VV570
Numerical Analysis

Assignment 3
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Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Cantor's set*
In this exercise we investigate a few properties of Cantor's set.

Calling C_0 the compact $[0,1]$ we trisect it and remove the middle open part and define C_1 as $[0,1/3] \cup [2/3,1]$. Recursively repeating the process we construct an infinite sequence $C_0 \supset C_1 \supset \cdots \supset C_i \supset \cdots$. We define the *Cantor set* as the intersection of all the nested C_i, $C = \bigcap_{i=0}^{\infty} C_i$.

* 1. Prove that C is compact.

2. Show that for any two elements $x < y$ of C, there exists $z \notin C$ such that $x < z < y$.
3. We estimate how "large" the set C is with respect to λ, the Lebesgue measure.
   a) What is the Lebesgue measure of a countable set?
   b) Show that for $n \in \mathbb{N}$, $\lambda(C_n) = \left(\frac{2}{3}\right)^n$, and conclude that C has Lebesgue measure 0.
4. We estimate how "large" the set C is with respect to cardinality.
   a) Show that C is not empty.
   * b) We write any $x \in [0,1]$ using the ternary expansion
   $$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \quad \text{with } a_i \in \{0,1,2\}.$$
   Describe the form of the x belonging to C_i, $i \in \mathbb{N}$.
   c) Clearly explain Cantor diagonal argument.
   * d) Using Cantor diagonal argument show that Cantor's set is uncountable.
5. From questions 3, Cantor's set has Lebesgue measure 0, and from question 4 it is uncountable. Explain how this is possible.

Ex. 2 — *Cantor's function*
We now define the Cantor function following the construction process of the Cantor set. Let f_1 be $1/2$ over the "removed interval" $(1/3, 2/3)$ and linear on C_1. Then define f_2 to be $1/4$ and $3/4$ on the two removed intervals, to coincide with f_1 in $1/3$ and $2/3$, while being linear on the remaining four intervals. The process is carried on as the Cantor set is built, defining the Cantor function f_C.

1. Show that the $(f_n)_{n\in\mathbb{N}}$ define a sequence of monotonically increasing continuous functions.
2. Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f_C : [0,1] \to [0,1]$.
3. Prove that Cantor function is
   a) Uniformly continuous;
   Hint: prove (or assume) that a continuous function on a compact is uniformly continuous.

b) Monotonically increasing;

* c) Differentiable almost everywhere, with $f'(x) = 0$;

4. Prove that f_C is not absolutely continuous.

Note: the Cantor function is one of the most simple function to be uniformly continuous but not absolutely continuous

Ex. 3 — *Taylor's theorem*

Reasoning by induction and applying the fundamental theorem of calculus, prove

1. Taylor's theorem as given in the slides (theorem 2.36).
2. Taylor's theorem with the remainder in the Lagrange form; Calling $P_n(x)$ the polynomial part of $f(x)$ in Taylor's theorem, show the existence of $c \in [a, x]$ such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Ex. 4 — *Convergence of rationals to irrationals*

Intuitively a *complete space* has "no point missing" anywhere. In particular it means that any Cauchy sequence converges inside the space. In this exercise we show that e is not rational while we find a Cauchy sequence of rationals converging to e.

1. Show that e is irrational.
2. Show that the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = \left(1 + \frac{1}{n}\right)^n$ is a Cauchy sequence converging to e.
3. Is $\mathbb{Q}$ complete? Explain.