Reduced Row-Echelon Matrices

Recall that an echelon matrix E is one that has the following two properties:

1. Every all-zero row of E lies beneath every row that contains a nonzero element.
2. The leading nonzero entry in each row lies to the right of the leading nonzero entry in the preceding row.

(It follows from Properties 1 and 2 that the elements beneath any leading entry in the same column are all zero.)

DEFINITION Reduced Echelon Matrix

A **reduced echelon matrix E** is an echelon matrix that has—in addition to Properties 1 and 2—the following properties:

3. Each leading entry of **E** is 1.

4. Each leading entry of **E** is the only nonzero element in its column.

Main Differences

	Echelon Matrix	Reduced Echelon Matrix
Leading entry	a nonzero number	1
Column of leading entry	below consists of 0s	below and above consists of 0s

The following matrices are reduced echelon matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

The echelon matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are not in *reduced* echelon form, because **A** does not have Property 3 and **B** does not have Property 4. ■

Reduced Row-Echelon Matrices

Example: Solve the system

$$\begin{aligned} x_1 + 2x_2 + \ \ x_3 &= \ \ 4 \\ 3x_1 + 8x_2 + 7x_3 &= 20 \\ 2x_1 + 7x_2 + 9x_3 &= 23, \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 \to (-2)R_3 + R_2]{R_1 \to (3)R_3 + R_1} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

$x_1 = 5$

$x_2 = -2$

$x_3 = 3$

$\{(5, -2, 3)\}$.

Use Gauss-Jordan elimination to solve the linear system

$$\begin{aligned} x_1 + \ x_2 + \ x_3 + \ x_4 &= 12 \\ x_1 + 2x_2 \qquad\quad + 5x_4 &= 17 \\ 3x_1 + 2x_2 + 4x_3 - \ x_4 &= 31, \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 12 \\ 1 & 2 & 0 & 5 & 17 \\ 3 & 2 & 4 & -1 & 31 \end{array}\right] \xrightarrow[R_3 \to (-3)R_1 + R_3]{R_2 \to (-1)R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & -1 & 1 & -4 & -5 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_2]{R_1 \to (-1)R_2 + R_1} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 7 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

$$\begin{aligned} x_1 \quad + 2x_3 - 3x_4 &= 7 \\ x_2 - x_3 + 4x_4 &= 5 \end{aligned}$$

Leading variables: x_1, x_2

Free variables: x_3, x_4

$x_3 = s$, $x_4 = t$.

$x_1 + 2x_3 - 3x_4 = 7$ $\qquad$ $x_3 = s$

$x_2 - x_3 + 4x_4 = 5$ $\qquad$ $x_4 = t$

$x_2 - x_3 + 4x_4 = 5 \Rightarrow x_2 = 5 - 4t + s$

$x_1 + 2x_3 - 3x_4 = 7 \Rightarrow x_1 = -2s + 3t + 7$

$\{(-2s + 3t + 7,\ 5 - 4t + s,\ s,\ t)\}.$

Homogeneous Systems

The linear system is called **homogeneous** provided that the constants b_1, b_2, $\ldots$, b_m on the right-hand side are all zero. Thus a homogeneous system of m equations in n variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Every homogeneous system obviously has at least the **trivial solution**

$$x_1 = 0, \quad x_2 = 0, \ldots, x_n = 0.$$

THEOREM 3 Homogeneous Systems with More Variables than Equations

Every homogeneous linear system with more variables than equations has infinitely many solutions.

The homogeneous linear system

$$
\begin{aligned}
47x_1 - 73x_2 + 56x_3 + 21x_4 &= 0 \\
19x_1 + 81x_2 - 17x_3 - 99x_4 &= 0 \\
53x_1 + 62x_2 + 39x_3 + 25x_4 &= 0
\end{aligned}
$$

of three equations in four unknowns necessarily has infinitely many solutions.

The situation is different for a *nonhomogeneous* system with more variables than equations. The simple example

$x_1 + x_2 + x_3 =$
$2x_1 + 2x_2 + 2x_3 = 2$

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
x_1 + x_2 + x_3 &= 1
\end{aligned}
\rightarrow
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
0 &= 1
\end{aligned}
$$

shows that such a system may be inconsistent. $\rightarrow$ $\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 0 &= 0 \end{aligned}$

Every nonhomogeneous system with more variables than equations either has no solution or has infinitely many solutions .

Equal Numbers of Equations and Variables

An especially important case in the theory of linear systems is that of a homogeneous system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\
&\;\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= 0
\end{aligned} \tag{8}
$$

with the *same* number n of variables and equations. The coefficient matrix $\mathbf{A} = [a_{ij}]$ then has the same number of rows and columns and thus is an $n \times n$ **square matrix**.

Here we are most interested in the situation when (8) has *only* the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. This occurs if and only if the reduced echelon system contains *no* free variables. That is, all n of the variables $x_1, x_2, \ldots, x_n$ must be leading variables. Because the system consists of exactly n equations, we conclude that the reduced echelon system is simply

$$\begin{aligned}
x_1 \qquad\qquad\qquad\qquad &= 0 \\
x_2 \qquad\qquad\qquad &= 0 \\
x_3 \qquad\qquad &= 0 \\
\ddots \qquad &\ \ \vdots \\
x_n &= 0,
\end{aligned}$$

and, therefore, that the reduced echelon form of the coefficient matrix **A** is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{9}$$

Such a (square) matrix, with ones on its **principal diagonal** (the one from upper left to lower right) and zeros elsewhere, is called an **identity matrix**

THEOREM 4 Homogeneous Systems with Unique Solutions

Let $\mathbf{A}$ be an $n \times n$ matrix. Then the homogeneous system with coefficient matrix $\mathbf{A}$ has only the trivial solution if and only if $\mathbf{A}$ is row equivalent to the $n \times n$ identity matrix.

Example:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 3x_1 + 8x_2 + 7x_3 &= 0 \\ 2x_1 + 7x_2 + 9x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 3 & 8 & 7 & 0 \\ 2 & 7 & 9 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 2 & 7 & 9 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The system has only the trivial solution $x_1 = x_2 = x_3 = 0$.

Matrix Operations

Two matrices **A** and **B** *of the same size*—the same number of rows and the same number of columns—are called **equal** provided that each element of **A** is equal to the corresponding element of **B**. Thus two matrices of the same size are equal provided they are *elementwise equal*, and we write $\mathbf{A} = \mathbf{B}$ to denote equality of the two matrices **A** and **B**.

If

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 8 \end{bmatrix},$$

then $\mathbf{A} \neq \mathbf{B}$ because $a_{22} = 6$, whereas $b_{22} = 7$, and $\mathbf{A} \neq \mathbf{C}$ because the matrices **A** and **C** are not of the same size. ■

DEFINITION Addition of Matrices

If $\mathbf{A} = [\, a_{ij} \,]$ and $\mathbf{B} = [\, b_{ij} \,]$ are matrices *of the same size*, then their **sum** $\mathbf{A}+\mathbf{B}$ is the matrix obtained by adding corresponding elements of the matrices **A** and **B**. That is,

$$\mathbf{A}+\mathbf{B} = [\, a_{ij} + b_{ij} \,], \tag{1}$$

where the notation on the right signifies that the element in the ith row and jth column of the matrix $\mathbf{A}+\mathbf{B}$ is $a_{ij} + b_{ij}$.

If

$$\mathbf{A} = \underset{2\times 3}{\begin{bmatrix} 3 & 0 & -1 \\ 2 & -7 & 5 \end{bmatrix}}, \quad \mathbf{B} = \underset{2\times 3}{\begin{bmatrix} 4 & -3 & 6 \\ 9 & 0 & -2 \end{bmatrix}}, \quad \text{and} \quad \mathbf{C} = \underset{2\times 2}{\begin{bmatrix} 3 & -2 \\ -1 & 6 \end{bmatrix}},$$

$$A+B = \begin{bmatrix} 3+4 & 0-3 & -1+6 \\ 2+9 & -7+0 & 5-2 \end{bmatrix} = \begin{bmatrix} 7 & -3 & 5 \\ 11 & -7 & 3 \end{bmatrix}$$

$A+C$ is not defined because A and C are not of the same size.

DEFINITION Multiplication of a Matrix by a Number

If $\mathbf{A} = [\, a_{ij} \,]$ is a matrix and c is a number, then $c\mathbf{A}$ is the matrix obtained by multiplying each element of $\mathbf{A}$ by c. That is,

$$c\mathbf{A} = [\, ca_{ij} \,]. \tag{2}$$

Using multiplication of a matrix by a scalar, we define the **negative** $-\mathbf{A}$ of the matrix $\mathbf{A}$ and the **difference** $\mathbf{A} - \mathbf{B}$ of the two matrices $\mathbf{A}$ and $\mathbf{B}$ by writing

$$-\mathbf{A} = (-1)\mathbf{A} \quad \text{and} \quad \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

Example:

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -7 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -3 & 6 \\ 9 & 0 & -2 \end{bmatrix}$$

$$3A = \begin{bmatrix} 9 & 0 & -3 \\ 6 & -21 & 15 \end{bmatrix} \quad -B = \begin{bmatrix} -4 & 3 & -6 \\ -9 & 0 & 2 \end{bmatrix} \quad 3A - B = \begin{bmatrix} 5 & 3 & -9 \\ -3 & -21 & 17 \end{bmatrix}$$

Vectors

A column vector (or simply vector) is merely an n×1 matrix, one having a single column

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = (a_1, a_2, \ldots, a_n).$$

A row vector is a 1×n matrix having a single row

$$[a_1 \quad a_2 \quad \cdots \quad a_n]. \neq (a_1, a_2, \ldots, a_n)$$

Now consider the linear system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

of m equations in n variables. We may regard a solution of this system as a *vector*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = (x_1, x_2, x_3, \ldots, x_n)$$

Consider the homogeneous system

$$\begin{aligned} x_1 + 3x_2 - 15x_3 + 7x_4 &= 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 &= 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 &= 0. \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -15 & 7 & 0 \\ 1 & 4 & -19 & 10 & 0 \\ 2 & 5 & -26 & 11 & 0 \end{array}\right] \xrightarrow[R_3 \to (-2)R_2 + R_3]{R_2 \to (-1)R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 3 & -15 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & -1 & 4 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_2 + R_3]{R_1 \to (-3)R_2 + R_1} \left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - 3x_3 - 2x_4 = 0$$
$$x_2 - 4x_3 + 3x_4 = 0$$

$\underbrace{x_1 \quad x_2}_{\text{leading variable}}$

free variables: x_3, x_4

$x_3 = s$, $\quad x_4 = t$

$x_1 - 3x_3 - 2x_4 = 0$, $x_3 = s$, $x_4 = t$

$x_2 - 4x_3 + 3x_4 = 0$

$x_2 = 4s - 3t$, $x_1 = 3s + 2t$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s+2t \\ 4s-3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3s \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 2t \\ -3t \\ 0 \\ t \end{bmatrix}$$

$$= s \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$= s \cdot (3,4,1,0) + t \cdot (2,-3,0,1)$$

$$X = s \cdot X_1 + t \cdot X_2$$

This equation expresses in vector form of the general solution of the system. It says that the vector x is a solution if and only if x is a linear combinations — a sum of multiples — of the particular solutions $X_1 = (3,4,1,0)$ and $X_2 = (2,-3,0,1)$

Matrix Multiplication

The first surprise is that matrices are *not* multiplied elementwise. The initial purpose of matrix multiplication is to simplify the notation for systems of linear equations. If we write

$$\mathbf{A} = \left[\, a_{ij} \,\right], \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

then $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$ are, respectively, the coefficient matrix, the unknown vector, and the constant vector for the linear system We want to define the matrix product $\mathbf{Ax}$ in such a way that the entire system of linear equations reduces to the single matrix equation

$$\mathbf{Ax} = \mathbf{b}.$$

The first step is to define the product of a *row* vector $\mathbf{a}$ and a *column* vector $\mathbf{b}$,

$$\mathbf{a} = \left[\, a_1 \quad a_2 \quad \cdots \quad a_n \,\right] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

each having n elements. In this case, the product **ab** is *defined* to be

$$\mathbf{ab} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

Thus **ab** is the *sum of products* of corresponding elements of **a** and **b**.

Example: $\begin{bmatrix} 3 & 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ -3 \\ 4 \end{bmatrix} = 3.5 + 0.2 + (-1)(-3) + 7.4 = 46$

Note that if

$a = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Then, $ax = a_1x_1 + a_2x_2 + \cdots + a_nx_n$

Hence the single equation $a x_1 + a_2x_2 + \cdots + a_nx_n = b$ reduces to the equation $ax = b$

DEFINITION Matrix Multiplication

Suppose that **A** is an $m \times p$ matrix and **B** is a $p \times n$ matrix. Then the **product** **AB** is the $m \times n$ matrix defined as follows: The element of **AB** in its ith row and jth column is the *sum of products* of corresponding elements in the ith row of **A** and the jth column of **B**.

If the ith row of A is $[a_{i1} \ a_{i2} \ \cdots \ a_{ip}]$ and jth column of B $\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$

Then the element in the ith row and jth column of AB is $a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{ip} \cdot b_{pj}$.

$C = AB$ and $C = [c_{ij}] \Rightarrow c_{ij} = \sum_{k=1}^{p} a_{ik} \cdot b_{kj}$

$$A_{m\times p} \text{ times } B_{p\times n} = AB_{m\times n}$$

$$\underbrace{p \quad p}_{\text{These "cancel"}}$$

If

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix},$$

$$AB = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}_{2\times 2} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 2.1 + (-1).3 & 2.5 + (-1).7 \\ (-4).1 + 3.3 & (-4)\,5 + 3.7 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix}$$

$$B.A = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 1.2 + 5.(-4) & 1.(-1) + 3.5 \\ 3.2 + 7(-4) & 3.(-1) + 7.3 \end{bmatrix}$$

$$= \begin{bmatrix} -18 & 14 \\ -22 & 18 \end{bmatrix}$$

$\Rightarrow \quad AB \neq BA$ This shows that the multiplication of matrices is not commutative!

Example: If A is a 3×2 matrix and B is a 2×3 matrix, then

$$A_{3\times 2} \text{ times } B_{2\times 3} = AB_{3\times 3}$$

$$B_{2\times 3} \text{ times } A_{3\times 2} = BA_{2\times 2}$$

If C is a 3×5 matrix and D is a 5×7 matrix, then

$$C_{3\times 5} \text{ times } D_{5\times 7} = CD_{3\times 7}$$

$$D_{5\times 7} \text{ times } C_{3\times 5} = \text{undefined.}$$

Matrix Equations

If $\mathbf{A} = [\, a_{ij} \,]$ is an $m \times n$ coefficient matrix and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an $n \times 1$ variable (column) matrix, then the product $\mathbf{Ax}$ is the $m \times 1$ matrix

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \overset{(?)}{=} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}.$$

We therefore see that

$$\mathbf{Ax} = \mathbf{b}$$

Matrix Equations

If $\mathbf{A} = [\, a_{ij} \,]$ is an $m \times n$ coefficient matrix and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an $n \times 1$ variable (column) matrix, then the product $\mathbf{Ax}$ is the $m \times 1$ matrix

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \overset{(?)}{=} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}.$$

We therefore see that

$$\mathbf{Ax} = \mathbf{b}$$

Matrix Algebra

The definitions of matrix addition and multiplication can be used to establish the rules of matrix algebra listed in the following theorem.

THEOREM 1 Rules of Matrix Algebra

If **A**, **B**, and **C** are matrices of appropriate sizes to make the indicated operations possible, then the following identities hold.

Commutative law of addition: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

Associative law of addition: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Associative law of multiplication: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

Distributive laws: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

and

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

1 – If a and b are real numbers, then rules such as

$$(a+b)\mathbf{C} = a\mathbf{C} + b\mathbf{C}, \quad (ab)\mathbf{C} = a(b\mathbf{C}), \quad a(\mathbf{BC}) = (a\mathbf{B})\mathbf{C}$$

2- In general $AB \neq BA$

$\mathbf{AB} \neq \mathbf{BA}$. Other exceptions are associated with zero matrices. A **zero matrix** is one whose elements are *all* zero, such as

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

We ordinarily denote a zero matrix (whatever its size) by $\mathbf{0}$. It should be clear that for any matrix $\mathbf{A}$,

$$\mathbf{0} + \mathbf{A} = \mathbf{A} = \mathbf{A} + \mathbf{0}, \quad \mathbf{A0} = \mathbf{0}, \quad \text{and} \quad \mathbf{0A} = \mathbf{0},$$

where in each case $\mathbf{0}$ is a zero matrix of appropriate size. Thus zero matrices appear to play a role in the arithmetic of matrices similar to the role of the real number 0 in ordinary arithmetic.

For real numbers, the following two rules are familiar:

- If $ab = ac$ and $a \neq 0$, then $b = c$
  (the "cancellation law").
- If $ad = 0$, then either $a = 0$ or $d = 0$.

But these rules do not generally hold for matrices!

If

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -2 & 7 \\ 3 & 1 & -1 & 5 \end{bmatrix}_{2\times 4}, \quad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 3 & -1 \\ -2 & 4 \\ 2 & -3 \end{bmatrix}_{4\times 2}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ -2 & 3 \\ 1 & -3 \end{bmatrix}_{4\times 2},$$

Then $B \neq C$. But

$$AB = \begin{bmatrix} 25 & -10 \\ 18 & -5 \end{bmatrix} = AC \qquad AB = AC$$

If $D = B - C = \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $AD = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$.

but neither A nor D is a zero matrix.