Laplace Transform Methods

Laplace Transforms and Inverse Transforms

DEFINITION The Laplace Transform

Given a function $f(t)$ defined for all $t \geqq 0$, the *Laplace transform* of f is the function F defined as follows:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \qquad (1)$$

for all values of s for which the improper integral converges.

Example: Find the Laplace transforms of the following functions:

a) $f(t) = e^{at}$ for $t \geq 0$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{(a-s)t}\, dt = \lim_{b \to \infty} \int_0^b e^{(a-s)t}\, dt$$

$$= \lim_{b\to\infty} \int_0^b e^{(a-s)t}\,dt = \lim_{b\to\infty} \begin{cases} \left.\dfrac{e^{(a-s)t}}{a-s}\right|_{t=0}^{b} &, s \neq a. \\ \left. t \right|_{t=0}^{b} &, s = a \end{cases}$$

$$= \lim_{b\to\infty} \begin{cases} \dfrac{e^{(a-s)b}}{a-s} - \dfrac{1}{a-s} &, s \neq a \\ b &, s = a \end{cases}$$

$$\mathcal{L}\{e^{at}\} = \begin{cases} \dfrac{1}{s-a} &, s > a \\ \infty &, s \leq a \end{cases}$$

Thus the improper integral converges for $s > a$.

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{for } s > a.$$

$*\ \mathcal{L}\{1\} = \frac{1}{s}$ for $s > 0$.

b) $f(t) = t^a$, where a is real and $a > -1$.

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st} t^a \, dt$$

Remark:

The Laplace transform $\mathcal{L}\{t^a\}$ of a power function is most conveniently expressed in terms of the **gamma function** $\Gamma(x)$, which is defined for $x > 0$ by the formula

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt. \tag{6}$$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \lim_{b\to\infty} \int_0^b e^{-t} dt = \lim_{b\to\infty} \left(-e^{-t}\right)\Big|_0^b$$

$$= \lim_{b\to\infty} \left(-e^{-b} + 1\right) = 1.$$

$$\Gamma(x+1) = x\,\Gamma(x).$$

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^x\, dt = \lim_{b\to\infty} \int_0^b e^{-t}\, t^x\, dt$$

$$\left[\begin{array}{l} u = t^x, \; dv = e^{-t}dt \\ du = x\,t^{x-1}dt \\ \quad v = -e^{-t} \end{array}\right]$$

$$= \lim_{b\to\infty} \left(-t^x e^{-t} \Big|_0^b + x \int_0^b e^{-t}\, t^{x-1}\, dt \right)$$

$$= \lim_{b\to\infty} \left(-b^x e^{-b} \right) + x \underbrace{\int_0^\infty e^{-t}\, t^{x-1}\, dt}_{\Gamma(x)}$$

Let $^{*}\, g(b) = b^x e^{-b}$, $\quad \ln g(b) = x\ln b - b\ln e = x\ln b - b$

$$\lim_{b\to\infty} \ln(g(b)) = \ln \lim_{b\to\infty} g(b) = \lim_{b\to\infty} (x\ln b - b) \quad (\infty - \infty)$$

$$= \lim_{b\to\infty} b\Big(x \underbrace{\frac{\ln b}{b}}_{\to 0} - 1\Big) = -\infty$$

$$* \lim_{b\to\infty} g(b) = 0$$

Hence, $\Gamma(x+1) = x\Gamma(x)$.

* If n is possitive integer, then

$$\Gamma(n+1) = n!$$

$$\begin{aligned}
\Gamma(n+1) &= n.\Gamma(n) \\
&= n.\ \Gamma((n-1)+1) \\
&= n.\ (n-1).\Gamma(n-1) \\
&= n.\ (n-1)\Gamma((n-2)+1) \\
&= n(n-1)(n-2)\Gamma(n-2) \\
&\ \vdots \\
&= n\ (n-1)(n-2)\ldots\ 2.1.\ \Gamma(1). \\
&= n.(n-1)\ \ldots\ 2.1. \\
&= n!.
\end{aligned}$$

* $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-t}\cdot t^{-1/2}\,dt=\lim_{b\to\infty}\int_0^b e^{-t}\cdot t^{-1/2}\,dt$$

$t=u^2 \quad u=\sqrt{t}$

$du=\dfrac{dt}{2\sqrt{t}}$

$t=0 \Rightarrow u=0$

$t=b \Rightarrow u=\sqrt{b}$

$$=\lim_{b\to\infty}2\int_0^{\sqrt{b}} e^{-u^2}\,du=2\cdot\underbrace{\int_0^\infty e^{-u^2}\,du}_{I}$$

$$I^2=\left(\int_0^\infty e^{-u^2}\,du\right)\left(\int_0^\infty e^{-v^2}\,dv\right)=\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv$$

$u=r\cos\theta$

$v=r\sin\theta$

$du\,dv=r\,dr\,d\theta$

$$=\int_0^{\pi/2}\int_0^\infty e^{-r^2}r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\left(\lim_{b\to\infty}\int_0^b e^{-r^2}r\,dr\right)d\theta=\int_0^{\pi/2}\lim_{b\to\infty}\left(\left(\frac{e^{-r^2}}{2}\right)\Bigg|_0^b\right)d\theta$$

$$= \int_0^{\pi/2} \lim_{b\to\infty} \left(\frac{e^{-b^2}}{2} + \frac{1}{2} \right) d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta$$

$$\Rightarrow I^2 = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}.$$

$$\rightsquigarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$\mathcal{L}\{t^a\} = \lim_{b\to\infty} \int_0^b e^{-st} t^a \, dt$$

$$\left[\begin{array}{ll} u = st \Rightarrow & du = s\,dt \\ t = \frac{u}{s},\ s>0 & dt = \frac{du}{s} \\ t=0 \Rightarrow u=0 & \\ t=b \Rightarrow u=sb & \end{array}\right]$$

$$= \lim_{b\to\infty} \int_0^{sb} e^{-u} \left(\frac{u}{s}\right)^a \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a \, du$$

$$\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}, \quad s>0.$$

Notice that if n is nonnegative integer,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad s>0 \ (\text{because } \Gamma(n+1)=n!)$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad s>0$$

$$\mathcal{L}\{t^{3/2}\} = \frac{\Gamma(3/2+1)}{s^{3/2+1}} = \frac{1}{s^{5/2}} \cdot \frac{3}{2}\Gamma(3/2)$$

$$= \frac{3}{2} \cdot \frac{1}{s^{5/2}} \cdot \Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2} \cdot \frac{1}{s^{5/2}} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{4} \cdot \frac{1}{s^{5/2}} \cdot \sqrt{\pi} \quad , \quad s > 0 .$$

THEOREM 1 Linearity of the Laplace Transform

If a and b are constants, then

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \tag{12}$$

for all s such that the Laplace transforms of the functions f and g both exist.

Example: $\mathcal{L}\{\cosh kt\} = ?$ $\quad k > 0.$

$$\mathcal{L}\{\cosh kt\} = \mathcal{L}\left\{\frac{e^{kt} + e^{-kt}}{2}\right\}$$

$$= \frac{1}{2} \cdot \mathcal{L}\{e^{kt}\} + \frac{1}{2} \cdot \mathcal{L}\{e^{-kt}\}$$

$$= \frac{1}{2} \cdot \underset{s > k}{\frac{1}{s-k}} + \frac{1}{2} \cdot \underset{s > -k}{\frac{1}{s+k}}$$

$$= \frac{1}{2} \cdot \frac{s+k+s-k}{s^2-k^2} = \frac{s}{s^2-k^2} \qquad s > k > 0.$$

similarly. $\mathcal{L}\left\{\sinh kt\right\} = \dfrac{k}{s^2-k^2} \qquad s > k > 0.$

Example: $\mathcal{L}\{\cos kt\} = ?$

$$e^{ikt} = \cos kt + i \sin kt$$

$$e^{-ikt} = \cos kt - i \sin kt$$

$$+ \overline{}$$

$$e^{ikt} + e^{-ikt} = 2\cos kt \quad \Rightarrow \quad \cos kt = \frac{e^{ikt} + e^{-ikt}}{2}$$

$$\mathcal{L}\{\cos kt\} = \mathcal{L}\left\{\frac{e^{ikt} + e^{-ikt}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{ikt}\} + \frac{1}{2}\cdot\mathcal{L}\{e^{-ikt}\}$$

$$= \frac{1}{2}\cdot\frac{1}{s-ik} + \frac{1}{2}\cdot\frac{1}{s+ik}$$

$$s > Re[ik] = 0 \qquad s > Re[-ik] = 0$$

$$= \frac{s+ik+s-ik}{2\left[s^2-(ik)^2\right]} = \frac{s}{s^2+k^2}$$

$$s > 0$$

Similarly, $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} \quad s>0.$

Example: $\mathcal{L}\{3e^{2t} + 2\sin^2 3t\} = 3\,\mathcal{L}\{e^{2t}\} + 2\,\mathcal{L}\{\sin^2 3t\}$

$$= 3 \cdot \underset{s>2}{\frac{1}{s-2}} + 2 \cdot \mathcal{L}\left\{\frac{1-\cos 6t}{2}\right\} = \frac{3}{s-2} + \mathcal{L}\{1\} - \mathcal{L}\{\cos 6t\}$$

$$= \underset{s>2}{\frac{3}{s-2}} + \underset{s>0}{\frac{1}{s}} - \underset{s>0}{\frac{s}{s^2+36}} = \frac{3s^2+144s-72}{s(s-2)(s^2+36)}, \quad s>2.$$

Theorem (Translation on the s- Axis)

If $F(s) = \mathcal{L}(f(t))$ exists for $s > c$, then

$\mathcal{L}\{e^{at} f(t)\}$ exists for $s > a+c$, and

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)..$$

Definition: Piecewise Continuous Functions

The function f(t) is said to be piecewise continuous on the bounded interval $a \le t \le b$ provided that $[a, b]$ can be subdivided into finitely many abutting subintervals in such a way that

1. f is continuous in the interior of each of these subintervals; and
2. $f(t)$ has a finite limit as t approaches each endpoint of each subinterval from its interior.

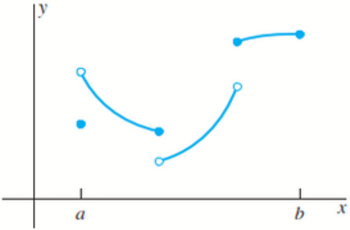


FIGURE 10.1.3. The graph of a piecewise continuous function; the solid dots indicate values of the function at discontinuities.

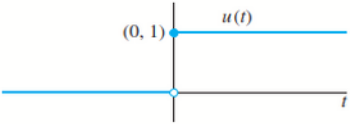


FIGURE 10.1.4. The graph of the unit step function.

Theorem 2 (Existence of Laplace Transforms)

If f is piecewise continuous and there exist nonnegative constants M, c and T such that

$$|f(t)| \leq M \cdot e^{ct} \quad \text{for } t \geq T$$

then $F(s)$ exists for all $s > c$, and $\lim_{s \to \infty} F(s) = 0$.

Example: $f(t) = \sinh kt = \dfrac{e^{kt} - e^{-kt}}{2}$, $t > 0$.

$$|f(t)| \leq \frac{1}{2}\left(e^{kt} + e^{-kt}\right) \leq \frac{1}{2}\left(e^{|k|t} + e^{|k|t}\right) = e^{|k|t}$$

$\Rightarrow M = 1, \quad c = |k|$

Thus, $F(s) = \dfrac{k}{s^2 - k^2}$ exist for all $s > |k|$

Moreover, $\lim\limits_{s \to \infty} F(s) = \lim\limits_{s \to \infty} \dfrac{k}{s^2 - k^2} = 0$

THEOREM 3 Uniqueness of Inverse Laplace Transforms

Suppose that the functions $f(t)$ and $g(t)$ satisfy the hypotheses of Theorem 2, so that their Laplace transforms $F(s)$ and $G(s)$ both exist. If $F(s) = G(s)$ for all $s > c$ (for some c), then $f(t) = g(t)$ wherever on $[0, +\infty)$ both f and g are continuous.

- Thus if $F(s)$ is the transform of some continuous function $f(t)$, then $f(t)$ is uniquely determined. This observation allows us to make the definition of inverse Laplace Transform.

Definition: If $F(s) = \mathcal{L}\{f(t)\}$, then we call $f(t)$ the inverse Laplace transform of $F(s)$ and write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Example: $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$

$* \ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

$n+1 = 5 \Rightarrow n = 4$.

Example: $\mathcal{L}^{-1}\left\{\frac{1}{s+9}\right\} = e^{-9t}$

$* \ \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$a = -9$

Example: $\mathcal{L}^{-1}\left\{\frac{2}{s^2+16}\right\} = \frac{2\sin 4t}{4}$

$* \ \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

$k = 4$ $\qquad = \frac{1}{2}\sin 4t$.

Example: Find the inverse Laplace transform of

$$R(s) = \frac{s^2+1}{s^3-2s^2-8s}$$

$$R(s) = \frac{s^2+1}{s(s+2)(s-4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-4}$$

$$s^2+1 = A\cdot(s+2)(s-4) + Bs(s-4) + Cs(s+2)$$

$s=0$: $\quad 1 = -8A \;\Rightarrow\; A = -1/8$

$s=-2$: $\quad 5 = 12B \;\Rightarrow\; B = 5/12$

$s=4$: $\quad 17 = 24C \;\Rightarrow\; C = 17/24$

$s > -2,\ a = -2 \qquad\qquad s > 4,\ a = 4$

$$\mathcal{L}^{-1}\{R(s)\} = -\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{5}{12}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{17}{24}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= -\frac{1}{8}\cdot(1) + \frac{5}{12}\cdot e^{-2t} + 17/24\, e^{4t}$$

Transformation of Initial Value Problems

We now discuss the application of Laplace transforms to solve a linear differential equation with constant coefficients

THEOREM 1 Transforms of Derivatives

Suppose that the function $f(t)$ is continuous and piecewise smooth for $t \geqq 0$ and is of exponential order as $t \to +\infty$, so that there exist nonnegative constants M, c, and T such that

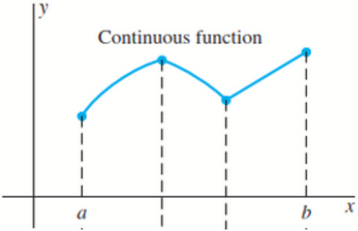
$$|f(t)| \leqq Me^{ct} \quad \text{for } t \geqq T. \tag{★}$$

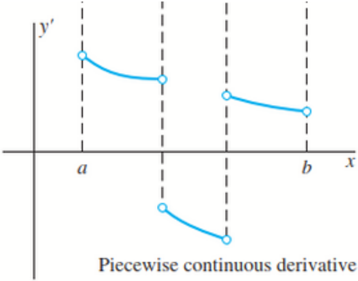
Then $\mathcal{L}\{f'(t)\}$ exists for $s > c$, and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0). \tag{4}$$

Remark

1) The function f is called **piecewise smooth** on the bounded interval $[a, b]$ if it is piecewise continuous on $[a, b]$ and differentiable except at finitely many points, with $f'(t)$ being piecewise continuous on $[a, b]$.





2) $$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]_{t=0}^{\infty} + s\int_0^\infty e^{-st} f(t)\,dt.$$

$u = e^{-st}$, $dv = f'(t)\,dt$ $\Rightarrow$ $du = -s e^{-st} dt$

$v = f(t)$

$$= \lim_{b \to \infty} \left\{ \left[e^{-st} f(t) \Big|_0^b \right] + s \int_0^b e^{-st} f(t)\, dt \right\}$$

$$= \lim_{b \to \infty} \left\{ e^{-sb} f(b) - f(0) + s \int_0^b e^{-st} f(t)\, dt \right\}$$

$$= -f(0) + s \cdot \mathcal{L}\{f(t)\}.$$

COROLLARY Transforms of Higher Derivatives

Suppose that the functions $f, f', f'', \ldots, f^{(n-1)}$ are continuous and piecewise smooth for $t \geqq 0$, and that each of these functions satisfies the conditions in (2) with the same values of M and c. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists when $s > c$, and

$$\begin{aligned}\mathcal{L}\{f^{(n)}(t)\} &= s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \\ &= s^n F(s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0). \qquad (7)\end{aligned}$$

$n = 1 \Rightarrow \mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$n = 2 \Rightarrow \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$n = 3 \Rightarrow \mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).$

Example: Solve the initial value problem

a) $x'' + 6x' + 34x = 0 \qquad x(0) = 3 \ , \quad x'(0) = 1$

$$\mathcal{L}\{x''\} + 6\cdot\mathcal{L}\{x'\} + 34\,\mathcal{L}\{x\} = 0$$

$$\left[s^2 X(s) - s\,x(0) - x'(0)\right] + 6\cdot\left[s\,X(s) - x(0)\right] + 34\,X(s) = 0$$

$$(s^2 + 6s + 34)\,X(s) - (s+6)\,x(0) - x'(0) = 0$$

$$(s^2 + 6s + 34)\,X(s) - 3(s+6) - 1 = 0 \qquad X(s) = \frac{3s+19}{s^2+6s+34}$$

$$X(s) = \frac{3s+19}{s^2+6s+34} = \frac{3s+19}{(s+3)^2+25} = \frac{3(s+3)}{(s+3)^2+25} + \frac{10}{(s+3)^2+25}$$

$$X(s)\ \frac{3(s+3)}{(s+3)^2+25} + \frac{10}{(s+3)^2+25}$$

$10 = 2.5$

$a=-3 \quad k=5 \qquad a=-3 \quad k=5$

$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2},\quad s>a$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2},\quad s>a$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 3\,.\,e^{-3t}\cos 5t + 2\,.\,e^{-3t}\sin 5t.$$

b) $x'' + 4x = \sin 3t \qquad x(0) = x'(0) = 0.$

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{\sin 3t\}$$

$\sin kt$	$\frac{k}{s^2+k^2}, \quad s > 0$

$$s^2X(s) - sX(0) - X'(0) + 4 \cdot X(s) = \frac{3}{s^2+9}$$

$$(s^2+4)\, X(s) = \frac{3}{s^2+9} \Rightarrow X(s) = \frac{3}{(s^2+4)(s^2+9)}.$$

$$X(s) = \frac{3}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$(As+B)(s^2+9) + (Cs+D)\cdot(s^2+4) = 3$$

$$\underbrace{(A+C)}_{=0} s^3 + \underbrace{(B+D)}_{=0} s^2 + \underbrace{(9A+4C)}_{=0} s + \underbrace{(9B+4D)}_{=3} = 3$$

$$A + C = 0 \qquad B + D = 0 \qquad 9A + 4C = 0 \qquad 9B + 4D = 3$$

$$\Rightarrow \quad A = 0 \quad , \quad C = 0 \quad , \quad B = \frac{3}{5} \; , \quad D = -\frac{3}{5} \, .$$

$$x(t) = \mathcal{L}^{-1}\left\{X(s)\right\} = \frac{3}{5} \cdot \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1 \cdot 2}{s^2+4}\right\} - \frac{3}{5} \cdot \frac{1}{3} \mathcal{L}^{-1}\left\{\frac{1 \cdot 3}{s^2+9}\right\}$$

$$= \frac{3}{10} \sin 2t - \frac{1}{5} \cdot \sin 3t \, .$$

$\sin kt$	$\frac{k}{s^2 + k^2}, \quad s > 0$

c) $y'' + 4y' + 4y = t^2 \qquad y(0) = y'(0) = 0.$

$$s^2 Y(s) - s y(0) - y'(0) + 4\left(s Y(s) - y(0)\right) + 4 Y(s) = \frac{2}{s^3}, \quad s>0$$

t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$

$$(s^2+4s+4)\, Y(s) = \frac{2}{s^3}$$

$$\Rightarrow \quad Y(s) = \frac{2}{s^3 (s+2)^2}$$

$$Y(s) = \frac{2}{s^3(s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+2} + \frac{E}{(s+2)^2}$$

i) $A \cdot s^2(s+2)^2 + B\, s\,(s+2)^2 + C\,(s+2)^2 + D(s+2)s^3 + E s^3 = 2$

$s=0$: $\quad 4C = 2 \Rightarrow C = 1/2$

$s=-2$: $\quad -8E = 2 \Rightarrow E = -1/4$

ii) $A[2s(s+2)^2 + 2s^2(s+2)] + B[(s+2)^2 + 2s(s+2)] +$

$2C(s+2) + D(s^3 + 3(s+2)s^2) + 3Es^2 = 0$

$s=0: \quad 4B + 4C = 0 \quad \Rightarrow B = -1/2$

$s=-2: \quad -8D + 12E = 0 \quad \Rightarrow D = -3/8$

iii) $A[2(s+2)^2 + 4s(s+2) + 4s(s+2) + 2s^2]$

$+ B[2(s+2) + 2(s+2) + 2s] + 2C$

$+ D[3s^2 + 3s^2 + 6(s+2).s] + 6Es = 0$

$s=0: \quad 8A + 8B + 2C = 0 \quad \Rightarrow A = 3/8$

$$Y(s) = \frac{3}{8s} - \frac{1}{2s^2} + \frac{1}{2s^3} - \frac{3}{8(s+2)} - \frac{1}{4(s+2)^2}$$

$n=0 \quad n=1 \quad n=3 \quad a=-2, n=0 \quad a=-2 \ n=1$

t^n	$\frac{n!}{s^{n+1}}, \ s > 0$
e^{at}	$\frac{1}{s-a}, \ s > a$

$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \ s > a$

$$Y(s) = \frac{3}{8s} - \frac{1}{2s^2} + \frac{1}{2s^3} - \frac{3}{8(s+2)} - \frac{1}{4(s+2)^2}$$

$n=0 \quad n=1 \quad n=3 \quad a=-2,\ n=0 \quad a=-2\ n=1$

t^n	$\frac{n!}{s^{n+1}}$,	$s > 0$
e^{at}	$\frac{1}{s-a}$,	$s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$,	$s > a$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{8} - \frac{t}{2} + \frac{1}{2}\cdot\frac{t^2}{2} - \frac{3}{8}e^{-2t} - \frac{1}{4}e^{-2t}\cdot t.$$

Additional Transform Techniques

Example:

Show that

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}.$$

If $f(t) = te^{at}$, then $f(0) = 0$, $f'(t) = e^{at} + at.e^{at}$

$$\Rightarrow \mathcal{L}\{f'(t)\} = \mathcal{L}\{e^{at}\} + a.\mathcal{L}\{te^{at}\}$$

$$s.\mathcal{L}\{f(t)\} - f(0) = \frac{1}{s-a} + a.\mathcal{L}\{f(t)\}$$

$s>a$

$$(s-a)\mathcal{L}\{f(t)\} = \frac{1}{s-a} \Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{(s-a)^2} \quad s>a.$$

Example: Find $\mathcal{L}\{t \sin kt\}$

Let $f(t) = t \cdot \sin kt \quad \Rightarrow \quad f(0) = 0$

$f'(t) = \sin kt + kt \cos kt$

$f'(0) = 0$

$f''(t) = 2k \cos kt - k^2 t \sin kt$

$$\mathcal{L}\{f''(t)\} = 2k\, \mathcal{L}\{\cos kt\} - k^2 \cdot \mathcal{L}\{t \sin kt\}$$

$$s^2 \mathcal{L}\{f(t)\} - s\underbrace{f(0)}_{=0} - \underbrace{f'(0)}_{=0} = 2k \cdot \frac{s}{s^2+k^2} - k^2 \mathcal{L}\{f(t)\}$$

$s>0$

$$(s^2+k^2)\,\mathcal{L}\{f(t)\} = \frac{2ks}{s^2+k^2}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2ks}{(s^2+k^2)^2} \quad s>0.$$

Similarly,

$$\mathcal{L}\{t\cos kt\} = \frac{s^2-k^2}{(s^2+k^2)^2}, \quad s>0$$

$$\mathcal{L}\{t\sinh kt\} = \frac{2ks}{(s^2-k^2)^2}, \quad s>|k|$$

$$\mathcal{L}\{t\cosh kt\} = \frac{s^2+k^2}{(s^2-k^2)^2}, \quad s>|k|.$$

Example:

a) Show that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^3}(\sin kt - kt\cos kt)$$

$$\mathcal{L}\left\{\frac{1}{2k^3}(\sin kt - kt\cos kt)\right\} = \frac{1}{2k^3}\left[\mathcal{L}\{\sin kt\} - k\,\mathcal{L}\{t\cos kt\}\right]$$

$$= \frac{1}{2k^3}\left[\underset{s>0}{\frac{k}{s^2+k^2}} - \underset{s>0}{\frac{s^2-k^2}{(s^2+k^2)^2}}\right] = \frac{1}{2k^3}\left[\frac{k(s^2+k^2) - k(s^2-k^2)}{(s^2+k^2)^2}\right]$$

$$= \frac{1}{2k^3}\cdot\frac{2k^3}{(s^2+k^2)^2} = \frac{1}{(s^2+k^2)^2} \quad s>0$$

b) Solve the I.V.P

$$x'' + a^2 x = b\sin ct \qquad x(0) = x'(0) = 0$$

$$s^2 X(s) - s\,x(0) - x'(0) + a^2 X(s) = b\,\frac{c}{s^2+c^2}$$

$$(s^2+a^2)\,X(s) = \frac{bc}{s^2+c^2} \Rightarrow X(s) = \frac{bc}{(s^2+a^2)(s^2+c^2)}$$

- $a^2 \neq c^2$

$$X(s) = \frac{-bc}{a^2-c^2}\cdot\left(\frac{1}{s^2+a^2} - \frac{1}{s^2+c^2}\right)$$

$$X(s) = \frac{-bc}{a^2-c^2}\left(\frac{1}{a}\cdot\sin at - \frac{1}{c}\cdot\sin ct\right)$$

- $a^2 = c^2$

$$X(s) = \frac{bc}{(s^2+a^2)^2} \Rightarrow \quad x(t) = \frac{bc}{2a^3}\left(\sin at - at\cos at\right).$$

THEOREM 2 Transforms of Integrals

If $f(t)$ is a piecewise continuous function for $t \geqq 0$ and satisfies the condition of exponential order $|f(t)| \leqq Me^{ct}$ for $t \geqq T$, then

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s} \tag{17}$$

for $s > c$. Equivalently,

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau. \tag{18}$$

proof: Because f is piecewise continuous function, the fundamental theorem of calculus implies that

$$g(t) = \int_0^t f(\tau)\,d\tau$$

is continuous and that $g'(t) = f(t)$ where f is continuous.

$\Rightarrow$ g is continuous and piecewise smooth for $t \geq 0$.

Furthermore,

$$|g(t)| \leq \int_0^t |f(\tau)| d\tau \leq M \cdot \int_0^t e^{c\tau} d\tau = \frac{M}{c} e^{c\tau} \Big|_0^t$$

$$|g(t)| \leq \frac{M}{c} (e^{ct} - 1) < \frac{M}{c} e^{ct}$$

Thus, $\mathcal{L}\{g(t)\}$ exists for $s > c$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$$

$$= s \cdot \mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} - \int_0^0 f(\tau) d\tau$$

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}\{f(t)\}.$$

Table of Laplace Transforms

Function	Transform	Function	Transform
$f(t)$	$F(s)$	1	$\frac{1}{s}, \; s > 0$
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	t	$\frac{1}{s^2}, \; s > 0$
$f'(t)$	$sF(s) - f(0)$	t^n	$\frac{n!}{s^{n+1}}, \; s > 0$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}, \; s > 0$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	$t^a, \; a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}, \; s > 0$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	e^{at}	$\frac{1}{s-a}, \; s > a$
$e^{at}f(t)$	$F(s-a)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \; s > a$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$\cos kt$	$\frac{s}{s^2+k^2}, \; s > 0$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\sin kt$	$\frac{k}{s^2+k^2}, \; s > 0$
$tf(t)$	$-F'(s)$	$\cosh kt$	$\frac{s}{s^2-k^2}, \; s > \lvert k \rvert$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	$\sinh kt$	$\frac{k}{s^2-k^2}, \; s > \lvert k \rvert$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}, \; s > 0$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}, \; s > a$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}, \; s > a$

THEOREM 2 Transforms of Integrals

If $f(t)$ is a piecewise continuous function for $t \geqq 0$ and satisfies the condition of exponential order $|f(t)| \leqq Me^{ct}$ for $t \geqq T$, then

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s} \tag{17}$$

for $s > c$. Equivalently,

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\, d\tau. \tag{18}$$

Example!

Find the inverse Laplace transform of

$$G(s) = \frac{1}{s^2(s-a)}.$$

$$\mathcal{L}^{-1}\{G(s)\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} d\tau.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} d\tau$$

$$= \int_0^t e^{a\tau} d\tau = \left.\frac{e^{a\tau}}{a}\right|_0^t = \frac{e^{at}-1}{a}$$

$$\mathcal{L}^{-1}\{G(s)\} = \int_0^t \frac{e^{a\tau}-1}{a} d\tau = \left.\left(\frac{e^{a\tau}}{a^2} - \frac{\tau}{a}\right)\right|_0^t$$

$$= \frac{e^{at}}{a^2} - \frac{t}{a} - \frac{1}{a^2}.$$

This technique is often more convenient way than the method of partial fraction for $\mathcal{L}^{-1}\left\{\frac{P(s)}{s^n Q(s)}\right\}$

Derivatives, Integrals, and Products of Transforms

what happens if $\mathcal{L}\{h(t)\} = F(s) \cdot G(s)$?

$$\mathcal{L}\{\cos t \cdot \sin t\} = \mathcal{L}\left\{\tfrac{1}{2}\sin 2t\right\} = \frac{1}{s^2+4}, \quad s>0$$

$\neq$

$$\mathcal{L}\{\cos t\} \cdot \mathcal{L}\{\sin t\} = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1} = \frac{s}{(s^2+1)^2}$$

DEFINITION The Convolution of Two Functions

The **convolution** $f * g$ of the piecewise continuous functions f and g is defined for $t \geqq 0$ as follows:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{3}$$

Remark: The convolution is commutative.

$$f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \qquad \left[\begin{array}{ll} t-\tau = u & \Rightarrow -d\tau = du \\ \tau = 0 & \Rightarrow u = t \\ \tau = t & \Rightarrow u = 0 \\ & \Rightarrow \tau = t-u \end{array}\right]$$

$$= -\int_t^0 f(t-u)\, g(u)\, du$$

$$= \int_0^t g(u)\, f(t-u)\, du = g(t) * f(t).$$

$$\Rightarrow \quad f(t) * g(t) = g(t) * f(t).$$

Example: $(\cos t) * (\sin t) = \int_0^t (\cos \tau) \cdot (\sin(t-\tau))\, d\tau$

$$\cos A \cdot \sin B = \frac{1}{2} \left[\sin(A+B) - \sin(A-B) \right]$$

$$(\cos t) * (\sin t) = \frac{1}{2} \cdot \int_0^t \left[\sin(\tau + t - \tau) - \sin(\tau - (t-\tau)) \right] d\tau$$

$$= \frac{1}{2} \int_0^t \left[\sin(t) - \sin(2\tau - t) \right] d\tau$$

$$= \frac{1}{2} \left[\tau \cdot \sin(t) + \frac{1}{2} \cos(2\tau - t) \right] \Big|_0^t$$

$$= \frac{1}{2} \left[t \sin t + \frac{1}{2} \cos t - \frac{1}{2} \cos(-t) \right] = \frac{t}{2} \sin t .$$

THEOREM 1 The Convolution Property

Suppose that $f(t)$ and $g(t)$ are piecewise continuous for $t \geqq 0$ and that $|f(t)|$ and $|g(t)|$ are bounded by Me^{ct} as $t \to +\infty$. Then the Laplace transform of the convolution $f(t) * g(t)$ exists for $s > c$; moreover,

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\} \tag{4}$$

and

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t). \tag{5}$$

***Proof of Theorem 1*:** The transforms $F(s)$ and $G(s)$ exist when $s > c$. For any $\tau > 0$ the definition of the Laplace transform gives

$$G(s) = \int_0^\infty e^{-su} g(u)\,du = \int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt \quad (u = t - \tau), \ du = dt$$

$u = 0 \Rightarrow t = \tau$

$u \to \infty \Rightarrow t \to \infty$

and therefore

$$G(s) = e^{s\tau} \int_0^\infty e^{-st} g(t-\tau)\,dt,$$

because we may *define* $f(t)$ and $g(t)$ to be zero for $t < 0$. Then

$$F(s)G(s) = G(s) \int_0^\infty e^{-s\tau} f(\tau)\,d\tau = \int_0^\infty e^{-s\tau} f(\tau) G(s)\,d\tau$$

$$= \int_0^\infty e^{-s\tau} f(\tau) \left(e^{s\tau} \int_0^\infty e^{-st} g(t-\tau)\,dt \right) d\tau$$

$$= \int_0^\infty \left(\int_0^\infty e^{-st} f(\tau) g(t-\tau)\,dt \right) d\tau.$$

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \left(\int_0^\infty e^{-st} f(\tau) g(t-\tau)\, d\tau \right) dt \\ &= \int_0^\infty e^{-st} \left(\int_0^t f(\tau) g(t-\tau)\, d\tau \right) dt \\ &= \int_0^\infty e^{-st} \left[f(t) * g(t) \right] dt, \end{aligned}$$

and therefore

$$F(s)G(s) = \mathcal{L}\{f(t) * g(t)\}.$$

We replace the upper limit of the inner integral with t because $g(t-\tau) = 0$ whenever $\tau > t$. This completes the proof of Theorem 1. ■

Example: $\mathcal{L}^{-1}\left\{\frac{2}{(s-1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$$= e^t * \sin(2t) = \sin(2t) * e^t = \int_0^t \sin(2\tau) \cdot e^{t-\tau}\, d\tau$$

$$= e^t \int_0^t \sin(2\tau)\, e^{-\tau}\, d\tau$$

$$= e^t \left[\frac{e^{-\tau}}{5} \cdot \left(-\sin 2\tau - 2\cos 2\tau\right) \right] \Big|_0^t$$

$$= e^t \left[\frac{e^{-t}}{5} \left(-\sin 2t - 2\cos 2t\right) + \frac{2}{5} \right]$$

$$= \frac{2}{5} e^t - \frac{1}{5} \sin 2t - \frac{2}{5} \cos 2t .$$

THEOREM 2 Differentiation of Transforms

If $f(t)$ is piecewise continuous for $t \geqq 0$ and $|f(t)| \leqq Me^{ct}$ as $t \to +\infty$, then

$$\mathcal{L}\{-tf(t)\} = F'(s) \tag{6}$$

for $s > c$. Equivalently,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}. \tag{7}$$

Repeated application of Eq. (6) gives

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \tag{8}$$

for $n = 1, 2, 3, \ldots$.

Example: $\mathcal{L}\{t^2 \sin kt\} = (-1)^2 \left[\mathcal{L}\{\sin kt\}\right]'' = \dfrac{d^2}{ds^2}\left(\dfrac{k}{s^2+k^2}\right).$

$s>0$

$$= \frac{d}{ds}\left(\frac{-2ks}{(s^2+k^2)^2}\right) = \frac{6ks^2-2k^3}{(s^2+k^2)^4}.$$

Example: Find $\mathcal{L}^{-1}\{\tan^{-1}(1/s)\}$

$$\Rightarrow \mathcal{L}^{-1}\{F'(s)\} = -tf(t) = -t\,\mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\,\mathcal{L}^{-1}\{F'(s)\}$$

$$\mathcal{L}^{-1}\{\tan^{-1}(\tfrac{1}{s})\} = -\frac{1}{t}\cdot\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\tan^{-1}\left(\tfrac{1}{s}\right)\right)\right\}$$

$$= -\frac{1}{t}\cdot\mathcal{L}^{-1}\left\{\frac{-1/s^2}{1+1/s^2}\right\} = +\frac{1}{t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{\sin t}{t}.$$

Integration of Transforms

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(\sigma)\,d\sigma \quad \text{if} \quad \lim_{t\to 0^+} \frac{f(t)}{t} \text{ exists and is finite.}$$

Example: $\mathcal{L}\left\{\frac{\sinh t}{t}\right\} = ?$

$$\lim_{t\to 0^+} \frac{\sinh t}{t} \overset{\left(\frac{0}{0}\right)}{=} \lim_{t\to 0^+} \frac{\cosh t}{1} = 1 \quad \checkmark$$

$$\mathcal{L}\left\{\frac{\sinh t}{t}\right\} = \int_s^{\infty} F(\sigma)\,d\sigma = \int_s^{\infty} \frac{d\sigma}{\sigma^2 - 1}$$

$* \ \mathcal{L}\{\sinh t\} = \frac{1}{s^2-1}$, $s > 1$.

$$\mathcal{L}\left\{\frac{\sinh t}{t}\right\} = \lim_{b\to\infty} \int_s^{b} \frac{1}{2}\left[\frac{1}{\sigma - 1} - \frac{1}{\sigma + 1}\right] d\sigma$$

$$\mathcal{L}\left\{\frac{\sinh t}{t}\right\} = \lim_{b\to\infty} \int_s^b \frac{1}{2}\left[\frac{1}{\sigma-1} - \frac{1}{\sigma+1}\right] d\sigma$$

$$= \lim_{b\to\infty} \frac{1}{2}\left[\ln|\sigma-1| - \ln|\sigma+1|\right]\Big|_s^b$$

$$= \frac{1}{2}\cdot \lim_{b\to\infty}\left[\ln\left|\frac{\sigma-1}{\sigma+1}\right|\right]\Big|_s^b$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{s-1}{s+1}\right|\right] = \frac{1}{2}\ln\left|\frac{s+1}{s-1}\right|.$$

Example: $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s^2-1)^2}\right\}$

$$\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\} = \frac{f(t)}{t} = \frac{\mathcal{L}^{-1}\{F(s)\}}{t}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = t\cdot\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}.$$

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\} = t\cdot\mathcal{L}^{-1}\left\{\int_s^\infty \frac{2\sigma}{(\sigma^2-1)^2}\,d\sigma\right\}$$

$$= t\cdot\mathcal{L}^{-1}\left\{\lim_{b\to\infty}\int_s^b \frac{2\sigma}{(\sigma^2-1)^2}\,d\sigma\right\}$$

$$= t\cdot\mathcal{L}^{-1}\left\{\lim_{b\to\infty}\left[\frac{-1}{\sigma^2-1}\right]\Big|_s^b\right\}$$

$$= t \cdot \mathcal{L}^{-1}\left\{ \lim_{b\to\infty} \left[\frac{-1}{b^2-1} + \frac{1}{s^2-1} \right] \right\}$$

$$= t \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s^2-1} \right\} = t \cdot \sinh t.$$

Unit step function and Piecewise continuous Input functions.

Definition (Unit step funtion).

$$u_a(t) = u(t-a) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t \geqq a. \end{cases}$$

$a>0$

The unit step function is piecewise continuous, since its jump occurs.

$$\mathcal{L}\{u_a(t)\}_{a>0} = \int_0^\infty e^{-st} \cdot u_a(t)\,dt$$

$$= \int_0^a e^{-st} \underbrace{u_a(t)}_{0}\,dt + \int_a^\infty e^{-st} \underbrace{u_a(t)}_{1}\,dt$$

$$= \int_a^\infty e^{-st}\,dt = \lim_{b\to\infty} \int_a^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} \left.\frac{e^{-st}}{-s}\right|_a^b = \lim_{b\to\infty} \frac{-e^{-sb} + e^{-sa}}{s} = \frac{e^{-sa}}{s}, \quad s>0$$

$$\mathcal{L}\{u_a(t)\}_{a>0} = \frac{e^{-as}}{s} \quad (a>0,\ s>0).$$

Example: $\mathcal{L}\{u_5(t)\} = \frac{e^{-5s}}{s}, \quad s>0.$

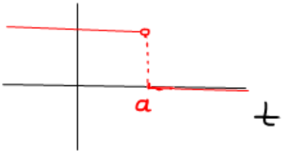
Example: $\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{s}\right\} = u_{10}(t).$

Example: $\mathcal{L}^{-1}\left\{\int_2^\infty e^{-st}\,dt\right\} = \mathcal{L}^{-1}\left\{\int_0^\infty e^{-st}\,u_2(t)\,dt\right\}$

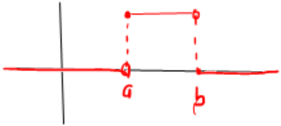
$$= \mathcal{L}^{-1}\left\{\mathcal{L}\{u_2(t)\}\right\}$$

$$= u_2(t).$$

$$1 - u_a(t) = \begin{cases} 1 & \text{for } t < a \\ 0 & \text{for } t \geq a \end{cases}$$



$$u_a(t) - u_b(t) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } a \leq t \leq b \\ 0 & \text{for } b \leq t \end{cases}$$



$$\mathcal{L}\{u_a(t) - u_b(t)\} = \mathcal{L}\{u_a(t)\} - \mathcal{L}\{u_b(t)\}$$

$$= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} \qquad s > 0.$$

Example: Express the function

$$f(t) = \begin{cases} 0, & t < 1 \\ 2 & 1 \leq t < 2 \\ 1 & 2 \leq t < 3 \\ 3 & 3 \leq t \end{cases}$$

in terms of unit step functions and then find its Laplace transform.

$$f(t) = 2.\begin{cases} 0, & t < 1 \\ 1 & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} + \begin{cases} 0 & t < 2 \\ 1 & 2 \leq t < 3 \\ 0 & 3 \leq t \end{cases} + 3.\begin{cases} 0 & t < 3 \\ 1 & t \geq 3. \end{cases}$$

$$= 2.\left[u_1(t) - u_2(t)\right] + \left[u_2(t) - u_3(t)\right] + 3.\, u_3(t).$$

$$= 2u_1(t) - u_2(t) + 2u_3(t).$$

$$f(t) = 2u_1(t) - u_2(t) + 2u_3(t).$$

$$\mathcal{L}\{f(t)\} = 2 \cdot \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} + 2 \cdot \frac{e^{-3s}}{s}, \quad s > 0.$$

THEOREM 1 Translation on the t-Axis

If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \tag{3a}$$

$u_a(t)$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a) \tag{3b}$$

for $s > c + a$.

Notice that,

$$u(t-a)f(t-a) = \begin{cases} 0 & \text{if } t < a, \\ f(t-a) & \text{if } t \geqq a. \end{cases}$$

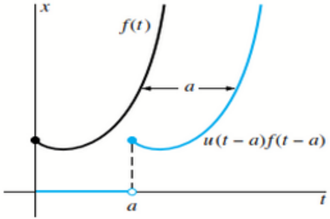


FIGURE 10.5.2. Translation of $f(t)$ a units to the right.

proof!

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau$$

$$= \int_0^\infty e^{-s(a+\tau)}f(\tau)\,d\tau = \lim_{b\to\infty}\int_0^b e^{-s(a+\tau)}f(\tau)\,d\tau.$$

$t = a+\tau \Rightarrow dt = d\tau, \quad \tau = t-a.$

$\tau = 0 \Rightarrow t = a, \quad \tau = b \Rightarrow t = a+b$

$$e^{-as}F(s) = \lim_{b\to\infty}\int_a^{a+b} e^{-st}f(t-a)\,dt = \int_a^\infty e^{-st}f(t-a)\,dt$$

$$= \int_0^\infty e^{-st}u(t-a)f(t-a)\,dt = \mathcal{L}\{u(t-a)\cdot f(t-a)\}.$$

Example: Find $\mathcal{L}\{g(t)\}$ if

$$g(t) = \begin{cases} 1 , & t < 3 \\ t^2 , & t \geq 3 \end{cases}$$

$$g(t) = \begin{cases} 1 , & t < 3 \\ 0 , & t \geq 3 \end{cases} + t^2 \cdot \begin{cases} 0 , & t < 3 \\ 1 , & t \geq 3 \end{cases}$$

$$= [1 - u_3(t)] + t^2 \cdot u_3(t).$$

$$= [1 - u_3(t)] + (t - 3 + 3)^2 u_3(t)$$

$$= 1 - u_3(t) + (t-3)^2 u_3(t) + 6(t-3) u_3(t) + 9 \cdot u_3(t).$$

$$= 1 + [(t-3)^2 + 6(t-3) + 8] u_3(t)$$

$$g(t) = 1 + \left[(t-3)^2 + 6(t-3) + 8 \right] u_3(t)$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{1\} + \mathcal{L}\left\{ \left[(t-3)^2 + 6(t-3) + 8 \right] u_3(t) \right\}$$

$$f(t-3) = (t-3)^2 + 6(t-3) + 8$$

$$f(t) = t^2 + 6t + 8 \Rightarrow \quad F(s) = \frac{2}{s^3} + \frac{6}{s^2} + \frac{8}{s}, \quad s>0.$$

$$\mathcal{L}\{g(t)\} = \frac{1}{s} + e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{8}{s} \right), \quad s>0.$$

Example: Solve the IVP

$$x'' - 4x = \begin{cases} 0, & t < 2\pi \\ \sin 2t, & t \geq 2\pi \end{cases}, \quad x(0) = x'(0) = 0.$$

$$x'' - 4x = (\sin 2t) \cdot u_{2\pi}(t)$$

$\sin\theta = \sin(\theta - 2\pi n)$

$n = 1, 2, \ldots$

$$x'' - 4x = \sin 2(t - 2\pi) \cdot u_{2\pi}(t)$$

$$\left[s^2 X(s) - s\,x(0) - x'(0)\right] - 4X(s) = e^{-2\pi s} \cdot \mathcal{L}\{\sin 2t\}$$

$$(s^2 - 4)X(s) = e^{-2\pi s} \cdot \frac{2}{s^2+4} \Rightarrow X(s) = \frac{2e^{-2\pi s}}{(s^2-4)(s^2+4)}$$

$$X(s) = \frac{2}{8} \cdot e^{-2\pi s} \cdot \left(\frac{1}{s^2-4} - \frac{1}{s^2+4}\right).$$

$$X(s) = \frac{1}{4} \cdot \left(\frac{e^{-2\pi s}}{s^2-4} - \frac{e^{-2\pi s}}{s^2+4} \right)$$

$$H(s) = \frac{1}{s^2-4} - \frac{1}{s^2+4} \Rightarrow h(t) = \frac{\sinh(2t)}{2} - \frac{\sin(2t)}{2} .$$

$$x(t) = \mathcal{L}^{-1}\left\{ X(s) \right\} = \frac{1}{4} \cdot u_{2\pi}(t)\, h(t-2\pi)$$

$$x(t) = \frac{1}{8} \cdot u_{2\pi}(t) \cdot \left[\sinh 2(t-2\pi) - \sin 2(t-2\pi) \right] .$$