DNSC 6212: Optimization Methods and Applications
Fall 2017 – Midterm Exam

Instructions

1. The exam was made available at 8:00 am on Friday 10/27, and is due by midnight on Sunday 10/29.
2. Please work independently on the exam. No collaboration, whatsoever, is permitted. *Make sure to include and sign a statement attesting that the exam was completed by you alone, without seeking help from others, and in accordance with the Code of Academic Integrity.*
3. I added a Discussion Forum in Blackboard for the exam. If you feel you need further clarification for any item, please post your question under the forum. If I decide that the question is of merit, I shall post a clarification for everyone's benefit. I suggest that you subscribe to the forum to stay aware of any questions and answers posted. We shall not be answering questions regarding the exam via e-mail.
4. Make sure to upload to Blackboard one zipped folder that contains your exam solution (pdf file, AMPL files, Excel, etc.). Please do not send us your exam solution via e-mail.

Question 1 (LP Algebraic Formulation – 20 Points)

A large paper manufacturing company has 10 mills from which it needs to supply 1,000 customers. It uses three alternative types of machines and four types of raw materials to make five different types of paper. The company needs to develop a detailed production-distribution plan for the upcoming month, with the objective of minimizing the total cost of producing and distributing the paper during the month. Specifically, it is necessary to determine the amount of each type of paper to be made at each paper mill on each type of machine *and* the amount of each type of paper to be shipped from each paper mill to each customer.

The relevant data can be expressed symbolically as follows:

D_{jk} = number of units of paper type k demanded by customer j.

r_{klm} = number of units of raw material m needed to produce 1 unit of paper type k on machine type l.

R_{im} = number of units of raw material m available at paper mill i.

c_{kl} = number of capacity units of machine type l that will produce 1 unit of paper type k.

C_{il} = number of capacity units of machine type l available at paper mill i.

p_{ikl} = production cost for each unit of paper type k produced on machine type l at paper mill i.

t_{ijk} = transportation cost for each unit of paper type k shipped from mill i to customer j.

(a) Using the above notation, provide a complete algebraic formulation. Make sure to clearly define the:

- Decision variables, (4 points)
- Constraints, and (10 points)
- Objective function. (3 points)

(b) Count the number of the actual decision variables and constraints by type for the model. (3 points)

Question 2 (AMPL and Excel Implementations – 40 Points)

(NOTE: For this question, you will be graded on both the *quality* and *validity* of your AMPL and Excel implementations).

Green Earth is an organization that operates a reclamation center that collects four types of solid waste materials and treats them so that they can be amalgamated into a salable product. Note that treating and amalgamating are separate processes. Three different grades of this product can be made, depending on the mix of materials used (see first column in Table 2a). Although there is some flexibility in the mix for each grade, quality standards may specify the minimum or maximum weight allowed for the proportion of a material in the product grade, whereby the proportion is the weight of the material expressed as percentage of the total weight for the product grade. For each of the two higher grades, a fixed percentage is specified for one of the materials. These specifications are given in Table 2a along with the cost of amalgamation and the selling price for each grade.

Table 2a: Product Data

Grade	Specification	Amalgamation Cost per Pound	Selling Price per Pound
A	Material 1: Not more than 30% of total Material 2: Not less than 40% of total Material 3: Not more than 50% of total Material 4: Exactly 20% of total	\$3.00	\$8.50
B	Material 1: Not more than 50% of total Material 2: Not less than 10% of total Material 4: Exactly 10% of total	\$2.50	\$7.00
C	Material 1: Not more than 70% of total	\$2.00	\$5.50

Table 2b: Solid Waste Materials Data

Material	Pounds Available per Week	Treatment Cost per Pound	Additional Restrictions
1	3,000	\$3.00	1. For each material, at least half of the pounds available per week should be collected and treated. 2. \$30,000 per week should be used for treating materials.
2	2,000	\$6.00	
3	4,000	\$4.00	
4	1,000	\$5.00	

The reclamation center collects its solid waste materials from regular sources and so is normally able to maintain a steady rate of such materials for treating them. Table 2b gives the quantities available for collection and treatment each week, as well as the cost of treatment, for each type of material. In addition, Good Earth has allocated \$30,000 per week to cover the entire treatment cost for the solid waste materials. These funds come from contribution and grants raised by the organization. The organization's mandate requires that at least half the amount available of each material to be actually collected and treated.

The organization's management wants to determine the amount of each product grade to produce and the exact mix of materials to be used for each grade. The objective is to maximize the net weekly income (total sales minus total amalgamation cost), <u>exclusive</u> of the fixed treatment cost of $30,000 per week that is being covered by gifts and grants. To achieve this, the following LP was formulated.

$$\max \quad 5.5(x_{A1}+x_{A2}+x_{A3}+x_{A4})+4.5(x_{B1}+x_{B2}+x_{B3}+x_{B4})+3.5(x_{C1}+x_{C2}+x_{C3}+x_{C4})$$

$$\begin{aligned}
\text{s.t.} \quad & x_{A1} \le 0.3(x_{A1}+x_{A2}+x_{A3}+x_{A4}) \\
& x_{A2} \ge 0.4(x_{A1}+x_{A2}+x_{A3}+x_{A4}) \\
& x_{A3} \le 0.5(x_{A1}+x_{A2}+x_{A3}+x_{A4}) \\
& x_{A4} = 0.2(x_{A1}+x_{A2}+x_{A3}+x_{A4}) \\
& x_{B1} \le 0.5(x_{B1}+x_{B2}+x_{B3}+x_{B4}) \\
& x_{B2} \ge 0.1(x_{B1}+x_{B2}+x_{B3}+x_{B4}) \\
& x_{B4} = 0.1(x_{B1}+x_{B2}+x_{B3}+x_{B4}) \\
& x_{C1} \le 0.7(x_{C1}+x_{C2}+x_{C3}+x_{C4})
\end{aligned}$$

$$\begin{aligned}
& x_{A1}+x_{B1}+x_{C1} \le 3{,}000 \\
& x_{A2}+x_{B2}+x_{C2} \le 2{,}000 \\
& x_{A3}+x_{B3}+x_{C3} \le 4{,}000 \\
& x_{A4}+x_{B4}+x_{C4} \le 1{,}000
\end{aligned}$$

$$\begin{aligned}
& x_{A1}+x_{B1}+x_{C1} \ge 1{,}500 \\
& x_{A2}+x_{B2}+x_{C2} \ge 1{,}000 \\
& x_{A3}+x_{B3}+x_{C3} \ge 2{,}000 \\
& x_{A4}+x_{B4}+x_{C4} \ge 500
\end{aligned}$$

$$3(x_{A1}+x_{B1}+x_{C1})+6(x_{A2}+x_{B2}+x_{C2})+4(x_{A3}+x_{B3}+x_{C3})+5(x_{A4}+x_{B4}+x_{C4})=30{,}000$$

$$x_{A1} \ge 0, x_{A2} \ge 0, \ldots, x_{C4} \ge 0$$

(a) Justify why the formulation is a valid one for the stated problem. Make sure to clearly explain the variables, objective function and constraints. (5 points)
(b) Implement and solve the model using AMPL. Follow the "good practices" demonstrated in the various examples we've done together. In particular, make sure that the model allows for changes in the data as well as the dimension (i.e., size) of the problem instance. Provide the .mod, .dat and .run files for the model. Describe the solution obtained. (20 points)
(c) Implement the model using Excel. Follow the spreadsheet implementation guidelines shown on slide 36 of the "Introduction to Optimization and LPs" slides. Make sure that you obtain consistent results with those in (b). (15 points)

Question 3 (Revised Simplex Method – 20 Points)

You are given the following LP:

$$
\begin{array}{lllllllll}
\max & 20x_1 + 6x_2 + 8x_3 & & & & & & & \\
\text{s.t.} & 8x_1 + 2x_2 + 3x_3 + x_4 & & & & & = & 200 \\
& 4x_1 + 3x_2 & & + x_5 & & & = & 100 \\
& 2x_1 \quad\quad + x_3 & & & + x_6 & & = & 50 \\
& \quad\quad\quad\quad\quad x_3 & & & & + x_7 & = & 20 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 & & & & & &
\end{array}
$$

Suppose that the Revised Simplex algorithm is utilized to solve the LP, and that at the current iteration, the basic variables are x_1, x_2, x_6 and x_7.

(a) Determine the current basis (**B**) and basis inverse ($\mathbf{B}^{-1}$) (Note: The Excel array function MINVERSE can be used to determine the inverse of a matrix). (2 points)
(b) Compute the solution corresponding to the current basis. (2 points)
(c) Compute the corresponding pricing vector. (2 points)
(d) Without generating the implied simplex directions, use your pricing vector to determine the reduced costs for all the non-basic variables. (3 points)
(e) Choose an entering column and determine the corresponding simplex direction. (3 points)
(f) Determine the column that should leave the matrix and the corresponding update matrix (**E**). (3 points)
(g) Based on the selected leaving and entering columns, determine the updated solution. (2 points)
(h) Use **E** to determine the basis inverse matrix corresponding to the updated basis. (3 points)

Question 4 (Sensitivity Analysis – 20 Points)

Silva and Sons Ltd (SSL) is the largest coconut processor in Sri Lanka. SSL buys coconuts at 300 rupees per thousand to produce two grades (fancy and granule) of desiccated (dehydrated) coconut, shell flour, and charcoal. Nuts are first sorted into those good enough for desiccated coconut (90%) versus those good only for their shells. Those dedicated to desiccated coconut production go to hatcheting/pairing to remove the meat and then through a drying process. Their shells pass on for use in flour and charcoal.

SSL has the capability to hatchet 300,000 nuts per month and dry 450 tons of desiccated coconut per month. Every 1,000 nuts suitable for processing in this way yields 0.16 ton of desiccated coconut, 18% of which in fancy grade and the rest granulated. Shell flour is ground from coconut shells: 1,000 shells yield 0.22 tons of flour. Charcoal also comes from shells: 1,000 shells yield 0.5 tons of charcoal.

SSL can sell fancy desiccated coconut at a net gross profit of 3,500 rupees per ton (i.e., while accounting for the processing costs), but the market is limited to 40 tons per month. A contract requires SSL delivery of at least 30 tons of granulated-quality desiccated coconut at net gross profit of 1,350 rupees per ton, but any larger amounts can also be sold at that price. The market for shell flour is limited to 50 tons per month at a net of 450 rupees per ton. Unlimited amounts of charcoal can be sold at a net of 250 rupees per ton.

The following LP was prepared for this production problem.

$$\max \quad 3{,}500 s_1 + 1{,}350 s_2 + 450 s_3 + 250 s_4 - 300 p_1 - 300 p_2$$

$$\text{s.t.} \quad p_1 = 0.9\left(p_1 + p_2\right) \Leftrightarrow 0.10 p_1 - 0.9 p_2 = 0 \qquad (1)$$

$$s_1 = 0.18\left(s_1 + s_2\right) \Leftrightarrow 0.82 s_1 - 0.18 s_2 = 0 \qquad (2)$$

$$p_1 \le 300 \qquad (3)$$

$$s_1 + s_2 \le 450 \qquad (4)$$

$$s_1 \le 40 \qquad (5)$$

$$s_2 \ge 30 \qquad (6)$$

$$s_3 \le 50 \qquad (7)$$

$$0.16 p_1 - s_1 - s_2 = 0 \qquad (8)$$

$$\frac{s_3}{0.22} + \frac{s_4}{0.5} = p_1 + p_2 \Leftrightarrow 0.11 p_1 + 0.11 p_2 - 0.5 s_3 - 0.22 s_4 = 0 \qquad (9)$$

$$p_1, p_2, s_1, s_2, s_3, s_4 \ge 0 \qquad (10)$$

The model was implemented is Excel and solved. The Excel implementation along with the sensitivity analysis reports are shown below. Note that Solver's sensitivity report shows allowable increases and decreases for coefficients, instead of the lower and upper range information that other software (such as AMPL) provide

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B15	Fancy (s1) Produced (tons)	8.64	0	3,500	1E+30	2,898.457
B16	Granulated (s2) Produced (tons)	39.36	0	1,350	1E+30	636.247
B17	Flour (s3) Produced (tons)	0	-118.182	450	118.182	1E+30
B18	Charcoal (s4) Produced (tons)	166.667	0	250	1E+30	52
B20	"Good" Coconut (p1) Purchased (1,000's)	300	0	-300	1E+30	83.476
B21	"Other" Coconut (p2) Purchased (1,000's)	33.333	0	-300	1E+30	751.28

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B24	Coconut Quality	0	194.444	0	30	1E+30
B25	Desiccated Coconut Quality	0	2150	0	9.36	8.64
B26	Limit on Hatcheting (1,000's)	300	83.476	300	1,088.889	71.341
B27	Drying Limit (tons)	48	0	450	1E+30	402
B28	Market Size - Fancy (tons)	8.64	0	40	1E+30	31.36
B29	Contracted - Granulated (tons)	39.36	0	30	9.36	1E+30
B30	Market Size - Flour (tons)	0	0	50	1E+30	50
B31	Material Balance I (Des. Coconuts)	0	-1737	0	11.415	174.222
B32	Material Balance II (Shells)	0	-1,136.364	0.000	36.667	1E+30

	A	B	C	D
1	**Some Input Data**			
2	Fancy Gross Profit (per ton)	$3,500		
3	Granulated Gross Profit (per ton)	$1,350		
4	Flour Gross Profit (per ton)	$450		
5	Charcoal Gross Profit (per ton)	$250		
6	Cost (per 1000)	$300		
7	"Good" Coconuts Ratio	90%		
8	Fancy Ratio	18%		
9	Des. Coconut Yield (ton per 1,000)	0.16		
10	Flour Yield (tons per 1,000)	0.22		
11	Charcoal Yield (tons per 1,000)	0.50		
12				
13	**Decision Variables**			
14		Produced (tons)		
15	Fancy (s_1)	8.640		
16	Granulated (s_2)	39.360		
17	Flour (s_3)	0.000		
18	Charcoal (s_4)	166.667		
19		Purchased (1,000's)		
20	"Good" Coconut (p_1)	300.000		
21	"Other" Coconut (p_2)	33.333		
22				
23	**Constraints**			
24	1. Coconut Quality	0	=	0
25	2. Desiccated Coconut Quality	0	=	0
26	3. Limit on Hatcheting (1,000's)	300.000	<=	300
27	4. Drying Limit (tons)	48.000	<=	450
28	5. Market Size - Fancy (tons)	8.640	<=	40
29	6. Contracted - Granulated (tons)	39.360	>=	30
30	7. Market Size - Flour (tons)	0	<=	50
31	8. Material Balance I (Des. Coconuts)	0	=	0
32	9. Material Balance II (Shells)	0	=	0
33				
34	**Objective**			
35	Selling Income	$125,042.67		
36	Purchase Cost	$100,000.00		
37	Net Income	$25,042.67	←	Maximize

Based on the stated model, the Excel screen capture and the sensitivity results shown, answer the following questions.

(a) Justify why the formulation is a valid one for the stated problem. Make sure to clearly explain the variables, objective function and constraints. (2 points)
(b) State the dual of the stated linear program. (2.5 points)
(c) What is the optimal primal solution and corresponding objective function? (1 point)
(d) What is the optimal dual solution? (1 point)
(e) Verify that the dual solution is feasible for the stated dual formulation and that it has the same optimal objective function value as the primal problem. (1 point)
(f) How much SSL should be willing to pay to increase hatcheting capacity by 1 unit (1,000 nuts per month)? (1 point)
(g) How much SSL should be willing to pay to increase drying capacity by 1 unit (1 ton per week)? (1 point)
(h) Determine or bound as well as possible the profit impact decreasing the hatcheting capacity (thousands of nuts per month) to 250. Do the same for a capacity of 200. (1.5 points)
(i) Determine or bound as well as possible the profit impact increasing the hatcheting capacity (thousands of nuts per month) to 1,000. Do the same for a capacity of 2,000. (1.5 points)
(j) The company now has excess drying capacity. How low can the capacity go before the plan gets affected? (1 point)
(k) The optimal solution now makes no shell flour. At what selling price per ton would it become economical to make and sell flour? (1 point)
(l) Determine or bound as well as possible the profit impact of decreasing the selling price of granulated desiccated coconut to 800 rupees per ton. Do the same for a decrease to 600 rupees. (1.5 points)
(m) Determine or bound as well as possible the profit impact of increasing the price of charcoal to 400 rupees. Do the same for an increase to 600 rupees. (1.5 points)
(n) Determine whether the optimal primal solution would change if we dropped the constraint drying capacity. (1 point)
(o) Determine whether the optimal primal solution would change if we added a new limitation that the total number of nuts available per month cannot exceed 400,000. Do the same for the total not to exceed 200,000. (1.5 points)