VV570

Numerical Analysis

Assignment 6

Jing and Manuel — UM-JI (Summer 2018)

Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Richardson extrapolation*

In this exercise we investigate Richardson extrapolation, a sequence acceleration method which can be used to improve the rate of convergence of a quadrature formula.

Let $a_0 \in \mathbb{R}$ be a value to be computed and $A(t),\ t > 0$ be such that

(i) $\lim_{t\to 0} A(t) = a_0$;

(ii) For all $n \leq 0$, there exist $a_1, \cdots, a_n$, and c_{n+1} such that

$$A(t) = a_0 + \sum_{i=1}^{n} a_i t^i + R_{n+1}(t), \qquad \text{with } |R_{n+1}(t)| \leq c_{n+1}(t);$$

Let $(A_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$\begin{cases} A_0(t) & = A(t) \\ A_n(t) & = \frac{r^n A_{n-1}(t) - A_{n-1}(rt)}{r^n - 1}, \qquad n \geq 1 \text{ and } r > 1 \text{ a constant.} \end{cases}$$

* 1. Prove by induction that for all $n \in \mathbb{N}$, $A_n(t) = a_0 + O(t^{n+1})$.

2. Fixing $t_0 > 0$ and $r_0 > 1$, we define a sequence $(t_m)_{m\geq 0}$ such that $t_m = r_0^{-m} t_0$.
   a) Show that when n is fixed then $\lim_{m\to\infty} A_n(t_m) = a_0$.
   b) Show that $A_n(t_m) = a_0 + O\left(r_0^{-m(n+1)}\right)$.

3. For m and n two integers, we define a matrix M whose entry at column n and row m is $A_{m,n} = A_n(t_m)$. Write the pseudocode of a clear algorithm generating the matrix M and returning a_0.

4. Romberg integration
   a) Understand box 1 and intuitively explain how Richardson extrapolation accelerates the convergence of the trapezium rule.
   b) For a function of your choice compare the trapezium rule to Romberg method.

Box 1: Romberg integration: an aplication of Richardson extrapolation

We consider Romberg method, the application of Richardson extrapolation to the case of the trapezium rule.

For $f \in C[\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$, the trapezium rule with constant step $h \in \mathbb{R}_+^*$ is given by

$$\int_\alpha^\beta f(x)\,\mathrm{d}x \approx T_h(f) = h\left(\frac{1}{2}f(\alpha) + f(\alpha+h) + \cdots + f(\beta-h) + \frac{1}{2}f(\beta)\right).$$

We assume the following result (which can be proven using the Bernoulli theorems).

> Let $f \in C^\infty[\alpha,\beta]$. Then for all $n \in \mathbb{N}^*$, there exists $(a_j)_{j\in\mathbb{N}}$ a sequence of real numbers such that
> $$T_h(f) = \int_\alpha^\beta f(x)\,\mathrm{d}x + \sum_{j=1}^{n} a_j h^{2j} + O(h^2 n).$$

From this theorem it suffices to set $a_0 = \int_\alpha^\beta f(x)\,\mathrm{d}x$, and $t = h^2$ to obtain

$$A(t) = a_0 + \sum_{j=1}^{n} a_j t^j + O(t^{n+1}), \forall n \geq 1.$$

Hence all the requirements for Richardson extrapolation are met. Applying it for instance with $r_0 = 4$ and $t_0 = (\beta - \alpha)^2$ yields

$$A_{m,n} = \int_\alpha^\beta f(x)\,\mathrm{d}x + O\left(\frac{1}{4^{m(n+1)}}\right).$$

In practice one chooses m and n with respect the expected precision level.

Ex. 2 — *Integration*
We consider the quadrature formula

$$\int_a^b f(x)\,\mathrm{d}x \approx (b-a)f\left(\frac{a+b}{2}\right). \tag{2.1}$$

1. Why does formula 2.1 fall under Peano's method?
2. Determine Peano kernel for this formula, and show that it keeps a constant sign.
3. Conclude on the existence of $\xi \in [a,b]$, such that for any $f \in C^2[a,b]$, the error is expressed as
$$E(f) = \frac{1}{24} f''(\xi)(b-a)^3.$$

Ex. 3 — *Gauss' method*
Let $(q_k)_{k\in\mathbb{N}}$ be a sequence of polynomials such that

$$q_k(x) = \frac{\sin(k+1)\theta}{sin\theta}, \quad \text{with } x = \cos\theta.$$

1. Let $w(x) = \sqrt{1-x^2}$ be a function over $(-1,1)$.
   a) Show that w is a weight function.
   b) Show that the $(q_k)_{k\in\mathbb{N}}$ define a sequence of orthogonal polynomials for the weight function w.
   c) Determine the orthonormal polynomials $(p_k)_k$ associated to $(q_k)_k$.

2. We consider the Gauss' method of order $2n+1$ defined by

$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x \approx \sum_{k=0}^{n} A_k f(x_k).$$

a) Determine all the x_k, $0 \leq k \leq n$.

b) Show that for all $0 \leq k \leq n$,

$$A_k = \frac{\pi}{n+2}\sin^2\frac{(k+1)\pi}{n+2}.$$

c) Assuming $f \in C^{2n+2}[-1,1]$, show that there exists $\xi \in (-1,1)$, such that the error of the method is given by

$$E_n(f) = c\frac{f^{(2n+2)}(\xi)}{(2n+2)!}, \quad \text{where } c \text{ is a constant to be determined.}$$