THEOREM 4 General Solutions of Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \tag{3}$$

on an open interval I, where the p_i are continuous. If Y is any solution whatsoever of Eq. (3), then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

for all x in I.

$\Rightarrow$ y_1 and y_2 are linearly independent solutions of

$$y'' - 4y = 0$$

But, $y_3(x) = \cosh 2x = \dfrac{e^{2x} + e^{-2x}}{2}$ and $y_4 = \sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2}$

are also solutions. Thus, the solution space of the differential equation $y'' - 4y = 0$ has two different bases $\{e^{2x}, e^{-2x}\}$ and $\{\cosh 2x, \sinh 2x\}$

$\Rightarrow$ $y(x) = c_1 e^{2x} + c_2 e^{-2x}$ or $y(x) = a\cosh 2x + b\sinh 2x$

are general solutions.

Homogeneous Equations with Constant Coefficients

The n-th order linear homogeneous equation with constant coefficient has the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \tag{1}$$

where the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real constants with $a_n \neq 0$.

Example: $y'' - 5y' + 6y = 0$ $\qquad y = e^{rx}, \quad y' = r e^{rx}, \quad y'' = r^2 e^{rx}$

$r^2 e^{rx} - 5r e^{rx} + 6 e^{rx} = 0$

$(r^2 - 5r + 6) \underbrace{e^{rx}}_{\neq 0} = 0 \quad \Rightarrow r^2 - 5r + 6 = 0, \; r_1 = 2, \; r_2 = 3$

Thus $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ are particular solutions.

$w(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$ $\qquad y(x) = c_1 e^{2x} + c_2 e^{3x}$ is a general solution.

$y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$, ----, $y^{(k)} = r^k e^{rx}$

$$\underbrace{a_n r^n e^{rx}}_{y^{(n)}} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_2 r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0;$$

that is,

$$\underbrace{e^{rx}}_{\neq 0} \left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 \right) = 0.$$

$$\Rightarrow \quad a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0. \tag{3}$$

This equation is called the "characteristic equation" or "auxiliary equation" of the diff. eq.

Distinct Real Roots

THEOREM 1 Distinct Real Roots

If the roots $r_1, r_2, \ldots, r_n$ of the characteristic equation in (3) are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x} \tag{4}$$

is a general solution of Eq. (1). Thus the n linearly independent functions $\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$ constitute a basis for the n-dimensional solution space of Eq. (1).

Example:

Solve the initial value problem

$$y^{(3)} + 3y'' - 10y' = 0;$$
$$y(0) = 7, \quad y'(0) = 0, \quad y''(0) = 70.$$

The characteristic equation: $r^3 + 3r^2 - 10r = 0$

$r(r^2 + 3r - 10) = 0 \Rightarrow r(r+5)(r-2) = 0$

$\Rightarrow r_1 = 0, \; r_2 = -5, \; r_3 = 2$

$y(x) = c_1 e^{0x} + c_2 e^{-5x} + c_3 e^{2x} = c_1 + c_2 e^{-5x} + c_3 e^{2x}$ (general solution)

$$\left.\begin{aligned} y(0) = 7 &\Rightarrow c_1 + c_2 + c_3 = 7 \\ y'(0) = 0 &\Rightarrow -5c_2 + 2c_3 = 0 \\ y''(0) = 70 &\Rightarrow 25c_2 + 4c_3 = 70 \end{aligned}\right\} \begin{aligned} c_1 &= 0 \\ c_3 &= 5 \\ c_2 &= 2 \end{aligned}$$

$y(x) = 2e^{-5x} + 5e^{2x}$ (particular solution)

Polynomial Differential Operators

If the roots of the characteristic equation in (3) are *not* distinct—there are repeated roots—then we cannot produce n linearly independent solutions of Eq. (1) by the method of Theorem 1. For example, if the roots are 1, 2, 2, and 2, we obtain only the *two* functions e^x and e^{2x}. The problem, then, is to produce the missing linearly independent solutions. For this purpose, it is convenient to adopt "operator notation" and write Eq. (1) in the form $Ly = 0$, where the **operator**

$$L = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0. \tag{5}$$

operates on the n times differentiable function $y(x)$ to produce the linear combination

$$Ly = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y^{(2)} + a_1 y' + a_0 y$$

of y and its first n derivatives. We also denote by $D = d/dx$ the operation of differentiation with respect to x, so that

$$Dy = y', \quad D^2 y = y'', \quad D^3 y = y^{(3)},$$

and so on. In terms of D, the operator L in (5) may be written

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_2 D^2 + a_1 D + a_0, \tag{6}$$

and we will find it useful to think of the right-hand side in Eq. (6) as a (formal) nth-degree polynomial in the "variable" D; it is a **polynomial differential operator.**

A first-degree polynomial operator with leading coefficient 1 has the form $D - a$, where a is a real number. It operates on a function $y = y(x)$ to produce

$$(D - a)y = Dy - ay = y' - ay.$$

The important fact about such operators is that any two of them *commute*:

$$(D - a)(D - b)y = (D - b)(D - a)y \tag{7}$$

for any twice differentiable function $y = y(x)$. The proof of the formula in (7) is the following computation:

$$\begin{aligned}(D - a)(D - b)y &= (D - a)(y' - by)\\ &= D(y' - by) - a(y' - by)\\ &= y'' - (b + a)y' + aby = y'' - (a + b)y' + bay\\ &= D(y' - ay) - b(y' - ay)\\ &= (D - b)(y' - ay) = (D - b)(D - a)y.\end{aligned}$$

We see here also that $(D - a)(D - b) = D^2 - (a + b)D + ab$. Similarly, it can be shown by induction on the number of factors that an operator product of the form $(D - a_1)(D - a_2) \cdot \cdots \cdot (D - a_n)$ expands—by multiplying out and collecting coefficients—in the same way as does an ordinary product $(x - a_1)(x - a_2) \cdot \cdots \cdot (x - a_n)$ of linear factors with x denoting a real variable. Consequently, the algebra of polynomial differential operators closely resembles the algebra of ordinary real polynomials.

Repeated Real Roots

THEOREM 2 Repeated Roots

If the characteristic equation in (3) has a repeated root r of multiplicity k, then the part of a general solution of the differential equation in (1) corresponding to r is of the form

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{rx}. \tag{14}$$

Example: Find a general solution of the fifth-order differential equation

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

The characteristic equation is $9r^5 - 6r^4 + r^3 = 0$

$r^3(9r^2 - 6r + 1) = 0 \qquad r^3(3r-1)^2 = 0$

$r_1 = 0 \qquad r_2 = \frac{1}{3}$

The triple root $r = 0$ contributes

$$(c_1 + c_2x + c_3x^2)e^0 = c_1 + c_2x + c_3x^2$$

The double root $r = \frac{1}{3}$ contributes

$$(c_4 + c_5x) \cdot e^{x/3}$$

$\Rightarrow y(x) = c_1 + c_2x + c_3x^2 + (c_4 + c_5x)\, e^{x/3}$ (general solution)

Complex Roots

Because we have assumed that the coefficients of the differential equation and its characteristic equation are real, any complex (nonreal) roots will occur in complex conjugate pairs $a \mp ib$ where a and b are real and $i = \sqrt{-1}$.

$$y(x) = c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x}.$$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots .$$

If we substitute $t = i\theta$ in this series and recall that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on, we get

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right). \end{aligned}$$

$e^{(a \pm ib)x}$

$e^{(a+ib)x} + e^{(a-ib)x}$

Because the two real series in the last line are the Taylor series for $\cos\theta$ and $\sin\theta$, respectively, this implies that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{15}$$

This result is known as **Euler's formula**. Because of it, we *define* the exponential function e^z, for $z = x + iy$ an arbitrary complex number, to be

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y). \tag{16}$$

THEOREM 3 Complex Roots

If the characteristic equation in (3) has an unrepeated pair of complex conjugate roots $a \pm bi$ (with $b \neq 0$), then the corresponding part of a general solution of Eq. (1) has the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx). \tag{21}$$

Thus the linearly independent solutions $e^{ax} \cos bx$ and $e^{ax} \sin bx$ (corresponding to the complex conjugate characteristic roots $a \pm bi$) generate a 2-dimensional subspace of the solution space of the differential equation.

Example!

$$y'' + b^2 y = 0 \quad (b > 0)$$

The characteristic equation is $r^2 + b^2 = 0$ $\quad r^2 = -b^2$

$\Rightarrow r = \mp bi$

$0 \mp bi$

$y(x) = c_1 \cos bx + c_2 \sin bx$

Example!

Find the particular solution of

$$y'' - 4y' + 5y = 0$$

for which $y(0) = 1$ and $y'(0) = 5$.

$r^2 - 4r + 5 = 0 \quad , \quad (r-2)^2 + 1 = 0$

$(r-2)^2 = -1$

$r - 2 = \mp i \Rightarrow r = 2 \mp i$

$y(x) = e^{2x} \cdot (c_1 \cos x + c_2 \sin x)$ (General solution)

$y'(x) = 2e^{2x} \cdot (c_1 \cos x + c_2 \sin x) + e^{2x} \cdot (-c_1 \sin x + c_2 \cos x)$

$\left.\begin{array}{l} y(0) = 1 \Rightarrow c_1 = 1 \\ y'(0) = 5 \Rightarrow 2c_1 + c_2 = 5 \end{array}\right\} \quad c_2 = 3$

$y(x) = e^{2x}(\cos x + 3 \sin x)$ particular solution

Example:

Find a general solution of the differential equation $y^{(4)} + 4y = 0$.

The characteristic equation is $r^4 + 4 = 0$

$(r^2)^2 - (2i)^2 = 0$ $\quad (r^2 - 2i)(r^2 + 2i) = 0$

$(r - \sqrt{2i})(r + \sqrt{2i})(r^2 + 2i) = 0$

$r_1 = \sqrt{2i}$, $r_2 = -\sqrt{2i}$, $r^2 + 2i = 0$, $r_{3,4} = \mp\sqrt{2i}$

Remark:

$z = x + iy = re^{i\theta}$ is the polar form

of the complex number z. This form follows from Euler's formula upon writing

$$z = r\left(\frac{x}{r} + i\frac{y}{r}\right) = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$x = r\cos\theta$
$y = r\sin\theta$

in terms of the **modulus** $r = \sqrt{x^2 + y^2} > 0$ of the number z and its **argument** θ

$\theta = \tan^{-1}\frac{y}{x}$.

$\sqrt{z} = \mp\sqrt{re^{i\theta}} = \mp\sqrt{r}\, e^{i\theta/2}$.

$$\sqrt{2i} = (2i)^{1/2} = \sqrt{2}\, e^{i\pi/4} = \sqrt{2} \cdot \left(\cos\frac{\pi}{4} + i \cdot \frac{\pi}{4}\right) = 1+i$$

$$\sqrt{-2i} = (-2i)^{1/2} = \sqrt{2} \cdot e^{3i\pi/4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= -1+i$$

$r_1 = 1+i$, $r_2 = -1-i$, $r_3 = -1+i$, $r_4 = 1-i$

$\underline{\underline{1 \mp i}}$ $\qquad$ $\underline{\underline{-1 \mp i}}$

$$y(x) = e^{1x} \cdot (c_1 \cos x + c_2 \sin x) + e^{-1 \cdot x} \cdot (c_3 \cos x + c_4 \sin x).$$

Repeated Complex Roots

Theorem 2 holds for repeated complex roots. If the conjugate pair $a \pm bi$ has multiplicity k, then the corresponding part of the general solution has the form

$$\begin{aligned}(A_1 + A_2 x + \cdots + A_k x^{k-1})e^{(a+bi)x} &+ (B_1 + B_2 x + \cdots + B_k x^{k-1})e^{(a-bi)x} \\ &= \sum_{p=0}^{k-1} x^p e^{ax}(c_p \cos bx + d_p \sin bx). \end{aligned} \tag{24}$$

It can be shown that the $2k$ functions

$$x^p e^{ax} \cos bx, \quad x^p e^{ax} \sin bx, \quad 0 \leqq p \leqq k-1$$

that appear in Eq. (24) are linearly independent, and therefore generate a $2k$-dimensional subspace of the solution space of the differential equation.

Example:

Find a general solution of $(D^2 + 6D + 13)^2 y = 0$.

$$(r^2 + 6r + 13)^2 = 0 \Rightarrow ((r+3)^2 + 4)^2 = 0$$

$$(r+3)^2 + 4 = 0 \Rightarrow (r+3)^2 = -4$$

$$r + 3 = \mp 2i$$

$$r = -3 \mp 2i$$ has multiplicity $k=2$

$$y(x) = e^{-3x} \cdot (c_1 \cos 2x + d_1 \sin 2x) + x e^{-3x} (c_2 \cos 2x + d_2 \sin 2x)$$

$$= e^{-3x} \left((c_1 + c_2 x) \cos 2x + (d_1 + d_2 x) \sin 2x \right)$$

Example! Find a general solution of

$$y^{(3)} + y' - 10y = 0$$

$$r^3 + r - 10 = 0$$

$r = 2$, $\quad 8 + 2 - 10 = 0$ ✓

$(r-2)$ is a factor of $r^3 + r - 10$.

$$\begin{array}{l|l} r^3 + r - 10 & r-2 \\ \underline{-\; r^3 - 2r^2} & r^2 + 2r + 5 \\ 2r^2 + r - 10 & \\ \underline{-\; 2r^2 - 4r} & \\ \quad 5r - 10 & \\ \underline{\quad 5r - 10} & \\ \quad\quad 0 & \end{array}$$

$$r^3 + r - 10 = (r-2)(r^2 + 2r + 5) = 0$$

$$= (r-2) \cdot ((r+1)^2 + 4) = 0$$

$r_1 = 2$, $\quad r_{2,3} = -1 \mp 2i$

$$y(x) = c_1 e^{2x} + e^{-x}(c_2 \cos 2x + c_3 \sin 2x)$$

Example:

The roots of the characteristic equation of a certain differential equation are 3, −5, 0, 0, 0, 0, −5, $2 \pm 3i$, and $2 \pm 3i$. Write a general solution of this homogeneous differential equation.

$r_1 = 3$ has multiplicity $k=1$

$r_2 = -5$ has multiplicity $k=2$

$r_3 = 0$ has multiplicity $k=4$

$r_{4,5} = 2 \mp 3i$ has multiplicity $k=2$

$$y(x) = c_1 \cdot e^{3x} + (c_2 + c_3 x)\, e^{-5x} + (c_4 + c_5 x + c_6 x^2 + c_7 x^3)\, e^{0x}$$

$$+ e^{2x}\left((c_8 + c_9 x) \cos 3x + (c_{10} + c_{11} x) \sin 3x \right).$$

HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \quad , \quad (a_n \neq 0)$$

$\downarrow$

Offer a solution of the form $\boxed{y = e^{rx}}$

$$\underbrace{(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0)}_{=0} \underset{\neq 0}{\underset{\downarrow}{e^{rx}}} = 0$$

$\downarrow$

Characteristic Equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

$r_1, r_2, \ldots, r_n$: roots of the char. eq.

Real and Distinct Roots

$r_1 \neq r_2 \neq \cdots \neq r_n$, all real

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$

$\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$

$\downarrow$

basis

Repeated Real Roots

r: repeated real root of multiplicity k

$y = (c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{rx}$

$\{e^{rx}, x e^{rx}, \ldots, x^{k-1} e^{rx}\}$

$\downarrow$

basis

Complex Roots

$r = a \mp ib \quad (b \neq 0)$

$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$\{e^{ax} \cos bx, e^{ax} \sin bx\}$

$\downarrow$

basis

Repeated Complex Roots

$r = a \mp ib$ of multiplicity of k

$y = e^{ax}[(b_1 + b_2 x + \cdots + b_k x^{k-1}) \cos bx + (c_1 + c_2 x + \cdots + c_k x^{k-1}) \sin bx]$

$\{x^p e^{ax} \cos bx, x^p e^{ax} \sin bx\}$

$p = 0, 1, \ldots, k-1$

$\downarrow$

basis

Nonhomogeneous Equations

THEOREM 5 Solutions of Nonhomogeneous Equations

Let y_p be a particular solution of the nonhomogeneous equation in (2) on an open interval I where the functions p_i and f are continuous. Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the associated homogeneous equation in (3). If Y is any solution whatsoever of Eq. (2) on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) \tag{16}$$

for all x in I.

We now consider the *nonhomogeneous* nth-order linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x) \tag{2}$$

with associated homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0. \tag{3}$$

Suppose that a single fixed particular solution y_p of the nonhomogeneous equation in (2) is known, and that Y is any other solution of Eq. (2). If $y_c = Y - y_p$, then substitution of y_c in the differential equation gives (using the linearity of differentiation)

$$\begin{aligned} y_c^{(n)} + p_1 y_c^{(n-1)} &+ \cdots + p_{n-1}y_c' + p_n y_c \\ &= (Y^{(n)} + p_1 Y^{(n-1)} + \cdots + p_{n-1}Y' + p_n Y) \\ &\quad - (y_p^{(n)} + p_1 y_p^{(n-1)} + \cdots + p_{n-1}y_p' + p_n y_p) \\ &= f(x) - f(x) = 0. \end{aligned}$$

Thus $y_c = Y - y_p$ is a solution of the associated homogeneous equation in (3). Then

$$Y = y_c + y_p, \tag{14}$$

and it follows from Theorem 4 that

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n, \tag{15}$$

where $y_1, y_2, \ldots, y_n$ are linearly independent solutions of the associated *homogeneous* equation. We call y_c a **complementary function** of the nonhomogeneous equation and have thus proved that a *general solution* of the nonhomogeneous equation in (2) is the sum of its complementary function y_c and a single particular solution y_p of Eq. (2).

Method of Undetermined coefficients

This method is a straightforward way of doing an intelligent guess as to the general form of y_p.

nth order nonhom. Lin. equations with const. coeff.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x) \qquad (1)$$

↓

Find $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

↓

Is $f(x)$ in the form of a polynomial $P_m(x)$, $P_m(x)e^{ax}$, $P_m(x)e^{ax}\cos bt$ or $P_m(x)e^{ax}\sin bt$?

Yes → Method of Undetermined Coeff.

No → Variation of Parameters

Method of Undetermined Coeff.

$f(x)$	y_p
$P_m(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$	$x^s(A_0 x^m + \cdots + A_m)$
$P_m(x)e^{ax}$	$x^s(A_0 x^m + \cdots + A_m)e^{ax}$
$P_m(x)e^{ax}$ $\cos bx$ or/and $\sin bx$	$x^s[(A_0 x^m + \cdots + A_m)\cos bx + (B_0 x^m + \cdots + B_m)\sin bx]e^{ax}$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary func. y_c.

* Then, replace y_p in (1) to determine the values of the coeff in y_p.

→ $y = y_c + y_p$ ←

Example:

Find a particular solution of $y'' + 3y' + 4y = 3x + 2$.

$y'' + 3y' + 4y = f(x)$

$f(x) = 3x + 2$. is a polynomial of degree 1, so our guess is that

$y_p = Ax + B$ since the derivatives of a polynomial are themselves polynomials of lower degree.

A and B undetermined coefficient.

$y_p = Ax + B$, $y_p' = A$, $y_p'' = 0$

$y_p'' + 3y_p' + 4y_p = 3x + 2$ for all x.

$0 + 3A + 4Ax + 4B = 3x + 2$

$4A = 3$, $3A + 4B = 2$

$A = 3/4$

$B = -1/16$

$y_p(x) = \frac{3}{4}x - \frac{1}{16}$.

Example:

Find a particular solution of $y'' - 4y = 2e^{3x}$.

Any derivative of e^{3x} is a constant multiple of e^{3x}, so it is reasonable to try $y_p(x) = Ae^{3x}$

$$y_p(x) = Ae^{3x}, \quad y_p' = 3Ae^{3x}, \quad y_p'' = 9Ae^{3x}$$

$$y_p'' - 4y_p = 2e^{3x}$$

$$9Ae^{3x} - 4Ae^{3x} = 2e^{3x}$$

$$5A = 2$$

$$A = 2/5$$

$$y_p(x) = \frac{2}{5}e^{3x}.$$

Example:

Find a particular solution of $3y'' + y' - 2y = 2\cos x$.

A first guess might be $y_p(x) = A\cos x$, but the presence of y' on the left-hand side signals that we need a term involving $\sin x$ as well. So we try

$$y_p(x) = A\cos x + B\sin x$$
$$y_p'(x) = -A\sin x + B\cos x$$
$$y_p''(x) = -A\cos x - B\sin x$$

$$3y_p'' + y_p' - 2y_p = 2\cos x$$

$$-3A\cos x - 3B\sin x - A\sin x + B\cos x - 2A\cos x - 2B\sin x = 2\cos x$$

$$(-3A + B - 2A)\cos x + (-3B - A - 2B)\sin x = 2\cos x$$

$$\underbrace{(-5A+B)}_{=2}\cos x + \underbrace{(-5B - A)}_{=0}\sin x = 2\cos x$$

$$\left.\begin{aligned} A + 5B &= 0 \\ -5A + B &= 2 \end{aligned}\right\} \quad B = \frac{1}{13}, \quad A = \frac{-5}{13}$$

$$y_p(x) = -\frac{5}{13}\cos x + \frac{1}{13}\sin x.$$

Example:

Find a particular solution of $y'' - 4y = 2e^{2x}$.

If we try $y_p(x) = Ae^{2x}$, we find that

$$y_p'' - 4y_p = 4Ae^{2x} - 4Ae^{2x} = 0 \neq 2e^{2x}$$

Infact, the preceding computation shows that Ae^{2x} satisfies instead the associated homogeneous equation. Therefore, we should begin with a trial function $y_p(x)$ whose derivative involves both e^{2x} and something else that can cancel upon substitution into the differential eq. to leave the e^{2x} term that we need. A reasonable guess is

$$y_p(x) = Axe^{2x}$$

$$y_p(x) = Axe^{2x}$$

$$y_p'(x) = Ae^{2x} + 2Axe^{2x}$$

$$y_p''(x) = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$$

$$y_p'' - 4y_p = 2e^{2x}$$

$$4Ae^{2x} + 4Axe^{2x} - 4Axe^{2x} = 2e^{2x}$$

$$4A = 2 \Rightarrow A = \frac{1}{2}$$

$$y_p(x) = \frac{1}{2}xe^{2x}$$

Example: solve the initial value problem

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x \quad y(0) = 1 \quad y'(0) = 2.$$

$$r^2 - 3r + 2 = 0 \quad (r-2)(r-1) = 0 \quad r_1 = 2,\ r_2 = 1$$

$$y_c(x) = c_1 e^{x} + c_2 e^{2x}$$

$$y_p(x) = A e^{-x} + B\cos 3x + C\sin 3x$$

$$y_p'(x) = -A e^{-x} + 3.B(-\sin 3x) + 3.C\cos 3x$$

$$y_p''(x) = A e^{-x} - 9B\cos 3x - 9C\sin 3x$$

$$y_p'' - 3y_p' + 2y_p = 3e^{-x} - 10\cos 3x$$

$$(A + 3A + 2A)e^{-x} + (-9B - 9C + 2B)\cos 3x + (-9C + 9B + 2C)\sin 3x = 3e^{-x} - 10\cos 3x$$

$$6A = 3 \qquad -7B - 9C = -10 \qquad -7C + 9B = 0$$

$$A = \frac{1}{2} \qquad B = \frac{7}{13}, \quad C = \frac{9}{13}$$

$$y_p(x) = \frac{1}{2} e^{-x} + \frac{7}{13} \cos 3x + \frac{9}{13} \sin 3x$$

$$y(x) = y_c(x) + y_p(x)$$

$$= c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{-x} + \frac{7}{13} \cos 3x + \frac{9}{13} \sin 3x$$

$$\left.\begin{array}{ll} y(0) = 1 \Rightarrow & c_1 + c_2 + \frac{1}{2} + \frac{7}{13} = 1 \\ y'(0) = 2 \Rightarrow & c_1 + 2c_2 - \frac{1}{2} + \frac{27}{13} = 2 \end{array}\right\} \quad \begin{array}{l} c_1 + c_2 = \frac{-1}{26} \\ c_1 + 2c_2 = \frac{11}{26} \end{array}$$

$$c_1 = \frac{-1}{2}, \quad c_2 = \frac{6}{13}$$

$$y(x) = -\frac{1}{2} e^{x} + \frac{6}{13} e^{2x} + \frac{1}{2} e^{-x} + \frac{7}{13} \cos 3x + \frac{9}{13} \sin 3x.$$

Example: Find the general form of a particular solution of the following differential equation:

a) $y^{(3)} + 9y' = x\sin x + x^2 e^{2x}$

$y^{(3)} + 9y' = 0 \Rightarrow$ The characteristic equation is

$$r^3 + 9r = 0$$

$$r(r^2+9) = 0 \qquad (r_{2,3} = \mp 3i)$$

$$r = 0, \quad r_2 = 3i \quad r_3 = -3i$$

$$y_c(x) = c_1 \cdot e^{0x} + e^{0x}(c_2 \cos 3x + c_3 \sin 3x)$$

$$= c_1 + c_2 \cos 3x + c_3 \sin 3x.$$

$y^{(3)} + 9y' = x\sin x + x^2 e^{2x}$ There is no duplication with the terms of the complementary function.

The trial solution takes the form:

$$y_p(x) = (A + Bx)\sin x + (C + Dx).\cos x$$
$$+ (E + Fx + Gx^2).e^{2x}$$

b) $y^{(3)} + y'' = 3e^x + 4x^2$

$y^{(3)} + y'' = 0 \Rightarrow$ The characteristic Equation is

$$r^3 + r^2 = 0$$
$$r^2(r+1) = 0$$
$$r_1 = r_2 = 0, \quad r_3 = -1. \qquad s = 2.$$

$$y_c(x) = c_1 e^{-x} + c_2 + c_3 x$$

$y^{(3)} + y'' = 3e^x + 4x^2$

There is a duplication with the term involving x^n.

$$y_p(x) = Ae^x + (B + Cx + Dx^2).x^2$$