# Problem 13

**Problem 1.** $\forall X \in \texttt{PNat}, \forall L \in \texttt{NatList}, \mathrm{count}(L, X) = \mathrm{count}(\mathrm{rev}(L), X)$.

*Proof.* By structural induction on $L$.

**(1) Base case**

What to show: $\mathrm{count}(nil, x) = \mathrm{count}(\mathrm{rev}(nil), x)$
where $x \in \texttt{PNat}$. Note that $x$ is a fresh constant[1].

$$\begin{aligned}
\underline{\mathrm{count}(nil, x)} &\longrightarrow 0 && \text{(by cnt1)} \\
\mathrm{count}(\underline{\mathrm{rev}(nil)}, x) &\longrightarrow \underline{\mathrm{count}(nil, x)} && \text{(by rev1)} \\
&\longrightarrow 0 && \text{(by cnt1)}
\end{aligned}$$

**(2) Induction case**

What to show: $\mathrm{count}(y \mid l, x) = \mathrm{count}(\mathrm{rev}(y \mid l), x)$
Induction hypothesis: $\mathrm{count}(l, x) = \mathrm{count}(\mathrm{rev}(l, x)$
where $x, y \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that $x, y, l$ are fresh constants.

We use case splitting for our proofs as follows:
**Case 1**: $y = x$

$$\begin{aligned}
\mathrm{count}(\underline{y} \mid l, x) &\longrightarrow \underline{\mathrm{count}(x \mid l, x)} && \text{(by case splitting)} \\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\mathrm{count}(l, x)) \text{ else } \mathrm{count}(l, x) \text{ fi} && \text{(by cnt2)} \\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } s(\mathrm{count}(l, x)) \text{ else } \mathrm{count}(l, x) \text{ fi}} && \text{(by equality)} \\
&\longrightarrow s(\underline{\mathrm{count}(l, x)}) && \text{(by if1)} \\
&\longrightarrow s(\mathrm{count}(\mathrm{rev}(l), x)) && \text{(by IH)} \\
\mathrm{count}(\mathrm{rev}(\underline{y} \mid l), x) &\longrightarrow \mathrm{count}(\underline{\mathrm{rev}(x \mid l)}, x) && \text{(by case splitting)} \\
&\longrightarrow \underline{\mathrm{count}(\mathrm{rev}(l) \mathbin{@} (x \mid nil), x)} && \text{(by rev2)}
\end{aligned}$$

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

$$
\begin{aligned}
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) + \underline{\mathrm{count}(x \mid nil, x)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 1)} \\
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) + \text{ if } \underline{(x = x)} \text{ then } s(\mathrm{count}(nil, x)) \\
&\quad \text{else } \mathrm{count}(nil, x) \text{ fi} \qquad\qquad \text{(by cnt2)} \\
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) + \underline{\text{if } true \text{ then } s(\mathrm{count}(nil, x))} \\
&\quad \underline{\text{else } \mathrm{count}(nil, x) \text{ fi}} \qquad\qquad \text{(by equality)} \\
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) + s(\underline{\mathrm{count}(nil, x)}) \qquad \text{(by if1)} \\
&\longrightarrow \underline{\mathrm{count}(\mathrm{rev}(l), x) + s(0)} \qquad \text{(by cnt1)} \\
&\longrightarrow \underline{s(0) + \mathrm{count}(\mathrm{rev}(l), x)} \qquad \text{(by comm+)} \\
&\longrightarrow s(\underline{0 + \mathrm{count}(\mathrm{rev}(l), x)}) \qquad \text{(by +2)} \\
&\longrightarrow s(\mathrm{count}(\mathrm{rev}(l), x)) \qquad \text{(by +1)}
\end{aligned}
$$

**Case 2**: $(y = x) = false$

$$
\begin{aligned}
\underline{\mathrm{count}(y \mid l, x)} &\longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\mathrm{count}(l, x)) \text{ else } \mathrm{count}(l, x) \text{ fi} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by cnt2)} \\
&\longrightarrow \underline{\text{if } false \text{ then } s(\mathrm{count}(l, x)) \text{ else } \mathrm{count}(l, x) \text{ fi}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \underline{\mathrm{count}(l, x)} \qquad \text{(by if2)} \\
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) \qquad \text{(by IH)} \\
\mathrm{count}(\underline{\mathrm{rev}(y \mid l)}, x) &\longrightarrow \underline{\mathrm{count}(\mathrm{rev}(l) \mathbin{@} (y \mid nil), x)} \qquad \text{(by rev2)} \\
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) + \underline{\mathrm{count}(y \mid nil, x)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 1)} \\
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) + \text{ if } \underline{(y = x)} \text{ then } s(\mathrm{count}(nil, x)) \\
&\quad \text{else } \mathrm{count}(nil, x) \text{ fi} \qquad\qquad \text{(by cnt2)} \\
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) + \underline{\text{if } false \text{ then } s(\mathrm{count}(nil, x))} \\
&\quad \underline{\text{else } \mathrm{count}(nil, x) \text{ fi}} \qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) + \underline{\mathrm{count}(nil, x)} \qquad \text{(by if2)} \\
&\longrightarrow \underline{\mathrm{count}(\mathrm{rev}(l), x) + 0} \qquad \text{(by cnt1)} \\
&\longrightarrow \underline{0 + \mathrm{count}(\mathrm{rev}(l), x)} \qquad \text{(by comm+)} \\
&\longrightarrow \mathrm{count}(\mathrm{rev}(l), x) \qquad \text{(by +1)}
\end{aligned}
$$

$\square$

**Lemma 1.** $\forall X \in \texttt{PNat}, \forall L1, L2 \in \texttt{NatList}, \text{count}(L1 \mathbin{@} L2, X) = \text{count}(L1, X) + \text{count}(L2, X)$.

*Proof.* By structural induction on $L1$.

**(1) Base case**

What to show: $\text{count}(nil \mathbin{@} l2, x) = \text{count}(nil, x) + \text{count}(l2, x)$
where $x \in \texttt{PNat}$ and $l2 \in \texttt{NatList}$. Note that $x, l2$ are fresh constants.

$$\begin{aligned}
\text{count}(\underline{nil \mathbin{@} l2}, x) &\longrightarrow \text{count}(l2, x) && \text{(by @1)}\\
\underline{\text{count}(nil, x)} + \text{count}(l2, x) &\longrightarrow \underline{0 + \text{count}(l2, x)} && \text{(by cnt1)}\\
&\longrightarrow \text{count}(l2, x) && \text{(by +1)}
\end{aligned}$$

**(2) Induction case**

What to show: $\text{count}((y \mid l1) \mathbin{@} l2, x) = \text{count}(y \mid l1, x) + \text{count}(l2, x)$
Induction hypothesis: $\text{count}(l1 \mathbin{@} l2, x) = \text{count}(l1, x) + \text{count}(l2, x)$
where $x, y \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$. Note that $x, y, l1, l2$ are fresh constants.

We use case splitting for our proofs as follows:
**Case 1**: $y = x$

$$\begin{aligned}
\text{count}((\underline{y} \mid l1) \mathbin{@} l2, x) &\longrightarrow \text{count}(\underline{(x \mid l1) \mathbin{@} l2}, x)\\
&\qquad\qquad \text{(by case splitting)}\\
&\longrightarrow \underline{\text{count}(x \mid (l1 \mathbin{@} l2), x)} \qquad \text{(by @2)}\\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))\\
&\qquad \text{else count}(l1 \mathbin{@} l2, x) \text{ fi} \qquad \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))}\\
&\qquad \underline{\text{else count}(l1 \mathbin{@} l2, x) \text{ fi}}\\
&\qquad\qquad \text{(by equality)}\\
&\longrightarrow s(\underline{\text{count}(l1 \mathbin{@} l2, x)}) \qquad \text{(by if1)}\\
&\longrightarrow s(\text{count}(l1, x) + \text{count}(l2, x))\\
&\qquad\qquad \text{(by IH)}\\
\text{count}(\underline{y} \mid l1, x) + \text{count}(l2, x) &\longrightarrow \underline{\text{count}(x \mid l1, x)} + \text{count}(l2, x)\\
&\qquad\qquad \text{(by case splitting)}\\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l1, x))\\
&\qquad \text{else count}(l1, x) \text{ fi } + \text{count}(l2, x)\\
&\qquad\qquad \text{(by cnt2)}
\end{aligned}$$

$$
\begin{aligned}
&\longrightarrow \underline{\text{if } \mathit{true} \text{ then } s(\text{count}(l1, x))}\\
&\quad \underline{\text{else count}(l1, x) \text{ fi}} + \text{count}(l2, x)\\
&\hfill \text{(by equality)}\\
&\longrightarrow \underline{s(\text{count}(l1, x)) + \text{count}(l2, x)}\\
&\hfill \text{(by if1)}\\
&\longrightarrow s(\text{count}(l1, x) + \text{count}(l2, x))\\
&\hfill \text{(by +2)}
\end{aligned}
$$

**Case 2**: $(y = x) = false$

$$
\begin{aligned}
\text{count}(\underline{(y \mid l1) \mathbin{@} l2}, x) &\longrightarrow \underline{\text{count}(y \mid (l1 \mathbin{@} l2), x)} \qquad \text{(by @2)}\\
&\longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))\\
&\quad\ \text{else count}(l1 \mathbin{@} l2, x) \text{ fi} \quad \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } \mathit{false} \text{ then } s(\text{count}(l1 \mathbin{@} l2, x))}\\
&\quad\ \underline{\text{else count}(l1 \mathbin{@} l2, x) \text{ fi}}\\
&\hfill \text{(by case splitting)}\\
&\longrightarrow \underline{\text{count}(l1 \mathbin{@} l2, x)} \qquad \text{(by if2)}\\
&\longrightarrow \text{count}(l1, x) + \text{count}(l2, x) \quad \text{(by IH)}\\
\underline{\text{count}(y \mid l1, x)} + \text{count}(l2, x) &\longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l1, x))\\
&\quad \text{count}(l1, x) \text{ fi } + \text{count}(l2, x)\\
&\hfill \text{(by cnt2)}\\
&\longrightarrow \underline{\text{if } \mathit{false} \text{ then } s(\text{count}(l1, x))}\\
&\quad\ \underline{\text{count}(l1, x) \text{ fi}} + \text{count}(l2, x)\\
&\hfill \text{(by case splitting)}\\
&\longrightarrow \text{count}(l1, x) + \text{count}(l2, x) \quad \text{(by if2)}
\end{aligned}
$$

$\square$