BLG202E

HOMEWORK 2

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Q1)

a)

Q1)

Newton's Method

a)

n	x_n	$f(x_n)$	$f'(x_n)$	$\lvert x_n - x_{n-1}\rvert$
0	1	-1	3	–
1	1.3333	-0.1827	2	0.3333
2	1.4247	-0.0089	1.8077	0.0914
3	1.4296	0	1.7980	0.0049
4	1.4296	0	1.7980	0
5	1.4296	0	1.7980	0
6	1.4296	0	1.7980	0

Secant Method

b)

n	x_n	$f(x_n)$	$f'(x_n)$	$\lvert x_n - x_{n-1}\rvert$
0	1	-1	3	–
1	2	0.772	1	1
2	1.564	0.225	1.557	0.436
3	1.384	-0.082	1.898	0.180
4	1.433	0.006	1.791	0.049
5	1.429	0.0001	1.798	0.004
6	1.429	0.0001	1.798	0
7	1.429	0.0001	1.798	0

b)

b) Error estimation for Newton's Method:

$$|x_6 - x_5| = |1.4296 - 1.4296| \sim 0$$

Error estimation for Secant Method:

$$|1.429 - 1.429| \sim 0$$

c)

c) error in the n'th approximation

$\text{Error } n = x_n - X \rightarrow$ real root

$$\text{Error }(n+1) = x_{n+1} - X$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x)} \quad \text{(Newton's Method)}$$

$$\text{Error }(n+1) + \cancel{X} = \text{Error } n + \cancel{X} - \frac{4 \cdot \ln(x_n)}{\frac{4}{x_n} - 1}$$

$$\boxed{\text{Error }(n+1) = \text{Error } n - \frac{4 \ln(x_n)}{\frac{4}{x_n} - 1}}$$

d)

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Enter initial value x_0: 1
Enter initial value x_1 (for secant method): 2
Enter the number of maximum iteration: 6

######-Newton's Method-######
x_1 is:1.3333333333333333
x_2 is:1.4246358550964382
x_3 is:1.4295983588957426
x_4 is:1.4296118246268645
x_5 is:1.4296118247255556
x_6 is:1.4296118247255556

######-Newton's Method Error Estimation-######
Error estimation for x_0 to x_1 is:0.33333333333333326
Error estimation for x_1 to x_2 is:0.09130252176310494
Error estimation for x_2 to x_3 is:0.004962503799304474
Error estimation for x_3 to x_4 is:1.3465731121931057e-05
Error estimation for x_4 to x_5 is:9.869105532800404e-11
Error estimation for x_5 to x_6 is:0.0
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######-Secant Method-######
x_2 is:1.5641466559351878
x_3 is:1.3848163538828964
x_4 is:1.4329154592164408
x_5 is:1.4296921139216703
x_6 is:1.429611680322313
x_7 is:1.4296118247318659

######-Secant Method Error Estimation-######
Error estimation for x_0 to x_1 is:1.0
Error estimation for x_1 to x_2 is:0.43585334406481224
Error estimation for x_2 to x_3 is:0.1793303020522914
Error estimation for x_3 to x_4 is:0.048099105333544445
Error estimation for x_4 to x_5 is:0.00322334529477053
Error estimation for x_5 to x_6 is:8.043359935738792e-05
Error estimation for x_6 to x_7 is:1.444095529823386e-07
PS C:\Users\mcanc>
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Q2)

a)

Q2)

a)

$$P_{12}A_1 = \begin{bmatrix} -4 & -1 & 3 \\ -1 & 3 & 0 \\ 0 & -4 & -1 \end{bmatrix}$$

$$M_1P_{12}A_1 = \begin{bmatrix} -4 & -1 & 3 \\ 0 & 13/4 & -3/4 \\ 0 & -4 & -1 \end{bmatrix}$$

$$P_{23}M_1P_{12}A_1 = \begin{bmatrix} -4 & -1 & 3 \\ 0 & -4 & -1 \\ 0 & 13/4 & -3/4 \end{bmatrix}$$

$$M_2P_{23}M_1P_{12}A_1 = \begin{bmatrix} -4 & -1 & 3 \\ 0 & -4 & -1 \\ 0 & 0 & -25/16 \end{bmatrix} = U$$

$$M_2P_{23}M_1P_{23}P_{23}P_{12}A_1 = U$$

$$P_{23}P_{12}A_1 = (M_2P_{23}M_1P_{23})^{-1}U$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 13/16 & 1 \end{bmatrix} \qquad M_2P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 13/16 \end{bmatrix}$$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_1P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_2P_{23}M_1P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & \frac{13}{16} & 1 \end{bmatrix}$$

$$(M_2P_{23}M_1P_{23})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/4 & -\frac{13}{16} & 1 \end{bmatrix}$$

$$P_{23}P_{12}A = (M_2 P_{23} M_1 P_{23})^{-1} \cdot U$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -1 & 3 & 0 \\ -4 & -1 & 3 \\ 0 & -4 & -1 \end{bmatrix}}_{A1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/4 & \frac{-13}{16} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -4 & -1 & 3 \\ 0 & -4 & -1 \\ 0 & 0 & -\frac{25}{16} \end{bmatrix}}_{U}$$

a) (cont.)

Q2)

a) (part 2)

$$P_{12}A_2 = \begin{bmatrix} 4 & 4 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$M_1P_{12}A_2 = \begin{bmatrix} 4 & 4 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$M_2M_1P_{12}A_2 = \begin{bmatrix} 4 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

$$P_{23}M_2M_1P_{12}A_2 = \begin{bmatrix} 4 & 4 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$P_{23}M_2M_1P_{12}A_2 = U$$

$$P_{23}M_2M_1P_{23}P_{23}P_{12}A_2 = U$$

$$P_{23}P_{12}A_2 = (P_{23}M_2M_1P_{23})^{-1}\, U$$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{23}M_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_1P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{23}M_2M_1P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix}$$

$$(P_{23}M_2M_1P_{23})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix}$$

$$P_{23}P_{12}A_2 = (P_{23}M_2M_1P_{12})^{-1}\,U$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 4 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{A_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 4 & 4 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

b)

(Q2)

b) $A_2 . x = b$

$$P\,b = L.\underbrace{U.x}_{y}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix} . y$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 1/2 & 1 & 0 & -2 \\ 1/4 & 0 & 1 & 3 \end{array}\right] \quad \left.\begin{array}{l} y_{11} = -4 \\ -2 + y_{21} = -2 \rightarrow y_{21} = 0 \\ -1 + y_{31} = 3 \rightarrow y_{31} = 4 \end{array}\right) \quad y = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 0 & -4 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$4x_{11} + 4 = -4 \qquad x_{11} = -2$

$-2x_{21} + 2 = 0 \rightarrow x_{21} = 1$

$x_{31} = 2$

$$\boxed{x = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}}$$

Q3)

Q3) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & -8/3 \end{bmatrix}$

$$E_2 E_1 A = U$$

$$A = (E_2 E_1)^{-1} U$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 2/3 & 1 \end{bmatrix}, \quad (E_2 E_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & -8/3 \end{bmatrix}}_{U}$$

$A \cdot X = I$ where $X = A^{-1}$

$A = L \cdot U$

$\underbrace{AX}_{I} = L \cdot \underbrace{U \cdot X}_{y} \rightarrow I = L \cdot y$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix} y$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2/3 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1 & 0 \\ 0 & -2/3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & +1/3 & +2/3 & 1 \end{array}\right]$$

$y = L^{-1}$

$U \cdot X = y$

$X = U^{-1} y$

$$U^{-1} \Rightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 1 & 0 \\ 0 & 0 & -8/3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -3/8 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/3 & -1/8 \\ 0 & 1 & 0 & 0 & 2/3 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & -3/8 \end{array}\right] \leftarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2/3 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & -3/8 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1/2 & 1/3 & -1/8 \\ 0 & 2/3 & -1/4 \\ 0 & 0 & -3/8 \end{bmatrix}}_{u^{-1}}$$

$$x = u^{-1} y$$

$$X = \begin{bmatrix} 1/2 & 1/3 & -1/8 \\ 0 & 2/3 & -1/4 \\ 0 & 0 & -3/8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 2/3 & 1 \end{bmatrix}$$

$$\boxed{X = \begin{bmatrix} 5/8 & 1/4 & -1/8 \\ 1/4 & 1/2 & -1/4 \\ -1/8 & -1/4 & -3/8 \end{bmatrix}}$$

Q4)

a)

Q4)

a) $\sum_{i=1}^{20} (y^i - \tilde{y}^i)^2$ has to be minimized.

if $\tilde{y}^i = a + b x^i$, the summation can be written as

$\sum_{i=1}^{20} (y^i - a - b x^i)^2$. To find coefficients, we have to

take derivatives of summation w.r.t. each coefficient, and make it equal to zero.

$$\frac{\partial \sum_{i=1}^{20} (y^i - a - bx^i)^2}{\partial a} = -2 \cdot \sum_{i=1}^{20} (y^i - a - bx^i)$$

$$① \quad \sum_{i=1}^{20} (y^i - a - bx^i) = 0$$

$$\frac{\partial \sum_{i=1}^{20} (y^i - a - bx^i)^2}{\partial b} = -2 \cdot \sum_{i=1}^{20} x^i (y^i - a - bx^i)$$

$$② \quad \sum_{i=1}^{20} x^i (y^i - a - bx^i) = 0$$

from ①, $\sum_{i=1}^{20} y^i = \sum_{i=1}^{20} a + \sum_{i=1}^{20} b x^i$

from ②, $\sum_{i=1}^{20} x^i y^i = \sum_{i=1}^{20} x^i a + \sum_{i=1}^{20} (x^i)^2 b$

matix representation :

$$\begin{bmatrix} 20 & \sum_{i=1}^{20} x^i \\ \sum_{i=1}^{20} x^i & \sum_{i=1}^{20} (x^i)^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{20} y^i \\ \sum_{i=1}^{20} x^i y^i \end{bmatrix}$$

$$\begin{bmatrix} 20 & 105 \\ 105 & 717.5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 90.99 \\ 560.87 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 20 & 105 & 90.99 \\ 105 & 717.5 & 560.87 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1.923 \\ 0 & 1 & 0.5 \end{array}\right]$$

$$\boxed{l(x) = 1.923 + 0.5x}$$

b)

b) Using same logic from part a, summation which has to be minimized is $\sum_{i=1}^{20}(y^i - c(x^i)^2 - dx^i - e)^2$.

$$\frac{\partial \sum_{i=1}^{20}(y^i - c(x^i)^2 - dx^i - e)^2}{\partial e} = -2 \cdot \sum_{i=1}^{20}(y^i - c(x^i)^2 - dx^i - e)$$

$$\sum_{i=1}^{20}(y^i - c(x^i)^2 - dx^i - e) = 0 \quad 1$$

$$\frac{\partial \sum_{i=1}^{20}(y^i - c(x^i)^2 - dx^i - e)^2}{\partial d} = -2 \sum_{i=1}^{20} x^i(y^i - c(x^i)^2 - dx^i - e)$$

$$\sum_{i=1}^{20} x^i\left(y^i - c(x^i)^2 - dx^i - e\right) = 0 \qquad (2)$$

$$\frac{\partial \sum_{i=1}^{20}\left(y^i - c(x^i)^2 - dx^i - e\right)^2}{\partial c} = -2\sum_{i=1}^{20}(x^i)^2\left(y^i - c(x^i)^2 - dx^i - e\right)$$

$$\sum_{i=1}^{20}(x^i)^2\left(y^i - c(x^i)^2 - dx^i - e\right) = 0 \qquad (3)$$

from (1), $\sum_{i=1}^{20} y^i = \sum_{i=1}^{20} c(x^i)^2 + \sum_{i=1}^{20} d(x^i) + \sum_{i=1}^{20} e$

from (2), $\sum_{i=1}^{20} y^i x^i = \sum_{i=1}^{20} c(x^i)^3 + \sum_{i=1}^{20} d(x^i)^2 + \sum_{i=1}^{20} e x^i$

from (3), $\sum_{i=1}^{20} y^i (x^i)^2 = \sum_{i=1}^{20} c(x^i)^4 + \sum_{i=1}^{20} d(x^i)^3 + \sum_{i=1}^{20} e(x^i)^2$

Matix representation :

$$\begin{bmatrix} \sum_{i=1}^{20}(x^i)^2 & \sum_{i=1}^{20} x^i & 20 \\ \sum_{i=1}^{20}(x^i)^3 & \sum_{i=1}^{20}(x^i)^2 & \sum_{i=1}^{20} x^i \\ \sum_{i=1}^{20}(x^i)^4 & \sum_{i=1}^{20}(x^i)^3 & \sum_{i=1}^{20}(x^i)^2 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{20} y^i \\ \sum_{i=1}^{20} x^i y^i \\ \sum_{i=1}^{20} (x^i)^2 y^i \end{bmatrix}$$

Substitute values:

$$\begin{bmatrix} 45166.63 & 5512.5 & 717.5 \\ 5512.5 & 717.5 & 105 \\ 717.5 & 105 & 20 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 4062.34 \\ 560.87 \\ 90.59 \end{bmatrix}$$

$$\boxed{P(x) = -0.068x^2 + 1.220x + 0.603}$$

c)

c) Using the same logic,

$$\sum_{i=1}^{20} \left(y^i - k - m\ln(x^i)\right)^2$$

$$\frac{\partial \sum_{i=1}^{20} \left(y^i - k - m\ln(x^i)\right)^2}{\partial k} = -2\sum_{i=1}^{20} \left(y^i - k - m\ln(x^i)\right)$$

$$\sum_{i=1}^{20} (y^i) = \sum_{i=1}^{20} k + \sum_{i=1}^{20} m\ln(x^i)$$

$$\frac{\partial \sum_{i=1}^{20} \left(y^i - k - m\ln(x^i)\right)^2}{\partial m} = -2\sum_{i=1}^{20} \ln(x^i)\left(y^i - k - m\ln(x^i)\right)$$

$$\sum_{i=1}^{20} y^i \ln(x^i) = \sum_{i=1}^{20} \ln(x^i)k + \sum_{i=1}^{20} m\left(\ln(x^i)\right)^2$$

Matrix representation:

$$\begin{bmatrix} 20 & \sum_{i=1}^{20} \ln(x^i) \\ \sum_{i=1}^{20} \ln(x^i) & \sum_{i=1}^{20} \ln(x^i)^2 \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{20} y^i \\ \sum_{i=1}^{20} y^i \cdot \ln(x^i) \end{bmatrix}$$

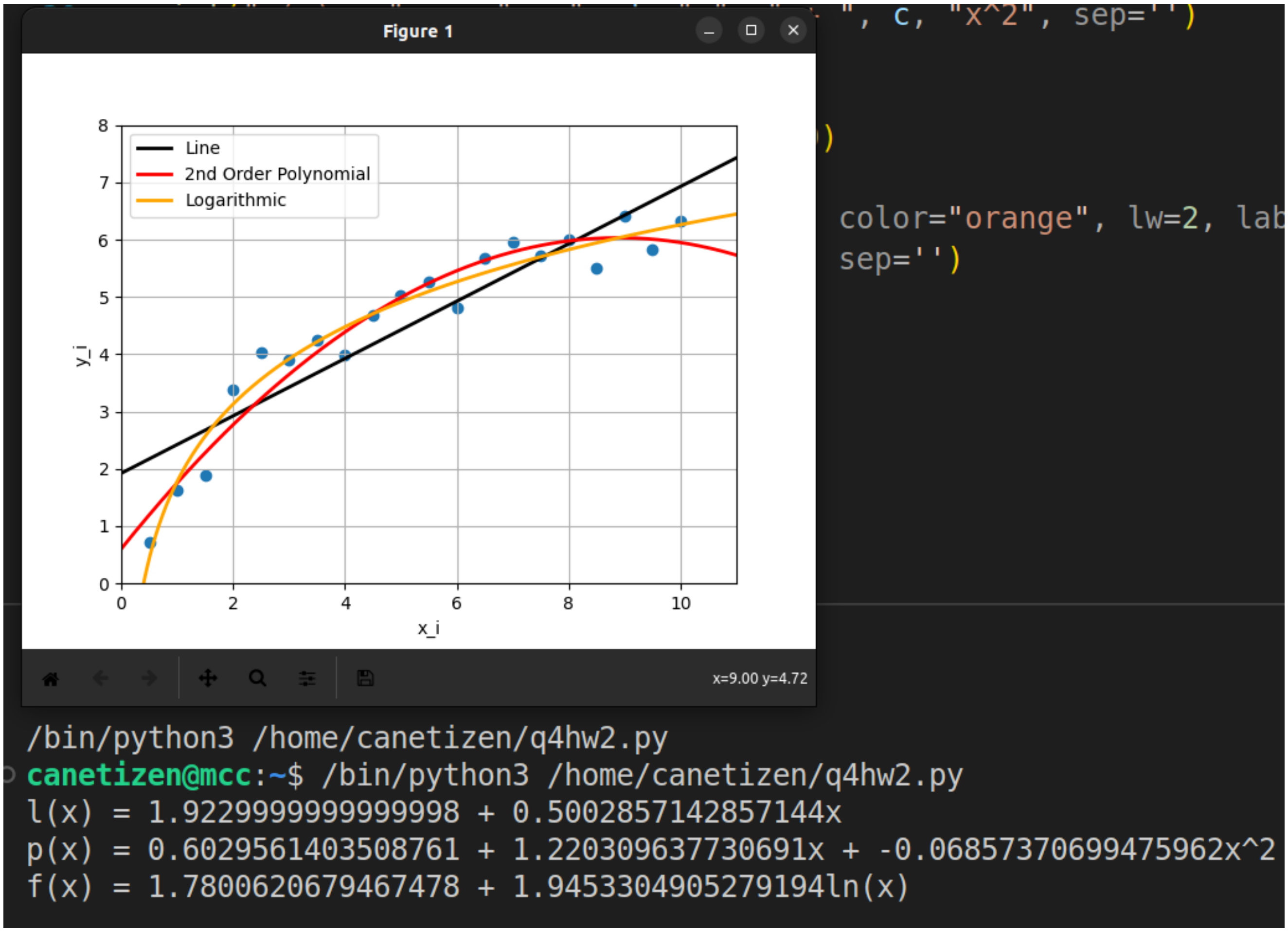
Substitute values:

$$\begin{bmatrix} 20 & 28.47 \\ 28.47 & 53.08 \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} = \begin{bmatrix} 90.99 \\ 153.95 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 20 & 28.47 & 90.99 \\ 28.47 & 53.08 & 153.95 \end{array}\right] \rightarrow \begin{array}{l} k = 1.78 \\ m = 1.95 \end{array}$$

$$f(x) = 1.78 + 1.95 \cdot \ln(x)$$

d)


", c, "x^2", sep='')
Figure 1
Line
2nd Order Polynomial
Logarithmic
y_i
x_i
x=9.00 y=4.72
color="orange", lw=2, lab
sep='')
/bin/python3 /home/canetizen/q4hw2.py
canetizen@mcc:~$ /bin/python3 /home/canetizen/q4hw2.py
l(x) = 1.9229999999999998 + 0.5002857142857144x
p(x) = 0.6029561403508761 + 1.220309637730691x + -0.06857370699475962x^2
f(x) = 1.7800620679467478 + 1.9453304905279194ln(x)