

Problem 14

Problem 1. $\forall X \in \text{PNat}, \forall L1, L2 \in \text{NatList}, \text{diff}(L1, X \mid L2) = \text{drop}(\text{diff}(L1, L2), X)$.

Proof. By structural induction on L .

(1) Base case

What to show: $\text{diff}(\text{nil}, x \mid l2) = \text{drop}(\text{diff}(\text{nil}, l2), x)$
where $x \in \text{PNat}$ and $l2 \in \text{NatList}$.

$$\begin{array}{ll} \frac{\text{diff}(\text{nil}, x \mid l2) \longrightarrow \text{nil}}{\text{drop}(\text{diff}(\text{nil}, l2), x) \longrightarrow \text{drop}(\text{nil}, x)} & \begin{array}{l} \text{(by diff1)} \\ \text{(by diff1)} \end{array} \\ \longrightarrow \text{nil} & \text{(by drop1)} \end{array}$$

(2) Induction case

What to show: $\text{diff}(y \mid l1, x \mid l2) = \text{drop}(\text{diff}(y \mid l1, l2), x)$
Induction hypothesis: $\text{diff}(l1, x \mid l2) = \text{drop}(\text{diff}(l1, l2), x)$
where $x, y \in \text{PNat}$ and $l1, l2 \in \text{NatList}$.

We use case splitting for our proofs as follows:

Case 1: $has(l2, y) = true$

$$\begin{aligned}
& \underline{diff(y \mid l1, x \mid l2)} \longrightarrow \text{if } \underline{has(x \mid l2, y)} \text{ then } diff(l1, x \mid l2) \\
& \quad \text{else } (y \mid diff(l1, x \mid l2)) \text{ fi} \quad (\text{by diff2}) \\
& \longrightarrow \text{if } ((y = x) \text{ or } \underline{has(l2, y)}) \text{ then } diff(l1, x \mid l2) \\
& \quad \text{else } (y \mid diff(l1, x \mid l2)) \text{ fi} \quad (\text{by has2}) \\
& \longrightarrow \text{if } ((y = x) \text{ or } \underline{true}) \text{ then } diff(l1, x \mid l2) \\
& \quad \text{else } (y \mid diff(l1, x \mid l2)) \text{ fi} \\
& \quad \quad (\text{by case splitting}) \\
& \longrightarrow \text{if } \underline{true} \text{ then } diff(l1, x \mid l2) \\
& \quad \underline{\text{else } (y \mid diff(l1, x \mid l2)) \text{ fi}} \quad (\text{by or}) \\
& \longrightarrow \underline{diff(l1, x \mid l2)} \quad (\text{by if1}) \\
& \longrightarrow \text{drop}(diff(l1, l2), x) \quad (\text{by IH}) \\
& \text{drop}(\underline{diff(y \mid l1, l2)}, x) \longrightarrow \text{drop}(\text{if } \underline{has(l2, y)} \text{ then } diff(l1, l2) \\
& \quad \text{else } (y \mid diff(l1, l2)) \text{ fi}, x) \quad (\text{by diff2}) \\
& \longrightarrow \text{drop}(\text{if } \underline{true} \text{ then } diff(l1, l2) \\
& \quad \underline{\text{else } (y \mid diff(l1, l2)) \text{ fi}}, x) \\
& \quad \quad (\text{by case splitting}) \\
& \longrightarrow \text{drop}(diff(l1, l2), x) \quad (\text{by if1})
\end{aligned}$$

Case 2: $has(l2, y) = false$

$$\begin{aligned}
& \underline{diff(y \mid l1, x \mid l2)} \longrightarrow \text{if } \underline{has(x \mid l2, y)} \text{ then } diff(l1, x \mid l2) \\
& \quad \text{else } (y \mid diff(l1, x \mid l2)) \text{ fi} \quad (\text{by diff2}) \\
& \longrightarrow \text{if } ((y = x) \text{ or } \underline{has(l2, y)}) \text{ then } diff(l1, x \mid l2) \\
& \quad \text{else } (y \mid diff(l1, x \mid l2)) \text{ fi} \quad (\text{by has2}) \\
& \longrightarrow \text{if } ((y = x) \text{ or } \underline{false}) \text{ then } diff(l1, x \mid l2) \\
& \quad \text{else } (y \mid diff(l1, x \mid l2)) \text{ fi} \\
& \quad \quad (\text{by case splitting}) \\
& \longrightarrow \text{if } (y = x) \text{ then } \underline{diff(l1, x \mid l2)} \\
& \quad \text{else } (y \mid diff(l1, x \mid l2)) \text{ fi} \quad (\text{by or}) \\
& \longrightarrow \text{if } (y = x) \text{ then } drop(diff(l1, l2), x) \\
& \quad \text{else } (y \mid \underline{diff(l1, x \mid l2)}) \text{ fi} \quad (\text{by IH}) \\
& \longrightarrow \text{if } (y = x) \text{ then } drop(diff(l1, l2), x) \\
& \quad \text{else } (y \mid drop(diff(l1, l2), x)) \text{ fi} \quad (\text{by IH}) \\
drop(\underline{diff(y \mid l1, l2)}, x) & \longrightarrow drop(\text{if } \underline{has(l2, y)} \text{ then } diff(l1, l2) \\
& \quad \text{else } (y \mid diff(l1, l2)) \text{ fi}, x) \quad (\text{by diff2}) \\
& \longrightarrow drop(\underline{\text{if } false \text{ then } diff(l1, l2)} \\
& \quad \underline{\text{else } (y \mid diff(l1, l2)) \text{ fi}, x}) \\
& \quad \quad (\text{by case splitting}) \\
& \longrightarrow \underline{drop(y \mid diff(l1, l2), x)} \quad (\text{by if2}) \\
& \longrightarrow \text{if } (y = x) \text{ then } drop(diff(l1, l2), x) \\
& \quad \text{else } (y \mid drop(diff(l1, l2), x)) \text{ fi} \quad (\text{by drop2})
\end{aligned}$$

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