

Introduction to Numerical Analysis

HW4

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June 18, 2018

1 LEGENDRE POLYNOMIALS

1. *Proof.* Let

$$\varphi(x) = (x^2 - 1)^n \quad (1.1)$$

then

$$Q_n(x) = \frac{1}{2^n n!} \varphi^{(n)}(x) \quad (1.2)$$

and

$$\varphi^{(k)}(1) = \varphi^{(k)}(-1) = 0 \quad (k = 0, 1, \dots, n-1) \quad (1.3)$$

Suppose $h(x) \in C^n(-1, 1)$, then performing integration by parts

$$\begin{aligned} \int_{-1}^1 P_n(x) h(x) dx &= \frac{1}{2^n n!} \int_{-1}^1 \varphi^{(n)}(x) h(x) dx \\ &= -\frac{1}{2^n n!} \int_{-1}^1 \varphi^{(n-1)}(x) h'(x) dx \\ &= \dots \\ &= \frac{(-1)^n}{2^n n!} \int_{-1}^1 \varphi(x) h^{(n)}(x) dx \end{aligned} \quad (1.4)$$

Thus, the proof can be discussed on 2 cases

a) If the order of $h(x)$ is less than n , then

$$h^{(n)}(x) = 0 \quad (1.5)$$

Thus

$$\int_{-1}^1 Q_n(x)Q_m(x)dx = 0 \quad (n \neq m) \quad (1.6)$$

b) If $h(x) = Q_n(x)$, then the $n - th$ derivative of $g(x)$ is

$$h^{(n)}(x) = Q^{(n)}(x) = \frac{(2n)!}{2^n n!} \quad (1.7)$$

Thus

$$\begin{aligned} \int_{-1}^1 Q_n(x)Q_m(x)dx &= \int_{-1}^1 Q_n^2(x)dx \quad (n = m) \\ &= \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} \int_{-1}^1 (x^2 - 1)^n dx \\ &= \frac{(2n)!}{2^{2n} (n!)^2} \int_{-1}^1 (1 - x^2)^n dx \\ &= \frac{(2n)!}{2^{2n} (n!)^2} \int_0^{\pi/2} \cos^{2n+1} t dt \\ &= \frac{(2n)!}{2^{2n} (n!)^2} \frac{2 \times 4 \times \dots \times (2n)}{1 \times 3 \times \dots \times (2n+1)} \\ &= \frac{2}{2n+1} \end{aligned} \quad (1.8)$$

Thus, $(Q_n)_{n \in \mathbb{N}}$ are a sequence of orthogonal polynomials. \square

2. *Proof.* Denote

$$\varphi(x) = (x^2 - 1)^n \quad (1.9)$$

then

$$Q_n(x) = \frac{1}{2^n n!} \varphi^{(n)}(x) \quad (1.10)$$

As the power of each item in $\varphi(x)$ is even when $\varphi(x)$ is extended, thus $\varphi^{(n)}(x)$ is even function if the order of derivative is even, and $\varphi^{(n)}(x)$ is odd function if the order of derivative is odd.

Therefore $Q_n(x)$ is even function if n is even, and $Q_n(x)$ is odd function if n is odd. So, it can be summarized as $Q_n(-x) = (-1)^n Q_n(x)$. \square

3. *Proof.* As $xQ_n(x)$ can be written as

$$xQ_n(x) = \sum_{i=0}^{n+1} a_i Q_i(x) \quad (1.11)$$

According to the orthogonality of Legendre polynomials, $a_i = 0$ for $i = 0, 1, \dots, n-2, n$, as

$$0 = \int_{-1}^1 xQ_n(x)Q_i(x)dx = a_i \int_{-1}^1 Q_i^2(x)dx \quad (i = 0, 1, \dots, n-2, n) \quad (1.12)$$

Thus, $xQ_n(x)$ can be written as

$$xQ_n(x) = a_{n-1}Q_{n-1}(x) + a_{n+1}Q_{n+1}(x) \quad (1.13)$$

Since the highest order item in $xQ_n(x)$ is x^{n+1} and its coefficient is $\frac{(2n)!}{2^n(n!)^2}$, thus

$$\frac{(2n)!}{2^n(n!)^2} = a_{n+1} \frac{(2n+2)!}{2^{n+1}(n+1)!^2} \quad (1.14)$$

which implies that

$$a_{n+1} = \frac{n+1}{2n+1} \quad (1.15)$$

Denote

$$I_n = \int_{-1}^1 xQ_{n-1}(x)Q_n(x)dx \quad (1.16)$$

Then

$$I_n = a_{n-1} \int_{-1}^1 Q_{n-1}^2(x)dx = a_{n-1} \frac{2}{2n-1} \quad (1.17)$$

As

$$I_{n+1} = a_{n+1} \int_{-1}^1 Q_{n+1}^2(x)dx = a_{n+1} \frac{2}{2n+3} = \frac{2(n+1)}{(2n+1)(2n+3)} \quad (1.18)$$

Thus

$$I_n = \frac{2n}{(2n-1)(2n+1)} \quad (1.19)$$

which implies that

$$a_{n-1} = \frac{n}{2n+1} \quad (1.20)$$

Hence

$$(2n+1)xQ_n(x) = nQ_{n-1}(x) + (n+1)Q_{n+1}(x) \quad (1.21)$$

□

4. *Proof.* As

$$\begin{aligned} Q_n(x) &= \frac{1}{2^n n!} [(x+1)^n (x-1)^n]^{(n)} \\ &= \frac{2^n}{n!} \left[\left(\frac{x+1}{2} \right)^n \left(\frac{x-1}{2} \right)^n \right]^{(n)} \end{aligned} \quad (1.22)$$

Denote

$$f(x) = \left(\frac{x+1}{2} \right)^n \quad (1.23)$$

and

$$g(x) = \left(\frac{x-1}{2} \right)^n \quad (1.24)$$

Then

$$f^{(k)}(x) = \frac{n!}{2^k (n-k)!} \left(\frac{x+1}{2} \right)^{n-k} \quad (1.25)$$

and

$$g^{(k)}(x) = \frac{n!}{2^k(n-k)!} \left(\frac{x-1}{2}\right)^{n-k} \quad (1.26)$$

Since

$$(fg)^{(n)} = \sum_{k=0}^n C_n^k f^{(k)} g^{(n-k)} \quad (1.27)$$

Thus

$$\begin{aligned} Q_n(x) &= \frac{2^n}{n!} \sum_{k=0}^n C_n^k f^{(k)}(x) g^{(n-k)}(x) \\ &= \frac{2^n}{n!} \sum_{k=0}^n C_n^k \frac{n!}{2^k(n-k)!} \left(\frac{x+1}{2}\right)^{n-k} \frac{n!}{2^{n-k}k!} \left(\frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^n C_n^k \frac{n!}{(n-k)!k!} \left(\frac{x+1}{2}\right)^{n-k} \left(\frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^n (C_n^k)^2 \left(\frac{x+1}{2}\right)^{n-k} \left(\frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k}^2 \left(\frac{1+x}{2}\right)^{n-k} \left(\frac{1-x}{2}\right)^k \end{aligned} \quad (1.28)$$

□

2 INTERPOLATION

$f(2)$ can be determined using the Lagrange interpolation scheme. As the lagrange interpolation polynomial can be written as below, and $n = 8$ in this case.

$$f(x) = \sum_{i=1}^n f(x_i) l_i(x) \quad (2.1)$$

$l_i(x)$ are the base functions that can be written as below.

$$l_i(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_{n-1})(x-x_n)}{(x_i-x_1)(x_i-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_{n-1})(x_i-x_n)} \quad (2.2)$$

$l_i(2)$ are calculated accordingly as below.

$$\begin{array}{llll} l_1(2) = -0.0006 & l_2(2) = 0.1224 & l_3(2) = -0.5600 & l_4(2) = 1.0606 \\ l_5(2) = 0.4167 & l_6(2) = -0.0400 & l_7(2) = 0.0012 & l_8(2) = -0.0003 \end{array}$$

Thus, $f(2)$ is calculated according to (2.1) as 11.0.

3 NEWTON'S FORM OF INTERPOLATION POLYNOMIAL

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.