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Turbulence
HW3

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1 EXERCISE1

When the gravity is considered, the N-S equation is written as

$$\rho\frac{\partial u_j}{\partial t}+\rho u_k\frac{\partial u_j}{\partial x_k}=-\frac{\partial p}{\partial x_j}+\mu\frac{\partial^2 u_j}{\partial x_k\partial x_k}+\rho g_j \tag{1.1}$$

Decompose the density into mean and fluctuation as $\rho=\rho_0+\tilde{\rho}$, and substitute into the equation above yields

$$\rho_0\frac{\partial u_j}{\partial t}+\tilde{\rho}\frac{\partial u_j}{\partial t}+\rho_0 u_k\frac{\partial u_j}{\partial x_k}+\tilde{\rho}\frac{\partial u_j}{\partial x_k}=-\frac{\partial p}{\partial x_j}+\mu\frac{\partial^2 u_j}{\partial x_k\partial x_k}+\rho_0 g_j+\tilde{\rho}g_j \tag{1.2}$$

The 2nd and 4th term in the LHS of (1.2) can be neglected as $\tilde{\rho}<<\rho_0$. Assuming $\frac{\mu}{\rho_0}\approxeq\frac{\mu}{\rho}=\nu$ yields

$$\frac{\partial u_j}{\partial t}+u_k\frac{\partial u_j}{\partial x_k}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x_j}+\nu\frac{\partial^2 u_j}{\partial x_k\partial x_k}+g_j+g_j\frac{\tilde{\rho}}{\rho_0} \tag{1.3}$$

When the EOS for ideal gas($p=\rho RT$) is adopted, ρT can be assumed to be constant when the change of p is negligible and velocity is small. Hence

$$(\rho_0+\tilde{\rho})(\bar{T}+\tilde{T})=\rho_0\bar{T} \tag{1.4}$$

Thus

$$\frac{\tilde{\rho}}{\rho_0}+\frac{\tilde{T}}{\bar{T}}=0 \tag{1.5}$$

This is achieved by neglecting the 2nd-order small quantities.

Substitute into (1.3) yields

$$\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_k \partial x_k} + g_j - g_j \frac{\tilde{T}}{\bar{T}} \tag{1.6}$$

Decompose the velocity and pressure into mean and fluctuation as

$$\begin{aligned} u_j &= U_j + u'_j \\ p &= P + p' \end{aligned} \tag{1.7}$$

Substitute into (1.6)

$$\frac{\partial (U_j + u'_j)}{\partial t} + (U_k + u'_k)\frac{\partial (U_j + u'_j)}{\partial x_k} = -\frac{1}{\rho_0}\frac{\partial (P + p')}{\partial x_j} + \nu \frac{\partial^2 (U_j + u'_j)}{\partial x_k \partial x_k} + g_j - g_j \frac{\tilde{T}}{\bar{T}} \tag{1.8}$$

Multiply u'_j and take ensemble average yields

$$LHS \triangleq \overline{u'_j \frac{\partial u'_j}{\partial t}} + U_k \overline{u'_j \frac{\partial u'_j}{\partial x_k}} + \overline{u'_j u'_k}\frac{\partial U_j}{\partial x_k} + \overline{u'_j u'_k \frac{\partial u'_j}{\partial x_k}} = -\frac{1}{\rho_0}\overline{u'_j \frac{\partial p'}{\partial x_j}} + \nu \overline{u'_j \frac{\partial^2 u'_j}{\partial x_k \partial x_k}} + \frac{g_j}{\bar{T}}\overline{u'_j \tilde{T}} \triangleq RHS \tag{1.9}$$

Denote $k_T = \frac{1}{2}\overline{u'_j u'_j}$ and follow from the continuity equation

$$LHS = \frac{\bar{D} k_T}{\bar{D} t} + \overline{u'_j u'_k}\frac{\partial U_j}{\partial x_k} + \frac{1}{2}\frac{\partial}{\partial x_k}\overline{u'_k u'_j u'_j} \tag{1.10}$$

where

$$\frac{\bar{D}}{\bar{D} t} = \frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k} \tag{1.11}$$

Since

$$\overline{u'_j \frac{\partial^2 u'_j}{\partial x_k \partial x_k}} = \overline{\frac{\partial}{\partial x_k}\Big(u'_j \frac{\partial u'_j}{\partial x_k}\Big)} - \overline{\frac{\partial u'_j}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} = \frac{\partial^2 k_T}{\partial x_k \partial x_k} - \overline{\frac{\partial u'_j}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} \tag{1.12}$$

then, also follows from the continuity equation

$$RHS = -\frac{1}{\rho_0}\frac{\partial \overline{u'_j p'}}{\partial x_j} + \nu \Big(\frac{\partial^2 k_T}{\partial x_k \partial x_k} - \overline{\frac{\partial u'_j}{\partial x_k}\frac{\partial u'_j}{\partial x_k}}\Big) + \frac{g_j}{\bar{T}}\overline{u'_j \tilde{T}} \tag{1.13}$$

Combining (1.10) and (1.13) yields the turbulent kinetic energy equation for the buoyancy case

$$\frac{\bar{D} k_T}{\bar{D} t} = -\overline{u'_j u'_k}\frac{\partial U_j}{\partial x_k} - \frac{1}{2}\frac{\partial}{\partial x_k}\overline{u'_k u'_j u'_j} - \frac{1}{\rho_0}\frac{\partial \overline{u'_j p'}}{\partial x_j} + \nu \Big(\frac{\partial^2 k_T}{\partial x_k \partial x_k} - \overline{\frac{\partial u'_j}{\partial x_k}\frac{\partial u'_j}{\partial x_k}}\Big) + \frac{g_j}{\bar{T}}\overline{u'_j \tilde{T}} \tag{1.14}$$

2 EXERCISE 2

The governing equation for the passive scalar ϕ is

$$\frac{D\phi}{Dt} = \Gamma\nabla^2\phi \tag{2.1}$$

The gradient of (2.1) is

$$\frac{\partial\nabla\phi}{\partial t} + \nabla(\vec{U}\cdot\nabla\phi) = \Gamma\nabla^2(\nabla\phi) \tag{2.2}$$

An vector operation identity is used

$$\nabla(\vec{f}\cdot\vec{g}) = (\vec{g}\cdot\nabla)\vec{f} + \vec{g}\times(\nabla\times\vec{f}) + (\vec{f}\cdot\nabla)\vec{g} + \vec{f}\times(\nabla\times\vec{g}) \tag{2.3}$$

Thus

$$\begin{aligned}\nabla(\vec{U}\cdot\nabla\phi) &= (\nabla\phi\cdot\vec{U})\vec{U} + \nabla\phi\times(\nabla\times\vec{U}) + (\vec{U}\cdot\nabla)\nabla\phi + \vec{U}\times(\nabla\times\nabla\phi)\\ &= (\nabla\phi\cdot\vec{U})\vec{U} + \nabla\phi\times(\nabla\times\vec{U}) + (\vec{U}\cdot\nabla)\nabla\phi\end{aligned} \tag{2.4}$$

Hence, the governing equation for $\nabla\phi$ is

$$\frac{D}{Dt}\nabla\phi = -(\vec{U}\cdot\nabla\phi)\vec{U} - \nabla\phi\times(\nabla\times\vec{U}) + \Gamma\nabla^2(\nabla\phi) \tag{2.5}$$

Denote $\vec{W} = \nabla\phi$, then (2.5) can be re-written as

$$\frac{D}{Dt}\vec{W} = -(\vec{U}\cdot\vec{W})\vec{U} - \vec{W}\times(\nabla\times\vec{U}) + \Gamma\nabla^2\vec{W} \triangleq RHS \tag{2.6}$$

Inner product with $\vec{W}$, the 2nd term in RHS vanishes, remaining parts yields

$$\frac{D}{Dt}\big(\frac{1}{2}|\vec{W}|^2\big) = -(\vec{U}\cdot\vec{W})^2 + \Gamma\vec{W}\nabla^2\vec{W} \tag{2.7}$$

since

$$\vec{W}\nabla^2\vec{W} = \nabla^2\big(\frac{1}{2}|\vec{W}|^2\big) - \nabla\vec{W}:\nabla\vec{W} \tag{2.8}$$

then, the governing equation for the scalar gradient energy $\nabla\phi\cdot\nabla\phi$ is

$$\frac{D}{Dt}\big(\frac{1}{2}|\vec{W}|^2\big) = -(\vec{U}\cdot\vec{W})^2 + \Gamma\left(\nabla^2\big(\frac{1}{2}|\vec{W}|^2\big) - \nabla\vec{W}:\nabla\vec{W}\right) \tag{2.9}$$

Characteristic scales of dissipation term is estimated as

$$\Gamma\nabla^2(\frac{1}{2}|\vec{W}|^2) \sim \Gamma\frac{\phi^2}{L^4} \tag{2.10}$$