Problem 12

Problem 1. $\forall L \in \texttt{NatList}, \mathrm{sum}(L) = \mathrm{sum}(\mathrm{rev}(L))$.

Proof. By structural induction on L.

(1) Base case

What to show: $\mathrm{sum}(nil) = \mathrm{sum}(\mathrm{rev}(nil))$.

$$\begin{aligned} \underline{\mathrm{sum}(nil)} &\longrightarrow 0 && \text{(by sum1)} \\ \mathrm{sum}(\underline{\mathrm{rev}(nil)}) &\longrightarrow \underline{\mathrm{sum}(nil)} && \text{(by rev1)} \\ &\longrightarrow 0 && \text{(by sum1)} \end{aligned}$$

(2) Induction case

What to show: $\mathrm{sum}(x \mid l) = \mathrm{sum}(\mathrm{rev}(x \mid l))$

Induction hypothesis: $\mathrm{sum}(l) = \mathrm{sum}(\mathrm{rev}(l))$

where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$.

$$\begin{aligned} \underline{\mathrm{sum(x \mid l)}} &\longrightarrow x + \mathrm{sum}(l) && \text{(by sum2)} \\ &\longrightarrow \underline{x + \mathrm{sum}(\mathrm{rev}(l))} && \text{(by IH)} \\ &\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + x && \text{(by comm+)} \\ \mathrm{sum}(\underline{\mathrm{rev}(x \mid l)}) &\longrightarrow \underline{\mathrm{sum}(\mathrm{rev}(l) \mathbin{@} (x \mid nil))} && \text{(by rev2)} \\ &\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + \underline{sum\ (x \mid nil)} && \text{(by Lemma 1)} \\ &\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + (x + \underline{sum\ (nil)}) && \text{(by sum2)} \\ &\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + (\underline{x + 0}) && \text{(by sum1)} \\ &\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + (\underline{0 + x}) && \text{(by comm+)} \\ &\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + x && \text{(by +1)} \end{aligned}$$

$\square$

Lemma 1. $\forall L1, L2 \in \texttt{NatList}, \mathrm{sum}(L1 \mathbin{@} L2) = \mathrm{sum}(L1) + \mathrm{sum}(L2)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\mathrm{sum}(nil \mathbin{@} l2) = \mathrm{sum}(nil) + \mathrm{sum}(l2)$
where $l2 \in \texttt{NatList}$.

$$
\begin{aligned}
\mathrm{sum}(\underline{nil \mathbin{@} l2}) &\longrightarrow \mathrm{sum}(l2) && \text{(by @1)}\\
\underline{\mathrm{sum}(nil)} + \mathrm{sum}(l2) &\longrightarrow \underline{0 + \mathrm{sum}(l2)} && \text{(by sum1)}\\
&\longrightarrow 0 && \text{(by +1)}
\end{aligned}
$$

(2) Induction case

What to show: $\mathrm{sum}((x \mid l1) \mathbin{@} l2) = \mathrm{sum}(x \mid l1) + \mathrm{sum}(l2)$
Induction hypothesis: $\mathrm{sum}(l1 \mathbin{@} l2) = \mathrm{sum}(l1) + \mathrm{sum}(l2)$
where $x \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$.

$$
\begin{aligned}
\mathrm{sum}(\underline{(x \mid l)) \mathbin{@} l2}) &\longrightarrow \underline{\mathrm{sum}(x \mid (l1 \mathbin{@} l2))} && \text{(by @2)}\\
&\longrightarrow x + \underline{\mathrm{sum}(l1 \mathbin{@} l2)} && \text{(by sum2)}\\
&\longrightarrow \underline{x + (\mathrm{sum}(l1) + \mathrm{sum}(l2))} && \text{(by IH)}\\
&\longrightarrow (x + \mathrm{sum}(l1)) + \mathrm{sum}(l2) && \text{(by assoc+)}\\
\underline{\mathrm{sum}(x \mid l1)} + \mathrm{sum}(l2) &\longrightarrow (x + \mathrm{sum}(l1)) + \mathrm{sum}(l2) && \text{(by sum2)}
\end{aligned}
$$

$\square$