AI VIETNAM
All-in-One Course

Linear Regression

A Simple Approach

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PhD in Computer Science

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Outline

- Simple version of Linear Regression
- Computational Graph
- Mini-batch Training
- Batch Training
- Generalization of Linear Regression
- Loss Functions (extension)

Linear Regression

Introduction

	Feature	Label
	area	price
	6.7	9.1
	4.6	5.9
	3.5	4.6
	5.5	6.7

House price data

Features			Label
TV	Radio	Newspaper	Sales
230.1	37.8	69.2	22.1
44.5	39.3	45.1	10.4
17.2	45.9	69.3	12
151.5	41.3	58.5	16.5
180.8	10.8	58.4	17.9

Advertising data

if area=6.0, price=?

if TV=55.0, Radio=34.0, and Newspaper=62.0, price=?

Features													Label
crim	zn	indus	chas	nox	rm	age	dis	rad	tax	ptratio	black	lstat	medv
0.00632	18	2.31	0	0.538	6.575	65.2	4.09	1	296	15.3	396.9	4.98	24
0.02731	0	7.07	0	0.469	6.421	78.9	4.9671	2	242	17.8	396.9	9.14	21.6
0.03237	0	2.18	0	0.458	6.998	45.8	6.0622	3	222	18.7	394.63	2.94	33.4
0.06905	0	2.18	0	0.458	7.147	54.2	6.0622	3	222	18.7	396.9	5.33	36.2
0.08829	12.5	7.87	0	0.524	6.012	66.6	5.5605	5	311	15.2	395.6	12.43	22.9

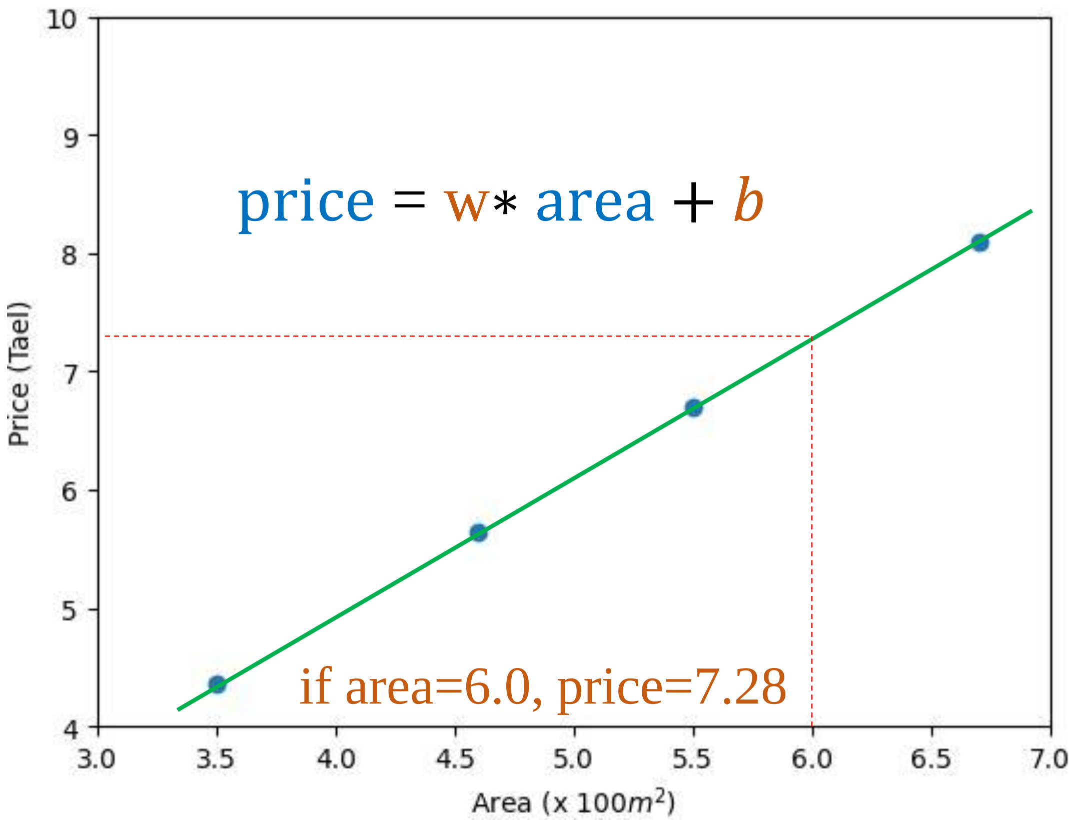
Boston House Price Data

House Price Prediction

House price data

Feature	Label
area	price
6.7	8.1
4.6	5.6
3.5	4.3
5.5	6.7

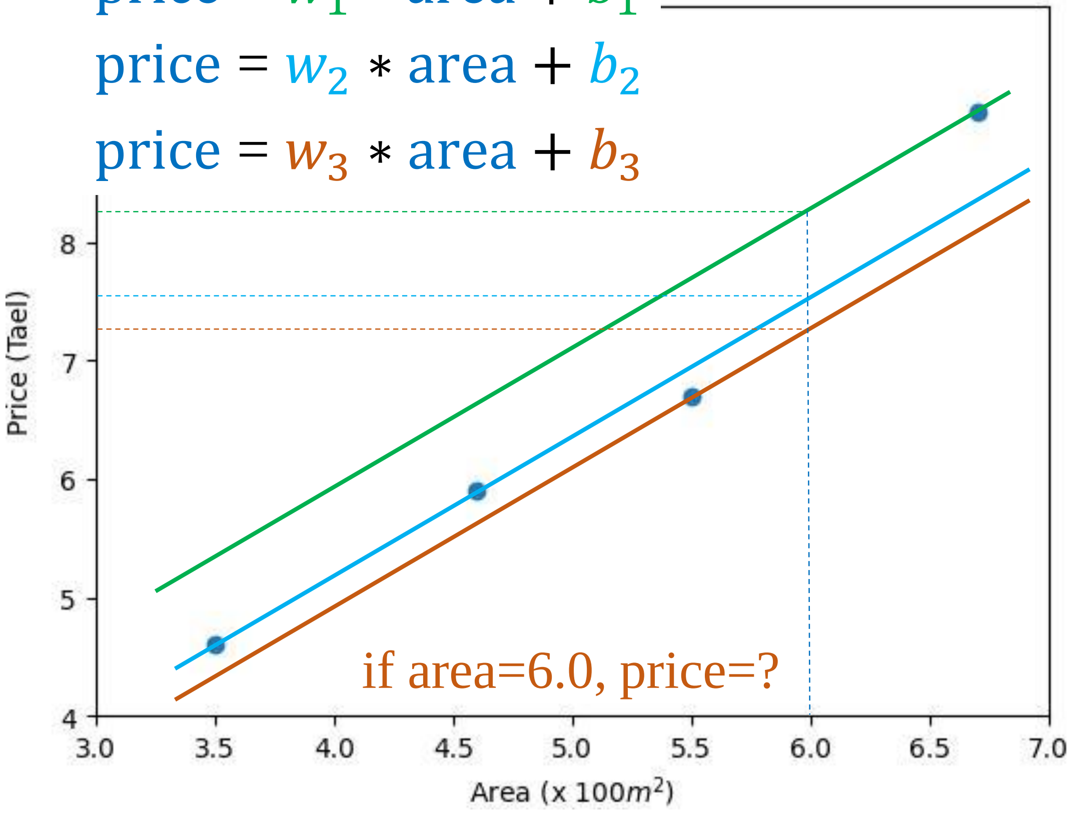
$$\text{price} = w * \text{area} + b$$



Feature	Label
area	price
6.7	9.1
4.6	5.9
3.5	4.6
5.5	6.7

House price data

$$\text{price} = w_1 * \text{area} + b_1$$
$$\text{price} = w_2 * \text{area} + b_2$$
$$\text{price} = w_3 * \text{area} + b_3$$

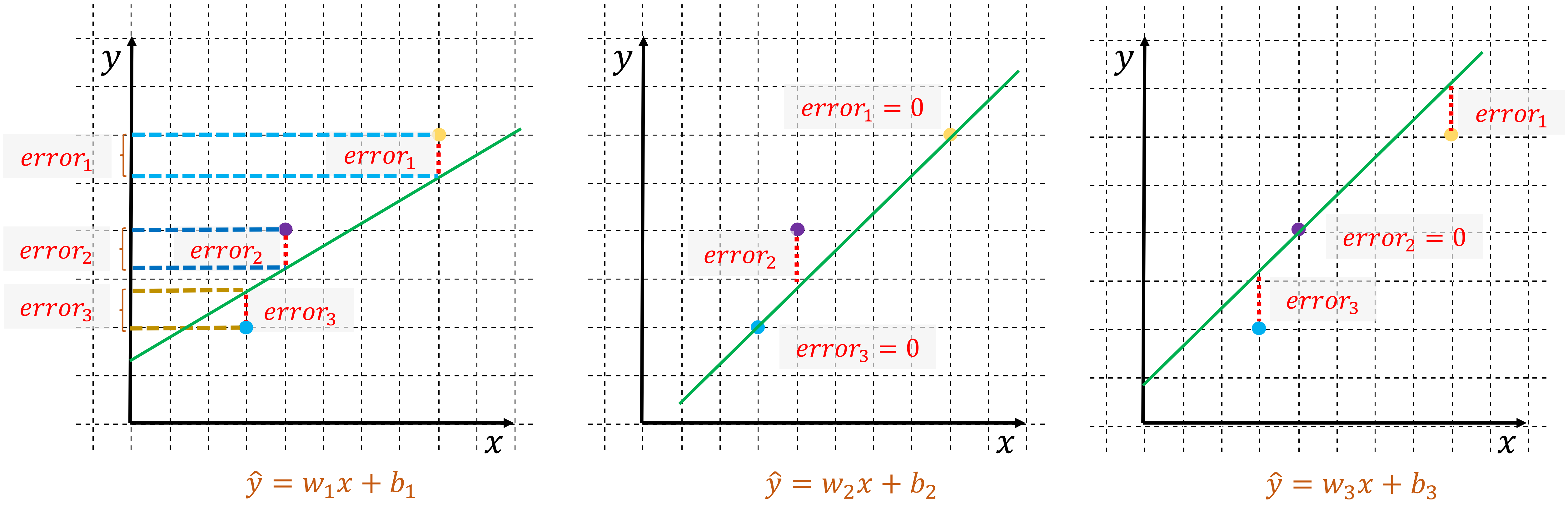


Linear Regression

❖ Area-based house price prediction

Training data

$$error_i = distance(\hat{y}_i, y_i)$$



Find a and b whose model has the smallest error, where $error = \sum_i error_i$ How?

Linear Regression

❖ Area-based house price prediction

$$\text{predicted\_price} = w * \text{area} + b$$

$$\text{error} = (\text{predicted\_price} - \text{real\_price})^2$$

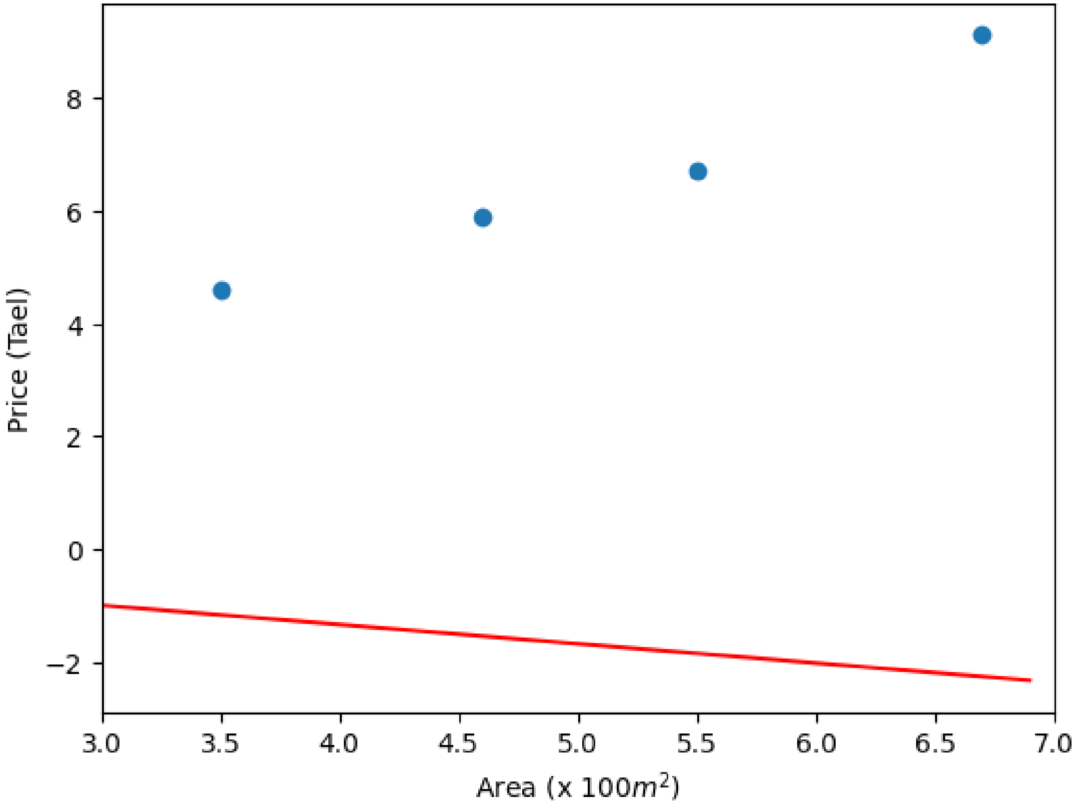
$$\hat{y}_i = wx_i + b$$

$$L(\hat{y}_i, y_i) = (\hat{y}_i - y_i)^2$$

area	price	predicted	error
6.7	9.1	-2.238	128.55
4.6	5.9	-1.524	55.11
3.5	4.6	-1.15	33.06
5.5	6.7	-1.83	72.76

w = -0.34

b = 0.04



Linear Regression

❖ Area-based house price prediction

$$\text{predicted\_price} = w * \text{area} + b$$

$$\text{error} = (\text{predicted\_price} - \text{real\_price})^2$$

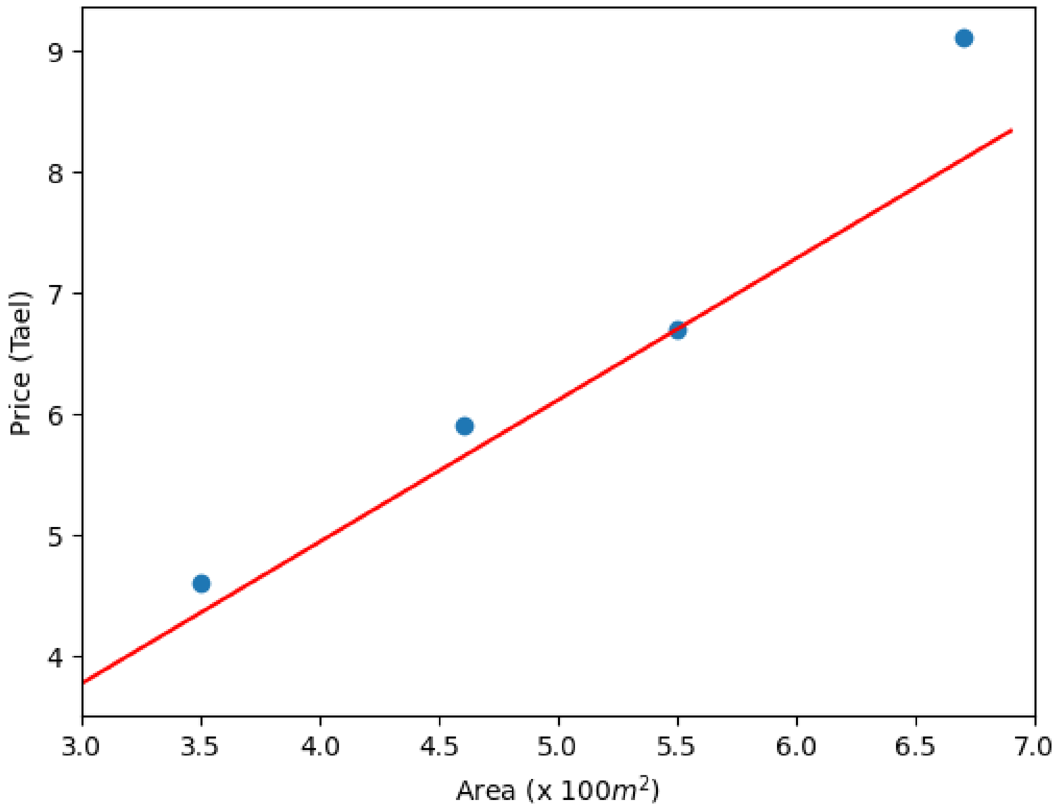
$$\hat{y}_i = wx_i + b$$

$$L(\hat{y}_i, y_i) = (\hat{y}_i - y_i)^2$$

area	price	predicted	error
6.7	9.1	8.099	1.002
4.6	5.9	5.642	0.066
3.5	4.6	4.355	0.06
5.5	6.7	6.695	0.00002

$w = 1.17$

$b = 0.26$



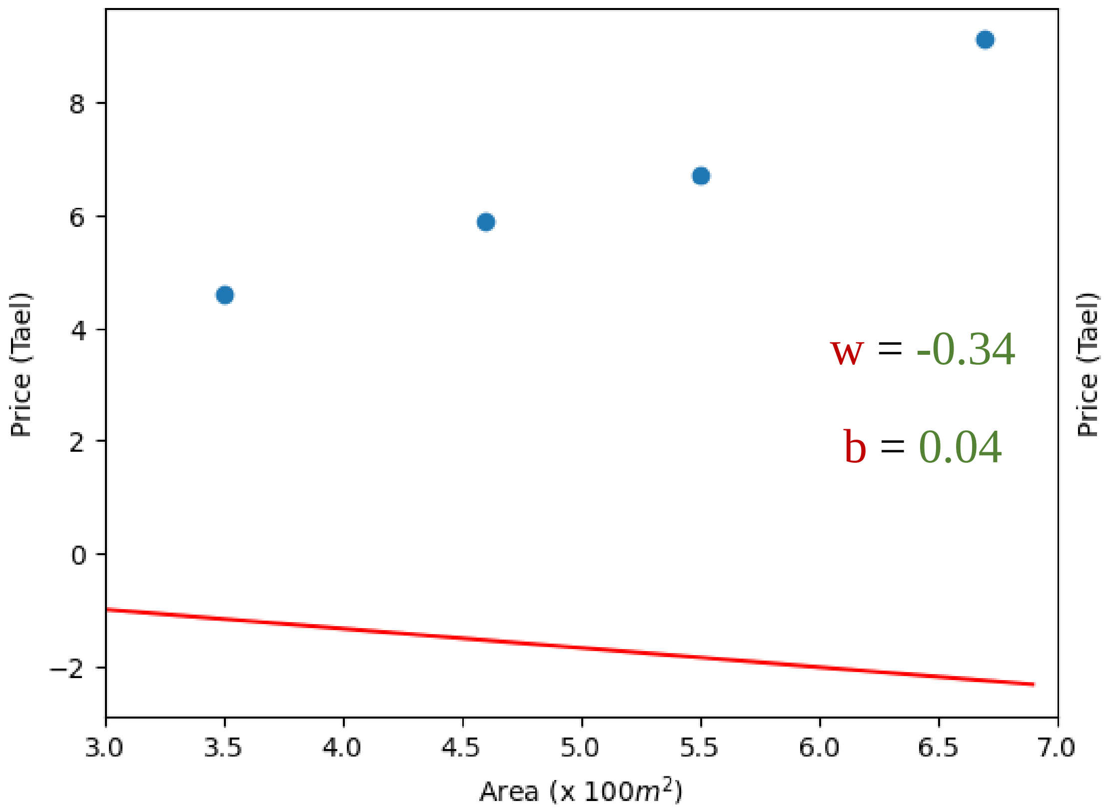
Linear Regression

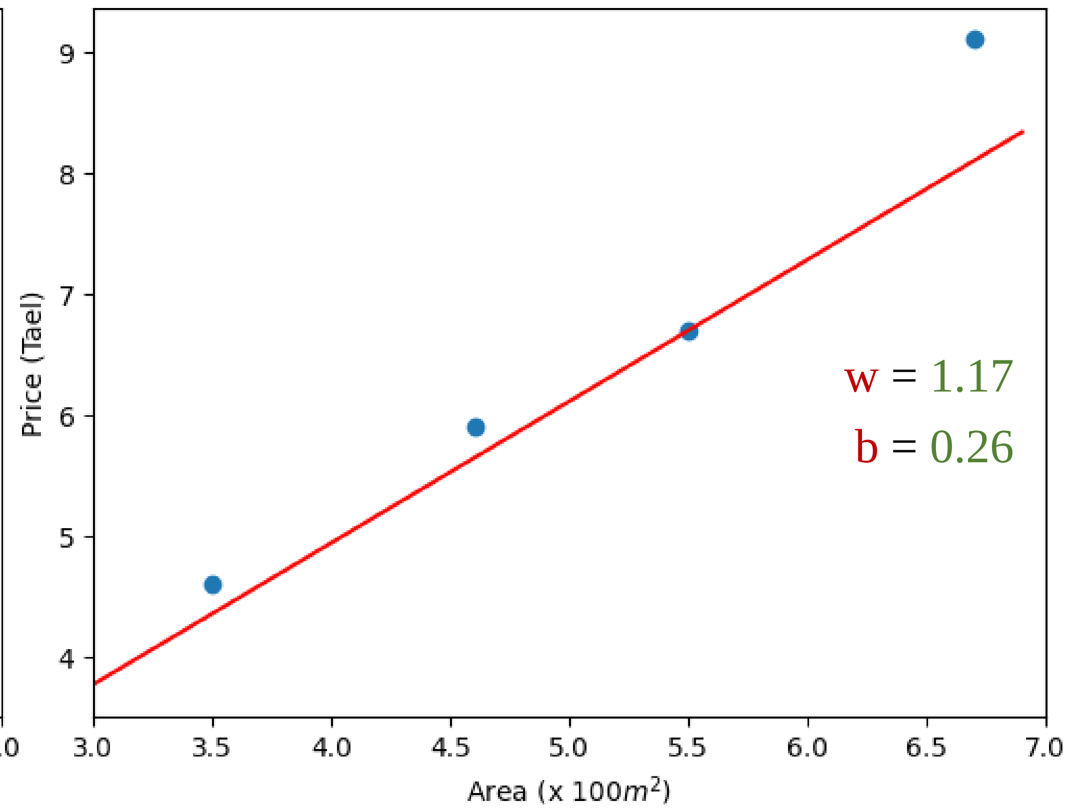
❖ **Area-based house price prediction**

$$\hat{y}_i = wx_i + b$$

$$L(\hat{y}_i, y_i) = (\hat{y}_i - y_i)^2$$

How to change w and b so that $L(\hat{y}_i, y_i)$ reduces





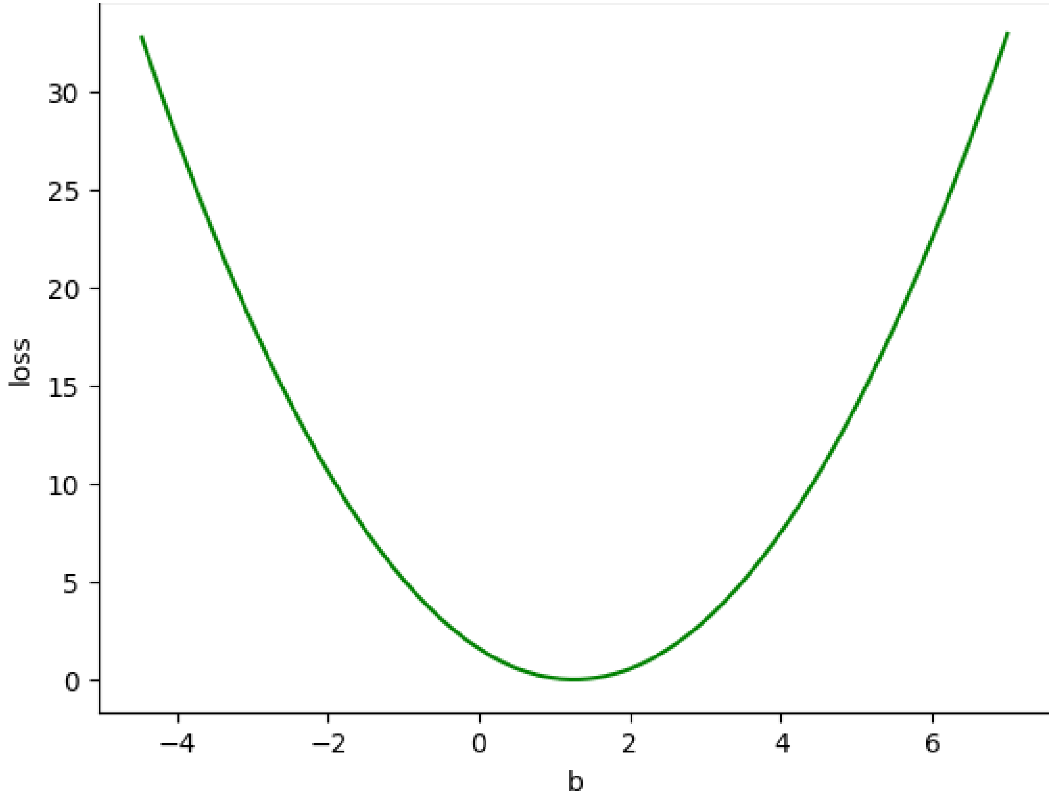
Linear Regression

$$\hat{y}_i = wx_i + b$$

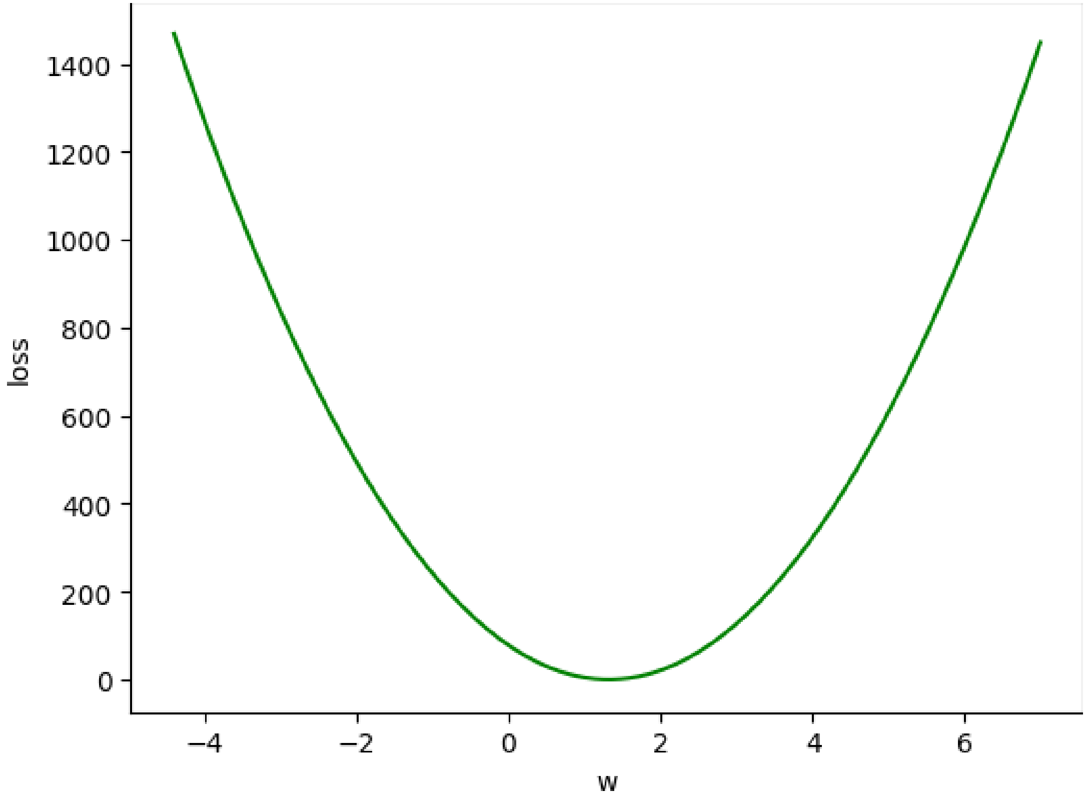
$$L(\hat{y}_i, y_i) = (\hat{y}_i - y_i)^2$$

❖ Understanding the loss function

How to change w and b so that $L(\hat{y}_i, y_i)$ reduces



Different b values with a fixed w value



Different w values with a fixed b value

Linear Regression

Linear equation

$$\hat{y} = wx + b$$

where $\hat{y}$ is a predicted value,

w and b are parameters

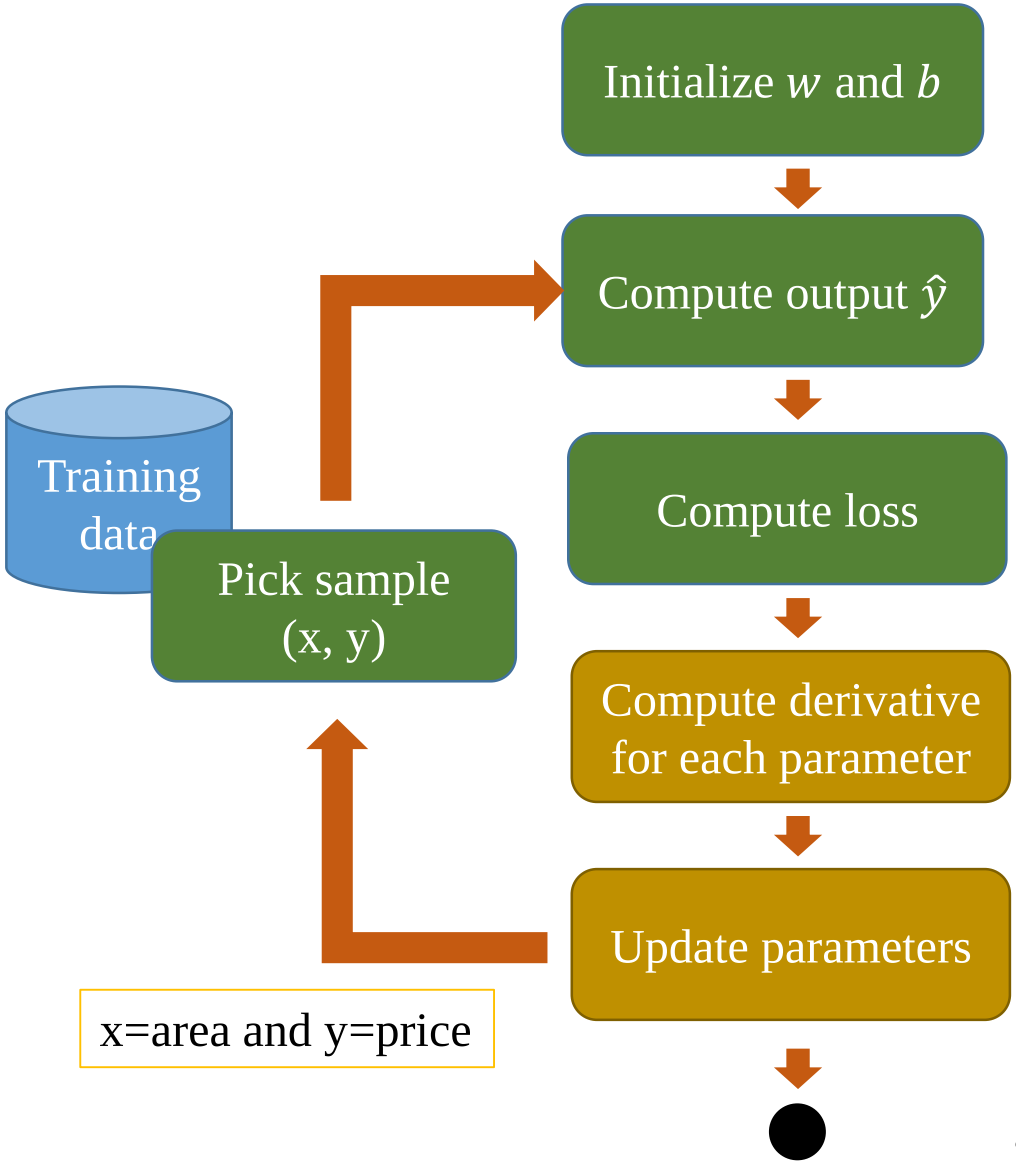
and x is input feature

Error (loss) computation

Idea: compare predicted values $\hat{y}$ and label values y

Squared loss

$$L(\hat{y}, y) = (\hat{y} - y)^2$$



Linear Regression

Linear equation

$$\hat{y} = wx + b$$

where $\hat{y}$ is a predicted value,

w and b are parameters

and x is input feature

Error (loss) computation

Idea: compare predicted values $\hat{y}$ and label values y

Squared loss

$$L(\hat{y}, y) = (\hat{y} - y)^2$$

Find better w and b

Use gradient descent to minimize the loss function

Compute derivate for each parameter

$$\frac{\partial L}{\partial w} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial w} = 2x(\hat{y} - y)$$

$$\frac{\partial L}{\partial b} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial b} = 2(\hat{y} - y)$$

Update parameters

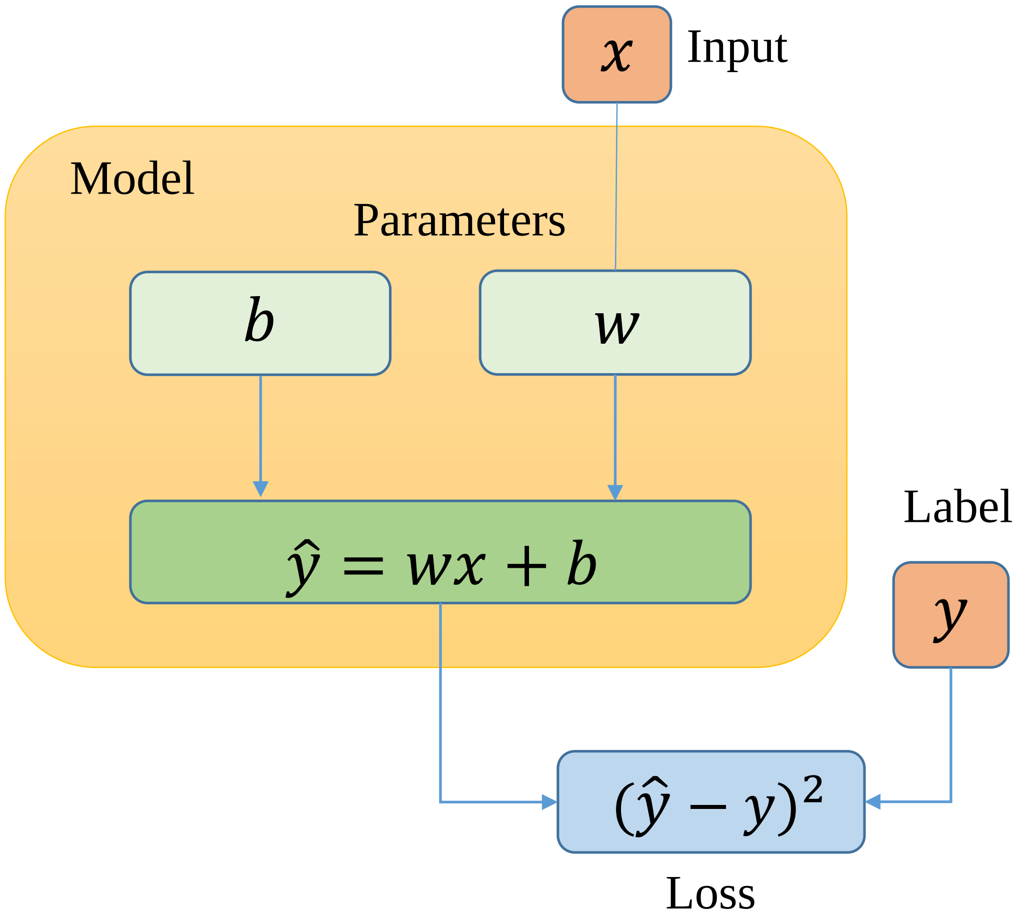
$$w = w - \eta \frac{\partial L}{\partial w} \qquad b = b - \eta \frac{\partial L}{\partial b}$$

η is learning rate

Linear Regression

❖ Toy example

Diagram



Cheat sheet

Compute the output $\hat{y}$

$$\hat{y} = wx + b$$

Compute the loss

$$L = (\hat{y} - y)^2$$

Compute derivative

$$\frac{\partial L}{\partial w} = 2x(\hat{y} - y)$$

$$\frac{\partial L}{\partial b} = 2(\hat{y} - y)$$

Update parameters

$$w = w - \eta \frac{\partial L}{\partial w}$$

$$b = b - \eta \frac{\partial L}{\partial b}$$

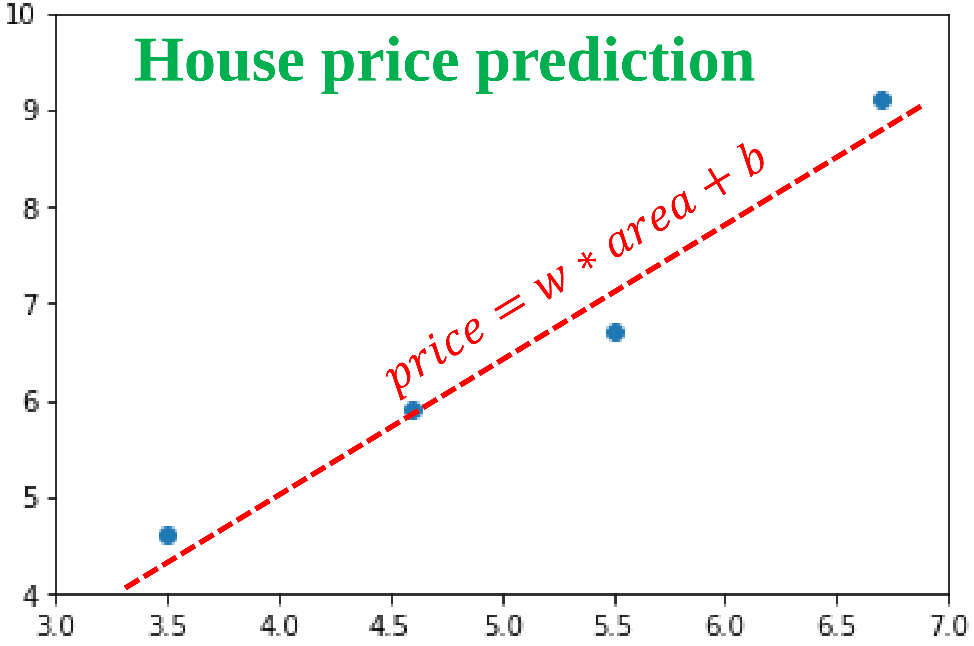
Linear Regression

1

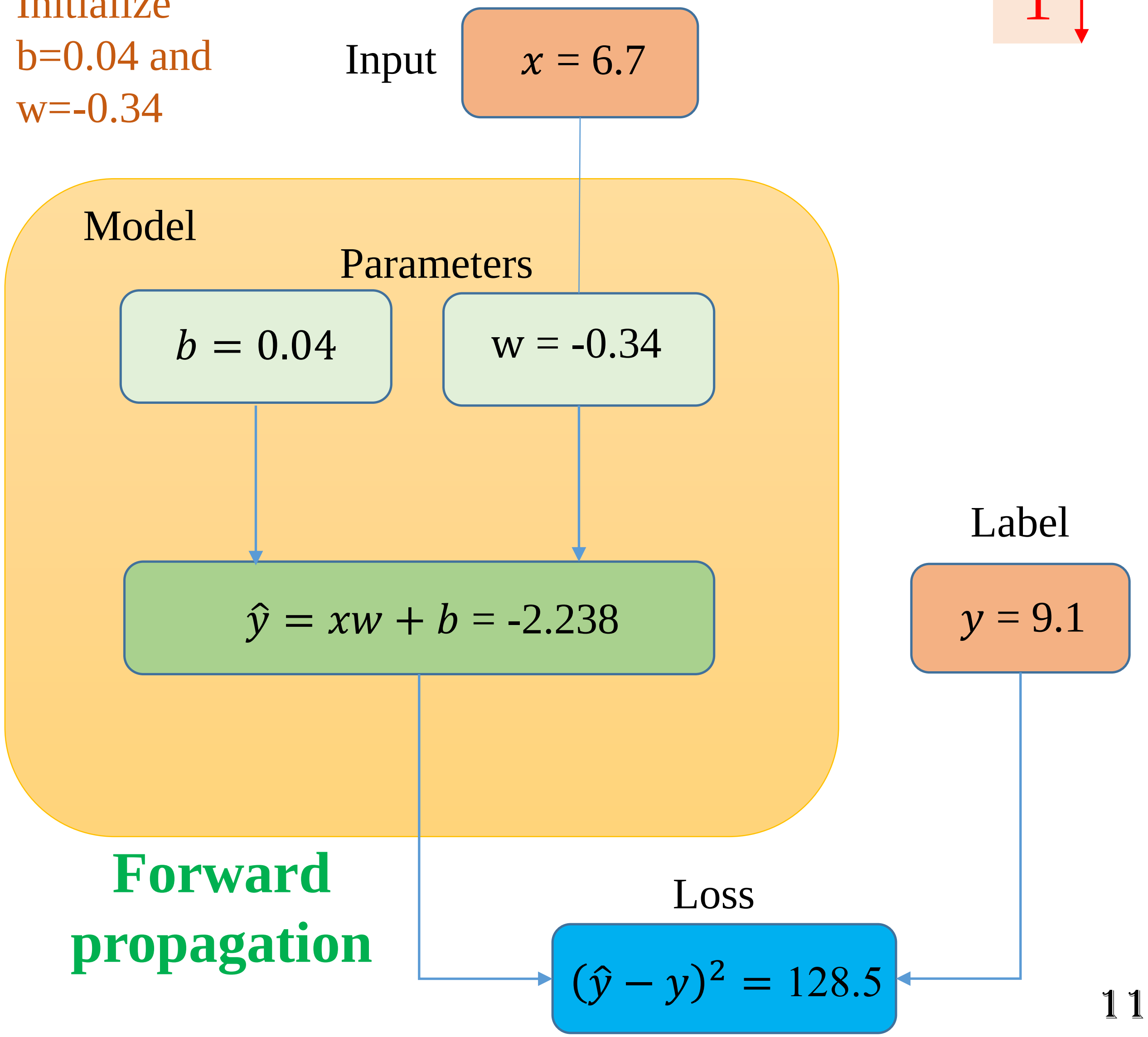
Feature | Label

Given sample data

	area	price
	6.7	9.1
	4.6	5.9
	3.5	4.6
	5.5	6.7

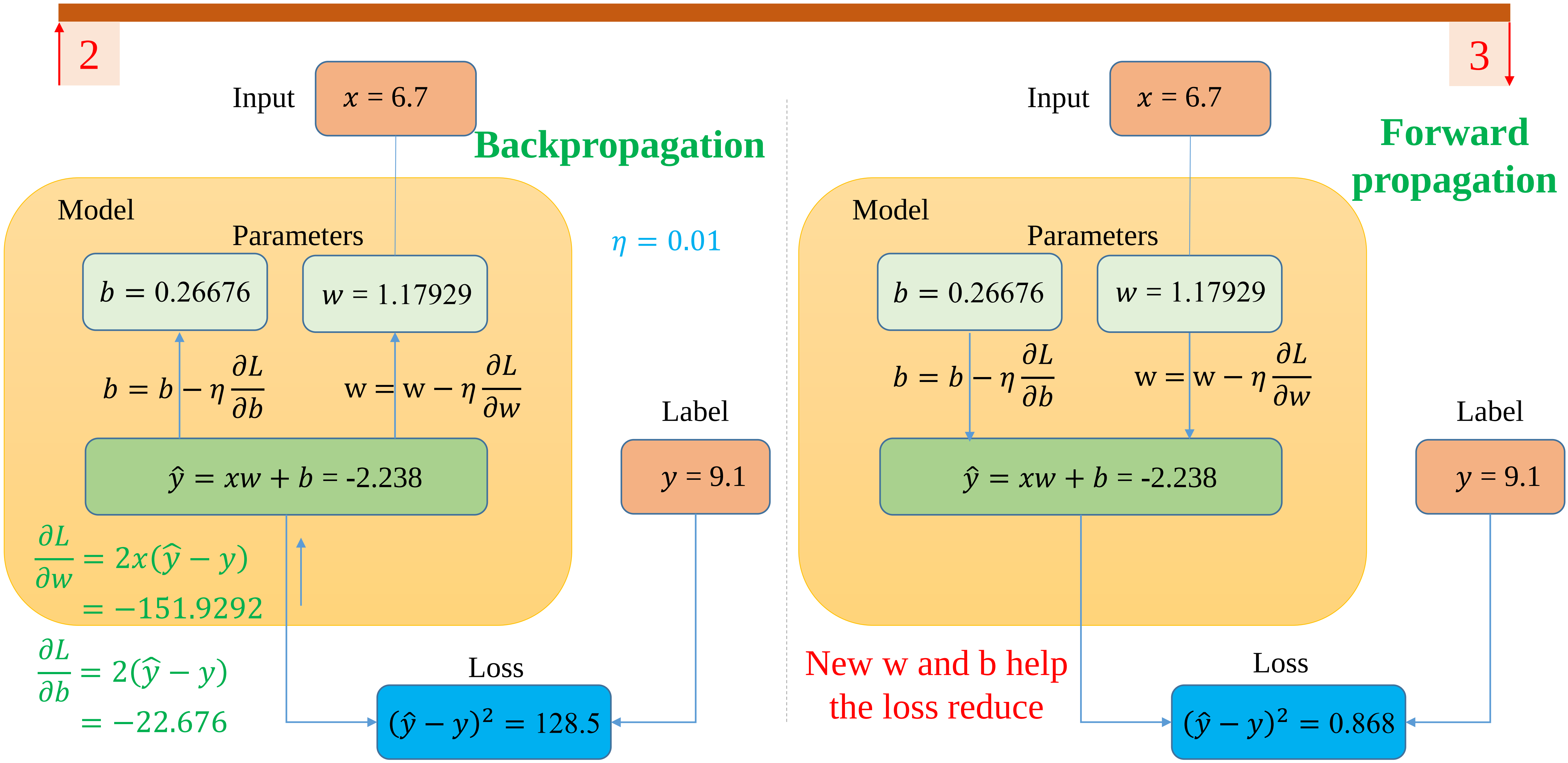


Initialize
b=0.04 and
w=-0.34



Forward propagation

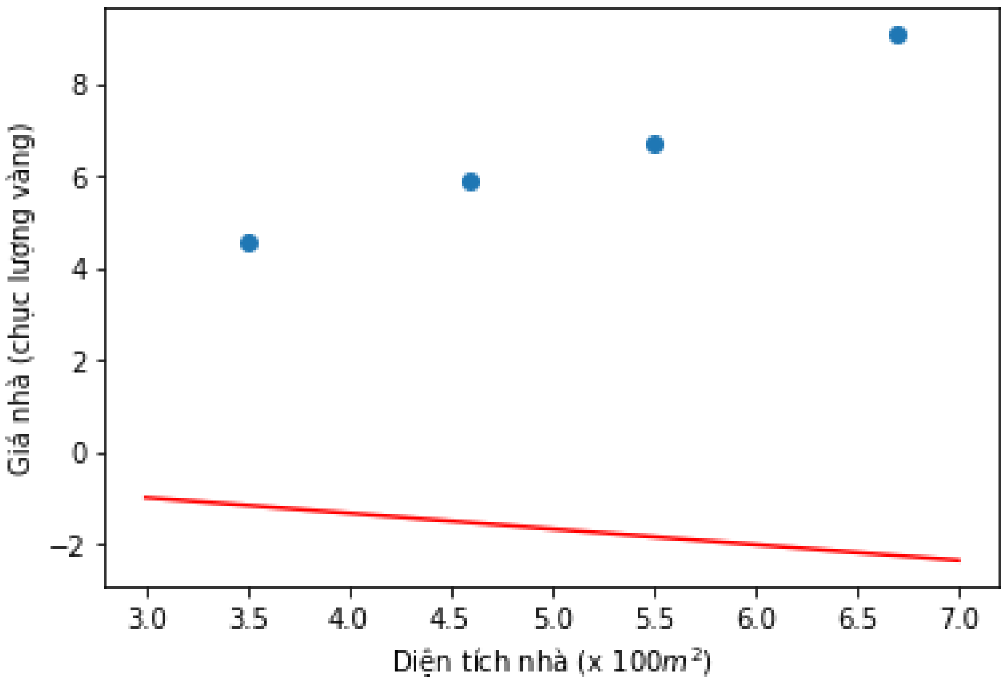
Linear Regression



Linear Regression

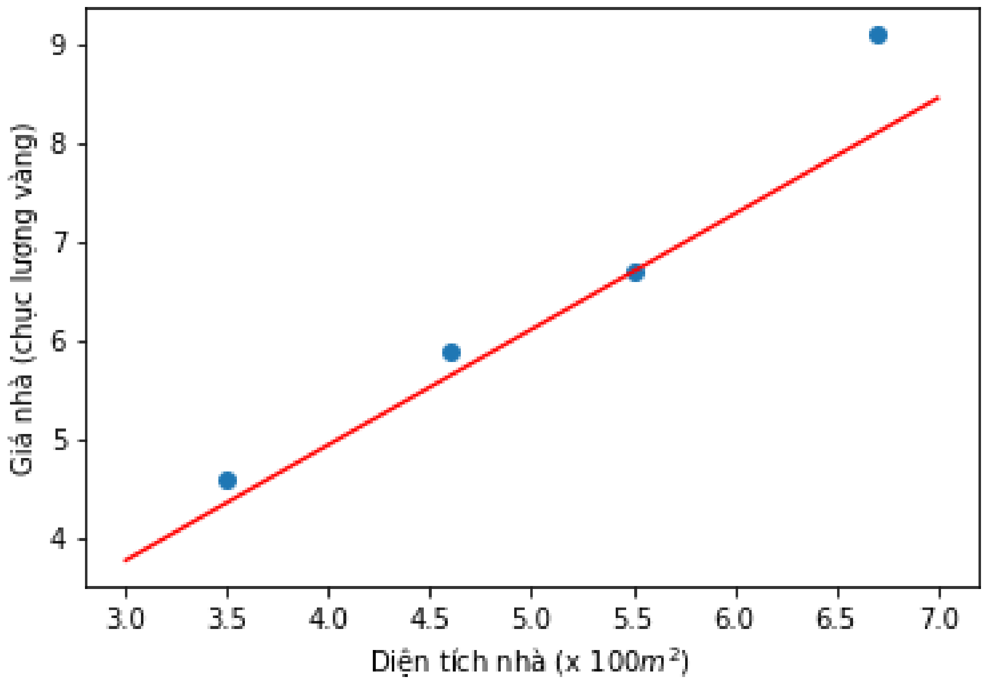
❖ Toy example

Model prediction before and after the first update



w = -0.34 b = 0.04 L = 128.55

Before updating

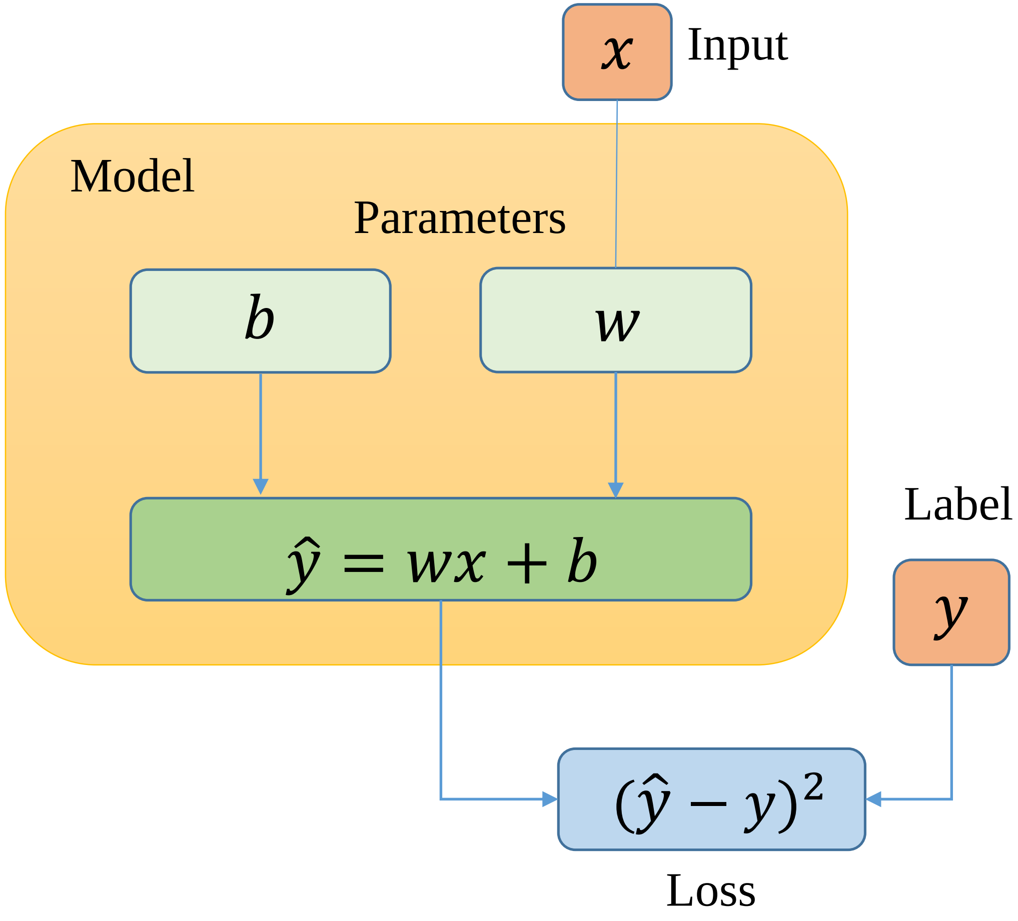


w = 1.179292 b = 0.26676 L = 0.868

After updating

Linear Regression

❖ Summary (simple version)



1) Pick a sample (x, y) from training data

2) Compute the output $\hat{y}$

$$\hat{y} = wx + b$$

3) Compute loss

$$L = (\hat{y} - y)^2$$

4) Compute derivative

$$\frac{\partial L}{\partial w} = 2x(\hat{y} - y) \qquad \frac{\partial L}{\partial b} = 2(\hat{y} - y)$$

5) Update parameters

$$w = w - \eta \frac{\partial L}{\partial w} \qquad b = b - \eta \frac{\partial L}{\partial b}$$

η is learning rate

Linear Regression

❖ Implementation

Cheat sheet

Compute the output $\hat{y}$

$$\hat{y} = wx + b$$

Compute the loss

$$L = (\hat{y} - y)^2$$

Compute derivative

$$\frac{\partial L}{\partial w} = 2x(\hat{y} - y)$$

$$\frac{\partial L}{\partial b} = 2(\hat{y} - y)$$

Update parameters

$$w = w - \eta \frac{\partial L}{\partial w}$$

$$b = b - \eta \frac{\partial L}{\partial b}$$

```python
# forward
def predict(x, w, b):
    return x*w + b

# compute gradient
def gradient(y_hat, y, x):
    dw = 2*x*(y_hat-y)
    db = 2*(y_hat-y)

    return (dw, db)

# update weights
def update_weight(w, b, lr, dw, db):
    w_new = w - lr*dw
    b_new = b - lr*db

    return (w_new, b_new)
```

Linear Regression

❖ Code for one update

```python
# forward
def predict(x, w, b):
    return x*w + b

# compute gradient
def gradient(y_hat, y, x):
    dw = 2*x*(y_hat-y)
    db = 2*(y_hat-y)

    return (dw, db)

# update weights
def update_weight(w, b, lr, dw, db):
    w_new = w - lr*dw
    b_new = b - lr*db

    return (w_new, b_new)
```

```python
# test sample
x = 6.7
y = 9.1

# init weights
b = 0.04
w = -0.34
lr = 0.01

# predict y_hat
y_hat = predict(x, w, b)
print('y_hat: ', y_hat)

# compute loss
loss = (y_hat-y)*(y_hat-y)
print('Loss: ', loss)

# compute gradient
(dw, db) = gradient(y_hat, y, x)
print('dw: ', dw)
print('db: ', db)

# update weights
(w, b) = update_weight(w, b, lr, dw, db)
print('w_new: ', w)
print('b_new: ', b)
```

Outline

- Simple version of Linear Regression
- Computational Graph
- Mini-batch Training
- Batch Training
- Generalization of Linear Regression
- Loss Functions

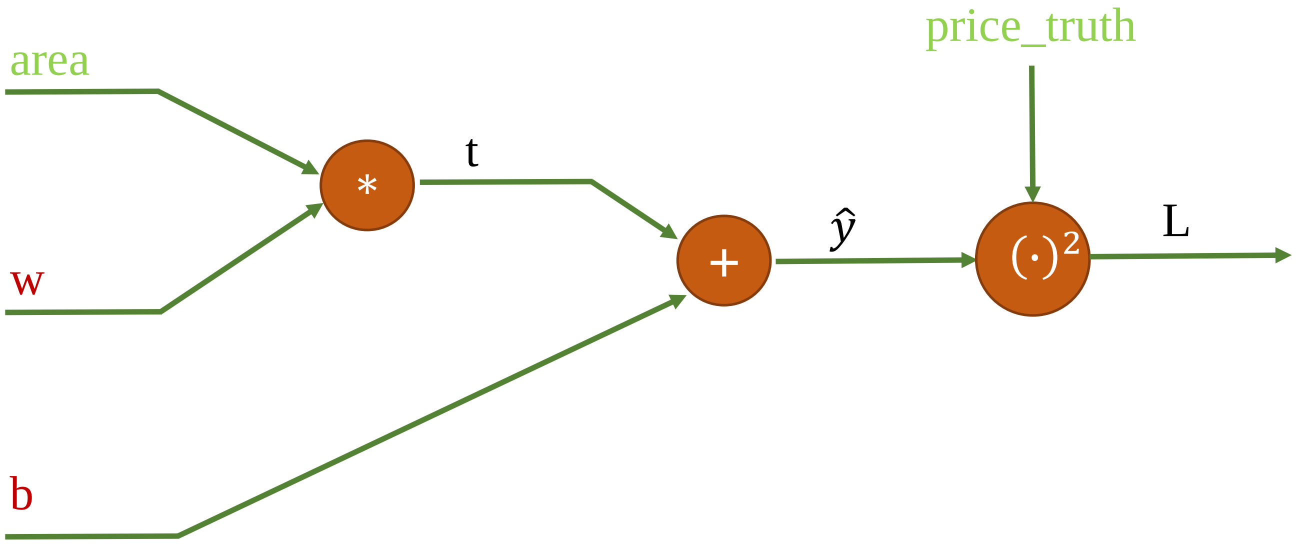
Computational graph

❖ House price prediction

❖ One-sample training

$$price = w * area + b$$

$$t = w * area$$

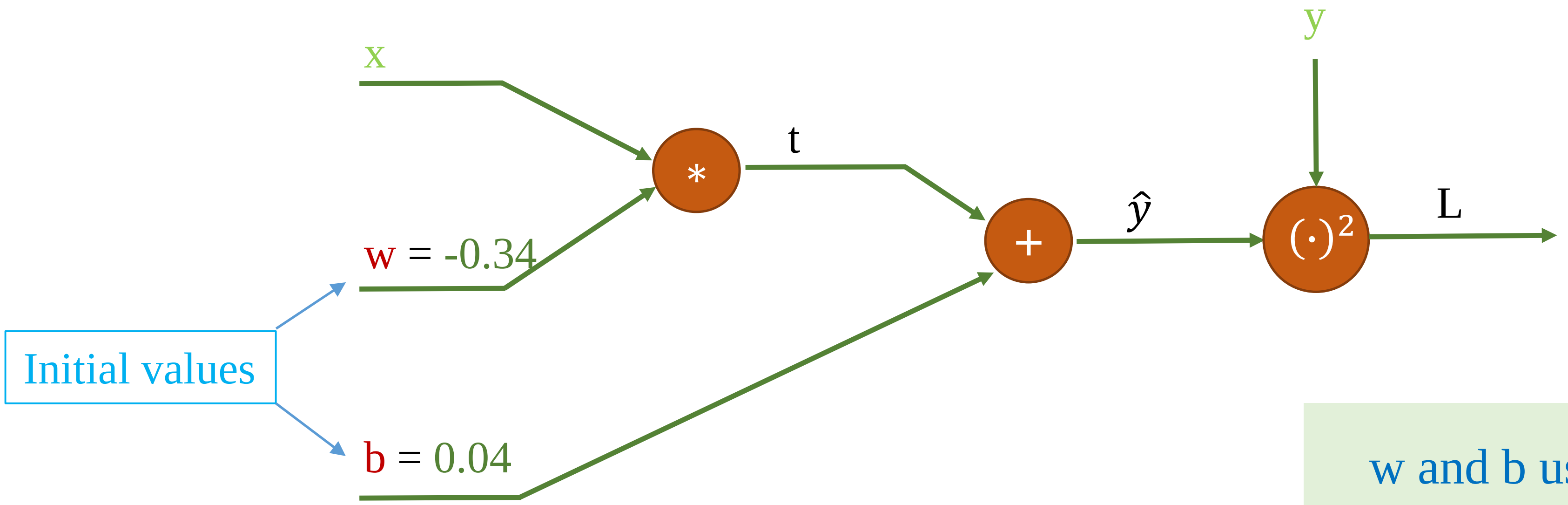


Computational graph

❖ House price prediction

❖ One-sample training

$t = wx$ $\qquad \hat{y} = t + b$ $\qquad L = (\hat{y} - y)^2$



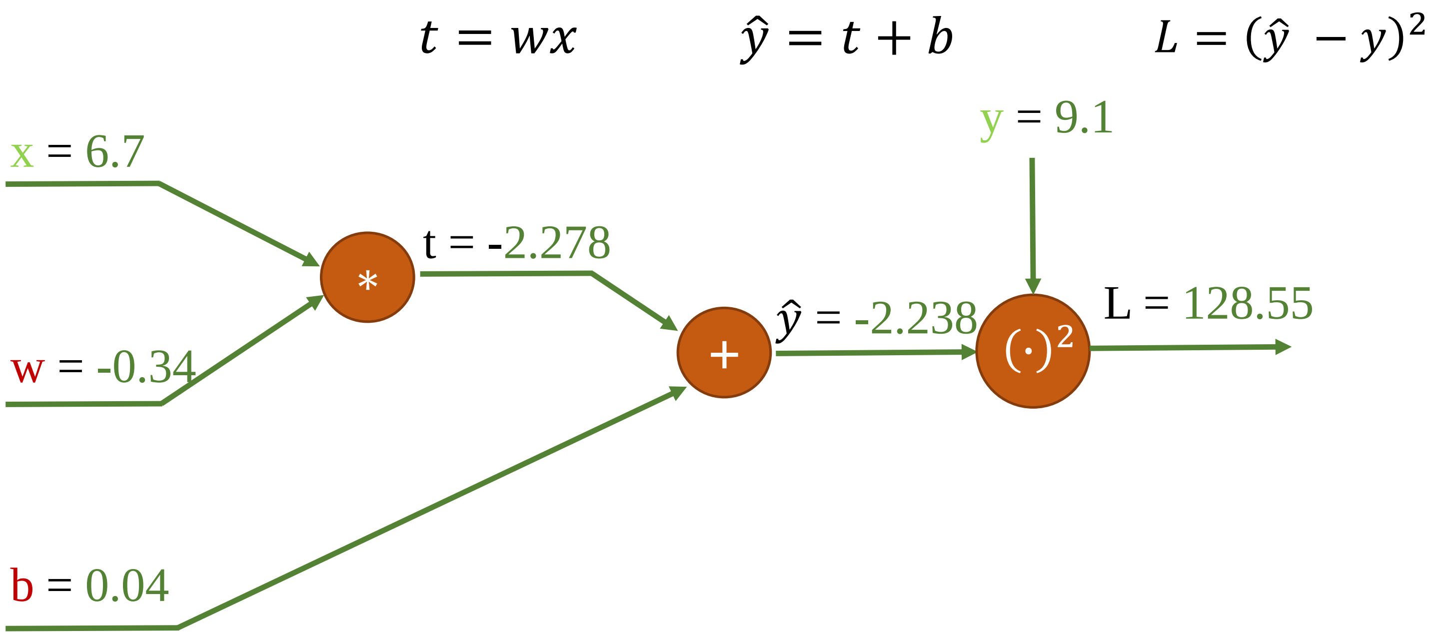
w and b usually be set by random values

For example, $\mathrm{N}(0, \sigma)$ where σ is small

Computational graph

❖ House price prediction

❖ One-sample training

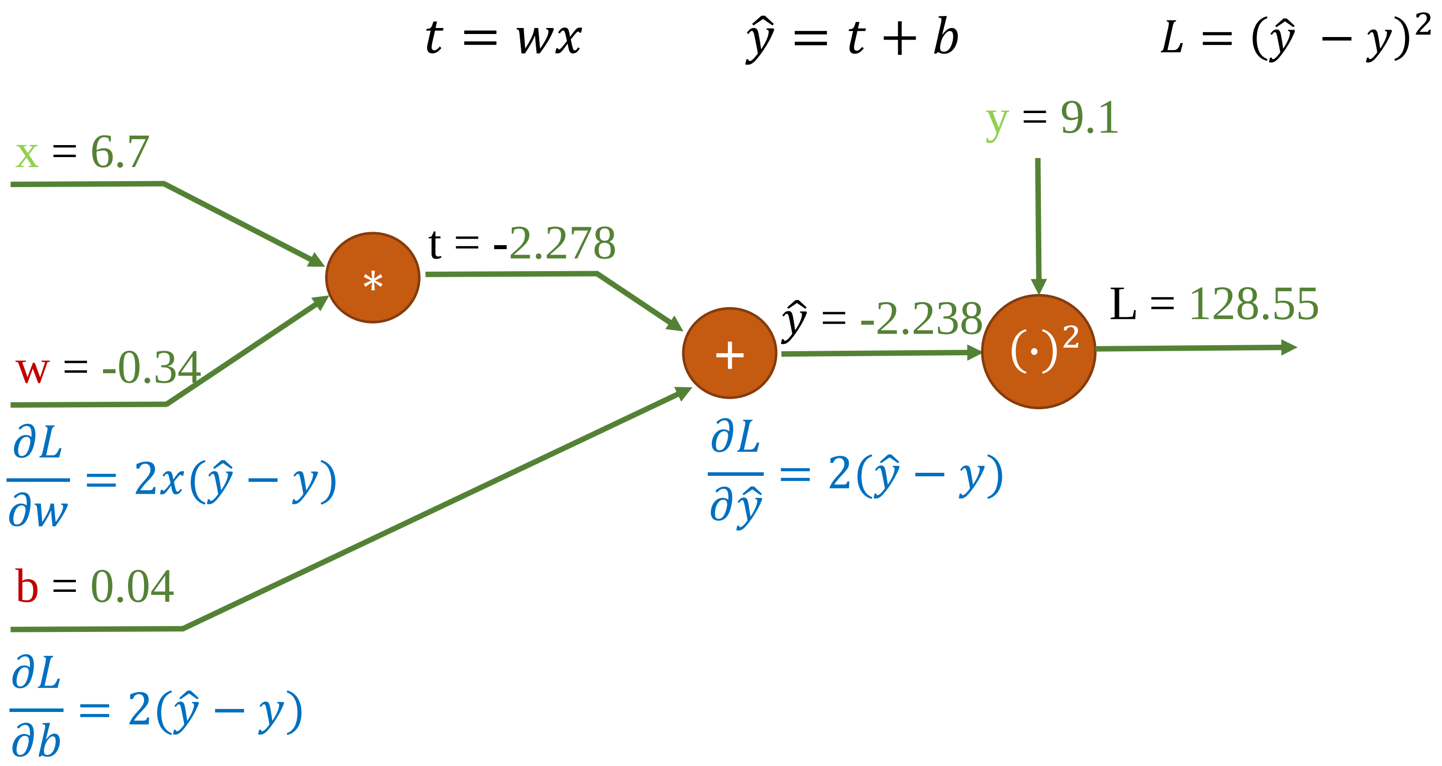


Feature	Label
area	price
6.7	9.1
4.6	5.9
3.5	4.6
5.5	6.7

Computational graph

❖ House price prediction

❖ One-sample training

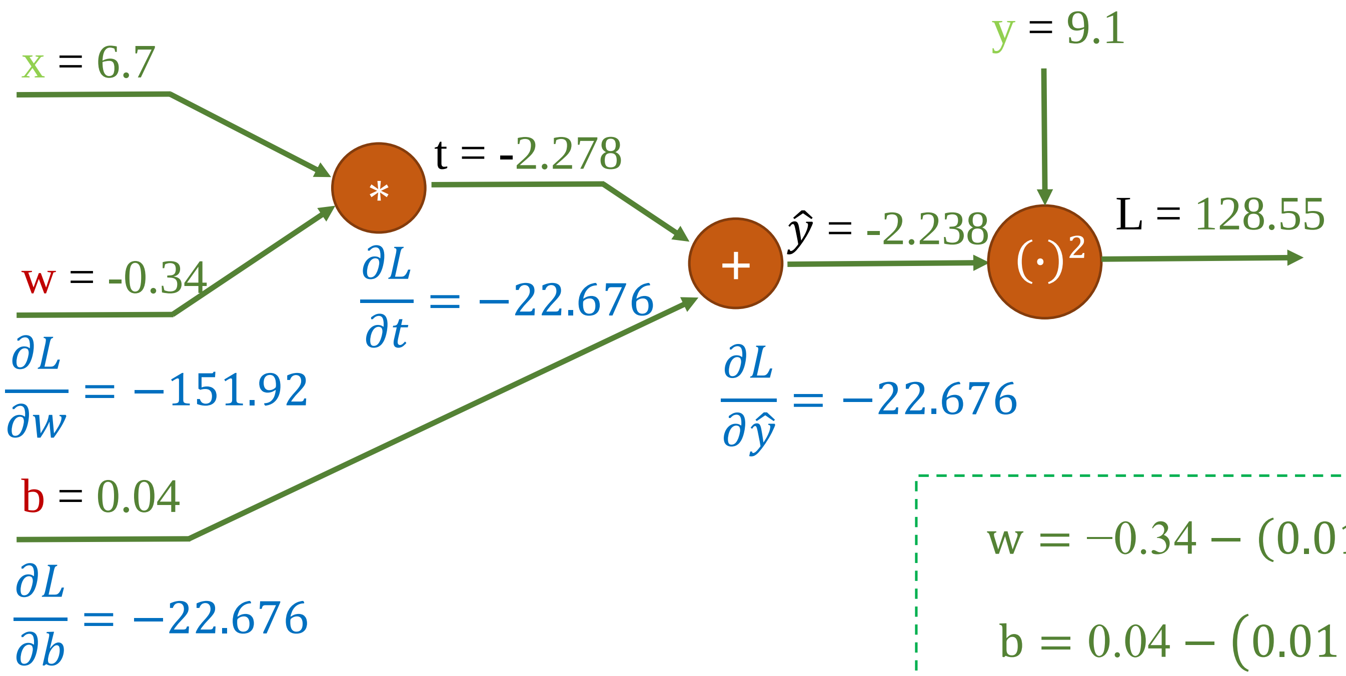


Computational graph

❖ House price prediction

❖ One-sample training

$$t = wx \qquad \hat{y} = t + b \qquad L = (\hat{y} - y)^2$$



$$w = -0.34 - (0.01 * (-151.9)) = 1.179$$

$$b = 0.04 - \left(0.01 * (-22.67)\right) = 0.266$$

Cách cập nhật a và b

$$w = w - \eta * \frac{\partial L}{\partial w}$$

$$b = b - \eta * \frac{\partial L}{\partial b}$$

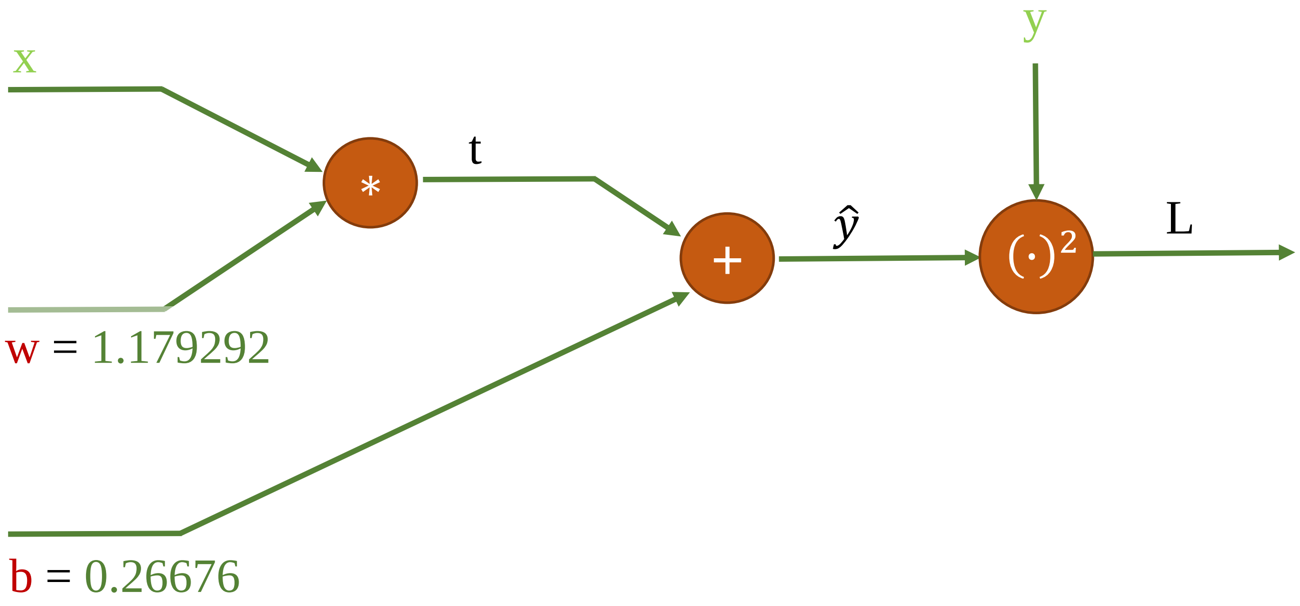
Learning rate $\eta = 0.01$

Computational graph

❖ House price prediction

❖ One-sample training

$t = wx$ $\hat{y} = t + b$ $L = (\hat{y} - y)^2$



Feature	Label
area	price
6.7	9.1
4.6	5.9
3.5	4.6
5.5	6.7

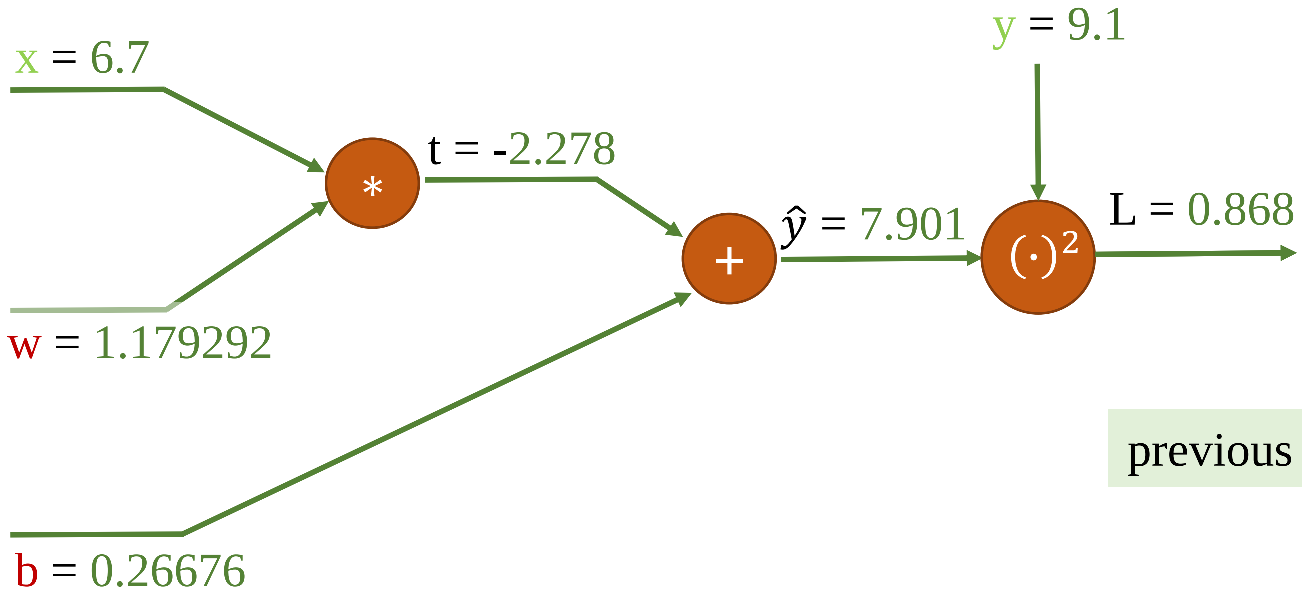
Computational graph

❖ House price prediction

❖ One-sample training

Feature	Label
area	price
6.7	9.1
4.6	5.9
3.5	4.6
5.5	6.7

$t = wx \qquad \hat{y} = t + b \qquad L = (\hat{y} - y)^2$



previous L = 128.55

Updated a and b values help to reduce the L value

Implementation

One-sample training

Feature	Label
area	price
6.7	9.1
4.6	5.9
3.5	4.6
5.5	6.7

column index=0 column index=1

```python
# data preparation
import numpy as np
import matplotlib.pyplot as plt

def get_column(data, index):
    result = [row[index] for row in data]
    return result

data = np.genfromtxt('data.csv',
                     delimiter=',').tolist()

x_data = get_column(data, 0)
y_data = get_column(data, 1)
N = len(x_data)

print(f'areas: {x_data}')
print(f'prices: {y_data}')
print(f'data_size: {N}')
```

```
areas: [6.7, 4.6, 3.5, 5.5]
prices: [9.1, 5.9, 4.6, 6.7]
data_size: 4
```

Implementation

❖ House price prediction

❖ One-sample training

1) Pick a sample (x, y) from training data

2) Compute the output $\hat{y}$

$$\hat{y} = wx + b$$

3) Compute loss

$$L = (\hat{y} - y)^2$$

4) Compute derivative

$$\frac{\partial L}{\partial w} = 2x(\hat{y} - y) \qquad \frac{\partial L}{\partial b} = 2(\hat{y} - y)$$

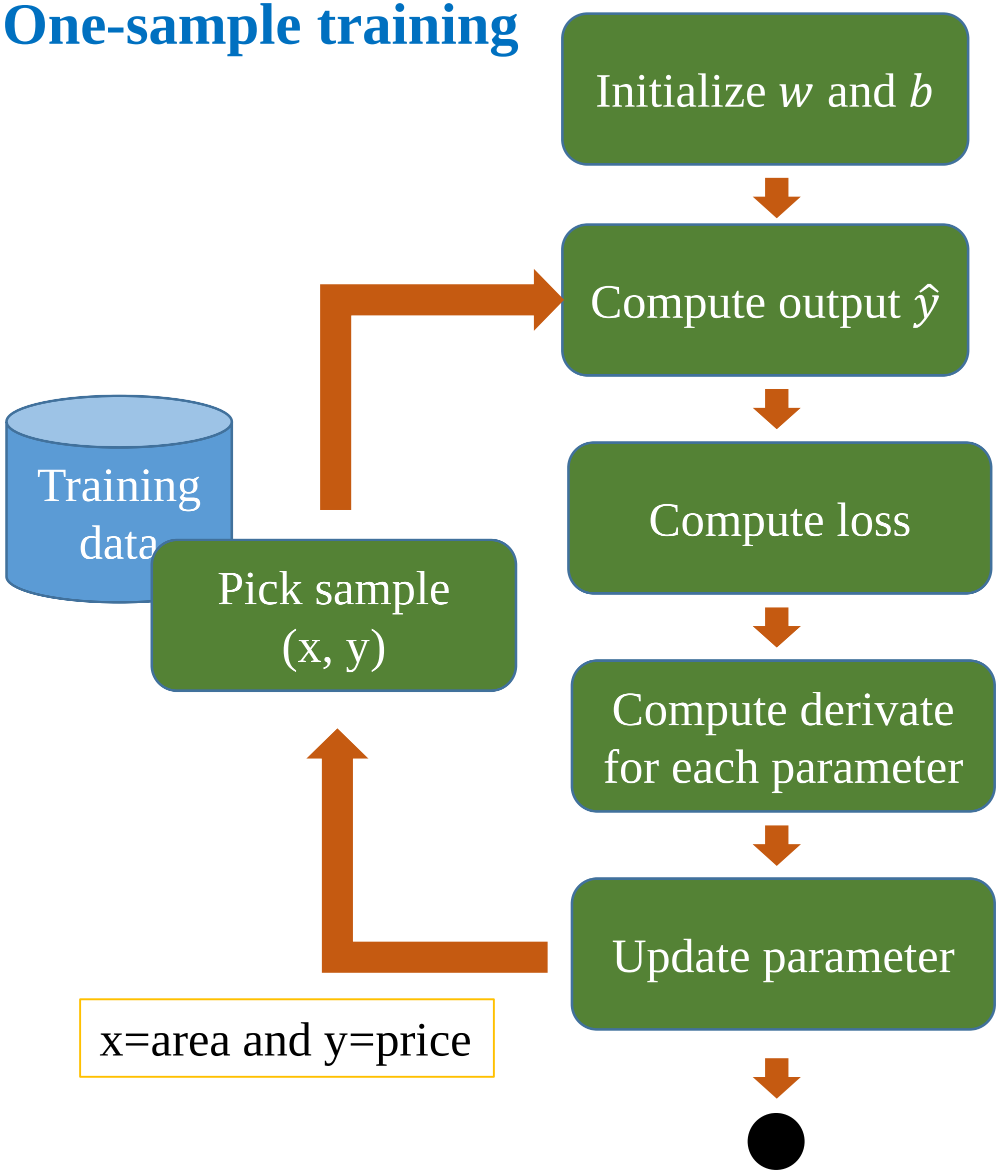
5) Update parameters

$$w = w - \eta \frac{\partial L}{\partial w} \qquad b = b - \eta \frac{\partial L}{\partial b}$$

```python
# forward
def predict(x, w, b):
    return x*w + b

# compute gradient
def gradient(y_hat, y, x):
    dw = 2*x*(y_hat-y)
    db = 2*(y_hat-y)

    return (dw, db)

# update weights
def update_weight(w, b, lr, dw, db):
    w_new = w - lr*dw
    b_new = b - lr*db

    return (w_new, b_new)
```

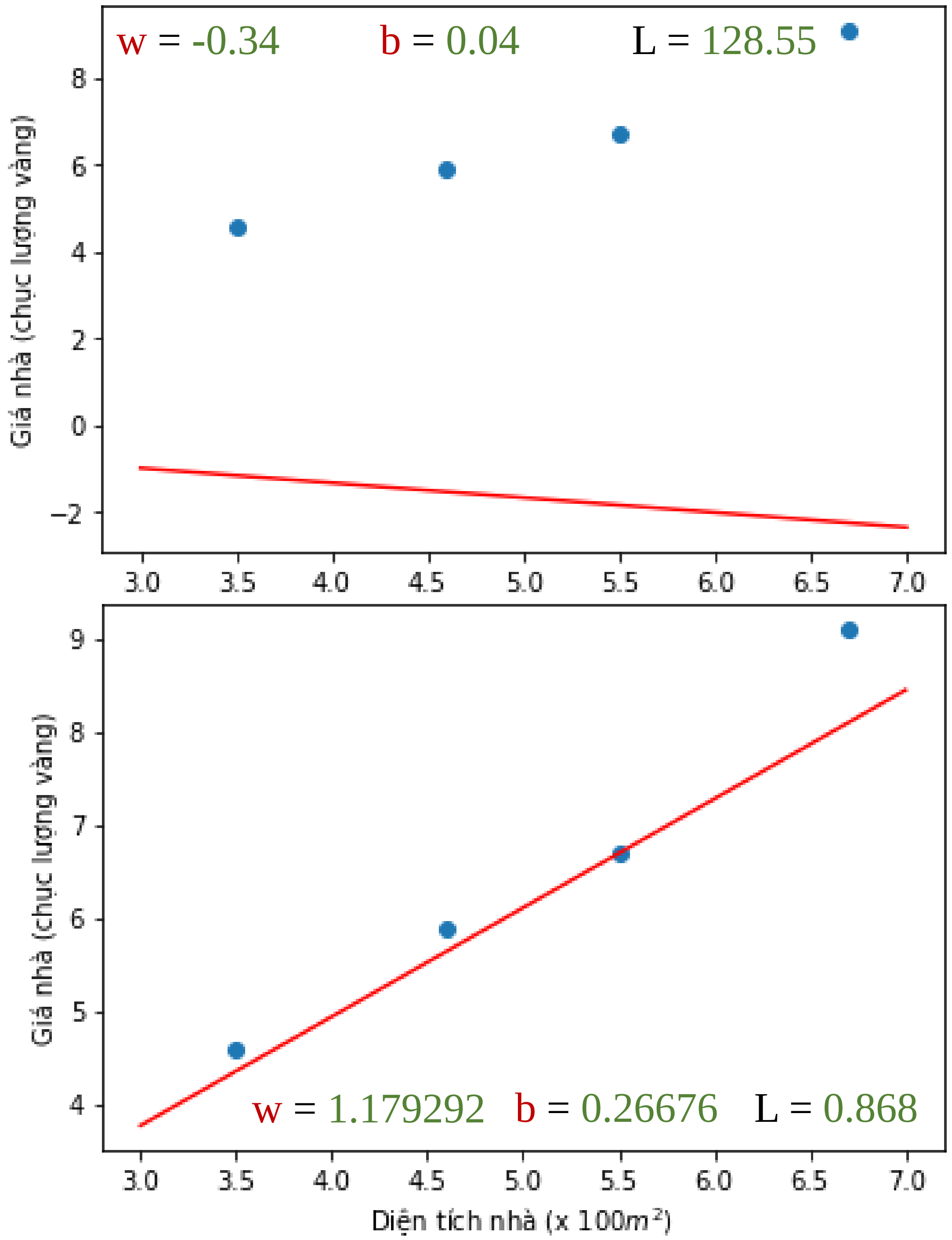
Implementation

One-sample training



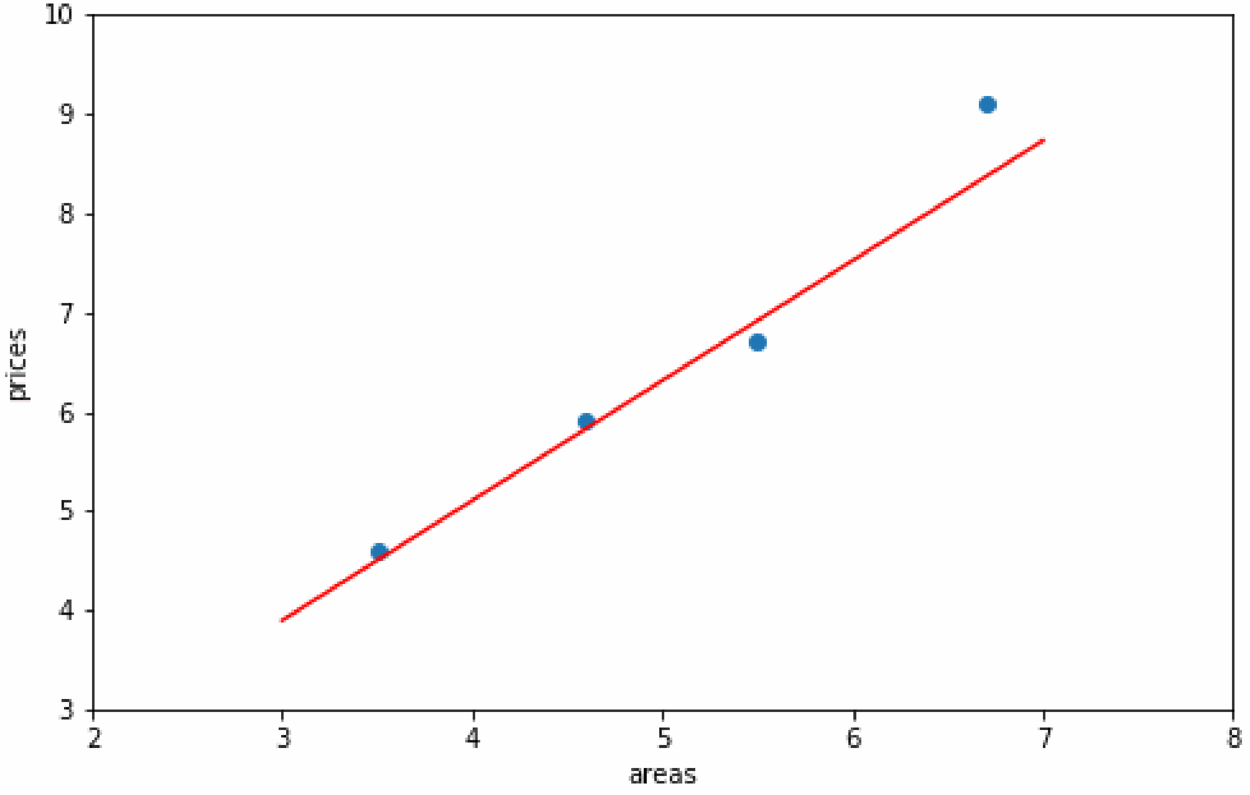

```python
# init weights
b = 0.04
w = -0.34
lr = 0.01

# how long
epoch_max = 10

for epoch in range(epoch_max):
    for i in range(data_size):
        # get a sample
        # ...

        # predict y_hat
        # ...

        # compute loss
        # ...

        # compute gradient
        # ...

        # update weights
        # ...

```

```python
# init weights
b = 0.04
w = -0.34
lr = 0.01

# how long
epoch_max = 10
data_size = 4

for epoch in range(epoch_max):
    for i in range(data_size):
        # get a sample
        x = areas[i]
        y = prices[i]

        # predict y_hat
        y_hat = predict(x, w, b)

        # compute loss
        loss = (y_hat-y)*(y_hat-y)

        # compute gradient
        (dw, db) = gradient(y_hat, y, x)

        # update weights
        (w, b) = update_weight(w, b, lr, dw, db)
```

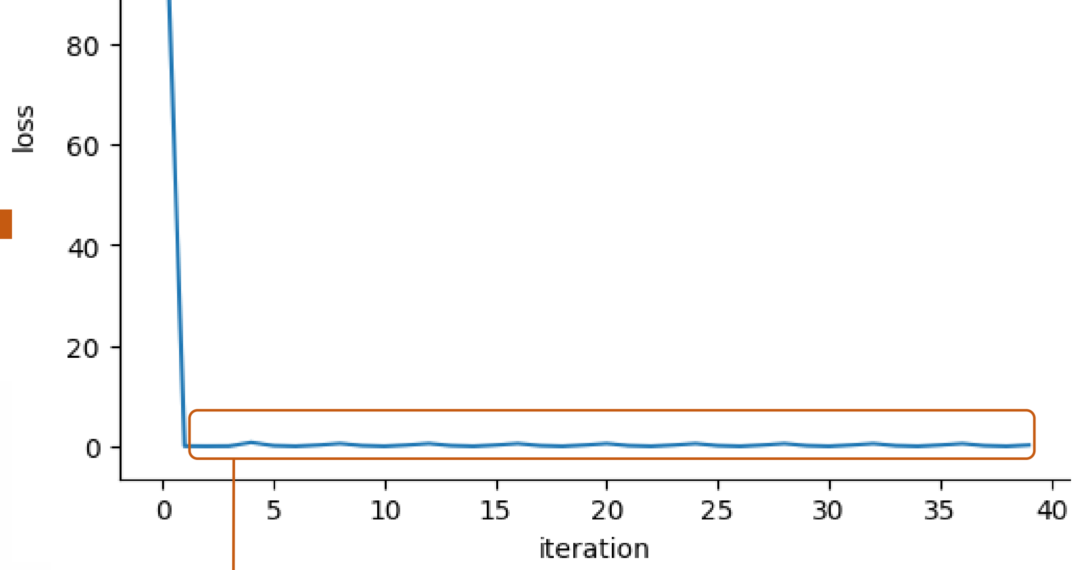


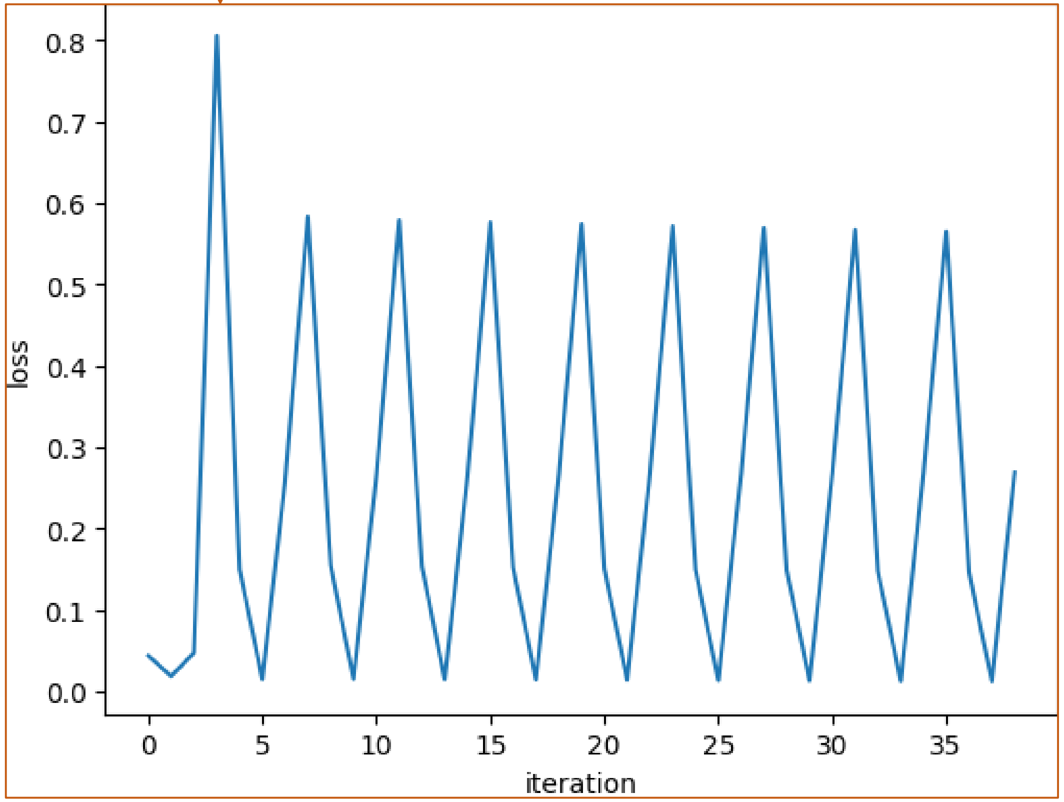
Computational graph

- ❖ **House price prediction**
  - ❖ **One-sample training**



Model after training





Quiz 1: Is it OK to use the following loss function?

$$L = \frac{1}{2}(\hat{y} - y)^2$$

Quiz 2: if so, construct formulas and change somehow so that the two models have the same output.

Quiz 3: What about the following loss function?

$$L = \frac{1}{2}(y - \hat{y})^2$$

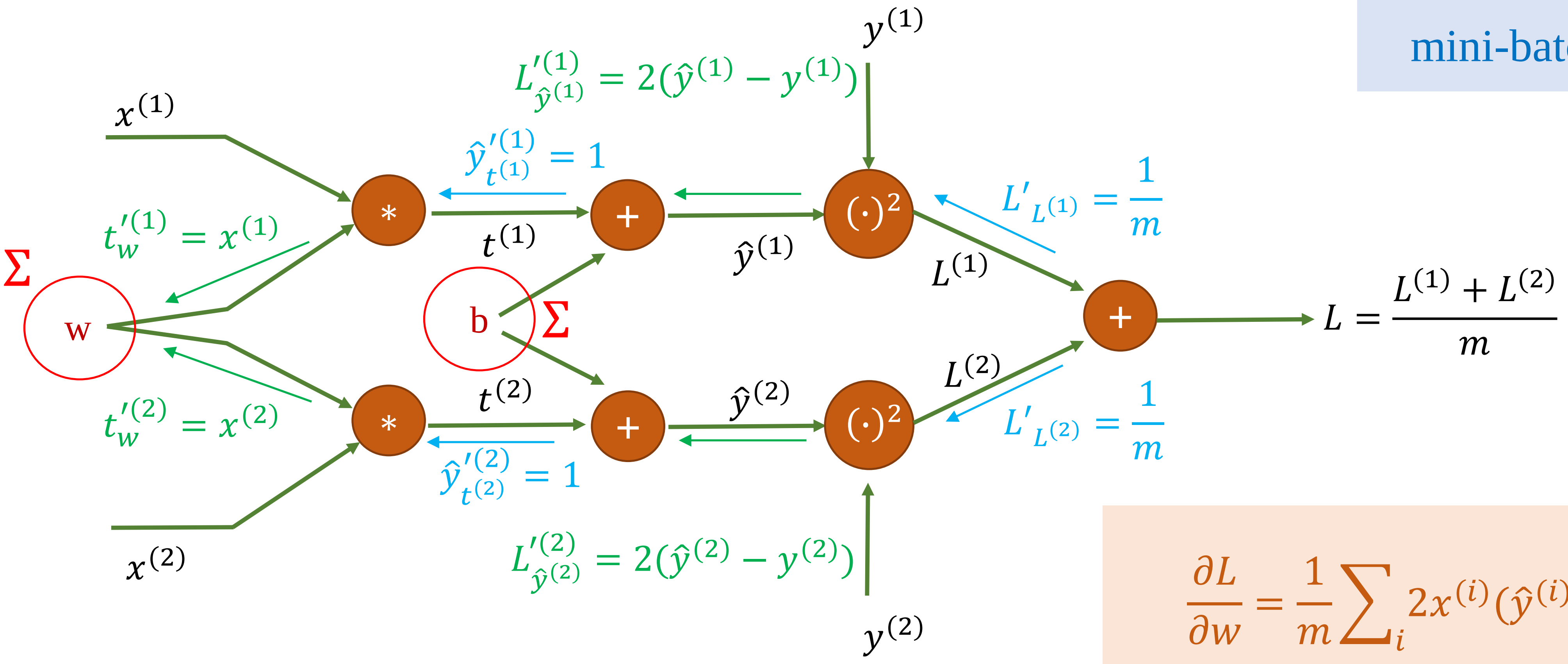
Outline

- Simple version of Linear Regression
- Computational Graph
- Mini-batch Training
- Batch Training
- Generalization of Linear Regression
- Loss Functions

Computational graph

❖ Compute derivate for w and b

mini-batch m = 2



$$\frac{\partial L}{\partial w} = \frac{1}{m}\sum_i 2x^{(i)}(\hat{y}^{(i)} - y^{(i)})$$

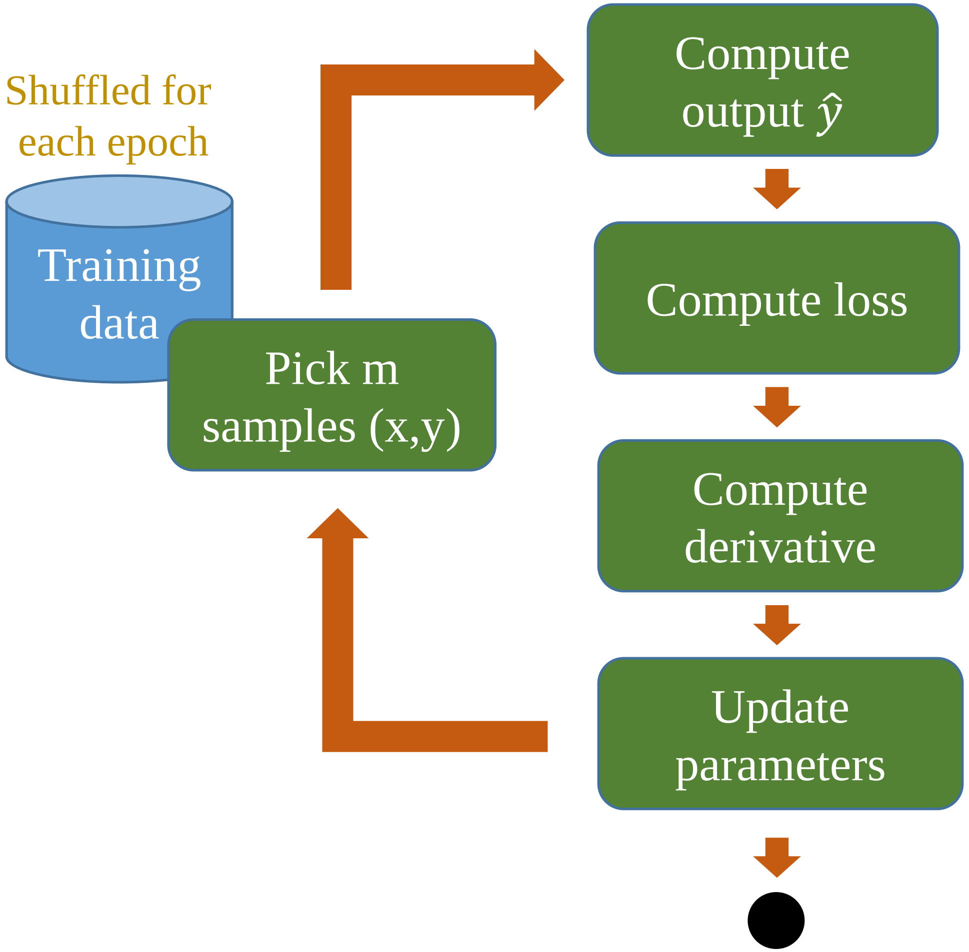
$$\frac{\partial L}{\partial b} = \frac{1}{m}\sum_i 2(\hat{y}^{(i)} - y^{(i)})$$

Computational graph

simplified

❖ House price prediction

❖ m-sample training (1<m<N)



1) Pick m samples $(x^{(i)}, y^{(i)})$ from training data

2) Tính output $\hat{y}^{(i)}$

$$\hat{y}^{(i)} = wx^{(i)} + b \qquad \text{for } 0 \le i < m$$

3) Tính loss

$$L^{(i)} = (\hat{y}^{(i)} - y^{(i)})^2 \quad \text{for } 0 \le i < m$$

4) Tính đạo hàm

$$\frac{\partial L^{(i)}}{\partial w} = 2x^{(i)}(\hat{y}^{(i)} - y^{(i)})$$

$$\frac{\partial L^{(i)}}{\partial b} = 2(\hat{y}^{(i)} - y^{(i)})$$

for $0 \le i < m$

5) Cập nhật tham số

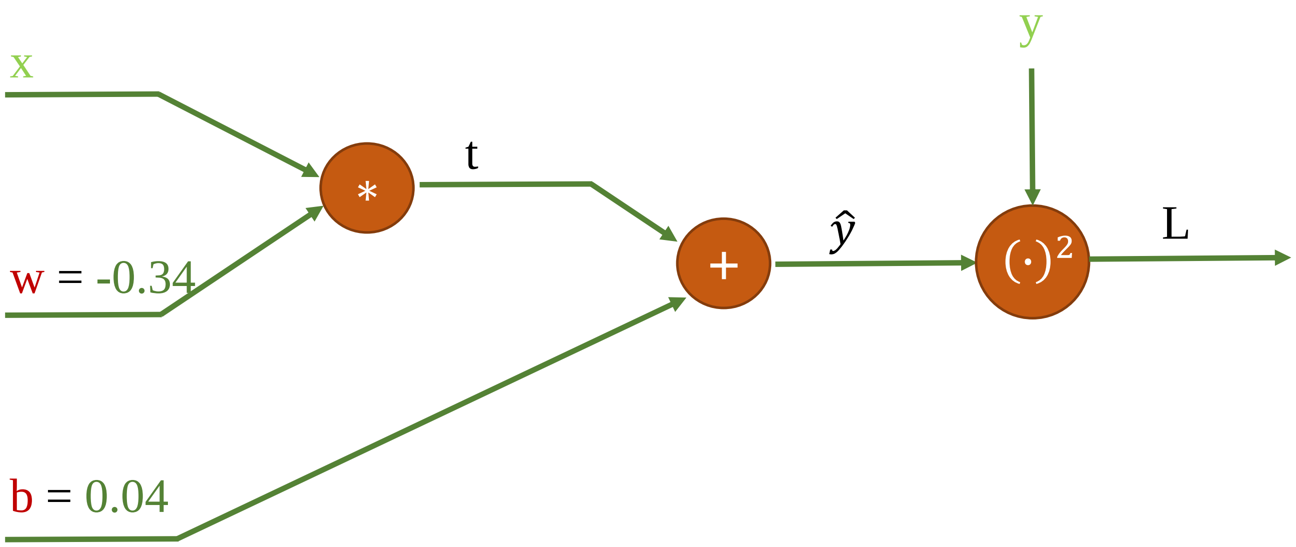
$$w = w - \eta \frac{\sum_i \frac{\partial L^{(i)}}{\partial w}}{m} \qquad b = b - \eta \frac{\sum_i \frac{\partial L^{(i)}}{\partial b}}{m}$$

Computational graph

❖ House price prediction

❖ m-sample training (1<m<N)

$t = wx \qquad \hat{y} = t + b \qquad L = (\hat{y} - y)^2$



Computational graph

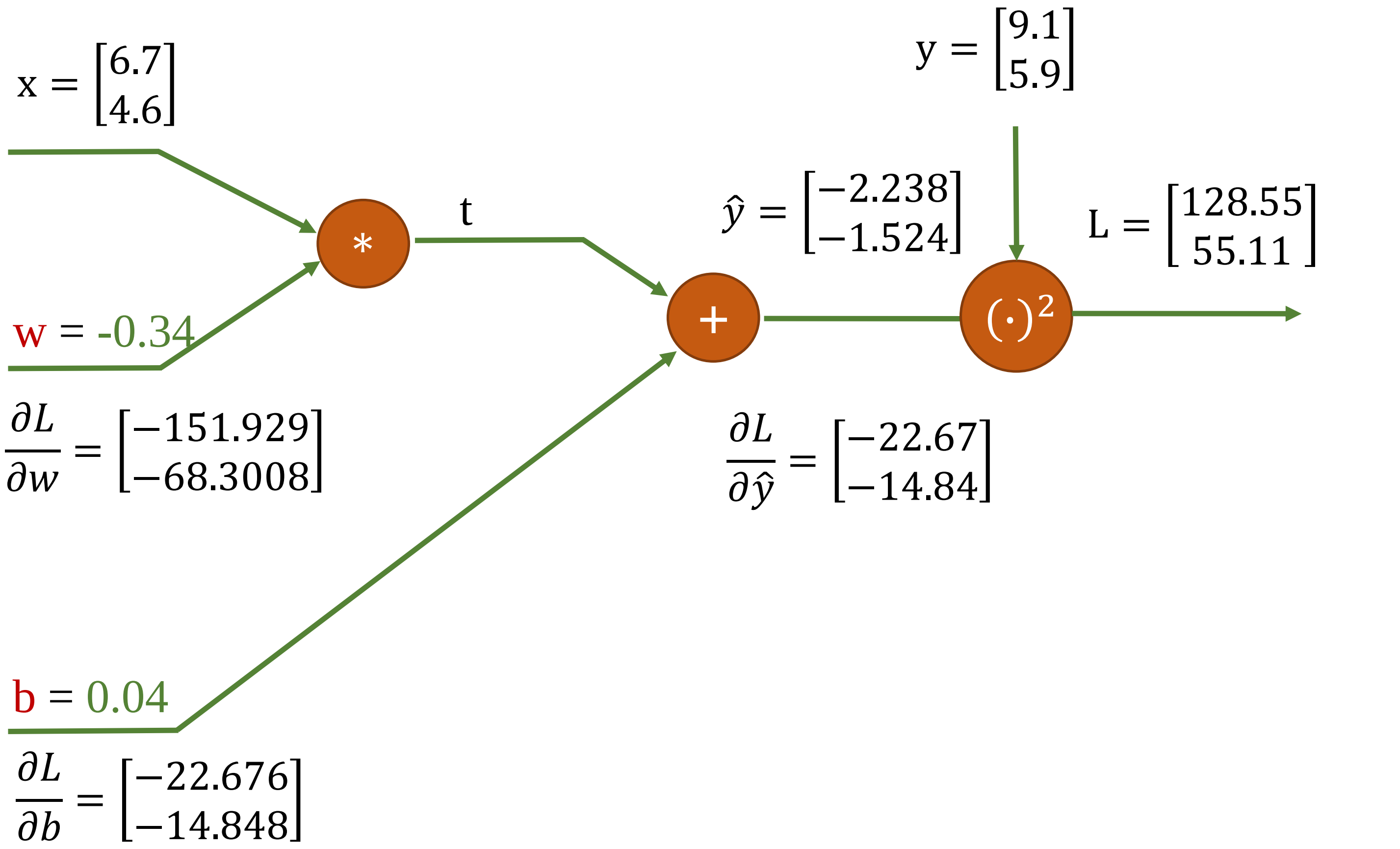
- ❖ **House price prediction**
  - ❖ **m-sample training (1<m<N)**

	Feature	Label
	area	price
	6.7	9.1
	4.6	5.9
	3.5	4.6
	5.5	6.7

m = 2

$$\frac{sum(L'_w)}{m} = -110.115$$

$$\frac{sum(L'_b)}{m} = -18.762$$



$t = wx \qquad \hat{y} = t + b \qquad L = (\hat{y} - y)^2$

Computational graph

- **House price prediction**
  - **m-sample training (1<m<N)**

	Feature	Label
	area	price
	6.7	9.1
	4.6	5.9
	3.5	4.6
	5.5	6.7

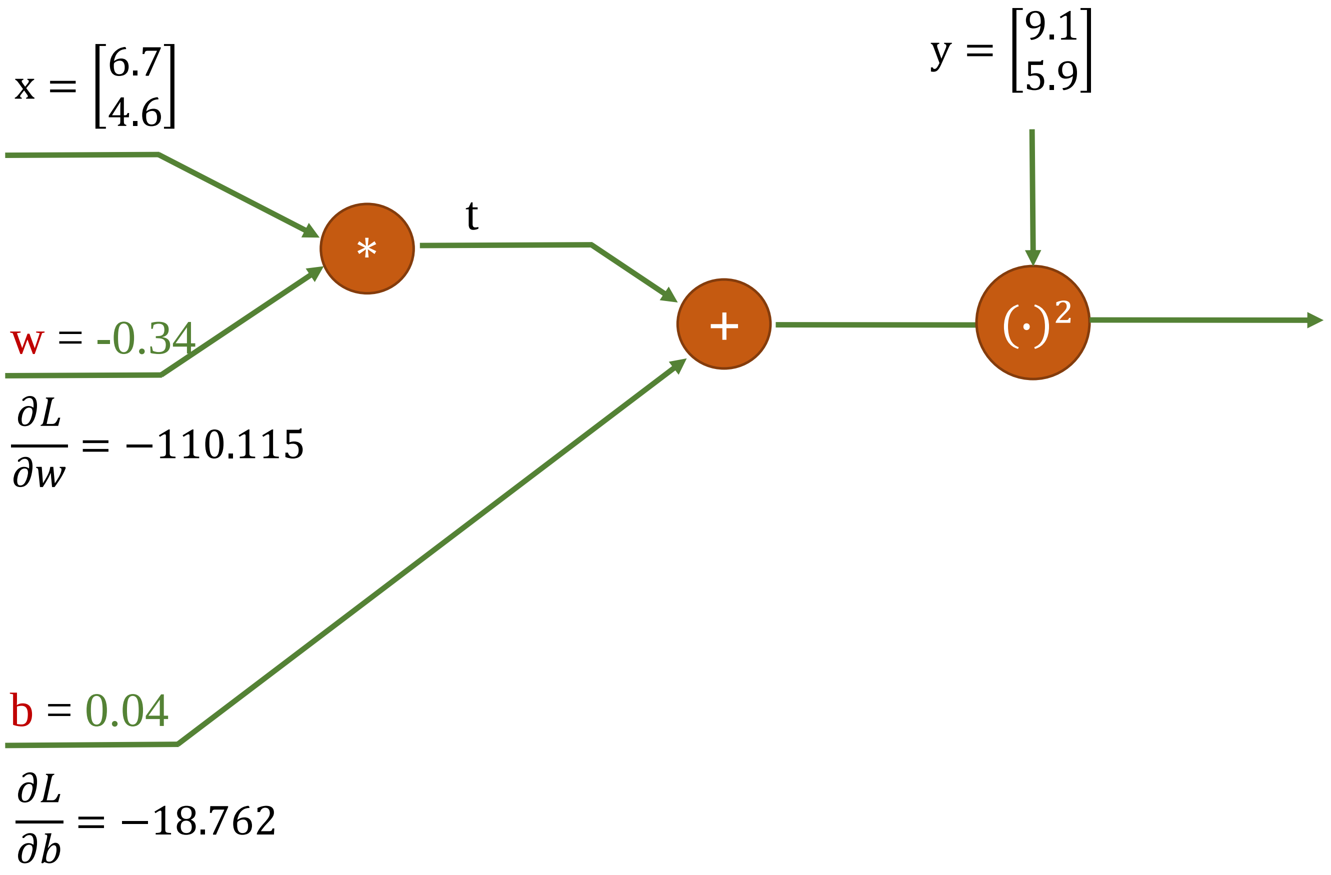
m = 2

Update w and b

$$w = w - \eta * \frac{\partial L}{\partial w}$$

$$b = b - \eta * \frac{\partial L}{\partial b}$$

Learning rate $\eta = 0.01$



replace

$w = -0.34 - (0.01 * (-110.115)) = \boxed{0.761}$

replace

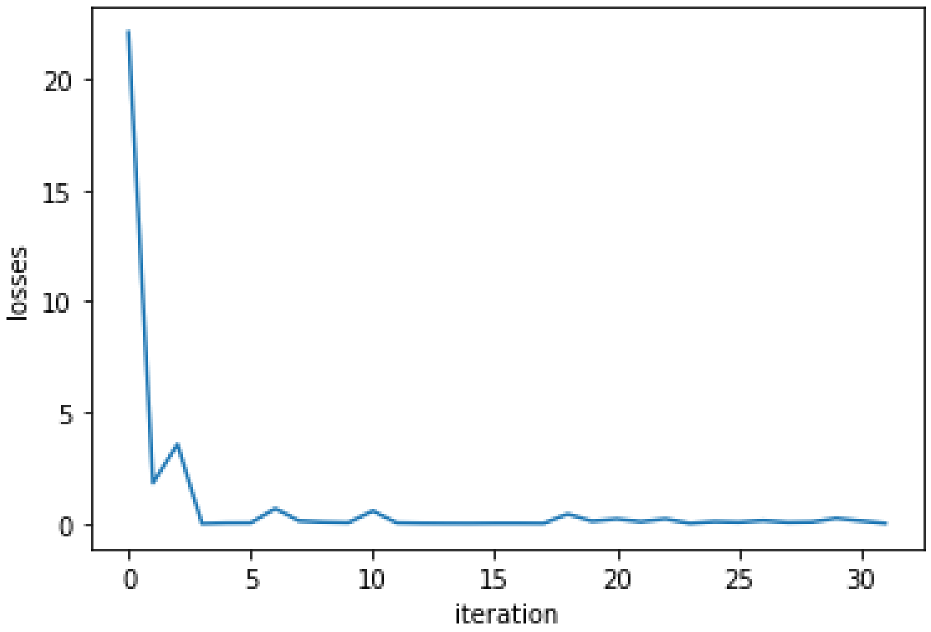
$b = 0.04 - \left(0.01 * (-18.762)\right) = \boxed{0.227}$

$t = wx$ $\quad \hat{y} = t + b$ $\quad L = (\hat{y} - y)^2$

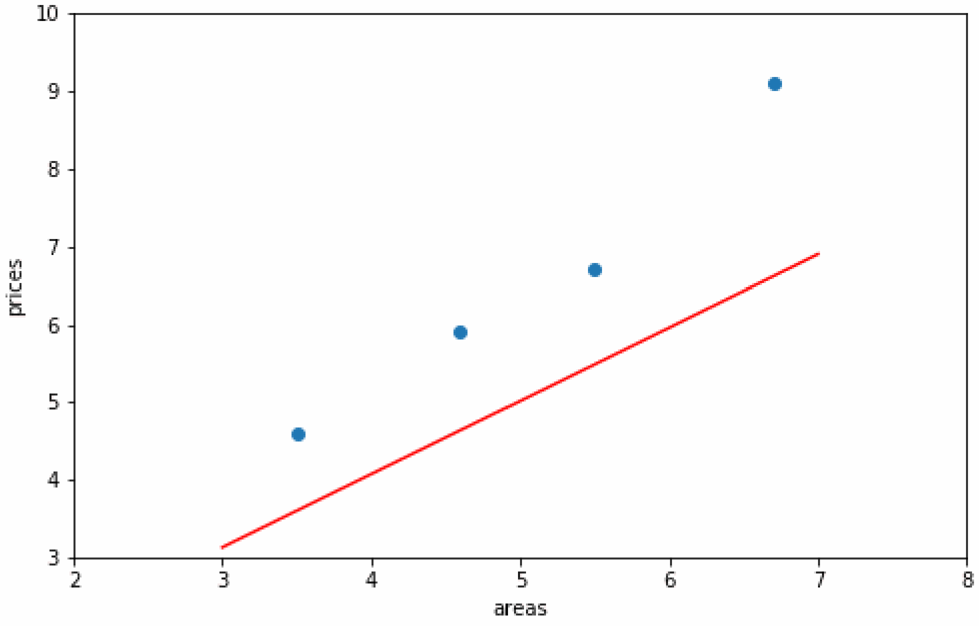
Computational graph

❖ House price prediction

❖ m-sample training (1<m<N)



Losses for 30 iterations



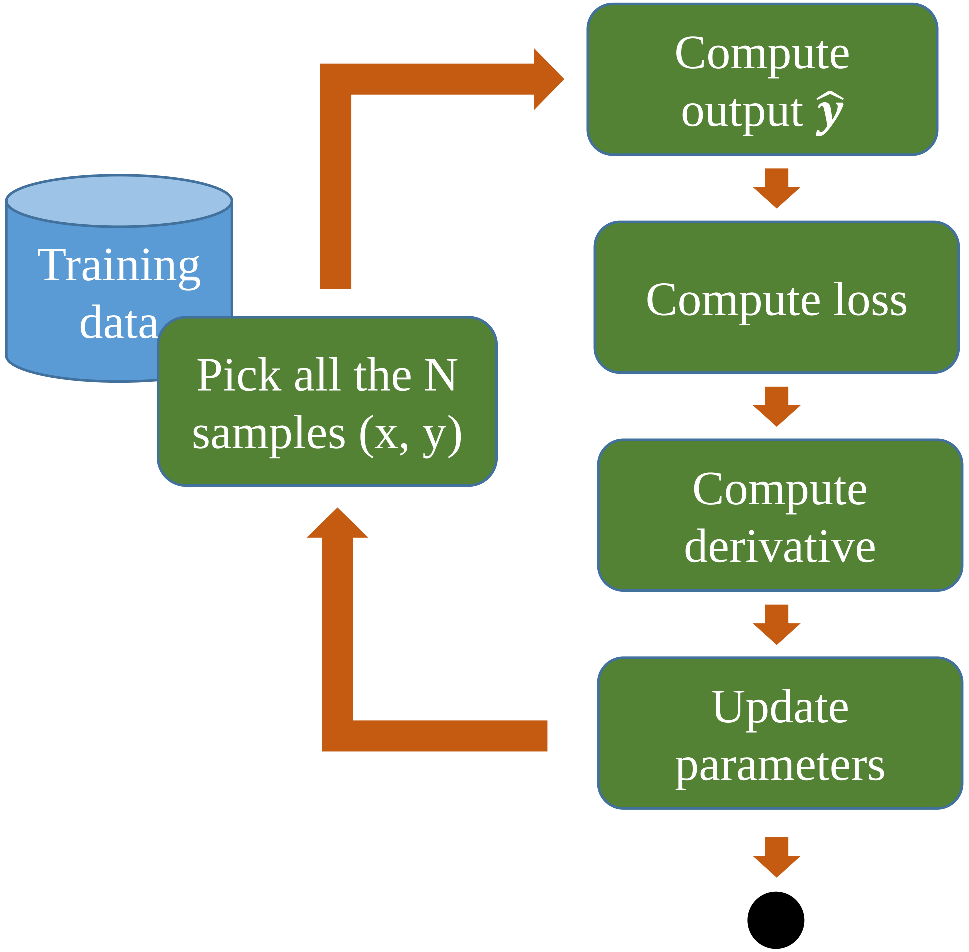
Model updating for different iterations

simplified

Computational graph

❖ House price prediction

❖ N-sample training



1) Pick all the N samples $(x^{(i)}, y^{(i)})$ from training data

2) Tính output $\hat{y}^{(i)}$

$$\hat{y}^{(i)} = wx^{(i)} + b \qquad \text{for } 0 \leq i < N$$

3) Tính loss

$$L^{(i)} = (\hat{y}^{(i)} - y^{(i)})^2 \quad \text{for } 0 \leq i < N$$

4) Tính đạo hàm

$$\frac{\partial L^{(i)}}{\partial w} = 2x^{(i)}(\hat{y}^{(i)} - y^{(i)})$$

$$\frac{\partial L^{(i)}}{\partial b} = 2(\hat{y}^{(i)} - y^{(i)}) \qquad \text{for } 0 \leq i < N$$

5) Cập nhật tham số

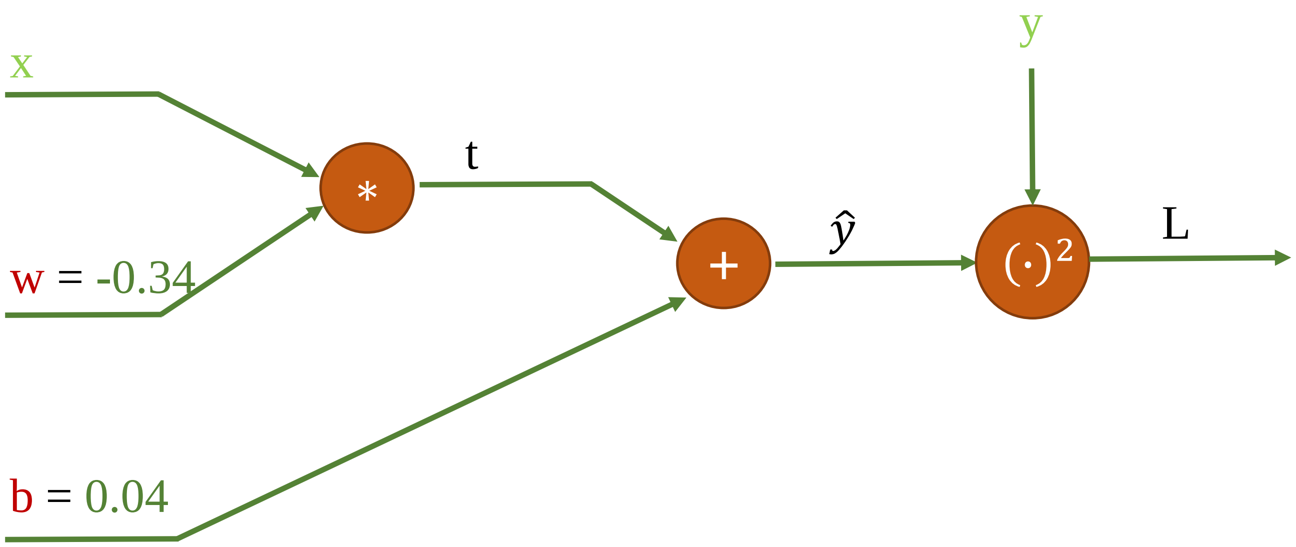
$$w = w - \eta \frac{\sum_i \frac{\partial L^{(i)}}{\partial w}}{N} \qquad b = b - \eta \frac{\sum_i \frac{\partial L^{(i)}}{\partial b}}{N}$$

Computational graph

❖ House price prediction

❖ N-sample training

$t = wx$ $\hat{y} = t + b$ $L = (\hat{y} - y)^2$

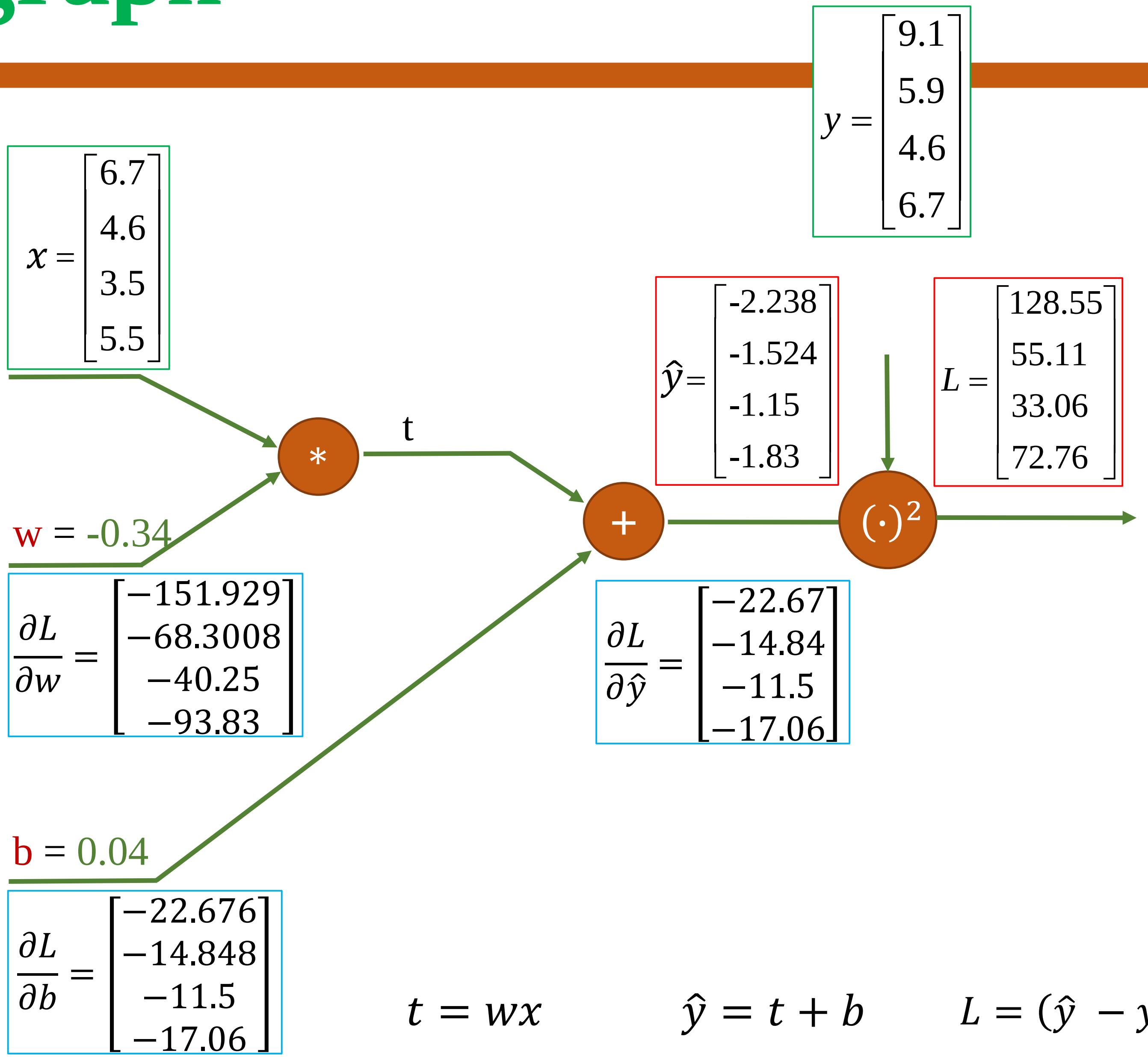


Computational graph

- ❖ **House price prediction**
  - ❖ **N-sample training**

$$\frac{sum(\frac{\partial L}{\partial w})}{4} = -88.5775$$

$$\frac{sum(\frac{\partial L}{\partial b})}{4} = -16.521$$



$t = wx \qquad \hat{y} = t + b \qquad L = (\hat{y} - y)^2$

Computational graph

❖ House price prediction

❖ N-sample training

Update w and b

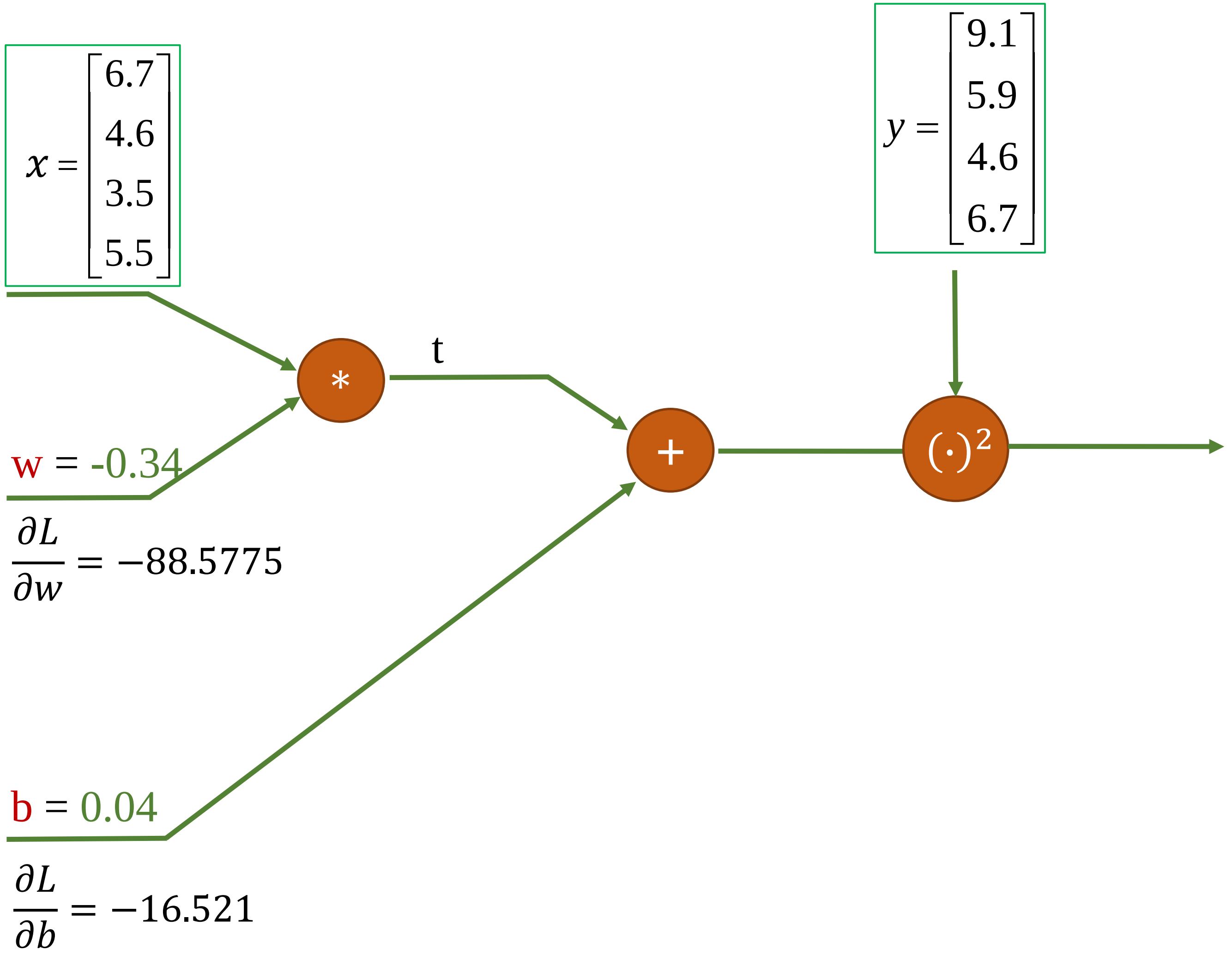
$$w = w - \eta * \frac{\partial L}{\partial w}$$

$$b = b - \eta * \frac{\partial L}{\partial b}$$

Learning rate $\eta = 0.01$

$$\mathrm{w} = -0.34 - (0.01 * (-88.5775)) = 0.54$$

$$\mathrm{b} = 0.04 - \big(0.01 * (-16.521)\big) = 0.205$$

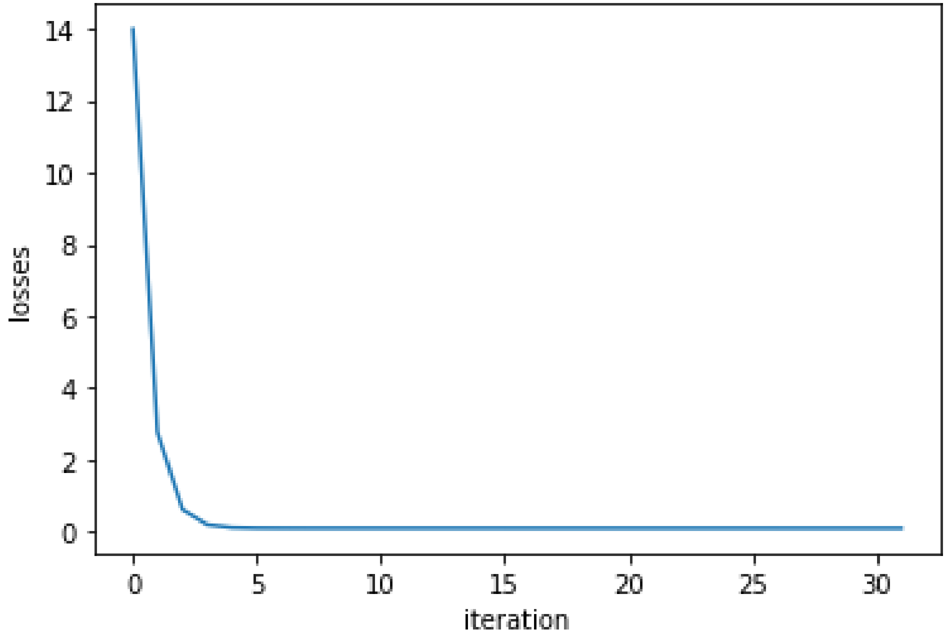


$t = wx$ $\quad \hat{y} = t + b$ $\quad L = (\hat{y} - y)^2$

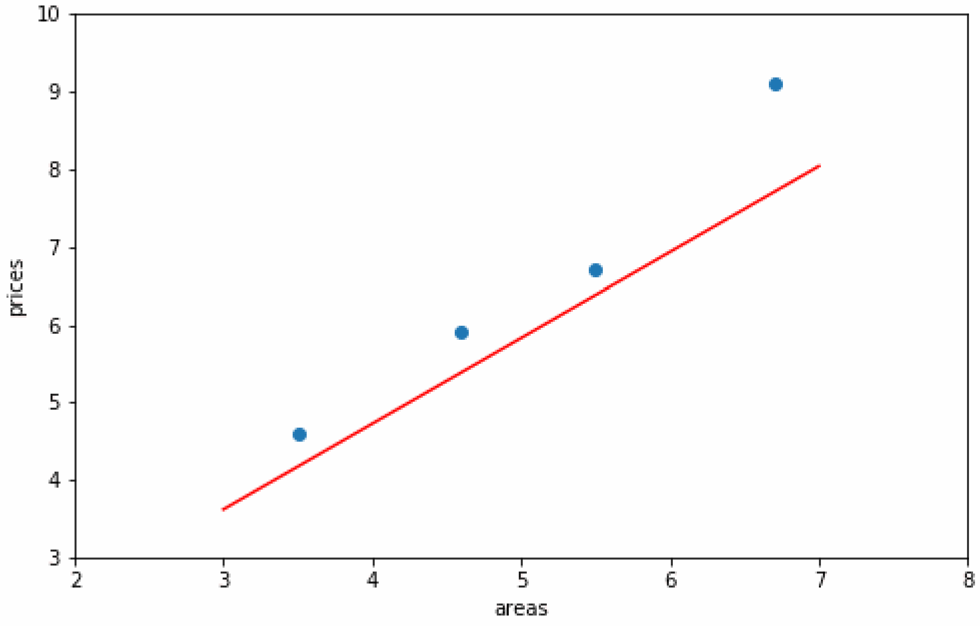
Computational graph

❖ House price prediction

❖ N-sample training



Losses for 30 iterations



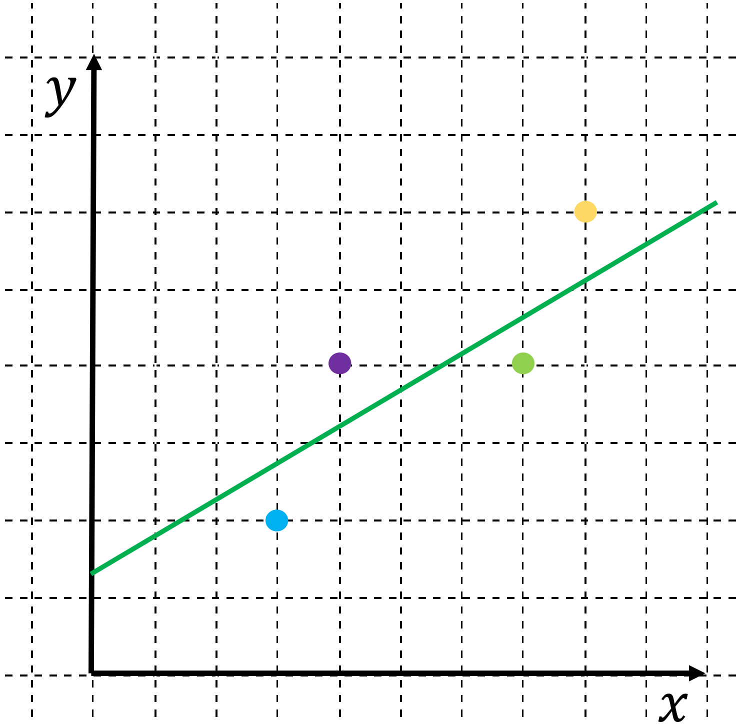
Model updating for different iterations

Outline

- Simple version of Linear Regression
- Computational Graph
- Mini-batch Training
- Batch Training
- Generalization of Linear Regression
- Loss Functions

Linear Regression

❖ General formula



Feature	Label
area	price
6.7	9.1
4.6	5.9
3.5	4.6
5.5	6.7

House price data

Linear regression models ← Linear equations

Linear equation = $w_1x_1 + w_2x_2 + \cdots + w_nx_n + b$

Linear Regression

❖ General formula

Feature	Label
area	price
6.7	9.1
4.6	5.9
3.5	4.6
5.5	6.7

House price data

Features			Label
TV	Radio	Newspaper	Sales
230.1	37.8	69.2	22.1
44.5	39.3	45.1	10.4
17.2	45.9	69.3	12
151.5	41.3	58.5	16.5
180.8	10.8	58.4	17.9

Advertising data

Model: $\hat{y} = w_1 x_1 + b$

price = a $* \, area + b$

Model: $\hat{y} = w_1 x_1 + w_2 x_2 + w_3 x_3 + b$

Sale = $w_1 * TV + w_2 * Radio + w_3 * Newspaper + b$

Linear Regression

❖ General formula

Features | Label

Boston House Price Data

crim	zn	indus	chas	nox	rm	age	dis	rad	tax	ptratio	black	lstat	medv
0.00632	18	2.31	0	0.538	6.575	65.2	4.09	1	296	15.3	396.9	4.98	24
0.02731	0	7.07	0	0.469	6.421	78.9	4.9671	2	242	17.8	396.9	9.14	21.6
0.03237	0	2.18	0	0.458	6.998	45.8	6.0622	3	222	18.7	394.63	2.94	33.4
0.06905	0	2.18	0	0.458	7.147	54.2	6.0622	3	222	18.7	396.9	5.33	36.2
0.08829	12.5	7.87	0	0.524	6.012	66.6	5.5605	5	311	15.2	395.6	12.43	22.9

$$\text{medv} = w_1 * x_1 + \cdots + w_{13} * x_{13} + b$$

Linear Regression

❖ Generalized formula

House price data

Feature	Label
area	price
6.7	9.1
4.6	5.9
3.5	4.6
5.5	6.7

Advertising data

Features			Label
TV	Radio	Newspaper	Sales
230.1	37.8	69.2	22.1
44.5	39.3	45.1	10.4
17.2	45.9	69.3	12
151.5	41.3	58.5	16.5
180.8	10.8	58.4	17.9

Model

$$\text{price} = \text{w} * area + b$$

$$\hat{y} = wx + b$$

Model (vectorization)

$$\hat{y} = \boldsymbol{\theta}^T \boldsymbol{x} \quad \text{where} \quad \boldsymbol{\theta}^T = [b \quad w]^T$$

$$\boldsymbol{x} = [x_0 \ \ area]^T$$

$$x_0 = 1$$

Model

$$\text{Sale} = w_1 * TV + w_2 * Radio + w_3 * Newspaper + b$$

$$\hat{y} = w_1 x_1 + w_2 x_2 + w_3 x_3 + b$$

Model (vectorization)

$$\hat{y} = \boldsymbol{\theta}^T \boldsymbol{x} \quad \text{where} \quad \boldsymbol{\theta}^T = [b \quad w_1 \quad w_2 \quad w_3]^T$$

$$\boldsymbol{x} = [x_0 \ \ TV \quad Radio \quad Newspaper]^T$$

$$x_0 = 1$$

Linear Regression

1) Pick a sample (x_1, x_2, x_3, y) from training data

2) Compute the output $\hat{y}$

$$\hat{y} = w_1 * TV + w_2 * R + w_3 * N + b$$

$$\hat{y} = w_1 * x_1 + w_2 * x_2 + w_3 * x_3 + b$$

3) Compute loss

$$L = (\hat{y} - y)^2$$

4) Compute derivative

$$\frac{\partial L}{\partial w_1} = 2x_1(\hat{y} - y) \qquad \frac{\partial L}{\partial w_3} = 2x_3(\hat{y} - y)$$

$$\frac{\partial L}{\partial w_2} = 2x_2(\hat{y} - y) \qquad \frac{\partial L}{\partial b} = 2(\hat{y} - y)$$

5) Update parameters

$$w_1 = w_1 - \eta \frac{\partial L}{\partial w_1} \qquad w_3 = w_3 - \eta \frac{\partial L}{\partial w_3}$$

$$w_2 = w_2 - \eta \frac{\partial L}{\partial w_2} \qquad b = b - \eta \frac{\partial L}{\partial b}$$

Features			Label
TV	**Radio**	**Newspaper**	**Sales**
230.1	37.8	69.2	22.1
44.5	39.3	45.1	10.4
17.2	45.9	69.3	12
151.5	41.3	58.5	16.5
180.8	10.8	58.4	17.9

Advertising data

Model

$$\text{Sale} = w_1 * TV + w_2 * Radio + w_3 * Newspaper + b$$

$$\hat{y} = w_1 x_1 + w_2 x_2 + w_3 x_3 + b$$

1) Pick a sample (x_1, x_2, x_3, y) from training data

2) Compute the output $\hat{y}$

$$\hat{y} = w_1 * TV + w_2 * R + w_3 * N + b$$

$$\hat{y} = w_1 * x_1 + w_2 * x_2 + w_3 * x_3 + b$$

3) Compute loss

$$L = (\hat{y} - y)^2$$

4) Compute derivative

$$\frac{\partial L}{\partial w_1} = 2x_1(\hat{y} - y) \qquad \frac{\partial L}{\partial w_3} = 2x_3(\hat{y} - y)$$

$$\frac{\partial L}{\partial w_2} = 2x_2(\hat{y} - y) \qquad \frac{\partial L}{\partial b} = 2(\hat{y} - y)$$

5) Update parameters

$$w_1 = w_1 - \eta \frac{\partial L}{\partial w_1} \qquad w_3 = w_3 - \eta \frac{\partial L}{\partial w_3}$$

$$w_2 = w_2 - \eta \frac{\partial L}{\partial w_2} \qquad b = b - \eta \frac{\partial L}{\partial b}$$

```python
# compute output and loss
def predict(x1, x2, x3, w1, w2, w3, b):
    return w1*x1 + w2*x2 + w3*x3 + b
def compute_loss(y_hat, y):
    return (y_hat - y)**2

# compute gradient
def compute_gradient_wi(xi, y, y_hat):
    dl_dwi = 2*xi*(y_hat-y)
    return dl_dwi
def compute_gradient_b(y, y_hat):
    dl_db = 2*(y_hat-y)
    return dl_db

# update weights
def update_weight_wi(wi, dl_dwi, lr):
    wi = wi - lr*dl_dwi
    return wi
def update_weight_b(b, dl_db, lr):
    b  = b - lr*dl_db
    return b
```

Features			Label
TV	**Radio**	**Newspaper**	**Sales**
230.1	37.8	69.2	22.1
44.5	39.3	45.1	10.4
17.2	45.9	69.3	12
151.5	41.3	58.5	16.5
180.8	10.8	58.4	17.9

```python
def initialize_params():
    w1 = random.gauss(mu=0.0, sigma=0.01)
    w2 = random.gauss(mu=0.0, sigma=0.01)
    w3 = random.gauss(mu=0.0, sigma=0.01)
    b  = 0

    return w1, w2, w3, b

# initialize model's parameters
w1, w2, w3, b = initialize_params()
print(w1, w2, w3, b)
```

```
0.01609506469549467 0.00607778501208891 0.0023344573891806507 0
```

```python
import numpy as np
import random

def get_column(data, index):
    result = [row[index] for row in data]
    return result

data = np.genfromtxt('advertising.csv',
                     delimiter=',',
                     skip_header=1).tolist()

# get tv (index=0)
tv_data = get_column(data, 0)

# get radio (index=1)
radio_data = get_column(data, 1)

# get newspaper (index=2)
newspaper_data = get_column(data, 2)

# get sales (index=0)
sales_data = get_column(data, 3)
```

Unnormalized data $\eta = 10^{-5}$

Features			Label
TV	Radio	Newspaper	Sales
230.1	37.8	69.2	22.1
44.5	39.3	45.1	10.4
17.2	45.9	69.3	12
151.5	41.3	58.5	16.5
180.8	10.8	58.4	17.9

```
0.01609506469549467 0.00607778501208891 0.0023344573891806507 0
```

```
x1: 230.1
x2: 37.8
x1: 69.2
y: 22.1
y_hat: 4.0947591112215855

dl_dw1: -8286.011857015827
dl_dw2: -1361.1962111916482
dl_dw3: -2491.925339006933
dl_db: -36.01048177755683

w1: 0.09895518326565295
w2: 0.019689747124005393
w3: 0.027253710779249984
b: 0.0003601048177755684
```

```python
for epoch in range(epoch_max):
    for i in range(N):
        # get a sample
        x1 = tv_data[i]
        x2 = radio_data[i]
        x3 = newspaper_data[i]
        y  = sales_data[i]

        # compute output
        y_hat = predict(x1, x2, x3, w1, w2, w3, b)

        # compute gradient w1, w2, w3, b
        dl_dw1 = compute_gradient_wi(x1, y, y_hat)
        dl_dw2 = compute_gradient_wi(x2, y, y_hat)
        dl_dw3 = compute_gradient_wi(x3, y, y_hat)
        dl_db  = compute_gradient_b(y, y_hat)

        # update parameters
        w1 = update_weight_wi(w1, dl_dw1, lr)
        w2 = update_weight_wi(w2, dl_dw2, lr)
        w3 = update_weight_wi(w3, dl_dw3, lr)
        b  = update_weight_b(b, dl_db, lr)
```

Normalized data | $\eta = 10^{-2}$

Features			**Label**
TV	**Radio**	**Newspaper**	**Sales**
230.1	37.8	69.2	22.1
44.5	39.3	45.1	10.4
17.2	45.9	69.3	12
151.5	41.3	58.5	16.5
180.8	10.8	58.4	17.9

```
0.01609506469549467 0.00607778501208891 0.0023344573891806507 0
```

```
x1: 0.5504267881241568
x2: -0.09835863697705782
x1: 0.007579284750337614
y: 22.1
y_hat: 0.008279045632653116

dl_dw1: -24.319750018095103
dl_dw2: 4.345823123098159
dl_dw3: -0.33487888747630074
dl_db: -44.1834419087347

w1: 0.2592925648764457
w2: -0.037380446218892686
w3: 0.005683246263943658
b: 0.441834419087347
```

$$x = \frac{x - x_{mean}}{x_{max} - x_{min}}$$

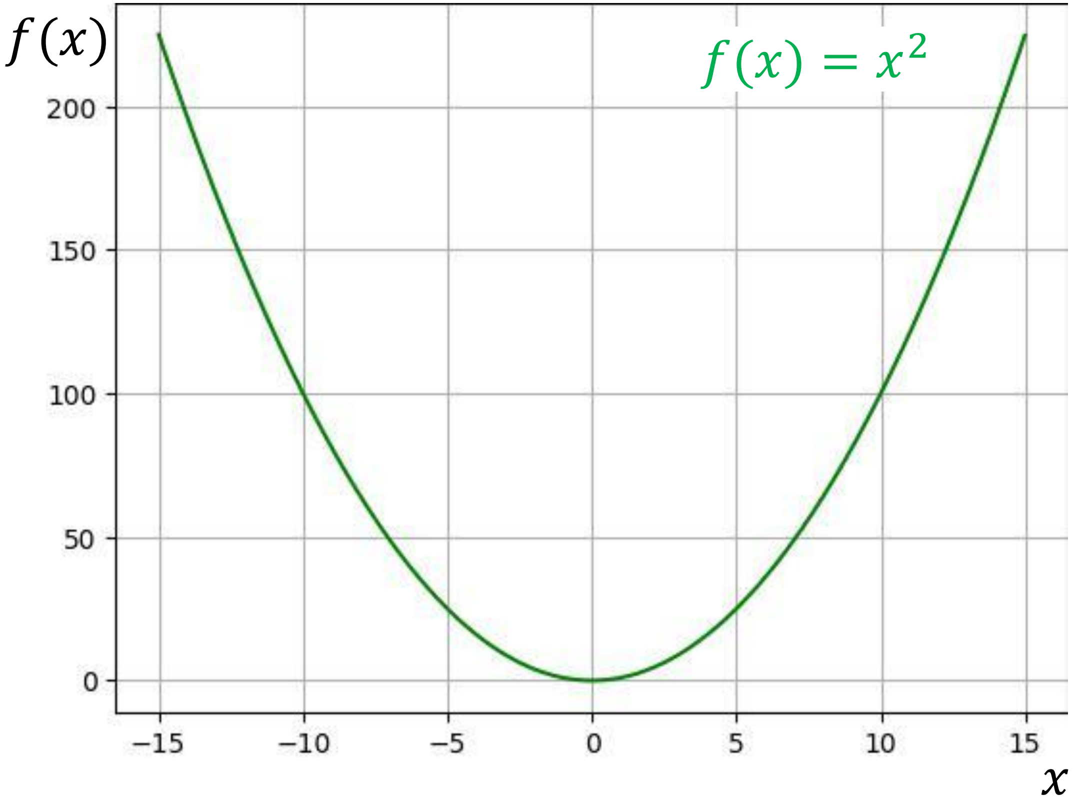
```python
for epoch in range(epoch_max):
    for i in range(N):
        # get a sample
        x1 = tv_data[i]
        x2 = radio_data[i]
        x3 = newspaper_data[i]
        y  = sales_data[i]

        # compute output
        y_hat = predict(x1, x2, x3, w1, w2, w3, b)

        # compute gradient w1, w2, w3, b
        dl_dw1 = compute_gradient_wi(x1, y, y_hat)
        dl_dw2 = compute_gradient_wi(x2, y, y_hat)
        dl_dw3 = compute_gradient_wi(x3, y, y_hat)
        dl_db  = compute_gradient_b(y, y_hat)

        # update parameters
        w1 = update_weight_wi(w1, dl_dw1, lr)
        w2 = update_weight_wi(w2, dl_dw2, lr)
        w3 = update_weight_wi(w3, dl_dw3, lr)
        b  = update_weight_b(b, dl_db, lr)
```

Outline

- Simple version of Linear Regression
- Computational Graph
- Mini-batch Training
- Batch Training
- Generalization of Linear Regression
- Loss Functions

Loss Functions

❖ **Mean Squared Error (MSE)**



$$f'(x) = 2x$$

One sample

$$L(\hat{y}, y) = (\hat{y} - y)^2$$

N samples

$$L(\hat{\boldsymbol{y}}, \boldsymbol{y}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{y}_i - y_i)^2$$

I area	y price	$\hat{y}$ prediction	L error
6.7	9.1	5.5	12.9
4.6	5.9	3.9	4.41
3.5	4.6	3.1	2.25
5.5	6.7	4.6	4.41

Loss Functions

	Feature	Label
	area	price
	6.7	9.1
	4.6	5.9
	3.5	4.6
	5.5	6.7

❖ Mean Squared Error (MSE)

1) Pick a sample (x, y) from training data

2) Compute the output $\hat{y}$

$$\hat{y} = wx + b$$

3) Compute loss

$$L = (\hat{y} - y)^2$$

4) Compute derivative

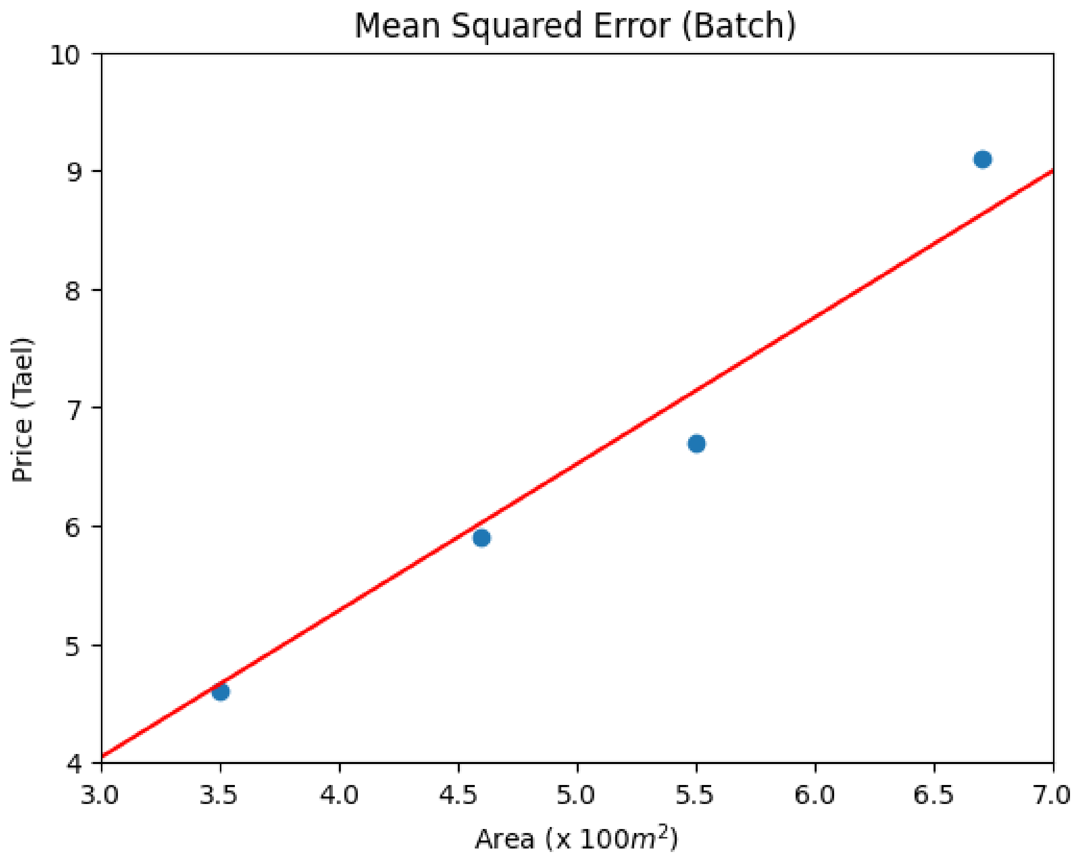
$$\frac{\partial L}{\partial w} = 2x(\hat{y} - y) \qquad \frac{\partial L}{\partial b} = 2(\hat{y} - y)$$

5) Update parameters

$$w = w - \eta \frac{\partial L}{\partial w} \qquad b = b - \eta \frac{\partial L}{\partial b}$$

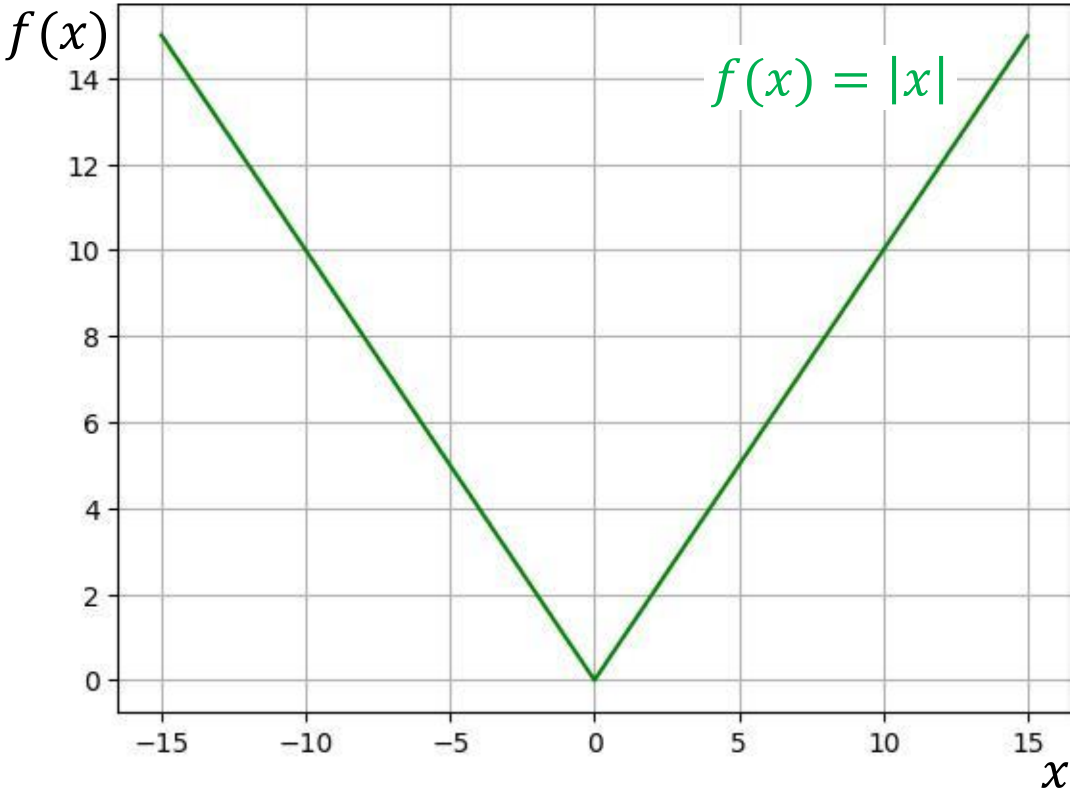
$w = 1.207$

$b = 0.251$



Loss Functions

❖ Mean Absolute Error (MAE)



$$f'(x) = \frac{x}{|x|} \quad \text{for } x \neq 0$$

One sample

$$L(\hat{y}, y) = |\hat{y} - y|$$

N samples

$$L(\hat{\boldsymbol{y}}, \boldsymbol{y}) = \frac{1}{N}\sum_{i=1}^{N}|\hat{y}_i - y_i|$$

I	y	$\hat{y}$	L
area	**price**	**prediction**	**error**
6.7	9.1	5.5	3.6
4.6	5.9	3.8	2.1
3.5	4.6	3.1	1.5
5.5	6.7	4.6	2.1

Loss Functions

	Feature	Label
	area	price
	6.7	9.1
	4.6	5.9
	3.5	4.6
	5.5	6.7

❖ Mean Absolute Error (MAE)

$w = 1.185$

$b = 0.340$

1) Pick a sample (x, y) from training data

2) Compute the output $\hat{y}$

$$\hat{y} = wx + b$$

3) Compute loss

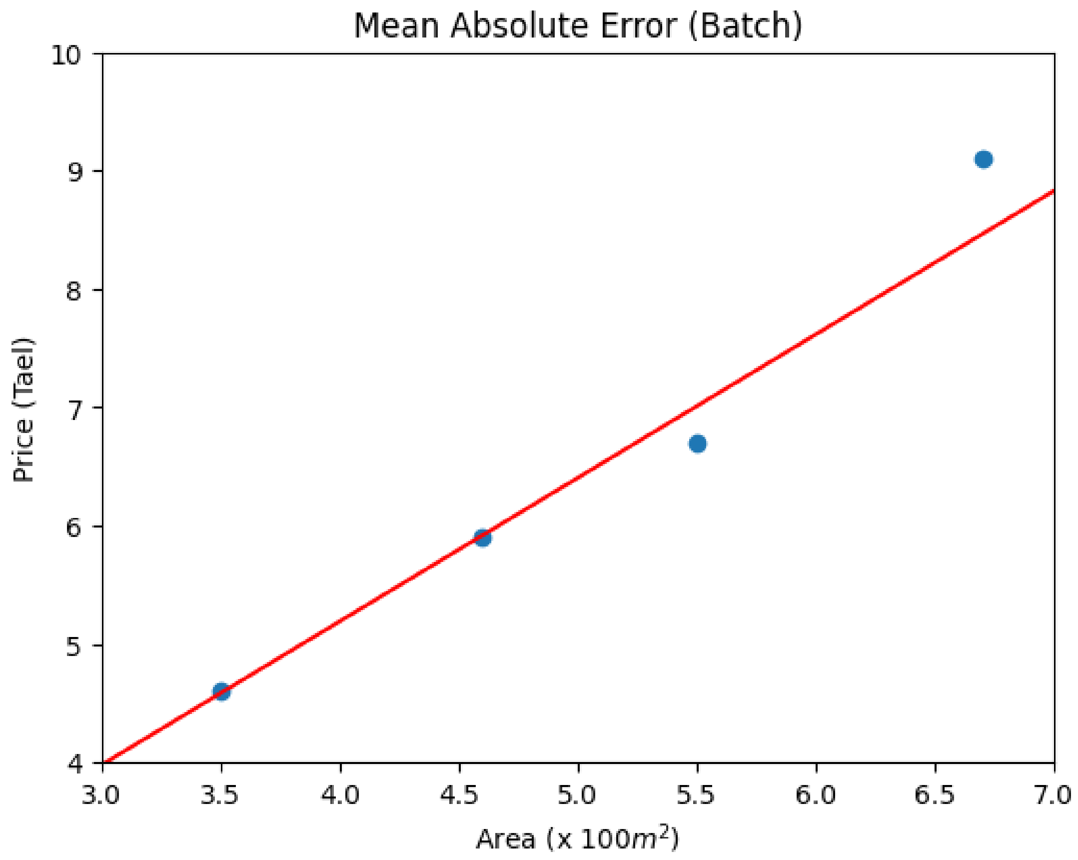
$$L = |\hat{y} - y|$$

4) Compute derivative

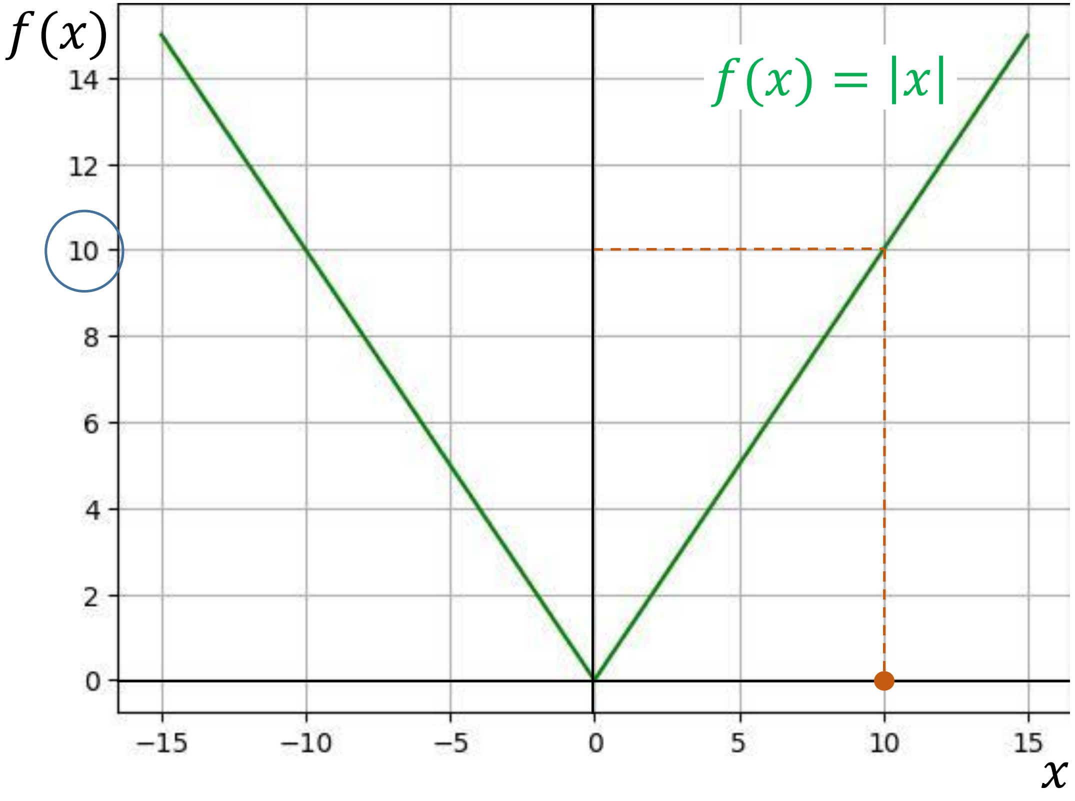
$$\frac{\partial L}{\partial w} = x\frac{(\hat{y} - y)}{|\hat{y} - y|} \qquad \frac{\partial L}{\partial b} = \frac{(\hat{y} - y)}{|\hat{y} - y|}$$

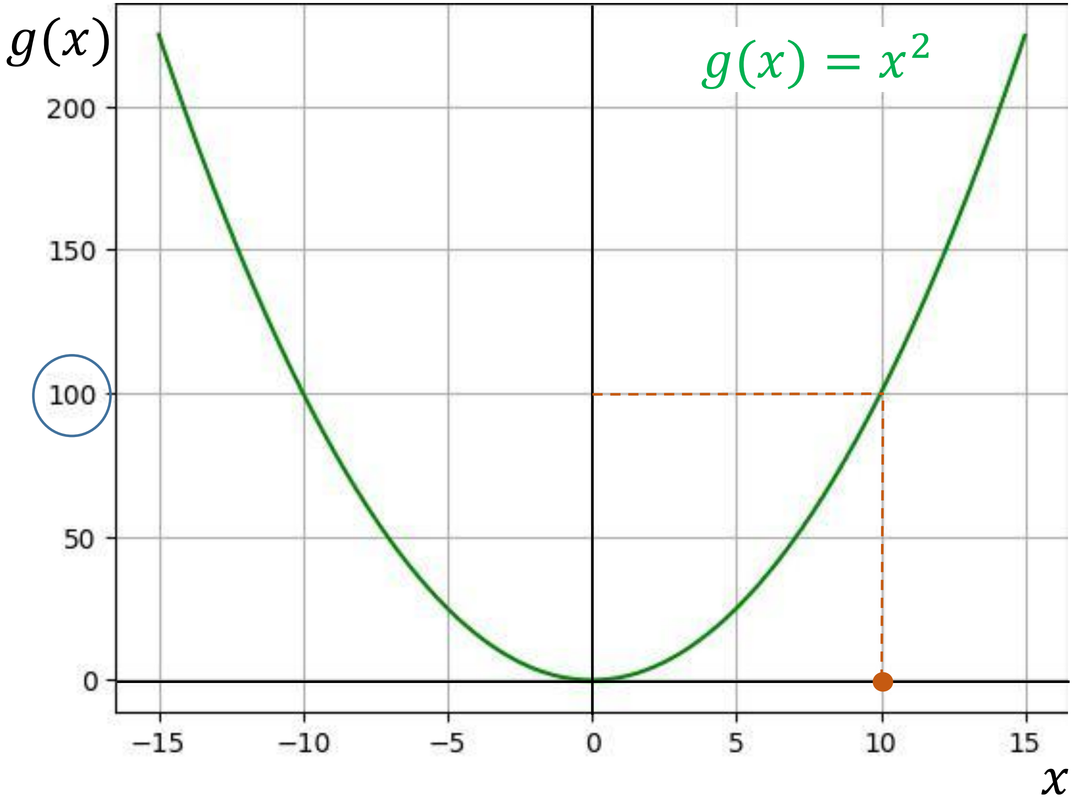
5) Update parameters

$$w = w - \eta\frac{\partial L}{\partial w} \qquad b = b - \eta\frac{\partial L}{\partial b}$$



Quiz 4: The pros and cons of MSE and MAE when data contain outliers?



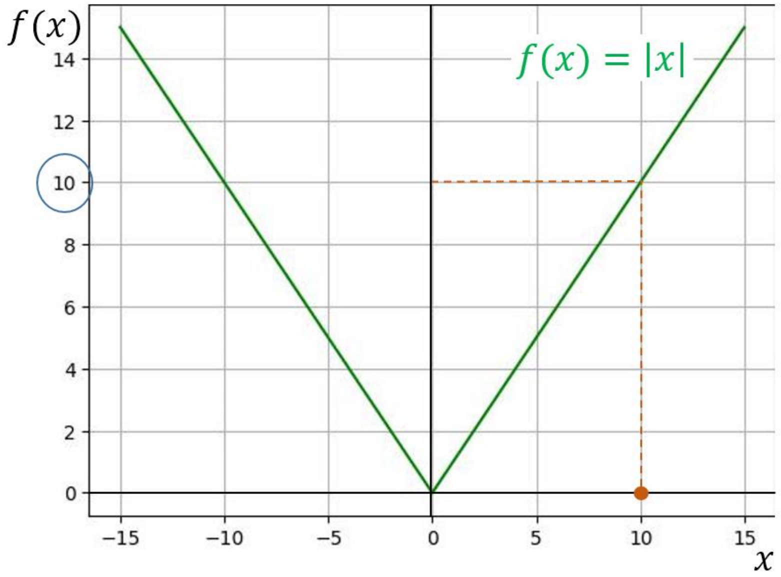


$g(10) \gg f(10)$

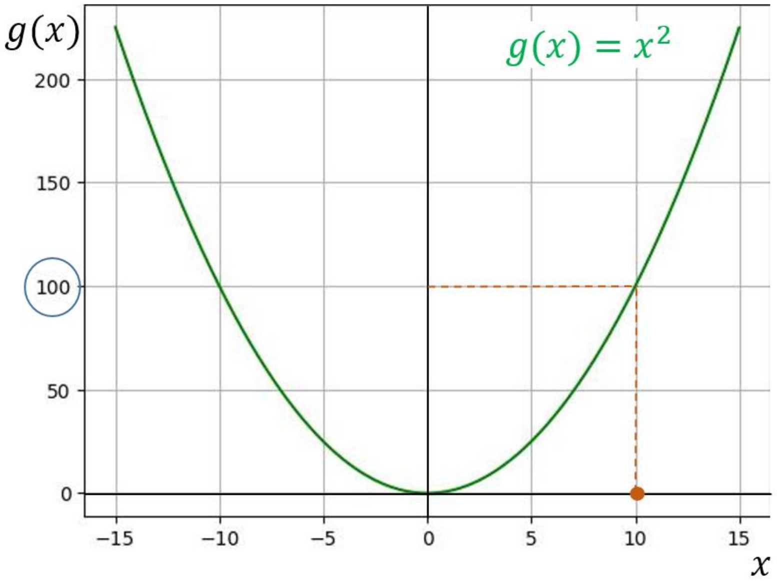
If $x = 10$ is an outlier, $g(10)$ has more negative effect

⇒ MAE is better to tolerate outliers

MAE

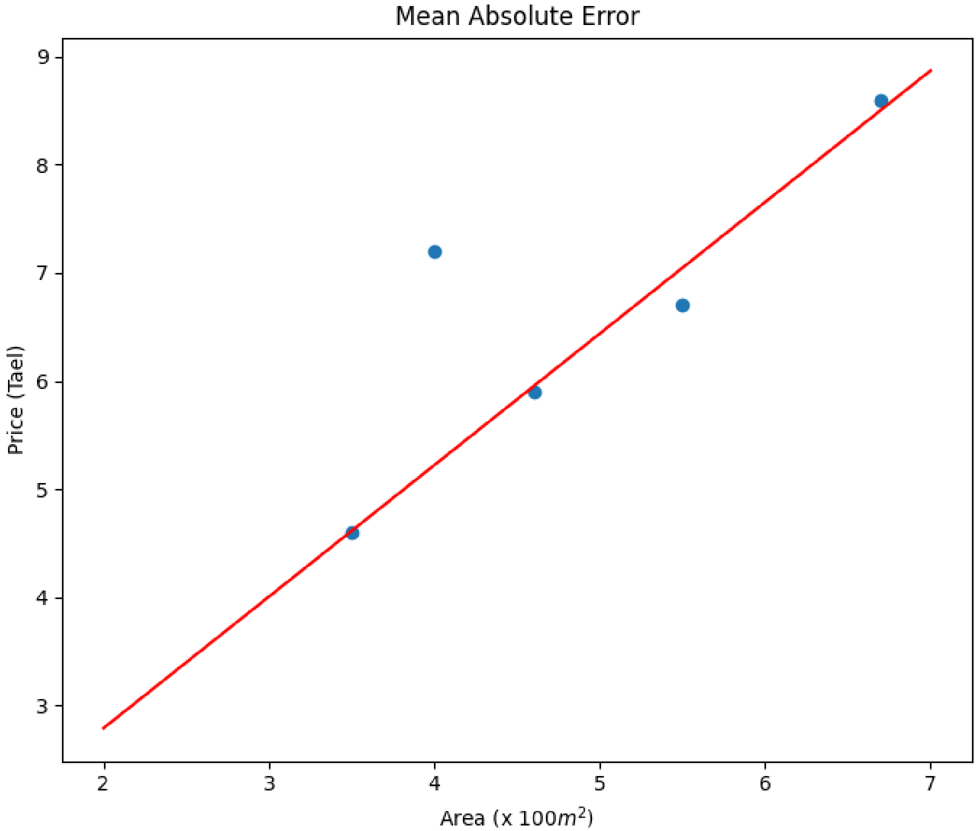


$f(x)$
$f(x) = |x|$
14
12
10
8
6
4
2
0
−15
−10
−5
0
5
10
15
x

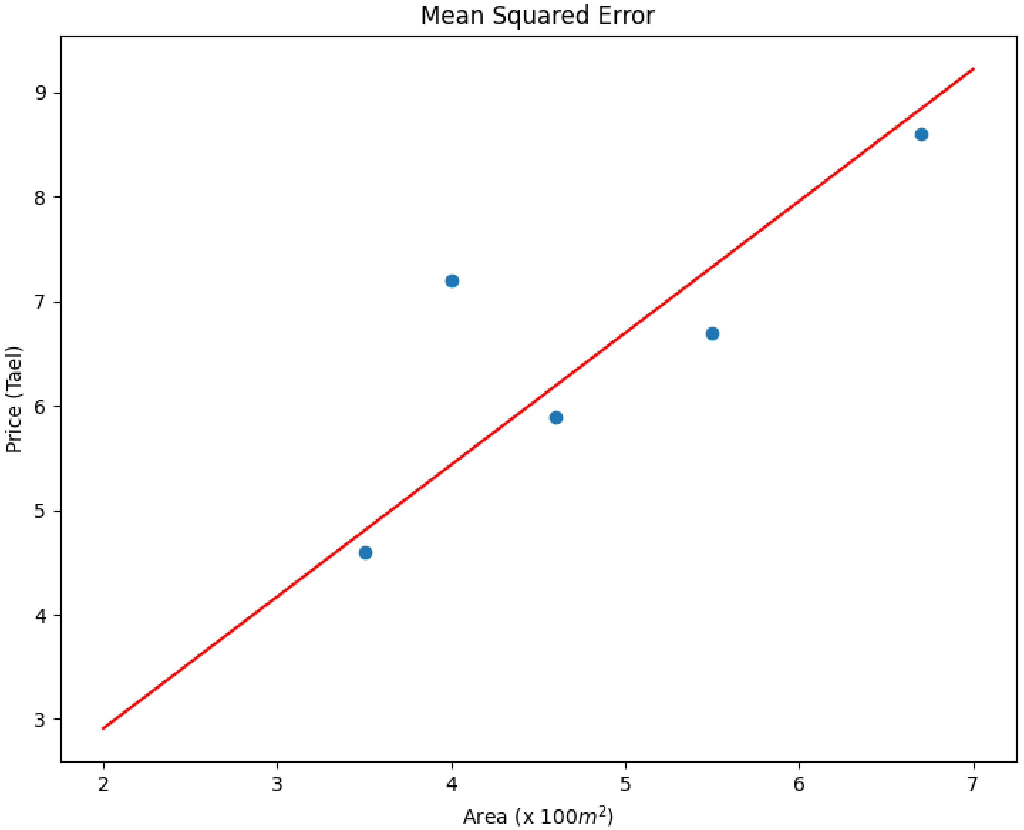


$g(x)$
$g(x) = x^2$
200
150
100
50
0
−15
−10
−5
0
5
10
15
x

MSE

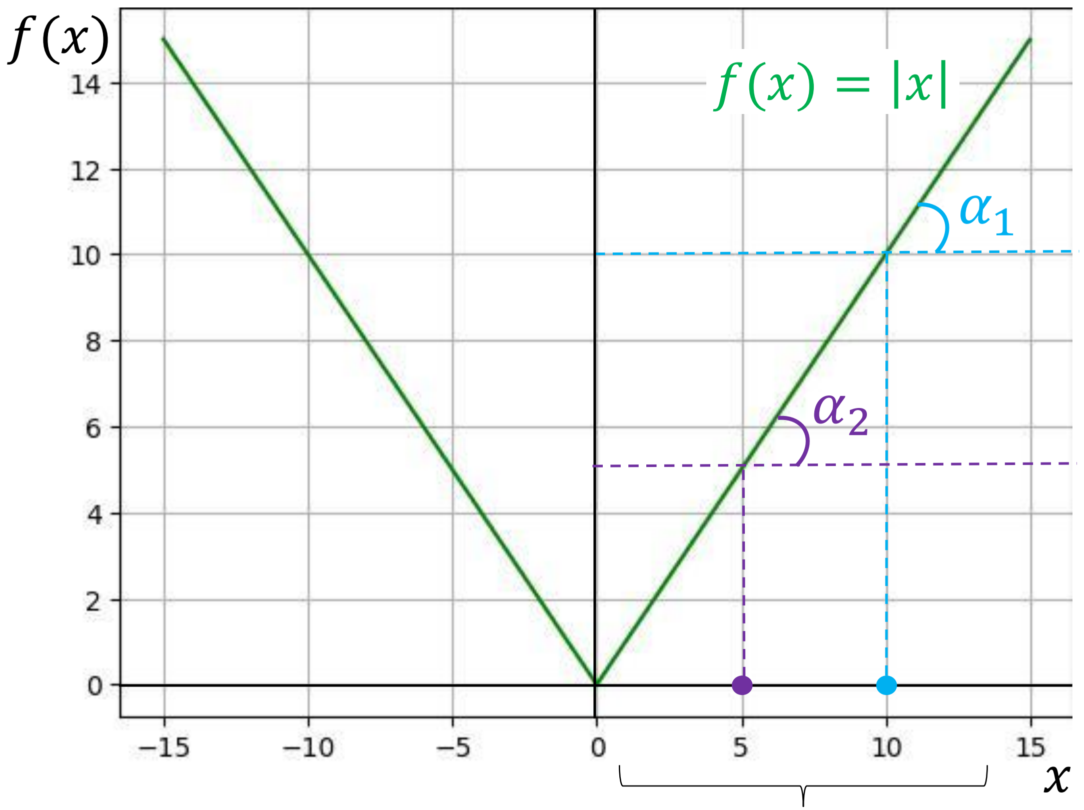


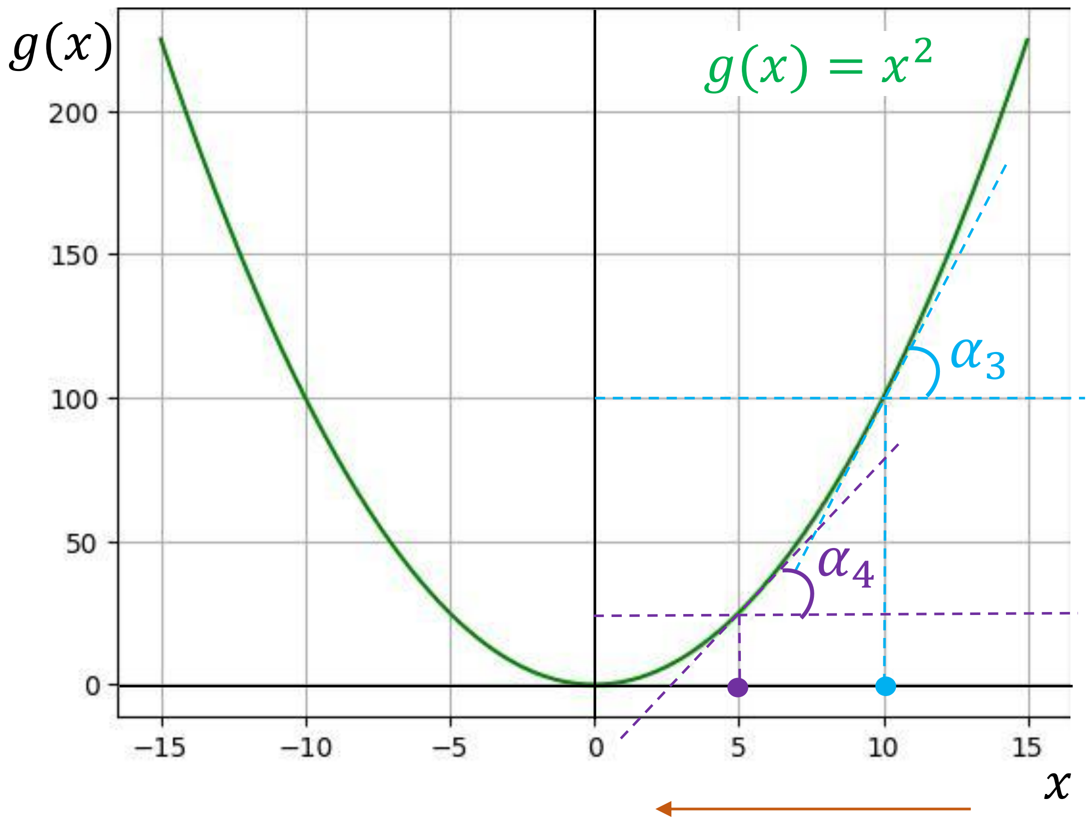
Mean Absolute Error
Price (Tael)
Area (x $100m^2$)
9
8
7
6
5
4
3
2
3
4
5
6
7



Mean Squared Error
Price (Tael)
Area (x $100m^2$)
9
8
7
6
5
4
3
2
3
4
5
6
7

Quiz 5: The pros and cons of MSE and MAE with a fixed learning rate η?





$\alpha_1 = \alpha_2$

$\alpha_3 > \alpha_4$

$\eta f'(x)$ values are constants

$\eta f'(x)$ values reduce

⇒ MSE is better when working with a fixed learning rate