Problem 17

Problem 1. $\forall L \in \texttt{NatList}, \mathrm{rmDup}(\mathrm{rmDup}(L)) = \mathrm{rmDup}(L)$.

Proof. By structural induction on L.

(1) Base case

What to show: $\quad \mathrm{rmDup}(\mathrm{rmDup}(nil)) = \mathrm{rmDup}(nil)$.

$$
\begin{aligned}
\mathrm{rmDup}(\underline{\mathrm{rmDup}(nil)}) &\longrightarrow \underline{\mathrm{rmDup}(nil)} && \text{(by rmDup1)}\\
&\longrightarrow nil && \text{(by rmDup1)}\\
\underline{\mathrm{rmDup}(nil)} &\longrightarrow nil && \text{(by rmDup1)}
\end{aligned}
$$

(2) Induction case

What to show: $\quad \mathrm{rmDup}(\mathrm{rmDup}(x \mid l)) = \mathrm{rmDup}(x \mid l)$
Induction hypothesis: $\quad \mathrm{rmDup}(\mathrm{rmDup}(l)) = \mathrm{rmDup}(l)$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants[1].

We use case splitting for our proofs as follows:
Case 1: $\mathrm{has}(l, x) = true$.

$$
\begin{aligned}
\mathrm{rmDup}(\underline{\mathrm{rmDup}(x \mid l)}) &\longrightarrow \mathrm{rmDup}(\text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{rmDup}(l)\\
&\qquad \text{else } (x \mid \mathrm{rmDup}(l)) \text{ fi}) && \text{(by rmDup2)}\\
&\longrightarrow \mathrm{rmDup}(\underline{\text{if } true \text{ then } \mathrm{rmDup}(l)}\\
&\qquad \underline{\text{else } (x \mid \mathrm{rmDup}(l)) \text{ fi}}) && \text{(by case splitting)}\\
&\longrightarrow \underline{\mathrm{rmDup}(\mathrm{rmDup}(l))} && \text{(by if1)}\\
&\longrightarrow \mathrm{rmDup}(l) && \text{(by IH)}\\
\underline{\mathrm{rmDup}(x \mid l)} &\longrightarrow \text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi}\\
& && \text{(by rmDup2)}\\
&\longrightarrow \underline{\text{if } true \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi}}\\
& && \text{(by case splitting)}\\
&\longrightarrow \mathrm{rmDup}(l) && \text{(by if1)}
\end{aligned}
$$

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

Case 2: $\text{has}(l, x) = false$.

$$
\begin{aligned}
\text{rmDup}(\underline{\text{rmDup}(x \mid l)}) &\longrightarrow \text{rmDup}(\text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(l) \\
&\quad \text{else } (x \mid \text{rmDup}(l)) \text{ fi}) \qquad \text{(by rmDup2)} \\
&\longrightarrow \text{rmDup}(\underline{\text{if } false \text{ then rmDup}(l)} \\
&\quad \underline{\text{else } (x \mid \text{rmDup}(l)) \text{ fi}}) \quad \text{(by case splitting)} \\
&\longrightarrow \underline{\text{rmDup}(x \mid \text{rmDup}(l))} \qquad \text{(by if2)} \\
&\longrightarrow \text{if } \underline{\text{has}(\text{rmDup}(l), x)} \text{ then rmDup}(\text{rmDup}(l)) \\
&\quad \text{else } (x \mid \text{rmDup}(\text{rmDup}(l))) \text{ fi} \\
&\qquad \text{(by rmDup2)} \\
&\longrightarrow \text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(\text{rmDup}(l)) \\
&\quad \text{else } (x \mid \text{rmDup}(\text{rmDup}(l))) \text{ fi} \\
&\qquad \text{(by Lemma 1)} \\
&\longrightarrow \underline{\text{if } false \text{ then rmDup}(\text{rmDup}(l))} \\
&\quad \underline{\text{else } (x \mid \text{rmDup}(\text{rmDup}(l))) \text{ fi}} \\
&\qquad \text{(by case splitting)} \\
&\longrightarrow x \mid \underline{\text{rmDup}(\text{rmDup}(l))} \qquad \text{(by if2)} \\
&\longrightarrow x \mid \text{rmDup}(l) \qquad \text{(by IH)} \\
\underline{\text{rmDup}(x \mid l)} &\longrightarrow \text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi} \\
&\qquad \text{(by rmDup2)} \\
&\longrightarrow \underline{\text{if } false \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}} \\
&\qquad \text{(by case splitting)} \\
&\longrightarrow x \mid \text{rmDup}(l) \qquad \text{(by if2)}
\end{aligned}
$$

□

Lemma 1. $\forall X \in \texttt{PNat}, \forall L \in \texttt{NatList}, \text{has}(L, X) = \text{has}(\text{rmDup}(L), X)$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{has}(nil, x) = \text{has}(\text{rmDup}(nil), x)$
where $x \in \texttt{PNat}$. Note that x is a fresh constant.

$$\underline{\text{has}(nil, x)} \longrightarrow nil \qquad \text{(by has1)}$$

$$
\begin{aligned}
\mathrm{has}(\underline{\mathrm{rmDup}(nil)}, x) &\longrightarrow \underline{\mathrm{has}(nil, x)} && \text{(by rmDup1)}\\
&\longrightarrow nil && \text{(by has1)}
\end{aligned}
$$

(2) Induction case

What to show: $\mathrm{has}(y \mid l, x) = \mathrm{has}(\mathrm{rmDup}(y \mid l), x)$
Induction hypothesis: $\mathrm{has}(l, x) = \mathrm{has}(\mathrm{rmDup}(l), x)$
where $x, y \in$ `PNat` and $l \in$ `NatList`. Note that x, y, l are fresh constants.

We use case splitting for our proofs as follows:
Case 1.1: $\mathrm{has}(l, y) = true$, $\mathrm{has}(\mathrm{rmDup}(l), x) = true$.

$$
\begin{aligned}
\underline{\mathrm{has}(y \mid l, x)} &\longrightarrow (x = y) \ or\ \underline{\mathrm{has}(l, x)} && \text{(by has2)}\\
&\longrightarrow (x = y) \ or\ \underline{\mathrm{has}(\mathrm{rmDup}(l), x)} && \text{(by IH)}\\
&\longrightarrow \underline{(x = y) \ or\ true} && \text{(by case splitting)}\\
&\longrightarrow true && \text{(by or)}\\
\mathrm{has}(\underline{\mathrm{rmDup}(y \mid l)}, x) &\longrightarrow \mathrm{has}(\text{if } \underline{\mathrm{has}(l, y)} \text{ then } \mathrm{rmDup}(l) \text{ else } (y \mid \mathrm{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by rmDup2)}\\
&\longrightarrow \mathrm{has}(\underline{\text{if } true \text{ then } \mathrm{rmDup}(l) \text{ else } (y \mid \mathrm{rmDup}(l)) \text{ fi}}, x)\\
& && \text{(by case splitting)}\\
&\longrightarrow \underline{\mathrm{has}(\mathrm{rmDup}(l), x)} && \text{(by if1)}\\
&\longrightarrow true && \text{(by case splitting)}
\end{aligned}
$$

Case 1.2.1: $\mathrm{has}(l, y) = true$, $\mathrm{has}(\mathrm{rmDup}(l), x) = false$, $y = x$.

$$
\begin{aligned}
\underline{\mathrm{has}(\underline{y} \mid l, x)} &\longrightarrow \underline{\mathrm{has}(x \mid l, x)} && \text{(case splitting)}\\
&\longrightarrow \underline{(x = x)} \ or\ \mathrm{has}(l, x) && \text{(by has2)}\\
&\longrightarrow \underline{true \ or\ \mathrm{has}(l, x)} && \text{(by equality)}\\
&\longrightarrow true && \text{(by or)}\\
\mathrm{has}(\underline{\mathrm{rmDup}(\underline{y} \mid l)}, x) &\longrightarrow \mathrm{has}(\underline{\mathrm{rmDup}(x \mid l)}, x) && \text{(by case splitting)}\\
&\longrightarrow \mathrm{has}(\text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by rmDup2)}\\
&\longrightarrow \mathrm{has}(\text{if } \underline{\mathrm{has}(\mathrm{rmDup}(l), x)} \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by IH)}\\
&\longrightarrow \mathrm{has}(\underline{\text{if } false \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi}}, x)\\
& && \text{(by case splitting)}
\end{aligned}
$$

$$
\begin{aligned}
&\longrightarrow \underline{\text{has}(x \mid \text{rmDup}(l), x)} && \text{(by if2)}\\
&\longrightarrow \underline{(x = x)}\ \textit{or}\ \text{has}(\text{rmDup}(l), x) && \text{(by has2)}\\
&\longrightarrow \underline{\textit{true or}\ \text{has}(\text{rmDup}(l), x)} && \text{(by equality)}\\
&\longrightarrow \textit{true} && \text{(by or)}
\end{aligned}
$$

Case 1.2.2: $\text{has}(l, y) = \textit{true}$, $\text{has}(\text{rmDup}(l), x) = \textit{false}$, $(y = x) = \textit{false}$.

$$
\begin{aligned}
\underline{\text{has}(y \mid l, x)} &\longrightarrow \underline{(x = y)}\ \textit{or}\ \text{has}(l, x) && \text{(by has2)}\\
&\longrightarrow \underline{\textit{false or}\ \text{has}(l, x)} && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{has}(l, x)} && \text{(by or)}\\
&\longrightarrow \underline{\text{has}(\text{rmDup}(l), x)} && \text{(by IH)}\\
&\longrightarrow \textit{false} && \text{(by case splitting)}\\
\text{has}(\underline{\text{rmDup}(y \mid l)}, x) &\longrightarrow \text{has}(\text{if } \underline{\text{has}(l, y)} \text{ then rmDup}(l) \text{ else } (y \mid \text{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by rmDup2)}\\
&\longrightarrow \text{has}(\underline{\text{if } \textit{true} \text{ then rmDup}(l) \text{ else } (y \mid \text{rmDup}(l)) \text{ fi}}, x)\\
& && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{has}(\text{rmDup}(l), x)} && \text{(by if1)}\\
&\longrightarrow \textit{false} && \text{(by case splitting)}
\end{aligned}
$$

Case 2: $\text{has}(l, y) = \textit{false}$

$$
\begin{aligned}
\underline{\text{has}(y \mid l, x)} &\longrightarrow (x = y)\ \textit{or}\ \underline{\text{has}(l, x)} && \text{(by has2)}\\
&\longrightarrow (x = y)\ \textit{or}\ \text{has}(\text{rmDup}(l), x) && \text{(by IH)}\\
\text{has}(\underline{\text{rmDup}(y \mid l)}, x) &\longrightarrow \text{has}(\text{if } \underline{\text{has}(l, y)} \text{ then rmDup}(l) \text{ else } (y \mid \text{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by rmDup2)}\\
&\longrightarrow \text{has}(\underline{\text{if } \textit{false} \text{ then rmDup}(l) \text{ else } (y \mid \text{rmDup}(l)) \text{ fi}}, x)\\
& && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{has}(y \mid \text{rmDup}(l), x)} && \text{(by if2)}\\
&\longrightarrow (x = y)\ \textit{or}\ \text{has}(\text{rmDup}(l), x) && \text{(by has2)}
\end{aligned}
$$

$\square$