SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Introduction to Numerical Analysis
HW4

Yu Cang
018370210001

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1 LEGENDRE POLYNOMIALS

1. *Proof.* Let

$$\varphi(x) = (x^2 - 1)^n \tag{1.1}$$

then

$$Q_n(x) = \frac{1}{2^n n!}\varphi^{(n)}(x) \tag{1.2}$$

and

$$\varphi^{(k)}(1) = \varphi^{(k)}(-1) = 0 \quad (k = 0, 1, ..., n-1) \tag{1.3}$$

Suppose $h(x) \in C^n(-1,1)$, then performing integration by parts

$$\begin{aligned}\int_{-1}^{1} P_n(x)h(x)dx &= \frac{1}{2^n n!}\int_{-1}^{1}\varphi^{(n)}(x)h(x)dx \\ &= -\frac{1}{2^n n!}\int_{-1}^{1}\varphi^{(n-1)}(x)h'(x)dx \\ &= ... \\ &= \frac{(-1)^n}{2^n n!}\int_{-1}^{1}\varphi(x)h^{(n)}(x)dx\end{aligned} \tag{1.4}$$

Thus, the proof can be discussed on 2 cases

a) If the order of $h(x)$ is less than n, then

$$h^{(n)}(x) = 0 \tag{1.5}$$

Thus

$$\int_{-1}^{1} Q_n(x)Q_m(x)dx = 0 \quad (n \neq m) \tag{1.6}$$

b) If $h(x) = Q_n(x)$, then the $n-th$ derivative of $g(x)$ is

$$h^{(n)}(x) = Q^{(n)}(x) = \frac{(2n)!}{2^n n!} \tag{1.7}$$

Thus

$$\begin{aligned}
\int_{-1}^{1} Q_n(x)Q_m(x)dx &= \int_{-1}^{1} Q_n^2(x)dx \quad (n = m) \\
&= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n dx \\
&= \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n dx \\
&= \frac{(2n)!}{2^{2n}(n!)^2}\int_{0}^{\pi/2} cos^{2n+1} t dt \\
&= \frac{(2n)!}{2^{2n}(n!)^2}\frac{2\times 4\times \ldots \times (2n)}{1\times 3\times \ldots \times (2n+1)} \\
&= \frac{2}{2n+1}
\end{aligned} \tag{1.8}$$

Thus, $(Q_n)_{n\in\mathbb{N}}$ are a sequence of orthogonal polynomials. □

2. *Proof.* Denote

$$\varphi(x) = (x^2-1)^n \tag{1.9}$$

then

$$Q_n(x) = \frac{1}{2^n n!}\varphi^{(n)}(x) \tag{1.10}$$

As the power of each item in $\varphi(x)$ is even when $\varphi(x)$ is extended, thus $\varphi^{(n)}(x)$ is even function if the order of derivative is even, and $\varphi^{(n)}(x)$ is odd function if the order of derivative is odd.
Therefore $Q_n(x)$ is even function if n is even, and $Q_n(x)$ is odd function if n is odd. So, it can be summarized as $Q_n(-x) = (-1)^n Q_n(x)$. □

3. *Proof.* As $xQ_n(x)$ can be written as

$$xQ_n(x) = \sum_{i=0}^{n+1} a_i Q_i(x) \tag{1.11}$$

According to the orthogonality of Legendre polynomials, $a_i = 0$ for $i = 0, 1, \ldots, n-2, n$, as

$$0 = \int_{-1}^{1} xQ_n(x)Q_i(x)dx = a_i\int_{-1}^{1} Q_i^2(x)dx \quad (i = 0, 1, \ldots, n-2, n) \tag{1.12}$$

Thus, $xQ_n(x)$ can be written as

$$xQ_n(x) = a_{n-1}Q_{n-1}(x) + a_{n+1}Q_{n+1}(x) \tag{1.13}$$

Since the highest order item in $xQ_n(x)$ is x^{n+1} and its coefficient is $\frac{(2n)!}{2^n(n!)^2}$, thus

$$\frac{(2n)!}{2^n(n!)^2} = a_{n+1}\frac{(2n+2)!}{2^{n+1}(n+1)!^2} \tag{1.14}$$

which implies that

$$a_{n+1} = \frac{n+1}{2n+1} \tag{1.15}$$

Denote

$$I_n = \int_{-1}^{1} xQ_{n-1}(x)Q_n(x)dx \tag{1.16}$$

Then

$$I_n = a_{n-1}\int_{-1}^{1} Q_{n-1}^2(x)dx = a_{n-1}\frac{2}{2n-1} \tag{1.17}$$

As

$$I_{n+1} = a_{n+1}\int_{-1}^{1} Q_{n+1}^2(x)dx = a_{n+1}\frac{2}{2n+3} = \frac{2(n+1)}{(2n+1)(2n+3)} \tag{1.18}$$

Thus

$$I_n = \frac{2n}{(2n-1)(2n+1)} \tag{1.19}$$

which implies that

$$a_{n-1} = \frac{n}{2n+1} \tag{1.20}$$

Hence

$$(2n+1)xQ_n(x) = nQ_{n-1}(x) + (n+1)Q_{n+1}(x) \tag{1.21}$$

□

4. *Proof.* As

$$\begin{aligned} Q_n(x) &= \frac{1}{2^n n!}[(x+1)^n(x-1)^n]^{(n)} \\ &= \frac{2^n}{n!}[(\frac{x+1}{2})^n(\frac{x-1}{2})^n]^{(n)} \end{aligned} \tag{1.22}$$

Denote

$$f(x) = (\frac{x+1}{2})^n \tag{1.23}$$

and

$$g(x) = (\frac{x-1}{2})^n \tag{1.24}$$

Then

$$f^{(k)}(x) = \frac{n!}{2^k(n-k)!}(\frac{x+1}{2})^{n-k} \tag{1.25}$$

and

$$g^{(k)}(x) = \frac{n!}{2^k(n-k)!}(\frac{x-1}{2})^{n-k} \tag{1.26}$$

Since

$$(fg)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)} \tag{1.27}$$

Thus

$$\begin{aligned}
Q_n(x) &= \frac{2^n}{n!}\sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x) \\
&= \frac{2^n}{n!}\sum_{k=0}^{n} C_n^k \frac{n!}{2^k(n-k)!}(\frac{x+1}{2})^{n-k}\frac{n!}{2^{n-k}k!}(\frac{x-1}{2})^k \\
&= \sum_{k=0}^{n} C_n^k \frac{n!}{(n-k)!k!}(\frac{x+1}{2})^{n-k}(\frac{x-1}{2})^k \\
&= \sum_{k=0}^{n} (C_n^k)^2(\frac{x+1}{2})^{n-k}(\frac{x-1}{2})^k \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 (\frac{1+x}{2})^{n-k}(\frac{1-x}{2})^k
\end{aligned} \tag{1.28}$$

□

2 Interpolation

$f(2)$ can be determined using the Lagrange interpolation scheme. As the lagrange interpolation polynomial can be written as below, and $n = 8$ in this case.

$$f(x) = \sum_{i=1}^{n} f(x_i) l_i(x) \tag{2.1}$$

$l_i(x)$ are the base functions that can be written as below.

$$l_i(x) = \frac{(x-x_1)(x-x_2)...(x-x_{i-1})(x-x_{i+1})...(x-x_{n-1})(x-x_n)}{(x_i-x_1)(x_i-x_2)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_{n-1})(x_i-x_n)} \tag{2.2}$$

$l_i(2)$ are calculated accordingly as below.

$$\begin{array}{llll}
l_1(2) = -0.0006 & l_2(2) = 0.1224 & l_3(2) = -0.5600 & l_4(2) = 1.0606 \\
l_5(2) = 0.4167 & l_6(2) = -0.0400 & l_7(2) = 0.0012 & l_8(2) = -0.0003
\end{array}$$

Thus, $f(2)$ is calculated according to (2.1) as 11.0.

3 NEWTON'S FORM OF INTERPOLATION POLYNOMIAL

1. a) *Proof.* Denote $P^1(x) = a_0 + a_1 x$, where a_0 and a_1 are coefficients to be determined.Then

$$\begin{cases} f(x_0) = a_0 + a_1 x_0 \\ f(x_1) = a_0 + a_1 x_1 \end{cases} \tag{3.1}$$

a_0 and a_1 are solved as below

$$\begin{cases} a_0 = \dfrac{x_1 f(x_0) - x_0 f(x_1)}{x_1 - x_0} \\ a_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} \end{cases} \tag{3.2}$$

Thus

$$\begin{aligned} P^1(x) &= \frac{x_1 f(x_0) - x_0 f(x_1)}{x_1 - x_0} + \frac{f(x_1) - f(x_0)}{x_1 - x_0} x \\ &= \frac{(x_1 - x_0) f(x_0) - x_0 (f(x_1) - f(x_0))}{x_1 - x_0} + \frac{f(x_1) - f(x_0)}{x_1 - x_0} x \\ &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) \\ &= P^0(x) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) \end{aligned} \tag{3.3}$$

□

b) If $P^2(x) = P^1(x) + R(x)$, then $R(x_0) = 0$ and $R(x_1) = 0$, thus $R(x) = c(x - x_1)(x - x_0)$, where c is the coefficient to be determined. Since

$$R(x_2) = P^2(x_2) - P^1(x_2) \tag{3.4}$$

Thus

$$c(x_2 - x_1)(x_2 - x_0) = f(x_2) - [f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)] \tag{3.5}$$

c is solved as

$$c = \frac{1}{x_2 - x_0}[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}] \tag{3.6}$$

Hence

$$R(x) = \frac{1}{x_2 - x_0}[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}](x - x_1)(x - x_0) \tag{3.7}$$

c) *Proof.* As have been shown above, the statement is valid for $j = 1, 2$. Suppose it's still valid for $j = n$, then

$$P^n(x) = P^{n-1}(x) + a_n \prod_{k=0}^{n-1} (x - x_k) \tag{3.8}$$

where a_n depends only on $a_0, a_1, ..., a_n$.
Let $P^{n+1}(x) = P^n(x) + R(x)$, then $R(x_k) = 0$ for $k = 0, 1, 2, ..., n$, thus

$$R(x) = a_{n+1} \prod_{k=0}^{n} (x - x_k) \tag{3.9}$$

Then a_{n+1} is solved as

$$a_{n+1} = \frac{f(x_{n+1}) - P^n(x_{n+1})}{\prod_{k=0}^{n}(x - x_k)} \tag{3.10}$$

As a_n only depends on $a_0, a_1, ..., a_n$, $P^n(x_{n+1})$ only depends on $x_0, x_1, ..., x_n, x_{n+1}$, thus a_{n+1} only depends on $a_0, a_1, ..., a_n, a_{n+1}$. □

2. *Proof.* As

$$P^j(x) = P^{j-1}(x) + a_j \prod_{k=0}^{j-1} (x - x_k) \tag{3.11}$$

Thus, by recursion, $P^j(x)$ can be written as

$$P^n(x) = P^0(x) + \sum_{j=1}^{n} a_j \prod_{k=0}^{j-1} (x - x_k) \tag{3.12}$$

□

3. *Proof.* (haven't figured out yet...) □

4. Algorithm that calculate the approximation of $f(x)$ is given as below

Algorithm 1 Calculation of the approximated value of $f(x)$

Input: x, n, $x[i]$, $f[i]$ $(i = 0, 1, ..., n)$
Output: $P^n(x) \approx f(x)$
$ret \leftarrow 0$
$prod \leftarrow 1$
for $i = 0 \rightarrow n$ **do**
  $coef[i] \leftarrow f[i]$
end for
for $i = 0 \rightarrow n$ **do**
  $ret \leftarrow coef[i] * prod$
  $prod \leftarrow prod * (x - x_i)$
  for $j = n \rightarrow i + 1$ **do**
    $coef[j] \leftarrow \frac{coef[j] - coef[j-1]}{x[j] - x[j-(i+1)]}$
  end for
end for
return ret

5. *Proof.* As

$$f[x_k, x_{k+1}] = \frac{f_{k+1} - f_k}{x_{k+1} - x_k} = \frac{\nabla f_k}{h} \tag{3.13}$$

The statement is valid for $m = 1$.
Assume it is still valid for $m = n$, then

$$f[x_i, x_{i+1}, ..., x_{i+n}] = \frac{\nabla^n f_i}{n! h^n} \tag{3.14}$$

Consider $m = n+1$, by the definition of $f[x_i, x_{i+1}, ..., x_{i+n+1}]$

$$f[x_i, x_{i+1}, ..., x_{i+n+1}] = \frac{f[x_{i+1}, ..., x_{i+n+1}] - f[x_i, x_{i+1}, ..., x_n]}{x_{i+n+1} - x_i} \tag{3.15}$$

By the induction hypothesis, it can be simplified as

$$f[x_i, x_{i+1}, ..., x_{i+n+1}] = \frac{1}{(n+1)h} \frac{1}{n! h^n} [\nabla^n f_{i+1} - \nabla^n f_i] \tag{3.16}$$

By the definition of operator ∇, it can be further simplified as

$$f[x_i, x_{i+1}, ..., x_{i+n+1}] = \frac{\nabla^{n+1} f_i}{(n+1)! h^{n+1}} \tag{3.17}$$

Thus, the statement is valid for all $m \geq 1$.

□

6. (haven't figured out yet...)

7. Algorithm that calculate the approximation of $f(x)$ with equidistant nodes is given as below

Algorithm 2 Calculation of the approximated value of $f(x)$ with equidistant nodes

Input: h, n, x, $f[i]$ $(i = 0, 1, \ldots, n)$
Output: $P^n(x) \approx f(x)$

$ret \leftarrow 0$
$prod \leftarrow 1$
$t \leftarrow \frac{x - x_0}{h}$
for $i = 0 \rightarrow n$ **do**
    $coef[i] \leftarrow f[i]$
end for
for $i = 0 \rightarrow n$ **do**
    $ret \leftarrow coef[i] \times prod$
    $prod \leftarrow prod \times \frac{t-i}{i+1}$
    for $j = n \rightarrow i+1$ **do**
        $coef[j] \leftarrow coef[j] - coef[j-1]$
    end for
end for
return ret