Homework 2 solutions

Intro to Robotics

1 Frames, translations and rotations

Given frames: Frame {A} = universe,

Frame {B} = $\{{}^A_B R_{Z'Y'X'}(45°, 100°, 75°), {}^A P_{Borg} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}\}$,

Frame {C} = $\{{}^B_C R_{Z'Y'X'}(90°, 0°, 15°) =, {}^B P_{Corg} = \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix}\}$

Given points: ${}^A P_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, ${}^C P_2 = \begin{bmatrix} -5 \\ 7 \\ 2 \end{bmatrix}$, ${}^B P_3 = \begin{bmatrix} 3.14 \\ -7.77 \\ 2.718 \end{bmatrix}$

1. Find ${}^C_B T$.
2. Find ${}^A P_2$.
3. Find ${}^B P_1$.
4. Find ${}^C P_3$.
5. Given ${}^A P_1$ rotate the point by ${}^A_B R_{Z'Y'X'}(50°, 150°, 200°)$ and translate by vector $\begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix}$.

Solution: Transformations available:

$$ {}^A_B T = \begin{bmatrix} {}^A_B R(45°, 100°, 75°) & & & {}^A P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.12 & .49 & .86 & 1 \\ -.12 & .86 & -.5 & -2 \\ -.98 & -.17 & -.04 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^B_C T = \begin{bmatrix} 0 & -.97 & .26 & -4 \\ 1 & 0 & 0 & 4 \\ 0 & .26 & .97 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

1. $$ {}^C_B T = {}^B_C T^{-1} = \begin{bmatrix} {}^B_C R^T & & & -({}^B_C R^T \, {}^B P_{Corg}) \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -4 \\ -.97 & 0 & .26 & -4.9 \\ .26 & 0 & .97 & -2.83 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

2. ${}^{A}P_2 = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P_2 = \begin{bmatrix} -.12 & .49 & .86 & 1 \\ -.12 & .86 & -.5 & -2 \\ -.98 & -.17 & -.04 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -.97 & .26 & -4 \\ 1 & 0 & 0 & 4 \\ 0 & .26 & .97 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 7 \\ 2 \end{bmatrix} =$
$\begin{bmatrix} 8.45 \\ -5.49 \\ 14.91 \\ 1 \end{bmatrix}$

3. ${}^{B}P_1 = {}^{B}_{A}T\,{}^{A}P_1 = \begin{bmatrix} -.12 & -.12 & -.98 & 4.8 \\ .49 & .86 & -.17 & 2.06 \\ .86 & -.5 & -.04 & -1.64 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.39 \\ 4.76 \\ -1.65 \\ 1 \end{bmatrix}$

4. ${}^{C}P_3 = {}^{C}_{B}T\,{}^{B}P_3 = \begin{bmatrix} 0 & 1 & 0 & -4 \\ -.97 & 0 & .26 & -4.9 \\ .26 & 0 & .97 & -2.83 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.14 \\ -7.77 \\ 2.718 \\ 1 \end{bmatrix} = \begin{bmatrix} -11.77 \\ -7.23 \\ .61 \\ 1 \end{bmatrix}$

5. ${}^{A}P_1' = T\,{}^{A}P_1$ where T is the transformation matrix: $T = \begin{bmatrix} {}^{A}_{B}R(50°, 150°, 200°) & & & {}^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

${}^{A}P_1' = T\,{}^{A}P_1 = \begin{bmatrix} -.56 & .61 & -.56 & -2 \\ -.66 & -.74 & -.14 & -3 \\ -.5 & .3 & .81 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} .36 \\ -4.71 \\ -6.35 \\ 1 \end{bmatrix}$

2 Solve for angles

Given frames: Frame {A} = universe,

Frame {B} = $\{{}^{A}_{B}R = \begin{bmatrix} .15 & .18 & -.97 \\ .09 & -.98 & -.16 \\ -.98 & -.06 & -.16 \end{bmatrix}, {}^{A}P_{Borg} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}\}$

1. Find the eulerian angles for ${}^{A}_{B}R_{Z'Y'X'}$.

2. Solve for the angle between $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} .5 \\ .5 \\ .71 \end{bmatrix}$

Solution: 1. ${}^{A}_{B}R = \begin{bmatrix} .15 & .18 & -.97 \\ .09 & -.98 & -.16 \\ -.98 & -.06 & -.16 \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & s\gamma s\beta c\alpha - c\gamma s\alpha & c\gamma s\beta c\alpha + s\gamma s\alpha \\ s\alpha c\beta & s\gamma s\beta s\alpha + c\gamma c\alpha & c\gamma s\beta s\alpha - s\gamma c\alpha \\ -s\beta & s\gamma c\beta & c\gamma c\beta \end{bmatrix} =$
$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

$\beta_1 = -arcsin(r_{31}) = -arcsin(-.98) = 78.52°$
$\beta_2 = 180° - \beta_1 = 180° - 78.52° = 101.48°$

$\alpha_1 = arctan(\frac{r_{21}}{r_{11}}) = arctan(\frac{.09}{.15}) = 30.96°$

since $\frac{r_{11}}{cos(\beta_1)} = \frac{.15}{cos(\beta_1)} = \frac{.15}{cos(\beta_1)} > 0$ (so no need to add anything)
$\alpha_2 = arctan(\frac{r_{21}}{r_{11}}) = arctan(\frac{.09}{.15}) = 30.96°, \frac{r_{11}}{cos(\beta_2)} = \frac{.15}{cos(\beta_2)} < 0$ and $-90° < 30.96° < 90°,$ we add 180° to $\alpha_2, \alpha_2 = \alpha_1 + 180 = 30.96 + 180 = 210.96°$

$\gamma_1 = arctan(\frac{r_{32}}{r_{33}}) = arctan(\frac{-.06}{-.16}) = 20.56°$
since $\frac{r_{33}}{cos(\beta_1)} = \frac{-.16}{.2} < 0$ and $-90° < 20.56° < 90°,$ we add 180° to $\gamma_1, \gamma_1 = 180° + 20.56° = 200.56°$
$\gamma_2 = arctan(\frac{r_{32}}{r_{33}}) = arctan(\frac{-.06}{-.16}) = 20.56°, \frac{r_{33}}{cos(\beta_2)} = \frac{-.16}{-.2} > 0$(so no need to add anything)

You will always have two sets of possible answers.
Set 1: $(\alpha_1, \beta_1, \gamma_1) = (30.96°, 78.52°, 200.56°)$
Set 2: $(\alpha_1, \beta_1, \gamma_1) = (210.96°, 101.48°, 20.56°)$

To verify, make sure:
$$ {}^A_BR = \begin{bmatrix} .15 & .18 & -.97 \\ .09 & -.98 & -.16 \\ -.98 & -.06 & -.16 \end{bmatrix} = \begin{bmatrix} c\alpha_1 & -s\alpha_1 & 0 \\ s\alpha_1 & c\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta_1 & 0 & s\beta_1 \\ 0 & 1 & 0 \\ -s\beta_1 & 0 & c\beta_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma_1 & -s\gamma_1 \\ 0 & s\gamma_1 & c\gamma_1 \end{bmatrix} = $$
$$ \begin{bmatrix} c\alpha_2 & -s\alpha_2 & 0 \\ s\alpha_2 & c\alpha_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta_2 & 0 & s\beta_2 \\ 0 & 1 & 0 \\ -s\beta_2 & 0 & c\beta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma_2 & -s\gamma_2 \\ 0 & s\gamma_2 & c\gamma_2 \end{bmatrix} $$

2. $\theta = arccos(\frac{v_1 \cdot v_2}{|v_1||v_2|}) = arccos(\frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} .5 \\ .5 \\ .71 \end{bmatrix}}{\sqrt{1^2+0^2+0^2}\sqrt{.5^2+.5^2+.71^2}}) = 60.01°$

3 Complex

1. rotate the 2d vector $P_1 = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$ by 110° using a complex rotation.

Solution: **1.** $v_r = e^{\theta i}v = (cos(\theta) + sin(\theta)i)v; let v = P_1 = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$
$(cos(\theta) + sin(\theta)i)v = (cos(110°) + sin(110°)i)(8 - 2i) = -.34 + .94i)(8 - 2i) = -2.72 + .68i + 7.52i + 1.88 = -.84 + 8.2i = (-.84, 8.2i)$

$$ v_r = \begin{bmatrix} -.84 \\ 8.2 \end{bmatrix} $$

4 Coding Portion

1.) Modify your frame class from homework 1, to also create 3d frames then check your answers for part 1.
2.) Write a function that takes a rotation matrix as input and solves for the angles in $R_{z'y'x'}(\alpha, \beta, \gamma)$

3.) Write a function to solve for the angle between two 3d vectors.

ALL CODE AVAILABLE ON BLACKBOARD