SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Introduction to Numerical Analysis
HW9

Yu Cang
018370210001

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1 QUESTION 1

(a) *Proof.* As $\mathscr{A}$ is convex, the mean value theorem can be applied in terms of y s.t.

$$\Phi(t, y_2) - \Phi(t, y_1) = (y_2 - y_1)\frac{\partial \Phi(t, y)}{\partial y}\Big|_{y=\xi} \tag{1.1}$$

where $\xi \in (y_1, y_2)$.
Since the there exists $c > 0$ s.t. for all $(t, y) \in \mathscr{A}$

$$\Big|\frac{\partial \Phi(t, y)}{\partial y}\Big| \le c \tag{1.2}$$

Thus

$$\begin{aligned}
&|\Phi(t, y_2) - \Phi(t, y_1)| \\
=&|y_2 - y_1|\Big|\frac{\partial \Phi(t, y)}{\partial y}\Big|_{y=\xi}\Big| \\
\le& c|y_2 - y_1|
\end{aligned} \tag{1.3}$$

which implies that $\Phi(t, y)$ satisfies Lipschitz condition in y on $\mathscr{A}$. $\square$

(b) *Proof.* Let $P_1 = (t_1, y_1)$ and $P_2 = (t_2, y_2)$. Then any point P' lies on the line segment joining P_1 and P_2 can be expressed as

$$\begin{aligned}
P &= (1-\alpha)P_1 + \alpha P_2 \\
&= ((1-\alpha)t_1 + \alpha t_2, (1-\alpha)y_1 + \alpha y_2) \\
&\triangleq (t', y')
\end{aligned} \tag{1.4}$$

where $\alpha \in [0,1]$.
It's clear that t_1 and t_2 lies between t_0 and T, and t' lies between t_1 and t_2. Thus, t' also lies between t_0 and T.
Further, it's also clear that $-\infty < y' < +\infty$.
Hence $P \in \mathscr{D}$, which implies that $\mathscr{D}$ is convex. □

(c) *Proof.* Let

$$\Phi(t,y) = \frac{4t^3 y}{1+t^4} \tag{1.5}$$

Then

$$\frac{\partial \Phi(t,y)}{\partial y} = \frac{4t^3}{1+t^4} = \frac{4t}{t^2+\frac{1}{t^2}} < \frac{4}{t^2+\frac{1}{t^2}} < \frac{4}{2\sqrt{t^2\frac{1}{t^2}}} = 2 \tag{1.6}$$

as $t \in (0,1)$.
which implies that $\Phi(t,y)$ satisfies a Lipschitz condition in y.
Thus, the given IVP problem has a unique solution. □

(d) Definitely not recommended.
As $\Phi(t,y) = 1 + y^2$, then

$$\frac{\partial \Phi}{\partial y} = 2y \triangleq \lambda y \tag{1.7}$$

Here $\lambda > 0$, and the Euler's method is not stable as the error will be amplified at each iteration step. Finally the calculation will diverge.

2 Question 2

(a) As $\Phi(t,y) = arctan(y)$, then

$$\left|\frac{\partial \Phi}{\partial y}\right| = \frac{1}{|1+y^2|} < 1 \triangleq c \tag{2.1}$$

(b) As $\dot{y} = \Phi(t,y) = arctan(y)$, then

$$|\ddot{y}| = \left|\frac{\dot{y}}{1+y^2}\right| = \frac{|arctan(y)|}{1+y^2} < \frac{\pi}{2(1+y^2)} \le \frac{\pi}{2} \tag{2.2}$$

(c) As the global error is bounded by

$$|e_k| \le \frac{\tau^*}{hc}[e^{c(t_k-t_0)} - 1] \tag{2.3}$$

where $\tau^* = \max_k |\tau_k|$, c is the Lipschitz constant and is taken as 1 as has been illustrated above.
Since

$$\begin{aligned}\frac{\tau^*}{h} &= \max_k \left|\frac{y(t_k) - y(t_{k-1}) - h\Phi(t_{k-1}, y(t_{k-1}))}{h}\right| \\ &\le \max_k \left|\frac{y(t_k) - y(t_{k-1})}{h}\right| + |\Phi(t_{k-1}, y(t_{k-1}))| \\ &= 2\max_k |\Phi(t_{k-1}, y(t_{k-1}))| < \pi\end{aligned} \tag{2.4}$$

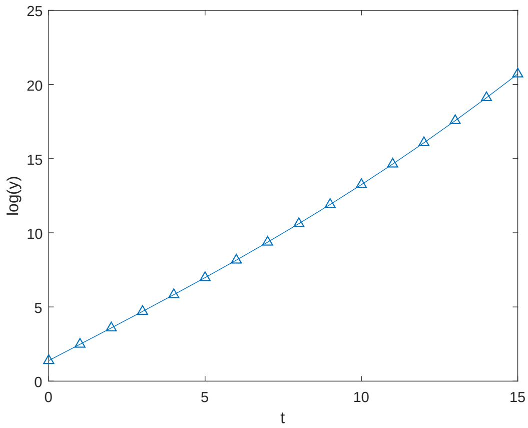
Thus, $|e_k| < \pi[e^{(t_k - t_0)} - 1]$.

3 Question 3

(a) With $y_0 = 4$, y_k are calculated using Euler's method as follows

$$y_k = y_{k-1} + h\Phi(t_{k-1}, y_{k-1}) \tag{3.1}$$

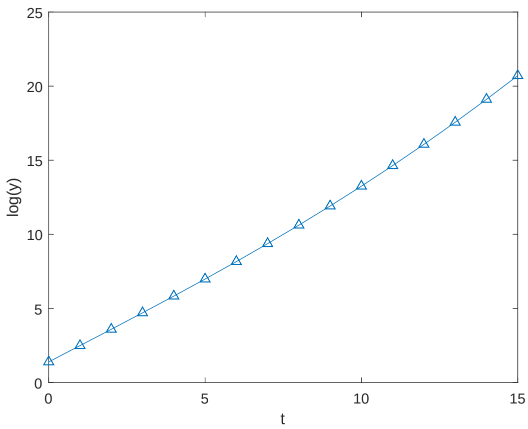
The calculation is carried out with matlab, and the results are given as below



(b) The initial value of $y_k^{(0)}$ is calculated using Euler's method.
Then $y_k^{(i)}$ are calculated in $i-th$ iteration, and the back-ward Euler's method is adopted.

$$y_k^{(i)} = y_{k-1}^{(i)} + h\Phi(t_k, y_k^{(i-1)}) \tag{3.2}$$

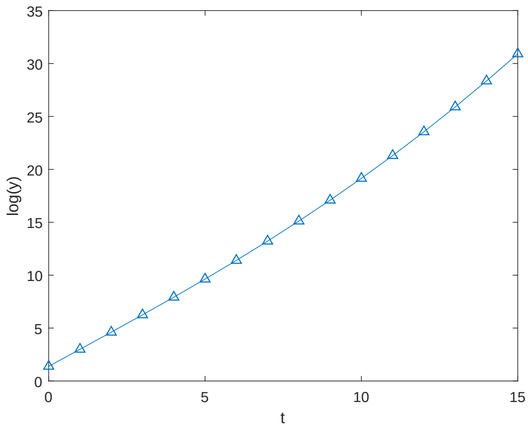
The calculation is carried out with matlab, and the results are given as below



(c) Iteration formula is given as

$$y_k = y_{k-1} + h\Phi(t_{k-1}, y_{k-1}) + \frac{h^2}{2}[\Phi_t(t_{k-1}, y_{k-1}) + \Phi_y(t_{k-1}, y_{k-1})\Phi(t_{k-1}, y_{k-1})] \tag{3.3}$$

The calculation is carried out with matlab, and the results are given as below

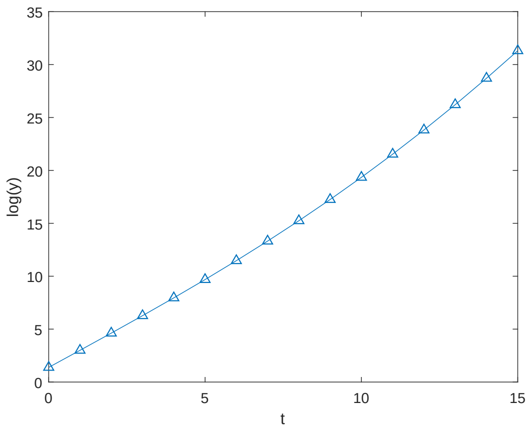


(d) The calculation is carried out with the prediction-correction method. Iteration formulas are given as

$$y^* = y_{k-1} + h\Phi(t_{k-1}, y_{k-1}) \tag{3.4}$$

$$y_k = y_{k-1} + \frac{h}{2}[\Phi(t_{k-1}, y_{k-1}) + \Phi(t_k, y^*)] \tag{3.5}$$

The calculation is carried out with matlab, and the results are given as below



(e) Here, an extra initial point(y_1) is required, and it is calculated using the Euler's method as

$$y_1 = y_0 + h\Phi(t_0, y_0) \tag{3.6}$$

Then, the iteration steps are taken as

$$y_k = y_{k-1} + \frac{h}{2}[3\Phi(t_{k-1}, y_{k-1}) - \Phi(t_{k-2}, y_{k-2})] \tag{3.7}$$

The calculation is carried out with matlab, and the results are given as below