Homework 1 solutions

Intro to Robotics

1 Matrix addition, scalar multiplication and transpose

$$5 * \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} =$$

What is the transpose of Matrix D ?

$$D = \begin{bmatrix} -1 & 9 & 17 \\ 44 & -122 & 8 \\ 24 & 0 & 1 \end{bmatrix}$$

Solution: **1.** $5 * \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 18 \\ 24 & 30 & 36 \\ 42 & 48 & 54 \end{bmatrix}$

2. $\begin{bmatrix} -1 & 9 & 17 \\ 44 & -122 & 8 \\ 24 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 44 & 24 \\ 9 & -122 & 0 \\ 17 & 8 & 1 \end{bmatrix}$

2 Matrix multiplication

$$\begin{bmatrix} 7 & 0 & 8 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 9 & 6 \\ 1 & 5 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 2 \\ 9 & 6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 & 8 \\ 2 & 4 & 3 \end{bmatrix} =$$

Explain why the matrix multiplication below is not possible.

$$\begin{bmatrix} 7 & 0 & 8 \\ 2 & 4 & 3 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 2 & 4 \end{bmatrix} =$$

Solution: **1.** $\begin{bmatrix} 7 & 0 & 8 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 9 & 6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 54 \\ 39 & 43 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 2 \\ 9 & 6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 & 8 \\ 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 6 \\ 25 & 24 & 90 \\ 17 & 20 & 23 \end{bmatrix}$

3. (3 x 3) (2 x 2) dimensions are not compatible for matrix multiplication.

3 Inverse and RREF

What is the inverse matrix of A?

$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 7 \\ 2 & 3 & 1 \end{bmatrix}$

Explain why the inverse of B does not exist?

$B = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 9 & 3 \\ -9 & -27 & -9 \end{bmatrix}$

Solution: **1.** $A^{-1} = \begin{bmatrix} .46 & -.04 & -.21 \\ -.54 & -.04 & .79 \\ .68 & .18 & -.93 \end{bmatrix}$

2. The inverse of B does not exist because B is a singular matrix. You can see this by the fact that column vector 1 and column vector 3 are equivalent(making them co-linear). This implies that the column vectors of B do not span 3d space, and thus the volume spanned by the column vectors is equal to 0(determinant is 0).

4 Determinant

What is the determinant of C? is C singular ?

$C = \begin{bmatrix} 3 & 7 & 1 \\ 1 & -4 & 6 \\ 8 & 8 & 8 \end{bmatrix}$

Solution: Matrix C is not singular because determinant is non-zero.

$det(\begin{bmatrix} 3 & 7 & 1 \\ 1 & -4 & 6 \\ 8 & 8 & 8 \end{bmatrix}) = 3(-4*8-6*8) - 7(1*8-6*8) + 1(1*8 - -4*8) = 80$

5 Cross product and normal vector

Compute the cross product of $v_1 \times v_2$

$v_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 8 \\ -4 \\ 3 \end{bmatrix}$

Let n be the normal vector. Draw v_1, v_2, and n with respect to each other. Show that the matrix M is non-singular without computing the determinant.

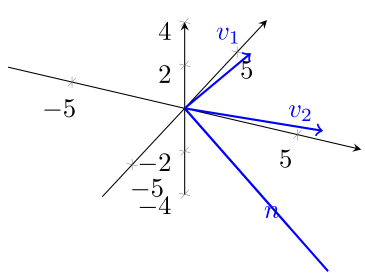
$M = [v_1 v_2 n]$

Solution: $v_1 \times v_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \times \begin{bmatrix} 8 \\ -4 \\ 3 \end{bmatrix} = det(\begin{bmatrix} i & j & k \\ 2 & 2 & 2 \\ 8 & -4 & 3 \end{bmatrix}) = i(2*3 - 2*-4) - j(2*3 - 2*8) + k(2*-4 - 2*8) = 14i + 10j - 24k$

thus, $n = \begin{bmatrix} 14 \\ 10 \\ -24 \end{bmatrix}$

$M = [v_1 v_2 n] = \begin{bmatrix} 2 & 8 & 14 \\ 2 & -4 & 10 \\ 2 & 3 & -24 \end{bmatrix}$

We can draw the column vectors and see they are not colinear, which implies the matrix M is not singular.



6 Dot product

Solve for the angle between v_1 and v_2

$v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

Solution: $arccos(\frac{v_1 \cdot v_2}{|v_1||v_2|}|) = arccos(\frac{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}}{|\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}||\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}|}) = arccos(\frac{1*4+-2*0+3*1}{\sqrt{1^2+-2^2+3^2}\sqrt{4^2+0^2+1^2}}) =$
$arccos(\frac{7}{\sqrt{14}\sqrt{17}}) = arccos(.45) = 63.26°$

7 Frames, translations and rotations

Given frames: Frame {A} = universe,

Frame {B} = $\{{}^A_B R = \begin{bmatrix} cos(135) & -sin(135) \\ sin(135) & cos(135) \end{bmatrix}, {}^A P_{Borg} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}\}$,

Frame {C} = $\{{}^B_C R = \begin{bmatrix} cos(-30) & -sin(-30) \\ sin(-30) & cos(-30) \end{bmatrix}, {}^B P_{Corg} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}\}$,

Frame {D} = $\{{}^A_D R = \begin{bmatrix} cos(0) & -sin(0) \\ sin(0) & cos(0) \end{bmatrix}, {}^A P_{Dorg} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}\}$,

Frame {E} = $\{{}^A_E R = \begin{bmatrix} cos(60) & -sin(60) \\ sin(60) & cos(60) \end{bmatrix}, {}^A P_{Eorg} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$,

Frame {F} = $\{{}^A_F R = \begin{bmatrix} cos(45) & -sin(45) \\ sin(45) & cos(45) \end{bmatrix}, {}^A P_{Forg} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}\}$,

Frame {G} = $\{{}^F_G R = \begin{bmatrix} cos(90) & -sin(90) \\ sin(90) & cos(90) \end{bmatrix}, {}^F P_{Gorg} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}\}$

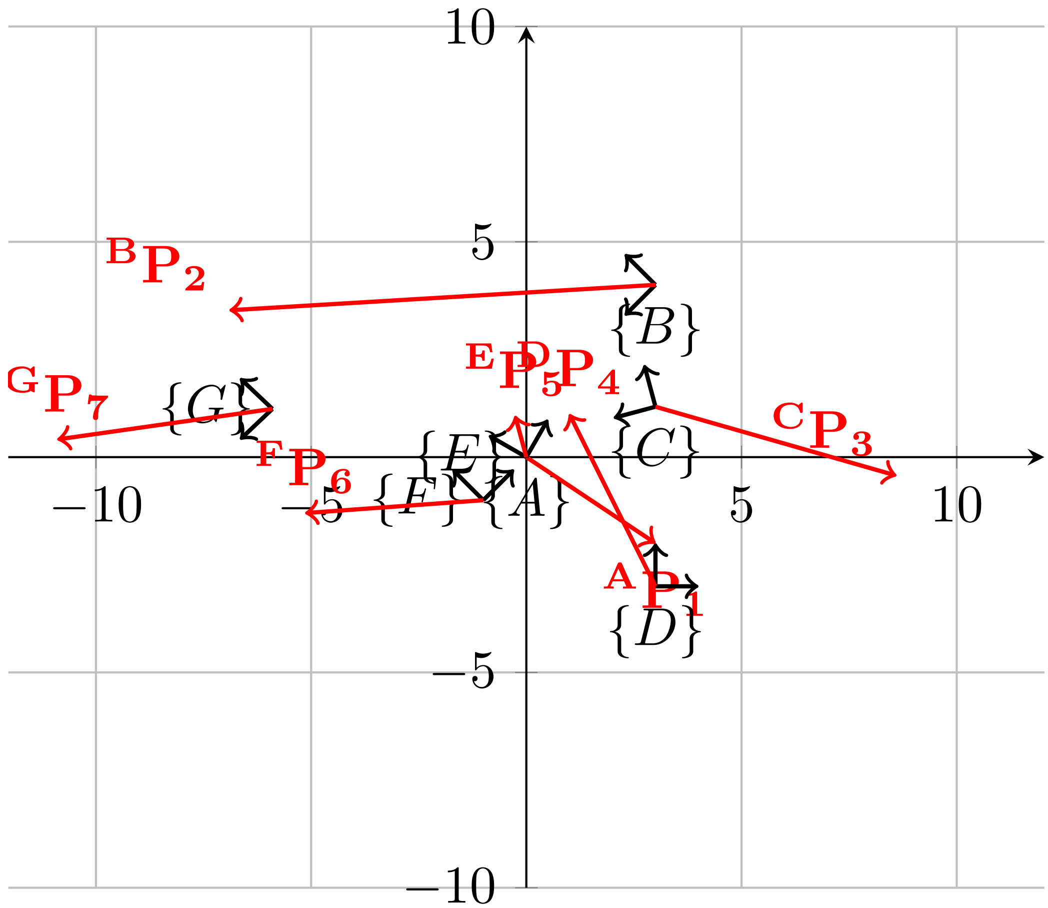
Given points: ${}^A P_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, ${}^B P_2 = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$, ${}^C P_3 = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$, ${}^D P_4 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, ${}^E P_5 = \begin{bmatrix} .7 \\ .7 \end{bmatrix}$,
${}^F P_6 = \begin{bmatrix} -3.14 \\ 2.718 \end{bmatrix}$, ${}^G P_7 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Questions:
1. Draw the frames and points given above.
2. Compute ${}^A_C T$ and ${}^A_G T$.
3. Find ${}^D P_1$ and ${}^A P_4$.
4. Find ${}^E P_1$ and ${}^A P_5$.
5. Find ${}^B P_1$ and ${}^A P_2$.
6. Find ${}^F P_1$ and ${}^A P_6$.
7. Find ${}^F P_2$ and ${}^B P_6$.
8. Find ${}^D P_5$ and ${}^E P_4$.
9. Find ${}^C P_7$ and ${}^G P_3$.

A. Apply a rotation of $-80°$ and translate by vector $v = \begin{bmatrix} -6 \\ 7 \end{bmatrix}$ to ${}^{A}P_1$.

B. Apply a rotation of $130°$ and translate by vector $v = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ to ${}^{C}P_7$.

Solution: **1.**



2. $${}^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T = \begin{bmatrix} c135 & -s135 & 3 \\ s135 & c135 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c-30 & -s-30 & -2 \\ s-30 & c-30 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.26 & -.97 & 3 \\ .97 & -.26 & 1.17 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{A}_{G}T = {}^{A}_{F}T\,{}^{F}_{G}T = \begin{bmatrix} c45 & -s45 & -1 \\ s45 & c45 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c90 & -s90 & -2 \\ s90 & c90 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.71 & -.71 & -5.95 \\ .71 & -.71 & 1.12 \\ 0 & 0 & 1 \end{bmatrix}$$

3. $${}^{D}P_1 = {}^{D}_{A}T\,{}^{A}P_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$${}^{A}P_4 = {}^{A}_{D}T\,{}^{D}P_4 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4. $${}^{E}P_1 = {}^{E}_{A}T\,{}^{A}P_1 = \begin{bmatrix} .5 & .87 & 0 \\ -.87 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -.23 \\ -3.6 \\ 1 \end{bmatrix}$$

$${}^{A}P_5 = {}^{A}_{E}T\,{}^{E}P_5 = \begin{bmatrix} .5 & -.87 & 0 \\ .87 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .7 \\ .7 \\ 1 \end{bmatrix} = \begin{bmatrix} -.26 \\ .96 \\ 1 \end{bmatrix}$$

5. $${}^{B}P_1 = {}^{B}_{A}T\,{}^{A}P_1 = \begin{bmatrix} -.71 & .71 & -.71 \\ -.71 & -.71 & 4.95 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4.24 \\ 4.24 \\ 1 \end{bmatrix}$$

$${}^{A}P_2 = {}^{A}_{B}T\,{}^{B}P_2 = \begin{bmatrix} c135 & .-s135 & 3 \\ s135 & c135 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -6.9 \\ 3.41 \\ 1 \end{bmatrix}$$

6. $${}^{F}P_1 = {}^{F}_{A}T\,{}^{A}P_1 = \begin{bmatrix} .71 & .71 & 1.41 \\ -.71 & .71 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.12 \\ -3.54 \\ 1 \end{bmatrix}$$

$${}^{A}P_6 = {}^{A}_{F}T\,{}^{F}P_6 = \begin{bmatrix} .71 & -.71 & -1 \\ .71 & .71 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3.14 \\ 2.718 \\ 1 \end{bmatrix} = \begin{bmatrix} -5.14 \\ -1.29 \\ 1 \end{bmatrix}$$

7. $${}^{F}P_2 = {}^{F}_{A}T\,{}^{A}P_2 = \begin{bmatrix} .71 & .71 & 1.41 \\ -.71 & .71 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6.9 \\ 5.41 \\ 1 \end{bmatrix} = \begin{bmatrix} .36 \\ 8.71 \\ 1 \end{bmatrix}$$

$${}^{B}P_6 = {}^{B}_{A}T\,{}^{A}P_6 = \begin{bmatrix} -.71 & .71 & -.71 \\ -.71 & -.71 & 4.95 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5.14 \\ -1.29 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.01 \\ 9.5 \\ 1 \end{bmatrix}$$

8. $${}^{D}P_5 = {}^{D}_{A}T\,{}^{A}P_5 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -.26 \\ .96 \\ 1 \end{bmatrix} = \begin{bmatrix} -3.26 \\ 3.96 \\ 1 \end{bmatrix}$$

$${}^{E}P_4 = {}^{E}_{A}T\,{}^{A}P_4 = \begin{bmatrix} .5 & .87 & 0 \\ -.87 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.37 \\ -.37 \\ 1 \end{bmatrix}$$

9. $${}^{C}P_7 = {}^{C}_{A}T\,{}^{A}_{G}T\,{}^{A}P_1 = \begin{bmatrix} -.26 & .97 & -.36 \\ -.97 & -.26 & 3.2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -.71 & -.71 & -3.95 \\ .71 & -.71 & 1.12 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.87 \\ 13.62 \\ 1 \end{bmatrix}$$

$${}^{G}P_3 = {}^{G}_{A}T\,{}^{A}_{C}T\,{}^{C}P_3 = \begin{bmatrix} -.71 & .71 & -5 \\ -.71 & -.71 & -3.41 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -.26 & -.97 & 3 \\ .97 & -.26 & 1.17 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6.39 \\ -12.19 \\ 1 \end{bmatrix}$$

A. $T = \begin{bmatrix} c-80 & -s-80 & -6 \\ s-80 & c-80 & 7 \\ 0 & 0 & 1 \end{bmatrix}$

${}^{A}P_1' = T\,{}^{A}P_1 = \begin{bmatrix} c-80 & -s-80 & -6 \\ s-80 & c-80 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7.45 \\ 3.7 \\ 1 \end{bmatrix}$

B. $T = \begin{bmatrix} c130 & -s130 & 4 \\ s130 & c130 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

${}^{C}P_7' = T\,{}^{C}P_7 = \begin{bmatrix} c130 & -s130 & 4 \\ s130 & c130 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2.87 \\ 13.62 \\ 1 \end{bmatrix} = \begin{bmatrix} -8.28 \\ -8.56 \\ 1 \end{bmatrix}$

8 Programming question 1

In python use numpy, opencv, or code your own functions to verify your answers are correct for parts 1-6. Or you can use c++ with eigen(https://eigen.tuxfamily.org/index.php?title=Main_Page) or install opencv in c++, to verify your answers for parts 1-6. **Submit as .py or .cpp file. I should be able to run your code and get the answers for parts 1-6.**

CODE ON BLACKBOARD

9 Programming question 2

Create a class in python or c++ to represent a 2D frame object. Make sure to include at least parent frame, child frame, origin, and orientation.

1. Write a function for multiplying frames. This should return a new frame object.

2. Write a function that takes as input a point, rotation, and translation, then returns a new point that has been rotated and translated.

3. Write a function that takes from_frame, to_frame, and a point in from_frame as input, and returns the point in to_frame.

4. Use your class to verify your answers from part 7.
submit the .py or .cpp files you created

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