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Methods of Applied Mathematics I
HW7

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1 EXERCISE7.1

1. *Proof.* Since $|\lambda| = 1$, then

$$||e_\lambda^{(N)}||_2 = \frac{1}{\sqrt{N+1}}\sqrt{\sum_{i=0}^{N}\lambda^{2i}} = \frac{1}{\sqrt{N+1}}\sqrt{N+1} = 1 \tag{1.1}$$

□

2. *Proof.* Since

$$\begin{aligned}(L-\lambda I)e_\lambda^{(N)} &= Le_\lambda^{(N)} - \lambda e_\lambda^{(N)}\\ &= \frac{1}{\sqrt{N+1}}\left[(\lambda,\lambda^2,\ldots,\lambda^N,0,0,\ldots)-(\lambda,\lambda^2,\ldots,\lambda^N,\lambda^{N+1},0,\ldots)\right]\\ &= \frac{-1}{\sqrt{N+1}}(0,0,\ldots,0,\lambda^{N+1},0,\ldots)\end{aligned} \tag{1.2}$$

Together with $|\lambda| = 1$, then

$$\lim_{N\to\infty}||(L-\lambda I)e_\lambda^{(N)}||_2 = \lim_{N\to\infty}\frac{|\lambda|^{N+1}}{\sqrt{N+1}} = \lim_{N\to\infty}\frac{1}{\sqrt{N+1}} = 0 \tag{1.3}$$

□

3. Not figured out...

4. $$Ru = \lambda u \tag{1.4}$$

   Denote $u = (a_0, a_1, ..., a_n, ...)$, then
   $$\begin{aligned} 0 &= \lambda a_0 \\ a_0 &= \lambda a_1 \\ &... \end{aligned} \tag{1.5}$$

2 EXERCISE7.2

1. *Proof.* Take a subset of the domain of L^{-1}, which is denoted as M and is defined as
   $$M = \left\{ u \in L^2\big([0,1]\big) \middle| \exists \xi > 0 \ \forall x \in [0,\xi] \ u(x) = 0 \right\} \tag{2.1}$$
   Given any $u(x) \in L^2$, then
   $$\int_0^1 u^2(x)dx < \infty \tag{2.2}$$
   Hence $u(x)$ is bounded over $[0,1]$.
   Let
   $$T \triangleq \sup_{x\in[0,1]} |u(x)| \tag{2.3}$$
   then, $\forall \epsilon > 0, \exists v \in M$ s.t.
   $$v(x) = \begin{cases} 0, & x \in [0,\delta] \\ u(x), & x \in (\delta, 1] \end{cases} \tag{2.4}$$
   where $\delta = \frac{\epsilon}{T^2}$. Therefore
   $$d(u,v) \triangleq ||u - v||_2 = \int_0^1 [u(x) - v(x)]^2 dx = \int_0^\delta u^2(x)dx \le T^2\delta = \epsilon \tag{2.5}$$
   which indicates that M is dense in L^2, so do the domain of L^{-1}. □

2. *Proof.*
   $$||L|| = \sup_{u\in L^2} \frac{||Lu||_2}{||u||_2} = \sup_{u\in L^2} \frac{||xu(x)||_2}{||u(x)||_2} = \sup_{u\in L^2} \frac{|x| ||u||_2}{||u||_2} = \sup_{x\in[0,1]} |x| = 1 \tag{2.6}$$
   and
   $$||L^{-1}|| = \sup_{u\in L^2} \frac{||L^{-1}u||_2}{||u||_2} = \sup_{x\in[0,1]} \frac{1}{|x|} = \infty \tag{2.7}$$
   hence L^{-1} is unbounded. □

3. The state of L is $(I, 1_n)$.
   The state of L^{-1} is $(I, 2_c)$.

4. Yes

5. Since
$$[(L-\lambda I)u](x) = (Lu)(x) - \lambda u(x) = (x-\lambda)u(x) \tag{2.8}$$
The inverse of $(L-\lambda I)$ always exists, therefore $\sigma(L) = \emptyset$.

3 EXERCISE7.3

1. *Proof.* It's clear that L^{-1} is the differentiate operator, and L^{-1} is unbounded. So L has unbounded inverse.
   Since the domain of L is composed of square-integrable functions over $[0,1]$, say
$$\int_0^1 f^2(x)dx < \infty \tag{3.1}$$
   An element within the range of L is
$$g(x) = \int_0^x f(t)dt \tag{3.2}$$
   Then, if $f(x)$ is a polynomial in L^2 , it must be bounded over $[0,1]$ as it is continous. Denote the supreme of $f(x)$ as M, then $g(x) \le Mx$. Hence $g(x)$ is square-integrable over $[0,1]$, say
$$\int_0^1 g^2(x)dx \le M^2 \int_0^1 x^2 dx < \infty \tag{3.3}$$
   Clearly, the domain of L doesn't contain all the polynomials and therefore the range of L is open and incomplete. The boundary of the range of L are the limits of squences like $f_n(x) = nx$ when $n \to \infty$.
   Hence, the state of L is $(III, 1_n)$. □

2.
$$L^* = L \tag{3.4}$$

4 EXERCISE7.4

Proof. For $p = 1$
$$\begin{aligned} RHS &\triangleq ||(a_n)||_1 \cdot ||(b_n)||_1 = \sum_{i=0}^{\infty} |a_i| \cdot \sum_{j=0}^{\infty} |b_j| = \sum_{n=0}^{\infty} \sum_{i+j=n} |a_i||b_j| \\ &\ge \sum_{n=0}^{\infty} |\sum_{i+j=n} a_i b_j| = \sum_{n=0}^{\infty} |c_n| = ||(c_n)||_1 \triangleq LHS \end{aligned} \tag{4.1}$$

For $p > 1$, take $q > 0$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$
Then, using the Holder's inequality

$$
\begin{aligned}
LHS \triangleq ||(c_n)||_p &= \Big(\sum_{n=0}^{\infty} |c_n|^p\Big)^{\frac{1}{p}} = \Big(\sum_{n=0}^{\infty} \Big|\sum_{i+j=n} a_i b_j\Big|^p\Big)^{\frac{1}{p}} \\
&= \Big(\sum_{n=0}^{\infty} \Big|\sum_{i+j=n} a_i b_j^{\frac{1}{p}} b_j^{\frac{1}{q}}\Big|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{n=0}^{\infty} \Big|\sum_{i+j=n} \Big(|a_i||b_j|^{\frac{1}{p}}\Big)|b_j|^{\frac{1}{q}}\Big|^p\Big)^{\frac{1}{p}} \\
&\le \Big(\sum_{n=0}^{\infty} \Big[\Big(\sum_{i+j=n} |a_i|^p |b_j|\Big)^{\frac{1}{p}} \Big(\sum_{j=0}^{n} |b_j|\Big)^{\frac{1}{q}}\Big]^p\Big)^{\frac{1}{p}} \\
&= \Big[\sum_{n=0}^{\infty} \Big(\sum_{i+j=n} |a_i|^p |b_j|\Big)\Big(\sum_{j=0}^{n} |b_j|\Big)^{\frac{p}{q}}\Big]^{\frac{1}{p}} \\
&\le \Big(\sum_{i=0}^{\infty} |a_i|^p\Big)^{\frac{1}{p}} \cdot \Big(\sum_{j=0}^{\infty} |b_j|\Big)^{\frac{1}{p}} \cdot \Big(\sum_{j=0}^{\infty} |b_j|\Big)^{\frac{1}{q}} \\
&= ||(a_n)||_p \cdot ||(b_n)||_1 \triangleq RHS
\end{aligned}
\tag{4.2}
$$

□