Problem 9

Problem 1. $\forall L1, L2 \in \texttt{NatList}, \text{diff}(L1, L2) = \text{rev}(\text{diff}(\text{rev}(L1), L2))$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{diff}(nil, l2) = \text{rev}(\text{diff}(\text{rev}(nil), l2))$
where $l2 \in \texttt{NatList}$.

$$\begin{aligned}
\underline{\text{diff}(nil, l2)} &\longrightarrow nil && \text{(by diff1)} \\
\text{rev}(\text{diff}(\underline{\text{rev}(nil)}, l2)) &\longrightarrow \text{rev}(\underline{\text{diff}(nil, l2)}) && \text{(by rev1)} \\
&\longrightarrow \text{rev}(nil) && \text{(by diff1)} \\
&\longrightarrow nil && \text{(by rev1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{diff}(x \mid l1, l2) = \text{rev}(\text{diff}(\text{rev}(x \mid l1), l2))$
Induction hypothesis: $\text{diff}(l1, l2) = \text{rev}(\text{diff}(\text{rev}(l1), l2))$
where $x \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$.
We use case splitting for our proofs as follows:

Case 1: $\mathrm{has}(l2, x) = true$

$$\begin{aligned}
\underline{\mathrm{diff}(x \mid l1, l2)} &\longrightarrow \text{if } \underline{\mathrm{has}(l2, x)} \text{ then } \mathrm{diff}(l1, l2) \text{ else } (x \mid \mathrm{diff}(l1, l2)) \text{ fi} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by diff2)} \\
&\longrightarrow \underline{\text{if } true \text{ then } \mathrm{diff}(l1, l2) \text{ else } (x \mid \mathrm{diff}(l1, l2)) \text{ fi}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by case splitting)} \\
&\longrightarrow \underline{\mathrm{diff}(l1, l2)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by if1)} \\
&\longrightarrow \mathrm{rev}(\mathrm{diff}(\mathrm{rev}(l1), l2)) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by IH)} \\
\mathrm{rev}(\mathrm{diff}(\underline{\mathrm{rev}(x \mid l1)}, l2)) &\longrightarrow \mathrm{rev}(\underline{\mathrm{diff}(\mathrm{rev}(l1) \mathbin{@} (x \mid nil), l2)}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by rev2)} \\
&\longrightarrow \mathrm{rev}(\mathrm{diff}(\mathrm{rev}(l1), l2) \mathbin{@} \underline{\mathrm{diff}(x \mid nil, l2)}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 1)} \\
&\longrightarrow \mathrm{rev}(\mathrm{diff}(\mathrm{rev}(l1), l2) \mathbin{@} \\
&\quad (\text{if } \underline{\mathrm{has}(l2, x)} \text{ then } \mathrm{diff}(nil, l2) \text{ else } (x \mid \mathrm{diff}(nil, l2)) \text{ fi})) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by diff2)} \\
&\longrightarrow \mathrm{rev}(\mathrm{diff}(\mathrm{rev}(l1), l2) \mathbin{@} \\
&\quad \underline{(\text{if } true \text{ then } \mathrm{diff}(nil, l2) \text{ else } (x \mid \mathrm{diff}(nil, l2)) \text{ fi})}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by diff2)} \\
&\longrightarrow \mathrm{rev}(\mathrm{diff}(\mathrm{rev}(l1), l2) \mathbin{@} \underline{\mathrm{diff}(nil, l2)}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by if1)} \\
&\longrightarrow \mathrm{rev}(\underline{\mathrm{diff}(\mathrm{rev}(l1), l2) \mathbin{@} nil}) \qquad\qquad \text{(by diff1)} \\
&\longrightarrow \mathrm{rev}(\mathrm{diff}(\mathrm{rev}(l1), l2)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2 from Problem 4)}
\end{aligned}$$

Case 2: $\text{has}(l2, x) = false$

$$\underline{\text{diff}(x \mid l1, l2)} \longrightarrow \text{if } \underline{\text{has}(l2, x)} \text{ then diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}$$
(by diff2)

$$\longrightarrow \underline{\text{if } false \text{ then diff}(l1, l2) \text{ else } (x \mid \text{diff}(l1, l2)) \text{ fi}}$$
(by case splitting)

$$\longrightarrow x \mid \underline{\text{diff}(l1, l2)} \quad \text{(by if2)}$$

$$\longrightarrow x \mid \text{rev}(\text{diff}(\text{rev}(l1), l2)) \quad \text{(by IH)}$$

$$\text{rev}(\text{diff}(\underline{\text{rev}(x \mid l1)}, l2)) \longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1) \mathbin{@} (x \mid nil), l2)})$$
(by rev2)

$$\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) \mathbin{@} \underline{\text{diff}(x \mid nil, l2)})$$
(by Lemma 1)

$$\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) \mathbin{@}$$
$$(\text{if } \underline{\text{has}(l2, x)} \text{ then diff}(nil, l2) \text{ else } (x \mid \text{diff}(nil, l2)) \text{ fi}))$$
(by diff2)

$$\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) \mathbin{@}$$
$$\underline{(\text{if } false \text{ then diff}(nil, l2) \text{ else } (x \mid \text{diff}(nil, l2)) \text{ fi})}$$
(by case splitting)

$$\longrightarrow \text{rev}(\text{diff}(\text{rev}(l1), l2) \mathbin{@} (x \mid \underline{\text{diff}(nil, l2)}))$$
(by if2)

$$\longrightarrow \text{rev}(\underline{\text{diff}(\text{rev}(l1), l2) \mathbin{@} (x \mid nil)})$$
(by diff1)

$$\longrightarrow \underline{\text{rev}(x \mid nil)} \mathbin{@} \text{rev}(\text{diff}(\text{rev}(l1), l2))$$
(by Lemma 1 from Problem 4)

$$\longrightarrow (\underline{\text{rev}(nil)} \mathbin{@} (x \mid nil)) \mathbin{@} \text{rev}(\text{diff}(\text{rev}(l1), l2))$$
(by rev2)

$$\longrightarrow (\underline{nil \mathbin{@} (x \mid nil)}) \mathbin{@} \text{rev}(\text{diff}(\text{rev}(l1), l2))$$
(by rev1)

$$\longrightarrow \underline{(x \mid nil) \mathbin{@} \text{rev}(\text{diff}(\text{rev}(l1), l2))} \quad \text{(by @1)}$$

$$\longrightarrow x \mid \underline{(nil \mathbin{@} \text{rev}(\text{diff}(\text{rev}(l1), l2)))} \quad \text{(by @2)}$$

$$\longrightarrow x \mid \text{rev}(\text{diff}(\text{rev}(l1), l2)) \quad \text{(by @1)}$$

□

Lemma 1. $\forall L1, L2, L3 \in \texttt{NatList}, \text{diff}(L1 \mathbin{@} L2, L3) = \text{diff}(L1, L3) \mathbin{@} \text{diff}(L2, L3)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\text{diff}(nil \mathbin{@} l2, l3) = \text{diff}(nil, l3) \mathbin{@} \text{diff}(l2, l3)$
where $l2, l3 \in \texttt{NatList}$.

$$\begin{aligned}
\text{diff}(\underline{nil \mathbin{@} l2}, l3) &\longrightarrow \text{diff}(l2, l3) && \text{(by @1)}\\
\underline{\text{diff}(nil, l3)} \mathbin{@} \text{diff}(l2, l3) &\longrightarrow \underline{nil \mathbin{@} \text{diff}(l2, l3)} && \text{(by diff1)}\\
&\longrightarrow \text{diff}(l2, l3) && \text{(by @1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{diff}((x \mid l1) \mathbin{@} l2, l3) = \text{diff}((x \mid l1), l3) \mathbin{@} \text{diff}(l2, l3)$
Induction hypothesis: $\text{diff}(l1 \mathbin{@} l2, l3) = \text{diff}(l1, l3) \mathbin{@} \text{diff}(l2, l3)$
where $x \in \texttt{PNat}$, and $l1, l2, l3 \in \texttt{NatList}$.

We use case splitting for our proofs as follows:
Case 1: $\text{has}(l3, x) = true$

$$\begin{aligned}
\text{diff}(\underline{(x \mid l1) \mathbin{@} l2}, l3) &\longrightarrow \underline{\text{diff}(x \mid (l1 \mathbin{@} l2), l3)} && \text{(by)}\\
&\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1 \mathbin{@} l2, l3)\\
&\quad \text{else } (x \mid \text{diff}(l1 \mathbin{@} l2, l3)) \text{ fi} && \text{(by diff2)}\\
&\longrightarrow \underline{\text{if } true \text{ then } \text{diff}(l1 \mathbin{@} l2, l3)}\\
&\quad \underline{\text{else } (x \mid \text{diff}(l1 \mathbin{@} l2, l3)) \text{ fi}}\\
&\qquad && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{diff}(l1 \mathbin{@} l2, l3)} && \text{(by if1)}\\
&\longrightarrow \text{diff}(l1, l3) \mathbin{@} \text{diff}(l2, l3) && \text{(by IH)}\\
\underline{\text{diff}(x \mid l1, l3)} \mathbin{@} \text{diff}(l2, l3) &\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi}\\
&\quad \mathbin{@} \text{diff}(l2, l3) && \text{(by diff2)}\\
&\longrightarrow \underline{\text{if } true \text{ then } \text{diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi}}\\
&\quad \underline{\mathbin{@} \text{diff}(l2, l3)} && \text{(by case splitting)}\\
&\longrightarrow \text{diff}(l1, l3) \mathbin{@} \text{diff}(l2, l3) && \text{(by if1)}
\end{aligned}$$

Case 2: $\text{has}(l3, x) = false$

$$
\begin{aligned}
\text{diff}(\underline{(x \mid l1) \mathbin{@} l2}, l3) &\longrightarrow \underline{\text{diff}(x \mid (l1 \mathbin{@} l2), l3)} && \text{(by @2)} \\
&\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then diff}(l1 \mathbin{@} l2, l3) \\
&\quad \text{else } (x \mid \text{diff}(l1 \mathbin{@} l2, l3)) \text{ fi} && \text{(by diff2)} \\
&\longrightarrow \underline{\text{if } false \text{ then diff}(l1 \mathbin{@} l2, l3)} \\
&\quad \underline{\text{else } (x \mid \text{diff}(l1 \mathbin{@} l2, l3)) \text{ fi}} \\
& && \text{(by case splitting)} \\
&\longrightarrow x \mid \underline{\text{diff}(l1 \mathbin{@} l2, l3)} && \text{(by if2)} \\
&\longrightarrow x \mid (\text{diff}(l1, l3) \mathbin{@} \text{diff}(l2, l3)) && \text{(by IH)} \\
\underline{\text{diff}(x \mid l1, l3)} \mathbin{@} \text{diff}(l2, l3) &\longrightarrow \text{if } \underline{\text{has}(l3, x)} \text{ then diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi} \\
&\quad \mathbin{@} \text{diff}(l2, l3) && \text{(by diff2)} \\
&\longrightarrow \underline{\text{if } false \text{ then diff}(l1, l3) \text{ else } (x \mid \text{diff}(l1, l3)) \text{ fi}} \\
&\quad \underline{\mathbin{@} \text{diff}(l2, l3)} && \text{(by case splitting)} \\
&\longrightarrow \underline{(x \mid \text{diff}(l1, l3)) \mathbin{@} \text{diff}(l2, l3)} && \text{(by if2)} \\
&\longrightarrow x \mid (\text{diff}(l1, l3) \mathbin{@} \text{diff}(l2, l3)) && \text{(by @2)}
\end{aligned}
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□