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Introduction to Numerical Analysis
HW7

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1 QUESTION 1

(a) For example

$$y = tan(x), \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \tag{1.1}$$

It's differentiable over $(-\frac{\pi}{2}, \frac{\pi}{2})$, and the derivative is

$$y' = \frac{1}{cos^2(x)} \tag{1.2}$$

It's obvious that $y' \to \infty$ when $x \to \frac{\pi}{2}$.

(b) *Proof.* Denote $g(x)$ over $[x_1, x_2]$ as

$$g(x) = f(x) - f(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) \tag{1.3}$$

where $a < x_1 < x_2 < b$. Then, $g(x_1) = 0$ and $g(x_2) = 0$.

Thus, from Rolle's theorem, there exists $\xi \in (x_1, x_2)$ s.t.

$$g'(\xi) = 0 \tag{1.4}$$

namely

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi) \tag{1.5}$$

Since f' is bounded, then $|f'(\xi)| \leq M$. Thus

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1| \tag{1.6}$$

which means that $f(x)$ is Lipschitz continous. $\square$

(c) For example

$$y = \frac{1}{x}, \quad x \in (0,1) \tag{1.7}$$

y is obviously differentiable and its derivative is

$$y' = -\frac{1}{x^2}, \quad x \in (0,1) \tag{1.8}$$

Suppose there exist a constant $c > 0$ s.t.

$$|y_2 - y_1| \leq c|x_2 - x_1| \tag{1.9}$$

is valid for all $0 < x_1 < x_2 < 1$. Let $y(b) - y(a) = c(b-a)$, where $a < b$then

$$c = \frac{1}{ab} \tag{1.10}$$

Take the mid-point of a, b, then

$$\frac{y(a) - y(\frac{b+a}{2})}{\frac{b+a}{2} - a} = \frac{1}{a(\frac{b+a}{2})} > \frac{1}{ab} = c \tag{1.11}$$

Thus, the assumption fails, which means that y is not Lipschitz continous.

(d) For example

$$y = |x|, \quad x \in (-1,1) \tag{1.12}$$

It's Lipschitz continous as for any $-1 < x_1 < x_2 < 1$

$$\frac{|y(x_2) - y(x_1)|}{|x_2 - x_1|} \leq 1 \tag{1.13}$$

but it is not differentiable at $x = 0$.

2 Question 2

1. *Proof.* Since $|g'(x)| < 1$, there exists $0 < L < 1$ s.t. $|g'(x)| \leq L$. Then, according to the Lagrange intermediate value theorem

$$\begin{aligned} |g(x_j) - g(x_i)| &= |g'(\xi)||x_j - x_i| \\ &\leq L|x_j - x_i| \end{aligned} \tag{2.1}$$

As $g(x^*) = x^*$, applying the fixed point iteration, then

$$\begin{aligned} |x_{k+1} - x^*| &= |g(x_k) - g(x^*)| \\ &\leq L|x_k - x^*| \\ &= L|(x_{k+1} - x^*) - (x_{k+1} - x_k)| \\ &\leq L|x_{k+1} - x^*| + L|x_{k+1} - x_k| \end{aligned} \tag{2.2}$$

Thus

$$\begin{aligned} |x_{k+1} - x^*| &\leq \frac{L}{1-L}|x_{k+1} - x_k| \\ &\leq \frac{L^2}{1-L}|x_k - x_{k-1}| \\ &\leq \dots \\ &\leq \frac{L^{k+1}}{1-L}|x_1 - x_0| \end{aligned} \tag{2.3}$$

Since $0 < L < 1$, and $|x_1 - x_0|$ is finite, it indicates that

$$\lim_{k\to\infty} |x_{k+1} - x^*| = 0 \tag{2.4}$$

Hence, the fixed-point iteration will converge to the unique fixed point x^*. □

2. *Proof.* Since $|g'(x)| > 1$, there exists $L > 1$ s.t. $|g'(x)| \geq L$. Then, according to the Lagrange intermediate value theorem

$$\begin{aligned} |g(x_j) - g(x_i)| &= |g'(\xi)||x_j - x_i| \\ &\geq L|x_j - x_i| \end{aligned} \tag{2.5}$$

As $g(x^*) = x^*$, applying the fixed point iteration, then

$$\begin{aligned} |x_{k+1} - x^*| &= |g(x_k) - g(x^*)| \\ &\geq L|x_k - x^*| \\ &= L|(x_{k+1} - x_k) - (x_{k+1} - x^*)| \\ &\geq L(|x_{k+1} - x_k| - |x_{k+1} - x^*|) \end{aligned} \tag{2.6}$$

Thus

$$\begin{aligned} |x_{k+1} - x^*| &\geq \frac{L}{1+L}|x_{k+1} - x_k| \\ &\geq \frac{L^2}{1+L}|x_k - x_{k-1}| \\ &\geq \dots \\ &\geq \frac{L^{k+1}}{1+L}|x_1 - x_0| \end{aligned} \tag{2.7}$$

Since $L > 1$, and $|x_1 - x_0|$ is finite, it indicates that

$$\lim_{k\to\infty} |x_{k+1} - x^*| = \infty \tag{2.8}$$

Hence, the fixed-point iteration will never converge to the unique fixed point x^*. □

3 QUESTION 3

1. For resolving the smallest positive root, the equation can be reformed as

$$x = tg^{-1}(4x) \triangleq g(x) \tag{3.1}$$

It can be easily observed that the root lies in $(1, \frac{\pi}{2})$.

Applying the fundamental inequality, $g'(x)$ can be further determined as

$$0 < g'(x) = \frac{4}{1+(4x)^2} = \frac{1}{\frac{1}{4}+4x^2} \leq \frac{1}{x} < 1 \tag{3.2}$$

Thus, the fixed-point iteration can be employed to resolve the result. The starting value is given as

$$x_0 = \frac{1+\frac{\pi}{2}}{2} \tag{3.3}$$

A python script is written to perform the numerical iteration, and the final result is

$$x^* \approx 1.393 \tag{3.4}$$

2. For resolving the second smallest positive root, the original coordinate axes Oxy can be transfromed to $O'xy$ to simplify to equation, where $(\pi, 0)$ in origial coordinate is selected as the transformed origin. Thus, it's equivalent to solve

$$tg(x) = 4(x+\pi) \tag{3.5}$$

For similar consideration, the equation is reformed as

$$x = tg^{-1}(4(x+\pi)) \triangleq g(x) \tag{3.6}$$

Obviously, the root is located in $(1, \frac{\pi}{2})$ in the transformed coordinate.Similarly, the range of $g'(x)$ is examined as

$$0 < g'(x) = \frac{4}{1+16(x+\pi)^2} = \frac{1}{\frac{1}{4}+4(x+\pi)^2} < \frac{1}{\frac{1}{4}+4x^2} < \frac{1}{x} < 1 \tag{3.7}$$

Thus, the fixed-point iteration can be employed to resolve the result. The starting value is given as

$$x'_0 = \frac{1+\frac{\pi}{2}}{2} \tag{3.8}$$

A python script is written to perform the numerical iteration, and the final result is

$$x^* = x'^* + \pi \approx 1.517 + \pi \approx 4.659 \tag{3.9}$$

4 QUESTION 6

1. For example, $e^x = x + 1$ over $[-1, 1]$ can be solved by bisection, but $g'(x) \triangleq (e^x - 1)' = e^x > 1$ when $x > 0$. Thus the fixed-point iteration will diverge if the initial value is given as $x_0 > 0$.

2. For example, $cos(x) = 0$ over $[0, \frac{\pi}{2}]$. Let $g'(x) \triangleq (cos(x) + x)' = 1 - sin(x)$, it's clear that $0 \leq g'(x) \leq 1$, thus, it can be resolved by the fixed-point iteration given an starting value $x_0 \in (0, \frac{\pi}{2})$. However, the Newton method would fail if x_0 is very close to 0. This is caused as $cos'(x) = sin(x) \to 0$, which lead to the further displacement of x_1 as $\frac{f(x_0)}{f'(x_0)} \to \infty$.