Problem 15

Problem 1. $\forall L \in \texttt{NatList}, \text{diff}(L, L) = nil$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{diff}(nil, nil) = nil$.

$$\text{diff}(nil, nil) \longrightarrow nil \qquad \text{(by diff1)}$$

(2) Induction case

What to show: $\text{diff}(x \mid l, x \mid l) = nil$
Induction hypothesis: $\text{diff}(l, l) = nil$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$.
We use case splitting for our proofs as follows:
Case 1: $has(l, x) = true$

$$\begin{aligned}
\underline{\text{diff}(x \mid l, x \mid l)} &\longrightarrow \text{drop}(\underline{\text{diff}(x \mid l, l)}, x) && \text{(by Problem 14)} \\
&\longrightarrow \text{drop}(\text{if } \underline{\text{has}(l, x)} \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}, x) \\
& && \text{(by diff2)} \\
&\longrightarrow \text{drop}(\underline{\text{if } true \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}}, x) \\
& && \text{(by case splitting)} \\
&\longrightarrow \text{drop}(\underline{\text{diff}(l, l)}, x) && \text{(by if1)} \\
&\longrightarrow \underline{\text{drop}(nil, x)} && \text{(by IH)} \\
&\longrightarrow nil && \text{(by drop1)}
\end{aligned}$$

Case 2: $has(l, x) = false$

$$\begin{aligned}
\underline{\text{diff}(x \mid l, x \mid l)} &\longrightarrow \text{drop}(\underline{\text{diff}(x \mid l, l)}, x) && \text{(by Problem 14)} \\
&\longrightarrow \text{drop}(\text{if } \underline{\text{has}(l, x)} \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}, x) \\
& && \text{(by diff2)} \\
&\longrightarrow \text{drop}(\underline{\text{if } false \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}}, x) \\
& && \text{(by case splitting)} \\
&\longrightarrow \text{drop}(x \mid \underline{\text{diff}(l, l)}, x) && \text{(by if2)} \\
&\longrightarrow \underline{\text{drop}(x \mid nil, x)} && \text{(by IH)} \\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then drop}(nil, x) \text{ else } (x \mid \text{drop}(nil, x)) \text{ fi} \\
& && \text{(by drop2)} \\
&\longrightarrow \underline{\text{if } true \text{ then drop}(nil, x) \text{ else } (x \mid \text{drop}(nil, x)) \text{ fi}} \\
& && \text{(by equality)} \\
&\longrightarrow \underline{\text{drop}(nil, x)} && \text{(by if1)} \\
&\longrightarrow nil && \text{(by drop1)}
\end{aligned}$$

□