Problem 15

Problem 1. $\forall L \in \texttt{NatList}, \text{diff}(L, L) = nil$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{diff}(nil, nil) = nil$.

$$\underline{\text{diff}(nil, nil)} \longrightarrow nil \qquad \text{(by diff1)}$$

(2) Induction case

What to show: $\text{diff}(x \mid l, x \mid l) = nil$
Induction hypothesis: $\text{diff}(l, l) = nil$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$. Note that x, l are fresh constants[1].

We use case splitting for our proofs as follows:
Case 1: $\text{has}(l, x) = true$

$$\begin{aligned}
\underline{\text{diff}(x \mid l, x \mid l)} &\longrightarrow \text{drop}(\underline{\text{diff}(x \mid l, l)}, x) && \text{(by Problem 14)} \\
&\longrightarrow \text{drop}(\text{if } \underline{\text{has}(l, x)} \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}, x) \\
& && \text{(by diff2)} \\
&\longrightarrow \text{drop}(\underline{\text{if } true \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}}, x) \\
& && \text{(by case splitting)} \\
&\longrightarrow \text{drop}(\underline{\text{diff}(l, l)}, x) && \text{(by if1)} \\
&\longrightarrow \underline{\text{drop}(nil, x)} && \text{(by IH)} \\
&\longrightarrow nil && \text{(by drop1)}
\end{aligned}$$

Case 2: $\text{has}(l, x) = false$

$$\underline{\text{diff}(x \mid l, x \mid l)} \longrightarrow \text{drop}(\underline{\text{diff}(x \mid l, l)}, x) \qquad \text{(by Problem 14)}$$

[1] A fresh constant of a sort denotes an arbitrary value of the sort, and has never been used before.

$\longrightarrow$ drop(if $\underline{\text{has}(l, x)}$ then diff(l, l) else $(x \mid \text{diff}(l, l))$ fi$, x)$
(by diff2)

$\longrightarrow$ drop($\underline{\text{if } \mathit{false} \text{ then diff}(l, l) \text{ else } (x \mid \text{diff}(l, l)) \text{ fi}}, x)$
(by case splitting)

$\longrightarrow$ drop$(x \mid \underline{\text{diff}(l, l)}, x)$ (by if2)

$\longrightarrow$ $\underline{\text{drop}(x \mid \mathit{nil}, x)}$ (by IH)

$\longrightarrow$ if $\underline{(x = x)}$ then drop(nil, x) else $(x \mid \text{drop}(\mathit{nil}, x))$ fi
(by drop2)

$\longrightarrow$ $\underline{\text{if } \mathit{true} \text{ then drop}(\mathit{nil}, x) \text{ else } (x \mid \text{drop}(\mathit{nil}, x)) \text{ fi}}$
(by equality)

$\longrightarrow$ $\underline{\text{drop}(\mathit{nil}, x)}$ (by if1)

$\longrightarrow$ nil (by drop1)

□