Relations

Recall that by identifying a function $f: A \to B$ with its graph $\{(x, f(x)) \mid x \in A\}$, we may define a function from A to B as a subset of $A \times B$ satisfying a certain condition. Much of the results/definitions for functions (such as composition, inverse, ---) can be generalized to arbitrary subsets of $A \times B$. Moreover, considering arbitrary functions can prove to be useful as you will see later.

<u>Definition</u>: Let A, B, C be sets

(1) Any subset R of $A \times B$ is called a <u>relation from A to B</u>. We sometimes write aRb, and read it as "a is R-related to b", to indicate that $(a,b) \in R$, and we write $a \not R b$ to indicate that $(a,b) \notin R$.

(2) If R is a relation from A to B (i.e., $R \subseteq A \times B$), we define

the <u>domain</u> of $R = \{a \in A \mid (a,b) \in R \text{ for some } b \in B\}$

the <u>range</u> of $R = \{b \in B \mid (a,b) \in R \text{ for some } a \in A\}$

(3) If R is a relation from A to B and S is a relation from B to C, then their <u>composition</u> $S \circ R$ is the relation from A to C defined by

$$S \circ R = \{(a,c) \in A \times C \mid (a,b) \in R \text{ and } (b,c) \in S \text{ for some } b \in B\}$$

$(R \subseteq A \times B,\ S = B \times C \Longrightarrow S \circ R \subseteq A \times C)$

(4) If R is a relation from A to B (i.e., $R \subseteq A \times B$), the <u>inverse</u> of R is the relation R^{-1} from B to A defined by

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

(5) A relation from A to the same set A is called a (binary) <u>relation on A</u>. So a relation on A is just a subset of $A \times A$.

Let R be a relation on a set A. As $R \subseteq A \times A$ and $(a,a) \in A \times A$ for all $a \in A$ and $(x,y), (y,x) \in A \times A$ for all $x, y \in A$, we may wonder whether elements of the form (a,a) or both elements of the form $(x,y), (y,x)$ are in R. Most of the relations we consider will contain elements of such forms.

<u>Definition</u>: Let R be a relation on a set A.

(1) R is called <u>reflexive</u> if $(a,a)\in R$ for all $a\in A$.

$$(\forall a\in A)\big(\underbrace{(a,a)\in R}_{aRa}\big) \quad \text{or} \quad (\forall a)\big(a\in A \rightarrow (a,a)\in R\big)$$

(2) R is called <u>symmetric</u> if, for any $a,b\in A$, $(a,b)\in R$ implies $(b,a)\in R$

$$(\forall a,b\in A)\big(\underbrace{(a,b)\in R}_{aRb} \rightarrow \underbrace{(b,a)\in R}_{bRa}\big)$$

(3) R is called <u>anti-symmetric</u> if, for any $a,b\in A$, $(a,b)\in R$ and $(b,a)\in R$ imply $a=b$.

$$(\forall a,b\in A)\big((a,b)\in R \land (b,a)\in R \rightarrow a=b\big)$$

(4) R is called <u>transitive</u> if, for any $a,b,c\in A$, $(a,b)\in R$ and $(b,c)\in R$ imply $(a,c)\in R$

$$(\forall a,b,c\in A)\big((a,b)\in R \land (b,c)\in R \rightarrow (a,c)\in R\big)$$

$$aRb \land bRc \rightarrow aRc$$

<u>Ex</u>: Let $A=\{1,2,3,4\}$ and $B=\{a,b,c,d\}$ and $R=\{(1,a),(1,b),(2,c),(4,c)\}$

R is a relation from A to B because $R\subseteq A\times B$.

As $(2,c)\in R$, we may write $2Rc$. As $(1,d)\in R$, we may write $1 \not R\, d$

The domain of R is $\{1,2,4\}$ and the range of R is $\{a,b,c\}$.

The inverse of R is $R^{-1}=\{(a,1),(b,1),(c,2),(c,4)\}$

As $(4,c)\in R$, 4 is R-related to c.

<u>Ex</u>: Let $A=\{1,2,3,4\}$, $B=\{a,b,c,d\}$, $C=\{\text{Ali}, \text{Ahmet}, \text{Ayşe}\}$,

$R=\{(1,a),(1,b),(2,c),(4,d)\}$, $S=\{(a,\text{Ali}),(c,\text{Ali}),(b,\text{Ayşe}),(b,\text{Ahmet})\}$

R is a relation from A to B and S is a relation from B to C, so $S\circ R$ is the following relation from A to C

$S\circ R=\{(x,z)\in A\times C \mid (x,y)\in R \text{ and } (y,z)\in S \text{ for some } y\in B\}$

$=\{(1,\text{Ali}),(1,\text{Ayşe}),(1,\text{Ahmet}),(2,\text{Ali})\}$

$(2,c)\in R$, $(c,\text{Ali})\in S$

$(2,\text{Ali})\in S\circ R$

Ex: Let P be the set of all people, and R be the relation on P defined by

$x R y$ iff x is the father of y (that is, $R = \{(x,y) \in P \times P \mid x \text{ is the father of } y\}$)

R is not a function from P to P.

The domain of $R = \{x \in P \mid xRy \ \exists y \in P\} = \{x \in P \mid x \text{ is the father of some people } y\}$

The range of $R = \{a \in P \mid bRa \ \exists b \in P\} = \{a \in P \mid b \text{ is the father of } a \text{ for some people } b\}$

a is the child of some b

$= P$

The inverse of R is $R^{-1} = \{(a,b) \in P \times P \mid bRa\} = \{(a,b) \in P \times P \mid b \text{ is the father of } a\}$

a is the child of some man b

Note that R^{-1} is a function from P to P.

R is not reflexive: Indeed, $(a,a) \notin R$ for all $a \in R$ because no one is the father of himself

R is not symmetric, not transitive

$R \circ R = \{(x,z) \in P \times P \mid (x,y) \in R \text{ and } (y,z) \in R \ \exists y \in P\} = \{(x,z) \in P \mid x \text{ is the grand father of } P\}$

x is the father of y y is the father of z

Ex: W = the set of all meaningful Turkish words. Consider the relation R defined on W as follows: $x_1 R x_2 \iff x_1$ and x_2 have at least one letter in common

(ana, baba) $\in R$, (yer, gök) $\notin R$

R is reflexive, symmetric, <u>not transitive</u>, not antisymmetric

As (çay, kahve) $\in R$ and (kahve, şeker) $\in R$ but (çay, şeker) $\notin R$,

R is <u>not</u> transitive

Ex: Let S be the set of all assertions. Consider the relation R defined on S as follows:

$(A, B) \in R$ if and only if $A \to B$ is true

R is reflexive because $A \to A$ is true for all assertions A.

R is not symmetric because if A is a false assertion and B is a true assertion, then $A \to B$ is true and $B \to A$ is false, implying that $(A,B) \in R$ but $(B,A) \notin R$. (Finding concrete examples are easy. For instance, let A: "$1 \neq 2$" and B: "$1 < 2$")

R is not antisymmetric because if A and B are distinct assertions that are both false, then $A \to B$ and $B \to A$ are both true, implying that $(A,B) \in R$ and $(B,A) \in R$ but $A \neq B$

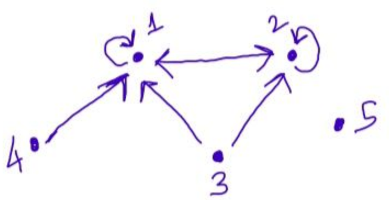
R is transitive: Let A, B, C be assertions such that $(A,B) \in R$ and $(B,C) \in R$. Then $A \to B$ and $B \to C$ are both true. We want to show that $(A,C) \in R$. That is, we want to prove that $A \to C$ is true. If A is false, then the implication $A \to C$ is true. So we may assume for the rest that A is true. As $A \to B$ and A are both true, B must be true. As $B \to C$ and B are both true, C must be true. Having justified that A and C are both true, we conclude that $A \to C$ is true, as desired. (Indeed, we have proved here that the deduction "$A \to B$, $B \to C$, $\therefore A \to C$" is valid. Give a two column proof)

Remark: We may represent a relation R on a set A by drawing a directed graph especially when A is a finite set: We draw a dot for each element of A. (These dots are called the vertices of the graph. So the vertices of the graph are the elements of A) For all vertices a and b we draw an arrow from a to b if $(a,b) \in R$. (These arrows are called the edges of the graph)

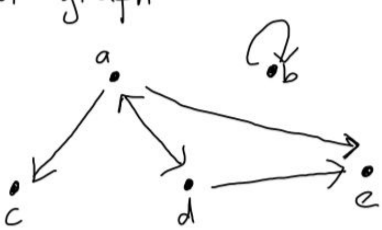
Ex: Let $A = \{1,2,3,4,5\}$ and $R = \{(x,y) \in A \times A \mid x^2 + y^3 < 20\}$.

Draw a directed graph representing the relation R on A.

Sol: Note that $R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2), (4,1)\}$



Ex: Write the relation R on $A = \{a,b,c,d,e\}$ represented by the following directed graph



Sol:

$R = \{(a,c), (a,d), (a,e), (b,b), (d,a), (d,e)\}$

Ex (Relations Defined by Functions)

Let $f: A \to B$ be a function where A and B sets. Let R be a relation on B. Consider the relation R_f on A defined as follows:

$$\forall a_1, a_2 \in A, \quad (a_1, a_2) \in R_f \text{ iff } (f(a_1), f(a_2)) \in R.$$

Show that:

(1) R_f is a relation on A

(2) If R is reflexive/symmetric/transitive, then so is R_f.

(3) Assuming that f is onto, the converse of (2) is true. (That is, assuming that f is onto, if R_f is reflexive/symmetric/transitive then so is R)

(4) If f is one to one and R is antisymmetric, then R_f is antisymmetric

Sol: Exercise.

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Consider the relation R on $\mathbb{R}$ defined for any elements x and y of $\mathbb{R}$ by

$$xRy \Leftrightarrow f(x) \leq f(y)$$

Show that if R is antisymmetric then f is injective.

Sol: Let $x, y \in \mathbb{R}$ be such that $f(x) = f(y)$. As $f(x) \leq f(y)$ and $f(y) \leq f(x)$, the definition of R implies that xRy and yRx. Since R is antisymmetric, $x = y$. So f is injective.

Ex: Let R be a relation on a set A. Then

(1) R is reflexive iff $\Delta_A \subseteq R$ where $\Delta_A = \{(a,a) \mid a \in A\}$

(2) R is symmetric iff $R = R^{-1}$

(3) R is transitive iff $R \circ R \subseteq R$

Sol: (1) and (2): Exercise.

(3): ($\Rightarrow$): Let $(a,b) \in R \circ R$. Then there is a $c \in A$ such that $(a,c) \in R$ and $(c,b) \in R$. As R is transitive, $(a,b) \in R$. Hence $R \circ R \subseteq R$

($\Leftarrow$): Let $x,y,z \in A$ such that $(x,y) \in R$ and $(y,z) \in R$. (We want to justify that $(x,z) \in R$). By the definition of the composition $(x,z) \in R \circ R$. As $R \circ R \subseteq R$, we see that $(x,z) \in R$, as desired. Hence R is transitive.

Equivalence Relations

Definition: Let R be a relation on a set A. Then,

(1) R is called an equivalence relation on A if R is reflexive, symmetric and transitive

(2) Let R be an equivalence relation on A. Then,

(i) For any $a \in A$, we define the $(R-)$ equivalence class of a (or containing a) as the set $[a]_R = \{x \in A \mid (x,a) \in R\} \left(\underset{\uparrow}{=} \{x \in A \mid (x,a) \in R\} \right)$

(As R is symmetric, $(x,a) \in R \Leftrightarrow (a,x) \in R$)

Other notations for the equivalence class $[a]_R$ of a are $[a]$ and $a/_R$

(ii) The set of all $(R-)$ equivalence classes is usually called the quotient set of A with respect to R (or modulo R) and denoted by A/R. Thus

$$A/_R = \{[a]_R \mid a \in A\}, \text{ the set of all } R\text{-equivalence classes.}$$

(Be careful here because it may happen that $[a_1]_R = [a_2]_R$ for some $a_1 \neq a_2$. See the next theorem)

Ex: Let $A = \{1,2,3,4,5,6,7\}$ and $R = \{(x,y) \mid x,y \in A \text{ and } x-y \text{ is even}\}$

R is reflexive: For any $a \in A$, as $a-a=0$ is even, $(a,a) \in R$.

R is symmetric: Let $a,b \in A$. Suppose that $(a,b) \in R$. Then $a-b$ is even. So $b-a$ is even, too. This implies that $(b,a) \in R$.

R is transitive: Let $a,b,c \in A$ such that $(a,b) \in R$ and $(b,c) \in R$. Then, $a-b$ and $b-c$ are both even. As the sum of even integers is even, $(a-b)+(b-c) = a-c$ is even. Therefore $(a,c) \in R$.

Hence R is an equivalence relation on R. What are the equivalence classes?

$[1]_R = \{x \in A \mid (1,x) \in R\} = \{x \in A \mid 1-x \text{ is even}\} = \{1,3,5,7\}$

$[3]_R = \{x \in A \mid (3,x) \in R\} = \{x \in A \mid 3-x \text{ is even}\} = \{1,3,5,7\}$

$[6]_R = \{x \in A \mid (6,x) \in R\} = \{x \in A \mid 6-x \text{ is even}\} = \{2,4,6\}$

We easily see that $[1] = [3] = [5] = [7] = \{1,3,5,7\}$

$[2] = [4] = [6] = \{2,4,6\}$.

The quotient set with respect to R is

$$A/R = \{[a]_R \mid a \in A\} = \{[1],[2]\} = \{[7],[4]\} = \{\{1,3,5,7\},\{2,4,6\}\}$$

Ex: Let $A = \{a,b,c,d\}$. Consider the relation R defined on the power set $\mathcal{P}(A)$ as follows: $\forall U,V \in \mathcal{P}(A)$, $(U,V) \in R \Leftrightarrow |U| = |V|$

We easily see that R is an equivalence relation on $\mathcal{P}(A)$. (Exercise).

Take any $U \in \mathcal{P}(A)$ and consider its equivalence class $[U]_R$.

$[U] = \{V \in \mathcal{P}(A) \mid (U,V) \in R\} = \{V \subseteq A \mid |U| = |V|\}$. Therefore,

$[\phi] = \{\phi\}$, $[\{a\}] = [\{b\}] = [\{c\}] = \{\{a\},\{b\},\{c\}\}$,

$[\{a,b\}] = [\{a,c\}] = [\{b,c\}] = \{\{a,b\},\{a,c\},\{b,c\}\}$, $[\{a,b,c\}] = \{\{a,b,c\}\}$,

$$A/R = \{[U]_R \mid U \in \mathcal{P}(A)\} = \{[\phi],[\{b\}],[\{a,c\}],[\{a,b,c\}]\}$$

$$= \{\{\phi\},\{\{a\},\{b\},\{c\}\},\{\{a,b\},\{a,c\},\{b,c\}\},\{\{a,b,c\}\}\}$$

the set of all subsets of A with 2 elements

Ex: Let A be the set of all students in our class. Consider the relation R on A defined by: $\forall p,q \in A$, pRq iff p and q have the same gender (i.e, they are both girl or boy)

We easily see that R is an equivalence relation, and there are two distinct equivalence classes, namely $\{\text{girls in } A\}$ and $\{\text{boys in } A\}$, and so $A/R = \{\{\text{girls in } A\}, \{\text{boys in } A\}\}$

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Ex : Let $f: A \to B$ be a function where A and B be sets. Consider the relation K_f on A defined by: $(\forall r, s \in A)$, $(r,s) \in K_f \Leftrightarrow f(r) = f(s)$

We easily see that K_f is an equivalence relation on A and the equivalence classes are $f^{-1}(b) = \{a \in A \mid f(a) = b\}$ where b is an arbitrary element of the range of f.

Indeed, for any $x \in A$, $[x]_{K_f} = \{a \in A \mid (a,x) \in K_f\} = \{a \in A \mid f(a) = f(x)\} = f^{-1}(b)$ where $b = f(x)$

The equivalence relation K_f is sometimes called the kernel of f.

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Ex: Consider the relation $|$ on $\mathbb{N}$ defined by

$\forall a, b \in \mathbb{N}$, $a | b$ iff a divides b (i.e., $b = ac$ for some $c \in \mathbb{N}$)

As 2 divides 6 but 6 does not divide 2, $(2,6) \in |$ but $(6,2) \notin |$, so $|$ is not symmetric. Hence, $|$ is not an equivalence relation o

Ex: (Construction of Rationals from Integers)

Define a relation R on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ as follows: $\forall (a,b), (m,n) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$

$$((a,b),(m,n)) \in R \Leftrightarrow na = mb$$

We easily see that R is an equivalence relation. Consider for instance the equivalence class containing $(1,2)$. It is

$$[(1,2)]_R = \{(m,n) \in \mathbb{Z} \times (\mathbb{Z} - \{0\}) \mid ((1,2),(m,n)) \in R\} = \{(m,n) \in \mathbb{Z} \times (\mathbb{Z} - \{0\}) \mid n = 2m\}$$

$$= \{(m, 2m) \mid m \in \mathbb{Z} - \{0\}\}$$

Similarly, for any $a, b \in \mathbb{Z}$ with $b \neq 0$, $[(a,b)]_R = \{(m,n) \in \mathbb{Z} \times (\mathbb{Z} - \{0\}) \mid na = mb\}$

If a and b are coprime (i.e., $\gcd(a,b)=1$), then $na=mb$ implies that a divides m and b divides n, and so $m=at_1$ and $n=bt_2$; moreover, from $na=mb$ we see that $t_1=t_2$. Consequently, if a and b are coprime, then

$$[(a,b)]_R = \{(at,bt) \mid t\in \mathbb{Z}-\{0\}\}.$$

As $(a,b)\,R\,(a\lambda,b\lambda)$ for all $\lambda\in\mathbb{Z}-\{0\}$, given any $(r,s)\in\mathbb{Z}\times(\mathbb{Z}-\{0\})$ there is a $(r',s')\in\mathbb{Z}\times(\mathbb{Z}-\{0\})$ such that $\gcd(r',s')=1$ and $[(r,s)]=[(r',s')]=\{(r't,s't) \mid t\in\mathbb{Z}-\{0\}\}$

(Exercise. Indeed, let let $d=\gcd(r,s)$ and $r'=\frac{r}{d}$ and $s'=\frac{s}{d}$)

So now the quotient set satisfies

$$\frac{\mathbb{Z}\times(\mathbb{Z}-\{0\})}{R} = \{[(r',s')] \mid r',s'\in\mathbb{Z},\ s'\neq 0,\ \gcd(r',s')=1\}$$

and the map $\psi: \frac{\mathbb{Z}\times(\mathbb{Z}-\{0\})}{R} \longrightarrow \mathbb{Q}$ given by $\psi([r',s'])=\frac{r'}{s'}$ is a (well-defined) bijective function.

Theorem: Let R be an equivalence relation on a nonempty set A. Let $x,y\in A$.

(1) $x\in[x]$. In particular, $[x]\neq\emptyset$. That is, any equivalence class is nonempty.

(2) $[x]=[y] \Leftrightarrow x\in[y] \Leftrightarrow y\in[x] \Leftrightarrow xRy \Leftrightarrow yRx$

(3) $(x,y)\in R \Leftrightarrow [x]=[y]$

$(x,y)\notin R \Leftrightarrow [x]\cap[y]=\emptyset$

(4) $[x]=[y]$ or $[x]\cap[y]=\emptyset$

(5) Any two equivalence class is either equal or disjoint.

(6) $A=\bigcup_{a\in A}[a]$

(7) $A=\biguplus_{[a]\in A/R}[a]$. That is, A can be written as the disjoint union of distinct equivalence classes

Proof: (1): As R is reflexive, $(x,x)\in R$. So $x\in[x]=\{a\in A \mid (a,x)\in R\}$.

(2): Let us use the following notations:

$$\underbrace{[x]=[y]}_{(i)} \Leftrightarrow \underbrace{x\in[y]}_{(ii)} \Leftrightarrow \underbrace{y\in[x]}_{(iii)} \Leftrightarrow \underbrace{xRy}_{(iv)} \Leftrightarrow \underbrace{yRx}_{(v)}$$

(iv) $\Leftrightarrow$ (v): This is because R is symmetric

(i) $\Rightarrow$ (ii): As $x\in[x]$ from part (1) and as $[x]=[y]$, it follows that $x\in[y]$.

(ii) $\Leftrightarrow$ (iv) and (iii) $\Leftrightarrow$ (v): Follows from the definition of the equivalence class. Recall that

$$[a]_R=\{b\in A \mid \underbrace{(a,b)\in R}_{aRb}\}=\{b\in A \mid \underbrace{(b,a)\in R}_{bRa}\}.$$ Thus

$$b\in[a]_R \Leftrightarrow (a,b)\in R \Leftrightarrow (b,a)\in R$$

(ii) $\Rightarrow$ (i): Assume that $x\in[y]$. From the previous sentence $(x,y)\in R$ and $(y,x)\in R$. We want to show that $[x]=[y]$.

Let $m\in[x]$. Then $(x,m)\in R$. As R is transitive and $(x,m),(y,x)\in R$, we see that $(y,m)\in R$. So $m\in[y]$. Thus $[x]\subseteq[y]$.

Show conversely that $[y]\subseteq[x]$ (Exercise).

(3): "$(x,y)\in R \Leftrightarrow [x]=[y]$" is contained in part (2).

($\Leftarrow$): Suppose that $[x]\cap[y]=\emptyset$. From part (1) we know that $[x]\neq\emptyset$. This implies that $[x]\neq[y]$. So $(x,y)\notin R$ by part (2).

($\Rightarrow$): Suppose that $(x,y)\notin R$. We want to prove that $[x]\cap[y]=\emptyset$. The proof will be by contradiction. Suppose for a contradiction that $[x]\cap[y]\neq\emptyset$. There is an element z such that $z\in[x]$ and $z\in[y]$. It then follows from part (2) that $(x,z)\in R$ and $(y,z)\in R$. As R is symmetric and transitive, $(x,y)\in R$. This is a contradiction.

(4) : Follows from part (3).

(5) : This is part (4) writt in plain English.

(6) : As $x \in [x]$ by part , $A \subseteq \bigcup_{\in A} [a]$. The result follows because $[a] \subseteq A$ for each $a \in A$.

(7) Deleting multiple copies of the quiv lasses in the union $A = \bigcup_{a \in A} [a]$, we get $A = \biguplus_{[a] \in A/R} [a]$. The index set and the disjointness of the union follow from part (5).

□

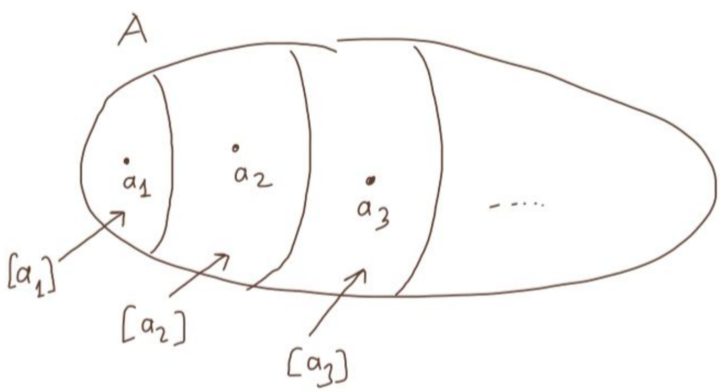
<u>Remark</u>: Let R be an equivalence relation on a nonempty set A.

(i.e., distinct elements of A/R)

(1) To find the distinct equivalence classes, choose an arbitrary $a_1 \in A$ and find $[a_1]$, then choose an arbitrary $a_2 \in A - [a_1]$ and find $[a_2]$, then choose an arbitrary $a_3 \in A - ([a_1] \cup [a_2])$ and find $[a_3]$, ...

Then $[a_1], [a_2], [a_3], \ldots\ldots$ are distinct equivalence classes. That is,

$A/R = \{[a_1], [a_2], [a_3], \ldots\}$



$a_1 \in A$

$a_2 \in A - [a_1]$; so $[a_2] \cap [a_1] = \phi$

$(z \in [a_1] \not\Rightarrow [a_1] = [z] \,!)$

$a_3 \in A - ([a_1] \cup [a_2])$; so

$[a_3] \cap [a_1] = \phi$, $[a_3] \cap [a_2] = \phi$

$\vdots$

(2) If there are $z_1, z_2, \ldots, z_n \in A$ such that $[z_i]$ are mutually disjoint and $\bigcup_{i=1}^{n} [z_i] = A$, then $[z_1], [z_2], \ldots, [z_n]$ are all the distinct equivalence classes, that is $A/R = \{[z_1], [z_2], \ldots, [z_n]\}$

<u>Proof</u>: (1) Follows from the previous Theorem. (Exercise)

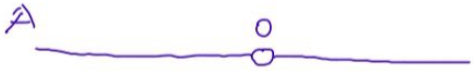
(2) Consider any equivalence class $[a]$ where $a \in A$. As $A = \bigcup_{i=1}^{n} [z_i]$, we see that $a \in [z_k]$ for some $k \in \{1, 2, \ldots, n\}$. Then part (2) of the previous theorem implies that $[a] = [z_k]$. Consequently, any equivalence class is one of $[z_1], [z_2], \ldots, [z_k]$. As $[z_i]$ are mutually disjoint, they are distinct.

$\square$

Ex: For each of the following show that R is an equivalence relation on A and describe the equivalence classes and the quotient set R/A.

EXERCISE!

(1) $A = \mathbb{R} - \{0\}$; $\forall x, y \in A$, xRy iff $x \cdot y > 0$



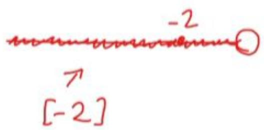
Take any $a_1 \in A$. Say for instance -2.

Consider $[a_1]$. Note that

$$[-2] = \{x \in A \mid x R(-2)\} = \{x \in \mathbb{R} - \{0\} \mid -2x > 0\}$$

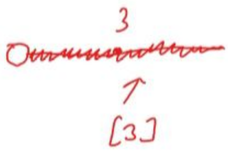
$\Updownarrow$

$x < 0$



$$= \{x \in \mathbb{R} - \{0\} \mid x < 0\}$$

Choose any $a_2 \in A - [a_1]$. So choose any $a_2 \in A - [-2]$. Say for instance $a_2 = 3$. Consider $[a_2]$. Note that



$$[a_2] = \{x \in A \mid x R 3\} = \{x \in \mathbb{R} - \{0\} \mid 3x > 0\}$$

$\Updownarrow$

$x > 0$

$$= \{x \in \mathbb{R} - \{0\} \mid x > 0\}$$

Note that if we had chosen a_2 from $[a_1]$ then we would follow that $[a_2] = [a_1]$
(Check for instance that $[-7] = [-2]$ and $[10] = [3]$)

As $[-2] \cup [3] = A$, it follows that $A/R = \{[-2], [3]\} = \{\mathbb{R}^-, \mathbb{R}^+\}$

negative reals positive reals

(Exercise: Is the relation R defined on $\mathbb{R}$ by "xRy iff $x \cdot y \geq 0$" an equivalence relation?)

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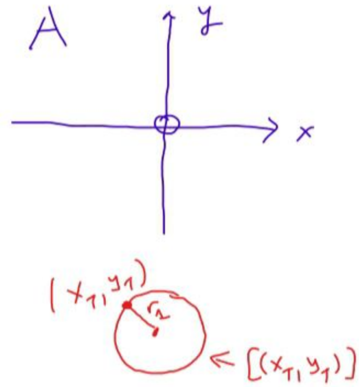
(2) $A = \mathbb{R} \times \mathbb{R} - \{(0,0)\}$; $\forall (a,b), (u,v) \in A$, $(a,b) R (u,v)$ iff $a^2 + b^2 = u^2 + v^2$

Note first that, for any $(x,y) \in \mathbb{R}^2$ and $r \in \mathbb{R}$, $x^2+y^2=r^2$ iff (x,y) lies on the circle of radius r centered at the origin.

A



Take any $a_1 \in A$. Let $a_1 = (x_1, y_1)$. Consider $[a_1]$.

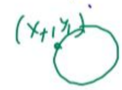
$$[(x_1,y_1)] = \{(x,y)\in A \mid (x,y)\, R\, (x_1,y_1)\} = \{(x,y)\in A \mid x^2+y^2 = x_1^2+y_1^2\}$$

$$= \{(x,y)\in A \mid x^2+y^2 = r_1^2\} \text{ where } r_1 = x_1^2+y_1^2$$

$=$ the circle of radius r_1 centered at the origin

So the equivalence classes are the circles of positive radius centered at the origin. R/A is an infinite set of circles,

Note that for any (u,v) chosen on the circle



$\leftarrow [(x_1,y_1)]$ we have $[(u,v)] = [(x_1,y_1)]$.

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(3) $A = \mathbb{Z}$; $\forall x,y \in \mathbb{Z}$, xRy iff 3 divides $x-y$ (i.e., $x-y=3k$, $\exists k \in \mathbb{Z}$)

Take any element of A, say 11. Consider $[11]$. Note that

$$[11] = \{x \in \mathbb{Z} \mid x\,R\,11\} = \{x\in\mathbb{Z} \mid x-11 = 3k \ \ \exists k\in\mathbb{Z}\} = \{11+3k \mid k\in\mathbb{Z}\}$$

$$= \{\dots, -4, -1, 2, 5, 8, 11, 14, \dots\}$$

Take any element of A not in $[11]$, say -3. Consider $[-3]$. Note that

$$[-3] = \{x\in\mathbb{Z} \mid x\,R\,{-3}\} = \{x\in\mathbb{Z} \mid x+3 = 3k \ \ \exists k\in\mathbb{Z}\} = \{-3+3k \mid k\in\mathbb{Z}\}$$

$$= \{\dots, -6, -3, 0, 3, 6, 9, \dots\dots\}$$

Take any element of A not in $[11]\cup[-3]$, say 1. Consider $[1]$. Note that

$$[1] = \{x\in\mathbb{Z} \mid x\,R\,1\} = \{x\in\mathbb{Z} \mid x-1 = 3k \ \ k\in\mathbb{Z}\} = \{1+3k \mid k\in\mathbb{Z}\}$$

$$= \{\dots, -5, -2, 1, 4, 7, 10, \dots\}$$

Note that $[11]\cup[-3]\cup[1] = \mathbb{Z} = A$, so $R/A = \{[11], [-3], [1]\}$ and there are 3 equivalence classes. Note for instance that $[1] = [2] = [5]$

(4) $A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (3,5), (5,3)\}$

$[1] = \{x \in A \mid (x,1) \in R\} = \{1, 2\}$; $(3 \in A - [1])$, $[3] = \{x \in A \mid (x,3) \in R\} = \{3, 5\}$;

$(4 \in A - ([1] \cup [3]))$, $[4] = \{x \in A \mid (x,4) \in R\} = \{4\}$.

Note that $[1] \cup [3] \cup [4] = A$, so $R/A = \{[1], [3], [4]\}$

$= \{\{1,2\}, \{3,5\}, \{4\}\}$

There are 3 equivalence classes.

Note that $[1] = [2]$, $[3] = [5]$

Partitions of a set

Definition: A partition of a set A is a collection (i.e., set) $\mathcal{F}$ of subsets of A (i.e, $\mathcal{F} \subseteq P(A)$) satisfying the following conditions:

(1) $X \neq \phi$ for any $X \in \mathcal{F}$.

(2) $X \cap Y = \phi$ for any $X, Y \in \mathcal{F}$ with $X \neq Y$

(3) $\bigcup_{X \in \mathcal{F}} X = A$

(In plain English, a partition of a set A is a collection of mutually disjoint nonempty subsets of A such that the union of all the sets in the collection is A).

Ex Let $A = \{1, 2, 3, 4, 5\}$

(1) $\mathcal{F} = \{\phi, \{1,2,3\}, \{4,5\}\}$ is not a partition of A because the empty set is in $\mathcal{F}$

(2) $\mathcal{F} = \{\{1\}, \{2,3,4\}, \{3,5\}\}$ is not a partition of A because sets in $\mathcal{F}$ are not mutually disjoint, namely $\{2,3,4\} \cap \{3,5\} = \{3\} \neq \phi$.

(3) $\mathcal{F} = \{\{1,2\}, \{4,5\}\}$ is not a partition of A because the union of all sets in $\mathcal{F}$ is not A, $\{1,2\} \cup \{4,5\} = \{1,2,3,4\} \neq A$.

(4) $\mathcal{F} = \{\{1\}, \{2\}, \{3,4,5\}\}$ is a partition of A.

Theorem: Let A be a set.

(1) For any equivalence relation R on A, the quotient set

$$A/R = \{ [a] \mid a \in R \}$$

is a partition of A (A/R is called the partition of A induced by R)

(2) For any partition $\mathcal{F}$ of A, the relation $\mathcal{E}_{\mathcal{F}}$ on A defined by,

$\forall x,y \in A$, $(x,y) \in \mathcal{E}_{\mathcal{F}}$ iff there is a $P \in \mathcal{F}$ such that $x \in P$ and $y \in P$,

is an equivalence relation on A. Moreover, the quotient set $A/\mathcal{E}_{\mathcal{F}}$ is $\mathcal{F}$. (The relation $\mathcal{E}_{\mathcal{F}}$ is called the equivalence relation induced by $\mathcal{F}$).

(3) The constructions in the first two parts are inverses of each others.

So there is a bijective correspondence between the set equivalence classes of A and the set of partitions of A given by the following inverse bijections:

$$\{\text{Equivalence Classes on } A\} \qquad\qquad \{\text{Partitions of } A\}$$

$$R \longmapsto A/R$$

$$\mathcal{E}_{\mathcal{F}} \longleftarrow\!\shortmid \mathcal{F}$$

Proof: (1) : Follows from the previous theorem.

(2): $\mathcal{E}_{\mathcal{F}}$ is reflexive: Let $a \in A$. As $\mathcal{F}$ is a partition of A, $A = \bigcup_{X \in \mathcal{F}} X$ so there is a $P \in \mathcal{F}$ such that $a \in P$. Hence $(a,a) \in \mathcal{E}_{\mathcal{F}}$ by the definition of $\mathcal{E}_A$.

$\mathcal{E}_{\mathcal{F}}$ is symmetric: Suppose that $(x,y) \in \mathcal{E}_{\mathcal{F}}$. Then there is a $P \in \mathcal{F}$ such that $x \in P$ and $y \in P$. So $y \in P$ and $x \in P$, implying that $(y,x) \in \mathcal{E}_{\mathcal{F}}$.

$\mathcal{E}_{\mathcal{F}}$ is transitive: Let $x,y,z \in A$. Suppose that $(x,y) \in \mathcal{E}_{\mathcal{F}}$ and $(y,z) \in \mathcal{E}_{\mathcal{F}}$. Then, by the definition of $\mathcal{E}_A$, there is a set $P \in \mathcal{F}$ such that $x \in P$ and $y \in P$, and there is a set $S \in \mathcal{F}$ such that $y \in S$ and $z \in S$. Thus, $y \in P \cap S$, implying that $P \cap S \neq \phi$. As distinct sets in a partition are disjoint, we see that $P = S$. Hence, $x \in S$ and $z \in S$. Consequently, $(x,z) \in \mathcal{E}_{\mathcal{F}}$.

Consider elements of the quotient set $A/\mathcal{E}_{\mathcal{F}}$ (i.e., equivalence classes of the equivalence relation $\mathcal{E}_{\mathcal{F}}$ on A). Take any $a \in A$ and consider $[a]_{\mathcal{E}_{\mathcal{F}}}$. Note that $[a]_{\mathcal{E}_{\mathcal{F}}} = \{x \in A \mid (x,a) \in \mathcal{E}_{\mathcal{F}}\} = \{x \in A \mid \exists P \in \mathcal{F} \text{ such that } x \in P \text{ and } a \in P\}$.

As $\mathcal{F}$ is a partition of A and $a \in A$, there is a unique set P in $\mathcal{F}$ such that $a \in P$. Therefore, $[a]_{\mathcal{E}_{\mathcal{F}}} = P$, the unique set in $\mathcal{F}$ containing a. Therefore, the $\mathcal{E}_{\mathcal{F}}$-equivalence classes are the sets in $\mathcal{F}$. So $A/\mathcal{E}_{\mathcal{F}} = \mathcal{F}$.

(3) By the last paragraph we know that $A/\mathcal{E}_{\mathcal{F}} = \mathcal{F}$. This means that

$\psi \circ \phi = \text{id}$ where
$$\phi : \{\text{Partitions on } A\} \longrightarrow \{\text{Equivalence Relations on } A\}$$
$$\mathcal{F} \longmapsto \mathcal{E}_{\mathcal{F}}$$
$$\psi : \{\text{Equivalence relations on } A\} \longrightarrow \{\text{Partitions on } A\}$$
$$R \longmapsto A/R$$

Let an equivalence relation R on A be given. Consider the equivalence relation

$\mathcal{E}_{R/A}$ on A induced by the partition R/A. Note for any $x,y \in A$ that

$(x,y) \in \mathcal{E}_{R/A}$ iff there is a $P \in \underset{\{[z]_R \mid z \in A\}}{\underset{\parallel}{R/A}}$ such that $x \in P$ and $y \in P$

iff there is a $z \in A$ such that $\underset{(x,z)\in R}{\underset{\Updownarrow}{x \in [z]_R}}$ and $\underset{(z,y)\in R}{\underset{\Updownarrow}{y \in [z]_R}}$

iff there is a $z \in A$ such that $(x,z) \in R$ and $(z,y) \in R$

iff $(x,y) \in R$;

(⟹): As R is transitive, $(x,z) \in R$ and $(z,y) \in R$ imply that $(x,y) \in R$

(⟸): Assume $(x,y) \in R$. Let $z = x$. As R is reflexive, $(x,z) = (x,x) \in R$. Note also that $(x,z) = (x,y)$

Hence, $\mathcal{E}_{A/R} = R$. This means that $\phi \circ \psi = id$.

□

Ex: (1) Let $A = \{1,2,3,4,5\}$ and $\mathcal{F} = \{\{1\},\{2,3\},\{4,5\}\}$. Note that $\mathcal{F}$ is a partition of A. Write $\mathcal{E}_{\mathcal{F}}$.

$$\mathcal{E}_{\mathcal{F}} = \{(x,y) \in A \times A \mid \underbrace{\exists P \in \mathcal{F}}_{P \text{ must be one of } \{1\},\{2,3\},\{4,5\}} \text{ such that } x \in P \text{ and } y \in P\}$$

$$= \{\underbrace{(1,1)}_{P=\{1\}},\ \underbrace{(2,2),(3,3),(2,3),(3,2)}_{P=\{2,3\}},\ \underbrace{(4,4),(5,5),(4,5),(5,4)}_{P=\{4,5\}}\}$$

---

(2) Let $A = \{1,2,3,4\}$ and $R = \{(1,1),(2,2),(3,3),(4,4),(1,3),(3,1)\}$. Note that R is an equivalence relation on A. Write the partition of A induced by R (i.e., write the quotient set A/R)

Note that $1 \in A$ and $[1] = \{x \in A \mid (x,1) \in R\} = \{1,3\}$.

Note that $2 \in A - [1]$ and $[2] = \{x \in A \mid (x,2) \in R\} = \{2\}$

Note that $4 \in A - ([1] \cup [2])$ and $[4] = \{x \in A \mid (x,4) \in R\} = \{4\}$.

Note that $[1] \cup [2] \cup [4] = A$. So $A/_{R} = \{[1],[2],[4]\} = \{\{1,3\},\{2\},\{4\}\}$

<u>Fact</u>: Let A, B, C, D be sets and $R \subseteq A \times B$, $S \subseteq B \times C$, $T \subseteq C \times D$ be relations. Then:

(1) $T \circ (S \circ R) = (T \circ S) \circ R$, (Relation composition is associative)

(2) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

<u>Proof</u>: (1): Take any element $(a,d) \in T \circ (S \circ R)$ $(\subseteq A \times D)$. By the definition of the composition of relations $S \circ R$ $(\subseteq A \times C)$ and T $(\subseteq C \times D)$, there is a $c \in C$ such that $(a,c) \in S \circ R$ and $\underbrace{(c,d) \in T}_{\text{I}}$. As $(a,c) \in S \circ R$, again by the definition of composition there is a $b \in B$ such that $\underbrace{(a,b) \in R}_{\text{II}}$ and $\underbrace{(b,c) \in S}_{\text{III}}$.

From I and III we see that $(b,d) \in T \circ S$. As $(a,b) \in R$ by II, it now follows from $(a,b) \in R$ and $(b,d) \in T \circ S$ that $(a,d) \in (T \circ S) \circ R$. Hence,

$T \circ (S \circ R) \subseteq (T \circ S) \circ R$ (by green lines). The reverse containment "$(T \circ S) \circ R \subseteq T \circ (S \circ R)$" can be proved similarly (Exercise).

(2): Note first that $(S \circ R)^{-1}$ and $R^{-1} \circ S^{-1}$ are both subsets of $C \times A$. Now,

$$
\begin{aligned}
(c,a) \in (S \circ R)^{-1} &\iff (a,c) \in S \circ R \iff (a,b) \in R \text{ and } (b,c) \in S \text{ for some } b \in B \\
&\iff (b,a) \in R^{-1} \text{ and } (c,b) \in S^{-1} \text{ for some } b \in B \\
&\iff (c,a) \in R^{-1} \circ S^{-1}
\end{aligned}
$$

Hence $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. $\square$

Exercise: Let A be the set of all people in our class. Show that each of the following relations are equivalence relations on A, and determine the equivalence classes

(1) $x R_1 y$ iff x and y were born in the same city

(2) $x R_2 y$ iff x and y are of the same age.

(3) $x R_3 y$ iff x and y are gratuated from the same high school.

Integers Modulo n, $\mathbb{Z}_n$

Fact:

Let n be a fixed positive integer. Consider the relation R on $\mathbb{Z}$ defined as follows:

$\forall x,y \in \mathbb{Z}$, $(x,y) \in R \Leftrightarrow n$ divides $x-y$ (i.e., $x-y = nk$ for some $k \in \mathbb{Z}$)

(1) R is an equivalence relation on $\mathbb{Z}$. (We usually use the notation "$x \equiv y \mod n$" for $(x,y) \in R$)

(2) For any $a \in \mathbb{Z}$, if r is the remainder of the division of a by n then $a \equiv r \mod n$

$$\left(\begin{array}{r|l} a & n \\ \hline - & q \leftarrow \text{quotient} \\ r \uparrow \text{remainder} & \end{array} \qquad \begin{array}{c} 0 \leq r < n \\ r \,\&\, n \text{ are unique} \end{array} \right)$$

(3) (The relation R is usually called the congruence relation modulo n, and the equivalence class of $a \in \mathbb{Z}$ is usually denoted by $[a]_n$) $\forall x,y \in \mathbb{Z}$,

$[x]_n = [y]_n \Leftrightarrow x \equiv y \mod n \Leftrightarrow$ The remainders of the divisions of x and y by n are the same.

In particular, $[x]_n = [r]_n$ where r is the remainder of the division of x by n.

(4) $[0]_n, [1]_n, [2]_n, \ldots, [n-1]_n$ are distinct

(5) (The quotient set $\mathbb{Z}/R$ is usually denoted by $\mathbb{Z}_n$)

$\mathbb{Z}_n = \{[0]_n, [1]_n, [2]_n, \ldots, [n-1]_n\}$ and so $|\mathbb{Z}_n| = n$.

Proof: (1): Exercise.

(2) If $\begin{array}{l|l} a & \underline{n} \\ \underline{-} & q \\ r & \end{array}$ then $a = nq + r$, so $a - r = nq$, implying that $(a, r) \in R$.

(3) Recall from a previous fact that "$[x] = [y] \iff x \in [y] \iff y \in [x] \iff xRy \iff yRx$"

So it follows that "$[x]_n = [y]_n \iff x \equiv y \mod n$". Let $\begin{array}{l|l} x & \underline{n} \\ \underline{-} & q_1 \\ r_1 & \end{array}$ & $\begin{array}{l|l} y & \underline{n} \\ \underline{-} & q_2 \\ r_2 & \end{array}$ where $0 \leq r_1, r_2 < n$. Then, by part (2),

$[x]_n = [r_1]_n$ and $[y]_n = [r_2]_n$. Assume that $r_1 \leq r_2$ (One of r_1 and r_2 must be smaller or equal than the other!). Then $0 \leq r_2 - r_1 < n$. Hence, "n divides $r_2 - r_1$ iff $r_2 - r_1 = 0$". So, "$r_1 \equiv r_2 \mod n$ iff $r_1 = r_2$". The result follows.

(4) Follows from the proof of the part (3).

(5) Follows from parts (3) and (4). $\square$