Question1 (8 points)
Recall if Φ is *continuous* and satisfies a *Lipschitz condition* in y on the set

$$\mathcal{D} = \{(t, y) \mid t_0 \leq t \leq T, -\infty < y < \infty\}$$

then

$$\dot{y} = \Phi(t, y), \qquad y(t_0) = y_0, \qquad \text{where} \qquad t_0 \leq t \leq T$$

has a unique solution.

(a) (2 points) Show Φ satisfies a Lipschitz condition in y on $\mathcal{A}$ with Lipschitz constant c if $\mathcal{A}$ is *convex* and there exists a $c > 0$ such that

$$\left|\frac{\partial}{\partial y}\Phi(t, y)\right| \leq c \qquad \text{for all} \quad (t, y) \in \mathcal{A}$$

[Hint: A set $\mathcal{A} \subset \mathbb{R}^2$ is said to be *convex* if the line segment joining any two points in $\mathcal{A}$ lies entirely in $\mathcal{A}$. You may also find the mean value theorem useful.]

(b) (2 points) Show for any constants t_0 and T, the set $\mathcal{D}$ is convex.

(c) (2 points) Use the above to show the following IVP has a unique solution.

$$\dot{y} = \frac{4t^3 y}{1 + t^4}, \qquad y(0) = 1, \qquad \text{where} \qquad 0 \leq t \leq 1$$

(d) (2 points) Do you think it is a good idea to solve the following IVP numerically?

$$\dot{y} = 1 + y^2, \qquad y(0) = 0, \qquad \text{where} \qquad 0 \leq t \leq 3$$

Justify your answer. Show Euler's method is going to fail miserably for this IVP.

Question2 (6 points)
Consider the following IVP

$$\dot{y} = \arctan(y), \qquad y(0) = y_0, \qquad \text{where} \qquad t_0 \leq t \leq T$$

(a) (2 points) Find a Lipschitz constant for $\arctan(y)$.

(b) (2 points) Find an upper bound on $|\ddot{y}|$ without solving the IVP.

(c) (2 points) Find an upper bound on the absolute global error

$$|e_k| = |\hat{y}_k - y(t_k)|\,, \qquad \text{where } \hat{y}_k \text{ is the Euler's approximation to } y(t_k),$$

in terms of step size and t_k.

Question3 (13 points)
Solve the following IVP using the step size $h = 1$

$$\dot{y} = (2 + 0.01t^2)y, \qquad y(0) = 4, \qquad \text{where} \qquad 0 \leq t \leq 15$$

(a) (1 point) By Euler's method.

(b) (2 points) By the backward Euler's method.

(c) (2 points) By the second-order Taylor's method.

(d) (1 point) By the Heun's method.

(e) (1 point) By the two-step Adams-Bashforth method.

(f) (2 points) It was mentioned in class that Heun's method, which is derived by applying the trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\Big(f(a)+f(b)\Big)$$

is one the simplest form of Runge-Kutta method. The other simple second-order Runge-Kutta method, which is also known as the modified Euler's method, uses the mid-point rule

$$\int_a^b f(x)\,dx \approx (b-a)\,f\Big(\frac{a+b}{2}\Big)$$

Use this information to derive this second-order Runge-Kutta method. Write a piece of pseudocode for it, then implement it to solve the above IVP.

(g) (1 point) The most widely used Runge-Kutta method is a fourth-order Runge-Kutta method, which uses four sequential evaluations of Φ during each time step, that is, it has four stages. Similar to the previous two Runge-Kutta, it can be understood from a quadrature rule. In this case, Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\Big(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\Big)$$

This scheme proceeds as follows:

$$\hat{y}_0 = y_0$$

$$\hat{y}_n = \hat{y}_{n-1} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\begin{aligned}
\text{where}\qquad \Phi_1 &= \Phi(t_{k-1}, \hat{y}_{k-1}) \\
\Phi_2 &= \Phi\left(t_{k-1}+\frac{h}{2}, \hat{y}_{k-1}+\frac{h}{2}\Phi_1\right) \\
\Phi_3 &= \Phi\left(t_{k-1}+\frac{h}{2}, \hat{y}_{k-1}+\frac{h}{2}\Phi_2\right) \\
\Phi_4 &= \Phi(t_{k-1}+h, \hat{y}_{k-1}+h\Phi_3)
\end{aligned}$$

Use this fourth-order Runge-Kutta method to solve the above IVP.

(h) (1 point) Compare all of the above approximations to the exact solution by plotting them on the same graph.

(i) (2 points) Use the approximation from Euler's method to find the value of y at

$$t = 9.625$$

by interpolation in Newton's form.

Question4 (3 points)
Use the classic fourth-order Runge-Kutta method to find the numerical solution of the following higher-order differential equation, and compare the results to the exact solution.

$$t^3\dddot{y} + t^2\ddot{y} - 2t\dot{y} + 2y = 8t^3 - 2, \qquad y(1) = 2, \quad \dot{y}(1) = 8, \quad \ddot{y}(1) = 6$$

for $1 \le t \le 2$ with $h = 0.1$. The exact solution is

$$y = -\frac{1}{t} - 1 + 2t + t^2 + t^3$$

Question5 (4 points)
Solve the following boundary value problem on the domain $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ with $h = \frac{\pi}{60}$

$$\ddot{y} = -\left(2(\dot{y})^3 + y^2\dot{y}\right)\sec t, \qquad y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt[4]{2}}, \quad y\left(\frac{\pi}{3}\right) = \frac{\sqrt[4]{12}}{2}$$

(a) (2 points) By the shoot method.

(b) (2 points) By the Finite-difference method.

Question6 (4 points)
Solve the following BVP on the domain $[0, 1]$ by using its variational form

$$\ddot{y} + \dot{y} + t = 0; \qquad y(0) = 0 \quad y(1) = 0$$

(a) (2 points) Assume a linear hat basis for the solution.

(b) (2 points) Assume a cubic polynomial basis for the solution.

Question7 (2 points)
Find the following integral using the 4-order Runge Kutta method.

$$\int_0^1 \left(\frac{1}{(x-0.3)^2 + 0.01} + \frac{1}{(x-0.9)^2 + 0.04} - 6\right) dx$$

Is equally spaced x_k the best option for this problem?

Question8 (0 points)
Consider the following BVP on the domain $[1, 3]$

$$x^3y^{(4)} + 6x^2y^{(3)} + 6xy'' - 10x = 0$$

The boundary conditions are

$$y(1) = y(3) = y'(1) = y'(3) = 0$$

(a) (1 point (bonus)) Find its variational form.

(b) (3 points (bonus)) Solve it using its variational form.

(c) (1 point (bonus)) Compare your solution and the derivative of your solution with the exact solution and its derivative obtained by writing the differential equation as

$$\left(x^3y''\right)'' = 10x$$