Problem 1

Problem 1. $\forall L \in \texttt{NatList}, \mathrm{fold}{*}(\mathrm{rev}(L)) = \mathrm{fold}{*}(L)$.

Proof. By structural induction on L.

(1) Base case

What to show: $\mathrm{fold}{*}(\mathrm{rev}(nil)) = \mathrm{fold}{*}(nil)$.

$$\mathrm{fold}{*}(\underline{\mathrm{rev}(nil)}) \longrightarrow \mathrm{fold}{*}(nil) \qquad \text{(by rev1)}$$

(2) Induction case

What to show: $\mathrm{fold}{*}(\mathrm{rev}(x \mid l)) = \mathrm{fold}{*}(x \mid l)$
Induction hypothesis: $\mathrm{fold}{*}(\mathrm{rev}(l)) = \mathrm{fold}{*}(l)$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants[1].

$$\begin{aligned}
\mathrm{fold}{*}(\underline{\mathrm{rev}(x \mid l)}) &\longrightarrow \underline{\mathrm{fold}{*}(\mathrm{rev}(l) \mathbin{@} (x \mid nil))} && \text{(by rev2)}\\
&\longrightarrow \underline{\mathrm{fold}{*}(\mathrm{rev}(l))} * \mathrm{fold}{*}(x \mid nil) && \text{(by Lemma 1)}\\
&\longrightarrow \mathrm{fold}{*}(l) * \underline{\mathrm{fold}{*}(x \mid nil)} && \text{(by IH)}\\
&\longrightarrow \mathrm{fold}{*}(l) * (x * \underline{\mathrm{fold}{*}(nil)}) && \text{(by fold*-2)}\\
&\longrightarrow \mathrm{fold}{*}(l) * (\underline{x * \mathrm{s}(0)}) && \text{(by fold*-1)}\\
&\longrightarrow \mathrm{fold}{*}(l) * (\underline{\mathrm{s}(0) * x}) && \text{(by comm*)}\\
&\longrightarrow \mathrm{fold}{*}(l) * (\underline{(0 * x)} + x) && \text{(by *2)}\\
&\longrightarrow \mathrm{fold}{*}(l) * \underline{(0 + x)} && \text{(by *1)}\\
&\longrightarrow \underline{\mathrm{fold}{*}(l) * x} && \text{(by +1)}\\
&\longrightarrow x * \mathrm{fold}{*}(l) && \text{(by comm*)}\\
\underline{\mathrm{fold}{*}(x \mid l)} &\longrightarrow x * \mathrm{fold}{*}(l) && \text{(fold*-2)}
\end{aligned}$$

□

Lemma 1. $\forall L1, L2 \in \texttt{NatList}, \mathrm{fold}{*}(L1 \mathbin{@} L2) = \mathrm{fold}{*}(L1) * \mathrm{fold}{*}(L2)$.

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{fold}{\ast}(nil \mathbin{@} l2) = \text{fold}{\ast}(nil) \ast \text{fold}{\ast}(l2)$
where $l2 \in \mathtt{NatList}$. Note that $l2$ is a fresh constant.

$$\begin{aligned}
\text{fold}{\ast}(\underline{nil \mathbin{@} l2}) &\longrightarrow \text{fold}{\ast}(l2) && \text{(by @1)}\\
\underline{\text{fold}{\ast}(nil)} \ast \text{fold}{\ast}(l2) &\longrightarrow \underline{\text{s}(0) \ast \text{fold}{\ast}(l2)} && \text{(by fold*-1)}\\
&\longrightarrow \underline{(0 \ast \text{fold}{\ast}(l2))} + \text{fold}{\ast}(l2) && \text{(by *2)}\\
&\longrightarrow \underline{0 + \text{fold}{\ast}(l2)} && \text{(by *1)}\\
&\longrightarrow \text{fold}{\ast}(l2) && \text{(by +1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{fold}{\ast}((x \mid l1) \mathbin{@} l2) = \text{fold}{\ast}(x \mid l1) \ast \text{fold}{\ast}(l2)$
Induction hypothesis: $\text{fold}{\ast}(l1 \mathbin{@} l2) = \text{fold}{\ast}(l1) \ast \text{fold}{\ast}(l2)$
where $x \in \mathtt{PNat}$ and $l1, l2 \in \mathtt{NatList}$. Note that $x, l1, l2$ are fresh constants.

$$\begin{aligned}
\text{fold}{\ast}(\underline{(x \mid l1) \mathbin{@} l2}) &\longrightarrow \underline{\text{fold}{\ast}(x \mid (l1 \mathbin{@} l2))} && \text{(by @2)}\\
&\longrightarrow x \ast \underline{\text{fold}{\ast}(l1 \mathbin{@} l2)} && \text{(by fold*-2)}\\
&\longrightarrow x \ast (\text{fold}{\ast}(l1) \ast \text{fold}{\ast}(l2)) && \text{(by IH)}\\
\underline{\text{fold}{\ast}(x \mid l1)} \ast \text{fold}{\ast}(l2) &\longrightarrow \underline{(x \ast \text{fold}{\ast}(l1)) \ast \text{fold}{\ast}(l2)} && \text{(by fold*-2)}\\
&\longrightarrow x \ast (\text{fold}{\ast}(l1) \ast \text{fold}{\ast}(l2)) && \text{(by assoc*)}
\end{aligned}$$

$\square$