THEOREM 1 Bases as Maximal Linearly Independent Sets

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for the vector space V. Then any set of more than n vectors in V is linearly dependent.

THEOREM 2 The Dimension of a Vector Space

Any two bases for a vector space consist of the same number of vectors.

A nonzero vector space V is called **finite dimensional** provided that there exists a basis for V consisting of a finite number of vectors from V. In this case the number n of vectors in each basis for V is called the **dimension** of V, denoted by $n = \dim V$. Then V is an *n-dimensional* vector space.

Note that the zero vector space $\{\mathbf{0}\}$ has no basis because it contains *no* linearly independent set of vectors. (Sometimes it is convenient to adopt the convention that the null set is a basis for $\{\mathbf{0}\}$.) Here we define $\dim\{\mathbf{0}\}$ to be zero. A nonzero vector space that has no finite basis is called **infinite dimensional**.

Let $\mathcal{P}$ be the set of all polynomials of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where the largest exponent $n \geq 0$ that appears is the *degree* of the polynomial $p(x)$, and the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real numbers. We add polynomials in $\mathcal{P}$ and multiply them by scalars in the usual way—that is, by collecting coefficients of like powers of x. For instance, if

$$p(x) = 3 + 2x + 5x^3 \quad \text{and} \quad q(x) = 7 + 4x + 3x^2 + 9x^4,$$

then

$$\begin{aligned}(p+q)(x) &= (3+7) + (2+4)x + (0+3)x^2 + (5+0)x^3 + (0+9)x^4 \\ &= 10 + 6x + 3x^2 + 5x^3 + 9x^4\end{aligned}$$

and

$$(7p)(x) = 7(3 + 2x + 5x^3) = 21 + 14x + 35x^3.$$

It is readily verified that, with these operations, $\mathcal{P}$ is a vector space. But $\mathcal{P}$ has no finite basis. For if $p_1, p_2, \ldots, p_n$ are elements of $\mathcal{P}$, then the degree of any linear combination of them is at most the maximum of *their* degrees. Hence no polynomial in $\mathcal{P}$ of higher degree lies in $\operatorname{span}\{p_1, p_2, \ldots, p_n\}$. Thus no finite subset of $\mathcal{P}$ spans $\mathcal{P}$, and therefore $\mathcal{P}$ is an infinite dimensional vector space. ■

THEOREM 3 Independent Sets, Spanning Sets, and Bases

Let V be an n-dimensional vector space and let S be a subset of V. Then

(a) If S is linearly independent and consists of n vectors, then S is a basis for V;

(b) If S spans V and consists of n vectors, then S is a basis for V;

(c) If S is linearly independent, then S is contained in a basis for V;

(d) If S spans V, then S contains a basis for V.

Bases for Solution Spaces

Example:

Find a basis for the solution space of the homogeneous linear system

$$\begin{aligned} 3x_1 + 6x_2 - x_3 - 5x_4 + 5x_5 &= 0 \\ 2x_1 + 4x_2 - x_3 - 3x_4 + 2x_5 &= 0 \\ 3x_1 + 6x_2 - 2x_3 - 4x_4 + x_5 &= 0. \end{aligned}$$

ALGORITHM A Basis for the Solution Space

To find a basis for the solution space W of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$, carry out the following steps.

1. Reduce the coefficient matrix $\mathbf{A}$ to echelon form.
2. Identify the r leading variables and the $k = n - r$ free variables. If $k = 0$, then $W = \{\mathbf{0}\}$.
3. Set the free variables equal to parameters $t_1, t_2, \ldots, t_k$, and then solve by back substitution for the leading variables in terms of these parameters.
4. Let $\mathbf{v}_j$ be the solution vector obtained by setting t_j equal to 1 and the other parameters equal to zero. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for W.

1. Reduce the coefficient matrix **A** to echelon form.

$$\begin{bmatrix} 3 & 6 & -1 & -5 & 5 \\ 2 & 4 & -1 & -3 & 2 \\ 3 & 6 & -2 & -4 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 2 & 4 & -1 & -3 & 2 \\ 3 & 6 & -2 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to (-3)R_1 + R_3]{R_2 \to (-2)R_1 + R_2} \begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & -1 & 1 & -4 \\ 0 & 0 & -2 & 2 & -8 \end{bmatrix} \xrightarrow{R_2 \to (-1)R_2} \begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & -2 & 2 & -8 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 2R_2 + R_3} \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

2. Identify the r leading variables and the $k = n - r$ free variables. If $k = 0$, then $W = \{\mathbf{0}\}$.

Leading variables: x_1, x_3

Free variables: x_2, x_4, x_5

3. Set the free variables equal to parameters $t_1, t_2, \ldots, t_k$, and then solve by back substitution for the leading variables in terms of these parameters.

$$x_1 + 2x_2 - 2x_4 + 3x_5 = 0$$
$$x_3 - x_4 + 4x_5 = 0$$

$x_2 = r$
$x_4 = s$
$x_5 = t$

$\Rightarrow x_3 = s - 4t \qquad x_1 = -2r + 2s - 3t$

$$x = (-2r + 2s - 3t, r, s - 4t, s, t)$$
$$= r \cdot (-2, 1, 0, 0, 0) + s \cdot (2, 0, 1, 1, 0) + t \cdot (-3, 0, -4, 0, 1)$$

4. Let $\mathbf{v}_j$ be the solution vector obtained by setting t_j equal to 1 and the other parameters equal to zero. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for W.

$r.(-2,1,0,0,0) + s.(2,0,1,1,0) + t.(-3,0,-4,0,1)$

$r=1,\ s=0,\ t=0 \qquad v_1 = (-2,1,0,0,0)$

$s=1,\ r=t=0 \qquad v_2 = (2,0,1,1,0)$

$t=1,\ r=s=0 \qquad v_3 = (-3,0,-4,0,1)$

$\{v_1, v_2, v_3\}$ is a basis for the solution space and its dimension is 3.

Row and Column Spaces

Row Space and Row Rank

The individual equations of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ correspond to the "row matrices"

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$
$$\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$
$$\vdots$$
$$\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

of the $m \times n$ matrix $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$. The **row vectors** of $\mathbf{A}$ are the m vectors

$$\begin{aligned} \mathbf{r}_1 &= (a_{11}, a_{12}, \ldots, a_{1n}) \\ \mathbf{r}_2 &= (a_{21}, a_{22}, \ldots, a_{2n}) \\ &\vdots \\ \mathbf{r}_m &= (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{aligned} \tag{1}$$

in $\mathbf{R}^n$.

$$\mathbf{r}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}^T$$

for $i = 1, 2, \ldots, m$, and thus actually are column vectors (despite being called "row vectors"). The subspace of $\mathbf{R}^n$ spanned by the m row vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ is called the **row space** $\text{Row}(\mathbf{A})$ of the matrix $\mathbf{A}$. The *dimension* of the row space $\text{Row}(\mathbf{A})$ is called the **row rank** of the matrix $\mathbf{A}$.

Column Space and Column Rank

Now we turn our attention from row vectors to column vectors. Given an $m \times n$ matrix $\mathbf{A} = [a_{ij}]$, the **column vectors** of $\mathbf{A}$ are the n vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \tag{4}$$

in $\mathbf{R}^m$. The subspace of $\mathbf{R}^m$ spanned by the n column vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ is called the **column space** $\mathrm{Col}(\mathbf{A})$ of the matrix $\mathbf{A}$. The *dimension* of the space $\mathrm{Col}(\mathbf{A})$ is called the **column rank** of the matrix $\mathbf{A}$.

Null Space: The solution space of the homogeneous system $Ax = 0$ is some times called the null space of A.

$$\mathrm{Null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$

Example: Given

$$A = \begin{bmatrix} 1 & -3 & 0 & -5 \\ -1 & 4 & 1 & 7 \\ 2 & 1 & 7 & 4 \\ 2 & -2 & 4 & -2 \end{bmatrix}$$

a) Find a basis for $\mathrm{Col}(A)$

b) Find a basis for $\mathrm{Row}(A)$

c) Find a basis for $\mathrm{Null}(A)$.

a) $\text{Col}(A) = \text{span}\{(1,-1,2,2),(-3,4,1,-2),(0,1,7,4),(-5,7,4,-2)\}$

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \\ 1 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 7 \\ 4 \end{bmatrix} + c_4 \begin{bmatrix} -5 \\ 7 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -3 & 0 & -5 \\ -1 & 4 & 1 & 7 \\ 2 & 1 & 7 & 4 \\ 2 & -2 & 4 & -2 \end{bmatrix} \xrightarrow[\substack{R_3 \to (-2)R_1 + R_3 \\ R_4 \to (-2)R_1 + R_4}]{R_2 \to R_1 + R_2} \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 7 & 7 & 14 \\ 0 & 4 & 4 & 8 \end{bmatrix}$$

$$\xrightarrow[R_4 \to (-4)R_2 + R_4]{R_3 \to (-7)R_2 + R_3} \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 4 \\ 1 \\ -2 \end{bmatrix} \text{ are linearly independent}$$

$\Rightarrow$ These vectors form a basis for col (A). The column rank of A is 2.

Note: 1)

ALGORITHM 2   A Basis for the Column Space

To find a basis for the column space of a matrix **A**, use elementary row operations to reduce **A** to an echelon matrix **E**. Then the column vectors of **A** that correspond to the pivot columns of **E** form a basis for Col(**A**).

2) Find a subset of the vectors $v_1 = (1,-1,2,2)$, $v_2 = (-3,4,1,2)$, $v_3 = (0,1,7,4)$ and $v_4 = (2,-2,4,-2)$ that forms a basis for the subspace W of $\mathbb{R}^4$ spanned by these four vectors. We can arrange the given vectors as a column vectors of the matrix A. The subspace for W is a basis obtained from these vectors for col(A)

b) $Row(A) = span\{(1,-3,0,-5),(-1,4,1,7),(2,1,7,4),(2,-2,4,2)\}$

$$\cdots \rightarrow \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(1,-3,0,-5)$ and $(0,1,1,2)$ are linearly independent and span Row(A). Thus these vectors form a basis for A.

The row rank of A is 2.

ALGORITHM 1 A Basis for the Row Space

To find a basis for the row space of a matrix **A**, use elementary row operations to reduce **A** to an echelon matrix **E**. Then the nonzero row vectors of **E** form a basis for Row(**A**).

Note:

1) The nonzero row vectors of an echelon matrix are linearly independent and therefore form a basis for its row space.

2) If two matrices **A** and **B** are equivalent, then they have the same row space.

3) Row rank and Column rank of any matrix are equal.

The common value of the row rank and column rank of the matrix A is called the rank of A and is denoted by Rank (A).

c) $\begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 - 3x_2 - 5x_4 = 0$

$x_2 + x_3 + 2x_4 = 0$

Leading variables: x_1, x_2

Free variables: x_3, x_4

$x_3 = t$, $x_4 = s$ $\Rightarrow$ $x_2 = -t - 2s$, $x_1 = -3t + s$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3t+s \\ -t-2s \\ t \\ s \end{bmatrix} = t \cdot \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$(-3, -1, 1, 0)$ and $(-1, -2, 0, 1)$ are linearly independent, so form a basis for Null(A).

$\dim \text{Null}(A) = 2$, $\text{Rank}(A) + \text{Null}(A) = 4$

$\text{Rank}(A) = 2$,

Note: $\text{Rank}(A) + \dim \text{Null}(A) = n$ for any $m \times \underline{\underline{n}}$ matrix A.

Ex: Consider the 4x5 matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & +1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$$

a) Find a basis for Row(A).

b) Find a basis for Col(A)

c) Find Rank(A).

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & -2 & 6 & -9 & 1 \\ 0 & -1 & 3 & -5 & 4 \\ 0 & -2 & 6 & -9 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 2R_3]{R_2 \to R_2 - 2R_3}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & -1 & 3 & -5 & 4 \\ 0 & 0 & 0 & 1 & -7 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_2]{R_3 \to -R_3} \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) $\{[1 \;\; 2 \;\; 1 \;\; 3 \;\; 2], [0 \;\; 1 \;\; -3 \;\; 5 \;\; -4],$
$[0 \;\; 0 \;\; 0 \;\; 1 \;\; -7]\}$

b) $\{(1,3,2,2), (2,4,3,2), (3,0,1,-3)\}$

c) $\text{Rank}(A) = 3$

a) Determine for which values of **k** the system

$$\mathbf{3x + 2y = 1}$$
$$\mathbf{6x + 4y = k}$$

i) has a unique solution? ii) has no solution? iii) has infinitely many solutions?
iv) Find the solution set for case (iii).

b) Let **A** and **B** be $\mathbf{3 \times 3}$ invertible matrices. If $\det \mathbf{A} = \mathbf{2}$ and $\det \mathbf{B} = \mathbf{-3}$, calculate following determinants.

$$\text{i)}\ \det(\mathbf{3A^TB^{-1}}) \qquad \text{ii)}\ \det(\mathbf{4AdjA + 2A^{-1}})$$

a) $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ k \end{bmatrix}$

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 6 & 4 & k \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 2/3 & 1/3 \\ 6 & 4 & k \end{array}\right] \xrightarrow{R_2 \to (-6)R_1 + R_2} \left[\begin{array}{cc|c} 1 & 2/3 & 1/3 \\ 0 & 0 & -2+k \end{array}\right]$$

i) No unique solution

ii) $-2+k \neq 0 \Rightarrow k \neq 2$, no solution

iii) $-2+k = 0 \Rightarrow k = 2$, infinitely many solutions.

iv) $\left[\begin{array}{cc|c} 1 & 2/3 & 1/3 \\ 0 & 0 & -2+k \end{array}\right]$ $k=2 \Rightarrow \left[\begin{array}{cc|c} 1 & 2/3 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$

$$x + \frac{2}{3}y = \frac{1}{3}$$

$$y = t \Rightarrow x = \frac{1}{3} - \frac{2}{3}t.$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 - \frac{2}{3}t \\ t \end{bmatrix} = t.\begin{bmatrix} -2/3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

b) $\det A = 2, \quad \det B = -3$

i) $\det(3A^TB^{-1}) = |3A^TB^{-1}| = 3^3.|A^T|.|B^{-1}| = 27.|A|.\frac{1}{|B|}$

$$= 27.2.\frac{1}{-3} = -18.$$

$$A^{-1} = \frac{\operatorname{Adj}A}{|A|} \Rightarrow \operatorname{Adj}A = A^{-1}.|A|.$$

ii) $\det(4\operatorname{Adj} + 2A^{-1})$

$\det(4.A^{-1}.\underbrace{|A|}_{2} + 2A^{-1}) = \det(10A^{-1}) = 10^3.|A^{-1}| = 10^3.\frac{1}{|A|}$

$$= \frac{10^3}{2} = 500$$

c) Let $\mathbf{W} = \left\{ \mathbf{v} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \in \mathbb{R}^3 \mid \mathbf{x} + \mathbf{y} = \mathbf{0} \right\}$. Determine whether the following statements are true or false.

i) **W** is a subspace of $\mathbb{R}^3$ and $\mathbf{dim}(\mathbf{W}) = 3$.

ii) **W** is a subspace of $\mathbb{R}^2$ and $\mathbf{dim}(\mathbf{W}) = 1$.

iii) **W** is not a subspace of $\mathbb{R}^3$.

iv) **W** is a subspace of $\mathbb{R}^3$ and $\mathbf{dim}(\mathbf{W}) = 2$.

$u, v \in W$, $\quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $u_1 + u_2 = 0$, $\quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $v_1 + v_2 = 0$

* $u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$ $\quad (u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2) = 0 + 0 = 0.$

$\Rightarrow u + v \in W.$

* $c.u = \begin{bmatrix} cu_1 \\ cu_2 \\ c.u_3 \end{bmatrix}$ $\quad (cu_1) + (cu_2) = c(u_1 + u_2) = c.0 = 0.$

$\Rightarrow c.u \in W$

$\Rightarrow W$ is a subspace of $\mathbb{R}^3$.

$v \in W$, $\quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $x + y = 0 \Rightarrow y = -x$

$$\Rightarrow v = \begin{bmatrix} x \\ -x \\ z \end{bmatrix} = x. \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z. \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ -x \\ z \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\dim W = 2.$

i) $\mathbf{W}$ is a subspace of $\mathbb{R}^3$ and $\mathbf{dim}(\mathbf{W}) = \mathbf{3}$. F

ii) $\mathbf{W}$ is a subspace of $\mathbb{R}^2$ and $\mathbf{dim}(\mathbf{W}) = \mathbf{1}$. F

iii) $\mathbf{W}$ is not a subspace of $\mathbb{R}^3$. F

iv) $\mathbf{W}$ is a subspace of $\mathbb{R}^3$ and $\mathbf{dim}(\mathbf{W}) = \mathbf{2}$. T.

Let $\mathbf{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}$.

[**05p**] **a)** Define the null space of the matrix **A**.
[**10p**] **b)** Find a basis for the null space of the matrix **A** and its dimension.
[**10p**] **c)** Find a basis for the column space of **A** and its rank.

a) $\text{Null}(A) = \{ X = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid AX = 0 \}$

b)
$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \xrightarrow[R_3 \to (-1)R_1 + R_3]{R_2 \to (-2)R_1 + R_2} \begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 6 & 6 & 18 \end{bmatrix}$$

$$\xrightarrow[R_3 \to \frac{1}{6} R_3]{R_2 \to \frac{1}{7} R_2} \begin{bmatrix} 1 & -4 & -3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad x_3 = t, \quad x_4 = s$$

$$\left.\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ x_2 + x_3 + 3x_4 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_2 &= -t - 3s \\ x_1 &= -t - 5s \end{aligned}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t-5s \\ -t-3s \\ t \\ s \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

The vectors $(-1,-1,1,0)$ and $(-5,-3,0,1)$ are linearly independent and $\{(-1,1,1,0),(5,-3,0,1)\}$ spans $Null(A)$.

Then, $\{(-1,1,1,0),(5,-3,0,1)\}$ is a basis for $Null(A)$.

$$\dim Null(A) = 2.$$

c) $\begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}$.

The first and the second columns contain the leading entries. Thus

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \right\}$ spans the column space and

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$ are linearly independent.

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \right\}$ is a basis for the column space of A.

Rank(Col(A)) = 2.

1. For which value of β is the system

$$\begin{aligned} x - 3y + z &= 1 \\ -3x + 7y - z &= -1 \\ \beta x - 2y + 2z &= -2 \end{aligned}$$

inconsistent?

(a) -4

(b) -2

(c) -1

(d) 0

(e) 2

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ -3 & 7 & -1 & -1 \\ \beta & -2 & 2 & -2 \end{array}\right] \xrightarrow[R_3 \to (-\beta) R_1 + R_3]{R_2 \to 3R_1 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & -2 & 2 & 2 \\ 0 & 3\beta-2 & 2-\beta & \beta-2 \end{array}\right] \xrightarrow{R_2 \to \left(-\frac{1}{2}\right) R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 3\beta-2 & 2-\beta & \beta-2 \end{array}\right] \xrightarrow[R_1 \to 3R_2 + R_1]{R_3 \to (2-3\beta) R_2 + R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2\beta & 4\beta-4 \end{array}\right]$$

$$2\beta = 0 \Rightarrow \beta = 0$$

3. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$. If $AA^T = 9I_3$, which of the following is true?

(a) $a = -1, \quad b = 2$

(b) $a = 1, \quad b = 2$

(c) $a = -2, \quad b = -1$

(d) $a = 2, \quad b = 1$

(e) $a = -2, \quad b = 1$

$$AA^T = 9 . I_3 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} \overbrace{1+4+4}^{9} & \overbrace{2+2-4}^{0} & \overbrace{a+4+2b}^{0} \\ \underbrace{2+2-4}_{0} & \underbrace{4+1+4}_{9} & 2a+2-2b^{\,0} \\ \underbrace{a+4b+2b}_{0} & \underbrace{2a+2-2b}_{0} & \underbrace{a^2+4+b^2}_{9} \end{bmatrix}$$

$$2a-2b+2=0$$
$$a+4+2b=0$$

$$a-b=-1$$
$$a+2b=-4$$

$$3b=-3 \Rightarrow b=-1$$
$$a=-2$$

5. Let $A = [a_{ij}]$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, and

$C = \begin{bmatrix} 2 & -5 & 8 \\ 7 & 6 & 4 \\ 1 & 5 & -3 \end{bmatrix}$. If $(A(BC)^{-1})^{-1} = (BC^T B^T)^T$,

which of the following is the matrix A ?

$$(A(BC)^{-1})^{-1} = (A . C^{-1} B^{-1})^{-1} = B C A^{-1}$$

$$(BC^T B^T)^T = B C B^T$$

$$BCA^{-1} = BCB^T$$

$$A^{-1} = B^T$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to (-1)R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$(A^{-1})^{-1} = A.$$