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Embedding even-length cycles in a hexagonal honeycomb mesh

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The existence and construction of cycles of various lengths in an interconnection network are important issues in efficiently executing ring-structured parallel algorithms in such a network. The hexagonal honeycomb mesh (HHM) is regarded as a promising candidate for interconnection networks. In this paper we address the problem of how to embed even-length cycles in an HHM. We prove that an HHM of order $t \geq 3$ admits a cycle of length l for each even number l such that $l = 6$ or $10 \leq l \leq 6t^2 - 2$. We also describe a systematic method for building these cycles.

Keywords: Interconnection network; Hexagonal honeycomb mesh; Cycle embedding

ACM CCS: B.4.3

1. Introduction

It is well known that many classical parallel algorithms possess a ring-structured task graph. In order to implement a ring-structured parallel algorithm efficiently on a specific multicomputer system, it is essential to map the tasks owned by the parallel algorithm to the nodes of the underlying interconnection network so that any two tasks that are adjacent in the ring are mapped to two adjacent nodes of the network [1]. If the number of tasks in the ring-structured parallel algorithm equals the number of nodes in the associated interconnection network, it is desirable for this network to have a Hamiltonian cycle (i.e. a cycle which passes every node of the network). In real situations, a ring-structured parallel algorithm with few tasks is often executed on an interconnection network with a larger number of nodes (this is especially the case for coarse-grained parallel algorithms), and efficient mapping demands that the network admit a cycle of some given length. An interconnection network which possesses cycles of various lengths is preferred for this purpose.

Stojmenovic [2] proposed two types of honeycomb structure as candidates for interconnection networks: the *honeycomb mesh* and the *honeycomb torus*. An important advantage

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of honeycomb networks over their mesh/torus counterparts is that the number of links per node is restricted to two or three, which can significantly decrease the complexity of their hardware implementation. Because of this attractive feature, honeycomb networks and their generalizations have received considerable attention [3–16].

Although the cycle-embedding capabilities of honeycomb torus networks and their generalizations have been studied intensively [6,7,9,12], these networks are difficult to implement in VLSI layout [2]. In contrast, the asymmetric honeycomb mesh can easily be implemented on a single board, which makes it a more appealing choice from the practical viewpoint [14]. Although a hexagonal honeycomb mesh (HHM) of order $t \geq 2$ is not Hamiltonian, its cycle-embedding capability is still worth studying, as indicated previously. To our knowledge, no such investigation has been reported in the literature.

In this paper we address the existence of cycles of various lengths in an HHM. We prove that an HHM of order $t \geq 3$ admits a length-l cycle for each even number l such that $l = 6$ or $10 \leq l \leq 6t^2 - 2$. A systematic method for building these cycles is also presented. The result obtained partially justifies the utility of HHM.

The remainder of the paper is organized as follows. Section 2 presents the definition of hexagonal honeycomb mesh, and describes a node-labelling scheme and a cycle-labelling scheme. Section 3 establishes the result. Finally, some conclusions are presented in section 4.

2. Preliminaries

For fundamental graph-theoretical terminologies the reader is referred to [17]. For a cycle C in a graph, let $E(C)$ denote the set of all the edges on C. An l-cycle is a cycle of length l. Given two sets S_1 and S_2, we let $S_1 \oplus S_2 = (S_1 - S_2) \cup (S_2 - S_1)$.

DEFINITION 1 *A hexagonal honeycomb mesh* (*HHM*) *of order t is defined recursively as follows:*

(i) *An HHM of order* 1 *is a* 6-*cycle drawn on the plane as shown in figure* 1(a).
(ii) *For* $t \geq 2$, *an HHM of order t is a graph obtained by adding a set of* $6(t-1)$ 6-*cycles around the border of an HHM of order* $t-1$ *in an edge-sharing manner* (*figures* 1(b)–1(d)).

An HHM of order t has a total of $6t^2$ nodes and $9t^2 - 3t$ edges. Figure 1 shows four small HHMs.

We now introduce some notation to enable us to label a node or a 6-cycle in an HHM.

DEFINITION 2 *Consider a* 6-*cycle in an HHM. The node that is on the north* (*south*, *northeast*, *southeast*, *northwest*, *southwest*) *corner of the* 6-*cycle is referred to as the N-node* (*S-node*, *NE-node*, *SE-node*, *NW-node*, *SW-node*) *of this* 6-*cycle* (*figure* 2).

DEFINITION 3 *Consider an HHM of order* $t \geq 2$. *A* 6-*cycle that is on the boundary of the HHM is referred to as a B cycle of the HHM.*

(i) *The B cycle of the HHM that is on the east* (*west*, *northeast*, *southeast*, *northwest*, *southwest*) *corner of the HHM is referred to as the E-B cycle* (*W-B cycle*, *NE-B cycle*, *SE-B cycle*, *NW-B cycle*, *SW-B cycle*) *of the HHM.*
(ii) *The rth B cycle of the HHM when counting in order from the NE-B cycle to the E-B cycle* (*from the SE-B cycle to the E-B cycle*, *from the NW-B cycle to the W-B cycle*, *from the SW-B cycle to the W-B cycle*, *from the NE-B cycle to the NW-B cycle*, *from the SE-B cycle to*

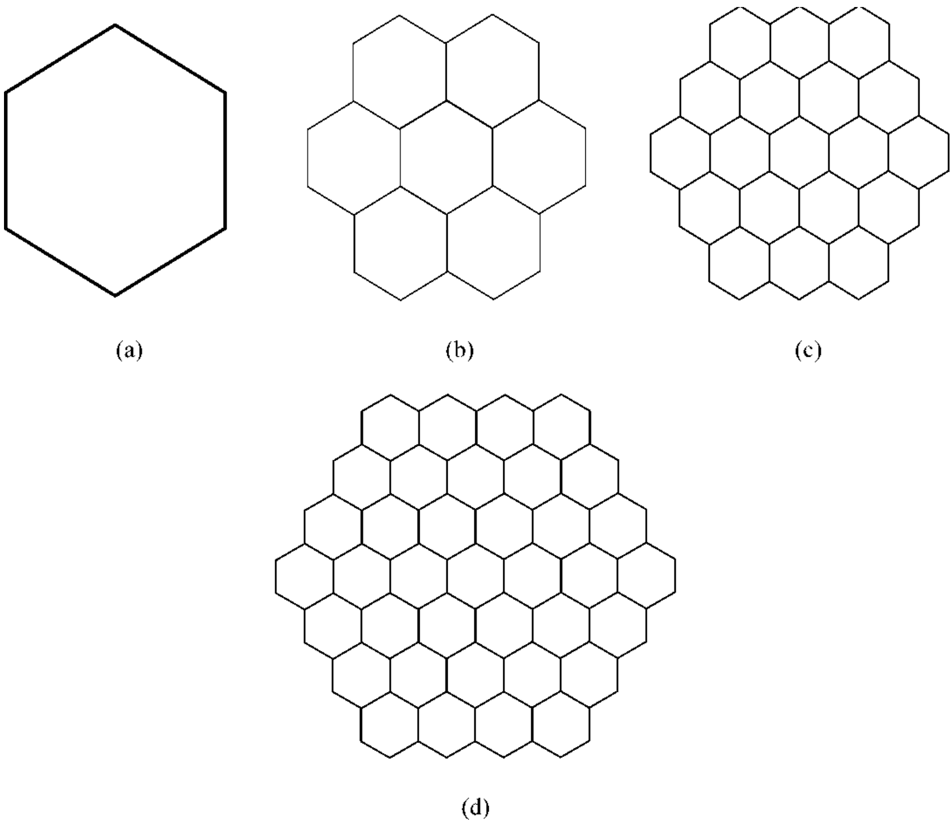

Figure 1. An HHM of order t: (a) $t = 1$; (b) $t = 2$; (c) $t = 3$; (d) $t = 4$.

the SW-B cycle) of the HHM is referred to as the NE-E(r)-B cycle (SE-E(r)-B cycle, NW-W(r)-B cycle, SW-W(r)-B cycle, NE-NW(r)-B cycle, SE-SW(r)-B cycle) of the HHM.

Figure 3 illustrates the terminologies introduced in Definition 3.

DEFINITION 4 *Consider an HHM of order t and an integer s where* $1 \leq s \leq t$. *The HHM of order s that is at the centre of the original HHM is referred to as the central sub-HHM of order s of the original HHM (see figure 4 for an illustration).*

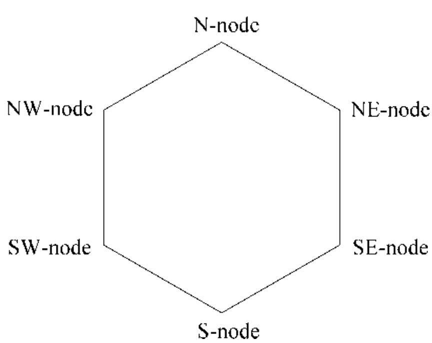

Figure 2. Labelling of the six nodes in a given 6-cycle of an HHM.

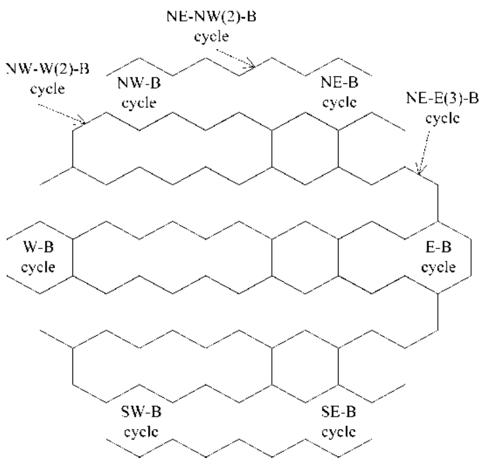

Figure 3. Labelling of some 6-cycles on the boundary of an HHM of order 4.

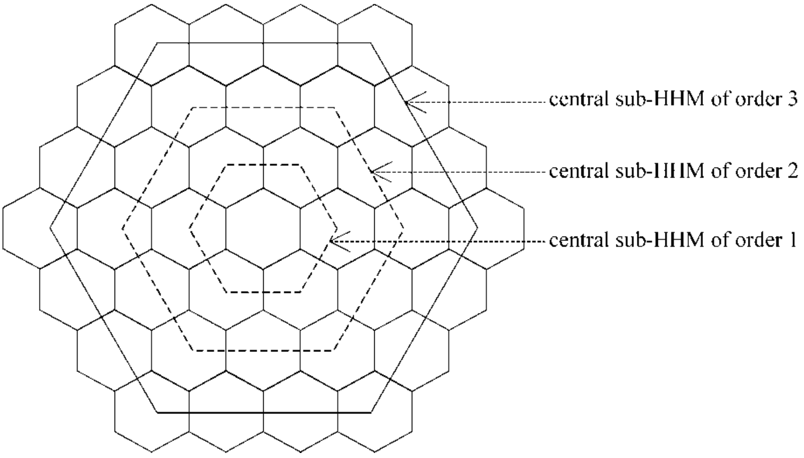

Figure 4. Three central sub-HHMs of an HHM of order 4.

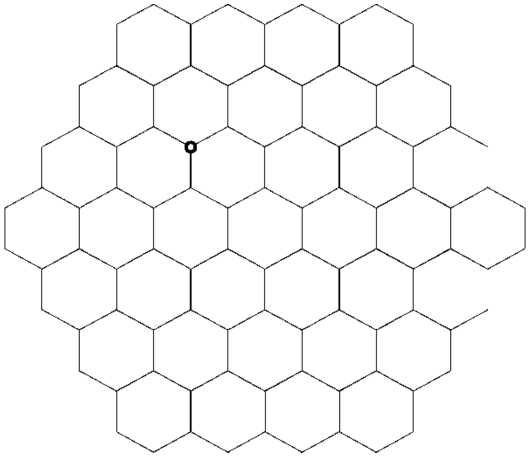

Figure 5. Labelling of a 6-cycle and a node in an HHM of order 4.

Using the notation defined above, it is easy to label a node or a 6-cycle in an HHM. For example, in figure 5 the shaded 6-cycle is labelled as the NW-B cycle of the central sub-HHM of order 2, and the encircled node is labelled as the NW-node on this 6-cycle.

3. Main result

The following facts are obvious.

- An HHM does not contain any cycles of odd length because it is a bipartite graph.
- An HHM does not admit any cycles of length $6t^2$ because it is not a Hamiltonian graph.
- An HHM has a 6-cycle.
- An HHM has neither a 4-cycle nor an 8-cycle.

The following question now arises: For any given even number l with $10 \leq l \leq 6t^2 - 2$, does an HHM of order t contain a cycle of length l? We now answer this question.

THEOREM 1 *Consider an HHM of order* $t \geq 3$. *Let* l *be an even number with* $10 \leq l \leq 6t^2 - 2$. *Then the HHM contains a cycle of length* l.

Proof Clearly, the HHM contains a 10-cycle. Next, let us assume that $12 \leq l \leq 6t^2 - 2$. We need to examine four possible cases with respect to the value of l.

Case 1 $l = 6t^2 - 2$. Look at two nodes: the NW-node on the NW-W(2)-B cycle of the HHM and the SW-node on the W-B cycle of the central sub-HHM of order 2. It can be seen that the graph obtained by deleting these two nodes from the HHM contains a unique Hamiltonian cycle (figure 6), which is a cycle of length $6t^2 - 2$ of the original HHM.

Case 2 $l = 6t^2 - 4k - 2$, where k is an integer with $1 \leq k \leq 1.5t^2 - 3.5$ (equivalently, $12 \leq l \leq 6t^2 - 6$ and $6t^2 - l \equiv 2 \pmod 4$)). We distinguish between two possibilities of the parity of t.

Case 2.1 t is even. Let C denote the cycle described in case 1, and let $E = E(C)$. Let $C_1, C_2, \ldots, C_k$ denote the first to kth 6-cycles when counting in the order determined by the set of arrows shown in figure 7. For i from 1 to k, we let $E = E \oplus E(C_i)$. When this procedure has been completed, a cycle of length $6t^2 - 4k - 2$ is obtained.

Case 2.2 t is odd. Let C denote the cycle given in case 1, and let $E = E(C)$. Let C_0 denote the central 6-cycle of the HHM. Let $C_1, C_2, \ldots, C_{k-1}$ denote the first to $(k-1)$th 6-cycles when counting in the order determined by the set of arrows shown in figure 8. For i from 0 to $k-1$, we let $E = E \oplus E(C_i)$. When this procedure has been completed, a cycle of length $6t^2 - 4k - 2$ is obtained.

Case 3 $l = 6t^2 - 4$. We proceed by considering the parity of t.

Case 3.1 t is odd. Look at four nodes: the NW-node on the NW-W(2)-B cycle of the HHM, and the N-node, the NW-node, and the SW-node on the W-B cycle of the central sub-HHM of order 3. It can be seen that the graph obtained by deleting these four nodes from the HHM contains a unique Hamiltonian cycle (figure 9), which is a cycle of length $6t^2 - 4$ of the original HHM.

Case 3.2 t is even. Look at four nodes: the N-node and the NW-node on the NW-B cycle of the HHM, the S-node on the NW-W(2)-B cycle of the HHM, and the SW-node on the W-B cycle of the central sub-HHM of order 2. It can be seen that the graph obtained by deleting these four nodes from the HHM

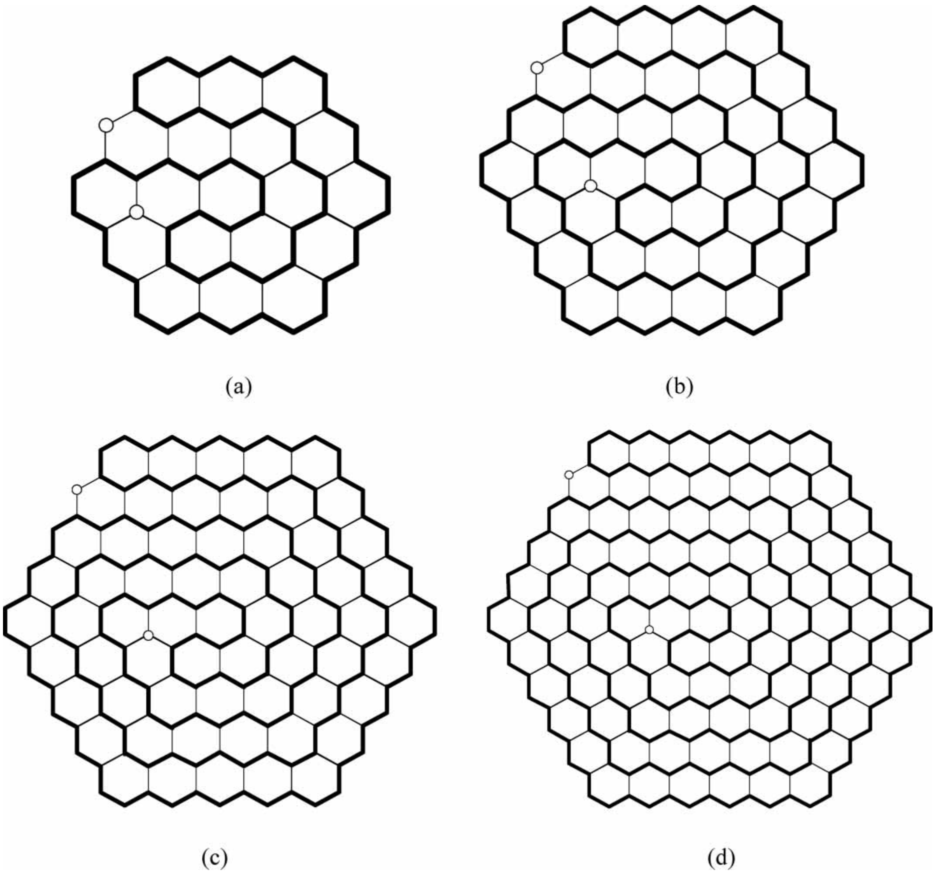

Figure 6. A cycle of length $6t^2 - 2$ in an HHM of size t: (a) $t = 3$; (b) $t = 4$; (c) $t = 5$; (d) $t = 6$.

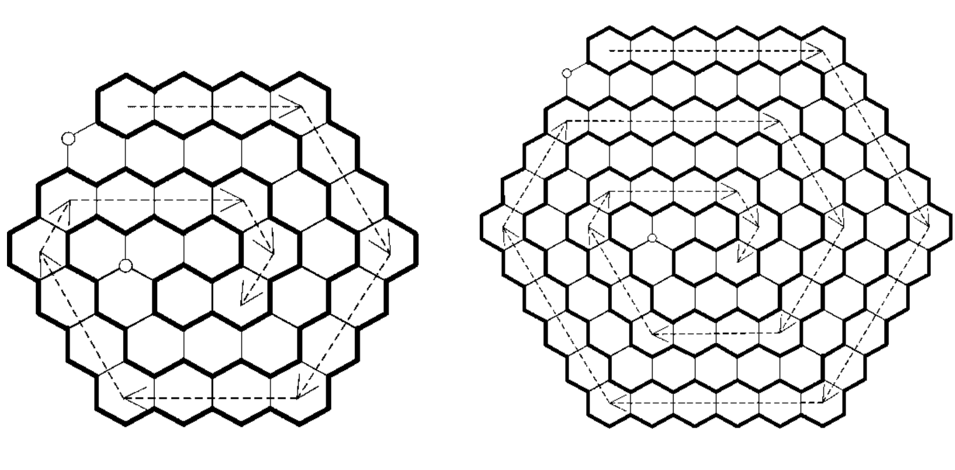

Figure 7. The order in which a set of k 6-cycles are selected to build a cycle of length $6t^2 - 4k - 2$ in an HHM of even size t: (a) $t = 4$; (b) $t = 6$.

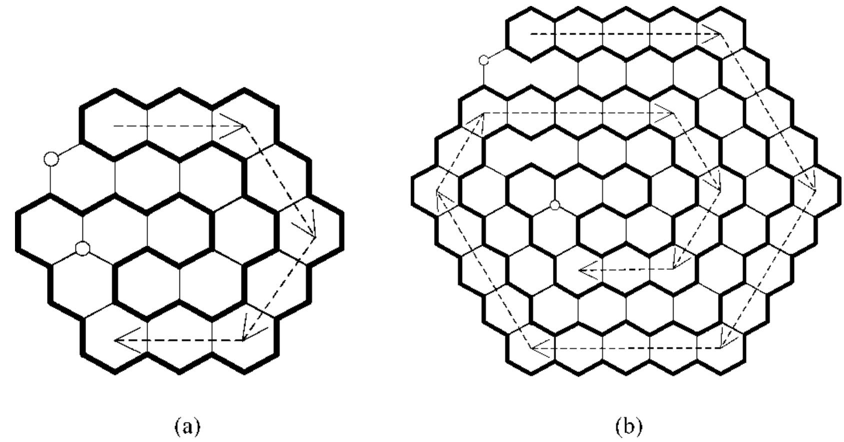

Figure 8. The order in which a set of k 6-cycles are selected to build a cycle of length $6t^2 - 4k - 2$ in an HHM of odd size t: (a) $t = 3$; (b) $t = 5$.

contains a unique Hamiltonian cycle (figure 10), which is a cycle of length $6t^2 - 4$ of the original HHM.

Case 4 $l = 6t^2 - 4k - 4$, where k is an integer with $1 \leq k \leq 1.5t^2 - 4$ (equivalently, $12 \leq l \leq 6t^2 - 8$ and $6t^2 - l \equiv 0 \pmod 4$)). Again, we need to consider the parity of t.

Case 4.1 t is odd. Let C denote the cycle presented in case 3.1, and let $E = E(C)$. Let C_1, C_2, …, C_k denote the first to kth 6-cycles when counting in the order determined by the set of arrows shown in figure 11. For i from 1 to k, we let $E = E \oplus E(C_i)$. On completion of this procedure, a cycle of length $6t^2 - 4k - 4$ is formed.

Case 4.2 t is even. Let C denote the cycle given in case 3.2, and let $E = E(C)$. Let C_1, C_2, …, C_k denote the first to kth 6-cycles when counting in the order determined by the set of outer arrows followed by the set of inner arrows shown in figure 12. For i from 1 to k, we let $E = E \oplus E(C_i)$. When this procedure has been completed, a cycle of length $6t^2 - 4k - 4$ is formed.

This completes the proof of Theorem 1. ■

For completeness and by brute force, we obtain the following theorem.

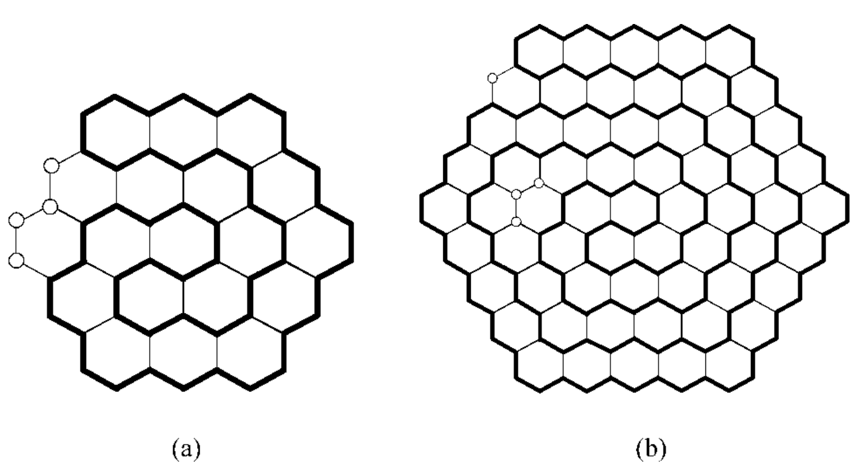

Figure 9. A cycle of length $6t^2 - 4$ in an HHM of odd order t: (a) $t = 3$; (c) $t = 5$.

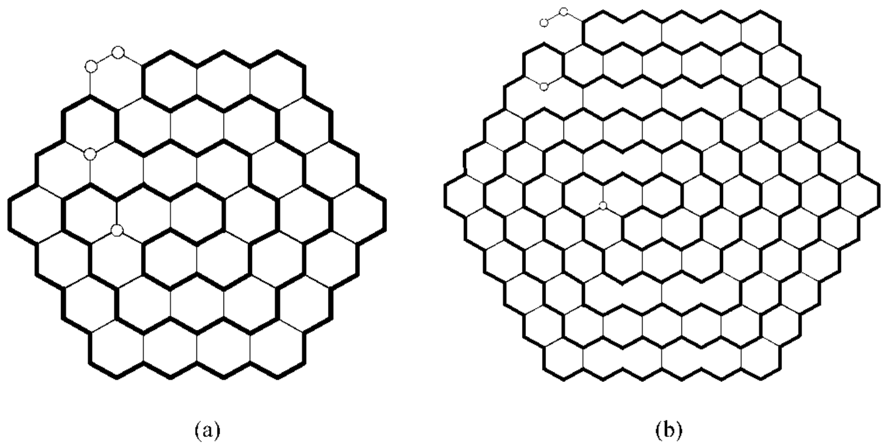

Figure 10. A cycle of length $6t^2 - 4$ in an HHM of even order t: (a) $t = 4$; (c) $t = 6$.

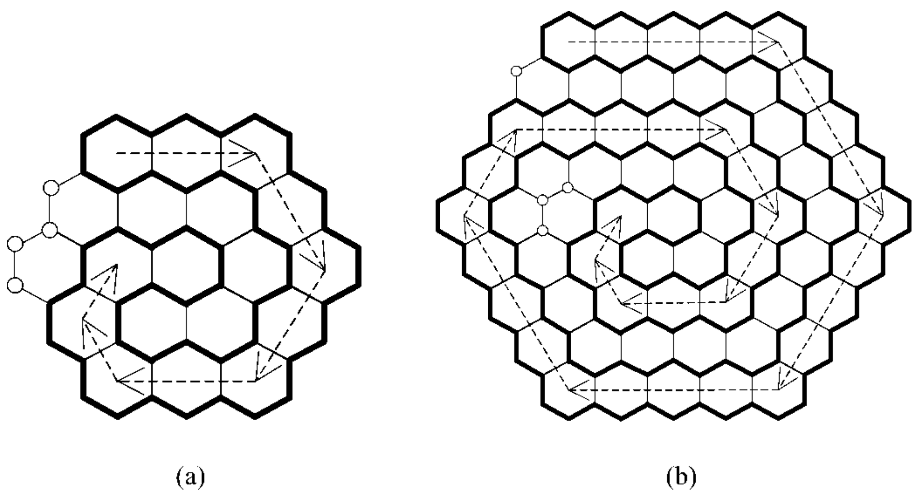

Figure 11. The order in which a set of k 6-cycles are selected to build a cycle of length $6t^2 - 4k - 4$ in an HHM of odd order t: (a) $t = 3$; (b) $t = 5$.

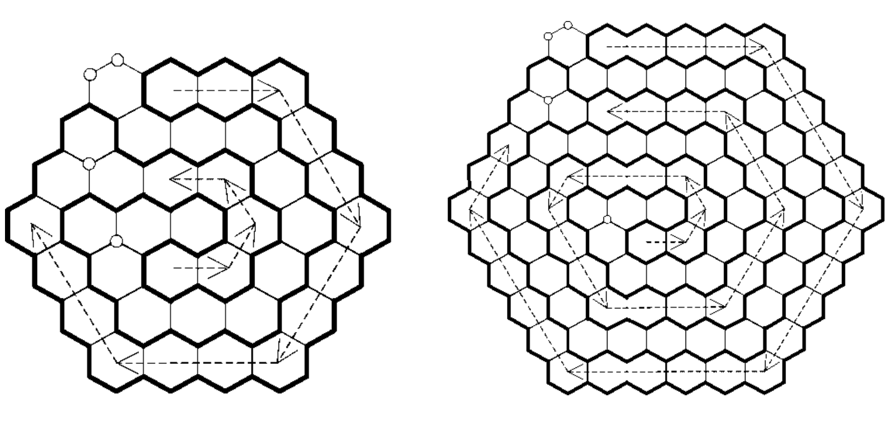

Figure 12. The order in which a set of 6-cycles are visited to build a cycle of length $6t^2 - 4k - 4$ in an HHM of even order t: (a) $t = 4$; (b) $t = 6$.

THEOREM 2 *An HHM of order* 2 *admits a cycle of length* $l \in \{10, 12, 14, 16, 18, 22\}$, *and admits no cycle of length* 20.

4. Conclusions

The existence of cycles of various lengths in an interconnection network is an important issue in evaluating how well a ring-structured parallel algorithm can be implemented on this network. The hexagonal honeycomb mesh is a promising candidate for interconnection networks. In this paper we have proved that a hexagonal honeycomb mesh possesses a cycle that has as length any value within a broad range. A systematic method for building these cycles has also been presented. The result obtained justifies the utility of this type of honeycomb mesh.

In future work we will extend the method used in this paper to establish similar results for other types of honeycomb network such as the *rectangular honeycomb mesh* and the *parallelogramic honeycomb mesh* [2].

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