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# Introduction to Numerical Analysis
# HW3

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## 1 CANTOR'S SET

1. *Proof.* As $C_i$ is closed and $C$ is the intersection of these closed sets, $C$ is closed.
   It's clear that $C$ is bounded. Thus, by Heine-Borel theorem, $C$ is compact. □

2. *Proof.* Suppose $x \in C_m$, $y \in C_n$ and $x < y$. There are $2^m$ subsets in $C_m$ and the length of each subset is $\frac{1}{3^m}$. Also, there are $2^n$ subsets in $C_n$ and the length of each subset is $\frac{1}{3^n}$.
   As $C \subset C_m \cap C_n$, suppose $m \leq n$, there must exist a subset in $C_n$ s.t. $x \in C_n$.
   If $x$ and $y$ lie in the same subset of $C_n$, with further division of the subset, there will be a gap between $x$ and $y$, thus, there exists an element $z$ lies in the gap and satisfies $x < z < y$.
   If $x$ and $y$ lie in different subsets of $C_n$, denoted as $C_{n,i}$ and $C_{n,j}$,it's obvious that such a $z$ exists in the gap between $C_{n,i}$ and $C_{n,j}$ and satisfies $x < z < y$. □

3. a) 0
   b) *Proof.* As there are $2^n$ subsets in $C_n$ and the length of each subset is $\frac{1}{3^n}$, thus the Lebesgue measure of $C_n$ is the sum of the Lebesgue measure of each closed subset. Thus $\lambda(C_n) = (\frac{2}{3})^n$.
   As $C = \bigcap_{n=1}^{\infty} C_n$, then $\lambda(C) \leq \lambda(C_n) = (\frac{2}{3})^n$, thus $\lambda(C) = 0$. □

4. a) *Proof.* As the end points of each subset in $C_n$ is not removed in any subdivision, thus $C$ is not empty. □

b)

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \text{ with } a_i \in \{0, 2\} \tag{1.1}$$

c) Consider the $i-th$ digit in elemnet $s_i$, it will be possible to construct such an element that the $i-th$ digit is different from that in $s_i$, thus $s$ is not included in the original list.

d) *Proof.* Suppose $C$ is countable, express each $x \in C$ in ternary form, then each digit in $x$ is either 0 or 2. Thus, the choice of each digit appears to be binary. Consider the $i-th$ digit in elemnet $x_i$, it will be possible to construct such an element $t$, whose $i-th$ digit is different from that in $x_i$(complementary). Thus $t$ is not included in the original list, which implies the assumption fails. □

5. Althouth $C$ is uncountable, but the measure of its complement is 1, which is illustrated below, thus the measure of $C$ is 0.

$$\begin{aligned}
\lambda(C^c) &= \frac{1}{3} + 2 * \frac{1}{9} + \dots \\
&= \frac{1}{3} \sum_{i=0}^{\infty} (\frac{2}{3})^n \\
&= \frac{1}{3} \lim_{n \to \infty} \frac{1 - (2/3)^n}{1 - (2/3)} \\
&= 1
\end{aligned} \tag{1.2}$$

# 2 Cantor's Function

1. *Proof.* It's easy to verify that $f_0, f_1$ are monotonically increasing continuous functions. Suppose $f_n$ are still monotonically increasing continuous functions for $n = k$.
For $n = k+1$, as $f_{k+1}$ is only different from $f_k$ on each closed subset of $C_k$, whose length is $\frac{1}{3^n}$ and is denoted as $I_{k,p}(1 \le p \le 2^k)$ here, it is left to prove that $f_{k+1}$ remains monotonically increasing continuous on each $I_{k,p}$ after the construction process.
Since $f_k$ is linear on each $I_{k,p}$, and the recursive construction process doesn't change the values on each ending points, on which the values of $f_k$ is denoted as $a, b$ recursively, the subset of $I_{k,p}$ is valid and the value of $f_{k+1}$ on the central part, whose length is $\frac{1}{3^{k+1}}$, is $\frac{a+b}{2}$. As the measure of the remaining parts of $I_{k,p}$ is not 0, the linear function connecting each ending points is obviously valid. Thus, the new function $f_{k+1}$ is still monotonically increasing continuous. □

2. *Proof.* Let $g_n(x) = |f_n(x) - f(x)|$, then $g_n(x)$ is positive on $C_n$ and 0 on other places. Further, $g_n(x)$ holds the same value on each closed subset of $C_n$.
Given any $\xi > 0$, it is left to find $N$ s.t. $g_n(x) < \xi$ when $n > N$.
As $g_n(x)$ reaches its extream $\frac{1}{6}\frac{1}{3^n}$ at the $\frac{1}{3}$ and $\frac{2}{3}$ of each compact subset of $C_n$, thus, given any $\xi$

$$\frac{1}{6}\frac{1}{3^n} < \xi \tag{2.1}$$

which is equivalent to

$$n > \frac{ln(\frac{1}{6\xi})}{ln(3)} \triangleq N \tag{2.2}$$

and the target $N$ is given as above. □

3. *Proof.* As $f_c(x)$ is continous over $[0,1]$, it is left to prove that a continous function on a compact is uniformly continous, which can be proved by contradiction.

   □

4.

# 3 Taylor's theorem

1.

2.

# 4 Convergence of rationals to irrationals

1. *Proof.* As $e$ can be written as

$$e = \sum_{i=0}^{\infty} \frac{1}{i!} \tag{4.1}$$

Suppose it is rational, then it can be written as below, where $p, q$ are prime to each other.

$$e = \frac{p}{q} \tag{4.2}$$

Thus

$$p(q-1)! = q!\sum_{i=0}^{q} \frac{1}{i!} + q!\sum_{i=q+1}^{\infty} \frac{1}{i!} \tag{4.3}$$

Since both LHS and the first part of RHS are integers, the second part of RHS(denoted as $RHS_2$) should be an integer as well.// $RHS_2$ can be further expanded as below

$$\begin{aligned} RHS_2 &= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \dots \\ &< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \dots \\ &= \frac{1}{q} \\ &< 1 \end{aligned} \tag{4.4}$$

Therefore $RHS_2$ is not an integer as $RHS_2 > 0$. Thus the assumption fails, and $e$ is irrational. □

2. *Proof.* It's clear that $u_n$ is increasing, and the limit is $e$, so the maximum distance between 2 element can be denoted as

$$d_{max} < e - (1 + \frac{1}{n})^n \tag{4.5}$$

For any $\xi > 0$, let $d_{max} < \xi$, it is satisfied when $n > N$, where $N$ is the index of the first element s.t. $u_N > e - \xi$.
Thus $u_n$ is a cauchy sequence converging to $e$. □

3. No, as the limit of a rational sequence may not converging to a rational.