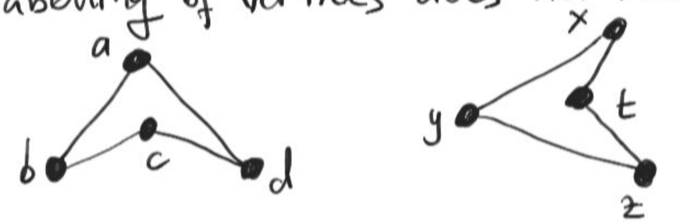
Isomorphism of Graphs

The same graph can be drawn in many different ways. All that matters is that the correct vertices are adjacent, not where the vertices are drawn on the paper

Also labelling of vertices does not matter



$a \leftrightarrow y$
$b \leftrightarrow z$
$c \leftrightarrow t$
$d \leftrightarrow x$

<u>Notation:</u> Let G be a graph and let u, v be vertices of G. If there is an undirected (respectively, directed) edge in G from u to v, we use the notation $u \overset{G}{—} v$ (respectively, $u \xrightarrow{G} v$).

<u>Definition:</u> Let $G = (V, E)$ and $H = (W, F)$ be simple undirected graphs.
vertices edges vertices edges no loops, no multiple edges

By an (graph) isomorphism from G to H we mean a <u>bijective</u> function $\phi: V \to W$ satisfying the following condition:

For all $v_1, v_2 \in V$, $\quad v_1 \overset{G}{—} v_2$ iff $\phi(v_1) \overset{H}{—} \phi(v_2)$

(That is, f is a bijective correspondence between the vertices of G and H such that any two adjacent vertices in G corresponds to two adjacent vertices in H, and conversely.)

We may also write and say that $\phi: G \to H$ is an isomorphism, if ϕ is an isomorphism from G to H.

<u>Remark:</u> (Properties Invariant Under Isomorphisms)
Isomorphic graphs have the same structure. The only difference between them is that their vertices may have been denoted by different names.

Let $\phi : G \to H$ be an isomorphism from a graph $G = (V, E)$ to a graph $H = (W, F)$. Then

(1) $|V| = |W|$ and $|E| = |F|$

(2) $\deg(v) = \deg(\phi(v))$ for any $v \in V$

(3) For any $k \in \mathbb{N}$,

G has a vertex of degree k iff H has a vertex of degree k

$$\begin{pmatrix}\text{The number of vertices} \\ \text{of } G \text{ of degree } k\end{pmatrix} = \begin{pmatrix}\text{The number of vertices} \\ \text{of } H \text{ of degree } k\end{pmatrix}$$

(4) $\phi(N_G(v)) = N_H(\phi(v))$ for any $v \in V$, where the set $N(v)$ of neighbours of a vertex v is defined in the next definition.

<u>Definition</u>: Let $G = (V, E)$ be a simple undirected graph.

(1) For any vertex v of G, the <u>neighbours</u> of v are the vertices of G that are adjacent to v. We denote by $N_G(v)$ the set of all neighbours of G. That is,

$$N_G(v) = \{u \in V \mid uv \in E\}$$

Note that $|N_G(v)| = \deg(v)$

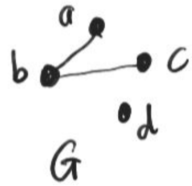
(2) G is called <u>complete</u> if any two distinct vertices are adjacent

Note that: G is complete $\iff N_G(v) = V - \{v\} \ \forall v \in V \iff \deg(v) = |V| - 1 \ \forall v \in V$

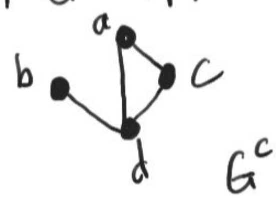
(3) The <u>complementary graph</u> or <u>the complement</u> of G is defined to be the following graph G^c (or $\overline{G}$):

- The vertices of G^c are the same with the vertices of G
- Two vertices are adjacent in G^c iff they are not adjacent in G

For instance, if



then



(4) G is called <u>empty graph</u> if it has no edges.

<u>Remark / Definition</u>: Neighbours, complements, graph isomorphisms can be defined for simple directed graphs by making minor obvious changes.

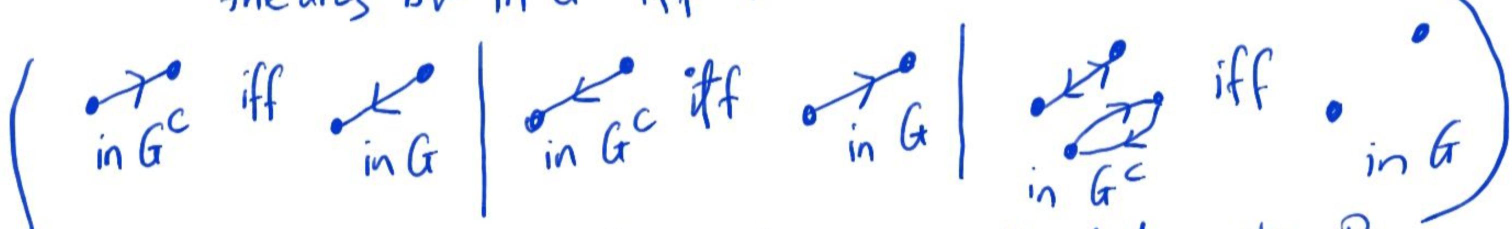
(1) Let $G=(V,E)$ be a simple directed graph. For any $v \in V$, we may define <u>in-neighbours</u> and <u>out-neighbours</u> of v as follows:

- $u \in V$ is called an in-neighbour of v if there is an arc $\overrightarrow{uv}$ from u to v.
- $u \in V$ is called an out-neighbour of v if there is an arc $\overrightarrow{vu}$ from v to u.

In a simple directed graph an arc of the form "  " is allowable. However, there cannot be more than one arc in the same direction between two distinct vertices.

(2) The complementary graph of a simple directed graph $G=(V,E)$ is defined to be the following simple directed graph G^c:

- The vertices of G^c is the same with the vertices of G.
- For any vertex u, v
  the arc $\overrightarrow{uv}$ in G^c iff the arc $\overrightarrow{vu}$ in G
  the arc $\overrightarrow{vu}$ in G^c iff the arc $\overrightarrow{uv}$ in G
  the arcs $\overleftrightarrow{uv}$ in G^c iff there is no arc in G between u and v.



(3) Let $G=(V,E)$ and $H=(W,F)$ be simple directed graphs. By a graph isomorphism ϕ from G to H we mean a bijective function $\phi: V \to W$ such that, $\forall v_1, v_2 \in V$, $v_1 \xrightarrow{G} v_2$ iff $\phi(v_1) \xrightarrow{H} \phi(v_2)$

Ex: (1) Let G be a simple undirected graph with n vertices. For any vertex v, compare degrees of v in G^c and G.

Sol: $\deg(v)$ in $G^c = (n-1) - \deg(v)$ in G

(2) (Exercise 19.33 in textbook) A certain simple undirected graph with 25 vertices has 250 edges. How many edges does it complement have?

Sol: Between 25 vertices we may draw $\binom{25}{2} = 300$ edges. So in the complement there are $300 - 250 = 50$ edges.

(3) (Exercise 19.34 in textbook) Let G be a simple undirected graph on n vertices, and suppose that G is isomorphic to its complement. Prove that $n \equiv 0, 1 \bmod 4$.

Sol: As we can draw $\binom{n}{2}$ edges between n vertices,

$$(\text{The number of edges in } G) + (\text{The number of edges in } G^c) = \binom{n}{2}$$

As G and G^c are isomorphic, they have the same number of edges. This implies that $\binom{n}{2}$ is an even integer. So $\binom{n}{2} = \frac{n(n-1)}{2} = 2k$ for some integer k. So $n(n-1) = 4k$, equivalently $n(n-1) \equiv 0 \bmod 4$. Note that if $n \equiv 2, 3 \bmod 4$ then $n(n-1) \equiv 2\cdot 1, 3\cdot 2 \not\equiv 0 \bmod 4$. Hence, $n \not\equiv 2, 3 \bmod 4$. If $n \equiv 0, 1 \bmod 4$ then $n(n-1) \equiv 0\cdot(-1), 1\cdot 0 \equiv 0 \bmod 4$. Consequently, the result follows.

Remark: Let A, B, C be simple (undirected) graphs.

(1) The identity map 1 on the set of vertices of A is an isomorphism from A to A.

(2) If $\phi: A \to B$ is an isomorphism, then the inverse map ϕ^{-1} of ϕ is an isomorphism $B \to A$ from B to A. (As ϕ is a bijective map from the vertex set of A to the vertex set of A, ϕ is an invertible function).

(3) If $\phi: \mathcal{A} \to \mathcal{B}$ and $\psi: \mathcal{B} \to \mathcal{C}$ are graph isomorphisms, then their composition $\psi \circ \phi: \mathcal{A} \to \mathcal{C}$ is a graph isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Proof: Exercise. □

Remark/Notation: From the previous remark we know that if there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ then there is an isomorphism from $\mathcal{B}$ to $\mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are graphs. Therefore, $\mathcal{A}$ is isomorphic to $\mathcal{B}$ iff $\mathcal{B}$ is isomorphic to $\mathcal{A}$. So we may simply say that "$\mathcal{A}$ and $\mathcal{B}$ are isomorphic" instead of "$\mathcal{A}$ is isomorphic to $\mathcal{B}$". We write $\mathcal{A} \cong \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic (i.e., if there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ or $\mathcal{B}$ to $\mathcal{A}$).

Corollary: "Being isomorphic" (i.e., $\cong$) is an equivalence relation on the set of all simple undirected (directed) graphs.

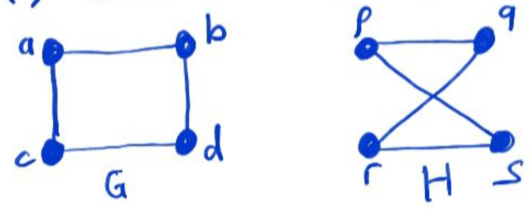
Proof: Exercise. □

Ex: Determine whether the given graphs are isomorphic. If so, write an explicit isomorphism between them.

(1)

They are not isomorphic because the first one has 4 edges but the second one has 5 edges.

(2) (Exercise 20.4, (1), in textbook)



They are isomorphic. We first check the number of vertices and edges in both graphs. They are the same. We then check degrees of vertices in both graphs. Each vertex has degree 2. We now begin to think that the given graphs are isomorphic. Let us try to write an isomorphism $\phi: G \to H$. Now $\phi: \{a,b,c,d\} \to \{p,q,r,s\}$ must be a bijection preserving adjacencies. $\phi(a)$ may be arbitrary, say $\phi(a) = p$. As $a \overset{G}{—} b$,

$p \overset{H}{—} \phi(b)$. So, $\phi(b)$ is an element of $\{p,q,r,s\}-\{p\}$ and p & $\phi(b)$ must be adjacent in H. The vertices of H adjacent to p are q and s. So $\phi(b)=q$ or s. Consider first the case in which $\phi(b)=q$. As there is an edge in G between c and a but there is no edge in G between c and b, there must be an edge in H between $\phi(c)$ and $\phi(a)=p$ but there must be no edge in H be-tween $\phi(c)$ and $\phi(b)=q$. The only vertex of H in $\{p,q,r,s\}-\{p,q\}$ satisfying these conditions is s. Therefore

$$\phi : \begin{Bmatrix} a \mapsto p \\ b \mapsto q \\ c \mapsto s \\ d \mapsto t \end{Bmatrix}.$$

We may easily check this map ϕ preserves adjacent verties, and it is a bijection between vertices of G and vertices of H. Consequently, $\phi : G \to H$ is a graph isomorphism.

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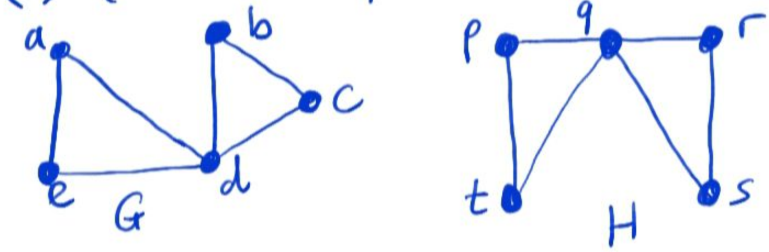
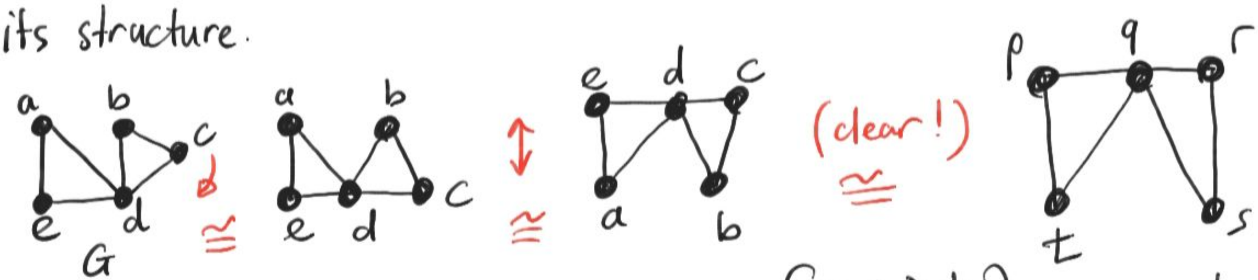
(2) (Exercise 20.4, (2), in textbook)



They are isomorphic. Note that in both graphs there is a unique vertex of degree 4, namely d and q. So any graph isomorphism from G to H must map d to q. However, if we look at the pictures of the given graphs, we see that one can be obtained from the other by rotation/reflection ..., without changing its structure.



It is now clear that the map

$$\phi : \begin{Bmatrix} a \mapsto t \\ b \mapsto s \\ c \mapsto r \\ d \mapsto q \\ e \mapsto p \end{Bmatrix}$$

is a graph isomorphism from G to H.

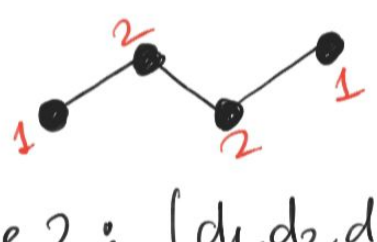
Ex (Exercise 20.5, (1) in textbook)

Draw all nonisomorphic simple, undirected graphs having exactly 4 vertices and 3 edges.

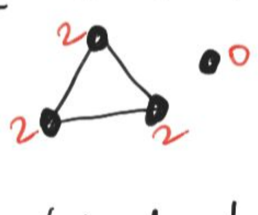
Sol: Let G be a simple undirected graph with n vertices. Each vertex of G has a degree. Suppose we write the degrees of its n vertices in a nondecreasing order $d_1 \le d_2 \le \cdots \le d_n$. This sequence is sometimes called the degree sequence of G. Note that $0 \le d_i \le n-1$ for all i because each vertex can be connected to at most all the other n-1 verties. Also recall from a previous theorem that $d_1 + d_2 + \cdots + d_n = 2|E|$ where E is the set of edges of G. Furthermore, if $d_n = n-1$ then each $d_i > 0$ (because one of the vertex is connected to the each of the other vertices)

Now consider the case $n=4$ and $|E|=3$. From the conditions "$0 \le d_1 \le d_2 \le d_3 \le d_4 \le 3$, $d_1+d_2+d_3+d_4 = 6$, $d_4 = 3 \Rightarrow d_i > 0$", we see that there are 3 possibilities for d_i's.

Case 1: $(d_1, d_2, d_3, d_4) = (1,1,2,2)$



Case 2: $(d_1, d_2, d_3, d_4) = (0,2,2,2)$



Case 3: $(d_1, d_2, d_3, d_4) = (1,1,1,3)$

Note that any of these three graphs are not isomorphic

Walks, trails, paths

Definition: Let $G=(V,E)$ be a simple undirected graph, and let $x,y \in V$ be vertices, not necessarilly distinct (i.e., it is possible that $x=y$).

(1) By a walk from x to y (or an x-y walk) we mean a finite sequence

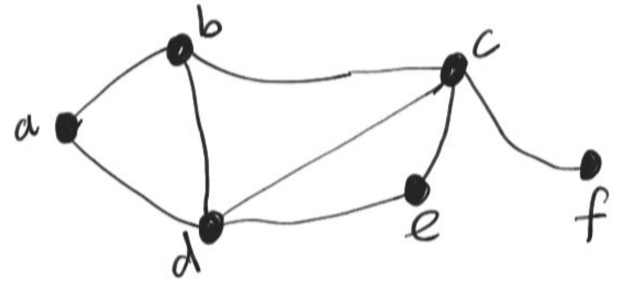
$$x=v_0, v_1, v_2, \dots, v_{n-1}, v_n = y$$

of vertices of G starting at x and ending at y such that each consecutive vertices v_{i-1}, v_i are adjacent in G (i.e, $v_{i-1}v_i \in E$ for all $i=1,2,\dots,n$)

The edges $v_0v_1, v_1v_2, v_3v_4, \dots, v_{n-1}v_n$ are called the edges of the walk, an the number n of the edges of the walk is called the length of the walk. We may also denote this walk by the notation

$$x=v_0 - v_1 - v_2 - \cdots - v_{n-1} - v_n = y.$$

If $x=y$, then the the walk is called a closed walk. For instance, in



a − b − d − c − e − d − b

is a walk from a to b of length 6.

c − e − d − c − b − a − d − c

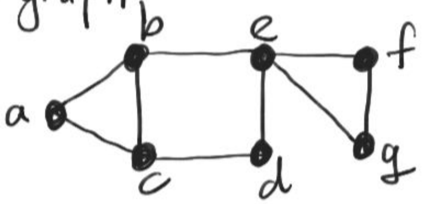
is a closed walk (a walk from c to c).

(2) If no edges in an walk from x to y is repeated, then the walk is called a trail from x to y (or an x-y trail). A closed trail is called a circuit. (Note that a circuit must contain at least 3 edges)

(3) If no vertex, except possibly the endpoints $x=y$, in a walk from x to y is repeated, then the walk is called a path from x to y. A closed path is called a cycle. (We assume that a cycle contains at least 3 edges so that a − b − a is not a cycle and any cycle is a circuit)

$$\underset{\text{end point}}{x} = v_0 - v_1 - v_2 - \cdots - v_{n-1} = v_n = \underset{\text{end point}}{y}$$

Ex: In the graph



(1) b − e − f − g − e − b − c − d is a walk from b to d, but not a trail because the edge be = eb is repeated.

(2) b − e − f − g − e − d is a trail from b to d, but not a path because the vertex e is repeated.

(3) b − e − d is a path from b to d.

(4) b − e − f − g − e − b is a closed walk but not a circuit because the edge be = eb is repeated.

(5) b − e − f − g − e − d − c − b is a circuit but not a cycle because the vertex e is repeated.

(6) b − a − c − b is a cycle

Remark: Walks, trails, paths for directed / multigraphs can be defined similarly by making obvious changes. Fortunetely, we will not consider directed graphs and multigraphs so musch.

Proposition: Let G be a simple undirected graph, and let x and y be vertices of G such that $x \neq y$. If there is a walk from x to y, then there is a path from x to y.

Proof: Among all walks from x to y choose one whose length is shortest, say $x = v_0 - v_1 - \cdots - v_{n-1} - v_n = y$. We claim that it must be a path from x to y. If not a vertex in it must be repeated. So $v_r = v_s$ for some r, s such that $0 \leq r < s \leq n$

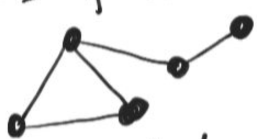
Erasing the part between $v_r = v_s$ we obtain a shorter walk

$$x = v_0 - v_1 - \cdots - v_r = v_s - \cdots - v_{n-1} - v_n = y$$

from x to y, which is a contradiction. $\square$

<u>Definition</u>: Let G be a simple undirected graph.

(1) G is called <u>connected</u> if there is a path between any two distinct vertices of G.



connected

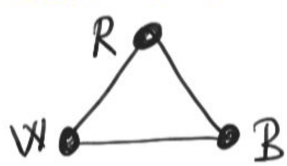
not connected.

(2) Assume that G is connected. The <u>distance</u> $d(x,y)$ between two distinct vertices x and y of G is defined to be the length of a shortest path between x and y. For any vertex v we define that $d(v,v) = 0$.

Vertex Coloring

Let $G = (V, E)$ be a simple undirected graph. Suppose we have k distinct colors. If it is possible to paint/color all vertices of G in such a way that the colors of the adjacent vertices are distinct, then any such situation is called a k-coloring of G.

a 2-coloring



a 3-coloring

(Note that a 2-coloring is not possible)

where R = Red, W = White, B = Blue.

If we denote the distinct k colors used by numbers $1, 2, 3, \ldots, k$, we may see a k-coloring by a function $f : V \longrightarrow \{1, 2, 3, \ldots, k\}$ where, for any $v \in V$, $f(v)$ is the color used to paint the vertex v.

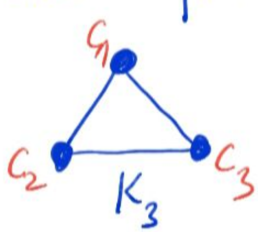
<u>Definition</u>: Let $G=(V,E)$ be a simple undirected graph. For any $k\in\mathbb{N}^+$, a <u>k-coloring of G</u> is a function $f: V\longrightarrow\{1,2,\dots,k\}$ satisfying

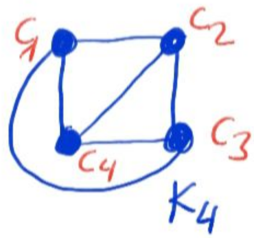
$$\forall v_1,v_2\in V,\quad v_1 \overset{G}{—} v_2 \Rightarrow f(v_1)\neq f(v_2)$$

If such a function exists, we say that <u>G can be colored by k colors</u>.
The smallest positive integer m such that G can be colored by m colors is called the <u>chromatic number of G</u>, and denoted by $\chi(G)$. (Thus, $\chi(G)=m \Leftrightarrow$ G can be colored by m colors but cannot be colored by $m-1$ colors).

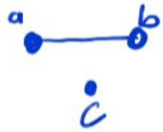
<u>Ex</u>: (1) The chromatic number of P_n is 2 where P_n denotes a path of length $n-1$

(2) The complete graph K_n on n vertices has a k-coloring $\Leftrightarrow$ $k\geq n$

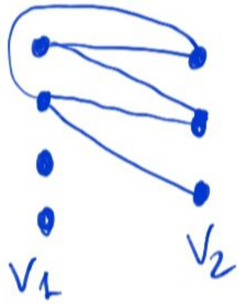




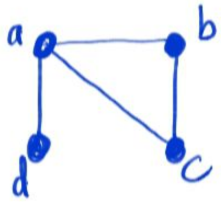
<u>Definition</u>: Let $G=(V,E)$ be a simple undirected graph. Then G is called <u>bipartite</u> if there is a partition of V into two sets V_1 and V_2 (i.e., $\phi\neq V_1\subseteq V$, $\phi\neq V_2\subseteq V$, $V_1\cap V_2=\phi$, $V=V_1\cup V_2$) such that every edge of G is of the form ab for some $a\in V_1$ and for some $b\in V_2$ (i.e., there is no edge between vertices in V_1, and there is no edge between the vertices in V_2).

Ex: (1) 

is a bipartite graph where $V = \{a, b, c\}$, $V_1 = \{a, c\}$ and $V_2 = \{b\}$ (or $V_1 = \{a\}$ and $V_2 = \{b, c\}$)

(2) 

is a bipartite graph

(3) 

is not a bipartite graph: Here $V = \{a, b, c, d\}$. Suppose for a moment that G is bipartite with partites V_1 and V_2. The vertex a must be in one of V_1 and V_2. Say without loss of generality that $a \in V_1$. All the vertices that are connected to a by an edge must be in the other partite V_2. So $V_2 = \{b, c, d\}$. But now, for instance, b and c are in V_2 and there is an edge between b and c. This is a contradiction.

Theorem: Let $G = (V, E)$ be a simple undirected graph. Assume that $|V| \geq 2$. Then, G is bipartite $\Leftrightarrow$ G has no odd cycle (i.e., a cycle of odd length)

Proof: ($\Rightarrow$) Suppose that G is bipartite. Let

$$x = v_0 - v_1 - v_2 - \cdots - v_{n-1} - v_n = x$$

be a cycle of length n. We want to show that n is not odd. As G is bipartite, $V = V_1 \cup V_2$ for some partition $\{V_1, V_2\}$ of V and any edge must have endpoints that are in different partites V_i. Now v_0 must be in one of V_1 and V_2, say $v_0 \in V_1$. Then

$v_1 \in V_2$, $v_3 \in V_1$, $v_4 \in V_2$, $\dots$ So $v_0, v_2, v_4, v_6, \dots, v_{2k}, \dots \in V_1$ and $v_1, v_3, v_5, \dots, v_{2k+1}, \dots \in V_2$. As $x = v_0 \in V_1$ and $x = v_n$, n must be even.

($\Leftarrow$) The proof of this part is skipped because it is technical

(When G is connected we may choose partition of the vertex set as follows: Choose any vertex u. Then the sets

$$V_1 = \{v \in V \mid d(u,v) \text{ is even}\} \text{ and } V_2 = \{v \in V \mid d(u,v) \text{ is odd}\}$$

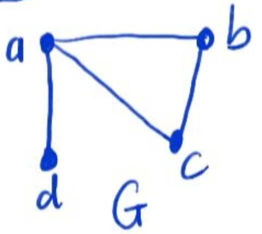
form partites of G) $\square$

Definition: let $G = (V, E)$ be a simple undirected graph with n vertices, say $V = \{v_1, v_2, \dots, v_n\}$. The adjacency matrix of G is the $n \times n$ matrix A_G defined by

$$\underbrace{(A_G)_{ij}}_{\text{the } (i,j)^{th} \text{ entry of } A_G} = \begin{cases} 1, & \text{if } v_i v_j \in E \\ 0, & \text{if } v_i v_j \notin E \end{cases}$$

Note that A_G depens on the order of elements of V. Note also that A_G is symmetric.

Ex:



$$A_G = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 1 & 0 & 0 \\ d & 1 & 0 & 0 & 0 \end{array}$$

Theorem: Let G be a simple undirected graph with $V = \{v_1, v_2, \dots, v_n\}$ and A be the adjacency matrix of G. For any $m \in \mathbb{N}^+$,

$$(A^m)_{ij} = \text{the number of walks from } v_i \text{ to } v_j \text{ of length } m$$

Proof: By induction on m.

For $m=1$, $A^m = A$ and so $(A^m)_{ij} = (A)_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E \\ 0 & \text{otherwise} \end{cases}$

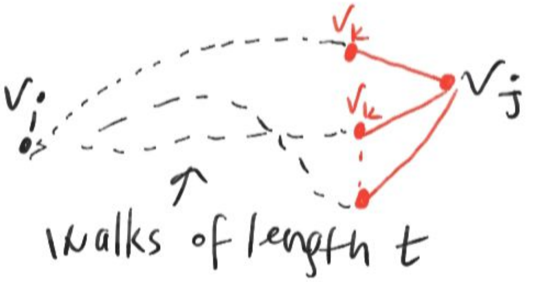
Note that there is a walk from v_i to v_j of length 1 iff v_i and v_j are adjacent. So the result is true for $m=1$.

Assume that the result is true for $m=t$ where $t \geq 1$ is an integer.

Consider $$(A^{t+1})_{ij} = (A^t A)_{ij} = \sum_{k=1}^{n} (A^t)_{ik} \underbrace{(A)_{kj}}_{0 \text{ or } 1 \iff v_kv_j \in E}$$

$$= \sum_{k \in S} (A^t)_{ik}$$

where $S = \{k \in \{1,2,\dots,n\} \mid v_kv_j \in E\}$. Note that $v_i - v_j$ walks of length $t+1$ can be counted as follows: Consider all vertices v_k that are adjacent to v_j. Any $v_i - v_j$ walk of length $t+1$ is obtained from a $v_i - v_k$ walk of length t by connecting to the edge v_kv_j where v_k is any vertex adjacent to v_j



Hence, $$|\{v_i - v_j \text{ walks of length } t+1\}| = \sum_{k \in S} \underbrace{|\{v_i - v_k \text{ walks of length } t\}|}_{= (A^t)_{ij}}$$

By the induction hypothesis

So the result is true for $m = t+1$.

$\square$

Finite sets, infinite sets and their cardinalities

Informally, the cardinality of a set is the number of elements that it contains. For instance, the number of the elements of the finite set $A=\{1,a,\#\}$ is 3, and we may consider $1, a, \#$ as 1^{st}, 2^{nd}, 3^{rd} elements of A, which defines a bijective function $f: A \longrightarrow \{1,2,3\}$ given by $f(1)=1$, $f(a)=2$ and $f(\#)=3$.

<u>Definition</u>: Let A be a set.

(1) If A has only finitely many elements, then we say that A is a <u>finite set</u>. Note by the definition that the empty set ϕ is a finite set.

(2) A is called an <u>infinite set</u> if it is not finite.

(3) Let A be a nonempty finite set. We say that the <u>cardinality of A</u> is n, and we write $|A|=n$ (or $\#A=n$), where $n \geq 1$ is an integer, if there is a bijection $f: A \longrightarrow \{1,2,\dots,n\}$. The cardinality of the empty set is defined to be 0. The cardinality of a nonempty set is ≥ 1.

<u>Remark</u>: The cardinality of a finite set is a unique natural number.

<u>Proof</u>: Let A be a finite set. By the definition, we may assume that $A \neq \phi$. Suppose for a contradiction that $|A|=m$ and $|A|=n$ for some distinct positive integers m and n. By the definition there are bijections

$$A \xrightarrow{f} \{1,2,\dots,m\} \quad \text{and} \quad A \xrightarrow{g} \{1,2,\dots,n\}$$

So $\{1,2,\dots,m\} \xrightarrow{g\circ f^{-1}} \{1,2,\dots,n\}$ is a bijection. Using the injectivity of $g\circ f^{-1}$ we see that $m \leq n$. Similarly, we may see that $n \leq m$. So $m=n$, a contradiction. $\square$

The following, one of our previous results, is useful to compare cardinalities of finite sets.

<u>Fact:</u> Let A and B be finite sets.

(1) If there is an injective function $A \to B$, then $|A| \leq |B|$.

(2) If there is a surjective function $A \to B$, then $|A| \geq |B|$.

We gave a proof of the above fact where we studied "functions". The textbook gives a slightly different proof in which the "pigeonhole principle" is used. Consider for instance part (1). Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ and $f: A \to B$ be an injection. Then, defining $B_i := \{a \in A \mid f(a) = b_i\}$, we see that

$$A \subseteq B_1 \cup B_2 \cup \cdots \cup B_n$$

So each element of A is in some B_i. If $|A| = m > n$, then in some B_j there must be more than one element of A. The last sentence, which is intiutively obvious, is a consequence of <u>the Pigeonhole Principle</u>, saying that "if there are more pigeons than pigeonholes, then some pigeonhole must contain at least two pigeons". In the above, elements of A are pigeons, and the sets $B_1, B_2, \ldots, B_n$ are pigeonholes.

The number of elements of an infinite set is infinite. So the cardinality of an infinite set is infinite. For instance, the cardinality of the sets $\mathbb{Z}$, $\mathbb{N}^+$ and $\mathbb{R}$ are all infinite. However, we will see soon that "an infinite may be greater than another infinite". (We will see that the cardinality of $\mathbb{R}$ is greater than the cardinality of $\mathbb{N}^+$, and that the cardinalities of $\mathbb{Z}$ and $\mathbb{N}^+$ are the same). We may compare infinities by using a version of the previous fact.

<u>Definition</u>: Let A and B sets (finite or infinite). We say that A and B have <u>the same cardinality</u> (or equinumerous, or equipotent) if there is a bijective map $A \to B$ from A to B. We write $A \sim B$ or $|A| = |B|$ to indicate that A and B have the same cardinality (Some books may prefer to use the notation $A \approx B$ instead of $A \sim B$)

<u>Remark</u>: Let A, B, C be sets. Then,

$$A \sim A, \quad A \sim B \Leftrightarrow B \sim A, \quad A \sim B \text{ and } B \sim C \Rightarrow A \sim C$$

So, $\sim$ is an equivalence relation on any set of sets.

<u>Proof</u>: As the identity map is bijective, $A \sim A$. As a bijective map is invertible and its inverse is bijective, we see that "$A \sim B \Leftrightarrow B \sim A$". The transitivity of $\sim$ follows from the fact that the composition of bijective maps is bijective. $\square$

<u>Remark</u>: Let A and B be sets. If there is a bijection from A to B, then there is a bijection from B to A. Thus, $A \sim B$ if there is a bijection from one set to another.

<u>Ex</u>: (1) $\mathbb{N}^+ \sim \mathbb{Z}$

Consider the map $f: \mathbb{N}^+ \to \mathbb{Z}$ defined by $f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$

We may easily check that f is bijective

$$\begin{array}{ccccccccccccccc}
 & & 0 & & -1 & & -2 & & -3 & & -4 & & -5 & & \\
f\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathbb{N}^+ = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & \cdots & \} \\
f\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
 & & & 1 & & 2 & & 3 & & 4 & & 5 & & &
\end{array}$$

(2) $\mathbb{Z} \sim 2\mathbb{Z}$ = the set of even integers

Consider the function $f: \mathbb{Z} \to 2\mathbb{Z}$ defined by $f(x) = 2x$. We may easily check that f is bijective.

$$\begin{array}{ccccccccccccc} \cdots, & -2, & -1, & 0, & 1, & 2, & 3, & 4, & 5, & 6, & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & -4 & -2 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & \end{array}$$

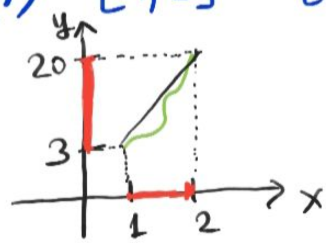
(3) $\mathbb{R} \sim (-1,1)$ ← interval

Consider the map $f: \mathbb{R} \to (-1,1)$ defined by $f(x) = \dfrac{x}{1+|x|}$. Show that f is bijective (Exercise).

(4) $\mathbb{N}^+ \times \mathbb{N}^+ \sim \mathbb{N}^+$

Any positive integer a can be written as $a = 2^r s$ for some unique integers $r \geq 0$ and $s > 0$ such that s is odd. (This is a consequence of so called the Fundamental Theorem of Arithmetic, saying that any number > 1 has a unique prime factorization). Using this we may easily see that the function $f: \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ defined by $f(m,n) = 2^{m-1}(2n-1)$ is bijective.

(5) $[1,2] \sim [3,20]$ where they are intervals



We want to construct a bijective map $f: [1,2] \to [3,20]$. We may simply take f whose graph is the line segment connecting the points $(1,3)$ and $(2,20)$. The line through $(1,3)$ and $(2,20)$ is given by $\underset{f(x)}{\underset{\shortparallel}{y}} - 3 = \dfrac{20-3}{2-1}(x-1)$. So

$f(x) = 17(x-1) + 3$ is a bijection from $[1,2]$ to $[3,20]$.

(6) $\mathbb{Q} \sim \mathbb{N}^+$

Any rational number q can be written as $q = \frac{z}{n}$ for some unique $z \in \mathbb{Z}$ and some unique $n \in \mathbb{N}^+$ such that z and n are coprime. (For instance, $-\frac{14}{6} = \frac{-7 \leftarrow z}{3 \leftarrow n}$). Therefore the map $f: \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}^+$ given by $f(q = \frac{z}{n}) = (z, n)$ is injective. By part (1), $\mathbb{N}^+ \sim \mathbb{Z}$. So there is a bijection $\mathbb{Z} \sim \mathbb{N}^+$. Composing this with the first component of f, we see that there is an injective function $\mathbb{Q} \to \mathbb{N}^+ \times \mathbb{N}^+$. By part(4), $\mathbb{N}^+ \times \mathbb{N}^+ \sim \mathbb{N}^+$. So there is a bijective function $\mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$. Composing this with the last injective map, we see that "there is an injective function $\mathbb{Q} \to \mathbb{N}^+$" (I)

Conversely, as $\mathbb{N}^+ \subseteq \mathbb{Q}$, the inclusion map $\mathbb{N}^+ \to \mathbb{Q}$, $x \mapsto x$ is injective. In particular, "there is an injective function $\mathbb{Q} \to \mathbb{N}^+$" (II)

Now, it follows from (I), (II) and the Schröder Bernstein Theorem that there is a bijective map $\mathbb{Q} \to \mathbb{N}^+$.