Problem 10

Problem 1. $\forall X \in \texttt{PNat}, \forall L \in \texttt{NatList}, \text{drop}(L, X) = \text{rev}(\text{drop}(\text{rev}(L), X))$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{drop}(nil, x) = \text{rev}(\text{drop}(\text{rev}(nil), x))$
where $x \in \texttt{PNat}$.

$$\begin{aligned}
\underline{\text{drop}(nil, x)} &\longrightarrow nil && \text{(by drop1)} \\
\text{rev}(\text{drop}(\underline{\text{rev}(nil)}, x)) &\longrightarrow \text{rev}(\underline{\text{drop}(nil, x)}) && \text{(by rev1)} \\
&\longrightarrow \underline{\text{rev}(nil)} && \text{(by drop1)} \\
&\longrightarrow nil && \text{(by rev1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{drop}(y \mid l, x) = \text{rev}(\text{drop}(\text{rev}(y \mid l), x))$
Induction hypothesis: $\text{drop}(l, x) = \text{rev}(\text{drop}(\text{rev}(l), x))$
where $x, y \in \texttt{PNat}$ and $l \in \texttt{NatList}$.
We use case splitting for our proofs as follows:
Case 1: $y = x$

$$\begin{aligned}
\text{drop}(\underline{y} \mid l, x) &\longrightarrow \underline{\text{drop}(x \mid l, x)} && \text{(by case splitting)} \\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then drop}(l, x) \text{ else } (x \mid \text{drop}(l, x)) \text{ fi} \\
& && \text{(by drop2)} \\
&\longrightarrow \underline{\text{if } \textit{true} \text{ then drop}(l, x) \text{ else } (x \mid \text{drop}(l, x)) \text{ fi}} \\
& && \text{(by equality)} \\
&\longrightarrow \underline{\text{drop}(l, x)} && \text{(by if1)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x)) && \text{(by IH)} \\
\text{rev}(\text{drop}(\text{rev}(\underline{y} \mid l), x)) &\longrightarrow \text{rev}(\text{drop}(\underline{\text{rev}(x \mid l)}, x)) \\
& && \text{(by case splitting)} \\
&\longrightarrow \text{rev}(\underline{\text{drop}(\text{rev}(l) \mathbin{@} (x \mid nil), x)}) && \text{(by rev2)}
\end{aligned}$$

$$\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) \mathbin{@} \underline{\text{drop}(x \mid nil, x)})$$
(by Lemma 1)

$$\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) \mathbin{@} \text{if } \underline{(x = x)} \text{ then drop}(nil, x) \text{ else } (x \mid \text{drop}(nil, x)) \text{ fi}))$$
(by drop2)

$$\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) \mathbin{@} \underline{(\text{if } true \text{ then drop}(nil, x) \text{ else } (x \mid \text{drop}(nil, x)) \text{ fi})})$$
(by equality)

$$\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) \mathbin{@} \underline{\text{drop}(nil, x)})$$
(by if1)

$$\longrightarrow \text{rev}(\underline{\text{drop}(\text{rev}(l), x) \mathbin{@} nil}) \quad \text{(by drop1)}$$

$$\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x))$$
(by Lemma 2 from Problem 4)

Case 2: $(y = x) = false$

$$\underline{\text{drop}(y \mid l, x)} \longrightarrow \text{if } \underline{(y = x)} \text{ then drop}(l, x) \text{ else } (y \mid \text{drop}(l, x)) \text{ fi}$$
(by drop2)

$$\longrightarrow \underline{\text{if } false \text{ then drop}(l, x) \text{ else } (y \mid \text{drop}(l, x)) \text{ fi}}$$
(by case splitting)

$$\longrightarrow y \mid \underline{\text{drop}(l, x)} \quad \text{(by if2)}$$

$$\longrightarrow y \mid \text{rev}(\text{drop}(\text{rev}(l), x)) \quad \text{(by IH)}$$

$$\text{rev}(\text{drop}(\underline{\text{rev}(y \mid l)}, x)) \longrightarrow \text{rev}(\underline{\text{drop}(\text{rev}(l) \mathbin{@} (y \mid nil), x)}) \quad \text{(by rev2)}$$

$$\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) \mathbin{@} \underline{\text{drop}((y \mid nil), x)})$$
(by Lemma 1)

$$\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) \mathbin{@} \text{if } \underline{(y = x)} \text{ then drop}(nil, x) \text{ else } (y \mid \text{drop}(nil, x)) \text{ fi})$$
(by drop2)

$$\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) \mathbin{@} \underline{\text{if } false \text{ then drop}(nil, x) \text{ else } (y \mid \text{drop}(nil, x)) \text{ fi}})$$
(by case splitting)

$$\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) \mathbin{@} (y \mid \underline{\text{drop}(nil, x)}))$$
(by if2)

$$\longrightarrow \underline{\text{rev}(\text{drop}(\text{rev}(l), x) \mathbin{@} (y \mid nil))}$$
(by drop1)

$$
\begin{aligned}
&\longrightarrow \underline{\mathrm{rev}(y \mid nil)} \mathbin{@} \mathrm{rev}(\mathrm{drop}(\mathrm{rev}(l), x)) \\
&\qquad\qquad \text{(by Lemma 1 from Problem 4)} \\
&\longrightarrow (\underline{\mathrm{rev}(nil)} \mathbin{@} (y \mid nil)) \mathbin{@} \mathrm{rev}(\mathrm{drop}(\mathrm{rev}(l), x)) \\
&\qquad\qquad \text{(by rev2)} \\
&\longrightarrow \underline{(nil \mathbin{@} (y \mid nil))} \mathbin{@} \mathrm{rev}(\mathrm{drop}(\mathrm{rev}(l), x)) \\
&\qquad\qquad \text{(by rev1)} \\
&\longrightarrow \underline{(y \mid nil) \mathbin{@} \mathrm{rev}(\mathrm{drop}(\mathrm{rev}(l), x))} \quad \text{(by @1)} \\
&\longrightarrow y \mid \underline{(nil \mathbin{@} \mathrm{rev}(\mathrm{drop}(\mathrm{rev}(l), x)))} \quad \text{(by @2)} \\
&\longrightarrow y \mid \mathrm{rev}(\mathrm{drop}(\mathrm{rev}(l), x)) \quad \text{(by @1)}
\end{aligned}
$$

□

Lemma 1. $\forall X \in \texttt{PNat}, \forall L1, L2 \in \texttt{NatList}, \mathrm{drop}(L1 \mathbin{@} L2, X) = \mathrm{drop}(L1, X) \mathbin{@} \mathrm{drop}(L2, X)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\mathrm{drop}(nil \mathbin{@} l2, x) = \mathrm{drop}(nil, x) \mathbin{@} \mathrm{drop}(l2, x)$
where $x \in \texttt{PNat}$ and $l2 \in \texttt{NatList}$.

$$
\begin{aligned}
\mathrm{drop}(\underline{nil \mathbin{@} l2}, x) &\longrightarrow \mathrm{drop}(l2, x) && \text{(by @1)} \\
\underline{\mathrm{drop}(nil, x)} \mathbin{@} \mathrm{drop}(l2, x) &\longrightarrow \underline{nil \mathbin{@} \mathrm{drop}(l2, x)} && \text{(by drop1)} \\
&\longrightarrow \mathrm{drop}(l2, x) && \text{(by @1)}
\end{aligned}
$$

(2) Induction case

What to show: $\mathrm{drop}((y \mid l1) \mathbin{@} l2, x) = \mathrm{drop}(y \mid l1, x) \mathbin{@} \mathrm{drop}(l2, x)$
Induction hypothesis: $\mathrm{drop}(l1 \mathbin{@} l2, x) = \mathrm{drop}(l1, x) \mathbin{@} \mathrm{drop}(l2, x)$
where $x, y \in \texttt{PNat}$, and $l1, l2 \in \texttt{NatList}$.

We use case splitting for our proofs as follows:
Case 1: $y = x$

$$
\begin{aligned}
\mathrm{drop}((\underline{y} \mid l1) \mathbin{@} l2, x) &\longrightarrow \mathrm{drop}(\underline{(x \mid l1) \mathbin{@} l2}, x) \\
&\qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \underline{\mathrm{drop}(x \mid (l1 \mathbin{@} l2), x)} && \text{(by @2)} \\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } \mathrm{drop}(l1 \mathbin{@} l2, x) \\
&\qquad \text{else } (x \mid \mathrm{drop}(l1 \mathbin{@} l2, x)) \text{ fi} \\
&\qquad\qquad \text{(by drop2)}
\end{aligned}
$$

$$\longrightarrow \underline{\text{if } true \text{ then drop}(l1 \mathbin{@} l2, x)} \quad \underline{\text{else } (x \mid \text{drop}(l1 \mathbin{@} l2, x))\ \text{fi}} \quad \text{(by equality)}$$

$$\longrightarrow \underline{\text{drop}(l1 \mathbin{@} l2, x)} \quad \text{(by if1)}$$

$$\longrightarrow \text{drop}(l1, x) \mathbin{@} \text{drop}(l2, x) \quad \text{(by IH)}$$

$$\text{drop}(\underline{y} \mid l1, x) \mathbin{@} \text{drop}(l2, x) \longrightarrow \underline{\text{drop}(x \mid l1, x)} \mathbin{@} \text{drop}(l2, x) \quad \text{(by case splitting)}$$

$$\longrightarrow \text{if } \underline{(x = x)} \text{ then drop}(l1, x) \text{ else } (x \mid \text{drop}(l1, x))\ \text{fi} \mathbin{@} \text{drop}(l2, x) \quad \text{(by drop2)}$$

$$\longrightarrow \underline{\text{if } true \text{ then drop}(l1, x) \text{ else } (x \mid \text{drop}(l1, x))\ \text{fi}} \mathbin{@} \text{drop}(l2, x) \quad \text{(by equality)}$$

$$\longrightarrow \text{drop}(l1, x) \mathbin{@} \text{drop}(l2, x) \quad \text{(by if1)}$$

Case 2: $(y = x) = false$

$$\text{drop}(\underline{(y \mid l1) \mathbin{@} l2}, x) \longrightarrow \underline{\text{drop}(y \mid (l1 \mathbin{@} l2), x)} \quad \text{(by @2)}$$

$$\longrightarrow \text{if } \underline{(y = x)} \text{ then drop}(l1 \mathbin{@} l2, x) \quad \text{else } (y \mid \text{drop}(l1 \mathbin{@} l2, x))\ \text{fi} \quad \text{(by drop2)}$$

$$\longrightarrow \underline{\text{if } false \text{ then drop}(l1 \mathbin{@} l2, x)} \quad \underline{\text{else } (y \mid \text{drop}(l1 \mathbin{@} l2, x))\ \text{fi}} \quad \text{(by case splitting)}$$

$$\longrightarrow y \mid \underline{\text{drop}(l1 \mathbin{@} l2, x)} \quad \text{(by if2)}$$

$$\longrightarrow y \mid \text{drop}(l1, x) \mathbin{@} \text{drop}(l2, x) \quad \text{(by IH)}$$

$$\underline{\text{drop}(y \mid l1, x)} \mathbin{@} \text{drop}(l2, x) \longrightarrow \text{if } \underline{(y = x)} \text{ then drop}(l1, x) \text{ else } (y \mid \text{drop}(l1, x))\ \text{fi} \mathbin{@} \text{drop}(l2, x) \quad \text{(by drop2)}$$

$$\longrightarrow \underline{\text{if } false \text{ then drop}(l1, x) \text{ else } (y \mid \text{drop}(l1, x))\ \text{fi}} \mathbin{@} \text{drop}(l2, x) \quad \text{(by case splitting)}$$

$$\longrightarrow \underline{(y \mid \text{drop}(l1, x)) \mathbin{@} \text{drop}(l2, x)} \quad \text{(by if2)}$$

$$\longrightarrow y \mid (\text{drop}(l1, x) \mathbin{@} \text{drop}(l2, x) \quad \text{(by @2)}$$

□