Problem 18

Problem 1. $\forall L1, L2 \in \texttt{NatList}, \text{rmDup}(L1 \mathbin{@} L2) = \text{rmDup}(\text{rmDup}(L1) \mathbin{@} \text{rmDup}(L2))$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{rmDup}(nil \mathbin{@} l2) = \text{rmDup}(\text{rmDup}(nil) \mathbin{@} \text{rmDup}(l2))$
where $l2 \in \texttt{NatList}$. Note that $l2$ is a fresh constant[1].

$$\begin{aligned}
\text{rmDup}(\underline{nil \mathbin{@} l2}) &\longrightarrow \text{rmDup}(l2) && \text{(by @1)}\\
\text{rmDup}(\underline{\text{rmDup}(nil)} \mathbin{@} \text{rmDup}(l2)) &\longrightarrow \text{rmDup}(\underline{nil \mathbin{@} \text{rmDup}(l2)}) && \text{(by rmDup1)}\\
&\longrightarrow \underline{\text{rmDup}(\text{rmDup}(l2))} && \text{(by @1)}\\
&\longrightarrow \text{rmDup}(l2) && \text{(by Problem 17)}
\end{aligned}$$

(2) Induction case

What to show: $\text{rmDup}((x \mid l1) \mathbin{@} l2) = \text{rmDup}(\text{rmDup}(x \mid l1) \mathbin{@} \text{rmDup}(l2))$
Induction hypothesis: $\text{rmDup}(l1 \mathbin{@} l2) = \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2))$
where $x \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$. Note that $x, l1, l2$ are fresh constants.

We use case splitting for our proofs as follows:
Case 1: $\text{has}(l1, x) = \mathit{true}$.

$$\begin{aligned}
\text{rmDup}(\underline{(x \mid l1) \mathbin{@} l2}) &\longrightarrow \underline{\text{rmDup}(x \mid (l1 \mathbin{@} l2))} && \text{(by @2)}\\
&\longrightarrow \text{if } \underline{\text{has}(l1 \mathbin{@} l2, x)} \text{ then } \text{rmDup}(l1 \mathbin{@} l2)\\
&\quad \text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi} && \text{(by rmDup2)}\\
&\longrightarrow \text{if } (\underline{\text{has}(l1, x)} \mathit{\ or\ } \text{has}(l2, x)) \text{ then } \text{rmDup}(l1 \mathbin{@} l2)\\
&\quad \text{else } (x \mid \text{rmDup}(l1 \mathbin{@} l2)) \text{ fi}\\
& && \text{(by Problem 6 - Lemma 1)}
\end{aligned}$$

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

$\longrightarrow$ if $\underline{(true \; or \; \text{has}(l2, x))}$ then rmDup($l1$ @ $l2$)
else (x | rmDup($l1$ @ $l2$)) fi
(by case splitting)
$\longrightarrow$ if $true$ then rmDup($l1$ @ $l2$)
else (x | rmDup($l1$ @ $l2$)) fi (by or)
$\longrightarrow$ rmDup($l1$ @ $l2$) (by if1)
$\longrightarrow$ rmDup(rmDup($l1$) @ rmDup($l2$)) (by IH)

rmDup(rmDup(x | $l1$) @ rmDup($l2$)) $\longrightarrow$ rmDup((if has($l1, x$) then rmDup($l1$)
else (x | rmDup($l1$)) fi) @ rmDup($l2$))
(by rmDup2)
$\longrightarrow$ rmDup((if $true$ then rmDup($l1$)
else (x | rmDup($l1$)) fi) @ rmDup($l2$))
(by case splitting)
$\longrightarrow$ rmDup(rmDup($l1$) @ rmDup($l2$))
(by if1)

Case 2: has($l1, x$) = $false$.

rmDup((x | $l1$) @ $l2$) $\longrightarrow$ rmDup(x | ($l1$ @ $l2$)) (by @2)
$\longrightarrow$ if has($l1$ @ $l2, x$) then rmDup($l1$ @ $l2$)
else (x | rmDup($l1$ @ $l2$)) fi (by rmDup2)
$\longrightarrow$ if (has($l1, x$) *or* has($l2, x$)) then rmDup($l1$ @ $l2$)
else (x | rmDup($l1$ @ $l2$)) fi
(by Problem 6 - Lemma 1)
$\longrightarrow$ if ($false$ *or* has($l2, x$)) then rmDup($l1$ @ $l2$)
else (x | rmDup($l1$ @ $l2$)) fi
(by case splitting)
$\longrightarrow$ if has($l2, x$) then rmDup($l1$ @ $l2$)
else (x | rmDup($l1$ @ $l2$)) fi (by or)
$\longrightarrow$ if has($l2, x$) then rmDup(rmDup($l1$) @ rmDup($l2$))
else (x | rmDup($l1$ @ $l2$)) fi (by IH)
$\longrightarrow$ if has($l2, x$) then rmDup(rmDup($l1$) @ rmDup($l2$))

else $(x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)))$ fi
(by IH)

$\text{rmDup}(\underline{\text{rmDup}(x \mid l1)} \mathbin{@} \text{rmDup}(l2)) \longrightarrow \text{rmDup}((\text{if } \underline{\text{has}(l1, x)} \text{ then rmDup}(l1)$
else $(x \mid \text{rmDup}(l1))$ fi) @ rmDup($l2$))
(by rmDup2)
$\longrightarrow \text{rmDup}((\underline{\text{if } \mathit{false} \text{ then rmDup}(l1)}$
$\underline{\text{else } (x \mid \text{rmDup}(l1))\text{ fi}) \mathbin{@} \text{rmDup}(l2)}$
(by case splitting)
$\longrightarrow \text{rmDup}(\underline{(x \mid \text{rmDup}(l1)) \mathbin{@} \text{rmDup}(l2)})$
(by if2)
$\longrightarrow \underline{\text{rmDup}(x \mid (\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)))}$
(by @2)
$\longrightarrow$ if $\underline{\text{has}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2), x)}$
then rmDup(rmDup($l1$) @ rmDup($l2$))
else $(x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)))$ fi
(by rmDup2)
$\longrightarrow$ if $\underline{\text{has}(\text{rmDup}(l1), x)}$ *or* has(rmDup($l2$), x)
then rmDup(rmDup($l1$) @ rmDup($l2$))
else $(x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)))$ fi
(by Problem 6 - Lemma 1)
$\longrightarrow$ if $\underline{\text{has}(l1, x)}$ *or* has(rmDup($l2$), x)
then rmDup(rmDup($l1$) @ rmDup($l2$))
else $(x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)))$ fi
(by Problem 17 - Lemma 1)
$\longrightarrow$ if $\underline{\mathit{false} \textit{ or } \text{has}(\text{rmDup}(l2), x)}$
then rmDup(rmDup($l1$) @ rmDup($l2$))
else $(x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)))$ fi
(by case splitting)
$\longrightarrow$ if $\underline{\text{has}(\text{rmDup}(l2), x)}$
then rmDup(rmDup($l1$) @ rmDup($l2$))
else $(x \mid \text{rmDup}(\text{rmDup}(l1) \mathbin{@} \text{rmDup}(l2)))$ fi
(by or)
$\longrightarrow$ if has($l2, x$)

then rmDup(rmDup($l1$) @ rmDup($l2$))
else (x | rmDup(rmDup($l1$) @ rmDup($l2$))) fi
(by Problem 17 - Lemma 1)

□