Problem 20

Problem 1. $\forall L \in \texttt{NatList}, \text{setEqual}(L, \text{rev}(L)) = true.$

Proof. By structural induction on L.

(1) Base case

What to show: $\text{setEqual}(nil, \text{rev}(nil)) = true$
where $l \in \texttt{NatList}$.

$$\begin{aligned}
\text{setEqual}(nil, \underline{\text{rev}(nil)}) &\longrightarrow \underline{\text{setEqual}(nil, nil)} && \text{(by rev1)}\\
&\longrightarrow (\underline{\text{diff}(nil, nil)} = nil)\ and\ (\text{diff}(nil, nil) = nil) && \text{(by setEq)}\\
&\longrightarrow \underline{(nil = nil)}\ and\ (\text{diff}(nil, nil) = nil) && \text{(by diff1)}\\
&\longrightarrow \underline{true\ and\ (\text{diff}(nil, nil) = nil)} && \text{(by equality)}\\
&\longrightarrow \underline{\text{diff}(nil, nil)} = nil && \text{(by and)}\\
&\longrightarrow \underline{nil = nil} && \text{(by diff1)}\\
&\longrightarrow true && \text{(by equality)}
\end{aligned}$$

(2) Induction case

What to show: $\text{setEqual}(x \mid l, \text{rev}(x \mid l)) = true$
Induction hypothesis: $\text{setEqual}(l, \text{rev}(l)) = true$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$.
We use case splitting for our proofs as follows:
Case 1: $\text{has}(\text{rev}(l), x) = true$.

$$\begin{aligned}
\text{setEqual}(x \mid l, \underline{\text{rev}(x \mid l)}) &\longrightarrow \underline{\text{setEqual}(x \mid l, \text{rev}(l)\ @\ (x \mid nil))} && \text{(by rev2)}\\
&\longrightarrow (\underline{\text{diff}(x \mid l, \text{rev}(l)\ @\ (x \mid nil))} = nil)\ and\\
&\quad (\text{diff}(\text{rev}(l)\ @\ (x \mid nil), x \mid l) = nil) && \text{(by setEq)}
\end{aligned}$$

$\longrightarrow$ $(\text{diff}(\underline{\text{diff}(x \mid l, \text{rev}(l))}, x \mid nil) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by Problem 7 - Lemma 1)

$\longrightarrow$ $(\text{diff}((\text{if } \underline{\text{has}(\text{rev}(l), x)} \text{ then diff}(l, \text{rev}(l))$
else $(x \mid \text{diff}(l, \text{rev}(l)))$ fi$), x \mid nil) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by diff2)

$\longrightarrow$ $(\text{diff}(\underline{(\text{if } true \text{ then diff}(l, \text{rev}(l))}$
$\underline{\text{else } (x \mid \text{diff}(l, \text{rev}(l)))\text{ fi})}, x \mid nil) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by case splitting)

$\longrightarrow$ $(\text{diff}(\underline{\text{diff}(l, \text{rev}(l))}, x \mid nil) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by if1)

$\longrightarrow$ $(\underline{\text{diff}(nil, x \mid nil)} = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by Problem 7)

$\longrightarrow$ $\underline{(nil = nil)}$ *and* $(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by diff1)

$\longrightarrow$ $\underline{true \textit{ and } (\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)}$
(by equality)

$\longrightarrow$ $\underline{\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l)} = nil$
(by and)

$\longrightarrow$ $(\underline{\text{diff}(\text{rev}(l), x \mid l)} \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil$
(by Problem 9 - Lemma 1)

$\longrightarrow$ $(\text{drop}(\underline{\text{diff}(\text{rev}(l), l)}, x) \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil$
(by Problem 14)

$\longrightarrow$ $(\underline{\text{drop}(nil, x)} \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil$
(by Problem 16)

$\longrightarrow$ $\underline{(nil \mathbin{@} \text{diff}(x \mid nil, x \mid l))} = nil$
(by drop1)

$\longrightarrow$ $\underline{\text{diff}(x \mid nil, x \mid l)} = nil$ (by @1)

$\longrightarrow$ $\text{drop}(\underline{\text{diff}(x \mid nil, l)}, x) = nil$
(by Problem 14)

$\longrightarrow$ drop((if $\underline{\text{has}(l, x)}$ then diff(nil, l)
else $(x \mid \text{diff}(nil, l))$ fi), $x) = nil$
(by diff2)

$\longrightarrow$ drop((if $\underline{\text{has}(\text{rev}(l), x)}$ then diff(nil, l)
else $(x \mid \text{diff}(nil, l))$ fi), $x) = nil$
(by Problem 6)

$\longrightarrow$ drop($\underline{(\text{if } true \text{ then diff}(nil, l)}$
$\underline{\text{else } (x \mid \text{diff}(nil, l)) \text{ fi})}, x) = nil$
(by case splitting)

$\longrightarrow$ drop($\underline{\text{diff}(nil, l)}, x) = nil$ (by if1)

$\longrightarrow$ $\underline{\text{drop}(nil, x)} = nil$ (by diff1)

$\longrightarrow$ $\underline{nil = nil}$ (by drop1)

$\longrightarrow$ $true$ (by equality)

Case 2: has(rev$(l), x) = false$.

setEqual$(x \mid l, \underline{\text{rev}(x \mid l)})$ $\longrightarrow$ $\underline{\text{setEqual}(x \mid l, \text{rev}(l) \text{ @ } (x \mid nil))}$
(by rev2)

$\longrightarrow$ $(\underline{\text{diff}(x \mid l, \text{rev}(l) \text{ @ } (x \mid nil))} = nil)$ *and*
$(\text{diff}(\text{rev}(l) \text{ @ } (x \mid nil), x \mid l) = nil)$
(by setEq)

$\longrightarrow$ $(\text{diff}(\underline{\text{diff}(x \mid l, \text{rev}(l))}, x \mid nil) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \text{ @ } (x \mid nil), x \mid l) = nil)$
(by Problem 7 - Lemma 1)

$\longrightarrow$ $(\text{diff}((\text{if } \underline{\text{has}(\text{rev}(l), x)} \text{ then diff}(l, \text{rev}(l))$
else $(x \mid \text{diff}(l, \text{rev}(l))) \text{ fi}), x \mid nil) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \text{ @ } (x \mid nil), x \mid l) = nil)$
(by diff2)

$\longrightarrow$ $(\text{diff}(\underline{(\text{if } false \text{ then diff}(l, \text{rev}(l))}$
$\underline{\text{else } (x \mid \text{diff}(l, \text{rev}(l))) \text{ fi})}, x \mid nil) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \text{ @ } (x \mid nil), x \mid l) = nil)$
(by case splitting)

$\longrightarrow$ $(\text{diff}(x \mid \underline{\text{diff}(l, \text{rev}(l))}, x \mid nil) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \text{ @ } (x \mid nil), x \mid l) = nil)$
(by if2)

$\longrightarrow$ ($\underline{\text{diff}(x \mid nil, x \mid nil)} = nil$) *and*
(diff(rev(l) @ ($x \mid nil$), $x \mid l$) = nil)
(by Problem 7)

$\longrightarrow$ (drop($\underline{\text{diff}(x \mid nil, l)}, x$) = nil) *and*
(diff(rev(l) @ ($x \mid nil$), $x \mid l$) = nil)
(by Problem 14)

$\longrightarrow$ (drop((if $\underline{\text{has}(l, x)}$ then diff(nil, l)
else ($x \mid$ diff(nil, l)) fi), x) = nil) *and*
(diff(rev(l) @ ($x \mid nil$), $x \mid l$) = nil)
(by diff2)

$\longrightarrow$ (drop((if $\underline{\text{has}(\text{rev}(l), x)}$ then diff(nil, l)
else ($x \mid$ diff(nil, l)) fi), x) = nil) *and*
(diff(rev(l) @ ($x \mid nil$), $x \mid l$) = nil)
(by Problem 6)

$\longrightarrow$ (drop($\underline{\text{(if } false \text{ then diff}(nil, l)}$
$\underline{\text{else } (x \mid \text{diff}(nil, l)) \text{ fi})}, x$) = nil) *and*
(diff(rev(l) @ ($x \mid nil$), $x \mid l$) = nil)
(by case splitting)

$\longrightarrow$ (drop($x \mid \underline{\text{diff}(nil, l)}, x$) = nil) *and*
(diff(rev(l) @ ($x \mid nil$), $x \mid l$) = nil)
(by if2)

$\longrightarrow$ ($\underline{\text{drop}(x \mid nil, x)} = nil$) *and*
(diff(rev(l) @ ($x \mid nil$), $x \mid l$) = nil)
(by diff1)

$\longrightarrow$ ((if $\underline{x = x}$ then drop(nil, x)
else ($x \mid$ drop(nil, x)) fi) = nil) *and*
(diff(rev(l) @ ($x \mid nil$), $x \mid l$) = nil)
(by drop2)

$\longrightarrow$ ($\underline{\text{(if } true \text{ then drop}(nil, x)}$
$\underline{\text{else } (x \mid \text{drop}(nil, x)) \text{ fi)}} = nil$) *and*
(diff(rev(l) @ ($x \mid nil$), $x \mid l$) = nil)
(by equality)

$\longrightarrow$ ($\underline{\text{drop}(nil, x)} = nil$) *and*

$$
\begin{aligned}
&(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by if1)} \\
\longrightarrow\ &\underline{(nil = nil)}\ \mathit{and}\ (\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by drop1)} \\
\longrightarrow\ &\underline{\mathit{true}\ \mathit{and}\ (\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by equality)} \\
\longrightarrow\ &\underline{\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l)} = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by and)} \\
\longrightarrow\ &(\underline{\text{diff}(\text{rev}(l), x \mid l)} \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Problem 9 - Lemma 1)} \\
\longrightarrow\ &(\text{drop}(\underline{\text{diff}(\text{rev}(l), l)}, x) \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Problem 14)} \\
\longrightarrow\ &(\underline{\text{drop}(nil, x)} \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Problem 16)} \\
\longrightarrow\ &\underline{(nil \mathbin{@} \text{diff}(x \mid nil, x \mid l))} = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by drop1)} \\
\longrightarrow\ &\underline{\text{diff}(x \mid nil, x \mid l)} = nil \qquad\qquad \text{(by @1)} \\
\longrightarrow\ &\text{drop}(\underline{\text{diff}(x \mid nil, l)}, x) = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Problem 14)} \\
\longrightarrow\ &\text{drop}((\text{if } \underline{\text{has}(l, x)} \text{ then } \text{diff}(nil, l) \\
&\quad \text{else } (x \mid \text{diff}(nil, l)) \text{ fi}), x) = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by diff2)} \\
\longrightarrow\ &\text{drop}((\text{if } \underline{\text{has}(\text{rev}(l), x)} \text{ then } \text{diff}(nil, l) \\
&\quad \text{else } (x \mid \text{diff}(nil, l)) \text{ fi}), x) = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Problem 6)} \\
\longrightarrow\ &\text{drop}(\underline{(\text{if } \mathit{false} \text{ then } \text{diff}(nil, l)} \\
&\quad \underline{\text{else } (x \mid \text{diff}(nil, l)) \text{ fi}}), x) = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)} \\
\longrightarrow\ &\text{drop}(x \mid \underline{\text{diff}(nil, l)}, x) = nil \qquad\qquad \text{(by if2)} \\
\longrightarrow\ &\underline{\text{drop}(x \mid nil, x)} = nil \qquad\qquad \text{(by diff1)} \\
\longrightarrow\ &(\text{if } \underline{x = x} \text{ then } \text{drop}(nil, x) \text{ else } (x \mid \text{drop}(nil, x)) \text{ fi}) = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by drop2)} \\
\longrightarrow\ &\underline{(\text{if } \mathit{true} \text{ then } \text{drop}(nil, x) \text{ else } (x \mid \text{drop}(nil, x)) \text{ fi})} = nil \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by equality)}
\end{aligned}
$$

$\longrightarrow \underline{\text{drop}(nil, x)} = nil$ (by if1)

$\longrightarrow \underline{nil = nil}$ (by drop1)

$\longrightarrow true$ (by equality)

□