

## Problem 9

**Problem 1.**  $\forall L1, L2 \in \text{NatList}, \text{diff}(L1, L2) = \text{rev}(\text{diff}(\text{rev}(L1), L2))$ .

*Proof.* By structural induction on  $L$ .

**(1) Base case**

What to show:  $\text{diff}(\text{nil}, l2) = \text{rev}(\text{diff}(\text{rev}(\text{nil}), l2))$   
 where  $l2 \in \text{NatList}$ . Note that  $l2$  is a fresh constant<sup>1</sup>.

$$\begin{aligned}
 \frac{}{\text{diff}(\text{nil}, l2) \longrightarrow \text{nil}} & \quad (\text{by diff1}) \\
 \text{rev}(\text{diff}(\text{rev}(\text{nil}), l2)) & \longrightarrow \text{rev}(\text{diff}(\text{nil}, l2)) & (\text{by rev1}) \\
 & \longrightarrow \text{rev}(\text{nil}) & (\text{by diff1}) \\
 & \longrightarrow \text{nil} & (\text{by rev1})
 \end{aligned}$$

**(2) Induction case**

What to show:  $\text{diff}(x \mid l1, l2) = \text{rev}(\text{diff}(\text{rev}(x \mid l1), l2))$   
 Induction hypothesis:  $\text{diff}(l1, l2) = \text{rev}(\text{diff}(\text{rev}(l1), l2))$   
 where  $x \in \text{PNat}$  and  $l1, l2 \in \text{NatList}$ . Note that  $x, l1, l2$  are fresh constants.

We use case splitting for our proofs as follows:

**Case 1:**  $\text{has}(l2, x) = \text{true}$

$$\begin{aligned}
 \frac{}{\text{diff}(x \mid l1, l2) \longrightarrow \text{if } \text{has}(l2, x) \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}} & \quad (\text{by diff2}) \\
 \longrightarrow \frac{\text{if } \text{true} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}}{} & \quad (\text{by case splitting}) \\
 \longrightarrow \text{diff}(l1, l2) & \quad (\text{by if1}) \\
 \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2)) & \quad (\text{by IH}) \\
 \text{rev}(\text{diff}(\text{rev}(x \mid l1), l2)) \longrightarrow \text{rev}(\text{diff}(\text{rev}(l1) @ (x \mid \text{nil}), l2)) & \quad (\text{by rev2})
 \end{aligned}$$

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<sup>1</sup>A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

$$\begin{aligned}
&\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \underline{\text{diff}(x \mid \text{nil}, l2)}) \\
&\hspace{15em} \text{(by Lemma 1)} \\
&\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
&(\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi})) \\
&\hspace{15em} \text{(by diff2)} \\
&\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
&\underline{(\text{if } \text{true} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi}))} \\
&\hspace{15em} \text{(by diff2)} \\
&\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \underline{\text{diff}(\text{nil}, l2)}) \\
&\hspace{15em} \text{(by if1)} \\
&\longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1), l2) @ \text{nil}}) \hspace{2em} \text{(by diff1)} \\
&\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2)) \\
&\hspace{10em} \text{(by Lemma 2 from Problem 4)}
\end{aligned}$$

**Case 2:**  $\text{has}(l2, x) = \text{false}$

$$\begin{aligned}
&\underline{\text{diff}(x \mid l1, l2)} \longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi} \\
&\hspace{15em} \text{(by diff2)} \\
&\longrightarrow \underline{\text{if } \text{false} \text{ then } \text{diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}} \\
&\hspace{15em} \text{(by case splitting)} \\
&\longrightarrow x \mid \underline{\text{diff}(l1, l2)} \hspace{10em} \text{(by if2)} \\
&\longrightarrow x \mid \text{rev}(\text{diff}(\text{rev}(l1), l2)) \hspace{10em} \text{(by IH)} \\
&\text{rev}(\text{diff}(\underline{\text{rev}(x \mid l1)}, l2)) \longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1) @ (x \mid \text{nil}), l2)}) \\
&\hspace{15em} \text{(by rev2)} \\
&\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \underline{\text{diff}(x \mid \text{nil}, l2)}) \\
&\hspace{15em} \text{(by Lemma 1)} \\
&\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
&(\text{if } \underline{\text{has}(l2, x)} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi})) \\
&\hspace{15em} \text{(by diff2)} \\
&\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ \\
&\underline{(\text{if } \text{false} \text{ then } \text{diff}(\text{nil}, l2) \text{ else } (x \mid \text{diff}(\text{nil}, l2)) \text{ fi}))} \\
&\hspace{15em} \text{(by case splitting)} \\
&\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) @ (x \mid \underline{\text{diff}(\text{nil}, l2)})) \\
&\hspace{15em} \text{(by if2)}
\end{aligned}$$

$$\begin{aligned}
&\longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1), l2) @ (x \mid \text{nil})}) && \text{(by diff1)} \\
&\longrightarrow \underline{\text{rev}(x \mid \text{nil})} @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) && \text{(by Lemma 1 from Problem 4)} \\
&\longrightarrow (\underline{\text{rev}(\text{nil})} @ (x \mid \text{nil})) @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) && \text{(by rev2)} \\
&\longrightarrow (\underline{\text{nil} @ (x \mid \text{nil})}) @ \text{rev}(\text{diff}(\text{rev}(l1), l2)) && \text{(by rev1)} \\
&\longrightarrow (x \mid \text{nil}) @ \underline{\text{rev}(\text{diff}(\text{rev}(l1), l2))} && \text{(by @1)} \\
&\longrightarrow x \mid (\underline{\text{nil} @ \text{rev}(\text{diff}(\text{rev}(l1), l2))}) && \text{(by @2)} \\
&\longrightarrow x \mid \text{rev}(\text{diff}(\text{rev}(l1), l2)) && \text{(by @1)}
\end{aligned}$$

□

**Lemma 1.**  $\forall L1, L2, L3 \in \text{NatList}, \text{diff}(L1 @ L2, L3) = \text{diff}(L1, L3) @ \text{diff}(L2, L3)$ .

*Proof.* By structural induction on  $L1$ .

**(1) Base case**

What to show:  $\text{diff}(\text{nil} @ l2, l3) = \text{diff}(\text{nil}, l3) @ \text{diff}(l2, l3)$   
 where  $l2, l3 \in \text{NatList}$ . Note that  $l2, l3$  are fresh constants.

$$\begin{aligned}
&\text{diff}(\text{nil} @ l2, l3) \longrightarrow \text{diff}(l2, l3) && \text{(by @1)} \\
&\underline{\text{diff}(\text{nil}, l3)} @ \text{diff}(l2, l3) \longrightarrow \underline{\text{nil} @ \text{diff}(l2, l3)} && \text{(by diff1)} \\
&\longrightarrow \text{diff}(l2, l3) && \text{(by @1)}
\end{aligned}$$

**(2) Induction case**

What to show:  $\text{diff}((x \mid l1) @ l2, l3) = \text{diff}((x \mid l1), l3) @ \text{diff}(l2, l3)$   
 Induction hypothesis:  $\text{diff}(l1 @ l2, l3) = \text{diff}(l1, l3) @ \text{diff}(l2, l3)$   
 where  $x \in \text{PNat}$  and  $l1, l2, l3 \in \text{NatList}$ . Note that  $x, l1, l2, l3$  are fresh constants.

We use case splitting for our proofs as follows:

**Case 1:**  $\text{has}(l3, x) = \text{true}$

$$\begin{aligned}
&\text{diff}(\underline{(x \mid l1) @ l2}, l3) \longrightarrow \underline{\text{diff}(x \mid (l1 @ l2), l3)} && \text{(by)} \\
&\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1 @ l2, l3) && \\
&\quad \text{else } (x \mid \text{diff}(l1 @ l2, l3)) \text{ fi} && \text{(by diff2)}
\end{aligned}$$

$$\begin{aligned}
& \longrightarrow \frac{\text{if } \underline{\text{true}} \text{ then } \text{diff}(l1 \text{ @ } l2, l3) \text{ else } (x \mid \text{diff}(l1 \text{ @ } l2, l3)) \text{ fi}}{\text{(by case splitting)}} \\
& \longrightarrow \underline{\text{diff}(l1 \text{ @ } l2, l3)} \quad \text{(by if1)} \\
& \longrightarrow \text{diff}(l1, l3) \text{ @ } \text{diff}(l2, l3) \quad \text{(by IH)} \\
\underline{\text{diff}(x \mid l1, l3) \text{ @ } \text{diff}(l2, l3)} & \longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi} \\
& \quad \text{@ } \text{diff}(l2, l3) \quad \text{(by diff2)} \\
& \longrightarrow \frac{\text{if } \underline{\text{true}} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi}}{\text{@ } \underline{\text{diff}(l2, l3)}} \quad \text{(by case splitting)} \\
& \longrightarrow \text{diff}(l1, l3) \text{ @ } \text{diff}(l2, l3) \quad \text{(by if1)}
\end{aligned}$$

**Case 2:**  $\text{has}(l3, x) = \text{false}$

$$\begin{aligned}
\underline{\text{diff}((x \mid l1) \text{ @ } l2, l3)} & \longrightarrow \underline{\text{diff}(x \mid (l1 \text{ @ } l2), l3)} \quad \text{(by @2)} \\
& \longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1 \text{ @ } l2, l3) \\
& \quad \text{else } (x \mid \text{diff}(l1 \text{ @ } l2, l3)) \text{ fi} \quad \text{(by diff2)} \\
& \longrightarrow \frac{\text{if } \underline{\text{false}} \text{ then } \text{diff}(l1 \text{ @ } l2, l3) \text{ else } (x \mid \text{diff}(l1 \text{ @ } l2, l3)) \text{ fi}}{\text{(by case splitting)}} \\
& \longrightarrow x \mid \underline{\text{diff}(l1 \text{ @ } l2, l3)} \quad \text{(by if2)} \\
& \longrightarrow x \mid (\text{diff}(l1, l3) \text{ @ } \text{diff}(l2, l3)) \quad \text{(by IH)} \\
\underline{\text{diff}(x \mid l1, l3) \text{ @ } \text{diff}(l2, l3)} & \longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi} \\
& \quad \text{@ } \text{diff}(l2, l3) \quad \text{(by diff2)} \\
& \longrightarrow \frac{\text{if } \underline{\text{false}} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi}}{\text{@ } \underline{\text{diff}(l2, l3)}} \quad \text{(by case splitting)} \\
& \longrightarrow \underline{(x \mid \text{diff}(l1, l3)) \text{ @ } \text{diff}(l2, l3)} \quad \text{(by if2)} \\
& \longrightarrow x \mid (\text{diff}(l1, l3) \text{ @ } \text{diff}(l2, l3)) \quad \text{(by @2)}
\end{aligned}$$

□