

Problem 10

Problem 1. $\forall X \in \mathbf{PNat}, L \in \mathbf{NatList}, \text{drop}(L, X) = \text{rev}(\text{drop}(\text{rev}(L), X))$.

Proof. By structural induction on L .

(1) Base case

What to show: $\text{drop}(\text{nil}, x) = \text{rev}(\text{drop}(\text{rev}(\text{nil}), x))$
where $x \in \mathbf{PNat}$.

$$\begin{aligned} \text{drop}(\text{nil}, x) &\longrightarrow \text{nil} && \text{(by drop1)} \\ \text{rev}(\text{drop}(\text{rev}(\text{nil}), x)) &\longrightarrow \text{rev}(\text{drop}(\text{nil}, x)) && \text{(by rev1)} \\ &\longrightarrow \text{rev}(\text{nil}) && \text{(by drop1)} \\ &\longrightarrow \text{nil} && \text{(by rev1)} \end{aligned}$$

(2) Induction case

What to show: $\text{drop}(y \mid l, x) = \text{rev}(\text{drop}(\text{rev}(y \mid l), x))$

Induction hypothesis: $\text{drop}(l, x) = \text{rev}(\text{drop}(\text{rev}(l), x))$

where $x, y \in \mathbf{PNat}$ and $l \in \mathbf{NatList}$.

We use case splitting for our proofs as follows:

Case 1: $y = x$

$$\begin{aligned}
& \text{drop}(\underline{y} \mid l, x) \longrightarrow \underline{\text{drop}(x \mid l, x)} && \text{(by case splitting)} \\
& \longrightarrow \text{if } \underline{(x = x)} \text{ then } \text{drop}(l, x) \text{ else } (x \mid \text{drop}(l, x)) \text{ fi} && \text{(by drop2)} \\
& \longrightarrow \underline{\text{if } true \text{ then } \text{drop}(l, x) \text{ else } (x \mid \text{drop}(l, x)) \text{ fi}} && \text{(by equality)} \\
& \longrightarrow \underline{\text{drop}(l, x)} && \text{(by if1)} \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x)) && \text{(by IH)} \\
\text{rev}(\text{drop}(\text{rev}(\underline{y} \mid l), x)) & \longrightarrow \text{rev}(\text{drop}(\underline{\text{rev}(x \mid l)}, x)) && \text{(by case splitting)} \\
& \longrightarrow \text{rev}(\underline{\text{drop}(\text{rev}(l) @ (x \mid nil), x)}) && \text{(by rev2)} \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \underline{\text{drop}(x \mid nil, x)}) && \text{(by Lemma 1)} \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \\
& \quad (\text{if } \underline{(x = x)} \text{ then } \text{drop}(nil, x) \text{ else } (x \mid \text{drop}(nil, x)) \text{ fi})) && \text{(by drop2)} \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \\
& \quad \underline{(\text{if } true \text{ then } \text{drop}(nil, x) \text{ else } (x \mid \text{drop}(nil, x)) \text{ fi})}) && \text{(by equality)} \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \underline{\text{drop}(nil, x)}) && \text{(by if1)} \\
& \longrightarrow \text{rev}(\underline{\text{drop}(\text{rev}(l), x) @ nil}) && \text{(by drop1)} \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x)) && \text{(by Lemma 2 from Problem 4)}
\end{aligned}$$

Case 2: $(y = x) = false$

$$\begin{aligned}
& \underline{\text{drop}(y \mid l, x)} \longrightarrow \text{if } \underline{(y = x)} \text{ then } \text{drop}(l, x) \text{ else } (y \mid \text{drop}(l, x)) \text{ fi} \\
& \hspace{15em} (\text{by drop2}) \\
& \longrightarrow \underline{\text{if } false \text{ then } \text{drop}(l, x) \text{ else } (y \mid \text{drop}(l, x)) \text{ fi}} \\
& \hspace{15em} (\text{by case splitting}) \\
& \longrightarrow y \mid \underline{\text{drop}(l, x)} \hspace{10em} (\text{by if2}) \\
& \longrightarrow y \mid \text{rev}(\text{drop}(\text{rev}(l), x)) \hspace{10em} (\text{by IH}) \\
\text{rev}(\text{drop}(\text{rev}(y \mid l), x)) & \longrightarrow \text{rev}(\underline{\text{drop}(\text{rev}(l) @ (y \mid nil), x)}) \hspace{5em} (\text{by rev2}) \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \underline{\text{drop}((y \mid nil), x)}) \\
& \hspace{15em} (\text{by Lemma 1}) \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \\
& \hspace{5em} \text{if } \underline{(y = x)} \text{ then } \text{drop}(nil, x) \text{ else } (y \mid \text{drop}(nil, x)) \text{ fi}) \\
& \hspace{15em} (\text{by drop2}) \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \\
& \hspace{5em} \underline{\text{if } false \text{ then } \text{drop}(nil, x) \text{ else } (y \mid \text{drop}(nil, x)) \text{ fi}}) \\
& \hspace{15em} (\text{by case splitting}) \\
& \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ (y \mid \underline{\text{drop}(nil, x)})) \\
& \hspace{15em} (\text{by if2}) \\
& \longrightarrow \underline{\text{rev}(\text{drop}(\text{rev}(l), x) @ (y \mid nil))} \hspace{5em} (\text{by drop1}) \\
& \longrightarrow \underline{\text{rev}(y \mid nil)} @ \text{rev}(\text{drop}(\text{rev}(l), x)) \\
& \hspace{10em} (\text{by Lemma 1 from Problem 4}) \\
& \longrightarrow \underline{(\text{rev}(nil) @ (y \mid nil))} @ \text{rev}(\text{drop}(\text{rev}(l), x)) \\
& \hspace{15em} (\text{by rev2}) \\
& \longrightarrow \underline{(nil @ (y \mid nil))} @ \text{rev}(\text{drop}(\text{rev}(l), x)) \\
& \hspace{15em} (\text{by rev1}) \\
& \longrightarrow \underline{(y \mid nil)} @ \text{rev}(\text{drop}(\text{rev}(l), x)) \hspace{5em} (\text{by @1}) \\
& \longrightarrow y \mid \underline{(nil @ \text{rev}(\text{drop}(\text{rev}(l), x)))} \hspace{5em} (\text{by @2}) \\
& \longrightarrow y \mid \text{rev}(\text{drop}(\text{rev}(l), x)) \hspace{10em} (\text{by @1})
\end{aligned}$$

□

Lemma 1. $\forall X \in \text{PNat}, L1, L2 \in \text{NatList}, \text{drop}(L1 @ L2, X) = \text{drop}(L1, X) @ \text{drop}(L2, X).$

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{drop}(\text{nil} @ l2, x) = \text{drop}(\text{nil}, x) @ \text{drop}(l2, x)$
 where $x \in \text{PNat}$ and $l2 \in \text{NatList}$.

$$\begin{aligned} & \text{drop}(\text{nil} @ l2, x) \longrightarrow \text{drop}(l2, x) && \text{(by @1)} \\ \text{drop}(\text{nil}, x) @ \text{drop}(l2, x) & \longrightarrow \text{nil} @ \text{drop}(l2, x) && \text{(by drop1)} \\ & \longrightarrow \text{drop}(l2, x) && \text{(by @1)} \end{aligned}$$

(2) Induction case

What to show: $\text{drop}((y | l1) @ l2, x) = \text{drop}(y | l1, x) @ \text{drop}(l2, x)$

Induction hypothesis: $\text{drop}(l1 @ l2, x) = \text{drop}(l1, x) @ \text{drop}(l2, x)$

where $x, y \in \text{PNat}$, and $l1, l2 \in \text{NatList}$.

We use case splitting for our proofs as follows:

Case 1: $y = x$

$$\begin{aligned} & \text{drop}((\underline{y} | l1) @ l2, x) \longrightarrow \text{drop}((x | l1) @ l2, x) && \text{(by case splitting)} \\ & \longrightarrow \text{drop}(x | (l1 @ l2), x) && \text{(by @2)} \\ & \longrightarrow \text{if } (\underline{x = x}) \text{ then } \text{drop}(l1 @ l2, x) \\ & \quad \text{else } (x | \text{drop}(l1 @ l2, x)) \text{ fi} && \text{(by drop2)} \\ & \longrightarrow \text{if } \text{true} \text{ then } \text{drop}(l1 @ l2, x) \\ & \quad \text{else } (x | \text{drop}(l1 @ l2, x)) \text{ fi} && \text{(by equality)} \\ & \longrightarrow \text{drop}(l1 @ l2, x) && \text{(by if1)} \\ & \longrightarrow \text{drop}(l1, x) @ \text{drop}(l2, x) && \text{(by IH)} \\ \text{drop}(\underline{y} | l1, x) @ \text{drop}(l2, x) & \longrightarrow \text{drop}(x | l1, x) @ \text{drop}(l2, x) && \text{(by case splitting)} \\ & \longrightarrow \text{if } (\underline{x = x}) \text{ then } \text{drop}(l1, x) \text{ else } (x | \text{drop}(l1, x)) \text{ fi} \\ & \quad @ \text{drop}(l2, x) && \text{(by drop2)} \\ & \longrightarrow \text{if } \text{true} \text{ then } \text{drop}(l1, x) \text{ else } (x | \text{drop}(l1, x)) \text{ fi} \\ & \quad @ \text{drop}(l2, x) && \text{(by equality)} \\ & \longrightarrow \text{drop}(l1, x) @ \text{drop}(l2, x) && \text{(by if1)} \end{aligned}$$

Case 2: $(y = x) = false$

$$\begin{aligned}
& \text{drop}(\underline{(y \mid l1) @ l2}, x) \longrightarrow \underline{\text{drop}(y \mid (l1 @ l2), x)} \quad (\text{by @2}) \\
& \longrightarrow \text{if } \underline{(y = x)} \text{ then } \text{drop}(l1 @ l2, x) \\
& \quad \text{else } (y \mid \text{drop}(l1 @ l2, x)) \text{ fi} \\
& \quad \quad (\text{by drop2}) \\
& \longrightarrow \text{if } \underline{false} \text{ then } \text{drop}(l1 @ l2, x) \\
& \quad \underline{\text{else } (y \mid \text{drop}(l1 @ l2, x)) \text{ fi}} \\
& \quad \quad (\text{by case splitting}) \\
& \longrightarrow y \mid \underline{\text{drop}(l1 @ l2, x)} \quad (\text{by if2}) \\
& \longrightarrow y \mid \text{drop}(l1, x) @ \text{drop}(l2, x) \quad (\text{by IH}) \\
& \underline{\text{drop}(y \mid l1, x) @ \text{drop}(l2, x)} \longrightarrow \text{if } \underline{(y = x)} \text{ then } \text{drop}(l1, x) \text{ else } (y \mid \text{drop}(l1, x)) \text{ fi} \\
& \quad @ \text{drop}(l2, x) \quad (\text{by drop2}) \\
& \longrightarrow \underline{\text{if } false \text{ then } \text{drop}(l1, x) \text{ else } (y \mid \text{drop}(l1, x)) \text{ fi}} \\
& \quad @ \text{drop}(l2, x) \quad (\text{by case splitting}) \\
& \longrightarrow \underline{(y \mid \text{drop}(l1, x)) @ \text{drop}(l2, x)} \\
& \quad \quad (\text{by if2}) \\
& \longrightarrow y \mid (\text{drop}(l1, x) @ \text{drop}(l2, x)) \\
& \quad \quad (\text{by @2})
\end{aligned}$$

□