SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Methods of Applied Mathematics I
HW5

Yu Cang 018370210001
Zhiming Cui 017370910006

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1 EXERCISE5.1

Proof. Suppose $\dim(U) = N$ and $\dim(V) = M$. Since all the norm are equivalent in finite-dimension spaces, the linear operator L can be represented in matrix form as $\{l_{ji}\}$. Then

$$\begin{aligned} ||Lu|| &= \sqrt{\sum_{j=1}^{M}\Big|\sum_{i=1}^{N} l_{ji}u_i\Big|^2} \\ &\le \sqrt{\sum_{j=1}^{M}\Big[\sum_{i=1}^{N} l_{ji}^2 \sum_{i=1}^{N} u_i^2\Big]} \quad \text{(Cauchy-Schwartz)} \\ &= \sqrt{\sum_{j=1}^{M} ||u||^2 \sum_{i=1}^{N} l_{ji}^2} = ||u||\sqrt{\sum_{j=1}^{M}\sum_{i=1}^{N} l_{ji}^2} \end{aligned} \tag{1.1}$$

Hence, L is bounded. □

2 EXERCISE5.2

Proof. Denote $w = \alpha u + \beta v$, then

$$(Tw)(x) = xw(x) = x(\alpha u(x) + \beta v(x)) = \alpha x u(x) + \beta x v(x) = \alpha(Tu)(x) + \beta(Tv)(x) \tag{2.1}$$

Hence, T is linear.
Further

$$||T|| = \sup_{u\in\mathbb{C}[0,1]} \frac{||Tu||}{||u||} = \sup_{u\in\mathbb{C}[0,1]} \frac{\sup_{x\in[0,1]}|xu(x)|}{\sup_{x\in[0,1]}|u(x)|} = \sup_{u\in\mathbb{C}[0,1]} \frac{\sup_{x\in[0,1]}|u(x)|}{\sup_{x\in[0,1]}|u(x)|} = 1 \tag{2.2}$$

□

3 EXERCISE5.3

1. *Proof.* Firstly, it's obvious that $||u|| \geq 0$. Meanwhile, $||u|| \geq \sup_{x\in[a,b]}|u(x)| > 0$ when $u(x) \neq 0$. Hence $||u|| = 0$ iff. $u(x) = 0$.
   Secondly, the linearity is valid as

   $$||\alpha \cdot u|| = \sup_{x\in[a,b]}|\alpha u(x)| + \sup_{x\in[a,b]}|\alpha u'(x)| = |\alpha|(\sup_{x\in[a,b]}|u(x)| + \sup_{x\in[a,b]}|u'(x)|) = |\alpha|||u|| \tag{3.1}$$

   Finally, the triangle inequality is justified as

   $$\begin{aligned} ||u+v|| &= \sup_{x\in[a,b]}|u(x)+v(x)| + \sup_{x\in[a,b]}|u'(x)+v'(x)| \\ &\leq \sup_{x\in[a,b]}|u(x)| + \sup_{x\in[a,b]}|u'(x)| + \sup_{x\in[a,b]}|v(x)| + \sup_{x\in[a,b]}|v'(x)| \\ &= ||u|| + ||v|| \end{aligned} \tag{3.2}$$

   Thus, $||u||$ defines a norm. □

2. For example, let $u(x) = x^n$, then

   $$||T|| = \sup_{u\in\mathbb{C}^1[0,1]} \frac{||Tu||}{||u||} = \sup_{u\in\mathbb{C}^1[0,1]} \frac{\sup_{x\in[0,1]}|u'(x)|}{\sup_{x\in[0,1]}|u(x)|} = n \tag{3.3}$$

   which indicates that T is not bounded, hence T is not continuous.