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Introduction to Numerical Analysis
HW5

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1 LEBESGUE CONSTANT FOR CHEBYSHEV NODES

1. a) *Proof.* Denote

$$
\begin{aligned}
LHS &\triangleq (x - x_i) l_i(x) \\
RHS &\triangleq \frac{T_{n+1}(x)}{T'_{n+1}(x_i)}
\end{aligned}
\tag{1.1}
$$

It is left to prove $LHS = RHS$.
The left part can be written as

$$LHS = c_l \omega(x) \tag{1.2}$$

where

$$\omega(x) = \prod_{i=0}^{n} (x - x_i) \tag{1.3}$$

and

$$c_l = \frac{1}{\prod_{k=0,k\neq i}^{n} (x_i - x_k)} \tag{1.4}$$

Since both LHS and RHS are polynomials of order $n+1$, they are equivalent iff. they have same roots and leading coefficients.
On the one hand,as for all x_i, where $i = 0, 1, ..., n$

$$T_{n+1}(x_i) = cos((n+1)\theta_i) = cos(\frac{2i+1}{2}\pi) = 0 \tag{1.5}$$

Thus, LHS and RHS have same roots. RHS can therefore be written as

$$RHS(x) = c_r \omega(x) \tag{1.6}$$

On the other hand, since

$$LHS'(x)|_{x=x_i} = (l_i(x) + (x - x_i) l_i'(x))|_{x=x_i} = 1 \tag{1.7}$$

and

$$RHS'(x)|_{x=x_i} = \frac{T'_{n+1}(x)}{T'_{n+1}(x_i)}\bigg|_{x=x_i} = 1 \tag{1.8}$$

Thus, the leading coefficients of LHS and RHS are equal, namely $c_l = c_r$.
Hence, $LHS = RHS$. □

b) *Proof.*

$$\begin{aligned} T'_{n+1}(x) &= (cos((n+1)arccos(x)))' \\ &= sin((n+1)arccos(x))(n+1)\frac{1}{\sqrt{1-x^2}} \\ &= \frac{n+1}{\sqrt{1-cos^2(\theta)}} sin((n+1)\theta) \end{aligned} \tag{1.9}$$

As $\theta_k = \frac{2k+1}{2(n+1)}\pi$, thus, $sin(\theta_k) > 0$, and

$$T'_{n+1}(x_k) = \frac{n+1}{sin(\theta_k)} sin(\frac{2k+1}{2}\pi) = (-1)^k \frac{n+1}{sin(\theta_k)} \tag{1.10}$$

□

c) *Proof.* As

$$T_{n+1}(1) = cos((n+1)arccos(1)) = 1 \tag{1.11}$$

Thus

$$\begin{aligned} \sum_{i=0}^{n} |l_i(1)| &= \sum_{i=0}^{n} \left| \frac{T_{n+1}(1)}{(1-x_i)T'_{n+1}(x_i)} \right| \\ &= \sum_{i=0}^{n} \frac{1}{\left|(1-x_i)T'_{n+1}(x_i)\right|} \\ &= \frac{1}{n+1}\sum_{i=0}^{n} \left| \frac{sin\theta_i}{(1-cos\theta_i)} \right| \\ &= \frac{1}{n+1}\sum_{i=0}^{n} \left| \frac{sin\theta_i}{2sin^2(\frac{\theta_i}{2})} \right| \\ &\geq \frac{1}{n+1}\sum_{i=0}^{n} cot(\frac{\theta_i}{2}) \end{aligned} \tag{1.12}$$

□

2. a) *Proof.* According to the mean value theorem, there exists $\theta \in [\frac{\theta_k}{2}, \frac{\theta_{k+1}}{2}]$, s.t.

$$\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}} cot(t)dt = \frac{\theta_{k+1}-\theta_k}{2} cot(\theta) \tag{1.13}$$

As $cot'(t) = -\frac{1}{sin^2(t)} < 0$, and $\theta_k \le \theta \le \theta_{k+1}$, thus

$$cot(\theta) \le cot(\theta_k) \tag{1.14}$$

Therefore

$$\int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}} cot(t)dt \le \frac{\theta_{k+1}-\theta_k}{2} cot(\theta_k) \tag{1.15}$$

□

b) *Proof.* As $\theta_{k+1} - \theta_k = \frac{\pi}{n+1}$ and according to the result that have been proved above

$$\begin{aligned} \sum_{k=0}^{n} \int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}} cot(t)dt &\le \sum_{k=0}^{n} \frac{\theta_{k+1}-\theta_k}{2} cot(\frac{\theta_k}{2}) \\ &= \frac{\pi}{2(n+1)} \sum_{k=0}^{n} cot(\frac{\theta_k}{2}) \end{aligned} \tag{1.16}$$

□

c) *Proof.* As $\theta_n = \frac{2n+1}{2n+2}\pi < \pi$, $\theta_{n+1} = \frac{2n+3}{2n+2} > \pi$, and $cot(x)$ is positive over $(0, \frac{\pi}{2})$, while negative otherwise. Thus

$$\int_{\frac{\theta_0}{2}}^{\frac{\pi}{2}} cot(t)dt \le \int_{\frac{\theta_0}{2}}^{\frac{\theta_n}{2}} cot(t)dt = \sum_{k=0}^{n-1} \int_{\frac{\theta_k}{2}}^{\frac{\theta_{k+1}}{2}} cot(t)dt \tag{1.17}$$

Hence

$$\int_{\frac{\theta_0}{2}}^{\frac{\pi}{2}} cot(t)dt \le \frac{\pi}{2(n+1)} \sum_{i=0}^{n} cot(\frac{\theta_i}{2}) \tag{1.18}$$

(... not fine) □

3. *Proof.*

$$\begin{aligned}
\Lambda_n &= \max_{x\in[a,b]} \sum_{i=0}^{n} |l_i(x)| \\
&\geq \sum_{i=0}^{n} |l_i(1)| \\
&\geq \frac{1}{n+1} \sum_{i=0}^{n} cot(\frac{\theta_i}{2}) \\
&\geq \frac{2}{\pi} \int_{\theta_0/2}^{\pi/2} cot(t)dt \\
&= \frac{2}{\pi} ln(|sin(t)|)\Big|_{\theta_0/2}^{\pi/2} \\
&= -\frac{2}{\pi} ln(sin(\frac{\theta_0}{2})) \\
&\geq \frac{2}{\pi} ln(\frac{2}{\theta_0}) = \frac{2}{\pi} ln(\frac{4(n+1)}{\pi}) \\
&\geq \frac{2}{\pi} ln(n)
\end{aligned} \tag{1.19}$$

□

2 INTERPOLATION

1. *Proof.* Since $\Phi(f) = 0$, $f(x_k) + f(y_k) = 0$ is valid for any k.
   Thus $f(x_k) = f(y_k) = 0$ or $f(x_k)$ and $f(y_k)$ have different signs.
   In the first case, take $\xi_k = x_k$ or $\xi_k = y_k$;
   In the second case, as f is continous over $[a,b]$, according to the intermediate value theorem, there exists $\xi_k \in [x_k, y_k]$ s.t. $f(\xi_k) = 0$. □

2. *Proof.* Let $h_1(x) \in \mathbb{R}_n[x]$, $h_2(x) \in \mathbb{R}_n[x]$ and $h_1(x) \neq h_2(x)$. They can be written as

$$h_1(x) = \sum_{i=0}^{n} a_i x^i \tag{2.1}$$

$$h_2(x) = \sum_{i=0}^{n} b_i x^i \tag{2.2}$$

Assume $\Phi(h_1) = \Phi(h_2)$, then

$$h_1(x_k) + h_1(y_k) = h_2(x_k) + h_2(y_k) \quad (k = 0, 1, ..., n) \tag{2.3}$$

Thus

$$\begin{bmatrix} (1+1) & (x_0+y_0) & (x_0^2+y_0^2) & \dots & (x_0^n+y_0^n) \\ (1+1) & (x_1+y_1) & (x_1^2+y_1^2) & \dots & (x_1^n+y_1^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1+1) & (x_n+y_n) & (x_n^2+y_n^2) & \dots & (x_n^n+y_n^n) \end{bmatrix} \begin{bmatrix} a_0-b_0 \\ a_1-b_1 \\ \vdots \\ a_n-b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.4}$$

The determinant of the coefficient matrix above is positive as it is the sum of 2^{n+1} determinents, each of these be positive as $x_0 < y_0 < x_1 < y_1 < ... < x_n < y_n$. Thus, solution to this linear system is $x = [0, 0, ..., 0]^T$, which means

$$a_i = b_i \ \ (i = 0, 1, ..., n) \tag{2.5}$$

So, the hypothsis fails as $h_1(x) = h_2(x)$. Hence, $\Phi(f) \neq \Phi(g)$, which means Φ is injective. Assume $\Phi(P1) = \Phi(f)$, $\Phi(P2) = \Phi(f)$, according to the converse-negative proposition of the injective property, there should be $P1 = P2$, thus the unicity is proved. □

3. *Proof.* Let $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, denote $h(x) = P_n(x) - f(x)$. As $\Phi(P_n) = \Phi(f)$, then

$$h(x_k) + h(y_k) = 0 \tag{2.6}$$

Thus, there exists $\xi_k \in [x_k, y_k]$ s.t. $h(\xi_k) = 0$. So

$$\sum_{i=0}^{n} a_i \xi_k^i = f(\xi_k) \ \ (k = 0, 1, ..., n) \tag{2.7}$$

which can be written as matrix form

$$\begin{bmatrix} 1 & \xi_0 & \xi_0^2 & ... & \xi_0^n \\ 1 & \xi_1 & \xi_1^2 & ... & \xi_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_n & \xi_n^2 & ... & \xi_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(\xi_0) \\ f(\xi_1) \\ \vdots \\ f(\xi_n) \end{bmatrix} \tag{2.8}$$

The determinant of the coefficient matrix is the so called Vandermonde determinant, and $\xi_j > \xi_i$ when $j > i$. Thus, determinant of the coefficient matrix is positive, which means there exist $n+1$ points in $[a, b]$, denoted as ξ_i, s.t. $P(\xi_i) = f(\xi_i)$. Thus P_n is the interpolation polynomial of f.

Applying the Taylor theorem, it's clear that

$$f(x) - P_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \tag{2.9}$$

Thus

$$\|f - p_n\|_\infty = \max_{x \in [a,b]} |f - P_n| \leq \frac{(b-a)^{n+1}}{(n+1)!} \sup_{x \in [a,b]} |f^{(n+1)}(x)| \tag{2.10}$$

□

3 Trigonometric polynomials

1. *Proof.* It's clear that the statement holds when $k = 0, 1$ as 1 and $cos\theta$ are in T_0 and T_1 respectively.
   Suppose the statement is still valid when $k = n-1$, thus, there exists Q_{n-1} s.t.

$$(cos\theta)^{n-1} = Q_{n-1} \tag{3.1}$$

When it comes to $k = n$, with the induction hypothesis,

$$\begin{aligned}
(cos\theta)^n &= Q_{n-1} \times cos\theta \\
&= (\frac{a_0}{\sqrt{2}} + \sum_{k=1}^{n-1} a_k cosk\theta)cos\theta \\
&= \frac{a_0}{\sqrt{2}} cos\theta + \sum_{k=1}^{n-1} a_k (\frac{cos(k+1)\theta + cos(k-1)\theta}{2}) \\
&= \frac{b_0}{\sqrt{2}} + \sum_{k=0}^{n} b_k cosk\theta \triangleq Q_n
\end{aligned} \tag{3.2}$$

Thus, $(cos\theta)^n$ is in T_n. Hence, for any $0 \le k \le n$, $(cos\theta)^k$ is in T_n as $T_k \subseteq T_n$.

(the bijection part haven't figured out yet ...)

□

2. *Proof.* As $Q_n(\theta_i) = F(\theta_i)$, the existence of $Q_n \in T_n$ is equivalent to the existence of solution of the linear system given as below

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & cos(\theta_0) & cos(2\theta_0) & \dots & cos(n\theta_0) \\ \frac{1}{\sqrt{2}} & cos(\theta_1) & cos(2\theta_1) & \dots & cos(n\theta_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}} & cos(\theta_n) & cos(2\theta_n) & \dots & cos(n\theta_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} F(\theta_0) \\ F(\theta_1) \\ F(\theta_2) \\ \vdots \\ F(\theta_n) \end{bmatrix} \tag{3.3}$$

Thus, it's left to prove the determinant of the coefficient matrix is non-zero, which can be denoted as $\det(A)$.
As $cos(k\theta)$ can be written as

$$cos(k\theta) = \sum_{n=0}^{k} b_n (cos\theta)^n \tag{3.4}$$

Thus, $\det(A)$ can be simplified as below, where C is an non-zero factor.

$$\det(A) = C \det \begin{bmatrix} 1 & cos(\theta_0) & cos^2(\theta_0) & \dots & cos^n(\theta_0) \\ 1 & cos(\theta_1) & cos^2(\theta_1) & \dots & cos^n(\theta_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & cos(\theta_n) & cos^2(\theta_n) & \dots & cos^n(\theta_n) \end{bmatrix} \tag{3.5}$$

With $\theta_i = \frac{2i+1}{2(n+1)}\pi$, it's clear that $\det(A) \neq 0$ as its Vandermon derterminant part is non-zero. Thus the existence of Q_n is valid. □

3. *Proof.* It has been exactly proved above. □

4. *Proof.* □