Vv556 Methods of Applied Mathematics I

Linear Operators

Assignment 6

Date Due: 12:10 PM, Thursday, the 1st of November 2018

This assignment has a total of **(8 Marks)**.

Exercise 6.1

Let $x \in \ell^2$, i.e., $x = (x_n)$ is a square-summable sequence. Define the operator $L\colon \ell^2 \to \ell^2$ by $Lx = y$,

$$y_n = \sum_{m=1}^{\infty} a_{nm} x_m, \qquad n = 1, 2, 3, \ldots$$

where the coefficients a_{nm}, $n, m \in \mathbb{N} \setminus \{0\}$ satisfy $\sum_{m,n=1}^{\infty} |a_{nm}|^2 =: M^2 < \infty$. Any such L is called a *Hilbert-Schmidt operator* on ℓ^2.

i) Verify that $y \in \ell^2$, i.e., that L is well-defined.
(2 Marks)

ii) Find the matrix elements $L_{ij} := \langle e_i, Le_j \rangle$ of L with respect to the standard basis $\{e_n\}_{n\in\mathbb{N}}$, with $e_n := (0, \ldots, 0, 1, 0, \ldots)$, where the 1 is in the nth position.
(1 Mark)

iii) Show that L is bounded with $\|L\| \le M$.
(2 Marks)

iv) Show that L defined by

$$Lx = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \ldots \right)$$

is a Hilbert-Schmidt operator. Find the operator norm $\|L\|$. What can you say about its relationship to M^2?
(3 Marks)