Problem 18

Problem 1. $\forall L1, L2 \in \texttt{NatList}, \text{rmDup}(L1 \mathbin{@} L2) = \text{rmDup}(\text{rmDup}(L1) \mathbin{@} \text{rmDup}(L2))$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{rmDup}(nil \mathbin{@} l2) = \text{rmDup}(\text{rmDup}(nil) \mathbin{@} \text{rmDup}(l2))$
where $l2 \in \texttt{NatList}$.

$$\begin{aligned}
\text{rmDup}(\underline{nil \mathbin{@} l2}) &\longrightarrow \text{rmDup}(l2) && \text{(by @1)} \\
\text{rmDup}(\underline{\text{rmDup}(nil)} \mathbin{@} \text{rmDup}(l2)) &\longrightarrow \text{rmDup}(\underline{nil \mathbin{@} \text{rmDup}(l2)}) \\
&&& \text{(by rmDup1)} \\
&\longrightarrow \underline{\text{rmDup}(\text{rmDup}(l2))} && \text{(by @1)} \\
&\longrightarrow \text{rmDup}(l2) && \text{(by Problem 17)}
\end{aligned}$$

(2) Induction case

What to show: $\text{rmDup}((x \mid l1) \mathbin{@} l2) = \text{rmDup}(\text{rmDup}(x \mid l1) \mathbin{@} \text{rmDup}(l2))$
Induction hypothesis: $\text{rmDup}(l1 \mathbin{@} l2) = \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))$
where $x \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$.
We use case splitting for our proofs as follows:

Case 1: $\text{has}(l1, x) = true$.

$$\begin{aligned}
\text{rmDup}(\underline{(x \mid l1) \mathbin{@} l2}) &\longrightarrow \underline{\text{rmDup}(x \mid (l1 \mathbin{@} l2))} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by @2)} \\
&\longrightarrow \text{if } \underline{\text{has}(l1 \mathbin{@} l2, x)} \text{ then rmDup}(l1 \mathbin{@} l2) \\
&\quad \text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by rmDup2)} \\
&\longrightarrow \text{if } (\underline{\text{has}(l1, x)} \textit{ or } \text{has}(l2, x)) \text{ then rmDup}(l1 \mathbin{@} l2) \\
&\quad \text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi} \\
&\qquad\qquad\qquad\qquad\qquad\quad \text{(by Problem 6 - Lemma 1)} \\
&\longrightarrow \text{if } \underline{(true \textit{ or } \text{has}(l2, x))} \text{ then rmDup}(l1 \mathbin{@} l2) \\
&\quad \text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \underline{\text{if } true \text{ then rmDup}(l1 \mathbin{@} l2)} \\
&\quad \underline{\text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by or)} \\
&\longrightarrow \underline{\text{rmDup}(l1 \mathbin{@} l2)} \qquad \text{(by if1)} \\
&\longrightarrow \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by IH)} \\
\text{rmDup}(\underline{\text{rmDup}(x \mid l1)} \mathbin{@} \text{rmDup}(l2)) &\longrightarrow \text{rmDup}((\text{if } \underline{\text{has}(l1, x)} \text{ then rmDup}(l1) \\
&\quad \text{else } (x \mid \text{rmDup}(l1)) \text{ fi}) \mathbin{@} \text{rmDup}(l2)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by rmDup2)} \\
&\longrightarrow \text{rmDup}((\underline{\text{if } true \text{ then rmDup}(l1)} \\
&\quad \underline{\text{else } (x \mid \text{rmDup}(l1)) \text{ fi}}) \mathbin{@} \text{rmDup}(l2)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by if1)}
\end{aligned}$$

Case 2: $\text{has}(l1, x) = false$.

$$
\begin{aligned}
\text{rmDup}(\underline{(x \mid l1) \mathbin{@} l2}) &\longrightarrow \underline{\text{rmDup}(x \mid (l1 \mathbin{@} l2))} && \text{(by @2)}\\
&\longrightarrow \text{if } \underline{\text{has}(l1 \mathbin{@} l2, x)} \text{ then rmDup}(l1 \mathbin{@} l2)\\
&\quad \text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi} && \text{(by rmDup2)}\\
&\longrightarrow \text{if } (\underline{\text{has}(l1, x)} \textit{ or } \text{has}(l2, x)) \text{ then rmDup}(l1 \mathbin{@} l2)\\
&\quad \text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi}\\
&\qquad\qquad \text{(by Problem 6 - Lemma 1)}\\
&\longrightarrow \text{if } \underline{(false \textit{ or } \text{has}(l2, x))} \text{ then rmDup}(l1 \mathbin{@} l2)\\
&\quad \text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi}\\
&\qquad\qquad\qquad \text{(by case splitting)}\\
&\longrightarrow \text{if has}(l2, x) \text{ then } \underline{\text{rmDup}(l1 \mathbin{@} l2)}\\
&\quad \text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi} && \text{(by or)}\\
&\longrightarrow \text{if has}(l2, x) \text{ then rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))\\
&\quad \text{else } (x \mid \underline{\text{rmDup}(l1 \mathbin{@} l2)}) \text{ fi} && \text{(by IH)}\\
&\longrightarrow \text{if has}(l2, x) \text{ then rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))\\
&\quad \text{else } (x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))) \text{ fi}\\
&\qquad\qquad\qquad \text{(by IH)}
\end{aligned}
$$

$\text{rmDup}(\underline{\text{rmDup}(x \mid l1)} \mathbin{@} \text{rmDup}(l2)) \longrightarrow \text{rmDup}((\text{if } \underline{\text{has}(l1, x)} \text{ then rmDup}(l1)$
$\text{else } (x \mid \text{rmDup}(l1)) \text{ fi}) \mathbin{@} \text{rmDup}(l2))$
(by rmDup2)
$\longrightarrow \text{rmDup}((\underline{\text{if } false \text{ then rmDup}(l1)}$
$\underline{\text{else } (x \mid \text{rmDup}(l1)) \text{ fi}}) \mathbin{@} \text{rmDup}(l2))$
(by case splitting)
$\longrightarrow \text{rmDup}(\underline{(x \mid \text{rmDup}(l1)) \mathbin{@} \text{rmDup}(l2)})$
(by if2)
$\longrightarrow \underline{\text{rmDup}(x \mid (\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)))}$
(by @2)
$\longrightarrow \text{if } \underline{\text{has}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2), x)}$
$\text{then rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))$
$\text{else } (x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))) \text{ fi}$
(by rmDup2)
$\longrightarrow \text{if } \underline{\text{has}(\text{rmDup}(l1), x)} \textit{ or } \text{has}(\text{rmDup}(l2), x)$
$\text{then rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))$
$\text{else } (x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))) \text{ fi}$
(by Problem 6 - Lemma 1)
$\longrightarrow \text{if } \underline{\text{has}(l1, x)} \textit{ or } \text{has}(\text{rmDup}(l2), x)$
$\text{then rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))$
$\text{else } (x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))) \text{ fi}$
(by Problem 17 - Lemma 1)
$\longrightarrow \text{if } \underline{false \textit{ or } \text{has}(\text{rmDup}(l2), x)}$
$\text{then rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))$
$\text{else } (x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))) \text{ fi}$
(by case splitting)
$\longrightarrow \text{if } \underline{\text{has}(\text{rmDup}(l2), x)}$
$\text{then rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))$
$\text{else } (x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))) \text{ fi}$
(by or)
$\longrightarrow \text{if has}(l2, x)$
$\text{then rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))$
$\text{else } (x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))) \text{ fi}$
(by Problem 17 - Lemma 1)

□