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Methods of Applied Mathematics I
HW6

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1 EXERCISE6.1

1. *Proof.* With the Cauchy-Schwartz inequality

$$\sum_{n=1}^{\infty}|y_n|^2=\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty}a_{nm}x_m)^2\leq\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}x_m^2\sum_{m=1}^{\infty}a_{nm}^2\right]=\left(\sum_{m=1}^{\infty}x_m^2\right)\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{nm}^2\right)<\infty \tag{1.1}$$

The last setp is valid as x is a square-summable sequence and $\sum_{m,n=1}^{\infty}a_{nm}^2=M^2<\infty$. Hence $y\in l^2$. □

2. According to the definition of L

$$Le_j=(a_{1j},a_{2j},...,a_{nj}) \tag{1.2}$$

Thus

$$L_{ij}=<e_i,Le_j>=a_{ij} \tag{1.3}$$

3. *Proof.* Follow from the first part

$$||y||_2\leq||x||_2\cdot M \tag{1.4}$$

Thus

$$||L||=\sup_{x\in l^2}\frac{||Lx||_2}{||x||_2}\leq\frac{||x||_2M}{||x||_2}=M \tag{1.5}$$

□

4. *Proof.* Denote (e_j) being the standard basis, then

$$Le_j = (0, 0, ..., \frac{1}{j}, 0, ..., 0) \tag{1.6}$$

Thus

$$a_{ij} \triangleq L_{ij} = \delta_{ij}\frac{1}{j} \tag{1.7}$$

Hence

$$\sum_{n,m=1}^{\infty} |a_{nm}|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1.8}$$

which indicates that L is a Hilbert-Schmidt operator.
Let $x_0 = (1, 0, 0, ..., 0)$, then

$$||L|| = \sup_{x\in l^2} \frac{||Lx||_2}{||x||_2} \geq \frac{||Lx_0||_2}{||x_0||_2} = \frac{||(1, 0, ...)||_2}{||(1, 0, ...)||_2} = 1 \tag{1.9}$$

Also, with the Cauchy-Schwartz inequality

$$||L|| = \sup_{x\in l^2} \frac{||Lx||_2}{||x||_2} = \sup_{x\in l^2} \frac{\sqrt{\sum_{n=1}^{\infty} (\frac{x_n}{n})^2}}{||x||_2} \leq \sup_{x\in l^2} \frac{\sqrt{\sum_{n=1}^{\infty} x_n^2}}{||x||_2} = \sup_{x\in l^2} \frac{||x||_2}{||x||_2} = 1 \tag{1.10}$$

Therefore $||L|| = 1$.
It can be seen that M is the upper bound of $||L||$. □