SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Turbulence
HW2

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1 EXERCISE1

An identity is used

$$\frac{du(t+s)}{dt} = \frac{du(t+s)}{d(t+s)} = \frac{du(t+s)}{ds} \tag{1.1}$$

Since the structure function and two-point correlation function are defined as

$$N_i\{\vec{x},t\} = \frac{\partial u_i'}{\partial t} + \overline{u_k}\frac{\partial u_i'}{\partial x_k} + u_k'\frac{\partial \overline{u_i}}{x_k} + \frac{\partial (u_i' u_k')}{\partial x_k} - \frac{\partial \overline{u_i' u_k'}}{\partial x_k} + \frac{1}{\rho}\frac{\partial p'}{\partial x_i} - \nu\frac{\partial^2 u_i'}{\partial x_k \partial x_k} - f_i' = 0 \tag{1.2}$$

and

$$R_{ij} = \overline{u_i'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)} \tag{1.3}$$

Thus, for $D^{(i)}\{R_{ij}\} = \overline{u_j'(\vec{x}+\vec{r},t+\tau)N_i\{\vec{x},t\}}$, components in the expansion are calculated as

$$\begin{aligned}
\overline{u_j'(\vec{x}+\vec{r},t+\tau)\frac{\partial u_i'(\vec{x},t)}{\partial t}} &= \overline{\frac{\partial (u_i'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau))}{\partial t} - u_i'(\vec{x},t)\frac{\partial u_j'(\vec{x}+\vec{r},t+\tau)}{\partial t}} \\
&= \frac{\partial \overline{u_i'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)}}{\partial t} - \overline{u_i'(\vec{x},t)\frac{\partial u_j'(\vec{x}+\vec{r},t+\tau)}{\partial \tau}} \\
&= \frac{\partial R_{ij}}{\partial t} - \frac{\partial \overline{u_i'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)}}{\partial \tau} \\
&= \frac{\partial R_{ij}}{\partial t} - \frac{\partial R_{ij}}{\partial \tau}
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
\overline{u'_j(\vec{x}+\vec{r},t+\tau)\overline{u_k}\frac{\partial u'_i}{\partial x_k}} &= \overline{u_k(\vec{x},t)}\,\overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial u'_i(\vec{x},t)}{\partial x_k}} \\
&= \overline{u_k(\vec{x},t)}\left[\frac{\partial \overline{u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_k} - \overline{u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial (x_k+r_k)}}\right] \\
&= \overline{u_k(\vec{x},t)}\left[\frac{\partial R_{ij}}{\partial x_k} - \overline{u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}}\right] \\
&= \overline{u_k(\vec{x},t)}\left[\frac{\partial R_{ij}}{\partial x_k} - \frac{\partial \overline{u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_k}\right] \\
&= \overline{u_k(\vec{x},t)}\left[\frac{\partial R_{ij}}{\partial x_k} - \frac{\partial R_{ij}}{\partial r_k}\right]
\end{aligned} \tag{1.5}$$

$$\overline{u'_j(\vec{x}+\vec{r},t+\tau)u'_k(\vec{x},t)\frac{\partial \overline{u_i(\vec{x},t)}}{\partial x_k}} = \overline{u'_k(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}\frac{\partial \overline{u_i(\vec{x},t)}}{\partial x_k} = R_{kj}\frac{\partial \overline{u_i(\vec{x},t)}}{\partial x_k} \tag{1.6}$$

$$\begin{aligned}
\overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial[u'_i(\vec{x},t)u'_k(\vec{x},t)]}{\partial x_k}} &= \frac{\partial \overline{[u'_i(\vec{x},t)u'_k(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}}{\partial x_k} - \overline{u'_i(\vec{x},t)u'_k(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}} \\
&= \frac{\partial R_{(ik)j}}{\partial x_k} - \overline{u'_i(\vec{x},t)u'_k(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}} \\
&= \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial \overline{[u'_i(\vec{x},t)u'_k(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}}{\partial r_k} \\
&= \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial R_{(ik)j}}{\partial r_k}
\end{aligned} \tag{1.7}$$

$$\overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial \overline{u'_i(\vec{x},t)u'_k(\vec{x},t)}}{\partial x_k}} = \overline{u'_j(\vec{x}+\vec{r},t+\tau)}\frac{\partial \overline{u'_i(\vec{x},t)u'_k(\vec{x},t)}}{\partial x_k} = 0 \tag{1.8}$$

$$\begin{aligned}
\overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial p'(\vec{x},t)}{\partial x_i}} &= \frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \overline{p'(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_i}} \\
&= \frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \overline{p'(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial (x_i+r_i)}} \\
&= \frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \overline{p'(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_i}} \\
&= \frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_i} \\
&= \frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_i}
\end{aligned} \tag{1.9}$$

$$
\begin{aligned}
&\overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial^2 u'_i(\vec{x},t)}{\partial x_k \partial x_k}} = \overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial}{\partial x_k}\left(\frac{\partial u'_i(\vec{x},t)}{\partial x_k}\right)} \\
&= \frac{\partial}{\partial x_k}\overline{\left(u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial u'_i(\vec{x},t)}{\partial x_k}\right)} - \overline{\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}\frac{\partial u'_i(\vec{x},t)}{\partial x_k}} \\
&= \frac{\partial}{\partial x_k}\overline{\left(\frac{\partial[u'_j(\vec{x}+\vec{r},t+\tau)u'_i(\vec{x},t)]}{\partial x_k} - u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}\right)} - \overline{\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}\frac{\partial u'_i(\vec{x},t)}{\partial x_k}} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - \overline{\frac{\partial}{\partial x_k}\left(u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}\right)} - \overline{\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}\frac{\partial u'_i(\vec{x},t)}{\partial x_k}} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - \overline{\frac{\partial}{\partial x_k}\frac{\partial[u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}{\partial r_k}} - \overline{\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}\frac{\partial u'_i(\vec{x},t)}{\partial x_k}} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - \frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} - \overline{\left(\frac{\partial}{\partial x_k}\left(u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}\right) - u'_i(\vec{x},t)\frac{\partial^2 u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k \partial r_k}\right)} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - \frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} - \overline{\left(\frac{\partial}{\partial x_k}\left(\frac{\partial[u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}{\partial r_k}\right) - u'_i(\vec{x},t)\frac{\partial^2 u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k \partial r_k}\right)} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + \overline{\frac{\partial^2[u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}{\partial r_k \partial r_k}} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + \frac{\partial^2 R_{ij}}{\partial r_k \partial r_k}
\end{aligned} \tag{1.10}
$$

then, detailed expression for $D^{(i)}\{R_{ij}\}$ is given as

$$
\begin{aligned}
D^{(i)}\{R_{ij}\} &= \frac{\partial R_{ij}}{\partial t} - \frac{\partial R_{ij}}{\partial \tau} + \overline{u_k(\vec{x},t)}\left[\frac{\partial R_{ij}}{\partial x_k} - \frac{\partial R_{ij}}{\partial r_k}\right] + R_{kj}\frac{\partial \overline{u_i(\vec{x},t)}}{\partial x_k} \\
&+ \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial R_{(ik)j}}{\partial r_k} + \frac{1}{\rho}\left(\frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_i}\right) \\
&- \nu\left(\frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + \frac{\partial^2 R_{ij}}{\partial r_k \partial r_k}\right) - \overline{u'_j(\vec{x}+\vec{r},t+\tau)f'_i(\vec{x},t)} = 0
\end{aligned} \tag{1.11}
$$

For $D^{(j)}\{R_{ij}\} = \overline{u'_i(\vec{x},t)N_j\{\vec{x}+\vec{r},t+\tau\}}$, components in the expansion are calculated as

$$
\overline{u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial(t+\tau)}} = \overline{u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial \tau}} = \frac{\partial \overline{u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial \tau} = \frac{\partial R_{ij}}{\partial \tau} \tag{1.12}
$$

$$
\begin{aligned}
&\overline{u'_i(\vec{x},t)\overline{u_k(\vec{x}+\vec{r},t+\tau)}\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial(x_k+r_k)}} = \overline{u_k(\vec{x}+\vec{r},t+\tau)}\,\overline{u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}} \\
&= \overline{u_k(\vec{x}+\vec{r},t+\tau)}\frac{\partial \overline{[u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}}{\partial r_k} = \overline{u_k(\vec{x}+\vec{r},t+\tau)}\frac{\partial R_{ij}}{\partial r_k}
\end{aligned} \tag{1.13}
$$

$$\overline{u_i'(\vec{x},t)\,u_k'(\vec{x}+\vec{r},t+\tau)\frac{\partial\overline{u_j(\vec{x}+\vec{r},t+\tau)}}{\partial(x_k+r_k)}} = R_{ik}\frac{\partial\overline{u_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_k} \tag{1.14}$$

$$\begin{aligned}\overline{u_i'(\vec{x},t)\frac{\partial[u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)]}{\partial(x_k+r_k)}} &= \overline{u_i'(\vec{x},t)\frac{\partial[u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)]}{\partial r_k}}\\ &= \frac{\partial\overline{[u_i'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)]}}{\partial r_k} = \frac{\partial R_{i(jk)}}{\partial r_k}\end{aligned} \tag{1.15}$$

$$\overline{u_i'(\vec{x},t)\frac{\partial\overline{u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)}}{\partial(x_k+r_k)}} = \overline{u_i'(\vec{x},t)}\frac{\partial\overline{u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)}}{\partial r_k} = 0 \tag{1.16}$$

$$\overline{u_i'(\vec{x},t)\frac{\partial p'(\vec{x}+\vec{r},t+\tau)}{\partial(x_j+r_j)}} = \overline{u_i'(\vec{x},t)\frac{\partial p'(\vec{x}+\vec{r},t+\tau)}{\partial r_j}} = \frac{\partial\overline{p'(\vec{x}+\vec{r},t+\tau)u_i'(\vec{x},t)}}{\partial r_j} \tag{1.17}$$

$$\begin{aligned}\overline{u_i'(\vec{x},t)\frac{\partial^2 u_j'(\vec{x}+\vec{r},t+\tau)}{\partial(x_k+r_k)\partial(x_k+r_k)}} &= \overline{u_i'(\vec{x},t)\frac{\partial^2 u_j'(\vec{x}+\vec{r},t+\tau)}{\partial r_k\partial r_k}}\\ &= \frac{\partial^2\overline{u_i'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)}}{\partial r_k\partial r_k} = \frac{\partial R_{ij}}{\partial r_k\partial r_k}\end{aligned} \tag{1.18}$$

then, detailed expression for $D^{(j)}\{R_{ij}\}$ is given as

$$\begin{aligned}D^{(j)}\{R_{ij}\} = \frac{\partial R_{ij}}{\partial\tau} + \overline{u_k(\vec{x}+\vec{r},t+\tau)}\frac{\partial R_{ij}}{\partial r_k} + R_{ik}\frac{\partial\overline{u_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_k} + \frac{\partial R_{i(jk)}}{\partial r_k}\\ +\frac{1}{\rho}\frac{\partial\overline{p'(\vec{x}+\vec{r},t+\tau)u_i'(\vec{x},t)}}{\partial r_j} - \nu\frac{\partial R_{ij}}{\partial r_k\partial r_k} - \overline{u_i'(\vec{x},t)f_j'(\vec{x}+\vec{r},t+\tau)} = 0\end{aligned} \tag{1.19}$$

Therefore, the two-point correlation equation is calculated as

$$\begin{aligned}D^{(i)}\{R_{ij}\} + D^{(j)}\{R_{ij}\} = \frac{\partial R_{ij}}{\partial t} + \overline{u_k(\vec{x},t)}\frac{\partial R_{ij}}{\partial x_k} + \left[\overline{u_k(\vec{x}+\vec{r},t+\tau)} - \overline{u_k(\vec{x},t)}\right]\frac{\partial R_{ij}}{\partial r_k}\\ +R_{kj}\frac{\partial\overline{u_i(\vec{x},t)}}{\partial x_k} + R_{ik}\frac{\partial\overline{u_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_k} + \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial[R_{(ik)j} - R_{i(jk)}]}{\partial r_k}\\ +\frac{1}{\rho}\left(\frac{\partial\overline{p'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \frac{\partial\overline{p'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)}}{\partial r_i} + \frac{\partial\overline{p'(\vec{x}+\vec{r},t+\tau)u_i'(\vec{x},t)}}{\partial r_j}\right)\\ -\nu\left(\frac{\partial^2 R_{ij}}{\partial x_k\partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k\partial r_k} + 2\frac{\partial^2 R_{ij}}{\partial r_k\partial r_k}\right) - \overline{u_j'(\vec{x}+\vec{r},t+\tau)f_i'(\vec{x},t)} - \overline{u_i'(\vec{x},t)f_j'(\vec{x}+\vec{r},t+\tau)} = 0\end{aligned} \tag{1.20}$$

2 Exercise 2