Problem 6

Problem 1. $\forall X \in \texttt{PNat}, \forall L \in \texttt{NatList}, \text{has}(L, X) = \text{has}(\text{rev}(L), X)$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{has}(nil, x) = \text{has}(\text{rev}(nil), x)$
where $x \in \texttt{PNat}$. Note that x is a fresh constant[1].

$$
\begin{aligned}
\underline{\text{has}(nil, x)} &\longrightarrow false && \text{(by has1)} \\
\text{has}(\underline{\text{rev}(nil)}, x) &\longrightarrow \underline{\text{has}(nil, x)} && \text{(by rev1)} \\
&\longrightarrow false && \text{(by has1)}
\end{aligned}
$$

(2) Induction case

What to show: $\text{has}(y \mid l, x) = \text{has}(\text{rev}(y \mid l), x)$
Induction hypothesis: $\text{has}(l, x) = \text{has}(\text{rev}(l), x)$
where $x, y \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, y, l are fresh constants.

$$
\begin{aligned}
\underline{\text{has}(y \mid l, x)} &\longrightarrow (x = y) \text{ or } \underline{\text{has}(l, x)} && \text{(by has2)} \\
&\longrightarrow (x = y) \text{ or } \text{has}(\text{rev}(l), x) && \text{(by IH)} \\
\text{has}(\underline{\text{rev}(y \mid l)}, x) &\longrightarrow \underline{\text{has}(\text{rev}(l) \mathbin{@} (y \mid nil)), x)} && \text{(by)} \\
&\longrightarrow \text{has}(\text{rev}(l), x) \text{ or } \underline{\text{has}(y \mid nil, x)} && \text{(by Lemma 1)} \\
&\longrightarrow \text{has}(\text{rev}(l), x) \text{ or } ((x = y) \text{ or } \underline{\text{has}(nil, x)}) && \\
& && \text{(by has2)} \\
&\longrightarrow \text{has}(\text{rev}(l), x) \text{ or } \underline{((x = y) \text{ or } false)} && \text{(by has1)} \\
&\longrightarrow \underline{\text{has}(\text{rev}(l), x) \text{ or } (x = y)} && \text{(by or)} \\
&\longrightarrow (x = y) \text{ or } \text{has}(\text{rev}(l), x) && \text{(by comm-or)}
\end{aligned}
$$

∎

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

Lemma 1. $\forall X \in \texttt{PNat}, \forall L1, L2 \in \texttt{NatList}, \text{has}(L1 \mathbin{@} L2, X) = \text{has}(L1, X)$ or $\text{has}(L2, X)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{has}(nil \mathbin{@} l2, x) = (\text{has}(nil, x) \text{ or } \text{has}(l2, x))$
where $x \in \texttt{PNat}$ and $l2 \in \texttt{NatList}$. Note that $x, l2$ are fresh constants.

$$\begin{aligned}
\text{has}(\underline{nil \mathbin{@} l2}, x) &\longrightarrow \text{has}(l2, x) && \text{(by @1)} \\
\underline{\text{has}(nil, x)} \text{ or } \text{has}(l2, x) &\longrightarrow \underline{false \text{ or } \text{has}(l2, x)} && \text{(by has1)} \\
&\longrightarrow \text{has}(l2, x) && \text{(by or)}
\end{aligned}$$

(2) Induction case

What to show: $\text{has}((y \mid l1) \mathbin{@} l2, x) = (\text{has}(y \mid l1, x) \text{ or } \text{has}(l2, x))$
Induction hypothesis: $\text{has}(l1 \mathbin{@} l2, x) = (\text{has}(l1, x) \text{ or } \text{has}(l2, x))$
where $x, y \in \texttt{PNat}$, and $l1, l2 \in \texttt{NatList}$. Note that $x, y, l1, l2$ are fresh constants.

$$\begin{aligned}
\text{has}(\underline{(y \mid l1) \mathbin{@} l2}, x) &\longrightarrow \underline{\text{has}(y \mid (l1 \mathbin{@} l2), x)} && \text{(by @2)} \\
&\longrightarrow (x = y) \text{ or } \underline{\text{has}(l1 \mathbin{@} l2, x)} && \text{(by has2)} \\
&\longrightarrow (x = y) \text{ or } (\text{has}(l1, x) \text{ or } \text{has}(l2, x)) && \text{(by IH)} \\
\underline{\text{has}(y \mid l1, x)} \text{ or } \text{has}(l2, x) &\longrightarrow \underline{((x = y) \text{ or } \text{has}(l1, x)) \text{ or } \text{has}(l2, x)} && \text{(by has2)} \\
&\longrightarrow (x = y) \text{ or } (\text{has}(l1, x) \text{ or } \text{has}(l2, x)) && \text{(by assoc-or)}
\end{aligned}$$

$\square$