

Problem 11

Problem 1. $\forall X \in \text{PNat}, \forall L \in \text{NatList}, \text{has}(\text{drop}(L, X), X) = \text{false}$.

Proof. By structural induction on L .

(1) Base case

What to show: $\text{has}(\text{drop}(\text{nil}, x), x) = \text{false}$
 where $x \in \text{PNat}$. Note that x is a fresh constant¹.

$$\begin{aligned} \text{has}(\underline{\text{drop}(\text{nil}, x)}, x) &\longrightarrow \underline{\text{has}(\text{nil}, x)} && \text{(by drop1)} \\ &\longrightarrow \text{false} && \text{(by has1)} \end{aligned}$$

(2) Induction case

What to show: $\text{has}(\text{drop}(y \mid l, x), x) = \text{false}$
 Induction hypothesis: $\text{has}(\text{drop}(l, x), x) = \text{false}$
 where $x, y \in \text{PNat}$ and $l \in \text{NatList}$. Note that x, y, l are fresh constants.

We use case splitting for our proofs as follows:

Case 1: $y = x$

$$\begin{aligned} \text{has}(\text{drop}(\underline{y} \mid l, x), x) &\longrightarrow \text{has}(\underline{\text{drop}(x \mid l, x)}, x) && \text{(by case splitting)} \\ &\longrightarrow \text{has}(\text{if } \underline{(x = x)} \text{ then } \text{drop}(l, x) \text{ else } (x \mid \text{drop}(l, x)) \text{ fi}, x) && \text{(by drop2)} \\ &\longrightarrow \text{has}(\text{if } \underline{\text{true}} \text{ then } \text{drop}(l, x) \text{ else } (x \mid \text{drop}(l, x)) \text{ fi}, x) && \text{(by equality)} \\ &\longrightarrow \underline{\text{has}(\text{drop}(l, x), x)} && \text{(by if1)} \\ &\longrightarrow \text{false} && \text{(by IH)} \end{aligned}$$

Case 2: $(y = x) = \text{false}$

$$\text{has}(\underline{\text{drop}(y \mid l, x)}, x) \longrightarrow \text{has}(\text{if } \underline{(y = x)} \text{ then } \text{drop}(l, x) \text{ else } (y \mid \text{drop}(l, x)) \text{ fi}, x) \quad \text{(by drop2)}$$

¹A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

$$\begin{aligned}
&\longrightarrow \text{has}(\text{if } \underline{\text{false}} \text{ then } \text{drop}(l, x) \text{ else } (y \mid \text{drop}(l, x)) \text{ fi}, x) \\
&\hspace{15em} \text{(by case splitting)} \\
&\longrightarrow \underline{\text{has}(y \mid \text{drop}(l, x), x)} \hspace{10em} \text{(by if2)} \\
&\longrightarrow \underline{(y = x)} \text{ or } \text{has}(\text{drop}(l, x), x) \hspace{10em} \text{(by has2)} \\
&\longrightarrow \underline{\text{false}} \text{ or } \text{has}(\text{drop}(l, x), x) \\
&\hspace{15em} \text{(by case splitting)} \\
&\longrightarrow \underline{\text{has}(\text{drop}(l, x), x)} \hspace{10em} \text{(by or)} \\
&\longrightarrow \text{false} \hspace{15em} \text{(by IH)}
\end{aligned}$$

□