Example:

Determine the appropriate form for a particular solution of

$$y'' + 6y' + 13y = e^{-3x}\cos 2x.$$

$r^2 + 6r + 13 = 0$ (Characteristic Eq.)

$(r+3)^2 + 4 = 0 \Rightarrow r+3 = \mp 2i \quad r_{1,2} = -3 \mp 2i$

$r_1 = -3 + 2i \quad r_2 = -3 - 2i$

$y_c(x) = e^{-3x} \cdot (c_1 \cos 2x + c_2 \sin 2x)$

$y'' + 6y + 13y = e^{-3x} \cos 2x$

There is a duplication with the term $e^{-3x}\cos 2x$, $s = 1$

$y_p(x) = x.\left(Ae^{-3x}\cos 2x + Be^{-3x}.\sin 2x\right)$

Example:

Determine the appropriate form for a particular solution of the fifth-order equation

$$(D-2)^3(D^2+9)y = x^2e^{2x} + x\sin 3x.$$

Characteristic Eq. : $(r-2)^3 \cdot (r^2+9) = 0$

$r_1 = 2 \quad s = 3 \quad , \quad r_{2,3} = \mp 3i \quad s = 1.$

$$y_c(x) = (c_1 + c_2x + c_3x^2)\, e^{2x} + (c_4\cos 3x + c_5 \sin 3x)$$

$$(D-2)^3(D^2+9)y = x^2e^{2x} + x\sin 3x$$

$$y_p(x) = x^3(A + Bx + Cx^2)\,e^{2x}$$
$$+ x\,((D + Ex)\sin 3x + (F + Gx)\cos 3x)$$

Variation of Parameters

Finally, let us point out the kind of situation in which the method of undetermined coefficients cannot be used. Consider, for example, the equation

$$y'' + y = \tan x, \tag{17}$$

which at first glance may appear similar to those considered in the preceding examples. Not so; the function $f(x) = \tan x$ has *infinitely many* linearly independent derivatives

$$\sec^2 x, \quad 2\sec^2 x \tan x, \quad 4\sec^2 x \tan^2 x + 2\sec^4 x, \quad \ldots .$$

We discuss here the method of **variation of parameters**, which—in principle (that is, if the integrals that appear can be evaluated)—can always be used to find a particular solution of the nonhomogeneous linear differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x), \tag{18}$$

provided that we already know the general solution

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \tag{19}$$

of the associated homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \tag{20}$$

The basic idea of the method of variation of parameters:

1) We replace the constants or parameters $c_1, c_2, \dots, c_n$ in the complementary function with variables functions $u_1, u_2, \dots, u_n$ of x.

$$c_1 \to u_1(x) \qquad c_2 \to u_2(x) \quad \dots \quad c_n \to u_n(x)$$

2) We ask whether it's possible to choose these functions in such way that the combination

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \dots + u_n(x)y_n(x)$$

is a particular solution of the non-homogeneous equation.

It turns out that this is always possible.

The method is essentially the same for all orders $n \geq 2$, but we will describe it in detail only for the case $n=2$.

$$L(y) = y'' + P(x)\, y' + Q(x)\, y = f(x)$$

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$L(y_p) = f$$

$$y_p' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$$

To avoid the appearance of the second derivatives u_1'' and u_2'', the additional condition that we now impose is that $u_1' y_1 + u_2' y_2 = 0$

Then, $y_p' = u_1 y_1' + u_2 y_2'$, $y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$

$$y_p'' + P y_p' + Q y_p = u_1' y_1' + u_2' y_2' + u_1 (y_1'' + P y_1' + Q y_1)$$
$$+ u_2 (y_2'' + P y_2' + Q y_2)$$

since both y_1 and y_2 satisfy the homogeneous equations

$$y_1'' + P y_1' + Q y_1 = 0 \quad \text{and} \quad y_2'' + P y_2' + Q y_2 = 0$$

$$y_p'' + P y_p' + Q y_p = u_1' y_1' + u_2' y_2' = f(x)$$

Hence we obtain a system

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$

!

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \neq 0$$

$$u_1' = \frac{1}{W(y_1, y_2)} \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} = \frac{-fy_2}{W(y_1,y_2)}$$

$$\frac{du_1}{dx} = \frac{-f(x)\,y_2(x)}{W(x)} \Rightarrow u_1 = -\int \frac{f(x)\,y_2(x)}{W(x)}\,dx + c_1$$

We take $c_1 = 0$ since we look for a particular solution y_p.

$$\frac{du_2}{dx} = \frac{1}{W(x)} \cdot \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} = \frac{y_1(x)\,f(x)}{W(x)}$$

$$u_2(x) = +\int \frac{y_1(x)\,f(x)}{W(x)}\,dx + c_2 \quad (c_2 = 0).$$

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

$$y_p(x) = -y_1(x) \cdot \int \frac{f(x)\,y_2(x)}{W(x)}\,dx + y_2(x) \int \frac{y_1(x)\,f(x)}{W(x)}\,dx.$$

THEOREM 1 Variation of Parameters

If the nonhomogeneous equation $y'' + P(x)y' + Q(x)y = f(x)$ has complementary function $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$, then a particular solution is given by

$$y_p(x) = \left(-\int \frac{y_2(x) f(x)}{W(x)}\,dx\right) y_1(x) + \left(\int \frac{y_1(x) f(x)}{W(x)}\,dx\right) y_2(x), \qquad (33)$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation.

Example:

Find a particular solution of the equation $y'' + y = \tan x$.

$y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r^2 = -1 \quad r_{1,2} = \mp i$

$y_c(x) = c_1 \cos x + c_2 \sin x$

$\rightarrow y_p(x) = u_1(x) \cos x + u_2(x) \sin x$

$u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = f(x)$

$u_1' \cos x + u_2' \sin x = 0$

$-u_1' \sin x + u_2' \cos x = \tan x$

$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$

$$u_1 = -\int \frac{f(x)\,y_2(x)}{W(x)}\,dx$$

$$u_1 = -\int \frac{\tan x \cdot \sin x}{1}\,dx = -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= \sin x - \ln|\sec x + \tan x| + c_1$$

$$(c_1 = 0)$$

$$u_2 = \int \frac{f(x)\,y_1(x)}{W(x)}\,dx = \int \frac{\tan x \cdot \cos x}{1}\,dx = \int \sin x = -\cos x + c_2$$

$$(c_2 = 0)$$

$$y_p(x) = u_1(x)\cos x + u_2 \sin x$$

$$= \cos x \cdot (\sin x - \ln|\sec x + \tan x|) \cdot - \sin x \cos x$$

$$= -\cos x \ln|\sec x + \tan x| .$$

Example: The complementary func. $y_c(x) = c_1 x^2 + c_2 x^3$ of the equation $x^2 y'' + 4x y' + 6y = x^3$ is given. Find its particular solution.

$$y'' + \frac{4}{x} y' + \frac{6}{x^2} y = x$$

$$y_c = c_1 x^2 + c_2 x^3$$

$$y_p(x) = u_1(x)\, x^2 + u_2(x)\, x^3$$

$$u_1' x^2 + u_2' x^3 = 0$$

$$2u_1' x + 3u_2' x^2 = x$$

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 \neq 0.$$

$$u_1' = \frac{1}{W} \begin{vmatrix} 0 & x^3 \\ x & 3x^2 \end{vmatrix} = \frac{-x^4}{x^4} = -1 \quad \Rightarrow \quad u_1(x) = -x + c_1 \qquad (c_1 = 0)$$

$$u_2' = \frac{1}{W} \begin{vmatrix} x^2 & 0 \\ 2x & x \end{vmatrix} = \frac{x^3}{x^4} = \frac{1}{x} \quad \Rightarrow \quad u_2(x) = \ln|x| + c_2 \qquad (c_2 = 0)$$

$$y_p(x) = u_1(x)\,x^2 + u_2(x)\,x^3$$

$$= (-x)\,x^2 + (\ln|x|)\,x^3$$

$$= x^3(\ln|x| - 1).$$

Eigenvalues and Eigenvectors

DEFINITION Eigenvalues and Eigenvectors

The number λ is said to be an **eigenvalue** of the $n \times n$ matrix $\mathbf{A}$ provided there exists a *nonzero* vector $\mathbf{v}$ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \tag{1}$$

in which case the vector $\mathbf{v}$ is called an **eigenvector** of the matrix $\mathbf{A}$. We also say that the eigenvector $\mathbf{v}$ is **associated** with the eigenvalue λ, or that the eigenvalue λ **corresponds** to the eigenvector $\mathbf{v}$.

Remark 1 If $\mathbf{v} = \mathbf{0}$, then the equation $\mathbf{Av} = \lambda\mathbf{v}$ holds for every scalar λ and hence is of no significance. This is why only *nonzero* vectors qualify as eigenvectors in the definition.

$Av = \lambda v$

Remark 2 Let λ and $\mathbf{v}$ be an eigenvalue and associated eigenvector of the matrix $\mathbf{A}$. If k is any nonzero scalar and $\mathbf{u} = k\mathbf{v}$, then

$$\begin{aligned}\mathbf{Au} &= \mathbf{A}(k\mathbf{v}) = k(\mathbf{Av}) \\ &= k(\lambda\mathbf{v}) = \lambda(k\mathbf{v}) = \lambda\mathbf{u},\end{aligned}$$

so $\mathbf{u} = k\mathbf{v}$ is also an eigenvector associated with λ. Thus, *any nonzero scalar multiple of an eigenvector is also an eigenvector and is associated with the same eigenvalue*. In Example 1, for instance, $\mathbf{u}_1 = -3\mathbf{v}_1 = \begin{bmatrix} -6 & -3 \end{bmatrix}^T$ is an eigenvector associated with $\lambda_1 = 2$, and $\mathbf{u}_2 = 4\mathbf{v}_2 = \begin{bmatrix} 12 & 8 \end{bmatrix}^T$ is an eigenvector associated with $\lambda_2 = 1$. ■

Remark: $Av = \lambda v \Rightarrow Av = \lambda I v$

$$\Rightarrow Av - \lambda I v = 0$$

$$(A - \lambda I)v = 0.$$

This system has a nontrivial solution $v \neq 0$ if and only if the determinant

$$\det(A - \lambda I) = |A - \lambda I|$$

is zero.

The equation $|A - \lambda I| = 0$ is called the **characteristic equation** of the square matrix A.

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

is obtained simply by subtracting λ from each diagonal element of $\mathbf{A}$. If we think of expanding the determinant by cofactors, we see that $|\mathbf{A} - \lambda\mathbf{I}|$ is a *polynomial* in the variable λ, and that the highest power of λ comes from the product of the diagonal elements of the matrix in (4). Therefore, the characteristic equation of the $n \times n$ matrix $\mathbf{A}$ takes the form

$$(-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0, \tag{5}$$

an nth-degree polynomial equation in λ.

THEOREM 1 The Characteristic Equation

The number λ is an eigenvalue of the $n \times n$ matrix $\mathbf{A}$ if and only if λ satisfies the characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0. \tag{3}$$

ALGORITHM Eigenvalues and Eigenvectors

To find the eigenvalues and associated eigenvectors of the $n \times n$ matrix $\mathbf{A}$:

1. First solve the characteristic equation

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

2. Then, for *each* eigenvalue λ thereby found, solve the linear system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

to find the eigenvectors associated with λ.

Example

Find the eigenvalues and associated eigenvectors of the matrices

a)
$$\mathbf{A} = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}.$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{vmatrix} = (5-\lambda)(-4-\lambda) + 14$$

$$= -20 - 5\lambda + 4\lambda + \lambda^2 + 14$$

$$= \lambda^2 - \lambda - 6 = 0$$

$$\Rightarrow (\lambda-3)(\lambda+2) = 0$$

Thus the matrix A has two eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 3$.

case 1: $\lambda_1 = -2$

$$(A - \lambda I)v = 0$$

$$(A + 2I)v = 0$$

$$\begin{bmatrix} 7 & 7 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left. \begin{array}{l} 7x + 7y = 0 \\ -2x - 2y = 0 \end{array} \right\} \begin{array}{l} x + y = 0 \\ y = t, x = -t \end{array}$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Hence, to within a constant multiple, the only eigenvector associated with $\lambda_1 = -2$ is $v_1 = [-1 \ 1]^T$

case 2: $\lambda_2 = 3$

$$(A - 3I)v = 0$$

$$\begin{bmatrix} 2 & 7 \\ -2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left. \begin{array}{l} 2x + 7y = 0 \\ -2x - 7y = 0 \end{array} \right\} \begin{array}{l} 2x + 7y = 0 \\ y = 2t \Rightarrow x = -7t \end{array}$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7t \\ 2t \end{bmatrix} = t \cdot \begin{bmatrix} -7 \\ 2 \end{bmatrix}$$

To within a constant multiple, the only eigenvector associated with $\lambda_2 = 3$ is $v_2 = [-7 \ 2]^T$

b)

$$\mathbf{A} = \begin{bmatrix} 0 & 8 \\ -2 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 8 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 16 = 0 \Rightarrow \lambda^2 = -16$$

$$\lambda_1 = 4i \quad \lambda_2 = -4i$$

case 1: $\lambda_1 = 4i$ $\quad (A - 4iI)v = 0$

$$\begin{bmatrix} -4i & 8 \\ -2 & -4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4ix + 8y = 0$

$2i /\ -2x - 4iy = 0$

$$\left\{ \begin{array}{l} -4ix + 8y = 0 \\ \downarrow \\ -ix + 2y = 0 \end{array} \right.$$

$2y = ix$

$y = it$, $x = 2t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t \\ it \end{bmatrix} = t \cdot \begin{bmatrix} 2 \\ i \end{bmatrix}$$

Thus $v_1 = [2 \quad i]^T$ is a complex eigenvector associated with the complex eigenvalue $\lambda_1 = 4i$.

Case 2: $\lambda_2 = -4i$

$(A + 4iI) \cdot v = 0$

$\begin{bmatrix} 4i & 8 \\ -2 & 4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{aligned} 4ix + 8y &= 0 \\ -2i / \ -2x + 4iy &= 0 \end{aligned}$ $\left.\right\} \begin{aligned} 4ix + 8y &= 0 \\ 2y &= -ix \end{aligned}$

$y = -it, \quad x = 2t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t \\ -it \end{bmatrix} = t \cdot \begin{bmatrix} 2 \\ -i \end{bmatrix}.$

Thus $v_2 = [2 \ -i]^T$ is a complex eigenvector associated with the complex eigenvalue $\lambda_2 = -4i$.

c) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$|I - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1.$$

$$(I - I).v = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The single eigenvalue $\lambda = 1$ corresponds to two linearly independent eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

d)

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0 \qquad \lambda_1 = \lambda_2 = 2.$$

$$(A - 2I).V = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} 3y = 0 \\ y = 0. \end{matrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x. \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigen valu $\lambda = 2$ corresponds to the single eigenvector $V = [1\ 0]^T$

Remark: If the eigenvectors $v_1, v_2, \ldots, v_k$ are assosiated with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of the matrix A, then these k eigenvectors are linearly independent.

! If $A = [a_{ij}]$ is a $n \times n$ matrix, then the trace of A is defined to be

$$Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Example: If A is a 2×2 matrix, then

$$\det(A - \lambda I) = (-\lambda)^2 + Tr(A)(-\lambda) + \det A.$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= ad - a\lambda - \lambda d + (-\lambda)^2 - bc$$

$$= (-\lambda)^2 + (a+d)(-\lambda) + ad - bc$$

Note that

$a + d = Tr(A) = \lambda_1 + \lambda_2$

$$= (-\lambda)^2 + Tr(A)(-\lambda) + \det(A).$$

Eigenspaces

Let λ be a fixed eigenvalue of the $n \times n$ matrix $\mathbf{A}$. Then the set of all eigenvectors associated with $\mathbf{A}$ is the set of all nonzero solution vectors of the system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

The solution space of this system is called the **eigenspace** of $\mathbf{A}$ associated with the eigenvalue λ.

Example: Find the eigenspace of the following matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$$

and state bases for the eigenspace.

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ -4 & 6-\lambda & 2 \\ 16 & -15 & -5-\lambda \end{vmatrix} = (3-\lambda)\cdot\begin{vmatrix} 6-\lambda & 2 \\ -15 & -5-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left((6-\lambda)(-5-\lambda)+30\right)$$

$$= \lambda(3-\lambda)(\lambda-1) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 3.$$

Case 1: $\lambda_1 = 0$. $\quad Av = 0 \quad$ Null $(A) = ?$

$$\begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} 3x &= 0 \\ -4x + 6y + 2z &= 0 \\ 16 \; -15y - 5z &= 0 \end{aligned}$$

$$x = 0 \Rightarrow \left.\begin{aligned} 6y + 2z &= 0 \\ -15y - 5z &= 0 \end{aligned}\right\} \begin{aligned} 3y + z &= 0 \\ 3y + z &= 0 \end{aligned} \qquad y = t \Rightarrow z = -3t$$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ -3t \end{bmatrix} = t \cdot \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \qquad \mathrm{Null}(A) = \left\{ t \cdot \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

is the eigenspace of A associated with the eigenvalue $\lambda_1 = 0$.

$\{(0,1,-3)\}$ is a basis for Null (A).
(eigenspace)

Case 2: $\lambda_2 = 1$. $(A-I)v = 0$ $Null(A-I) = ?$

$$\begin{bmatrix} 2 & 0 & 0 \\ -4 & 5 & 2 \\ 16 & -15 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x = 0$$
$$-4x + 5y + 2z = 0$$
$$16x - 15y - 6z = 0$$

$$x = 0 \Rightarrow \left.\begin{array}{l} 5y + 2z = 0 \\ -15y - 6z = 0 \end{array}\right\} \left.\begin{array}{l} 5y + 2z = 0 \\ 5y + 2z = 0 \end{array}\right\} \quad z = 5t \Rightarrow y = -2t$$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2t \\ 5t \end{bmatrix} = t \cdot \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

$Null(A-I) = \left\{ t \cdot \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} \mid t \in \mathbb{R} \right\}$ is the eigenspace of A associated with the eigenvalue $\lambda_2 = 1$.

$\{(0, -2, 5)\}$ is a basis for $Null(A-I)$.
(eigenspace)

Case 3: $\lambda_3 = 3$ $\quad (A - 3I).v = 0 \quad \text{Null}(A-3I) \overset{?}{=}$

$$\begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & 2 \\ 16 & -15 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x + 3y + 2z = 0$

$16x - 15y - 8z = 0$

$$\begin{bmatrix} -4 & 3 & 2 \\ 16 & -15 & -8 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to 4R_1 + R_2} \begin{bmatrix} -4 & 3 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} -4 & 0 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \to -\frac{1}{3} R_2]{R_1 \to -\frac{1}{4} R_1} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x - \frac{z}{2} = 0 \qquad z = 2t \Rightarrow x = t$

$y = 0$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\text{Null}(A - 3I) = \left\{ t \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \;\middle|\; t \in \mathbb{R} - \{0\} \right\}$

is the eigenspace of A associated with the eigenvalue $\lambda_3 = 3$.

$\{ (1,0,2) \}$ is a basis for $\text{Null}(A-3I)$.
(eigenspace)

Remark: Substitution of $\lambda = 0$ in the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ yields $|\mathbf{A}| = 0$. Therefore, $\lambda = 0$ is an eigenvalue of the matrix $\mathbf{A}$ if and only if $\mathbf{A}$ is singular: $|\mathbf{A}| = 0$. ■