Introduction to Numerical Analysis HW1

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1 METRIC SPACE

1. *Proof.* Since there's no elements in $\emptyset$, so $\emptyset$ is open, as $X^\complement = \emptyset$, so X is closed.
   X is open as it contains all the elemnets, since $\emptyset^\complement = X$, thus $\emptyset$ is closed. □

2. *Proof.* Since
$$T \triangleq (U_1 \cap U_2 \cap ... \cap U_m)^\complement = U_1^\complement \cup U_2^\complement \cup ... \cup U_m^\complement \tag{1.1}$$

   And $U_i^\complement$ is open, as U_i is closed.
   Thus, T is open, and therefore the intersection is closed. □

3. *Proof.* Since
$$T \triangleq (U_1 \cup U_2 \cup ... \cup U_m)^\complement = U_1^\complement \cap U_2^\complement \cap ... \cap U_m^\complement \tag{1.2}$$

   And $U_i^\complement$ is open, as U_i is closed.
   Thus, T is open, and therefore the intersection is closed. □

2 CONTINUITY

1. $y = \frac{1}{x}$ is continuous over $(0, +\infty)$, but not uniform continuous.

   Proof. Let $x_1 = \frac{1}{n+1}$, $x_2 = \frac{1}{n}$. Then $\lim_{n \to +\infty} |x_2 - x_1| = \frac{1}{n(n+1)} = 0$. But $\lim_{n \to +\infty} |y_2 - y_1| = 1 \neq 0$.
   Thus it is not uniform continuous. □

2. $y = \sqrt{x}$ is uniform continuous over $(0, +\infty)$ but not Lipschitz continuous.
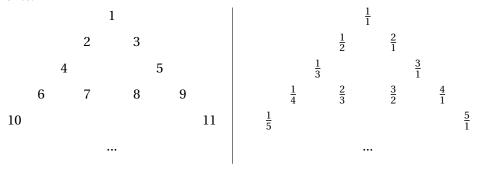
Proof. As $y\prime = \frac{1}{2\sqrt{x}}$, the slope tends towards infty when x approaches 0, thus it is not Lipschitz continuous. □

3 CARDINALTY

1. *Proof.* The following function is a one-to-one mapping from N to Z, thus N and Z have the same number of elements.

$$f(n) = \begin{cases} -\frac{n}{2} & \text{n is even} \\ \frac{n+1}{2} & \text{n is odd} \end{cases} \tag{3.1}$$

Arange the elements in N and Q in the pascal triangle style as below, and it is clear enough to see the one-to-one mapping from N to Q with rationals are grouped according to the sum of dividend and divisor. Thus N and Q have the same number of elements.

			1							$\frac{1}{1}$		
		2		3					$\frac{1}{2}$		$\frac{2}{1}$	
	4				5			$\frac{1}{3}$				$\frac{3}{1}$
	6	7	8	9			$\frac{1}{4}$	$\frac{2}{3}$		$\frac{3}{2}$		$\frac{4}{1}$
10					11	$\frac{1}{5}$						$\frac{5}{1}$
		...							...			

□

2. *Proof.* The following function is a one-to-one mapping from [0, 1] to R, thus they have the same number of elements.

$$f(x) = \begin{cases} -\infty & \text{x=0} \\ tan[\pi(x-0.5)] & 0 < x < 1 \\ +\infty & \text{x=1} \end{cases} \tag{3.2}$$

□

3. *Proof.* Suppose all the real numbers can be listed, and each one is marked as r_i, where $i = 1, 2, 3....$ Given a real number $r\prime$ such that its **i-th** digit is different from that in r_i, it's obvious that $r\prime$ is not included in the real numbers listed above, which is contradictory to the hypothesis. Thus real numbers can not be listed and it contains more elements than N. □

4 Slides

1. *Proof.* Let E be an inner product space over $\mathbb{C}$, and $u, v \in E$. Given Y defined as below, where $\lambda \in \mathbb{R}$.

$$Y = |u - \lambda v|^2 \tag{4.1}$$

It's obvious that $Y \geq 0$. Expand the squares according to the definition of inner products, the inequality can be written as below.

$$<v,v>\lambda^2 - (<u,v>+<v,u>)\lambda + <u,u> \geq 0 \tag{4.2}$$

The LHS of the inequality can be viewed as a quadratic function where λ is the variable. Thus, the discriminant is semi-negative definite, which can be written as below.

$$\Delta = (<u,v>+<v,u>)^2 - 4<u,u><v,v> \leq 0 \tag{4.3}$$

As $<v,u> = \overline{<u,v>}$, the inequality above can be simplified as below.

$$<u,v>^2 + \overline{<u,v>}^2 + 2|<u,v>|^2 \leq 4\|u\|^2\|v\|^2 \tag{4.4}$$

For $x \in \mathbb{C}$, the following equality is justified.

$$x^2 + \bar{x}^2 = \frac{1}{2}[(x+\bar{x})^2 + (x-\bar{x})^2] = \frac{1}{2}[(2Re(x))^2 + (2Im(x))^2] = 2|x|^2 \tag{4.5}$$

Since $<u,v>$ is an complex number, apply (4.5) into (4.4) and eliminate the constant 4, the Cauchy-Schwarz inequality is obtained at last.

$$|<u,v>|^2 \leq \|u\|^2\|v\|^2 \tag{4.6}$$

$\square$

2. *Proof.* $d(x,y)$ is non-negative follows from the definition of metric space.

$$
\begin{aligned}
d(x,y) &= \frac{1}{2}(d(x,y)+d(x,y)) & \\
&= \frac{1}{2}(d(x,y)+d(y,x)) & \text{By symmetry} \\
&\geq \frac{1}{2}d(x,x) & \text{By triangle inequality} \\
&= 0 & \text{By identity of indiscernible}
\end{aligned}
\tag{4.7}
$$

$\square$

5 Linear Algebra

1. *Proof.* Suppose $\{u_1, u_2, ..., u_m\}$ forms the basis of $\text{Ker}(f)$, and it can be extended to form the basis of V_1, which is denoted as $\{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$. It's clear that the dimension of $\text{Ker}(f)$ is m and the dimension of V_1 is $m + n$. So, it is left to prove that the dimension of V_2 is n.

The dimension of V_2 is n means $\{f(v_1), f(v_2), ..., f(v_n)\}$ forms a basis of V_2. To see that, let w be an arbitary vector in V_1, thus, there exist unique scalars a_i, b_j such that

$$w = a_1 u_1 + a_2 u_2 + ... + a_m u_m + b_1 v_1 + b_2 v_2 + ... + b_n v_n \tag{5.1}$$

The image of w under mapping f is given as below.

$$\begin{aligned} f(w) &= \sum_{i=1}^{m} a_i f(u_i) + \sum_{j=1}^{m} b_j f(v_j) \\ &= \sum_{j=1}^{m} b_j f(v_j) \quad \text{As } f(u_i) = 0 \end{aligned} \tag{5.2}$$

Thus $\{f(v_1), f(v_2), ..., f(v_n)\}$ spans V_2. It is left to show that they are linearly independent, which means there's no redundancy in the list.

Given coefficents c_i such that

$$c_1 f(v_1) + c_2 f(v_2) + ... + c_n f(v_n) = 0 \tag{5.3}$$

Since f is a linear mapping, (5.3) can be grouped as below.

$$f(c_1 v_1 + c_2 v_2 + ... + c_n v_n) = 0 \tag{5.4}$$

Thus $c_1 v_1 + c_2 v_2 + ... + c_n v_n \in Ker(f)$. As $\{u_1, u_2, ..., u_m\}$ forms the basis of $\text{Ker}(f)$, there exists coefficients d_i such that

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = d_1 u_1 + d_2 u_2 + ... + d_m u_m \tag{5.5}$$

Since $\{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$ forms the basis of V_1, all the coefficients in (5.5) should be 0. Thus, (5.3) is valid if and only if all the coefficents c_i is 0, which implies that $\{f(v_1), f(v_2), ..., f(v_n)\}$ are linearly independent.

Thus, $\{f(v_1), f(v_2), ..., f(v_n)\}$ forms the basis of V_2, and the dimension of V_2 is therefore n. The rank-nullity theorem written as below is proved.

$$dim(V_1) = dim(ker(f)) + dim(V_2) \tag{5.6}$$

$\square$

2. *Proof.* Let U, V, W be vector spaces over the same field K, function f be a linear map from U to V and function g be a linear map from V to W. It's left to show that the composition of f and g, denoted as h, which maps a vector in U to W, is still a linear map.

Given $u_1, u_2 \in U$ and any scalar $c \in \mathbb{K}$, then

$$
\begin{aligned}
h(u_1 + u_2) &= g(f(u_1 + u_2)) \\
&= g(f(u_1) + f(u_2)) \\
&= g(f(u_1)) + g(f(u_2)) \\
&= h(u_1) + h(u_2)
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
h(cu_1) &= g(f(cu_1)) \\
&= g(cf(u_1)) \\
&= cg(f(u_1)) \\
&= ch(u_1)
\end{aligned}
\tag{5.8}
$$

Thus, the composition of two linear maps is still a linear map. $\qquad\square$

3. *Proof.* Let U, V be vector spaces over the same field K, function f be linear maps from U to V. It's left to show that the inverse of f, denoted as f^{-1}, which maps a vector in V to U is still a linear map.

By inverse, it means that
$$
f(f^{-1}(x)) = f^{-1}(f(x)) = x
\tag{5.9}
$$

Given $u_1, u_2 \in U$ and any scalar $c \in \mathbb{K}$, denote the counterpart of u_1, u_2 in V as v_1, v_2 under the linear mapping function f. Then

$$
\begin{aligned}
f^{-1}(v_1 + v_2) &= f^{-1}(f(u_1) + f(u_2)) \\
&= f^{-1}(f(u_1 + u_2)) \\
&= u_1 + u_2 \\
&= f^{-1}(v_1) + f^{-1}(v_2)
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
f^{-1}(cv_1) &= f^{-1}(cf(u_1)) \\
&= f^{-1}(f(cu_1)) \\
&= cu_1 \\
&= cf^{-1}(v_1)
\end{aligned}
\tag{5.11}
$$

Thus, the inverse of a linear map is still a linear map. $\qquad\square$

6 DISCONTINUOUS LINEAR MAPS

1. *Proof.* The k times derivative of $f_n(x)$ can be written as below.

$$f_n^{(k)}(x) = n^{k-1} sin(nx + \frac{k\pi}{2}) \quad k = 0,1,2,3,...$$ (6.1)

It's clear that $f_n^{(k)}(x)$ is continous for any k, thus $f_n(x) \in C^\infty(\mathbb{R})$. □

2.
$$df_n(x) = cos(nx)dx$$ (6.2)

3. *Proof.* Let $x_0 = 0, x_1 = \frac{\pi}{2n}$, then

$$\lim_{n->+\infty} |x_1 - x_0| = 0$$ (6.3)

$$\lim_{n->+\infty} |df_n(x_1) - df_n(x_0)| = |0 - dx| = |dx| \neq 0$$ (6.4)

Thus, the differential is not continous when n tends to infty. □

4. Sorry, haven't figured out yet...

7 PI

1. The main program(see Algorithm1) adopts Machin's formula to calculate π.

$$\pi = 4[4atan(\frac{1}{5}) - atan(\frac{1}{239})]$$ (7.1)

Subroutine(see Algorithm2) calculating $atan(\frac{1}{x})$ is used by the main program. Since

$$atan(x) = \frac{1}{1}x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + ...$$ (7.2)

Therefore
$$atan(\frac{1}{x}) = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + ...$$ (7.3)

Algorithm 1 Calculation of π using Machin's formula

Input: None.
Output: The value of π with approximation.
  1: **return** $4[4atan(1/5) - atan(1/239)]$

Algorithm 2 Calculation of $atan(\frac{1}{x})$

Input: x
Output: $atan(\frac{1}{x})$

1: $ret \leftarrow 0$
2: $e \leftarrow 1/x$
3: $s \leftarrow -x^2$
4: $c \leftarrow 1$
5: **while** $e \neq 0$ **do**
6:     $ret \leftarrow ret + e/c$
7:     $e \leftarrow e/s$
8:     $c \leftarrow c + 2$
9: **end while**
10: **return** ret

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2. The matlab code is given as below.

```
function [pi] = pi_machin()
pi = 4*(4*atan_inv(5) - atan_inv(239));

function [r] = atan_inv(x)
r = 0;
e = 1/x;
s = - x*x;
c = 1;
while (e ~= 0)
r = r + e/c;
e = e/s;
c = c+2;
end
```