SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Methods of Applied Mathematics I
HW2

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1 EXERCISE2.1

1. *Proof.*

$$
\begin{aligned}
RHS &= \frac{1}{4}(||x+y||^2 - ||x-y||^2) \\
&= \frac{1}{4}(<x+y, x+y> - <x-y, x-y>) \\
&= \frac{1}{4}(<x+y, x> + <x+y, y> - <x-y, x> - <x-y, -y>) \\
&= \frac{1}{4}(\overline{<x, x+y>} + \overline{<y, x+y>} - \overline{<x, x-y>} + \overline{<y, x-y>}) \\
&= \frac{1}{4}(\overline{<x, x>} + \overline{<x, y>} + \overline{<y, x>} + \overline{<y, y>} - \overline{<x, x>} + \overline{<x, y>} + \overline{<y, x>} - \overline{<y, y>}) \\
&= \frac{1}{2}(\overline{<x, y>} + \overline{<y, x>}) \\
&= \frac{1}{2}(<x, y> + <y, x>) \\
&= <x, y> = LHS
\end{aligned}
\tag{1.1}
$$

The last line is valid as the inner-product is defined on real space s.t. $<x, y> = <y, x>$.
□

2. *Proof.* As have been proved aboved

$$\begin{aligned}&\frac{1}{4}(||x+y||^2-||x-y||^2)\\ =&\frac{1}{2}(<x,y>+<y,x>)\end{aligned} \tag{1.2}$$

Also, replace y with iy yields

$$\begin{aligned}&\frac{i}{4}(||x+iy||^2-||x-iy||^2)\\ =&\frac{i}{2}(<x,iy>+<iy,x>)\\ =&\frac{i}{2}(i<x,y>+\bar{i}<y,x>)\\ =&\frac{1}{2}(-<x,y>+<y,x>)\end{aligned} \tag{1.3}$$

Thus

$$\begin{aligned}RHS &= \frac{1}{4}(||x+y||^2-||x-y||^2)-\frac{i}{4}(||x+iy||^2-||x-iy||^2)\\ &= \frac{1}{2}(<x,y>+<y,x>)-\frac{1}{2}(-<x,y>+<y,x>)\\ &=<x,y>=LHS\end{aligned} \tag{1.4}$$

□

3. *Proof.*

$$\begin{aligned}LHS &= ||x+y||^2+||x-y||^2\\ &=<x+y,x+y>+<x-y,x-y>\\ &=<x+y,x>+<x+y,y>+<x-y,x>-<x-y,y>\\ &=\overline{<x,x+y>}+\overline{<y,x+y>}+\overline{<x,x-y>}-\overline{<y,x-y>}\\ &=\overline{<x,x>}+\overline{<x,y>}+\overline{<y,x>}+\overline{<y,y>}+\overline{<x,x>}-\overline{<x,y>}-\overline{<y,x>}+\overline{<y,y>}\\ &=2(<x,x>+<y,y>)\\ &=2(||x||^2+||y||^2)=RHS\end{aligned} \tag{1.5}$$

□

4. *Proof.* As any norm induced by inner product should satisfy the parallelogram law, it's easy to show that the norm $||.||_\infty : f \mapsto \sup_{x\in[a,b]}|f(x)|$ on $C[a,b]$ can not be induced by any inner product since it does not satisfy the parallelogram law. For example, let

$$f(x)=\frac{x-a}{b-a} \tag{1.6}$$

and

$$g(x)=(\frac{x-a}{b-a})^2 \tag{1.7}$$

then

$$||f+g|| = \max_{x\in[a,b]} |(f+g)(x)| = |(f+g)(b)| = 2 \tag{1.8}$$

$$||f-g|| = \max_{x\in[a,b]} |(f-g)(x)| = |(f-g)(\frac{a+b}{2})| = \frac{1}{4} \tag{1.9}$$

$$||f|| = \max_{x\in[a,b]} |f(x)| = |f(b)| = 1 \tag{1.10}$$

$$||g|| = \max_{x\in[a,b]} |g(x)| = |g(b)| = 1 \tag{1.11}$$

It's obvious that

$$||f+g||^2 + ||f-g||^2 = \frac{65}{4} \neq 2(||f||^2 + ||g||^2) = 4 \tag{1.12}$$

□

5. Assume the norm satisfies the parallelogram law, that is

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2) \tag{1.13}$$

Let

$$< x, y > \triangleq \frac{1}{4}(||x+y||^2 - ||x-y||^2) \tag{1.14}$$

it's left to prove that $< x, y >$ is an inner product.
Firstly, $< x, y+z >$ can be expanded according to the definition as follows

$$\begin{aligned} < x, y+z > &= \frac{1}{4}(||x+y+z||^2 - ||x-(y+z)||^2) \\ &= \frac{1}{4}(||y+z+x||^2 - ||y+z-x||^2) \end{aligned} \tag{1.15}$$

Since

$$||y+z+x||^2 + ||y-z+x||^2 = 2(||y+x||^2 + ||z||^2) \tag{1.16}$$

Then

$$||y+z+x||^2 = 2(||y+x||^2 + ||z||^2) - ||y-z+x||^2 \tag{1.17}$$

Swap y and z yields

$$||x+y+z||^2 = 2(||z+x||^2 + ||y||^2) - ||z-y+x||^2 \tag{1.18}$$

Adding the two equations above and divided by 2 yields

$$||y+z+x||^2 = ||y||^2 + ||z||^2 + ||x+y||^2 + ||x+z||^2 - \frac{1}{2}(||y-z+x||^2 + ||z-y+x||^2) \tag{1.19}$$

Replace x with $-x$

$$||y+z-x||^2 = ||y||^2 + ||z||^2 + ||x-y||^2 + ||x-z||^2 - \frac{1}{2}(||y-z-x||^2 + ||z-y-x||^2) \tag{1.20}$$

Hence

$$||y+z+x||^2 + ||y+z-x||^2 = (||x+y||^2 - ||x-y||^2) + (||x+z||^2 - ||x-z||^2) \tag{1.21}$$

As

$$< x, y >= \frac{1}{4}(||x+y||^2 - ||x-y||^2) \tag{1.22}$$

and

$$< x, z >= \frac{1}{4}(||x+z||^2 - ||x-z||^2) \tag{1.23}$$

It's therefore can be concluded that

$$< x, y+z >=< x, y > + < x, z > \tag{1.24}$$

Secondly,

2 EXERCISE2.2

Denote $c_0 = \{(a_n)_{n\in\mathbb{N}} : \lim\limits_{n\to\infty} a_n = 0\}$ and $l^1 = \{(b_n)_{n\in\mathbb{N}} : \sum_{n=0}^{\infty} |b_n| \le \infty\}$

1. YES

   Proof. For any $(a_n) \in c_0$ and $\epsilon > 0$, $\exists N_1$ s.t. when $n > N_1$, $|a_n| \le \frac{\epsilon}{2}$.
   Also, a sequence $(b_n) \in l^1$ can be found s.t. $|b_n| \le \frac{\epsilon}{2}$ when $n > N_2$, and $b_n = a_n$ for $n \le N_3 \triangleq \max(N_1, N_2)$. This does not violate requirement for (b_n) as the sum of $|b_n|$ here is finite.
   Thus, $||(a_n) - (b_n)||_\infty = \sup_{n\in\mathbb{N}} |a_n - b_n| \le |a_n| + |b_n| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ when $n > N_3$. And it implies that l^1 is dense in c_0 under the $||.||_\infty$ norm. □

2. NO

   Proof. For example, pick a sequence $(a_n) \in c_0$ s.t. $a_n = \frac{1}{n+1}$, for any sequence $(b_n) \in l^1$. The norm $||(a_n) - (b_n)|| = \sum_{n=0}^{\infty} |a_n - b_n| \ge \sum_{n=0}^{\infty} (|a_n| - |b_n|) = (\sum_{n=0}^{\infty} |a_n|) - (\sum_{n=0}^{\infty} |b_n|)$ diverges as $\sum_{n=0}^{\infty} |a_n|$ goes infinite and $\sum_{n=0}^{\infty} |b_n|$ is finite. □