SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Introduction to Numerical Analysis HW4

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1 LEGENDRE POLYNOMIALS

1. *Proof.* Let

$$\varphi(x) = (x^2 - 1)^n \tag{1.1}$$

then

$$Q_n(x) = \frac{1}{2^n n!}\varphi^{(n)}(x) \tag{1.2}$$

and

$$\varphi^{(k)}(1) = \varphi^{(k)}(-1) = 0 \quad (k = 0, 1, ..., n-1) \tag{1.3}$$

Suppose $h(x) \in C^n(-1,1)$, then performing integration by parts

$$\begin{aligned} \int_{-1}^{1} P_n(x)h(x)dx &= \frac{1}{2^n n!}\int_{-1}^{1}\varphi^{(n)}(x)h(x)dx \\ &= -\frac{1}{2^n n!}\int_{-1}^{1}\varphi^{(n-1)}(x)h'(x)dx \\ &= ... \\ &= \frac{(-1)^n}{2^n n!}\int_{-1}^{1}\varphi(x)h^{(n)}(x)dx \end{aligned} \tag{1.4}$$

Thus, the proof can be discussed on 2 cases

a) If the order of $h(x)$ is less than n, then

$$h^{(n)}(x) = 0 \tag{1.5}$$

Thus

$$\int_{-1}^{1} Q_n(x)Q_m(x)dx = 0 \ \ (n \neq m) \tag{1.6}$$

b) If $h(x) = Q_n(x)$, then the $n-th$ derivative of $g(x)$ is

$$h^{(n)}(x) = Q^{(n)}(x) = \frac{(2n)!}{2^n n!} \tag{1.7}$$

Thus

$$\begin{aligned}
\int_{-1}^{1} Q_n(x)Q_m(x)dx &= \int_{-1}^{1} Q_n^2(x)dx \ \ (n = m) \\
&= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n dx \\
&= \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n dx \\
&= \frac{(2n)!}{2^{2n}(n!)^2}\int_{0}^{\pi/2} cos^{2n+1} t dt \\
&= \frac{(2n)!}{2^{2n}(n!)^2}\frac{2 \times 4 \times ... \times (2n)}{1 \times 3 \times ... \times (2n+1)} \\
&= \frac{2}{2n+1}
\end{aligned} \tag{1.8}$$

Thus, $(Q_n)_{n\in\mathbb{N}}$ are a sequence of orthogonal polynomials. □

2. *Proof.* Denote

$$\varphi(x) = (x^2-1)^n \tag{1.9}$$

then

$$Q_n(x) = \frac{1}{2^n n!}\varphi^{(n)}(x) \tag{1.10}$$

As the power of each item in $\varphi(x)$ is even when $\varphi(x)$ is extended, thus $\varphi^{(n)}(x)$ is even function if the order of derivative is even, and $\varphi^{(n)}(x)$ is odd function if the order of derivative is odd.
Therefore $Q_n(x)$ is even function if n is even, and $Q_n(x)$ is odd function if n is odd. So, it can be summarized as $Q_n(-x) = (-1)^n Q_n(x)$. □

3. *Proof.* As $xQ_n(x)$ can be written as

$$xQ_n(x) = \sum_{i=0}^{n+1} a_i Q_i(x) \tag{1.11}$$

According to the orthogonality of Legendre polynomials, $a_i = 0$ for $i = 0, 1, ..., n-2, n$, as

$$0 = \int_{-1}^{1} xQ_n(x)Q_i(x)dx = a_i\int_{-1}^{1} Q_i^2(x)dx \ \ (i = 0, 1, ..., n-2, n) \tag{1.12}$$

Thus, $xQ_n(x)$ can be written as

$$xQ_n(x) = a_{n-1}Q_{n-1}(x) + a_{n+1}Q_{n+1}(x) \tag{1.13}$$

Since the highest order item in $xQ_n(x)$ is x^{n+1} and its coefficient is $\frac{(2n)!}{2^n(n!)^2}$, thus

$$\frac{(2n)!}{2^n(n!)^2} = a_{n+1}\frac{(2n+2)!}{2^{n+1}(n+1)!^2} \tag{1.14}$$

which implies that

$$a_{n+1} = \frac{n+1}{2n+1} \tag{1.15}$$

Denote

$$I_n = \int_{-1}^{1} xQ_{n-1}(x)Q_n(x)dx \tag{1.16}$$

Then

$$I_n = a_{n-1}\int_{-1}^{1} Q_{n-1}^2(x)dx = a_{n-1}\frac{2}{2n-1} \tag{1.17}$$

As

$$I_{n+1} = a_{n+1}\int_{-1}^{1} Q_{n+1}^2(x)dx = a_{n+1}\frac{2}{2n+3} = \frac{2(n+1)}{(2n+1)(2n+3)} \tag{1.18}$$

Thus

$$I_n = \frac{2n}{(2n-1)(2n+1)} \tag{1.19}$$

which implies that

$$a_{n-1} = \frac{n}{2n+1} \tag{1.20}$$

Hence

$$(2n+1)xQ_n(x) = nQ_{n-1}(x) + (n+1)Q_{n+1}(x) \tag{1.21}$$

□

4. *Proof.* □

2 INTERPOLATION

$f(2)$ can be determined using the Lagrange interpolation scheme. As the lagrange interpolation polynomial can be written as below, and $n = 8$ in this case.

$$f(x) = \sum_{i=1}^{n} f(x_i)l_i(x) \tag{2.1}$$

$l_i(x)$ are the base functions that can be written as below.

$$l_i(x) = \frac{(x-x_1)(x-x_2)...(x-x_{i-1})(x-x_{i+1})...(x-x_{n-1})(x-x_n)}{(x_i-x_1)(x_i-x_2)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_{n-1})(x_i-x_n)} \tag{2.2}$$

$l_i(2)$ are calculated accordingly as below.

$$l_1(2) = -0.0006 \quad l_2(2) = 0.1224 \quad l_3(2) = -0.5600 \quad l_4(2) = 1.0606$$
$$l_5(2) = 0.4167 \quad l_6(2) = -0.0400 \quad l_7(2) = 0.0012 \quad l_8(2) = -0.0003$$

Thus, $f(2)$ is calculated according to (2.1) as 11.0.

3 Newton's form of interpolation polynomial

1.
2.
3.
4.
5.
6.
7.