MAT281E Linear Algebra and Applications HW 1

Mustafa Can Çalışkan, 150200097

1)

1)

1)

$$A_{3\times3}\,x_{3\times1} = \underbrace{\begin{bmatrix}1&1&-3\\0&1&-1\\-1&1&1\end{bmatrix}}_{3\times3}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{3\times1} = b_{3\times1} = \underbrace{\begin{bmatrix}4\\3\\2\end{bmatrix}}_{3\times1}$$

$$\begin{bmatrix}1&1&-3&4\\0&1&-1&3\\-1&1&1&2\end{bmatrix} \xrightarrow[E_1]{r_1+r_3} \begin{bmatrix}1&1&-3&4\\0&1&-1&3\\0&2&-2&6\end{bmatrix} \xrightarrow[E_2]{-2r_2\to r_3} \begin{bmatrix}1&1&-3&4\\0&1&-1&3\\0&0&0&0\end{bmatrix}$$

$$\downarrow E_3 \quad -r_2 \to r_1$$

System is consistent, There are infinitely many solutions

$$\underbrace{\begin{bmatrix}1&0&-2&1\\0&1&-1&3\\0&0&0&0\end{bmatrix}}_{r.r.e.f.}$$

1)

2)

$$A_{3\times3}X_{3\times1}=\begin{bmatrix}2&1&-3\\1&0&1\\-1&2&-4\end{bmatrix}_{3\times3}\begin{bmatrix}x\\y\\z\end{bmatrix}_{3\times1}=b_{3\times1}=\begin{bmatrix}1\\1\\5\end{bmatrix}_{3\times1}$$

1)

2)

$$\begin{bmatrix}2&1&-3&1\\1&0&1&1\\-1&2&-4&5\end{bmatrix}\xrightarrow[r_2\leftrightarrow r_1]{E_1}\begin{bmatrix}1&0&1&1\\2&1&-3&1\\-1&2&-4&5\end{bmatrix}\xrightarrow[-2r_1\to r_2]{E_2}\begin{bmatrix}1&0&1&1\\0&1&-5&-1\\-1&2&-4&5\end{bmatrix}$$

$$\downarrow E_3 \quad r_1\to r_3$$

$$\begin{bmatrix}1&0&1&1\\0&1&-5&-1\\0&0&1&\frac{8}{7}\end{bmatrix}\xleftarrow[1/7r_3]{E_5}\begin{bmatrix}1&0&1&1\\0&1&-5&-1\\0&0&7&8\end{bmatrix}\xleftarrow[-2r_2\to r_3]{E_4}\begin{bmatrix}1&0&1&1\\0&1&-5&-1\\0&2&-3&6\end{bmatrix}$$

$$\downarrow E_6=5r_3\to r_2$$

$$\begin{bmatrix}1&0&1&1\\0&1&0&33/7\\0&0&1&8/7\end{bmatrix}\xrightarrow[-r_3\to r_1]{E_7}\begin{bmatrix}1&0&0&-1/7\\0&1&0&33/7\\0&0&1&8/7\end{bmatrix}$$

$x=-1/7$

$y=33/7$

$z=8/7$

1)

3)

$$\underset{2\times4}{A}\ \underset{4\times1}{X} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix}}_{2\times4} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{4\times1} = \underset{2\times1}{b} = \underbrace{\begin{bmatrix} 1 \\ 5 \end{bmatrix}}_{2\times1}$$

1)

3)

$$\left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 & 5 \end{array}\right] \xrightarrow[-2r_1 \to r_2]{E_1} \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 0 & 3 \end{array}\right] \xrightarrow[-1/2\, r_2]{E_2} \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1/2 & 0 & -3/2 \end{array}\right]$$

$$\downarrow E_3 \quad -r_2 \to r_1$$

$$\underbrace{\left[\begin{array}{ccccc} 1 & 0 & 1/2 & 1 & 5/2 \\ 0 & 1 & 1/2 & 0 & -3/2 \end{array}\right]}_{r.r.e.f.}$$

System is consistent

There are inf. many solutions

x3=k

x4=m

x1+k+m=5/2

x2+k=-3/2

2)

2)

$$\begin{bmatrix} 2 & 2 & 4 \\ -1 & 2 & 1 \\ -2 & -1 & 4 \end{bmatrix} \xrightarrow[\text{by } 1/2]{\text{1st row was multiplied}} \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ -2 & -1 & 4 \end{bmatrix} \xrightarrow[\text{added to 2nd row}]{\text{1 times 1st row was}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ -2 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow[\text{added to 3th row}]{\text{2 times the 1st row was}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 3 \\ 0 & 1 & 8 \end{bmatrix} \xrightarrow[\text{by } 1/3]{\text{2nd row was multiplied}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 8 \end{bmatrix} \xrightarrow[\text{was added to 3th row}]{\text{-1 times the 2nd row}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\xrightarrow[\text{by } 1/7]{\text{3th row was multiplied}} \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Row echelon form}} \xrightarrow[\text{was added to 2nd row}]{\text{-1 times the 3th row}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{was added to 1st row}]{\text{-2 times the 3th row}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{added to 1st row}]{\text{-1 times the 2nd row was}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Reduced row echelon form}}$$

3)

3)

Echelon form properties:

a) If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. It is called as leader 1.

b) If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

c) In any two successive rows that do not consist entirely of zeros, the leader 1 in the lower row occurs farther to the right than the leader 1 in the higher row.

d) Each column that contains a leader 1 has zeros everywhere else.

A matrix that has all properties is said to be in reduced row echelon form.

A matrix that has the first three properties is said to be in row echelon form.

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix has just a, b, c properties. Each column that contains leader 1 does not have zeros everywhere else. (3th column). Thus, the matrix is in row echelon form.

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix has all properties. Therefore the matrix is in reduced row echelon form.
(1 in 2nd column is not leader 1)

4)

4) If $A_{(m\times n)}$, then A^T is defined to be the $(n\times m)$ matrix that results from interchanging the rows and columns of A.

$$A_{(m\times n)} A^T_{(n\times m)} = (AA^T)_{(m\times m)} \text{ is exist}$$

$$\rightarrow \#\text{cols}(A) = \#\text{rows}(A^T)$$

$$A^T_{(n\times m)} A_{(m\times n)} = (A^T A)_{(n\times n)} \text{ is exist.}$$

$$\#\text{cols}(A^T) = \#\text{rows}(A)$$

5)

5)

$$C_{2\times 3} A_{3\times 4} = (CA)_{2\times 4}$$

$$(CA)_{2\times 4} B^T_{4\times 2} = (CAB^T)_{2\times 2}$$

If $(CAB^T)^{-1}$ is exist, it must have the same size as $(CAB^T)_{2\times 2}$ $(A^{-1}A = AA^{-1} = I)$ $\xrightarrow{2\times 2}$

6)

6)

Let $Ax=b$ and A is invertible

$\downarrow$

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b$$

$$I\begin{bmatrix}1\\1\\1\end{bmatrix} \neq A^{-1}\begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow I\begin{bmatrix}1\\1\\1\end{bmatrix} \neq \begin{bmatrix}0\\0\\0\end{bmatrix}$$

7)

7)

a)

$$A = \left[\begin{array}{c:c} A_1 & A_2 \end{array}\right] \qquad AB = \left[A_1B_1 + A_2B_2 \right]$$

$$B = \left[\begin{array}{c} B_1 \\ \hdashline B_2 \end{array}\right]$$

$$A_1B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A_2B_2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 2 & 1 \end{bmatrix}$$

7)

a)

$$A_1B_1 + A_2B_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 2 & 2 \end{bmatrix} = AB$$

7)

$$BA = \underset{2\times 1}{\begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} A_1 \vdots A_2 \end{bmatrix}} = \begin{bmatrix} B_1A_1 & B_1A_2 \\ B_2A_1 & B_2A_2 \end{bmatrix}$$

$$B_1A_1 = \underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$$

$$B_1A_2 = \underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 0 & 1 & 4 \\ 0 & 1 & 1 \end{bmatrix}}$$

$$B_2A_1 = \underset{3\times 2}{\begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}}$$

$$B_2A_2 = \underset{3\times 2}{\begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & -1 \\ 0 & 2 & 8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 \\ -1 & 2 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & -1 \\ 2 & 2 & 0 & 2 & 8 \end{bmatrix}$$

8) (1)

8) (1)

Ex:

$$\left[\begin{array}{cccc|cccc} 1&1&1&1&1&0&0&0\\ 0&1&1&1&0&1&0&0\\ 0&0&1&1&0&0&1&0\\ 0&0&0&1&0&0&0&1 \end{array}\right] \xrightarrow[-r_2 \to r_1]{E_1} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&-1&0&0\\ 0&1&1&1&0&1&0&0\\ 0&0&1&1&0&0&1&0\\ 0&0&0&1&0&0&0&1 \end{array}\right]$$

$$\downarrow E_2 \quad -r_3 \to r_2$$

$$\left[\begin{array}{cccc|cccc} 1&0&0&0&1&-1&0&0\\ 0&1&0&0&0&1&-1&0\\ 0&0&1&0&0&0&1&-1\\ 0&0&0&1&0&0&0&1 \end{array}\right] \xleftarrow[-r_4 \to r_3]{E_4} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&-1&0&0\\ 0&1&0&0&0&1&-1&0\\ 0&0&1&1&0&0&1&0\\ 0&0&0&1&0&0&0&1 \end{array}\right]$$

Both A and A^{-1} are upper triangular matrices.

A has to be proper to $\dfrac{\operatorname{adj}(A)}{\det(A)}$ for existence of A^{-1}

$\det(A)$ cannot be 0

8) (2)

8) (2)

$$A = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1\end{array}\right] \xrightarrow[1/a \cdot r_1]{E_1} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1\end{array}\right]$$

$$\downarrow E_2 \quad -cr_1 \to r_2$$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \left(\frac{-c}{ad-cb}\right) & \left(\frac{a}{ad-cb}\right)\end{array}\right] \xleftarrow[\frac{a}{ad-cb}\cdot r_2]{E_3} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \left(d-\frac{cb}{a}\right) & -c/a & 1\end{array}\right]$$

$$\downarrow E_4 = -\frac{b}{a} r_2 \to r_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \left(\frac{bc}{a(ad-cb)}+\frac{1}{a}\right) & \left(\frac{-b}{ad-cb}\right) \\ 0 & 1 & \left(\frac{-c}{ad-cb}\right) & \left(\frac{a}{ad-cb}\right)\end{array}\right]$$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

9)

9)

$$(A-I)^2 = -A$$

$$AA - 2AI + I = -A$$

$$A^{-1}(AA - 2AI + I) = -A^{-1}A$$

$$A - 2I + A^{-1} = -I$$

$$A^{-1} = I - A$$

10)

10) $AA^{-1} = A^{-1}A = I$

$$AA^{-1} = \begin{bmatrix} r_1 \\ \vdots \\ r_2 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} A^{-1} = \begin{bmatrix} r_1 A^{-1} \\ \vdots \\ r_2 A^{-1} \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 1 \end{bmatrix}$$

11)

11) $(A+B)^2 = (A+B)(A+B)$

$$= AA + AB + BA + BB$$

$$\cancel{AA} + \cancel{AB} + BA + \cancel{BB} = \cancel{AA} + \cancel{AB} + AB + \cancel{BB}$$

$$BA = AB \, ?$$

- ✓ Either B or A can be I ($IA = AI$)
- ✓ Both A and B can be I ($II = II$)
- ✓ Either A or B can be zero matrix ($OA = AO$)
- ✓ Both A and B can be zero matrices ($OO = OO$)
- A and B can be inverses of each other ($AB = BA = I$)

It is not <u>necessarily</u> true

12)

12)

1)
$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} \xrightarrow{E_{13}(-1)} \begin{bmatrix} 1 & -5 & 0 \\ 1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} \xrightarrow{E_{22}(5)} \begin{bmatrix} 6 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}$$

$$\downarrow E_1(1/6)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \xleftarrow{E_{31}(-1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} \xleftarrow{E_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}$$

$$\downarrow E_{32}(-5)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sequence is not unique (1,2 examples)

2)
$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} \xrightarrow{E_{31}(2)} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 5 & 5 & 3 \end{bmatrix} \xrightarrow{E_{32}(-5)} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{E_3(\frac{1}{3})} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow (E_{13}(-1))$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{E_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{E_1(\frac{1}{2})} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

13)

13)

Ex:

$$A = \begin{bmatrix} a & b & c \\ 0 & 0 & c \\ a & b & 0 \end{bmatrix}$$

$$\det(A) = (-1)^5 \cdot c \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

the matrix is not invertible.

14)

14)

a)

$$\left[\begin{array}{cccc|cccc} 2&0&0&2&1&0&0&0\\ 1&2&0&0&0&1&0&0\\ 0&1&1&0&0&0&1&0\\ 0&1&0&0&0&0&0&1 \end{array}\right] \xrightarrow[r_2 \leftrightarrow r_1]{E_1} \left[\begin{array}{cccc|cccc} 1&2&0&0&0&1&0&0\\ 2&0&0&2&1&0&0&0\\ 0&1&1&0&0&0&1&0\\ 0&1&0&0&0&0&0&1 \end{array}\right]$$

$$\downarrow E_2 \quad -2r_1 \rightarrow r_2$$

$$\left[\begin{array}{cccc|cccc} 1&2&0&0&0&1&0&0\\ 0&0&0&2&1&-2&0&4\\ 0&1&1&0&0&0&1&0\\ 0&1&0&0&0&0&0&1 \end{array}\right] \xleftarrow[4r_4 \rightarrow r_2]{E_3} \left[\begin{array}{cccc|cccc} 1&2&0&0&0&1&0&0\\ 0&-4&0&2&1&-2&0&0\\ 0&1&1&0&0&0&1&0\\ 0&1&0&0&0&0&0&1 \end{array}\right]$$

$$\downarrow E_4 \quad r_2 \leftrightarrow r_4$$

$$\left[\begin{array}{cccc|cccc} 1&2&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&1\\ 0&1&1&0&0&0&1&0\\ 0&0&0&2&1&-2&0&4 \end{array}\right] \xrightarrow[-r_2 \rightarrow r_3]{E_5} \left[\begin{array}{cccc|cccc} 1&2&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&1&-1\\ 0&0&0&2&1&-2&0&4 \end{array}\right]$$

$$\downarrow E_6 \quad 1/2\, r_4$$

$$\left[\begin{array}{cccc|cccc} 1&0&0&0&0&1&0&-2\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&1&-1\\ 0&0&0&1&1/2&-1&0&2 \end{array}\right] \xleftarrow[-2r_2 \rightarrow r_1]{E_7} \left[\begin{array}{cccc|cccc} 1&2&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&1&-1\\ 0&0&0&1&1/2&-1&0&2 \end{array}\right]$$

$$\underbrace{\begin{array}{cccc} 0&1&0&-2\\ 0&0&0&1\\ 0&0&1&-1\\ 1/2&-1&0&2 \end{array}}_{A^{-1}}$$

14)

b)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-r_1 \to r_3]{E_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-1/2 r_2 \to r_3]{E_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1/2 & 1 \end{array}\right]$$

det = 0 ← the matrix is singular.

c)

$$\left[\begin{array}{ccc|ccc} c & 0 & a & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ c & b & a & 0 & 0 & 1 \end{array}\right] \xrightarrow[1/c\, r_1]{E_1} \left[\begin{array}{ccc|ccc} 1 & 0 & a/c & 1/c & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ c & b & a & 0 & 0 & 1 \end{array}\right] \xrightarrow[-cr_1 \to r_3]{E_2} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{a}{c} & 1/c & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & b & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-r_2 \to r_3]{E_3} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{a}{c} & 1/c & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

det = 0 ← the matrix is singular.

15)

15)

1)

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ cannot be turned to $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ matrix by elementary row operations. Transpose is not an elementary row operation.

2)

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow[-3r_2 \to r_1]{E_1} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow[r_1 \to r_2]{E_2} \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \xrightarrow[1/2\, r_1]{E_3} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow[-1/2\, r_1 \to r_2]{(-1/2)\; E_4} \begin{bmatrix} 0 & 1 \\ 1 & 3/2 \end{bmatrix} \xrightarrow[2r_2]{E_5} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

16)

16) If B is an $n \times n$ triangular matrix, det(B) is the product of the entries on the main diagonal of the matrix. This matrix is invertible because of $\det(A) \neq 0$

16)

$$\begin{bmatrix} 2 & 0 & 4 & -4 & -1 \\ 0 & 3 & 0 & -2 & 2 \\ 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(E_1=\frac{1}{2})} \begin{bmatrix} 1 & 0 & 2 & -2 & -1/2 \\ 0 & 3 & 0 & -2 & 2 \\ 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow (E_2=\frac{1}{3})$$

$$\begin{bmatrix} 1 & 0 & 2 & -2 & -1/2 \\ 0 & 1 & 0 & -2/3 & 2/3 \\ 0 & 0 & 1 & 0 & 1/5 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xleftarrow{(E_3=1/5)} \begin{bmatrix} 1 & 0 & 2 & -2 & -1/2 \\ 0 & 1 & 0 & -2/3 & 2/3 \\ 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow E_4=-1/3\,r_4$$

$$\begin{bmatrix} 1 & 0 & 2 & -2 & -1/2 \\ 0 & 1 & 0 & -2/3 & 2/3 \\ 0 & 0 & 1 & 0 & 1/5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \rightarrow \text{R.e.f.} \\ \rightarrow \text{No row of zeroes.} \end{array}$$

17)

17) If $\det(A) = 0$, A^{-1} does not exist.

$$a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24} \neq 0$$

$$-1\underbrace{\begin{vmatrix}0&1&1\\0&1&0\\2&4&1\end{vmatrix}}_{\begin{vmatrix}0&1\\2&1\end{vmatrix}} + x\underbrace{\begin{vmatrix}1&1&1\\0&1&0\\x+1&4&1\end{vmatrix}}_{1\cdot\begin{vmatrix}1&1\\x+1&1\end{vmatrix}} - 3\underbrace{\begin{vmatrix}1&0&1\\0&0&0\\x+1&2&1\end{vmatrix}}_{0} - 1\underbrace{\begin{vmatrix}1&0&1\\0&0&1\\x+1&2&4\end{vmatrix}}_{-1\begin{vmatrix}1&0\\x+1&2\end{vmatrix}}$$

$\underbrace{-1\cdot -2}_{2}$

$x\cdot 1(1-(x+1)) = -x^2$

$2, -2$

$-1\cdot -1\cdot 2 = 2$

$4 - x^2 = 0$

$x \neq 2$

$x \neq -2$

18)

18)

1) $\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

18)

*= any integer

2)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

19)

19) Using smart choice,

$$\det(A) = \underbrace{a_{21}C_{21}}_{0} + a_{22}C_{22} + \underbrace{a_{23}C_{23}}_{0} + \underbrace{a_{24}C_{24}}_{0}$$

$$\det(A) = 3\begin{vmatrix} 2 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 3\left(\underbrace{a_{21}C_{21}}_{0} + a_{22}C_{22} + a_{23}C_{23}\right)$$

$$\underbrace{\underbrace{2\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}}_{2.2=4} - \underbrace{3\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}}_{+(-3.2)=-6}}_{-2}$$

$$\det(A) = -2.3 = -6$$

21)

21)

$$AA = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\underbrace{AA}_{I}AA \dots = A^{120}$$

$$A \; AA \dots = A^{118}$$

$$\underbrace{\underbrace{AA}_{I}AA}_{A} = A^{4}$$

$$AA = I$$

I