
Methods of Applied Mathematics I

HW9

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1 EXERCISE9.1

1. *Proof.* Since

$$\begin{aligned}(LKu)(x) &= -\frac{d^2}{dx^2} \int_0^1 g(x, \xi) u(\xi) d\xi \\&= -\frac{d^2}{dx^2} \left(\int_0^x g(x, \xi) u(\xi) d\xi + \int_x^1 g(x, \xi) u(\xi) d\xi \right) \\&= -\frac{d^2}{dx^2} \left(\int_0^x \xi(1-x) u(\xi) d\xi + \int_x^1 x(1-\xi) u(\xi) d\xi \right) \\&= -\frac{d^2}{dx^2} \left((1-x) \int_0^x \xi u(\xi) d\xi + x \int_x^1 (1-\xi) u(\xi) d\xi \right) \\&= -\frac{d}{dx} \left(-\int_0^x \xi u(\xi) d\xi + \int_x^1 (1-\xi) u(\xi) d\xi \right) \\&= xu(x) + (1-x)u(x) = u(x)\end{aligned}\tag{1.1}$$

Thus, $LK = I$, which means $L = K^{-1}$. □

2. *Proof.* (\Rightarrow) Suppose $\frac{1}{\lambda}$ is an eigenvalue for K , then

$$ku = \frac{1}{\lambda} u\tag{1.2}$$

Thus

$$u = \lambda Ku \quad (1.3)$$

Take the operator L on it

$$Lu = \lambda LKu = \lambda Lu = \lambda u \quad (1.4)$$

which indicates that λ is an eigenvalue of L .

(\Leftarrow) Suppose $\lambda \neq 0$ is an eigenvalue of L , then

$$Lu = \lambda u \quad (1.5)$$

Thus

$$u = \frac{1}{\lambda} Lu \quad (1.6)$$

As $K = L^{-1}$ is known, take the operator K onto it yields

$$Ku = \frac{1}{\lambda} KLu = \frac{1}{\lambda} Lu = \frac{1}{\lambda} u \quad (1.7)$$

which indicates that $\frac{1}{\lambda}$ is an eigenvalue of K .

It's clear that for the given λ , K and L share the same eigenfunction $u(x)$. \square

3. *Proof.* It's obvious that L is bounded over M .

Since

$$\begin{aligned} \langle u, Lv \rangle &= - \int_0^1 u(x) v''(x) dx = - \left(u(x) v'(x) \Big|_0^1 - \int_0^1 u'(x) v'(x) dx \right) \\ &= \int_0^1 u'(x) v'(x) dx = \int_0^1 u'(x) dv(x) = u'(x) v(x) \Big|_0^1 - \int_0^1 v(x) u''(x) dx \\ &= - \int_0^1 u''(x) v(x) dx = \langle Lu, v \rangle \end{aligned} \quad (1.8)$$

which indicates that L is self-adjoint. Thus the eigenvalues of L are real numbers.

The eigenfunction $\psi_n(x)$ of L together with λ_n satisfy

$$\psi_n''(x) + \lambda_n \psi_n(x) = 0 \quad (1.9)$$

and

$$\psi_n(0) = \psi_n(1) = 0 \quad (1.10)$$

Solution of this ODE depends on the sign of λ_n .

a) If $\lambda_n > 0$, then

$$\psi_n(x) = C_1 \sin(\sqrt{\lambda_n} x) + C_2 \cos(\sqrt{\lambda_n} x) \quad (1.11)$$

apply the boundary condition yields

$$C_2 = 0 \quad (1.12)$$

and

$$C_1 \sin(\sqrt{\lambda_n}) = 0 \quad (1.13)$$

Thus

$$\lambda_n = (n\pi)^2 \quad (1.14)$$

and

$$\psi_n(x) = C_2 \sin(n\pi x) \quad (C_2 \neq 0) \quad (1.15)$$

b) If $\lambda_n = 0$, then

$$\psi_n(x) = C_0 + C_1 x \quad (1.16)$$

Apply the B.C. yields $C_0 = C_1 = 0$, which indicates 0 is not an eigenvalue for L .

c) If $\lambda_n < 0$, then

$$\psi_n(x) = C_1 e^{\sqrt{-\lambda_n}x} + C_2 e^{-\sqrt{-\lambda_n}x} \quad (1.17)$$

Apply the B.C. yields $C_0 = C_1 = 0$, which indicates $\lambda_n < 0$ are not eigenvalues for L .

□

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