

Methods of Applied Mathematics I

HW3

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1 EXERCISE3.1

Proof. Let $\phi(x) = (x^2 - 1)^n$, then

$$\phi(\pm 1) = \phi^{(1)}(\pm 1) = \cdots = \phi^{(n-1)}(\pm 1) = 0 \quad (1.1)$$

The L^2 norm of $P_n(x)$ is

$$\begin{aligned} \|P_n\|_2 &= \sqrt{\int_{-1}^1 P_n^2(x) dx} \\ &= \sqrt{\left(\frac{1}{2^n n!}\right)^2 \int_{-1}^1 [\phi^{(n)}(x)]^2 dx} \\ &= \sqrt{\left(\frac{1}{2^n n!}\right)^2 (-1)^n \int_{-1}^1 \phi^{(2n)}(x) \phi(x) dx} \quad (\text{Integrate by parts}) \\ &= \sqrt{\left(\frac{1}{2^n n!}\right)^2 (-1)^n (2n)! \int_{-1}^1 (x^2 - 1)^n dx} \\ &= \sqrt{\left(\frac{1}{2^n n!}\right)^2 (-1)^n (2n)! \cdot 2 \int_0^{\frac{\pi}{2}} (\sin^2(x) - 1)^n \cos(x) dx} \\ &= \sqrt{\left(\frac{1}{2^n n!}\right)^2 (2n)! \cdot 2 \int_0^{\frac{\pi}{2}} \cos^{2n+1}(x) dx} \end{aligned} \quad (1.2)$$

Since

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1}(x) dx = \frac{2 \times 4 \times \cdots \times (2n)}{1 \times 3 \times \cdots \times (2n+1)} \quad (1.3)$$

substitute it into the norm equation of P_n yields

$$\|P_n\|_2 = \sqrt{\left(\frac{1}{2^n n!}\right)^2 (2n)! \cdot 2 \cdot \frac{2 \times 4 \times \cdots \times (2n)}{1 \times 3 \times \cdots \times (2n+1)}} = \sqrt{\frac{2}{2n+1}} \quad (1.4)$$

□

2 EXERCISE3.2

1. Consider the projection of $f(x)$ onto $P_0(x)$, $P_1(x)$ and $P_2(x)$ correspondingly

$$(f, P_0) = \int_{-1}^1 e^x dx = e - \frac{1}{e} \quad (2.1)$$

$$(f, P_1) = \int_{-1}^1 x e^x dx = \frac{2}{e} \quad (2.2)$$

$$(f, P_2) = \int_{-1}^1 \frac{3x^2 - 1}{2} e^x dx = e - \frac{7}{e} \quad (2.3)$$

As $(P_i, P_j) = 0$ when $i \neq j$, $p(x)$ can be written as linear combination of P_i

$$p(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) \quad (2.4)$$

with coefficient a_i given as

$$a_i = \frac{(f, P_i)}{(P_i, P_i)} = \frac{2i+1}{2} (f, P_i) \quad (2.5)$$

Finally

$$p(x) = 1.1752 P_0(x) + 1.1036 P_1(x) + 0.3578 P_2(x) \quad (2.6)$$

2. The plot(Fig 2.1) is provided for comparsion.

The distinction is quite obvious and Legendre polynomials does better than taylor series expansion at zero point.

Further analysis indicates that the approximation error of $p(x)$ is given by

$$\|\delta_2(x)\|_2 = \|e^x - p(x)\|_2 = \sqrt{\int_{-1}^1 e^{2x} dx - \sum_{i=0}^2 \frac{2a_i^2}{2i+1}} = 0.0388 \quad (2.7)$$

3 EXERCISE3.3

- 1.
- 2.

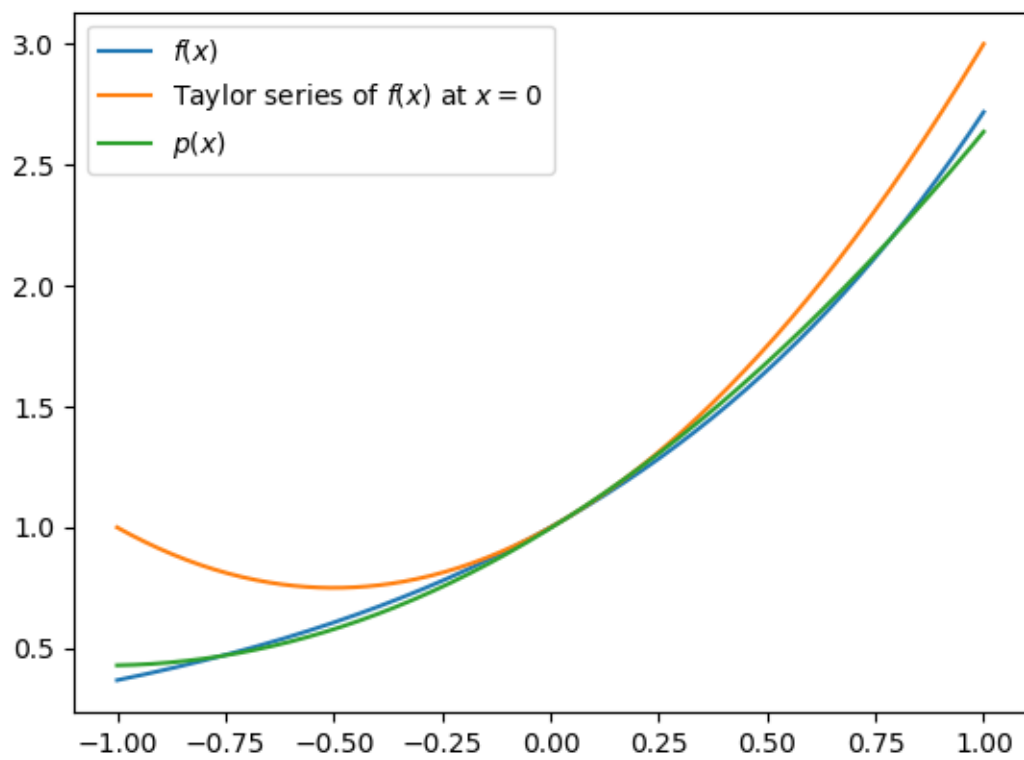


Figure 2.1: Comparison of different approximation for e^x