Chapter 14

Partial Derivatives

14.1

Functions and Several Variables

DEFINITIONS **Function of n Independent Variables**

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

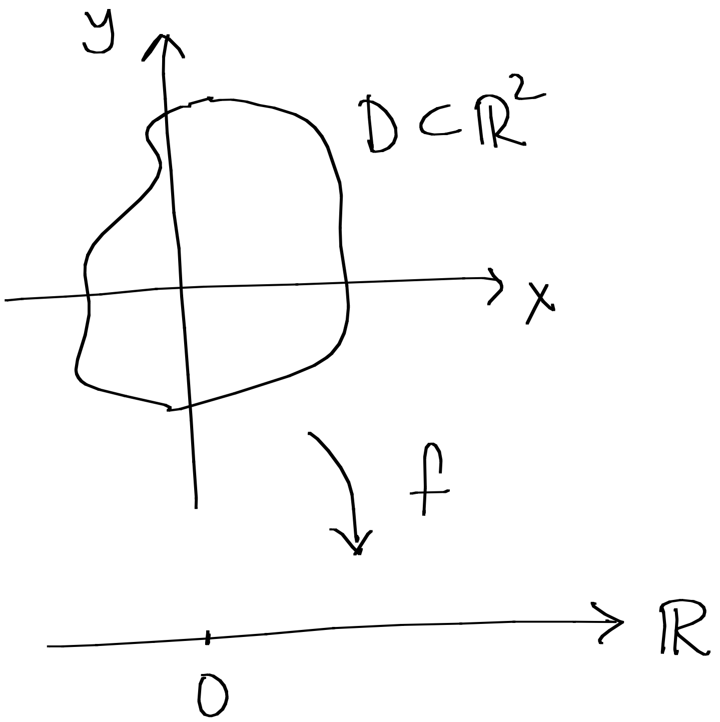
$f: D \subseteq \mathbb{R}^n \longrightarrow R \subseteq \mathbb{R}$

$(x_1, x_2, \dots, x_n) \longmapsto w = f(x_1, x_2, \dots, x_n)$

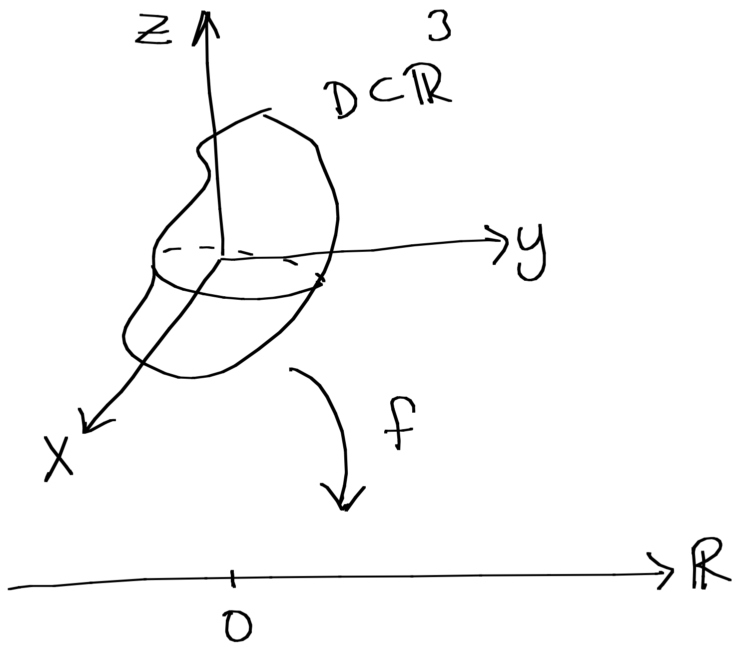
$x_1, x_2, \dots, x_n$: independent variables

w: dependent variable

<u>$n=2$</u> ; $f: D \subseteq \mathbb{R}^2 \longrightarrow R \subseteq \mathbb{R}$



<u>$n=3$</u> : $f: D \subset \mathbb{R}^3 \to R \subset \mathbb{R}$



EXAMPLE 1 Evaluating a Function

The value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 0, 4)$ is

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.$$

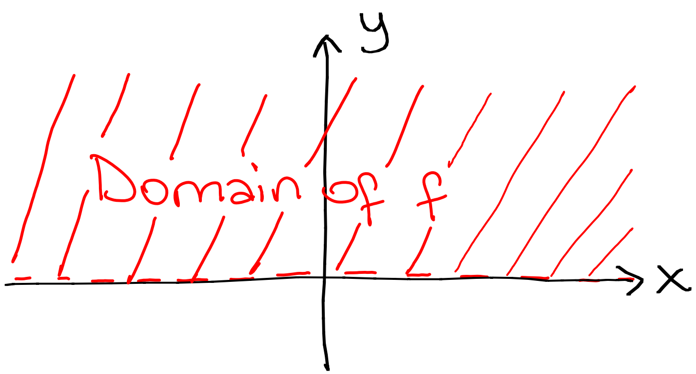
From Section 12.1, we recognize f as the distance function from the origin to the point (x, y, z) in Cartesian space coordinates. ■

Domain of f is $\mathbb{R}^3$, range of f is $[0, \infty)$.

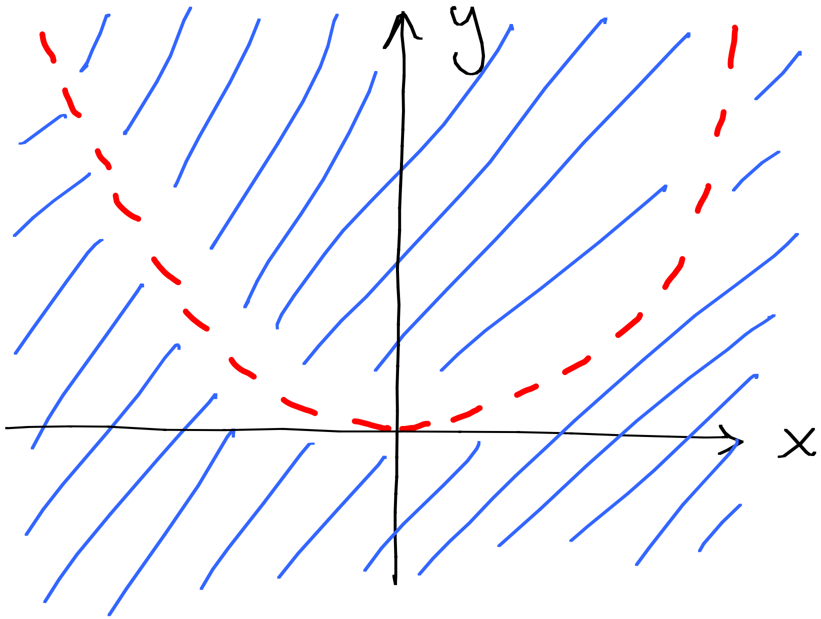
Example: Let $f(x,y) = x \cdot \ln(y)$

Domain of f: $\{(x,y) \in \mathbb{R}^2 : y > 0\}$

Range of f: $\mathbb{R} = (-\infty, \infty)$



Example: Let $f(x,y) = \dfrac{2x}{y-x^2}$.



Domain of f: $\{(x,y) \in \mathbb{R}^2 : y \neq x^2\}$

which is the complement of the parabola $y=x^2$ in $\mathbb{R}^2$.

Range of f: $\mathbb{R} = (-\infty, \infty)$.

Example: Let $f(x,y,z) = \dfrac{1}{xyz}$.

Domain of f: $\{(x,y,z) \in \mathbb{R}^3 : xyz \neq 0\}$

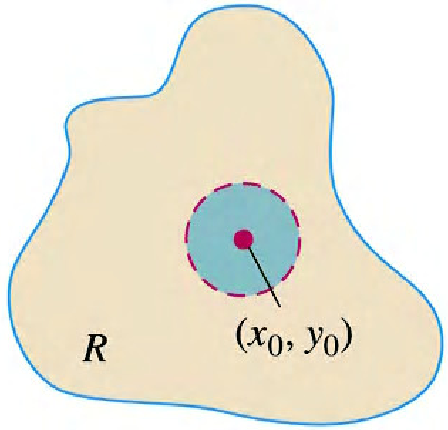
$xyz = 0 \Leftrightarrow x=0$ (yz-plane), or $y=0$ (xz-plane), or $z=0$ (xy-plane)

So the domain is the complement of the union of coordinate planes in $\mathbb{R}^3$. Range of f is $\mathbb{R} - \{0\}$.

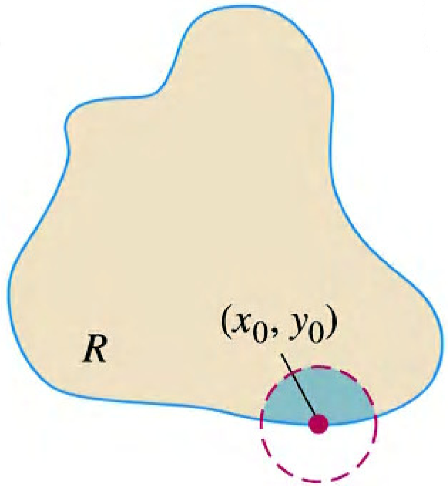
DEFINITIONS Interior and Boundary Points, Open, Closed

A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.1). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.2).

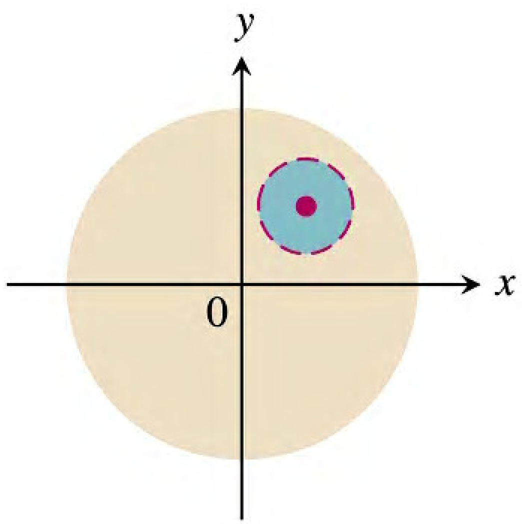


(a) Interior point



(b) Boundary point

FIGURE 14.1 Interior points and boundary points of a plane region R. An interior point is necessarily a point of R. A boundary point of R need not belong to R.



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)

$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk.
Contains all boundary points.

FIGURE 14.2 Interior points and boundary points of the unit disk in the plane.

DEFINITIONS **Bounded and Unbounded Regions in the Plane**

A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

$D = \{(x, y) : x^2 + y^2 \le 1\}$ is a bounded region.

$R = \{(x, y) : x \ge 0, y \ge 0\}$ is an unbounded region.

Example 3: Identify domain and range of

$$f(x,y) = \sqrt{y - x^2}$$

$f : \mathbb{R}^2 \to \mathbb{R}$

Domain of f: $D = \{ (x,y) \mid y - x^2 \geq 0 \} \subseteq \mathbb{R}^2$

Since $\sqrt{\cdot} \geq 0$, the range consists of nonnegative real numbers.

Range of f: $[0, \infty) \subset \mathbb{R}$

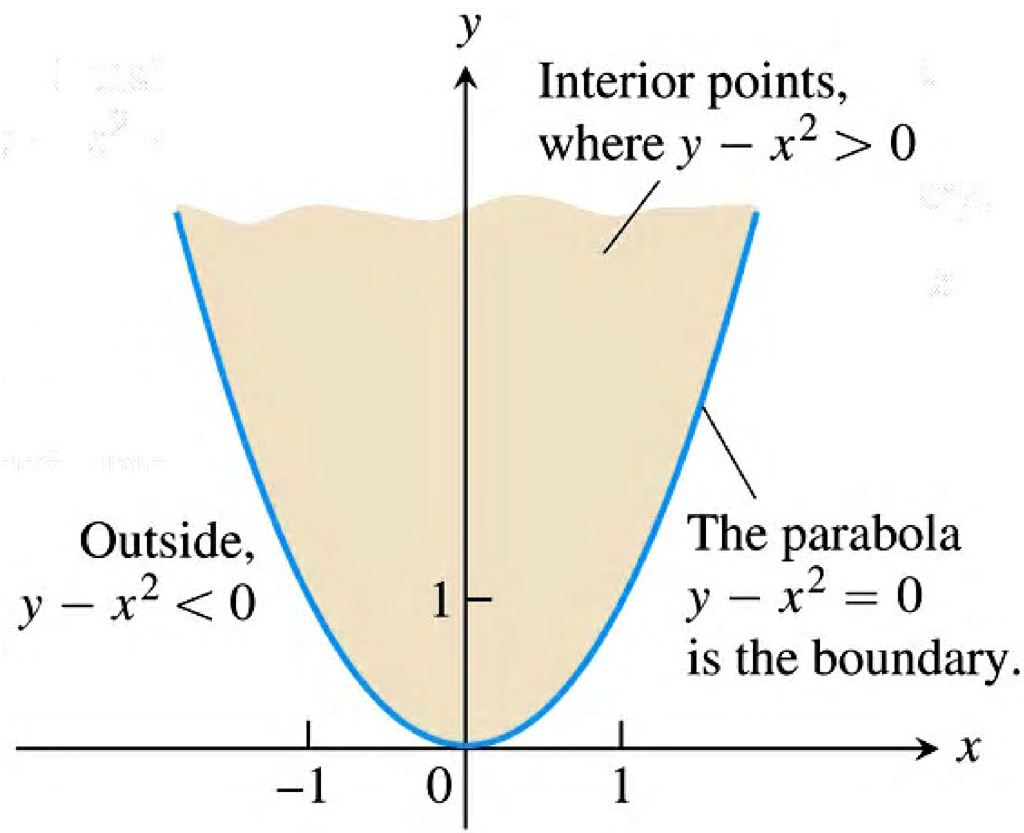


FIGURE 14.3 The domain of $f(x, y) = \sqrt{y - x^2}$ consists of the shaded region and its bounding parabola $y = x^2$ (Example 3). It is a closed and unbounded region.

DEFINITIONS Level Curve, Graph, Surface

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

Example

Let $f(x,y) = 100 - x^2 - y^2$

For $c = 0$:

$f(x,y) = 100 - x^2 - y^2 = 0$

$\Leftrightarrow x^2 + y^2 = 100$

(outer circle)

For $c = 51$:

$f(x,y) = 100 - x^2 - y^2 = 51$

$\Leftrightarrow x^2 + y^2 = 49$

(middle circle)

For $c = 75$:

$x^2 + y^2 = 25$ (inner circle)

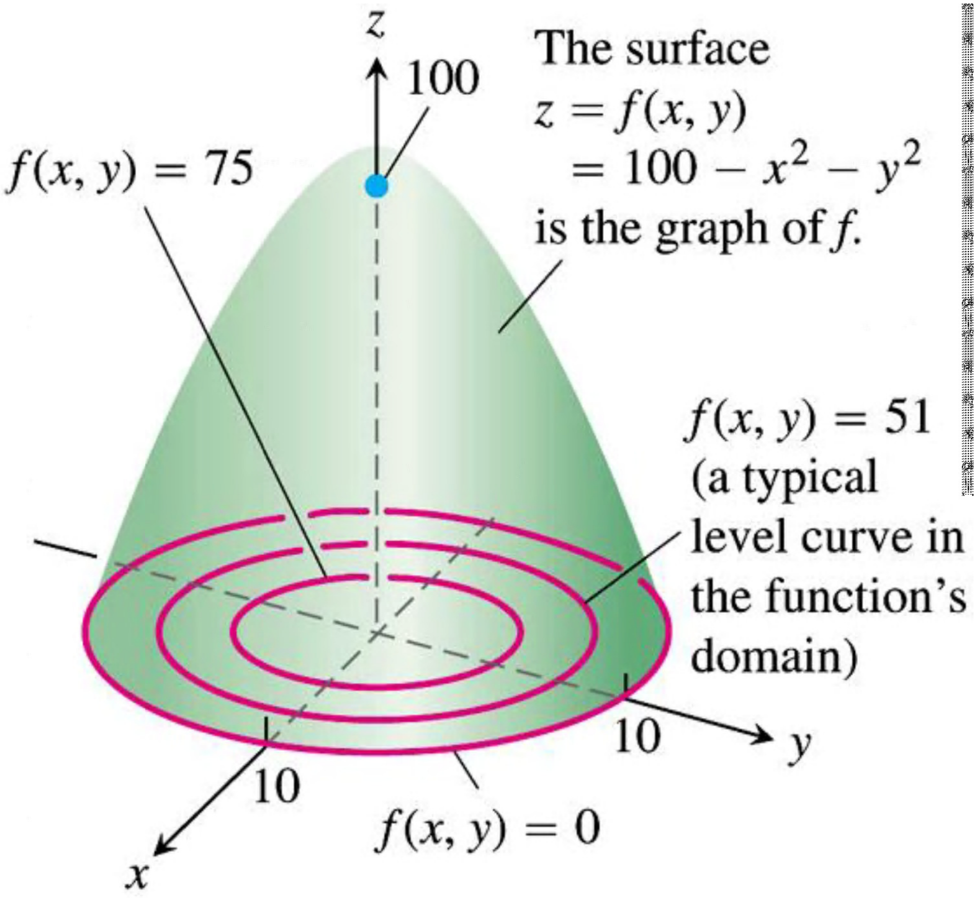


FIGURE 14.4 The graph and selected level curves of the function $f(x, y) = 100 - x^2 - y^2$ (Example 4).

To each level curve $f(x,y)=c$, there corresponds a contour curve which is the intersection of the plane $z=c$ with the surface $z=f(x,y)$.

Level curves are projections of contour curves on the xy-plane.

Example

Let $f(x,y)=100-x^2-y^2$

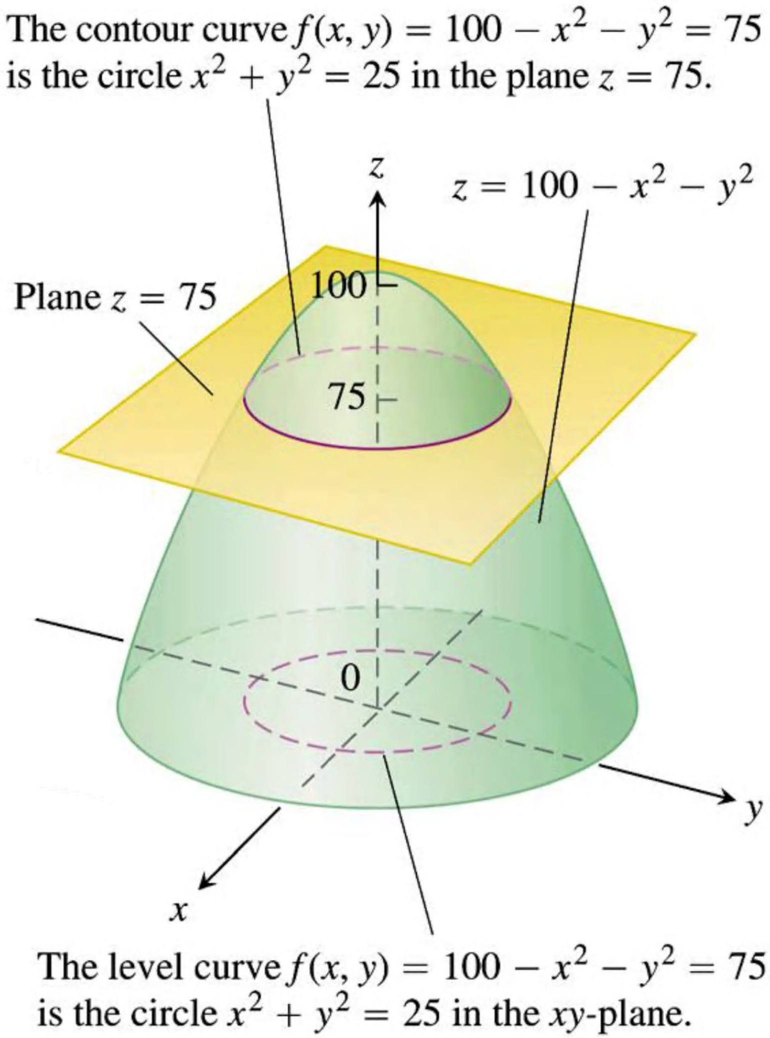


FIGURE 14.5 A plane $z = c$ parallel to the xy-plane intersecting a surface $z = f(x, y)$ produces a contour curve.

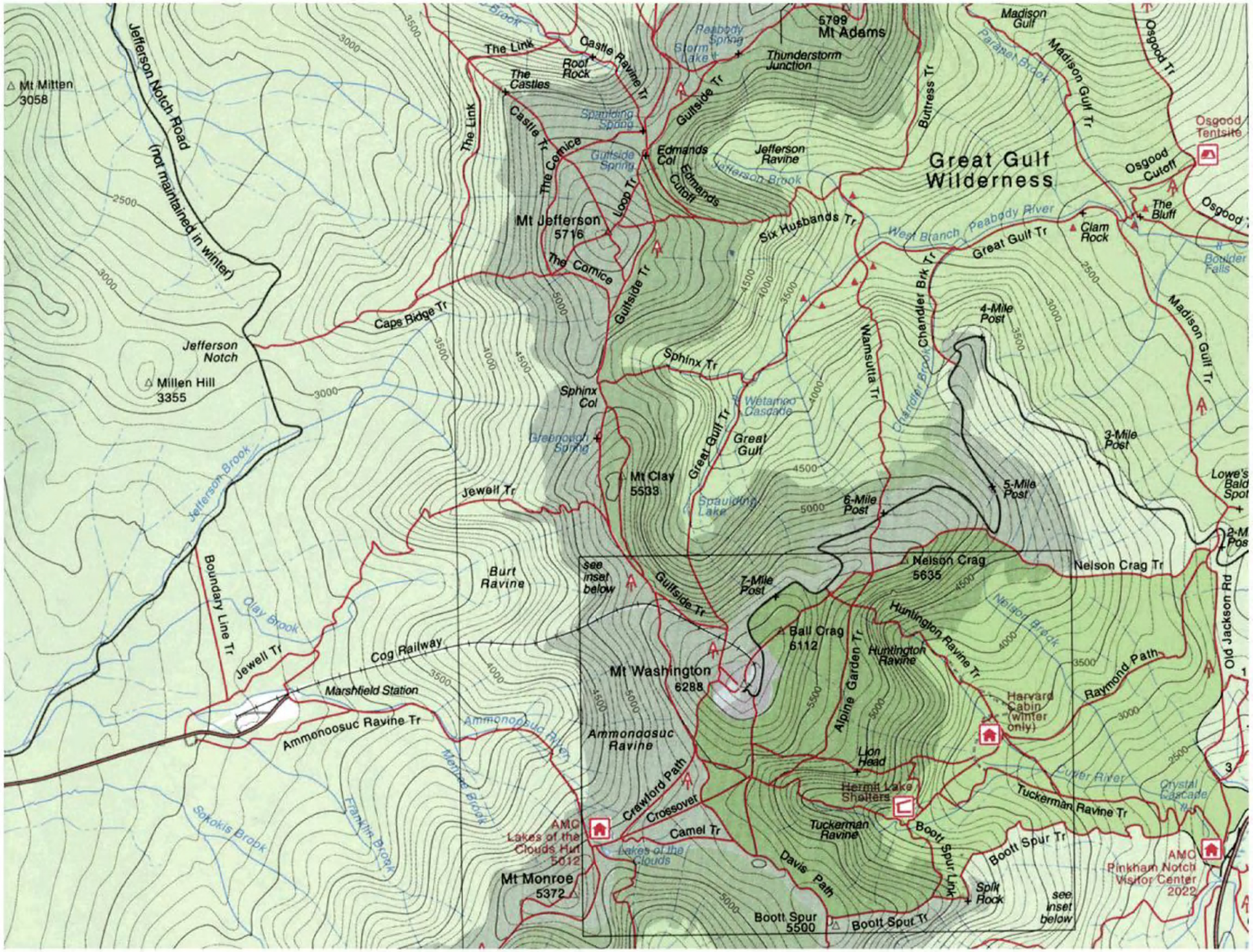


FIGURE 14.6 Contours on Mt. Washington in New Hampshire. (Reproduced by permission from the Appalachian Mountain Club.)

DEFINITION Level Surface

The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

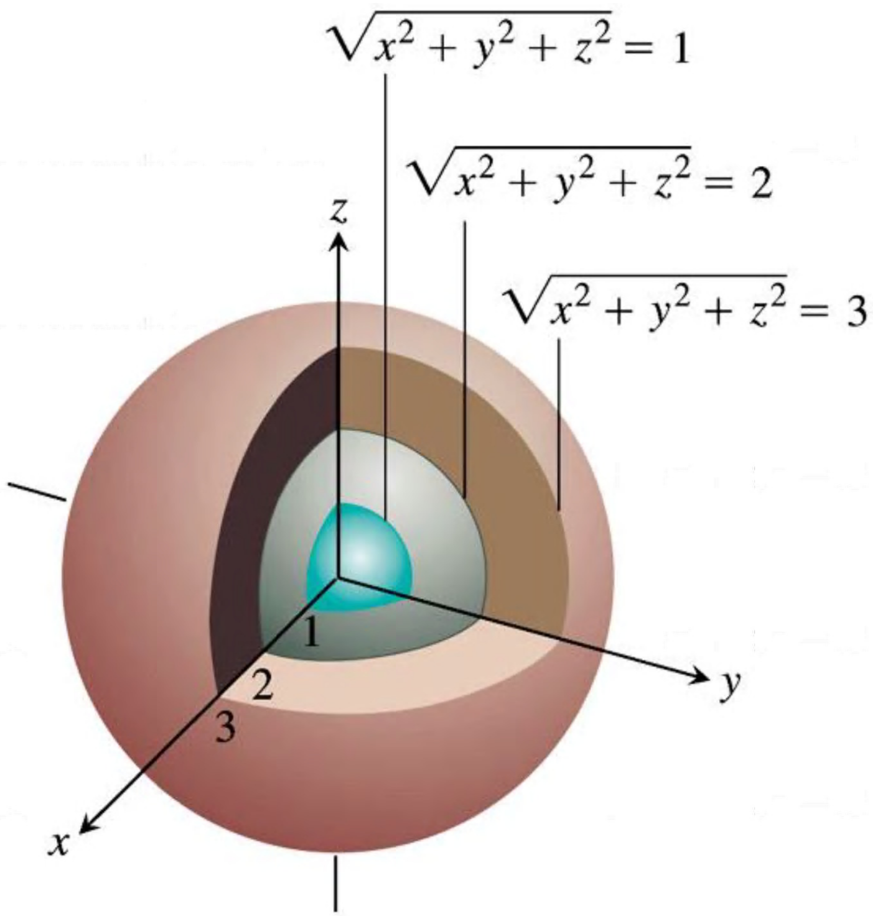


FIGURE 14.7 The level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ are concentric spheres (Example 5).

Observe that any surface can be realized as a level surface of a 3-variable function.

Example: The surface $z=f(x,y)$ is a level surface of $g(x,y,z)=f(x,y)-z$ at $c=0$ level.

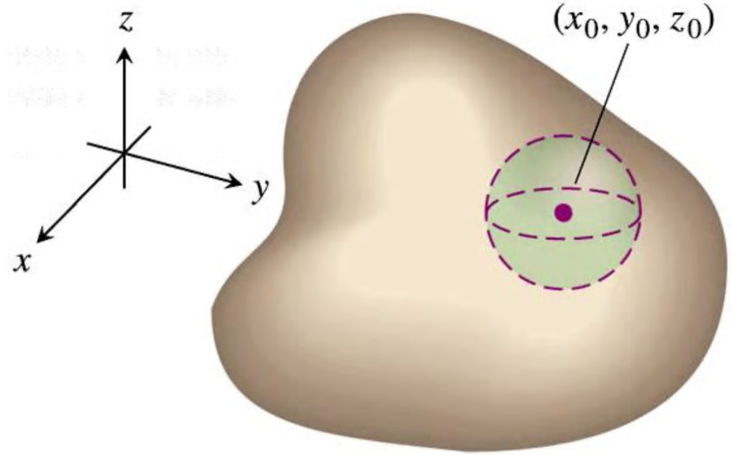
Example: The sphere $x^2+(y-1)^2+(z+2)^2=5$ is a level surface of $f(x,y,z)=x^2+(y-1)^2+(z+2)^2$ at $c=5$ level

OR $g(x,y,z)=x^2+(y-1)^2+(z+2)^2-5$ at $c=0$ level.

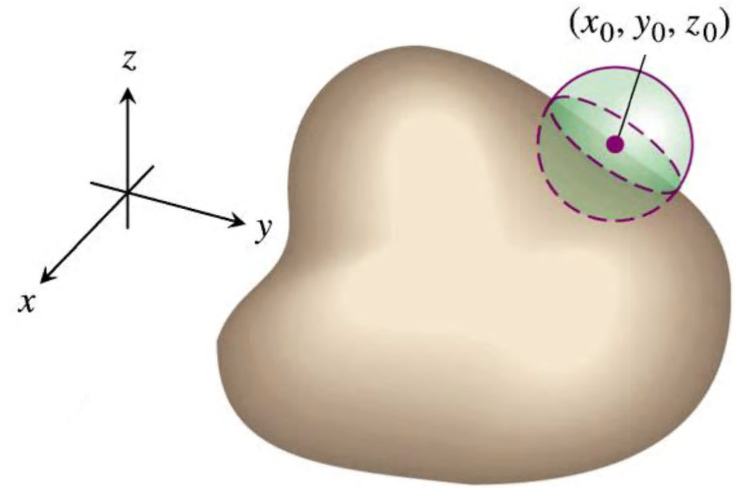
DEFINITIONS Interior and Boundary Points for Space Regions

A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if it is the center of a solid ball that lies entirely in R (Figure 14.8a). A point (x_0, y_0, z_0) is a **boundary point** of R if every sphere centered at (x_0, y_0, z_0) encloses points that lie outside of R as well as points that lie inside R (Figure 14.8b). The **interior** of R is the set of interior points of R. The **boundary** of R is the set of boundary points of R.

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.

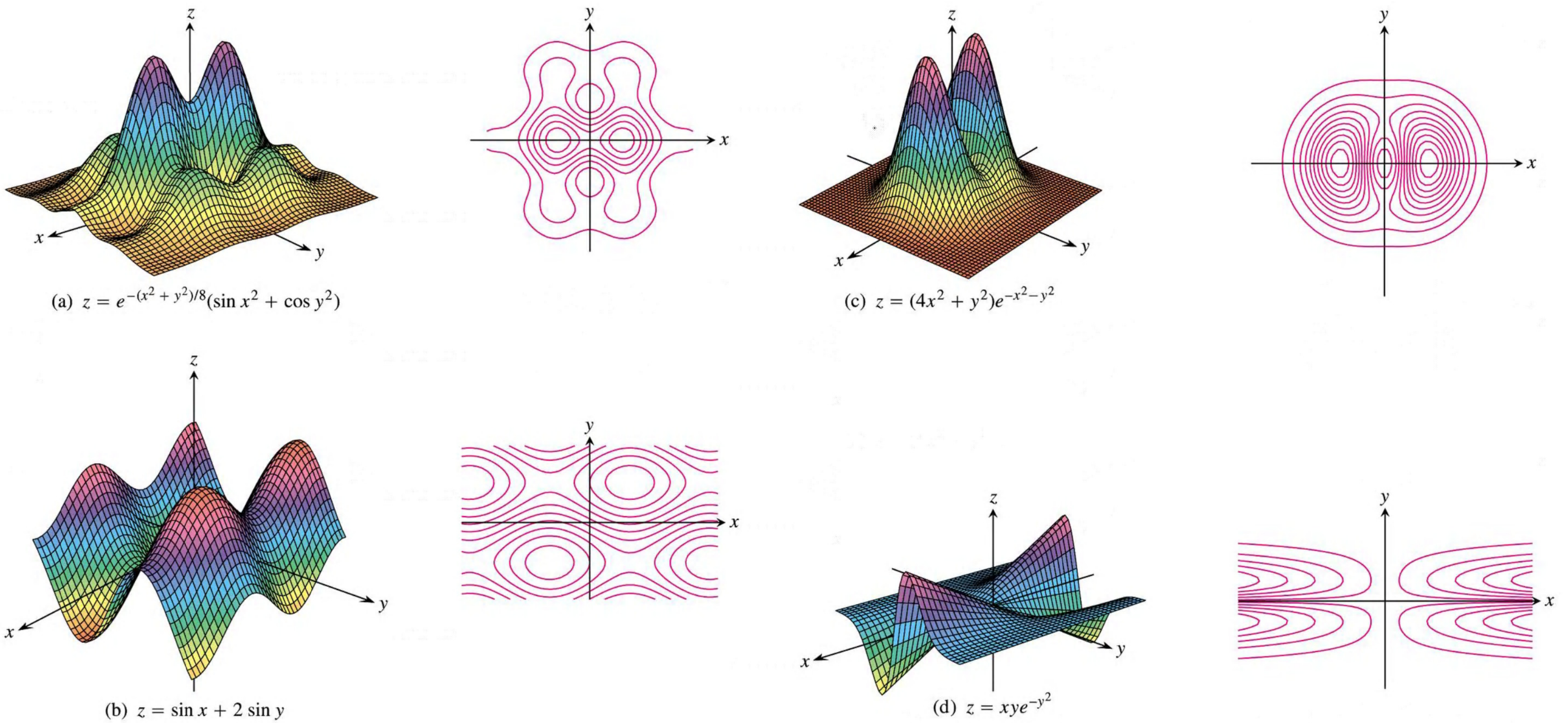


(a) Interior point



(b) Boundary point

FIGURE 14.8 Interior points and boundary points of a region in space.



(a) $z = e^{-(x^2 + y^2)/8}(\sin x^2 + \cos y^2)$

(b) $z = \sin x + 2 \sin y$

(c) $z = (4x^2 + y^2)e^{-x^2 - y^2}$

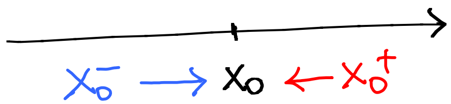
(d) $z = xye^{-y^2}$

FIGURE 14.10 Computer-generated graphs and level ~~surfaces~~ curves of typical functions of two variables.

14.2

Limits and Continuity in Higher Dimensions

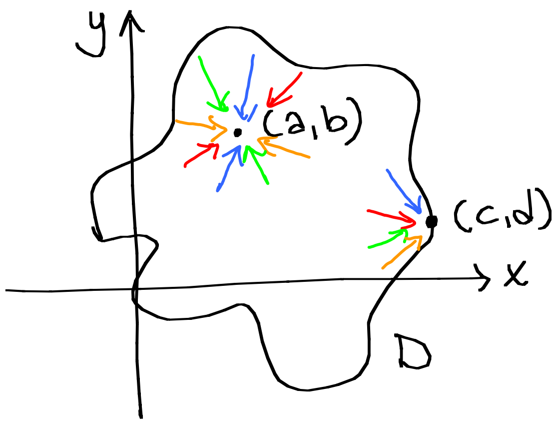
⊛ If $f: D \subset \mathbb{R} \to \mathbb{R}$ is a function of one variable, then to evaluate limit at a point $x_0 \in \mathbb{R}$ we check $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$:



x approaches to x_0 from left hand side and right hand side.

⊛ If $f: D \subset \mathbb{R}^n \to \mathbb{R}$ is a function of n variables with $n \geq 2$, then there are infinitely many different ways of approaching to a point $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

<u>For $n=2$:</u>



(a,b) : interior point of D

(c,d) : boundary point of D

Informal Definition: If the values of $f(x,y)$ lie arbitrarily close to a unique finite number L for all points (x,y) sufficiently close to a point (x_0,y_0), then we say that f approaches the limit L as (x,y) approaches to (x_0,y_0).

FORMAL
DEFINITION **Limit of a Function of Two Variables**

We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

The point (x_0,y_0) can be a boundary point as well as an interior point of the domain of f. The only requirement is that the point (x,y) remain in the domain of f at all times.

THEOREM 1 Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x, y)\to(x_0, y_0)} g(x, y) = M.$$

1. *Sum Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$
2. *Difference Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$
3. *Product Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$
4. *Constant Multiple Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (kf(x, y)) = kL \qquad \text{(any number } k\text{)}$$
5. *Quotient Rule:* $$\lim_{(x, y)\to(x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \qquad M \neq 0$$
6. *Power Rule:* If r and s are integers with no common factors, and $s \neq 0$, then

$$\lim_{(x, y)\to(x_0, y_0)} (f(x, y))^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Using the definition of limit we can obtain the following results.

* $$\lim_{(x,y)\to(x_0,y_0)} x = x_0$$
* $$\lim_{(x,y)\to(x_0,y_0)} y = y_0$$
* $$\lim_{(x,y)\to(x_0,y_0)} k = k, \text{ for any } k\in\mathbb{R}$$

Theorem 1 together with the above results imply that we can evaluate the limits of polynomials, rational functions, power functions and their combinations by direct substitution, whenever defined.

EXAMPLE 1 Calculating Limits

(a) $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

a) $$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y} = \frac{\lim\limits_{(x,y)\to(0,1)} (x-xy+3)}{\lim\limits_{(x,y)\to(0,1)} (x^2y+5xy-y)} = \frac{\lim\limits_{(x,y)\to(0,1)} (x) - \lim\limits_{(x,y)\to(0,1)} (xy) + \lim\limits_{(x,y)\to(0,1)} (3)}{\lim\limits_{(x,y)\to(0,1)} (x^2y) + \lim\limits_{(x,y)\to(0,1)} (5xy) - \lim\limits_{(x,y)\to(0,1)} (y)}$$

EXAMPLE 2 Calculating Limits

Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{\left(x^2 - xy\right)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x - y)\left(\sqrt{x} + \sqrt{y}\right)}{x - y} \qquad \text{Algebra}$$

$$= \lim_{(x,y)\to(0,0)} x\left(\sqrt{x} + \sqrt{y}\right) \qquad \text{Cancel the nonzero factor } (x - y).$$

$$= 0\left(\sqrt{0} + \sqrt{0}\right) = 0$$

We can cancel the factor $(x - y)$ because the path $y = x$ (along which $x - y = 0$) is *not* in the domain of the function

$$f(x,y) = \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

■

DEFINITION Continuous Function of Two Variables

A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Recall from MAT103 that the standard functions (polynomials, rational functions, power functions, trigonometric functions and their inverses, exponential functions, logarithm functions) are continuous on their domain.

Theorem 1 implies that algebraic combinations of continuous functions are continuous wherever they are defined.

Example: Let $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

The domain of $\dfrac{2xy}{x^2+y^2}$ is $\mathbb{R}^2 - \{(0,0)\}$. So $f(x,y)$ is continuous at any point $(x,y) \neq (0,0)$.

$f(0,0) = 0$. To be continuous at $(0,0)$, the limit of $f(x,y)$ at $(0,0)$ should be 0. Along the line $y=x$ with $x \neq 0$, f has a constant value:

$$f(x,y)\Big|_{y=x} = f(x,x) = \frac{2x.x}{x^2+x^2} = \frac{2x^2}{2x^2} = 1 \quad (\text{since } x \neq 0)$$

So, $$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x}} f(x,y) = \lim_{(x,y)\to(0,0)} \left[f(x,y)\Big|_{y=x} \right] = 1 \neq 0$$

Therefore, f is not continuous at $(0,0)$.

In fact f has no limit at $(0,0)$:

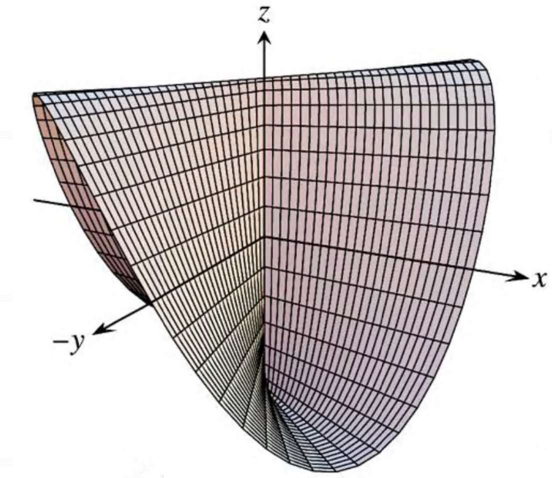
For any $m \in \mathbb{R}$, f has a constant value along the line $y = mx$ with $x \neq 0$:

$$f(x,y)\Big|_{y=mx} = \frac{2xy}{x^2+y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2+(mx)^2} = \frac{2mx^2}{(1+m^2)x^2} = \frac{2m}{1+m^2} \quad (\text{since } x \neq 0).$$

So, $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \lim_{(x,y)\to(0,0)} \left[f(x,y)\Big|_{y=mx} \right] = \frac{2m}{1+m^2}$. The limit depends on m

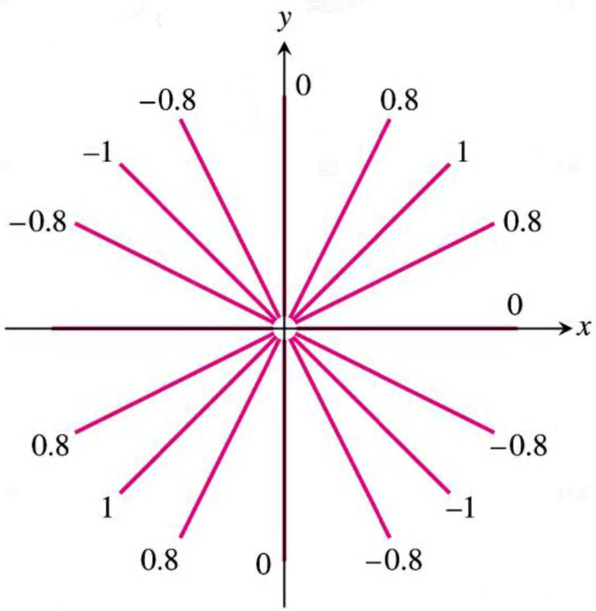
Approaching to $(0,0)$ along different lines give different limit candidates, which contradicts to the uniqueness property of limit.

So $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.



(a)

Exercise: Show that the range of $f(x,y)$ is $[-1,1]$.



(b)

FIGURE 14.11 (a) The graph of

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

The function is continuous at every point except the origin. (b) The level curves of f (Example 4).

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0), then $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.

EXAMPLE 5 Applying the Two-Path Test

Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.12) has no limit as (x, y) approaches $(0, 0)$.

Solution The limit cannot be found by direct substitution, which gives the form 0/0. We examine the values of f along curves that end at $(0, 0)$. Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y)\bigg|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\bigg|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x, y)\to(0,0) \\ \text{along } y=kx^2}} f(x, y) = \lim_{(x, y)\to(0,0)} \left[f(x, y)\bigg|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches $(0, 0)$ along the x-axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches $(0, 0)$.

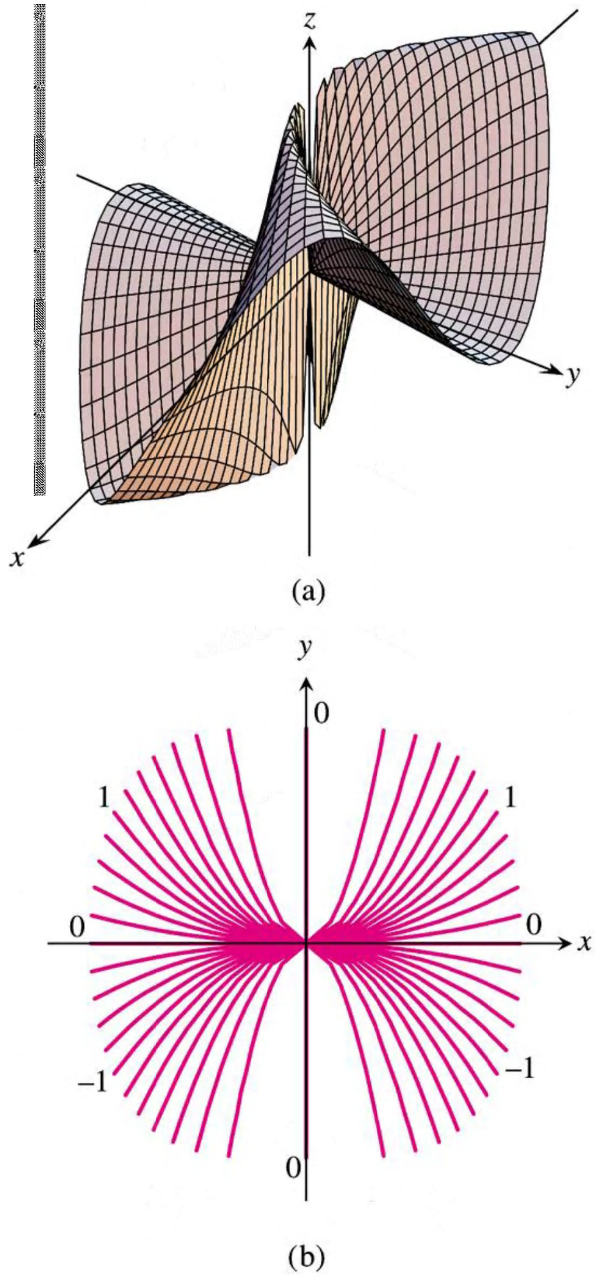


FIGURE 14.12 (a) The graph of $f(x, y) = 2x^2y/(x^4 + y^2)$. As the graph suggests and the level-curve values in part (b) confirm, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist (Example 5).

Exercise: Show that the following limits do not exist by using the two-path test:

a) $$\lim_{(x,y)\to(1,0)} \frac{y}{x+y-1}$$

b) $$\lim_{(x,y)\to(1,-2)} \frac{(x-1)(y+2)}{(x-1)^2+(y+2)^2}$$

c) $$\lim_{(x,y)\to(1,0)} \frac{x^2+y^2-2x+1}{y^2-x^2+2x-1}$$

d) $$\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$$

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

Examples: $f(x,y) = e^{x-y}$, $g(x,y) = \cos \frac{xy}{x^2+1}$, and $h(x,y) = \ln(1+x^2y^2)$ are continuous at every point $(x,y) \in \mathbb{R}^2$.

The Sandwich Theorem

Let f, g and h be functions of two variables that satisfies

$g(x,y) \le f(x,y) \le h(x,y)$ for all $(x,y) \neq (x_0,y_0)$ in a disk centered

at (x_0,y_0) and suppose that $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = L$.

Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

Example: Evaluate $\lim\limits_{(x,y)\to(0,0)} x.\cos\dfrac{1}{y}$.

$$-1 \le \cos\frac{1}{y} \le 1 \quad \text{for all } y \in \mathbb{R} - \{0\}$$

$$\left.\begin{array}{llll} \text{If} & x>0 & \text{then} & -x \le x.\cos\frac{1}{y} \le x \\ \text{If} & x<0 & \text{then} & x \le x.\cos\frac{1}{y} \le -x \end{array}\right\} \quad \lim_{(x,y)\to(0,0)} x = \lim_{(x,y)\to(0,0)} -x = 0$$

By the Sandwich Theorem $\lim\limits_{(x,y)\to(0,0)} x.\cos\dfrac{1}{y} = 0$.

Example: Evaluate $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}$.

Note that $\displaystyle 0 \le \frac{(x-1)^2}{(x-1)^2+y^2} \le 1$ for all $(x,y) \neq (1,0)$

If $x>1$ then $\ln x>0$, so $\displaystyle 0 \le \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} \le \ln x$

If $0<x<1$ then $\ln x<0$, so $\displaystyle \ln x \le \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} \le 0$

Since $\displaystyle\lim_{(x,y)\to(1,0)} \ln x = \lim_{(x,y)\to(1,0)} 0 = 0$, by Sandwich Theorem

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0.$$

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln(x + y + z) \qquad \text{and} \qquad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P \to (1,0,-1)} \frac{e^{x+z}}{z^2 + \cos\sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z), may be found by direct substitution.

The Sandwich Theorem also holds for functions of more than two variables.

If you cannot evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$ in rectangular coordinates then you may change the problem into polar coordinates.

Substitute $x = r.\cos\theta$, $y = r.\sin\theta$, and investigate the limit of the resulting function as $r \to 0$. That is, try to determine whether there exists a unique finite number L satisfying the following:

For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

If $|r| < \delta$ then $|f(r\cos\theta, r\sin\theta) - L| < \varepsilon$ $\quad (*)$

If such an L exists then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = L$

Example: Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$, if exists.

$$f(x,y) = \frac{x^3}{x^2+y^2} \Rightarrow f(r.\cos\theta, r.\sin\theta) = \frac{r^3.\cos^3\theta}{r^2.\cos^2\theta + r^2.\sin^2\theta} = r.\cos^3\theta \quad (r \neq 0)$$

$$\lim_{r\to 0} f(r.\cos\theta, r.\sin\theta) = \lim_{r\to 0} r.\cos^3\theta = 0 \text{ for any } \theta \in \mathbb{R}.$$

But we need to verify the above limit by showing that (*) is satisfied. Suppose that we are given $\varepsilon > 0$. We want to find a $\delta > 0$ such that for all r and θ if $|r| < \delta$ then $|r.\cos^3\theta - 0| < \varepsilon$.

$|r.\cos^3\theta - 0| = |r|.|\cos^3\theta| \leq |r|$. So, choose any δ with $0 < \delta \leq \varepsilon$ so that if $|r| < \delta$ then $|r.\cos^3\theta - 0| \leq |r| < \delta \leq \varepsilon$.

Example: Evaluate the $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$ if exists.

$$f(x,y) = \frac{x^2}{x^2+y^2} \Rightarrow f(r.\cos\theta, r.\sin\theta) = \frac{r^2\cos^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \cos^2\theta$$: depends on θ only.

$$\lim_{r\to 0} f(r.\cos\theta, r.\sin\theta) = \lim_{r\to 0} \cos^2\theta$$ does not exist

So $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$ does not exist.

Example: Evaluate the $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$, if exists.

$$f(x,y) = \frac{2x^2y}{x^4+y^2} \Rightarrow f(r.\cos\theta, r.\sin\theta) = \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}$$

If $r \neq 0$ and $\theta = 0$ or π then $f(r.\cos\theta, r.\sin\theta) = 0$

so $\lim\limits_{\substack{r\to 0\\ \theta=0,\pi}} f(r.\cos\theta, r.\sin\theta) = 0$

If $r \neq 0$ and $\theta \neq 0$ or π then

$$\lim_{r\to 0} f(r.\cos\theta, r.\sin\theta) = \lim_{r\to 0} \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta} = 0$$

"0" seems to be a candidate for the limit but it is not!

On the path $y=x^2$, we have $r\sin\theta = r^2\cos^2\theta$ and so

$$f(r.\cos\theta, r.\sin\theta) = \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta} = \frac{2r\cos^2\theta\, r\cos^2\theta}{r^2\cos^4\theta + r^2\cos^4\theta} = 1 \neq 0$$

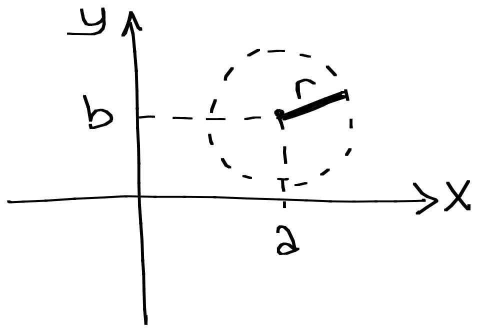
Thus, the limit $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ does not exist.

As in this example, not to jump into false conclusions we need to either verify that the limit candidate is actually the limit by showing that $(*)$ is satisfied or show that the limit does not exist by using two path test.

To use polar coordinates for $\lim_{(x,y)\to(a,b)} f(x,y)$

(limits at arbitrary points $(x_0, y_0) = (a,b)$)

substitute $x = a + r\cos\theta$, $y = b + r\sin\theta$ in $f(x,y)$



$(x,y) \to (a,b)$ implies $r \to 0$

Then $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{r\to 0} f(a + r\cos\theta, b + r\sin\theta)$, if exists.

14.3

Partial Derivatives

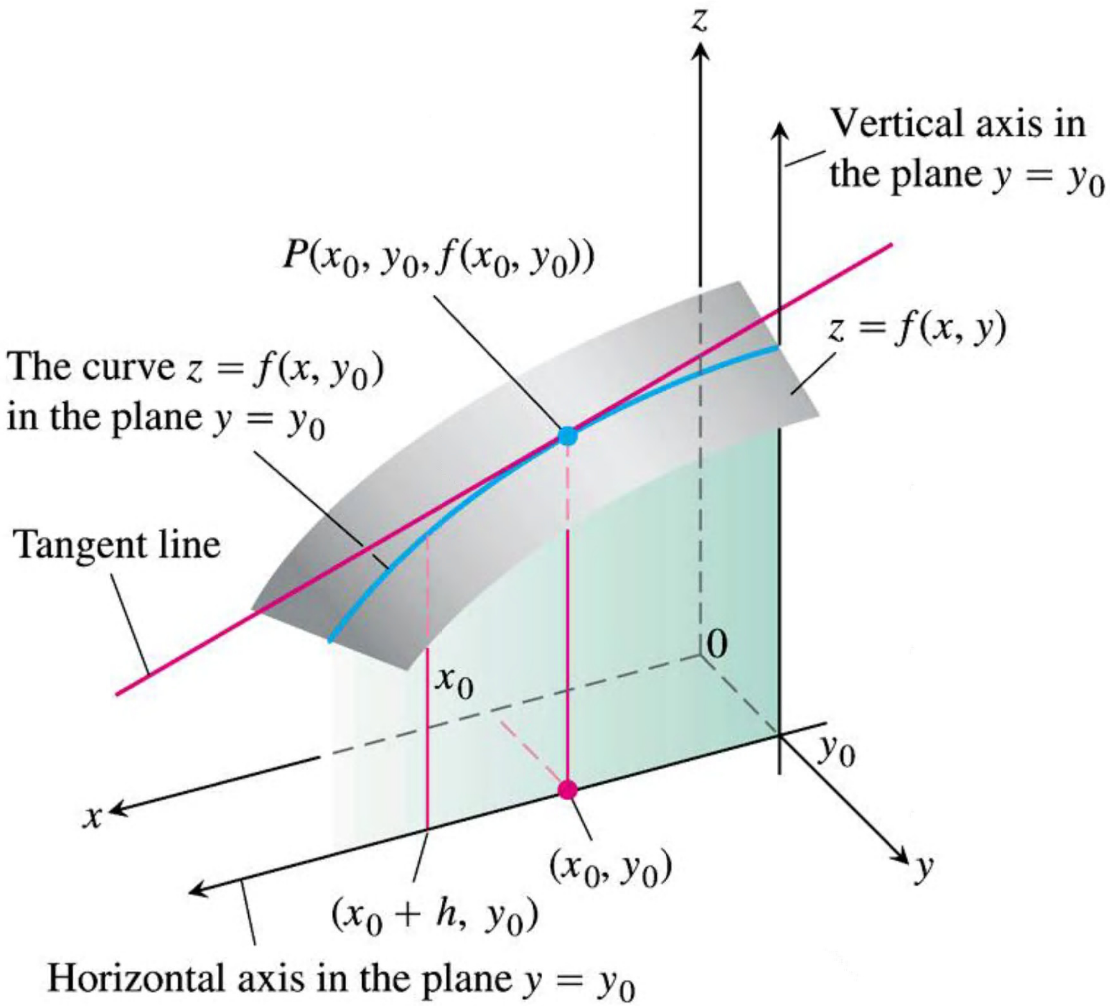


FIGURE 14.13 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

DEFINITION **Partial Derivative with Respect to x**

The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = \left.\frac{d}{dx} f(x, y_0)\right|_{x = x_0}$$

provided the limit exists.

The partial derivative of $f(x,y)$ with respect to x at the point (x_0, y_0) is the slope of the tangent line to the curve $z = f(x, y_0)$ in the plane $y = y_0$ at the point $P(x_0, y_0, f(x_0, y_0))$.

$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}$ gives the rate of change of f with respect to x at (x_0, y_0).

DEFINITION **Partial Derivative with Respect to *y***

The **partial derivative of $f(x, y)$ with respect to y** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

The partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0) is the slope of the tangent line to the curve $z=f(x_0,y)$ in the plane $x=x_0$ at the point $P(x_0,y_0,f(x_0,y_0))$.

$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ gives the rate of change of f with respect to y at (x_0,y_0)

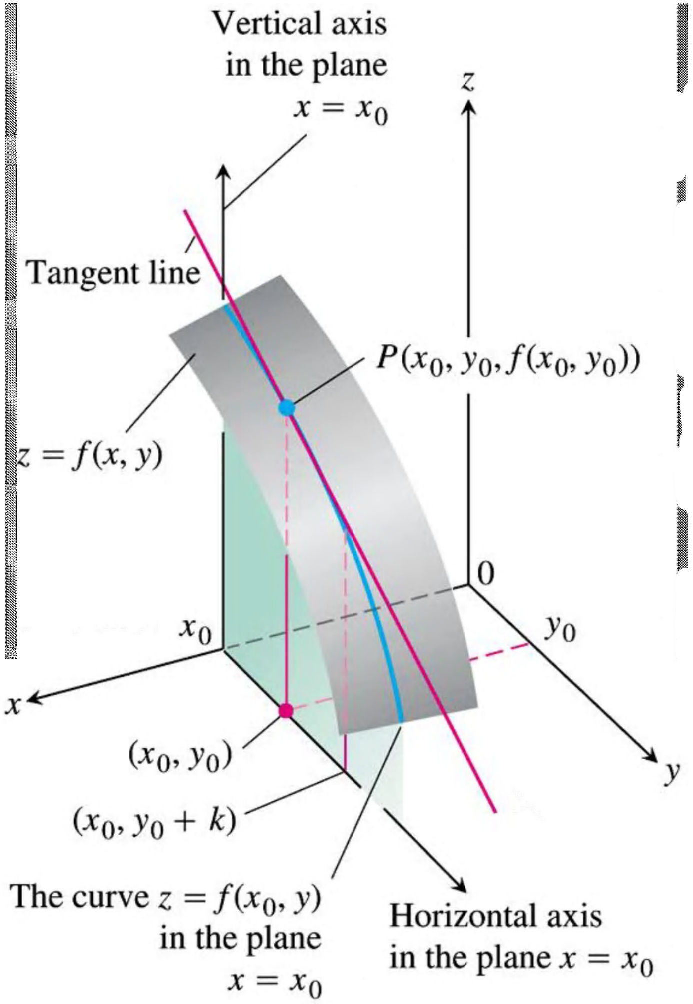


FIGURE 14.14 The intersection of the plane $x = x_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

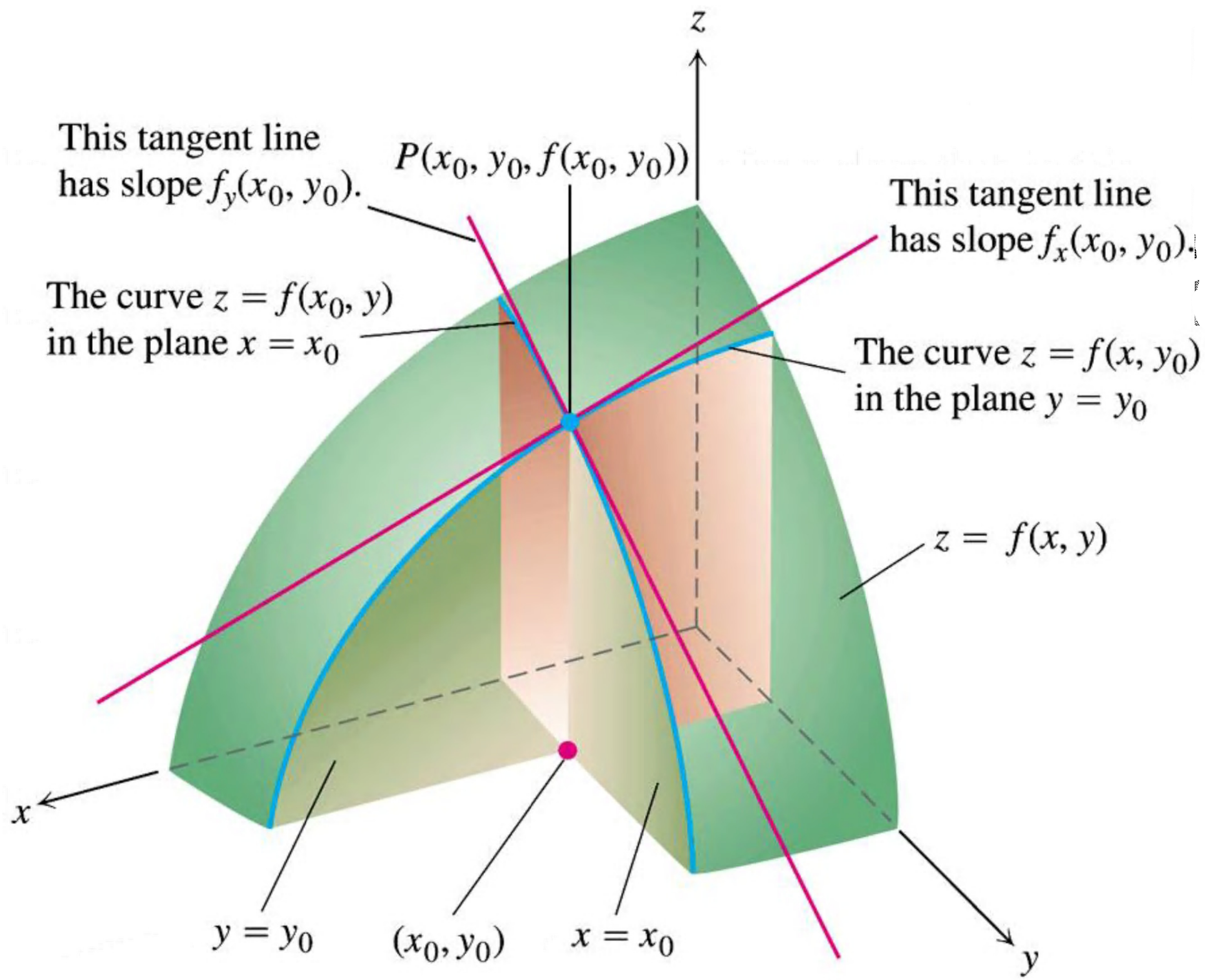


FIGURE 14.15 Figures 14.13 and 14.14 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

Alternative Notations For $z=f(x,y)$:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \frac{d}{dx} f(x,y_0)\Big|_{x=x_0} = \frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0) = \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = z_x(x_0,y_0)$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \frac{d}{dy} f(x_0,y)\Big|_{y=y_0} = \frac{\partial f}{\partial y}(x_0,y_0) = f_y(x_0,y_0) = \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} = z_y(x_0,y_0)$$

$\frac{\partial f}{\partial x} = f_x = z_x = \frac{\partial z}{\partial x}$: Partial derivatives of f with respect to x (as a function)

$\frac{\partial f}{\partial y} = f_y = z_y = \frac{\partial z}{\partial y}$: Partial derivatives of f with respect to y (as a function)

EXAMPLE 1 Finding Partial Derivatives at a Point

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$.

EXAMPLE 2 Finding a Partial Derivative as a Function

Find $\partial f/\partial y$ if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

Exercise: Find all partial derivatives of f, if

a) $f(x,y) = e^{x} \ln(x^2+y^2+1)$

b) $f(x,y) = \ln(\sec(xy) + \tan(xy))$

c) $f(x,y,z) = (xy)^{\sin z}$

Example: Find $\frac{\partial z}{\partial x}$ if the equation $yz - \ln z = x + y$ defines z as a function of the two independent variables x and y.

We will use implicit differentiation: We differentiate both sides of the equation with respect to x, holding y as a constant and treating z as a differentiable function of x.

$$\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x+y) \Rightarrow \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln z) = \frac{\partial x}{\partial x} + \underbrace{\frac{\partial y}{\partial x}}_{=0}$$

$$\Rightarrow \underbrace{\frac{\partial y}{\partial x}}_{=0} \cdot z + y \cdot \frac{\partial z}{\partial x} - \frac{1}{z} \cdot \frac{\partial z}{\partial x} = 1 + 0 \Rightarrow \left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{z}{yz - 1}$$

Exercise: Find $\frac{\partial z}{\partial y}$.

Example: The plane $x=1$ intersects the paraboloid $z=x^2+y^2$ in a parabola $z=1+y^2$ in the plane $x=1$. Find the slope of the tangent to this parabola at $(1,2,5)$.

The slope of the tangent is $\left.\frac{\partial z}{\partial y}\right|_{(1,2)}$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}\left(x^2+y^2\right)\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2)\right|_{(1,2)} + \left.\frac{\partial}{\partial y}(y^2)\right|_{(1,2)}$$

$$= 0 + \left.2y\right|_{(1,2)} = 2.2 = 4.$$

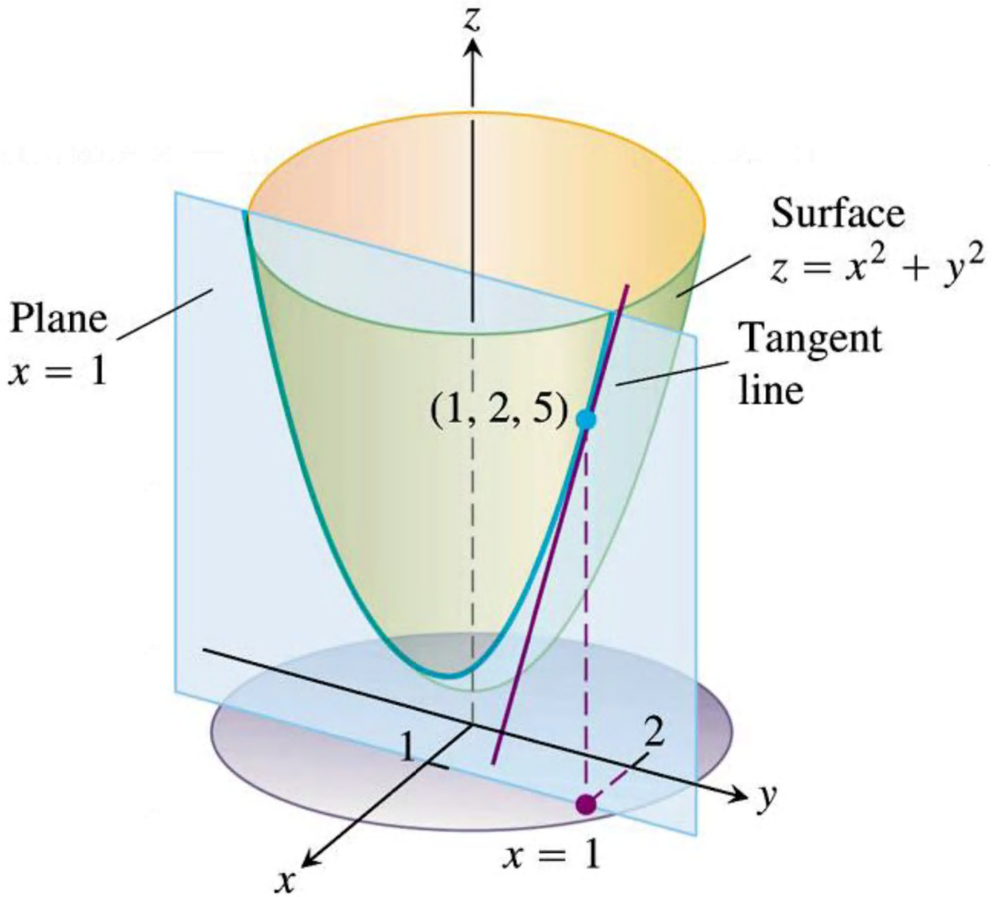


FIGURE 14.16 The tangent to the curve of intersection of the plane $x = 1$ and surface $z = x^2 + y^2$ at the point (1, 2, 5) (Example 5).

EXAMPLE 6 A Function of Three Variables

If x, y, and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z)$$

$$= x \cos(y + 3z) \frac{\partial}{\partial z}(y + 3z) = 3x \cos(y + 3z).$$

Evaluate $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

Example: Let $f(x,y)=\begin{cases} 0, & xy\neq 0 \\ 1, & xy=0 \end{cases}$

a) Is $f(x,y)$ continuous at $(0,0)$?

$$f(0,0)=1, \quad \lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} f(x,y) = \lim_{x\to 0} f(x,x) \underset{\substack{\uparrow\\(\text{Since } x\neq 0)}}{=} \lim_{x\to 0} 0 = 0 \neq 1 = f(0,0)$$

$\Rightarrow$ f is not continuous at $(0,0)$ (In fact, f has no limit at $(0,0)$)

b) Do the partial derivatives f_x and f_y exist at $(0,0)$?

$$f_x(0,0) = \frac{d}{dx} f(x,0) = \frac{d}{dx}(1) = 0 \Rightarrow f_x(0,0)=0 \text{ exists.}$$

$$f_y(0,0) = \frac{d}{dy} f(0,y) = \frac{d}{dy}(1) = 0 \Rightarrow f_y(0,0)=0 \text{ exists.}$$

(*) f is not continuous at $(0,0)$ but it has partial derivatives at $(0,0)$. So having partial derivatives is not sufficient for differentiability, which will imply continuity.

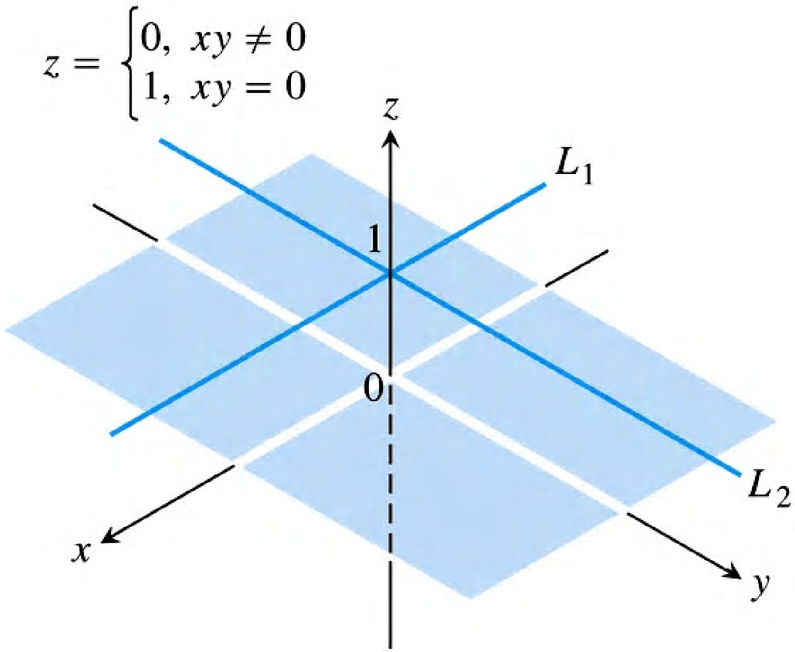


FIGURE 14.18 The graph of

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

consists of the lines L_1 and L_2 and the four open quadrants of the xy-plane. The function has partial derivatives at the origin but is not continuous there (Example 8).

Second Order Partial Derivatives

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad , \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad , \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

EXAMPLE 9 Finding Second-Order Partial Derivatives

If $f(x, y) = x\cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x)$$

$$= \cos y + ye^x$$

So

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x)$$

$$= -x\sin y + e^x$$

So

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x = f_{yx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y. = f_{yy}$$

THEOREM 2 The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Example: If $z = xy + \dfrac{e^y}{y^2+1}$ then find $\dfrac{\partial^2 z}{\partial x \partial y} = z_{yx}$.

z_{yx} : differentiate z first with respect to y then with respect to x.

Differentiating z with respect to x is easier than differentiating z with respect to y. $z = f(x,y)$ is an algebraic combination of polynomials and exponentials, which are defined and continuous on $\mathbb{R}^2$. From the partial derivatives of z we obtain same type of functions which are continuous on $\mathbb{R}^2$. So by the Mixed Derivative Theorem, $z_{yx} = z_{xy}$

$$z_x = y \Rightarrow z_{xy} = 1 = z_{yx}$$

Show the equivalence by evaluating z_{yx}.

Higher Order Partial Derivatives

We use similar notations to the second order partials with the same logic. For instance:

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right)$$

$$\frac{\partial^4 f}{\partial x \partial y^2 \partial x} = f_{xyyx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)\right)$$

Example: Let $f(x,y,z) = 1 - 2xy^2z + x^2y$. Find f_{yxyz}.

$$f_y = -4xyz + x^2 \Rightarrow f_{yx} = -4yz + 2x \Rightarrow f_{yxy} = -4z$$

$$\Rightarrow f_{yxyz} = -4$$

Exercise: Compute

a) f_{xyy} for $f(x,y) = \cos(xy) - x^3 + y^4$

b) f_{yzx} for $f(x,y,z) = \sqrt{xy^3z} + 4x^2y$

c) f_{zyx} for $f(x,y,z) = \sin(xy) + \cos(xz) + \tan(yz)$

Theorem: If the partial derivatives f_x and f_y of a function $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

Theorem: If a function $f(x,y)$ is differentiable at (x_0,y_0), then f is continuous at (x_0,y_0)

14.4

The Chain Rule

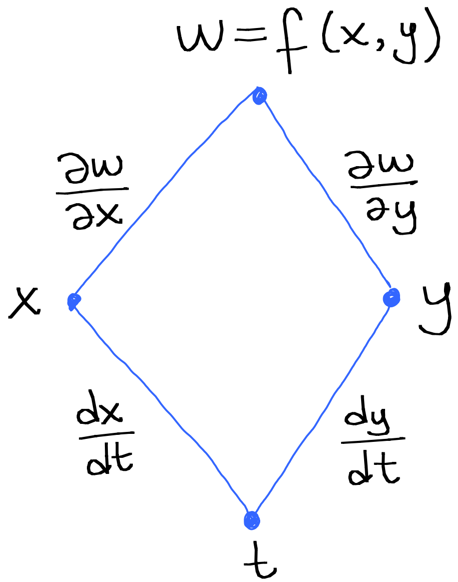
THEOREM 5 Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{df}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$



Dependent variable

Intermediate variables

Independent variable

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

Example: Use the Chain Rule to find the derivative of $w = f(x,y) = e^{xy} + y\sin x$ with respect to t along the path $x = t^2+1$, $y = 1-t$.

$$\begin{tikzcd} & w \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & t & \end{tikzcd}$$

$$\frac{\partial w}{\partial x} = ye^{xy} + y\cos x \,, \qquad \frac{\partial w}{\partial y} = xe^{xy} + \sin x$$

$$\frac{dx}{dt} = 2t \,, \qquad \frac{dy}{dt} = -1$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} = (ye^{xy} + y\cos x)\,2t + (xe^{xy} + \sin x).(-1)$$

$$\frac{dw}{dt} = \left[(1-t)e^{(t^2+1)(1-t)} + (1-t)\cos(t^2+1)\right]2t - \left[(t^2+1)\,e^{(t^2+1)(1-t)} + \sin(t^2+1)\right]$$

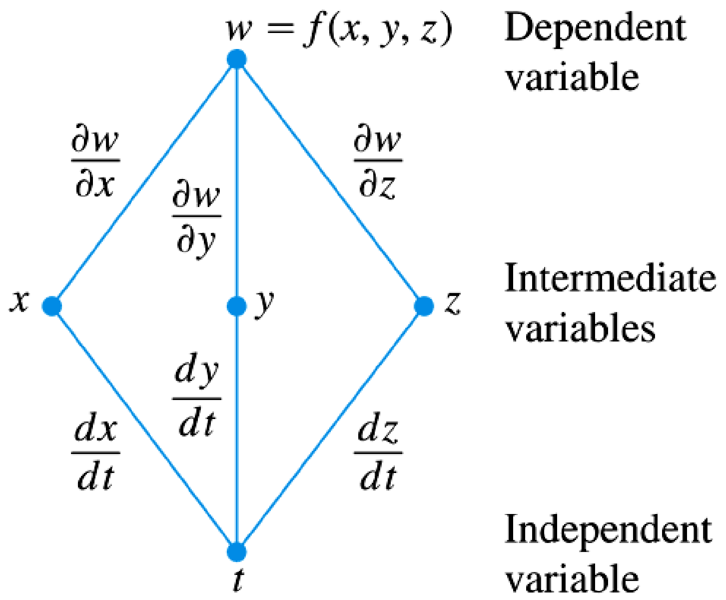
THEOREM 6 Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

EXAMPLE 2 Changes in a Function's Values Along a Helix

Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

In this example the values of w are changing along the path of a helix (Section 13.1). What is the derivative's value at $t = 0$?

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\
&= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t.
\end{aligned}$$

Substitute for the intermediate variables.

$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2.$$

■

THEOREM 7 **Chain Rule for Two Independent Variables and Three Intermediate Variables**

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

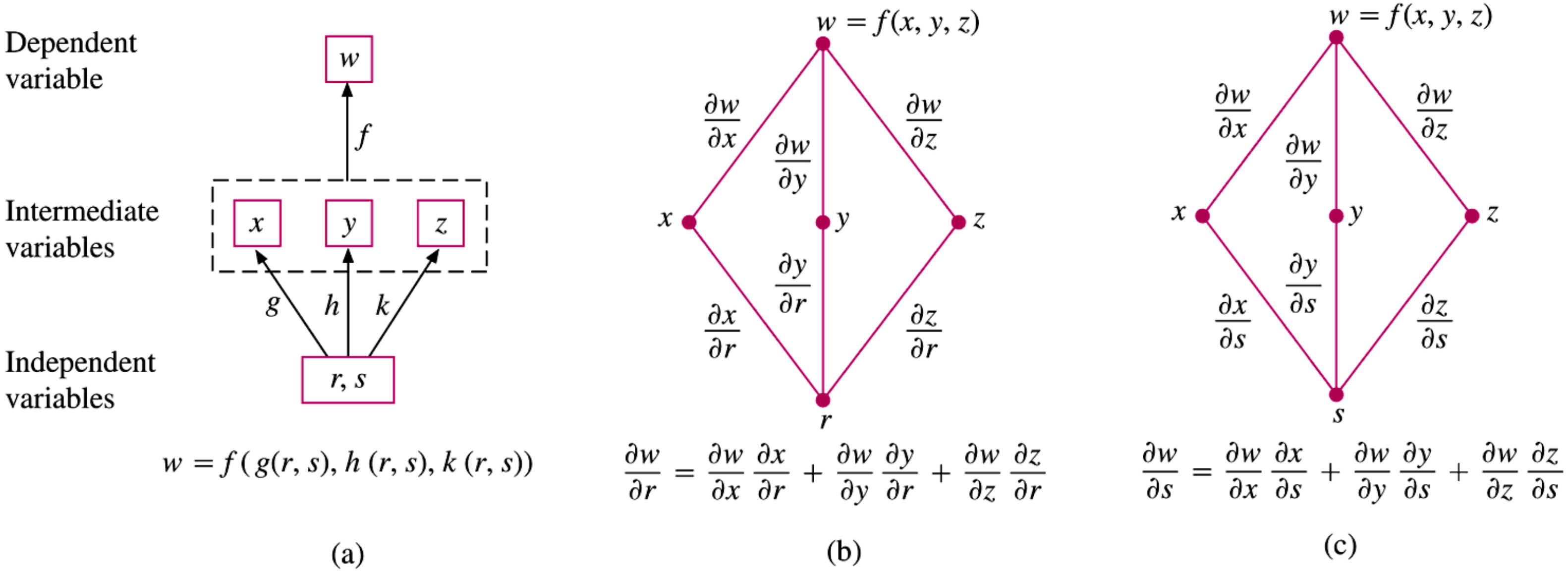


FIGURE 14.19 Composite function and tree diagrams for Theorem 7.

EXAMPLE 3 Partial Derivatives Using Theorem 7

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\
&= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r
\end{aligned}$$

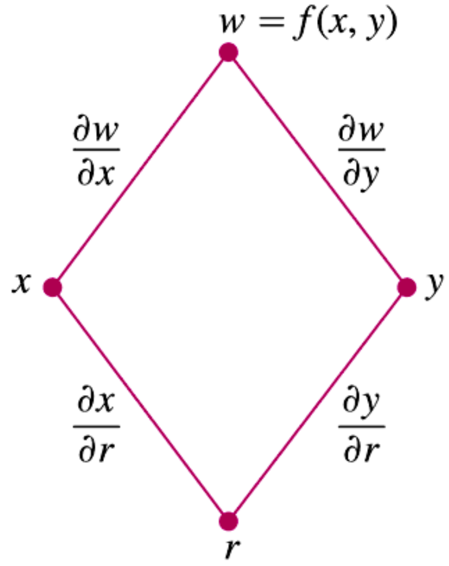
Substitute for intermediate variable z.

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}
\end{aligned}$$

If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \qquad \text{and} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

FIGURE 14.20 Tree diagram for the equation

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.$$

Exercise: Draw the tree diagram for

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Example: Let $w = e^{\frac{x}{y}}$, $x = r\cos(s)$, $y = r\sin(s)$. Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s.

Solution: $\frac{\partial w}{\partial x} = \frac{1}{y} e^{\frac{x}{y}}$, $\frac{\partial w}{\partial y} = \left(-\frac{x}{y^2}\right) e^{\frac{x}{y}}$, $\frac{\partial x}{\partial r} = \cos(s)$, $\frac{\partial x}{\partial s} = -r\sin(s)$

$\frac{\partial y}{\partial r} = \sin(s)$, $\frac{\partial y}{\partial s} = r\cos(s)$

$$\frac{\partial w}{\partial r} = \frac{1}{y} e^{\frac{x}{y}} \cdot \cos(s) + \left(-\frac{x}{y^2}\right) e^{\frac{x}{y}} \cdot \sin(s) = \frac{\cot(s)}{r} e^{\cot(s)} - \frac{\cot(s)}{r} e^{\cot(s)} = 0$$

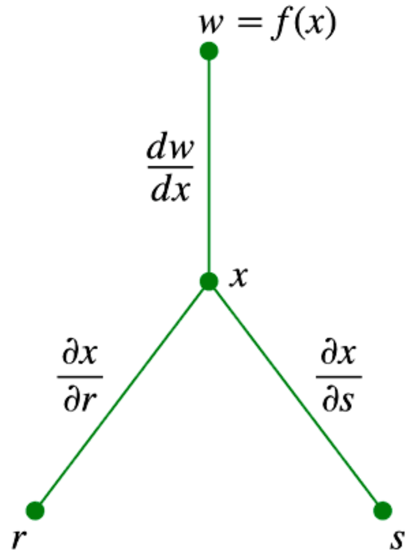
$$\frac{\partial w}{\partial s} = \frac{1}{y} e^{\frac{x}{y}} (-r\sin(s)) + \left(-\frac{x}{y^2}\right) e^{\frac{x}{y}} \cdot r\cos(s) = -e^{\cot(s)} - \cot^2(s) e^{\cot(s)}$$

$$= -\csc^2(s) e^{\cot(s)}$$

If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \qquad \text{and} \qquad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$

Chain Rule



Dependent Variable

Intermediate Variable

Independent variables

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

FIGURE 14.21 Tree diagram for differentiating f as a composite function of r and s with one intermediate variable.

Suppose that

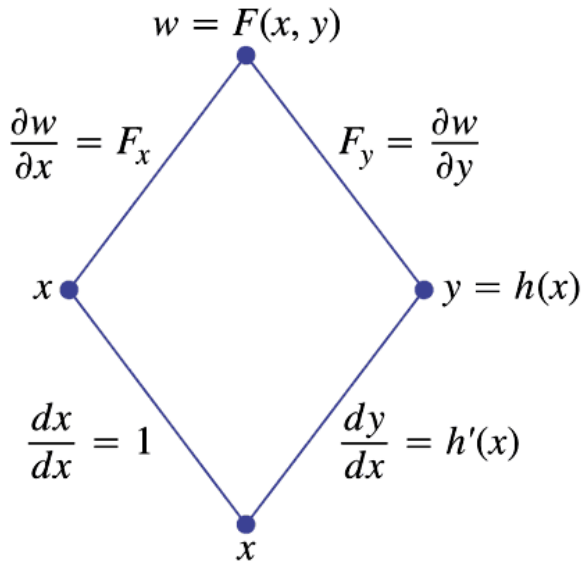
1. $F(x,y)$ is differentiable
2. The equation

   $F(x,y)=0$ defines

   y implicitly as a

   differentiable function

   of x, say $y=h(x)$,



$$\frac{dw}{dx} = F_x \cdot 1 + F_y \cdot \frac{dy}{dx}$$

FIGURE 14.22 Tree diagram for differentiating $w = F(x, y)$ with respect to x. Setting $dw/dx = 0$ leads to a simple computational formula for implicit differentiation (Theorem 8).

Since $w = F(x,y) = 0$,

the derivative $\frac{dw}{dx} = 0$

$$\Rightarrow 0 = F_x \cdot 1 + F_y \cdot \frac{dy}{dx}.$$

If $F_y = \frac{\partial w}{\partial y} \neq 0$ then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

THEOREM 8 A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

EXAMPLE 5 Implicit Differentiation

Use Theorem 8 to find dy/dx if $y^2 - x^2 - \sin xy = 0$.

Solution Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

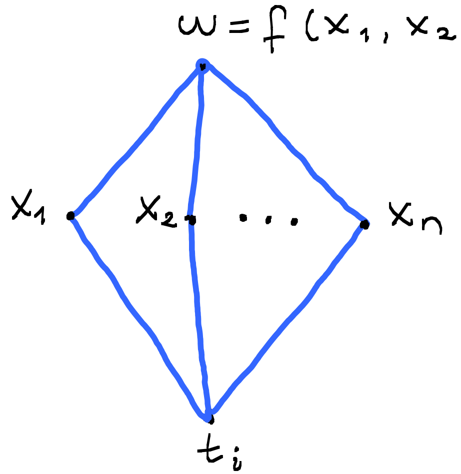
$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy}$$

$$= \frac{2x + y\cos xy}{2y - x\cos xy}.$$

This calculation is significantly shorter than the single-variable calculation with which we found dy/dx in Section 3.6, Example 3. ■

In general, suppose that $w = f(x_1, x_2, \dots, x_n)$ is a differentiable function of the variables $x_1, x_2, \dots, x_n$ and $x_1, x_2, \dots, x_n$ are differentiable functions of $t_1, t_2, \dots, t_m$. Then w is a differentiable function of the variables $t_1, t_2, \dots, t_m$ and

$$\frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial w}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial w}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}, \quad \text{for all } i \in \{1, 2, \dots, m\}$$



Exercise: 1) Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ if $z = e^{xy}$, $x = 3u\sin v$ and $y = 4v^2u$.

2) Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ if $z = e^{x+y}$, $x = u\sin v$ and $y = v\cos u$

3) For a differentiable function f on $\mathbb{R}$, show that $z = f\left(\frac{x}{y}\right)$ satisfies $x . z_x + y . z_y = 0$.

4) For a differentiable function $f(x,y)$ with $x = r.\cos\theta$ and $y = r.\sin\theta$, show that $f_r = f_x \cdot \cos\theta + f_y \cdot \sin\theta$ and

$f_{rr} = f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta$.

5) Given $z = f(u,v)$ where $u = xy$, $v = x^2 - y^2$ and $f_u(1,0) = 2$

$f_v(1,0) = 3$, $f_{uu}(1,0) = 0$, $f_{uv}(1,0) = 1$, $f_{vu}(1,0) = 1$, $f_{vv}(1,0) = -2$.

Find a) z_x, b) z_{xy} at the point $x = 1$, $y = 1$.

6) $f(x,y)$ is called a Harmonic function if it satisfies the Laplace equation $f_{xx} + f_{yy} = 0$. Show that $f(x,y) = e^{kx} \cdot \cos(ky)$ and $g(x,y) = e^{kx} \cdot \sin(ky)$ are Harmonic functions.