Diagonalization of Matrices

Something very nice happens when the $n \times n$ matrix $\mathbf{A}$ does have n linearly independent eigenvectors. Suppose that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) of $\mathbf{A}$ correspond to the n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, respectively. Let

$$\mathbf{P} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

be the $n \times n$ matrix having these eigenvectors as its *column* vectors. Then

$$\mathbf{AP} = \mathbf{A}\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \cdots & \mathbf{A}\mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \\ | & | & & | \end{bmatrix},$$

because $\mathbf{A}\mathbf{v}_j = \lambda_j\mathbf{v}_j$ for each $j = 1, 2, \ldots, n$. Thus the product matrix $\mathbf{AP}$ has column vectors $\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2, \ldots, \lambda_n\mathbf{v}_n$.

Now consider the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

whose diagonal elements are the eigenvalues corresponding (in the same order) to the eigenvectors forming the columns of **P**. Then

$$\mathbf{PD} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \\ | & | & & | \end{bmatrix},$$

$$\mathbf{AP} = \mathbf{PD}.$$

But the matrix **P** is invertible, because its n column vectors are linearly independent. So we may multiply on the right by $\mathbf{P}^{-1}$ to obtain

$$\mathbf{A} = \mathbf{PDP}^{-1}. \qquad \mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}.$$

Similarity and Diagonalization

DEFINITION Similar Matrices

The $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are called **similar** provided that there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}. \tag{7}$$

A is called diagonalizable if it is similar to a diagonal matrix D.

THEOREM 1 Criterion for Diagonalizability

The $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if it has n linearly independent eigenvectors.

$A - P \; P^{-1}$

Example: Determine whether or not given matrix A is diagonalizable.

a) $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ $\lambda_1 = \lambda_2 = 2$ $v = [1 \ \ 0]^T$

Since the matrix A has not 2 linearly independent eigenvectors, A is not diagonalizable.

b) $A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$

$\lambda_1 = 0$ $v_1 = (0, 1, -3)$

$\lambda_2 = 1$ $v_2 = (0, -2, 5)$

$\lambda_3 = 3$ $v_3 = (1, 0, 2)$

Since the matrix has 3 linearly independent eigenvectors, A is diagonalizable.

Theorem: If the $n \times n$ matrix A has n distinct eigenvalues, then it is diagonalizable.

Important note! This theorem does not say that if the $n \times n$ matrix k $(k<n)$ distinct eigenvalues, that it is not diagonalizabe. Be careful!

Example:

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}, \quad \lambda_1 = 2 \quad (\text{mult.}\,2) \quad v_1^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\lambda_2 = 3 \Rightarrow v^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

There are 2 eigenvalues but 3 linearly independent eigevectors

$\Rightarrow A$ is diagonalizable.

Powers of Matrices

Let A be diagonalizable

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1},$$

where

$$\mathbf{P} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

and

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

$$\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}$$

because $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$. More generally, for each positive integer k,

$$\begin{aligned} \mathbf{A}^k &= (\mathbf{PDP}^{-1})^k \\ &= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\cdots(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) \\ &= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}\cdots(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1}; \\ \mathbf{A}^k &= \mathbf{PD}^k\mathbf{P}^{-1}. \end{aligned}$$

But the kth power $\mathbf{D}^k$ of the diagonal matrix $\mathbf{D}$ is easily computed:

$$\mathbf{D}^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}.$$

Example: Find A^5 if

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

$\lambda_1 = 3 \quad v^{(1)} = [1 \quad 1 \quad 1]^T$

$\lambda_2 = 2 \quad v_1^{(2)} = [1 \quad 1 \quad 0]^T, \quad v_2^{(2)} = [-1 \quad 0 \quad 2]^T$

(mult. 2)

$$\Rightarrow P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^5 = P D^5 P^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_1 + 3R_3]{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\underbrace{\begin{matrix} 2 & -2 & 1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{matrix}}_{= P^{-1}}$$

$$D^5 = \begin{bmatrix} 3^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 2^5 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 243 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 243 & 32 & 32 \\ 243 & 32 & 0 \\ 243 & 0 & 64 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & 1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 454 & -422 & 211 \\ 422 & -390 & 211 \\ 422 & -422 & 243 \end{bmatrix}.$$

THEOREM 1 Cayley-Hamilton

If the $n \times n$ matrix $\mathbf{A}$ has the characteristic polynomial

$$p(\lambda) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0,$$

then

$$p(\mathbf{A}) = (-1)^n \mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + \cdots + c_2\mathbf{A}^2 + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}. \qquad (18)$$

Example: Verify that the Cayley-Hamilton Theorem for the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. Then compute A^2, A^3 and A^{-1}.

$$p(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = \lambda^2 - 4\lambda + 4.$$

$$p(A) = A^2 - 4A + 4I \overset{?}{=} 0$$

$$P(A) = A^2 - 4A + 4I$$

$$= \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} - 4 \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 12 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -8 & -12 \\ 0 & -8 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^2 - 4A + 4I = 0 \Rightarrow \underline{A^2 = 4A - 4I} = 4 \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 4 \cdot \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 0 & 4 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = A(4A - 4I) = 4A^2 - 4A = 4(4A - 4I) - 4A$$

$$= 16A - 16I - 4A$$

$$= 12A - 16I.$$

$$A^3 = 12A - 16I$$

$$= 12 \cdot \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} - 16 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 24-16 & 36 \\ 0 & 24-16 \end{bmatrix} = \begin{bmatrix} 8 & 36 \\ 0 & 8 \end{bmatrix}.$$

$$A^2 = 4A - 4I \Rightarrow A^{-1} \cdot A^2 = A^{-1}(4A - 4I)$$

$$\Rightarrow A = 4I - 4A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{4}(4I - A) = I - \frac{1}{4}A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -3/4 \\ 0 & 1/2 \end{bmatrix}.$$

Linear Systems of Differential Equations

Matrices and Linear Systems

We discuss here the general system of n first-order linear equations

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t),\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t),\\
x_3' &= p_{31}(t)x_1 + p_{32}(t)x_2 + \cdots + p_{3n}(t)x_n + f_3(t),\\
&\ \ \vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t).
\end{aligned}
\qquad (*)
$$

We say that this system is homogeneous if the functions $f_1, f_2, \ldots, f_n$ are all identically zero; otherwise it is nonhomogeneous.

A solution of the system $(*)$ is an n-tuple of functions $x_1(t), x_2(t), \ldots, x_n(t)$ that (on some interval) identically satisfy each of the equations in $(*)$

Example: $\begin{cases} 20x' = -6x + y \\ 20y' = 6x + 3y \end{cases}$ is a linear first-order system (homogeneous)

$$\begin{cases} x_1' = x_2 \\ 2x_2' = -6x_1 + 2x_3 \\ x_3' = x_4 \\ x_4' = 2x_1 - 2x_3 + 40\sin 3t \end{cases}$$

is a non-homogeneous linear first-order system.

$\begin{cases} x_1' = x_2 \\ x_2' = (x_1)^3 + (x_2)^3 \end{cases}$ is a non-linear first order system.

$\begin{cases} x'' = -6x + 2y \\ y'' = 2x - y \end{cases}$ is a second-order system.

Example: Transform the third-order equation

$$x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$$

into an equivalent system of first-order differential equations.

$$x_1 = x$$

$$x_2 = x' \Rightarrow x_1' = x_2$$

$$x_3 = x'' \Rightarrow x_2' = x_3$$

$$x''' + 3x'' + 2x' - 5x = \sin 2t$$

$$x_3' + 3x_3 + 2x_1 - 5x_1 = \sin 2t$$

$$x_3' = 5x_1 - 2x_2 - 3x_3 + \sin 2t$$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 5x_1 - 2x_2 - 3x_3 + \sin 2t \end{cases}$$

THEOREM 1 Existence and Uniqueness for Linear Systems

Suppose that the functions $p_{11}, p_{12}, \ldots, p_{nn}$ and the functions $f_1, f_2, \ldots, f_n$ are continuous on the open interval I containing the point a. Then, given the n numbers $b_1, b_2, \ldots, b_n$, the system in (★) has a unique solution on the entire interval I that satisfies the n initial conditions

$$x_1(a) = b_1, \quad x_2(a) = b_2, \quad \ldots, \quad x_n(a) = b_n. \tag{21}$$

Example: Find the interval in which the initial value problem

$$tx' = 2x + y + 1$$

$$y' = x + \cos t$$

$$x(1) = 2, \quad y(1) = -1$$

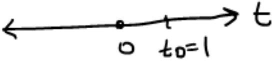
has a unique solution.

$$x' = \frac{2}{t}x + \frac{1}{t}y + \frac{1}{t}$$

$$y' = x + \cos t$$

$p_{11}(t) = \frac{2}{t}$, $p_{12}(t) = \frac{1}{t}$ and $f_1(t) = \frac{1}{t}$ are discontinuous at $t = 0$.

$p_{21}(t) = 1$, $p_{22}(t) = 0$ and $f_2(t) = \cos t$ are continuous everywhere.



$I = (0, \infty)$ $\quad [t_0 = 1 \in I]$

A system of differential equations often can be simplified by expressing it as a single differential equation involving a matrix-valued function. A **matrix-valued function**, or simply **matrix function**, is a matrix such as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = [x_j(t)]$$

$$P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) & \cdots & P_{1n}(t) \\ P_{21}(t) & P_{22}(t) & \cdots & P_{2n}(t) \\ \vdots & & & \\ P_{n1}(t) & P_{n2}(t) & & P_{nn}(t) \end{bmatrix} = [P_{ij}(t)] \qquad f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} = [f_i(t)].$$

We say that the matrix function P(t) is continuous (or differentiable) at a point (or on an interval) if each of its elements has the same property. The derivative of a differentiable matrix function is defined by elementwise differentiation; that is,

$$P'(t) = \frac{dP}{dt} = \left[\frac{dP_{ij}}{dt}\right]$$

Diffrentiation Rules: Let A and B be matrix function, and c is a (constant) real number and C is a constant matrix, then

$$\frac{d}{dt}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$$

$$\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\mathbf{B}$$

$$\frac{d}{dt}(c\mathbf{A}) = c\frac{d\mathbf{A}}{dt}, \quad \frac{d}{dt}(\mathbf{CA}) = \mathbf{C}\frac{dA}{dt}, \quad \text{and} \quad \frac{d}{dt}(\mathbf{AC}) = \frac{d\mathbf{A}}{dt}\mathbf{C}.$$

Example: If $A(t) = \begin{bmatrix} \sin t & 1 \\ t & \cos t \end{bmatrix}$ and $C = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, then

verify $\dfrac{d}{dt}(AC) = \dfrac{dA}{dt} \cdot C$

$$AC = \begin{bmatrix} \sin t & 1 \\ t & \cos t \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3\sin t - 1 \\ 3t - \cos t \end{bmatrix}$$

$$\frac{d}{dt}(AC) = \begin{bmatrix} 3\cos t \\ 3 + \sin t \end{bmatrix}$$

$$\frac{dA}{dt} \cdot C = \begin{bmatrix} \cos t & 0 \\ 1 & -\sin t \end{bmatrix}$$

$$\frac{dA}{dt} \cdot C = \begin{bmatrix} \cos t & 0 \\ 1 & -\sin t \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3\cos t \\ 3 + \sin t \end{bmatrix}$$

$$\Rightarrow \frac{d}{dt}(AC) = \frac{dA}{dt} \cdot C.$$

Notice that CA is not defined.

The system (*) takes the form of a matrix equation

$$\frac{dX}{dt} = P(t).X + f(t),$$

where $P(t) = [P_{ij}]$ is a coefficient matrix; and $X = [x_i]$ and $f(t) = [f_i(t)]$ are column vectors.

Example: Write the system

$$x_1' = 4x_1 - 3x_2,$$
$$x_2' = 6x_1 - 7x_2$$

in the form $X' = P(t)X + f(t)$. Then verify that the vector functions $X_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ and $X_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$ are both solutions of the matrix differential equation with the coeffient matrix P.

$$x_1' = 4x_1 - 3x_2,$$
$$x_2' = 6x_1 - 7x_2$$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad P(t) = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \qquad X'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

$$f(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X' = P(t).X$$

$$\left.\begin{array}{l} X_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \Rightarrow X'(t) = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \\ P(t).X_1(t) = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} . \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \end{array}\right\} X_1'(t) = P(t).X_1(t).$$

$$X_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} \Rightarrow X'(t) = \begin{bmatrix} -5.e^{-5t} \\ -15e^{-5t} \end{bmatrix},$$

$$P(t).X_2(t) = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix}$$

$$\Rightarrow X_2'(t) = P(t).X_2(t).$$

THEOREM 1 Principle of Superposition

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n solutions of the homogeneous linear equation $\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x}$ the open interval I. If $c_1, c_2, \ldots, c_n$ are constants, then the linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

is also a solution on I.

***Proof*:** We know that $\mathbf{x}_i' = \mathbf{P}(t)\mathbf{x}_i$ for each i $(1 \leqq i \leqq n)$, so it follows immediately that

$$\begin{aligned}\mathbf{x}' &= c_1\mathbf{x}_1' + c_2\mathbf{x}_2' + \cdots + c_n\mathbf{x}_n' \\ &= c_1\mathbf{P}(t)\mathbf{x}_1 + c_2\mathbf{P}(t)\mathbf{x}_2 + \cdots + c_n\mathbf{P}(t)\mathbf{x}_n \\ &= \mathbf{P}(t)(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n).\end{aligned}$$

That is, $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, as desired. The remarkable simplicity of this proof demonstrates clearly one advantage of matrix notation. ■

Independence and General Solutions

The vector-valued functions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$, *not all zero*, such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0} \tag{12}$$

for all t in I. Otherwise, they are **linearly independent.** Equivalently, they are linearly independent provided that no one of them is a linear combination of the others.

Example: Determine whether the following matrix functions are linearly independent or not.

a) $x_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$. $\quad x_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

$$c_1 x_1(t) + c_2 x_2(t) = 0$$

$$c_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ 3e^{5t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 9e^{-3t} - 2e^{-3t} = 7e^{-3t} \neq 0$$

Thus $x_1(t)$ and $x_2(t)$ are linearly independent.

b) $x_1(t) = \begin{bmatrix} 2\sin t\cos t \\ \sin t \end{bmatrix}$, $\quad x_2(t) = \begin{bmatrix} \sin 2t \\ \sin t \end{bmatrix}$

Since $x_1(t) = x_2(t)$, they are linearly dependent.

Definition: If $x_1, x_2, \dots, x_n$ are matrix-valued column vectors, then their wronskian is the $n \times n$ determinat

$$W(x_1, x_2, \dots, x_n)(t) = W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix},$$

THEOREM 2 Wronskians of Solutions

Suppose that $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are n solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I. Suppose also that $\mathbf{P}(t)$ is continuous on I. Let

$$W = W(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n).$$

Then

- If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are linearly dependent on I, then $W = 0$ at every point of I.
- If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are linearly independent on I, then $W \neq 0$ at each point of I.

Thus there are only two possibilities for solutions of homogeneous systems: Either $W = 0$ at *every* point of I, or $W = 0$ at *no* point of I.

The general solution of the homogeneous n × n system x '= P(t)x is a linear combination

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

of any n given linearly independent solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$.

The general solution [redacted] of the homogeneous linear system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ can be written in the form

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c},$$

where

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{bmatrix}$$

is the $n \times n$ matrix whose *column vectors* are the linearly independent solutions $\mathbf{x}_1$, $\mathbf{x}_2, \ldots, \mathbf{x}_n$, and where $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}^T$ is the vector of coefficients in the linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t).$$

THEOREM 3 General Solutions of Homogeneous Systems

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{15}$$

for all t in I.

Example: $x(t) = c_1 e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a general solution of the system

$$x'(t) = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} x(t).$$

***Proof*:** Let a be a fixed point of I. We first show that there exist numbers c_1, $c_2, \ldots, c_n$ such that the solution

$$\mathbf{y}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{16}$$

has the same initial values at $t = a$ as does the given solution $\mathbf{x}(t)$; that is, such that

$$c_1\mathbf{x}_1(a) + c_2\mathbf{x}_2(a) + \cdots + c_n\mathbf{x}_n(a) = \mathbf{x}(a). \tag{17}$$

Let $\mathbf{X}(t)$ be the $n \times n$ matrix with column vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, and let $\mathbf{c}$ be the column vector with components $c_1, c_2, \ldots, c_n$. Then Eq. (17) may be written in the form

$$\mathbf{X}(a)\mathbf{c} = \mathbf{x}(a). \tag{18}$$

The Wronskian determinant $W(a) = |\mathbf{X}(a)|$ is nonzero because the solutions $\mathbf{x}_1$, $\mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent. Hence the matrix $\mathbf{X}(a)$ has an inverse matrix $\mathbf{X}(a)^{-1}$. Therefore the vector $\mathbf{c} = \mathbf{X}(a)^{-1}\mathbf{x}(a)$ satisfies Eq. (18), as desired.

Finally, note that the given solution $\mathbf{x}(t)$ and the solution $\mathbf{y}(t)$ of Eq. (16)—with the values of c_i determined by the equation $\mathbf{c} = \mathbf{X}(a)^{-1}\mathbf{x}(a)$—have the same initial values (at $t = a$). It follows from the existence-uniqueness theorem that $\mathbf{x}(t) = \mathbf{y}(t)$ for all t in I. This establishes Eq. (15). ■

Initial Value Problems

Suppose now that we wish to solve the *initial value problem*

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x}, \quad \mathbf{x}(a) = \mathbf{b},$$

where the initial vector $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$ is given.

Example: a) verify that

$$x_1(t) = [2e^{t} \;\; 2e^{t} \;\; e^{t}]^T, \quad x_2(t) = [2e^{3t} \;\; 0 \;\; -e^{3t}]^T$$

and $x_3(t) = [2e^{5t} \;\; -2e^{5t} \;\; e^{5t}]^T$ are solutions of the

equation $\frac{dX}{dt} = P.X$, where $P = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}$

b) Write the general solution of the system in (a)

c) Solve the initial value problem

$$\frac{dx}{dt} = Px \qquad x(0) = [0 \quad 2 \quad 6]^T$$

(a) $$\frac{dX_1}{dt} = \begin{bmatrix} 2e^t \\ 2e^t \\ t \end{bmatrix} \qquad PX_1 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2e^t \\ 2e^t \\ t \end{bmatrix} = \begin{bmatrix} 2e^t \\ 2e^t \\ t \end{bmatrix} = X_1'$$

$$\frac{dX_2}{dt} = \begin{bmatrix} 6e^{3t} \\ 0 \\ -3e^{3t} \end{bmatrix} \qquad PX_2 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix} = \begin{bmatrix} 6e^{3t} \\ 0 \\ -3e^{3t} \end{bmatrix} = X_2'$$

$$\frac{dX_3}{dt} = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix} \qquad PX_3 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix} = X_3'$$

(b)

$$W(X_1, X_2, X_3) \begin{vmatrix} 2e^{t} & 2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & -e^{3t} & e^{5t} \end{vmatrix} = e^{t} \cdot e^{3t} \cdot e^{5t} \begin{vmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\overset{C_3 \to C_1 + C_3}{=} e^{9t} \cdot \begin{vmatrix} 2 & 2 & 4 \\ 2 & 0 & 0 \\ 1 & -1 & 2 \end{vmatrix} = e^{9t} \cdot (-2) \cdot \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = -2e^{9t}(4+4)$$

$$= -16 \cdot e^{9t} \neq 0$$

X_1, X_2, X_3 are linearly independent.

$X(t) = C_1 X_1(t) + C_2 X_2(t) + C_3 X_3(t)$ is a general solution.

(c)

$$X(t) = \begin{bmatrix} 2e^{t} & 2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & -e^{3t} & e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & -1 & 1 & 6 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{2}R_2]{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 6 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & -2 & 0 & 6 \end{array}\right] \xrightarrow[R_3 \to \left(\frac{-1}{2}\right)R_3]{R_2 \to (-1)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3/2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$c_1 = 2, \quad c_2 = -3, \quad c_3 = 1$$

$$X(t) = \begin{bmatrix} 2e^{t} & 2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & -e^{3t} & e^{5t} \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$