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Methods of Applied Mathematics I
HW3

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1 EXERCISE3.1

Proof. Let $\phi(x) = (x^2-1)^n$, then

$$\phi(\pm 1) = \phi^{(1)}(\pm 1) = \cdots = \phi^{(n-1)}(\pm 1) = 0 \tag{1.1}$$

The L^2 norm of $P_n(x)$ is

$$\begin{aligned}
||P_n||_2 &= \sqrt{\int_{-1}^{1} P_n^2(x)dx} \\
&= \sqrt{(\frac{1}{2^n n!})^2 \int_{-1}^{1} [\phi^{(n)}(x)]^2 dx} \\
&= \sqrt{(\frac{1}{2^n n!})^2 (-1)^n \int_{-1}^{1} \phi^{(2n)}(x)\phi(x)dx} \quad \text{(Integrate by parts)} \\
&= \sqrt{(\frac{1}{2^n n!})^2 (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n dx} \\
&= \sqrt{(\frac{1}{2^n n!})^2 (-1)^n (2n)! \cdot 2 \int_{0}^{\frac{\pi}{2}} (sin^2(x)-1)^n cos(x)dx} \\
&= \sqrt{(\frac{1}{2^n n!})^2 (2n)! \cdot 2 \int_{0}^{\frac{\pi}{2}} cos^{2n+1}(x)dx}
\end{aligned} \tag{1.2}$$

Since

$$\int_0^{\frac{\pi}{2}} cos^{2n+1}(x)dx = \frac{2\times 4\times\cdots\times(2n)}{1\times 3\times\cdots\times(2n+1)} \tag{1.3}$$

substitute it into the norm equation of P_n yields

$$||P_n||_2 = \sqrt{(\frac{1}{2^n n!})^2(2n)!\cdot 2\cdot\frac{2\times 4\times\cdots\times(2n)}{1\times 3\times\cdots\times(2n+1)}} = \sqrt{\frac{2}{2n+1}} \tag{1.4}$$

□

2 EXERCISE3.2

1. Consider the projection of $f(x)$ onto $P_0(x)$, $P_1(x)$ and $P_2(x)$ correspondingly

$$(f,P_0) = \int_{-1}^{1} e^x dx = e - \frac{1}{e} \tag{2.1}$$

$$(f,P_1) = \int_{-1}^{1} xe^x dx = \frac{2}{e} \tag{2.2}$$

$$(f,P_2) = \int_{-1}^{1} \frac{3x^2-1}{2} e^x dx = e - \frac{7}{e} \tag{2.3}$$

As $(P_i,P_j)=0$ when $i\neq j$, $p(x)$ can be written as linear combination of P_i

$$p(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) \tag{2.4}$$

with coefficient a_i given as

$$a_i = \frac{(f,P_i)}{(P_i,P_i)} = \frac{2i+1}{2}(f,P_i) \tag{2.5}$$

Finally

$$p(x) = 1.1752P_0(x) + 1.1036P_1(x) + 0.3578P_2(x) \tag{2.6}$$

2. The plot(Fig 2.1) is provided for comparsion.

The distinction is quite obvious and Legendre polynomials does better than taylor series expansion at zero point.
Further analysis indicates that the approximation error of $p(x)$ is given by

$$||\delta_2(x)||_2 = ||e^x - p(x)||_2 = \sqrt{\int_{-1}^{1} e^2(x)dx - \sum_{i=0}^{2}\frac{2a_i^2}{2i+1}} = 0.0388 \tag{2.7}$$

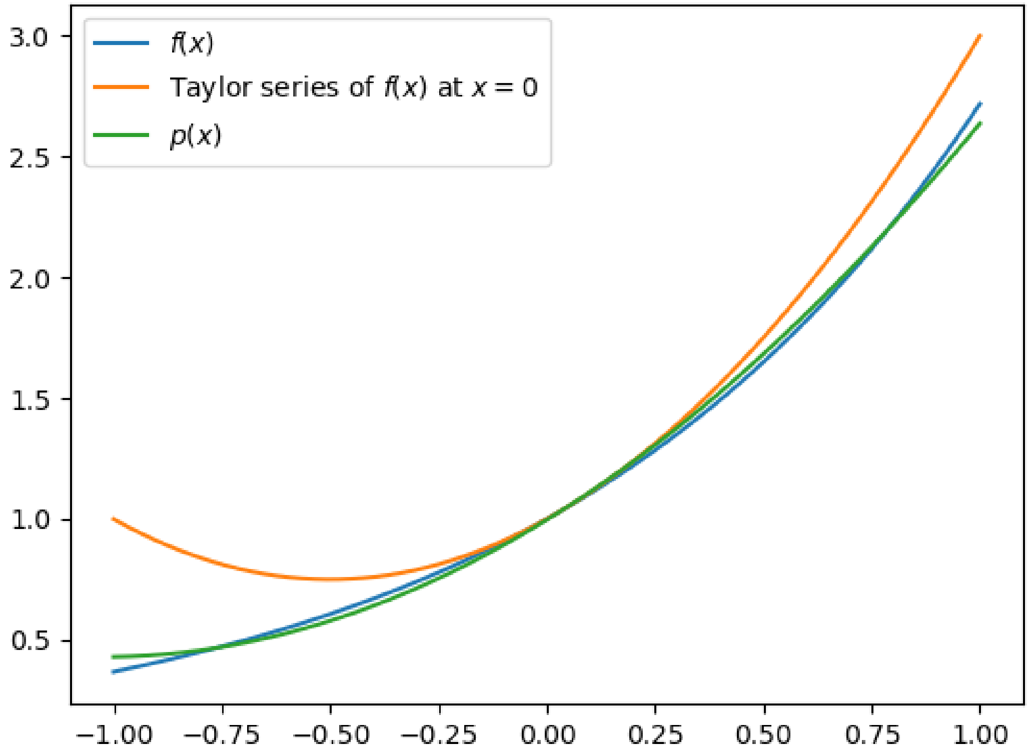

Figure 2.1: Comparsion of different approximation for e^x

3 EXERCISE3.3

1. *Proof.* Firstly

$$\int_{-1}^{1} \frac{1}{\sqrt{2}} cos(n\pi x) dx = \frac{1}{\sqrt{2} n\pi} \int_{-n\pi}^{n\pi} cos(x) dx = 0 \tag{3.1}$$

$$\int_{-1}^{1} \frac{1}{\sqrt{2}} sin(n\pi x) dx = \frac{1}{\sqrt{2} n\pi} \int_{-n\pi}^{n\pi} sin(x) dx = 0 \tag{3.2}$$

$$\int_{-1}^{1} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} dx = \frac{1}{2} \int_{-1}^{1} dx = 1 \tag{3.3}$$

Secondly, when $n \neq m$

$$\begin{aligned}
&\int_{-1}^{1} cos(n\pi x)cos(m\pi x)dx \\
=&\frac{1}{2}\int_{-1}^{1} cos[(n+m)\pi x] + cos[(n-m)\pi x]dx \\
=&\frac{1}{2}[\frac{1}{(n+m)\pi}\int_{-(n+m)\pi}^{(n+m)\pi} cos(x)dx + \frac{1}{(n-m)\pi}\int_{-(n-m)\pi}^{(n-m)\pi} cos(x)dx] = 0
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&\int_{-1}^{1} sin(n\pi x)sin(m\pi x)dx \\
=&-\frac{1}{2}\int_{-1}^{1} cos[(n+m)\pi x] - cos[(n-m)\pi x]dx \\
=&-\frac{1}{2}[\frac{1}{(n+m)\pi}\int_{-(n+m)\pi}^{(n+m)\pi} cos(x)dx - \frac{1}{(n-m)\pi}\int_{-(n-m)\pi}^{(n-m)\pi} cos(x)dx] = 0
\end{aligned} \tag{3.5}$$

Then, when $n = m$ it's obvious that

$$\int_{-1}^{1} cos(n\pi x)cos(m\pi x)dx = \int_{-1}^{1} cos^2(n\pi x)dx > 0 \tag{3.6}$$

and

$$\int_{-1}^{1} sin(n\pi x)sin(m\pi x)dx = \int_{-1}^{1} sin^2(n\pi x)dx > 0 \tag{3.7}$$

Finally, no matter n equals m or not

$$\begin{aligned}
&\int_{-1}^{1} sin(n\pi x)cos(m\pi x)dx \\
=&\frac{1}{2}[\int_{-1}^{1} sin[(n+m)\pi x]dx + \int_{-1}^{1} sin[(n-m)\pi x]dx] \\
=&\frac{1}{2}[\int_{-(n+m)\pi}^{(n+m)\pi} sin(x)dx + \int_{-(n-m)\pi}^{(n-m)\pi} sin(x)dx] \\
=&0
\end{aligned} \tag{3.8}$$

□

2. *Proof.* Let

$$t = \frac{x-(b+a)/2}{(b-a)/2} \tag{3.9}$$

Then $dt = \frac{2}{b-a}dx$. With $\{e_n\}$ being an orthonormal system

$$\begin{aligned}
\int_a^b \tilde{e}_i(x)\tilde{e}_j(x)dx &= \frac{2}{b-a}\int_a^b e_i(\frac{x-(b+a)/2}{(b-a)/2})e_j(\frac{x-(b+a)/2}{(b-a)/2})dx \\
&= \int_{-1}^{1} e_i(t)e_j(t)dt = \delta_{ij}
\end{aligned} \tag{3.10}$$

□