Problem 2

Problem 1. $\forall X \in \texttt{PNat}, \mathrm{rev}(\mathrm{mkl1}(X)) = \mathrm{mkl2}(X)$.

Proof. By structural induction on X.

(1) Base case

What to show: $\mathrm{rev}(\mathrm{mkl1}(0)) = \mathrm{mkl2}(0)$.

$$\begin{aligned}
\mathrm{rev}(\underline{\mathrm{mkl1}(0)}) &\longrightarrow \underline{\mathrm{rev}(\mathit{nil})} && \text{(by mkl1-1)}\\
&\longrightarrow \mathit{nil} && \text{(by rev1)}\\
\underline{\mathrm{mkl2}(0)} &\longrightarrow \underline{\mathrm{smkl2}(0, \mathit{nil})} && \text{(by mkl2)}\\
&\longrightarrow \mathit{nil} && \text{(by smkl2-1)}
\end{aligned}$$

(2) Induction case

What to show: $\mathrm{rev}(\mathrm{mkl1}(\mathrm{s}(x))) = \mathrm{mkl2}(\mathrm{s}(x))$

Induction hypothesis: $\mathrm{rev}(\mathrm{mkl1}(x)) = \mathrm{mkl2}(x)$

where $x \in \texttt{PNat}$. Note that x is a fresh constant[1].

$$\begin{aligned}
\mathrm{rev}(\underline{\mathrm{mkl1}(\mathrm{s}(x))}) &\longrightarrow \underline{\mathrm{rev}(\mathrm{s}(x) \mid \mathrm{mkl1}(x))} && \text{(by mkl1-2)}\\
&\longrightarrow \underline{\mathrm{rev}(\mathrm{mkl1}(x))} \mathbin{@} (\mathrm{s}(x) \mid \mathit{nil}) && \text{(by rev2)}\\
&\longrightarrow \underline{\mathrm{mkl2}(x)} \mathbin{@} (\mathrm{s}(x) \mid \mathit{nil}) && \text{(by IH)}\\
&\longrightarrow \mathrm{smkl2}(x, \mathit{nil}) \mathbin{@} (\mathrm{s}(x) \mid \mathit{nil}) && \text{(by mkl2)}\\
\underline{\mathrm{mkl2}(\mathrm{s}(x))} &\longrightarrow \underline{\mathrm{smkl2}(\mathrm{s}(x), \mathit{nil})} && \text{(by mkl2)}\\
&\longrightarrow \underline{\mathrm{smkl2}(x, \mathrm{s}(x) \mid \mathit{nil})} && \text{(by smkl2-2)}\\
&\longrightarrow \mathrm{smkl2}(x, \mathit{nil}) \mathbin{@} (\mathrm{s}(x) \mid \mathit{nil}) && \text{(by Lemma 1)}
\end{aligned}$$

□

Lemma 1. $\forall X \in \texttt{PNat}, L \in \texttt{NatList}, \mathrm{smkl2}(X, L) = \mathrm{smkl2}(X, \mathit{nill}) \mathbin{@} L$.

Proof. By structural induction on X.

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

(1) Base case

What to show: $\mathrm{smkl2}(0, l) = \mathrm{smkl2}(0, nil) \mathbin{@} l$
where $l \in \texttt{NatList}$. Note that l is a fresh constant.

$$\begin{aligned}
\underline{\mathrm{smkl2}(0, l)} &\longrightarrow l && \text{(by smkl2-1)} \\
\underline{\mathrm{smkl2}(0, nil)} \mathbin{@} l &\longrightarrow \underline{nil \mathbin{@} l} && \text{(by smkl2-1)} \\
&\longrightarrow l && \text{(by @1)}
\end{aligned}$$

(2) Induction case

What to show: $\mathrm{smkl2}(\mathrm{s}(x), l) = \mathrm{smkl2}(\mathrm{s}(x), nil) \mathbin{@} l$
Induction hypothesis: $\mathrm{smkl2}(x, l) = \mathrm{smkl2}(x, nil) \mathbin{@} l$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants.

$$\begin{aligned}
\underline{\mathrm{smkl2}(\mathrm{s}(x), l)} &\longrightarrow \underline{\mathrm{smkl2}(x,\ \mathrm{s}(x) \mid l)} && \text{(by smkl2-2)} \\
&\longrightarrow \mathrm{smkl2}(x,\ nil) \mathbin{@} (\mathrm{s}(x) \mid l) && \text{(by IH)} \\
\underline{\mathrm{smkl2}(\mathrm{s}(x), nil)} \mathbin{@} l &\longrightarrow \underline{\mathrm{smkl2}(x, \mathrm{s}(x) \mid nil)} \mathbin{@} l && \text{(by smkl2-2)} \\
&\longrightarrow \underline{(\mathrm{smkl2}(x, nil) \mathbin{@} (\mathrm{s}(x) \mid nil)) \mathbin{@} l} && \text{(by IH)} \\
&\longrightarrow \mathrm{smkl2}(x, nil) \mathbin{@} \underline{((\mathrm{s}(x) \mid nil) \mathbin{@} l)} && \\
& && \text{(by assoc@)} \\
&\longrightarrow \mathrm{smkl2}(x, nil) \mathbin{@} (\mathrm{s}(x) \mid \underline{(nil \mathbin{@} l)}) && \text{(by @2)} \\
&\longrightarrow \mathrm{smkl2}(x, nil) \mathbin{@} (\mathrm{s}(x) \mid l) && \text{(by @1)}
\end{aligned}$$

□