Problem 15

Problem 1. $\forall L \in \texttt{NatList}, \text{diff}(L, L) = nil$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{diff}(nil, nil) = nil$.

$$\underline{\text{diff}(nil, nil)} \longrightarrow nil \qquad \text{(by diff1)}$$

(2) Induction case

What to show: $\text{diff}(x \mid l, x \mid l) = nil$
Induction hypothesis: $\text{diff}(l, l) = nil$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$.
We use case splitting for our proofs as follows:
Case 1: $\text{has}(l, x) = true$

$$\begin{aligned}
\underline{\text{diff}(x \mid l, x \mid l)} &\longrightarrow \text{drop}(\underline{\text{diff}(x \mid l, l)}, x) && \text{(by Problem 14)} \\
&\longrightarrow \text{drop}(\text{if } \underline{\text{has}(l, x)} \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}, x) \\
& && \text{(by diff2)} \\
&\longrightarrow \text{drop}(\underline{\text{if } true \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}}, x) \\
& && \text{(by case splitting)} \\
&\longrightarrow \text{drop}(\underline{\text{diff}(l, l)}, x) && \text{(by if1)} \\
&\longrightarrow \underline{\text{drop}(nil, x)} && \text{(by IH)} \\
&\longrightarrow nil && \text{(by drop1)}
\end{aligned}$$

Case 2: $\text{has}(l, x) = false$

$$\begin{aligned}
\underline{\text{diff}(x \mid l, x \mid l)} &\longrightarrow \text{drop}(\underline{\text{diff}(x \mid l, l)}, x) && \text{(by Problem 14)} \\
&\longrightarrow \text{drop}(\text{if } \underline{\text{has}(l, x)} \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}, x) \\
& && \text{(by diff2)}
\end{aligned}$$

$\longrightarrow$ drop(if $false$ then diff(l, l) else $(x \mid \text{diff}(l, l))$ fi$, x)$
(by case splitting)

$\longrightarrow$ drop$(x \mid \text{diff}(l, l), x)$ (by if2)

$\longrightarrow$ drop$(x \mid nil, x)$ (by IH)

$\longrightarrow$ if $(x = x)$ then drop(nil, x) else $(x \mid \text{drop}(nil, x))$ fi
(by drop2)

$\longrightarrow$ if $true$ then drop(nil, x) else $(x \mid \text{drop}(nil, x))$ fi
(by equality)

$\longrightarrow$ drop(nil, x) (by if1)

$\longrightarrow$ nil (by drop1)

□