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Methods of Applied Mathematics I
HW10

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1 EXERCISE10.1

1. *Proof.* Since

$$\begin{aligned} < u, Lu > &= \int_0^1 u(x)u''(x)dx \\ &= -\left[uu'|_0^1 - \int_0^1 (u'(x))^2 dx\right] \\ &= \int_0^1 (u'(x))^2 dx \geq 0 \end{aligned} \tag{1.1}$$

Thus, the operator L is positive definite. □

2. *Proof.* Denote λ as the eigenvalue, and u being the corresponding eigenfunction, then

$$Lu = \lambda u \tag{1.2}$$

As has been proved before, the positive definite property of L implies that

$$< u, Lu >=< u, \lambda u >= \lambda < u, u >= \lambda ||u||^2 \geq 0 \tag{1.3}$$

Thus

$$\lambda > 0 \tag{1.4}$$

as the case $\lambda = 0$ is trival.
Then, the solution of this ODE is

$$u(x) = asin(\sqrt{\lambda}x) + bcos(\sqrt{\lambda}x) \tag{1.5}$$

With the B.C. at endpoints, it follows that

$$b = 0 \tag{1.6}$$

and

$$a(sin(\sqrt{\lambda}) + \sqrt{\lambda}cos(\sqrt{\lambda})) = 0 \tag{1.7}$$

Thus

$$sin(\sqrt{\lambda}) + \sqrt{\lambda}cos(\sqrt{\lambda}) = 0 \tag{1.8}$$

which indicates that the eigenvalue λ satisfies

$$\sqrt{\lambda} = -tan(\sqrt{\lambda}) \tag{1.9}$$

□

3. From the plot of $y_1(x) = -x$ and $y_2(x) = tan(x)$, the first 2 intersections in $x > 0$ region lie in the gap $(\frac{1}{2}\pi, \pi)$ and $(\frac{3}{2}\pi, 2\pi)$.
   With the aid of MATLAB, the 2 lowest eigenvalues λ_1 and λ_2 are found as

$$\lambda_1 = (2.0288)^2 = 4.1160 \tag{1.10}$$

   and

$$\lambda_2 = (4.9132)^2 = 24.1395 \tag{1.11}$$

4. Denote $p(x) = a_0 + a_1 x + a_2 x^2$, the B.V. at endpoints implies that

$$V_1 = \left\{p(x) | p(x) = a_2(x^2 - \frac{3}{2}x)\right\} \tag{1.12}$$

   Thus, for any $v \in V_1$

$$\begin{aligned} R(v) &= \frac{< v, Lv >}{< v, v >} \\ &= \frac{\int_0^1 a_2(x^2 - \frac{3}{2}x)(-2a_2)dx}{\int_0^1 a_2^2(x^2 - \frac{3}{2}x)^2 dx} \\ &= -2\frac{\int_0^1 (x^2 - \frac{3}{2}x)dx}{\int_0^1 (x^2 - \frac{3}{2}x)^2 dx} \\ &= 4.167 \end{aligned} \tag{1.13}$$

5. Denote $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, the B.V. at endpoints implies that

$$V_2 = \left\{p(x) | p(x) = a_2(x^2 - \frac{3}{2}x) + a_3(x^3 - 2x)\right\} \tag{1.14}$$

   Thus, for any $v \in V_2$

$$\begin{aligned} R(v) &= \frac{< v, Lv >}{< v, v >} \\ &= \frac{\int_0^1 [a_2(x^2 - \frac{3}{2}x) + a_3(x^2 - 2x)][2a_2 + 6a_3 x]dx}{\int_0^1 [a_2(x^2 - \frac{3}{2}x) + a_3(x^2 - 2x)]^2 dx} \\ &= \frac{175a_2^2 + 630a_2 a_3 + 588a_3^2}{42a_2^2 + 154a_2 a_3 + 142a_3^2} \end{aligned} \tag{1.15}$$

As the leading coefficient $a_3 \neq 0$, then, denote $t \triangleq \frac{a_2}{a_3}$

$$R(v) = \frac{175t^2 + 630t + 588}{42t^2 + 154t + 142} \tag{1.16}$$

It's obvious that $R(v)$ goes to its minimal when $t \to \infty$, thus $\min R(v) = \frac{175}{42} = 4.1667$.

6. Firstly, determine the general form of each element $v \in V_3$.
   Denote $v(x) = a_2(x^2 - \frac{3}{2}x) + a_3(x^3 - 2x)$ as $v \in V_2$, then, as v also lies in $V_1^{\perp}$, it means

$$\int_0^1 (x^2 - \frac{3}{2}x)v(x)dx = 0 \tag{1.17}$$

namely

$$a_2 \int_0^1 (x^2 - \frac{3}{2}x)^2 dx + a_3 \int_0^1 (x^2 - \frac{3}{2}x)(x^3 - 2x)dx = 0 \tag{1.18}$$

Thus

$$a_2 = -\frac{16}{3}a_3 \tag{1.19}$$

Hence

$$v(x) = a_3(x^3 - \frac{16}{3}x^2 + 6x) \tag{1.20}$$

and

$$v''(x) = a_3(6x - \frac{32}{3}) \tag{1.21}$$

Thus, for any $v \in V_3$, the second eigenvalue is estimated as

$$\begin{aligned} R(v) &= \frac{< v, Lv >}{< v, v >} \\ &= -\frac{\int_0^1 (x^3 - \frac{16}{3}x^2 + 6x)(6x - \frac{32}{3})dx}{\int_0^1 (x^3 - \frac{16}{3}x^2 + 6x)^2 dx} \\ &= 4.2803 \end{aligned} \tag{1.22}$$