SHANGHAI JIAO TONG UNIVERSITY, UM-SJTU JOINT INSTITUTE

Methods of Applied Mathematics I
HW8

Yu Cang 018370210001
Zhiming Cui 017370910006

November 20, 2018

1 EXERCISE8.1

1. *Proof.* Since
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|^2=\sum_{k=1}^{\infty}\frac{1}{k}\to\infty \tag{1.1}$$
Thus, L is not a Hilbert-Schmidt operator. □

2. *Proof.* $\forall x,y\in l^2$, the inner product can be expressed as
$$\begin{aligned}<x,Ly>&=<(x_1,x_2,...,x_n,...),(y_1,\frac{y_2}{\sqrt{2}},...,\frac{y_n}{\sqrt{n}},...)>\\&=\sum_{i=1}^{\infty}\frac{x_iy_i}{\sqrt{i}}\\&=<(x_1,\frac{x_2}{\sqrt{2}},...,\frac{x_n}{\sqrt{n}},...),(y_1,y_2,...,y_n,...)>\\&=<Lx,y>\end{aligned} \tag{1.2}$$
Thus, L is self adjoint. □

3. *Proof.* Denote $L_n:l^2\to l^2$ by
$$L_n(x_n)=(x_1,\frac{x_2}{\sqrt{2}},\frac{x_3}{\sqrt{3}},...,\frac{x_n}{\sqrt{n}},0,0,...) \tag{1.3}$$

L_n is a Hilbert-Schmidt operator as

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|^2 = \sum_{k=1}^{n}\frac{1}{k} < \infty \tag{1.4}$$

Thus, L_n is compact. Then, L is compact if (L_n) converges to L in norm, i.e.

$$\lim_{n\to\infty}||L_n - L|| = 0 \tag{1.5}$$

To show this, expand the operator norm as

$$\begin{aligned} ||L_n - L|| &= \sup_{x\in l^2}\frac{||(L_n - L)x||}{||x||} \\ &= \sup_{x\in l^2}\frac{||(0,...,0,\frac{x_{n+1}}{\sqrt{n+1}},\frac{x_{n+2}}{\sqrt{n+2}},...)||}{||x||} \\ &\le \frac{1}{\sqrt{n+1}}\frac{||(0,...,0,x_{n+1},x_{n+2},...)||}{||x||} \\ &\le \frac{1}{\sqrt{n+1}} \end{aligned} \tag{1.6}$$

So, $\lim_{n\to\infty}||L_n - L|| \le \lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0$.
According to the positivity of norm, $\lim_{n\to\infty}||L_n - L|| \ge 0$.
Hence $\lim_{n\to\infty}||L_n - L|| = 0$. □

4. Since

$$||L|| = \sup_{x\in l^2}\frac{||Lx||}{||x||} \ge \frac{||Le_1||}{||e_1||} = 1 \tag{1.7}$$

and

$$||L|| = \sup_{x\in l^2}\frac{||Lx||}{||x||} \le \frac{||x||}{||x||} = 1 \tag{1.8}$$

Thus $||L|| = 1$.
As the norm of operator is a bound for all its eigenvalues, then $|\lambda| \le 1$. Thus the upper bound for the spcetrum is found.
For the lower bound, Rayleigh Quotient is used. For $x \neq 0$,

$$\lambda \ge L_T \triangleq \inf_{x\in dom(L)} R(x) = \inf_{x\in dom(L)}\frac{< x, Lx >}{||x||^2} = \lim_{n\to\infty}\frac{< e_n, Le_n >}{||e_n||^2} = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0 \tag{1.9}$$

Thus the lower bound for the spcetrum is found.

5. Clearly, $\{1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, ...\}$ is the point specturm as they are eigenvalues of L.

2 EXERCISE8.2

1. *Proof.* $\forall u \in M$, with the B.C. of u, the transform by operator KL yields

$$\begin{aligned}
(KL)u(x) &= \int_0^1 g(x,\xi)(Lu)(\xi)d\xi \\
&= \int_0^x g(x,\xi)(Lu)(\xi)d\xi + \int_x^1 g(x,\xi)(Lu)(\xi)d\xi \\
&= \int_0^x \xi(x-1)u''(\xi)d\xi + \int_x^1 x(\xi-1)u''(\xi)d\xi \\
&= \int_0^x \xi(x-1)du'(\xi) + \int_x^1 x(\xi-1)du'(\xi) \\
&= (\xi(x-1)u')\Big|_0^x - \int_0^x u'(x-1)d\xi + (x(\xi-1)u')\Big|_x^1 - \int_x^1 u' x d\xi \\
&= x(x-1)u'(x) - (x-1)(u(x)-u(0)) - x(x-1)u'(x) - x(u(1)-u(x)) \\
&= u(x)
\end{aligned} \tag{2.1}$$

Thus, $KL = I$ on M. □

2. *Proof.* For example, take $u(x) = \sqrt{x}$, then

$$||L|| \geq \frac{||Lu||}{||u||} = \frac{||x^{-\frac{3}{2}}||}{4||x^{\frac{1}{2}}||} \to \infty \tag{2.2}$$

Thus, L is unbounded. □

3.

4. *Proof.* Swap ξ and x, then

$$g(\xi,x) = \begin{cases} \xi(1-x) & \xi < x \\ x(1-\xi) & \xi \geq x \end{cases} = \begin{cases} x(1-\xi) & x < \xi \\ \xi(1-x) & x \geq \xi \end{cases} = g(x,\xi) \tag{2.3}$$

Thus, $\forall u, v \in M$,

$$\begin{aligned}
< u, Kv > &= \int_0^1 u(x)\left(\int_0^1 g(x,\xi)v(\xi)d\xi\right)dx \\
&= \int_0^1\int_0^1 u(x)g(\xi,x)v(\xi)d\xi dx \\
&= \int_0^1\left(\int_0^1 g(\xi,x)u(x)dx\right)v(\xi)d\xi \\
&=< Ku, v >
\end{aligned} \tag{2.4}$$

Hence, K is self-adjoint. □

5.

$$L_T = \inf_{u\in M}\frac{< u, Ku >}{< u, u >} = \inf_{u\in M}\frac{< Ku, u >}{< u, u >} \tag{2.5}$$

and

$$U_T = \sup_{u \in M} \frac{< u, Ku >}{< u, u >} = \sup_{u \in M} \frac{< Ku, u >}{< u, u >} \tag{2.6}$$