

# Introduction to Numerical Analysis

## HW4

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### 1 LEGENDRE POLYNOMIALS

1. *Proof.* Let

$$\varphi(x) = (x^2 - 1)^n \quad (1.1)$$

then

$$Q_n(x) = \frac{1}{2^n n!} \varphi^{(n)}(x) \quad (1.2)$$

and

$$\varphi^{(k)}(1) = \varphi^{(k)}(-1) = 0 \quad (k = 0, 1, \dots, n-1) \quad (1.3)$$

Suppose  $h(x) \in C^n(-1, 1)$ , then performing integration by parts

$$\begin{aligned} \int_{-1}^1 P_n(x) h(x) dx &= \frac{1}{2^n n!} \int_{-1}^1 \varphi^{(n)}(x) h(x) dx \\ &= -\frac{1}{2^n n!} \int_{-1}^1 \varphi^{(n-1)}(x) h'(x) dx \\ &= \dots \\ &= \frac{(-1)^n}{2^n n!} \int_{-1}^1 \varphi(x) h^{(n)}(x) dx \end{aligned} \quad (1.4)$$

Thus, the proof can be discussed on 2 cases

a) If the order of  $g(x)$  is less than  $n$ , then

$$g^{(n)}(x) = 0 \quad (1.5)$$

Thus

$$\int_{-1}^1 Q_n(x)Q_m(x)dx = 0 \quad (n \neq m) \quad (1.6)$$

b) If  $g(x) = Q_n(x)$ , the the  $n - th$  derivative of  $g(x)$  is

$$g^{(n)}(x) = Q^{(n)}(x) = \frac{(2n)!}{2^n n!} \quad (1.7)$$

Thus

$$\begin{aligned} \int_{-1}^1 Q_n(x)Q_m(x)dx &= \int_{-1}^1 Q_n^2(x)dx \\ &= \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} \int_{-1}^1 (x^2 - 1)^n dx \\ &= \frac{(2n)!}{2^{2n} (n!)^2} \int_{-1}^1 (1 - x^2)^n dx \\ &= \frac{(2n)!}{2^{2n} (n!)^2} \int_0^{\pi/2} \cos^{2n+1} t dt \\ &= \frac{(2n)!}{2^{2n} (n!)^2} \frac{2 \times 4 \times \dots \times (2n)}{1 \times 3 \times \dots \times (2n+1)} \\ &= \frac{2}{2n+1} \quad (n = m) \end{aligned} \quad (1.8)$$

Thus,  $(Q_n)_{n \in \mathbb{N}}$  are a sequence of orthogonal polynomials.

□

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3.

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## 2 INTERPOLATION

$f(2)$  can be determined using the Lagrange interpolation scheme. As the lagrange interpolation polynomial can be written as below, and  $n = 8$  in this case.

$$f(x) = \sum_{i=1}^n f(x_i) l_i(x) \quad (2.1)$$

$l_i(x)$  are the base functions that can be written as below.

$$l_i(x) = \frac{(x - x_1)(x - x_2) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_{n-1})(x - x_n)}{(x_i - x_1)(x_i - x_2) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_{n-1})(x_i - x_n)} \quad (2.2)$$

$l_i(2)$  are calculated accordingly as below.

$$\begin{array}{llll} l_1(2) = -0.0006 & l_2(2) = 0.1224 & l_3(2) = -0.5600 & l_4(2) = 1.0606 \\ l_5(2) = 0.4167 & l_6(2) = -0.0400 & l_7(2) = 0.0012 & l_8(2) = -0.0003 \end{array}$$

Thus,  $f(2)$  is calculated according to (2.1) as 11.0.

### 3 NEWTON'S FORM OF INTERPOLATION POLYNOMIAL

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.