Reducible Second-Order Equations

A *second-order differential equation* involves the second derivative of the unknown function $y(x)$, and thus has the general form

$$F(x, y, y', y'') = 0. \tag{32}$$

If *either* the dependent variable y *or* the independent variable x is missing from a second-order equation, then it is easily reduced by a simple substitution to a first-order equation that may be solvable by the methods of this chapter.

Dependent variable y missing. If y is missing, then Eq. (32) takes the form

$$F(x, y', y'') = 0. \tag{33}$$

Then the substitution

$$p = y' = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx} \tag{34}$$

results in the *first-order* differential equation

$$F(x, p, p') = 0.$$

If we can solve this equation for a general solution $p(x, C_1)$ involving an arbitrary constant C_1, then we need only write

$$y(x) = \int y'(x)\,dx = \int p(x, C_1)\,dx + C_2$$

to get a solution of Eq. (33) that involves two arbitrary constants C_1 and C_2 (as is to be expected in the case of a second-order differential equation).

Example:

Solve the equation $xy'' + 2y' = 6x$ in which the dependent variable y is missing.

$p = y' \qquad y'' = p' \qquad \Rightarrow \quad xp' + 2p = 6x$

$y + P(x)y = Q(x)$

$\Rightarrow \quad p' + \frac{2}{x}p = 6$ linear 1st order ODE.

$\rho(x) = e^{\int P(x)dx}$

$$\rho(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$x^2 p' + 2x\,p = 6x^2$

$D_x(x^2 \cdot p) = 6x^2 \quad \Rightarrow \quad x^2 p = 2x^3 + C_1 \Rightarrow p = 2x + \frac{C_1}{x^2}$

$p = y' \Rightarrow \quad y(x) = \int p\,dx = x^2 - \frac{C_1}{x} + C_2 .$

Independent variable x missing. If x is missing, then Eq. (32) takes the form

$$F(y, y', y'') = 0. \tag{35}$$

Then the substitution

$$p = y' = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy} \tag{36}$$

$p = P(y)$

results in the *first-order* differential equation

$$F\left(y, p, p\frac{dp}{dy}\right) = 0$$

for p as a function of y. If we can solve this equation for a general solution $p(y, C_1)$ involving an arbitrary constant C_1, then (assuming that $y' \neq 0$) we need only write

$$x(y) = \int \frac{dx}{dy}\,dy = \int \frac{1}{dy/dx}\,dy = \int \frac{1}{p}\,dy = \int \frac{dy}{p(y, C_1)} + C_2.$$

If the final integral $P = \int (1/p)\,dy$ can be evaluated, the result is an implicit solution $x(y) = P(y, C_1) + C_2$ of our second-order differential equation.

Example:

$, \; y > 0 \quad y' > 0 .$

Solve the equation $yy'' = (y')^2$ in which the independent variable x is missing.

$p = y' , \quad y'' = p \cdot \frac{dp}{dy}$

$y \cdot p \frac{dp}{dy} = p^2 \quad , \; p \neq 0$

$\frac{dp}{p} = \frac{dy}{y} \quad \Rightarrow \quad \ln p = \ln y + \ln c_1$

$p = c_1 \cdot y$

$p = y' \Rightarrow \frac{dy}{dx} = c \cdot y \quad \Rightarrow \quad \frac{dy}{y} = c_1 dx$

$\Rightarrow \quad \ln y = c_1 x + \ln c_2$

$y = c_2 e^{c_1 x} .$

1st order DE $\quad \frac{dy}{dx} = f(x,y)$

Seperable

$y' = F(x)\cdot G(y)$

$$\int \frac{dy}{G(y)} = \int \frac{dx}{F(x)}$$

Linear

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\rho(x) = e^{\int P(x)dx}$$

$$D_x(\rho(x)\cdot y) = \rho(x)Q(x)$$

$$\rho(x)\cdot y = \int \rho(x)Q(x)\,dx$$

Substitution

Homogeneous

$y' = f\left(\frac{y}{x}\right)$

$v = \frac{y}{x}, \; y = vx$

$v + xv' = f(v)$

$$\frac{dv}{f(v)-v} = \frac{dx}{x}$$

↓

seperable

Bernoulli

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$v = y^{1-n}$

$v' = (1-n)y^{-n}\cdot y'$

$v' + (1-n)Pv = (1-n)Q$

↓

Linear 1st order

Exact

→ Exact : $M(x,y)\,dx + N(x,y)\,dy = 0$

$M_y = N_x$. Find $F(x,y)$ such that $F_x = M$, $F_y = N$.

$F_x = M \Rightarrow F(x,y) = \int M(x,y)\,dx + g(y)$ $\quad F_y = N \Rightarrow \frac{d}{dy}\left(\int M(x,y)\,dx + g(y)\right) = N$

$F(x,y) = C.$

Review Problems

Find general solutions of the following differential equations.

1. $2x^2y - x^3y' = y^3$

$$y' = \frac{2x^2y - y^3}{x^3} = 2\cdot\frac{y}{x} - \left(\frac{y}{x}\right)^3 \qquad \text{homogeneous.}$$

$$v = \frac{y}{x}, \quad y = vx \qquad \frac{dy}{dx} = \frac{dv}{dx}\cdot x + v = 2v - v^3$$

$$x\frac{dv}{dx} = v - v^3 \Rightarrow \frac{dv}{v-v^3} = \frac{dx}{x} \qquad \frac{dv}{v(v-1)(v+1)} = -\frac{dx}{x}$$

seperable

$$\frac{1}{v(v-1)(v+1)} = \frac{A}{v} + \frac{B}{v-1} + \frac{C}{v+1}$$

$$1 = A\cdot(v^2-1) + B(v^2+v) + C(v^2-v)$$

$$v=0 \Rightarrow A=-1, \quad v=1 \Rightarrow B=1/2, \quad v=-1 \Rightarrow C=1/2$$

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k},\ k=1,2,3,\ldots$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k},\ k=1,2,3,\ldots$

$$-\int \frac{dv}{v} + \frac{1}{2}\int \frac{dv}{v-1} + \frac{1}{2}\int \frac{dv}{v+1} = -\int \frac{dx}{x}$$

$$-\ln|v| + \frac{1}{2}\ln|v-1| + \frac{1}{2}\ln|v+1| = -\ln|x| + \frac{\ln C}{2}.$$

$$-2\ln|v| + \ln|v-1| + \ln|v+1| = -2\ln|x| + \ln C.$$

$$\ln\frac{|v^2-1|}{v^2} = \ln\frac{C}{x^2} \Rightarrow \frac{|v^2-1|}{v^2} = \frac{C}{x^2}$$

$$\left|x^2\left(1-\frac{1}{v^2}\right)\right| = C \Rightarrow x^2\left(1-\frac{1}{v^2}\right) = D \quad ; \quad C = \mp D$$

$$v = \frac{y}{x} \Rightarrow x^2\cdot\left(1-\frac{x^2}{y^2}\right) = D \Rightarrow x^2(y^2-x^2) = y^2 D$$

$$\Rightarrow (x^2-D)\,y^2 = x^4$$

$$\Rightarrow y^2 = \frac{x^4}{x^2-D} \Rightarrow y = \mp\sqrt{\frac{x^4}{x^2-D}} \Rightarrow y(x) = \mp\frac{x^2}{\sqrt{x^2-D}}.$$

OR

$$2x^2y - x^3y' = y^3$$

$$y' - \frac{2}{x}y = -\frac{1}{x^3}y^3 \qquad \text{Bernoulli Equation.}$$

$$v = y^{1-n} = y^{-2} \qquad \frac{dv}{dx} = -2 \cdot y^{-3} \cdot y'$$

$$\frac{dv}{dx} + \frac{4}{x}v = \frac{2}{x^3} \qquad \text{(Linear Equation)}$$

$$f(x) = \exp\left(\int \frac{4}{x}dx\right) = \exp(4\ln x) = \exp(\ln x^4) = x^4$$

$$x^4\frac{dv}{dx} + 4x^3v = 2x \quad , \quad \frac{d}{dx}(x^4 \cdot v) = 2x \Rightarrow x^4 \cdot v = x^2 + C$$

$$v(x) = \frac{1}{x^2} + \frac{c}{x^4}$$

$$\frac{1}{(y(x))^2} = \frac{1}{x^2} + \frac{c}{x^4} \Rightarrow (y(x))^{-2} = \frac{x^2+c}{x^4} \Rightarrow y(x) = \mp\sqrt{\frac{x^4}{x^2+c}}$$

$$\rightsquigarrow y(x) = \mp\frac{x^2}{\sqrt{x^2+c}}.$$

2. $e^y + y\cos x + (xe^y + \sin x)\, y' = 0$

$$\underbrace{(e^y + y\cos x)}_{M}\, dx + (\underbrace{x e^y + \sin x}_{N})\, dy = 0$$

$M = e^y + y\cos x$, $N = xe^y + \sin x$

$M_y = e^y + \cos x$, $N_x = e^y + \cos x$ $\Rightarrow$ $M_y = N_x$ exact.

Find $F(x,y)$ such that $F_x = M$, $F_y = N$.

$F_x = M = e^y + y\cos x \Rightarrow F(x,y) = xe^y + y\sin x + g(y)$

$F_y = xe^y + \sin x + g'(y) = N = xe^y + \sin x \Rightarrow g'(y) = 0$

$g(y) = c_1$

$\Rightarrow xe^y + y\sin x = c.$ $(F(x,y) = c)$.

3 $(x^2-1)y' + (x-1)y = 1$

$$y' + \underbrace{\frac{1}{x+1}}_{p(x)} y = \underbrace{\frac{1}{x^2-1}}_{Q(x)} , \quad x \neq \pm 1 \quad (\text{linear})$$

$$p(x) = \frac{1}{x+1} \qquad \rho(x) = \exp\left(\int p(x)\,dx \right) = \exp\left(\int \frac{1}{x+1}\,dx \right)$$

$$= e^{\ln(x+1)} = x+1$$

$$(x+1)y' + y = \frac{1}{x-1} .$$

$$\frac{d}{dx}\left((x+1) \cdot y \right) = \frac{1}{x-1} \Rightarrow (x+1)y = \ln|x-1| + C$$

$$y(x) = \frac{\ln|x-1| + C}{x+1} .$$

4 $y' = \sqrt{x+y}$

$$v = x+y \Rightarrow \frac{dv}{dx} = 1 + \frac{dy}{dx} = 1 + \sqrt{x+y} = 1+\sqrt{v}$$

$$\frac{dv}{dx} = 1+\sqrt{v} \Rightarrow \frac{dv}{1+\sqrt{v}} = dx$$ (seperable)

$$\frac{\sqrt{v}}{1+\sqrt{v}} \cdot \frac{dv}{\sqrt{v}} = dx \qquad \sqrt{v} = t \Rightarrow \frac{dv}{2\sqrt{v}} = dt \Rightarrow \frac{dv}{\sqrt{v}} = 2dt$$

$$\frac{t}{1+t} 2dt = dx \Rightarrow 2\left(1 - \frac{1}{1+t}\right) dt = dx$$

$$\Rightarrow 2(t - \ln|1+t|) = x + c.$$

$$\Rightarrow 2(\sqrt{v} - \ln|1+\sqrt{v}|) = x+c \Rightarrow 2.(\sqrt{x+y} + \ln|1+\sqrt{x+y}|) = x+c.$$

⑤ $$\frac{dy}{dx} = \frac{2xy+2x}{x^2+1}$$

$$\frac{dy}{dx} - \frac{2x}{x^2+1}\, y = \frac{2x}{(x^2+1)^2}$$ linear

$$\rho(x) = \exp\left(-\int \frac{2x}{x^2+1}\, dx\right) = \exp\left(-\ln(x^2+1)\right) = \frac{1}{x^2+1}.$$

$$\frac{1}{x^2+1}\frac{dy}{dx} - \frac{2x}{(x^2+1)^2}\, y = \frac{2x}{(x^2+1)^2} \Rightarrow \frac{d}{dx}\left(\frac{1}{x^2+1}\, y\right) = \frac{2x}{(x^2+1)^2}$$

$$\Rightarrow \frac{y}{x^2+1} = -\frac{1}{x^2+1} + C. \Rightarrow y(x) = -1 + C(x^2+1).$$

or $\frac{dy}{dx} = \frac{2x}{x^2+1}(1+y) \Rightarrow \frac{dy}{y+1} = \frac{2x}{x^2+1}\,dx$ seperable

$$\Rightarrow \ln|y+1| = \ln|x^2+1| + \ln C.$$

$$|y+1| = C \cdot |x^2+1|$$

$$\Rightarrow \quad y+1 = D \cdot (x^2+1) \text{ , where } D \mp \mp C.$$

$$y(x) = -1 + D(x^2+1).$$

(6) $\dfrac{dy}{dx} = \dfrac{x+3y}{y-3x}$

$\dfrac{dy}{dx} = \dfrac{1+\ 3(y/x)}{(y/x)\ -3}$ homogeneous.

$y = vx \Rightarrow \dfrac{dy}{dx} = v + x\cdot\dfrac{dv}{dx} = \dfrac{1+3v}{v-3} \Rightarrow x\dfrac{dv}{dx} = \dfrac{1+6v-v^2}{v-3}$

$\Rightarrow \dfrac{v-3}{v^2+6v-v^2}\,dv = -\dfrac{dx}{x} \Rightarrow \dfrac{1}{2}\cdot\dfrac{2v-6}{v^2-6v-1}\,dv = -\dfrac{dx}{x}.$

$\Rightarrow \dfrac{1}{2}\ln|v^2-6v-1| = -\ln|x| + \dfrac{\ln c}{2}$

$\Rightarrow \ln|x^2\cdot(v^2-6v-1)| = \ln c$

$\Rightarrow |x^2(v^2-6v-1)| = c$

$\Rightarrow x^2(v^2-6v-1) = D \quad , D = \mp c$

$\Rightarrow x^2\left(\left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right) - 1\right) = D \rightsquigarrow y^2 - 6xy - x^2 = D.$

OR $$\frac{dy}{dx} = \frac{x+3y}{y-3x}$$

$$\Rightarrow (y-3x)\,dy - (x+3y)\,dx = 0$$

$$\Rightarrow \underbrace{(x+3y)}_{M}\,dx + \underbrace{(3x-y)}_{N}dy = 0$$

$M = x+3y \qquad N = 3x-y$

$M_y = 3 \quad = \quad N = 3 \qquad$ Exact

$$F_x = M = x+3y \Rightarrow F(x,y) = \frac{x^2}{2} + 3xy + g(y)$$

$$F_y(x,y) = 3x + g'(y) = N = 3x-y$$

$$\Rightarrow g'(y) = -y \Rightarrow g(y) = -\frac{y^2}{2} + C_1$$

$$F(x,y) = \frac{x^2}{2} + 3xy - \frac{y^2}{2} + C_1 \Rightarrow x^2 + 6xy - y^2 = C. \quad (F(x,y) = C)$$

Higher-Order Linear Differential Equations

General Solutions of Linear Equations

Consider the general nth-order linear differential equation of the form

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x). \tag{1}$$

Unless otherwise noted, we will always assume that the coefficient functions $P_i(x)$ and $F(x)$ are continuous on some open interval I (perhaps unbounded), where we wish to solve the equation. Under the additional assumption that $P_0(x) \neq 0$ at each point of I, we can divide each term in Eq. (1) by $P_0(x)$ to obtain an equation with leading coefficient 1, of the form

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x). \tag{2}$$

The **homogeneous** linear equation **associated with** Eq. (2) is

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0. \tag{3}$$

THEOREM 1 Existence and Uniqueness for Linear Equations

Suppose that the functions $p_1, p_2, \ldots, p_n$, and f are continuous on the open interval I containing the point a. Then, given n numbers $b_0, b_1, \ldots, b_{n-1}$, the nth-order linear equation [Eq. (2)]

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the n initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}. \tag{5}$$

Note that Theorem 1 implies that the trivial solution $y(x) = 0$ is the only solution of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

that satisfies the trivial initial conditions

$$y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0.$$

Example: $x^2 y'' - 4xy' + 6y = 0 \qquad y(0) = y'(0) = 0$

$$y'' - \frac{4}{x} y' + \frac{6}{x^2} y = 0$$

$p(x) = -\frac{4}{x}$, $q(x) = \frac{6}{x^2}$ are not continuous on an open interval containing the point $x = 0$.

On the other hand, $y_1(x) = x^2$ and $y_2(x) = x^3$ are two different solutions. This is not contradict the uniqueness part of Theorem 1.

THEOREM 2 Principle of Superposition for Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear equation in (3) on the interval I. If $c_1, c_2, \ldots, c_n$ are constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \tag{4}$$

is also a solution of Eq. (3) on I.

<u>proof in case $n=2$:</u> Let y_1 and y_2 be solutions of the equation

$$y'' + p(x)y' + q(x)y = 0$$

and $y = c_1 y_1 + c_2 y_2$

$\Rightarrow \quad y' = c_1 y_1' + c_2 y_2', \quad y'' = c_1 y_1'' + c_2 y_2''$

$$y'' + p(x)\,y' + q(x)\,y = (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2')$$
$$+ q(x)(c_1 y_1 + c_2 y_2)$$
$$= c_1 (\underbrace{y_1'' + p(x) y_1' + q(x) y_1}_{=0})$$
$$+ c_2 (\underbrace{y_2'' + p(x) y_2' + q(x) y_2}_{=0})$$

because y_1 and y_2 are solutions

$$\Rightarrow \quad y'' + p(x) y' + q(x) y = 0$$

$\Rightarrow y$ is also a solution.

Remark: The set S of all solutions of the homogeneous linear equation is a subspace of the vector space of all functions on the real line $\mathbb{R}$. Therefore S is called the solution space of the differential equation.

Example: Verify that three functions

$$y_1(x) = e^{-3x}, \quad y_2(x) = \cos^2 x \quad \text{and} \quad y_3(x) = \sin^2 x$$

are solutions of the homogeneous third-order equation

$$y^{(3)} + 3y'' + 4y' + 12y = 0$$

on the entire real line.

$$y_1(x) = e^{-3x}, \quad y_1'(x) = -3.e^{-3x}, \quad y_1''(x) = 9e^{-3x}, \quad y_1^{(3)}(x) = -27e^{-3x}$$

$$y_1^{(3)} + 3y_1'' + 4y_1' + 12y = -27e^{-3x} + 3.9.3^{-3x} + 4.(-3).e^{-3x} + 12.e^{-3x} = 0$$

Similarly, it is possible to show that

$$y_2^{(3)} + 3y_2'' + 4y_2' + 12y_2 = 0 \quad \text{and} \quad y_3^{(3)} + 3y_3'' + 4y_3' + 12y_3 = 0.$$

Remark: 1) $y(x) = -3y_1(x) + 3y_2(x) - 2y_3(x)$

$$= 3e^{-3x} + 3\cos 2x - 2\sin 2x$$

is also a solution on the entire real line by Principle of superposition. Thus a general solution is given by

$$y(x) = c_1 e^{-3x} + c_2\cos 2x + c_3 \sin 2x.$$

2) Because its general solution involves the three arbitrary constants c_1, c_2 and c_3, the third-order equation has a "three fold infinity" of solutions.

$y(x) = c_1 e^{-3x}$, $y(x) = c_2\cos 2x$, $y(x) = c_3\sin 2x$.

3) The particular solution $y(x) = -3e^{-3x} + 3\cos 2x - 2\sin 2x$ has initial values $y(0)=0$, $y'(0) = 9e^{-3x} - 4\cos 2x - 6\sin 2x\Big|_{x=0} = 5$

$y''(0) = -27e^{-3x} + 8\cos 2x - 12\cos 2x\Big|_{x=0} = -39$. Thus the solution of I.V.P. $y^{(3)} + 3y'' + 4y' + 12y = 0$ $y(0)=0$, $y'(0)=5$, $y''(0) = -39$

is $y(x) = -3e^{-3x} + 3\cos 2x - 2\sin 2x$.

Linear Independent Solutions :

DEFINITION Linear Dependence of Functions

The n functions $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \tag{7}$$

on I; that is,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x in I.

They are linearly independent on I provided that the identity

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

holds on I only in the trivial case

$$c_1 = c_2 = \cdots = c_n = 0.$$

Example: The functions

$f_1(x) = \sin 2x \qquad f_2(x) = \sin x . \cos x$ and $f_3(x) = e^x$

are linearly dependent on the real line because.

$(1).f_1 + (-2) f_2 + (0) f_3 = 0 \qquad (\sin 2x = 2\sin x \cos x)$

Example: Show that the functions $y_1(x) = e^{-3x}$

$y_2(x) = \cos 2x$ and $y_3(x) = \sin 2x$ are linearly independent.

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

$$c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin 2x = 0 \qquad \text{for all } x \in \mathbb{R}.$$

$$-3c_1 e^{-3x} - 2c_2 \sin 2x + 2c_3 \cos 2x = 0$$

$$9c_1 e^{-3x} - 4c_2 \cos 2x - 4c_3 \sin 2x = 0.$$

$$\begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= e^{-3x} \cdot \begin{vmatrix} -2\sin 2x & 2\cos 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} - (-3)e^{-3x} \cdot \begin{vmatrix} \cos 2x & \sin 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$+ 9 \cdot e^{-3x} \cdot \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 8e^{-3x} + 0 + 18e^{-3x}$$

$$= 26e^{-3x} \neq 0$$

Because $W \neq 0$ everywhere, $c_1 = c_2 = c_3 = 0$.

y_1, y_2, y_3 are linearly independent.

Theorem 3: Wronskians of Solutions

Suppose that $y_1, y_2, \ldots, y_n$ are n solutions of the homogeneous nth-order linear equation

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_{n-1}(x)\, y' + P_n(x)\, y = 0$$

on an open interval I where each P_j is continuous.

Let

$$W = W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

(Wronskian of n functions $y_1, y_2, \ldots, y_n$)

a) If $y_1, y_2, \ldots, y_n$ are linearly dependent, then $W = 0$ on I.

b) If $y_1, y_2, \ldots, y_n$ are linearly independent, then $W \neq 0$ at each point of I.

Remark: To show that the functions $f_1, f_2, \dots, f_n$ are linearly independent on the interval I, it suffices to show that their wronskian is nonzero at just one point of I.

proof:

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$
$$c_1 y_1' + c_2 y_2' + \cdots + c_n y_n' = 0$$
$$\vdots$$
$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \cdots + c_n y_n^{(n-1)} = 0$$

(a) A homogeneous $n \times n$ linear system of equations has a nontrivial solution If and only if $|W| = 0$.

(b) Assume that $W(a) = 0$ at some point of I. Show that are linearly dependent This implies the solutions $y_1, y_2, \dots, y_n$ are linearly dependent

$\Rightarrow$ $c_1, c_2, \dots, C_n$ are not all zero.

Define $Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ (particular solution)

$$\Rightarrow \quad Y(a) = Y'(a) = \cdots = Y^{(n-1)}(a) = 0 \quad \text{on } I$$

$$\Rightarrow \quad Y(x) = 0.$$

Example: Show that the three solutions

$$y_1(x) = x \quad , \quad y_2(x) = x \ln x \; , \; y_3(x) = x^2$$

of the third-order equation

$$x^3 y^{(3)} - x^2 y'' + 2xy' - 2y = 0$$

are linearly independent on the open interval $x > 0$. Then find a particular solution that satisfies the initial conditions

$$y(1) = 3 , \quad y'(1) = 2 , \quad y''(1) = 1 .$$

$$W = \begin{vmatrix} x & x\ln x & x^2 \\ 1 & \ln x + x \cdot \frac{1}{x} & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix} = \begin{vmatrix} x & x\ln x & x^2 \\ 1 & \ln x + 1 & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix}$$

$$= x \cdot \begin{vmatrix} \ln x + 1 & 2x \\ \frac{1}{x} & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} x\ln x & x^2 \\ \frac{1}{x} & 2 \end{vmatrix} = x \cdot \left(2\ln x + 2 - 2 \cdot x \cdot \frac{1}{x}\right)$$

$$- 1 \cdot \left(2x \ln x - \frac{1}{x} x^2\right)$$

$$= 2x \ln x - 2x \ln x + x = x > 0$$

$$\rightsquigarrow W \neq 0$$

Thus y_1, y_2 and y_3 are linearly independent on the interval $x > 0$.

$$y(x) = c_1 x + c_2 x \ln x + c_3 x^2 \xRightarrow{y(1)=3} c_1 + c_3 = 3$$

$$y'(x) = c_1 + c_2(1 + \ln x) + 2c_3 x \xRightarrow{y'(1)=2} c_1 + c_2 + 2c_3 = 2$$

$$y''(x) = \frac{c_2}{x} + 2c_3 \xRightarrow{y''(1)=1} c_2 + 2c_3 = 1$$

$$c_1 + 1 = 2 \Rightarrow c_1 = 1$$

$$c_1 + c_3 = 3 \Rightarrow c_3 = 2$$

$$c_2 + 2c_3 = 1 \Rightarrow c_2 = -3.$$

$$y(x) = x - 2x \ln x + 2x^2.$$

THEOREM 4 General Solutions of Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \qquad (3)$$

on an open interval I, where the p_i are continuous. If Y is any solution whatsoever of Eq. (3), then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

for all x in I.