Problem 3

Problem 1. $\forall X \in \texttt{PNat}, \text{fact}(X) = \text{fold}*(\text{mkl2}(X))$.

Proof. By direct proof.
What to show: $\text{fact}(x) = \text{fold}*(\text{mkl2}(x))$
where $x \in \texttt{PNat}$.

$$\begin{aligned}
\underline{\text{fact}(x)} &\longrightarrow \underline{\text{fold}*(\text{mkl1}(x))} && \text{(by Lemma 1)} \\
&\longrightarrow \text{fold}*(\underline{\text{rev}(\text{mkl1}(x))}) && \text{(by Problem 1)} \\
&\longrightarrow \text{fold}*(\text{mkl2}(x)) && \text{(by Problem 2)}
\end{aligned}$$

□

Lemma 1. $\forall X \in \texttt{PNat}, \text{fact}(X) = \text{fold}*(\text{mkl1}(X))$.

Proof. By structural induction on X.

(1) Base case
What to show: $\text{fact}(0) = \text{fold}*(\text{mkl1}(0))$.

$$\begin{aligned}
\underline{\text{fact}(0)} &\longrightarrow \text{s}(0) && \text{(by fact1)} \\
\text{fold}*(\underline{\text{mkl1}(0)}) &\longrightarrow \underline{\text{fold}*(\mathit{nil})} && \text{(by mkl1-1)} \\
&\longrightarrow \text{s}(0) && \text{(by fold*-1)}
\end{aligned}$$

(2) Induction case
What to show: $\text{fact}(\text{s}(x)) = \text{fold}*(\text{mkl1}(\text{s}(x)))$
Induction hypothesis: $\text{fact}(x) = \text{fold}*(\text{mkl1}(x))$
where $x \in \texttt{PNat}$.

$$\begin{aligned}
\underline{\text{fact}(\text{s}(x))} &\longrightarrow \underline{\text{s}(x) * \text{fact}(x)} && \text{(by fact2)} \\
&\longrightarrow (x * \underline{\text{fact}(x)}) + \underline{\text{fact}(x)} && \text{(by *2)} \\
&\longrightarrow (x * \text{fold}*(\text{mkl1}(x))) + \text{fold}*(\text{mkl1}(x)) && \text{(by IH)} \\
\text{fold}*(\underline{\text{mkl1}(\text{s}(x))}) &\longrightarrow \underline{\text{fold}*(\text{s}(x) \mid \text{mkl1}(x))} && \text{(by mkl1-2)} \\
&\longrightarrow \text{s}(x) * \text{fold}*(\text{mkl1}(x)) && \text{(by fold*-2)} \\
&\longrightarrow (x * \text{fold}*(\text{mkl1}(x))) + \text{fold}*(\text{mkl1}(x)) && \text{(by *2)}
\end{aligned}$$

□