SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Introduction to Numerical Analysis
HW4

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1 LEGENDRE POLYNOMIALS

1. *Proof.* Let

$$\varphi(x) = (x^2 - 1)^n \tag{1.1}$$

then

$$Q_n(x) = \frac{1}{2^n n!}\varphi^{(n)}(x) \tag{1.2}$$

and

$$\varphi^{(k)}(1) = \varphi^{(k)}(-1) = 0 \quad (k = 0, 1, ..., n-1) \tag{1.3}$$

Suppose $h(x) \in C^n(-1,1)$, then performing integration by parts

$$\begin{aligned}\int_{-1}^{1} P_n(x)h(x)dx &= \frac{1}{2^n n!}\int_{-1}^{1}\varphi^{(n)}(x)h(x)dx \\ &= -\frac{1}{2^n n!}\int_{-1}^{1}\varphi^{(n-1)}(x)h'(x)dx \\ &= ... \\ &= \frac{(-1)^n}{2^n n!}\int_{-1}^{1}\varphi(x)h^{(n)}(x)dx\end{aligned} \tag{1.4}$$

Thus, the proof can be discussed on 2 cases

a) If the order of $g(x)$ is less than n, then

$$g^{(n)}(x) = 0 \tag{1.5}$$

Thus

$$\int_{-1}^{1} Q_n(x)Q_m(x)dx = 0 \quad (n \neq m) \tag{1.6}$$

b) If $g(x) = Q_n(x)$, then the $n-th$ derivative of $g(x)$ is

$$g^{(n)}(x) = Q^{(n)}(x) = \frac{(2n)!}{2^n n!} \tag{1.7}$$

Thus

$$\begin{aligned}\int_{-1}^{1} Q_n(x)Q_m(x)dx &= \int_{-1}^{1} Q_n^2(x)dx \\ &= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n dx \\ &= \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n dx \\ &= \frac{(2n)!}{2^{2n}(n!)^2}\int_{0}^{\pi/2} cos^{2n+1} t dt \\ &= \frac{(2n)!}{2^{2n}(n!)^2}\frac{2\times 4\times \ldots \times (2n)}{1\times 3\times \ldots \times (2n+1)} \\ &= \frac{2}{2n+1} \quad (n = m)\end{aligned} \tag{1.8}$$

Thus, $(Q_n)_{n\in\mathbb{N}}$ are a sequence of orthogonal polynomials. □

2. *Proof.* Denote

$$\varphi(x) = (x^2-1)^n \tag{1.9}$$

then

$$Q_n(x) = \frac{1}{2^n n!}\varphi^{(n)}(x) \tag{1.10}$$

As the power of each item in $\varphi(x)$ is even when $\varphi(x)$ is extended, thus $\varphi^{(n)}(x)$ is even function if the order of derivative is even, and $\varphi^{(n)}(x)$ is odd function if the order of derivative is odd.
Therefore $Q_n(x)$ is even function if n is even, and $Q_n(x)$ is odd function if n is odd. So, it can be summarized as $Q_n(-x) = (-1)^n Q_n(x)$. □

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2 Interpolation

$f(2)$ can be determined using the Lagrange interpolation scheme. As the lagrange interpolation polynomial can be written as below, and $n = 8$ in this case.

$$f(x) = \sum_{i=1}^{n} f(x_i) l_i(x) \tag{2.1}$$

$l_i(x)$ are the base functions that can be written as below.

$$l_i(x) = \frac{(x - x_1)(x - x_2)...(x - x_{i-1})(x - x_{i+1})...(x - x_{n-1})(x - x_n)}{(x_i - x_1)(x_i - x_2)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_{n-1})(x_i - x_n)} \tag{2.2}$$

$l_i(2)$ are calculated accordingly as below.

$$\begin{array}{llll} l_1(2) = -0.0006 & l_2(2) = 0.1224 & l_3(2) = -0.5600 & l_4(2) = 1.0606 \\ l_5(2) = 0.4167 & l_6(2) = -0.0400 & l_7(2) = 0.0012 & l_8(2) = -0.0003 \end{array}$$

Thus, $f(2)$ is calculated according to (2.1) as 11.0.

3 Newton's form of interpolation polynomial

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