

Problem 10

Problem 1. $\forall X \in \mathbf{PNat}, \forall L \in \mathbf{NatList}, \text{drop}(L, X) = \text{rev}(\text{drop}(\text{rev}(L), X))$.

Proof. By structural induction on L .

(1) Base case

What to show: $\text{drop}(\text{nil}, x) = \text{rev}(\text{drop}(\text{rev}(\text{nil}), x))$
 where $x \in \mathbf{PNat}$. Note that x is a fresh constant¹.

$$\begin{array}{ll}
 \underline{\text{drop}(\text{nil}, x)} \longrightarrow \text{nil} & \text{(by drop1)} \\
 \text{rev}(\text{drop}(\underline{\text{rev}(\text{nil})}, x)) \longrightarrow \text{rev}(\underline{\text{drop}(\text{nil}, x)}) & \text{(by rev1)} \\
 \longrightarrow \underline{\text{rev}(\text{nil})} & \text{(by drop1)} \\
 \longrightarrow \text{nil} & \text{(by rev1)}
 \end{array}$$

(2) Induction case

What to show: $\text{drop}(y \mid l, x) = \text{rev}(\text{drop}(\text{rev}(y \mid l), x))$
 Induction hypothesis: $\text{drop}(l, x) = \text{rev}(\text{drop}(\text{rev}(l), x))$
 where $x, y \in \mathbf{PNat}$ and $l \in \mathbf{NatList}$. Note that x, y, l are fresh constants.

We use case splitting for our proofs as follows:

Case 1: $y = x$

$$\begin{array}{ll}
 \text{drop}(\underline{y} \mid l, x) \longrightarrow \underline{\text{drop}(x \mid l, x)} & \text{(by case splitting)} \\
 \longrightarrow \text{if } \underline{(x = x)} \text{ then } \text{drop}(l, x) \text{ else } (x \mid \text{drop}(l, x)) \text{ fi} & \\
 & \text{(by drop2)} \\
 \longrightarrow \underline{\text{if } \text{true} \text{ then } \text{drop}(l, x) \text{ else } (x \mid \text{drop}(l, x)) \text{ fi}} & \\
 & \text{(by equality)} \\
 \longrightarrow \underline{\text{drop}(l, x)} & \text{(by if1)} \\
 \longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x)) & \text{(by IH)}
 \end{array}$$

¹A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

$$\begin{aligned}
\text{rev}(\text{drop}(\text{rev}(\underline{y} \mid l), x)) &\longrightarrow \text{rev}(\text{drop}(\underline{\text{rev}(x \mid l)}, x)) \\
&\quad \text{(by case splitting)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l) @ (x \mid \text{nil}), x)) \quad \text{(by rev2)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \underline{\text{drop}(x \mid \text{nil}, x)}) \\
&\quad \text{(by Lemma 1)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \\
&\quad \text{(if } \underline{(x = x)} \text{ then } \text{drop}(\text{nil}, x) \text{ else } (x \mid \text{drop}(\text{nil}, x)) \text{ fi)}) \\
&\quad \text{(by drop2)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \\
&\quad \text{(if } \text{true} \text{ then } \text{drop}(\text{nil}, x) \text{ else } (x \mid \text{drop}(\text{nil}, x)) \text{ fi)}) \\
&\quad \text{(by equality)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \underline{\text{drop}(\text{nil}, x)}) \\
&\quad \text{(by if1)} \\
&\longrightarrow \text{rev}(\underline{\text{drop}(\text{rev}(l), x) @ \text{nil}}) \quad \text{(by drop1)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x)) \\
&\quad \text{(by Lemma 2 from Problem 4)}
\end{aligned}$$

Case 2: $(y = x) = \text{false}$

$$\begin{aligned}
\underline{\text{drop}(y \mid l, x)} &\longrightarrow \text{if } \underline{(y = x)} \text{ then } \text{drop}(l, x) \text{ else } (y \mid \text{drop}(l, x)) \text{ fi} \\
&\quad \text{(by drop2)} \\
&\longrightarrow \underline{\text{if } \text{false} \text{ then } \text{drop}(l, x) \text{ else } (y \mid \text{drop}(l, x)) \text{ fi}} \\
&\quad \text{(by case splitting)} \\
&\longrightarrow y \mid \underline{\text{drop}(l, x)} \quad \text{(by if2)} \\
&\longrightarrow y \mid \text{rev}(\text{drop}(\text{rev}(l), x)) \quad \text{(by IH)} \\
\text{rev}(\text{drop}(\underline{\text{rev}(y \mid l)}, x)) &\longrightarrow \text{rev}(\underline{\text{drop}(\text{rev}(l) @ (y \mid \text{nil}), x)}) \quad \text{(by rev2)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \underline{\text{drop}((y \mid \text{nil}), x)}) \\
&\quad \text{(by Lemma 1)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \\
&\quad \text{if } \underline{(y = x)} \text{ then } \text{drop}(\text{nil}, x) \text{ else } (y \mid \text{drop}(\text{nil}, x)) \text{ fi}) \\
&\quad \text{(by drop2)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ \\
&\quad \underline{\text{if } \text{false} \text{ then } \text{drop}(\text{nil}, x) \text{ else } (y \mid \text{drop}(\text{nil}, x)) \text{ fi}}) \\
&\quad \text{(by case splitting)}
\end{aligned}$$

$$\begin{aligned}
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ (y | \text{drop}(\text{nil}, x))) && \text{(by if2)} \\
&\longrightarrow \text{rev}(\text{drop}(\text{rev}(l), x) @ (y | \text{nil})) && \text{(by drop1)} \\
&\longrightarrow \text{rev}(y | \text{nil}) @ \text{rev}(\text{drop}(\text{rev}(l), x)) && \text{(by Lemma 1 from Problem 4)} \\
&\longrightarrow (\text{rev}(\text{nil}) @ (y | \text{nil})) @ \text{rev}(\text{drop}(\text{rev}(l), x)) && \text{(by rev2)} \\
&\longrightarrow (\text{nil} @ (y | \text{nil})) @ \text{rev}(\text{drop}(\text{rev}(l), x)) && \text{(by rev1)} \\
&\longrightarrow (y | \text{nil}) @ \text{rev}(\text{drop}(\text{rev}(l), x)) && \text{(by @1)} \\
&\longrightarrow y | (\text{nil} @ \text{rev}(\text{drop}(\text{rev}(l), x))) && \text{(by @2)} \\
&\longrightarrow y | \text{rev}(\text{drop}(\text{rev}(l), x)) && \text{(by @1)}
\end{aligned}$$

□

Lemma 1. $\forall X \in \text{PNat}, \forall L1, L2 \in \text{NatList}, \text{drop}(L1 @ L2, X) = \text{drop}(L1, X) @ \text{drop}(L2, X)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{drop}(\text{nil} @ l2, x) = \text{drop}(\text{nil}, x) @ \text{drop}(l2, x)$
 where $x \in \text{PNat}$ and $l2 \in \text{NatList}$. Note that $x, l2$ are fresh constants.

$$\begin{aligned}
&\text{drop}(\text{nil} @ l2, x) \longrightarrow \text{drop}(l2, x) && \text{(by @1)} \\
&\text{drop}(\text{nil}, x) @ \text{drop}(l2, x) \longrightarrow \text{nil} @ \text{drop}(l2, x) && \text{(by drop1)} \\
&\hspace{15em} \longrightarrow \text{drop}(l2, x) && \text{(by @1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{drop}((y | l1) @ l2, x) = \text{drop}(y | l1, x) @ \text{drop}(l2, x)$
 Induction hypothesis: $\text{drop}(l1 @ l2, x) = \text{drop}(l1, x) @ \text{drop}(l2, x)$
 where $x, y \in \text{PNat}$, and $l1, l2 \in \text{NatList}$. Note that $x, y, l1, l2$ are fresh constants.

We use case splitting for our proofs as follows:

Case 1: $y = x$

$$\begin{aligned}
&\text{drop}((y | l1) @ l2, x) \longrightarrow \text{drop}(\text{nil} @ l2, x) \\
&\hspace{15em} \text{(by case splitting)}
\end{aligned}$$

$$\begin{aligned}
& \longrightarrow \underline{\text{drop}(x \mid (l1 \ @ \ l2), x)} && \text{(by @2)} \\
& \longrightarrow \text{if } \underline{(x = x)} \text{ then drop}(l1 \ @ \ l2, x) \\
& \quad \text{else } (x \mid \text{drop}(l1 \ @ \ l2, x)) \text{ fi} && \text{(by drop2)} \\
& \longrightarrow \text{if } \underline{true} \text{ then drop}(l1 \ @ \ l2, x) \\
& \quad \underline{\text{else } (x \mid \text{drop}(l1 \ @ \ l2, x)) \text{ fi}} && \text{(by equality)} \\
& \longrightarrow \underline{\text{drop}(l1 \ @ \ l2, x)} && \text{(by if1)} \\
& \longrightarrow \text{drop}(l1, x) \ @ \ \text{drop}(l2, x) && \text{(by IH)} \\
\text{drop}(\underline{y \mid l1}, x) \ @ \ \text{drop}(l2, x) & \longrightarrow \underline{\text{drop}(x \mid l1, x)} \ @ \ \text{drop}(l2, x) && \text{(by case splitting)} \\
& \longrightarrow \text{if } \underline{(x = x)} \text{ then drop}(l1, x) \text{ else } (x \mid \text{drop}(l1, x)) \text{ fi} \\
& \quad @ \ \text{drop}(l2, x) && \text{(by drop2)} \\
& \longrightarrow \underline{\text{if } true \text{ then drop}(l1, x) \text{ else } (x \mid \text{drop}(l1, x)) \text{ fi}} \\
& \quad @ \ \text{drop}(l2, x) && \text{(by equality)} \\
& \longrightarrow \text{drop}(l1, x) \ @ \ \text{drop}(l2, x) && \text{(by if1)}
\end{aligned}$$

Case 2: $(y = x) = false$

$$\begin{aligned}
\text{drop}(\underline{(y \mid l1)} \ @ \ l2, x) & \longrightarrow \underline{\text{drop}(y \mid (l1 \ @ \ l2), x)} && \text{(by @2)} \\
& \longrightarrow \text{if } \underline{(y = x)} \text{ then drop}(l1 \ @ \ l2, x) \\
& \quad \text{else } (y \mid \text{drop}(l1 \ @ \ l2, x)) \text{ fi} && \text{(by drop2)} \\
& \longrightarrow \underline{\text{if } false \text{ then drop}(l1 \ @ \ l2, x)} \\
& \quad \underline{\text{else } (y \mid \text{drop}(l1 \ @ \ l2, x)) \text{ fi}} && \text{(by case splitting)} \\
& \longrightarrow y \mid \underline{\text{drop}(l1 \ @ \ l2, x)} && \text{(by if2)} \\
& \longrightarrow y \mid \text{drop}(l1, x) \ @ \ \text{drop}(l2, x) && \text{(by IH)} \\
\underline{\text{drop}(y \mid l1, x)} \ @ \ \text{drop}(l2, x) & \longrightarrow \text{if } \underline{(y = x)} \text{ then drop}(l1, x) \text{ else } (y \mid \text{drop}(l1, x)) \text{ fi} \\
& \quad @ \ \text{drop}(l2, x) && \text{(by drop2)} \\
& \longrightarrow \underline{\text{if } false \text{ then drop}(l1, x) \text{ else } (y \mid \text{drop}(l1, x)) \text{ fi}} \\
& \quad @ \ \text{drop}(l2, x) && \text{(by case splitting)} \\
& \longrightarrow \underline{(y \mid \text{drop}(l1, x)) \ @ \ \text{drop}(l2, x)} && \text{(by if2)}
\end{aligned}$$

$$\longrightarrow y \mid (\text{drop}(l1, x) @ \text{drop}(l2, x))$$

(by @2)

□