Problem 4

Problem 1. $\forall L \in \texttt{NatList}, \text{rev}(\text{rev}(L)) = L.$

Proof. By structural induction on L.

(1) Base case

What to show: $\text{rev}(\text{rev}(nil)) = nil.$

$$
\begin{aligned}
\text{rev}(\underline{\text{rev}(nil)}) &\longrightarrow \underline{\text{rev}(nil)} && \text{(by rev1)} \\
&\longrightarrow nil && \text{(by rev1)}
\end{aligned}
$$

(2) Induction case

What to show: $\text{rev}(\text{rev}(x \mid l)) = x \mid l$

Induction hypothesis: $\text{rev}(\text{rev}(l)) = l$

where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants[1].

$$
\begin{aligned}
\text{rev}(\underline{\text{rev}(x \mid l)}) &\longrightarrow \underline{\text{rev}(\text{rev}(l) \mathbin{@} (x \mid nil))} && \text{(by rev2)} \\
&\longrightarrow \underline{\text{rev}(x \mid nil)} \mathbin{@} \text{rev}(\text{rev}(l)) && \text{(by Lemma 1)} \\
&\longrightarrow (\underline{\text{rev}(nil)} \mathbin{@} (x \mid nil)) \mathbin{@} \text{rev}(\text{rev}(l)) && \text{(by rev2)} \\
&\longrightarrow \underline{(nil \mathbin{@} (x \mid nil))} \mathbin{@} \text{rev}(\text{rev}(l)) && \text{(by rev1)} \\
&\longrightarrow \underline{(x \mid nil) \mathbin{@} \text{rev}(\text{rev}(l))} && \text{(by @1)} \\
&\longrightarrow x \mid \underline{(nil \mathbin{@} \text{rev}(\text{rev}(l)))} && \text{(by @2)} \\
&\longrightarrow x \mid \underline{\text{rev}(\text{rev}(l))} && \text{(by @1)} \\
&\longrightarrow x \mid l && \text{(by IH)}
\end{aligned}
$$

$\square$

Lemma 1. $\forall L1, L2 \in \texttt{NatList}, \text{rev}(L1 \mathbin{@} L2) = \text{rev}(L2) \mathbin{@} \text{rev}(L1).$

Proof. By structural induction on $L1$.

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

(1) Base case

What to show: $\text{rev}(nil \mathbin{@} l2) = \text{rev}(l2) \mathbin{@} \text{rev}(nil)$
where $l2 \in \texttt{NatList}$. Note that $l2$ is a fresh constant.

$$
\begin{aligned}
\underline{\text{rev}(nil \mathbin{@} l2)} &\longrightarrow \text{rev}(l2) && \text{(by @1)}\\
\text{rev}(l2) \mathbin{@} \underline{\text{rev}(nil)} &\longrightarrow \underline{\text{rev}(l2) \mathbin{@} nil} && \text{(by rev1)}\\
&\longrightarrow \text{rev}(l2) && \text{(by Lemma 2)}
\end{aligned}
$$

(2) Induction case

What to show: $\text{rev}((x \mid l1) \mathbin{@} l2) = \text{rev}(l2) \mathbin{@} \text{rev}(x \mid l1)$
Induction hypothesis: $\text{rev}(l1 \mathbin{@} l2) = \text{rev}(l2) \mathbin{@} \text{rev}(l1)$
where $x \in \texttt{PNat}$, and $l1, l2 \in \texttt{NatList}$. Note that $x, l1, l2$ are fresh constants.

$$
\begin{aligned}
\text{rev}(\underline{(x \mid l1) \mathbin{@} l2}) &\longrightarrow \underline{\text{rev}(x \mid (l1 \mathbin{@} l2))} && \text{(by @2)}\\
&\longrightarrow \underline{\text{rev}(l1 \mathbin{@} l2)} \mathbin{@} (x \mid nil) && \text{(by rev2)}\\
&\longrightarrow \underline{(\text{rev}(l2) \mathbin{@} \text{rev}(l1)) \mathbin{@} (x \mid nil)} && \text{(by IH)}\\
&\longrightarrow \text{rev}(l2) \mathbin{@} (\text{rev}(l1) \mathbin{@} (x \mid nil))\\
& && \text{(by Lemma 2 in Problem 2)}\\
\text{rev}(l2) \mathbin{@} \underline{\text{rev}(x \mid l1)} &\longrightarrow \text{rev}(l2) \mathbin{@} (\text{rev}(l1) \mathbin{@} (x \mid nil)) && \text{(by rev2)}
\end{aligned}
$$

□

Lemma 2. $\forall L \in \texttt{NatList}, L \mathbin{@} nil = L.$

Proof. By structural induction on L.

(1) Base case

What to show: $nil \mathbin{@} nil = nil.$

$$
\underline{nil \mathbin{@} nil} \longrightarrow nil \qquad \text{(by @1)}
$$

(2) Induction case

What to show: $(x \mid l) \mathbin{@} nil = x \mid l$
Induction hypothesis: $l \mathbin{@} nil = l$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants.

$$
\begin{aligned}
\underline{(x \mid l) \mathbin{@} nil} &\longrightarrow x \mid \underline{(l \mathbin{@} nil)} && \text{(by @2)}\\
&\longrightarrow x \mid l && \text{(by IH)}
\end{aligned}
$$

□