

## Problem 13

**Problem 1.**  $\forall X \in \mathbf{PNat}, \forall L \in \mathbf{NatList}, \text{count}(L, X) = \text{count}(\text{rev}(L), X)$ .

*Proof.* By structural induction on  $L$ .

**(1) Base case**

What to show:  $\text{count}(\text{nil}, x) = \text{count}(\text{rev}(\text{nil}), x)$   
where  $x \in \mathbf{PNat}$ .

$$\begin{array}{ll} \frac{}{\text{count}(\text{nil}, x) \longrightarrow 0} & \text{(by cnt1)} \\ \text{count}(\text{rev}(\text{nil}), x) \longrightarrow \frac{}{\text{count}(\text{nil}, x)} & \text{(by rev1)} \\ & \longrightarrow 0 \quad \text{(by cnt1)} \end{array}$$

**(2) Induction case**

What to show:  $\text{count}(y \mid l, x) = \text{count}(\text{rev}(y \mid l), x)$

Induction hypothesis:  $\text{count}(l, x) = \text{count}(\text{rev}(l), x)$

where  $x, y \in \mathbf{PNat}$  and  $l \in \mathbf{NatList}$ .

We use case splitting for our proofs as follows:

**Case 1:**  $y = x$

$$\begin{aligned}
\text{count}(\underline{y} \mid l, x) &\longrightarrow \underline{\text{count}(x \mid l, x)} && \text{(by case splitting)} \\
&\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l, x)) \text{ else } \text{count}(l, x) \text{ fi} && \text{(by cnt2)} \\
&\longrightarrow \underline{\text{if } true \text{ then } s(\text{count}(l, x)) \text{ else } \text{count}(l, x) \text{ fi}} && \text{(by equality)} \\
&\longrightarrow s(\underline{\text{count}(l, x)}) && \text{(by if1)} \\
&\longrightarrow s(\text{count}(\text{rev}(l), x)) && \text{(by IH)} \\
\text{count}(\text{rev}(\underline{y} \mid l), x) &\longrightarrow \text{count}(\underline{\text{rev}(x \mid l)}, x) && \text{(by case splitting)} \\
&\longrightarrow \underline{\text{count}(\text{rev}(l) @ (x \mid nil), x)} && \text{(by rev2)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{count}(x \mid nil, x)} && \text{(by Lemma 1)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \text{if } \underline{(x = x)} \text{ then } s(\text{count}(nil, x)) && \text{(by cnt2)} \\
&\quad \text{else } \text{count}(nil, x) \text{ fi} && \text{(by cnt2)} \\
&\longrightarrow \underline{\text{count}(\text{rev}(l), x) + \text{if } true \text{ then } s(\text{count}(nil, x))} && \text{(by equality)} \\
&\quad \text{else } \text{count}(nil, x) \text{ fi}} && \text{(by equality)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + s(\underline{\text{count}(nil, x)}) && \text{(by if1)} \\
&\longrightarrow \underline{\text{count}(\text{rev}(l), x) + s(0)} && \text{(by cnt1)} \\
&\longrightarrow \underline{s(0) + \text{count}(\text{rev}(l), x)} && \text{(by comm+)} \\
&\longrightarrow s(\underline{0 + \text{count}(\text{rev}(l), x)}) && \text{(by +2)} \\
&\longrightarrow s(\text{count}(\text{rev}(l), x)) && \text{(by +1)}
\end{aligned}$$

**Case 2:**  $(y = x) = false$

$$\begin{aligned}
\underline{\text{count}(y \mid l, x)} &\longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l, x)) \text{ else } \text{count}(l, x) \text{ fi} && \text{(by cnt2)} \\
&\longrightarrow \underline{\text{if } false \text{ then } s(\text{count}(l, x)) \text{ else } \text{count}(l, x) \text{ fi}} && \text{(by case splitting)} \\
&\longrightarrow \underline{\text{count}(l, x)} && \text{(by if2)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) && \text{(by IH)} \\
\text{count}(\underline{\text{rev}(y \mid l)}, x) &\longrightarrow \underline{\text{count}(\text{rev}(l) @ (y \mid nil), x)} && \text{(by rev2)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{count}(y \mid nil, x)} && \text{(by Lemma 1)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \text{if } \underline{(y = x)} \text{ then } s(\text{count}(nil, x)) && \\
&\quad \text{else } \text{count}(nil, x) \text{ fi} && \text{(by cnt2)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{if } false \text{ then } s(\text{count}(nil, x))} && \\
&\quad \underline{\text{else } \text{count}(nil, x) \text{ fi}} && \text{(by case splitting)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) + \underline{\text{count}(nil, x)} && \text{(by if2)} \\
&\longrightarrow \underline{\text{count}(\text{rev}(l), x) + 0} && \text{(by cnt1)} \\
&\longrightarrow \underline{0 + \text{count}(\text{rev}(l), x)} && \text{(by comm+)} \\
&\longrightarrow \text{count}(\text{rev}(l), x) && \text{(by +1)}
\end{aligned}$$

□

**Lemma 1.**  $\forall X \in \text{PNat}, \forall L1, L2 \in \text{NatList}, \text{count}(L1 @ L2, X) = \text{count}(L1, X) + \text{count}(L2, X)$ .

*Proof.* By structural induction on  $L1$ .

**(1) Base case**

What to show:  $\text{count}(nil @ l2, x) = \text{count}(nil, x) + \text{count}(l2, x)$   
 where  $x \in \text{PNat}$  and  $l2 \in \text{NatList}$ .

$$\begin{aligned}
&\text{count}(\underline{nil @ l2}, x) \longrightarrow \text{count}(l2, x) && \text{(by @1)} \\
\underline{\text{count}(nil, x) + \text{count}(l2, x)} &\longrightarrow \underline{0 + \text{count}(l2, x)} && \text{(by cnt1)} \\
&\longrightarrow \text{count}(l2, x) && \text{(by +1)}
\end{aligned}$$

**(2) Induction case**

What to show:  $\text{count}((y \mid l1) @ l2, x) = \text{count}(y \mid l1, x) + \text{count}(l2, x)$

Induction hypothesis:  $\text{count}(l1 \text{ @ } l2, x) = \text{count}(l1, x) + \text{count}(l2, x)$   
 where  $x, y \in \text{PNat}$  and  $l1, l2 \in \text{NatList}$ .

We use case splitting for our proofs as follows:

**Case 1:**  $y = x$

$$\begin{aligned}
 \text{count}(\underline{y} \mid l1) \text{ @ } l2, x &\longrightarrow \text{count}(\underline{x \mid l1} \text{ @ } l2, x) && \text{(by case splitting)} \\
 &\longrightarrow \underline{\text{count}(x \mid (l1 \text{ @ } l2), x)} && \text{(by @2)} \\
 &\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l1 \text{ @ } l2, x)) \\
 &\quad \text{else } \text{count}(l1 \text{ @ } l2, x) \text{ fi} && \text{(by cnt2)} \\
 &\longrightarrow \underline{\text{if } true \text{ then } s(\text{count}(l1 \text{ @ } l2, x))} \\
 &\quad \underline{\text{else } \text{count}(l1 \text{ @ } l2, x) \text{ fi}} && \text{(by equality)} \\
 &\longrightarrow s(\underline{\text{count}(l1 \text{ @ } l2, x)}) && \text{(by if1)} \\
 &\longrightarrow s(\text{count}(l1, x) + \text{count}(l2, x)) && \text{(by IH)} \\
 \text{count}(\underline{y} \mid l1, x) + \text{count}(l2, x) &\longrightarrow \underline{\text{count}(x \mid l1, x) + \text{count}(l2, x)} && \text{(by case splitting)} \\
 &\longrightarrow \text{if } \underline{(x = x)} \text{ then } s(\text{count}(l1, x)) \\
 &\quad \text{else } \text{count}(l1, x) \text{ fi} + \text{count}(l2, x) && \text{(by cnt2)} \\
 &\longrightarrow \underline{\text{if } true \text{ then } s(\text{count}(l1, x))} \\
 &\quad \underline{\text{else } \text{count}(l1, x) \text{ fi} + \text{count}(l2, x)} && \text{(by equality)} \\
 &\longrightarrow \underline{s(\text{count}(l1, x)) + \text{count}(l2, x)} && \text{(by if1)} \\
 &\longrightarrow s(\text{count}(l1, x) + \text{count}(l2, x)) && \text{(by +2)}
 \end{aligned}$$

**Case 2:**  $(y = x) = false$

$$\begin{aligned}
& \text{count}(\underline{(y \mid l1) @ l2}, x) \longrightarrow \text{count}(y \mid (l1 @ l2), x) \quad (\text{by @2}) \\
& \longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l1 @ l2, x)) \\
& \quad \text{else count}(l1 @ l2, x) \text{ fi} \quad (\text{by cnt2}) \\
& \longrightarrow \underline{\text{if } false \text{ then } s(\text{count}(l1 @ l2, x))} \\
& \quad \underline{\text{else count}(l1 @ l2, x) \text{ fi}} \\
& \quad \quad (\text{by case splitting}) \\
& \longrightarrow \underline{\text{count}(l1 @ l2, x)} \quad (\text{by if2}) \\
& \longrightarrow \text{count}(l1, x) + \text{count}(l2, x) \quad (\text{by IH}) \\
& \underline{\text{count}(y \mid l1, x) + \text{count}(l2, x)} \longrightarrow \text{if } \underline{(y = x)} \text{ then } s(\text{count}(l1, x)) \\
& \quad \text{count}(l1, x) \text{ fi} + \text{count}(l2, x) \\
& \quad \quad (\text{by cnt2}) \\
& \longrightarrow \underline{\text{if } false \text{ then } s(\text{count}(l1, x))} \\
& \quad \underline{\text{count}(l1, x) \text{ fi} + \text{count}(l2, x)} \\
& \quad \quad (\text{by case splitting}) \\
& \longrightarrow \text{count}(l1, x) + \text{count}(l2, x) \quad (\text{by if2})
\end{aligned}$$

□