Problem 20

Problem 1. $\forall L \in \texttt{NatList}, \text{setEqual}(L, \text{rev}(L)) = \mathit{true}$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{setEqual}(\mathit{nil}, \text{rev}(\mathit{nil})) = \mathit{true}$.

$$\begin{aligned}
\text{setEqual}(\mathit{nil}, \underline{\text{rev}(\mathit{nil})}) &\longrightarrow \underline{\text{setEqual}(\mathit{nil}, \mathit{nil})} && \text{(by rev1)}\\
&\longrightarrow (\underline{\text{diff}(\mathit{nil}, \mathit{nil})} = \mathit{nil})\ \mathit{and}\ (\text{diff}(\mathit{nil}, \mathit{nil}) = \mathit{nil}) && \text{(by setEq)}\\
&\longrightarrow \underline{(\mathit{nil} = \mathit{nil})}\ \mathit{and}\ (\text{diff}(\mathit{nil}, \mathit{nil}) = \mathit{nil}) && \text{(by diff1)}\\
&\longrightarrow \underline{\mathit{true}\ \mathit{and}\ (\text{diff}(\mathit{nil}, \mathit{nil}) = \mathit{nil})} && \text{(by equality)}\\
&\longrightarrow \underline{\text{diff}(\mathit{nil}, \mathit{nil})} = \mathit{nil} && \text{(by and)}\\
&\longrightarrow \underline{\mathit{nil} = \mathit{nil}} && \text{(by diff1)}\\
&\longrightarrow \mathit{true} && \text{(by equality)}
\end{aligned}$$

(2) Induction case

What to show: $\text{setEqual}(x \mid l, \text{rev}(x \mid l)) = \mathit{true}$

Induction hypothesis: $\text{setEqual}(l, \text{rev}(l)) = \mathit{true}$

where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants[1].

We use case splitting for our proofs as follows:

Case 1: $\text{has}(\text{rev}(l), x) = \mathit{true}$.

$$\begin{aligned}
\text{setEqual}(x \mid l, \underline{\text{rev}(x \mid l)}) &\longrightarrow \underline{\text{setEqual}(x \mid l, \text{rev}(l)\ @\ (x \mid \mathit{nil}))} && \text{(by rev2)}\\
&\longrightarrow (\underline{\text{diff}(x \mid l, \text{rev}(l)\ @\ (x \mid \mathit{nil}))} = \mathit{nil})\ \mathit{and}
\end{aligned}$$

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

$$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$$
(by setEq)

$\longrightarrow (\text{diff}(\underline{\text{diff}(x \mid l, \text{rev}(l))}, x \mid nil) = nil)\ and$
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by Problem 7 - Lemma 1)

$\longrightarrow (\text{diff}((\text{if } \underline{\text{has}(\text{rev}(l), x)} \text{ then } \text{diff}(l, \text{rev}(l))$
$\text{else } (x \mid \text{diff}(l, \text{rev}(l))) \text{ fi}), x \mid nil) = nil)\ and$
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by diff2)

$\longrightarrow (\text{diff}(\underline{(\text{if } true \text{ then } \text{diff}(l, \text{rev}(l))}$
$\underline{\text{else } (x \mid \text{diff}(l, \text{rev}(l))) \text{ fi})}, x \mid nil) = nil)\ and$
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by case splitting)

$\longrightarrow (\text{diff}(\underline{\text{diff}(l, \text{rev}(l))}, x \mid nil) = nil)\ and$
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by if1)

$\longrightarrow (\underline{\text{diff}(nil, x \mid nil)} = nil)\ and$
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by Problem 7)

$\longrightarrow \underline{(nil = nil)}\ and\ (\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by diff1)

$\longrightarrow \underline{true\ and\ (\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)}$
(by equality)

$\longrightarrow \underline{\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l)} = nil$
(by and)

$\longrightarrow (\underline{\text{diff}(\text{rev}(l), x \mid l)} \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil$
(by Problem 9 - Lemma 1)

$\longrightarrow (\text{drop}(\underline{\text{diff}(\text{rev}(l), l)}, x) \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil$
(by Problem 14)

$\longrightarrow (\underline{\text{drop}(nil, x)} \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil$
(by Problem 16)

$\longrightarrow \underline{(nil \mathbin{@} \text{diff}(x \mid nil, x \mid l))} = nil$
(by drop1)

$\longrightarrow \underline{\text{diff}(x \mid nil, x \mid l)} = nil$ (by @1)

$\longrightarrow \text{drop}(\underline{\text{diff}(x \mid nil, l)}, x) = nil$
(by Problem 14)

$\longrightarrow \text{drop}((\text{if } \underline{\text{has}(l, x)} \text{ then diff}(nil, l)$
$\text{else } (x \mid \text{diff}(nil, l)) \text{ fi}), x) = nil$
(by diff2)

$\longrightarrow \text{drop}((\text{if } \underline{\text{has}(\text{rev}(l), x)} \text{ then diff}(nil, l)$
$\text{else } (x \mid \text{diff}(nil, l)) \text{ fi}), x) = nil$
(by Problem 6)

$\longrightarrow \text{drop}(\underline{(\text{if } true \text{ then diff}(nil, l)}$
$\underline{\text{else } (x \mid \text{diff}(nil, l)) \text{ fi}}), x) = nil$
(by case splitting)

$\longrightarrow \text{drop}(\underline{\text{diff}(nil, l)}, x) = nil$ (by if1)

$\longrightarrow \underline{\text{drop}(nil, x)} = nil$ (by diff1)

$\longrightarrow \underline{nil = nil}$ (by drop1)

$\longrightarrow true$ (by equality)

Case 2: $\text{has}(\text{rev}(l), x) = false$.

$\text{setEqual}(x \mid l, \underline{\text{rev}(x \mid l)}) \longrightarrow \underline{\text{setEqual}(x \mid l, \text{rev}(l) \mathbin{@} (x \mid nil))}$
(by rev2)

$\longrightarrow (\underline{\text{diff}(x \mid l, \text{rev}(l) \mathbin{@} (x \mid nil))} = nil)\ and$
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by setEq)

$\longrightarrow (\text{diff}(\underline{\text{diff}(x \mid l, \text{rev}(l))}, x \mid nil) = nil)\ and$
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by Problem 7 - Lemma 1)

$\longrightarrow (\text{diff}((\text{if } \underline{\text{has}(\text{rev}(l), x)} \text{ then diff}(l, \text{rev}(l))$
$\text{else } (x \mid \text{diff}(l, \text{rev}(l))) \text{ fi}), x \mid nil) = nil)\ and$
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by diff2)

$\longrightarrow (\text{diff}(\underline{(\text{if } false \text{ then diff}(l, \text{rev}(l))}$
$\underline{\text{else } (x \mid \text{diff}(l, \text{rev}(l))) \text{ fi}}), x \mid nil) = nil)\ and$
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by case splitting)

$\longrightarrow (\text{diff}(x \mid \underline{\text{diff}(l, \text{rev}(l))}, x \mid nil) = nil)\ and$

$$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$$
(by if2)

$\longrightarrow$ $(\underline{\text{diff}(x \mid nil, x \mid nil)} = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by Problem 7)

$\longrightarrow$ $(\text{drop}(\underline{\text{diff}(x \mid nil, l)}, x) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by Problem 14)

$\longrightarrow$ $(\text{drop}((\text{if } \underline{\text{has}(l, x)} \text{ then diff}(nil, l)$
$\text{else } (x \mid \text{diff}(nil, l)) \text{ fi}), x) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by diff2)

$\longrightarrow$ $(\text{drop}((\text{if } \underline{\text{has}(\text{rev}(l), x)} \text{ then diff}(nil, l)$
$\text{else } (x \mid \text{diff}(nil, l)) \text{ fi}), x) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by Problem 6)

$\longrightarrow$ $(\text{drop}(\underline{(\text{if } false \text{ then diff}(nil, l)}$
$\underline{\text{else } (x \mid \text{diff}(nil, l)) \text{ fi})}, x) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by case splitting)

$\longrightarrow$ $(\text{drop}(x \mid \underline{\text{diff}(nil, l)}, x) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by if2)

$\longrightarrow$ $(\underline{\text{drop}(x \mid nil, x)} = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by diff1)

$\longrightarrow$ $((\text{if } \underline{x = x} \text{ then drop}(nil, x)$
$\text{else } (x \mid \text{drop}(nil, x)) \text{ fi}) = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by drop2)

$\longrightarrow$ $(\underline{(\text{if } true \text{ then drop}(nil, x)}$
$\underline{\text{else } (x \mid \text{drop}(nil, x)) \text{ fi})} = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by equality)

$\longrightarrow$ $(\underline{\text{drop}(nil, x)} = nil)$ *and*
$(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by if1)

$\longrightarrow$ $\underline{(nil = nil)}$ *and* $(\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)$
(by drop1)

$\longrightarrow$ $\underline{true \textit{ and } (\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l) = nil)}$
(by equality)

$\longrightarrow$ $\underline{\text{diff}(\text{rev}(l) \mathbin{@} (x \mid nil), x \mid l)} = nil$
(by and)

$\longrightarrow$ $(\underline{\text{diff}(\text{rev}(l), x \mid l)} \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil$
(by Problem 9 - Lemma 1)

$\longrightarrow$ $(\text{drop}(\underline{\text{diff}(\text{rev}(l), l)}, x) \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil$
(by Problem 14)

$\longrightarrow$ $(\underline{\text{drop}(nil, x)} \mathbin{@} \text{diff}(x \mid nil, x \mid l)) = nil$
(by Problem 16)

$\longrightarrow$ $\underline{(nil \mathbin{@} \text{diff}(x \mid nil, x \mid l))} = nil$
(by drop1)

$\longrightarrow$ $\underline{\text{diff}(x \mid nil, x \mid l)} = nil$ (by @1)

$\longrightarrow$ $\text{drop}(\underline{\text{diff}(x \mid nil, l)}, x) = nil$
(by Problem 14)

$\longrightarrow$ $\text{drop}((\text{if } \underline{\text{has}(l, x)} \text{ then } \text{diff}(nil, l)$
$\text{else } (x \mid \text{diff}(nil, l)) \text{ fi}), x) = nil$
(by diff2)

$\longrightarrow$ $\text{drop}((\text{if } \underline{\text{has}(\text{rev}(l), x)} \text{ then } \text{diff}(nil, l)$
$\text{else } (x \mid \text{diff}(nil, l)) \text{ fi}), x) = nil$
(by Problem 6)

$\longrightarrow$ $\text{drop}(\underline{(\text{if } false \text{ then } \text{diff}(nil, l)}$
$\underline{\text{else } (x \mid \text{diff}(nil, l)) \text{ fi})}, x) = nil$
(by case splitting)

$\longrightarrow$ $\text{drop}(x \mid \underline{\text{diff}(nil, l)}, x) = nil$ (by if2)

$\longrightarrow$ $\underline{\text{drop}(x \mid nil, x)} = nil$ (by diff1)

$\longrightarrow$ $(\text{if } \underline{x = x} \text{ then } \text{drop}(nil, x) \text{ else } (x \mid \text{drop}(nil, x)) \text{ fi}) = nil$
(by drop2)

$$\longrightarrow \underline{(\text{if } \mathit{true} \text{ then } \mathrm{drop}(\mathit{nil}, x) \text{ else } (x \mid \mathrm{drop}(\mathit{nil}, x)) \text{ fi})} = \mathit{nil}$$
(by equality)

$$\longrightarrow \underline{\mathrm{drop}(\mathit{nil}, x)} = \mathit{nil} \qquad \text{(by if1)}$$

$$\longrightarrow \underline{\mathit{nil} = \mathit{nil}} \qquad \text{(by drop1)}$$

$$\longrightarrow \mathit{true} \qquad \text{(by equality)}$$

□