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Introduction to Numerical Analysis
HW3

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1 CANTOR'S SET

1. *Proof.* As C_i is closed and C is the intersection of these closed sets, C is closed.
   It's clear that C is bounded. Thus, by Heine-Borel theorem, C is compact. □

2. *Proof.* Suppose $x \in C_m$, $y \in C_n$ and $x < y$. There are 2^m subsets in C_m and the length of each subset is $\frac{1}{3^m}$. Also, there are 2^n subsets in C_n and the length of each subset is $\frac{1}{3^n}$.
   As $C \subset C_m \cap C_n$, suppose $m \leq n$, there must exist a subset in C_n s.t. $x \in C_n$.
   If x and y lie in the same subset of C_n, with further division of the subset, there will be a gap between x and y, thus, there exists an element z lies in the gap and satisfies $x < z < y$.
   If x and y lie in different subsets of C_n, denoted as $C_{n,i}$ and $C_{n,j}$,it's obvious that such a z exists in the gap between $C_{n,i}$ and $C_{n,j}$ and satisfies $x < z < y$. □

3. a) 0
   b) *Proof.* As there are 2^n subsets in C_n and the length of each subset is $\frac{1}{3^n}$, thus the Lebesgue measure of C_n is the sum of the Lebesgue measure of each closed subset. Thus $\lambda(C_n) = (\frac{2}{3})^n$.
   As $C = \bigcap_{n=1}^{\infty} C_n$, then $\lambda(C) \leq \lambda(C_n) = (\frac{2}{3})^n$, thus $\lambda(C) = 0$. □

4. a) *Proof.* As the end points of each subset in C_n is not removed in any subdivision, thus C is not empty. □

b)

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \text{ with } a_i \in \{0, 2\} \tag{1.1}$$

c) Consider the $i-th$ digit in elemnet s_i, it will be possible to construct such an element that the $i-th$ digit is different from that in s_i, thus s is not included in the original list.

d) *Proof.* Suppose C is countable, express each $x \in C$ in ternary form, then each digit in x is either 0 or 2. Thus, the choice of each digit appears to be binary. Consider the $i-th$ digit in elemnet x_i, it will be possible to construct such an element t, whose $i-th$ digit is different from that in x_i(complementary). Thus t is not included in the original list, which implies the assumption fails. □

5. Althouth C is uncountable, but the measure of its complement is 1, which is illustrated below, thus the measure of C is 0.

$$\begin{aligned}\lambda(C^c) &= \frac{1}{3} + 2 * \frac{1}{9} + ... \\ &= \frac{1}{3}\sum_{i=0}^{\infty}(\frac{2}{3})^n \\ &= \frac{1}{3}\lim_{n\to\infty}\frac{1-(2/3)^n}{1-(2/3)} = 1\end{aligned} \tag{1.2}$$

2 Cantor's Function

1. *Proof.* It's easy to verify that f_0, f_1 are monotonically increasing continuous functions. Suppose f_n are still monotonically increasing continuous functions for $n = k$.
For $n = k+1$, as f_{k+1} is only different from f_k on each closed subset of C_k, whose length is $\frac{1}{3^n}$ and is denoted as $I_{k,p}(1 \le p \le 2^k)$ here, it is left to prove that f_{k+1} remains monotonically increasing continuous on each $I_{k,p}$ after the construction process.
Since f_k is linear on each $I_{k,p}$, and the recursive construction process doesn't change the values on each ending points, on which the values of f_k is denoted as a, b recursively, the subset of $I_{k,p}$ is valid and the value of f_{k+1} on the central part, whose length is $\frac{1}{3^{k+1}}$, is $\frac{a+b}{2}$. As the measure of the remaining parts of $I_{k,p}$ is not 0, the linear function connecting each ending points is obviously valid. Thus, the new function f_{k+1} is still monotonically increasing continuous. □

2. *Proof.* Let $g_n(x) = |f_n(x) - f(x)|$, then $g_n(x)$ is positive on C_n and 0 on other places. Further, $g_n(x)$ holds the same value on each closed subset of C_n.
Given any $\xi > 0$, it is left to find N s.t. $g_n(x) < \xi$ when $n > N$.
As $g_n(x)$ reaches its extream $\frac{1}{6}\frac{1}{3^n}$ at the $\frac{1}{3}$ and $\frac{2}{3}$ of each compact subset of C_n, thus, given any ξ

$$\frac{1}{6}\frac{1}{3^n} < \xi \tag{2.1}$$

which is equivalent to

$$n > \frac{ln(\frac{1}{6\xi})}{ln(3)} \triangleq N \tag{2.2}$$

and the target N is given as above. □

3. a) *Proof.* As $f_c(x)$ is continous over $[0,1]$, it is left to prove that a continous function on a compact is uniformly continous, which can be proved by contradiction. (the proof haven't figured out yet...) □

   b) *Proof.* For any $x \in (0,1] \cap C$, $f_c(x)$ can be expressed as below

   $$f_c(x) = sup\{f_c(y) | y < x, y \in [0,1] \backslash C\} \tag{2.3}$$

   Thus $f_c(x)$ is monotonically increasing. □

   c) *Proof.* As $f_c(x)$ is constant on each segment dropped by C, thus $f_c'(x)$ is almost 0 everywhere. □

4. *Proof.* Assume $f_c(x)$ is absolutely continous, then its Riemann integral exists, which can be expressed as

   $$\int_0^1 f_c'(x)dx = f_c(1) - f_c(0) \tag{2.4}$$

   However, the LHS of (2.4) is 0 as $f_c'(x)$ is 0 almost everywhere, while the RHS of (2.4) is 1. Thus its riemann integral doesn't exist, and therefore the assumption fails.

   □

3 Taylor's theorem

1. *Proof.* For $n = 0$, apply the fundamental theorem of calculus to $f(x)$ and its derivatives, $f(x)$ can be written as below. It's clear that the theorem is valid in such case.

   $$f(x) = f(a) + \int_a^x f'(t)dt \tag{3.1}$$

   Suppose the theorem is still valid for $n = k$, then $f(x)$ can be written as

   $$f(x) = \sum_{i=0}^{k} \frac{f^i(a)}{i!}(x-a)^i + \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)dt \tag{3.2}$$

For $n = k+1$, as $f^{(k+1)}$ is absolutely continous, apply the fundamental theorem of calculus to $f^{(k+1)}$, the second part of RHS of $f(x)$ can be written as below

$$\begin{aligned} RHS_2 &= \frac{1}{k!}\int_a^x (x-t)^k [f^{(k+1)}(a) + \int_a^t f^{(k+2)}(u)du]dt \\ &= \frac{f^{(k+1)}(a)}{k!}\int_a^x (x-t)^k dt + \frac{1}{k!}\int_a^x (x-t)^k \int_a^t f^{(k+2)}(u)dudt \\ &= \frac{f^{k+1}(a)}{(k+1)!}(x-a)^{k+1} + \frac{1}{k!}\int_a^x (x-t)^k \int_a^t f^{(k+2)}(u)dudt \end{aligned} \tag{3.3}$$

Let

$$g(t) = \int_a^t f^{(k+2)}(u)du \tag{3.4}$$

RHS_2 can be further expressed as

$$\begin{aligned} RHS_2 &= \frac{f^{k+1}(a)}{(k+1)!}(x-a)^{k+1} + \frac{1}{k!}\int_a^x (x-t)^k g(t)dt \\ &= \frac{f^{k+1}(a)}{(k+1)!}(x-a)^{k+1} - \frac{1}{(k+1)!}\int_a^x g(t)d(x-t)^{k+1} \\ &= \frac{f^{k+1}(a)}{(k+1)!}(x-a)^{k+1} - \frac{1}{(k+1)!}[(x-t)^{k+1}g(t)\Big|_a^x - \int_a^x (x-t)^{k+1}g'(t)dt] \end{aligned} \tag{3.5}$$

As $g(a) = 0$ and $g'(t) = f^{(k+2)}(t)$, RHS_2 can be simplified as

$$RHS_2 = \frac{f^{k+1}(a)}{(k+1)!}(x-a)^{k+1} + \frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)dt \tag{3.6}$$

Substitude (3.6) into (3.2), $f(x)$ can be expressed as below when $n = k+1$

$$\begin{aligned} f(x) &= \sum_{i=0}^{k+1} \frac{f^i(a)}{i!}(x-a)^i + \frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)dt \\ &= \sum_{i=0}^{n} \frac{f^i(a)}{i!}(x-a)^i + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt \end{aligned} \tag{3.7}$$

Thus the theorem is still valid when n is extended to $k+1$. □

2. *Proof.* It is left to prove that there exists $c \in [a, x]$ s.t.

$$\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{3.8}$$

which is equal to

$$(n+1)\int_a^x (x-t)^n f^{(n+1)}(t)dt = f^{(n+1)}(c)(x-a)^{n+1} \tag{3.9}$$

Denote $m = min(f^{(n+1)}(x))$ and $M = max(f^{(n+1)}(x))$, apply the first medium value theorem, there exists $c \in (m, M)$ s.t.

$$\int_a^x (x-t)^n f^{(n+1)}(t)dt = f^{(n+1)}(c)\int_a^x (x-t)^n dt \tag{3.10}$$

Thus, (3.9) is verified. □

4 Convergence of rationals to irrationals

1. *Proof.* As e can be written as
$$e = \sum_{i=0}^{\infty} \frac{1}{i!} \tag{4.1}$$
Suppose it is rational, then it can be written as below, where p, q are prime to each other.
$$e = \frac{p}{q} \tag{4.2}$$
Thus
$$p(q-1)! = q! \sum_{i=0}^{q} \frac{1}{i!} + q! \sum_{i=q+1}^{\infty} \frac{1}{i!} \tag{4.3}$$
Since both LHS and the first part of RHS are integers, the second part of RHS(denoted as RHS_2) should be an integer as well.// RHS_2 can be further expanded as below
$$\begin{aligned} RHS_2 &= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \ldots \\ &< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \ldots \\ &= \frac{1}{q} < 1 \end{aligned} \tag{4.4}$$
Therefore RHS_2 is not an integer as $RHS_2 > 0$. Thus the assumption fails, and e is irrational. □

2. *Proof.* It's clear that u_n is increasing, and the limit is e, so the maximum distance between 2 element can be denoted as
$$d_{max} < e - (1 + \frac{1}{n})^n \tag{4.5}$$
For any $\xi > 0$, let $d_{max} < \xi$, it is satisfied when $n > N$, where N is the index of the first element s.t. $u_N > e - \xi$.
Thus u_n is a cauchy sequence converging to e. □

3. No, as the limit of a rational sequence may not converging to a rational.