Problem 17

Problem 1. $\forall L \in \texttt{NatList}, \text{rmDup}(\text{rmDup}(L)) = \text{rmDup}(L)$.

Proof. By structural induction on L.

(1) Base case

What to show: $\text{rmDup}(\text{rmDup}(nil)) = \text{rmDup}(nil)$.

$$\begin{aligned}
\text{rmDup}(\underline{\text{rmDup}(nil)}) &\longrightarrow \underline{\text{rmDup}(nil)} && \text{(by rmDup1)}\\
&\longrightarrow nil && \text{(by rmDup1)}\\
\underline{\text{rmDup}(nil)} &\longrightarrow nil && \text{(by rmDup1)}
\end{aligned}$$

(2) Induction case

What to show: $\text{rmDup}(\text{rmDup}(x \mid l)) = \text{rmDup}(x \mid l)$

Induction hypothesis: $\text{rmDup}(\text{rmDup}(l)) = \text{rmDup}(l)$

where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$.

We use case splitting for our proofs as follows:

Case 1: $\text{has}(l, x) = true$.

$$\begin{aligned}
\text{rmDup}(\underline{\text{rmDup}(x \mid l)}) &\longrightarrow \text{rmDup}(\text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(l)\\
&\qquad \text{else } (x \mid \text{rmDup}(l)) \text{ fi}) && \text{(by rmDup2)}\\
&\longrightarrow \text{rmDup}(\underline{\text{if } true \text{ then rmDup}(l)}\\
&\qquad \underline{\text{else } (x \mid \text{rmDup}(l)) \text{ fi}}) && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{rmDup}(\text{rmDup}(l))} && \text{(by if1)}\\
&\longrightarrow \text{rmDup}(l) && \text{(by IH)}\\
\underline{\text{rmDup}(x \mid l)} &\longrightarrow \text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}\\
& && \text{(by rmDup2)}\\
&\longrightarrow \underline{\text{if } true \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}}\\
& && \text{(by case splitting)}\\
&\longrightarrow \text{rmDup}(l) && \text{(by if1)}
\end{aligned}$$

Case 2: $\mathrm{has}(l, x) = false$.

$$
\begin{aligned}
\mathrm{rmDup}(\underline{\mathrm{rmDup}(x \mid l)}) &\longrightarrow \mathrm{rmDup}(\text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{rmDup}(l) \\
&\qquad \text{else } (x \mid \mathrm{rmDup}(l)) \text{ fi}) && \text{(by rmDup2)} \\
&\longrightarrow \mathrm{rmDup}(\underline{\text{if } false \text{ then } \mathrm{rmDup}(l)} \\
&\qquad \underline{\text{else } (x \mid \mathrm{rmDup}(l)) \text{ fi}}) && \text{(by case splitting)} \\
&\longrightarrow \underline{\mathrm{rmDup}(x \mid \mathrm{rmDup}(l))} && \text{(by if2)} \\
&\longrightarrow \text{if } \underline{\mathrm{has}(\mathrm{rmDup}(l), x)} \text{ then } \mathrm{rmDup}(\mathrm{rmDup}(l)) \\
&\qquad \text{else } (x \mid \mathrm{rmDup}(\mathrm{rmDup}(l))) \text{ fi} \\
&&& \text{(by rmDup2)} \\
&\longrightarrow \text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{rmDup}(\mathrm{rmDup}(l)) \\
&\qquad \text{else } (x \mid \mathrm{rmDup}(\mathrm{rmDup}(l))) \text{ fi} \\
&&& \text{(by Lemma 1)} \\
&\longrightarrow \underline{\text{if } false \text{ then } \mathrm{rmDup}(\mathrm{rmDup}(l))} \\
&\qquad \underline{\text{else } (x \mid \mathrm{rmDup}(\mathrm{rmDup}(l))) \text{ fi}} \\
&&& \text{(by case splitting)} \\
&\longrightarrow x \mid \underline{\mathrm{rmDup}(\mathrm{rmDup}(l))} && \text{(by if2)} \\
&\longrightarrow x \mid \mathrm{rmDup}(l) && \text{(by IH)} \\
\underline{\mathrm{rmDup}(x \mid l)} &\longrightarrow \text{if } \underline{\mathrm{has}(l, x)} \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi} \\
&&& \text{(by rmDup2)} \\
&\longrightarrow \underline{\text{if } false \text{ then } \mathrm{rmDup}(l) \text{ else } (x \mid \mathrm{rmDup}(l)) \text{ fi}} \\
&&& \text{(by case splitting)} \\
&\longrightarrow x \mid \mathrm{rmDup}(l) && \text{(by if2)}
\end{aligned}
$$

$\square$

Lemma 1. $\forall X \in \texttt{PNat}, \forall L \in \texttt{NatList}, \mathrm{has}(L, X) = \mathrm{has}(\mathrm{rmDup}(L), X)$.

Proof. By structural induction on L.

(1) Base case

What to show: $\quad \mathrm{has}(nil, x) = \mathrm{has}(\mathrm{rmDup}(nil), x)$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$.

$$
\begin{aligned}
\underline{\mathrm{has}(nil, x)} &\longrightarrow nil && \text{(by has1)} \\
\mathrm{has}(\underline{\mathrm{rmDup}(nil)}, x) &\longrightarrow \underline{\mathrm{has}(nil, x)} && \text{(by rmDup1)} \\
&\longrightarrow nil && \text{(by has1)}
\end{aligned}
$$

(2) Induction case

What to show: $\text{has}(y \mid l, x) = \text{has}(\text{rmDup}(y \mid l), x)$

Induction hypothesis: $\text{has}(l, x) = \text{has}(\text{rmDup}(l), x)$

where $x, y \in$ `PNat` and $l \in$ `NatList`.

We use case splitting for our proofs as follows:

Case 1.1: $\text{has}(l, y) = \mathit{true}$, $\text{has}(\text{rmDup}(l), x) = \mathit{true}$.

$$\begin{aligned}
\underline{\text{has}(y \mid l, x)} &\longrightarrow (x = y) \ \mathit{or} \ \underline{\text{has}(l, x)} && \text{(by has2)}\\
&\longrightarrow (x = y) \ \mathit{or} \ \underline{\text{has}(\text{rmDup}(l), x)} && \text{(by IH)}\\
&\longrightarrow \underline{(x = y) \ \mathit{or} \ \mathit{true}} && \text{(by case splitting)}\\
&\longrightarrow \mathit{true} && \text{(by or)}\\
\text{has}(\underline{\text{rmDup}(y \mid l)}, x) &\longrightarrow \text{has}(\text{if } \underline{\text{has}(l, y)} \text{ then rmDup}(l) \text{ else } (y \mid \text{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by rmDup2)}\\
&\longrightarrow \text{has}(\underline{\text{if } \mathit{true} \text{ then rmDup}(l) \text{ else } (y \mid \text{rmDup}(l)) \text{ fi}}, x)\\
& && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{has}(\text{rmDup}(l), x)} && \text{(by if1)}\\
&\longrightarrow \mathit{true} && \text{(by case splitting)}
\end{aligned}$$

Case 1.2.1: $\text{has}(l, y) = \mathit{true}$, $\text{has}(\text{rmDup}(l), x) = \mathit{false}$, $y = x$.

$$\begin{aligned}
\underline{\text{has}(\underline{y} \mid l, x)} &\longrightarrow \underline{\text{has}(x \mid l, x)} && \text{(case splitting)}\\
&\longrightarrow \underline{(x = x)} \ \mathit{or} \ \text{has}(l, x) && \text{(by has2)}\\
&\longrightarrow \underline{\mathit{true} \ \mathit{or} \ \text{has}(l, x)} && \text{(by equality)}\\
&\longrightarrow \mathit{true} && \text{(by or)}\\
\text{has}(\underline{\text{rmDup}(\underline{y} \mid l)}, x) &\longrightarrow \text{has}(\underline{\text{rmDup}(x \mid l)}, x) && \text{(by case splitting)}\\
&\longrightarrow \text{has}(\text{if } \underline{\text{has}(l, x)} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by rmDup2)}\\
&\longrightarrow \text{has}(\text{if } \underline{\text{has}(\text{rmDup}(l), x)} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by IH)}\\
&\longrightarrow \text{has}(\underline{\text{if } \mathit{false} \text{ then rmDup}(l) \text{ else } (x \mid \text{rmDup}(l)) \text{ fi}}, x)\\
& && \text{(by case splitting)}\\
&\longrightarrow \underline{\text{has}(x \mid \text{rmDup}(l), x)} && \text{(by if2)}\\
&\longrightarrow \underline{(x = x)} \ \mathit{or} \ \text{has}(\text{rmDup}(l), x) && \text{(by has2)}\\
&\longrightarrow \underline{\mathit{true} \ \mathit{or} \ \text{has}(\text{rmDup}(l), x)} && \text{(by equality)}\\
&\longrightarrow \mathit{true} && \text{(by or)}
\end{aligned}$$

Case 1.2.2: $\mathrm{has}(l, y) = true$, $\mathrm{has}(\mathrm{rmDup}(l), x) = false$, $(y = x) = false$.

$$
\begin{aligned}
\underline{\mathrm{has}(y \mid l, x)} &\longrightarrow \underline{(x = y)}\ \mathit{or}\ \mathrm{has}(l, x) && \text{(by has2)}\\
&\longrightarrow \underline{false\ \mathit{or}\ \mathrm{has}(l, x)} && \text{(by case splitting)}\\
&\longrightarrow \underline{\mathrm{has}(l, x)} && \text{(by or)}\\
&\longrightarrow \underline{\mathrm{has}(\mathrm{rmDup}(l), x)} && \text{(by IH)}\\
&\longrightarrow false && \text{(by case splitting)}\\
\mathrm{has}(\underline{\mathrm{rmDup}(y \mid l)}, x) &\longrightarrow \mathrm{has}(\text{if } \underline{\mathrm{has}(l, y)} \text{ then } \mathrm{rmDup}(l) \text{ else } (y \mid \mathrm{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by rmDup2)}\\
&\longrightarrow \mathrm{has}(\underline{\text{if } true \text{ then } \mathrm{rmDup}(l) \text{ else } (y \mid \mathrm{rmDup}(l)) \text{ fi}}, x)\\
& && \text{(by case splitting)}\\
&\longrightarrow \underline{\mathrm{has}(\mathrm{rmDup}(l), x)} && \text{(by if1)}\\
&\longrightarrow false && \text{(by case splitting)}
\end{aligned}
$$

Case 2: $\mathrm{has}(l, y) = false$

$$
\begin{aligned}
\underline{\mathrm{has}(y \mid l, x)} &\longrightarrow (x = y)\ \mathit{or}\ \underline{\mathrm{has}(l, x)} && \text{(by has2)}\\
&\longrightarrow (x = y)\ \mathit{or}\ \mathrm{has}(\mathrm{rmDup}(l), x) && \text{(by IH)}\\
\mathrm{has}(\underline{\mathrm{rmDup}(y \mid l)}, x) &\longrightarrow \mathrm{has}(\text{if } \underline{\mathrm{has}(l, y)} \text{ then } \mathrm{rmDup}(l) \text{ else } (y \mid \mathrm{rmDup}(l)) \text{ fi}, x)\\
& && \text{(by rmDup2)}\\
&\longrightarrow \mathrm{has}(\underline{\text{if } false \text{ then } \mathrm{rmDup}(l) \text{ else } (y \mid \mathrm{rmDup}(l)) \text{ fi}}, x)\\
& && \text{(by case splitting)}\\
&\longrightarrow \mathrm{has}(y \mid \mathrm{rmDup}(l), x) && \text{(by if2)}\\
&\longrightarrow (x = y)\ \mathit{or}\ \mathrm{has}(\mathrm{rmDup}(l), x) && \text{(by has2)}
\end{aligned}
$$

□