Problem 7

Problem 1. $\forall L \in \texttt{NatList}, \mathrm{diff}(L, \mathrm{rev}(L)) = nil$.

Proof. By structural induction on L.

(1) Base case

What to show: $\mathrm{diff}(nil, \mathrm{rev}(nil)) = nil$.

$$\underline{\mathrm{diff}(nil, \mathrm{rev}(nil))} \longrightarrow nil \qquad \text{(by diff1)}$$

(2) Induction case

What to show: $\mathrm{diff}(x \mid l, \mathrm{rev}(x \mid l)) = nil$

Induction hypothesis: $\mathrm{diff}(l, \mathrm{rev}(l)) = nil$

where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants[1].

$$\begin{aligned}
\mathrm{diff}(x \mid l, \underline{\mathrm{rev}(x \mid l)}) &\longrightarrow \underline{\mathrm{diff}(x \mid l, \mathrm{rev}(l) \mathbin{@} (x \mid nil))} && \text{(by rev2)}\\
&\longrightarrow \text{if } \underline{\mathrm{has}(\mathrm{rev}(l) \mathbin{@} (x \mid nil), x)} \text{ then}\\
&\qquad \mathrm{diff}(l, \mathrm{rev}(l) \mathbin{@} (x \mid nil))\\
&\qquad \text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \mathbin{@} (x \mid nil)) \text{ fi} && \text{(by diff2)}\\
&\longrightarrow \text{if } \mathrm{has}(\mathrm{rev}(l), x) \text{ or } \underline{\mathrm{has}(x \mid nil, x)} \text{ then}\\
&\qquad \mathrm{diff}(l, \mathrm{rev}(l) \mathbin{@} (x \mid nil))\\
&\qquad \text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \mathbin{@} (x \mid nil)) \text{ fi}\\
&\qquad\qquad \text{(by Lemma 1 from Problem 6)}\\
&\longrightarrow \text{if } \mathrm{has}(\mathrm{rev}(l), x) \text{ or } ((x = x) \text{ or } \underline{\mathrm{has}(nil, x)}) \text{ then}\\
&\qquad \mathrm{diff}(l, \mathrm{rev}(l) \mathbin{@} (x \mid nil))\\
&\qquad \text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \mathbin{@} (x \mid nil)) \text{ fi} && \text{(by has2)}\\
&\longrightarrow \text{if } \mathrm{has}(\mathrm{rev}(l), x) \text{ or } \underline{(true \text{ or } false)} \text{ then}\\
&\qquad \mathrm{diff}(l, \mathrm{rev}(l) \mathbin{@} (x \mid nil))\\
&\qquad \text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \mathbin{@} (x \mid nil)) \text{ fi} && \text{(by has1)}
\end{aligned}$$

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

$$
\begin{aligned}
&\longrightarrow \text{if } \underline{\text{has}(\text{rev}(l), x) \text{ or } \mathit{true}} \text{ then} \\
&\quad \text{diff}(l, \text{rev}(l) \mathbin{@} (x \mid \mathit{nil})) \\
&\quad \text{else } x \mid \text{diff}(l, \text{rev}(l) \mathbin{@} (x \mid \mathit{nil})) \text{ fi} && \text{(by or)} \\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then diff}(l, \text{rev}(l) \mathbin{@} (x \mid \mathit{nil}))} \\
&\quad \underline{\text{else } x \mid \text{diff}(l, \text{rev}(l) \mathbin{@} (x \mid \mathit{nil})) \text{ fi}} && \text{(by or)} \\
&\longrightarrow \underline{\text{diff}(l, \text{rev}(l) \mathbin{@} (x \mid \mathit{nil}))} && \text{(by if1)} \\
&\longrightarrow \text{diff}(\underline{\text{diff}(l, \text{rev}(l))}, x \mid \mathit{nil}) && \text{(by Lemma 1)} \\
&\longrightarrow \underline{\text{diff}(\mathit{nil}, x \mid \mathit{nil})} && \text{(by IH)} \\
&\longrightarrow \mathit{nil} && \text{(by diff1)}
\end{aligned}
$$

□

Lemma 1. $\forall L1, L2, L3 \in \texttt{NatList}, \text{diff}(L1, L2 \mathbin{@} L3) = \text{diff}(\text{diff}(L1, L2), L3)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{diff}(\mathit{nil}, l2 \mathbin{@} l3) = \text{diff}(\text{diff}(\mathit{nil}, l2), l3)$
where $l2, l3 \in \texttt{NatList}$. Note that $l2, l3$ are fresh constants.

$$
\begin{aligned}
\underline{\text{diff}(\mathit{nil}, l2 \mathbin{@} l3)} &\longrightarrow \mathit{nil} && \text{(by diff1)} \\
\text{diff}(\underline{\text{diff}(\mathit{nil}, l2)}, l3) &\longrightarrow \underline{\text{diff}(\mathit{nil}, l3)} && \text{(by diff1)} \\
&\longrightarrow \mathit{nil} && \text{(by diff1)}
\end{aligned}
$$

(2) Induction case

What to show: $\text{diff}(x \mid l1, l2 \mathbin{@} l3) = \text{diff}(\text{diff}(x \mid l1, l2), l3)$
Induction hypothesis: $\text{diff}(l1, l2 \mathbin{@} l3) = \text{diff}(\text{diff}(l1, l2), l3)$
where $x \in \texttt{PNat}$, and $l1, l2, l3 \in \texttt{NatList}$. Note that $x, l1, l2, l3$ are fresh constants.

We use case splitting for our proofs as follows:
Case 1: $\text{has}(l2, x) = \mathit{true}$

$$
\begin{aligned}
\underline{\text{diff}(x \mid l1, l2 \mathbin{@} l3)} &\longrightarrow \text{if } \underline{\text{has}(l2 \mathbin{@} l3, x)} \text{ then diff}(l1, l2 \mathbin{@} l3) \\
&\quad \text{else } (x \mid \text{diff}(l1, l2 \mathbin{@} l3))) \text{ fi} && \text{(by)} \\
&\longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ or has}(l3, x) \text{ then diff}(l1, l2 \mathbin{@} l3) \\
&\quad \text{else } (x \mid \text{diff}(l1, l2 \mathbin{@} l3))) \text{ fi} \\
&\qquad\qquad \text{(by Lemma 1 from Problem 6)}
\end{aligned}
$$

$$\longrightarrow \text{if } \underline{true \text{ or } \text{has}(l3, x)} \text{ then } \text{diff}(l1, l2 \mathbin{@} l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 \mathbin{@} l3))) \text{ fi}$$
(by case splitting)

$$\longrightarrow \underline{\text{if } true \text{ then } \text{diff}(l1, l2 \mathbin{@} l3)}$$
$$\underline{\text{else } (x \mid \text{diff}(l1, l2 \mathbin{@} l3))) \text{ fi}} \quad \text{(by or)}$$

$$\longrightarrow \underline{\text{diff}(l1, l2 \mathbin{@} l3)} \quad \text{(by if1)}$$

$$\longrightarrow \text{diff}(\text{diff}(l1, l2), l3) \quad \text{(by IH)}$$

$$\text{diff}(\underline{\text{diff}(x \mid l1, l2)}, l3) \longrightarrow \text{diff}(\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}, l3)$$
(by diff2)

$$\longrightarrow \text{diff}(\underline{\text{if } true \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}}, l3)$$
(by case splitting)

$$\longrightarrow \text{diff}(\text{diff}(l1, l2), l3) \quad \text{(by if1)}$$

Case 2: $\text{has}(l2, x) = false$

$$\underline{\text{diff}(x \mid l1, l2 \mathbin{@} l3)} \longrightarrow \text{if } \underline{\text{has}(l2 \mathbin{@} l3, x)} \text{ then } \text{diff}(l1, l2 \mathbin{@} l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 \mathbin{@} l3)) \text{ fi} \quad \text{(by)}$$

$$\longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ or } \text{has}(l3, x) \text{ then } \text{diff}(l1, l2 \mathbin{@} l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 \mathbin{@} l3)) \text{ fi}$$
(by Lemma 1 from Problem 6)

$$\longrightarrow \text{if } \underline{false \text{ or } \text{has}(l3, x)} \text{ then } \text{diff}(l1, l2 \mathbin{@} l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 \mathbin{@} l3)) \text{ fi}$$
(by case splitting)

$$\longrightarrow \text{if } \text{has}(l3, x) \text{ then } \underline{\text{diff}(l1, l2 \mathbin{@} l3)}$$
$$\text{else } (x \mid \underline{\text{diff}(l1, l2 \mathbin{@} l3)}) \text{ fi} \quad \text{(by or)}$$

$$\longrightarrow \text{if } \text{has}(l3, x) \text{ then } \text{diff}(\text{diff}(l1, l2), l3)$$
$$\text{else } (x \mid \text{diff}(\text{diff}(l1, l2), l3))) \text{ fi} \quad \text{(by IH)}$$

$$\text{diff}(\underline{\text{diff}(x \mid l1, l2)}, l3) \longrightarrow \text{diff}(\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}, l3)$$
(by diff2)

$$\longrightarrow \text{diff}(\underline{\text{if } false \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}}, l3)$$
(by case splitting)

$$\longrightarrow \underline{\text{diff}(x \mid \text{diff}(l1, l2), l3)} \quad \text{(by if2)}$$

$$\longrightarrow \text{if } \text{has}(l3, x) \text{ then } \text{diff}(\text{diff}(l1, l2), l3)$$
$$\text{else } (x \mid \text{diff}(\text{diff}(l1, l2), l3))) \text{ fi} \quad \text{(by diff2)}$$

□