Note that if there is a bijective map $f: \mathbb{N}^+ \to A$ where A is an infinite set A, then we can enumerate and count the elements of A by letting for instance $a_i := f(i)$. In this case,

$A = \{a_1, a_2, a_3, \dots, a_n, \dots\}$ so that the n^{th} element of A is a_n and we can count the elements $a_1, a_2, \dots, a_n, \dots$ of A as $1, 2, \dots, n, \dots$

<u>Definition:</u> Let A be a set.

(1) A is called <u>countable</u> if either A is finite or $A \sim \mathbb{N}^+$.

(2) A is called <u>uncountable</u> if A is not countable.

(3) Some books may prefer to say that A is <u>denumerable</u> if A is infinite and countable (that is, $A \sim \mathbb{N}^+$)

The following theorem is useful to show that a given set countable

<u>Theorem:</u> Let A be a nonempty set. The following conditions are equal:

① A is countable.

② A is a subset of a countable set.

③ There is an injective function $A \to \mathbb{N}^+$.

④ There is a surjective function $\mathbb{N}^+ \to A$.

<u>Proof:</u>

③ $\Leftrightarrow$ ④: This follows from some exercises we solved where we studied functions. Recall that:

"$f: X \to Y$ is injective $\Leftrightarrow$ f has a left inverse (i.e., there is a function $g: Y \to X$ such that $g \circ f = 1_X$) (I)

"$g: Y \to X$ is surjective $\Leftrightarrow$ g has a right inverse (i.e., there is a function $f: X \to Y$ such that $g \circ f = 1_X$) (II)

① $\Rightarrow$ ②: Follows easily because $A \subseteq A$.

② $\Rightarrow$ ③: Suppose that A is a subset of a countable set B. We have two cases to consider. B is finite or infinite.

<u>Case I</u>: B is finite. As $B \neq \emptyset$, $|B| = n$ for some $n \in \mathbb{N}^+$. So there is bijective function $f: B \to \{1, 2, \dots, n\}$. Composing this with the inclusions $\underset{x \mapsto x}{i: A \to B}$ and $\mu: \{1, 2, \dots, n\} \to \mathbb{N}^+$ we see that $\mu \circ f \circ i: A \to \mathbb{N}^+$ is an injective function.

Case II : B is infinite. In this case, $B \sim \mathbb{N}^+$ by the definition of countable set. So there is a bijective function $f: B \to \mathbb{N}^+$. Composing this with the inclusion map $\begin{array}{c} A \to B \\ x \mapsto x \end{array}$, we see that there is an injective function $A \to \mathbb{N}^+$.

③ ⇒ ①: We may assume that A is infinite because any finite set is countable. From ③ there is an injective function $f: A \to \mathbb{N}^+$. So $f: A \to f(A)$ is a bijection, implying that $f(A)$ is an infinite subset of $\mathbb{N}^+$. Using the Well Ordering Principle (i.e., any nonempty subset of $f(A)$ has the smallest element), we enumerate the elements of $f(A)$ as follows:

$m_1 :=$ the smallest element of $f(A)$

$m_2 :=$ the smallest element of $f(A) - \{m_1\}$

$m_3 :=$ the smallest element of $f(A) - \{m_1, m_2\}$

$\vdots$

$m_k :=$ the smallest element of $f(A) - \{m_1, m_2, \dots, m_{k-1}\}$

$\vdots$

Thus $f(A) = \{m_1, m_2, m_3, \dots, m_k, \dots\} \subseteq \mathbb{N}^+$ and $m_1 < m_2 < m_3 < \dots$

Note that the map $h: f(A) \to \mathbb{N}^+$ given by $h(m_k) = k$ is bijective. So the composition $h \circ f: A \to \mathbb{N}^+$ is bijective.

Hence, A is countable. □

Note that the elements of an infinite countable set A can be listed as an infinite sequence $a_1, a_2, a_3, a_4, \dots, a_n, \dots$ of distinct objects indexed by $\mathbb{N}^+$ so that $A = \{a_1, a_2, a_3, \dots\}$ because there is a bijection $f: \mathbb{N}^+ \to A$ and we may let $a_i = f(i) \in A$.

Corollary: Let A and B be sets, and $f: A \to B$ be a function.

(1) If f is injective and B is countable, then A is countable.

(2) If f is surjective and A is countable, then B is countable.

Proof: (1): As B is countable, part "① ⟺ ③" of the previous theorem gives that there is an injective map $\phi: B \to \mathbb{N}^+$. So $\phi \circ f: A \to \mathbb{N}^+$ is an injective map. Again part "① ⟺ ③" of the previous theorem implies that A is countable.

(2): Exercise. $\square$

Corollary: Any subset of a countable set is countable.

Proof: This is a restatement of part "① ⟺ ②" of the previous theorem. $\square$

Intuitively, you may think as follows. If a function $f: A \to B$ from a set A to a set B is injective (respectively, surjective), then $|A| \leq |B|$ (respectively, $|A| \geq |B|$), even the case that the sets A and B are infinite. And any set whose cardinality is smaller than or equal to the cardinality of a countable set is countable.

In part (6) of the previous example we saw by using Schröder-Berstein Theorem that $\mathbb{Q} \sim \mathbb{N}^+$. Therefore, $\mathbb{Q}$ is an infinite countable set.

Theorem: $\mathbb{Q}$ is countable.

Proof: We proved this in part (6) of the previous example: we proved there must be a bijective function $\mathbb{Q} \to \mathbb{N}^+$ without giving an explicit bijection. Here we want to give an explicit by bijection $\mathbb{Q}^+ \to \mathbb{N}^+$ which is due to Cantor. We first enumerate positive rational numbers in an infinite array whose rows and columns are indexed by positive integers as follows:

$$
\begin{array}{cccccccc}
1/1 & 1/2 & 1/3 & 1/4 & 1/5 & \cdots & 1/m & \cdots \\
2/1 & 2/2 & 2/3 & 2/4 & 2/5 & \cdots & 2/m & \cdots \\
3/1 & 3/2 & 3/3 & 3/4 & 3/5 & \cdots & 3/m & \cdots \\
4/1 & 4/2 & 4/3 & 4/4 & 4/5 & \cdots & 4/m & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\
n/1 & n/2 & n/3 & n/4 & n/5 & \cdots & n/m & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots &
\end{array}
$$

It is clear that every element of $\mathbb{Q}^+$ appears somewhere in the above array. Indeed, a positive rational number appears infinitely many times, for instance, $1/2 = 2/4 = 3/6 = \cdots$ appears in each row once. We can enumerate (i.e., list) distinct elements of the above array (i.e., elements of $\mathbb{Q}^+$) by following the red directed curve. But if a rational number appeared before we omit the new appearence of the number:

$$
\begin{array}{lcccccccccccccc}
\mathbb{Q}^+: & 1/1, & 2/1, & 1/2, & 1/3, & \cancel{2/2}, & 3/1, & 4/1, & 3/2, & 2/3, & 1/4, & 1/5, & \cancel{2/4}, & \cdots & \\
& \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \cdots & \downarrow \\
\mathbb{N}^+: & 1 & 2 & 3 & 4 & & 5 & 6 & 7 & 8 & 9 & 10 & & &
\end{array}
$$

The red arrows define a bijection $f: \mathbb{Q}^+ \to \mathbb{N}^+$. For instance,

$f(2/3)=8$. It is now easy to modify f to write an explicit bijection $F: \mathbb{Q} \to \mathbb{N}^+$. For instance, we may let

$$F(q) = \begin{cases} 2f(q), & \text{if } q>0 \\ 1, & \text{if } q=0 \\ 2f(-q)+1, & \text{if } q<0 \end{cases}$$

As an exercise prove that $F: \mathbb{Q} \to \mathbb{N}^+$ is bijective. $\square$

<u>Theorem</u> (Cantor)

(1) The interval $(0,1)$ is uncountable

(2) $\mathbb{R}$ is uncountable.

<u>Proof:</u>

(1): The proof we will give is due to Cantor, and known as "Cantor's Diagonalization Argument". Suppose for a contradiction that $(0,1)$ is countable. So we can list real numbers in $(0,1)$ as an infinite sequence $x_1, x_2, x_3, \dots, x_n, \dots$. Each real number x_i has a unique decimal expansion $x_i = 0.x_{i,1}\, x_{i,2}\, x_{i,3}\, x_{i,4} \cdots$ where (For instance, $1/2 = 0.50000\cdots$, $1/3 = 0.3333\cdots$, $\sqrt{2/3} = 0.8164\dots$) $x_{i,n}$ denotes the nth decimal digit of x_i. Now, consider the real number defined by

$$d = 0.d_1 d_2 d_3 \cdots d_n \cdots \quad \text{where} \quad d_i = \begin{cases} 1, & \text{if } x_{i,i} \neq 1 \\ 7, & \text{if } x_{i,i} = 1 \end{cases}$$

Note that $d \in (0,1)$. As $d_i \neq x_{i,i}$ for each i, the ith decimal digits of d and x_i are different. So $d \neq x_i$ for all i. This is a contradiction, because $d \in (0,1)$ implies that d must be in the list $x_1, x_2, x_3, \dots, x_n, \dots$ (Note that if we write the decimal digits of $x_1, x_2, x_3, \dots, x_n, \dots$ as the rows of an array, then letting f be the real number whose decimal digits are the diagonal entries of the array we choose d in such a way that kth decimal digits of f and d are different for all k

$$f = 0.$$

$$\begin{aligned}
x_1 &= 0.\, x_{1,1}\; x_{1,2}\; x_{1,3}\; x_{1,4}\; x_{1,5} \cdots \\
x_2 &= 0.\, x_{2,1}\; x_{2,2}\; x_{2,3}\; x_{2,4}\; x_{2,5} \cdots \\
x_3 &= 0.\, x_{3,1}\; x_{3,2}\; x_{3,3}\; x_{3,4}\; x_{3,5} \cdots \\
x_4 &= 0.\, x_{4,1}\; x_{4,2}\; x_{4,3}\; x_{4,4}\; x_{4,5} \cdots \\
x_5 &= 0.\, x_{5,1}\; x_{5,2}\; x_{5,3}\; x_{5,4}\; x_{5,5} \cdots \\
&\vdots
\end{aligned}$$

).

(2): As a subset of a countable set is countable and $(0,1) \subseteq \mathbb{R}$, it follows from part (1) that $\mathbb{R}$ is uncountable. □

Intiutively we may consider countable sets as being small even they are infinite. Finite sets and countable sets share many common properties

Theorem:

(1) Every subset of a countable set is countable.

(2) The intersection of a countable set with any set is countable. That is, if A is countable set then $A \cap B$ is countable for any set B.

(3) The union of countably many countable sets is countable. That is, Let $\{A_i \mid i \in I\}$ be a family of sets. If each A_i is countable and the index set I is countable, then the union $\bigcup_{i \in I} A_i$ is countable.

(4) The cartesian product of finitely many countable sets is countable. That is, if $A_1, A_2, \dots, A_n$ are finitely many countably sets, then the set $A_1 \times A_2 \times \dots \times A_n$ is countable.

Proof:

(1): This is just a restatement of a previous result.

(2): As $A \cap B$ is a subset of A and A is countable, $A \cap B$ is countable by part (1).

(3): Let $\{A_i \mid i \in I\}$ be a family of sets such that each A_i is countable and I is countable. As I is countable, we may write $I = \{i_1, i_2, i_3, \dots\}$ (i.e., we may list elements of I as a sequence $i_1, i_2, i_3, \dots$, and it is possible that I is finite). We will change the sets A_i to pairwisely disjoint sets without changing their union. Consider the sets B_{i_k} defined as follows:

$$B_{i_1} = A_{i_1}$$

$$B_{i_2} = A_{i_2} - A_{i_1}$$

$$B_{i_3} = A_{i_3} - (A_{i_1} \cup A_{i_2})$$

$$\vdots$$

$$B_{i_k} = A_{i_k} - (A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_{k-1}})$$

$$\vdots$$

By their constructions we easily see that B_{i_k}'s are pairwisely disjoint and $\left(\bigcup_{i \in I} A_i =\right) \bigcup_k A_{i_k} = \bigcup_k B_{i_k}$.

As A_{i_k} is countable and $B_{i_k} \subseteq A_{i_k}$, it follows from part (1) that B_{i_k} is countable. Hence there is an injective function $f_{i_k}: B_{i_k} \longrightarrow \mathbb{N}^+$ for each k. Consider now the function

$$\psi: \bigcup_k B_{i_k} \longrightarrow \mathbb{N}^+ \times \mathbb{N}^+, \quad \psi(x) = \left(i_x, f_{i_x}(x)\right)$$

where i_x is the unique index such that $x \in B_{i_x}$. Note that as B_{i_k} are pairwisely disjoint, any $x \in \bigcup_k B_{i_k}$ is in exactly one of the sets $B_{i_1}, B_{i_2}, B_{i_3}, \ldots$ We may easily check that ψ is injective (exercise). As $\mathbb{N}^+ \times \mathbb{N}^+$ is countable (because $\mathbb{N}^+ \times \mathbb{N}^+ \sim \mathbb{N}^+$ by a previous example) and as there is an injective map $\bigcup_k B_{i_k} \longrightarrow \mathbb{N}^+ \times \mathbb{N}^+$, it follows from a previous corollary that $\bigcup_{i \in I} A_i = \bigcup_k B_{i_k}$ is countable.

Second proof: As each A_{i_k} is countable, we may write

$$A_{i_k} = \{a_{i_k}^1, a_{i_k}^2, a_{i_k}^3, \ldots\}$$

, i.e., we may list its elements as a sequence (possibly finite)

We may write elements of A_{i_k} as rows an array as follows:

$$\begin{matrix} a_{i_1}^1 & a_{i_1}^2 & a_{i_1}^3 & a_{i_1}^4 & a_{i_1}^5 & \cdots \\ a_{i_2}^1 & a_{i_2}^2 & a_{i_2}^3 & a_{i_2}^4 & a_{i_2}^5 & \cdots \\ a_{i_3}^1 & a_{i_3}^2 & a_{i_3}^3 & a_{i_3}^4 & a_{i_3}^5 & \cdots \\ a_{i_4}^1 & a_{i_4}^2 & a_{i_4}^3 & a_{i_4}^4 & a_{i_4}^5 & \cdots \\ & & \vdots & & & \end{matrix}$$

Each element of $\bigcup_{i \in I} A_i$ appears somewhere in the above array, possibly appears more than once. Following the red arrows we may list the elements of array as follows:

$$\begin{array}{ccccccc} & a_{i_1}^{1}, & a_{i_1}^{2}, & a_{i_2}^{1}, & a_{i_1}^{3}, & a_{i_2}^{2}, & a_{i_3}^{1}, \dots \\ f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \quad \dots \\ & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

We may define a function $f: \bigcup_k A_{i_k} \to \mathbb{N}^+$ as indicated above. Note that f is injective. So $\bigcup_k A_{i_k}$ is countable.

(4): Let $A_1, A_2, \dots, A_n$ be finitely many countable sets. We will prove by induction on n that $A_1 \times A_2 \times \cdots \times A_n$ is countable. The key result that will be used in the proof is that "cartesian product two countable sets is countable". Let us prove this key result first:

"If U and V are both countable sets, then $U \times V$ is countable"

Proof: There are injective maps $f: U \to \mathbb{N}^+$ and $g: V \to \mathbb{N}^+$. The map $\phi: U \times V \to \mathbb{N}^+ \times \mathbb{N}^+$ given by $\phi(u,v) = (f(u), g(v))$ is injective. As $\mathbb{N}^+ \times \mathbb{N}^+ \sim \mathbb{N}^+$ by a previous example, there is a bijective map $\psi: \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$. Now the composition $\psi \circ \phi: U \times V \to \mathbb{N}^+$ is injective. So $U \times V$ is countable.

Now let us turn back to the induction. The result is true for $n=1$ because A_1 is countable. Assume that the result is true for $n=k$. So we assumed that $A_1 \times A_2 \times \dots \times A_k$ is countable. We want to show that the result is true for $n=k+1$. That is, we want to show that $A_1 \times A_2 \times \dots \times A_{k+1}$ is countable. Note that

$$A_1 \times A_2 \times \dots \times A_{k+1} = \underbrace{(A_1 \times A_2 \times \dots \times A_k)}_{\text{countable by the induction hypothesis}} \times \underbrace{A_{k+1}}_{\text{countable}} = \text{the cartesian product of two countable sets}$$

which is countable by the key result above. $\square$

<u>Remark:</u> "The countability of the cartesian product of two countable sets" may also be proved by an "array argument". Let A and B be countable sets. We may write

$$A=\{a_1, a_2, a_3, \dots\} \quad \text{and} \quad B=\{b_1, b_2, b_3, \dots\}$$

(i.e., we may write elements of A and B as sequences because they are countable). It is possible that A or B (or both) are finite. We may list the elements of $A\times B$ as the following array:

$$\begin{array}{cccccc}
(a_1,b_1) & (a_1,b_2) & (a_1,b_3) & (a_1,b_4) & (a_1,b_5) & \cdots \\
(a_2,b_1) & (a_2,b_2) & (a_2,b_3) & (a_2,b_4) & (a_2,b_5) & \cdots \\
(a_3,b_1) & (a_3,b_2) & (a_3,b_3) & (a_3,b_4) & (a_3,b_5) & \cdots \\
(a_4,b_1) & (a_4,b_2) & (a_4,b_3) & (a_4,b_4) & (a_4,b_5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Using the red arrows we may list the elements of the array as

$$\begin{array}{ccccccc}
(a_1,b_1), & (a_1,b_2), & (a_2,b_1), & (a_1,b_3), & (a_2,b_2), & (a_3,b_1), & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & 2 & 3 & 4 & 5 & 6 &
\end{array}$$

and this allows us to define an injective function $A\times B \to \mathbb{N}^+$ as indicated above.

<u>Ex</u>: Let $A\subseteq B$. Show that if B is uncountable but A is countable, then $B-A$ is uncountable.

<u>Sol</u>: Note that $B = A\cup(B-A)$. If $B-A$ were countable, then it would imply that B is countable too because the union of countably many countable sets is countable and B is the union of A and $B-A$.

Theorem: Any infinite set contains an infinite countable subset.

Proof: Let A be an infinite set. As A is infinite, "$A-X \neq \emptyset$ for any finite subset X of A". So we may take an element of $A-X$ for any finite subset X of A. We successively take elements as follows:

Take any $a_1 \in A$, take any $a_2 \in A-\{a_1\}$, take any $a_3 \in A-\{a_1, a_2\}$, ...,

take any $a_k \in A-\{a_1, a_2, a_3, \dots, a_{k-1}\}$, ...

As a_i are distinct, $i \mapsto a_i$ defines a bijection $\mathbb{N}^+ \to \{a_1, a_2, \dots, a_k, \dots\}$

Thus $\{a_1, a_2, \dots, a_k, \dots\}$ is an infinite countable subset of A. $\square$

Ex (Shifting Elements of Countable Sets)

(1) Let $A = \{a_1, a_2, a_3, \dots, a_n, \dots\}$ be an infinite countable set and B be a finite subset of A. Construct a bijection $A-B \to A$.

Sol: The numbering of elements of A helps us to order elements of of any nonempty subset of A. For instance, for $\{a_5, a_7, a_{15}, a_{20}\}$ we say that its $1^{st}, 2^{nd}, 3^{rd}, 4^{th}$ elements are a_5, a_7, a_{15}, a_{20}.

Now it is clear that the map $A-B \to A$, sending the k^{th} element of $A-B$ to the k^{th} element of A for each k, is a bijection.

(2) Construct a bijection $\mathbb{N}-\{5,6,9\} \to \mathbb{N}$

Sol:

$$\begin{array}{rcccccccccccccc}
\mathbb{N}-\{5,6,9\} = & 0, & 1, & 2, & 3, & 4, & 7, & 8, & 10, & 11, & 12, & 13, & 14, & \dots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbb{N} = & 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & \dots
\end{array}$$

$\underbrace{7, 8, 9, 10, 11, \dots}_{x \mapsto x-3 \text{ for } x \geq 10}$, so we shift the elements 3 units

For instance, $f(x) = \begin{cases} x & \text{if } x=0,1,2,3,4 \\ 5 & \text{if } x=7 \\ 6 & \text{if } x=8 \\ x-3 & \text{if } x \geq 10 \end{cases}$ is a bijection $\mathbb{N}-\{5,6,9\} \to \mathbb{N}$

The previous theorem and the example help us to construct explicit bijections between some sets. See the next example.

Ex:

(1) Let $B \subseteq A$ be sets. Suppose that B is countable and $A-B$ is infinite. Construct a bijection $A-B \to A$

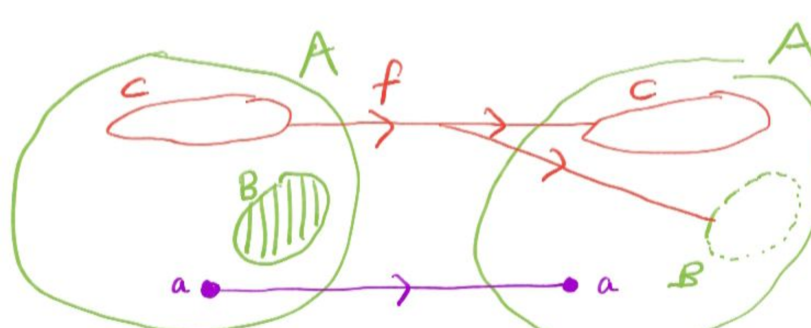
Sol: As $A-B$ is infinite, it follows from the previous theorem that $A-B$ contains an infinite countable subset $C=\{c_1, c_2, c_3, \dots\}$. As B is countable, either B is finite or $B \sim \mathbb{N}^+$.

Case I: B is finite. Then $B=\{b_1, b_2, \dots, b_m\}$ for some m. Note that $C \subseteq A-B$ and $(A-B)-C = A-(B\cup C)$. By shifting elements we may find a bijection $f: C \to B\cup C$ because C and $B\cup C$ are both infinite countable sets. If such an f is constructed, then the map $\phi: A-B \to A$ defined by

$$\phi(x) = \begin{cases} f(x), & \text{if } x \in C \\ x, & \text{if } x \in (A-B)-C \end{cases}$$

will be a desired bijection. Note that we may define f, for instance, as

$$f: \left\{ \begin{array}{c} c_1 \mapsto b_1 \\ c_2 \mapsto b_2 \\ \vdots \\ c_m \mapsto b_m \\ \hline c_{m+1} \mapsto c_1 \\ c_{m+2} \mapsto c_2 \\ \vdots \\ c_{m+k} \mapsto c_k \\ \vdots \end{array} \right\}$$



Case II : B is infinite. Then $B = \{b_1, b_2, b_3, \cdots\}$ as B is countable. Note that $C \subseteq A-B$ and $(A-B)-C = A-(B \cup C)$. As in the previous, if we can find a bijection $f: C \to B \cup C$, which is possible because C and $B \cup C$ are both infinite countable sets, then the map $\psi: A-B \to A$ given by

$$\psi(x) = \begin{cases} f(x), & \text{if } x \in C \\ x, & \text{if } x \in A-(B \cup C) \end{cases}$$

will be a bijection. We may define f, for instance, as

$$f: \left\{ \begin{array}{l} c_{2k} \mapsto c_k \\ c_{2k-1} \mapsto b_k \end{array} \right\}$$
for each k

---

(2) Construct a bijection $[0,1] \to (0,1)$

Sol: Note that $(0,1) \subseteq [0,1]$. If we take an infinite countable subset $C \subseteq (0,1)$, then as $[0,1] - (C \cup \{0,1\}) = (0,1) - C$ we can construct a bijection $\psi: [0,1] \to (0,1)$ by

$$\psi(x) = \begin{cases} f(x), & \text{if } x \in C \cup \{0,1\} \\ x, & \text{if } x \in [0,1] - (C \cup \{0,1\}) \end{cases}$$

where $f: C \cup \{0,1\} \to C$ is a bijection, which can be constructed by shifting elements of $C \cup \{0,1\}$.

We may take, for instance, $C = \left\{ \frac{1}{n+1} \,\middle|\, n = 1,2,3,4,\dots \right\}$ which is an infinite countable subset of $(0,1)$. Then f can be defined as

$$\begin{array}{rl} C \cup \{0,1\} = & \{0, 1, 1/2, 1/3, 1/4, 1/5, 1/6, \cdots\} \\ & \quad\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow \\ C = & \{1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, \cdots\} \end{array} \quad \Bigg| \quad f: \left\{ \begin{array}{l} 0 \mapsto 1/2 \\ 1 \mapsto 1/3 \\ 1/n+1 \mapsto 1/n+3 \end{array} \right\}$$

Now the map $\psi:[0,1]\to(0,1)$ defined by

$$\psi(x)=\begin{cases} 1/2, & \text{if } x=0\\ 1/3, & \text{if } x=1\\ 1/n+2, & \text{if } x=\frac{1}{n+1} \text{ for some } n\in\mathbb{N}^+\\ x, & \text{otherwise}\end{cases}$$

is a bijection. $\circ$

(3) Construct a bijective map $\mathbb{R}-\mathbb{N}\to\mathbb{R}$

Sol: We imitate part (1). As $\mathbb{R}-\mathbb{N}\subseteq\mathbb{R}$, we take an infinite countable subset C of $\mathbb{R}-\mathbb{N}$. As $(\mathbb{R}-\mathbb{N})-C=\mathbb{R}-(\mathbb{N}\cup C)$, we first construct a bijection $f: C\to\mathbb{N}\cup C$.

We may take, for instance, $C=\{-1,-2,-3,\ldots,-n,\ldots\}$ which is an infinite countable subset of $\mathbb{R}-\mathbb{N}$. Now, $\mathbb{N}\cup C=\mathbb{Z}$ and we may let, for instance, $f: C\to\mathbb{N}\cup C$ be the map

$$f(-n)=\begin{cases} -k & \text{if } n=2k\\ k-1 & \text{if } n=2k-1\end{cases}$$

which is a bijection.

$$\begin{array}{cccccccccccccc} -1, & -2, & -3, & -4, & -5, & -6, & -7, & -8, & -9, & -10, & -11, & \cdots & : & C\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \downarrow f\\ 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 4 & -5 & 5 & \cdots & : & \mathbb{N}\cup C\end{array}$$

Thus the map $\psi:\mathbb{R}-\mathbb{N}\longrightarrow\mathbb{R}$ defined by

$$\psi(x)=\begin{cases} f(x) & \text{if } x\in C\\ x & \text{if } x\notin C\end{cases} = \begin{cases} -k, & \text{if } x=-2k\ \exists k\in\mathbb{N}^+\\ k-1, & \text{if } x=-(2k-1)\ \exists k\in\mathbb{N}^+\\ x, & \text{if } x\notin\mathbb{Z}\end{cases}$$

is a bijection. $\circ$

Ex: Let A be a set. Show that A is infinite iff there is a proper subset B of A (i.e., $B \subseteq A$ and $B \neq A$) such that $B \sim A$

Sol: ($\Rightarrow$): Suppose that A is infinite. There is an infinite countable subset $C = \{c_1, c_2, c_3, \dots\}$ of A. Put $B = A - \{c_1\}$. As $B - \{c_2, c_3, c_4, \dots\} = A - \{c_1, c_2, c_3, \dots\}$, we may argue as in the previous example to construct a bijection $\psi: B \to A$. For this we must first construct a bijection $f: \{c_2, c_3, c_4, \dots\} \to \{c_1, c_2, c_3, \dots\}$. The obvious choice for f is to define $f(c_k) = c_{k-1}$ for each $k = 2, 3, 4, \dots$

Then $\psi: B \to A$ given by

$$\psi(b) = \left\{\begin{array}{ll} f(b), & \text{if } b \in \{c_2, c_3, c_4, \dots\} \\ b, & \text{if } b \notin \{c_2, c_3, c_4, \dots\} \end{array}\right\} = \begin{cases} c_{k-1}, & \text{if } b = c_k \ \exists k \geq 2 \\ b, & \text{otherwise} \end{cases}$$

is a bijection.

($\Leftarrow$): Let $B \subseteq A$ and $B \neq A$ and $B \sim A$. Suppose for a contradiction that A is finite. Then $|A| = |B| = n$ for some positive integer n (Note that as A has a proper subset, A cannot be the empty set). But then $|A - B| = 0$, implying that $A = B$. This is a contradiction, because B is supposed to be a proper subset of A.

Let $A \subseteq B$ be sets such that B is uncountable but A is countable. From a previous example we know that $B - A$ is uncountable. As any finite set is countable, any uncountable set is infinite. In particular, $B - A$ is infinite. Now, $A \subseteq B$ and A is countable and $B - A$ is infinite. It then follows from another previous example that $B - A \sim B$ (indeed, in the mentioned example we learned how to construct a specific bijection $B - A \to B$). In particular, $\mathbb{R}-\mathbb{Q} \sim \mathbb{R}$. The elements of $\mathbb{R}-\mathbb{Q}$ are called irrational real numbers. "$\mathbb{R}-\mathbb{Q} \sim \mathbb{R}$" implies that there are uncountably many irrational real numbers.

Exercise: Show that $\sqrt{2}$ is irrational. More generally, for any prime number p show that $\sqrt{p}$ is irrational.

Cardinality of power sets

Theorem (Cantor): Let A be any set. There is not any surjective function $f: A \to \mathcal{P}(A)$ where $\mathcal{P}(A)$ denotes the power set of A.

Proof: Suppose for a moment that there is a surjective function $f: A \to \mathcal{P}(A)$. Consider the following element of $\mathcal{P}(A)$ (i.e, subset of A)

$$B = \{ a \in A \mid a \notin f(a) \}$$

a subset of A

As $f: A \to \mathcal{P}(A)$ is onto and $B \in \mathcal{P}(A)$, there is an $a_0 \in A$ such that $f(a_0) = B$. As $B \subseteq A$ and $a_0 \in A$, we must have either $a_0 \in B$ or $a_0 \notin B$. But we will see that each of these two cannot be true.

Case I: Assume $a_0 \in B$. By the definition of the set, $a_0 \notin f(a_0)$. As $f(a_0) = B$, we see that $a_0 \notin B$. A contradiction.

Case II: Assume $a_0 \notin B$. As $f(a_0) = B$, this implies that $a_0 \notin f(a_0)$. But then it follows from the definition of B that $a_0 \in B$. A contradiction.

$\square$

Let A and B be two sets (possibly infinite). It is reasonable to say that the cardinality of A is less than or equal to the cardinality of B if there is a subset X of B such that X and A have the same cardinality (i.e., there is a bijection $A \to X$). It is clear that "there is a bijection from A to a subset X of B iff there is an injection from A to B" (Exercise). Therefore, the following is quite natural.

Definition: Let A and B be sets.

(1) We say that the cardinality of A is less than or equal to the cardinality of B, and we write $|A| \leq |B|$, if there is an injective function $A \to B$ (or equivalently there is a surjective function $B \to A$).

(2) Instead of saying the cardinality of A is less than or equal to the cardinality of B, we may say that the cardinality of B is greater than or equal to the cardinality of A, and write $|B| \geq |A|$.

(3) The notation $|A| < |B|$ means that $|A| \leq |B|$ but $|A| \neq |B|$. So $|A| < |B|$ means that there is an injective function $A \to B$ but there is no bijective function $A \to B$.

Remark: Let A and B be sets. It follows from Schröder-Bernstein Thm that "$|A| \leq |B|$ and $|B| \leq |A| \Rightarrow |A| = |B|$"

The consistency of the parts of the previous definition comes from an exercise we solved where we studied functions: The result says that "$f: A \to B$ is injective $\Leftrightarrow$ there is a $g: B \to A$ such that $g \circ f = 1_A$", and "$g: B \to A$ is surjective $\Leftrightarrow$ there is an $f: A \to B$ such that $g \circ f = 1_A$".

Remark: Let A be any set (possibly infinite), and $\mathcal{P}(A)$ be the power set of A. It is clear that the map $A \to \mathcal{P}(A)$, given for any $a \in A$ by $a \mapsto \{a\}$, is injective. So $|A| \leq |\mathcal{P}(A)|$. On the other hand, the previous theorem due to Cantor implies that $|A| \neq |\mathcal{P}(A)|$. Thus $|\mathcal{P}(A)| > |A|$, i.e., the cardinality of $\mathcal{P}(A)$ is strictly greater than the cardinality of A. Using this we may obtain infinite cardinalities strictly bigger than any given infinite cardinality. For instance, $|\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \cdots\cdots$

Theorem (Cantor): $\mathcal{P}(\mathbb{N}^+) \sim \mathbb{R}$. That is $|\mathcal{P}(\mathbb{N}^+)| = |\mathbb{R}|$

Proof: (We will find injective maps $\mathcal{P}(\mathbb{N}^+) \to \mathbb{R}$ and $\mathbb{R} \to \mathcal{P}(\mathbb{N}^+)$ from which the result follows by Schröder Bernstein Theorem).

Let $f: \mathcal{P}(\mathbb{N}^+) \to \mathbb{R}$ be the map defined for any $A \in \mathcal{P}(\mathbb{N}^+)$ by $f(A) = 0.d_1^A d_2^A \cdots d_n^A \cdots \in (0,1)$ where the n^{th} decimal digit of $f(A)$ is given by $d_n^A = \begin{cases} 2, & \text{if } n \in A \\ 3, & \text{if } n \notin A \end{cases}$

For instance, if $A = \{1\}$ then $f(A) = 0.2333\cdots$, and if $A = \{4,7\}$ then $f(A) = 0.333233233\cdots$. It is clear that f is injective.

Consider now the map $g: \mathbb{R} \to \mathcal{P}(\mathbb{Q})$ defined for any $x \in \mathbb{R}$ by $g(x) = \{q \in \mathbb{Q} \mid q < x\}$. We claim that g is injective: Suppose for a moment that $g(x) = g(y)$ for some $x, y \in \mathbb{R}$ such that $x \neq y$. Assume $x < y$. We may find a rational number p such that $x < p < y$. Then, $p \in g(y)$ but $p \notin g(x)$. Hence $g(x) \neq g(y)$, a contradiction.

As $\mathbb{Q} \sim \mathbb{N}^+$, we may find a bijective map $\mathbb{Q} \to \mathbb{N}^+$ and so find a bijective map $h: \mathcal{P}(\mathbb{Q}) \to \mathcal{P}(\mathbb{N}^+)$ (Exercise).

Now the composition $h \circ g: \mathbb{R} \to \mathcal{P}(\mathbb{N}^+)$ is an injective map. $\square$