

Problem 9

Problem 1. $\forall L1, L2 \in \text{NatList}, \text{diff}(L1, L2) = \text{rev}(\text{diff}(\text{rev}(L1), L2)).$

Proof. By structural induction on L .

(1) Base case

What to show: $\text{diff}(\text{nil}, l2) = \text{rev}(\text{diff}(\text{rev}(\text{nil}), l2))$
 where $l2 \in \text{NatList}$.

$$\begin{aligned}
 \text{diff}(\text{nil}, l2) &\longrightarrow \text{nil} && \text{(by diff1)} \\
 \text{rev}(\text{diff}(\text{rev}(\text{nil}), l2)) &\longrightarrow \text{rev}(\text{diff}(\text{nil}, l2)) && \text{(by rev1)} \\
 &\longrightarrow \text{rev}(\text{nil}) && \text{(by diff1)} \\
 &\longrightarrow \text{nil} && \text{(by rev1)}
 \end{aligned}$$

(2) Induction case

What to show: $\text{diff}(x \mid l1, l2) = \text{rev}(\text{diff}(\text{rev}(x \mid l1), l2))$

Induction hypothesis: $\text{diff}(l1, l2) = \text{rev}(\text{diff}(\text{rev}(l1), l2))$

where $x \in \text{PNat}$ and $l1, l2 \in \text{NatList}$.

We use case splitting for our proofs as follows:

Case 1: $\text{has}(l2, x) = \text{true}$

$$\begin{aligned}
 \text{diff}(x \mid l1, l2) &\longrightarrow \text{if } \text{has}(l2, x) \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi} \\
 &\hspace{15em} \text{(by diff2)} \\
 &\longrightarrow \text{if } \text{true} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi} \\
 &\hspace{10em} \text{(by case splitting)} \\
 &\longrightarrow \text{diff}(l1, l2) && \text{(by if1)} \\
 &\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2)) && \text{(by IH)} \\
 \text{rev}(\text{diff}(\text{rev}(x \mid l1), l2)) &\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1) @ (x \mid \text{nil}), l2)) \\
 &\hspace{15em} \text{(by rev2)} \\
 &\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \text{diff}(x \mid \text{nil}, l2)) \\
 &\hspace{10em} \text{(by Lemma 1)} \\
 &\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @
 \end{aligned}$$

$$\begin{aligned}
& (\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi}) \\
& \quad \text{(by diff2)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
& \underline{(\text{if } \text{true} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi})}) \\
& \quad \text{(by diff2)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \underline{\text{diff}(\text{nil}, l2)}) \\
& \quad \text{(by if1)} \\
& \longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1), l2) @ \text{nil}}) \quad \text{(by diff1)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2)) \\
& \quad \text{(by Lemma 2 from Problem 4)}
\end{aligned}$$

Case 2: $\text{has}(l2, x) = \text{false}$

$$\begin{aligned}
& \underline{\text{diff}(x \mid l1, l2)} \longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi} \\
& \quad \text{(by diff2)} \\
& \longrightarrow \underline{\text{if } \text{false} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}} \\
& \quad \text{(by case splitting)} \\
& \longrightarrow x \mid \underline{\text{diff}(l1, l2)} \quad \text{(by if2)} \\
& \longrightarrow x \mid \text{rev}(\text{diff}(\text{rev}(l1), l2)) \quad \text{(by IH)} \\
& \text{rev}(\text{diff}(\underline{\text{rev}(x \mid l1)}, l2)) \longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1) @ (x \mid \text{nil}), l2)}) \\
& \quad \text{(by rev2)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \underline{\text{diff}(x \mid \text{nil}, l2)}) \\
& \quad \text{(by Lemma 1)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
& (\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi})) \\
& \quad \text{(by diff2)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
& \underline{(\text{if } \text{false} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi})}) \\
& \quad \text{(by case splitting)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ (x \mid \underline{\text{diff}(\text{nil}, l2)})) \\
& \quad \text{(by if2)} \\
& \longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1), l2) @ (x \mid \text{nil})}) \\
& \quad \text{(by diff1)} \\
& \longrightarrow \underline{\text{rev}(x \mid \text{nil})} @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) \\
& \quad \text{(by Lemma 1 from Problem 4)}
\end{aligned}$$

$$\begin{aligned}
&\longrightarrow \underline{\text{rev}(\text{nil})} @ (x \mid \text{nil}) @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) && \text{(by rev2)} \\
&\longrightarrow \underline{\text{nil} @ (x \mid \text{nil})} @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) && \text{(by rev1)} \\
&\longrightarrow \underline{(x \mid \text{nil}) @ \text{rev}(\text{diff}(\text{rev}(l1), l2))} && \text{(by @1)} \\
&\longrightarrow x \mid \underline{(\text{nil} @ \text{rev}(\text{diff}(\text{rev}(l1), l2)))} && \text{(by @2)} \\
&\longrightarrow x \mid \text{rev}(\text{diff}(\text{rev}(l1), l2)) && \text{(by @1)}
\end{aligned}$$

□

Lemma 1. $\forall L1, L2, L3 \in \text{NatList}, \text{diff}(L1 @ L2, L3) = \text{diff}(L1, L3) @ \text{diff}(L2, L3)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{diff}(\text{nil} @ l2, l3) = \text{diff}(\text{nil}, l3) @ \text{diff}(l2, l3)$
where $l2, l3 \in \text{NatList}$.

$$\begin{aligned}
&\text{diff}(\text{nil} @ l2, l3) \longrightarrow \text{diff}(l2, l3) && \text{(by @1)} \\
&\underline{\text{diff}(\text{nil}, l3)} @ \text{diff}(l2, l3) \longrightarrow \underline{\text{nil} @ \text{diff}(l2, l3)} && \text{(by diff1)} \\
&\longrightarrow \text{diff}(l2, l3) && \text{(by @1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{diff}((x \mid l1) @ l2, l3) = \text{diff}((x \mid l1), l3) @ \text{diff}(l2, l3)$

Induction hypothesis: $\text{diff}(l1 @ l2, l3) = \text{diff}(l1, l3) @ \text{diff}(l2, l3)$

where $x \in \text{PNat}$, and $l1, l2, l3 \in \text{NatList}$.

We use case splitting for our proofs as follows:

Case 1: $\text{has}(l3, x) = \text{true}$

$$\begin{aligned}
&\text{diff}(\underline{(x \mid l1) @ l2}, l3) \longrightarrow \underline{\text{diff}(x \mid (l1 @ l2), l3)} && \text{(by)} \\
&\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1 @ l2, l3) && \\
&\quad \text{else } (x \mid \text{diff}(l1 @ l2, l3)) \text{ fi} && \text{(by diff2)} \\
&\longrightarrow \text{if } \underline{\text{true}} \text{ then } \text{diff}(l1 @ l2, l3) && \\
&\quad \underline{\text{else } (x \mid \text{diff}(l1 @ l2, l3)) \text{ fi}} && \\
&\quad \quad \quad \text{(by case splitting)} \\
&\longrightarrow \underline{\text{diff}(l1 @ l2, l3)} && \text{(by if1)} \\
&\longrightarrow \text{diff}(l1, l3) @ \text{diff}(l2, l3) && \text{(by IH)}
\end{aligned}$$

$$\begin{aligned}
\underline{\text{diff}(x \mid l1, l3) @ \text{diff}(l2, l3)} &\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi} \\
&\quad @ \text{diff}(l2, l3) \quad (\text{by diff2}) \\
&\longrightarrow \text{if } \underline{\text{true}} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi} \\
&\quad \underline{@ \text{diff}(l2, l3)} \quad (\text{by case splitting}) \\
&\longrightarrow \text{diff}(l1, l3) @ \text{diff}(l2, l3) \quad (\text{by if1})
\end{aligned}$$

Case 2: $\text{has}(l3, x) = \text{false}$

$$\begin{aligned}
\underline{\text{diff}((x \mid l1) @ l2, l3)} &\longrightarrow \underline{\text{diff}(x \mid (l1 @ l2), l3)} \quad (\text{by @2}) \\
&\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1 @ l2, l3) \\
&\quad \text{else } (x \mid \text{diff}(l1 @ l2, l3)) \text{ fi} \quad (\text{by diff2}) \\
&\longrightarrow \underline{\text{if } \text{false} \text{ then } \text{diff}(l1 @ l2, l3)} \\
&\quad \underline{\text{else } (x \mid \text{diff}(l1 @ l2, l3)) \text{ fi}} \\
&\quad (\text{by case splitting}) \\
&\longrightarrow x \mid \underline{\text{diff}(l1 @ l2, l3)} \quad (\text{by if2}) \\
&\longrightarrow x \mid (\text{diff}(l1, l3) @ \text{diff}(l2, l3)) \quad (\text{by IH}) \\
\underline{\text{diff}(x \mid l1, l3) @ \text{diff}(l2, l3)} &\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi} \\
&\quad @ \text{diff}(l2, l3) \quad (\text{by diff2}) \\
&\longrightarrow \underline{\text{if } \text{false} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi}} \\
&\quad \underline{@ \text{diff}(l2, l3)} \quad (\text{by case splitting}) \\
&\longrightarrow \underline{(x \mid \text{diff}(l1, l3)) @ \text{diff}(l2, l3)} \quad (\text{by if2}) \\
&\longrightarrow x \mid (\text{diff}(l1, l3) @ \text{diff}(l2, l3)) \quad (\text{by @2})
\end{aligned}$$

□