Problem 7

Problem 1. $\forall L \in \texttt{NatList}, \text{diff}(L, \text{rev}(L)) = nil.$

Proof. By structural induction on L.

(1) Base case

What to show: $\text{diff}(nil, \text{rev}(nil)) = nil.$

$$\underline{\text{diff}(nil, \text{rev}(nil))} \longrightarrow nil \qquad\qquad \text{(by diff1)}$$

(2) Induction case

What to show: $\text{diff}(x \mid l, \text{rev}(x \mid l)) = nil$

Induction hypothesis: $\text{diff}(l, \text{rev}(l)) = nil$

where $x \in$ `PNat` and $l \in$ `NatList`.

$$\mathrm{diff}(x \mid l, \underline{\mathrm{rev}(x \mid l)}) \longrightarrow \underline{\mathrm{diff}(x \mid l, \mathrm{rev}(l) \ @ \ (x \mid nil))} \qquad \text{(by rev2)}$$

$$\longrightarrow \text{if } \underline{\mathrm{has}(\mathrm{rev}(l) \ @ \ (x \mid nil), x)} \text{ then}$$
$$\mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil))$$
$$\text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil)) \text{ fi} \quad \text{(by diff2)}$$

$$\longrightarrow \text{if } \mathrm{has}(\mathrm{rev}(l), x) \text{ or } \underline{\mathrm{has}(x \mid nil, x)} \text{ then}$$
$$\mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil))$$
$$\text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil)) \text{ fi}$$
$$\text{(by Lemma 1 from Problem 6)}$$

$$\longrightarrow \text{if } \mathrm{has}(\mathrm{rev}(l), x) \text{ or } ((x = x) \text{ or } \underline{\mathrm{has}(nil, x)}) \text{ then}$$
$$\mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil))$$
$$\text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil)) \text{ fi} \quad \text{(by has2)}$$

$$\longrightarrow \text{if } \mathrm{has}(\mathrm{rev}(l), x) \text{ or } \underline{(true \text{ or } false)} \text{ then}$$
$$\mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil))$$
$$\text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil)) \text{ fi} \quad \text{(by has1)}$$

$$\longrightarrow \text{if } \underline{\mathrm{has}(\mathrm{rev}(l), x) \text{ or } true} \text{ then}$$
$$\mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil))$$
$$\text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil)) \text{ fi} \quad \text{(by or)}$$

$$\longrightarrow \underline{\text{if } true \text{ then } \mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil))}$$
$$\underline{\text{else } x \mid \mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil)) \text{ fi}} \quad \text{(by or)}$$

$$\longrightarrow \underline{\mathrm{diff}(l, \mathrm{rev}(l) \ @ \ (x \mid nil))} \qquad \text{(by if1)}$$

$$\longrightarrow \mathrm{diff}(\underline{\mathrm{diff}(l, \mathrm{rev}(l))}, x \mid nil) \qquad \text{(by Lemma 1)}$$

$$\longrightarrow \underline{\mathrm{diff}(nil, x \mid nil)} \qquad \text{(by IH)}$$

$$\longrightarrow nil \qquad \text{(by diff1)}$$

$\square$

Lemma 1. $\forall L1, L2, L3 \in$ `NatList`, $\mathrm{diff}(L1, L2 \ @ \ L3) = \mathrm{diff}(\mathrm{diff}(L1, L2), L3)$.

Proof. By structural induction on $L1$.

(1) Base case
What to show: $\mathrm{diff}(nil, l2 \ @ \ l3) = \mathrm{diff}(\mathrm{diff}(nil, l2), l3)$

where $l2, l3 \in$ `NatList`.

$$\underline{\text{diff}(nil, l2 @ l3)} \longrightarrow nil \qquad\qquad \text{(by diff1)}$$
$$\text{diff}(\underline{\text{diff}(nil, l2)}, l3) \longrightarrow \underline{\text{diff}(nil, l3)} \qquad\qquad \text{(by diff1)}$$
$$\longrightarrow nil \qquad\qquad \text{(by diff1)}$$

(2) Induction case

What to show:　　diff$(x \mid l1, l2 @ l3) = $diff$($diff$(x \mid l1, l2), l3)$
Induction hypothesis:　　diff$(l1, l2 @ l3) = $diff$($diff$(l1, l2), l3)$
where $x \in$ `PNat`, and $l1, l2, l3 \in$ `NatList`.

We use case splitting for our proofs as follows:
Case 1: has$(l2, x) = true$

$$\underline{\text{diff}(x \mid l1, l2 @ l3)} \longrightarrow \text{if } \underline{\text{has}(l2 @ l3, x)} \text{ then diff}(l1, l2 @ l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 @ l3))) \text{ fi} \qquad\qquad \text{(by)}$$
$$\longrightarrow \text{if } \underline{\text{has}(l2, x) \text{ or has}(l3, x)} \text{ then diff}(l1, l2 @ l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 @ l3))) \text{ fi}$$
$$\text{(by Lemma 1 from Problem 6)}$$
$$\longrightarrow \text{if } \underline{true \text{ or has}(l3, x)} \text{ then diff}(l1, l2 @ l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 @ l3))) \text{ fi}$$
$$\text{(by case splitting)}$$
$$\longrightarrow \underline{\text{if } true \text{ then diff}(l1, l2 @ l3)}$$
$$\underline{\text{else } (x \mid \text{diff}(l1, l2 @ l3))) \text{ fi}} \qquad\qquad \text{(by or)}$$
$$\longrightarrow \underline{\text{diff}(l1, l2 @ l3)} \qquad\qquad \text{(by if1)}$$
$$\longrightarrow \text{diff}(\text{diff}(l1, l2), l3) \qquad\qquad \text{(by IH)}$$
$$\text{diff}(\underline{\text{diff}(x \mid l1, l2)}, l3) \longrightarrow \text{diff}(\text{if } \underline{\text{has}(l2, x)} \text{ then diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}, l3)$$
$$\text{(by diff2)}$$
$$\longrightarrow \text{diff}(\underline{\text{if } true \text{ then diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}}, l3)$$
$$\text{(by case splitting)}$$
$$\longrightarrow \text{diff}(\text{diff}(l1, l2), l3) \qquad\qquad \text{(by if1)}$$

Case 2: $\text{has}(l2, x) = false$

$$\underline{\text{diff}(x \mid l1, l2 \ @ \ l3)} \longrightarrow \text{if } \underline{\text{has}(l2 \ @ \ l3, x)} \text{ then } \text{diff}(l1, l2 \ @ \ l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 \ @ \ l3)) \text{ fi} \qquad \text{(by)}$$
$$\longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ or } \text{has}(l3, x) \text{ then } \text{diff}(l1, l2 \ @ \ l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 \ @ \ l3)) \text{ fi}$$
$$\text{(by Lemma 1 from Problem 6)}$$
$$\longrightarrow \text{if } \underline{false \text{ or } \text{has}(l3, x)} \text{ then } \text{diff}(l1, l2 \ @ \ l3)$$
$$\text{else } (x \mid \text{diff}(l1, l2 \ @ \ l3)) \text{ fi}$$
$$\text{(by case splitting)}$$
$$\longrightarrow \text{if } \text{has}(l3, x) \text{ then } \underline{\text{diff}(l1, l2 \ @ \ l3)}$$
$$\text{else } (x \mid \underline{\text{diff}(l1, l2 \ @ \ l3)}) \text{ fi} \qquad \text{(by or)}$$
$$\longrightarrow \text{if } \text{has}(l3, x) \text{ then } \text{diff}(\text{diff}(l1, l2), l3)$$
$$\text{else } (x \mid \text{diff}(\text{diff}(l1, l2), l3)) \text{ fi} \qquad \text{(by IH)}$$
$$\text{diff}(\underline{\text{diff}(x \mid l1, l2)}, l3) \longrightarrow \text{diff}(\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}, l3)$$
$$\text{(by diff2)}$$
$$\longrightarrow \text{diff}(\underline{\text{if } false \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}}, l3)$$
$$\text{(by case splitting)}$$
$$\longrightarrow \underline{\text{diff}(x \mid \text{diff}(l1, l2), l3)} \qquad \text{(by if2)}$$
$$\longrightarrow \text{if } \text{has}(l3, x) \text{ then } \text{diff}(\text{diff}(l1, l2), l3)$$
$$\text{else } (x \mid \text{diff}(\text{diff}(l1, l2), l3)) \text{ fi} \qquad \text{(by diff2)}$$

$$\square$$