

## Problem 19

**Problem 1.**  $\forall L \in \text{NatList}, \text{setEqual}(L, \text{rmDup}(L)) = \text{true}$ .

*Proof.* By structural induction on  $L$ .

**(1) Base case**

What to show:  $\text{setEqual}(\text{nil}, \text{rmDup}(\text{nil})) = \text{true}$   
 where  $l \in \text{NatList}$ .

$$\begin{aligned}
 \text{setEqual}(\text{nil}, \underline{\text{rmDup}(\text{nil})}) &\longrightarrow \underline{\text{setEqual}(\text{nil}, \text{nil})} && \text{(by rmDup1)} \\
 &\longrightarrow \underline{(\text{diff}(\text{nil}, \text{nil}) = \text{nil})} \text{ and } (\text{diff}(\text{nil}, \text{nil}) = \text{nil}) && \text{(by setEq)} \\
 &\longrightarrow \underline{(\text{nil} = \text{nil})} \text{ and } (\text{diff}(\text{nil}, \text{nil}) = \text{nil}) && \text{(by diff1)} \\
 &\longrightarrow \underline{\text{true and } (\text{diff}(\text{nil}, \text{nil}) = \text{nil})} && \text{(by equality)} \\
 &\longrightarrow \underline{\text{diff}(\text{nil}, \text{nil}) = \text{nil}} && \text{(by and)} \\
 &\longrightarrow \underline{\text{nil} = \text{nil}} && \text{(by diff1)} \\
 &\longrightarrow \text{true} && \text{(by equality)}
 \end{aligned}$$

**(2) Induction case**

What to show:  $\text{setEqual}(x \mid l, \text{rmDup}(x \mid l)) = \text{true}$

Induction hypothesis:  $\text{setEqual}(l, \text{rmDup}(l)) = \text{true}$

where  $x \in \text{PNat}$  and  $l \in \text{NatList}$ .

We use case splitting for our proofs as follows:

**Case 1:**  $\text{has}(l, x) = \text{true}$ .

$$\begin{aligned}
 \text{setEqual}(x \mid l, \underline{\text{rmDup}(x \mid l)}) &\longrightarrow \text{setEqual}(x \mid l, \\
 &\quad \text{if } \underline{\text{has}(l, x)} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi} \\
 &\quad \text{(by rmDup2)} \\
 &\longrightarrow \text{setEqual}(x \mid l,
 \end{aligned}$$

$$\begin{aligned}
& \frac{\text{if } \text{true} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}}{\text{(by case splitting)}} \\
\longrightarrow & \text{setEqual}(x \mid l, \text{rmDup}(l)) \quad (\text{by if1}) \\
\longrightarrow & \frac{(\text{diff}(x \mid l, \text{rmDup}(l)) = \text{nil}) \text{ and}}{(\text{diff}(\text{rmDup}(l), x \mid l) = \text{nil})} \\
& \quad (\text{by setEq}) \\
\longrightarrow & ((\text{if } \text{has}(\text{rmDup}(l), x) \text{ then } \text{diff}(l, \text{rmDup}(l)) \\
& \quad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}) = \text{nil}) \text{ and} \\
& \quad (\text{diff}(\text{rmDup}(l), x \mid l) = \text{nil}) \\
& \quad (\text{by diff2}) \\
\longrightarrow & ((\text{if } \text{has}(l, x) \text{ then } \text{diff}(l, \text{rmDup}(l)) \\
& \quad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}) = \text{nil}) \text{ and} \\
& \quad (\text{diff}(\text{rmDup}(l), x \mid l) = \text{nil}) \\
& \quad (\text{by Problem 17 - Lemma 1}) \\
\longrightarrow & \frac{((\text{if } \text{true} \text{ then } \text{diff}(l, \text{rmDup}(l)) \\
& \quad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}) = \text{nil}) \text{ and}}{(\text{diff}(\text{rmDup}(l), x \mid l) = \text{nil})} \\
& \quad (\text{by case splitting}) \\
\longrightarrow & \frac{(\text{diff}(l, \text{rmDup}(l)) = \text{nil}) \text{ and}}{(\text{diff}(\text{rmDup}(l), x \mid l) = \text{nil})} \quad (\text{by if1}) \\
\longrightarrow & \frac{(\text{nil} = \text{nil}) \text{ and } (\text{diff}(\text{rmDup}(l), x \mid l) = \text{nil})}{(\text{by Lemma 1})} \\
\longrightarrow & \frac{\text{true and } (\text{diff}(\text{rmDup}(l), x \mid l) = \text{nil})}{(\text{by equality})} \\
\longrightarrow & \text{diff}(\text{rmDup}(l), x \mid l) = \text{nil} \quad (\text{by and}) \\
\longrightarrow & \text{drop}(\text{diff}(\text{rmDup}(l), l), x) = \text{nil} \\
& \quad (\text{by Problem 14}) \\
\longrightarrow & \text{drop}(\text{nil}, x) = \text{nil} \quad (\text{by Lemma 1}) \\
\longrightarrow & \text{nil} = \text{nil} \quad (\text{by drop1}) \\
\longrightarrow & \text{true} \quad (\text{by equality})
\end{aligned}$$

**Case 2:**  $\text{has}(l, x) = \text{false}$ .

$$\text{setEqual}(x \mid l, \text{rmDup}(x \mid l)) \longrightarrow \text{setEqual}(x \mid l,$$

$$\begin{aligned}
& \text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi} \\
& \quad \text{(by rmDup2)} \\
\longrightarrow & \text{setEqual}(x \mid l, \\
& \quad \underline{\text{if } \text{false} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}} \\
& \quad \text{(by case splitting)} \\
\longrightarrow & \underline{\text{setEqual}(x \mid l, x \mid \text{rmDup}(l))} \\
& \quad \text{(by if2)} \\
\longrightarrow & (\underline{\text{diff}(x \mid l, x \mid \text{rmDup}(l)) = \text{nil}}) \text{ and} \\
& \quad (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}) \\
& \quad \text{(by setEq)} \\
\longrightarrow & ((\underline{\text{if } \text{has}(x \mid \text{rmDup}(l), x) \text{ then diff}(l, x \mid \text{rmDup}(l))} \\
& \quad \text{else } (x \mid \text{diff}(l, x \mid \text{rmDup}(l))) \text{ fi} = \text{nil}) \text{ and} \\
& \quad (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}) \\
& \quad \text{(by diff2)} \\
\longrightarrow & ((\underline{\text{if } (x = x) \text{ or } \text{has}(\text{rmDup}(l), x)} \\
& \quad \text{then diff}(l, x \mid \text{rmDup}(l)) \\
& \quad \text{else } (x \mid \text{diff}(l, x \mid \text{rmDup}(l))) \text{ fi} = \text{nil}) \text{ and} \\
& \quad (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}) \\
& \quad \text{(by has2)} \\
\longrightarrow & ((\underline{\text{if } \text{true or } \text{has}(\text{rmDup}(l), x)} \\
& \quad \text{then diff}(l, x \mid \text{rmDup}(l)) \\
& \quad \text{else } (x \mid \text{diff}(l, x \mid \text{rmDup}(l))) \text{ fi} = \text{nil}) \text{ and} \\
& \quad (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}) \\
& \quad \text{(by equality)} \\
\longrightarrow & ((\underline{\text{if } \text{true} \text{ then diff}(l, x \mid \text{rmDup}(l))} \\
& \quad \underline{\text{else } (x \mid \text{diff}(l, x \mid \text{rmDup}(l))) \text{ fi} = \text{nil}}) \text{ and} \\
& \quad (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}) \\
& \quad \text{(by or)} \\
\longrightarrow & (\underline{\text{diff}(l, x \mid \text{rmDup}(l)) = \text{nil}}) \text{ and} \\
& \quad (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}) \\
& \quad \text{(by if1)} \\
\longrightarrow & (\underline{\text{drop}(\text{diff}(l, \text{rmDup}(l)), x) = \text{nil}}) \text{ and} \\
& \quad (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}) \\
& \quad \text{(by Problem 14)}
\end{aligned}$$

$$\begin{aligned}
&\longrightarrow \underline{\text{drop}(\text{nil}, x) = \text{nil}} \text{ and} \\
&\quad (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}) \\
&\quad \quad \quad \text{(by Lemma 1)} \\
&\longrightarrow \underline{\text{nil} = \text{nil}} \text{ and} \\
&\quad (\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}) \\
&\quad \quad \quad \text{(by drop1)} \\
&\longrightarrow \text{true and} \\
&\quad \underline{(\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil})} \\
&\quad \quad \quad \text{(by equality)} \\
&\longrightarrow \underline{\text{diff}(x \mid \text{rmDup}(l), x \mid l) = \text{nil}} \\
&\quad \quad \quad \text{(by and)} \\
&\longrightarrow (\text{if } \underline{\text{has}(x \mid l, x)} \text{ then } \text{diff}(\text{rmDup}(l), x \mid l) \\
&\quad \quad \text{else } (x \mid \text{diff}(\text{rmDup}(l), x \mid l)) \text{ fi}) = \text{nil} \\
&\quad \quad \quad \text{(by diff2)} \\
&\longrightarrow (\text{if } \underline{(x = x)} \text{ or } \text{has}(l, x) \text{ then } \text{diff}(\text{rmDup}(l), x \mid l) \\
&\quad \quad \text{else } (x \mid \text{diff}(\text{rmDup}(l), x \mid l)) \text{ fi}) = \text{nil} \\
&\quad \quad \quad \text{(by has2)} \\
&\longrightarrow (\text{if } \underline{\text{true or has}(l, x)} \text{ then } \text{diff}(\text{rmDup}(l), x \mid l) \\
&\quad \quad \text{else } (x \mid \text{diff}(\text{rmDup}(l), x \mid l)) \text{ fi}) = \text{nil} \\
&\quad \quad \quad \text{(by equality)} \\
&\longrightarrow (\text{if } \underline{\text{true}} \text{ then } \text{diff}(\text{rmDup}(l), x \mid l) \\
&\quad \quad \underline{\text{else } (x \mid \text{diff}(\text{rmDup}(l), x \mid l)) \text{ fi}} = \text{nil} \\
&\quad \quad \quad \text{(by or)} \\
&\longrightarrow \underline{\text{diff}(\text{rmDup}(l), x \mid l) = \text{nil}} \quad \text{(by if1)} \\
&\longrightarrow \underline{\text{drop}(\text{diff}(\text{rmDup}(l), l), x) = \text{nil}} \\
&\quad \quad \quad \text{(by Problem 14)} \\
&\longrightarrow \underline{\text{drop}(\text{nil}, x) = \text{nil}} \quad \text{(by Lemma 2)} \\
&\longrightarrow \underline{\text{nil} = \text{nil}} \quad \text{(by drop1)} \\
&\longrightarrow \text{true} \quad \text{(by equality)}
\end{aligned}$$

□

**Lemma 1.**  $\forall X \in \text{PNat}, \forall L \in \text{NatList}, \text{diff}(L, \text{rmDup}(L)) = \text{nil}$ .

*Proof.* By structural induction on  $L$ .

**(1) Base case**

What to show:  $\text{diff}(\text{nil}, \text{rmDup}(\text{nil})) = \text{nil}$ .

$$\underline{\text{diff}(\text{nil}, \text{rmDup}(\text{nil}))} \longrightarrow \text{nil} \quad (\text{by diff1})$$

**(2) Induction case**

What to show:  $\text{diff}(x \mid l, \text{rmDup}(x \mid l)) = \text{nil}$

Induction hypothesis:  $\text{diff}(l, \text{rmDup}(l)) = \text{nil}$

where  $x \in \text{PNat}$  and  $l \in \text{NatList}$ .

We use case splitting for our proofs as follows:

**Case 1:**  $\text{has}(l, x) = \text{true}$ .

$$\begin{aligned} \text{diff}(x \mid l, \underline{\text{rmDup}(x \mid l)}) &\longrightarrow \text{diff}(x \mid l, \text{if } \underline{\text{has}(l, x)} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}) \\ &\quad (\text{by rmDup2}) \\ &\longrightarrow \text{diff}(x \mid l, \underline{\text{if } \text{true} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}}) \\ &\quad (\text{by case splitting}) \\ &\longrightarrow \underline{\text{diff}(x \mid l, \text{rmDup}(l))} \quad (\text{by if1}) \\ &\longrightarrow \text{if } \underline{\text{has}(\text{rmDup}(l), x)} \text{ then } \text{diff}(l, \text{rmDup}(l)) \\ &\quad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi} \quad (\text{by diff2}) \\ &\longrightarrow \text{if } \underline{\text{has}(l, x)} \text{ then } \text{diff}(l, \text{rmDup}(l)) \\ &\quad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi} \\ &\quad (\text{by Problem 17 - Lemma 1}) \\ &\longrightarrow \underline{\text{if } \text{true} \text{ then } \text{diff}(l, \text{rmDup}(l))} \\ &\quad \underline{\text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}} \\ &\quad (\text{by case splitting}) \\ &\longrightarrow \underline{\text{diff}(l, \text{rmDup}(l))} \quad (\text{by if1}) \\ &\longrightarrow \text{nil} \quad (\text{by IH}) \end{aligned}$$

**Case 2:**  $\text{has}(l, x) = \text{false}$ .

$$\begin{aligned} \text{diff}(x \mid l, \underline{\text{rmDup}(x \mid l)}) &\longrightarrow \text{diff}(x \mid l, \text{if } \underline{\text{has}(l, x)} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}) \\ &\quad (\text{by rmDup2}) \\ &\longrightarrow \text{diff}(x \mid l, \underline{\text{if } \text{false} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}}) \\ &\quad (\text{by case splitting}) \\ &\longrightarrow \underline{\text{diff}(x \mid l, x \mid \text{rmDup}(l))} \quad (\text{by if2}) \\ &\longrightarrow \text{drop}(\underline{\text{diff}(x \mid l, \text{rmDup}(l))}, x) \\ &\quad (\text{by Problem 14}) \end{aligned}$$

$$\begin{aligned}
&\longrightarrow \text{drop}((\text{if } \underline{\text{has}(\text{rmDup}(l), x)} \text{ then } \text{diff}(l, \text{rmDup}(l)) \\
&\quad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}), x) \\
&\hspace{15em} (\text{by diff2}) \\
&\longrightarrow \text{drop}((\text{if } \underline{\text{has}(l, x)} \text{ then } \text{diff}(l, \text{rmDup}(l)) \\
&\quad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}), x) \\
&\hspace{10em} (\text{by Problem 17 - Lemma 1}) \\
&\longrightarrow \text{drop}((\text{if } \underline{\text{false}} \text{ then } \text{diff}(l, \text{rmDup}(l)) \\
&\quad \text{else } (x \mid \text{diff}(l, \text{rmDup}(l))) \text{ fi}), x) \\
&\hspace{15em} (\text{by case splitting}) \\
&\longrightarrow \text{drop}(x \mid \underline{\text{diff}(l, \text{rmDup}(l))}, x) \hspace{2em} (\text{by if2}) \\
&\longrightarrow \underline{\text{drop}(x \mid \text{nil}, x)} \hspace{10em} (\text{by IH}) \\
&\longrightarrow \text{if } \underline{x = x} \text{ then } \text{drop}(\text{nil}, x) \text{ else } (x \mid \text{drop}(\text{nil}, x)) \text{ fi} \\
&\hspace{15em} (\text{by drop2}) \\
&\longrightarrow \underline{\text{if } \text{true} \text{ then } \text{drop}(\text{nil}, x) \text{ else } (x \mid \text{drop}(\text{nil}, x)) \text{ fi}} \\
&\hspace{15em} (\text{by equality}) \\
&\longrightarrow \underline{\text{drop}(\text{nil}, x)} \hspace{10em} (\text{by if1}) \\
&\longrightarrow \text{nil} \hspace{15em} (\text{by drop1})
\end{aligned}$$

□

**Lemma 2.**  $\forall X \in \text{PNat}, \forall L \in \text{NatList}, \text{diff}(\text{rmDup}(L), L) = \text{nil}$ .

*Proof.* By structural induction on  $L$ .

**(1) Base case**

What to show:  $\text{diff}(\text{rmDup}(\text{nil}), \text{nil}) = \text{nil}$ .

$$\begin{aligned}
\text{diff}(\underline{\text{rmDup}(\text{nil})}, \text{nil}) &\longrightarrow \underline{\text{diff}(\text{nil}, \text{nil})} && (\text{by rmDup1}) \\
&\longrightarrow \text{nil} && (\text{by diff1})
\end{aligned}$$

**(2) Induction case**

What to show:  $\text{diff}(\text{rmDup}(x \mid l), x \mid l) = \text{nil}$

Induction hypothesis:  $\text{diff}(\text{rmDup}(l), l) = \text{nil}$

where  $x \in \text{PNat}$  and  $l \in \text{NatList}$ .

We use case splitting for our proofs as follows:

**Case 1:**  $\text{has}(l, x) = \text{true}$ .

$$\begin{aligned}
\text{diff}(\underline{\text{rmDup}(x \mid l)}, x \mid l) &\longrightarrow \text{diff}((\text{if } \underline{\text{has}(l, x)} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l))) \text{ fi}), x \mid l) \\
&\hspace{15em} (\text{by rmDup2})
\end{aligned}$$

$$\begin{aligned}
&\longrightarrow \text{diff}(\underline{(\text{if } \text{true} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi})}, x \mid l) \\
&\hspace{15em} (\text{by case splitting}) \\
&\longrightarrow \underline{\text{diff}(\text{rmDup}(l), x \mid l)} \hspace{10em} (\text{by if1}) \\
&\longrightarrow \text{drop}(\underline{\text{diff}(\text{rmDup}(l), l)}, x) \\
&\hspace{15em} (\text{by Problem 14}) \\
&\longrightarrow \underline{\text{drop}(\text{nil}, x)} \hspace{10em} (\text{by IH}) \\
&\longrightarrow \text{nil} \hspace{15em} (\text{by drop1})
\end{aligned}$$

**Case 2:**  $\text{has}(l, x) = \text{false}$ .

$$\begin{aligned}
\text{diff}(\underline{\text{rmDup}(x \mid l)}, x \mid l) &\longrightarrow \text{diff}(\underline{(\text{if } \underline{\text{has}(l, x)} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi})}, x \mid l) \\
&\hspace{15em} (\text{by rmDup2}) \\
&\longrightarrow \text{diff}(\underline{(\text{if } \text{false} \text{ then } \text{rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi})}, x \mid l) \\
&\hspace{15em} (\text{by case splitting}) \\
&\longrightarrow \underline{\text{diff}(x \mid \text{rmDup}(l), x \mid l)} \hspace{10em} (\text{by if2}) \\
&\longrightarrow \text{drop}(\underline{\text{diff}(x \mid \text{rmDup}(l), l)}, x) \\
&\hspace{15em} (\text{by Problem 14}) \\
&\longrightarrow \text{drop}(\underline{(\text{if } \underline{\text{has}(l, x)} \text{ then } \text{diff}(\text{rmDup}(l), l) \text{ else } (x \mid \text{diff}(\text{rmDup}(l), l)) \text{ fi})}, x) \\
&\hspace{15em} (\text{by diff2}) \\
&\longrightarrow \text{drop}(\underline{(\text{if } \text{false} \text{ then } \text{diff}(\text{rmDup}(l), l) \text{ else } (x \mid \text{diff}(\text{rmDup}(l), l)) \text{ fi})}, x) \\
&\hspace{15em} (\text{by case splitting}) \\
&\longrightarrow \text{drop}(x \mid \underline{\text{diff}(\text{rmDup}(l), l)}, x) \hspace{5em} (\text{by if2}) \\
&\longrightarrow \underline{\text{drop}(x \mid \text{nil}, x)} \hspace{10em} (\text{by IH}) \\
&\longrightarrow \text{if } \underline{x = x} \text{ then } \text{drop}(\text{nil}, x) \text{ else } (x \mid \text{drop}(\text{nil}, x)) \text{ fi} \\
&\hspace{15em} (\text{by drop2}) \\
&\longrightarrow \underline{\text{if } \text{true} \text{ then } \text{drop}(\text{nil}, x) \text{ else } (x \mid \text{drop}(\text{nil}, x)) \text{ fi}} \\
&\hspace{15em} (\text{by equality}) \\
&\longrightarrow \underline{\text{drop}(\text{nil}, x)} \hspace{10em} (\text{by if1}) \\
&\longrightarrow \text{nil} \hspace{15em} (\text{by drop1})
\end{aligned}$$

□