VV570
Numerical Analysis

Assignment 2

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Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Connected space*
In this exercise we provide alternative definitions for a connected space. Let X be a metric space. We want to prove that the following conditions are equivalent

(i) The only subsets that are both open and closed in X are X and $\emptyset$;
(ii) It is impossible to write X as the union of two disjoint, non-empty open subsets;
(iii) It is impossible to write X as the union of two disjoint non-empty closed subsets;
(iv) There is no continuous, surjective application from X into $\{0,1\} \subset \mathbb{R}$;

1. Prove that (i), (ii), and (iii) are equivalent.
* 2. Assume (iv) is false and show that (iii) is false.
* 3. Assume (iii) is false and show that (iv) is false.

Ex. 2 — *Intermediate value theorem*
In this exercise we prove the intermediate value theorem following the sketch of proof provided in the slides (proof 2.11). From a high level perspective the result relies on the connectedness property of the real numbers.

1. Let X and Y be two metric spaces, A be a connected subset of X, and $f : X \to Y$ be a continuous map. Show that $f(A)$ is connected.
   Hint: use one of the characterisations of a connected set from exercise 1.
2. Let A be a subset of $\mathbb{R}$. We want to prove that A is connected if and only if A is an interval.
   a) Show that it is true for the empty set, and for all the subsets of $\mathbb{R}$ composed of a single element.
   b) Assuming that A is not an interval prove that A is not connected.
   c) We now prove the converse.
      i – Let I and J be two non-empty open intervals of $\mathbb{R}$. Show that there exists a continuous bijection from I into J whose inverse is also a continuous bijection.
      ii – Using question 1, show that it suffices to prove that $\mathbb{R}$ is connected.
      * iii – Let U be a subset of $\mathbb{R}$, different from $\mathbb{R}$, that is both open and closed in $\mathbb{R}$. Find a contradiction and conclude that $\mathbb{R}$ is connected.
      Hint: observe that a closed and non-empty set having an infimum has a minimum.
3. Conclude the proof of the intermediate value theorem in the case of the real numbers.

Ex. 3 — *Rolle's theorem*

Reasoning by induction and applying the extreme values theorem, prove Role's theorem (theorem 2.18).

Ex. 4 — *Extreme value theorem*

In this exercise we prove the extreme value theorem following the sketch of proof provided in the lecture slides (proof 2.14).

1. Let X be a metric space and A a subset of X.
   a) Show that if A is a compact subset of X then A is closed in X.
   b) Prove that if a subset of $\mathbb{R}$ is compact then it is closed and bounded.
2. Keeping the same notations we now prove the converse.
   a) Prove that if X is compact and A is closed in X, then A is a compact subset of X.
   * b) We want to prove that for any $a \leq b \in \mathbb{R}$, $[a, b]$ is compact in $\mathbb{R}$. Let $L = [a, b]$ et $(U_i)_{i \in I}$ be a family of opens from $\mathbb{R}$ covering L. We define A as the set of $x \in L$ such that $[a, x]$ is covered by a finite number of U_i.
      i – Prove the result for $a = b$.
      ii – We now treat the case where $a \in A$ and $a < b$. Let m be the supremum of A. Show that $m \in A$.
      iii – Assume $m < b$, and show the existence of $y \in A$ such that $y > m$.
      iv – Conclude that if A is closed and bounded then it is compact.
3. Complete the proof of the extreme value theorem.

Ex. 5 — *Continuity*

In this exercise we provide alternative characterisations for a continuous function and in particular complete proof 2.10. Let X and Y be two metric spaces, f be a function from X into Y, and $a \in X$. We want to prove that the following conditions are equivalent

(i) For all $\varepsilon \in \mathbb{R}_+^*$, there exists $\eta \in \mathbb{R}_+^*$ such that $f(B(a, \eta)) \subset B(f(a), \varepsilon)$;
(ii) For all $\varepsilon \in \mathbb{R}_+^*$, there exists $\eta \in \mathbb{R}_+^*$ such that $d(f(x), f(a)) < \varepsilon$ when $d(a, x) < \eta$, for $x \in X$;
(iii) For any neighborhood V of $f(a)$, there exists a neighborhood U of a such that $f(U) \subset V$;
(iv) For any neighborhood V of $f(a)$, $f^{-1}(V)$ is a neighborhood of a;

1. Show that (i) and (ii) are equivalent;
* 2. Consider a neighborhood of $f(a)$ and prove that (i) implies (iii).
* 3. Observe that if V is a neighborhood of $a \in X$, then any subset of X containing V is a neighborhood of a. Conclude that (iii) implies (iv).
4. Consider $V = B(f(a), \varepsilon)$, for some ε, and prove that (iv) implies (i).