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Methods of Applied Mathematics I
HW7

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1 EXERCISE7.1

1. *Proof.* Since $|\lambda| = 1$, then

$$||e_\lambda^{(N)}||_2 = \frac{1}{\sqrt{N+1}}\sqrt{\sum_{i=0}^{N}\lambda^{2i}} = \frac{1}{\sqrt{N+1}}\sqrt{N+1} = 1 \tag{1.1}$$

□

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2 EXERCISE7.2

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3 EXERCISE7.3

1. *Proof.* It's clear that L^{-1} is the differentiate operator, and L^{-1} is unbounded. So L has unbounded inverse.
   Since the domain of L is composed of square-integrable functions over $[0,1]$, say
   $$\int_0^1 f^2(x)dx < \infty \tag{3.1}$$
   An element within the range of L is
   $$g(x) = \int_0^x f(t)dt \tag{3.2}$$
   Then, if $f(x)$ is a polynomial in L^2 , it must be bounded over $[0,1]$ as it is continous. Denote the supreme of $f(x)$ as M, then $g(x) \le Mx$. Hence $g(x)$ is square-integrable over $[0,1]$, say
   $$\int_0^1 g^2(x)dx \le M^2 \int_0^1 x^2 dx < \infty \tag{3.3}$$
   Clearly, the domain of L doesn't contain all the polynomials and therefore the range of L is open and incomplete. The boundary of the range of L are the limits of squences like $f_n(x) = nx$ when $n \to \infty$.
   Hence, the state of L is $(III, 1_n)$. □

2. $$L^* = L \tag{3.4}$$

4 EXERCISE7.4

Proof. For $p = 1$

$$\begin{aligned} RHS &\triangleq ||(a_n)||_1 \cdot ||(b_n)||_1 = \sum_{i=0}^{\infty} |a_i| \cdot \sum_{j=0}^{\infty} |b_j| = \sum_{n=0}^{\infty} \sum_{i+j=n} |a_i||b_j| \\ &\ge \sum_{n=0}^{\infty} |\sum_{i+j=n} a_i b_j| = \sum_{n=0}^{\infty} |c_n| = ||(c_n)||_1 \triangleq LHS \end{aligned} \tag{4.1}$$

For $p > 1$, take $q > 0$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$
Then, using the Holder's inequality

$$
\begin{aligned}
LHS \triangleq ||(c_n)||_p &= \Big(\sum_{n=0}^{\infty} |c_n|^p\Big)^{\frac{1}{p}} = \Big(\sum_{n=0}^{\infty}\Big|\sum_{i+j=n} a_i b_j\Big|^p\Big)^{\frac{1}{p}} \\
&= \Big(\sum_{n=0}^{\infty}\Big|\sum_{i+j=n} a_i b_j^{\frac{1}{p}} b_j^{\frac{1}{q}}\Big|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{n=0}^{\infty}\Big|\sum_{i+j=n} \Big(|a_i||b_j|^{\frac{1}{p}}\Big)|b_j|^{\frac{1}{q}}\Big|^p\Big)^{\frac{1}{p}} \\
&\le \Big(\sum_{n=0}^{\infty}\Big[\Big(\sum_{i+j=n}|a_i|^p|b_j|\Big)^{\frac{1}{p}}\Big(\sum_{j=0}^{n}|b_j|\Big)^{\frac{1}{q}}\Big]^p\Big)^{\frac{1}{p}} \\
&= \Big[\sum_{n=0}^{\infty}\Big(\sum_{i+j=n}|a_i|^p|b_j|\Big)\Big(\sum_{j=0}^{n}|b_j|\Big)^{\frac{p}{q}}\Big]^{\frac{1}{p}} \\
&\le \Big(\sum_{i=0}^{\infty}|a_i|^p\Big)^{\frac{1}{p}} \cdot \Big(\sum_{j=0}^{\infty}|b_j|\Big)^{\frac{1}{p}} \cdot \Big(\sum_{j=0}^{\infty}|b_j|\Big)^{\frac{1}{q}} \\
&= ||(a_n)||_p \cdot ||(b_n)||_1 \triangleq RHS
\end{aligned}
\tag{4.2}
$$

□