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Introduction to Numerical Analysis HW1

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1 Metric Space

1. *Proof.* Since there's no elements in $\emptyset$, so $\emptyset$ is open, as $X^{\complement} = \emptyset$, so X is closed.
   X is open as it contains all the elemnets, since $\emptyset^{\complement} = X$, thus $\emptyset$ is closed. □

2. *Proof.* Since
   $$T \triangleq (U_1 \cap U_2 \cap \dots \cap U_m)^{\complement} = U_1^{\complement} \cup U_2^{\complement} \cup \dots \cup U_m^{\complement} \tag{1.1}$$
   And ${U_i}^{\complement}$ is open, as U_i is closed.
   Thus, T is open, and therefore the intersection is closed. □

3. *Proof.* Since
   $$T \triangleq (U_1 \cup U_2 \cup \dots \cup U_m)^{\complement} = U_1^{\complement} \cap U_2^{\complement} \cap \dots \cap U_m^{\complement} \tag{1.2}$$
   And ${U_i}^{\complement}$ is open, as U_i is closed.
   Thus, T is open, and therefore the intersection is closed. □

2 Continuity

1. $y = \frac{1}{x}$ is continuous over $(0, +\infty)$, but not uniform continuous.

   Proof. Let $x_1 = \frac{1}{n+1}$, $x_2 = \frac{1}{n}$. Then $\lim\limits_{n\to+\infty} |x_2 - x_1| = \frac{1}{n(n+1)} = 0$. But $\lim\limits_{n\to+\infty} |y_2 - y_1| = 1 \neq 0$.
   Thus it is not uniform continuous. □

2. $y = \sqrt{x}$ is uniform continuous over $(0, +\infty)$ but not Lipschitz continuous.

   Proof. As $y\prime = \frac{1}{2\sqrt{x}}$, the slope tends towards infty when x approaches 0, thus it is not Lipschitz continuous. $\square$

3 Cardinalty

1. *Proof.* The following function is a one-to-one mapping from N to Z, thus N and Z have the same number of elements.

$$f(n) = \begin{cases} -\frac{n}{2} & \text{n is even} \\ \frac{n+1}{2} & \text{n is odd} \end{cases} \tag{3.1}$$

   Arange the elements in N and Q in the pascal triangle style as below, and it is clear enough to see the one-to-one mapping from N to Q with rationals are grouped according to the sum of dividend and divisor. Thus N and Q have the same number of elements.

$$\begin{array}{ccccccccc|ccccccccc}
 & & & & 1 & & & & & & & & & \frac{1}{1} & & & & \\
 & & & 2 & & 3 & & & & & & & \frac{1}{2} & & \frac{2}{1} & & & \\
 & & 4 & & & & 5 & & & & & \frac{1}{3} & & & & \frac{3}{1} & & \\
 & 6 & & 7 & & 8 & & 9 & & & \frac{1}{4} & & \frac{2}{3} & & \frac{3}{2} & & \frac{4}{1} & \\
10 & & & & & & & & 11 & \frac{1}{5} & & & & & & & & \frac{5}{1} \\
 & & & & \cdots & & & & & & & & & \cdots & & & &
\end{array}$$

$\square$

2. *Proof.* The following function is a one-to-one mapping from [0, 1] to R, thus they have the same number of elements.

$$f(x) = \begin{cases} -\infty & \text{x=0} \\ tan[\pi(x-0.5)] & 0 < \text{x} < 1 \\ +\infty & \text{x=1} \end{cases} \tag{3.2}$$

$\square$

3. *Proof.* Suppose all the real numbers can be listed, and each one is marked as r_i, where $i = 1, 2, 3....$ Given a real number $r\prime$ such that its **i-th** digit is different from that in r_i, it's obvious that $r\prime$ is not included in the real numbers listed above, which is contradictory to the hypothesis. Thus real numbers can not be listed and it contains more elements than N. $\square$

4 SLIDES

1. Here I give 2 proofs, the first one comes directly from the class, the other from previous reading.(thanks to Ran Yi for pointing out that)

   Proof. Let E be an inner product space over $\mathbb{C}$, and $u, v \in E$.
   Given Y defined as below, where $\lambda \in \mathbb{R}$.

   $$Y = |u - \lambda v|^2 \tag{4.1}$$

   It's obvious that $Y \geq 0$. Expand the squares according to the definition of inner products, the inequality can be written as below.

   $$< v, v > \lambda^2 - (< u, v > + < v, u >)\lambda + < u, u > \ \geq \ 0 \tag{4.2}$$

   The LHS of the inequality can be viewed as a quadratic function where λ is the variable. Thus, the discriminant is semi-negative definite, which can be written as below.

   $$\Delta = (< u, v > + < v, u >)^2 - 4 < u, u >< v, v > \ \leq \ 0 \tag{4.3}$$

   As $< v, u >= \overline{< u, v >}$, the inequality above can be simplified as below.

   $$< u, v >^2 + \overline{< u, v >}^2 + 2| < u, v > |^2 \ \leq \ 4\|u\|^2\|v\|^2 \tag{4.4}$$

   For $x \in \mathbb{C}$, the following equality is justified.

   $$x^2 + \bar{x}^2 = \frac{1}{2}[(x + \bar{x})^2 + (x - \bar{x})^2] = \frac{1}{2}[(2Re(x))^2 + (2Im(x))^2] = 2|x|^2 \tag{4.5}$$

   Since $< u, v >$ is an complex number, apply (4.5) into (4.4) and eliminate the constant 4, the Cauchy-Schwarz inequality is obtained at last.

   $$| < u, v > |^2 \leq \|u\|^2\|v\|^2 \tag{4.6}$$

   □

   Proof. Let E be an inner product space over $\mathbb{C}$, and $u, v \in E$.

   Given Y defined as below, where $\lambda \in \mathbb{R}$.

   $$\begin{aligned} Y &= |u - \lambda v|^2 \\ &= < v, v > \lambda^2 - (< u, v > + < v, u >)\lambda + < u, u > \end{aligned} \tag{4.7}$$

It's obvious that $Y \geq 0$ for any λ.
Given λ as

$$\lambda = \frac{<u,v>}{<v,v>} \tag{4.8}$$

Substitude it into (4.7), the inequality reads as

$$<u,u><v,v> - <u,v>(<u,v>+<v,u>) + <u,v>^2 \geq 0 \tag{4.9}$$

It can be further simplified as

$$\begin{aligned} <u,u> \cdot <v,v> &= ||u||^2 \cdot ||v||^2 \\ &\geq <u,v> \cdot <v,u> \\ &= <u,v> \cdot \overline{<u,v>} \\ &= |<u,v>|^2 \end{aligned} \tag{4.10}$$

Thus, the Cauchy-Schwarz inequality got proved. □

2. *Proof.* $d(x,y)$ is non-negative follows from the definition of metric space.

$$\begin{aligned} d(x,y) &= \frac{1}{2}(d(x,y)+d(x,y)) \\ &= \frac{1}{2}(d(x,y)+d(y,x)) && \text{By symmetry} \\ &\geq \frac{1}{2}d(x,x) && \text{By triangle inequality} \\ &= 0 && \text{By identity of indiscernible} \end{aligned} \tag{4.11}$$

□

5 LINEAR ALGEBRA

1. *Proof.* Suppose $\{u_1, u_2, ..., u_m\}$ forms the basis of Ker(f), and it can be extended to form the basis of V_1, which is denoted as $\{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$. It's clear that the dimension of Ker(f) is m and the dimension of V_1 is $m+n$. So, it is left to prove that the dimension of V_2 is n.

   The dimension of V_2 is n means $\{f(v_1), f(v_2), ..., f(v_n)\}$ forms a basis of V_2. To see that, let w be an arbitary vector in V_1, thus, there exist unique scalars a_i, b_j such that

$$w = a_1u_1 + a_2u_2 + ... + a_mu_m + b_1v_1 + b_2v_2 + ... + b_nv_n \tag{5.1}$$

   The image of w under mapping f is given as below.

$$\begin{aligned} f(w) &= \sum_{i=1}^{m} a_i f(u_i) + \sum_{j=1}^{m} b_j f(v_j) \\ &= \sum_{j=1}^{m} b_j f(v_j) \quad \text{As } f(u_i) = 0 \end{aligned} \tag{5.2}$$

Thus $\{f(v_1), f(v_2), ..., f(v_n)\}$ spans V_2. It is left to show that they are linearly independent, which means there's no redundancy in the list.

Given coefficents c_i such that

$$c_1 f(v_1) + c_2 f(v_2) + ... + c_n f(v_n) = 0 \tag{5.3}$$

Since f is a linear mapping, (5.3) can be grouped as below.

$$f(c_1 v_1 + c_2 v_2 + ... + c_n v_n) = 0 \tag{5.4}$$

Thus $c_1 v_1 + c_2 v_2 + ... + c_n v_n \in Ker(f)$. As $\{u_1, u_2, ..., u_m\}$ forms the basis of Ker(f), there exists coefficients d_i such that

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = d_1 u_1 + d_2 u_2 + ... + d_m u_m \tag{5.5}$$

Since $\{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$ forms the basis of V_1, all the coefficients in (5.5) should be 0. Thus, (5.3) is valid if and only if all the coefficents c_i is 0, which implies that $\{f(v_1), f(v_2), ..., f(v_n)\}$ are linearly independent.

Thus, $\{f(v_1), f(v_2), ..., f(v_n)\}$ forms the basis of V_2, and the dimension of V_2 is therefore n. The rank-nullity theorem written as below is proved.

$$dim(V_1) = dim(ker(f)) + dim(V_2) \tag{5.6}$$

□

2. *Proof.* Let U, V, W be vector spaces over the same field K, function f be a linear map from U to V and function g be a linear map from V to W. It's left to show that the composition of f and g, denoted as h, which maps a vector in U to W, is still a linear map.

   Given $u_1, u_2 \in U$ and any scalar $c \in \mathbb{K}$, then

$$\begin{aligned} h(u_1 + u_2) &= g(f(u_1 + u_2)) \\ &= g(f(u_1) + f(u_2)) \\ &= g(f(u_1)) + g(f(u_2)) \\ &= h(u_1) + h(u_2) \end{aligned} \tag{5.7}$$

$$\begin{aligned} h(cu_1) &= g(f(cu_1)) \\ &= g(cf(u_1)) \\ &= cg(f(u_1)) \\ &= ch(u_1) \end{aligned} \tag{5.8}$$

   Thus, the composition of two linear maps is still a linear map. □

3. *Proof.* Let U, V be vector spaces over the same field K, function f be linear maps from U to V. It's left to show that the inverse of f, denoted as f^{-1}, which maps a vector in V to U is still a linear map.

   By inverse, it means that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x \tag{5.9}$$

   Given $u_1, u_2 \in U$ and any scalar $c \in \mathbb{K}$, denote the counterpart of u_1, u_2 in V as v_1, v_2 under the linear mapping function f. Then

$$\begin{aligned} f^{-1}(v_1 + v_2) &= f^{-1}(f(u_1) + f(u_2)) \\ &= f^{-1}(f(u_1 + u_2)) \\ &= u_1 + u_2 \\ &= f^{-1}(v_1) + f^{-1}(v_2) \end{aligned} \tag{5.10}$$

$$\begin{aligned} f^{-1}(cv_1) &= f^{-1}(cf(u_1)) \\ &= f^{-1}(f(cu_1)) \\ &= cu_1 \\ &= cf^{-1}(v_1) \end{aligned} \tag{5.11}$$

   Thus, the inverse of a linear map is still a linear map. □

6 Discontinuous linear maps

1. *Proof.* The k times derivative of $f_n(x)$ can be written as below.

$$f_n^{(k)}(x) = n^{k-1} sin(nx + \frac{k\pi}{2}) \quad k = 0, 1, 2, 3, ... \tag{6.1}$$

   It's clear that $f_n^{(k)}(x)$ is continous for any k, thus $f_n(x) \in C^{\infty}(\mathbb{R})$. □

2.

$$df_n(x) = cos(nx)dx \tag{6.2}$$

3. *Proof.* Let $x_0 = 0, x_1 = \frac{\pi}{2n}$, then

$$\lim_{n->+\infty} |x_1 - x_0| = 0 \tag{6.3}$$

$$\lim_{n->+\infty} |df_n(x_1) - df_n(x_0)| = |0 - dx| = |dx| \neq 0 \tag{6.4}$$

   Thus, the differential is not continous when n tends to infty. □

4. Sorry, haven't figured out yet...

7 PI

1. The main program(see Algorithm1) adopts Machin's formula to calculate π.

$$\pi = 4[4atan(\frac{1}{5}) - atan(\frac{1}{239})] \tag{7.1}$$

Subroutine(see Algorithm2) calculating $atan(\frac{1}{x})$ is used by the main program. Since

$$atan(x) = \frac{1}{1}x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + ... \tag{7.2}$$

Therefore

$$atan(\frac{1}{x}) = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + ... \tag{7.3}$$

Algorithm 1 Calculation of π using Machin's formula

Input: None.
Output: The value of π with approximation.

return $4[4atan(1/5) - atan(1/239)]$

Algorithm 2 Calculation of $atan(\frac{1}{x})$

Input: x
Output: $atan(\frac{1}{x})$

$ret \leftarrow 0$
$e \leftarrow 1/x$
$s \leftarrow -x^2$
$c \leftarrow 1$
while $e \neq 0$ **do**
  $ret \leftarrow ret + e/c$
  $e \leftarrow e/s$
  $c \leftarrow c + 2$
end while
return ret

2. The matlab code is given as below.

```
function [pi] = pi_machin()
pi = 4*(4*atan_inv(5) - atan_inv(239));

function [r] = atan_inv(x)
r = 0;
e = 1/x;
s = - x*x;
c = 1;
while (e ~= 0)
r = r + e/c;
e = e/s;
c = c+2;
end
```