

# Problem 1

**Problem 1.**  $\forall L \in \text{NatList}, \text{fold}*(\text{rev}(L)) = \text{fold}*(L)$ .

*Proof.* By structural induction on  $L$ .

**(1) Base case**

What to show:  $\text{fold}*(\text{rev}(\text{nil})) = \text{fold}*(\text{nil})$

$$\text{fold}*(\underline{\text{rev}(\text{nil})}) \longrightarrow \text{fold}*(\text{nil}) \quad (\text{by rev1})$$

**(2) Induction case**

What to show:  $\text{fold}*(\text{rev}(x \mid l)) = \text{fold}*(x \mid l)$

Induction hypothesis:  $\text{fold}*(\text{rev}(l)) = \text{fold}*(l)$

where  $x \in \text{PNat}$  and  $l \in \text{NatList}$ .

$$\begin{aligned} \text{fold}*(\underline{\text{rev}(x \mid l)}) &\longrightarrow \underline{\text{fold}*(\text{rev}(l) @ (x \mid \text{nil}))} && (\text{by rev2}) \\ &\longrightarrow \underline{\text{fold}*(\text{rev}(l))} * \text{fold}*(x \mid \text{nil}) && (\text{by Lemma 1}) \\ &\longrightarrow \text{fold}*(l) * \underline{\text{fold}*(x \mid \text{nil})} && (\text{by IH}) \\ &\longrightarrow \text{fold}*(l) * (x * \underline{\text{fold}*(\text{nil})}) && (\text{by fold*-2}) \\ &\longrightarrow \text{fold}*(l) * (\underline{x * s(0)}) && (\text{by fold*-1}) \\ &\longrightarrow \text{fold}*(l) * (\underline{s(0) * x}) && (\text{by comm*}) \\ &\longrightarrow \text{fold}*(l) * ((\underline{0 * x}) + x) && (\text{by *2}) \\ &\longrightarrow \text{fold}*(l) * (\underline{0 + x}) && (\text{by *1}) \\ &\longrightarrow \underline{\text{fold}*(l) * x} && (\text{by +1}) \\ &\longrightarrow x * \text{fold}*(l) && (\text{by comm*}) \\ \underline{\text{fold}*(x \mid l)} &\longrightarrow x * \text{fold}*(l) && (\text{fold*-2}) \end{aligned}$$

□

**Lemma 1.**  $\forall L1, L2 \in \text{NatList}, \text{fold}*(L1 @ L2) = \text{fold}*(L1) * \text{fold}*(L2)$ .

*Proof.* By structural induction on  $L1$ .

**(1) Base case**

What to show:  $\text{fold}^*(\text{nil} @ l2) = \text{fold}^*(\text{nil}) * \text{fold}^*(l2)$   
where  $l2 \in \text{NatList}$ .

$$\begin{aligned} \text{fold}^*(\text{nil} @ l2) &\longrightarrow \text{fold}^*(l2) && \text{(by @1)} \\ \underline{\text{fold}^*(\text{nil}) * \text{fold}^*(l2)} &\longrightarrow \underline{s(0) * \text{fold}^*(l2)} && \text{(by fold*-1)} \\ &\longrightarrow \underline{(0 * \text{fold}^*(l2))} + \text{fold}^*(l2) && \text{(by *2)} \\ &\longrightarrow \underline{0 + \text{fold}^*(l2)} && \text{(by *1)} \\ &\longrightarrow \text{fold}^*(l2) && \text{(by +1)} \end{aligned}$$

**(2) Induction case**

What to show:  $\text{fold}^*((x | l1) @ l2) = \text{fold}^*(x | l1) * \text{fold}^*(l2)$   
Induction hypothesis:  $\text{fold}^*(l1 @ l2) = \text{fold}^*(l1) * \text{fold}^*(l2)$   
where  $x \in \text{PNat}$  and  $l1, l2 \in \text{NatList}$ .

$$\begin{aligned} \text{fold}^*((x | l1) @ l2) &\longrightarrow \underline{\text{fold}^*(x | (l1 @ l2))} && \text{(by @2)} \\ &\longrightarrow x * \underline{\text{fold}^*(l1 @ l2)} && \text{(by fold*-2)} \\ &\longrightarrow x * (\text{fold}^*(l1) * \text{fold}^*(l2)) && \text{(by IH)} \\ \underline{\text{fold}^*(x | l1) * \text{fold}^*(l2)} &\longrightarrow \underline{(x * \text{fold}^*(l1)) * \text{fold}^*(l2)} && \text{(by fold*-2)} \\ &\longrightarrow x * (\text{fold}^*(l1) * \text{fold}^*(l2)) && \text{(by assoc*)} \end{aligned}$$

□