Vv556 Methods of Applied Mathematics I

Linear Operators

Assignment 8

Date Due: 12:10 PM, Thursday, the 22nd of November 2018

This assignment has a total of **(20 Marks)**.

Exercise 8.1
Define the operator $L\colon \ell^2 \to \ell^2$ by

$$L(x_n) = \left(x_1, \frac{1}{\sqrt{2}}x_2, \frac{1}{\sqrt{3}}x_3, \frac{1}{\sqrt{4}}x_4, \dots\right).$$

i) Show that L is not a Hilbert-Schmidt operator.
(1 Mark)

ii) Show that L is self-adjoint.
(1 Mark)

iii) Show that L is compact.
(2 Marks)

iv) Find upper and lower bounds for the spectrum of L.
(2 Marks)

v) Find the spectrum of L.
(4 Marks)

Exercise 8.2
Let $M := \{u \in L^2([0,1])\colon u \in C^2(0,1),\ u(0) = u(1) = 0\}$ and

$$L = -\frac{d^2}{dx^2}.$$

Let $K\colon L^2([0,1]) \to L^2([0,1])$ be given by

$$(Ku)(x) := \int_0^1 g(x,\xi)u(\xi)\,d\xi$$

with

$$g(x,\xi) := \begin{cases} x(1-\xi) & x < \xi, \\ \xi(1-x) & x \geq \xi. \end{cases}$$

i) Show that K is the inverse of L, i.e., $KL = I$ on M. (This requires some elementary calculations with the integral.)
(2 Marks)

ii) Show that L is unbounded.
(2 Marks)

iii) Show that K is compact.
(2 Marks)

iv) Show that $g(x,\xi) = g(\xi,x)$ and deduce that K is self-adjoint.
(2 Marks)

v) Find the upper and lower bounds of the Rayleigh quotient for K.
(2 Marks)