Problem 13

Problem 1. $\forall X \in \texttt{PNat}, \forall L \in \texttt{NatList}, \text{count}(L, X) = \text{count}(\text{rev}(L), X)$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{count}(nil, x) = \text{count}(\text{rev}(nil), x)$
where $x \in \texttt{PNat}$.

$$\begin{aligned}
\underline{\text{count}(nil, x)} &\longrightarrow 0 && \text{(by cnt1)}\\
\text{count}(\underline{\text{rev}(nil)}, x) &\longrightarrow \underline{\text{count}(nil, x)} && \text{(by rev1)}\\
&\longrightarrow 0 && \text{(by cnt1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{count}(y \mid l, x) = \text{count}(\text{rev}(y \mid l), x)$
Induction hypothesis: $\text{count}(l, x) = \text{count}(\text{rev}(l, x)$
where $x, y \in \texttt{PNat}$ and $l \in \texttt{NatList}$.
We use case splitting for our proofs as follows:
Case 1: $y = x$

$$\begin{aligned}
\text{count}(\underline{y} \mid l, x) &\longrightarrow \underline{\text{count}(x \mid l, x)} && \text{(by case splitting)}\\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l, x)) \text{ else count}(l, x) \text{ fi} && \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } s(\text{count}(l, x)) \text{ else count}(l, x) \text{ fi}} && \text{(by equality)}\\
&\longrightarrow s(\underline{\text{count}(l, x)}) && \text{(by if1)}\\
&\longrightarrow s(\text{count}(\text{rev}(l), x)) && \text{(by IH)}\\
\text{count}(\text{rev}(\underline{y} \mid l), x) &\longrightarrow \text{count}(\underline{\text{rev}(x \mid l)}, x) && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{count}(\text{rev}(l) \mathbin{@} (x \mid nil), x)} && \text{(by rev2)}\\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{count}(x \mid nil, x)} && \text{(by Lemma 1)}\\
&\longrightarrow \text{count}(\text{rev}(l), x) + \text{if } \underline{(x = x)} \text{ then } s(\text{count}(nil, x))
\end{aligned}$$

$$
\begin{aligned}
& \text{else } \mathrm{count}(nil, x) \text{ fi} && \text{(by cnt2)} \\
\longrightarrow\ & \mathrm{count}(\mathrm{rev}(l), x) + \underline{\text{if } true \text{ then } s(\mathrm{count}(nil, x))} \\
& \underline{\text{else } \mathrm{count}(nil, x) \text{ fi}} && \text{(by equality)} \\
\longrightarrow\ & \mathrm{count}(\mathrm{rev}(l), x) + s(\underline{\mathrm{count}(nil, x)}) && \text{(by if1)} \\
\longrightarrow\ & \underline{\mathrm{count}(\mathrm{rev}(l), x) + s(0)} && \text{(by cnt1)} \\
\longrightarrow\ & \underline{s(0) + \mathrm{count}(\mathrm{rev}(l), x)} && \text{(by comm+)} \\
\longrightarrow\ & s(\underline{0 + \mathrm{count}(\mathrm{rev}(l), x)}) && \text{(by +2)} \\
\longrightarrow\ & s(\mathrm{count}(\mathrm{rev}(l), x)) && \text{(by +1)}
\end{aligned}
$$

Case 2: $(y = x) = false$

$$
\begin{aligned}
\underline{\mathrm{count}(y \mid l, x)} \longrightarrow\ & \text{if } \underline{(y = x)} \text{ then } s(\mathrm{count}(l, x)) \text{ else } \mathrm{count}(l, x) \text{ fi} \\
& && \text{(by cnt2)} \\
\longrightarrow\ & \underline{\text{if } false \text{ then } s(\mathrm{count}(l, x)) \text{ else } \mathrm{count}(l, x) \text{ fi}} \\
& && \text{(by case splitting)} \\
\longrightarrow\ & \underline{\mathrm{count}(l, x)} && \text{(by if2)} \\
\longrightarrow\ & \mathrm{count}(\mathrm{rev}(l), x) && \text{(by IH)} \\
\mathrm{count}(\underline{\mathrm{rev}(y \mid l)}, x) \longrightarrow\ & \underline{\mathrm{count}(\mathrm{rev}(l) \mathbin{@} (y \mid nil), x)} && \text{(by rev2)} \\
\longrightarrow\ & \mathrm{count}(\mathrm{rev}(l), x) + \underline{\mathrm{count}(y \mid nil, x)} \\
& && \text{(by Lemma 1)} \\
\longrightarrow\ & \mathrm{count}(\mathrm{rev}(l), x) + \text{if } \underline{(y = x)} \text{ then } s(\mathrm{count}(nil, x)) \\
& \text{else } \mathrm{count}(nil, x) \text{ fi} && \text{(by cnt2)} \\
\longrightarrow\ & \mathrm{count}(\mathrm{rev}(l), x) + \underline{\text{if } false \text{ then } s(\mathrm{count}(nil, x))} \\
& \underline{\text{else } \mathrm{count}(nil, x) \text{ fi}} && \text{(by case splitting)} \\
\longrightarrow\ & \mathrm{count}(\mathrm{rev}(l), x) + \underline{\mathrm{count}(nil, x)} && \text{(by if2)} \\
\longrightarrow\ & \underline{\mathrm{count}(\mathrm{rev}(l), x) + 0} && \text{(by cnt1)} \\
\longrightarrow\ & \underline{0 + \mathrm{count}(\mathrm{rev}(l), x)} && \text{(by comm+)} \\
\longrightarrow\ & \mathrm{count}(\mathrm{rev}(l), x) && \text{(by +1)}
\end{aligned}
$$

□

Lemma 1. $\forall X \in \texttt{PNat}, \forall L1, L2 \in \texttt{NatList}, \mathrm{count}(L1 \mathbin{@} L2, X) = \mathrm{count}(L1, X) + \mathrm{count}(L2, X)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{count}(nil \mathbin{@} l2, x) = \text{count}(nil, x) + \text{count}(l2, x)$
where $x \in \texttt{PNat}$ and $l2 \in \texttt{NatList}$.

$$\begin{aligned}
\text{count}(\underline{nil \mathbin{@} l2}, x) &\longrightarrow \text{count}(l2, x) && \text{(by @1)}\\
\underline{\text{count}(nil, x)} + \text{count}(l2, x) &\longrightarrow \underline{0 + \text{count}(l2, x)} && \text{(by cnt1)}\\
&\longrightarrow \text{count}(l2, x) && \text{(by +1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{count}((y \mid l1) \mathbin{@} l2, x) = \text{count}(y \mid l1, x) + \text{count}(l2, x)$
Induction hypothesis: $\text{count}(l1 \mathbin{@} l2, x) = \text{count}(l1, x) + \text{count}(l2, x)$
where $x, y \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$.
We use case splitting for our proofs as follows:
Case 1: $y = x$

$$\begin{aligned}
\text{count}((\underline{y} \mid l1) \mathbin{@} l2, x) &\longrightarrow \text{count}(\underline{(x \mid l1) \mathbin{@} l2, x)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)}\\
&\longrightarrow \underline{\text{count}(x \mid (l1 \mathbin{@} l2), x)} && \text{(by @2)}\\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))\\
&\qquad \text{else count}(l1 \mathbin{@} l2, x) \text{ fi} && \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))}\\
&\qquad \underline{\text{else count}(l1 \mathbin{@} l2, x) \text{ fi}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by equality)}\\
&\longrightarrow s(\underline{\text{count}(l1 \mathbin{@} l2, x)}) && \text{(by if1)}\\
&\longrightarrow s(\text{count}(l1, x) + \text{count}(l2, x))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by IH)}\\
\text{count}(\underline{y} \mid l1, x) + \text{count}(l2, x) &\longrightarrow \underline{\text{count}(x \mid l1, x)} + \text{count}(l2, x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)}\\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l1, x))\\
&\qquad \text{else count}(l1, x) \text{ fi} + \text{count}(l2, x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } s(\text{count}(l1, x))}\\
&\qquad \underline{\text{else count}(l1, x) \text{ fi}} + \text{count}(l2, x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by equality)}\\
&\longrightarrow \underline{s(\text{count}(l1, x)) + \text{count}(l2, x)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by if1)}
\end{aligned}$$

$$\longrightarrow s(\text{count}(l1, x) + \text{count}(l2, x)) \quad \text{(by +2)}$$

Case 2: $(y = x) = false$

$$\begin{aligned}
\text{count}(\underline{(y \mid l1) \mathbin{@} l2}, x) &\longrightarrow \underline{\text{count}(y \mid (l1 \mathbin{@} l2), x)} && \text{(by @2)}\\
&\longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))\\
&\quad \text{else count}(l1 \mathbin{@} l2, x) \text{ fi} && \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } false \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))}\\
&\quad \underline{\text{else count}(l1 \mathbin{@} l2, x) \text{ fi}}\\
& && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{count}(l1 \mathbin{@} l2, x)} && \text{(by if2)}\\
&\longrightarrow \text{count}(l1, x) + \text{count}(l2, x) && \text{(by IH)}\\
\underline{\text{count}(y \mid l1, x)} + \text{count}(l2, x) &\longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l1, x))\\
&\quad \text{count}(l1, x) \text{ fi} + \text{count}(l2, x)\\
& && \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } false \text{ then } s(\text{count}(l1, x))}\\
&\quad \underline{\text{count}(l1, x) \text{ fi}} + \text{count}(l2, x)\\
& && \text{(by case splitting)}\\
&\longrightarrow \text{count}(l1, x) + \text{count}(l2, x) && \text{(by if2)}
\end{aligned}$$

□