SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Introduction to Numerical Analysis
HW3

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1 CANTOR'S SET

1. *Proof.* As C_i is closed and C is the intersection of these closed sets, C is closed. It's clear that C is bounded. Thus, by Heine-Borel theorem, C is compact. □

2. *Proof.* Suppose $x \in C_m$, $y \in C_n$ and $x < y$. There are 2^m subsets in C_m and the length of each subset is $\frac{1}{3^m}$. Also, there are 2^n subsets in C_n and the length of each subset is $\frac{1}{3^n}$. As $C \subset C_m \cap C_n$, suppose $m \le n$, there must exist a subset in C_n s.t. $x \in C_n$.
   If x and y lie in the same subset of C_n, with further division of the subset, there will be a gap between x and y, thus, there exists an element z lies in the gap and satisfies $x < z < y$.
   If x and y lie in different subsets of C_n, denoted as $C_{n,i}$ and $C_{n,j}$,it's obvious that such a z exists in the gap between $C_{n,i}$ and $C_{n,j}$ and satisfies $x < z < y$. □

3. a) 0
   b) *Proof.* As there are 2^n subsets in C_n and the length of each subset is $\frac{1}{3^n}$, thus the Lebesgue measure of C_n is the sum of the Lebesgue measure of each closed subset. Thus $\lambda(C_n) = (\frac{2}{3})^n$.
   As $C = \bigcap\limits_{n=1}^{\infty} C_n$, then $\lambda(C) \le \lambda(C_n) = (\frac{2}{3})^n$, thus $\lambda(C) = 0$. □

4. a) *Proof.* As the end points of each subset in C_n is not removed in any subdivision, thus C is not empty. □

b)
$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \text{ with } a_i \in \{0, 2\} \tag{1.1}$$

c) Consider the $i-th$ digit in elemnet s_i, it will be possible to construct such an element that the $i-th$ digit is different from that in s_i, thus s is not included in the original list.

d) *Proof.* Suppose C is countable, express each $x \in C$ in ternary form, then each digit in x is either 0 or 2. Thus, the choice of each digit appears to be binary. Consider the $i-th$ digit in elemnet x_i, it will be possible to construct such an element t, whose $i-th$ digit is different from that in x_i(complementary). Thus t is not included in the original list, which implies the assumption fails. □

5. Althouth C is uncountable, but the measure of its complement is 1, which is illustrated below, thus the measure of C is 0.

$$\begin{aligned} \lambda(C^c) &= \frac{1}{3} + 2 * \frac{1}{9} + ... \\ &= \frac{1}{3} \sum_{i=0}^{\infty} (\frac{2}{3})^n \\ &= \frac{1}{3} \lim_{n\to\infty} \frac{1-(2/3)^n}{1-(2/3)} \\ &= 1 \end{aligned} \tag{1.2}$$

2 Cantor's Function

1. *Proof.* It's easy to verify that f_0, f_1 are monotonically increasing continuous functions. Suppose f_n are still monotonically increasing continuous functions for $n = k$.
For $n = k+1$, as f_{k+1} is only different from f_k on each closed subset of C_k, whose length is $\frac{1}{3^n}$ and is denoted as $I_{k,p}(1 \le p \le 2^k)$ here, it is left to prove that f_{k+1} remains monotonically increasing continuous on each $I_{k,p}$ after the construction process.
Since f_k is linear on each $I_{k,p}$, and the recursive construction process doesn't change the values on each ending points, on which the values of f_k is denoted as a, b recursively, the subset of $I_{k,p}$ is valid and the value of f_{k+1} on the central part, whose length is $\frac{1}{3^{k+1}}$, is $\frac{a+b}{2}$. As the measure of the remaining parts of $I_{k,p}$ is not 0, the linear function connecting each ending points is obviously valid. Thus, the new function f_{k+1} is still monotonically increasing continuous. □

2. *Proof.* Let $g_n(x) = |f_n(x) - f(x)|$, then $g_n(x)$ is positive on C_n and 0 on other places. Further, $g_n(x)$ is symmetric around $x = 1/2$, thus, only $[0, 1/2]$ is considered for simplicity. Given any $\xi > 0$, it is left to find N s.t. $g_n(x) < \xi$ when $n > N$. □

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3 Taylor's theorem

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4 Convergence of rationals to irrationals

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3.