THOMAS' CALCULUS

MEDIA UPGRADE

Chapter 11

Infinite Sequences and Series

11.1

Sequences

<u>Def'n</u>: A sequence is an ordered list of some real numbers, called terms. i.e.

$$\{a_n\} = \{a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots \mid a_i \in \mathbb{R},\ i = 1, 2, 3 \dots\}$$

a_1 : 1^{st} term

a_2 : 2^{nd} term

$\vdots$

a_n : n^{th} term (or general term)

$\vdots$

$\mathbb{N} = \mathbb{Z}_+ = \{1, 2, 3, \dots\}$: set of natural numbers or positive integers

We use $\mathbb{N}$ to index the terms of sequences: the index of the n^{th} term a_n is the integer n.

Alternatively, a sequence $\{a_n\}$ can be seen as a function

$$a_n : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longmapsto a_n \in \mathbb{R}$$

DEFINITION **Infinite Sequence**

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

<u>Ex</u> $a_n = \sqrt{n}$ <u>or</u> $\{a_n\} = \{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots, \sqrt{n}, \ldots\}$

<u>or</u> $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$

<u>Ex</u> $a_n = (-1)^{n+1}\frac{1}{n}$ <u>or</u> $\{a_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\}$

Sequences can be described by $\boxed{\text{general terms}}$ or $\boxed{\text{listing terms.}}$

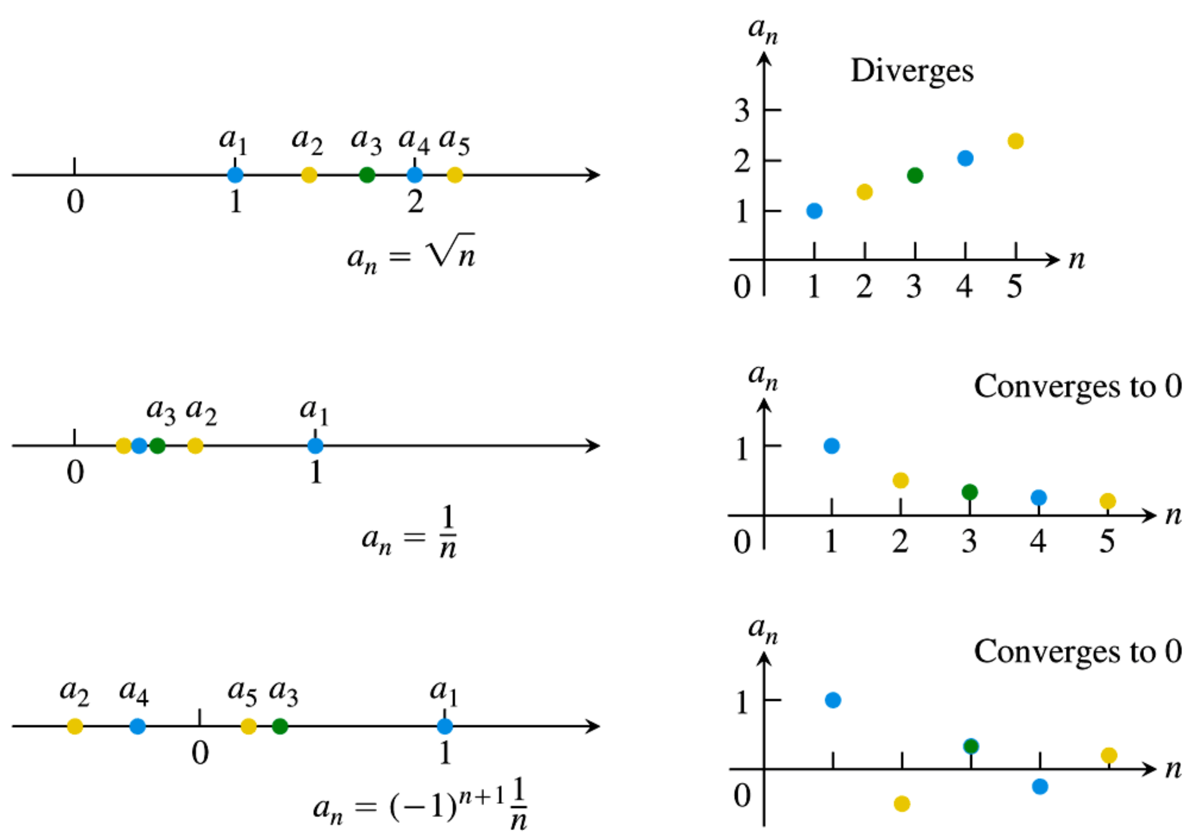


FIGURE 11.1 Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

Convergence – Divergence – Limit (Informal)

If the numbers a_n approach a single finite value $L \in \mathbb{R}$ as n increases, then the sequence $\{a_n\}$ is said to converge to $L \in \mathbb{R}$ and we write $\lim\limits_{n\to\infty} a_n = \lim a_n = L$.

Ex $a_n = \frac{1}{n} \Rightarrow \{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ then

$\lim a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$. Thus $\{a_n\}$ is convergent, and it converges to 0.

If there is no such $L \in \mathbb{R}$, then $\{a_n\}$ is divergent.

Ex $a_n = (-1)^{n+1} = \cos[(n-1)\pi]$ so that

$\{a_n\} = \{1, -1, 1, -1, \dots\}$, then $\lim a_n$ DNE

and $\{a_n\}$ is a divergent sequence.

Ex $a_n = \sqrt{n}$ $\Longrightarrow$ $\lim a_n = \lim\limits_{n\to\infty} \sqrt{n} = \infty \notin \mathbb{R}$

The sequence $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$ is divergent.

In this case we may also say it diverges to infinity.

DEFINITIONS **Converges, Diverges, Limit**

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \qquad \Rightarrow \qquad |a_n - L| < \epsilon .$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 11.2). or $\lim a_n = L$ (as we only evaluate limit at ∞)

<u>Ex:</u> Show that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Let $\varepsilon > 0$ be given. We want to show that there exists an integer N such that for all n, if $n > N$ then $\left|\frac{1}{n} - 0\right| < \varepsilon$. (*)

The inequality $\left|\frac{1}{n} - 0\right| < \varepsilon$ holds if $n > \frac{1}{\varepsilon}$. So, choose N to be any integer greater than $\frac{1}{\varepsilon}$, so that the above statement (*) holds.

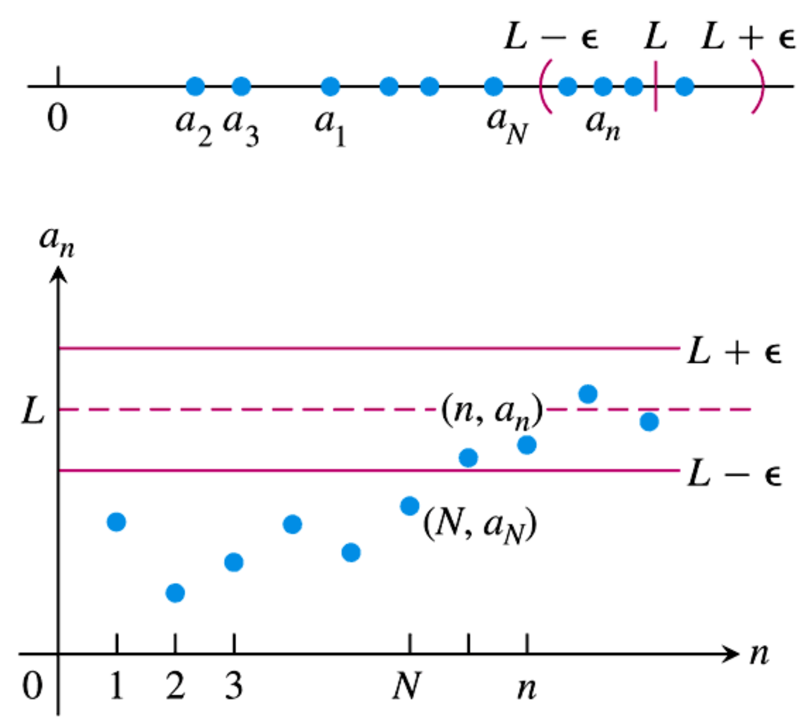


FIGURE 11.2 $a_n \longrightarrow L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n's after a_N lie within ϵ of L.

Ex: $\{a_n\} = \{(-1)^{n+1}\} = \{1, -1, 1, -1, 1, -1, \dots\}$ is a divergent seq.

$\{b_n\} = \left\{\frac{1+(-1)^{n+1}}{2} n\right\} = \{1, 0, 3, 0, 5, 0, 7, \dots\}$ is a divergent seq.

$\{c_n\} = \{(-1)^{n+1} \cdot n\} = \{1, -2, 3, -4, 5, -6, \dots\}$ is a divergent seq.

DEFINITION Diverges to Infinity

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty.$$

In the above example $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are divergent sequences but they don't diverge to infinity or neg. infinity.

$\{d_n\} = \left\{\frac{n^2+1}{n}\right\} = \left\{2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \dots\right\}$ diverges to infinity.

THEOREM 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ if $B \neq 0$

The theorem holds when $\{a_n\}$ and $\{b_n\}$ are convergent sequences.

From this theorem we cannot arive to some conclusions such as:

* If $\{a_n + b_n\}$ is a convergent sequence then both $\{a_n\}$ and $\{b_n\}$ are convergent: False!

Ex: $\{a_n\} = \{n\}$ and $\{b_n\} = \{-n\}$ are divergent, but $\{a_n + b_n\} = \{0\} \to 0$.

EXAMPLE 3 Applying Theorem 1

By combining Theorem 1 with the limits of Example 1, we have:

(a) $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty}\frac{1}{n} = -1 \cdot 0 = 0$ Constant Multiple Rule and Example 1a

(b) $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1 - 0 = 1$ Difference Rule and Example 1a

(c) $\lim_{n\to\infty}\frac{5}{n^2} = 5 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ Product Rule

(d) $\lim_{n\to\infty}\frac{4-7n^6}{n^6+3} = \lim_{n\to\infty}\frac{(4/n^6)-7}{1+(3/n^6)} = \frac{0-7}{1+0} = -7.$ Sum and Quotient Rules

THEOREM 2 The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

EXAMPLE 4 Applying the Sandwich Theorem

Since $1/n \to 0$, we know that

(a) $\dfrac{\cos n}{n} \to 0$ because $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$;

(b) $\dfrac{1}{2^n} \to 0$ because $0 \le \dfrac{1}{2^n} \le \dfrac{1}{n}$;

(c) $(-1)^n \dfrac{1}{n} \to 0$ because $-\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}$.

THEOREM 3 The Continuous Function Theorem for Sequences
Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

By this theorem $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$

<u>Ex:</u> Show that $\sqrt{\frac{n+1}{n}} \to 1$

$a_n = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1$ as $\frac{1}{n} \to 0$. Let $f(x) = \sqrt{x}$. f is cont. at $x = 1$, then by Cont. Funct. Thm $f(a_n) = \sqrt{\frac{n+1}{n}} \to f(1) = \sqrt{1} = 1$

<u>Ex</u> Is $\{2^{1/n}\}$ convergent?

We know that the sequence $\{\frac{1}{n}\}$ converges to 0.

Since $f(x) = 2^x$ is a continuous function at $x=0$,

by above theorem

$$\lim 2^{1/n} = \lim f\left(\frac{1}{n}\right) = f\left(\lim \frac{1}{n}\right) = f(0) = 2^0 = 1.$$

Therefore the sequence $\{2^{1/n}\}$ converges to 1.

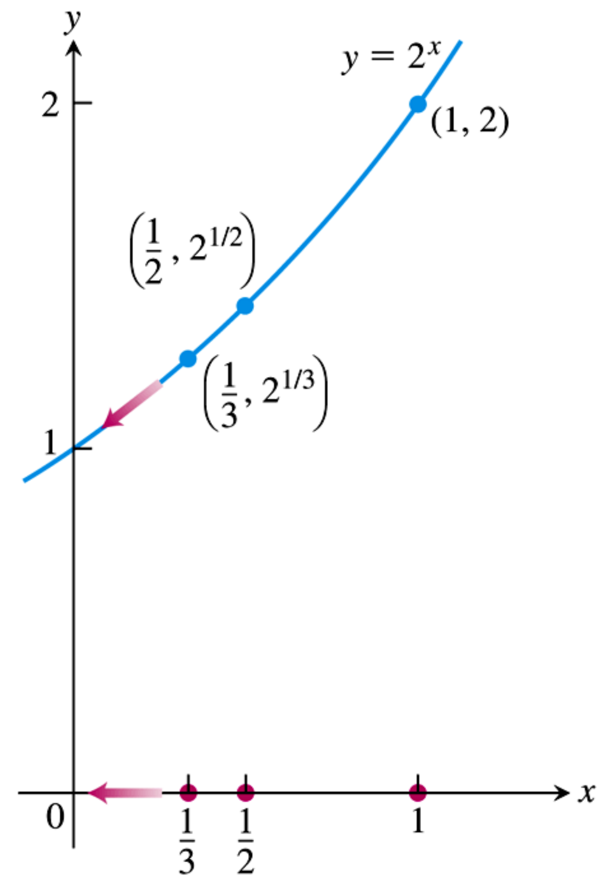


FIGURE 11.3 As $n \longrightarrow \infty$, $1/n \longrightarrow 0$ and $2^{1/n} \longrightarrow 2^0$ (Example 6).

THEOREM 4
Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x \to \infty} f(x) = L \quad \Rightarrow \quad \lim_{n \to \infty} a_n = L.$$

The converse of the theorem is false. That is, if $\lim_{n \to \infty} a_n = L$ it need not be true that $\lim_{x \to \infty} f(x) = L$.

For example, $\lim_{n \to \infty} \cos(2\pi n) = 1$ since $\cos(2\pi n) = 1$ for all $n \in \mathbb{N}$.

But $\lim_{x \to \infty} \cos(2\pi x)$ does not exist.

With this theorem, we can apply L'Hospital's Rule to evaluate the limits of sequences.

Ex Show that $\left\{ \frac{\ln n}{n} \right\}$ converges to 0.

The function $f(x) = \frac{\ln x}{x}$ is defined for all $x \geq 1$ and

$$a_n = \frac{\ln n}{n} = f(n) \quad \text{for all } n \in \mathbb{N}.$$

Since $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \frac{\ln x}{x} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{L'H}{=}} \lim\limits_{x \to \infty} \frac{1/x}{1} = 0$,

we conclude that $\lim\limits_{n \to \infty} a_n = \lim\limits_{x \to \infty} f(x) = 0$.

<u>Ex</u> Is $\left\{ \left(\frac{n+1}{n-1}\right)^n \right\}_{n=2}^{\infty}$ convergent?

Let $f(x) = \left(\frac{x+1}{x-1}\right)^x$ so that $f(n) = a_n = \left(\frac{n+1}{n-1}\right)^n$ for $n = 2, 3, \dots$

$\lim_{x \to \infty} \left(\frac{x+1}{x-1}\right)^x$ has 1^∞ indeterminate form.

Taking $\ln$ of $f(x)$ changes to $\infty \cdot 0$:

$$\lim_{x \to +\infty} \ln f(x) = \lim_{x \to +\infty} x \ln\left(\frac{x+1}{x-1}\right) \overset{[\infty \cdot 0]}{=} \lim_{x \to +\infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{1/x} : \left[\frac{0}{0}\right]$$

$$\overset{L'H}{=} \lim_{x \to \infty} \frac{-2/(x^2-1)}{-1/x^2} = \lim_{x \to +\infty} \frac{2x^2}{x^2-1} = 2$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} e^{\ln f(x)} = e^2 \implies \lim a_n = \lim_{x \to +\infty} f(x) = e^2$$

THEOREM 5

The following six sequences converge to the limits listed below:

1. $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\displaystyle\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\displaystyle\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\displaystyle\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$
7. $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Proof of this theorem is mostly based on indeterminate forms and L'H. Rule. Part 6 & 7 are proved by using the Sandwich Theorem.

Notation: $n!$ means the product $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$

$$1! = 1$$
$$2! = 1 \cdot 2 = 2$$
$$3! = 1 \cdot 2 \cdot 3 = 6$$

We define $0! = 1$.

Note that $(n+1)! = (n+1) \cdot (n!)$.

The above thm part 6 suggests that factorials grow even faster than exponentials (with any base!).

Ex $\lim\limits_{n\to\infty} \dfrac{100^n}{n!} = 0$ (6)

Ex $\lim \left(\dfrac{n-2}{n}\right)^n = \lim \left(1 + \dfrac{(-2)}{n}\right)^n = e^{-2}$ (5)

Ex $\lim \left(-\dfrac{1}{2}\right)^n = 0$ (4)

Ex $\lim \sqrt[n]{3n} = \lim \left[\left(3^{1/n}\right) \cdot \left(n^{1/n}\right)\right] = 1 \cdot 1 = 1$ (3) & (2)

Ex $\lim \sqrt[n]{n^2} = \lim n^{2/n} = \lim \left(n^{1/n}\right)^2 = 1$ (2)

Recursively defined Sequences:

Sometimes sequences are defined by giving

(i) initial terms, and

(ii) recursion formula : a rule for calculating the terms by the preceding terms.

Ex: $a_1=1$, $a_2=1$, $a_{n+1}=a_n+a_{n-1}$ for all $n>2$.

Then $\{a_n\}=\{1,1,2,3,5,8,13,\dots\}$: Fibonacci

Ex: $a_1=1$, $a_{n+1}=\dfrac{a_n}{1+a_n}$, $n\geq 1$.

Then $\{a_n\}=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\right\}=\left\{\frac{1}{n}\right\}$

This sequence can also be described by:

$a_1=1$, $a_{n+1}=\dfrac{n.a_n}{n+1}$, $n\geq 1$.

DEFINITION **Nondecreasing Sequence**

A sequence $\{a_n\}$ with the property that $a_n \leq a_{n+1}$ for all n is called a **nondecreasing sequence**.

Similarly, a sequence $\{a_n\}$ with the property that $a_n \geq a_{n+1}$ for all n is called a nonincreasing sequence.

Ex: $\{a_n\} = \left\{\frac{n}{n+1}\right\} = \left\{1 - \frac{1}{n+1}\right\}$ is a nondecreasing seq. as $1 - \frac{1}{n+1} \leq 1 - \frac{1}{n+2}$ for all n.

$\{b_n\} = \left\{\frac{1}{n}\right\}$ is a nonincreasing sequence since $\frac{1}{n} \geq \frac{1}{n+1}$ for all n.

A sequence $\{a_n\}$ is said to be monotonic if it is either nondecreasing or nonincreasing.

DEFINITIONS **Bounded, Upper Bound, Least Upper Bound**

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

Similarly, $\{a_n\}$ is bounded from below if there exists a number N such that $a_n \geq N$ for all n. The number N is a lower bound for $\{a_n\}$. If N is a lower bound for $\{a_n\}$ but no number greater than N is a lower bound for $\{a_n\}$, then N is the greatest lower bound for $\{a_n\}$.

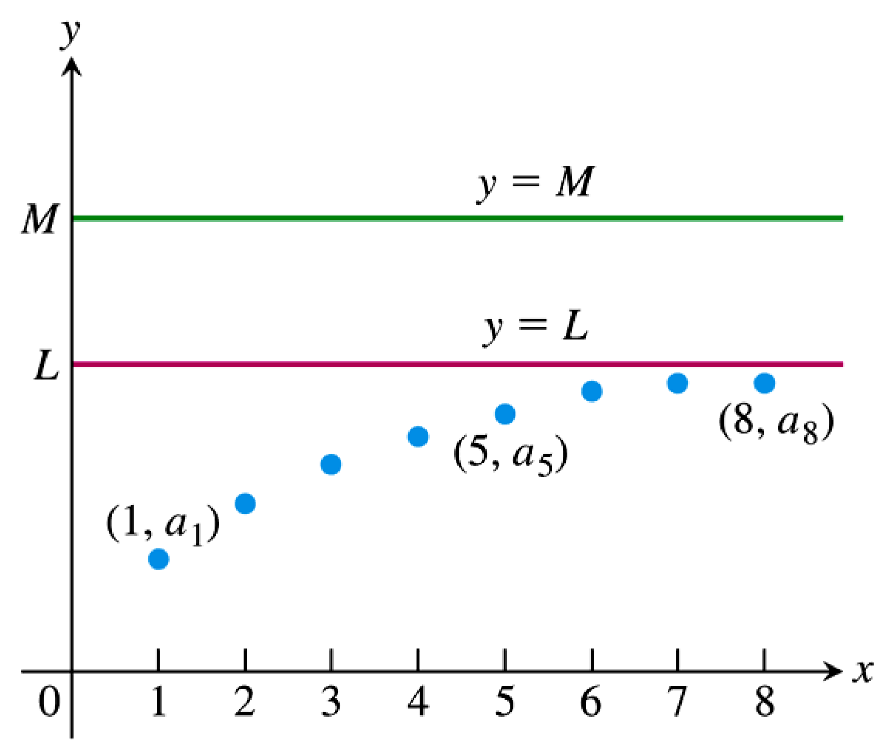


FIGURE 11.4 If the terms of a nondecreasing sequence have an upper bound M, they have a limit $L \leq M$.

Example: $\{a_n\} = \{n\}$ is a nondecreasing sequence. It is bounded from below by $M=1$ as $n \geq 1$ for all n. It is not bounded from above.

$\{b_n\} = \left\{\frac{n}{n+1}\right\} = \left\{1 - \frac{1}{n+1}\right\}$ is a nondecreasing seq. that is bounded from above by $M=1$ (least upper bound) as $1 - \frac{1}{n+1} \leq 1$ for all $n \geq 1$, and is bounded from below by $N = \frac{1}{2}$ as $\frac{n}{n+1} \geq \frac{1}{2}$ for all $n \geq 1$

$\{c_n\} = \left\{\frac{1}{n}\right\}$ is a nonincreasing seq. that is bounded from below by $N=0$ (greatest lower bound) as $\frac{1}{n} \geq 0$ for all n, and is bounded from above by $M=1$ as $\frac{1}{n} \leq 1$ for all n.

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

I. A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

II. A nonincreasing sequence of real numbers converges if and only if it is bounded from below. If a nonincreasing sequence converges, it converges to its greatest lower bound.

__Ex__ Let $\{a_n\} = \left\{ \frac{(n!)^2}{(2n)!} \right\} = \left\{ \frac{(1\cdot 2\cdot 3 \cdots n)\cdot(1\cdot 2\cdot 3\cdots n)}{1\cdot 2\cdot 3\cdots n\cdot(n+1)\cdots (2n)} \right\}$

__Claim__ $\{a_n\}$ is bounded below

(Pf) This is obvious since $a_n \geq 0$ for all $n \in \mathbb{N}$, the sequence is bdd. below by 0 (lower bound).

__Claim__ $\{a_n\}$ is nonincreasing i.e. $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.

(Pf) We either show $a_n - a_{n+1} \geq 0$ or $\frac{a_{n+1}}{a_n} \leq 1$

$$a_n - a_{n+1} = \frac{(n!)^2}{(2n)!} - \frac{[(n+1)!]^2}{[2(n+1)]!} = \frac{(n!)\cdot(n!)}{(2n)!} - \frac{(n+1)^2 (n!)\cdot(n!)}{(2n+2)(2n+1)[(2n)!]}$$

$$= \frac{(4n^2+6n+2)-(n^2+2n+1)}{(2n+2)!}\cdot(n!)^2 = (3n^2+4n+1)\frac{(n!)^2}{(2n+2)!} \geq 0$$

Thus by Monotonic Sequence Theorem $\{a_n\}$ is convergent.

<u>Claim:</u> $\lim a_n = 0$

(Pf) $$0 \le a_n = \frac{(1 \cdot 2 \cdot 3 \cdots n) \cdot (1 \cdot 2 \cdot 3 \cdots n)}{(1 \cdot 2 \cdot 3 \cdots n) \cdot (n+1)(n+2) \cdots (2n)}$$

$$= \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdots \cdots \frac{n}{2n}$$

$$\le \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \cdots \frac{1}{2} = \frac{1}{2^n}$$

Since $\lim \frac{1}{2^n} = 0$, by Sandwich thm $\lim a_n = 0$.

So 0 is the greatest lower bound for the nonincreasing sequence $\{a_n\}$, and $\{a_n\}$ converges to 0 as Mon. Seq. Thm. suggests.

Proof by Induction Method:

A mathematical statement that depends on $n \in \mathbb{N}$ can be proved by this proof technique.

Ex: The statement "$1+2+3+\ldots+n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$" can be proved by this method.

The procedure:

1) Check that the formula holds for $n=1$ (or minimum n)
2) Prove that if the formula holds for any positive integer $n=k$, then it also holds for $n=k+1$.

"If the first domino falls, and the k^{th} domino always knocks over the $(k+1)^{st}$ when it falls, all the dominoes fall.

We can use this technique to prove statements about recursively defined sequences.

Example

a) A sequence is given by $\mathbf{a_1 = 2}$, $\mathbf{a_{n+1} = \frac{1}{3 - a_n}}$.

i) Investigate whether this is a monotonic sequence or not.

ii) If any, find the upper and lower bounds.

iii) Is the sequence convergent or divergent? Explain your reason by using the relevant theorem.

iv) If it exits, find its limit; if it does not, explain the reason.

Solution: $a_1 = 2$, $a_2 = \frac{1}{3-a_1} = \frac{1}{3-2} = 1$, $a_3 = \frac{1}{3-a_2} = \frac{1}{3-1} = \frac{1}{2}$

$a_4 = \frac{1}{3-a_3} = \frac{1}{3-\frac{1}{2}} = \frac{2}{5}$

(i) Claim: $\{a_n\}$ is non-increasing, i.e. $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$

Proof by induction: * $a_2 = 1 \leq a_1 = 2$ ✓

* assume that $a_{k+1} \leq a_k$ for any $k > 1$

* prove that $a_{k+2} \leq a_{k+1}$

$a_{k+1} \leq a_k \Rightarrow -a_{k+1} \geq -a_k \Rightarrow 3-a_{k+1} \geq 3-a_k > 0$

$\Rightarrow a_{k+2} = \dfrac{1}{3-a_{k+1}} \leq \dfrac{1}{3-a_k} = a_{k+1} \Rightarrow a_{k+2} \leq a_{k+1}$

(ii) $a_1 = 2$ and $a_{n+1} \leq a_n \Rightarrow a_n \leq 2$ for all n.

$\Rightarrow a_{n+1} = \dfrac{1}{\underbrace{3-a_n}_{\geq 1}} > 0$ for all n

lower bound　　upper bound

So $0 < a_n \leq 2$ for all n.

(iii) $\{a_n\}$ is a nonincreasing sequence with a lower bound 0, hence it is convergent by the Monotonic Seq. Thm.

Let $\lim\limits_{n\to\infty} a_n = L$, then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_{n+2} = \lim\limits_{k\to\infty} a_k = L$

In the recursion formula, take the limit of both sides as $n \to \infty$:

$$a_{n+1} = \frac{1}{3-a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n}$$

Then $L = \frac{1}{3-L} \Rightarrow L^2 - 3L + 1 = 0$

$$L_{1,2} = \frac{3 \mp \sqrt{9-4.1.1}}{2} = \frac{3 \mp \sqrt{5}}{2}$$

$L_1 = \frac{3+\sqrt{5}}{2} > 2$ not possible since $0 < a_n \leq 2$ for all n

So $L = L_2 = \frac{3-\sqrt{5}}{2} = \lim_{n\to\infty} a_n$

11.2

Infinite Series

An infinite series is the sum of an infinite sequence

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

<u>Question</u>: Is this sum a finite number?

If so, can we find this sum?

For an infinite series there is a corresponding sequence of partial sums $\{s_n\}$, where s_n is the sum of the first n terms:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$s_n = a_1 + a_2 + \cdots + a_n \qquad (n^{th} \text{ partial sum})$$

$$\vdots$$

<u>Ex</u> Consider the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{n-1}} + \cdots$

Partial sum		Suggestive expression for partial sum	Value
First:	$s_1 = 1$	$2 - 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$	$\frac{7}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
nth:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

$$\lim s_n = \lim_{n \to \infty} 2 - \frac{1}{2^{n-1}} = 2$$

We say the sum of the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots \quad \text{is} \quad 2.$$

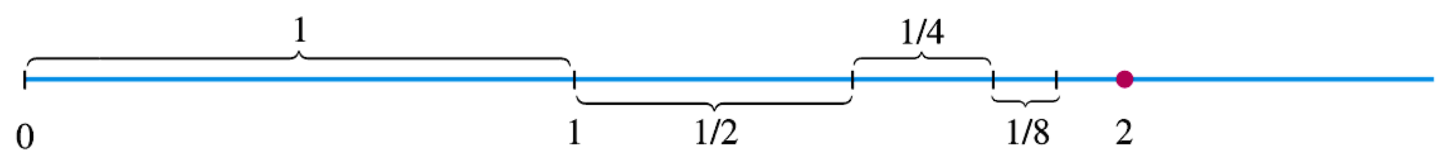
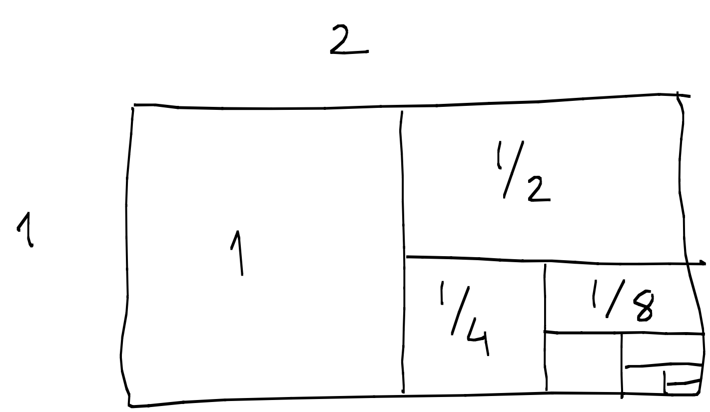


FIGURE 11.5 As the lengths $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ are added one by one, the sum approaches 2.



As the areas $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ are added, the sum approaches 2

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2$$

DEFINITIONS **Infinite Series, *n*th Term, Partial Sum, Converges, Sum**

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Our aim is to find a formula that expresses s_n as a function of n.

__Geometric Series__ are of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=0}^{\infty} a \cdot r^n = \sum_{n=1}^{\infty} a r^{n-1}$$

where a and r (ratio) are fixed.

__$r=1$__ : $s_n = a + ar + ar^2 + \cdots + ar^{n-1} = n \cdot a$

in this case the series diverges because $\lim s_n = \mp\infty$ depending on the sign of a

__$r=-1$__ : $s_n = a - a + a - a \cdots + (-1)^{n-1} a = \begin{cases} 0 & n: \text{even} \\ a & n: \text{odd} \end{cases}$

Thus, $\lim s_n$ DNE and hence the series diverges.

$\underline{|r| \neq 1}$: $s_n = a + ar + \cdots + ar^{n-1}$

$$r \cdot s_n = \underbrace{ar + ar^2 + \cdots + ar^{n-1}}_{s_n - a} + ar^n = s_n - a + ar^n$$

$$\Rightarrow s_n (r-1) = a(r^n - 1)$$

$$\Rightarrow s_n = a \frac{(r^n - 1)}{r-1}$$

$$\lim s_n = \lim_{n \to \infty} a \left(\frac{1-r^n}{1-r} \right) = \begin{cases} \frac{a}{1-r} & |r| < 1 \\ \text{diverges} & |r| > 1 \end{cases}$$

Hence, $\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & |r| < 1 \\ \text{diverges} & |r| \geq 1 \end{cases}$

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

EXAMPLE 1 Index Starts with $n = 1$

The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

EXAMPLE 2 Index Starts with $n = 0$

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$

EXAMPLE 3 A Bouncing Ball

You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive but less than 1. Find the total distance the ball travels up and down (Figure 11.6).

Solution The total distance is

$$s = a + \underbrace{2ar + 2ar^2 + 2ar^3 + \cdots}_{\text{This sum is } 2ar/(1-r).} = a + \frac{2ar}{1-r} = a\frac{1+r}{1-r}.$$

If $a = 6$ m and $r = 2/3$, for instance, the distance is

$$s = 6\frac{1 + (2/3)}{1 - (2/3)} = 6\left(\frac{5/3}{1/3}\right) = 30 \text{ m}.$$

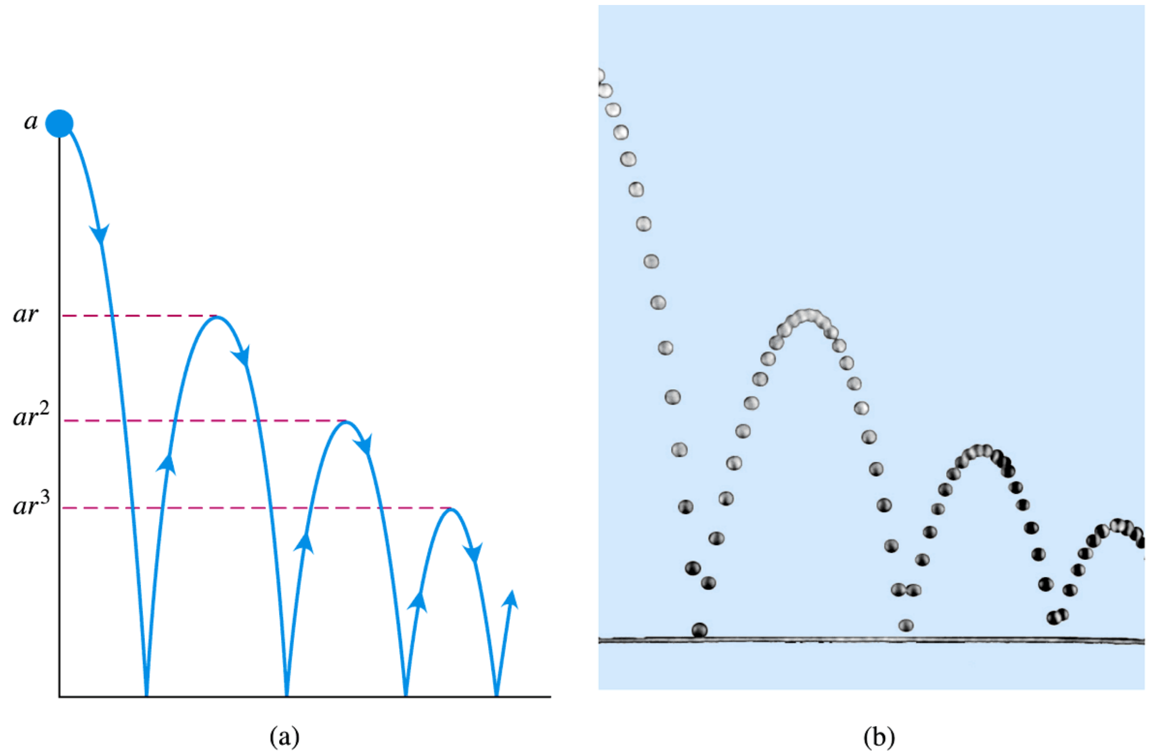


FIGURE 11.6 (a) Example 3 shows how to use a geometric series to calculate the total vertical distance traveled by a bouncing ball if the height of each rebound is reduced by the factor r. (b) A stroboscopic photo of a bouncing ball.

Ex $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots, \quad -1 < x < 1$

Ex $$\sum_{n=2}^{\infty} \frac{(-5)^n}{8^{2n}} = \sum_{n=2}^{\infty} \left(\frac{-5}{64}\right)^n = \sum_{n=0}^{\infty} \left(\frac{-5}{64}\right)^{n+2} = \sum_{n=0}^{\infty} \underbrace{\left(-\frac{5}{64}\right)^2}_{=a} \cdot \underbrace{\left(\frac{-5}{64}\right)^n}_{=r^n}$$

is convergent since $|r| = \left|\frac{-5}{64}\right| < 1$. It converges to

$$\frac{a}{1-r} = \left(-\frac{5}{64}\right)^2 \cdot \frac{1}{1-\left(\frac{-5}{64}\right)} = \frac{25}{(64)^2} \cdot \frac{64}{69} = \frac{25}{64 \cdot 69}$$

Ex (Repeating Decimals) $0.323232\ldots = 0.\overline{32}$

$$= \frac{32}{100} + \frac{32}{10000} + \frac{32}{10^6} + \ldots = \sum_{n=1}^{\infty} \frac{32}{100}\left(\frac{1}{100}\right)^{n-1} = \frac{32}{100} \cdot \frac{1}{1-\frac{1}{100}} = \frac{32}{99}$$

__Ex__ (Telescoping Series) Find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

The partial fraction decomposition of the general term gives:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} - \frac{1}{n+1}.$$ Thus

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\cdots} + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

Thus $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

Divergent Series

One reason that a series may fail to converge is that its terms don't become small.

EXAMPLE 6 Partial Sums Outgrow Any Number

(a) The series

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + \cdots + n^2 + \cdots$$

diverges because the partial sums grow beyond every number L. After $n = 1$, the partial sum $s_n = 1 + 4 + 9 + \cdots + n^2$ is greater than n^2.

(b) The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} + \cdots$$

diverges because the partial sums eventually outgrow every preassigned number. Each term is greater than 1, so the sum of n terms is greater than n. ■

THEOREM 7

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

<u>Proof:</u> Let S_n be the n^{th} partial sum and $S = \sum_{k=1}^{\infty} a_k$. Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = S$ and

$a_n = S_n - S_{n-1}$. So $\lim_{n\to\infty} a_n = S - S = 0$.

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Warning: The converse of the theorem is not true!

Ex Although $\lim_{n\to\infty} \frac{1}{n} = 0$, we are going to show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$ is divergent.

<u>Ex</u> $\sum_{n=1}^{\infty} n^2$ diverges since $\lim_{n\to\infty} n^2 = \infty \neq 0$.

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges since $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

<u>Ex</u> $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges since $\lim_{n\to\infty} (-1)^{n+1}$ DNE.

Combining Series

THEOREM 8
If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA$ (Any number k).

As corollaries of Theorem 8, we have

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

EXAMPLE 9 Find the sums of the following series.

(a)

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} &= \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} && \text{Difference Rule} \\
&= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} && \text{Geometric series with } a = 1 \text{ and } r = 1/2,\ 1/6 \\
&= 2 - \frac{6}{5} \\
&= \frac{4}{5}
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{4}{2^n} &= 4\sum_{n=0}^{\infty} \frac{1}{2^n} && \text{Constant Multiple Rule} \\
&= 4\left(\frac{1}{1-(1/2)}\right) && \text{Geometric series with } a = 1,\ r = 1/2 \\
&= 8
\end{aligned}$$

Adding or Deleting Terms

We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence this will usually change the sum. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$ and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n .$$

Conversely, if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$, then $\sum_{n=1}^{\infty} a_n$ converges. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

and

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n}\right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125} .$$

Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index h units, replace the n in the formula for a_n by $n - h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots .$$

To lower the starting value of the index h units, replace the n in the formula for a_n by $n + h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + a_3 + \cdots .$$

It works like a horizontal shift. We saw this in starting a geometric series with the index $n = 0$ instead of the index $n = 1$, but we can use any other starting index value as well. We usually give preference to indexings that lead to simple expressions.

EXAMPLE 10 Reindexing a Geometric Series

We can write the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \qquad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \qquad \text{or even} \qquad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}} .$$

The partial sums remain the same no matter what indexing we choose.

11.3

The Integral Test

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$:

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots.$$

Since the partial sums form a nondecreasing seq. then by the Monotonic Sequence Theorem we have:

Corollary of Theorem 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

<u>Ex</u> The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by above thm.

We group the terms of the series in the following way:

$$1+\frac{1}{2}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>2\cdot\frac{1}{4}=\frac{1}{2}}+\underbrace{\left(\frac{1}{5}+\cdots+\frac{1}{8}\right)}_{>4\cdot\frac{1}{8}=\frac{1}{2}}+\underbrace{\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)}_{>8\cdot\frac{1}{16}=\frac{1}{2}}+\cdots\underbrace{\left(\frac{1}{2^k+1}+\cdots+\frac{1}{2^{k+1}}\right)}_{>2^k\cdot\frac{1}{2^{k+1}}=\frac{1}{2}}+\cdots$$

$\Rightarrow$ $S_{2^{k+1}} > \frac{k+1}{2}$ which implies $\{s_n\}$ is not bounded from above. $\lim_{n\to\infty} s_n = \infty$

$\Rightarrow$ The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

<u>Ex</u> Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

$$s_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

If $f(x) = \frac{1}{x^2}$ then $s_n = f(1) + f(2) + \cdots + f(n)$.

$$\Rightarrow s_n < f(1) + \int_1^n \frac{dx}{x^2} < 1 + \int_1^{\infty} \frac{dx}{x^2} = 1 + 1 = 2$$

$$\left(\int_1^{\infty} \frac{dx}{x^2} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^2} = \lim_{R\to\infty} \left(-\frac{1}{x} \Big|_1^R \right) = \lim_{R\to\infty} \left(-\frac{1}{R} + 1 \right) = 1 \right)$$

Since $\{s_n\}$ is bdd above, by the above thm the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

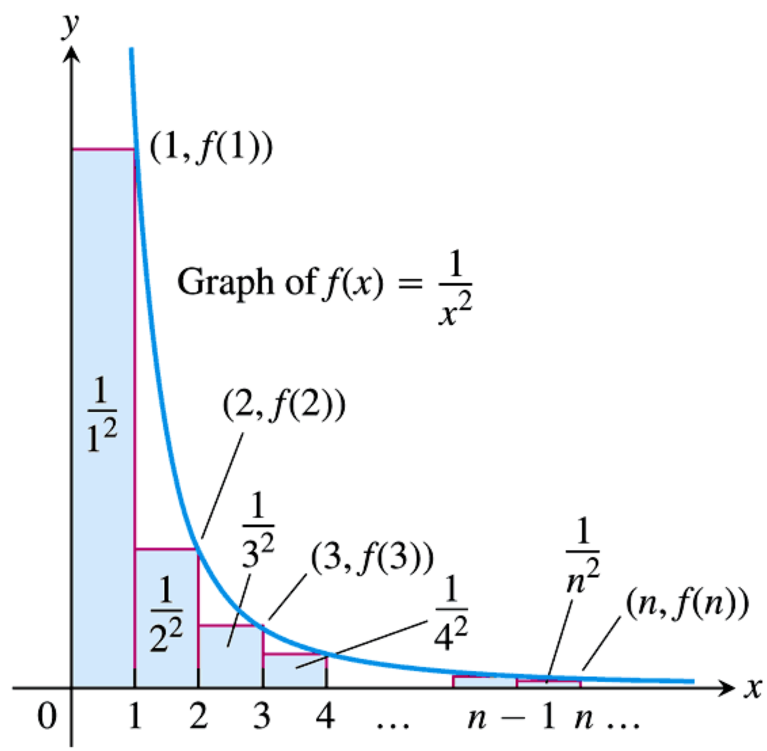


FIGURE 11.7 The sum of the areas of the rectangles under the graph of $f(x) = 1/x^2$ is less than the area under the graph (Example 2).

THEOREM 9 The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

(Pf)

$s_n = a_1 + a_2 + \cdots + a_n$

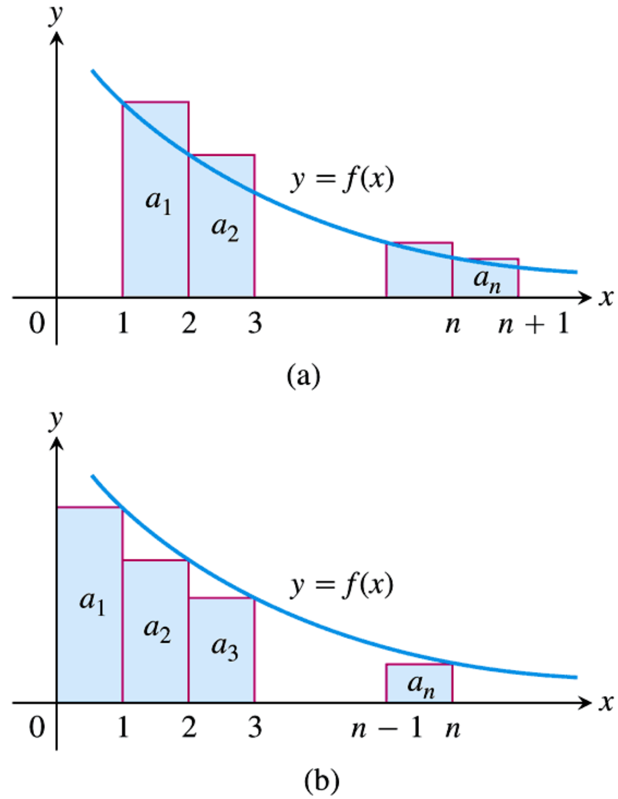


FIGURE 11.8 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

$$\int_1^{n+1} f(x)\,dx \le s_n$$

$+$

$$s_n \le a_1 + \int_1^{n} f(x)\,dx$$

$||$

$$\int_1^{n+1} f(x)\,dx \le s_n \le a_1 + \int_1^{n} f(x)\,dx$$

EXAMPLE 3 The p-Series

Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p}\, dx &= \int_1^{\infty} x^{-p}\, dx = \lim_{b \to \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b \\ &= \frac{1}{1-p} \lim_{b \to \infty} \left(\frac{1}{b^{p-1}} - 1 \right) \\ &= \frac{1}{1-p}(0-1) = \frac{1}{p-1}, \end{aligned}$$

$b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$.

the series converges by the Integral Test. We emphasize that the sum of the p-series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \frac{1}{1-p} \lim_{b \to \infty} (b^{1-p} - 1) = \infty .$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots .$$

We have convergence for $p > 1$ but divergence for every other value of p.

Ex $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent since $\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{R\to\infty} \left[\arctan x\right]_1^R$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ is convergent

Ex $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$ is divergent.

Consider $\int_1^{\infty} \frac{dx}{2^{\ln x}} = \int_0^{\infty} \frac{e^u}{2^u}\, du$ if
$u = \ln x$
$\Rightarrow x = e^u$
$\Rightarrow dx = e^u\, du$

Then,
$$= \int_0^{\infty} \left(\frac{e}{2}\right)^u du$$
$$= \lim_{R\to\infty} \frac{1}{\ln\left(\frac{e}{2}\right)} \left[\left(\frac{e}{2}\right)^R - 1\right] = \infty.$$

11.4

Comparison Tests

We have seen how to determine the convergence of geometric series, p-series, and a few others. We can test the convergence of many more series by comparing their terms to those of a series whose convergence is known.

THE COMPARISON TEST Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

EXAMPLE 1 Applying the Comparison Test

(a) The series

$$\sum_{n=1}^{\infty} \frac{5}{5n - 1}$$

diverges because its nth term

$$\frac{5}{5n - 1} = \frac{1}{n - \frac{1}{5}} > \frac{1}{n}$$

is greater than the nth term of the divergent harmonic series.

(b) The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots.$$

The geometric series on the left converges and we have

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1 - (1/2)} = 3.$$

The fact that 3 is an upper bound for the partial sums of $\sum_{n=0}^{\infty} (1/n!)$ does not mean that the series converges to 3. As we will see in Section 11.9, the series converges to e.

THEOREM 11 Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EXAMPLE 2 Using the Limit Comparison Test

Which of the following series converge, and which diverge?

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n + 1}{(n + 1)^2} = \sum_{n=1}^{\infty} \frac{2n + 1}{n^2 + 2n + 1}$

(b) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

(c) $\frac{1 + 2 \ln 2}{9} + \frac{1 + 3 \ln 3}{14} + \frac{1 + 4 \ln 4}{21} + \cdots = \sum_{n=2}^{\infty} \frac{1 + n \ln n}{n^2 + 5}$

Solution

(a) Let $a_n = (2n + 1)/(n^2 + 2n + 1)$. For large n, we expect a_n to behave like $2n/n^2 = 2/n$ since the leading terms dominate for large n, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2n^2 + n}{n^2 + 2n + 1} = 2,$$

$\sum a_n$ diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

Example 2 continued

(b) Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2^n}{2^n - 1} \\ &= \lim_{n\to\infty} \frac{1}{1 - (1/2^n)} \\ &= 1,\end{aligned}$$

Σa_n converges by Part 1 of the Limit Comparison Test.

(c) Let $a_n = (1 + n \ln n)/(n^2 + 5)$. For large n, we expect a_n to behave like $(n \ln n)/n^2 = (\ln n)/n$, which is greater than $1/n$ for $n \geq 3$, so we take $b_n = 1/n$. Since

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + n^2 \ln n}{n^2 + 5} = \infty,$$

$\sum a_n$ diverges by Part 3 of the Limit Comparison Test. ■

EXAMPLE 3 Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Solution Because $\ln n$ grows more slowly than n^c for any positive constant c (Section 11.1, Exercise 91), we would expect to have

$$\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

for n sufficiently large. Indeed, taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} \\ &= \lim_{n\to\infty} \frac{1/n}{(1/4)n^{-3/4}} \qquad \text{l'Hôpital's Rule} \\ &= \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0. \end{aligned}$$

Since $\sum b_n = \sum(1/n^{5/4})$ (a p-series with $p > 1$) converges, $\sum a_n$ converges by Part 2 of the Limit Comparison Test. ■

11.5

The Ratio and Root Tests

THEOREM 12 The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho .$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

In proof of part (a) we compare the series by a convergent geometric series. In part (b) we observe that if $\rho > 1$ then $\lim a_n \neq 0$

(c) $\boldsymbol{\rho = 1}$**.** The two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

show that some other test for convergence must be used when $\rho = 1$.

$$\text{For } \sum_{n=1}^{\infty} \frac{1}{n}: \qquad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1.$$

$$\text{For } \sum_{n=1}^{\infty} \frac{1}{n^2}: \qquad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1^2 = 1.$$

In both cases, $\rho = 1$, yet the first series diverges, whereas the second converges. ■

The Ratio Test is often effective when the terms of a series contain factorials of expressions involving n or expressions raised to a power involving n.

EXAMPLE 1 Applying the Ratio Test

Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

Solution

(a) For the series $\sum_{n=0}^{\infty} (2^n + 5)/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right) \to \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1 - (2/3)} + \frac{5}{1 - (1/3)} = \frac{21}{2}.$$

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \to 4.$$

The series diverges because $\rho = 4$ is greater than 1.

(c) If $a_n = 4^n n!n!/(2n)!$, then

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{4^n n!n!}$$

$$= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{2(n+1)}{2n+1} \to 1.$$

Because the limit is $\rho = 1$, we cannot decide from the Ratio Test whether the series converges. When we notice that $a_{n+1}/a_n = (2n+2)/(2n+1)$, we conclude that a_{n+1} is always greater than a_n because $(2n+2)/(2n+1)$ is always greater than 1. Therefore, all terms are greater than or equal to $a_1 = 2$, and the nth term does not approach zero as $n \to \infty$. The series diverges. ■

$$\frac{a_{n+1}}{a_n} = \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > 1 \quad \Rightarrow a_n \text{ is increasing}$$

$$\Rightarrow \lim a_n \neq 0.$$

EXAMPLE 2 Let $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$ Does $\sum a_n$ converge?

Solution We write out several terms of the series:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{3}{2^3} + \frac{1}{2^4} + \frac{5}{2^5} + \frac{1}{2^6} + \frac{7}{2^7} + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{3}{8} + \frac{1}{16} + \frac{5}{32} + \frac{1}{64} + \frac{7}{128} + \cdots.\end{aligned}$$

Clearly, this is not a geometric series. The nth term approaches zero as $n \to \infty$, so we do not know if the series diverges. The Integral Test does not look promising. The Ratio Test produces

$$\frac{a_{n+1}}{a_n} = \begin{cases} \dfrac{1}{2n}, & n \text{ odd} \\[2ex] \dfrac{n+1}{2}, & n \text{ even.} \end{cases}$$

As $n \to \infty$, the ratio is alternately small and large and has no limit.

A test that will answer the question (the series converges) is the Root Test. ■

THEOREM 13 The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

In proof of part (a) we compare the series by a convergent geometric series. In part (b) we observe that if $\rho > 1$ then $\lim a_n \neq 0$

EXAMPLE 3 Applying the Root Test

Which of the following series converges, and which diverges?

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ **(c)** $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1}{2} < 1$.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges because $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{\left(\sqrt[n]{n}\right)^2} \to \frac{2}{1} > 1$.

(c) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1$.

11.6

Alternating Series, Absolute and Conditional Convergence

A series in which the terms are alternately positive and negative is an **alternating series**.

Here are three examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \tag{1}$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \tag{2}$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots \tag{3}$$

Series (1), called the **alternating harmonic series**, converges, as we will see in a moment. Series (2) a geometric series with ratio $r = -1/2$, converges to $-2/[1 + (1/2)] = -4/3$. Series (3) diverges because the nth term does not approach zero.

We prove the convergence of the alternating harmonic series by applying the Alternating Series Test.

THEOREM 14 The Alternating Series Test (Leibniz's Theorem)

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

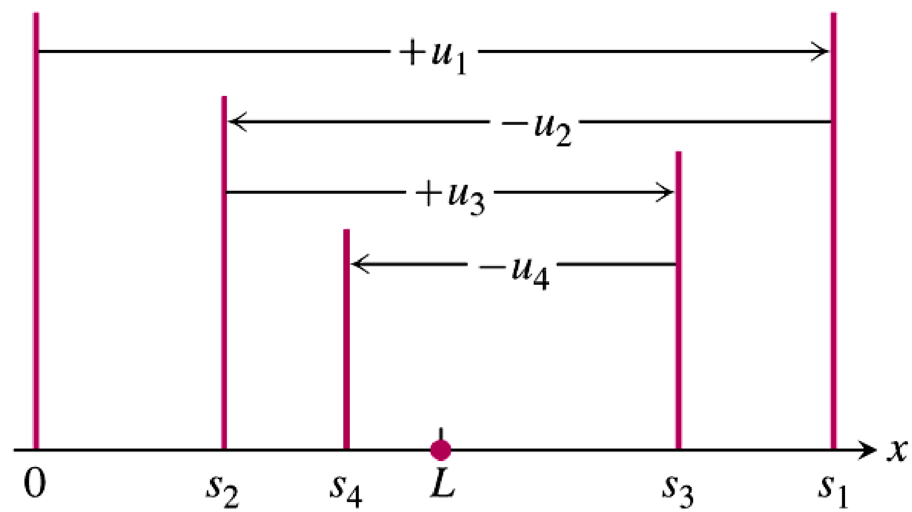


FIGURE 11.9 The partial sums of an alternating series that satisfies the hypotheses of Theorem 14 for $N = 1$ straddle the limit from the beginning.

EXAMPLE 1 The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

satisfies the three requirements of Theorem 14 with $N = 1$; it therefore converges.

Example 2 The series $\sum_{n=1}^{\infty}(-1)^n \frac{3n}{4n-1}$ is an alternating series. $u_n = \frac{3n}{4n-1} > 0$ for all $n \in \mathbb{N}$.

$u_n = \frac{3}{4} + \frac{3/4}{4n-1} \geq u_{n+1} = \frac{3}{4} + \frac{3/4}{4n+3}$

But $\lim_{n\to\infty} u_n = \frac{3}{4} \neq 0$ 3^{rd} condition is not satisfied.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{3n}{4n-1}$ does not exist, so by the n^{th} term test, the series diverges.

Example 3 Is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ convergent or divergent?

The series is an alternating series with $u_n = \frac{n^2}{n^3+1}$

(i) $u_n > 0$ for all n

(ii) let $f(x) = \frac{x^2}{x^3+1}$ then $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0$

if $x \geq 2$. Since $u_n = f(n)$, $u_n \geq u_{n+1}$ for $n \geq 2$

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$

Then, by Alt. Ser. Test, the series converges.

THEOREM 15 The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 14, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the numerical value of the first unused term. Furthermore, the remainder, $L - s_n$, has the same sign as the first unused term.

$$|\text{Error}| = \Big| \underbrace{\Big(\sum_{n=1}^{\infty} (-1)^{n+1} u_n \Big)}_{\text{actual value}} - \underbrace{s_n}_{\text{approx. value}} \Big| \leq \underbrace{u_{n+1}}_{\text{first unused term}}$$

EXAMPLE 2 We try Theorem 15 on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} \;\vdots\; + \frac{1}{256} - \cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than $1/256$. The sum of the first eight terms is 0.6640625. The sum of the series is

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

The difference, $(2/3) - 0.6640625 = 0.0026041666\ldots$, is positive and less than $(1/256) = 0.00390625$.

DEFINITION Absolutely Convergent

A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

DEFINITION Conditionally Convergent

A series that converges but does not converge absolutely **converges conditionally**.

THEOREM 16 The Absolute Convergence Test

If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Proof For each n,

$$-|a_n| \le a_n \le |a_n|, \qquad \text{so} \qquad 0 \le a_n + |a_n| \le 2|a_n|.$$

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ converges and, by the Direct Comparison Test, the nonnegative series $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express $\sum_{n=1}^{\infty} a_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges. ■

EXAMPLE 3 Applying the Absolute Convergence Test

(a) For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

The original series converges because it converges absolutely.

(b) For $\sum_{n=1}^{\infty}\frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty}(1/n^2)$ because $|\sin n| \le 1$ for every n. The original series converges absolutely; therefore it converges.

EXAMPLE 4 Alternating p-Series

If p is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \qquad p > 0$$

converges.

If $p > 1$, the series converges absolutely. If $0 < p \leq 1$, the series converges conditionally.

$$\text{Conditional convergence:} \qquad 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

$$\text{Absolute convergence:} \qquad 1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \cdots$$

1. **The *n*th-Term Test:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$, since absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

EXAMPLE 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence. □

EXAMPLE 2 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\Sigma\, b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$ □

EXAMPLE 3 $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^{\infty} xe^{-x^2}\,dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works. □

EXAMPLE 4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test. □

EXAMPLE 5 $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Since the series involves $k!$, we use the Ratio Test. □

EXAMPLE 6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\Sigma\, 1/3^n$, we use the Comparison Test. □

11.7

Power Series

DEFINITIONS Power Series, Center, Coefficients

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \qquad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

EXAMPLE 1 A Geometric Series

Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1 - x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{3}$$

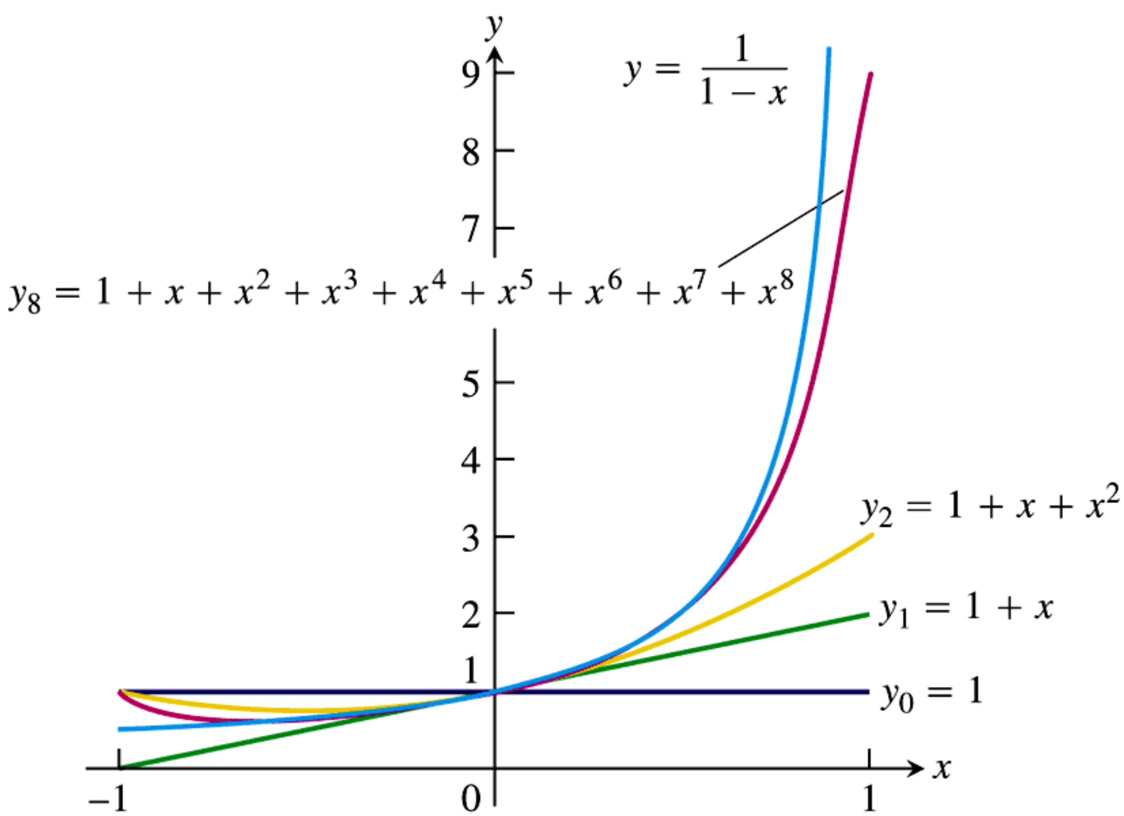


FIGURE 11.10 The graphs of $f(x) = 1/(1 - x)$ and four of its polynomial approximations (Example 1).

EXAMPLE 2 A Geometric Series

The power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots \tag{4}$$

matches Equation (2) with $a = 2, c_0 = 1, c_1 = -1/2, c_2 = 1/4, \ldots, c_n = (-1/2)^n$. This is a geometric series with first term 1 and ratio $r = -\frac{x-2}{2}$. The series converges for $\left|\frac{x-2}{2}\right| < 1$ or $0 < x < 4$. The sum is

$$\frac{1}{1-r} = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x},$$

so

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \qquad 0 < x < 4.$$

Series (4) generates useful polynomial approximations of $f(x) = 2/x$ for values of x near 2:

$$P_0(x) = 1$$

$$P_1(x) = 1 - \frac{1}{2}(x-2) = 2 - \frac{x}{2}$$

$$P_2(x) = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 = 3 - \frac{3x}{2} + \frac{x^2}{4},$$

and so on (Figure 11.11). ■

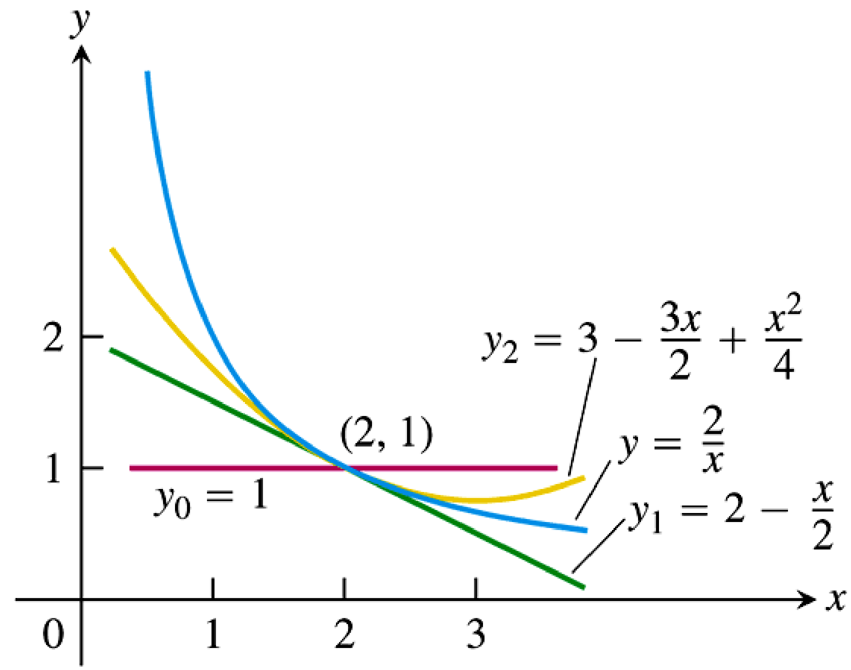


FIGURE 11.11 The graphs of $f(x) = 2/x$ and its first three polynomial approximations (Example 2).

EXAMPLE 3 Testing for Convergence Using the Ratio Test

For what values of x do the following power series converge?

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

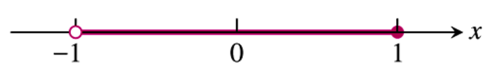
(c) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(d) $\displaystyle\sum_{n=0}^{\infty}n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum|u_n|$, where u_n is the nth term of the series in question.

(a) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \frac{n}{n+1}|x| \to |x|.$

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the nth term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$ we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series; it diverges. Series (a) converges for $-1 < x \leq 1$ and diverges elsewhere.



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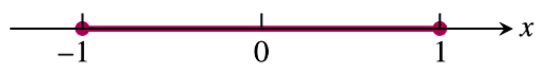
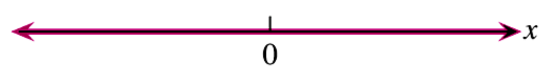
(b) $\left|\frac{u_{n+1}}{u_n}\right| = \frac{2n - 1}{2n + 1}x^2 \to x^2.$

The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$ because the nth term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \leq x \leq 1$ and diverges elsewhere.



(c) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n + 1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n + 1} \to 0$ for every x.

The series converges absolutely for all x.

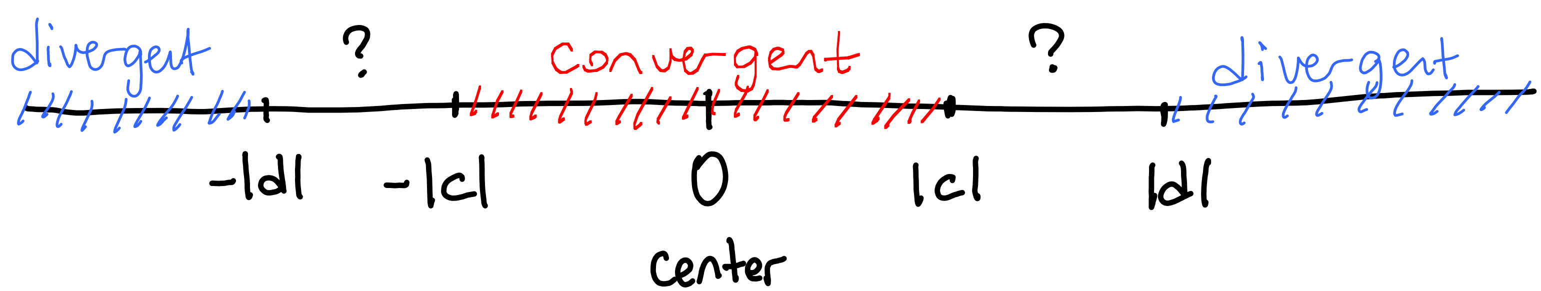


(d) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n + 1)!x^{n+1}}{n!x^n}\right| = (n + 1)|x| \to \infty$ unless $x = 0$.

The series diverges for all values of x except $x = 0$.

THEOREM 18 The Convergence Theorem for Power Series

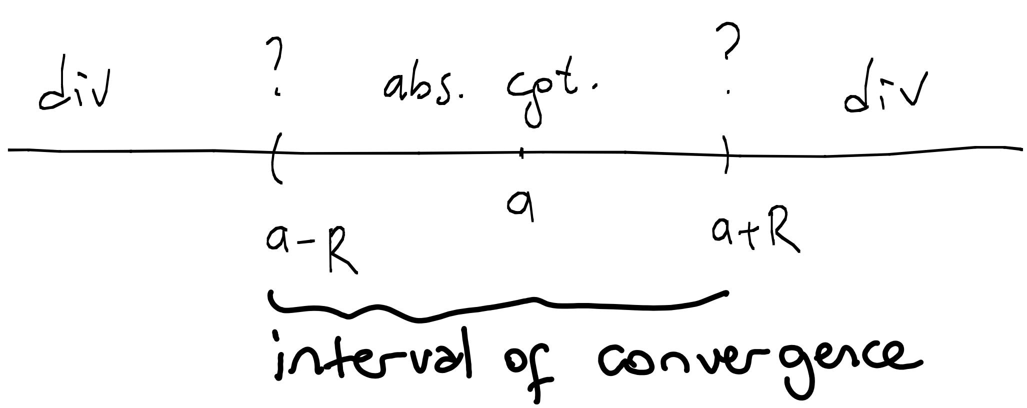
If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.



COROLLARY TO THEOREM 18

The convergence of the series $\sum c_n(x - a)^n$ is described by one of the following three possibilities:

1. There is a positive number R such that the series diverges for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.



R : radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

How to Test a Power Series for Convergence

1. *Use the Ratio Test (or nth-Root Test) to find the interval where the series converges absolutely*. Ordinarily, this is an open interval

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint*, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. *If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$* (it does not even converge conditionally), because the nth term does not approach zero for those values of x.

THEOREM 19 The Term-by-Term Differentiation Theorem

If $\sum c_n(x-a)^n$ converges for $a - R < x < a + R$ for some $R > 0$, it defines a function f:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad a - R < x < a + R.$$

Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

CAUTION Term-by-term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=1}^{\infty} \frac{n!\cos(n!x)}{n^2},$$

which diverges for all x. This is not a power series, since it is not a sum of positive integer powers of x.

EXAMPLE 4 Applying Term-by-Term Differentiation

Find series for $f'(x)$ and $f''(x)$ if

$$\begin{aligned} f(x) &= \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots \\ &= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1 \end{aligned}$$

Solution

$$\begin{aligned} f'(x) &= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1 \\ f''(x) &= \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1 \end{aligned}$$

THEOREM 20 The Term-by-Term Integration Theorem

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

converges for $a - R < x < a + R \ (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\, dx = \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} + C$$

for $a - R < x < a + R$.

EXAMPLE 5 A Series for $\tan^{-1} x$, $-1 \le x \le 1$

Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1.$$

Solution We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1 - (-x^2)} = \frac{1}{1 + x^2}.$$

We can now integrate $f'(x) = 1/(1 + x^2)$ to get

$$\int f'(x)\, dx = \int \frac{dx}{1 + x^2} = \tan^{-1} x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \qquad -1 < x < 1. \tag{7}$$

In Section 11.10, we will see that the series also converges to $\tan^{-1} x$ at $x = \pm 1$. *

EXAMPLE 6 A Series for $\ln(1 + x)$, $-1 < x \le 1$

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \qquad \text{Theorem 20} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1.\end{aligned}$$

It can also be shown that the series converges at $x = 1$ to the number $\ln 2$, but that was not guaranteed by the theorem.

THEOREM 21 The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

EXAMPLE 7 Multiply the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}, \qquad \text{for } |x| < 1,$$

by itself to get a power series for $1/(1-x)^2$, for $|x| < 1$.

Solution Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

and

$$c_n = \underbrace{a_0 b_n + a_1 b_{n-1} + \cdots + a_k b_{n-k} + \cdots + a_n b_0}_{n+1 \text{ terms}}$$

$$= \underbrace{1 + 1 + \cdots + 1}_{n+1 \text{ ones}} = n + 1.$$

Then, by the Series Multiplication Theorem,

$$A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots$$

is the series for $1/(1-x)^2$. The series all converge absolutely for $|x| < 1$.

Notice that Example 4 gives the same answer because

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

11.8

Taylor and Maclaurin Series

__Aim__: Given a differentiable function $f(x)$, find a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ that converges to $f(x)$ for $|x-a| < R$.

__Idea__: If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ is a power series representation of f, then any derivative of f can be represented by a series that is obtained by term-by-term differentiation.

Let $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1 (x-a) + a_2 (x-a)^2 + \cdots$

Then $f'(x) = a_1 + 2 \cdot a_2 (x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots$

$$f''(x) = 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 (x-a) + 4 \cdot 3 \cdot a_4 (x-a)^2 + \cdots$$

$$f'''(x) = 3 \cdot 2 \cdot 1 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4 \cdot (x-a) + \cdots$$

$$\vdots$$

$$f^{(n)}(x) = n!\, a_n + (n+1)!\, a_{n+1}(x-a) + K(x-a)^2 + \cdots$$

$$\vdots$$

This means: for $x=a$ we have $f^{(n)}(a) = n!\, a_n$

or $$\boxed{a_n = \frac{f^{(n)}(a)}{n!}}$$

DEFINITIONS Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

EXAMPLE 1 Finding a Taylor Series

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$
\begin{aligned}
f(x) &= x^{-1}, & f(2) &= 2^{-1} = \frac{1}{2}, \\
f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{2^2}, \\
f''(x) &= 2!x^{-3}, & \frac{f''(2)}{2!} &= 2^{-3} = \frac{1}{2^3}, \\
f'''(x) &= -3!x^{-4}, & \frac{f'''(2)}{3!} &= -\frac{1}{2^4}, \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^n n!x^{-(n+1)}, & \frac{f^{(n)}(2)}{n!} &= \frac{(-1)^n}{2^{n+1}}.
\end{aligned}
$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$. ■

DEFINITION **Taylor Polynomial of Order n**

Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We say "order" not "degree" since $f^{(n)}(a)$ may be zero.

Note that $P_1(x) = f(a) + f'(a)(x-a) = y$ is the tangent line to f at $x=a$ which we defined as the linearization of f.

Ex Let $f(x) = e^x$. Find Maclaurin series generated by $f(x)$.

$\Rightarrow f^{(n)}(x) = e^x$ for all $n \geq 1$

$$\Rightarrow a_n = \frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!}$$

$$\Rightarrow P_n(x) = \sum_{k=0}^{n} a_k (x-0)^k = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!}$$

is the Taylor polynomial of order n.

The Maclaurin series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$

We will see that the series converges to e^x for $x \in \mathbb{R}$.

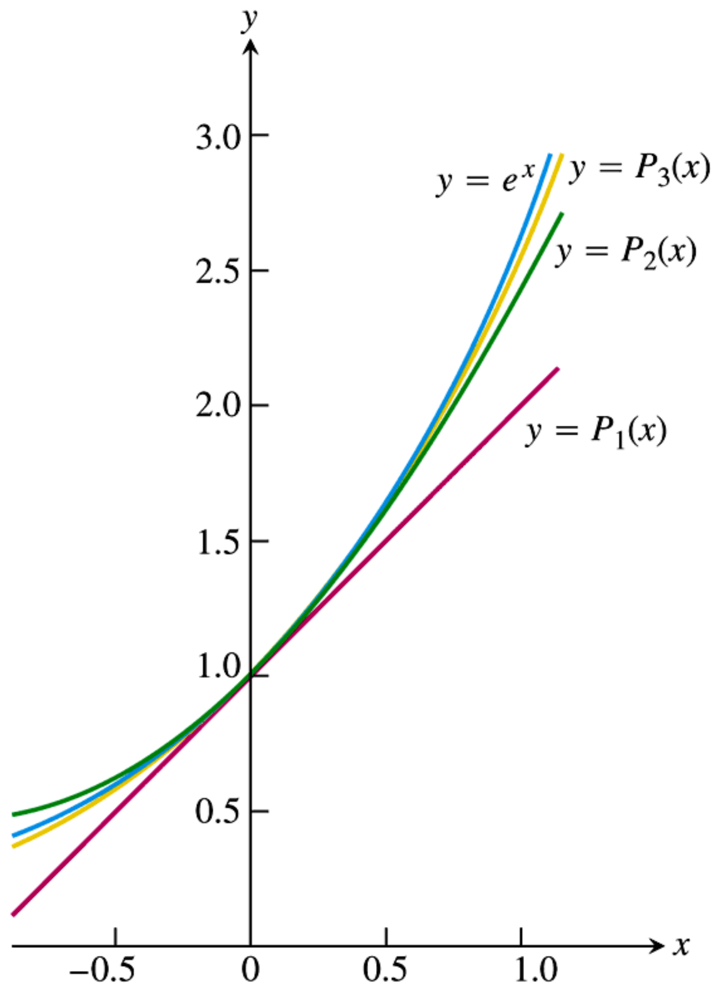


FIGURE 11.12 The graph of $f(x) = e^x$ and its Taylor polynomials

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + (x^2/2!)$

$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.

Notice the very close agreement near the center $x = 0$ (Example 2).

EXAMPLE 3 Finding Taylor Polynomials for $\cos x$

Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$\begin{aligned}
f(x) &= \cos x, & f'(x) &= -\sin x, \\
f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\
&\vdots & &\vdots \\
f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x.
\end{aligned}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$\begin{aligned}
f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
&= 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.
\end{aligned}$$

This is also the Maclaurin series for $\cos x$. In Section 11.9, we will see that the series converges to $\cos x$ at every x.

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n + 1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Figure 11.13 shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y-axis.

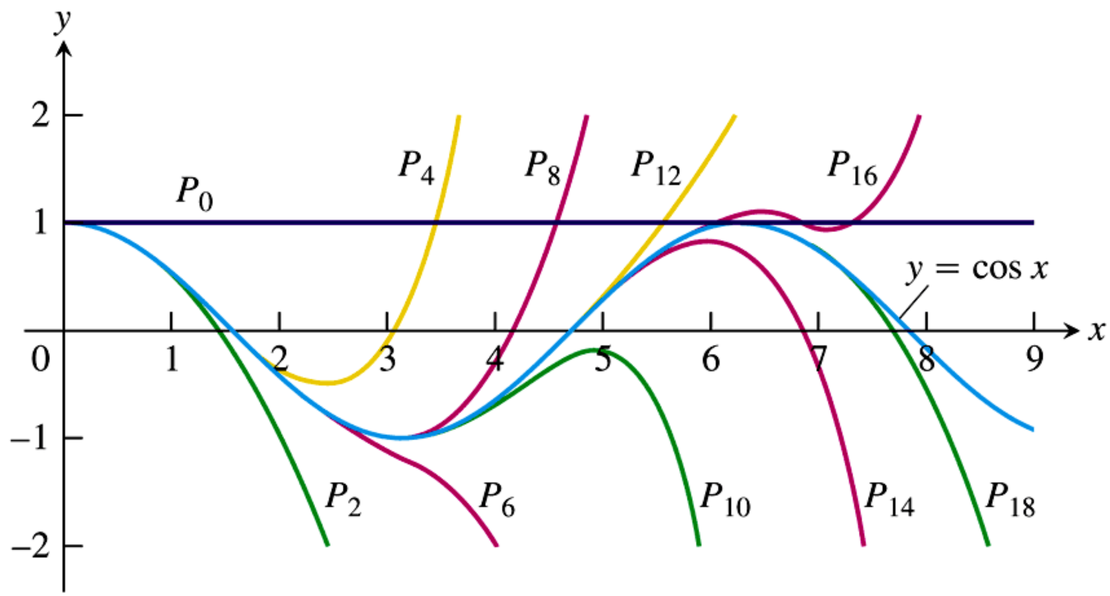


FIGURE 11.13 The polynomials

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

converge to $\cos x$ as $n \to \infty$. We can deduce the behavior of $\cos x$ arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

EXAMPLE 4 **A Function f Whose Taylor Series Converges at Every x but Converges to $f(x)$ Only at $x = 0$**

It can be shown (though not easily) that

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

(Figure 11.14) has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all n. This means that the Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 0 + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n + \cdots \\ = 0 + 0 + \cdots + 0 + \cdots. \end{aligned}$$

The series converges for every x (its sum is 0) but converges to $f(x)$ only at $x = 0$. ■

Two questions still remain.

1. For what values of x can we normally expect a Taylor series to converge to its generating function?
2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

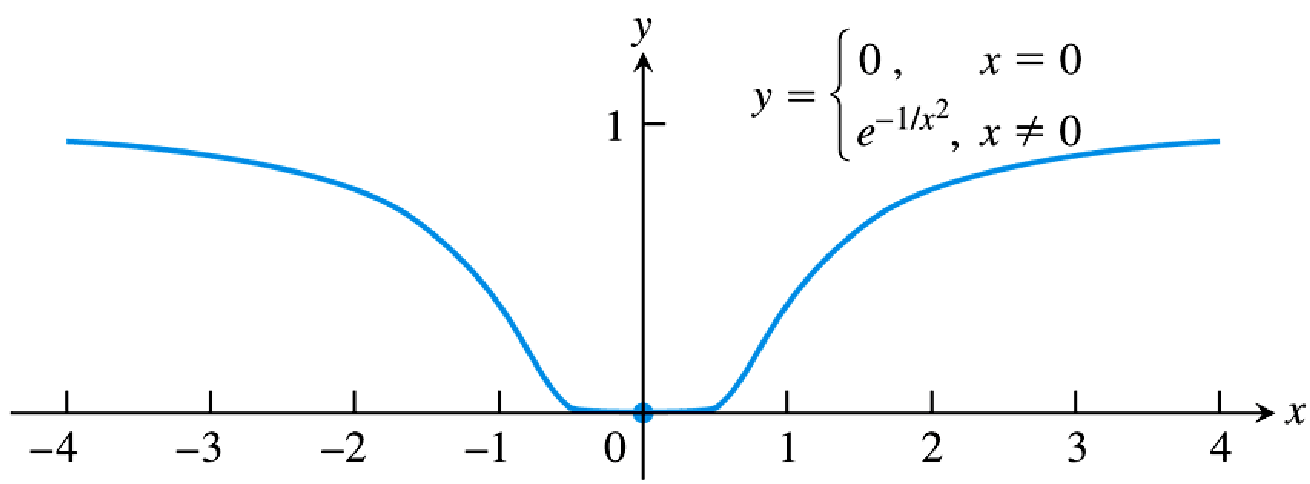


FIGURE 11.14 The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4).

11.9

Convergence of Taylor Series;

Error Estimates

THEOREM 22 Taylor's Theorem

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots$$
$$+ \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}.$$

Taylor's Theorem is a generalization of the Mean Value Theorem .

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \tag{1}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } a \text{ and } x. \tag{2}$$

When we state Taylor's theorem this way, it says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x).$$

The function $R_n(x)$ is determined by the value of the $(n+1)$st derivative $f^{(n+1)}$ at a point c that depends on both a and x, and which lies somewhere between them. For any value of n we want, the equation gives both a polynomial approximation of f of that order and a formula for the error involved in using that approximation over the interval I.

Equation (1) is called **Taylor's formula**. The function $R_n(x)$ is called the **remainder of order n** or the **error term** for the approximation of f by $P_n(x)$ over I. If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Often we can estimate R_n without knowing the value of c, as the following example illustrates.

EXAMPLE 1 The Taylor Series for e^x Revisited

Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Solution The function has derivatives of all orders throughout the interval $I = (-\infty, \infty)$. Equations (1) and (2) with $f(x) = e^x$ and $a = 0$ give

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

Polynomial from Section 11.8, Example 2

and

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1} \qquad \text{for some } c \text{ between 0 and } x.$$

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. When x is negative, so is c, and $e^c < 1$. When x is zero, $e^x = 1$ and $R_n(x) = 0$. When x is positive, so is c, and $e^c < e^x$. Thus,

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \qquad \text{when } x \le 0,$$

and

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \qquad \text{when } x > 0.$$

Finally, because

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \qquad \text{for every } x,$$

Section 11.1

$\lim_{n\to\infty} R_n(x) = 0$, and the series converges to e^x for every x. Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots. \qquad (3)$$

■

THEOREM 23 The Remainder Estimation Theorem

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M\frac{|x - a|^{n+1}}{(n + 1)!}.$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

EXAMPLE 2 The Taylor Series for $\sin x$ at $x = 0$

Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

Solution The function and its derivatives are

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, \\ f''(x) &= -\sin x, & f'''(x) &= -\cos x, \\ &\vdots & &\vdots \\ f^{(2k)}(x) &= (-1)^k \sin x, & f^{(2k+1)}(x) &= (-1)^k \cos x, \end{aligned}$$

so

$$f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k.$$

The series has only odd-powered terms and, for $n = 2k + 1$, Taylor's Theorem gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x).$$

All the derivatives of $\sin x$ have absolute values less than or equal to 1, so we can apply the Remainder Estimation Theorem with $M = 1$ to obtain

$$|R_{2k+1}(x)| \le 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}.$$

Since $(|x|^{2k+2}/(2k+2)!) \to 0$ as $k \to \infty$, whatever the value of x, $R_{2k+1}(x) \to 0$, and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x. Thus,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \qquad (4)$$

■

EXAMPLE 3 The Taylor Series for $\cos x$ at $x = 0$ Revisited

Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$ for every value of x.

Solution We add the remainder term to the Taylor polynomial for $\cos x$ (Section 11.8, Example 3) to obtain Taylor's formula for $\cos x$ with $n = 2k$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k}(x)| \leq 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x, $R_{2k} \to 0$ as $k \to \infty$. Therefore, the series converges to $\cos x$ for every value of x. Thus,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \qquad (5)$$

EXAMPLE 4 Finding a Taylor Series by Substitution

Find the Taylor series for $\cos 2x$ at $x = 0$.

Solution We can find the Taylor series for $\cos 2x$ by substituting $2x$ for x in the Taylor series for $\cos x$:

$$\begin{aligned}\cos 2x &= \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots && \text{Equation (5) with } 2x \text{ for } x \\ &= 1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k}x^{2k}}{(2k)!}.\end{aligned}$$

Equation (5) holds for $-\infty < x < \infty$, implying that it holds for $-\infty < 2x < \infty$, so the newly created series converges for all x. Exercise 45 explains why the series is in fact the Taylor series for $\cos 2x$. ■

EXAMPLE 5 Finding a Taylor Series by Multiplication

Find the Taylor series for $x \sin x$ at $x = 0$.

Solution We can find the Taylor series for $x \sin x$ by multiplying the Taylor series for $\sin x$ (Equation 4) by x:

$$\begin{aligned} x \sin x &= x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots. \end{aligned}$$

The new series converges for all x because the series for $\sin x$ converges for all x. Exercise 45 explains why the series is the Taylor series for $x \sin x$. ■

EXAMPLE 6 Calculate e with an error of less than 10^{-6}.

Solution We can use the result of Example 1 with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

with

$$R_n(1) = e^c \frac{1}{(n+1)!} \qquad \text{for some } c \text{ between 0 and 1.}$$

For the purposes of this example, we assume that we know that $e < 3$. Hence, we are certain that

$$\frac{1}{(n+1)!} < R_n(1) < \frac{3}{(n+1)!}$$

because $1 < e^c < 3$ for $0 < c < 1$.

By experiment we find that $1/9! > 10^{-6}$, while $3/10! < 10^{-6}$. Thus we should take $(n + 1)$ to be at least 10, or n to be at least 9. With an error of less than 10^{-6},

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{9!} \approx 2.718282. \qquad \blacksquare$$

EXAMPLE 7 For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ with an error of magnitude no greater than 3×10^{-4}?

Solution Here we can take advantage of the fact that the Taylor series for $\sin x$ is an alternating series for every nonzero value of x. According to the Alternating Series Estimation Theorem (Section 11.6), the error in truncating

$$\sin x = x - \frac{x^3}{3!} \;\vdots\; + \frac{x^5}{5!} - \cdots$$

after $(x^3/3!)$ is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

Therefore the error will be less than or equal to 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \qquad \text{or} \qquad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514.$$

Rounded down, to be safe

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: namely, that the estimate $x - (x^3/3!)$ for $\sin x$ is an underestimate when x is positive because then $x^5/120$ is positive.

Figure 11.15 shows the graph of $\sin x$, along with the graphs of a number of its approximating Taylor polynomials. The graph of $P_3(x) = x - (x^3/3!)$ is almost indistinguishable from the sine curve when $-1 \leq x \leq 1$.

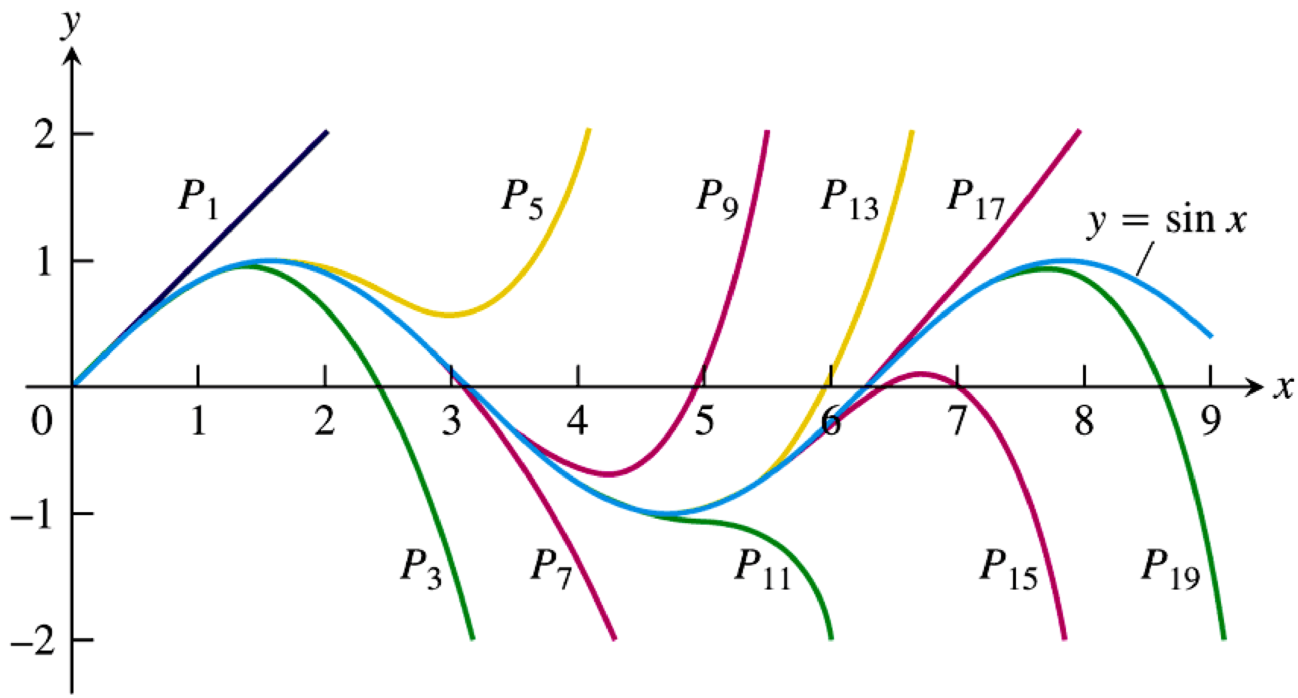


FIGURE 11.15 The polynomials

You might wonder how the estimate given by the Remainder Estimation Theorem compares with the one just obtained from the Alternating Series Estimation Theorem. If we write

$$\sin x = x - \frac{x^3}{3!} + R_3,$$

then the Remainder Estimation Theorem gives

$$|R_3| \le 1 \cdot \frac{|x|^4}{4!} = \frac{|x|^4}{24},$$

which is not as good. But if we recognize that $x - (x^3/3!) = 0 + x + 0x^2 - (x^3/3!) + 0x^4$ is the Taylor polynomial of order 4 as well as of order 3, then

$$\sin x = x - \frac{x^3}{3!} + 0 + R_4,$$

and the Remainder Estimation Theorem with $M = 1$ gives

$$|R_4| \le 1 \cdot \frac{|x|^5}{5!} = \frac{|x|^5}{120}.$$

This is what we had from the Alternating Series Estimation Theorem. ■

Combining Taylor Series

On the intersection of their intervals of convergence, Taylor series can be added, subtracted, and multiplied by constants, and the results are once again Taylor series. The Taylor series for $f(x) + g(x)$ is the sum of the Taylor series for $f(x)$ and $g(x)$ because the nth derivative of $f + g$ is $f^{(n)} + g^{(n)}$, and so on. Thus we obtain the Taylor series for $(1 + \cos 2x)/2$ by adding 1 to the Taylor series for $\cos 2x$ and dividing the combined results by 2, and the Taylor series for $\sin x + \cos x$ is the term-by-term sum of the Taylor series for $\sin x$ and $\cos x$.

11.10

Applications of Power Series

EXAMPLE 5 Express $\int \sin x^2 \, dx$ as a power series.

Solution From the series for $\sin x$ we obtain

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots.$$

Therefore,

$$\int \sin x^2 \, dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{10}}{19 \cdot 9!} - \cdots.$$

EXAMPLE 6 Estimating a Definite Integral

Estimate $\int_0^1 \sin x^2\, dx$ with an error of less than 0.001.

Solution From the indefinite integral in Example 5,

$$\int_0^1 \sin x^2\, dx = \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} - \cdots.$$

The series alternates, and we find by experiment that

$$\frac{1}{11 \cdot 5!} \approx 0.00076$$

is the first term to be numerically less than 0.001. The sum of the preceding two terms gives

$$\int_0^1 \sin x^2\, dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

With two more terms we could estimate

$$\int_0^1 \sin x^2\, dx \approx 0.310268$$

with an error of less than 10^{-6}. With only one term beyond that we have

$$\int_0^1 \sin x^2\, dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \frac{1}{6894720} \approx 0.310268303,$$

with an error of about 1.08×10^{-9}. To guarantee this accuracy with the error formula for the Trapezoidal Rule would require using about 8000 subintervals.

Arctangents

In Section 11.7, Example 5, we found a series for $\tan^{-1} x$ by differentiating to get

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

and integrating to get

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

However, we did not prove the term-by-term integration theorem on which this conclusion depended. We now derive the series again by integrating both sides of the finite formula

$$\frac{1}{1 + t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1 + t^2}, \tag{12}$$

in which the last term comes from adding the remaining terms as a geometric series with first term $a = (-1)^{n+1} t^{2n+2}$ and ratio $r = -t^2$. Integrating both sides of Equation (12) from $t = 0$ to $t = x$ gives

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n + 1} + R_n(x),$$

where

$$R_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1 + t^2}\, dt.$$

The denominator of the integrand is greater than or equal to 1; hence

$$|R_n(x)| \le \int_0^{|x|} t^{2n+2}\, dt = \frac{|x|^{2n+3}}{2n + 3}.$$

If $|x| \le 1$, the right side of this inequality approaches zero as $n \to \infty$. Therefore $\lim_{n\to\infty} R_n(x) = 0$ if $|x| \le 1$ and

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1.$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad |x| \le 1 \tag{13}$$

We take this route instead of finding the Taylor series directly because the formulas for the higher-order derivatives of $\tan^{-1} x$ are unmanageable. When we put $x = 1$ in Equation (13), we get **Leibniz's formula**:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots.$$

Because this series converges very slowly, it is not used in approximating π to many decimal places. The series for $\tan^{-1} x$ converges most rapidly when x is near zero. For that reason, people who use the series for $\tan^{-1} x$ to compute π use various trigonometric identities.

For example, if

$$\alpha = \tan^{-1}\frac{1}{2} \qquad \text{and} \qquad \beta = \tan^{-1}\frac{1}{3},$$

then

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1 = \tan\frac{\pi}{4}$$

and

$$\frac{\pi}{4} = \alpha + \beta = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}.$$

Now Equation (13) may be used with $x = 1/2$ to evaluate $\tan^{-1}(1/2)$ and with $x = 1/3$ to give $\tan^{-1}(1/3)$. The sum of these results, multiplied by 4, gives π.

EXAMPLE 7 Limits Using Power Series

Evaluate

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

Solution We represent $\ln x$ as a Taylor series in powers of $x - 1$. This can be accomplished by calculating the Taylor series generated by $\ln x$ at $x = 1$ directly or by replacing x by $x - 1$ in the series for $\ln(1 + x)$ in Section 11.7, Example 6. Either way, we obtain

$$\ln x = (x - 1) - \frac{1}{2}(x - 1)^2 + \cdots,$$

from which we find that

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \left(1 - \frac{1}{2}(x - 1) + \cdots\right) = 1.$$

EXAMPLE 8 Limits Using Power Series

Evaluate

$$\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}.$$

Solution The Taylor series for $\sin x$ and $\tan x$, to terms in x^5, are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

Hence,

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

and

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x - \tan x}{x^3} &= \lim_{x \to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right) \\ &= -\frac{1}{2}. \end{aligned}$$

EXAMPLE 9 Approximation Formula for $\csc x$

Find $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

Solution

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}=\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}{x\cdot\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}$$

$$=\frac{x^3\left(\frac{1}{3!}-\frac{x^2}{5!}+\cdots\right)}{x^2\left(1-\frac{x^2}{3!}+\cdots\right)}=x\,\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}.$$

Therefore,

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(x\,\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}\right)=0.$$

From the quotient on the right, we can see that if $|x|$ is small, then

$$\frac{1}{\sin x}-\frac{1}{x}\approx x\cdot\frac{1}{3!}=\frac{x}{6}\qquad\text{or}\qquad \csc x\approx\frac{1}{x}+\frac{x}{6}.$$

TABLE 11.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\ln \frac{1+x}{1-x} = 2 \tanh^{-1} x = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

Binomial Series

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)x^k}{k!} + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \qquad |x| < 1,$$

where

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!}, \qquad \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!} \qquad \text{for } k \ge 3.$$

Note: To write the binomial series compactly, it is customary to define $\binom{m}{0}$ to be 1 and to take $x^0 = 1$ (even in the usually excluded case where $x = 0$), yielding $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$. If m is a *positive integer*, the series terminates at x^m and the result converges for all x.