变分原理及有限元大作业

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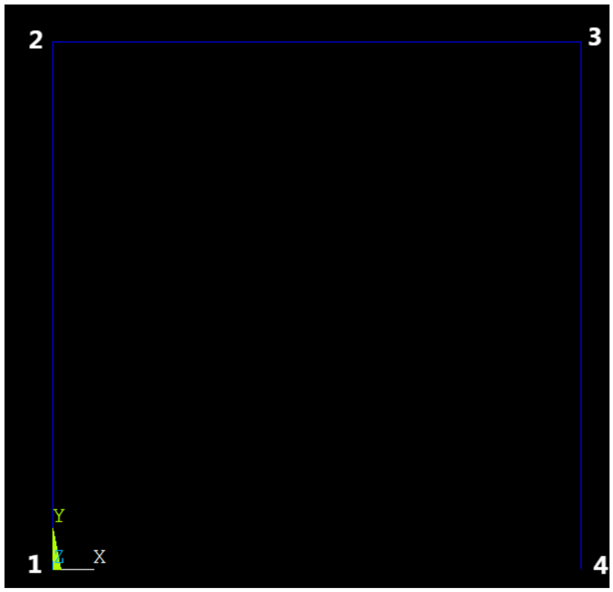
2016 年 5 月 16 日

目录

1 问题描述

求解如图所示的平面钢架的节点位移和内力，各杆材料和几何尺寸均相同。$E = 2 \times 10^7 N/cm^2$，$l = 100cm$，$A = 10cm^2$，$Iz = 25cm^4$，$P = 10000N$。



2 计算局部刚度矩阵

节点位移包括节点沿X、Y方向的位移和绕Z轴的转角，采用2节点的插值方式。局部刚度矩阵如下：

$$\overline{K^e} = \begin{pmatrix} \frac{EA}{l} & 0 & 0 & \frac{-EA}{l} & 0 & 0 \\ 0 & \frac{12EIz}{l^3} & \frac{6Iz}{l^2} & 0 & \frac{-12EIz}{l^3} & \frac{6EIz}{l^2} \\ 0 & \frac{6EIz}{l^2} & \frac{4EIz}{l} & 0 & \frac{-6EIz}{l^2} & \frac{2EIz}{l} \\ \frac{-EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & \frac{-12EIz}{l^3} & \frac{-6EIz}{l^2} & 0 & \frac{12EIz}{l^3} & \frac{-6EIz}{l^2} \\ 0 & \frac{6EIz}{l^2} & \frac{2EIz}{l} & 0 & \frac{-6EIz}{l^2} & \frac{4EIz}{l} \end{pmatrix}$$

将各个参数代入，计算结果如下：

$$\overline{K^1} = \overline{K^2} = \overline{K^3} = \begin{pmatrix} 2000000 & 0 & 0 & -2000000 & 0 & 0 \\ 0 & 6000 & 300000 & 0 & -6000 & 300000 \\ 0 & 300000 & 20000000 & 0 & -300000 & 10000000 \\ -2000000 & 0 & 0 & 2000000 & 0 & 0 \\ 0 & -6000 & -300000 & 0 & 6000 & -300000 \\ 0 & 300000 & 10000000 & 0 & -300000 & 20000000 \end{pmatrix}$$

3 计算全局刚度矩阵

首先确定全局坐标系在各个局部坐标系下的方向余弦系数，得到的变换矩阵如下：

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lambda^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lambda^3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

根据局部刚度矩阵与全局刚度矩阵之间的转换公式$K^e = \lambda^T \times \overline{K^e} \times \lambda$，得到各个单元在全局坐标系下的刚度矩阵为如下：

$$K^1 = \begin{pmatrix} 6000 & 0 & -300000 & -6000 & 0 & -300000 \\ 0 & 2000000 & 0 & 0 & -2000000 & 0 \\ -300000 & 0 & 20000000 & 300000 & 0 & 10000000 \\ -6000 & 0 & 300000 & 6000 & 0 & 300000 \\ 0 & -2000000 & 0 & 0 & 2000000 & 0 \\ -300000 & 0 & 10000000 & 300000 & 0 & 20000000 \end{pmatrix}$$

$$K^2 = \begin{pmatrix} 2000000 & 0 & 0 & -2000000 & 0 & 0 \\ 0 & 6000 & 300000 & 0 & -6000 & 300000 \\ 0 & 300000 & 20000000 & 0 & -300000 & 10000000 \\ -2000000 & 0 & 0 & 2000000 & 0 & 0 \\ 0 & -6000 & -300000 & 0 & 6000 & -300000 \\ 0 & 300000 & 10000000 & 0 & -300000 & 20000000 \end{pmatrix}$$

$$K^3 = \begin{pmatrix} 6000 & 0 & 300000 & -6000 & 0 & 300000 \\ 0 & 2000000 & 0 & 0 & -2000000 & 0 \\ 300000 & 0 & 20000000 & -300000 & 0 & 10000000 \\ -6000 & 0 & -300000 & 6000 & 0 & -300000 \\ 0 & -2000000 & 0 & 0 & 2000000 & 0 \\ 300000 & 0 & 10000000 & -300000 & 0 & 20000000 \end{pmatrix}$$

4 组集结构刚度矩阵

由于本例结构较为简单，将各个单元的全局刚度矩阵按照对应的节点外载荷组集起来时只需要沿对角线放置上去即可，结构刚度矩阵K初始全设为0，将单元1的刚度矩阵加在从（1,1）到（6,6）的对角方阵上，单元2的刚度矩阵加在从（4,4）到（9,9）的对角方阵上，单元3的刚度矩阵放在从（7,7）到（12,12）的对角方阵上。最终得到的结果如下所示：

$$K=\begin{pmatrix}
6000 & 0 & -300000 & -6000 & 0 & -300000 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2000000 & 0 & 0 & -2000000 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-300000 & 0 & 20000000 & 300000 & 0 & 10000000 & 0 & 0 & 0 & 0 & 0 & 0\\
-60000 & 300000 & 2006000 & 0 & 300000 & -2000000 & 0 & 0 & 0 & 0 & 0 & \\
0 & -2000000 & 0 & 0 & 2006000 & 300000 & 0 & -6000 & 300000 & 0 & 0 & 0\\
-300000 & 0 & 10000000 & 300000 & 300000 & 40000000 & 0 & -300000 & 10000000 & 0 & 0 & 0\\
0 & 0 & 0 & -2000000 & 0 & 0 & 2006000 & 0 & 300000 & -60000 & 300000 & \\
0 & 0 & 0 & 0 & -6000 & -300000 & 0 & 2006000 & -300000 & 0 & -2000000 & 0\\
0 & 0 & 0 & 0 & 300000 & 10000000 & 300000 & -300000 & 40000000 & -300000 & 0 & 10000000\\
0 & 0 & 0 & 0 & 0 & 0 & -6000 & 0 & -300000 & 6000 & 0 & -300000\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2000000 & 0 & 0 & 2000000 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 300000 & 0 & 10000000 & -300000 & 0 & 20000000
\end{pmatrix}$$

5　求解节点位移与应力

求得上述结构刚度矩阵后就可以代入已有载荷算出在节点2和节点3出的位移了，继而可算出在节点1和节点4处的约束反力。

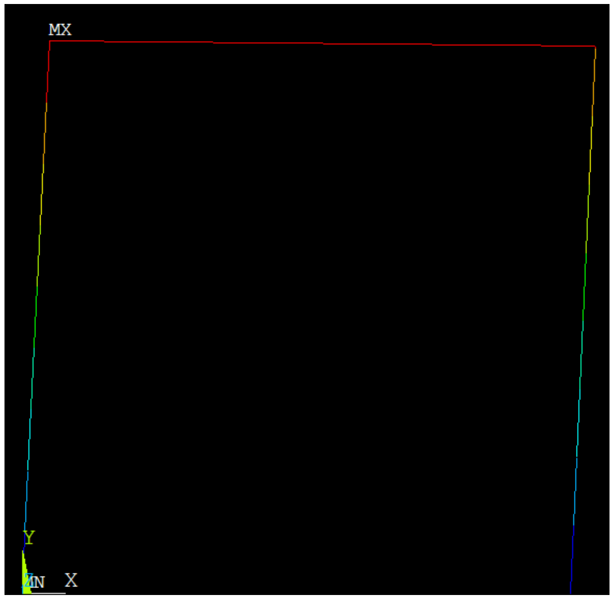
最终得到的位移和如下：

$$\begin{gathered}
u_2 = 1.19356041537694\\
v_2 = 0.00214102198115903\\
\varphi_2 = -0.00719205100884593\\
u_3 = 1.19106228897174\\
v_3 = -0.00214102198115903\\
\varphi_3 = -0.00716706974479397
\end{gathered}$$

最终得到的约束反力如下：

$$\begin{gathered}
F_1 x = -5003.74718960785\\
F_1 y = -4282.04396231806\\
M_1 = 286147.614524622\\
F_2 x = -4996.25281039226\\
F_2 y = 4282.04396231806\\
M_2 = 285647.989243583
\end{gathered}$$

通过Ansys验算得到的钢架位移结果如下所示：



6 Matlab程序

```
E=2e7;
l=100;
A=10;
Iz=25;
P=10000;

Ke=[E*A/l,0,0,-E*A/l,0,0;
    0,12*E*Iz/l^3,6*E*Iz/l^2,0,-12*E*Iz/l^3,6*E*Iz/l^2;
    0,6*E*Iz/l^2,4*E*Iz/l,0,-6*E*Iz/l^2,2*E*Iz/l;
    -E*A/l,0,0,E*A/l,0,0;
    0,-12*E*Iz/l^3,-6*E*Iz/l^2,0,12*E*Iz/l^3,-6*E*Iz/l^2;
    0,6*E*Iz/l^2,2*E*Iz/l,0,-6*E*Iz/l^2,4*E*Iz/l];

Ke_local=cat(3,zeros(6),zeros(6),zeros(6));
for k=1:3
    Ke_local(:,:,k)=Ke;
```

```
end

Lambada(:,:,1)=[0,1,0,0,0,0;
                -1,0,0,0,0,0;
                0,0,1,0,0,0;
                0,0,0,0,1,0;
                0,0,0,-1,0,0;
                0,0,0,0,0,1];

Lambada(:,:,2)=[1,0,0,0,0,0;
                0,1,0,0,0,0;
                0,0,1,0,0,0;
                0,0,0,1,0,0;
                0,0,0,0,1,0;
                0,0,0,0,0,1];

Lambada(:,:,3)=[0,-1,0,0,0,0;
                1,0,0,0,0,0;
                0,0,1,0,0,0;
                0,0,0,0,-1,0;
                0,0,0,1,0,0;
                0,0,0,0,0,1];

Ke_global=cat(3,zeros(6),zeros(6),zeros(6));
for k=1:3
   Ke_global(:,:,k)=Lambada(:,:,k)'*Ke_local(:,:,k)*Lambada(:,:,k);
end

K=zeros(12);
for elem=1:3
    for r=1+(elem-1)*3:6+(elem-1)*3
        for c=1+(elem-1)*3:6+(elem-1)*3
            K(r,c)=K(r,c)+Ke_global(r-(elem-1)*3,c-(elem-1)*3,elem);
```

```
        end
    end
end

load=[0,0,0,P,0,0,0,0,0,0,0,0]';
delta=zeros(12,1);

delta(4:9,1:1)=linsolve(K(4:9,4:9),load(4:9,1:1));
load=K*delta;
for t=5:9
    load(t)=0;
end
```