

Problem 6

Problem 1. $\forall X \in \text{PNat}, \forall L \in \text{NatList}, \text{has}(L, X) = \text{has}(\text{rev}(L), X)$.

Proof. By structural induction on L .

(1) Base case

What to show: $\text{has}(\text{nil}, x) = \text{has}(\text{rev}(\text{nil}), x)$
 where $x \in \text{PNat}$.

$$\begin{aligned} \text{has}(\text{nil}, x) &\longrightarrow \text{false} && \text{(by has1)} \\ \text{has}(\text{rev}(\text{nil}), x) &\longrightarrow \text{has}(\text{nil}, x) && \text{(by rev1)} \\ &\longrightarrow \text{false} && \text{(by has1)} \end{aligned}$$

(2) Induction case

What to show: $\text{has}(y \mid l, x) = \text{has}(\text{rev}(y \mid l), x)$
 Induction hypothesis: $\text{has}(l, x) = \text{has}(\text{rev}(l), x)$
 where $x, y \in \text{PNat}$ and $l \in \text{NatList}$.

$$\begin{aligned} \text{has}(y \mid l, x) &\longrightarrow (x = y) \text{ or } \text{has}(l, x) && \text{(by has2)} \\ &\longrightarrow (x = y) \text{ or } \text{has}(\text{rev}(l), x) && \text{(by IH)} \\ \text{has}(\text{rev}(y \mid l), x) &\longrightarrow \text{has}(\text{rev}(l) @ (y \mid \text{nil}), x) && \text{(by)} \\ &\longrightarrow \text{has}(\text{rev}(l), x) \text{ or } \text{has}(y \mid \text{nil}, x) && \text{(by Lemma 1)} \\ &\longrightarrow \text{has}(\text{rev}(l), x) \text{ or } ((x = y) \text{ or } \text{has}(\text{nil}, x)) && \text{(by has2)} \\ &\longrightarrow \text{has}(\text{rev}(l), x) \text{ or } ((x = y) \text{ or } \text{false}) && \text{(by has1)} \\ &\longrightarrow \text{has}(\text{rev}(l), x) \text{ or } (x = y) && \text{(by or)} \\ &\longrightarrow (x = y) \text{ or } \text{has}(\text{rev}(l), x) && \text{(by comm-or)} \end{aligned}$$

□

Lemma 1. $\forall X \in \text{PNat}, \forall L1, L2 \in \text{NatList}, \text{has}(L1 @ L2, X) = \text{has}(L1, X) \text{ or } \text{has}(L2, X)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{has}(\text{nil} @ l2, x) = (\text{has}(\text{nil}, x) \text{ or } \text{has}(l2, x))$
 where $x \in \text{PNat}$ and $l2 \in \text{NatList}$.

$$\begin{aligned} \text{has}(\text{nil} @ l2, x) &\longrightarrow \text{has}(l2, x) && \text{(by @1)} \\ \underline{\text{has}(\text{nil}, x) \text{ or } \text{has}(l2, x)} &\longrightarrow \text{false or } \text{has}(l2, x) && \text{(by has1)} \\ &\longrightarrow \text{has}(l2, x) && \text{(by or)} \end{aligned}$$

(2) Induction case

What to show: $\text{has}((y | l1) @ l2, x) = (\text{has}(y | l1, x) \text{ or } \text{has}(l2, x))$
 Induction hypothesis: $\text{has}(l1 @ l2, x) = (\text{has}(l1, x) \text{ or } \text{has}(l2, x))$
 where $x, y \in \text{PNat}$, and $l1, l2 \in \text{NatList}$.

$$\begin{aligned} \text{has}(\underline{(y | l1) @ l2}, x) &\longrightarrow \underline{\text{has}(y | (l1 @ l2), x)} && \text{(by @2)} \\ &\longrightarrow (x = y) \text{ or } \underline{\text{has}(l1 @ l2, x)} && \text{(by has2)} \\ &\longrightarrow (x = y) \text{ or } (\text{has}(l1, x) \text{ or } \text{has}(l2, x)) && \text{(by IH)} \\ \underline{\text{has}(y | l1, x) \text{ or } \text{has}(l2, x)} &\longrightarrow \underline{((x = y) \text{ or } \text{has}(l1, x)) \text{ or } \text{has}(l2, x)} && \text{(by has2)} \\ &\longrightarrow (x = y) \text{ or } (\text{has}(l1, x) \text{ or } \text{has}(l2, x)) && \text{(by assoc-or)} \end{aligned}$$

□