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Introduction to Numerical Analysis
HW2

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1 CONNECTED SPACE

1. *Proof.* a) (i) ⇒ (ii)
   Suppose (ii) is not true, which means $X = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, $U_1 \neq \emptyset$, $U_2 \neq \emptyset$, and both U_1 and U_2 are open.
   Thus, U_1 and U_2 are closed as $U_1 = U_2^{\complement}$ and $U_2 = U_1^{\complement}$.
   So, U_1 and U_2 are both open and closed in X, which is contradictory to (i).
   Thus the assumption fails and (ii) is true when (i) is true.

   b) (ii) ⇒ (i)
   Suppose (i) is not true, which means there exists U s.t. $U \subset X$, $U \neq \emptyset$ and U is both open and closed in X.
   Thus, $U^{\complement}$ is open as U is closed.
   As $X = U \cup U^{\complement}$, then X can be written as the union of two disjoint, non-empty open subsets, which is contradictory to (ii).
   Thus the assumption fails and (i) holds true when (ii) is true.

   c) (i)⇒(iii)
   Suppose (iii) is not true, which means $X = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, $U_1 \neq \emptyset$, $U_2 \neq \emptyset$ and both U_1 and U_2 are closed.
   Thus, U_1 and U_2 are open as $U_1 = U_2^{\complement}$ and $U_2 = U_1^{\complement}$.
   So, U_1 and U_2 are both open and closed in X, which is contradictory to (i).
   Thus the assumption fails and (iii) is true when (i) is true.

d) (iii) $\Rightarrow$ (i)
   Suppose (i) is not true, which means there exists U s.t. $U \subset X$, $U \neq \emptyset$ and U is both open and closed in X.
   Thus, $U^{\complement}$ is closed as U is open.
   As $X = U \cup U^{\complement}$, then X can be written as the union of two disjoint, non-empty closed subsets, which is contradictory to (iii).
   Thus the assumption fails and (i) holds true when (iii) is true.

$\square$

2. *Proof.* If (iv) is false, then there exists a continuous, surjective application from X into $[0,1] \subset U$, which can be denoted as f.
   $[0,1]$ can be written as $[0,a) \cup [a,1] \triangleq V_1 \cup V_2$, where $0 < a < 1$, V_1 and V_2 are closed.
   Denote $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$.
   As f is surjective, it follows that $U_1 \neq \emptyset$, $U_2 \neq \emptyset$ and $U_1 \cap U_2 = \emptyset$.
   As f is continous, it follows that U_1 and U_2 are also closed, $U_1 \cap U_2 = X$.
   Thus, it is contradictory to (iii) as X can be written as the union of two disjoint, non-empty closed subsets.
   So, if (iv) is not true then (iii) is also false. $\square$

3. *Proof.* If (iii) is false, then $X = U_1 \cup U_2$, where U_1 and U_2 are two disjoint, non-empty closed subsets.
   (haven't figured out yet...) $\square$

2 Intermediate value theorem

1. *Proof.* Suppose $f(A) = V_1 \cup V_2$, where V_1 and V_2 are two disjoint, non-empty open subsets. Denote $U_1 = f^{-1}(V_1)$, $U_2 = f^{-1}(V_2)$. $A = U_1 \cup U_2$ as each element in A is mapped to either V_1 or V_2. Further, U_1 and U_2 are open as f is a continous map. Thus A can be written as the union of two disjoint, non-empty open subsets, which is contradictory to the fact that A is a connected space. Therefore, $f(A)$ is connected. $\square$

2. *Proof.* a) It's clear that $\emptyset$ is connected as X is itself.
   For A containing only 1 element, it is connected as it can no be written as the union of two disjoint non-empty closed subsets.

   b) If A is not an interval and the corner cases in a) are excluded, then it can be written as union of non-empty, disjoint closed subsets. Thus A is not connected.

   c) i. The continuous bijection mapping f can be given as

$$f(x) = \frac{x - I_{min}}{I_{max} - I_{min}}(J_{max} - Jmin) \tag{2.1}$$

   where $I_{min}, I_{max}, J_{max}, Jmin$ are the limits of corresponding interval.
   The inverse continuous bijection can be constructed in the same way, which can be given as

$$f^{-1}(y) = \frac{y - J_{min}}{J_{max} - J_{min}}(I_{max} - I_{min}) \tag{2.2}$$

ii. Consider open interval $X = (0,1)$, it is clear that X is connected. As there exists a continous bijection mapping which maps X to $\mathbb{R}$, thus $f(A) = \mathbb{R}$ is connected as well.

iii. If U is both open and closed, then $U^{\complement}$ is also both open and closed. As there must exist a minimum for a closed and non-empty set, $R = U \cup U^{\complement}$ is then bounded, which is false. Thus, the only subsets that are both open and closed in $\mathbb{R}$ are $\mathbb{R}$ and $\emptyset$, which is equivalent to say $\mathbb{R}$ is connected.

□

3. *Proof.* For any connected set A, as is indicated above, $f(A)$ is also connected, where f is a continuous function.
And the connected subsets of $\mathbb{R}$ are all intervals.
Then $f(X)$ is an interval of $\mathbb{R}$, which contains both $f(a)$ and $f(b)$.
Thus, $f(X)$ contains both $f(a)$ and $f(b)$. □

3 Rolle's theorem

Proof. 1. For $n = 1$, if $f(x)$ has 2 distinct roots in $[a, b]$, then there exists the maximum M and minimum m between $[a, b]$ according to the extream value theorem.
If $M = m$, then $f(x)$ is constant, and it's obvious that for any $c \in [a, b]$, $f\prime(c) = 0$;
If $M \neq m$, then $\exists \xi \in (a, b)$, s.t. $f(\xi)$ reaches its extream, and equals to 0.

2. As induction hypothesis, assume the statement is true for $n = k$.

3. For $n = k+1$, where $f(x)$ has $k+2$ distinct roots denoted as $c_0 < c_1 < ... < c_k < c_{k+1}$, applying the results for $n = 1$ on each gap $[c_i, c_{i+1}]$ $(i = 0, 1, ..., k)$, then $g(x) \triangleq f'(x)$ has $k+1$ roots in $[c_0, c_{k+1}]$. By induction hypothesis, there exists $c \in [c_0, c_{k+1}]$ s.t. $g^{(k)}(c) = f^{(k+1)}(c) = f^{(n)}(c) = 0$. Thus the statement holds true for $n = k+1$.

□

4 Extreme value theorem

1. *Proof.* □

2. *Proof.* a) Given an open covering $\mathscr{U}$ of A, an open covering of X by throwing in the open subset $U_0 = X /\ A$. Since X is compact, there exists finitely many sets $U_1, U_2, U_3, ..., U_n \in \mathscr{U}$ s.t. $X = U_0 \cup U_1 \cup ... \cup U_n$. Then $A \subseteq U_1 \cup ... \cup U_n$, proving that A is compact.

   b) (haven't figured out yet...)

□

3. *Proof.* □

5 Continuity

1. *Proof.* (i)$\Rightarrow$(ii): For each $y \in B(f(a),\xi)$, there exists $U_x \subset X, U_x \neq \emptyset$ s.t. $y = f(U_x)$. Thus, $d(f(x), f(a)) < \xi$ is valid for any $x \in U \triangleq \bigcup_{x\in X} U_x$. As indicated by (i), there exists η s.t. $B(a,\eta) \subset U$. Thus, η is valid, and $d(a,x)$ for $x \in B(a,\eta)$ is less than η.
(ii)$\Rightarrow$(i): As X and Y are two metrix spaces, then the set containing all the elements in $d(x,a) < \eta$ is equivalent to the ball $B(a,\eta) \subset X$. It suffices to show that the η in (i) exists. □

2. *Proof.* Given ξ where $B(f(a),\xi) \subset V$, then it is indicated by (i) that there exists η where $f(B(a,\eta)) \in B(f(a),\xi)$. Denote $U = B(a,\eta)$, then $f(U) \subset B(f(a),\xi) \subset V$. □

3. *Proof.* As indicated by (iii), U is a neighborhood of a and $f(U) \subset V$. Since $U \subset f^{-1}(V)$, thus, by observation, $f^{-1}(V)$ is a neighborhood of a. □

4. *Proof.* For any $\xi \in \mathbb{R}_+$, take the neighborhood V of $f(a)$ s.t. $V \subset B(f(a),\xi)$. Then, by (iv), $f^{-1}(V)$ is a neighborhood of a. Thus, there exists $\eta \in \mathbb{R}_+$ s.t. $B(a,\eta) \subset f^{-1}(V)$, and it is obvious that $f(B(a,\eta)) \subset B(f(a),\xi)$. □