Problem 13

Problem 1. $\forall X \in \texttt{PNat}, \forall L \in \texttt{NatList}, \text{count}(L, X) = \text{count}(\text{rev}(L), X)$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{count}(nil, x) = \text{count}(\text{rev}(nil), x)$
where $x \in \texttt{PNat}$.

$$\begin{aligned} \underline{\text{count}(nil, x)} &\longrightarrow 0 && \text{(by cnt1)} \\ \text{count}(\underline{\text{rev}(nil)}, x) &\longrightarrow \underline{\text{count}(nil, x)} && \text{(by rev1)} \\ &\longrightarrow 0 && \text{(by cnt1)} \end{aligned}$$

(2) Induction case

What to show: $\text{count}(y \mid l, x) = \text{count}(\text{rev}(y \mid l), x)$
Induction hypothesis: $\text{count}(l, x) = \text{count}(\text{rev}(l, x)$
where $x, y \in \texttt{PNat}$ and $l \in \texttt{NatList}$.
We use case splitting for our proofs as follows:

Case 1: $y = x$

$$
\begin{aligned}
\text{count}(\underline{y} \mid l, x) &\longrightarrow \underline{\text{count}(x \mid l, x)} && \text{(by case splitting)}\\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l, x)) \text{ else count}(l, x) \text{ fi} && \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } s(\text{count}(l, x)) \text{ else count}(l, x) \text{ fi}} && \text{(by equality)}\\
&\longrightarrow s(\underline{\text{count}(l, x)}) && \text{(by if1)}\\
&\longrightarrow s(\text{count}(\text{rev}(l), x)) && \text{(by IH)}\\
\text{count}(\text{rev}(\underline{y} \mid l), x) &\longrightarrow \text{count}(\underline{\text{rev}(x \mid l)}, x) && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{count}(\text{rev}(l) \mathbin{@} (x \mid \mathit{nil}), x)} && \text{(by rev2)}\\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{count}(x \mid \mathit{nil}, x)} && \text{(by Lemma 1)}\\
&\longrightarrow \text{count}(\text{rev}(l), x) + \text{if } \underline{(x = x)} \text{ then } s(\text{count}(\mathit{nil}, x))\\
&\quad \text{else count}(\mathit{nil}, x) \text{ fi} && \text{(by cnt2)}\\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{if } \mathit{true} \text{ then } s(\text{count}(\mathit{nil}, x))}\\
&\quad \underline{\text{else count}(\mathit{nil}, x) \text{ fi}} && \text{(by equality)}\\
&\longrightarrow \text{count}(\text{rev}(l), x) + s(\underline{\text{count}(\mathit{nil}, x)}) && \text{(by if1)}\\
&\longrightarrow \underline{\text{count}(\text{rev}(l), x) + s(0)} && \text{(by cnt1)}\\
&\longrightarrow \underline{s(0) + \text{count}(\text{rev}(l), x)} && \text{(by comm+)}\\
&\longrightarrow s(\underline{0 + \text{count}(\text{rev}(l), x)}) && \text{(by +2)}\\
&\longrightarrow s(\text{count}(\text{rev}(l), x)) && \text{(by +1)}
\end{aligned}
$$

Case 2: $(y = x) = false$

$$\begin{aligned}
\underline{\text{count}(y \mid l, x)} &\longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l, x)) \text{ else count}(l, x) \text{ fi} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by cnt2)} \\
&\longrightarrow \underline{\text{if } false \text{ then } s(\text{count}(l, x)) \text{ else count}(l, x) \text{ fi}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \underline{\text{count}(l, x)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by if2)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by IH)} \\
\text{count}(\underline{\text{rev}(y \mid l)}, x) &\longrightarrow \underline{\text{count}(\text{rev}(l) \mathbin{@} (y \mid nil), x)} \qquad\qquad\qquad\qquad \text{(by rev2)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{count}(y \mid nil, x)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 1)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \text{if } \underline{(y = x)} \text{ then } s(\text{count}(nil, x)) \\
&\quad\ \text{else count}(nil, x) \text{ fi} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by cnt2)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{if } false \text{ then } s(\text{count}(nil, x))} \\
&\quad\ \underline{\text{else count}(nil, x) \text{ fi}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{count}(nil, x)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by if2)} \\
&\longrightarrow \underline{\text{count}(\text{rev}(l), x) + 0} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by cnt1)} \\
&\longrightarrow \underline{0 + \text{count}(\text{rev}(l), x)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by comm+)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by +1)}
\end{aligned}$$

□

Lemma 1. $\forall X \in \texttt{PNat}, \forall L1, L2 \in \texttt{NatList}, \text{count}(L1 \mathbin{@} L2, X) = \text{count}(L1, X) + \text{count}(L2, X)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{count}(nil \mathbin{@} l2, x) = \text{count}(nil, x) + \text{count}(l2, x)$
where $x \in \texttt{PNat}$ and $l2 \in \texttt{NatList}$.

$$\begin{aligned}
\text{count}(\underline{nil \mathbin{@} l2}, x) &\longrightarrow \text{count}(l2, x) \qquad\qquad \text{(by @1)} \\
\underline{\text{count}(nil, x)} + \text{count}(l2, x) &\longrightarrow \underline{0 + \text{count}(l2, x)} \qquad\qquad \text{(by cnt1)} \\
&\longrightarrow \text{count}(l2, x) \qquad\qquad \text{(by +1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{count}((y \mid l1) \mathbin{@} l2, x) = \text{count}(y \mid l1, x) + \text{count}(l2, x)$

Induction hypothesis: $\text{count}(l1 \mathbin{@} l2, x) = \text{count}(l1, x) + \text{count}(l2, x)$
where $x, y \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$.
We use case splitting for our proofs as follows:
Case 1: $y = x$

$$
\begin{aligned}
\text{count}((\underline{y} \mid l1) \mathbin{@} l2, x) &\longrightarrow \text{count}(\underline{(x \mid l1) \mathbin{@} l2}, x) \\
&\qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \underline{\text{count}(x \mid (l1 \mathbin{@} l2), x)} \qquad \text{(by @2)} \\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x)) \\
&\quad \text{else count}(l1 \mathbin{@} l2, x) \text{ fi} \qquad \text{(by cnt2)} \\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))} \\
&\quad \underline{\text{else count}(l1 \mathbin{@} l2, x) \text{ fi}} \\
&\qquad\qquad \text{(by equality)} \\
&\longrightarrow s(\underline{\text{count}(l1 \mathbin{@} l2, x)}) \qquad \text{(by if1)} \\
&\longrightarrow s(\text{count}(l1, x) + \text{count}(l2, x)) \\
&\qquad\qquad \text{(by IH)} \\
\text{count}(\underline{y} \mid l1, x) + \text{count}(l2, x) &\longrightarrow \underline{\text{count}(x \mid l1, x)} + \text{count}(l2, x) \\
&\qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l1, x)) \\
&\quad \text{else count}(l1, x) \text{ fi} + \text{count}(l2, x) \\
&\qquad\qquad \text{(by cnt2)} \\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } s(\text{count}(l1, x))} \\
&\quad \underline{\text{else count}(l1, x) \text{ fi}} + \text{count}(l2, x) \\
&\qquad\qquad \text{(by equality)} \\
&\longrightarrow \underline{s(\text{count}(l1, x)) + \text{count}(l2, x)} \\
&\qquad\qquad \text{(by if1)} \\
&\longrightarrow s(\text{count}(l1, x) + \text{count}(l2, x)) \\
&\qquad\qquad \text{(by +2)}
\end{aligned}
$$

Case 2: $(y = x) = false$

$$
\begin{aligned}
\text{count}(\underline{(y \mid l1) \mathbin{@} l2}, x) &\longrightarrow \text{count}(y \mid (l1 \mathbin{@} l2), x) && \text{(by @2)}\\
&\longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))\\
&\quad \text{else count}(l1 \mathbin{@} l2, x) \text{ fi} && \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } false \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))}\\
&\quad \underline{\text{else count}(l1 \mathbin{@} l2, x) \text{ fi}}\\
& && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{count}(l1 \mathbin{@} l2, x)} && \text{(by if2)}\\
&\longrightarrow \text{count}(l1, x) + \text{count}(l2, x) && \text{(by IH)}\\
\underline{\text{count}(y \mid l1, x)} + \text{count}(l2, x) &\longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l1, x))\\
&\quad \text{count}(l1, x) \text{ fi} + \text{count}(l2, x)\\
& && \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } false \text{ then } s(\text{count}(l1, x))}\\
&\quad \underline{\text{count}(l1, x) \text{ fi}} + \text{count}(l2, x)\\
& && \text{(by case splitting)}\\
&\longrightarrow \text{count}(l1, x) + \text{count}(l2, x) && \text{(by if2)}
\end{aligned}
$$

□