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Introduction to Numerical Analysis
HW5

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1 LEBESGUE CONSTANT FOR CHEBYSHEV NODES

1. a) *Proof.* Denote

$$LHS \triangleq (x - x_i) l_i(x)$$
$$RHS \triangleq \frac{T_{n+1}(x)}{T'_{n+1}(x_i)} \tag{1.1}$$

It is left to prove $LHS = RHS$.
The left part can be written as

$$LHS = c_l \omega(x) \tag{1.2}$$

where

$$\omega(x) = \prod_{i=0}^{n} (x - x_i) \tag{1.3}$$

and

$$c_l = \frac{1}{\prod_{k=0, k\neq i}^{n} (x_i - x_k)} \tag{1.4}$$

Since both LHS and RHS are polynomials of order $n+1$, they are equivalent iff. they have same roots and leading coefficients.
On the one hand,as for all x_i, where $i = 0, 1, ..., n$

$$T_{n+1}(x_i) = cos((n+1)\theta_i) = cos(\frac{2i+1}{2}\pi) = 0 \tag{1.5}$$

Thus, LHS and RHS have same roots. RHS can therefore be written as

$$RHS(x) = c_r \omega(x) \tag{1.6}$$

On the other hand, since

$$LHS'(x)|_{x=x_i} = (l_i(x) + (x - x_i) l'_i(x))|_{x=x_i} = 1 \tag{1.7}$$

and

$$RHS'(x)|_{x=x_i} = \frac{T'_{n+1}(x)}{T'_{n+1}(x_i)}\Big|_{x=x_i} = 1 \tag{1.8}$$

Thus, the leading coefficients of LHS and RHS are equal, namely $c_l = c_r$.
Hence, $LHS = RHS$. □

b) *Proof.*

$$\begin{aligned} T'_{n+1}(x) &= (cos((n+1)arccos(x)))' \\ &= sin((n+1)arccos(x))(n+1)\frac{1}{\sqrt{1-x^2}} \\ &= \frac{n+1}{\sqrt{1-cos^2(\theta)}} sin((n+1)\theta) \end{aligned} \tag{1.9}$$

As $\theta_k = \frac{2k+1}{2(n+1)}\pi$, thus, $sin(\theta_k) > 0$, and

$$T'_{n+1}(x_k) = \frac{n+1}{sin(\theta_k)} sin(\frac{2k+1}{2}\pi) = (-1)^k \frac{n+1}{sin(\theta_k)} \tag{1.10}$$

□

c) *Proof.* As

$$T_{n+1}(1) = cos((n+1)arccos(1)) = 1 \tag{1.11}$$

Thus

$$\begin{aligned} \sum_{i=0}^{n} |l_i(1)| &= \sum_{i=0}^{n} \left| \frac{T_{n+1}(1)}{(1-x_i)T'_{n+1}(x_i)} \right| \\ &= \sum_{i=0}^{n} \frac{1}{\left|(1-x_i)T'_{n+1}(x_i)\right|} \\ &= \frac{1}{n+1} \sum_{i=0}^{n} \left| \frac{sin\theta_i}{(1-cos\theta_i)} \right| \\ &= \frac{1}{n+1} \sum_{i=0}^{n} \left| \frac{sin\theta_i}{2sin^2(\frac{\theta_i}{2})} \right| \\ &\geq \frac{1}{n+1} \sum_{i=0}^{n} cot(\frac{\theta_i}{2}) \end{aligned} \tag{1.12}$$

□

2. a) *Proof.* According to the mean value theorem, there exists $\theta \in [\frac{\theta_k}{2}, \frac{\theta_{k+1}}{2}]$, s.t.

$$\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}} cot(t)dt = \frac{\theta_{k+1}-\theta_k}{2} cot(\theta) \tag{1.13}$$

As $cot'(t) = -\frac{1}{sin^2(t)} < 0$, and $\theta_k \leq \theta \leq \theta_{k+1}$, thus

$$cot(\theta) \leq cot(\theta_k) \tag{1.14}$$

Therefore

$$\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}} cot(t)dt \leq \frac{\theta_{k+1}-\theta_k}{2} cot(\theta_k) \tag{1.15}$$

□

b) *Proof.* As $\theta_{k+1} - \theta_k = \frac{\pi}{n+1}$ and according to the result that have been proved above

$$\begin{aligned}\sum_{k=0}^{n} \int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}} cot(t)dt &\leq \sum_{k=0}^{n} \frac{\theta_{k+1}-\theta_k}{2} cot(\frac{\theta_k}{2}) \\ &= \frac{\pi}{2(n+1)} \sum_{k=0}^{n} cot(\frac{\theta_k}{2})\end{aligned} \tag{1.16}$$

□

c) *Proof.* As $\theta_n = \frac{2n+1}{2n+2}\pi < \pi$, $\theta_{n+1} = \frac{2n+3}{2n+2} > \pi$, and $cot(x)$ is positive over $(0, \frac{\pi}{2})$, while negative otherwise. Thus

$$\int_{\frac{\theta_0}{2}}^{\frac{\pi}{2}} cot(t)dt \leq \int_{\frac{\theta_0}{2}}^{\frac{\theta_n}{2}} cot(t)dt = \sum_{k=0}^{n-1} \int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}} cot(t)dt \tag{1.17}$$

Hence

$$\int_{\frac{\theta_0}{2}}^{\frac{\pi}{2}} cot(t)dt \leq \frac{\pi}{2(n+1)} \sum_{i=0}^{n} cot(\frac{\theta_i}{2}) \tag{1.18}$$

(... not fine) □

3. *Proof.*

$$
\begin{aligned}
\Lambda_n &= \max_{x\in[a,b]} \sum_{i=0}^{n} |l_i(x)| \\
&\geq \sum_{i=0}^{n} |l_i(1)| \\
&\geq \frac{1}{n+1} \sum_{i=0}^{n} cot(\frac{\theta_i}{2}) \\
&\geq \frac{2}{\pi} \int_{\theta_0/2}^{\pi/2} cot(t)dt \\
&= \frac{2}{\pi} ln(|sin(t)|)\Big|_{\theta_0/2}^{\pi/2} \\
&= -\frac{2}{\pi} ln(sin(\frac{\theta_0}{2})) \\
&\geq \frac{2}{\pi} ln(\frac{2}{\theta_0}) = \frac{2}{\pi} ln(\frac{4(n+1)}{\pi}) \\
&\geq \frac{2}{\pi} ln(n)
\end{aligned}
\tag{1.19}
$$

□

2 Interpolation

3 Trigonometric polynomials