SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Methods of Applied Mathematics I
HW4

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1 EXERCISE4.1

Let $f(x)$ be extended as

$$f(x) = \begin{cases} x(\pi - x) x \in [2n\pi, (2n+1)\pi] \\ -x(\pi - x) x \in [-(2n-1)\pi, 2n\pi] \end{cases} \tag{1.1}$$

Then $f(x)$ is both odd and periodic. Thus fouier-sine series can be employed.

$$f(x) = \sum_{n=1}^{\infty} b_n sin(nx) \tag{1.2}$$

Coefficients b_n are calculated by

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x) sin(nx) dx \\ &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) sin(nx) dx \\ &= \frac{4[1-(-1)^n]}{n^3\pi} \quad \text{(Integrate by parts)} \end{aligned} \tag{1.3}$$

Thus

$$f(x) = \sum_{k=0}^{\infty} \frac{8 sin(2k+1)x}{\pi(2k+1)^3} \tag{1.4}$$

Taking $x = \frac{\pi}{2}$ yields

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32} \tag{1.5}$$

2 EXERCISE4.2

1. *Proof.* The orthogonal property is justified as

$$\int_0^{\pi} (\frac{1}{\sqrt{\pi}})^2 dx = \frac{1}{\pi}\int_0^{\pi} dx = 1 \tag{2.1}$$

$$\int_0^{\pi} (\sqrt{\frac{2}{\pi}} cos(nx))^2 dx = \frac{2}{\pi}\int_0^{\pi} cos^2(nx) dx = \frac{1}{\pi}\int_0^{\pi} (cos(2nx)+1) dx = 1 \tag{2.2}$$

$$\int_0^{\pi} \frac{1}{\sqrt{\pi}} \sqrt{\frac{2}{\pi}} cos(nx) dx = 0 \tag{2.3}$$

$$\int_0^{\pi} \sqrt{\frac{2}{\pi}} cos(nx) \sqrt{\frac{2}{\pi}} cos(mx) dx = \frac{2}{\pi}\int_0^{\pi} cos(nx)cos(mx) dx = 0 \tag{2.4}$$

□

2. *Proof.* It's trival to show both $K = 0$ and $K = 1$ are valid, and $K = 2$ is also justified as

$$cos^2(x) = \frac{1+cos(2x)}{2} \tag{2.5}$$

Assume the proposition is also valid for $K = n$, that means

$$span\{1, cos(x), cos(2x), ...cos(nx)\} = span\{1, cos(x), cos^2(x), ...cos^n(x)\} \tag{2.6}$$

which indicates that $\exists a_k^{(n)}$ and $b_k^{(n)} (k = 0, 1, ..., n)$ s.t.

$$cos^n(x) = \sum_{k=0}^{n} a_k^{(n)} \cdot cos(kx) \tag{2.7}$$

and

$$cos(nx) = \sum_{k=0}^{n} b_k^{(n)} \cdot cos^k(x) \tag{2.8}$$

When $K = n+1$, the proposition is still valid as

$$
\begin{aligned}
cos^{n+1}(x) &= cos(x)cos^n(x)\\
&= cos(x)\sum_{k=0}^{n} a_k^{(n)} cos(kx)\\
&= a_0^{(n)} cos(x) + cos(x)\sum_{k=1}^{n-1} a_k^{(n)} cos(kx) + a_n^{(n)} cos(x)cos(nx)\\
&= a_0^{(n)} cos(x) + \sum_{k=1}^{n-1} \frac{a_k^{(n)}}{2}[cos(k-1)x + cos(k+1)x] + \frac{a_n^{(n)}}{2}[cos(n-1)x + cos(n+1)x]\\
&= \frac{a_1^{(n)}}{2} + (a_0^{(n)} + \frac{a_2^{(n)}}{2})cos(x) + \sum_{k=2}^{n-1} \frac{a_{k-1}^{(n)} + a_{k+1}^{(n)}}{2} cos(kx) + \frac{a_{n-1}^{(n)}}{2} cos(nx) + \frac{a_n^{(n)}}{2} cos(n+1)x\\
&\triangleq \sum_{k=0}^{n+1} a_k^{(n+1)} cos(kx)
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
cos(n+1)x &= 2cos(nx)cos(x) - cos(n-1)x\\
&= 2cos(x)\sum_{k=0}^{n} b_k^{(n)} cos^k(x) - \sum_{k=0}^{n-1} b_k^{(n-1)} cos^k(x)\\
&= 2\sum_{k=0}^{n} b_k^{(n)} cos^{k+1}(x) - \sum_{k=0}^{n-1} b_k^{(n-1)} cos^k(x)\\
&= 2\sum_{k=1}^{n+1} b_{k-1}^{(n)} cos^k(x) - [b_0^{(n-1)} + \sum_{k=1}^{n-1} b_k^{(n-1)} cos^k(x)]\\
&= -b_0^{(n-1)} + \sum_{k=1}^{n-1} (2b_{k-1}^{(n)} - b_k^{(n-1)}) cos^k(x) + 2b_{n-1}^{(n)} cos^n(x) + 2b_n^{(n)} cos^{n+1}(x)\\
&\triangleq \sum_{k=0}^{n+1} b_k^{(n+1)} cos^k(x)
\end{aligned}
\tag{2.10}
$$

Thus

$$span\{1, cos(x), cos(2x), ..., cos(n+1)x\} = span\{1, cos(x), cos^2(x), ..., cos^{n+1}(x)\} \tag{2.11}$$

Hence, by induction hypothesis, the proposition is valid for all $k \in N$. □

3. *Proof.* Given any continous function f over $[-1, 1]$, by Weierstras Approximation Theorem, $\exists p(x)$ over $[-1, 1]$ s.t.

$$||f - p||_\infty < \epsilon \tag{2.12}$$

for any $\epsilon > 0$.
Assume p can be writte as

$$p(x) = \sum_{i=0}^{N} a_i x^i \tag{2.13}$$

Then with the change of variables $y = cos(x)$

$$f(x) = \tilde{f}(y) \approxeq p(y) = \sum_{i=0}^{N} a_i y^i = \sum_{i=0}^{N} a_i cos^i(x) \tag{2.14}$$

As $cos^i(x)$ can be expressed as the sum of $1, cos(x), cos(2x), ..., cos(ix)$ with proper coefficients, $f(x)$ is therefore can be approximated uniformly by finite linear combinations of $1, cos(x), cos(2x), ...$ □

4. *Proof.* As have been proved above, for any continous function f over $[0,\pi]$, it can be approximated by linear combinations of $1, cos(x), cos(2x), ...$, which means that $\exists g \in span\{B\}$ s.t.

$$||f - g||_\infty < \epsilon \tag{2.15}$$

for any $\epsilon > 0$.
Thus, span{B} is dense in $C([0,\pi])$ in the $||\cdot||_\infty$ norm. □

5. *Proof.* As have been proved aboved, $\forall f \in C[a,b], \exists p \in \mathbb{P}[a,b]$ s.t.

$$||f - p||_\infty = \sup_{x\in[a,b]} |f(x) - p(x)| < \frac{\epsilon}{\sqrt{b-a}} \tag{2.16}$$

for any $\epsilon > 0$.
Then for the L^2 norm

$$||f - p||_2 = \sqrt{\int_a^b (f(x) - p(x))^2 dx} \leq \sqrt{(b-a)\cdot \max_{x\in[a,b]} (f(x) - p(x))^2} < \epsilon \tag{2.17}$$

Thus, span{B} is dense in $C([0,\pi])$ in the $||\cdot||_2$ norm. □

6. Not figured out...