Problem 19

Problem 1. $\forall L \in \texttt{NatList}, \text{setEqual}(L, \text{rmDup}(L)) = \mathit{true}.$

Proof. By structural induction on L.

(1) Base case

What to show: $\text{setEqual}(\mathit{nil}, \text{rmDup}(\mathit{nil})) = \mathit{true}.$

$$\begin{aligned}
\text{setEqual}(\mathit{nil}, \underline{\text{rmDup}(\mathit{nil})}) &\longrightarrow \underline{\text{setEqual}(\mathit{nil}, \mathit{nil})} && \text{(by rmDup1)}\\
&\longrightarrow (\underline{\text{diff}(\mathit{nil}, \mathit{nil})} = \mathit{nil}) \mathbin{\mathit{and}} (\text{diff}(\mathit{nil}, \mathit{nil}) = \mathit{nil}) && \text{(by setEq)}\\
&\longrightarrow \underline{(\mathit{nil} = \mathit{nil})} \mathbin{\mathit{and}} (\text{diff}(\mathit{nil}, \mathit{nil}) = \mathit{nil}) && \text{(by diff1)}\\
&\longrightarrow \underline{\mathit{true} \mathbin{\mathit{and}} (\text{diff}(\mathit{nil}, \mathit{nil}) = \mathit{nil})} && \text{(by equality)}\\
&\longrightarrow \underline{\text{diff}(\mathit{nil}, \mathit{nil})} = \mathit{nil} && \text{(by and)}\\
&\longrightarrow \underline{\mathit{nil} = \mathit{nil}} && \text{(by diff1)}\\
&\longrightarrow \mathit{true} && \text{(by equality)}
\end{aligned}$$

(2) Induction case

What to show: $\text{setEqual}(x \mid l, \text{rmDup}(x \mid l)) = \mathit{true}$

Induction hypothesis: $\text{setEqual}(l, \text{rmDup}(l)) = \mathit{true}$

where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants[1].

We use case splitting for our proofs as follows:

Case 1: $\text{has}(l, x) = \mathit{true}.$

$$\begin{aligned}
\text{setEqual}(x \mid l, \underline{\text{rmDup}(x \mid l)}) &\longrightarrow \text{setEqual}(x \mid l,\\
&\qquad \text{if } \underline{\text{has}(l, x)} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}) && \text{(by rmDup2)}\\
&\longrightarrow \text{setEqual}(x \mid l,
\end{aligned}$$

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

$$
\begin{aligned}
&\underline{\text{if } \mathit{true} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}}) \\
&\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)} \\
\longrightarrow\ & \underline{\text{setEqual}(x \mid l, \text{rmDup}(l))} \quad \text{(by if1)} \\
\longrightarrow\ & (\underline{\text{diff}(x \mid l, \text{rmDup}(l))} = \mathit{nil}) \ \mathit{and} \\
& (\text{diff}(\text{rmDup}(l), x \mid l) = \mathit{nil}) \\
&\qquad\qquad\qquad\qquad\qquad \text{(by setEq)} \\
\longrightarrow\ & ((\text{if } \underline{\text{has}(\text{rmDup}(l), x)} \text{ then diff}(l, \text{rmDup}(l)) \\
& \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}) = \mathit{nil}) \ \mathit{and} \\
& (\text{diff}(\text{rmDup}(l), x \mid l) = \mathit{nil}) \\
&\qquad\qquad\qquad\qquad\qquad \text{(by diff2)} \\
\longrightarrow\ & ((\text{if } \underline{\text{has}(l, x)} \text{ then diff}(l, \text{rmDup}(l)) \\
& \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}) = \mathit{nil}) \ \mathit{and} \\
& (\text{diff}(\text{rmDup}(l), x \mid l) = \mathit{nil}) \\
&\qquad\qquad\qquad\qquad\qquad \text{(by Problem 17 - Lemma 1)} \\
\longrightarrow\ & \underline{((\text{if } \mathit{true} \text{ then diff}(l, \text{rmDup}(l))} \\
& \underline{\text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}) = \mathit{nil})} \ \mathit{and} \\
& (\text{diff}(\text{rmDup}(l), x \mid l) = \mathit{nil}) \\
&\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)} \\
\longrightarrow\ & (\underline{\text{diff}(l, \text{rmDup}(l)) = \mathit{nil}}) \ \mathit{and} \\
& (\text{diff}(\text{rmDup}(l), x \mid l) = \mathit{nil}) \quad \text{(by if1)} \\
\longrightarrow\ & \underline{(\mathit{nil} = \mathit{nil})} \ \mathit{and} \ (\text{diff}(\text{rmDup}(l), x \mid l) = \mathit{nil}) \\
&\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 1)} \\
\longrightarrow\ & \underline{\mathit{true} \ \mathit{and} \ (\text{diff}(\text{rmDup}(l), x \mid l) = \mathit{nil})} \\
&\qquad\qquad\qquad\qquad\qquad \text{(by equality)} \\
\longrightarrow\ & \underline{\text{diff}(\text{rmDup}(l), x \mid l)} = \mathit{nil} \quad \text{(by and)} \\
\longrightarrow\ & \text{drop}(\underline{\text{diff}(\text{rmDup}(l), l)}, x) = \mathit{nil} \\
&\qquad\qquad\qquad\qquad\qquad \text{(by Problem 14)} \\
\longrightarrow\ & \underline{\text{drop}(\mathit{nil}, x)} = \mathit{nil} \quad \text{(by Lemma 1)} \\
\longrightarrow\ & \underline{\mathit{nil} = \mathit{nil}} \quad \text{(by drop1)} \\
\longrightarrow\ & \mathit{true} \quad \text{(by equality)}
\end{aligned}
$$

Case 2: $\text{has}(l, x) = \mathit{false}$.

$$\text{setEqual}(x \mid l, \underline{\text{rmDup}(x \mid l)}) \longrightarrow \text{setEqual}(x \mid l,$$

$$\text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi})$$

(by rmDup2)

$$\longrightarrow \text{setEqual}(x \mid l, \underline{\text{if } false \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}})$$

(by case splitting)

$$\longrightarrow \underline{\text{setEqual}(x \mid l, x \mid \text{rmDup}(l))}$$

(by if2)

$$\longrightarrow (\underline{\text{diff}(x \mid l, x \mid \text{rmDup}(l))} = nil) \ and \ (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = nil)$$

(by setEq)

$$\longrightarrow ((\text{if } \underline{\text{has}(x \mid \text{rmDup}(l), x)} \text{ then diff}(l, x \mid \text{rmDup}(l)) \text{ else } (x \mid \text{diff}(l, x \mid \text{rmDup}(l))) \text{ fi}) = nil) \ and \ (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = nil)$$

(by diff2)

$$\longrightarrow ((\text{if } \underline{(x = x)} \ or \ \text{has}(\text{rmDup}(l), x) \text{ then diff}(l, x \mid \text{rmDup}(l)) \text{ else } (x \mid \text{diff}(l, x \mid \text{rmDup}(l))) \text{ fi}) = nil) \ and \ (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = nil)$$

(by has2)

$$\longrightarrow ((\text{if } \underline{true \ or \ \text{has}(\text{rmDup}(l), x)} \text{ then diff}(l, x \mid \text{rmDup}(l)) \text{ else } (x \mid \text{diff}(l, x \mid \text{rmDup}(l))) \text{ fi}) = nil) \ and \ (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = nil)$$

(by equality)

$$\longrightarrow (\underline{(\text{if } true \text{ then diff}(l, x \mid \text{rmDup}(l)) \text{ else } (x \mid \text{diff}(l, x \mid \text{rmDup}(l))) \text{ fi})} = nil) \ and \ (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = nil)$$

(by or)

$$\longrightarrow (\underline{\text{diff}(l, x \mid \text{rmDup}(l))} = nil) \ and \ (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = nil)$$

(by if1)

$$\longrightarrow (\text{drop}(\underline{\text{diff}(l, \text{rmDup}(l))}, x) = nil) \ and \ (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = nil)$$

(by Problem 14)

$$
\begin{aligned}
&\longrightarrow (\underline{\mathrm{drop}(nil, x)} = nil) \ and \\
&\quad (\mathrm{diff}(x \mid \mathrm{rmDup}(l), x \mid l) = nil) \\
&\hfill \text{(by Lemma 1)} \\
&\longrightarrow \underline{(nil = nil)} \ and \\
&\quad (\mathrm{diff}(x \mid \mathrm{rmDup}(l), x \mid l) = nil) \\
&\hfill \text{(by drop1)} \\
&\longrightarrow true \ and \\
&\quad \underline{(\mathrm{diff}(x \mid \mathrm{rmDup}(l), x \mid l)} = nil) \\
&\hfill \text{(by equality)} \\
&\longrightarrow \underline{\mathrm{diff}(x \mid \mathrm{rmDup}(l), x \mid l) = nil} \\
&\hfill \text{(by and)} \\
&\longrightarrow (\text{if } \underline{\mathrm{has}(x \mid l, x)} \text{ then } \mathrm{diff}(\mathrm{rmDup}(l), x \mid l) \\
&\quad \text{else } (x \mid \mathrm{diff}(\mathrm{rmDup}(l), x \mid l)) \text{ fi}) = nil \\
&\hfill \text{(by diff2)} \\
&\longrightarrow (\text{if } \underline{(x = x)} \ or \ \mathrm{has}(l, x) \text{ then } \mathrm{diff}(\mathrm{rmDup}(l), x \mid l) \\
&\quad \text{else } (x \mid \mathrm{diff}(\mathrm{rmDup}(l), x \mid l)) \text{ fi}) = nil \\
&\hfill \text{(by has2)} \\
&\longrightarrow (\text{if } \underline{true \ or \ \mathrm{has}(l, x)} \text{ then } \mathrm{diff}(\mathrm{rmDup}(l), x \mid l) \\
&\quad \text{else } (x \mid \mathrm{diff}(\mathrm{rmDup}(l), x \mid l)) \text{ fi}) = nil \\
&\hfill \text{(by equality)} \\
&\longrightarrow \underline{(\text{if } true \text{ then } \mathrm{diff}(\mathrm{rmDup}(l), x \mid l)} \\
&\quad \underline{\text{else } (x \mid \mathrm{diff}(\mathrm{rmDup}(l), x \mid l)) \text{ fi})} = nil \\
&\hfill \text{(by or)} \\
&\longrightarrow \underline{\mathrm{diff}(\mathrm{rmDup}(l), x \mid l)} = nil \quad \text{(by if1)} \\
&\longrightarrow \mathrm{drop}(\underline{\mathrm{diff}(\mathrm{rmDup}(l), l)}, x) = nil \\
&\hfill \text{(by Problem 14)} \\
&\longrightarrow \underline{\mathrm{drop}(nil, x)} = nil \quad \text{(by Lemma 2)} \\
&\longrightarrow \underline{nil = nil} \quad \text{(by drop1)} \\
&\longrightarrow true \quad \text{(by equality)}
\end{aligned}
$$

□

Lemma 1. $\forall X \in \mathtt{PNat}, \forall L \in \mathtt{NatList}, \mathrm{diff}(L, \mathrm{rmDup}(L)) = nil.$

Proof. By structural induction on L.

(1) Base case

What to show: $\text{diff}(nil, \text{rmDup}(nil)) = nil$.

$$\underline{\text{diff}(nil, \text{rmDup}(nil))} \longrightarrow nil \qquad \text{(by diff1)}$$

(2) Induction case

What to show: $\text{diff}(x \mid l, \text{rmDup}(x \mid l)) = nil$

Induction hypothesis: $\text{diff}(l, \text{rmDup}(l)) = nil$

where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants.

We use case splitting for our proofs as follows:

Case 1: $\text{has}(l, x) = true$.

$$\begin{aligned}
\text{diff}(x \mid l, \underline{\text{rmDup}(x \mid l)}) &\longrightarrow \text{diff}(x \mid l, \text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}) \\
&\qquad \text{(by rmDup2)} \\
&\longrightarrow \text{diff}(x \mid l, \underline{\text{if } true \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}}) \\
&\qquad \text{(by case splitting)} \\
&\longrightarrow \underline{\text{diff}(x \mid l, \text{rmDup}(l))} \qquad \text{(by if1)} \\
&\longrightarrow \text{if } \underline{\text{has}(\text{rmDup}(l), x)} \text{ then diff}(l, \text{rmDup}(l)) \\
&\qquad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi} \qquad \text{(by diff2)} \\
&\longrightarrow \text{if } \underline{\text{has}(l, x)} \text{ then diff}(l, \text{rmDup}(l)) \\
&\qquad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi} \\
&\qquad \text{(by Problem 17 - Lemma 1)} \\
&\longrightarrow \underline{\text{if } true \text{ then diff}(l, \text{rmDup}(l))} \\
&\qquad \underline{\text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}} \\
&\qquad \text{(by case splitting)} \\
&\longrightarrow \underline{\text{diff}(l, \text{rmDup}(l))} \qquad \text{(by if1)} \\
&\longrightarrow nil \qquad \text{(by IH)}
\end{aligned}$$

Case 2: $\text{has}(l, x) = false$.

$$\begin{aligned}
\text{diff}(x \mid l, \underline{\text{rmDup}(x \mid l)}) &\longrightarrow \text{diff}(x \mid l, \text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}) \\
&\qquad \text{(by rmDup2)} \\
&\longrightarrow \text{diff}(x \mid l, \underline{\text{if } false \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}}) \\
&\qquad \text{(by case splitting)} \\
&\longrightarrow \underline{\text{diff}(x \mid l, x \mid \text{rmDup}(l))} \qquad \text{(by if2)} \\
&\longrightarrow \text{drop}(\underline{\text{diff}(x \mid l, \text{rmDup}(l))}, x) \\
&\qquad \text{(by Problem 14)}
\end{aligned}$$

$$
\begin{aligned}
&\longrightarrow \mathrm{drop}((\text{if } \underline{\mathrm{has}(\mathrm{rmDup}(l), x)} \text{ then } \mathrm{diff}(l, \mathrm{rmDup}(l)) \\
&\qquad \text{else } (x \mid \mathrm{diff}(l, \mathrm{rmDup}(l))) \text{ fi}), x) \\
&\hfill \text{(by diff2)} \\
&\longrightarrow \mathrm{drop}((\text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{diff}(l, \mathrm{rmDup}(l)) \\
&\qquad \text{else } (x \mid \mathrm{diff}(l, \mathrm{rmDup}(l))) \text{ fi}), x) \\
&\hfill \text{(by Problem 17 - Lemma 1)} \\
&\longrightarrow \mathrm{drop}(\underline{(\text{if } false \text{ then } \mathrm{diff}(l, \mathrm{rmDup}(l))} \\
&\qquad \underline{\text{else } (x \mid \mathrm{diff}(l, \mathrm{rmDup}(l))) \text{ fi})}, x) \\
&\hfill \text{(by case splitting)} \\
&\longrightarrow \mathrm{drop}(x \mid \underline{\mathrm{diff}(l, \mathrm{rmDup}(l))}, x) \qquad \text{(by if2)} \\
&\longrightarrow \underline{\mathrm{drop}(x \mid nil, x)} \qquad \text{(by IH)} \\
&\longrightarrow \text{if } \underline{x = x} \text{ then } \mathrm{drop}(nil, x) \text{ else } (x \mid \mathrm{drop}(nil, x)) \text{ fi} \\
&\hfill \text{(by drop2)} \\
&\longrightarrow \underline{\text{if } true \text{ then } \mathrm{drop}(nil, x) \text{ else } (x \mid \mathrm{drop}(nil, x)) \text{ fi}} \\
&\hfill \text{(by equality)} \\
&\longrightarrow \underline{\mathrm{drop}(nil, x)} \qquad \text{(by if1)} \\
&\longrightarrow nil \qquad \text{(by drop1)}
\end{aligned}
$$

□

Lemma 2. $\forall X \in \texttt{PNat}, \forall L \in \texttt{NatList}, \mathrm{diff}(\mathrm{rmDup}(L), L) = nil.$

Proof. By structural induction on L.

(1) Base case

What to show: $\mathrm{diff}(\mathrm{rmDup}(nil), nil) = nil$.

$$
\begin{aligned}
\mathrm{diff}(\underline{\mathrm{rmDup}(nil)}, nil) &\longrightarrow \underline{\mathrm{diff}(nil, nil)} && \text{(by rmDup1)} \\
&\longrightarrow nil && \text{(by diff1)}
\end{aligned}
$$

(2) Induction case

What to show: $\mathrm{diff}(\mathrm{rmDup}(x \mid l), x \mid l) = nil$
Induction hypothesis: $\mathrm{diff}(\mathrm{rmDup}(l), l) = nil$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants.

We use case splitting for our proofs as follows:
Case 1: $\mathrm{has}(l, x) = true$.

$$
\mathrm{diff}(\underline{\mathrm{rmDup}(x \mid l)}, x \mid l) \longrightarrow \mathrm{diff}((\text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi}), x \mid l) \qquad \text{(by rmDup2)}
$$

$$
\begin{aligned}
&\longrightarrow \mathrm{diff}(\underline{(\text{if } \mathit{true} \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi})}, x \mid l) \\
&\hspace{16em} \text{(by case splitting)} \\
&\longrightarrow \underline{\mathrm{diff}(\mathrm{rmDup}(l), x \mid l)} \qquad\qquad \text{(by if1)} \\
&\longrightarrow \mathrm{drop}(\underline{\mathrm{diff}(\mathrm{rmDup}(l), l)}, x) \\
&\hspace{16em} \text{(by Problem 14)} \\
&\longrightarrow \underline{\mathrm{drop}(\mathit{nil}, x)} \qquad\qquad \text{(by IH)} \\
&\longrightarrow \mathit{nil} \qquad\qquad \text{(by drop1)}
\end{aligned}
$$

Case 2: $\mathrm{has}(l, x) = \mathit{false}$.

$$
\begin{aligned}
\mathrm{diff}(\underline{\mathrm{rmDup}(x \mid l)}, x \mid l) &\longrightarrow \mathrm{diff}((\text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi}), x \mid l) \\
&\hspace{16em} \text{(by rmDup2)} \\
&\longrightarrow \mathrm{diff}(\underline{(\text{if } \mathit{false} \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi})}, x \mid l) \\
&\hspace{16em} \text{(by case splitting)} \\
&\longrightarrow \underline{\mathrm{diff}(x \mid \mathrm{rmDup}(l), x \mid l)} \qquad\qquad \text{(by if2)} \\
&\longrightarrow \mathrm{drop}(\underline{\mathrm{diff}(x \mid \mathrm{rmDup}(l), l)}, x) \\
&\hspace{16em} \text{(by Problem 14)} \\
&\longrightarrow \mathrm{drop}((\text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{diff}(\mathrm{rmDup}(l), l) \\
&\qquad \text{else } (x \mid \mathrm{diff}(\mathrm{rmDup}(l), l)) \text{ fi}), x) \\
&\hspace{16em} \text{(by diff2)} \\
&\longrightarrow \mathrm{drop}(\underline{(\text{if } \mathit{false} \text{ then } \mathrm{diff}(\mathrm{rmDup}(l), l)} \\
&\quad \underline{\text{else } (x \mid \mathrm{diff}(\mathrm{rmDup}(l), l)) \text{ fi})}, x) \\
&\hspace{16em} \text{(by case splitting)} \\
&\longrightarrow \mathrm{drop}(x \mid \underline{\mathrm{diff}(\mathrm{rmDup}(l), l)}, x) \qquad\qquad \text{(by if2)} \\
&\longrightarrow \underline{\mathrm{drop}(x \mid \mathit{nil}, x)} \qquad\qquad \text{(by IH)} \\
&\longrightarrow \text{if } \underline{x = x} \text{ then } \mathrm{drop}(\mathit{nil}, x) \text{ else } (x \mid \mathrm{drop}(\mathit{nil}, x)) \text{ fi} \\
&\hspace{16em} \text{(by drop2)} \\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } \mathrm{drop}(\mathit{nil}, x) \text{ else } (x \mid \mathrm{drop}(\mathit{nil}, x)) \text{ fi}} \\
&\hspace{16em} \text{(by equality)} \\
&\longrightarrow \underline{\mathrm{drop}(\mathit{nil}, x)} \qquad\qquad \text{(by if1)} \\
&\longrightarrow \mathit{nil} \qquad\qquad \text{(by drop1)}
\end{aligned}
$$

$\square$