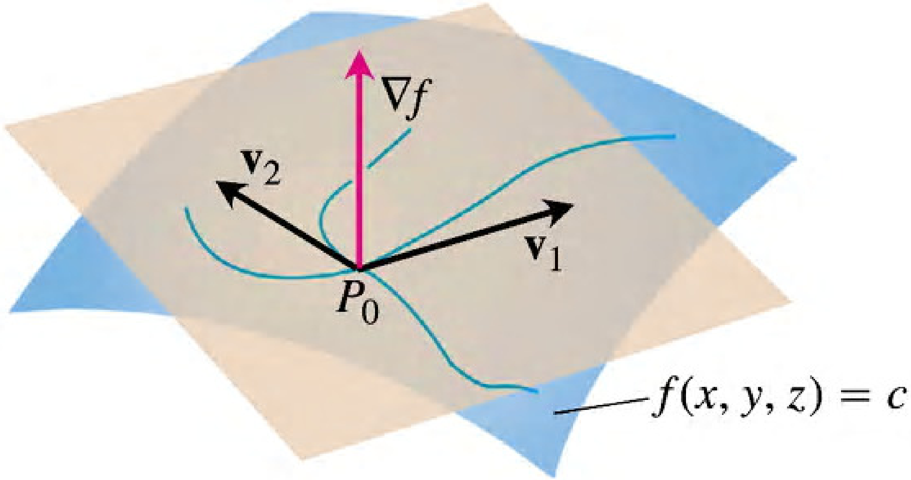


FIGURE 14.30 The gradient ∇f is orthogonal to the velocity vector of every smooth curve in the surface through P_0. The velocity vectors at P_0 therefore lie in a common plane, which we call the tangent plane at P_0.

DEFINITIONS Tangent Plane, Normal Line

The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \qquad (2)$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t \qquad (3)$$

Example: Find the tangent plane and normal line of the surface $z = 9 - x^2 - y^2$ (circular paraboloid) at the point $P_0(1,2,4)$.

Solution: The surface $z = 9 - x^2 - y^2$ is the level surface of the funct.

$f(x,y,z) = x^2 + y^2 + z - 9$ at level 0.

$$\nabla f|_{P_0} = (2xi + 2yj + k)\Big|_{(1,2,4)} = 2i + 4j + k$$

An equation of the tangent plane:

$$2(x-1) + 4(y-2) + 1.(z-4) = 0 \iff 2x + 4y + z = 14$$

An equation of the normal line:

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t, \quad t \in \mathbb{R}$$

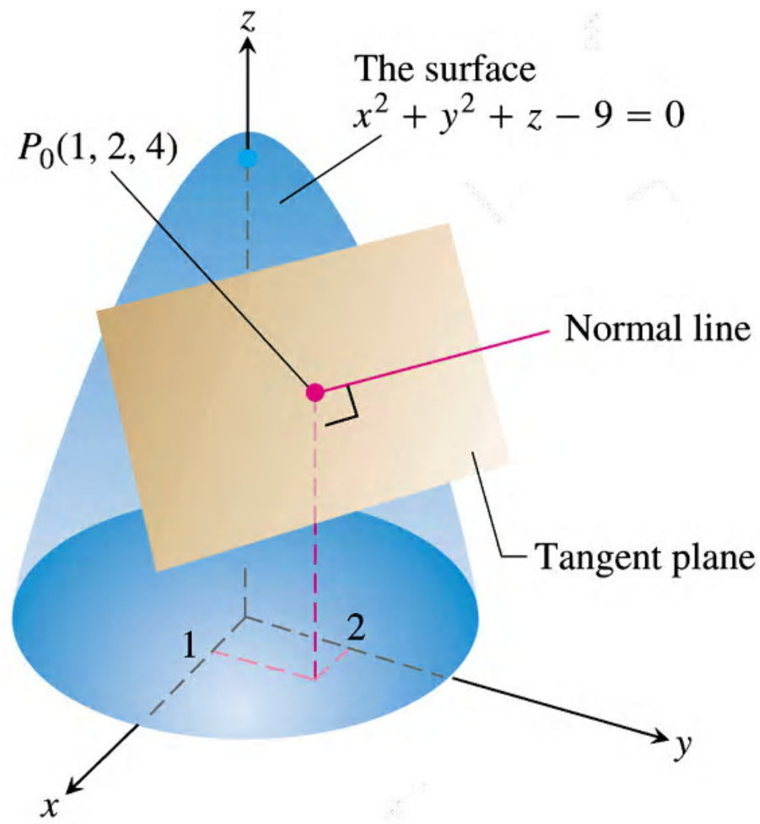


FIGURE 14.31 The tangent plane and normal line to the surface $x^2 + y^2 + z - 9 = 0$ at $P_0(1, 2, 4)$ (Example 1).

The surface $z=f(x,y)$ is the level surface of $g(x,y,z)=f(x,y)-z$ at level 0. $\nabla g = f_x \mathbf{i} + f_y \mathbf{j} - 1.\mathbf{k}$

Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0. \tag{4}$$

Example: Find the plane tangent to the surface $z = x.\cos y - ye^{x}$ at $P(0,0,0)$.

<u>Solution:</u> $z = f(x,y) = x\cos y - ye^{x}$

$f_x(0,0) = (\cos y - ye^{x})\big|_{(0,0)} = 1 - 0.1 = 1$

$f_y(0,0) = (-x\sin y - e^{x})\big|_{(0,0)} = 0 - 1 = -1$

$1.(x-0) - 1(y-0) .-1(z-0) = 0 \iff x - y - z = 0$: eqn. of tg. plane at $(0,0,0)$

Example: At what points on the paraboloid $y=x^2+z^2$, is the tangent plane parallel to the plane $x+2y+3z=1$.

<u>Solution:</u> The paraboloid $y=x^2+z^2$ is the level surface of the function $f(x,y,z)=y-x^2-z^2$ at level 0.

$\nabla f=-2xi+j-2zk$. ∇f is normal to the tangent plane of the paraboloid at the point of tangency. Assume that the tangent plane at $P_0(x_0,y_0,z_0)$ is parallel to $x+2y+3z=1$.

A normal vector to $x+2y+3z=1$ is $n=i+2j+3k$.

So $\nabla f|_{P_0}$ is parallel to n. So $\nabla f|_{P_0}=c.n$

$$-2x_0 i+j-2z_0 k=c(i+2j+3k) \Rightarrow c=\frac{1}{2},\quad x_0=-\frac{1}{4},\quad z_0=-\frac{3}{4}$$

$$y_0=x_0^2+z_0^2=\left(-\frac{1}{4}\right)^2+\left(-\frac{3}{4}\right)^2=\frac{5}{8} \Rightarrow P_0\left(-\frac{1}{4},\frac{5}{8},-\frac{3}{4}\right)$$

Example: The cylinder $x^2+y^2=2$ and the plane $x+z=4$ meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1,1,3)$.

<u>Solution:</u> The cylinder and the plane are the level surfaces of $f(x,y,z)=x^2+y^2-2$ and $g(x,y,z)=x+z-4$ at level 0, respectively. The tangent line to E at P_0 is orthogonal to both $\nabla f|_{P_0}$ and $\nabla g|_{P_0}$ and so it is parallel to $(\nabla f|_{P_0})\times(\nabla g|_{P_0})$.

$\nabla f|_{P_0}=(2xi+2yj)|_{P_0}=2i+2j$, $\quad \nabla g|_{P_0}=(1.i+0j+1.k)_{P_0}=i+k$

$$v=(\nabla f|_{P_0})\times(\nabla g|_{P_0})=\begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}=2i-2j-2k$$

$\Rightarrow$ tg. line: $x=1+2t$, $y=1-2t$, $z=3-2t$, $t\in\mathbb{R}$.

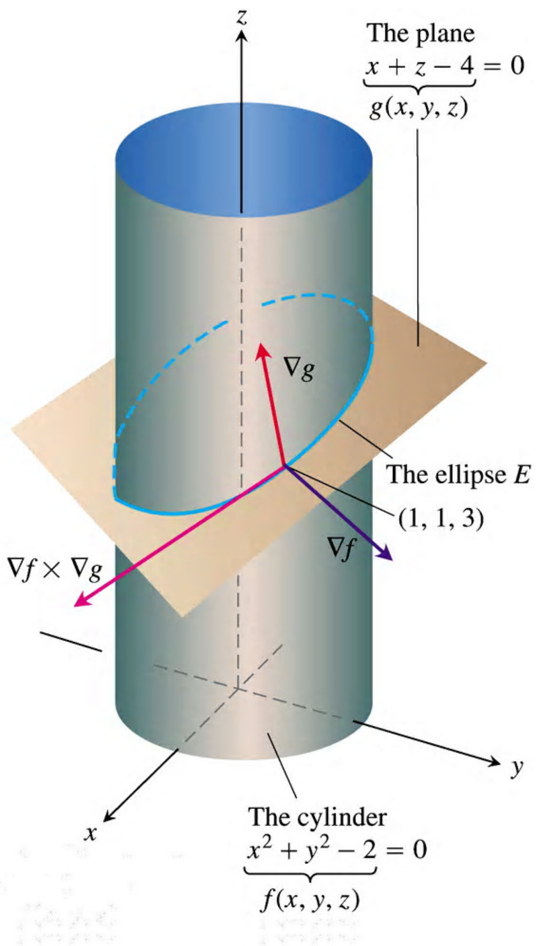
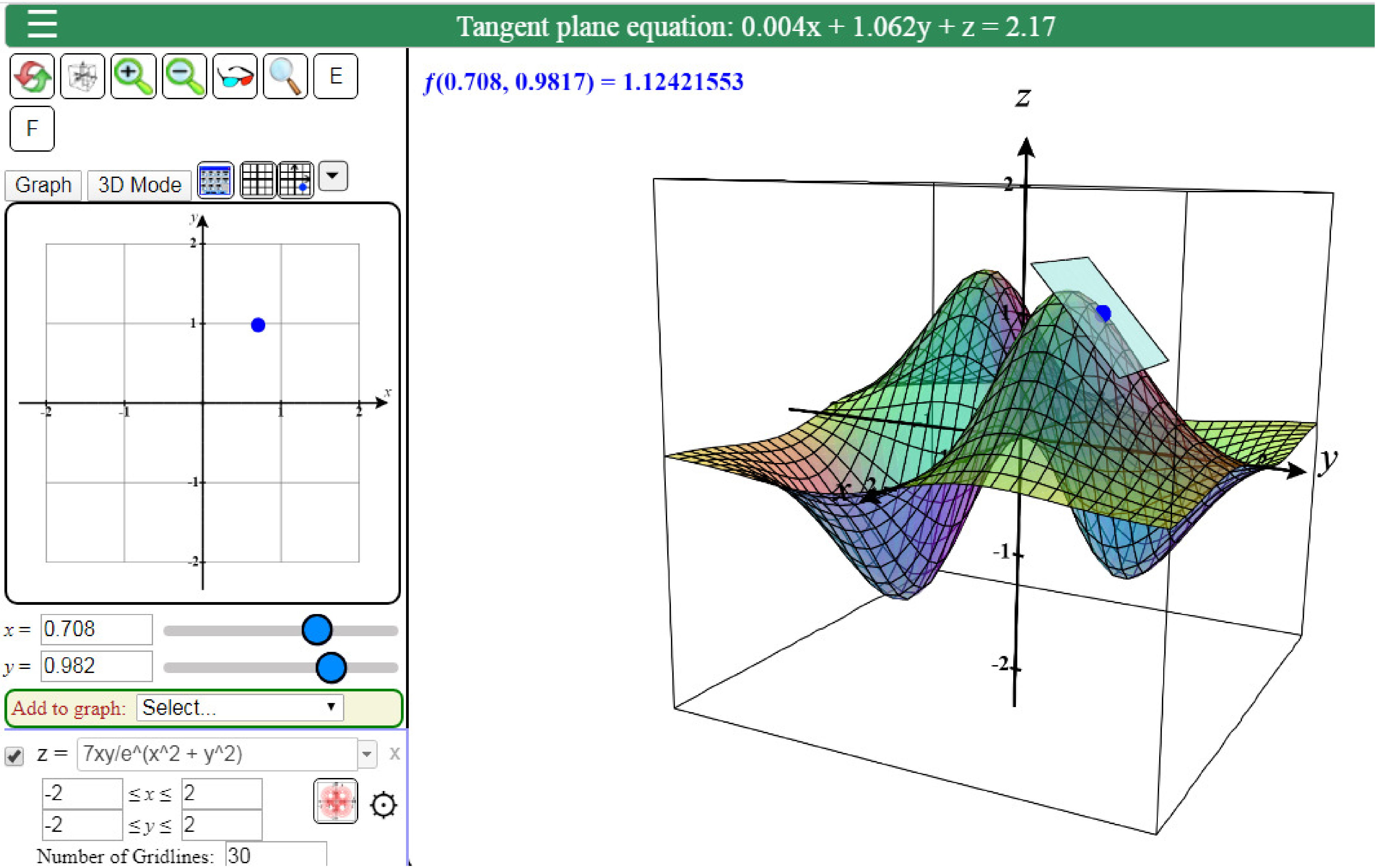


FIGURE 14.32 The cylinder $f(x, y, z) = x^2 + y^2 - 2 = 0$ and the plane $g(x, y, z) = x + z - 4 = 0$ intersect in an ellipse E (Example 3).

Visit the following page to see the tangent planes and normal vectors to the surface $z = \dfrac{7xy}{e^{x^2+y^2}}$ and the gradient vectors of $f(x,y) = \dfrac{7xy}{e^{x^2+y^2}}$. You can observe these geometric features for different functions that can be selected from the dropdown menu.

https://www.monroecc.edu/faculty/paulseeburger/calcnsf/CalcPlot3D/



Tangent plane equation: 0.004x + 1.062y + z = 2.17
f(0.708, 0.9817) = 1.12421553
E
F
Graph
3D Mode
x = 0.708
y = 0.982
Add to graph: Select...
z = 7xy/e^(x^2 + y^2)
-2 ≤ x ≤ 2
-2 ≤ y ≤ 2
Number of Gridlines: 30
z
y
2
1
-1
-2

Recall that if $y = f(x)$ then $dy = df = f'(x)\,dx$.

Estimating the Change in f in a Direction u

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\substack{\text{Directional}\\ \text{derivative}}} \cdot \underbrace{ds}_{\substack{\text{Distance}\\ \text{increment}}}$$

EXAMPLE 4 Estimating Change in the Value of $f(x, y, z)$

Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Solution We first find the derivative of f at P_0 in the direction of the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of f at P_0 is

$$\nabla f|_{(0,1,0)} = ((y \cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

Therefore,

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

The change df in f that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\mathbf{u}$ is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit.}$$

■

DEFINITION Total Differential

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

in the linearization of f is called the **total differential of f**.

Example: Suppose that a cylindrical can is designed to have a radius of 1 cm and a height of 5 cm, but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$ Estimate the resulting absolute change in the volume of the can.

<u>Solution:</u> $V = \pi r^2 h$, $r_0 = 1$, $h_0 = 5$, $V_r = 2\pi r h$, $V_h = \pi r^2$

$$\Delta V \approx dV = V_r(r_0, h_0)\, dr + V_h(r_0, h_0)\, dh$$

$$= 2\pi \cdot 1 \cdot 5 \cdot (0.03) + \pi \cdot 1^2 \cdot (-0.1) = 0.2\pi \approx 0.63$$

$$\Longrightarrow \Delta V \approx 0.63 \text{ cm}^3$$

Let $f(x,y,z)$ be a differentiable function with continuous second order partial derivatives. If x, y, and z change from x_0, y_0, and z_0 by small amounts dx, dy, and dz, the total differential

$$df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$$

gives a good approximation of the resulting change Δf in f.

Exercises

1) Find equations for the tangent plane and the normal line to the surface $\cos(\pi x) - x^2 y + e^{xz} + yz = 4$ at the point $P(0,1,2)$.

2) Find the maximum directional derivative of $f(x,y,z) = e^x \sin y + e^y \sin z + e^z \cos x$ at $P(0,0,0)$

3) Find the directional derivative of $f(x,y,z) = x^4 + 2xy^3 - z^4 y + 8xyz$ at $P(1,0,1)$ in a direction normal to the surface $3x^2 - 2y^2 + 4z^2 = 7$.

14.7

Extreme Values and Saddle Points

Recall: Local / Absolute extrema for functions of one variable $f(x)$

Candidates: 1) critical points : $f' = 0$ or undefined

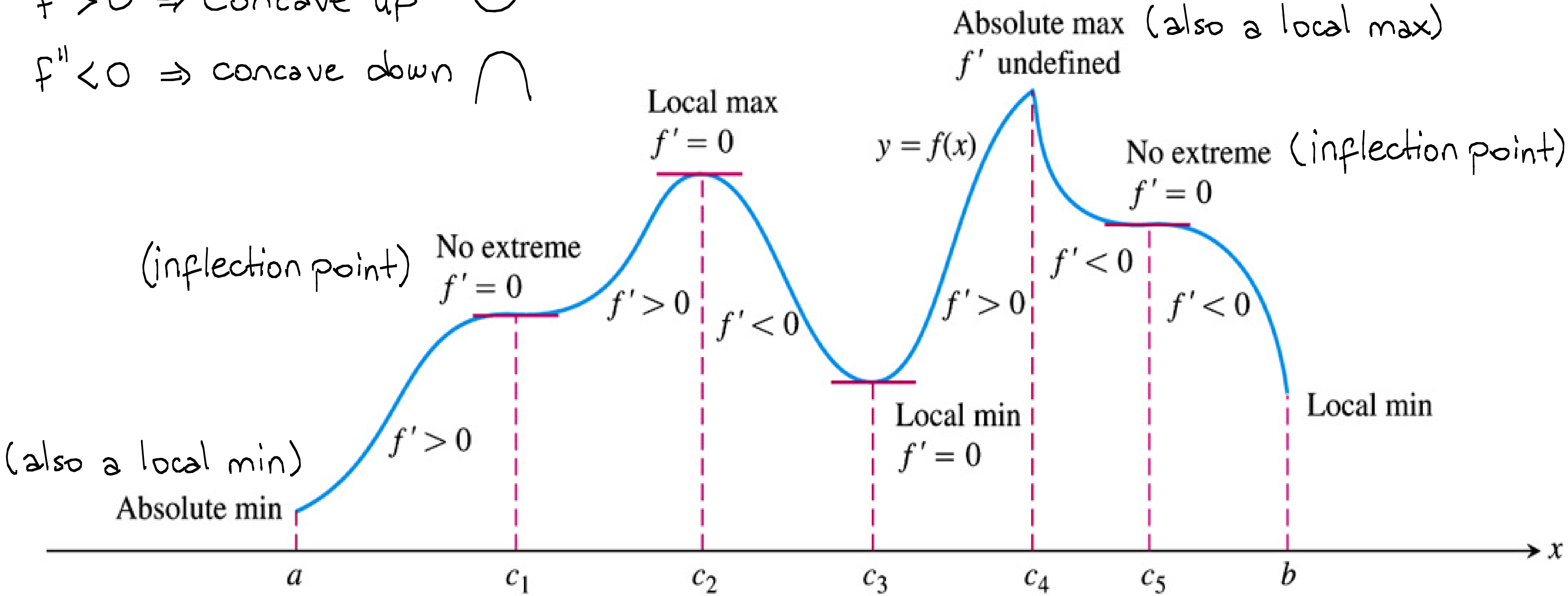
2) end points : $x = a$ and $x = b$

Critical points need to be tested for local extrema.

A function $f(x)$ continuous on a finite closed interval has absolute extrema.

$f'' > 0 \Rightarrow$ concave up $\cup$

$f'' < 0 \Rightarrow$ concave down $\cap$



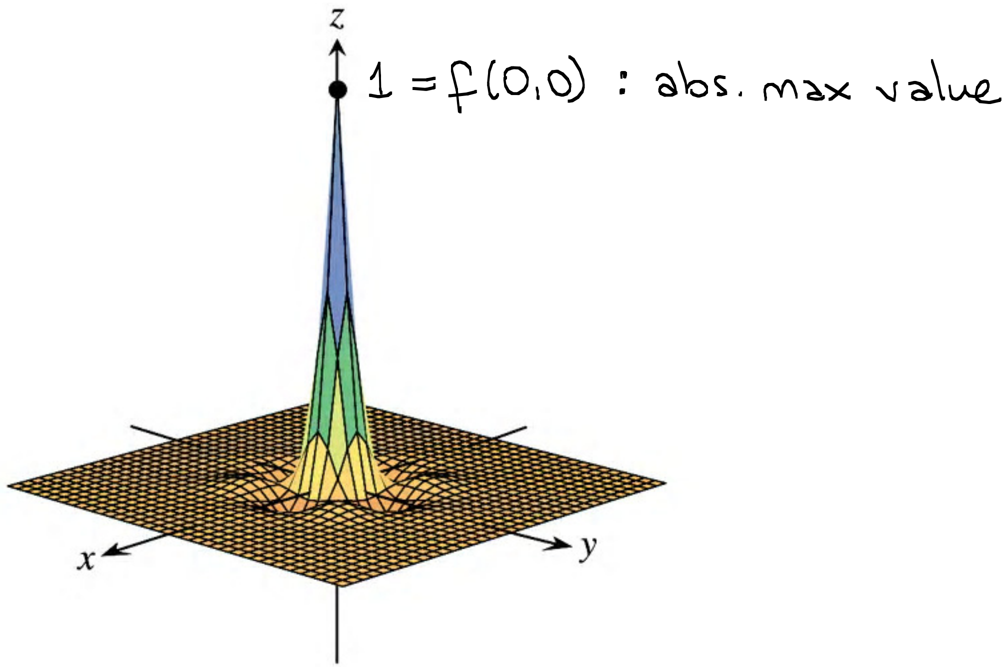


FIGURE 14.36 The function

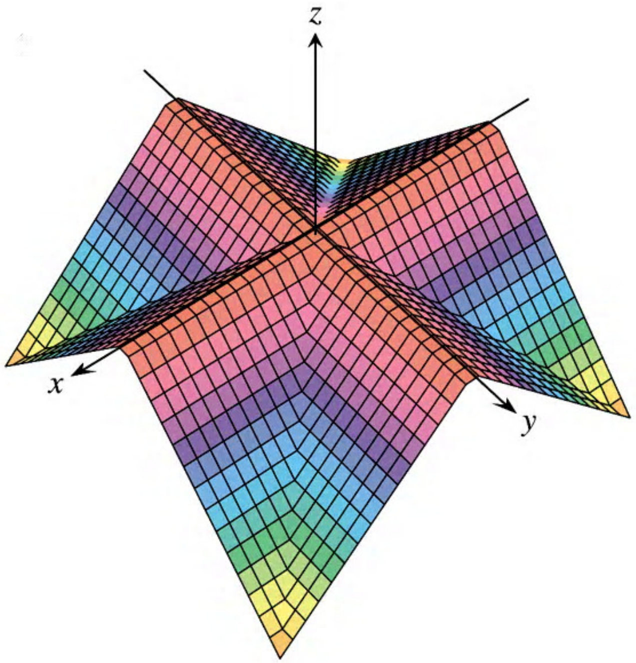
$$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}} = f(x,y)$$

has a maximum value of 1 and a minimum value of about -0.067 on the square region $|x| \leq 3\pi/2, |y| \leq 3\pi/2$.

DEFINITIONS Local Maximum, Local Minimum

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).



$f(x,0) = f(0,y) = 0$

0: absolute maximum

$f(\mp a, \mp a) = -a$

$-a$: absolute minimum

on the square region

$|x| \leq a$ & $|y| \leq a$

FIGURE 14.37 The "roof surface"

$$z = \frac{1}{2}\left(\big||x| - |y|\big| - |x| - |y|\right) = f(x,y)$$

viewed from the point (10, 15, 20). The defining function has a maximum value of 0 and a minimum value of $-a$ on the square region $|x| \leq a, |y| \leq a$.

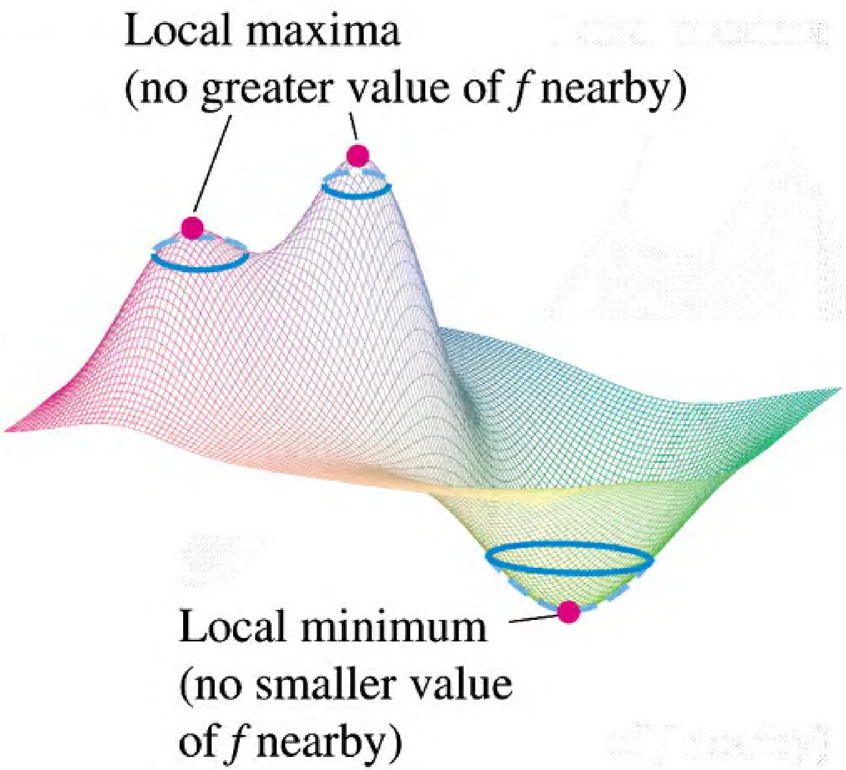


FIGURE 14.38 A local maximum is a mountain peak and a local minimum is a valley low.

THEOREM 10 First Derivative Test for Local Extreme Values

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proof: If f has a local extrema at (a,b), then $g(x) = f(x,b)$ has a local extrema at $x = a$. Thus $g'(a) = 0$. But since $g'(a) = f_x(a,b)$ we have $f_x(a,b) = 0$. Similarly $h(y) = f(a,y)$ has a local extrema at $y = b$, so $h'(b) = 0$. But since $h'(b) = f_y(a,b)$ we have $f_y(a,b) = 0$. An equation of the tangent plane to $z = f(x,y)$ at $P(a,b)$ is $z = f(a,b)$, so the tangent plane is horizontal. (parallel to the xy-plane)

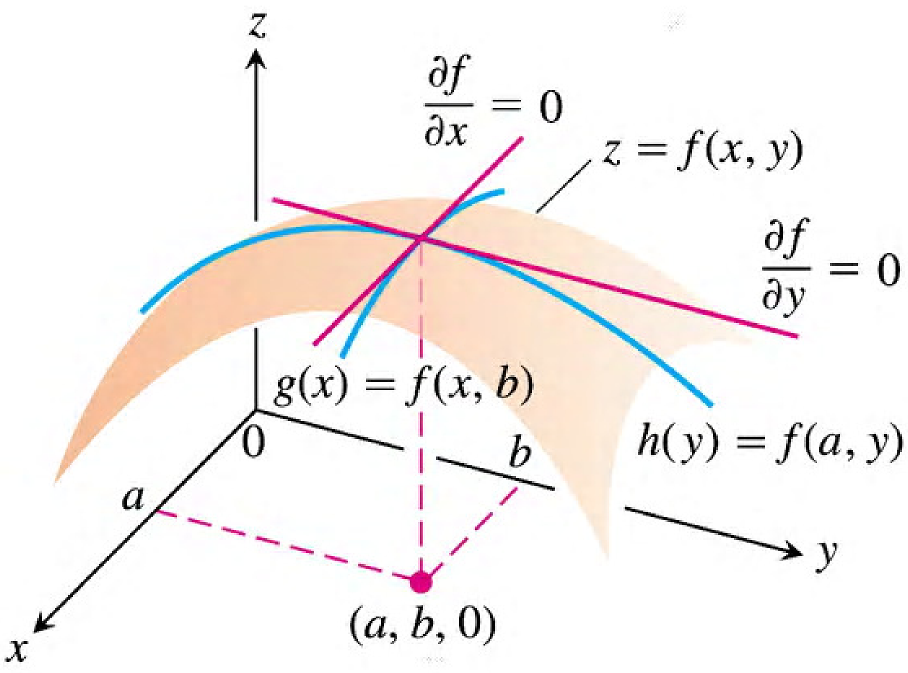


FIGURE 14.39 If a local maximum of f occurs at $x = a, y = b$, then the first partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are both zero.

DEFINITION Critical Point

An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

According to Theorem 10, the only points where a function $f(x,y)$ can assume extreme values are critical points and boundary points.

Not every critical point gives rise to a local extremum.

So the critical points and the boundary points are the candidates to be tested.

DEFINITION **Saddle Point**

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.40).

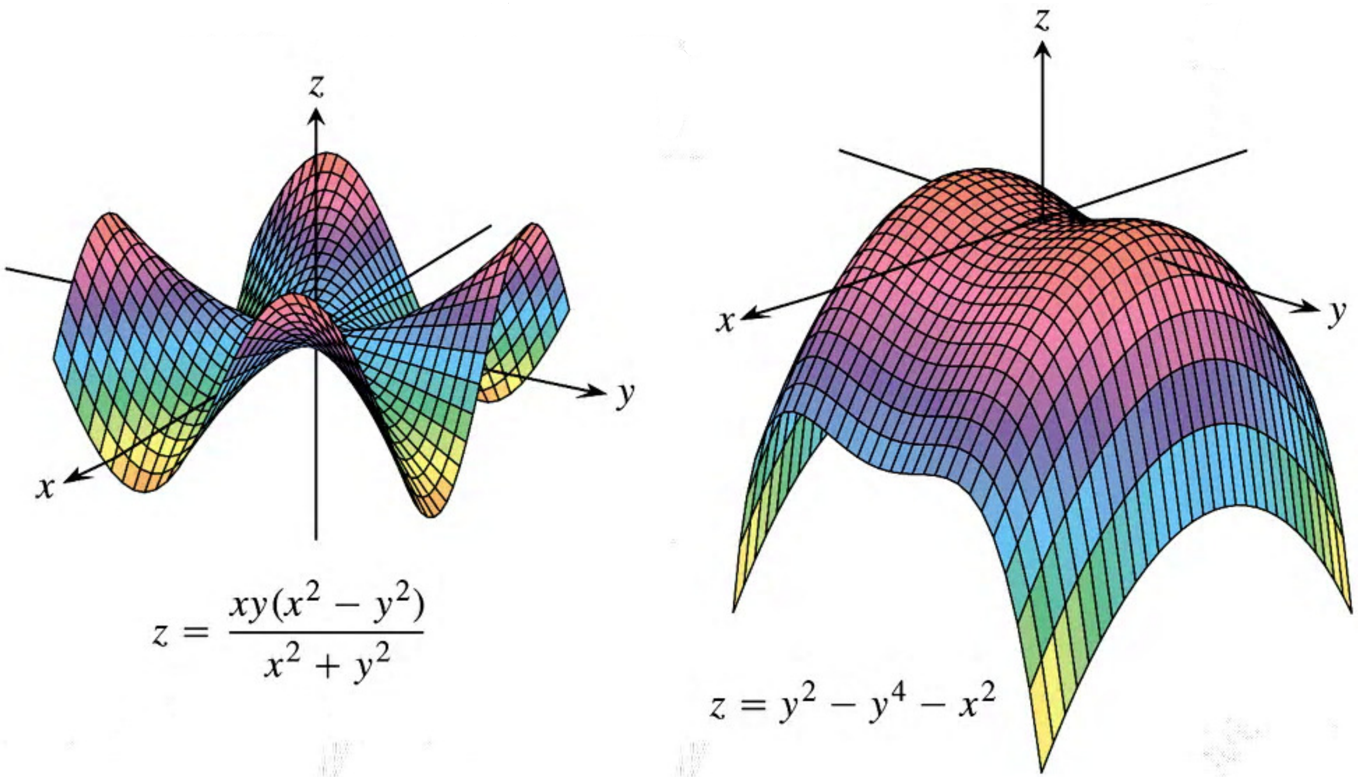


FIGURE 14.40 Saddle points at the origin.

EXAMPLE 1 Finding Local Extreme Values

Find the local extreme values of $f(x, y) = x^2 + y^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \qquad \text{and} \qquad f_y = 2y = 0.$$

The only possibility is the origin, where the value of f is zero. Since f is never negative, we see that the origin gives a local minimum (Figure 14.41). ■

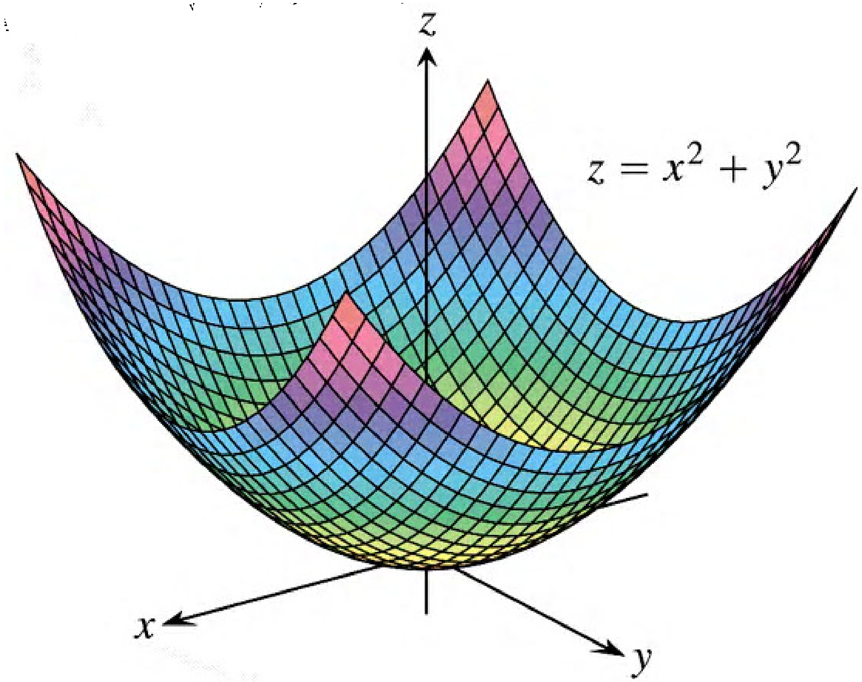


FIGURE 14.41 The graph of the function $f(x, y) = x^2 + y^2$ is the paraboloid $z = x^2 + y^2$. The function has a local minimum value of 0 at the origin (Example 1).

THEOREM 11 Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i. f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii. f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii. f has a **saddle point** at (a, b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv. **The test is inconclusive** at (a, b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The expression $f_{xx} \cdot f_{yy} - f_{xy}^2$ is called the discriminant or Hessian of f.

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} \quad : \text{Determinant form}$$

$f_{xx}\cdot f_{yy}-f_{xy}^2>0 \Leftrightarrow f_{xx}\cdot f_{yy}>f_{xy}^2\geq 0 \Leftrightarrow f_{xx}$ and f_{yy} have the same sign.

So in the above theorem, " $f_{xx}>0$ " can be replaced by " $f_{yy}>0$ " and " $f_{xx}<0$ " can be replaced by " $f_{yy}<0$ ". Because of the conditions of the above theorem, Mixed Derivative Theorem holds for $f(x,y)$, i.e $f_{xy}=f_{yx}$, so in the theorem "f_{xy}" can be replaced by "f_{yx}".

Example: Find the local extreme values (if any) of $f(x,y)=y^2-x^2$.

Solution: Domain of $f(x,y)$ is $\mathbb{R}^2$ (since f is a polynomial), so there is no boundary point. The partial derivatives $f_x=-2x$ and $f_y=2y$ exist and continuous everywhere.

$f_x=f_y=0 \Leftrightarrow (x,y)=(0,0)$: only critical point

$f_{xx}=-2$, $f_{yy}=2$, $f_{xy}=f_{yx}=0$

So at (0,0), the discriminant / Hessian of f : $f_{xx} \cdot f_{yy} - f_{xy}^2 = -4 < 0$.

By the second derivative test, f has a saddle point at (0,0).

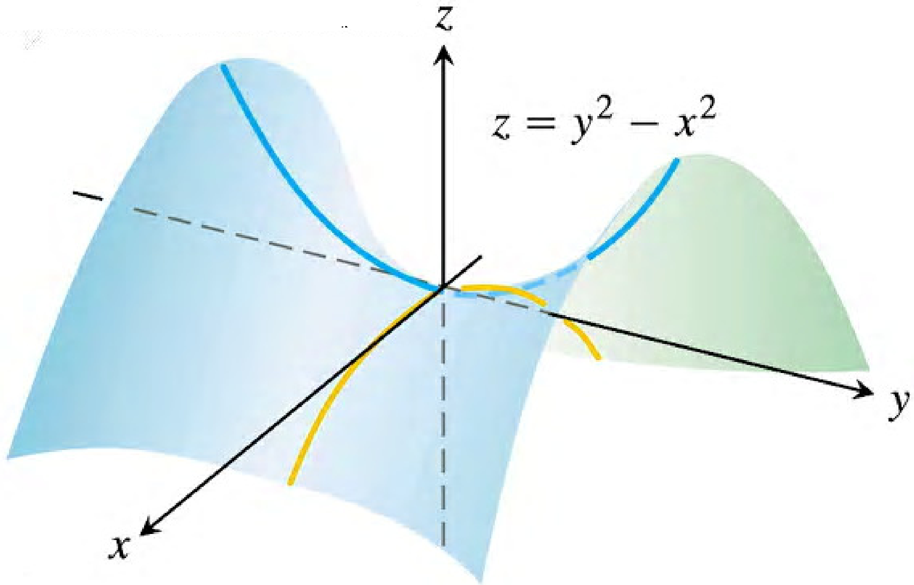


FIGURE 14.42 The origin is a saddle point of the function $f(x, y) = y^2 - x^2$. There are no local extreme values

EXAMPLE 3 Finding Local Extreme Values

Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$. ■

Example: $f(x,y) = xy$

$f_x = y \ , \quad f_y = x$

$f_x = f_y = 0 \Leftrightarrow (x,y) = (0,0)$ is the only critical point

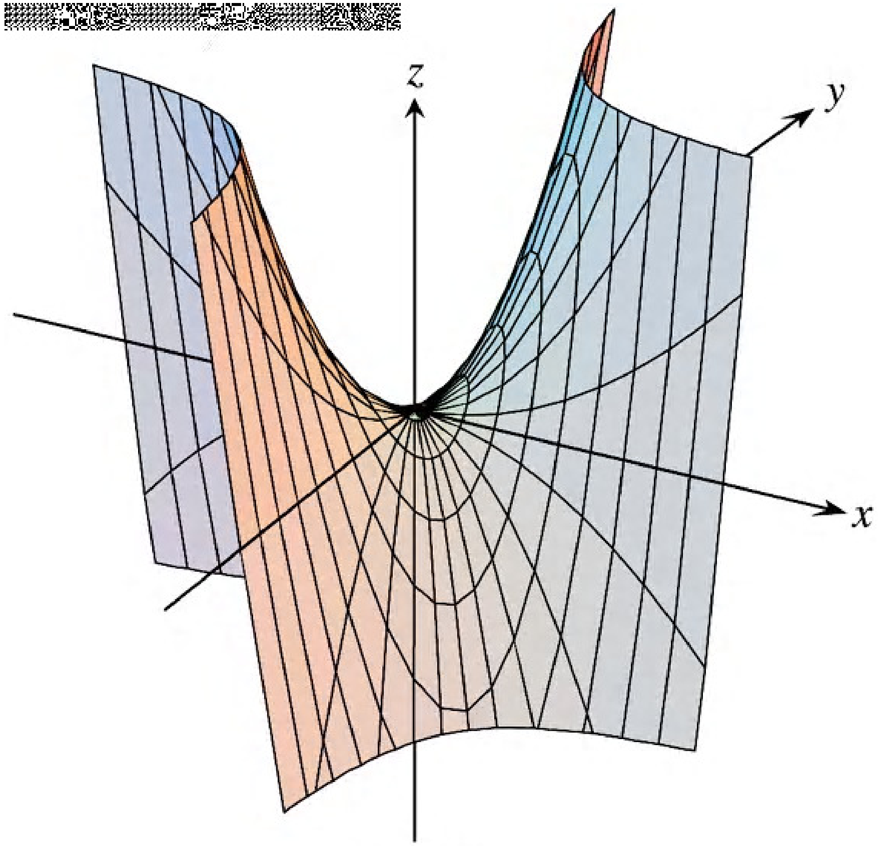
$f_{xx} = 0 \ , \quad f_{yy} = 0$

$f_{xy} = f_{yx} = 1$

The discriminant is

$f_{xx} \cdot f_{yy} - f_{xy}^2 = -1 < 0$

$\Rightarrow f$ has a saddle point at $(0,0)$ which is origin $(0,0,0)$.



$z = xy$

FIGURE 14.43 The surface $z = xy$ has a saddle point at the origin

Theorem (Extreme Value Theorem): Suppose that $f(x,y)$ is continuous on the closed and bounded region $D \subset \mathbb{R}^2$. Then f has both an absolute maximum and an absolute minimum on D. Further, an absolute extremum may only occur at a critical point in D or at a boundary point of D.

Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

1. *List the interior points of R* where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.
2. *List the boundary points of R* where f has local maxima and minima and evaluate f at these points. We show how to do this shortly.
3. *Look through the lists* for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists made in Steps 1 and 2.

EXAMPLE 5 Finding Absolute Extrema

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0,$ $y = 9 - x$.

Solution Since f is differentiable, the only places where f can assume these values are points inside the triangle (Figure 14.44) where $f_x = f_y = 0$ and points on the boundary.

(a) Interior points. For these we have

$$f_x = 2 - 2x = 0, \qquad f_y = 2 - 2y = 0,$$

yielding the single point $(x, y) = (1, 1)$. The value of f there is

$$f(1, 1) = 4.$$

(b) Boundary points. We take the triangle one side at a time:

(i) On the segment OA, $y = 0$. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of x defined on the closed interval $0 \leq x \leq 9$. Its extreme values (we know from Chapter 4) may occur at the endpoints

$$x = 0 \qquad \text{where} \qquad f(0, 0) = 2$$

$$x = 9 \qquad \text{where} \qquad f(9, 0) = 2 + 18 - 81 = -61$$

and at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$

(ii) On the segment OB, $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 2y - y^2.$$

We know from the symmetry of f in x and y and from the analysis we just carried out that the candidates on this segment are

$$f(0, 0) = 2, \qquad f(0, 9) = -61, \qquad f(0, 1) = 3.$$

(iii) We have already accounted for the values of f at the endpoints of AB, so we need only look at the interior points of AB. With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2 = -61 + 18x - 2x^2.$$

Setting $f'(x, 9 - x) = 18 - 4x = 0$ gives

$$x = \frac{18}{4} = \frac{9}{2}.$$

At this value of x,

$$y = 9 - \frac{9}{2} = \frac{9}{2} \qquad \text{and} \qquad f(x, y) = f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}.$$

Summary We list all the candidates: 4, 2, -61, 3, $-(41/2)$. The maximum is 4, which f assumes at (1, 1). The minimum is -61, which f assumes at (0, 9) and (9, 0). ■

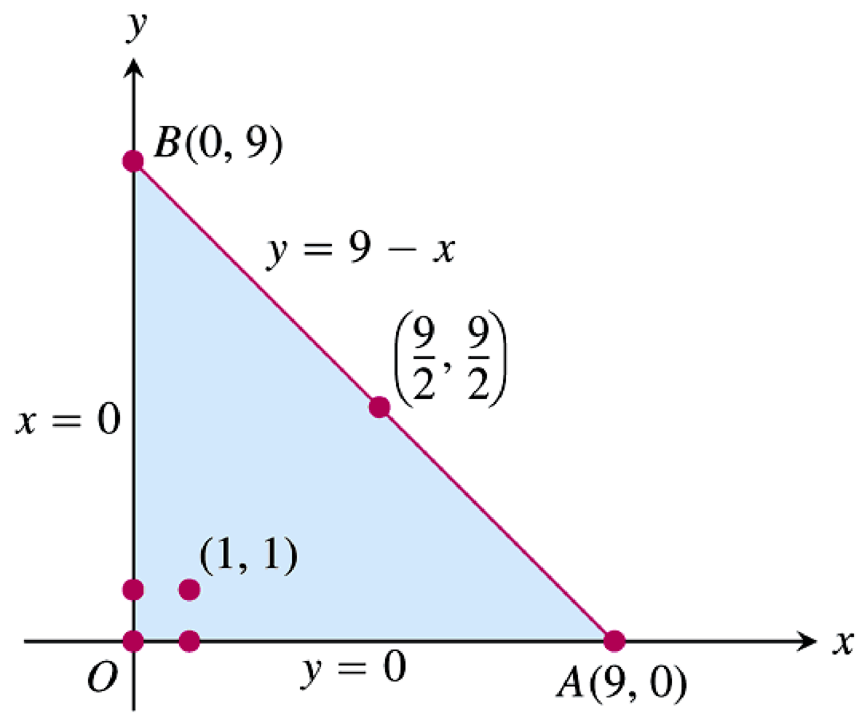


FIGURE 14.44 This triangular region is the domain of the function in Example 5.

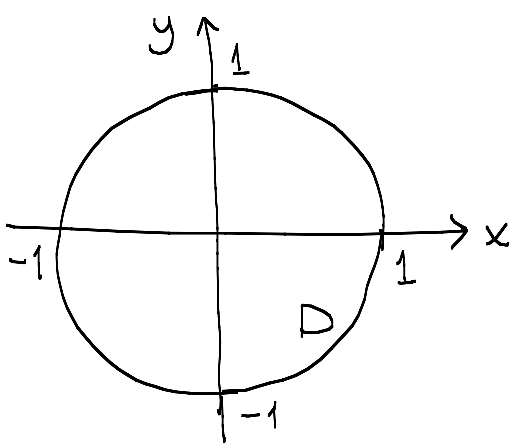
Example: Find the absolute maximum and minimum values of $f(x,y) = x^2 - y^2$ on the region $D = \{(x,y) : x^2+y^2 \leq 1\}$

Solution: a) Interior points:

$f_x = 2x$, $f_y = -2y$, $f_x = 0 = f_y \Leftrightarrow (x,y) = (0,0)$: only critical point in D

b) Boundary points: $\{(x,y) : x^2+y^2 = 1\}$

$y^2 = 1 - x^2 \Rightarrow f(x, \mp\sqrt{1-x^2}) = g(x) = x^2 - (1-x^2) = 2x^2 - 1$, $x \in [-1,1]$



$g'(x) = 4x = 0 \Rightarrow x = 0$: interior critical point of boundary

$x = 0 \Rightarrow y^2 = 1 - x^2 = 1 - 0 \Leftrightarrow y = \mp 1$

$\Rightarrow (x,y) = (0, \mp 1)$

$x = \mp 1 \Rightarrow y^2 = 1 - x^2 = 1 - 1 = 0 \Leftrightarrow y = 0$

$\Rightarrow (x,y) = (\mp 1, 0)$: end points of boundary

$f(0,0) = 0$, $f(\mp 1, 0) = 1$: abs. max value, $f(0, \mp 1) = -1$: abs. min. value.

Example: Find the point $P(x,y,z)$ closest to the origin on the plane $2x+y-z=5$

Solution:

The distance from $P(x,y,z)$ to origin $(0,0,0)$ is $d=\sqrt{x^2+y^2+z^2}$

Since P lies on the plane $2x+y-z=5$, we have $z=2x+y-5$

So $d=\sqrt{x^2+y^2+(2x+y-5)^2}$. We can minimize d by minimizing the simpler expression $d^2=x^2+y^2+(2x+y-5)^2=f(x,y)$, since we want to avoid square roots. Domain of f is $\mathbb{R}^2$, no boundary pts.

The only candidates for max min values are obtained from critical points

$$\left.\begin{array}{l} f_x=2x+2(2x+y-5).2=10x+4y-20=0 \\ f_y=2y+2(2x+y-5)=4x+4y-10=0 \end{array}\right\} \Leftrightarrow x=\frac{5}{3}, \quad y=\frac{5}{6}$$

The only critical point of f is $\left(\frac{5}{3},\frac{5}{6}\right)$

$f_{xx} = 10$, $f_{xy} = f_{yx} = 4$, $f_{yy} = 4 \Rightarrow f_{xx} \cdot f_{yy} - f_{xy}^2 = 24 > 0$, $f_{xx} > 0$

By the 2nd Derivative Test $f(x,y)$ has a local minimum at $\left(\frac{5}{3}, \frac{5}{6}\right)$

But this local minimum is actually an absolute minimum since we know the existence of a point on the given plane closest to origin.

$x = \frac{5}{3}$, $y = \frac{5}{6} \Rightarrow z = 2 \cdot \frac{5}{3} + \frac{5}{6} - 5 = -\frac{5}{6}$

$\Rightarrow$ Closest point $\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$, distance $d = \frac{5}{\sqrt{6}} \approx 2.04$

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i. **boundary points** of the domain of f

ii. **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i. $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local maximum**

ii. $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local minimum**

iii. $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) $\Rightarrow$ **saddle point**

iv. $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) $\Rightarrow$ **test is inconclusive**.

Exercise: 1) Find all the local maxima, local minima, and saddle points of the following functions:

a) $f(x,y) = 12xy - 2x^3 - 3y^2$

b) $f(x,y) = x^3 + y^3 + 3x^2y - 15y^2 + 2$

2) Find the absolute maxima and minima of the following functions in the given domains:

a) $f(x,y) = 5 + 4x - 2x^2 + 3y - y^2$, $D = \{(x,y) : -y \le x \le y,\ 0 \le y \le 2\}$

b) $f(x,y) = x^2 + y^2 + 4x - 6y$, $D = \{(x,y) : x^2 + y^2 \le 16\}$

(Hint: On the boundary of D: $x^2 + y^2 = 16$, use the parametrization $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$)

c) $f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$, $D = \{(x,y) : 0 \le y \le 2-2x,\ 0 \le x \le 1\}$

Visit the following page to see how the gradient vectors of $f(x,y)=\dfrac{7xy}{e^{x^2+y^2}}$ vanish $(\nabla f = 0 \Leftrightarrow f_x = 0 \text{ \& } f_y = 0)$ at the points where $f(x,y)$ attains its extreme values and at the saddle points on $z = \dfrac{7xy}{e^{x^2+y^2}}$. You can observe these geometric features for different functions that can be selected from the dropdown menu.

https://www.monroecc.edu/faculty/paulseeburger/calcnsf/CalcPlot3D/

14.8

Lagrange Multipliers

EXAMPLE 2 Finding a Minimum with Constraint

Find the points closest to the origin on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$.

Solution 1 The cylinder is shown in Figure 14.50. We seek the points on the cylinder closest to the origin. These are the points whose coordinates minimize the value of the function

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \text{Square of the distance}$$

subject to the constraint that $x^2 - z^2 - 1 = 0$. If we regard x and y as independent variables in the constraint equation, then

$$z^2 = x^2 - 1$$

and the values of $f(x, y, z) = x^2 + y^2 + z^2$ on the cylinder are given by the function

$$h(x, y) = x^2 + y^2 + (x^2 - 1) = 2x^2 + y^2 - 1.$$

To find the points on the cylinder whose coordinates minimize f, we look for the points in the xy-plane whose coordinates minimize h. The only extreme value of h occurs where

$$h_x = 4x = 0 \qquad \text{and} \qquad h_y = 2y = 0,$$

that is, at the point $(0, 0)$. But there are no points on the cylinder where both x and y are zero. What went wrong?

Substituting $z^2 = x^2 - 1$, we should assume that $x^2 \geq 1$,

otherwise $z^2 < 0$, so the equation won't have a real solution.

So the domain of $h(x,y) = 2x^2 + y^2 - 1$ is:

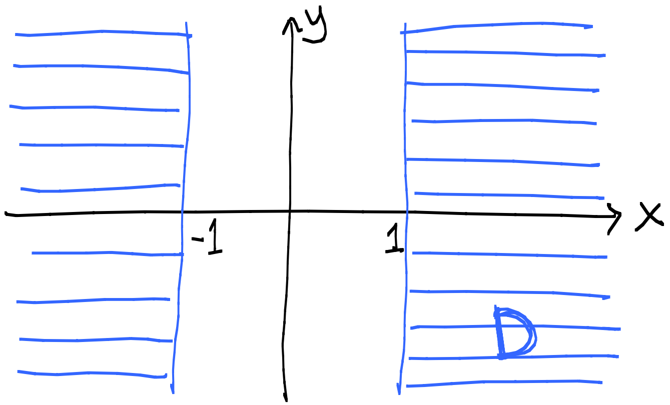
$D = \{(x,y) : y \in \mathbb{R},\ x^2 \geq 1\}$.

To find the minimum value of $h(x,y)$ we should consider the boundary of D as well as the critical points of $h(x,y)$ in D.

$(0,0)$ is the only point that satisfies $h_x = h_y = 0$,

but $(0,0) \notin D$! So it is not a critical point.

On the boundary of D, where $x^2=1$ (the lines $x=1$ and $x=-1$):



$$h(\mp 1, y) = 2 + y^2 - 1 = 1 + y^2 = g(y), \; y \in \mathbb{R}$$

$$g'(y) = 2y = 0 \Leftrightarrow y = 0$$

We obtain two candidates $(1,0)$ and $(-1,0)$ at which $h(x,y)$ may have a minimum. $h(1,0) = h(-1,0) = 1$.

$x = \mp 1$, $y=0 \Rightarrow z = 0 \Rightarrow P_1(1,0,0)$, $P_2(-1,0,0)$

1 is the only candidate for a min. value of $h(x,y) = 2x^2+y^2-1$.

Since $x^2 \geq 1$, $h(x,y) \geq y^2+1 \geq 1$.

So 1 is actually the minimum distance and P_1 and P_2 are the closest points on the hyperbolic cylinder to origin.

We can avoid this problem if we treat y and z as independent variables (instead of x and y) and express x in terms of y and z as

$$x^2 = z^2 + 1.$$

With this substitution, $f(x, y, z) = x^2 + y^2 + z^2$ becomes

$$k(y, z) = (z^2 + 1) + y^2 + z^2 = 1 + y^2 + 2z^2$$

and we look for the points where k takes on its smallest value. The domain of k in the yz-plane now matches the domain from which we select the y- and z-coordinates of the points (x, y, z) on the cylinder. Hence, the points that minimize k in the plane will have corresponding points on the cylinder. The smallest values of k occur where

$$k_y = 2y = 0 \qquad \text{and} \qquad k_z = 4z = 0,$$

or where $y = z = 0$. This leads to

$$x^2 = z^2 + 1 = 1, \qquad x = \pm 1.$$

The corresponding points on the cylinder are $(\pm 1, 0, 0)$. We can see from the inequality

$$k(y, z) = 1 + y^2 + 2z^2 \geq 1$$

that the points $(\pm 1, 0, 0)$ give a minimum value for k. We can also see that the minimum distance from the origin to a point on the cylinder is 1 unit.

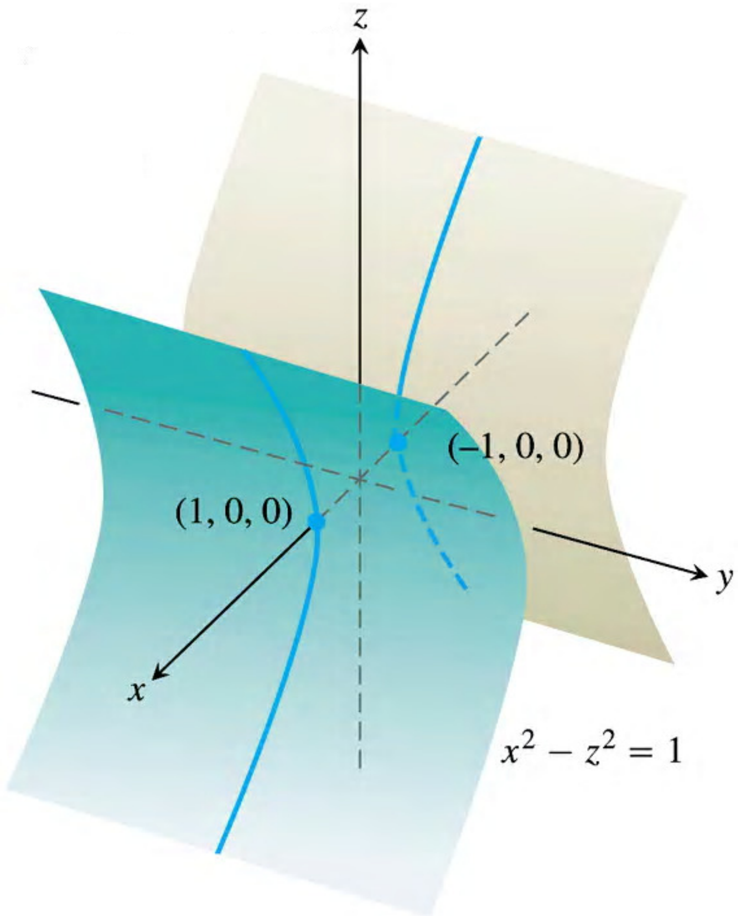


FIGURE 14.50 The hyperbolic cylinder $x^2 - z^2 - 1 = 0$ in Example 2.

The hyperbolic cylinder $x^2 - z^2 = 1$

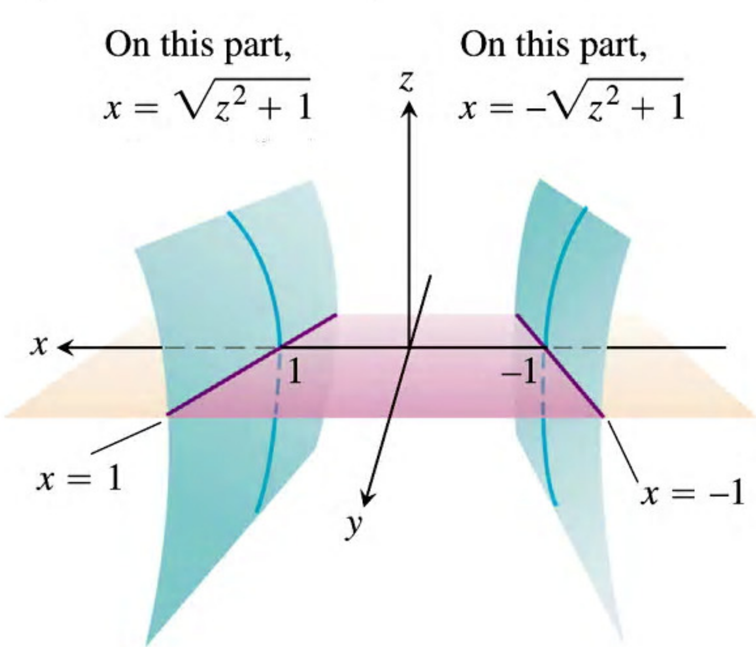


FIGURE 14.51 The region in the xy-plane from which the first two coordinates of the points (x, y, z) on the hyperbolic cylinder $x^2 - z^2 = 1$ are selected excludes the band $-1 < x < 1$ in the xy-plane (Example 2).

Solution 2 Another way to find the points on the cylinder closest to the origin is to imagine a small sphere centered at the origin expanding like a soap bubble until it just touches the cylinder (Figure 14.52). At each point of contact, the cylinder and sphere have the same tangent plane and normal line. Therefore, if the sphere and cylinder are represented as the level surfaces obtained by setting

$$f(x, y, z) = x^2 + y^2 + z^2 - a^2 \qquad \text{and} \qquad g(x, y, z) = x^2 - z^2 - 1$$

equal to 0, then the gradients ∇f and ∇g will be parallel where the surfaces touch. At any point of contact, we should therefore be able to find a scalar λ ("lambda") such that

$$\nabla f = \lambda \nabla g,$$

or

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2z\mathbf{k}).$$

Thus, the coordinates x, y, and z of any point of tangency will have to satisfy the three scalar equations

$$2x = 2\lambda x, \qquad 2y = 0, \qquad 2z = -2\lambda z.$$

For what values of λ will a point (x, y, z) whose coordinates satisfy these scalar equations also lie on the surface $x^2 - z^2 - 1 = 0$? To answer this question, we use our knowledge that no point on the surface has a zero x-coordinate to conclude that $x \neq 0$. Hence, $2x = 2\lambda x$ only if

$$2 = 2\lambda, \qquad \text{or} \qquad \lambda = 1.$$

For $\lambda = 1$, the equation $2z = -2\lambda z$ becomes $2z = -2z$. If this equation is to be satisfied as well, z must be zero. Since $y = 0$ also (from the equation $2y = 0$), we conclude that the points we seek all have coordinates of the form

$$(x, 0, 0).$$

What points on the surface $x^2 - z^2 = 1$ have coordinates of this form? The answer is the points $(x, 0, 0)$ for which

$$x^2 - (0)^2 = 1, \qquad x^2 = 1, \qquad \text{or} \qquad x = \pm 1.$$

The points on the cylinder closest to the origin are the points $(\pm 1, 0, 0)$. ■

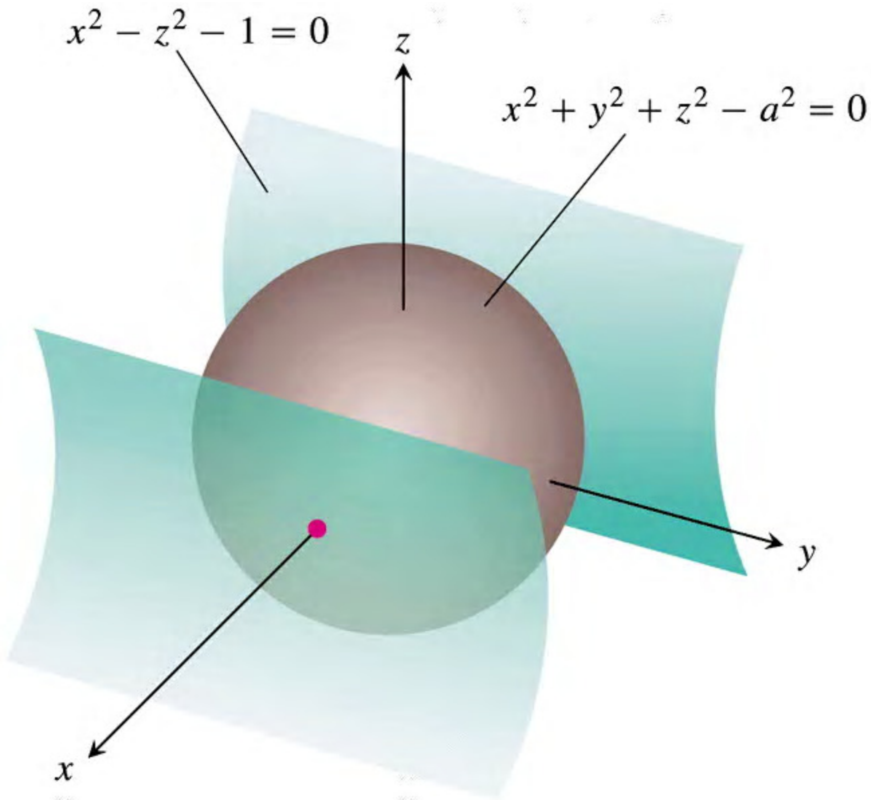


FIGURE 14.52 A sphere expanding like a soap bubble centered at the origin until it just touches the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ (Example 2).

2nd solution is using the Method of Lagrange Multipliers.

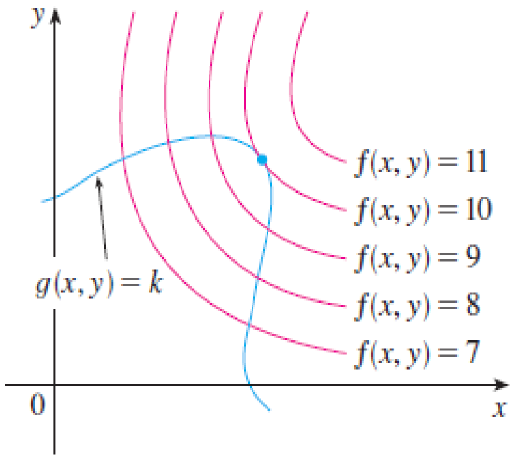


FIGURE 1

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$. Figure 1 shows this curve together with several level curves of f. These have the equations $f(x, y) = c$, where $c = 7$, 8, 9, 10, 11. To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.) This means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$. Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

THEOREM 12 The Orthogonal Gradient Theorem

Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}.$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

Proof We show that ∇f is orthogonal to the curve's velocity vector at P_0. The values of f on C are given by the composite $f(g(t), h(t), k(t))$, whose derivative with respect to t is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = \nabla f \cdot \mathbf{v}.$$

At any point P_0 where f has a local maximum or minimum relative to its values on the curve, $df/dt = 0$, so

$$\nabla f \cdot \mathbf{v} = 0. \qquad \blacksquare$$

By dropping the z-terms in Theorem 12, we obtain a similar result for functions of two variables.

COROLLARY OF THEOREM 12

At the points on a smooth curve $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{v} = 0$, where $\mathbf{v} = d\mathbf{r}/dt$.

Theorem 12 is the key to the method of Lagrange multipliers. Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and that P_0 is a point on the surface $g(x, y, z) = 0$ where f has a local maximum or minimum value relative to its other values on the surface. Then f takes on a local maximum or minimum at P_0 relative to its values on every differentiable curve through P_0 on the surface $g(x, y, z) = 0$. Therefore, ∇f is orthogonal to the velocity vector of every such differentiable curve through P_0. So is ∇g, moreover (because ∇g is orthogonal to the level surface $g = 0$, as we saw in Section 14.5). Therefore, at P_0, ∇f is some scalar multiple λ of ∇g.

Theorem *Suppose that $f(x, y, z)$ and $g(x, y, z)$ are functions with continuous first order partial derivatives and $\nabla g(x, y, z) \neq 0$ on the surface $g(x, y, z) = 0$. Suppose that either*

(i) *the minimum value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ occurs at (x_0, y_0, z_0); or*

(ii) *the maximum value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ occurs at (x_0, y_0, z_0).*

*Then $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$, for some constant λ (called a **Lagrange multiplier**).*

For functions of two independent variables, the condition is similar, but without the variable z.

METHOD OF LAGRANGE MULTIPLIERS To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

If we write the vector equation $\nabla f = \lambda \, \nabla g$ in terms of its components, then the equations in step (a) become

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.

For functions of two variables the method of Lagrange multipliers is similar to the method just described. To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \, \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

EXAMPLE 3 Using the Method of Lagrange Multipliers

Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.53)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Solution We want the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of x, y, and λ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4}x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \qquad 4y^2 + 4y^2 = 8 \qquad \text{and} \qquad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

The Geometry of the Solution

The level curves of the function $f(x, y) = xy$ are the hyperbolas $xy = c$ (Figure 14.54). The farther the hyperbolas lie from the origin, the larger the absolute value of f. We want

to find the extreme values of $f(x, y)$, given that the point (x, y) also lies on the ellipse $x^2 + 4y^2 = 8$. Which hyperbolas intersecting the ellipse lie farthest from the origin? The hyperbolas that just graze the ellipse, the ones that are tangent to it, are farthest. At these points, any vector normal to the hyperbola is normal to the ellipse, so $\nabla f = y\mathbf{i} + x\mathbf{j}$ is a multiple $(\lambda = \pm 2)$ of $\nabla g = (x/4)\mathbf{i} + y\mathbf{j}$. At the point $(2, 1)$, for example,

$$\nabla f = \mathbf{i} + 2\mathbf{j}, \qquad \nabla g = \frac{1}{2}\mathbf{i} + \mathbf{j}, \qquad \text{and} \qquad \nabla f = 2\nabla g.$$

At the point $(-2, 1)$,

$$\nabla f = \mathbf{i} - 2\mathbf{j}, \qquad \nabla g = -\frac{1}{2}\mathbf{i} + \mathbf{j}, \qquad \text{and} \qquad \nabla f = -2\nabla g.$$

■

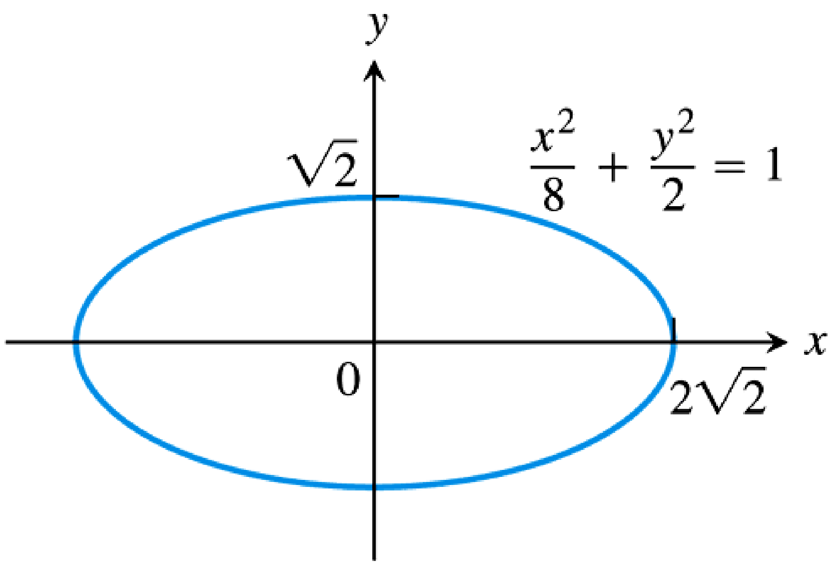


FIGURE 14.53 Example 3 shows how to find the largest and smallest values of the product xy on this ellipse.

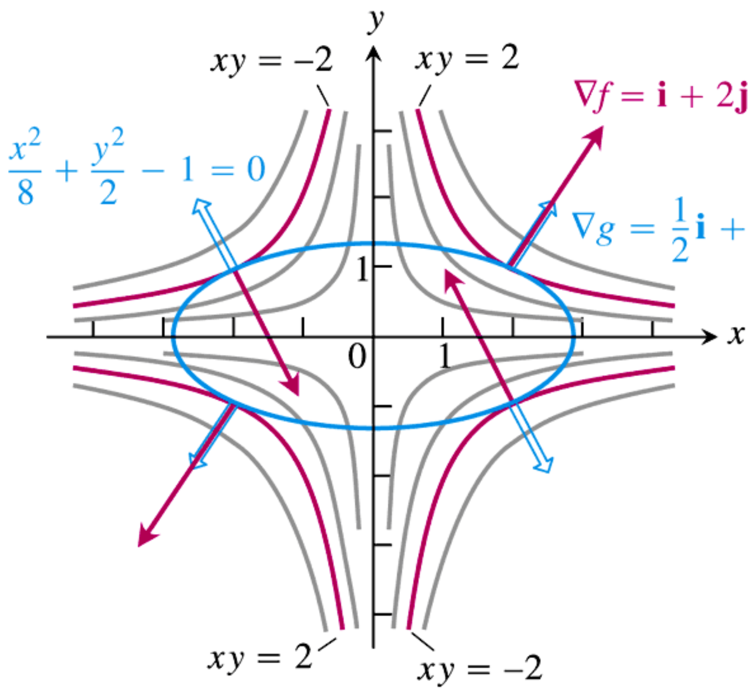


FIGURE 14.54 When subjected to the constraint $g(x, y) = x^2/8 + y^2/2 - 1 = 0$, the function $f(x, y) = xy$ takes on extreme values at the four points $(\pm 2, \pm 1)$. These are the points on the ellipse when ∇f (red) is a scalar multiple of ∇g (blue) (Example 3).

EXAMPLE 4 Finding Extreme Function Values on a Circle

Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$$

and look for the values of x, y, and λ that satisfy the equations

$$\nabla f = \lambda \nabla g: \quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j}$$

$$g(x, y) = 0: \quad x^2 + y^2 - 1 = 0.$$

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that x and y have the same sign. With these values for x and y, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

The Geometry of the Solution

The level curves of $f(x, y) = 3x + 4y$ are the lines $3x + 4y = c$ (Figure 14.55). The farther the lines lie from the origin, the larger the absolute value of f. We want to find the extreme values of $f(x, y)$ given that the point (x, y) also lies on the circle $x^2 + y^2 = 1$. Which lines intersecting the circle lie farthest from the origin? The lines tangent to the circle are farthest. At the points of tangency, any vector normal to the line is normal to the circle, so the gradient $\nabla f = 3\mathbf{i} + 4\mathbf{j}$ is a multiple $(\lambda = \pm 5/2)$ of the gradient $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$. At the point $(3/5, 4/5)$, for example,

$$\nabla f = 3\mathbf{i} + 4\mathbf{j}, \qquad \nabla g = \frac{6}{5}\mathbf{i} + \frac{8}{5}\mathbf{j}, \qquad \text{and} \qquad \nabla f = \frac{5}{2}\nabla g. \qquad \blacksquare$$

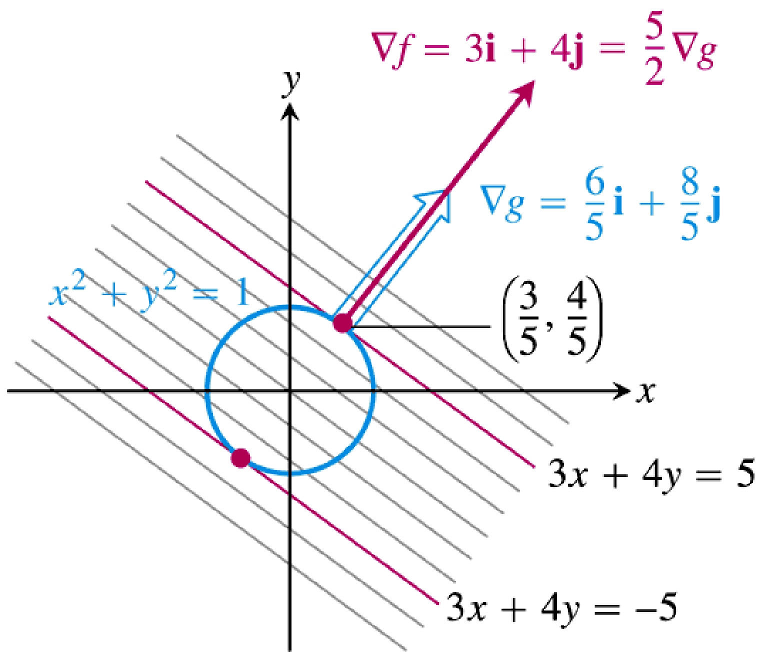


FIGURE 14.55 The function $f(x, y) = 3x + 4y$ takes on its largest value on the unit circle $g(x, y) = x^2 + y^2 - 1 = 0$ at the point $(3/5, 4/5)$ and its smallest value at the point $(-3/5, -4/5)$ (Example 4). At each of these points, ∇f is a scalar multiple of ∇g. The figure shows the gradients at the first point but not the second.

V EXAMPLE A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION we let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x \qquad V_y = \lambda g_y \qquad V_z = \lambda g_z \qquad 2xz + 2yz + xy = 12$$

which become

$$yz = \lambda(2z + y) \tag{2}$$

$$xz = \lambda(2z + x) \tag{3}$$

$$xy = \lambda(2x + 2y) \tag{4}$$

$$2xz + 2yz + xy = 12 \tag{5}$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply (2) by x, (3) by y, and (4) by z, then the left sides of these equations will be identical. Doing this, we have

$$xyz = \lambda(2xz + xy) \tag{6}$$

$$xyz = \lambda(2yz + xy) \tag{7}$$

$$xyz = \lambda(2xz + 2yz) \tag{8}$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from (2), (3), and (4) and this would contradict (5). Therefore, from (6) and (7), we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From (7) and (8) we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in (5), we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$.

$\Rightarrow V = x \cdot y \cdot z = 2 \cdot 2 \cdot 1 = 4\ m^3$ is the maximum volume of the box

EXAMPLE Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

SOLUTION The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d = \sqrt{(x - 3)^2 + (y - 1)^2 + (z + 1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

According to the method of Lagrange multipliers, we solve $\nabla f = \lambda \nabla g$, $g = 4$. This gives

$$\boxed{12} \qquad 2(x - 3) = 2x\lambda$$

$$\boxed{13} \qquad 2(y - 1) = 2y\lambda$$

$$\boxed{14} \qquad 2(z + 1) = 2z\lambda$$

$$\boxed{15} \qquad x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for x, y, and z in terms of λ from (12), (13), and (14), and then substitute these values into (15). From (12) we have

$$x - 3 = x\lambda \qquad \text{or} \qquad x(1 - \lambda) = 3 \qquad \text{or} \qquad x = \frac{3}{1 - \lambda}$$

[Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from (12).] Similarly, (13) and (14) give

$$y = \frac{1}{1 - \lambda} \qquad z = -\frac{1}{1 - \lambda}$$

Therefore, from (15), we have

$$\frac{3^2}{(1 - \lambda)^2} + \frac{1^2}{(1 - \lambda)^2} + \frac{(-1)^2}{(1 - \lambda)^2} = 4$$

which gives $(1 - \lambda)^2 = \frac{11}{4}$, $1 - \lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \qquad \text{and} \qquad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11})$ and the farthest is $(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11})$. □

EXAMPLE 3 Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leqslant 1$.

SOLUTION Critical points & boundary points : compare the values of f at the critical points with values at the points on the boundary. Since $f_x = 2x$ and $f_y = 4y$, the only critical point is $(0, 0)$. We compare the value of f at that point with the extreme values on the boundary.

Find the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

$$2x = 2x\lambda \tag{9}$$

$$4y = 2y\lambda \tag{10}$$

$$x^2 + y^2 = 1 \tag{11}$$

From (9) we have $x = 0$ or $\lambda = 1$. If $x = 0$, then (11) gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from (10), so then (11) gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we

$$f(0, 0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2$$

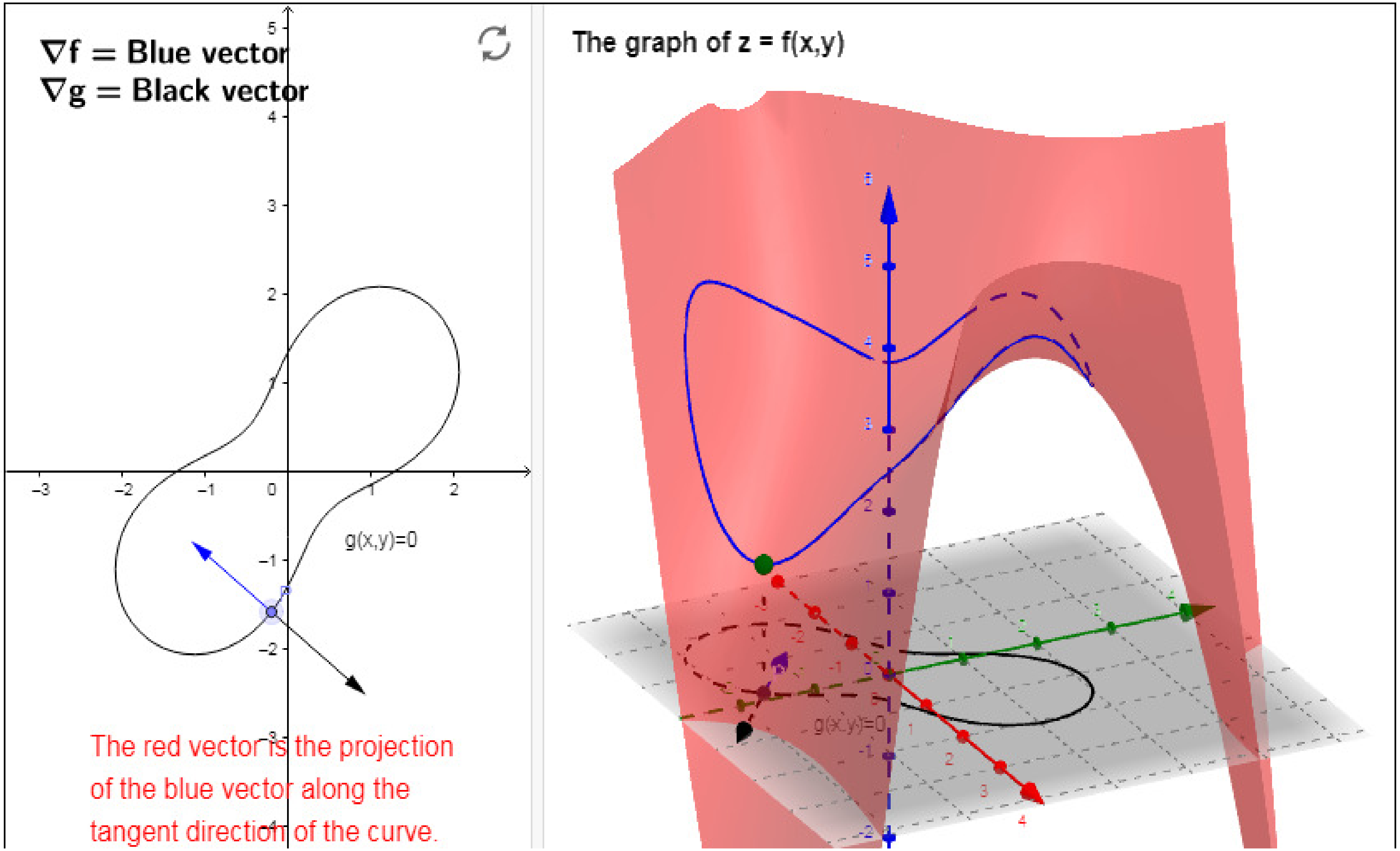
Therefore the maximum value of f on the disk $x^2 + y^2 \leqslant 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$. □

Visit the website https://www.geogebra.org/m/gXyun8mD to observe how the gradients ∇f and ∇g are parallel at the points where the function $f(x,y)$ attains its extreme values subject to a constraint $g(x,y) = 0$.

GeoGebra

Lagrange Multiplier

Yazar: Ku, Yin Bon (Albert)



Exercises: 1) Find the extreme values of $f(x,y)=x^2y$ subject to the constraint $2x^2+y^2=3$.

2) Let $T(x,y)=x^2+2x+y^2$ be the temperature at the point (x,y) on an elliptical metal plate defined by $x^2+4y^2\leq 24$. Find the minimum and maximum temperatures on the plate.

3) Find the maximum and minimum values of $f(x,y,z)=2x-3y+z$ on the sphere $x^2+y^2+z^2=14$.

4) Find the extreme values of $f(x,y,z)=xy+yz$ subject to the constraint $x^2+y^2+z^2=8$.