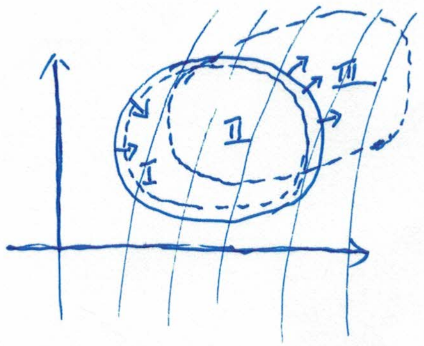
{ Review of the Reynolds Transport Theory:

* Link between different perspective / Relation between Euler and Lagrange Description.

Consider some physical quantity ϕ within a closed surface:



— t | —, C.V. not change

---- $t+\Delta t$ | ----, SYS. deform with time.

At $T=t$:

$$N \triangleq \int_{CV} \phi \, dv = \int_{SYS} \phi \, dv \Rightarrow N_{CV}(t) = N_{SYS}(t)$$

At $T=t+\Delta t$.

$$N_{CV} = \int_{I} \phi \, dv + \int_{II} \phi \, dv \triangleq N_{I}(t+\Delta t) + N_{II}(t+\Delta t) = N_{CV}(t+\Delta t)$$

$$N_{SYS} = \int_{II} \phi \, dv + \int_{III} \phi \, dv \triangleq N_{II}(t+\Delta t) + N_{III}(t+\Delta t)$$

$$\therefore \frac{D}{Dt} N = \lim_{\Delta t \to 0} \frac{N_{sys}(t+\Delta t) - N_{sys}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{[N_{CV}(t+\Delta t) + N_{III}(t+\Delta t) - N_{I}(t+\Delta t)] - N_{CV}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{N_{CV}(t+\Delta t) - N_{CV}(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{N_{III}(t+\Delta t)}{\Delta t} - \lim_{\Delta t \to 0} \frac{N_{I}(t+\Delta t)}{\Delta t}$$

$$= \frac{\partial}{\partial t} \int_{I+II} \phi \, dv + \int_{\partial III} \phi \vec{V} \cdot d\vec{s} - \int_{\partial I} \phi \vec{V} \cdot d\vec{s}$$ (carry in norm direction)

$$= \int_{I+II} \frac{\partial}{\partial t} \phi \, dv + \int_{\partial II} \phi \vec{V} \cdot d\vec{s}$$

§ Derivation of Mass conservation:

Consider a group of mass, with density ρ.

$$M = \oint \rho \, dv \Rightarrow \frac{D}{Dt} M = 0$$

$$\xRightarrow{RTT} \oint_V \frac{\partial \rho}{\partial t} dv + \oint_{\partial V} \rho \vec{V} \cdot d\vec{s} = 0$$

$$\xRightarrow{\text{Divergence Theorem}} \oint \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V})\right] dv = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

This is achieved unde the assumption that ρ is smooth enough to take derivatives.

§ Derivation of Momentum Conservation:

Let $\phi = \rho \vec{V}$, the momentum within is:

$$M = \oint \phi \, dv = \oint \rho \vec{V} \cdot dv \Rightarrow \frac{D}{Dt} M = \Sigma \text{forces}$$

$$\xRightarrow{RTT} \oint \frac{\partial}{\partial t}(\rho \vec{V}) dv + \underline{\oint (\rho \vec{V}) \cdot \vec{V} \cdot d\vec{s}} = \oint \rho \vec{g} \, dv + \int \bar{\bar{\tau}} \, d\vec{s}$$

$\bar{\bar{\tau}}$ is the stress tensor including normal pressure.

$$\bar{\bar{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

Understanding $\oint (\underbrace{\rho \vec{V})\, \vec{V}}_{\text{not dot product}} \underbrace{\cdot}_{\text{dot product}} d\vec{s}$ $\forall$

$$= \oint \rho \vec{V} (\vec{n} \cdot \vec{V}) \cdot dA = \oint \vec{n} \cdot \underbrace{\rho \vec{V} \vec{V}}_{\text{Dyad}} ds$$

$$\Downarrow \qquad\qquad\qquad \Uparrow$$

$$\begin{matrix} \rho u (n_x u + n_y v + n_z w) \\ \rho v (\quad \cdots \quad) \\ \rho w (\quad \cdots \quad) \end{matrix} = \begin{matrix} \vec{n} \cdot [\rho uu \;\; \rho uv \;\; \rho uw] \\ \vec{n} \cdot [\rho vu \;\; \rho vv \;\; \rho vw] \\ \vec{n} \cdot [\rho wu \;\; \rho wv \;\; \rho ww] \end{matrix}$$

$$\xRightarrow{\text{Divergence Theorem}} \oint \frac{\partial}{\partial t}(\rho \vec{V}) dv + \oint \nabla \cdot (\rho \vec{V} \vec{V}) dv = \oint \rho \vec{g} \, dv + \oint \nabla \cdot \bar{\bar{\tau}} \, dv$$

The divergence of a dyad is calculated as:

$$\nabla\cdot(\vec{f}\vec{g}) = (\nabla\cdot\vec{f})\vec{g} + (\vec{f}\cdot\nabla)\vec{g}$$

Thus: $\nabla\cdot(\rho\vec{V}\vec{V}) = [\nabla\cdot(\rho\vec{V})]\vec{V} + (\rho\vec{V}\cdot\nabla)\vec{V}$

Hence: $\cancel{\frac{\partial\rho}{\partial t}\vec{V}} + \frac{\partial\vec{V}}{\partial t}\rho + \cancel{\vec{V}\cdot[\nabla\cdot(\rho\vec{V})]} + \rho(\vec{V}\cdot\nabla)\vec{V} = \rho\vec{g} + \nabla\cdot\bar{\bar{\tau}}$

since. $\frac{\partial\rho}{\partial t}\vec{V} + \vec{V}\cdot[\nabla\cdot(\rho\vec{V})] = \vec{V}\cdot\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] = \vec{V}\cdot 0 = 0$

Thus: $\rho\left[\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] = \rho\vec{g} + \nabla\cdot\bar{\bar{\tau}}$

Namely: $\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} + \nabla\cdot\bar{\bar{\tau}}$

$\xRightarrow{\vec{V}\cdot}$ Kinetic Energy Equation:

$$\rho\vec{V}\cdot\frac{D\vec{V}}{Dt} = \rho\vec{g}\cdot\vec{V} + \vec{V}(\nabla\cdot\bar{\bar{\tau}})$$

$$\Leftrightarrow \quad \rho\frac{D(\frac{V^2}{2})}{Dt} = \rho\vec{g}\vec{V} + \vec{V}\cdot(\nabla\cdot\bar{\bar{\tau}})$$

§ Derivation of the Energy Equation:

$$dE = \delta Q + \delta W$$

| d: State variable
| δ: Process variable.

$$E = \oint \rho(e + \tfrac{1}{2}V^2)\, dv$$

Q: Only Conduction is counted, radiation is neglected. | W: 表面力功与体力功

$$\frac{D}{Dt}E \xlongequal{RTT} \frac{\partial}{\partial t}\oint \rho(e+\tfrac{1}{2}V^2)\,dv + \oint \rho(e+\tfrac{1}{2}V^2)\,\vec{V}\cdot d\vec{s}$$

$$\xrightarrow[\text{Theorem}]{\text{Divergence}} \oint \left\{ \frac{\partial}{\partial t}\rho(e+\tfrac{1}{2}V^2) + \nabla\cdot\left[\rho(e+\tfrac{1}{2}V^2)\vec{V}\right] \right\} dv.$$

$$\delta Q = -\int \vec{q}\cdot d\vec{s} = -\int \nabla\cdot\vec{q}\cdot dv.$$

$$\delta W = \int \rho\vec{g}\cdot\vec{V}\,dv + \int \vec{V}(\bar{\bar{\tau}}\cdot d\vec{s})$$

Understanding / Details about the Surface Work term:

$$\tau = \begin{bmatrix} \tau_{xx}-p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy}-p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz}-p \end{bmatrix}$$

$$\int \vec{V}(\bar{\bar{\tau}}\cdot d\vec{s}) = \int \vec{V}\cdot(\vec{n}\cdot\bar{\bar{\tau}})\,ds \xRightarrow[\text{as follows.}]{\text{Explained.}} \int \nabla\cdot(\vec{V}\cdot\bar{\bar{\tau}})\,dv.$$

$$\vec{V}\cdot(\vec{n}\cdot\bar{\bar{\tau}}) = \vec{V}\cdot\begin{pmatrix} n_x\tau_{xx}+n_y\tau_{yx}+n_z\tau_{zx} \\ n_x\tau_{xy}+n_y\tau_{yy}+n_z\tau_{zy} \\ n_x\tau_{xz}+n_y\tau_{yz}+n_z\tau_{zz} \end{pmatrix} = u(n_x\tau_{xx}+n_y\tau_{yx}+n_z\tau_{zx}) + v(n_x\tau_{xy}+n_y\tau_{yy}+n_z\tau_{zy}) + w(n_x\tau_{xz}+n_y\tau_{yz}+n_z\tau_{zz}).$$

$$\vec{n}\cdot(\vec{V}\bar{\bar{\tau}}) = \vec{n}\cdot\begin{pmatrix} u\tau_{xx}+v\tau_{yx}+w\tau_{zx} \\ u\tau_{xy}+v\tau_{yy}+w\tau_{zy} \\ u\tau_{xz}+v\tau_{yz}+w\tau_{zz} \end{pmatrix} = n_x(u\tau_{xx}+v\tau_{yx}+w\tau_{zx}) + n_y(u\tau_{xy}+v\tau_{yy}+w\tau_{zy}) + n_z(u\tau_{xz}+v\tau_{yz}+w\tau_{zz})$$

$$= u(n_x\tau_{xx}+n_y\tau_{xy}+n_z\tau_{xz}) + v(n_x\tau_{yx}+n_y\tau_{yy}+n_z\tau_{yz}) + w(n_x\tau_{zx}+n_y\tau_{zy}+n_z\tau_{zz})$$

Compare the simplified results of $\vec{V}\cdot(\vec{n}\cdot\bar{\bar{\tau}})$ and $\vec{n}\cdot(\vec{V}\bar{\bar{\tau}})$, It can be observed that

$\boxed{\vec{V}\cdot(\vec{n}\cdot\bar{\bar{\tau}}) = \vec{n}\cdot(\vec{V}\cdot\bar{\bar{\tau}}) \text{ as } \bar{\bar{\tau}} \text{ is Symmetric!}}$ Thus the divergence theorem can be applied.

Thus, the energy equation is given as: (with enough smoothness)

$$\frac{\partial}{\partial t}\oint \rho(e+\tfrac{1}{2}V^2)\,dv + \oint \nabla\cdot[\rho(e+\tfrac{1}{2}V^2)\vec{V}]\,dv = -\int \nabla\cdot\vec{q}\,dv + \int \rho\vec{g}\cdot\vec{V}\,dv + \int \nabla\cdot(\vec{V}\cdot\bar{\bar{\tau}})\,dv$$

$$\Leftrightarrow \frac{\partial}{\partial t}\rho(e+\tfrac{1}{2}V^2) + \nabla\cdot[\rho(e+\tfrac{1}{2}V^2)\vec{V}] = -\nabla\cdot\vec{q} + \rho\vec{g}\cdot\vec{V} + \nabla(\vec{V}\cdot\bar{\bar{\tau}})$$

Expand products with partial derivatives:

$$\text{LHS} = \frac{\partial\rho}{\partial t}(e+\tfrac{1}{2}V^2) + \rho\frac{\partial}{\partial t}(e+\tfrac{1}{2}V^2) + [\nabla\rho(e+\tfrac{1}{2}V^2)]\cdot\vec{V} + \rho(e+\tfrac{1}{2}V^2)\nabla\cdot\vec{V}$$

$$= \cancel{\frac{\partial\rho}{\partial t}(e+\tfrac{1}{2}V^2)} + \rho\frac{\partial}{\partial t}(e+\tfrac{1}{2}V^2) + [\rho\nabla(e+\tfrac{1}{2}V^2) + \cancel{(e+\tfrac{1}{2}V^2)\nabla\rho}]\cdot\vec{V} + \cancel{\rho(e+\tfrac{1}{2}V^2)\cdot\nabla\cdot\vec{V}}$$

$$= (e+\tfrac{1}{2}V^2)\left[\frac{\partial\rho}{\partial t} + \nabla\rho\cdot\vec{V} + \rho\cdot\nabla\cdot\vec{V}\right] + \rho\left[\frac{\partial}{\partial t}(e+\tfrac{1}{2}V^2) + \vec{V}\cdot\nabla(e+\tfrac{1}{2}V^2)\right]$$

$$= (e+\tfrac{1}{2}V^2)\underbrace{\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right]}_{\text{Mass conservation!} = 0} + \rho\frac{D}{Dt}(e+\tfrac{1}{2}V^2) \quad = \rho\frac{D}{Dt}(e+\tfrac{1}{2}V^2).$$

$$\Leftrightarrow \rho\frac{D}{Dt}(e+\tfrac{1}{2}V^2) = -\nabla\vec{q} + \rho\vec{g}\cdot\vec{V} + \nabla\cdot(\vec{V}\cdot\bar{\bar{\tau}})$$

With Kinetic Energy Equation: $\rho\frac{D}{Dt}(\tfrac{1}{2}V^2) = \rho\vec{g}\vec{V} + \vec{V}\cdot(\nabla\cdot\bar{\bar{\tau}})$

$$\Rightarrow \rho\frac{De}{Dt} = -\nabla\cdot\vec{q} + \nabla\cdot(\vec{V}\cdot\bar{\bar{\tau}}) - \vec{V}\cdot(\nabla\cdot\bar{\bar{\tau}})$$

$$\nabla\cdot(\vec{V}\cdot\bar{\bar{\tau}}) = \nabla\cdot\begin{pmatrix} u\tau_{xx}+v\tau_{yx}+w\tau_{zx} \\ u\tau_{xy}+v\tau_{yy}+w\tau_{zy} \\ u\tau_{xz}+v\tau_{yz}+w\tau_{zz} \end{pmatrix} = \frac{\partial}{\partial x}(u\tau_{xx}+v\tau_{yx}+w\tau_{zx}) + \frac{\partial}{\partial y}(u\tau_{xy}+v\tau_{yy}+w\tau_{zy}) + \frac{\partial}{\partial z}(u\tau_{xz}+v\tau_{yz}+w\tau_{zz}).$$

$$= \left[\left(\frac{\partial u}{\partial x}\tau_{xx} + \frac{\partial v}{\partial y}\tau_{yy} + \frac{\partial w}{\partial z}\tau_{zz}\right) + \left(\frac{\partial v}{\partial x}\tau_{yx} + \frac{\partial u}{\partial y}\tau_{xy}\right) + \left(\frac{\partial w}{\partial y}\tau_{zy} + \frac{\partial v}{\partial z}\tau_{yz}\right) + \left(\frac{\partial u}{\partial z}\tau_{xz} + \frac{\partial w}{\partial x}\tau_{zx}\right)\right]$$

$$+ \left[u\left(\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}\right) + v\left(\frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z}\right) + w\left(\frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{zy}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z}\right)\right]$$

$$\vec{V}\cdot(\nabla\cdot\bar{\bar{\tau}}) = \vec{V}\cdot\begin{pmatrix} \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} \\ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} \\ \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z} \end{pmatrix} = u\left(\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right) + v\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z}\right) + w\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \dots\right.$$

From the expansion, it can be seen that $\nabla\cdot(\vec{V}\cdot\bar{\bar{\tau}}) - \vec{V}\cdot(\nabla\cdot\bar{\bar{\tau}})$ can be simplified only when $\bar{\bar{\tau}} = \bar{\bar{\tau}}^T$!

Thus, the final form of energy equation in terms of internal energy is:

$$\boxed{\rho\frac{De}{Dt} = -\nabla\cdot\vec{q} + \underbrace{\tau_{ij}\frac{\partial u_j}{\partial x_i}}_{(9\text{ items})}} \quad \left(\text{if } \bar{\bar{\tau}} = \begin{bmatrix} -p+\tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p+\tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p+\tau_{zz} \end{bmatrix}\right)$$

$$= -\nabla\cdot\vec{q} - p\nabla\cdot\vec{V} + \tau_{ij}\frac{\partial u_j}{\partial x_i} \quad \left(\text{if } \bar{\bar{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}\right)$$

Volumetric force contribution vanishes! Means it does not change internal energy

Also, the energy equation can also be expressed in terms of enthalpy:

$$\rho\frac{Dh}{Dt} = \rho\frac{D}{Dt}(e + p/\rho) = \rho\frac{De}{Dt} + \rho\frac{D}{Dt}(p/\rho) = \rho\frac{De}{Dt} + \frac{Dp}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt}$$

$$\overset{\text{continuity}}{\underset{\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{V}=0}{=\!=}} \rho\frac{De}{Dt} + \frac{Dp}{Dt} - \frac{p}{\rho}(-\rho\nabla\cdot\vec{V}) = \rho\frac{De}{Dt} + \frac{Dp}{Dt} + p\nabla\cdot\vec{V} = -\nabla\cdot\vec{q} + \frac{Dp}{Dt} + \underbrace{\tau_{ij}\frac{\partial u_i}{\partial x_j}}_{\triangleq\phi}$$

Since radiation is volumetric, if it should be considered, an additional term q_R should be appended to the equations above directly as follows:

$$\rho\frac{De}{Dt} = -\nabla\cdot\vec{q} - p\nabla\cdot\vec{V} + \tau_{ij}\frac{\partial u_j}{\partial x_i} + q_R$$

$$\rho\frac{Dh}{Dt} = -\nabla\cdot\vec{q} + \tau_{ij}\frac{\partial u_j}{\partial x_i} + \frac{Dp}{Dt} + q_R$$

§ Species Equation:

In mult-component System, change of some species within a material surface is due to component diffusion and chemical reactions.

Component diffusion is described by Fick's Law:

$$-\oint \vec{J}_i \cdot d\vec{A} = -\int \nabla \cdot \vec{J}_i \cdot dv$$

Chemical reaction source : $\oint w_i \, dv$

$$\frac{D}{Dt}\oint \rho Y_i \, dv \xrightarrow{RTT} \frac{\partial}{\partial t}\int \rho Y_i \cdot dv + \oint \rho Y_i \vec{V} \cdot d\vec{s} = -\oint \vec{J}_i d\vec{s} + \oint w_i \, dv$$

Assumig enough smoothness:

$$\frac{\partial}{\partial t}(\rho Y_i) + \nabla \cdot (\rho Y_i \vec{V}) = -\nabla \cdot \vec{J}_i + w_i$$

$$LHS = \frac{\partial \rho}{\partial t} Y_i + \rho \frac{\partial Y_i}{\partial t} + \nabla(\rho Y_i) \cdot \vec{V} + \rho Y_i \cdot \nabla \vec{V}$$

$$= \cancel{\frac{\partial \rho}{\partial t} Y_i} + \rho \frac{\partial Y_i}{\partial t} + [\cancel{\nabla \rho Y_i} + \rho \nabla Y_i] \cdot \vec{V} + \cancel{\rho Y_i \nabla \vec{V}}$$

$$= Y_i \underbrace{\left[\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V}\right]}_{\text{Continuity Equation! } =0} + \rho \underbrace{\left[\frac{\partial Y_i}{\partial t} + \vec{V} \cdot \nabla Y_i\right]}_{\text{Mass Derivative}}$$

$$= \rho \frac{DY_i}{Dt}$$

Thus: $\rho \frac{DY_i}{Dt} = -\nabla \cdot \vec{J}_i + w_i$

§ Energy Equation for multi-component systems.

$$h_i = e_i + p/\rho = h_{i,ref} + \int_{T_{ref}}^{T} c_{p_i} \cdot dT$$

$$h = \sum Y_i h_i \Rightarrow \frac{Dh_i}{Dt} = \frac{\partial h_i}{\partial t} + \vec{V} \cdot \nabla h_i = c_{p_i} \frac{\partial T}{\partial t} + \vec{V} \cdot \nabla c_{p_i} T = c_{p_i} \frac{DT}{Dt}$$

Thus: $\rho \frac{Dh}{Dt} = \rho \frac{D}{Dt} \sum Y_i h_i = \rho \cdot \sum \left[\frac{DY_i}{Dt} h_i + Y_i \frac{Dh_i}{Dt} \right] = \rho \sum \left[h_i \frac{DY_i}{Dt} + C_{p_i} Y_i \frac{DT}{Dt} \right]$

with $\rho \frac{DY_i}{Dt} = -\nabla \cdot \vec{j}_i + w_i$ (Species equation).

$$\Rightarrow \rho \left[\left(\sum C_{p_i} Y_i \right) \frac{DT}{Dt} + \sum \left(-\nabla \cdot \vec{j}_i + w_i \right) h_i \right] = -\nabla \cdot \vec{q} + \phi + \frac{Dp}{Dt} + q_R \quad \text{(Apply Energy Equation)}$$

Simplification: $p = \text{const}$, $q_R = 0$, $C_{p_i} = C_p = \text{const}$, $\vec{q} = \lambda \nabla T$ (Fourier's Law)

$\sum Y_i = 1$.

$$\Rightarrow \rho C_p \frac{DT}{Dt} - \sum (\nabla \cdot \vec{j}_i) h_i = -\nabla (\lambda \nabla T) + \phi - \sum h_i w_i.$$

since $(\nabla \cdot \vec{j}_i) h_i = \nabla \cdot (h_i \vec{j}_i) - \vec{j}_i \cdot \nabla h_i$,

as $\nabla h_i = \nabla T \cdot C_{p_i}$. when C_{p_i} is assumed to be constant as above.

$$\sum (\nabla \cdot \vec{j}_i) h_i = \sum \nabla \cdot (h_i \vec{j}_i) - \underbrace{\left(\sum \vec{j}_i \right)}_{=0!} C_p \cdot \nabla T$$

$$= \sum \nabla \cdot (h_i \vec{j}_i)$$

$$\Rightarrow \rho C_p \frac{DT}{Dt} = \sum \nabla \cdot (h_i \vec{j}_i) - \nabla (\lambda \nabla T) + \phi - \sum h_i w_i.$$