

## Problem 9

**Problem 1.**  $\forall L1, L2 \in \text{NatList}, \text{diff}(L1, L2) = \text{rev}(\text{diff}(\text{rev}(L1), L2))$ .

*Proof.* By structural induction on  $L$ .

**(1) Base case**

What to show:  $\text{diff}(\text{nil}, l2) = \text{rev}(\text{diff}(\text{rev}(\text{nil}), l2))$   
where  $l2 \in \text{NatList}$ .

$$\begin{aligned} \text{diff}(\text{nil}, l2) &\longrightarrow \text{nil} && \text{(by diff1)} \\ \text{rev}(\text{diff}(\text{rev}(\text{nil}), l2)) &\longrightarrow \text{rev}(\text{diff}(\text{nil}, l2)) && \text{(by rev1)} \\ &\longrightarrow \text{rev}(\text{nil}) && \text{(by diff1)} \\ &\longrightarrow \text{nil} && \text{(by rev1)} \end{aligned}$$

**(2) Induction case**

What to show:  $\text{diff}(x \mid l1, l2) = \text{rev}(\text{diff}(\text{rev}(x \mid l1), l2))$

Induction hypothesis:  $\text{diff}(l1, l2) = \text{rev}(\text{diff}(\text{rev}(l1), l2))$

where  $x \in \text{PNat}$  and  $l1, l2 \in \text{NatList}$ .

We use case splitting for our proofs as follows:

**Case 1:**  $\text{has}(l2, x) = \text{true}$

$$\begin{aligned}
& \underline{\text{diff}(x \mid l1, l2)} \longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi} \\
& \hspace{15em} \text{(by diff2)} \\
& \longrightarrow \underline{\text{if } \text{true} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}} \\
& \hspace{15em} \text{(by case splitting)} \\
& \longrightarrow \underline{\text{diff}(l1, l2)} \hspace{15em} \text{(by if1)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2)) \hspace{15em} \text{(by IH)} \\
\text{rev}(\text{diff}(\underline{\text{rev}(x \mid l1)}, l2)) & \longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1) @ (x \mid \text{nil}), l2)}) \\
& \hspace{15em} \text{(by rev2)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \underline{\text{diff}(x \mid \text{nil}, l2)}) \\
& \hspace{15em} \text{(by Lemma 1)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
& (\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi})) \\
& \hspace{15em} \text{(by diff2)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
& (\underline{\text{if } \text{true} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi}})) \\
& \hspace{15em} \text{(by diff2)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \underline{\text{diff}(\text{nil}, l2)}) \\
& \hspace{15em} \text{(by if1)} \\
& \longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1), l2) @ \text{nil}}) \hspace{15em} \text{(by diff1)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2)) \\
& \hspace{15em} \text{(by Lemma 2 from Problem 4)}
\end{aligned}$$

**Case 2:**  $\text{has}(l2, x) = \text{false}$

$$\begin{aligned}
& \underline{\text{diff}(x \mid l1, l2)} \longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi} \\
& \hspace{15em} \text{(by diff2)} \\
& \longrightarrow \underline{\text{if } \text{false} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}} \\
& \hspace{15em} \text{(by case splitting)} \\
& \longrightarrow x \mid \underline{\text{diff}(l1, l2)} \hspace{10em} \text{(by if2)} \\
& \longrightarrow x \mid \text{rev}(\text{diff}(\text{rev}(l1), l2)) \hspace{10em} \text{(by IH)} \\
\text{rev}(\text{diff}(\text{rev}(x \mid l1), l2)) & \longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1) @ (x \mid \text{nil}), l2)}) \\
& \hspace{15em} \text{(by rev2)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \underline{\text{diff}(x \mid \text{nil}, l2)}) \\
& \hspace{15em} \text{(by Lemma 1)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
& \text{(if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi)}) \\
& \hspace{15em} \text{(by diff2)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
& \text{(if } \text{false} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi)}) \\
& \hspace{15em} \text{(by case splitting)} \\
& \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ (x \mid \underline{\text{diff}(\text{nil}, l2)})) \\
& \hspace{15em} \text{(by if2)} \\
& \longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1), l2) @ (x \mid \text{nil})}) \\
& \hspace{15em} \text{(by diff1)} \\
& \longrightarrow \underline{\text{rev}(x \mid \text{nil})} @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) \\
& \hspace{10em} \text{(by Lemma 1 from Problem 4)} \\
& \longrightarrow (\underline{\text{rev}(\text{nil})} @ (x \mid \text{nil})) @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) \\
& \hspace{15em} \text{(by rev2)} \\
& \longrightarrow (\underline{\text{nil} @ (x \mid \text{nil})}) @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) \\
& \hspace{15em} \text{(by rev1)} \\
& \longrightarrow (x \mid \text{nil}) @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) \hspace{2em} \text{(by @1)} \\
& \longrightarrow x \mid (\underline{\text{nil} @ \text{rev}(\text{diff}(\text{rev}(l1), l2))}) \hspace{2em} \text{(by @2)} \\
& \longrightarrow x \mid \text{rev}(\text{diff}(\text{rev}(l1), l2)) \hspace{10em} \text{(by @1)}
\end{aligned}$$

□

**Lemma 1.**  $\forall L1, L2, L3 \in \text{NatList}, \text{diff}(L1 @ L2, L3) = \text{diff}(L1, L3) @ \text{diff}(L2, L3).$

*Proof.* By structural induction on  $L1$ .

**(1) Base case**

What to show:  $\text{diff}(\text{nil} @ l2, l3) = \text{diff}(\text{nil}, l3) @ \text{diff}(l2, l3)$   
 where  $l2, l3 \in \text{NatList}$ .

$$\begin{aligned} \text{diff}(\text{nil} @ l2, l3) &\longrightarrow \text{diff}(l2, l3) && \text{(by @1)} \\ \underline{\text{diff}(\text{nil}, l3) @ \text{diff}(l2, l3)} &\longrightarrow \underline{\text{nil} @ \text{diff}(l2, l3)} && \text{(by diff1)} \\ &\longrightarrow \text{diff}(l2, l3) && \text{(by @1)} \end{aligned}$$

**(2) Induction case**

What to show:  $\text{diff}((x | l1) @ l2, l3) = \text{diff}((x | l1), l3) @ \text{diff}(l2, l3)$   
 Induction hypothesis:  $\text{diff}(l1 @ l2, l3) = \text{diff}(l1, l3) @ \text{diff}(l2, l3)$   
 where  $x \in \text{PNat}$ , and  $l1, l2, l3 \in \text{NatList}$ .

We use case splitting for our proofs as follows:

**Case 1:**  $\text{has}(l3, x) = \text{true}$

$$\begin{aligned} \text{diff}(\underline{(x | l1) @ l2}, l3) &\longrightarrow \underline{\text{diff}(x | (l1 @ l2), l3)} && \text{(by)} \\ &\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1 @ l2, l3) \\ &\quad \text{else } (x | \text{diff}(l1 @ l2, l3)) \text{ fi} && \text{(by diff2)} \\ &\longrightarrow \underline{\text{if } \text{true} \text{ then } \text{diff}(l1 @ l2, l3)} \\ &\quad \underline{\text{else } (x | \text{diff}(l1 @ l2, l3)) \text{ fi}} \\ &\quad \quad \quad \text{(by case splitting)} \\ &\longrightarrow \underline{\text{diff}(l1 @ l2, l3)} && \text{(by if1)} \\ &\longrightarrow \text{diff}(l1, l3) @ \text{diff}(l2, l3) && \text{(by IH)} \\ \underline{\text{diff}(x | l1, l3) @ \text{diff}(l2, l3)} &\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1, l3) \text{ else } (x | \text{diff}(l1, l3)) \text{ fi} \\ &\quad @ \text{diff}(l2, l3) && \text{(by diff2)} \\ &\longrightarrow \underline{\text{if } \text{true} \text{ then } \text{diff}(l1, l3) \text{ else } (x | \text{diff}(l1, l3)) \text{ fi}} \\ &\quad \underline{@ \text{diff}(l2, l3)} && \text{(by case splitting)} \\ &\longrightarrow \text{diff}(l1, l3) @ \text{diff}(l2, l3) && \text{(by if1)} \end{aligned}$$

**Case 2:**  $\text{has}(l3, x) = \text{false}$

$$\begin{aligned}
& \text{diff}(\underline{(x \mid l1) @ l2}, l3) \longrightarrow \underline{\text{diff}(x \mid (l1 @ l2), l3)} \quad (\text{by } @2) \\
& \longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1 @ l2, l3) \\
& \quad \text{else } (x \mid \text{diff}(l1 @ l2, l3)) \text{ fi} \quad (\text{by diff2}) \\
& \longrightarrow \underline{\text{if } \text{false} \text{ then } \text{diff}(l1 @ l2, l3)} \\
& \quad \underline{\text{else } (x \mid \text{diff}(l1 @ l2, l3)) \text{ fi}} \\
& \quad \quad (\text{by case splitting}) \\
& \longrightarrow x \mid \underline{\text{diff}(l1 @ l2, l3)} \quad (\text{by if2}) \\
& \longrightarrow x \mid (\text{diff}(l1, l3) @ \text{diff}(l2, l3)) \quad (\text{by IH}) \\
& \underline{\text{diff}(x \mid l1, l3) @ \text{diff}(l2, l3)} \longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi} \\
& \quad @ \text{diff}(l2, l3) \quad (\text{by diff2}) \\
& \longrightarrow \underline{\text{if } \text{false} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi}} \\
& \quad \underline{@ \text{diff}(l2, l3)} \quad (\text{by case splitting}) \\
& \longrightarrow \underline{(x \mid \text{diff}(l1, l3)) @ \text{diff}(l2, l3)} \quad (\text{by if2}) \\
& \longrightarrow x \mid (\text{diff}(l1, l3) @ \text{diff}(l2, l3)) \quad (\text{by } @2)
\end{aligned}$$

□