Vv556 Methods of Applied Mathematics I
Linear Operators

Date Due: 12:10 PM, Thursday, the 27th of September 2018

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This assignment has a total of **(18 Marks)**.

Exercise 2.1

i) Let $(V, \langle\cdot,\cdot\rangle_{\mathbb{R}})$ be a real inner product space and $\|x\| = \sqrt{\langle x, x\rangle}$ the norm induced by the inner product. Prove the *real polarisation identity*:

$$\langle x, y\rangle_{\mathbb{R}} = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

(2 Marks)

ii) Let $(V, \langle\cdot,\cdot\rangle_{\mathbb{C}})$ be a complex inner product space and $\|x\| = \sqrt{\langle x, x\rangle}$ the norm induced by the inner product. Prove the *complex polarisation identity*:

$$\langle x, y\rangle_{\mathbb{C}} = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big) + \frac{i}{4}\big(\|x-iy\|^2 - \|x+iy\|^2\big)$$

(2 Marks)

iii) Let V be a real or complex vector space. Show that every norm on V, if it is induced by some inner product, satsfies the *parallelogram rule*:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad \text{for all } x, y \in V$$

(2 Marks)

iv) Prove that the norm $\|\cdot\|_\infty \colon f \mapsto \sup_{x\in[a,b]} |f(x)|$ on $C([a,b])$ is not induced by an inner product, i.e., there exists no inner product $\langle\cdot,\cdot\rangle$ such that $\|\cdot\|_\infty = \sqrt{\langle\cdot,\cdot\rangle}$.
(2 Marks)

v) Show that every norm that satisfies the parallelogram rule is induced by an inner product. For simplicity, consider a real vector space only. *Instructions:*

- Use the polarization identity to define an inner product from the norm.
- Show that the so-defined inner product satisfies $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$.
- Then deduce that $\langle x, \lambda y\rangle = \lambda\langle x, y\rangle$ for rational $\lambda \in \mathbb{Q}$.
- Use the continuity of the norm to conclude that the equality holds in fact for $\lambda \in \mathbb{R}$.
- Verify the other properties for an inner product.

(4 Marks)

Exercise 2.2

We define the following spaces of complex-valued sequences $(a_n)_{n\in\mathbb{N}}$:

$$\ell^1 = \Big\{(a_n)_{n\in\mathbb{N}} \colon \sum_{n=0}^{\infty} |a_n| < \infty\Big\}, \qquad c_0 = \Big\{(a_n)_{n\in\mathbb{N}} \colon \lim_{n\to\infty} a_n = 0\Big\},$$

and norms

$$\|(a_n)\|_1 = \sum_{n=0}^{\infty} |a_n|, \qquad \|(a_n)\|_\infty = \sup_{n\in\mathbb{N}} |a_n|.$$

i) Is ℓ^1 dense in c_0 in the $\|\cdot\|_\infty$ norm? Why or why not? Explain!
(3 Marks)

ii) Is ℓ^1 dense in c_0 in the $\|\cdot\|_1$ norm? Why or why not? Explain!
(3 Marks)