AI VIETNAM
All-in-One Course

Basic Calculus

Limit, Derivative, and their Applications

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PhD in Computer Science

Year 2023

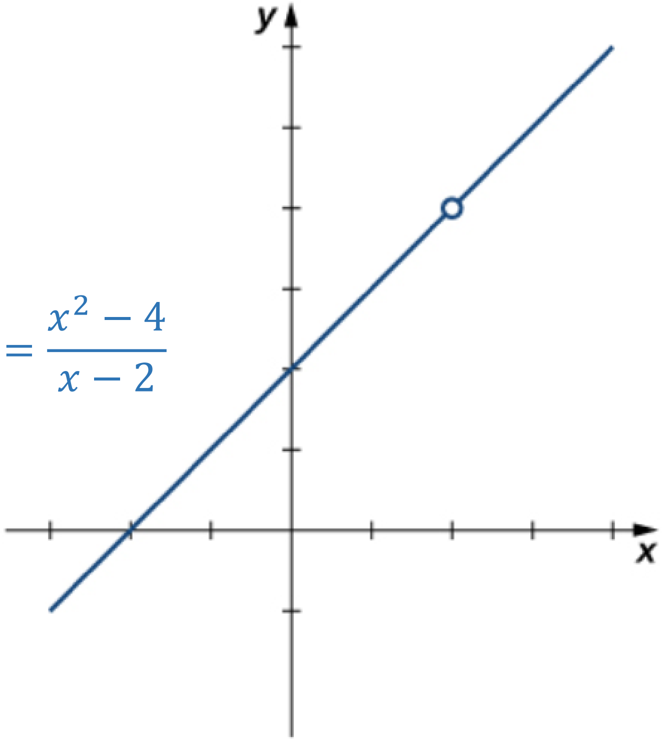
Outline

- Limit
- Area Computation Using Limit
- Derivative
- Newton’s Method

Limit

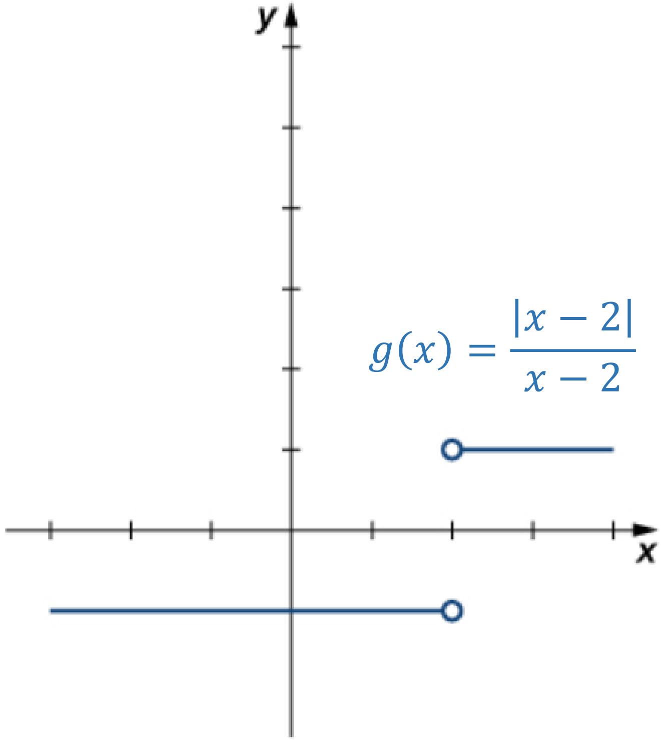
❖ Definition

from the reference book



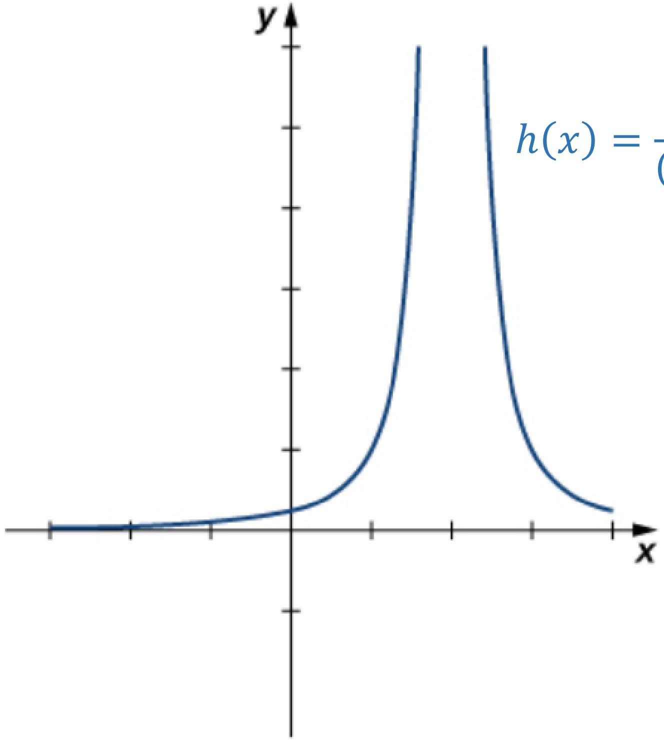
$$f(x) = \frac{x^2 - 4}{x - 2}$$

$$\lim_{x \to 2} f(x) = ?$$



$$g(x) = \frac{|x - 2|}{x - 2}$$

$$\lim_{x \to 2} g(x) = ?$$



$$h(x) = \frac{1}{(x - 2)^2}$$

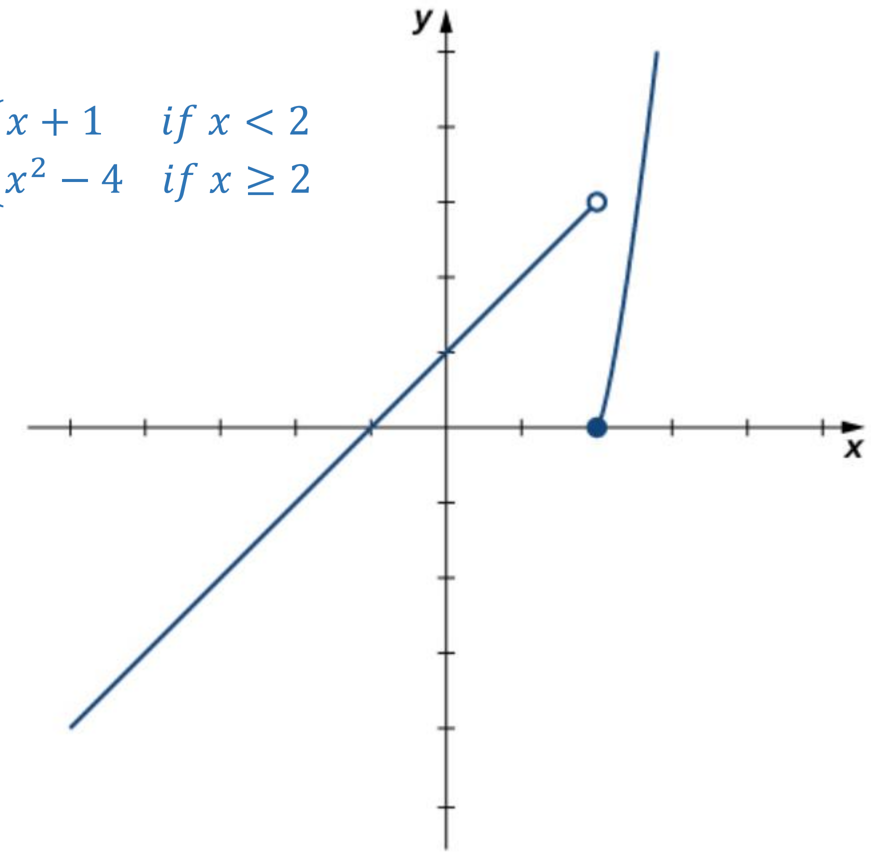
$$\lim_{x \to 2} h(x) = ?$$

Limit

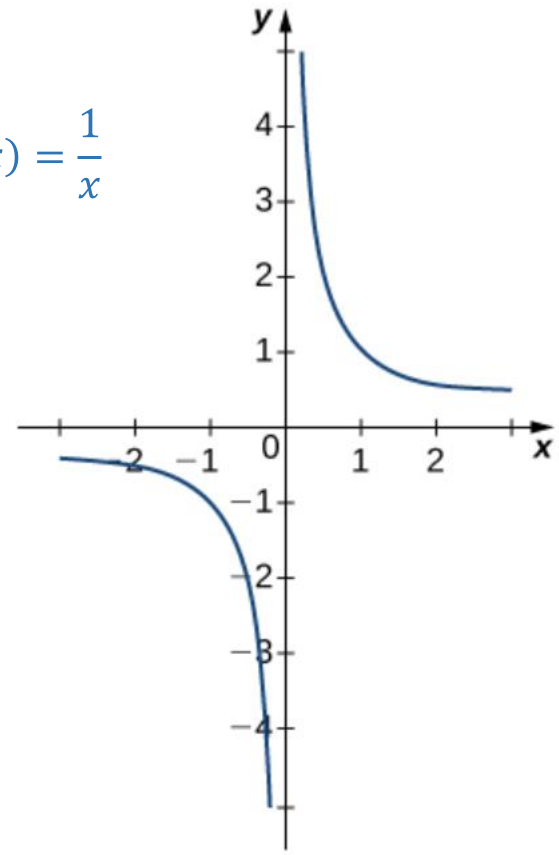
❖ Definition

from the reference book

$$f(x) = \begin{cases} x+1 & if\ x < 2 \\ x^2 - 4 & if\ x \geq 2 \end{cases}$$



$$g(x) = \frac{1}{x}$$



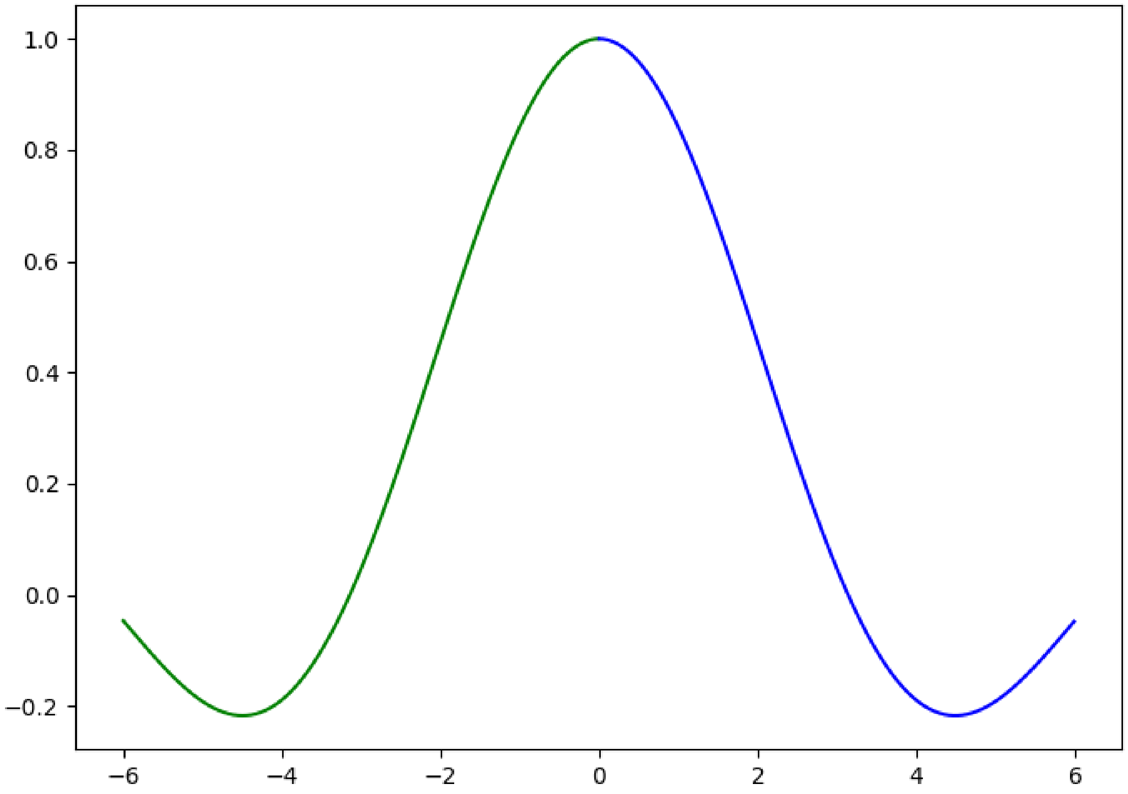
$$\lim_{x \to 2} f(x) = ?$$

$$\lim_{x \to 0} g(x) = ?$$

Limit

❖ Example

$$f(x) = \frac{\sin(x)}{x}$$

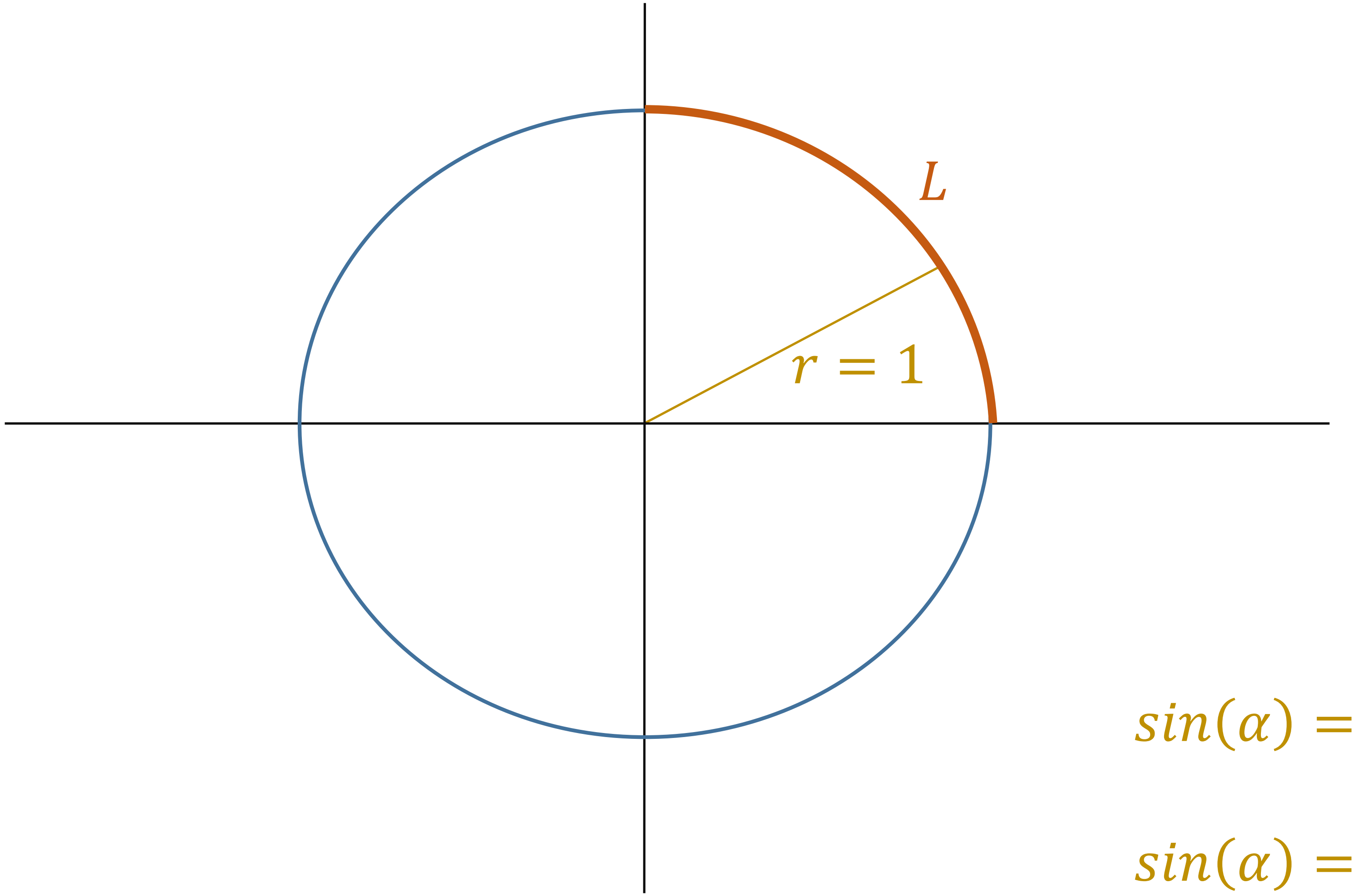


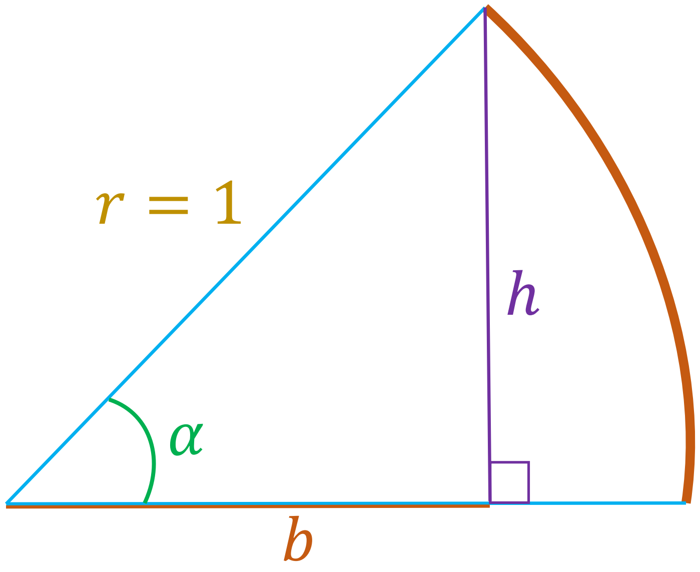

```python
# sin(x) / x
import math

def func(x):
    y = math.sin(x) / x
    return y
```

```python
import matplotlib.pyplot as plt
import numpy as np

left_x = np.arange(-6, 0, 0.01).tolist()
left_y = [func(x) for x in left_x]
plt.plot(left_x, left_y, 'g')

right_x = np.arange(0 + 1e-6, 6, 0.01).tolist()
right_y = [func(x) for x in right_x]
plt.plot(right_x, right_y, 'b')
```

Limit

- Compute the circumference of a unit circle





$$sin(\alpha) = \frac{h}{r}$$

$$sin(\alpha) = h$$

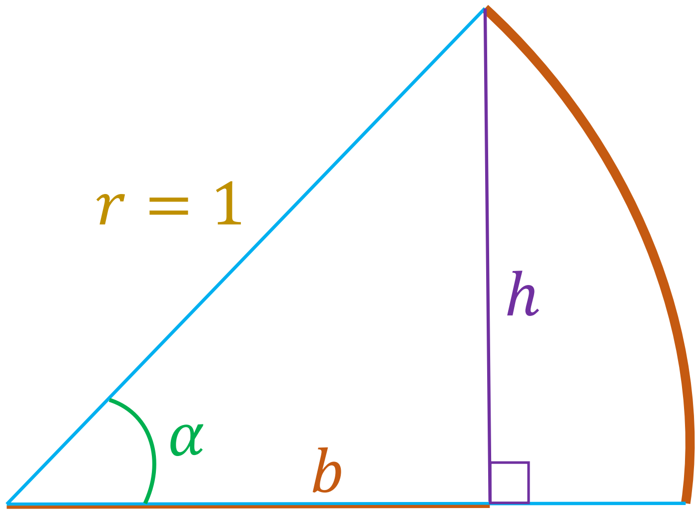
Divide the length into m segments

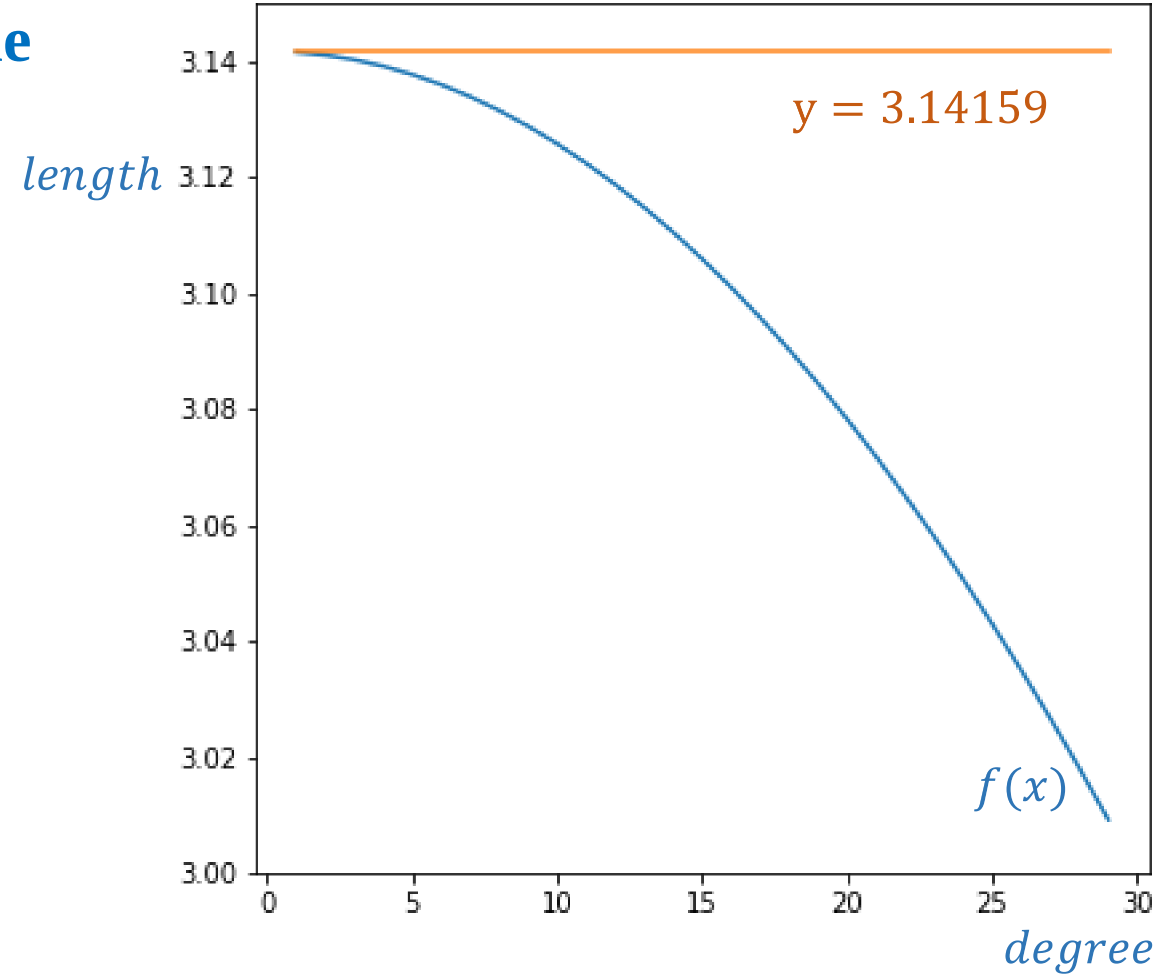
$$L = m * sin(\alpha)$$

Limit

❖ Compute the circumference of a unit circle

```
def compute_length(selected_degree):
    # sin value for the selected_degree
    # <--> the length of a part
    sin = math.sin(math.radians(selected_degree))

    # summarize (360/selected_degree) parts
    length = sin * (360/selected_degree)

    return length
```



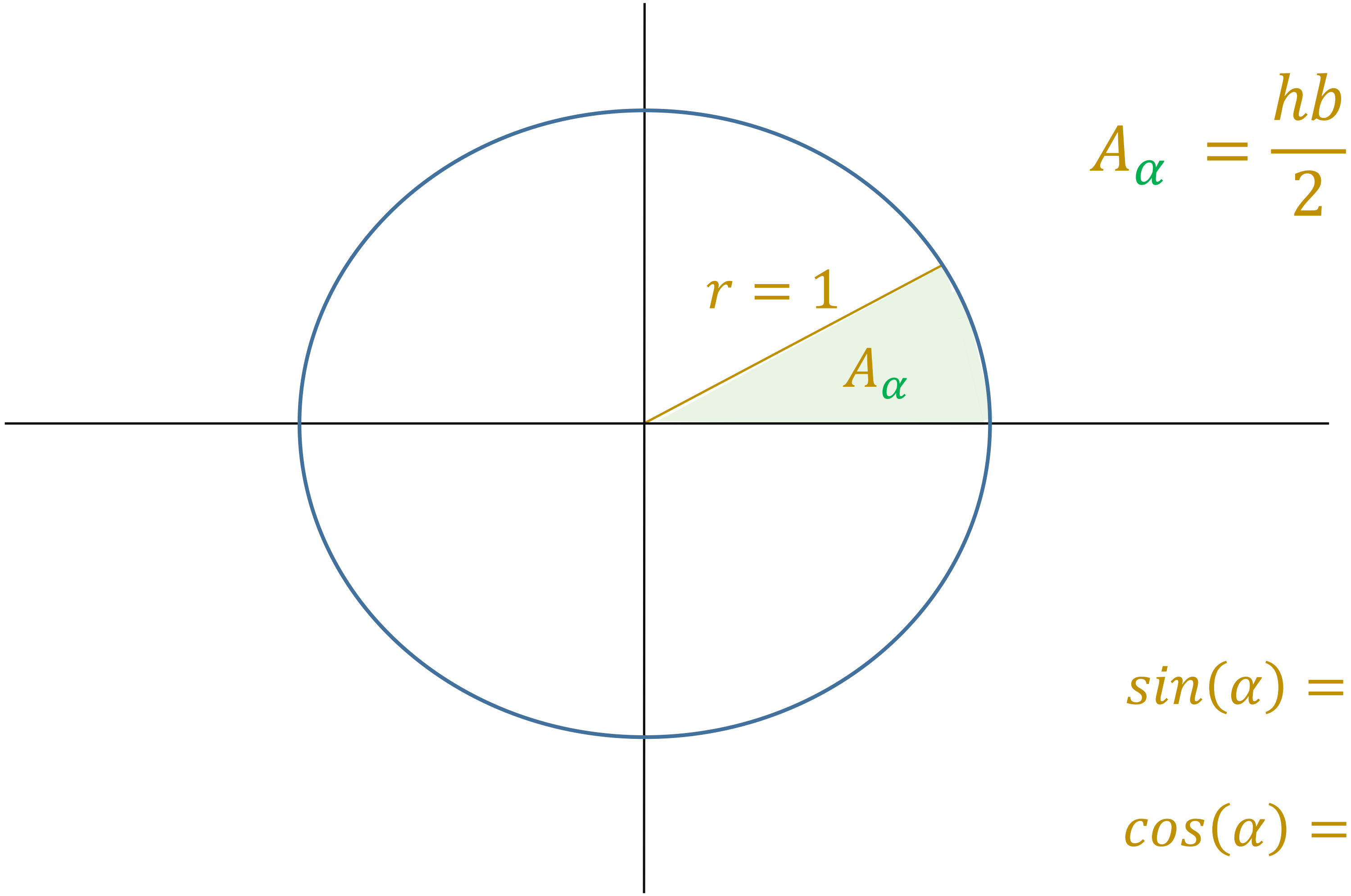


Divide the length into m segments

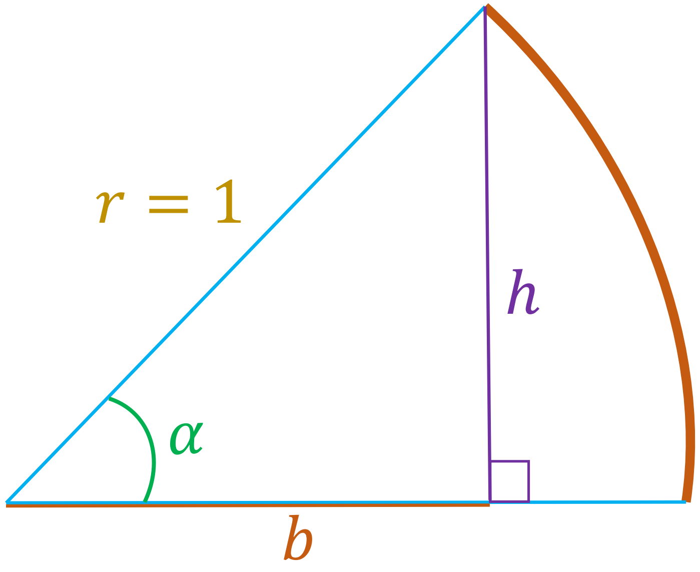
$$L = m * sin(\alpha)$$

Limit

❖ Compute the area of a unit circle



$$A_\alpha = \frac{hb}{2}$$



$$sin(\alpha) = \frac{h}{r}$$

$$cos(\alpha) = \frac{b}{r}$$

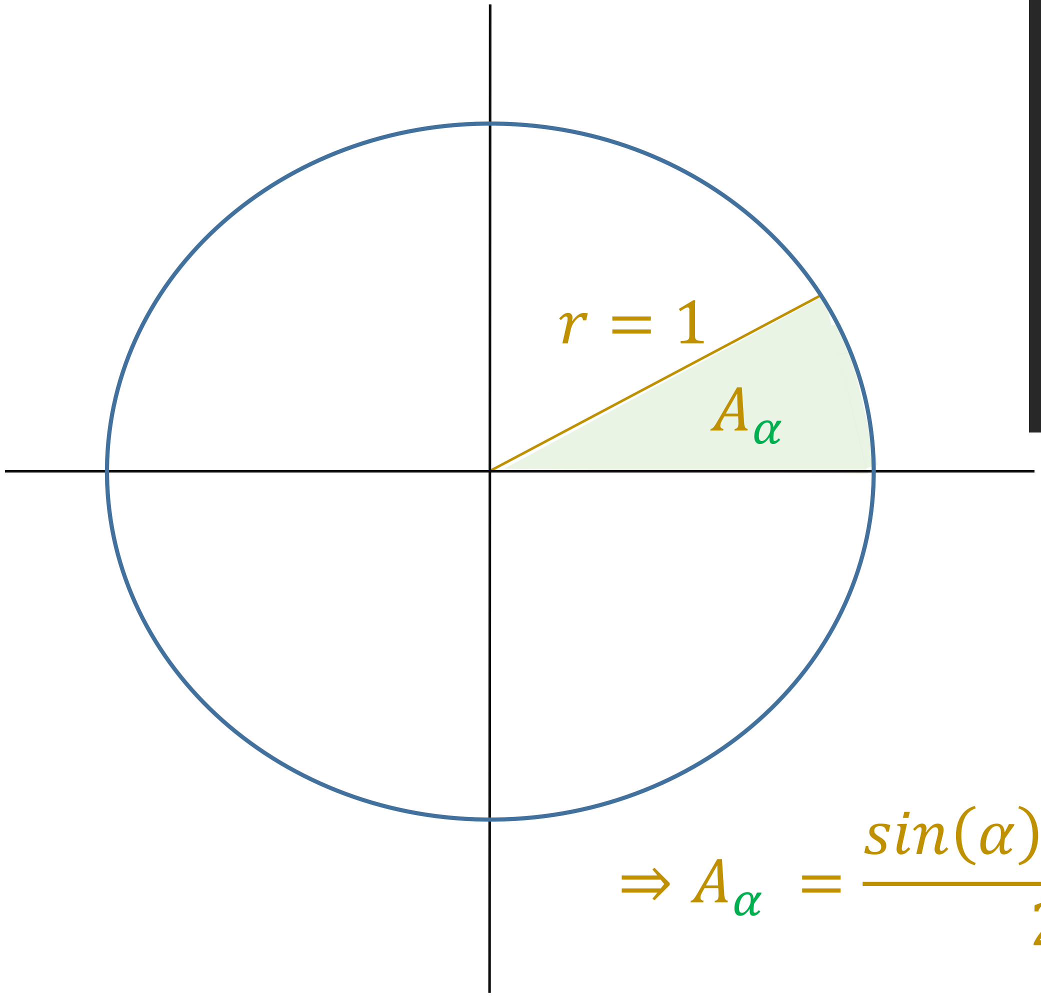
$$\Rightarrow A_\alpha = \frac{sin(\alpha)cos(\alpha)}{2}$$

$$\Rightarrow A = m * A_\alpha$$

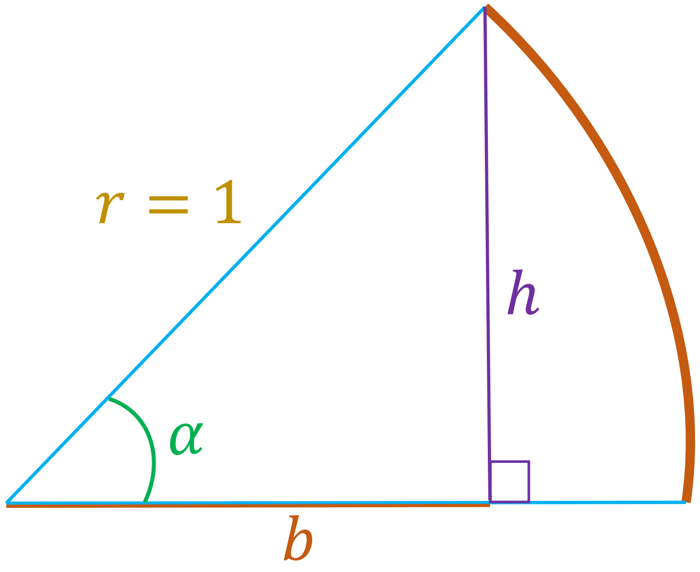
where m is a number of parts

Limit

❖ Compute the area of a unit circle




```python
def compute_area(selected_degree):
    # sin and cosine values for the selected_degree
    sin_degree = math.sin(math.radians(selected_degree))
    cos_degree = math.cos(math.radians(selected_degree))

    # compute area for a part
    area_degree = sin_degree*cos_degree/2

    # summarize (360/selected_degree) parts
    area = area_degree*(360/selected_degree)
    return area
```

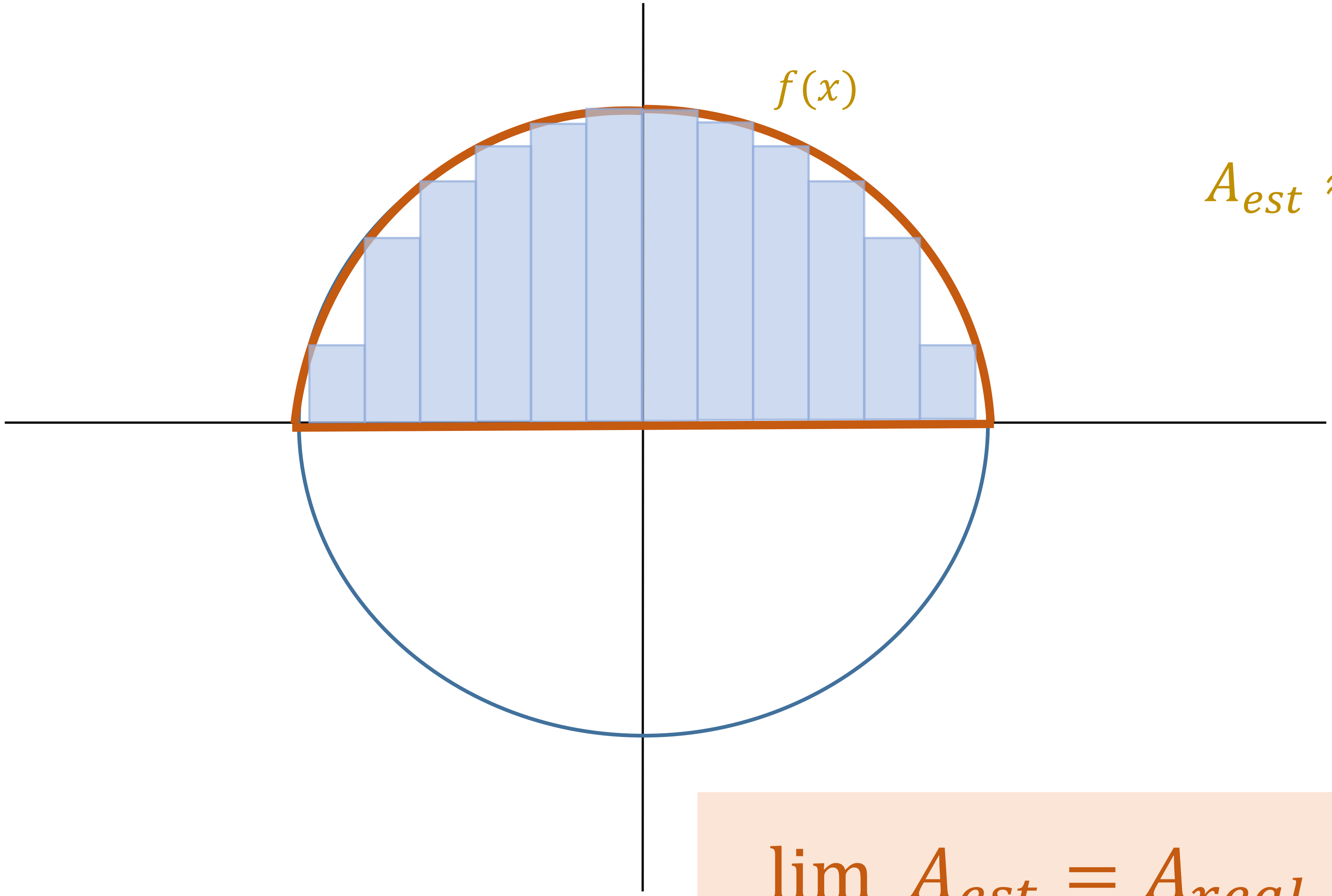


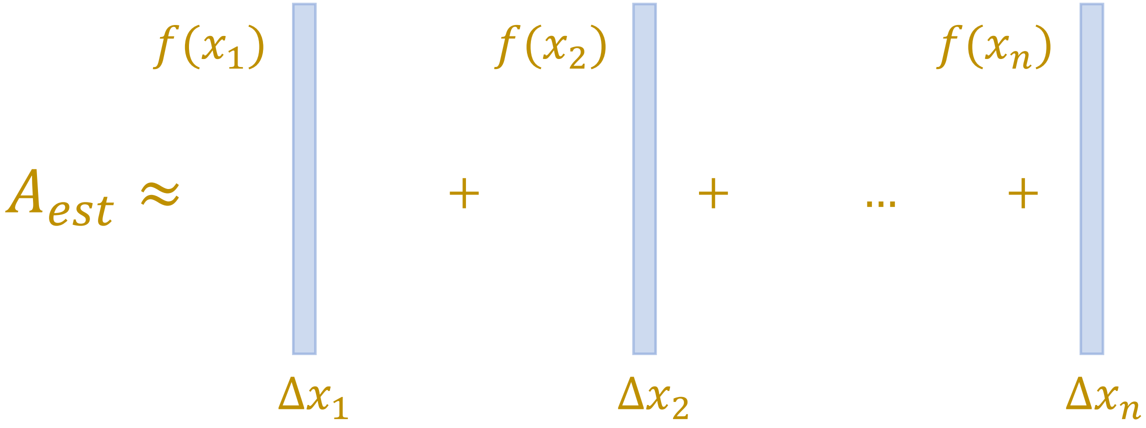
$$\Rightarrow A_\alpha = \frac{sin(\alpha)cos(\alpha)}{2}$$

$$\Rightarrow A = m * A_\alpha$$

Limit

❖ Compute the area of a unit circle





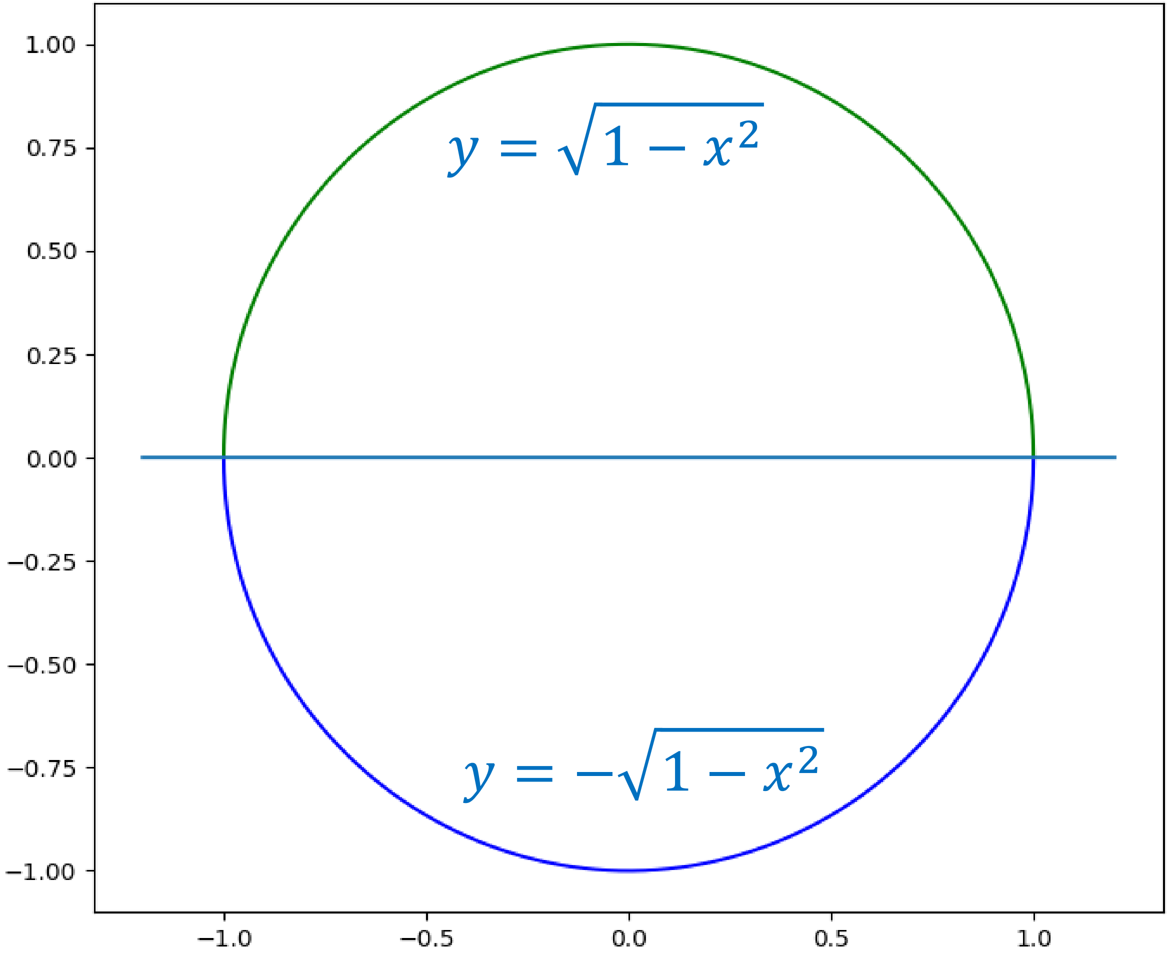
$$A_{est} \approx \sum_{i=1}^{n} f(x_i)\Delta x_i$$

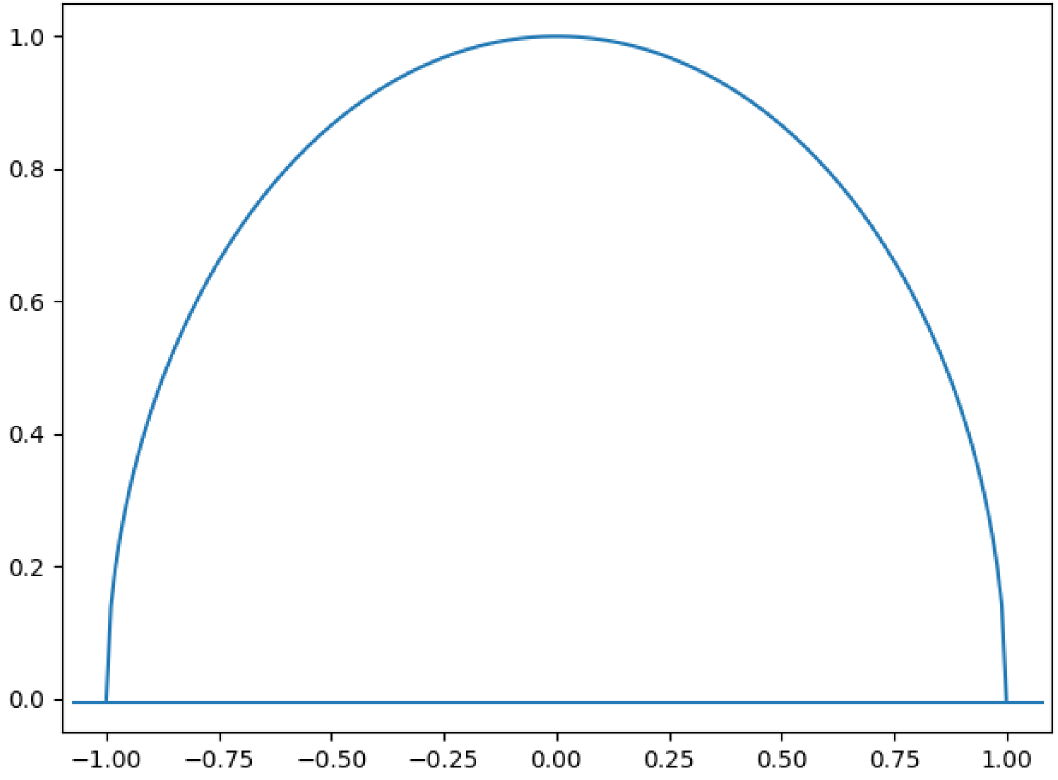
$$\lim_{\Delta x \to 0} A_{est} = A_{real}$$

Limit

❖ Compute the area of a unit circle

```python
def compute_y(x):
    return math.sqrt(1 - x*x)

data_x = np.arange(-1, 1, 1e-5).tolist()
data_y = [compute_y(x) for x in data_x]

plt.plot(data_x, data_y)
```





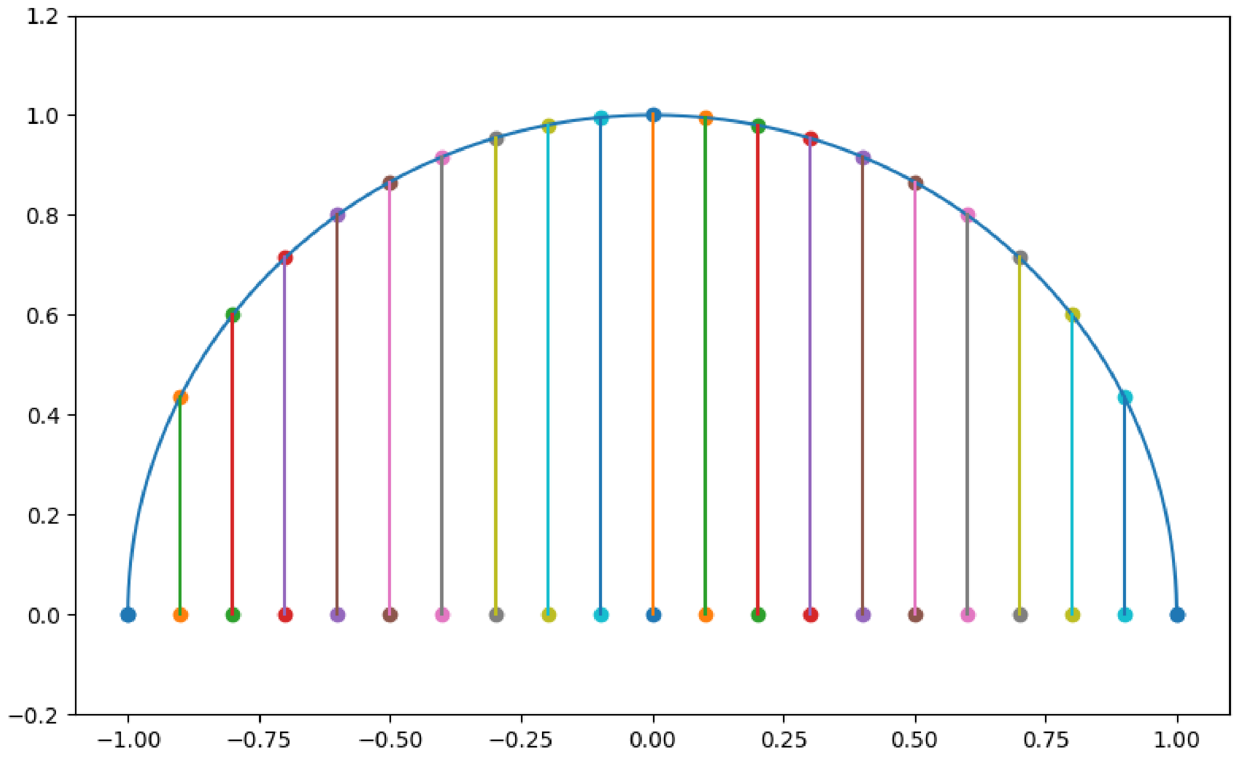
Limit

❖ Compute the area of a unit circle

math.pi=3.141592

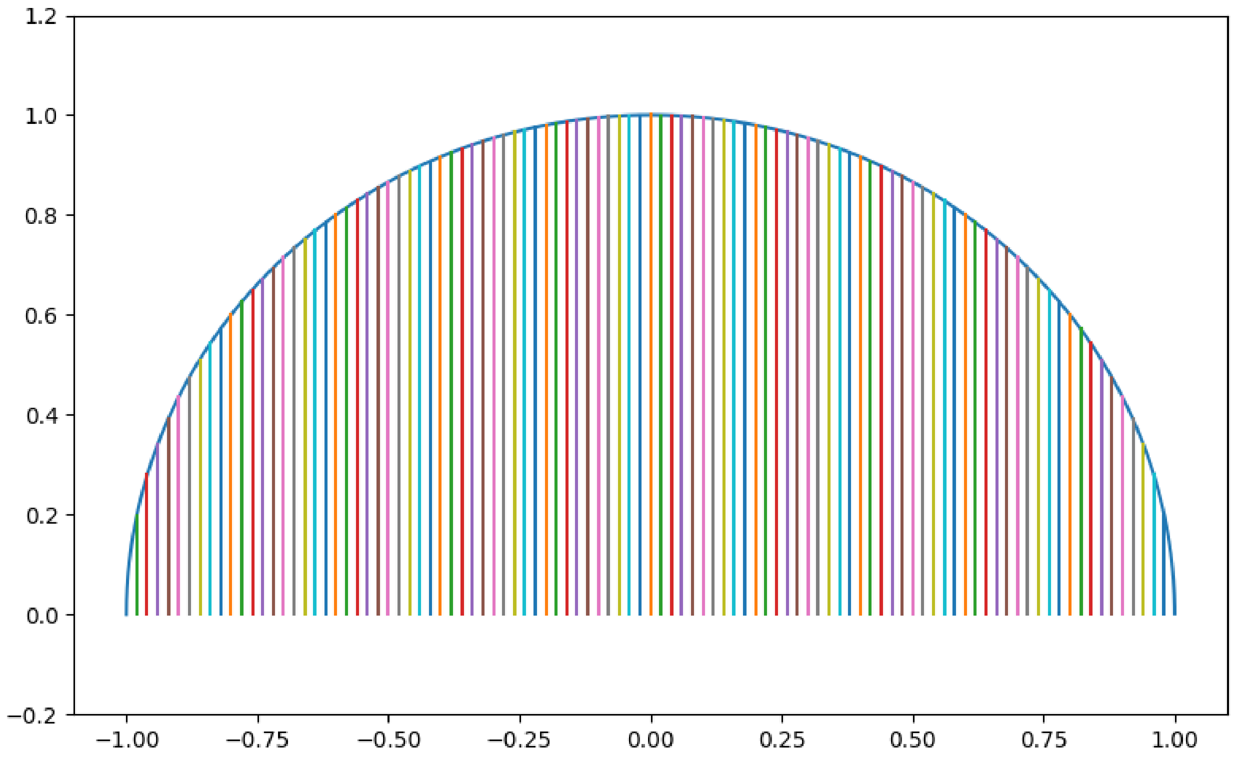
$n = 20$

$A_{est} = 3.1045$



$n = 200$

$A_{est} = 3.1404$



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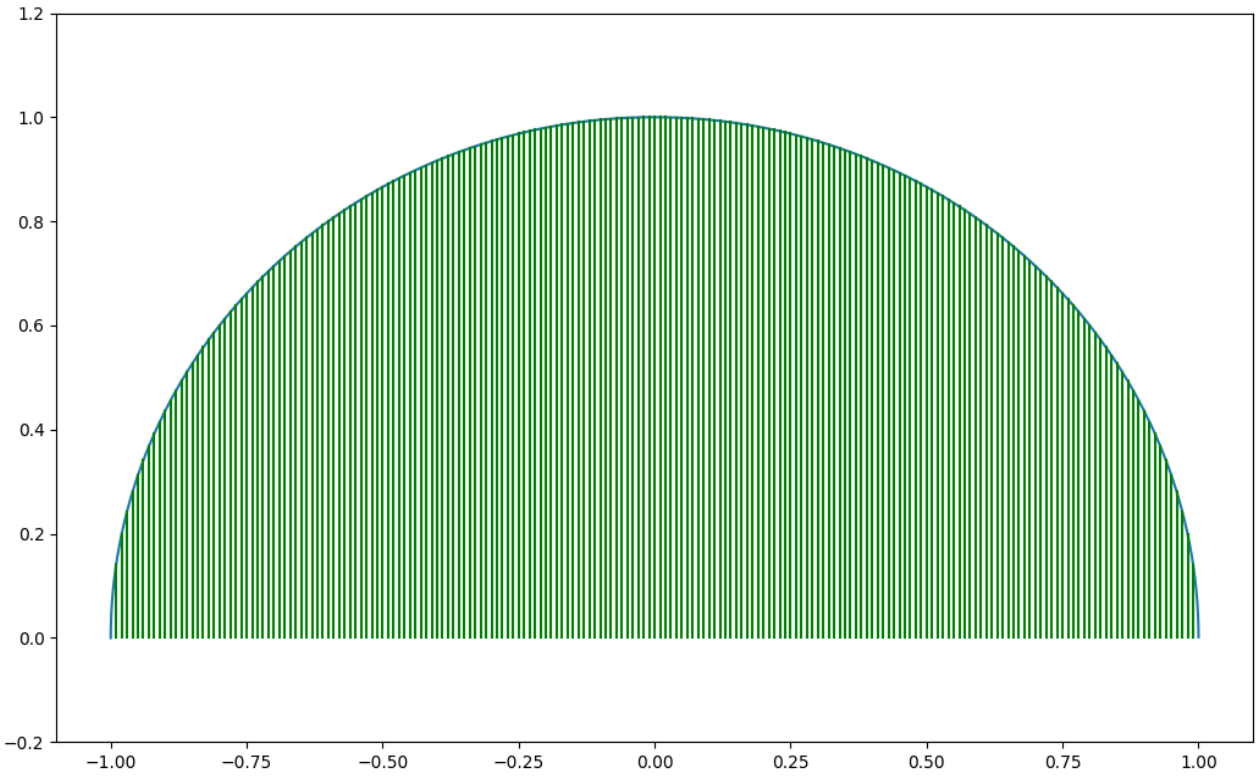
Limit

❖ Compute the area of a unit circle

math.pi=3.141592

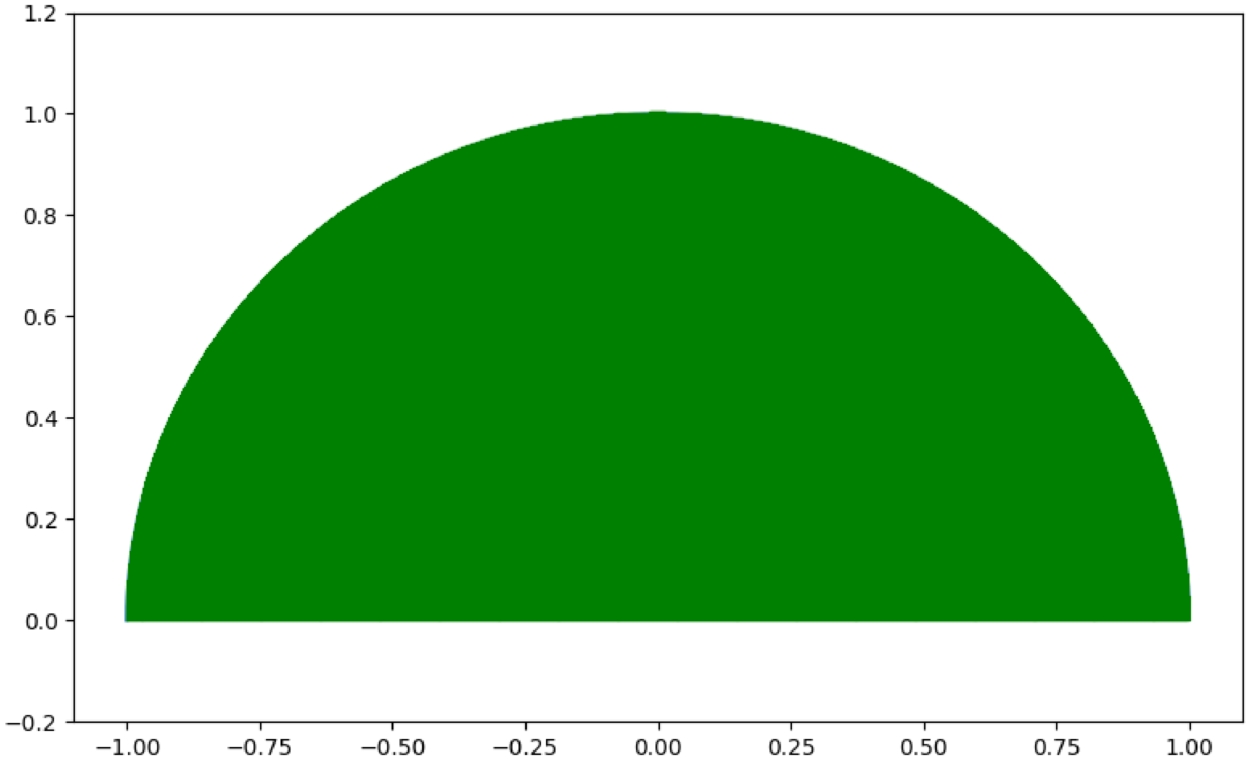
$n = 200$

$A_{est} = 3.1404$



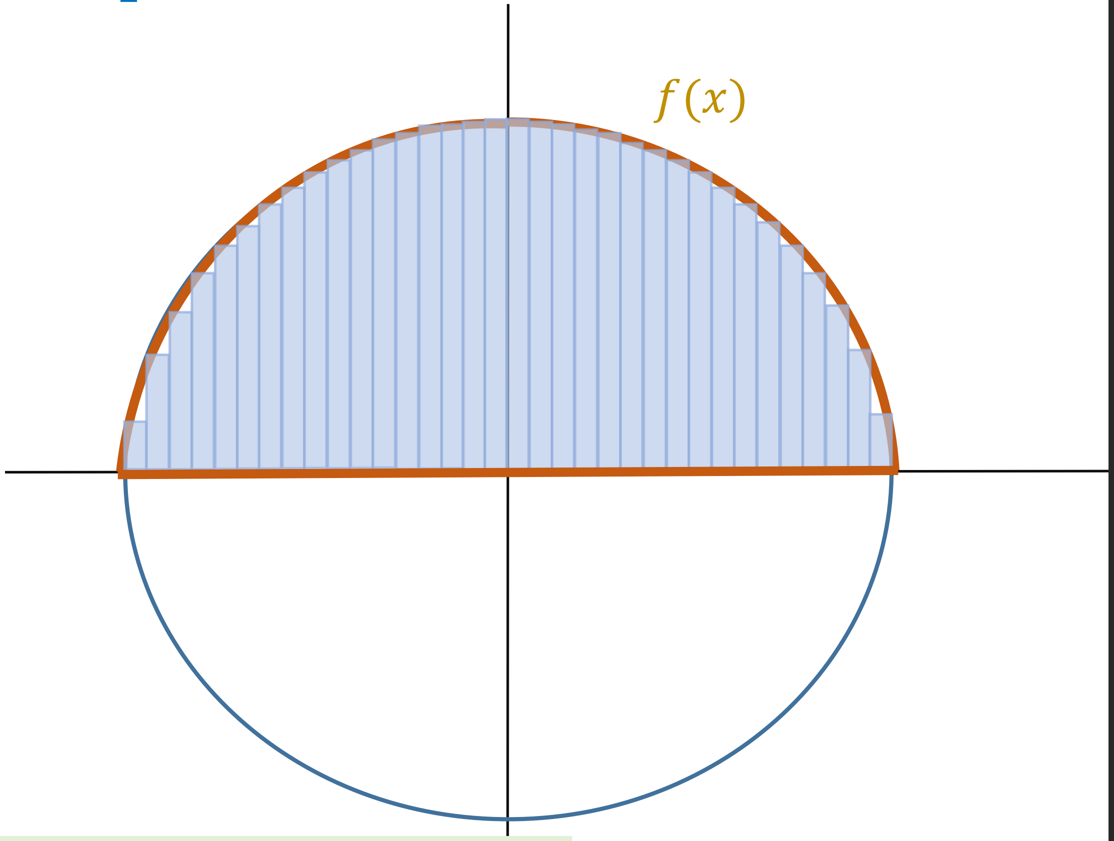
$n = 2000$

$A_{est} = 3.14155$



Limit

❖ Compute the area of a unit circle



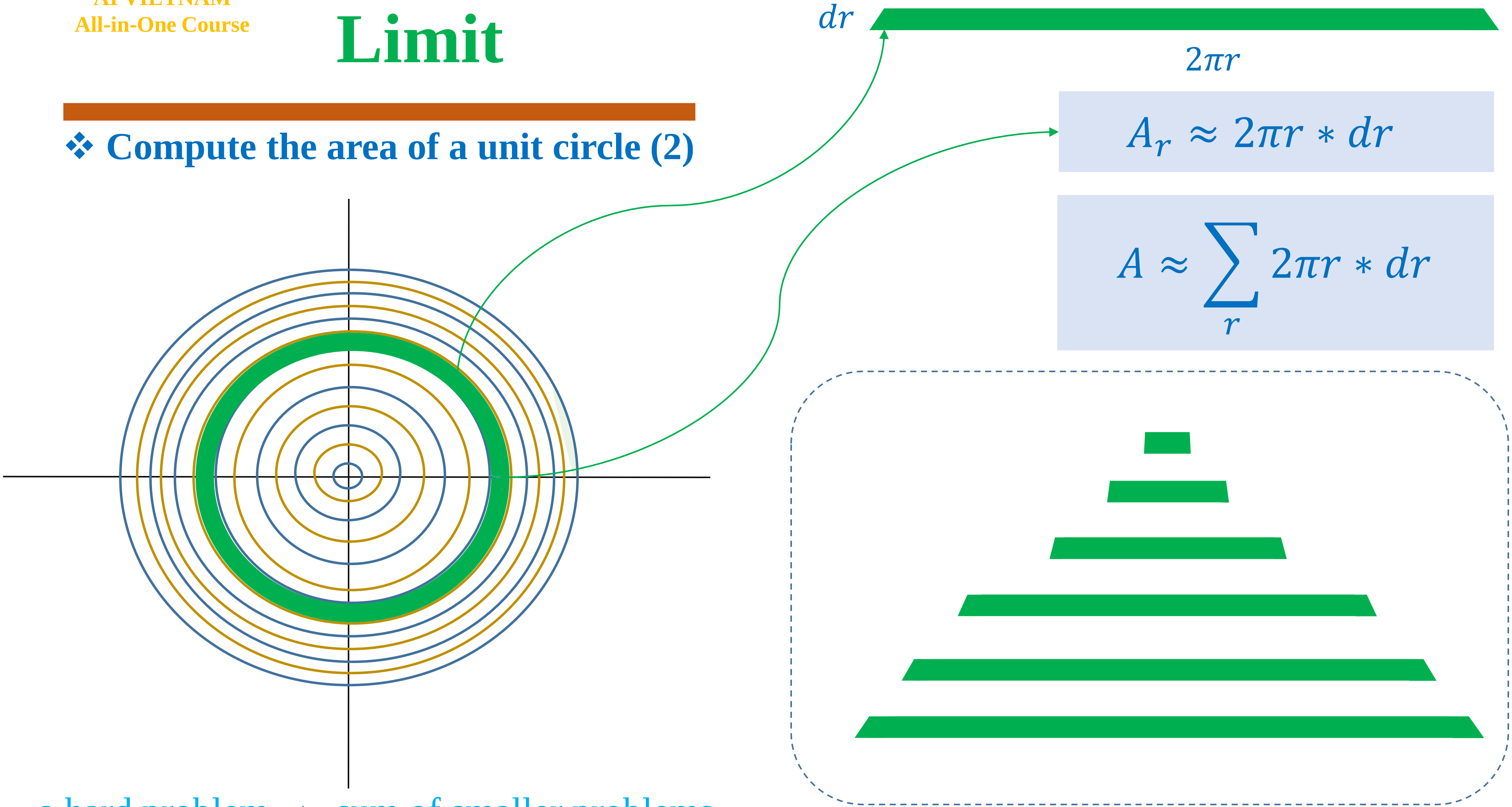
$$A_{est} \approx \sum_{i=1}^{n} f(x_i)\Delta x_i$$

```python
import math
import numpy as np

def get_y(x):
    y = math.sqrt(1 - x*x)
    return y

# create x, y
step = 1e-5
x_data = np.arange(-1, 1, step).tolist()
y_data = [get_y(x) for x in x_data]

# compute area
areas = [y*step for y in y_data]
area = sum(areas)
print(area*2)
```

```
3.141592616416181
```

Limit

❖ **Compute the area of a unit circle (2)**



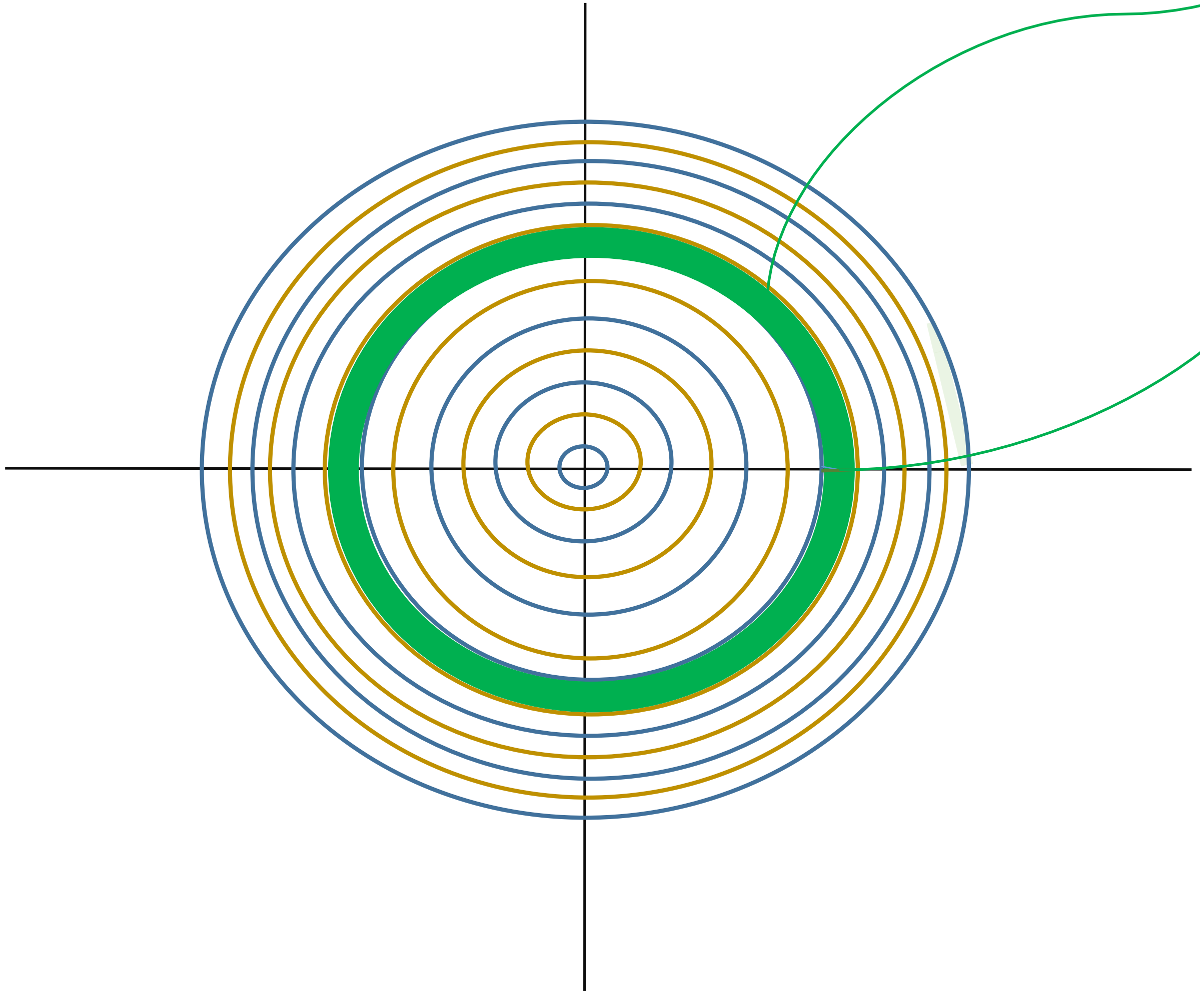
$$A_r \approx 2\pi r * dr$$

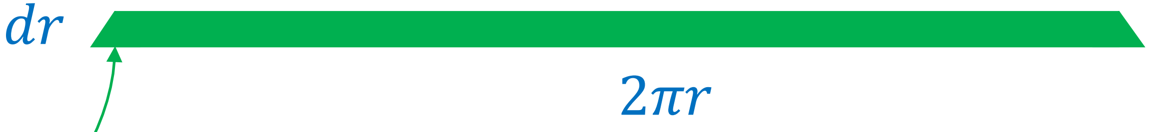
$$A \approx \sum_r 2\pi r * dr$$

a hard problem ➡ sum of smaller problems

Limit

❖ Compute the area of a unit circle (2)





$$A_r \approx 2\pi r * dr$$

$$A \approx \sum_{r} 2\pi r * dr$$

```python
import math
import numpy as np

step = 1e-5
radii = np.arange(0, 1, step).tolist()

areas = [math.pi*2*radius*step for radius in radii]
area = sum(areas)
print(f'area is {area}')
```

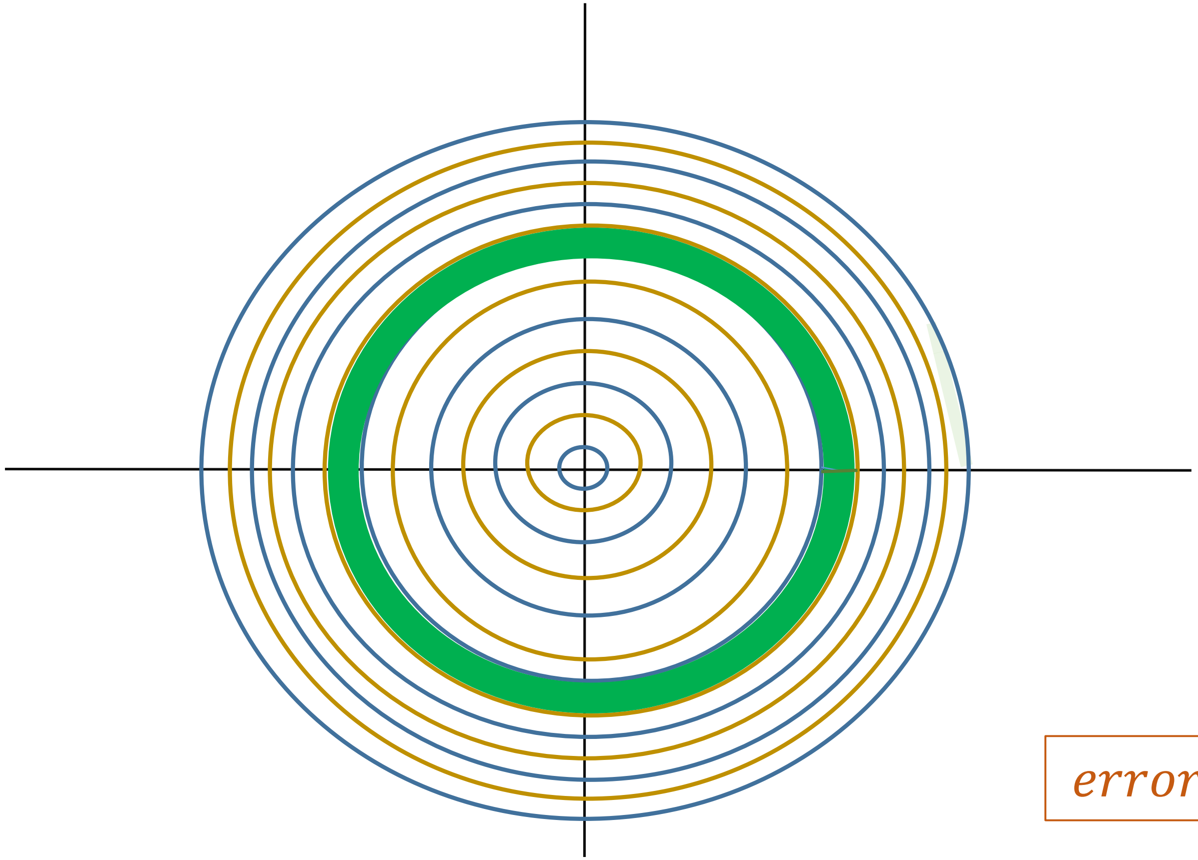
```
area is 3.1415612376632573
```

a hard problem ➡ sum of smaller problems

Limit

$$A \approx \sum_r 2\pi r * dr$$

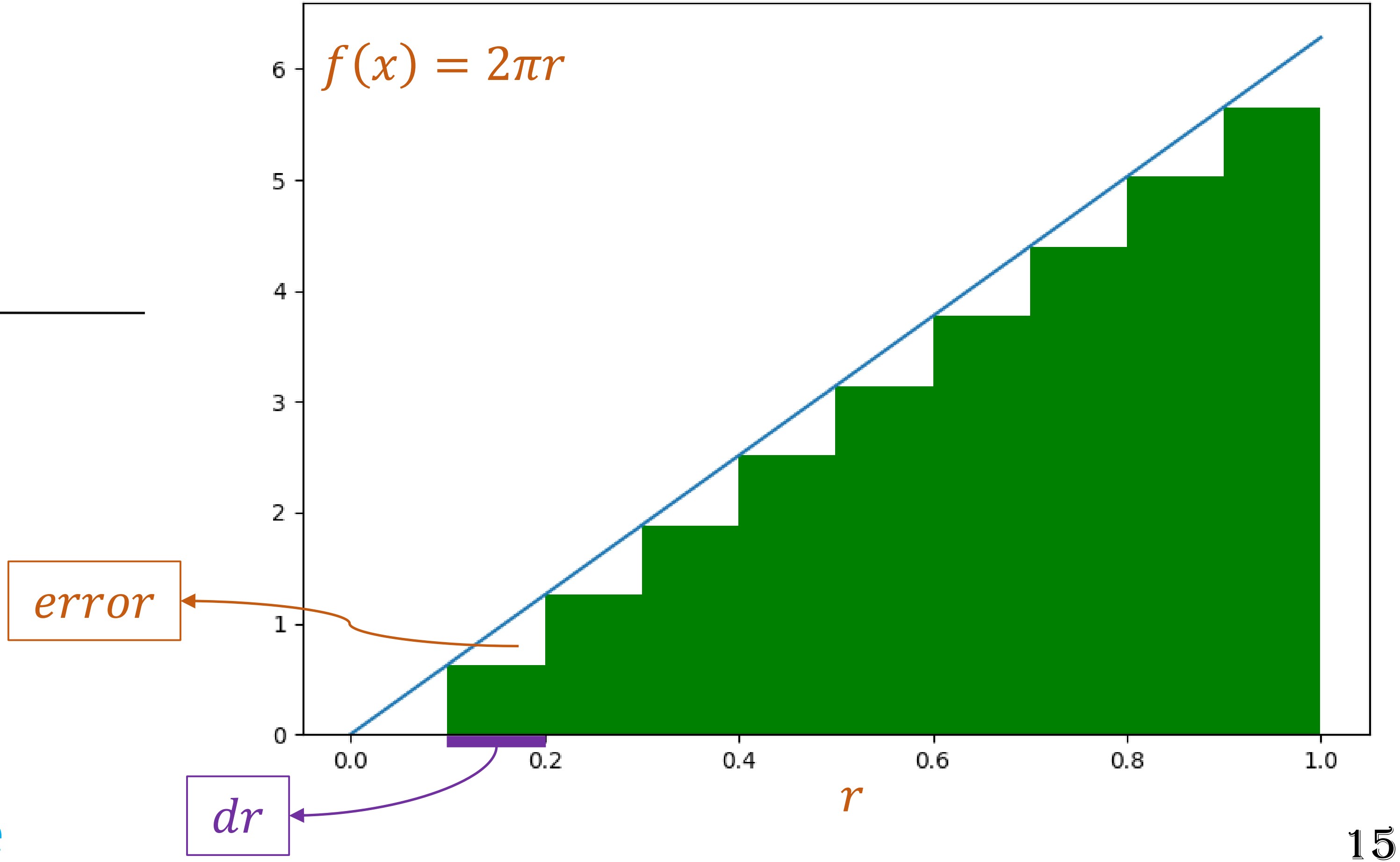
❖ **Compute the area of a unit circle (2)**



sum of smaller problems

⇩

area under a function within a range

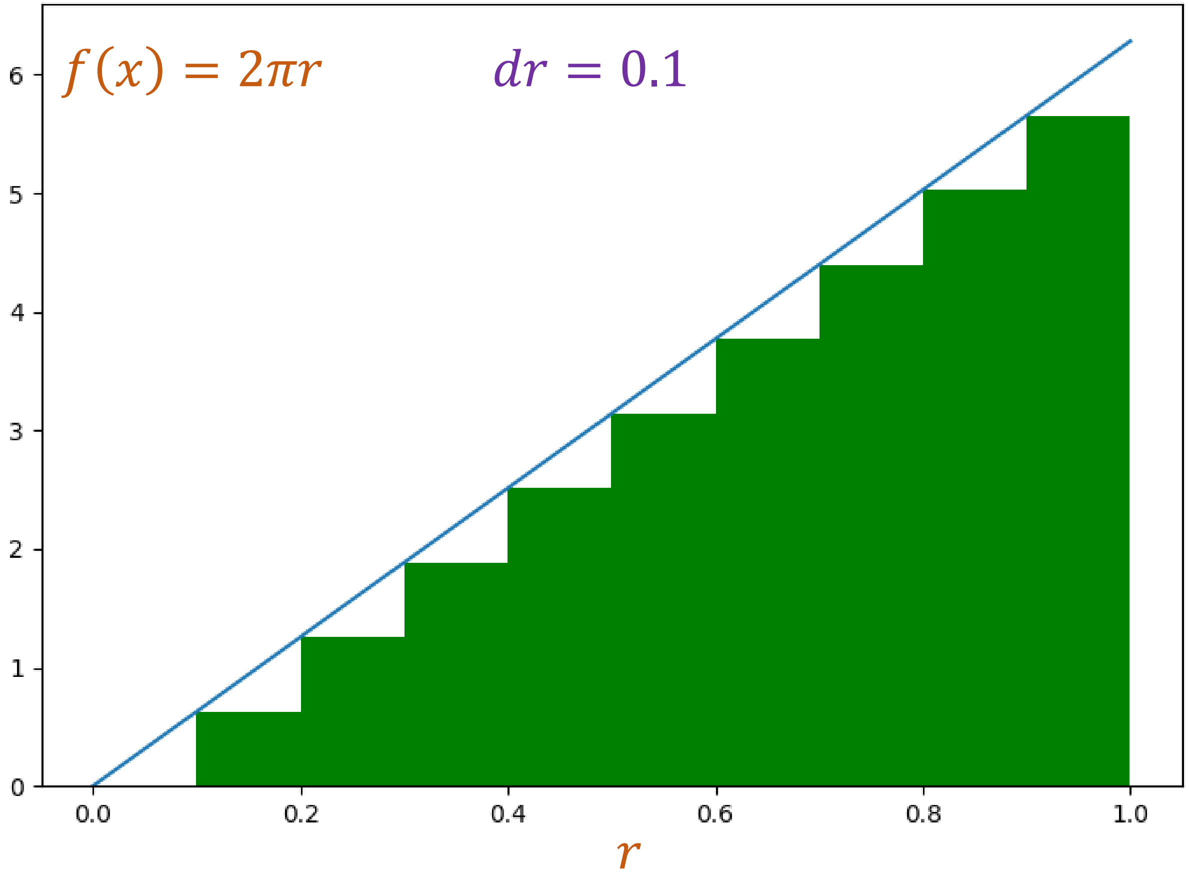


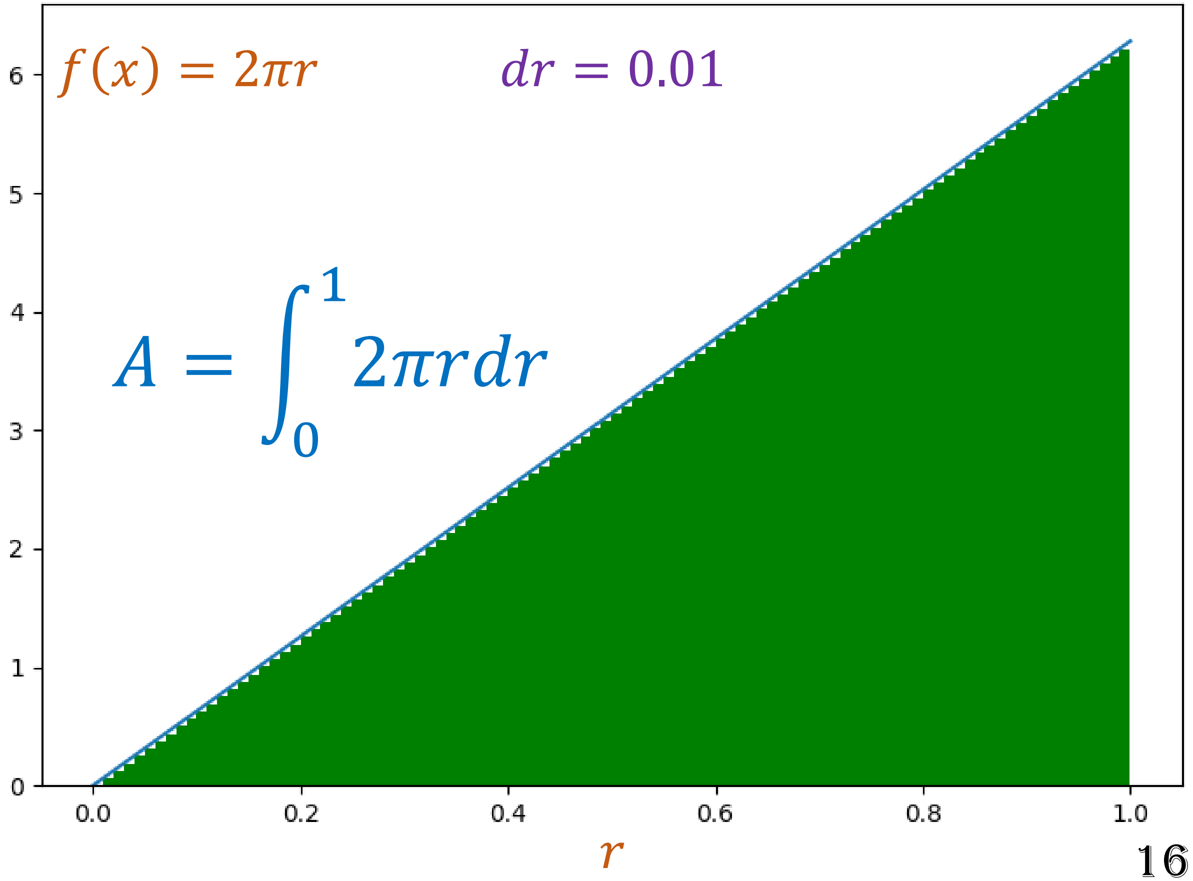
Limit

$$A \approx \sum_r 2\pi r * dr$$

❖ Compute the area of a unit circle (2)

a hard problem ➡ sum of smaller problems ➡ area under a function within a range



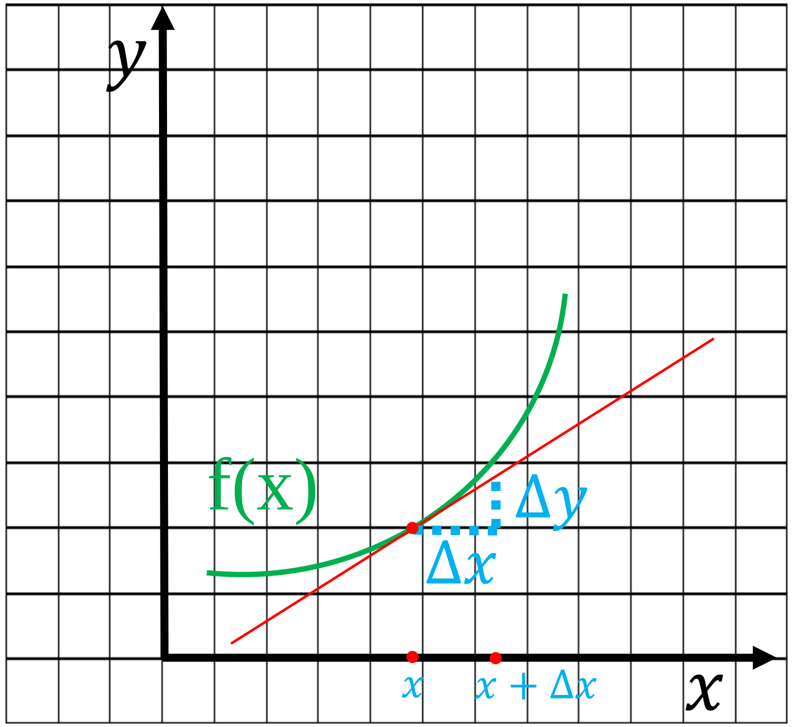


Outline

- Limit
- Area Computation Using Limit
- Derivative
- Newton’s Method

Derivative

Đạo hàm cho hàm liên tục

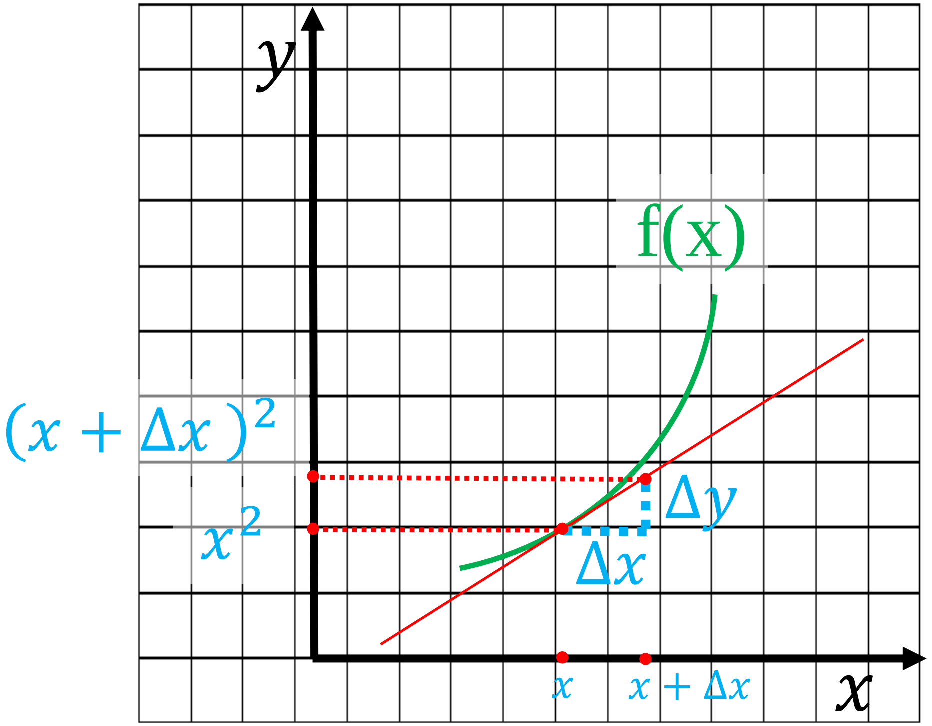


$$\frac{d}{dx}f(x), \frac{dy}{dx}, y', f'(x)$$

$$\text{Đạo hàm} = \frac{\textit{Thay đổi theo y}}{\textit{Thay đổi theo x}} = \frac{\Delta y}{\Delta x}$$

$$\frac{d}{dx}f(x) = \underbrace{\lim_{\Delta x \to 0}} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

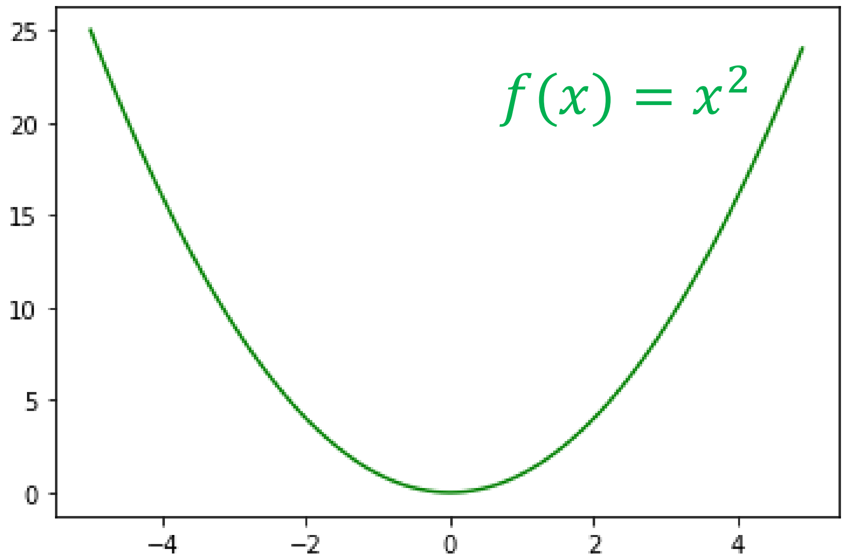
Δx cần tiến về 0 để đường tiếp tuyến tiến về hàm f(x) trong vùng lân cận tại x



$$\text{Đạo hàm} = \frac{\textit{Thay đổi theo } y}{\textit{Thay đổi theo } x} = \frac{\Delta y}{\Delta x}$$

$$\frac{d}{dx} f(x) = \underbrace{\lim_{\Delta x \to 0}} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Δx cần tiến về 0 để đường tiếp tuyến tiến về hàm f(x) trong vùng lân cận tại x



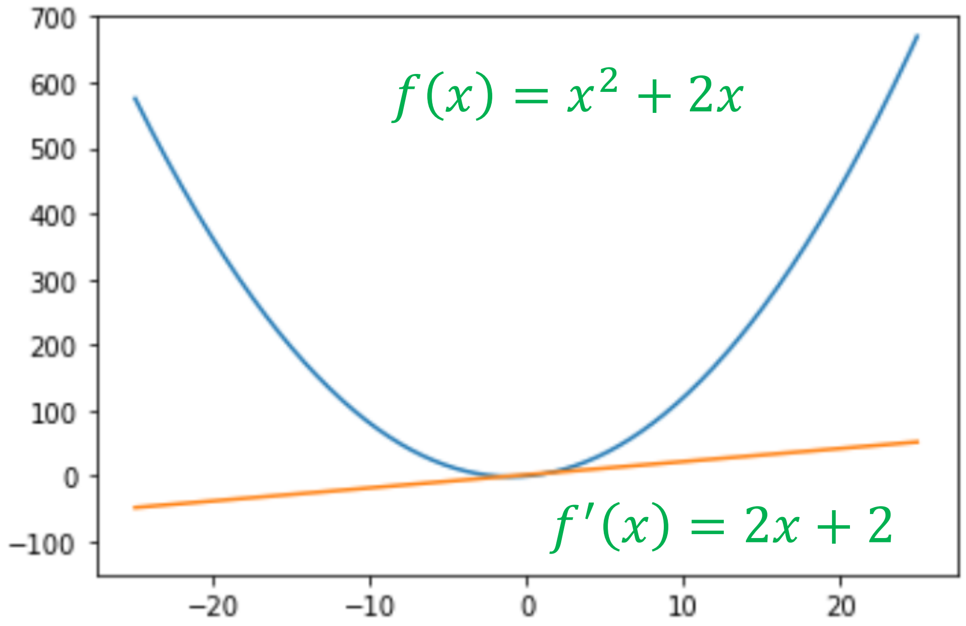
$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

Derivative

❖ Implementation




```python
# python code

def func(x):
    return x**2 + 2*x

def func_derivative(x):
    return 2*x + 2
```

```python
d_value = func_derivative(2.0)
print('f\'(x=2) is', d_value)
```

```
f'(x=2) is 6.0
```

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + 2(x + \Delta x) - (x^2 + 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x + 2 = 2x + 2$$

Derivative

$$f(x) = kx$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{k(x+\Delta x) - kx}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{kx + k\Delta x - kx}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{k\Delta x}{\Delta x}$$

$$= k$$

$$f(x) = \frac{1}{x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{1}{(x+\Delta x)} - \frac{1}{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{x - (x+\Delta x)}{(x+\Delta x)x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x (x+\Delta x)x}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x)x}$$

$$= \frac{-1}{x^2}$$

$$f(x) = \sqrt{x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} * \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(\sqrt{x+\Delta x})^2 - (\sqrt{x})^2}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Derivative

$f(x) = e^x$

$$\lim_{t\to 0}\frac{a^t - 1}{t} = ln(a)$$

$$f'(x) = \lim_{\Delta x\to 0}\frac{e^{(x+\Delta x)} - e^x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{e^x e^{\Delta x} - e^x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{e^x(e^{\Delta x}-1)}{\Delta x}$$

$$= e^x \lim_{\Delta x\to 0}\frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x ln(e) = e^x$$

$f(x) = a^x$

$$\lim_{t\to 0}\frac{a^t - 1}{t} = ln(a)$$

$$f'(x) = \lim_{\Delta x\to 0}\frac{a^{(x+\Delta x)} - a^x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{a^x a^{\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{a^x(a^{\Delta x}-1)}{\Delta x}$$

$$= a^x \lim_{\Delta x\to 0}\frac{a^{\Delta x} - 1}{\Delta x}$$

$$= a^x ln(a)$$

$f(x) = ln(x)$

$$\lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$$

$$f'(x) = \lim_{\Delta x\to 0}\frac{ln(x+\Delta x)\ - n(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{ln(\frac{x+\Delta x}{x})}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}ln(1+\frac{\Delta x}{x})$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\frac{\Delta x}{x}\frac{x}{\Delta x}ln(1+\frac{\Delta x}{x})$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\frac{\Delta x}{x}ln(1+\frac{\Delta x}{x})^{\frac{x}{\Delta x}}$$

$$let\ \ t = \frac{\Delta x}{x}, \lim_{\Delta x\to 0} \Rightarrow \lim_{t\to 0}.$$

$$= \lim_{t\to 0}\frac{1}{x}ln(1+t)^{\frac{1}{t}}$$

$$= \frac{1}{x}ln(\lim_{t\to 0}(1+t)^{\frac{1}{t}}) = \frac{1}{x}ln(e) = \frac{1}{x}$$

$f(x) = sin(x)$

$sin(a+b) = sin(a)cos(b) + sin(b)cos(a)$

$\lim_{x\to 0}\frac{sin(x)}{x} = 1$ $\quad\quad$ $\lim_{x\to 0}\frac{1-cos(x)}{x} = 0$

$$f'(x) = \lim_{\Delta x\to 0}\frac{sin(x+\Delta x) - sin(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{sin(x)cos(\Delta x) + sin(\Delta x)cos(x) - in(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{sin(x)(cos(\Delta x) - 1) + sin(\Delta x)cos(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{sin(x)(cos(\Delta x) - 1)}{\Delta x} + \lim_{\Delta x\to 0}\frac{sin(\Delta x)cos(x)}{\Delta x}$$

$$= sin(x)\lim_{\Delta x\to 0}\frac{(cos(\Delta x) - 1)}{\Delta x} + cos(x)\lim_{\Delta x\to 0}\frac{sin(\Delta x)}{\Delta x}$$

$$= sin(x) * 0 + cos(x) * 1$$

$$= cos(x)$$

$f(x) = \log_a x => \frac{\ln(x)}{\ln(a)}$ $\quad$ vì $\log_a b = \frac{\log_c b}{\log_c a}$

$\lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$

$$f'(x) = \lim_{\Delta x\to 0}\frac{1}{ln(a)}\frac{ln(x+\Delta x) - ln(x)}{\Delta x}$$

$$= \frac{1}{ln(a)}\lim_{\Delta x\to 0}\frac{ln(\frac{x+\Delta x}{x})}{\Delta x}$$

$$= \frac{1}{ln(a)}\lim_{\Delta x\to 0}\frac{1}{\Delta x}ln(1+\frac{\Delta x}{x})$$

$$= \frac{1}{ln(a)}\lim_{\Delta x\to 0}\frac{1}{\Delta x}\frac{\Delta x}{x}\frac{x}{\Delta x}ln(1+\frac{\Delta x}{x})$$

$$= \frac{1}{ln(a)}\lim_{\Delta x\to 0}\frac{1}{\Delta x}\frac{\Delta x}{x}ln(1+\frac{\Delta x}{x})^{\frac{x}{\Delta x}}$$

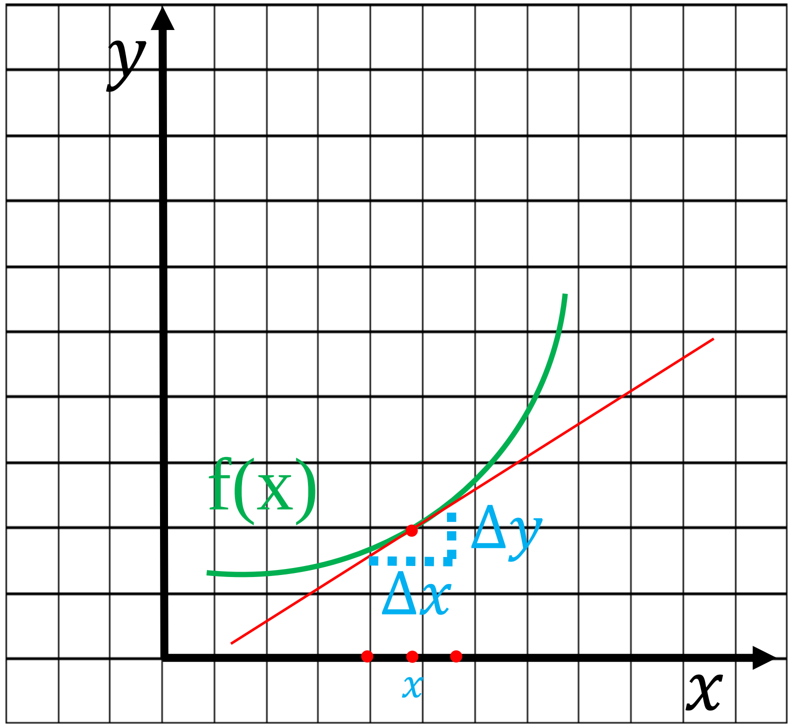
$let\ t = \frac{\Delta x}{x}, \lim_{\Delta x\to 0} => \lim_{t\to 0}.$

$$= \frac{1}{ln(a)}\lim_{t\to 0}\frac{1}{x}ln(1+t)^{\frac{1}{t}}$$

$$= \frac{1}{ln(a)}\frac{1}{x}ln(\lim_{t\to 0}(1+t)^{\frac{1}{t}}) = \frac{1}{ln(a)}\frac{1}{x}ln(e) = \frac{1}{ln(a)}\frac{1}{x}$$

Derivative and Applications

Đạo hàm trung tâm cho hàm liên tục

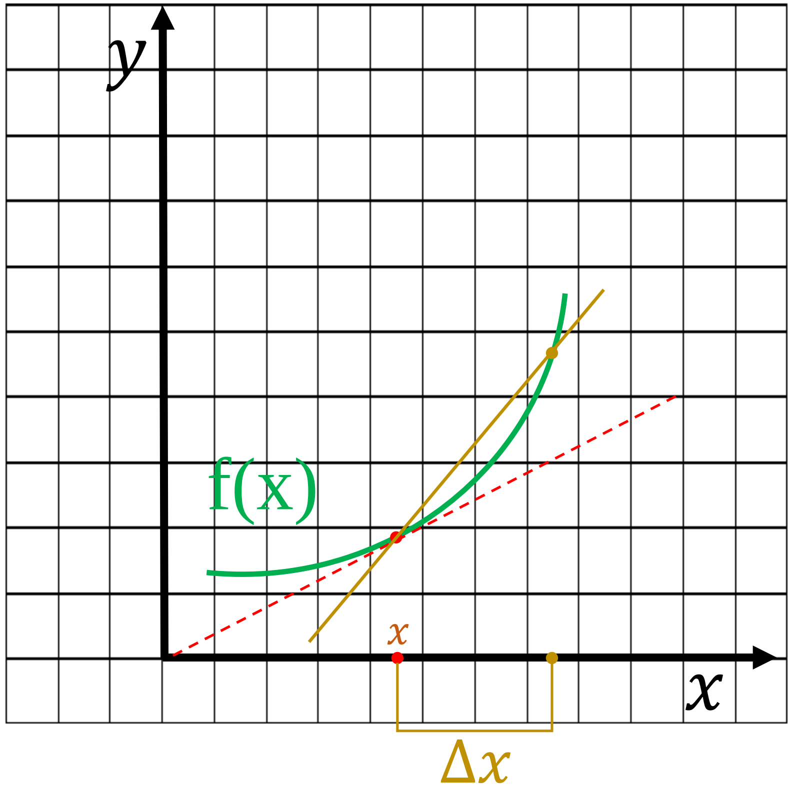


$$\text{Đạo hàm} = \frac{Thay\ đổi\ theo\ y}{Thay\ đổi\ theo\ x} = \frac{\Delta y}{\Delta x}$$

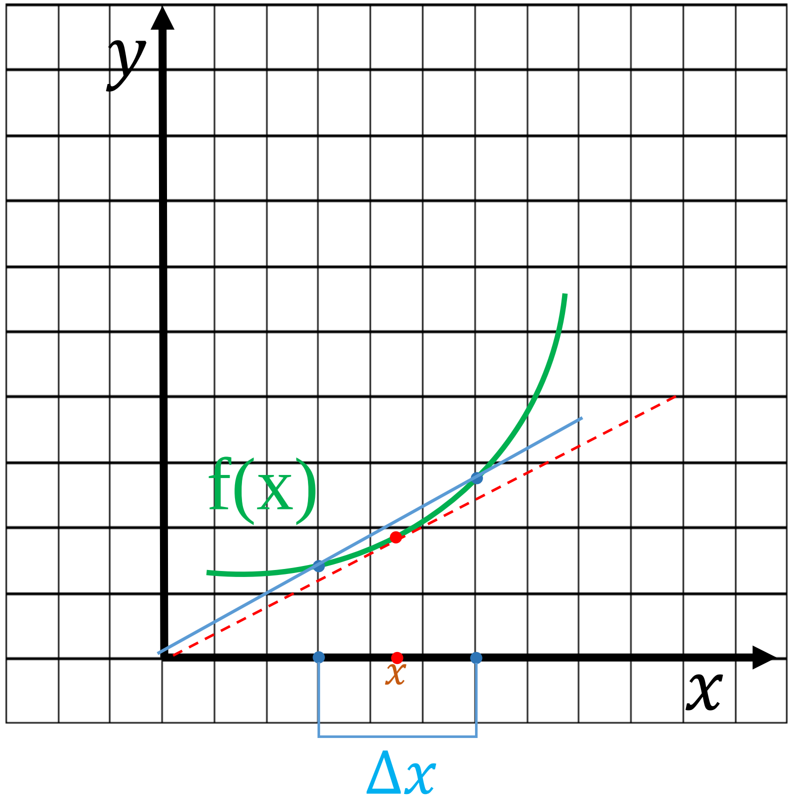
$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)}{\Delta x}$$

Derivative and Applications

Đạo hàm trung tâm cho hàm liên tục



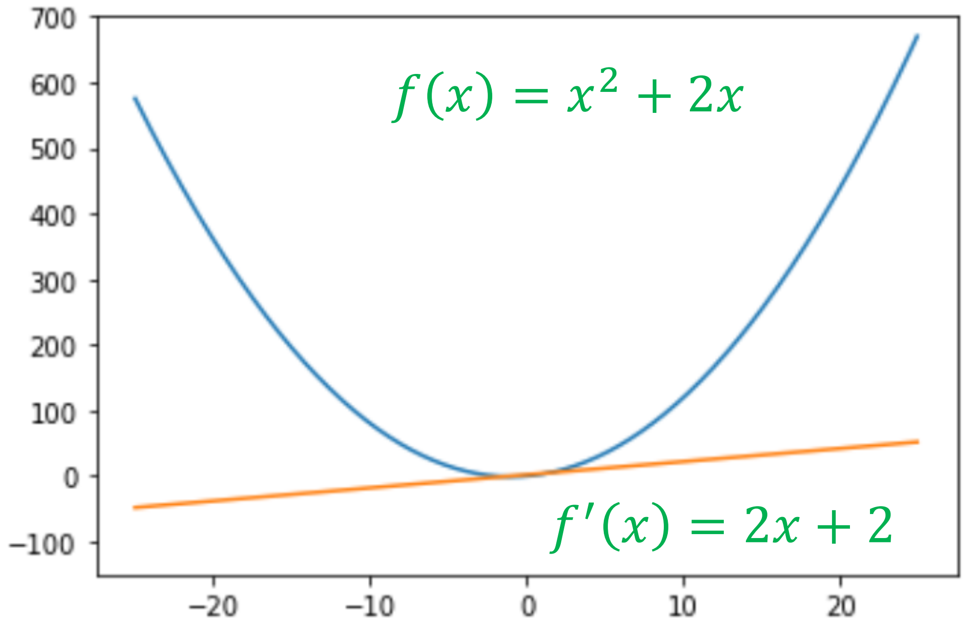
Đạo hàm một bên



Đạo hàm trung tâm

Derivative and Applications

❖ Implementation




```python
# python code

def func(x):
    return x**2 + 2*x

def func_derivative(x):
    return 2*x + 2
```

```python
d_value = func_derivative(2.0)
print('f\'(x=2) is', d_value)
```

```
f'(x=2) is 6.0
```

$$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x/2)^2 + 2(x + \Delta x/2) - ((x - \Delta x/2)^2 + 2(x - \Delta x/2))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + 2 = 2x + 2$$

Gradient in Python

Cho hàm số f(x)

$$f(x) = x^2 + 2x$$

Công thức đạo hàm

$$f'(x) = 2x + 2$$

Công thức đạo hàm một bên

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Công thức đạo hàm trung tâm

$$f'(x) = \lim_{\Delta x \to 0} \frac{f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)}{\Delta x}$$

```python
# python code:

def func(x):
    return x**2 + 2*x

def func_derivative(x):
    return 2*x + 2
```

```python
print(f'f\'(x=2) is {func_derivative(2.0)}')
```

```
f'(x=2) is 6.0
```

Theo lý thuyết đạo hàm, epsilon càng nhỏ thì giá trị đạo hàm tại một điểm càng chính xác!

Derivative

❖ Implementation

Cho hàm số f(x)

$$f(x) = x^2 + 2x$$

Công thức đạo hàm

$$f'(x) = 2x + 2$$

Công thức đạo hàm một bên

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Công thức đạo hàm trung tâm

$$f'(x) = \lim_{\Delta x \to 0} \frac{f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)}{\Delta x}$$

```
# đạo hàm một bên
def gradient(f, x, epsilon):
    return (f(x + epsilon) - f(x)) / epsilon

def func(x):
    return x**2 + 2*x

print(f'(e=1.0e2 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e2)}')
print(f'(e=1.0e1 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e1)}')
print(f'(e=1.0e0 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e0)}')
print(f'(e=1.0e-1 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e-1)}')
print(f'(e=1.0e-2 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e-2)}')
print(f'(e=1.0e-3 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e-3)}')
print(f'(e=1.0e-4 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e-4)}')
```

```
(e=1.0e2 and x=2)  : 106.0
(e=1.0e1 and x=2)  : 16.0
(e=1.0e0 and x=2)  : 7.0
(e=1.0e-1 and x=2)  : 6.099999999999994
(e=1.0e-2 and x=2)  : 6.009999999999849
(e=1.0e-3 and x=2)  : 6.000999999999479
(e=1.0e-4 and x=2)  : 6.000100000012054
```

Derivative

❖ Implementation

Cho hàm số f(x)

$$f(x) = x^2 + 2x$$

Công thức đạo hàm

$$f'(x) = 2x + 2$$

Công thức đạo hàm một bên

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Công thức đạo hàm trung tâm

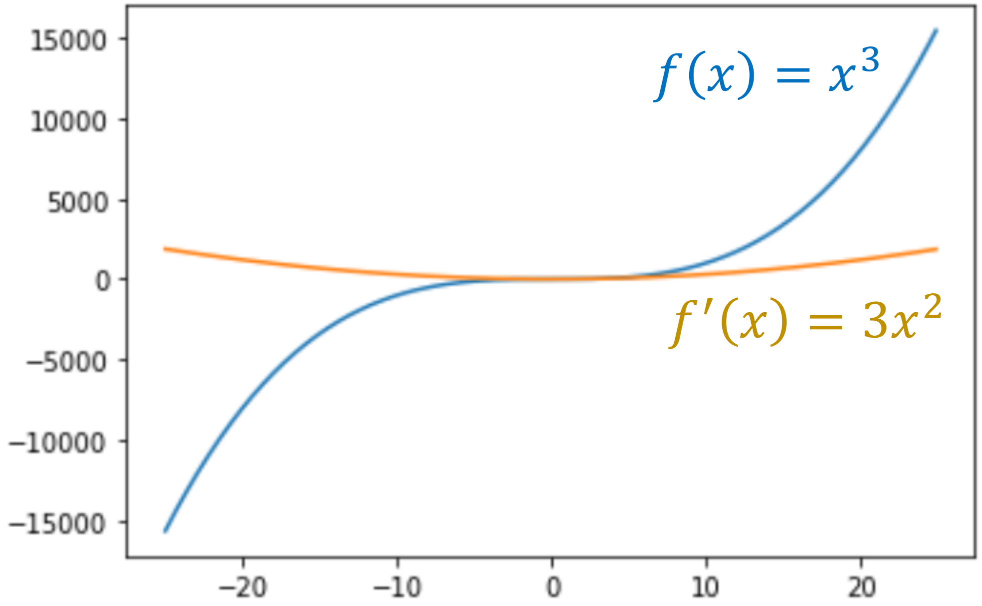
$$f'(x) = \lim_{\Delta x \to 0} \frac{f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)}{\Delta x}$$

```python
# đạo hàm trung tâm
def gradient(f, x, epsilon):
    return (f(x + epsilon/2) - f(x - epsilon/2)) / epsilon

def func(x):
    return x**2 + 2*x

print(f'(e=1.0e2 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e2)}')
print(f'(e=1.0e1 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e1)}')
print(f'(e=1.0e0 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e0)}')
print(f'(e=1.0e-1 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e-1)}')
print(f'(e=1.0e-2 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e-2)}')
print(f'(e=1.0e-3 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e-3)}')
print(f'(e=1.0e-4 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e-4)}')
```

```
(e=1.0e2 and x=2) : 6.0
(e=1.0e1 and x=2) : 6.0
(e=1.0e0 and x=2) : 6.0
(e=1.0e-1 and x=2) : 5.999999999999988
(e=1.0e-2 and x=2) : 5.999999999999783
(e=1.0e-3 and x=2) : 6.000000000011156
(e=1.0e-4 and x=2) : 6.00000000000378
```

Derivative

Implementation




```python
# python code

def func(x):
    return x**3

def func_derivative(x):
    return 3*x**2
```

```python
d_value = func_derivative(2.0)
print('f\'(x=2) is', d_value)
```

```
f'(x=2) is 12.0
```

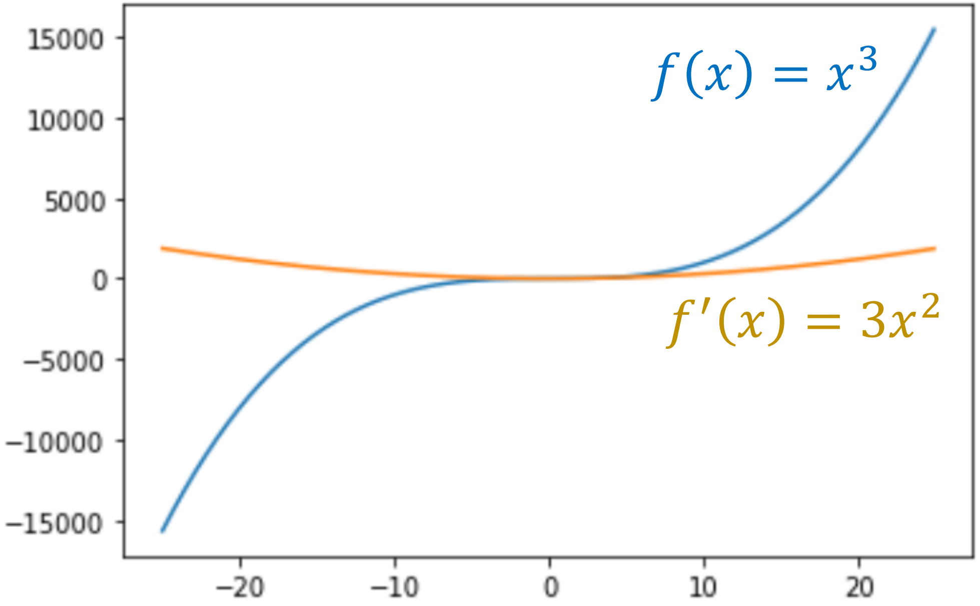
$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x((x + \Delta x)^2 + (x + \Delta x)x + x^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (x + \Delta x)^2 + (x + \Delta x)x + x^2 = x^2 + xx + x^2 = 3x^2$$

Derivative

Implementation




```python
# python code

def func(x):
    return x**3

def func_derivative(x):
    return 3*x**2
```

```python
d_value = func_derivative(2.0)
print('f\'(x=2) is', d_value)
```

```
f'(x=2) is 12.0
```

$$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x/2\,)^3 - (x - \Delta x/2\,)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x((x + \Delta x/2)^2 + (x + \Delta x/2)(x - \Delta x/2) + (x - \Delta x/2)^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (x + \Delta x/2)^2 + (x + \Delta x/2)(x - \Delta x/2) + (x - \Delta x/2)^2$$

$$= x^2 + xx + x^2 = 3x^2$$

Derivative

❖ Implementation

Cho hàm số f(x)

$$f(x) = x^3$$

Công thức đạo hàm

$$f'(x) = 3x^2$$

Công thức đạo hàm một bên

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Công thức đạo hàm trung tâm

$$f'(x) = \lim_{\Delta x \to 0} \frac{f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)}{\Delta x}$$

```python
# đạo hàm một bên
def gradient(f, x, epsilon):
    return (f(x + epsilon) - f(x)) / epsilon

def func(x):
    return x**3

print(f'(e=1.0e2 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e2)}')
print(f'(e=1.0e1 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e1)}')
print(f'(e=1.0e0 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e0)}')
print(f'(e=1.0e-1 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e-1)}')
print(f'(e=1.0e-2 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e-2)}')
print(f'(e=1.0e-3 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e-3)}')
print(f'(e=1.0e-4 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e-4)}')
```

```
(e=1.0e2 and x=2)  : 10612.0
(e=1.0e1 and x=2)  : 172.0
(e=1.0e0 and x=2)  : 19.0
(e=1.0e-1 and x=2)  : 12.610000000000001
(e=1.0e-2 and x=2)  : 12.060099999999707
(e=1.0e-3 and x=2)  : 12.006000999997823
(e=1.0e-4 and x=2)  : 12.000600010022566
```

Derivative

❖ Implementation

Cho hàm số f(x)

$$f(x) = x^3$$

Công thức đạo hàm

$$f'(x) = 3x^2$$

Công thức đạo hàm một bên

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Công thức đạo hàm trung tâm

$$f'(x) = \lim_{\Delta x \to 0} \frac{f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)}{\Delta x}$$

```python
# đạo hàm trung tâm
def gradient(f, x, epsilon):
    return (f(x + epsilon/2) - f(x - epsilon/2)) / epsilon

def func(x):
    return x**3

print(f'(e=1.0e2 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e2)}')
print(f'(e=1.0e1 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e1)}')
print(f'(e=1.0e0 and x=2)  : {gradient(f=func, x=2.0, epsilon=1.0e0)}')
print(f'(e=1.0e-1 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e-1)}')
print(f'(e=1.0e-2 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e-2)}')
print(f'(e=1.0e-3 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e-3)}')
print(f'(e=1.0e-4 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e-4)}')
print(f'(e=1.0e-5 and x=2) : {gradient(f=func, x=2.0, epsilon=1.0e-5)}')
```

```
(e=1.0e2 and x=2)  : 2512.0
(e=1.0e1 and x=2)  : 37.0
(e=1.0e0 and x=2)  : 12.25
(e=1.0e-1 and x=2) : 12.002499999999978
(e=1.0e-2 and x=2) : 12.000024999999681
(e=1.0e-3 and x=2) : 12.000000250001364
(e=1.0e-4 and x=2) : 12.000000002503342
(e=1.0e-5 and x=2) : 11.999999999900977
```

Outline

- Limit
- Area Computation Using Limit
- Derivative
- Newton's Method

Exercise

❖ Implement the following two formulas

Tìm xấp xỉ giá trị $\sqrt{N}$

- Gọi x là giá trị cần tìm $\sqrt{N} \approx x$
- Gọi $f(x) = x^2 - N$
- Khởi tạo $x_0 = N$
- $f'(x) = 2x$

1

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

2

$$x_{i+1} = \frac{x_i + \frac{N}{x_i}}{2}$$

Review

❖ Point-slope equation

from the reference book

Definition

Consider a line passing through the point (x_1, y_1) with slope m. The equation

$$y - y_1 = m(x - x_1)$$

is the **point-slope equation** for that line.

Consider a line with slope m and y-intercept $(0, b)$. The equation

$$y = mx + b$$

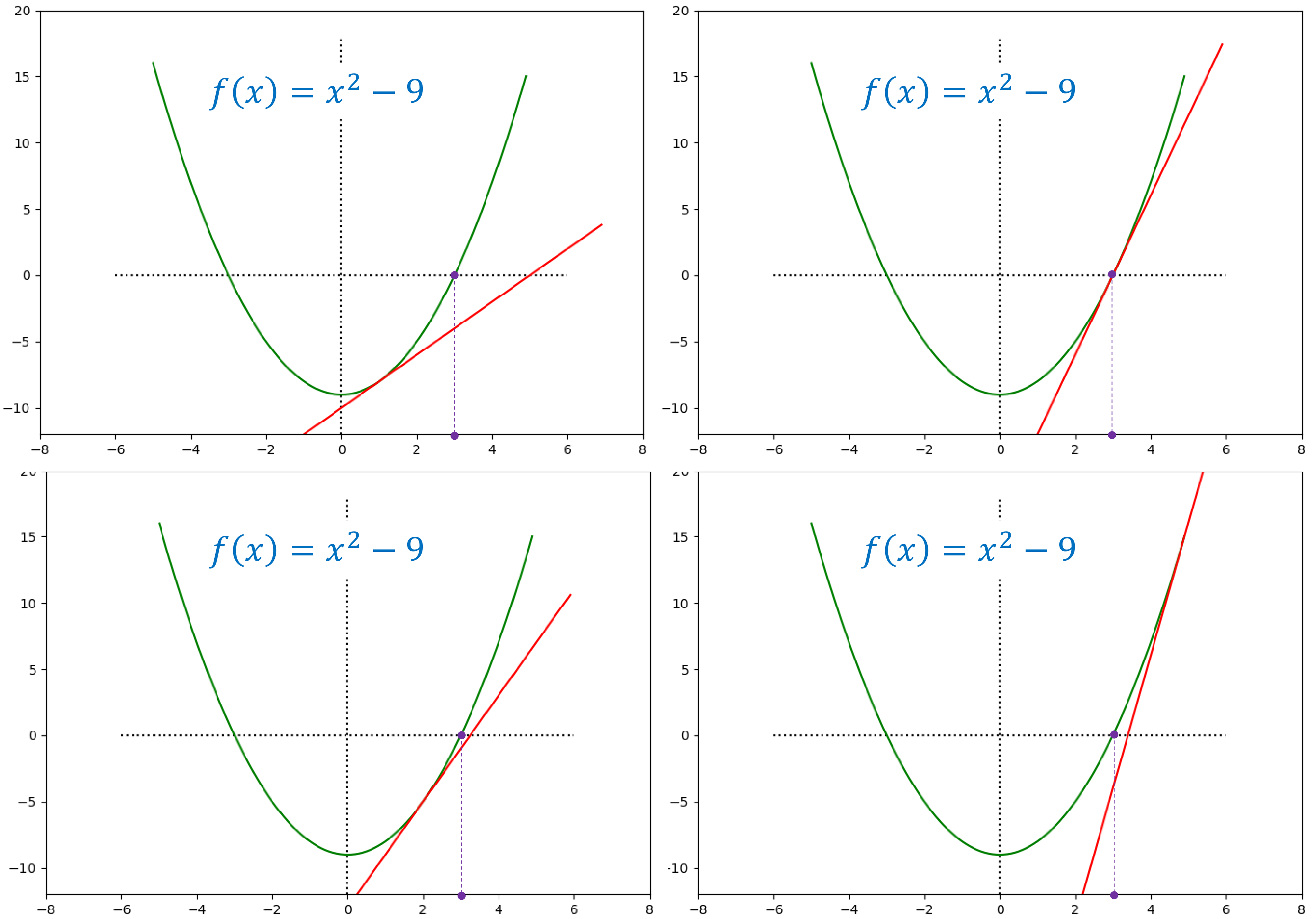
is an equation for that line in **slope-intercept form**.

The **standard form of a line** is given by the equation

$$ax + by = c,$$

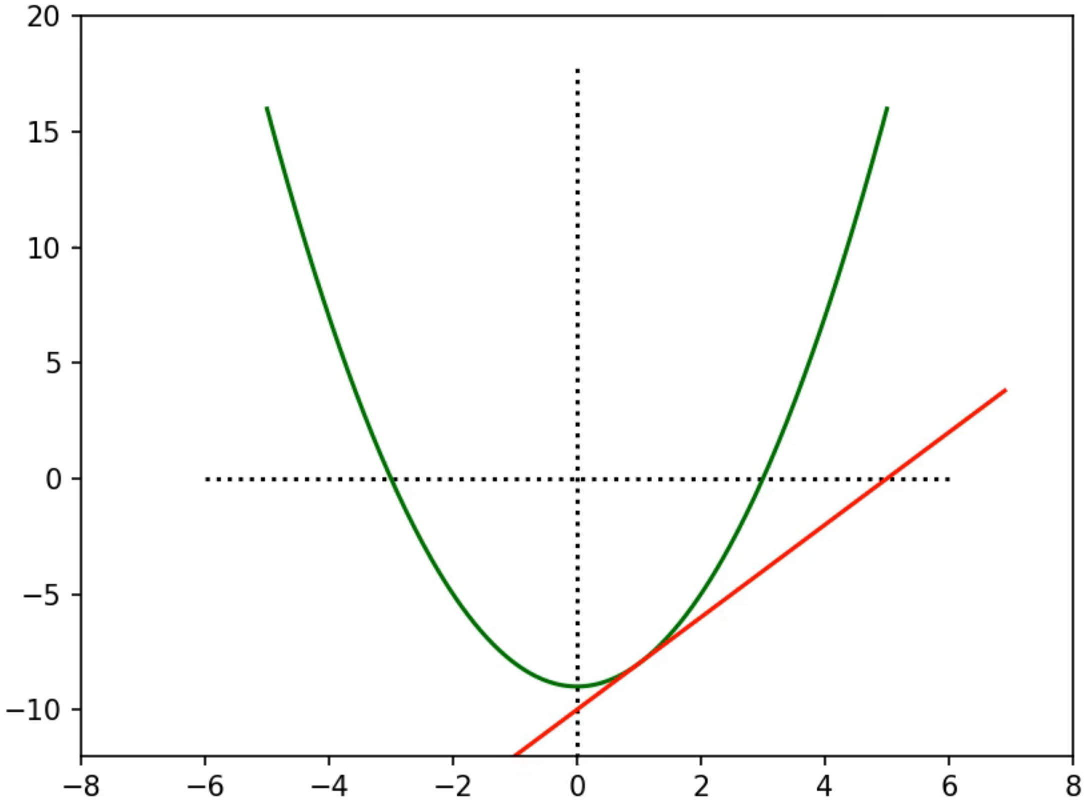
where a and b are both not zero. This form is more general because it allows for a vertical line, $x = k$.

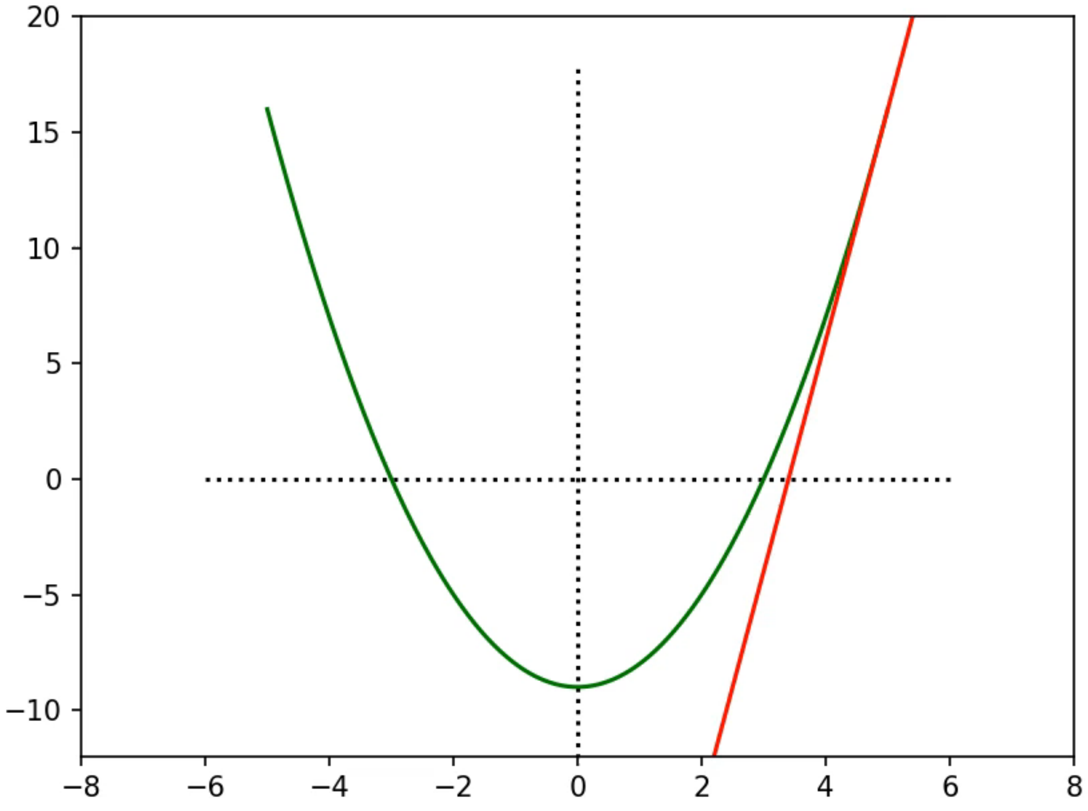
Let’s observe from these figures



Exercise

❖ Animation





Exercise

❖ Implement the following two formulas

Tìm xấp xỉ giá trị $\sqrt{N}$

- Gọi x là giá trị cần tìm $\sqrt{N} \approx x$
- Gọi $f(x) = x^2 - N$
- Khởi tạo $x_0 = N$
- $f'(x) = 2x$

1

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

2

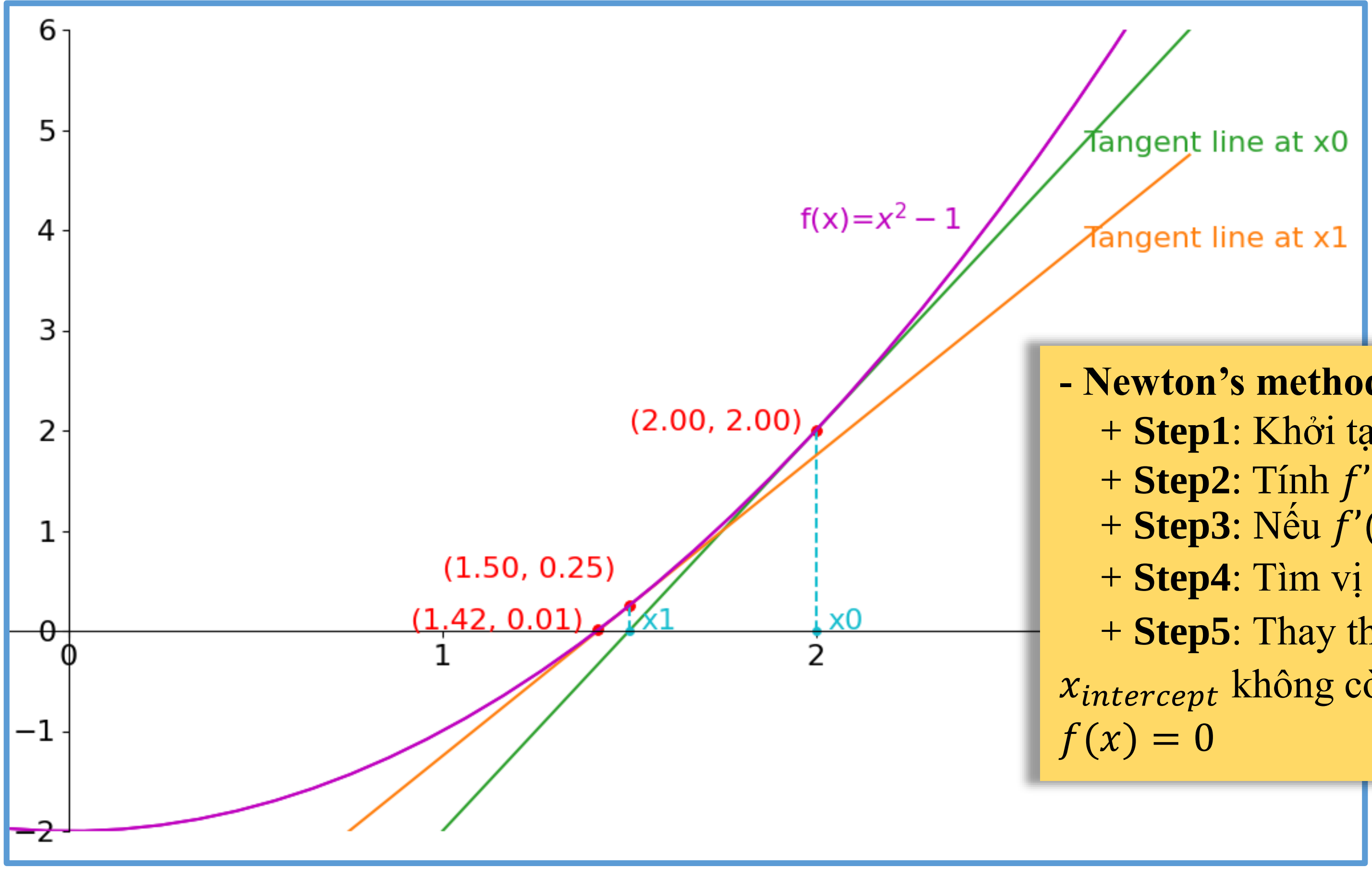
$$x_{i+1} = \frac{x_i + \frac{N}{x_i}}{2}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^2 - N}{2x_i} = \frac{2x_i^2 - x_i^2 + N}{2x_i}$$

$$= \frac{x_i^2 + N}{2x_i} = \frac{\frac{x_i^2 + N}{2x_i}}{\frac{2x_i}{x_i}}$$

$$= \frac{x_i + \frac{N}{x_i}}{2}$$



tangent line = $f(a) + f'(a)(x - a)$

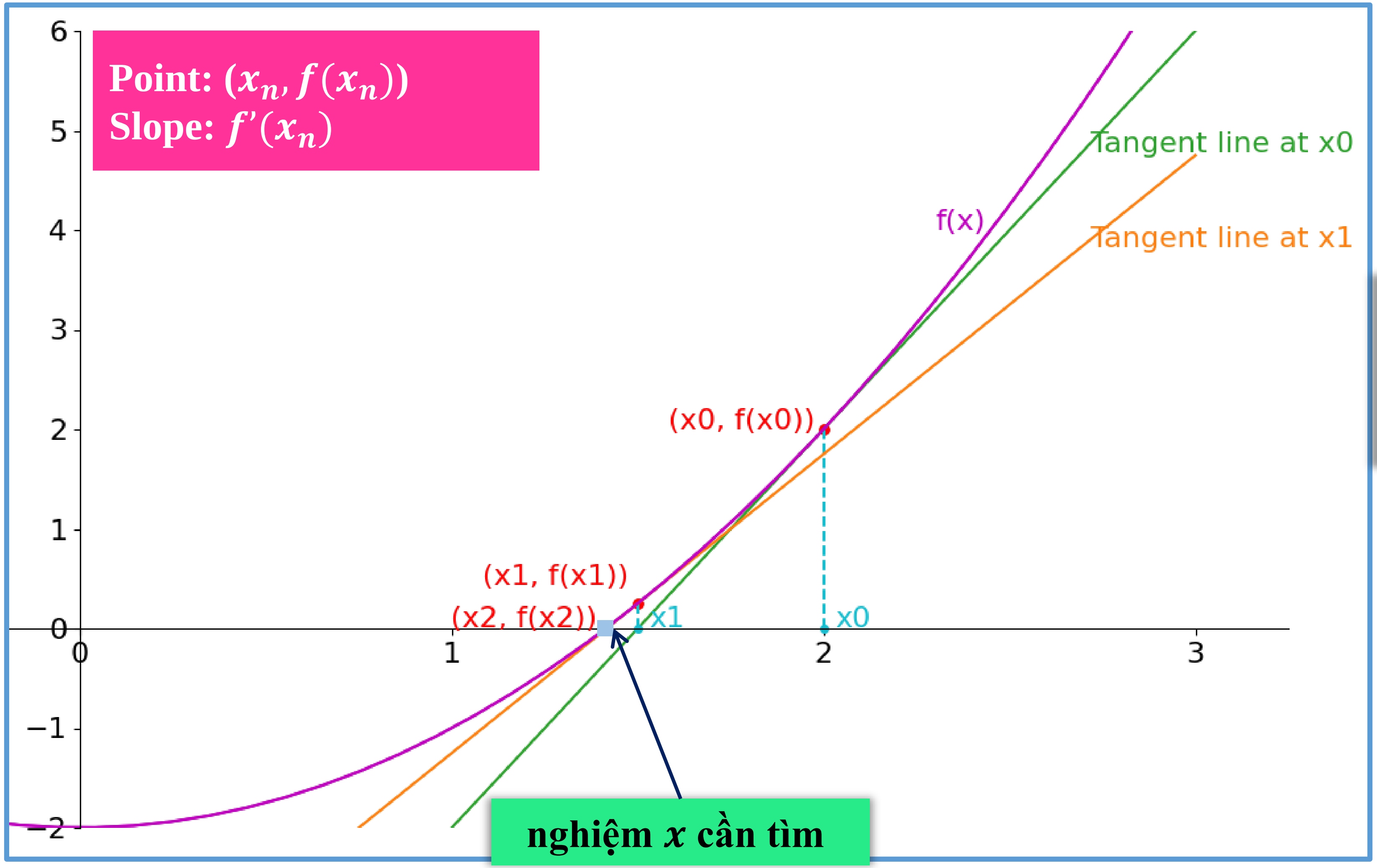
- **Newton's method**: gồm các bước sau
  + **Step1**: Khởi tạo giá trị x_0 bất kỳ theo $f(x)$ thu được một điểm $(x_0, f(x_0))$
  + **Step2**: Tính $f'(x_0)$
  + **Step3**: Nếu $f'(x_0) \neq 0$. Tìm tangent line.
  + **Step4**: Tìm vị trí tangent line cắt trục tọa độ x. $(x_{intercept}, 0)$
  + **Step5**: Thay thế $x_{intercept}$ vào x_0 ở **Step1**. Việc này được lặp cho đến khi $x_{intercept}$ không còn thay đổi nhiều giữa các lần lặp thì đó chính là solution để $f(x) = 0$

Example:

$f(x) = x^2 - 2$

- **Step1**: Khởi tạo $x_0 = 2$ thu được một điểm $(2, 2)$
- **Step2**: $f'(x) = 2x$ => $f'(x_0)$=4.
- **Step3**: tangent line = 2+4(x-2).
- **Step4**: Tìm vị trí tangent line cắt trục x (tangent line = $0 = 2 + 4(x - 2)$)=> $x_{intercept} = 1.5$.
- Gọi $x_{intercept}$ là $x_1$và thay vào x_0. Để thực hiện vòng lặp mới. thu được $x_2 \approx 1.42$. Lúc này $f(x_2) \approx 0.01 \approx 0$. Có thể tạm gọi $x = 1.42$ là nghiệm xấp xỉ để $f(x) = 0$.

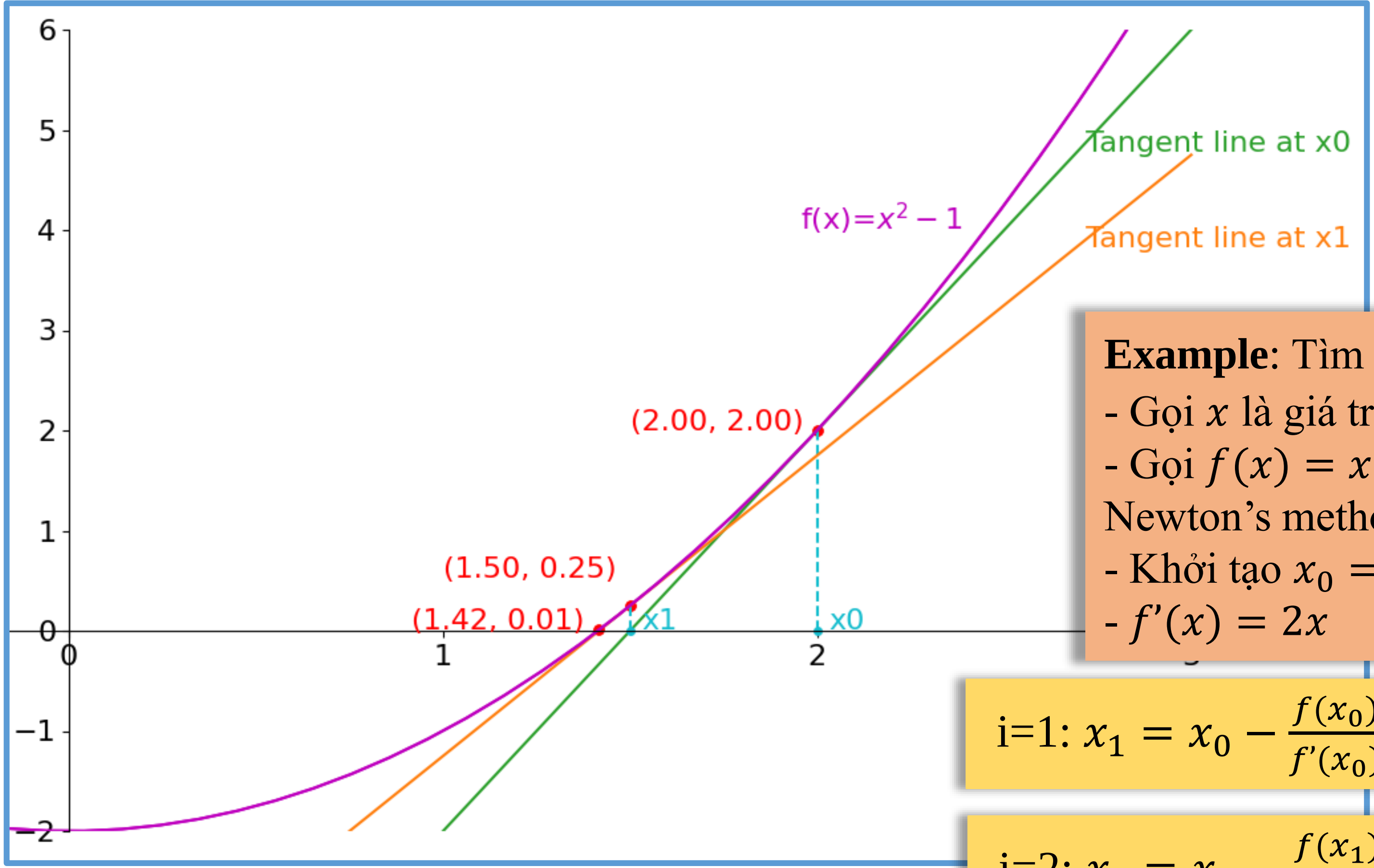
=> Có thể thực hiện thêm nhiều vòng lặp để tìm được kết qủa chính xác hơn



Newton's method: ước lượng solution của một equation $f(x) = 0$, bằng cách sử dụng các tangent line của graph $y = f(x)$ gần những điểm mà làm cho f bằng 0.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

tangent line $= f(a) + f'(a)(x - a)$ → $0 = f(a) + f'(a)(x - a)$ → $\frac{-f(a)}{f'(a)} + a = x$



Example: Tìm xấp xỉ giá trị $\sqrt{2}$ Newton's method với 5 lần lặp (N=5).
- Gọi x là giá trị cần tìm $\sqrt{2} \approx x$. Biết được $x^2 \approx 2$ => $x^2 - 2 \approx 0$.
- Gọi $f(x) = x^2 - 2$. Vậy nhiệm đi tìm nghiệm x để $f(x) = 0$. Dùng Newton's method với 5 lần lặp
- Khởi tạo $x_0 = 2$. (khởi tạo giá trị bất kỳ ở đây chọn 2)
- $f'(x) = 2x$

i=1: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ =1.5000000000143778.

i=2: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ =1.4166666666598111.

i=3: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ =1.4142156862742763.

i=4: $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$ =1.4142135623746896.

i=5: $x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$ =1.4142135623730951.

$\sqrt{2}$ ≈1.4142135623730951

$$x_{i+1} = x_i - \frac{{x_i}^2 - 2}{2x_i}$$

$$x_{i+1} = \frac{x_i + \frac{2}{x_i}}{2}$$

Tổng quát $\sqrt{N}$

$$x_{i+1} = \frac{x_i + \frac{N}{x_i}}{2}$$

Newton's method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$