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Methods of Applied Mathematics I
HW9

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1 EXERCISE9.1

1. *Proof.* Since

$$\begin{aligned}
(LKu)(x) &= -\frac{d^2}{dx^2}\int_0^1 g(x,\xi)u(\xi)d\xi \\
&= -\frac{d^2}{dx^2}\left(\int_0^x g(x,\xi)u(\xi)d\xi + \int_x^1 g(x,\xi)u(\xi)d\xi\right) \\
&= -\frac{d^2}{dx^2}\left(\int_0^x \xi(1-x)u(\xi)d\xi + \int_x^1 x(1-\xi)u(\xi)d\xi\right) \\
&= -\frac{d^2}{dx^2}\left((1-x)\int_0^x \xi u(\xi)d\xi + x\int_x^1 (1-\xi)u(\xi)d\xi\right) \\
&= -\frac{d}{dx}\left(-\int_0^x \xi u(\xi)d\xi + \int_x^1 (1-\xi)u(\xi)d\xi\right) \\
&= xu(x) + (1-x)u(x) = u(x)
\end{aligned} \tag{1.1}$$

Thus, $LK = I$, which means $L = K^{-1}$. □

2. *Proof.* (⇒) Suppose $\frac{1}{\lambda}$ is an eigenvalue for K, then

$$ku = \frac{1}{\lambda}u \tag{1.2}$$

Thus

$$u = \lambda K u \tag{1.3}$$

Take the operator L on it

$$Lu = \lambda LKu = \lambda Iu = \lambda u \tag{1.4}$$

which indicates that λ is an eigenvalue of L.
($\Leftarrow$) Suppose $\lambda \neq 0$ is an eigenvalue of L, then

$$Lu = \lambda u \tag{1.5}$$

Thus

$$u = \frac{1}{\lambda} Lu \tag{1.6}$$

As $K = L^{-1}$ is known, take the operator K onto it yields

$$Ku = \frac{1}{\lambda} KLu = \frac{1}{\lambda} Iu = \frac{1}{\lambda} u \tag{1.7}$$

which indicates that $\frac{1}{\lambda}$ is an eigenvalue of K.
It's clear that for the given λ, K and L share the same eigenfunction $u(x)$. □

3. *Proof.* It's obvious that L is bounded over M.
Since

$$\begin{aligned}
< u, Lv > &= -\int_0^1 u(x)v''(x)dx = -\left(u(x)v'(x)\Big|_0^1 - \int_0^1 u'(x)v'(x)dx \right) \\
&= \int_0^1 u'(x)v'(x)dx = \int_0^1 u'(x)dv(x) = u'(x)v(x)\Big|_0^1 - \int_0^1 v(x)u''(x)dx \\
&= -\int_0^1 u''(x)v(x)dx = < Lu, v >
\end{aligned} \tag{1.8}$$

which indicates that L is self-adjoint. Thus the eigenvalues of L are real numbers.
The eigenfunction $\psi_n(x)$ of L together with λ_n satisfy

$$\psi_n''(x) + \lambda_n \psi_n(x) = 0 \tag{1.9}$$

and

$$\psi_n(0) = \psi_n(1) = 0 \tag{1.10}$$

Solution of this ODE depends on the sign of λ_n.

a) If $\lambda_n > 0$, then

$$\psi_n(x) = C_1 sin(\sqrt{\lambda_n} x) + C_2 cos(\sqrt{\lambda_n} x) \tag{1.11}$$

apply the boundary condition yields

$$C_2 = 0 \tag{1.12}$$

and

$$C_1 sin(\sqrt{\lambda_n}) = 0 \tag{1.13}$$

Thus

$$\lambda_n = (n\pi)^2 \tag{1.14}$$

and

$$\psi_n(x) = C_2 sin(n\pi x) \quad (C_2 \neq 0) \tag{1.15}$$

b) If $\lambda_n = 0$, then

$$\psi_n(x) = C_0 + C_1 x \tag{1.16}$$

Apply the B.C. yields $C_0 = C_1 = 0$, which indicates 0 is not an eigenvalue for L.

c) If $\lambda_n < 0$, then

$$\psi_n(x) = C_1 e^{\sqrt{-\lambda_n}x} + C_2 e^{-\sqrt{-\lambda_n}x} \tag{1.17}$$

Apply the B.C. yields $C_0 = C_1 = 0$, which indicates $\lambda_n < 0$ are not eigenvalues for L.

□

4. It's obvious that, from previous proof, the eigenvalues of K are

$$\sigma_{point}(K) = \{\frac{1}{(n\pi)^2}, n = 1,2,3,....\} \tag{1.18}$$

As K is compact and self-adjoint, it's obvious bounded and

$$\sigma_{compression}(K) = \sigma_{point}(K) \tag{1.19}$$

If 0 is an eigenvalue of K, then

$$Ku = 0 \tag{1.20}$$

thus

$$LKu = 0 \quad \Rightarrow \quad u = 0 \tag{1.21}$$

which indicates that 0 is not an eigenvalue for K.
Hence,

$$\sigma_{continous}(K) = 0 \tag{1.22}$$

as $K^{-1} = L$ is unbounded.

5. It's obvious that, from previous proof, the eigenfunctions of K are

$$u_n(x) = sin(n\pi x) \quad n = 1,2,... \tag{1.23}$$

which form the Fouier basis.