
Methods of Applied Mathematics I

HW2

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1 EXERCIS2.1

1. *Proof.*

$$\begin{aligned} RHS &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \\ &= \frac{1}{4}(\langle x+y, x+y \rangle - \langle x-y, x-y \rangle) \\ &= \frac{1}{4}(\langle x+y, x \rangle + \langle x+y, y \rangle - \langle x-y, x \rangle - \langle x-y, y \rangle) \\ &= \frac{1}{4}(\overline{\langle x, x+y \rangle} + \overline{\langle y, x+y \rangle} - \overline{\langle x, x-y \rangle} + \overline{\langle y, x-y \rangle}) \\ &= \frac{1}{4}(\overline{\langle x, x \rangle} + \overline{\langle x, y \rangle} + \overline{\langle y, x \rangle} + \overline{\langle y, y \rangle} - \overline{\langle x, x \rangle} + \overline{\langle x, y \rangle} + \overline{\langle y, x \rangle} - \overline{\langle y, y \rangle}) \\ &= \frac{1}{2}(\overline{\langle x, y \rangle} + \overline{\langle y, x \rangle}) \\ &= \frac{1}{2}(\langle x, y \rangle + \langle y, x \rangle) \\ &= \langle x, y \rangle = LHS \end{aligned}$$

(1.1)

The last line is valid as the inner-product is defined on real space s.t. $\langle x, y \rangle = \langle y, x \rangle$.

□

2. *Proof.* As have been proved aboved

$$\begin{aligned} & \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \\ &= \frac{1}{2}(\langle x, y \rangle + \langle y, x \rangle) \end{aligned} \quad (1.2)$$

Also, replace y with iy yields

$$\begin{aligned} & \frac{i}{4}(\|x+iy\|^2 - \|x-iy\|^2) \\ &= \frac{i}{2}(\langle x, iy \rangle + \langle iy, x \rangle) \\ &= \frac{i}{2}(i\langle x, y \rangle + \bar{i}\langle y, x \rangle) \\ &= \frac{1}{2}(-\langle x, y \rangle + \langle y, x \rangle) \end{aligned} \quad (1.3)$$

Thus

$$\begin{aligned} RHS &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) - \frac{i}{4}(\|x+iy\|^2 - \|x-iy\|^2) \\ &= \frac{1}{2}(\langle x, y \rangle + \langle y, x \rangle) - \frac{1}{2}(-\langle x, y \rangle + \langle y, x \rangle) \\ &= \langle x, y \rangle = LHS \end{aligned} \quad (1.4)$$

□

3. *Proof.*

$$\begin{aligned} LHS &= \|x+y\|^2 + \|x-y\|^2 \\ &= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \\ &= \langle x+y, x \rangle + \langle x+y, y \rangle + \langle x-y, x \rangle - \langle x-y, y \rangle \\ &= \overline{\langle x, x+y \rangle} + \overline{\langle y, x+y \rangle} + \overline{\langle x, x-y \rangle} - \overline{\langle y, x-y \rangle} \\ &= \overline{\langle x, x \rangle} + \overline{\langle x, y \rangle} + \overline{\langle y, x \rangle} + \overline{\langle y, y \rangle} + \overline{\langle x, x \rangle} - \overline{\langle x, y \rangle} - \overline{\langle y, x \rangle} + \overline{\langle y, y \rangle} \\ &= 2(\langle x, x \rangle + \langle y, y \rangle) \\ &= 2(\|x\|^2 + \|y\|^2) = RHS \end{aligned} \quad (1.5)$$

□

4. *Proof.* As any norm induced by inner product should satisfy the parallelogram law, it's easy to show that the norm $\|\cdot\|_\infty : f \mapsto \sup_{x \in [a, b]} |f(x)|$ on $C[a, b]$ can not be induced by any inner product since it does not satisfy the parallelogram law. For example, let

$$f(x) = \frac{x-a}{b-a} \quad (1.6)$$

and

$$g(x) = \left(\frac{x-a}{b-a}\right)^2 \quad (1.7)$$

then

$$\|f + g\| = \max_{x \in [a, b]} |(f + g)(x)| = |(f + g)(b)| = 2 \quad (1.8)$$

$$\|f - g\| = \max_{x \in [a, b]} |(f - g)(x)| = |(f - g)(\frac{a+b}{2})| = \frac{1}{4} \quad (1.9)$$

$$\|f\| = \max_{x \in [a, b]} |f(x)| = |f(b)| = 1 \quad (1.10)$$

$$\|g\| = \max_{x \in [a, b]} |g(x)| = |g(b)| = 1 \quad (1.11)$$

It's obvious that

$$\|f + g\|^2 + \|f - g\|^2 = \frac{65}{4} \neq 2(\|f\|^2 + \|g\|^2) = 4 \quad (1.12)$$

□

5. Assume the norm satisfies the parallelogram law, that is

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (1.13)$$

Let

$$\langle x, y \rangle \triangleq \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) \quad (1.14)$$

it's left to prove that $\langle x, y \rangle$ is an inner product.

Firstly, $\langle x, y + z \rangle$ can be expanded according to the definition as follows

$$\begin{aligned} \langle x, y + z \rangle &= \frac{1}{4} (\|x + y + z\|^2 - \|x - (y + z)\|^2) \\ &= \frac{1}{4} (\|y + z + x\|^2 - \|y + z - x\|^2) \end{aligned} \quad (1.15)$$

Since

$$\|y + z + x\|^2 + \|y - z + x\|^2 = 2(\|y + x\|^2 + \|z\|^2) \quad (1.16)$$

Then

$$\|y + z + x\|^2 = 2(\|y + x\|^2 + \|z\|^2) - \|y - z + x\|^2 \quad (1.17)$$

Swap y and z yields

$$\|x + y + z\|^2 = 2(\|z + x\|^2 + \|y\|^2) - \|z - y + x\|^2 \quad (1.18)$$

Adding the two equations above and divided by 2 yields

$$\|y + z + x\|^2 = \|y\|^2 + \|z\|^2 + \|x + y\|^2 + \|x + z\|^2 - \frac{1}{2} (\|y - z + x\|^2 + \|z - y + x\|^2) \quad (1.19)$$

Replace x with $-x$

$$\|y + z - x\|^2 = \|y\|^2 + \|z\|^2 + \|x - y\|^2 + \|x - z\|^2 - \frac{1}{2} (\|y - z - x\|^2 + \|z - y - x\|^2) \quad (1.20)$$

Hence

$$\|y+z+x\|^2 + \|y+z-x\|^2 = (\|x+y\|^2 - \|x-y\|^2) + (\|x+z\|^2 - \|x-z\|^2) \quad (1.21)$$

As

$$\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \quad (1.22)$$

and

$$\langle x, z \rangle = \frac{1}{4}(\|x+z\|^2 - \|x-z\|^2) \quad (1.23)$$

It's therefore can be concluded that

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle \quad (1.24)$$

2 EXERCISE2.2

1.