SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Methods of Applied Mathematics I
HW1

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1 EXERCISE1.1

1.
$$\dim U = 2 \tag{1.1}$$
As there are 4 components within x, and 2 constraints.

2.
$$\dim U = 3 \tag{1.2}$$
As there are 4 components within x, and 1 constraint.

3.
$$\begin{aligned}\dim U + V &= \dim U + \dim V - \dim U \cap V \\ &= 2 + 3 - 2 \\ &= 3\end{aligned} \tag{1.3}$$
The dimension of $U \cap V$ can be observed from
$$\begin{aligned}U \cap V &= \{x \in \mathbb{R}^4 | x_1 + x_2 + x_3 = 0, x_1 + 3x_2 = x_4, x_1 = x_4\} \\ &= \{x \in \mathbb{R}^4 | x_2 = 0, x_1 + x_3 = 0\}\end{aligned} \tag{1.4}$$

2 EXERCISE1.2

1. The pointwise limit is given as follows

$$f = \begin{cases} 0, x = 0 \\ 1, x \in (0,1] \end{cases} \tag{2.1}$$

   It's of uniform convergence as

$$\begin{aligned} &\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| \\ &= \begin{cases} 0, & x = 0 \\ \lim_{x\to 0, n\to\infty} |x^{\frac{1}{n}} - 1|, & x \in (0,1] \end{cases} \\ &= 0 \end{aligned} \tag{2.2}$$

2. The pointwise limit is given as

$$f(x) = x \tag{2.3}$$

   It's not of uniform convergence.

3. The pointwise limit is given as

$$f(x) = 0 \tag{2.4}$$

   It's of uniform convergence as

$$\begin{aligned} &\lim_{n\to\infty} \sup_{x\in(0,\infty)} |f_n(x) - f(x)| \\ &= \lim_{n\to\infty} \sup_{x\in(0,\infty)} \left| \sqrt{\frac{1}{n} + x} - \sqrt{x} \right| \\ &= \lim_{n\to\infty} \sup_{x\in(0,\infty)} \left| \frac{1}{n(\sqrt{\frac{1}{n} + x} + \sqrt{x})} \right| \\ &= \lim_{n\to\infty, x\to 0} \frac{1}{n(\sqrt{\frac{1}{n} + x} + \sqrt{x})} \\ &= \lim_{n\to\infty, x\to 0} \frac{1}{2n\sqrt{x}} = 0 \end{aligned} \tag{2.5}$$

3 EXERCISE1.3

1. *Proof.* Given a sequence $(a_n) \in l^p$, then $\lim_{n\to\infty} |a_n|^p = 0$.
   Thus, $\lim_{n\to\infty} |a_n|^{q-p} = 0$, which indicates that $|a_n|^{q-p}$ is bounded.
   Hence $\exists C > 0$ s.t. $|a_n|^{q-p} < C$, which results in

$$\sum_{n=0}^{\infty} |a_n|^q = \sum_{n=0}^{\infty} |a_n|^p \cdot |a_n|^{q-p} < C \sum_{n=0}^{\infty} |a_n|^p < \infty \tag{3.1}$$

   Therefore $(a_n) \in l^q$. □

2. For example

$$a_n = \frac{1}{n} \tag{3.2}$$

3. Consider a sequence constructed as follows

$$\begin{aligned} & x_1 = \frac{1}{1} && (1^1 = 1\ item) \\ & x_2 = ... = x_5 = \frac{1}{2} && (2^2 = 4\ items) \\ & x_6 = ... = x_{32} = \frac{1}{3} && (3^3 = 27\ items) \\ & ... \end{aligned} \tag{3.3}$$

This sequence certainly converges to 0. But not in L^p for any $p >= 1$ as

$$\sum_{n=1}^{\infty} = 1 + \frac{2^2}{2^p} + \frac{3^3}{3^p} + ... + \frac{n^n}{n^p} + ... \tag{3.4}$$

for any $n > p$, $\frac{n^n}{n^p} > 1$.