

Problem 12

Problem 1. $\forall L \in \text{NatList}, \text{sum}(L) = \text{sum}(\text{rev}(L)).$

Proof. By structural induction on L .

(1) Base case

What to show: $\text{sum}(\text{nil}) = \text{sum}(\text{rev}(\text{nil})).$

$$\begin{array}{ll} \underline{\text{sum}(\text{nil})} \longrightarrow 0 & \text{(by sum1)} \\ \text{sum}(\underline{\text{rev}(\text{nil})}) \longrightarrow \underline{\text{sum}(\text{nil})} & \text{(by rev1)} \\ \longrightarrow 0 & \text{(by sum1)} \end{array}$$

(2) Induction case

What to show: $\text{sum}(x \mid l) = \text{sum}(\text{rev}(x \mid l))$

Induction hypothesis: $\text{sum}(l) = \text{sum}(\text{rev}(l))$

where $x \in \text{PNat}$ and $l \in \text{NatList}$.

$$\begin{array}{ll} \underline{\text{sum}(x \mid l)} \longrightarrow x + \text{sum}(l) & \text{(by sum2)} \\ \longrightarrow \underline{x + \text{sum}(\text{rev}(l))} & \text{(by IH)} \\ \longrightarrow \text{sum}(\text{rev}(l)) + x & \text{(by comm+)} \\ \text{sum}(\underline{\text{rev}(x \mid l)}) \longrightarrow \underline{\text{sum}(\text{rev}(l) @ (x \mid \text{nil}))} & \text{(by rev2)} \\ \longrightarrow \text{sum}(\text{rev}(l)) + \underline{\text{sum}(x \mid \text{nil})} & \text{(by Lemma 1)} \\ \longrightarrow \text{sum}(\text{rev}(l)) + (x + \underline{\text{sum}(\text{nil})}) & \text{(by sum2)} \\ \longrightarrow \text{sum}(\text{rev}(l)) + (x + \underline{0}) & \text{(by sum1)} \\ \longrightarrow \text{sum}(\text{rev}(l)) + (\underline{0} + x) & \text{(by comm+)} \\ \longrightarrow \text{sum}(\text{rev}(l)) + x & \text{(by +1)} \end{array}$$

□

Lemma 1. $\forall L1, L2 \in \text{NatList}, \text{sum}(L1 @ L2) = \text{sum}(L1) + \text{sum}(L2).$

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{sum}(\text{nil} @ l2) = \text{sum}(\text{nil}) + \text{sum}(l2)$
where $l2 \in \text{NatList}$.

$$\begin{aligned} \text{sum}(\text{nil} @ l2) &\longrightarrow \text{sum}(l2) && \text{(by @1)} \\ \underline{\text{sum}(\text{nil})} + \text{sum}(l2) &\longrightarrow \underline{0 + \text{sum}(l2)} && \text{(by sum1)} \\ &\longrightarrow 0 && \text{(by +1)} \end{aligned}$$

(2) Induction case

What to show: $\text{sum}((x \mid l1) @ l2) = \text{sum}(x \mid l1) + \text{sum}(l2)$
Induction hypothesis: $\text{sum}(l1 @ l2) = \text{sum}(l1) + \text{sum}(l2)$
where $x \in \text{PNat}$ and $l1, l2 \in \text{NatList}$.

$$\begin{aligned} \text{sum}(\underline{(x \mid l)}) @ l2 &\longrightarrow \underline{\text{sum}(x \mid (l1 @ l2))} && \text{(by @2)} \\ &\longrightarrow x + \underline{\text{sum}(l1 @ l2)} && \text{(by sum2)} \\ &\longrightarrow x + (\underline{\text{sum}(l1) + \text{sum}(l2)}) && \text{(by IH)} \\ &\longrightarrow (x + \text{sum}(l1)) + \text{sum}(l2) && \text{(by assoc+)} \\ \underline{\text{sum}(x \mid l1)} + \text{sum}(l2) &\longrightarrow (x + \text{sum}(l1)) + \text{sum}(l2) && \text{(by sum2)} \end{aligned}$$

□