Homework 3 solutions

Intro to Robotics

1 Quaternions

1. rotate the 3d vector $P_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ by 175° on xz plane and then 35° on the yz plane using quaternions.

Solution: Using -> $\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$ for multiplication we get:

$$q_{rot} = e^{\frac{35^\circ}{2}n_x} e^{\frac{175^\circ}{2}n_y} = \begin{bmatrix} c\frac{35^\circ}{2} & -s\frac{35^\circ}{2} & 0 & 0 \\ s\frac{35^\circ}{2} & c\frac{35^\circ}{2} & 0 & 0 \\ 0 & 0 & c\frac{35^\circ}{2} & -s\frac{35^\circ}{2} \\ 0 & 0 & s\frac{35^\circ}{2} & c\frac{35^\circ}{2} \end{bmatrix} \begin{bmatrix} c\frac{175}{2} \\ 0 \\ s\frac{175}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} .04 \\ .01 \\ 0.95 \\ 0.3 \end{bmatrix}$$

$$P_{1r} = q_{rot} P_1 q^*_{rot} = \begin{bmatrix} .04 & -.01 & -.95 & -.3 \\ .01 & .04 & -.3 & .95 \\ .95 & .3 & .04 & -.01 \\ .3 & -.95 & .01 & .04 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2.79 \\ 2.22 \\ -.26 \\ 1.1 \end{bmatrix}$$

$$\begin{bmatrix} -2.79 & -2.22 & .26 & -1.1 \\ 2.22 & -2.79 & -1.1 & -.26 \\ -.26 & 1.1 & -2.79 & -2.22 \\ 1.1 & .26 & 2.22 & -2.79 \end{bmatrix} \begin{bmatrix} .04 \\ -.01 \\ -0.95 \\ -0.3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.26 \\ 3.3 \\ -1.23 \end{bmatrix}$$

2. Convert ${}^A_B R_{Z'Y'X'}(50^\circ, 150^\circ, 200^\circ)$ to quaternion coordinates.

Solution: $e^{\frac{50^\circ}{2}n_z} e^{\frac{150^\circ}{2}n_y} e^{\frac{200^\circ}{2}n_x} =$

$$\begin{bmatrix} c\frac{50^\circ}{2} & 0 & 0 & -s\frac{50^\circ}{2} \\ 0 & c\frac{50^\circ}{2} & s\frac{50^\circ}{2} & 0 \\ 0 & s\frac{50^\circ}{2} & c\frac{50^\circ}{2} & 0 \\ s\frac{50^\circ}{2} & 0 & 0 & c\frac{50^\circ}{2} \end{bmatrix} \begin{bmatrix} c\frac{150^\circ}{2} & 0 & -s\frac{150^\circ}{2} & 0 \\ 0 & c\frac{150^\circ}{2} & -s\frac{150^\circ}{2} & 0 \\ s\frac{150^\circ}{2} & 0 & c\frac{150^\circ}{2} & 0 \\ 0 & s\frac{150^\circ}{2} & 0 & c\frac{150^\circ}{2} \end{bmatrix} \begin{bmatrix} c\frac{200^\circ}{2} & -s\frac{200^\circ}{2} & 0 & 0 \\ s\frac{200^\circ}{2} & c\frac{200^\circ}{2} & 0 & 0 \\ 0 & 0 & c\frac{200^\circ}{2} & -s\frac{200^\circ}{2} \\ 0 & 0 & s\frac{200^\circ}{2} & c\frac{200^\circ}{2} \end{bmatrix} =$$

$$\begin{bmatrix} .36 \\ .3 \\ -.04 \\ -.88 \end{bmatrix}$$

3. Convert rotation matrix $R = \begin{bmatrix} .28 & .77 & .57 \\ -.94 & .34 & 0 \\ -.19 & -.54 & .82 \end{bmatrix}$ to quaternion coordinates.

Solution: First convert Rotation matrix to z'y'x' euler angles:
$\beta_2 = 180° - \beta_1 = 180° - -arcsin(-.19) = 180° - 10.95 = 169.05°$
$\alpha_1 = arctan(\frac{r_{21}}{r_{11}}) = arctan(\frac{-.18}{-.18}) = 45°$
$\alpha_2 = arctan(\frac{r_{21}}{r_{11}}) = arctan(\frac{-.94}{.28}) = -73.41° = -73.41+180 = 106.59°$, $\frac{r_{11}}{cos(\beta_2)} = \frac{.28}{-.98} < 0$(add 180)
$\gamma_2 = arctan(\frac{r_{32}}{r_{33}}) = arctan(\frac{-.54}{.82}) = 45° = -33.37 + 180 = 146.63°$, $\frac{r_{33}}{cos(\beta_2)} = \frac{.82}{-.98} < 0$(add 180)

You will always have two sets of possible answers.
Set 2: $(\alpha_1, \beta_1, \gamma_1) = (106.59°, 169.05°, 146.63°)$

$$e^{\frac{106.59°}{2}n_z} e^{\frac{150°}{2}n_y} e^{\frac{200°}{2}n_x} =$$
$$\begin{bmatrix} c\frac{106.59°}{2} & 0 & 0 & -s\frac{106.59°}{2} \\ 0 & c\frac{106.59°}{2} & s\frac{106.59°}{2} & 0 \\ 0 & s\frac{106.59°}{2} & c\frac{50°}{2} & 0 \\ s\frac{106.59°}{2} & 0 & 0 & c\frac{50°}{2} \end{bmatrix} \begin{bmatrix} c\frac{169.05°}{2} & 0 & -s\frac{169.05°}{2} & 0 \\ 0 & c\frac{169.05°}{2} & -s\frac{169.05°}{2} & 0 \\ s\frac{169.05°}{2} & 0 & c\frac{169.05°}{2} & 0 \\ 0 & s\frac{169.05°}{2} & 0 & c\frac{169.05°}{2} \end{bmatrix}$$
$$\begin{bmatrix} c\frac{146.63°}{2} & -s\frac{146.63°}{2} & 0 & 0 \\ s\frac{146.63°}{2} & c\frac{146.63°}{2} & 0 & 0 \\ 0 & 0 & c\frac{146.63°}{2} & -s\frac{146.63°}{2} \\ 0 & 0 & s\frac{146.63°}{2} & c\frac{146.63°}{2} \end{bmatrix} = \begin{bmatrix} .78 \\ -.17 \\ .24 \\ -.55 \end{bmatrix}$$

4. consider quaternion q_1 = (0.845 +0.191i +0.462j +0.191k), convert to $R_{Z'Y'X'}(\alpha, \beta, \gamma)$.

Solution:
$alpha = arctan2(2 * (0.845 * 0.191 + 0.462 * 0.191), 1 - (2 * (0.191^2 + 0.462^2)))$
$beta = arcsin(2 * (0.845 * 0.462 - 0.191^2))$
$gamma = arctan2(2 * (0.845 * 0.191 + 0.191 * 0.462), 1 - (2 * (0.462^2 + 0.191^2)))$

5. Write a function that rotates 3d vectors using quaternions to verify your answers for 1-3.

View Attached in hw3.py and eulerangles.py on blackboard.

6. Write a function that takes a rotation matrix as input and returns the equivalent quaternions.

View Attached in hw3.py and eulerangles.py on blackboard.

7. Write a function that takes a quaternions as input and returns the equivalent rotation matrix.
View Attached in hw3.py and eulerangles.py on blackboard.

2 Kinematics 1

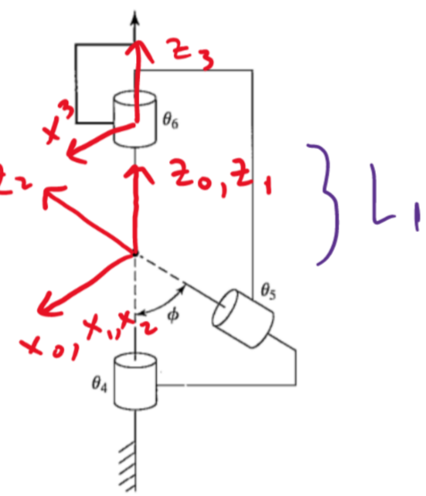

Figure 1:

1. Compute the frames, dh table and rotation matrices for the given schematic.

$a_0 = dist(z_0, z_1)$ in $x_0 = 0$
$a_1 = dist(z_1, z_2)$ in $x_1 = 0$
$a_2 = dist(z_2, z_3)$ in $x_2 = 0$

$\alpha_0 = angle(z_0, z_1)$ in $x_0 = 0$
$\alpha_1 = angle(z_1, z_2)$ in $x_1 = \phi$
$\alpha_2 = angle(z_2, z_3)$ in $x_2 = -\phi$

$d_1 = dist(x_0, x_1)$ in $z_1 = 0$
$d_2 = dist(x_1, x_2)$ in $z_2 = 0$
$d_3 = dist(x_2, x_3)$ in $z_3 = L_1$

$\theta_1 = angle(x_0, x_1)$ in $z_1 = \theta_4$
$\theta_2 = angle(x_1, x_2)$ in $z_2 = \theta_5$
$\theta_3 = angle(x_2, x_3)$ in $z_3 = \theta_6$

DH table				
Frame {i}	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_4
2	0	ϕ	0	θ_5
3	0	$-\phi$	L_1	θ_6

Using ${}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$ we get. Tip-Each row corresponds to one transformation matrix:

$$
{}^{0}_{1}T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & 0 \\ s\theta_4 & c\theta_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ s\theta_5 c\phi & c\theta_5 c\phi & -s\phi & 0 \\ s\theta_5 s\phi & c\theta_5 s\phi & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 c-\phi & c\theta_6 c-\phi & -s-\phi & -s-\phi L_1 \\ s\theta s-\phi & c\theta s-\phi & c-\phi & c-\phi L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

3 Kinematics 2

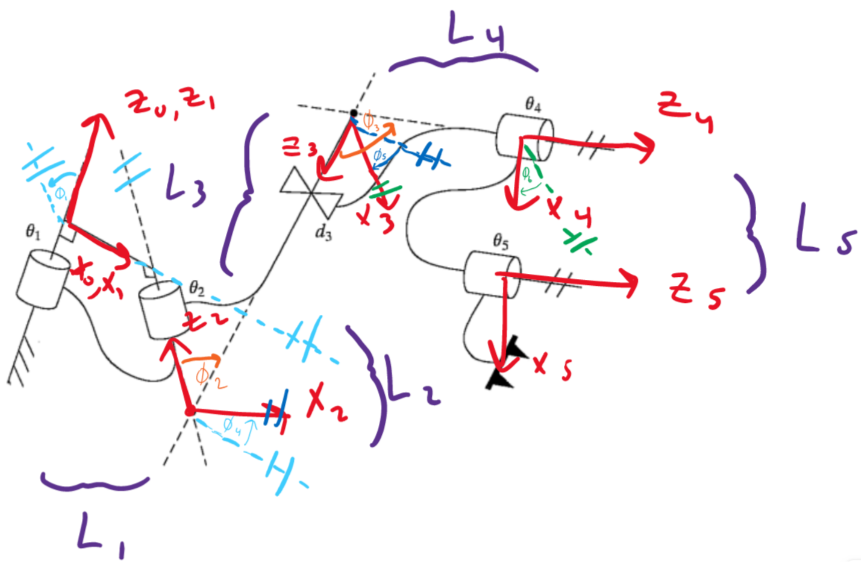

Figure 2:

1. Compute the frames, dh table and rotation matrices for the given schematic.

$a_0 = dist(z_0, z_1)$ in $x_0 = 0$
$a_1 = dist(z_1, z_2)$ in $x_1 = L_1$
$a_2 = dist(z_2, z_3)$ in $x_2 = 0$
$a_3 = dist(z_3, z_4)$ in $x_3 = 0$
$a_4 = dist(z_4, z_5)$ in $x_4 = L_5$

$\alpha_0 = angle(z_0, z_1)$ in $x_0 = 0$
$\alpha_1 = angle(z_1, z_2)$ in $x_1 = \phi_1$
$\alpha_2 = angle(z_2, z_3)$ in $x_2 = \phi_2$
$\alpha_3 = angle(z_3, z_4)$ in $x_3 = \phi_3$
$\alpha_4 = angle(z_4, z_5)$ in $x_4 = 0$

$d_1 = dist(x_0, x_1)$ in $z_1 = 0$
$d_2 = dist(x_1, x_2)$ in $z_2 = L_2$
$d_3 = dist(x_2, x_3)$ in $z_3 = L_3 - d_3$
$d_4 = dist(x_3, x_4)$ in $z_4 = L_4$
$d_5 = dist(x_4, x_5)$ in $z_5 = 0$

$\theta_1 = angle(x_0, x_1)$ in $z_1 = \theta_1$
$\theta_2 = angle(x_1, x_2)$ in $z_2 = \theta_2 - \phi_4$

$\theta_3 = angle(x_2, x_3)$ in $z_3 = \phi_5$
$\theta_4 = angle(x_3, x_4)$ in $z_4 = \theta_4 - \phi_6$
$\theta_5 = angle(x_4, x_5)$ in $z_5 = \theta_5$

DH table				
Frame {i}	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	L_1	ϕ_1	L_2	$\theta_2 - \phi_4$
3	0	ϕ_2	$L_3 - d_3$	ϕ_5
4	0	ϕ_3	L_4	$\theta_4 - \phi_6$
5	L_5	0	0	θ_5

Using ${}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$ we get:

$${}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}_{2}T = \begin{bmatrix} c(\theta_2 - \phi_4) & -s(\theta_2 - \phi_4) & 0 & L_1 \\ s(\theta_2 - \phi_4)c\phi_1 & c(\theta_2 - \phi_4)c\phi_1 & -s\phi_1 & -s\phi_1 L_2 \\ s(\theta_2 - \phi_4)s\phi_1 & c(\theta_2 - \phi_4)s\phi_1 & c\phi_1 & c\phi_1 L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{2}_{3}T = \begin{bmatrix} c\phi_5 & -s\phi_5 & 0 & 0 \\ s\phi_5 c\phi_2 & c\phi_5 c\phi_2 & -s\phi_2 & -s\phi_2 * (L_3 - d_3) \\ s\phi_5 s\phi_2 & c\phi_5 s\phi_2 & c\phi_2 & c\phi_2 * (L_3 - d_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{3}_{4}T = \begin{bmatrix} c(\theta_4 - \phi_6) & -s(\theta_4 - \phi_6) & 0 & 0 \\ s(\theta_4 - \phi_6)c\phi_3 & c(\theta_4 - \phi_6)c\phi_3 & -s\phi_3 & -s\phi_3 L_4 \\ s(\theta_4 - \phi_6)s\phi_3 & c(\theta_4 - \phi_6)s\phi_3 & c\phi_3 & c\phi_3 L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{4}_{5}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & L_5 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$