Proof Problems

Problem 1. $\forall L \in \texttt{NatList}, \mathrm{fold}{*}(\mathrm{rev}(L)) = \mathrm{fold}{*}(L)$.

Proof. By structural induction on L

(1) Base case

What to show: $\quad \mathrm{fold}{*}(\mathrm{rev}(nil)) = \mathrm{fold}{*}(nil)$

$$\mathrm{fold}{*}(\underline{\mathrm{rev}(nil)}) \longrightarrow \mathrm{fold}{*}(nil) \qquad \text{(by rev1)}$$

(2) Induction case

What to show: $\quad \mathrm{fold}{*}(\mathrm{rev}(x \mid l)) = \mathrm{fold}{*}(x \mid l)$
Induction hypothesis: $\quad \mathrm{fold}{*}(\mathrm{rev}(l)) = \mathrm{fold}{*}(l)$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$.

$$\begin{aligned}
\mathrm{fold}{*}(\underline{\mathrm{rev}(x \mid l)}) &\longrightarrow \underline{\mathrm{fold}{*}(\mathrm{rev}(l) \mathbin{@} (x \mid nil))} && \text{(by rev2)} \\
&\longrightarrow \underline{\mathrm{fold}{*}(\mathrm{rev}(l))} * \mathrm{fold}{*}(x \mid nil) && \text{(by Lemma 1)} \\
&\longrightarrow \mathrm{fold}{*}(l) * \underline{\mathrm{fold}{*}(x \mid nil)} && \text{(by IH)} \\
&\longrightarrow \mathrm{fold}{*}(l) * (x * \underline{\mathrm{fold}{*}(nil)}) && \text{(by fold*-2)} \\
&\longrightarrow \mathrm{fold}{*}(l) * (\underline{x * \mathrm{s}(0)}) && \text{(by fold*-1)} \\
&\longrightarrow \mathrm{fold}{*}(l) * (\underline{\mathrm{s}(0) * x}) && \text{(by comm*)} \\
&\longrightarrow \mathrm{fold}{*}(l) * (\underline{(0 * x)} + x) && \text{(by *2)} \\
&\longrightarrow \mathrm{fold}{*}(l) * \underline{(0 + x)} && \text{(by *1)} \\
&\longrightarrow \underline{\mathrm{fold}{*}(l) * x} && \text{(by +1)} \\
&\longrightarrow x * \mathrm{fold}{*}(l) && \text{(by comm*)}
\end{aligned}$$

□

Lemma 1. $\forall L1, L2 \in \texttt{NatList}, \mathrm{fold}{*}(L1 \mathbin{@} L2) = \mathrm{fold}{*}(L1) * \mathrm{fold}{*}(L2)$.

Proof. By structural induction on $L1$

(1) Base case

What to show: $\text{fold}*(nil \mathbin{@} l2) = \text{fold}*(nil) * \text{fold}*(l2)$
where $l2 \in \texttt{NatList}$.

$$\begin{aligned}
\text{fold}*(\underline{nil \mathbin{@} l2}) &\longrightarrow \text{fold}*(l2) && \text{(by @1)}\\
\underline{\text{fold}*(nil)} * \text{fold}*(l2) &\longrightarrow \underline{\text{s}(0) * \text{fold}*(l2)} && \text{(by fold*-1)}\\
&\longrightarrow \underline{(0 * \text{fold}*(l2))} + \text{fold}*(l2) && \text{(by *2)}\\
&\longrightarrow \underline{0 + \text{fold}*(l2)} && \text{(by *1)}\\
&\longrightarrow \text{fold}*(l2) && \text{(by +1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{fold}*((e \mid l1) \mathbin{@} l2) = \text{fold}*(e \mid l1) * \text{fold}*(l2)$
Induction hypothesis: $\text{fold}*(l1 \mathbin{@} l2) = \text{fold}*(l1) * \text{fold}*(l2)$
where $e \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$.

$$\begin{aligned}
\text{fold}*(\underline{(e \mid l1) \mathbin{@} l2}) &\longrightarrow \underline{\text{fold}*(e \mid (l1 \mathbin{@} l2))} && \text{(by @2)}\\
&\longrightarrow e * \underline{\text{fold}*(l1 \mathbin{@} l2)} && \text{(by fold*-2)}\\
&\longrightarrow e * (\text{fold}*(l1) * \text{fold}*(l2)) && \text{(by IH)}\\
\underline{\text{fold}*(e \mid l1)} * \text{fold}*(l2) &\longrightarrow \underline{(e * \text{fold}*(l1)) * \text{fold}*(l2)} && \text{(by fold*-2)}\\
&\longrightarrow e * (\text{fold}*(l1) * \text{fold}*(l2)) && \text{(by assoc*)}
\end{aligned}$$

□