

## Problem 7

**Problem 1.**  $\forall L \in \text{NatList}, \text{diff}(L, \text{rev}(L)) = \text{nil}$ .

*Proof.* By structural induction on  $L$ .

**(1) Base case**

What to show:  $\text{diff}(\text{nil}, \text{rev}(\text{nil})) = \text{nil}$ .

$$\underline{\text{diff}(\text{nil}, \text{rev}(\text{nil}))} \longrightarrow \text{nil} \quad (\text{by diff1})$$

**(2) Induction case**

What to show:  $\text{diff}(x \mid l, \text{rev}(x \mid l)) = \text{nil}$

Induction hypothesis:  $\text{diff}(l, \text{rev}(l)) = \text{nil}$

where  $x \in \text{PNat}$  and  $l \in \text{NatList}$ .

$$\begin{aligned}
\text{diff}(x \mid l, \text{rev}(x \mid l)) &\longrightarrow \text{diff}(x \mid l, \text{rev}(l) @ (x \mid \text{nil})) && \text{(by rev2)} \\
&\longrightarrow \text{if } \text{has}(\text{rev}(l) @ (x \mid \text{nil}), x) \text{ then} \\
&\quad \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \\
&\quad \text{else } x \mid \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \text{ fi} && \text{(by diff2)} \\
&\longrightarrow \text{if } \text{has}(\text{rev}(l), x) \text{ or } \text{has}(x \mid \text{nil}, x) \text{ then} \\
&\quad \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \\
&\quad \text{else } x \mid \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \text{ fi} \\
&\quad \text{(by Lemma 1 from Problem 6)} \\
&\longrightarrow \text{if } \text{has}(\text{rev}(l), x) \text{ or } ((x = x) \text{ or } \text{has}(\text{nil}, x)) \text{ then} \\
&\quad \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \\
&\quad \text{else } x \mid \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \text{ fi} && \text{(by has2)} \\
&\longrightarrow \text{if } \text{has}(\text{rev}(l), x) \text{ or } (\text{true or false}) \text{ then} \\
&\quad \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \\
&\quad \text{else } x \mid \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \text{ fi} && \text{(by has1)} \\
&\longrightarrow \text{if } \text{has}(\text{rev}(l), x) \text{ or } \text{true} \text{ then} \\
&\quad \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \\
&\quad \text{else } x \mid \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \text{ fi} && \text{(by or)} \\
&\longrightarrow \text{if } \text{true} \text{ then } \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \\
&\quad \text{else } x \mid \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) \text{ fi} && \text{(by or)} \\
&\longrightarrow \text{diff}(l, \text{rev}(l) @ (x \mid \text{nil})) && \text{(by if1)} \\
&\longrightarrow \text{diff}(\text{diff}(l, \text{rev}(l)), x \mid \text{nil}) && \text{(by Lemma 1)} \\
&\longrightarrow \text{diff}(\text{nil}, x \mid \text{nil}) && \text{(by IH)} \\
&\longrightarrow \text{nil} && \text{(by diff1)}
\end{aligned}$$

□

**Lemma 1.**  $\forall L1, L2, L3 \in \text{NatList}, \text{diff}(L1, L2 @ L3) = \text{diff}(\text{diff}(L1, L2), L3)$ .

*Proof.* By structural induction on  $L1$ .

**(1) Base case**

What to show:  $\text{diff}(\text{nil}, l2 @ l3) = \text{diff}(\text{diff}(\text{nil}, l2), l3)$

where  $l2, l3 \in \text{NatList}$ .

$$\begin{aligned}
& \frac{\text{diff}(\text{nil}, l2 @ l3)}{\text{diff}(\text{diff}(\text{nil}, l2), l3)} \longrightarrow \text{nil} && \text{(by diff1)} \\
& \longrightarrow \text{diff}(\text{nil}, l3) && \text{(by diff1)} \\
& \longrightarrow \text{nil} && \text{(by diff1)}
\end{aligned}$$

## (2) Induction case

What to show:  $\text{diff}(x \mid l1, l2 @ l3) = \text{diff}(\text{diff}(x \mid l1, l2), l3)$

Induction hypothesis:  $\text{diff}(l1, l2 @ l3) = \text{diff}(\text{diff}(l1, l2), l3)$

where  $x \in \text{PNat}$ , and  $l1, l2, l3 \in \text{NatList}$ .

We use case splitting for our proofs as follows:

**Case 1:**  $\text{has}(l2, x) = \text{true}$

$$\begin{aligned}
& \frac{\text{diff}(x \mid l1, l2 @ l3)}{\text{diff}(\text{diff}(x \mid l1, l2), l3)} \longrightarrow \text{if } \underline{\text{has}(l2 @ l3, x)} \text{ then } \text{diff}(l1, l2 @ l3) \\
& \quad \text{else } (x \mid \text{diff}(l1, l2 @ l3)) \text{ fi} && \text{(by)} \\
& \longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ or } \text{has}(l3, x) \text{ then } \text{diff}(l1, l2 @ l3) \\
& \quad \text{else } (x \mid \text{diff}(l1, l2 @ l3)) \text{ fi} \\
& \quad \text{(by Lemma 1 from Problem 6)} \\
& \longrightarrow \text{if } \underline{\text{true}} \text{ or } \text{has}(l3, x) \text{ then } \text{diff}(l1, l2 @ l3) \\
& \quad \text{else } (x \mid \text{diff}(l1, l2 @ l3)) \text{ fi} \\
& \quad \text{(by case splitting)} \\
& \longrightarrow \text{if } \underline{\text{true}} \text{ then } \text{diff}(l1, l2 @ l3) \\
& \quad \text{else } (x \mid \text{diff}(l1, l2 @ l3)) \text{ fi} && \text{(by or)} \\
& \longrightarrow \underline{\text{diff}(l1, l2 @ l3)} && \text{(by if1)} \\
& \longrightarrow \text{diff}(\text{diff}(l1, l2), l3) && \text{(by IH)} \\
& \text{diff}(\underline{\text{diff}(x \mid l1, l2)}, l3) \longrightarrow \text{diff}(\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}, l3) \\
& \quad \text{(by diff2)} \\
& \longrightarrow \text{diff}(\underline{\text{if } \text{true} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}}, l3) \\
& \quad \text{(by case splitting)} \\
& \longrightarrow \text{diff}(\text{diff}(l1, l2), l3) && \text{(by if1)}
\end{aligned}$$

**Case 2:**  $\text{has}(l2, x) = \text{false}$

$$\begin{aligned}
& \underline{\text{diff}(x \mid l1, l2 @ l3)} \longrightarrow \text{if } \underline{\text{has}(l2 @ l3, x)} \text{ then } \text{diff}(l1, l2 @ l3) \\
& \quad \text{else } (x \mid \text{diff}(l1, l2 @ l3)) \text{ fi} \quad \text{(by)} \\
& \longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ or } \text{has}(l3, x) \text{ then } \text{diff}(l1, l2 @ l3) \\
& \quad \text{else } (x \mid \text{diff}(l1, l2 @ l3)) \text{ fi} \\
& \quad \text{(by Lemma 1 from Problem 6)} \\
& \longrightarrow \text{if } \underline{\text{false}} \text{ or } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1, l2 @ l3) \\
& \quad \text{else } (x \mid \text{diff}(l1, l2 @ l3)) \text{ fi} \\
& \quad \text{(by case splitting)} \\
& \longrightarrow \text{if } \text{has}(l3, x) \text{ then } \underline{\text{diff}(l1, l2 @ l3)} \\
& \quad \text{else } (x \mid \underline{\text{diff}(l1, l2 @ l3)}) \text{ fi} \quad \text{(by or)} \\
& \longrightarrow \text{if } \text{has}(l3, x) \text{ then } \text{diff}(\text{diff}(l1, l2), l3) \\
& \quad \text{else } (x \mid \text{diff}(\text{diff}(l1, l2), l3)) \text{ fi} \quad \text{(by IH)} \\
& \text{diff}(\underline{\text{diff}(x \mid l1, l2)}, l3) \longrightarrow \text{diff}(\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}, l3) \\
& \quad \text{(by diff2)} \\
& \longrightarrow \text{diff}(\underline{\text{if } \text{false} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}}, l3) \\
& \quad \text{(by case splitting)} \\
& \longrightarrow \underline{\text{diff}(x \mid \text{diff}(l1, l2), l3)} \quad \text{(by if2)} \\
& \longrightarrow \text{if } \text{has}(l3, x) \text{ then } \text{diff}(\text{diff}(l1, l2), l3) \\
& \quad \text{else } (x \mid \text{diff}(\text{diff}(l1, l2), l3)) \text{ fi} \quad \text{(by diff2)}
\end{aligned}$$

□