Problem 1

Problem 1. $\forall L \in \texttt{NatList}, \text{fold}*(\text{rev}(L)) = \text{fold}*(L)$.

Proof. By structural induction on L.

(1) Base case

What to show: $\quad \text{fold}*(\text{rev}(nil)) = \text{fold}*(nil)$.

$$\text{fold}*(\underline{\text{rev}(nil)}) \longrightarrow \text{fold}*(nil) \qquad \text{(by rev1)}$$

(2) Induction case

What to show: $\quad \text{fold}*(\text{rev}(x \mid l)) = \text{fold}*(x \mid l)$
Induction hypothesis: $\quad \text{fold}*(\text{rev}(l)) = \text{fold}*(l)$
where $x \in \texttt{PNat}$ and $l \in \texttt{NatList}$.

$$\begin{aligned}
\text{fold}*(\underline{\text{rev}(x \mid l)}) &\longrightarrow \underline{\text{fold}*(\text{rev}(l) \mathbin{@} (x \mid nil))} && \text{(by rev2)} \\
&\longrightarrow \underline{\text{fold}*(\text{rev}(l))} * \text{fold}*(x \mid nil) && \text{(by Lemma 1)} \\
&\longrightarrow \text{fold}*(l) * \underline{\text{fold}*(x \mid nil)} && \text{(by IH)} \\
&\longrightarrow \text{fold}*(l) * (x * \underline{\text{fold}*(nil)}) && \text{(by fold*-2)} \\
&\longrightarrow \text{fold}*(l) * (\underline{x * \text{s}(0)}) && \text{(by fold*-1)} \\
&\longrightarrow \text{fold}*(l) * (\underline{\text{s}(0) * x}) && \text{(by comm*)} \\
&\longrightarrow \text{fold}*(l) * (\underline{(0 * x)} + x) && \text{(by *2)} \\
&\longrightarrow \text{fold}*(l) * \underline{(0 + x)} && \text{(by *1)} \\
&\longrightarrow \underline{\text{fold}*(l) * x} && \text{(by +1)} \\
&\longrightarrow x * \text{fold}*(l) && \text{(by comm*)} \\
\underline{\text{fold}*(x \mid l)} &\longrightarrow x * \text{fold}*(l) && \text{(fold*-2)}
\end{aligned}$$

□

Lemma 1. $\forall L1, L2 \in \texttt{NatList}, \text{fold}*(L1 \mathbin{@} L2) = \text{fold}*(L1) * \text{fold}*(L2)$.

Proof. By structural induction on $L1$.

(1) Base case

What to show: $\quad \text{fold}{*}(nil \mathbin{@} l2) = \text{fold}{*}(nil) * \text{fold}{*}(l2)$
where $l2 \in \texttt{NatList}$.

$$\begin{array}{rcll}
\text{fold}{*}(\underline{nil \mathbin{@} l2}) & \longrightarrow & \text{fold}{*}(l2) & \text{(by @1)} \\
\underline{\text{fold}{*}(nil)} * \text{fold}{*}(l2) & \longrightarrow & \underline{\text{s}(0) * \text{fold}{*}(l2)} & \text{(by fold*-1)} \\
& \longrightarrow & \underline{(0 * \text{fold}{*}(l2))} + \text{fold}{*}(l2) & \text{(by *2)} \\
& \longrightarrow & \underline{0 + \text{fold}{*}(l2)} & \text{(by *1)} \\
& \longrightarrow & \text{fold}{*}(l2) & \text{(by +1)}
\end{array}$$

(2) Induction case

What to show: $\quad \text{fold}{*}((x \mid l1) \mathbin{@} l2) = \text{fold}{*}(x \mid l1) * \text{fold}{*}(l2)$
Induction hypothesis: $\quad \text{fold}{*}(l1 \mathbin{@} l2) = \text{fold}{*}(l1) * \text{fold}{*}(l2)$
where $x \in \texttt{PNat}$ and $l1, l2 \in \texttt{NatList}$.

$$\begin{array}{rcll}
\text{fold}{*}(\underline{(x \mid l1) \mathbin{@} l2}) & \longrightarrow & \underline{\text{fold}{*}(x \mid (l1 \mathbin{@} l2))} & \text{(by @2)} \\
& \longrightarrow & x * \underline{\text{fold}{*}(l1 \mathbin{@} l2)} & \text{(by fold*-2)} \\
& \longrightarrow & x * (\text{fold}{*}(l1) * \text{fold}{*}(l2)) & \text{(by IH)} \\
\underline{\text{fold}{*}(x \mid l1)} * \text{fold}{*}(l2) & \longrightarrow & \underline{(x * \text{fold}{*}(l1)) * \text{fold}{*}(l2)} & \text{(by fold*-2)} \\
& \longrightarrow & x * (\text{fold}{*}(l1) * \text{fold}{*}(l2)) & \text{(by assoc*)}
\end{array}$$

$\square$