SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Methods of Applied Mathematics I
HW4

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1 EXERCISE4.1

Let $f(x)$ be extended as

$$f(x)=\begin{cases} x(\pi-x) x\in[2n\pi,(2n+1)\pi] \\ -x(\pi-x) x\in[-(2n-1)\pi,2n\pi] \end{cases} \tag{1.1}$$

Then $f(x)$ is both odd and periodic. Thus fouier-sine series can be employed.

$$f(x)=\sum_{n=1}^{\infty} b_n sin(nx) \tag{1.2}$$

Coefficients b_n are calculated by

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x) sin(nx) dx \\ &= \frac{2}{\pi}\int_0^{\pi} x(\pi-x) sin(nx) dx \\ &= \frac{4[1-(-1)^n]}{n^3\pi} \quad \text{(Integrate by parts)} \end{aligned} \tag{1.3}$$

Thus

$$f(x)=\sum_{k=0}^{\infty} \frac{8 sin(2k+1)x}{\pi(2k+1)^3} \tag{1.4}$$

Taking $x = \frac{\pi}{2}$ yields

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32} \tag{1.5}$$

2 EXERCISE4.2

1. *Proof.* The orthogonal property is justified as

$$\int_0^{\pi} (\frac{1}{\sqrt{\pi}})^2 dx = \frac{1}{\pi}\int_0^{\pi} dx = 1 \tag{2.1}$$

$$\int_0^{\pi} (\sqrt{\frac{2}{\pi}} cos(nx))^2 dx = \frac{2}{\pi}\int_0^{\pi} cos^2(nx)dx = \frac{1}{\pi}\int_0^{\pi} (cos(2nx)+1)dx = 1 \tag{2.2}$$

$$\int_0^{\pi} \frac{1}{\sqrt{\pi}}\sqrt{\frac{2}{\pi}} cos(nx)dx = 0 \tag{2.3}$$

$$\int_0^{\pi} \sqrt{\frac{2}{\pi}} cos(nx)\sqrt{\frac{2}{\pi}} cos(mx)dx = \frac{2}{\pi}\int_0^{\pi} cos(nx)cos(mx)dx = 0 \tag{2.4}$$

□

2. *Proof.* It's trival to show both $K = 0$ and $K = 1$ are valid, and $K = 2$ is also justified as

$$cos^2(x) = \frac{1+cos(2x)}{2} \tag{2.5}$$

Assume the proposition is also valid for $K = n$, that means

$$span\{1, cos(x), cos(2x), ...cos(nx)\} = span\{1, cos(x), cos^2(x), ...cos^n(x)\} \tag{2.6}$$

which indicates that $\exists a_k$ and $b_k (k = 0, 1, ..., n)$ s.t.

$$cos^n(x) = \sum_{k=0}^{n} a_k^{(n)} \cdot cos(kx) \tag{2.7}$$

$$cos(nx) = \sum_{k=0}^{n} b_k^{(n)} \cdot cos^k(x) \tag{2.8}$$

When $K = n+1$, the proposition is still valid as

$$\begin{aligned} cos^{n+1}(x) &= cos(x)\sum_{k=0}^{n} a_k cos(kx) \\ &= a_0 cos(x) + cos(x)\sum_{k=1}^{n-1} a_k cos(kx) + a_n cos(x)cos(nx) \\ &= a_0 cos(x) + \sum_{k=1}^{n-1} \frac{a_k}{2}[cos(k-1)x + cos(k+1)x] + \frac{a_n}{2}[cos(n-1)x + cos(n+1)x] \\ &= \frac{a_1}{2} + (a_0 + \frac{a_2}{2})cos(x) + \sum_{k=2}^{n-1} \frac{a_{k-1}+a_{k+1}}{2} cos(kx) + \frac{a_n}{2} cos(n+1)x \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} cos(n+1)x &= 2cos(nx)cos(x) - cos(n-1)x \\ &= 2cos(x)\sum_{k=0}^{n} b_k^{(n)} cos^k(x) - \sum_{k=0}^{n-1} b_k^{(n-1)} cos^k(x) \\ &= -b_0^{(n-1)} + \sum_{k=0}^{n-1}(2b_k^{(n)} - b_k^{(n-1)})cos^{k+1}(x) + 2b_k^{(n)} cos^{n+1}(x) \end{aligned} \tag{2.10}$$

□

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