Problem 12

Problem 1. $\forall L \in \texttt{NatList}, \mathrm{sum}(L) = \mathrm{sum}(\mathrm{rev}(L))$.

Proof. By structural induction on L.

(1) Base case

What to show: $\mathrm{sum}(nil) = \mathrm{sum}(\mathrm{rev}(nil))$.

$$\begin{aligned}
\underline{\mathrm{sum}(nil)} &\longrightarrow 0 && \text{(by sum1)}\\
\mathrm{sum}(\underline{\mathrm{rev}(nil)}) &\longrightarrow \underline{\mathrm{sum}(nil)} && \text{(by rev1)}\\
&\longrightarrow 0 && \text{(by sum1)}
\end{aligned}$$

(2) Induction case

What to show: $\mathrm{sum}(x \mid l) = \mathrm{sum}(\mathrm{rev}(x \mid l))$

Induction hypothesis: $\mathrm{sum}(l) = \mathrm{sum}(\mathrm{rev}(l))$

where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants[1].

$$\begin{aligned}
\underline{\mathrm{sum}(\mathrm{x} \mid \mathrm{l})} &\longrightarrow x + \mathrm{sum}(l) && \text{(by sum2)}\\
&\longrightarrow \underline{x + \mathrm{sum}(\mathrm{rev}(l))} && \text{(by IH)}\\
&\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + x && \text{(by comm+)}\\
\mathrm{sum}(\underline{\mathrm{rev}(x \mid l)}) &\longrightarrow \underline{\mathrm{sum}(\mathrm{rev}(l)\ @\ (x \mid nil))} && \text{(by rev2)}\\
&\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + \underline{sum\ (x \mid nil)} && \text{(by Lemma 1)}\\
&\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + (x + \underline{sum\ (nil)}) && \text{(by sum2)}\\
&\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + (\underline{x + 0}) && \text{(by sum1)}\\
&\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + (\underline{0 + x}) && \text{(by comm+)}\\
&\longrightarrow \mathrm{sum}(\mathrm{rev}(l)) + x && \text{(by +1)}
\end{aligned}$$

□

Lemma 1. $\forall L1, L2 \in \texttt{NatList}, \mathrm{sum}(L1\ @\ L2) = \mathrm{sum}(L1) + \mathrm{sum}(L2)$.

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{sum}(nil \mathbin{@} l2) = \text{sum}(nil) + \text{sum}(l2)$
where $l2 \in \texttt{NatList}$. Note that $l2$ is a fresh constant.

$$\begin{aligned}
\text{sum}(\underline{nil \mathbin{@} l2}) &\longrightarrow \text{sum}(l2) && \text{(by @1)}\\
\underline{\text{sum}(nil)} + \text{sum}(l2) &\longrightarrow \underline{0 + \text{sum}(l2)} && \text{(by sum1)}\\
&\longrightarrow 0 && \text{(by +1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{sum}((x \mid l1) \mathbin{@} l2) = \text{sum}(x \mid l1) + \text{sum}(l2)$
Induction hypothesis: $\text{sum}(l1 \mathbin{@} l2) = \text{sum}(l1) + \text{sum}(l2)$
where $x \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$. Note that $x, l1, l2$ are fresh constants.

$$\begin{aligned}
\text{sum}(\underline{(x \mid l)) \mathbin{@} l2}) &\longrightarrow \underline{\text{sum}(x \mid (l1 \mathbin{@} l2))} && \text{(by @2)}\\
&\longrightarrow x + \underline{\text{sum}(l1 \mathbin{@} l2)} && \text{(by sum2)}\\
&\longrightarrow \underline{x + (\text{sum}(l1) + \text{sum}(l2))} && \text{(by IH)}\\
&\longrightarrow (x + \text{sum}(l1)) + \text{sum}(l2) && \text{(by assoc+)}\\
\underline{\text{sum}(x \mid l1)} + \text{sum}(l2) &\longrightarrow (x + \text{sum}(l1)) + \text{sum}(l2) && \text{(by sum2)}
\end{aligned}$$

□