Vv556 Methods of Applied Mathematics I

Linear Operators

Assignment 9

Date Due: 12:10 PM, Thursday, the 29th of November 2018

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This assignment has a total of **(14 Marks)**.

Exercise 9.1

Let $M := \{u \in L^2([0,1]) \colon u \in C^2(0,1),\ u(0) = u(1) = 0\}$ and define

$$L = -\frac{d^2}{dx^2}$$

on $M \subset L^2([0,1])$. Let $K \colon L^2([0,1]) \to L^2([0,1])$ be given by

$$(Ku)(x) := \int_0^1 g(x,\xi)u(\xi)\,d\xi$$

with

$$g(x,\xi) := \begin{cases} x(1-\xi) & x < \xi, \\ \xi(1-x) & x \geq \xi. \end{cases}$$

It is known that K is compact and self-adjoint. Furthermore, $K = L^{-1}$.

i) Show that $L = K^{-1}$ if K is restricted to M. (You will have to perform some careful differentiations; use the chain rule.)
(2 Marks)

ii) Consider the eigenvalue equation $Lu = \lambda u$ for $u \in M$. Show that $\lambda \neq 0$ is an eigenvalue for L if and only if $1/\lambda$ is an eigenvalue for K. How are the eigenfunctions of K and L related?
(2 Marks)

iii) Find the eigenvalues $\lambda_n \in \mathbb{C}$ and eigenfunctions $\psi_n \in M$ of L by solving an ordinary differential equation. You will have to prove first that $\lambda_n \in \mathbb{R}$ and then consider the cases $\lambda_n > 0$, $\lambda_n = 0$ and $\lambda_n < 0$.
(5 Marks)

iv) Find the spectrum of K. Is zero an eigenvalue? Find the sequence of eigenvalues (λ_n) such that $\lambda_n \searrow 0$.
(3 Marks)

v) According to the spectral theorem, $L^2([0,1])$ has an orthonormal basis consisting of eigenfunctions of K. Use this to deduce that a certain Fourier basis is actually a basis.
(2 Marks)