SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Turbulence
HW2

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1 EXERCISE1

Consider the N-S equation

$$\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_k \partial x_k} + f_j \tag{1.1}$$

With the ensemble average, each quantity can be decomposed into mean and fluctuation. Namely

$$\begin{aligned} u_j &= U_j + u'_j \\ p &= P + p' \\ f_j &= F_j + f'_j \end{aligned} \tag{1.2}$$

Thus, the N-S equation can be expanded as

$$\left(\frac{\partial U_j}{\partial t} + \frac{\partial u'_j}{\partial t}\right) + \left(U_k \frac{\partial U_j}{\partial x_k} + u'_k \frac{\partial U_j}{\partial x_k} + U_k \frac{\partial u'_j}{\partial x_k} + u'_k \frac{\partial u'_j}{\partial x_k}\right) = -\frac{1}{\rho}\left(\frac{\partial P}{\partial x_j} + \frac{\partial p'}{\partial x_j}\right) + \nu\left(\frac{\partial^2 U_j}{\partial x_k \partial x_k} + \frac{\partial^2 u'_j}{\partial x_k \partial x_k}\right) + (F_j + f'_j) \tag{1.3}$$

Taking ensemble average on the equation above yiels the so called Reynolds equation

$$\frac{\partial U_j}{\partial t} + U_k \frac{\partial U_j}{\partial x_k} + \overline{u'_k \frac{\partial u'_j}{\partial x_k}} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu \frac{\partial^2 U_j}{\partial x_k \partial x_k} + F_j \tag{1.4}$$

Substract the Reynolds equation by N-S equation that has been expaneded yields

$$\frac{\partial u'_j}{\partial t} + u'_k \frac{\partial U_j}{\partial x_k} + U_k \frac{\partial u'_j}{\partial x_k} + u'_k \frac{\partial u'_j}{\partial x_k} - \overline{u'_k \frac{\partial u'_i}{\partial x_k}} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \nu \frac{\partial^2 u'_j}{\partial x_k \partial x_k} + f'_j \tag{1.5}$$

With the continuity equation for incompressible flow

$$\frac{\partial u'_k}{\partial x_k} = 0 \tag{1.6}$$

the substracted equation can be re-written as

$$N_j\{\vec{x}, t\} \triangleq \frac{\partial u'_j}{\partial t} + u'_k \frac{\partial \overline{u_j}}{x_k} + \overline{u_k}\frac{\partial u'_j}{\partial x_k} + \frac{\partial [u'_j u'_k]}{\partial x_k} - \frac{\partial \overline{u'_j u'_k}}{\partial x_k} + \frac{1}{\rho}\frac{\partial p'}{\partial x_j} - \nu \frac{\partial^2 u'_j}{\partial x_k \partial x_k} - f'_j = 0 \tag{1.7}$$

Also, an identity is frequently used within the exercise

$$\frac{du(t+s)}{dt} = \frac{du(t+s)}{d(t+s)} = \frac{du(t+s)}{ds} \tag{1.8}$$

Since the two-point correlation function is defined as

$$R_{ij} = \overline{u'_i(\vec{x}, t) u'_j(\vec{x}+\vec{r}, t+\tau)} \tag{1.9}$$

Thus, for $D^{(i)}\{R_{ij}\} = \overline{u'_j(\vec{x}+\vec{r}, t+\tau) N_i\{\vec{x}, t\}}$, components in the expansion are calculated as

$$\begin{aligned}
\overline{u'_j(\vec{x}+\vec{r}, t+\tau)\frac{\partial u'_i(\vec{x}, t)}{\partial t}} &= \overline{\frac{\partial (u'_i(\vec{x}, t) u'_j(\vec{x}+\vec{r}, t+\tau))}{\partial t}} - \overline{u'_i(\vec{x}, t)\frac{\partial u'_j(\vec{x}+\vec{r}, t+\tau)}{\partial t}} \\
&= \frac{\partial \overline{u'_i(\vec{x}, t) u'_j(\vec{x}+\vec{r}, t+\tau)}}{\partial t} - \overline{u'_i(\vec{x}, t)\frac{\partial u'_j(\vec{x}+\vec{r}, t+\tau)}{\partial \tau}} \\
&= \frac{\partial R_{ij}}{\partial t} - \frac{\partial \overline{u'_i(\vec{x}, t) u'_j(\vec{x}+\vec{r}, t+\tau)}}{\partial \tau} \\
&= \frac{\partial R_{ij}}{\partial t} - \frac{\partial R_{ij}}{\partial \tau}
\end{aligned} \tag{1.10}$$

$$\begin{aligned}
\overline{u'_j(\vec{x}+\vec{r}, t+\tau)\overline{u_k}\frac{\partial u'_i}{\partial x_k}} &= \overline{u_k(\vec{x}, t)}\,\overline{u'_j(\vec{x}+\vec{r}, t+\tau)\frac{\partial u'_i(\vec{x}, t)}{\partial x_k}} \\
&= \overline{u_k(\vec{x}, t)}\left[\frac{\partial \overline{u'_i(\vec{x}, t) u'_j(\vec{x}+\vec{r}, t+\tau)}}{\partial x_k} - \overline{u'_i(\vec{x}, t)\frac{\partial u'_j(\vec{x}+\vec{r}, t+\tau)}{\partial (x_k + r_k)}}\right] \\
&= \overline{u_k(\vec{x}, t)}\left[\frac{\partial R_{ij}}{\partial x_k} - \overline{u'_i(\vec{x}, t)\frac{\partial u'_j(\vec{x}+\vec{r}, t+\tau)}{\partial r_k}}\right] \\
&= \overline{u_k(\vec{x}, t)}\left[\frac{\partial R_{ij}}{\partial x_k} - \frac{\partial \overline{u'_i(\vec{x}, t) u'_j(\vec{x}+\vec{r}, t+\tau)}}{\partial r_k}\right] \\
&= \overline{u_k(\vec{x}, t)}\left[\frac{\partial R_{ij}}{\partial x_k} - \frac{\partial R_{ij}}{\partial r_k}\right]
\end{aligned} \tag{1.11}$$

$$
\overline{u'_j(\vec{x}+\vec{r},t+\tau)\,u'_k(\vec{x},t)\frac{\partial \overline{u_i(\vec{x},t)}}{\partial x_k}} = \overline{u'_k(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)}\frac{\partial \overline{u_i(\vec{x},t)}}{\partial x_k} = R_{kj}\frac{\partial \overline{u_i(\vec{x},t)}}{\partial x_k} \tag{1.12}
$$

$$
\begin{aligned}
\overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial [u'_i(\vec{x},t)\,u'_k(\vec{x},t)]}{\partial x_k}} &= \overline{\frac{\partial [u'_i(\vec{x},t)\,u'_k(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)]}{\partial x_k}} - \overline{u'_i(\vec{x},t)\,u'_k(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}} \\
&= \frac{\partial R_{(ik)j}}{\partial x_k} - \overline{u'_i(\vec{x},t)\,u'_k(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}} \\
&= \frac{\partial R_{(ik)j}}{\partial x_k} - \overline{\frac{\partial [u'_i(\vec{x},t)\,u'_k(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)]}{\partial r_k}} \\
&= \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial R_{(ik)j}}{\partial r_k}
\end{aligned} \tag{1.13}
$$

$$
\overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial \overline{u'_i(\vec{x},t)\,u'_k(\vec{x},t)}}{\partial x_k}} = \overline{u'_j(\vec{x}+\vec{r},t+\tau)}\frac{\partial \overline{u'_i(\vec{x},t)\,u'_k(\vec{x},t)}}{\partial x_k} = 0 \tag{1.14}
$$

$$
\begin{aligned}
\overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial p'(\vec{x},t)}{\partial x_i}} &= \overline{\frac{\partial p'(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_i}} - \overline{p'(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_i}} \\
&= \frac{\partial \overline{p'(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \overline{p'(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial (x_i+r_i)}} \\
&= \frac{\partial \overline{p'(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \overline{p'(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_i}} \\
&= \frac{\partial \overline{p'(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \overline{\frac{\partial p'(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_i}} \\
&= \frac{\partial \overline{p'(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \frac{\partial \overline{p'(\vec{x},t)\,u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_i}
\end{aligned} \tag{1.15}
$$

$$
\begin{aligned}
&\overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial^2 u'_i(\vec{x},t)}{\partial x_k \partial x_k}} = \overline{u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial}{\partial x_k}\left(\frac{\partial u'_i(\vec{x},t)}{\partial x_k}\right)} \\
&= \frac{\partial}{\partial x_k}\overline{\left(u'_j(\vec{x}+\vec{r},t+\tau)\frac{\partial u'_i(\vec{x},t)}{\partial x_k}\right)} - \overline{\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}\frac{\partial u'_i(\vec{x},t)}{\partial x_k}} \\
&= \overline{\frac{\partial}{\partial x_k}\left(\frac{\partial[u'_j(\vec{x}+\vec{r},t+\tau)u'_i(\vec{x},t)]}{\partial x_k} - u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}\right)} - \overline{\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}\frac{\partial u'_i(\vec{x},t)}{\partial x_k}} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - \overline{\frac{\partial}{\partial x_k}\left(u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}\right)} - \overline{\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k}\frac{\partial u'_i(\vec{x},t)}{\partial x_k}} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - \overline{\frac{\partial}{\partial x_k}\frac{\partial[u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}{\partial r_k}} - \overline{\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}\frac{\partial u'_i(\vec{x},t)}{\partial x_k}} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - \frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} - \overline{\left(\frac{\partial}{\partial x_k}\left(u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}\right) - u'_i(\vec{x},t)\frac{\partial^2 u'_j(\vec{x}+\vec{r},t+\tau)}{\partial x_k \partial r_k}\right)} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - \frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} - \overline{\left(\frac{\partial}{\partial x_k}\left(\frac{\partial[u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}{\partial r_k}\right) - u'_i(\vec{x},t)\frac{\partial^2 u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k \partial r_k}\right)} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + \overline{\frac{\partial^2[u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}{\partial r_k \partial r_k}} \\
&= \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + \frac{\partial^2 R_{ij}}{\partial r_k \partial r_k}
\end{aligned} \tag{1.16}
$$

then, detailed expression for $D^{(i)}\{R_{ij}\}$ is given as

$$
\begin{aligned}
D^{(i)}\{R_{ij}\} = \frac{\partial R_{ij}}{\partial t} - \frac{\partial R_{ij}}{\partial \tau} + \overline{u_k(\vec{x},t)}\left[\frac{\partial R_{ij}}{\partial x_k} - \frac{\partial R_{ij}}{\partial r_k}\right] + R_{kj}\frac{\partial \overline{u_i(\vec{x},t)}}{\partial x_k} \\
+ \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial R_{(ik)j}}{\partial r_k} + \frac{1}{\rho}\left(\frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \frac{\partial \overline{p'(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_i}\right) \\
-\nu\left(\frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + \frac{\partial^2 R_{ij}}{\partial r_k \partial r_k}\right) - \overline{u'_j(\vec{x}+\vec{r},t+\tau)f'_i(\vec{x},t)} = 0
\end{aligned} \tag{1.17}
$$

For $D^{(j)}\{R_{ij}\} = \overline{u'_i(\vec{x},t)N_j\{\vec{x}+\vec{r},t+\tau\}}$, components in the expansion are calculated as

$$
\overline{u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial(t+\tau)}} = \overline{u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial \tau}} = \frac{\partial \overline{u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)}}{\partial \tau} = \frac{\partial R_{ij}}{\partial \tau} \tag{1.18}
$$

$$
\begin{aligned}
\overline{u'_i(\vec{x},t)\overline{u_k(\vec{x}+\vec{r},t+\tau)}\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial(x_k+r_k)}} = \overline{u_k(\vec{x}+\vec{r},t+\tau)}\,\overline{u'_i(\vec{x},t)\frac{\partial u'_j(\vec{x}+\vec{r},t+\tau)}{\partial r_k}} \\
= \overline{u_k(\vec{x}+\vec{r},t+\tau)}\frac{\partial \overline{[u'_i(\vec{x},t)u'_j(\vec{x}+\vec{r},t+\tau)]}}{\partial r_k} = \overline{u_k(\vec{x}+\vec{r},t+\tau)}\frac{\partial R_{ij}}{\partial r_k}
\end{aligned} \tag{1.19}
$$

$$\overline{u_i'(\vec{x},t)u_k'(\vec{x}+\vec{r},t+\tau)\frac{\partial \overline{u_j(\vec{x}+\vec{r},t+\tau)}}{\partial(x_k+r_k)}} = R_{ik}\frac{\partial \overline{u_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_k} \tag{1.20}$$

$$\begin{aligned}\overline{u_i'(\vec{x},t)\frac{\partial[u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)]}{\partial(x_k+r_k)}} &= \overline{u_i'(\vec{x},t)\frac{\partial[u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)]}{\partial r_k}} \\ &= \frac{\partial\overline{[u_i'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)]}}{\partial r_k} = \frac{\partial R_{i(jk)}}{\partial r_k}\end{aligned} \tag{1.21}$$

$$\overline{u_i'(\vec{x},t)\frac{\partial\overline{u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)}}{\partial(x_k+r_k)}} = \overline{u_i'(\vec{x},t)}\frac{\partial\overline{u_j'(\vec{x}+\vec{r},t+\tau)u_k'(\vec{x}+\vec{r},t+\tau)}}{\partial r_k} = 0 \tag{1.22}$$

$$\overline{u_i'(\vec{x},t)\frac{\partial p'(\vec{x}+\vec{r},t+\tau)}{\partial(x_j+r_j)}} = \overline{u_i'(\vec{x},t)\frac{\partial p'(\vec{x}+\vec{r},t+\tau)}{\partial r_j}} = \frac{\partial\overline{p'(\vec{x}+\vec{r},t+\tau)u_i'(\vec{x},t)}}{\partial r_j} \tag{1.23}$$

$$\begin{aligned}\overline{u_i'(\vec{x},t)\frac{\partial^2 u_j'(\vec{x}+\vec{r},t+\tau)}{\partial(x_k+r_k)\partial(x_k+r_k)}} &= \overline{u_i'(\vec{x},t)\frac{\partial^2 u_j'(\vec{x}+\vec{r},t+\tau)}{\partial r_k\partial r_k}} \\ &= \frac{\partial^2\overline{u_i'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)}}{\partial r_k\partial r_k} = \frac{\partial R_{ij}}{\partial r_k\partial r_k}\end{aligned} \tag{1.24}$$

then, detailed expression for $D^{(j)}\{R_{ij}\}$ is given as

$$\begin{aligned}D^{(j)}\{R_{ij}\} = \frac{\partial R_{ij}}{\partial\tau} &+ \overline{u_k(\vec{x}+\vec{r},t+\tau)}\frac{\partial R_{ij}}{\partial r_k} + R_{ik}\frac{\partial\overline{u_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_k} + \frac{\partial R_{i(jk)}}{\partial r_k} \\ &+ \frac{1}{\rho}\frac{\partial\overline{p'(\vec{x}+\vec{r},t+\tau)u_i'(\vec{x},t)}}{\partial r_j} - \nu\frac{\partial R_{ij}}{\partial r_k\partial r_k} - \overline{u_i'(\vec{x},t)f_j'(\vec{x}+\vec{r},t+\tau)} = 0\end{aligned} \tag{1.25}$$

Therefore, the two-point correlation equation is calculated as

$$\begin{aligned}D^{(i)}\{R_{ij}\} + D^{(j)}\{R_{ij}\} = \frac{\partial R_{ij}}{\partial t} &+ \overline{u_k(\vec{x},t)}\frac{\partial R_{ij}}{\partial x_k} + \left[\overline{u_k(\vec{x}+\vec{r},t+\tau)} - \overline{u_k(\vec{x},t)}\right]\frac{\partial R_{ij}}{\partial r_k} \\ &+ R_{kj}\frac{\partial\overline{u_i(\vec{x},t)}}{\partial x_k} + R_{ik}\frac{\partial\overline{u_j(\vec{x}+\vec{r},t+\tau)}}{\partial r_k} + \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial[R_{(ik)j} - R_{i(jk)}]}{\partial r_k} \\ + \frac{1}{\rho}\Bigg(&\frac{\partial\overline{p'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)}}{\partial x_i} - \frac{\partial\overline{p'(\vec{x},t)u_j'(\vec{x}+\vec{r},t+\tau)}}{\partial r_i} + \frac{\partial\overline{p'(\vec{x}+\vec{r},t+\tau)u_i'(\vec{x},t)}}{\partial r_j}\Bigg) \\ -\nu\left(\frac{\partial^2 R_{ij}}{\partial x_k\partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k\partial r_k} + 2\frac{\partial^2 R_{ij}}{\partial r_k\partial r_k}\right) &- \overline{u_j'(\vec{x}+\vec{r},t+\tau)f_i'(\vec{x},t)} - \overline{u_i'(\vec{x},t)f_j'(\vec{x}+\vec{r},t+\tau)} = 0\end{aligned} \tag{1.26}$$

2 Exercise 2

Refer to (1.7), kinetic energy equation of each velocity component is given as

$$\begin{aligned} u_1' N_1 &= 0 \\ u_2' N_2 &= 0 \\ u_3' N_3 &= 0 \end{aligned} \tag{2.1}$$

In the pure shear flow case, assumptions can be made as

$$\begin{aligned} \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_3} &= 0 \\ U_2 = U_3 &= 0 \\ U_1 &= f(x_2) \end{aligned} \tag{2.2}$$

Hence, (2.1) can be simplified as

$$\begin{aligned} \frac{\overline{D}}{\overline{D}t}\left(\frac{u_1'^2}{2}\right) &= -u_2' u_1' \frac{\partial U_1}{\partial x_2} - u_1' \frac{\partial [u_1' u_2']}{\partial x_2} + u_1' \frac{\partial \overline{u_1' u_2'}}{\partial x_2} + u_1' \nu \frac{\partial^2 u_1'}{\partial x_2 \partial x_2} \\ \frac{\overline{D}}{\overline{D}t}\left(\frac{u_2'^2}{2}\right) &= -u_2' \frac{\partial [u_2' u_2']}{\partial x_2} + u_2' \frac{\partial \overline{u_2' u_2'}}{\partial x_2} - \frac{u_2'}{\rho} \frac{\partial p'}{\partial x_2} + u_2' \nu \frac{\partial^2 u_2'}{\partial x_2 \partial x_2} \\ \frac{\overline{D}}{\overline{D}t}\left(\frac{u_3'^2}{2}\right) &= -u_3' \frac{\partial [u_3' u_2']}{\partial x_2} + u_3' \frac{\partial \overline{u_3' u_2'}}{\partial x_2} + u_3' \nu \frac{\partial^2 u_3'}{\partial x_2 \partial x_2} \end{aligned} \tag{2.3}$$

where

$$\frac{\overline{D}}{\overline{D}t} = \frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k} \tag{2.4}$$

and it should be noted that the $U_k \frac{\partial}{\partial x_k}$ term vanishes in (2.3).
Estimations on each term can be made respectively.
For the first equation in (2.3)

$$\begin{aligned} u_2' u_1' \frac{\partial U_1}{\partial x_2} &\sim \frac{u'^2 U}{L} \\ u_1' \frac{\partial [u_1' u_2']}{\partial x_2} &\sim \frac{u'^3}{L} \\ u_1' \frac{\partial \overline{u_1' u_2'}}{\partial x_2} &\sim u' \nu \frac{\partial U_1}{\partial x_2} \sim \nu \frac{u' U}{L} \\ u_1' \nu \frac{\partial^2 u_1'}{\partial x_2 \partial x_2} &\sim \nu \frac{u'^2}{L^2} \end{aligned} \tag{2.5}$$

For the second equation in (2.3)

$$
\begin{aligned}
u_2'\frac{\partial[u_2'u_2']}{\partial x_2} &\sim \frac{u'^3}{L}\\
u_2'\frac{\partial\overline{u_2'u_2'}}{\partial x_2} &\sim \frac{u'^3}{L}\\
\frac{u_2'}{\rho}\frac{\partial p'}{\partial x_2} &\sim \frac{u'}{\rho}\rho\\
u_2'\nu\frac{\partial^2 u_2'}{\partial x_2\partial x_2} &\sim \nu\frac{u'^2}{L^2}
\end{aligned}
\tag{2.6}
$$

For the third equation in (2.3)

$$
\begin{aligned}
u_3'\frac{\partial[u_3'u_2']}{\partial x_2} &\sim \frac{u'^3}{L}\\
u_3'\frac{\partial\overline{u_3'u_2'}}{\partial x_2} &\sim \frac{u'^3}{L}\\
u_3'\nu\frac{\partial^2 u_3'}{\partial x_2\partial x_2} &\sim \nu\frac{u'^2}{L^2}
\end{aligned}
\tag{2.7}
$$