

# Homework 3 solutions

## Intro to Robotics

### 1 Quaternions

1. rotate the 3d vector  $P_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$  by  $175^\circ$  on xz plane and then  $35^\circ$  on the yz plane using quaternions.

**Solution:** Using  $\rightarrow \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$  for multiplication we get:

$$q_{rot} = e^{\frac{35^\circ}{2}n_x} e^{\frac{175^\circ}{2}n_y} = \begin{bmatrix} c\frac{35^\circ}{2} & -s\frac{35^\circ}{2} & 0 & 0 \\ s\frac{35^\circ}{2} & c\frac{35^\circ}{2} & 0 & 0 \\ 0 & 0 & c\frac{35^\circ}{2} & -s\frac{35^\circ}{2} \\ 0 & 0 & s\frac{35^\circ}{2} & c\frac{35^\circ}{2} \end{bmatrix} \begin{bmatrix} c\frac{175^\circ}{2} \\ 0 \\ s\frac{175^\circ}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} .04 \\ .01 \\ 0.95 \\ 0.3 \end{bmatrix}$$

$$P_{1r} = q_{rot} P_1 q_{rot}^* = \begin{bmatrix} .04 & -.01 & -.95 & -.3 \\ .01 & .04 & -.3 & .95 \\ .95 & .3 & .04 & -.01 \\ .3 & -.95 & .01 & .04 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2.79 \\ 2.22 \\ -.26 \\ 1.1 \end{bmatrix}$$

$$\begin{bmatrix} -2.79 & -2.22 & .26 & -1.1 \\ 2.22 & -2.79 & -1.1 & -.26 \\ -.26 & 1.1 & -2.79 & -2.22 \\ 1.1 & .26 & 2.22 & -2.79 \end{bmatrix} \begin{bmatrix} .04 \\ -.01 \\ -0.95 \\ -0.3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.26 \\ 3.3 \\ -1.23 \end{bmatrix}$$

2. Convert  ${}^A_B R_{Z'Y'X'}(50^\circ, 150^\circ, 200^\circ)$  to quaternion coordinates.

**Solution:**  $e^{\frac{50^\circ}{2}n_z} e^{\frac{150^\circ}{2}n_y} e^{\frac{200^\circ}{2}n_x} =$

$$\begin{bmatrix} c\frac{50^\circ}{2} & 0 & 0 & -s\frac{50^\circ}{2} \\ 0 & c\frac{50^\circ}{2} & s\frac{50^\circ}{2} & 0 \\ 0 & s\frac{50^\circ}{2} & c\frac{50^\circ}{2} & 0 \\ s\frac{50^\circ}{2} & 0 & 0 & c\frac{50^\circ}{2} \end{bmatrix} \begin{bmatrix} c\frac{150^\circ}{2} & 0 & -s\frac{150^\circ}{2} & 0 \\ 0 & c\frac{150^\circ}{2} & -s\frac{150^\circ}{2} & 0 \\ s\frac{150^\circ}{2} & 0 & c\frac{150^\circ}{2} & 0 \\ 0 & s\frac{150^\circ}{2} & 0 & c\frac{150^\circ}{2} \end{bmatrix} \begin{bmatrix} c\frac{200^\circ}{2} & -s\frac{200^\circ}{2} & 0 & 0 \\ s\frac{200^\circ}{2} & c\frac{200^\circ}{2} & 0 & 0 \\ 0 & 0 & c\frac{200^\circ}{2} & -s\frac{200^\circ}{2} \\ 0 & 0 & s\frac{200^\circ}{2} & c\frac{200^\circ}{2} \end{bmatrix} =$$

$$\begin{bmatrix} .36 \\ .3 \\ -.04 \\ -.88 \end{bmatrix}$$

3. Convert rotation matrix  $R = \begin{bmatrix} .28 & .77 & .57 \\ -.94 & .34 & 0 \\ -.19 & -.54 & .82 \end{bmatrix}$  to quaternion coordinates.

**Solution:** First convert Rotation matrix to z'y'x' euler angles:

$$\beta_2 = 180^\circ - \beta_1 = 180^\circ - -\arcsin(-.19) = 180^\circ - 10.95 = 169.05^\circ$$

$$\alpha_1 = \arctan\left(\frac{r_{21}}{r_{11}}\right) = \arctan\left(\frac{-.18}{-.18}\right) = 45^\circ$$

$$\alpha_2 = \arctan\left(\frac{r_{21}}{r_{11}}\right) = \arctan\left(\frac{-.94}{.28}\right) = -73.41^\circ = -73.41 + 180 = 106.59^\circ, \frac{r_{11}}{\cos(\beta_2)} = \frac{.28}{-.98} < 0 (\text{add } 180)$$

$$\gamma_2 = \arctan\left(\frac{r_{32}}{r_{33}}\right) = \arctan\left(\frac{-.54}{.82}\right) = 45^\circ = -33.37 + 180 = 146.63^\circ, \frac{r_{33}}{\cos(\beta_2)} = \frac{.82}{-.98} < 0 (\text{add } 180)$$

You will always have two sets of possible answers.

Set 2:  $(\alpha_1, \beta_1, \gamma_1) = (106.59^\circ, 169.05^\circ, 146.63^\circ)$

$$e^{\frac{106.59^\circ}{2} n_z} e^{\frac{150^\circ}{2} n_y} e^{\frac{200^\circ}{2} n_x} = \begin{bmatrix} c\frac{106.59^\circ}{2} & 0 & 0 & -s\frac{106.59^\circ}{2} \\ 0 & c\frac{106.59^\circ}{2} & s\frac{106.59^\circ}{2} & 0 \\ 0 & s\frac{106.59^\circ}{2} & c\frac{50^\circ}{2} & 0 \\ s\frac{106.59^\circ}{2} & 0 & 0 & c\frac{50^\circ}{2} \end{bmatrix} \begin{bmatrix} c\frac{169.05^\circ}{2} & 0 & -s\frac{169.05^\circ}{2} & 0 \\ 0 & c\frac{169.05^\circ}{2} & -s\frac{169.05^\circ}{2} & 0 \\ s\frac{169.05^\circ}{2} & 0 & c\frac{169.05^\circ}{2} & 0 \\ 0 & s\frac{169.05^\circ}{2} & 0 & c\frac{169.05^\circ}{2} \end{bmatrix} \begin{bmatrix} c\frac{146.63^\circ}{2} & -s\frac{146.63^\circ}{2} & 0 & 0 \\ s\frac{146.63^\circ}{2} & c\frac{146.63^\circ}{2} & 0 & 0 \\ 0 & 0 & c\frac{146.63^\circ}{2} & -s\frac{146.63^\circ}{2} \\ 0 & 0 & s\frac{146.63^\circ}{2} & c\frac{146.63^\circ}{2} \end{bmatrix} = \begin{bmatrix} .78 \\ -.17 \\ .24 \\ -.55 \end{bmatrix}$$

4. consider quaternion  $q_1 = (0.845 + 0.191i + 0.462j + 0.191k)$ , convert to  $R_{Z'Y'X'}(\alpha, \beta, \gamma)$ .

**Solution:**

$$\alpha = \arctan2(2 * (0.845 * 0.191 + 0.462 * 0.191), 1 - (2 * (0.191^2 + 0.462^2)))$$

$$\beta = \arcsin(2 * (0.845 * 0.462 - 0.191^2))$$

$$\gamma = \arctan2(2 * (0.845 * 0.191 + 0.191 * 0.462), 1 - (2 * (0.462^2 + 0.191^2)))$$

5. Write a function that rotates 3d vectors using quaternions to verify your answers for 1-3.

**View Attached in hw3.py and eulerangles.py on blackboard.**

6. Write a function that takes a rotation matrix as input and returns the equivalent quaternions.

**View Attached in hw3.py and eulerangles.py on blackboard.**

7. Write a function that takes a quaternions as input and returns the equivalent rotation matrix.

**View Attached in hw3.py and eulerangles.py on blackboard.**

## 2 Kinematics 1

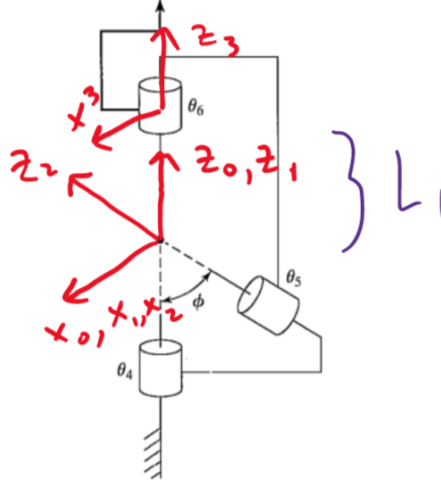


Figure 1:

1. Compute the frames, dh table and rotation matrices for the given schematic.

$$a_0 = \text{dist}(z_0, z_1) \text{ in } x_0 = 0$$

$$a_1 = \text{dist}(z_1, z_2) \text{ in } x_1 = 0$$

$$a_2 = \text{dist}(z_2, z_3) \text{ in } x_2 = 0$$

$$\alpha_0 = \text{angle}(z_0, z_1) \text{ in } x_0 = 0$$

$$\alpha_1 = \text{angle}(z_1, z_2) \text{ in } x_1 = \phi$$

$$\alpha_2 = \text{angle}(z_2, z_3) \text{ in } x_2 = -\phi$$

$$d_1 = \text{dist}(x_0, x_1) \text{ in } z_1 = 0$$

$$d_2 = \text{dist}(x_1, x_2) \text{ in } z_2 = 0$$

$$d_3 = \text{dist}(x_2, x_3) \text{ in } z_3 = L_1$$

$$\theta_1 = \text{angle}(x_0, x_1) \text{ in } z_1 = \theta_4$$

$$\theta_2 = \text{angle}(x_1, x_2) \text{ in } z_2 = \theta_5$$

$$\theta_3 = \text{angle}(x_2, x_3) \text{ in } z_3 = \theta_6$$

DH table				
Frame {i}	$a_{i-1}$	$\alpha_{i-1}$	$d_i$	$\theta_i$
<b>1</b>	0	0	0	$\theta_4$
<b>2</b>	0	$\phi$	0	$\theta_5$
<b>3</b>	0	$-\phi$	$L_1$	$\theta_6$

Using  ${}^i_{i-1}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$  we get. Tip-Each row corresponds to one transformation matrix:

$${}^0_1T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & 0 \\ s\theta_4 & c\theta_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ s\theta_5 c\phi & c\theta_5 c\phi & -s\phi & 0 \\ s\theta_5 s\phi & c\theta_5 s\phi & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 c - \phi & c\theta_6 c - \phi & -s - \phi & -s - \phi L_1 \\ s\theta_6 s - \phi & c\theta_6 s - \phi & c - \phi & c - \phi L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

### 3 Kinematics 2

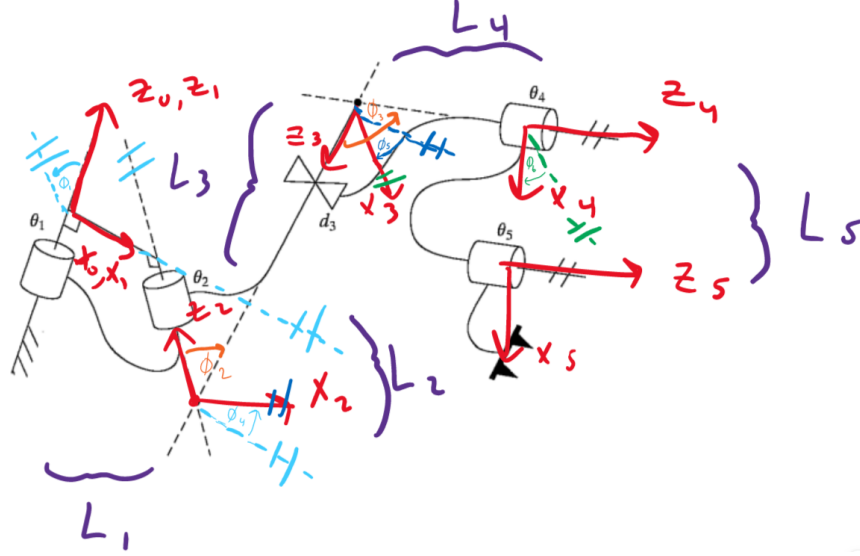


Figure 2:

1. Compute the frames, dh table and rotation matrices for the given schematic.

$$\begin{aligned}
 a_0 &= \text{dist}(z_0, z_1) \text{ in } x_0 = 0 \\
 a_1 &= \text{dist}(z_1, z_2) \text{ in } x_1 = L_1 \\
 a_2 &= \text{dist}(z_2, z_3) \text{ in } x_2 = 0 \\
 a_3 &= \text{dist}(z_3, z_4) \text{ in } x_3 = 0 \\
 a_4 &= \text{dist}(z_4, z_5) \text{ in } x_4 = L_5
 \end{aligned}$$

$$\begin{aligned}
 \alpha_0 &= \text{angle}(z_0, z_1) \text{ in } x_0 = 0 \\
 \alpha_1 &= \text{angle}(z_1, z_2) \text{ in } x_1 = \phi_1 \\
 \alpha_2 &= \text{angle}(z_2, z_3) \text{ in } x_2 = \phi_2 \\
 \alpha_3 &= \text{angle}(z_3, z_4) \text{ in } x_3 = \phi_3 \\
 \alpha_4 &= \text{angle}(z_4, z_5) \text{ in } x_4 = 0
 \end{aligned}$$

$$\begin{aligned}
 d_1 &= \text{dist}(x_0, x_1) \text{ in } z_1 = 0 \\
 d_2 &= \text{dist}(x_1, x_2) \text{ in } z_2 = L_2 \\
 d_3 &= \text{dist}(x_2, x_3) \text{ in } z_3 = L_3 - d_3 \\
 d_4 &= \text{dist}(x_3, x_4) \text{ in } z_4 = L_4 \\
 d_5 &= \text{dist}(x_4, x_5) \text{ in } z_5 = 0
 \end{aligned}$$

$$\begin{aligned}
 \theta_1 &= \text{angle}(x_0, x_1) \text{ in } z_1 = \theta_1 \\
 \theta_2 &= \text{angle}(x_1, x_2) \text{ in } z_2 = \theta_2 - \phi_4
 \end{aligned}$$

$\theta_3 = \text{angle}(x_2, x_3)$  in  $z_3 = \phi_5$   
 $\theta_4 = \text{angle}(x_3, x_4)$  in  $z_4 = \theta_4 - \phi_6$   
 $\theta_5 = \text{angle}(x_4, x_5)$  in  $z_5 = \theta_5$

DH table				
Frame {i}	$a_{i-1}$	$\alpha_{i-1}$	$d_i$	$\theta_i$
<b>1</b>	0	0	0	$\theta_1$
<b>2</b>	$L_1$	$\phi_1$	$L_2$	$\theta_2 - \phi_4$
<b>3</b>	0	$\phi_2$	$L_3 - d_3$	$\phi_5$
<b>4</b>	0	$\phi_3$	$L_4$	$\theta_4 - \phi_6$
<b>5</b>	$L_5$	0	0	$\theta_5$

Using  ${}^{i-1}_i T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$  we get:

$$\begin{aligned}
{}^0_1 T &= \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^1_2 T &= \begin{bmatrix} c(\theta_2 - \phi_4) & -s(\theta_2 - \phi_4) & 0 & L_1 \\ s(\theta_2 - \phi_4)c\phi_1 & c(\theta_2 - \phi_4)c\phi_1 & -s\phi_1 & -s\phi_1 L_2 \\ s(\theta_2 - \phi_4)s\phi_1 & c(\theta_2 - \phi_4)s\phi_1 & c\phi_1 & c\phi_1 L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^2_3 T &= \begin{bmatrix} c\phi_5 & -s\phi_5 & 0 & 0 \\ s\phi_5 c\phi_2 & c\phi_5 c\phi_2 & -s\phi_2 & -s\phi_2 * (L_3 - d_3) \\ s\phi_5 s\phi_2 & c\phi_5 s\phi_2 & c\phi_2 & c\phi_2 * (L_3 - d_3) \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^3_4 T &= \begin{bmatrix} c(\theta_4 - \phi_6) & -s(\theta_4 - \phi_6) & 0 & 0 \\ s(\theta_4 - \phi_6)c\phi_3 & c(\theta_4 - \phi_6)c\phi_3 & -s\phi_3 & -s\phi_3 L_4 \\ s(\theta_4 - \phi_6)s\phi_3 & c(\theta_4 - \phi_6)s\phi_3 & c\phi_3 & c\phi_3 L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^4_5 T &= \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & L_5 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$