Kinematics of Phantom Model 1.0

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I. Introduction

This paper represents the calculation of forward and inverse kinematics of the Phantom Model 1.0 robot. The robot has 3 DoF with three reveloute joints. The parameters of the reveloute joints are theta one, two and three are given for forward kinematics and cartesian space parameters x,y,z calculated. Same theta parameters are calculated according to given x,y,z parameters in the inverse kinematics application. The base frame of the application is given in Figure 1 as tip point, and the base vectors are represented. The rest of the calculations will be relatively this frame, and Denavit–Hartenberg (DH) parameters will be used to calculate frame transformations.

II. Discussion

A. Forward Kinematics

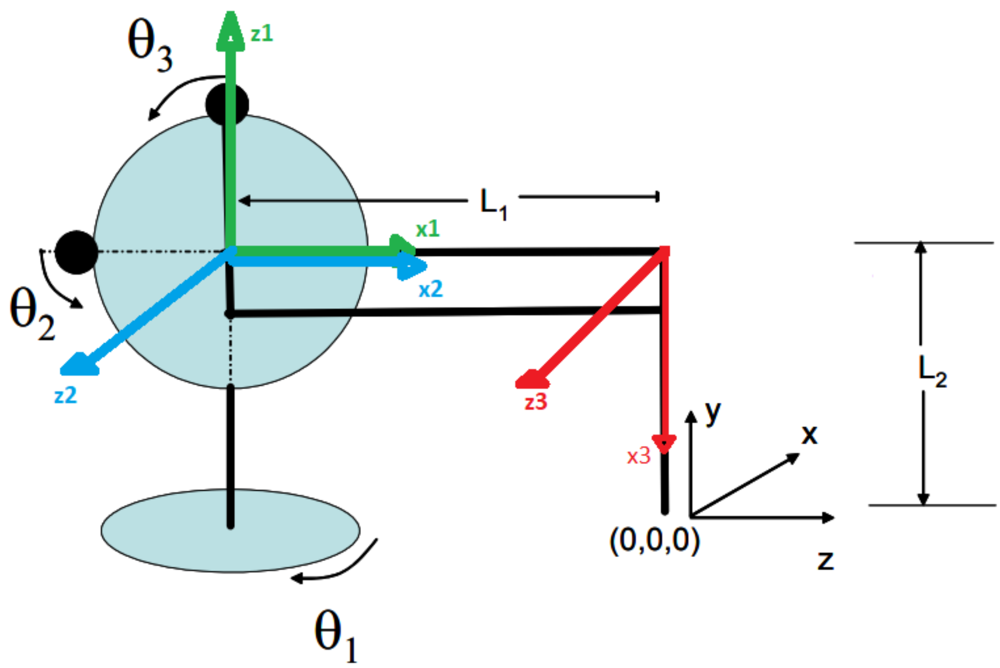

Figure 1 Frame assignments for 3 revolute joints

i	α_{i-1}	a_{i-1}	d_i	Θ_i
1	-90°	0	0	- Θ_1 -90
2	90°	0	0	Θ_2
3	0°	L1	0	Θ_3 -90

Figure 2 DH Parameters for frame assignments

The transition between frame zero and frame 1 is hard to represent in DH parameters since it has a component along y0 axis, it can be directly represented as transition along z0 as -L1 and along y0 as L2. In Figure 2 the α and d values are not appropriate in the first row, because representing the transformation is harder to represent in these parameters for this application, but the mentioned transition can be observed in transformation matrix from zero to one in Figure 3. The rotation between them has three steps rotation about the x0 with -90°, rotation about z1 as -90° and -theta1. The theta1 has negative sign because the represented theta rotation in Figure one is about the negative z1 axis for my frame1. In the end these rotations are applied according to Euler's rotation convention and the transformation matrix from 1 to 2 is represented in Figure 3.

```
[-sin(theta1), cos(theta1), 0,    0]
[           0,           0, 1,  140]
[ cos(theta1), sin(theta1), 0, -140]
[           0,           0, 0,    1]
```

Figure 3 The transformation matrix from frame 0 to frame 1

From frame 1 to frame 2 there are two consecutive rotations, first one is 90° about x1 and theta2 degrees about the z2 axises. The rotation is Euler rotation and Figure 4 and Figure 5 are the rotation matrixis accordingly. There is no transition between the frame 2 and 3, because they have common origin. So, the total rotation values and no transition values are merged in a transformation matrix and it is represented in Figure 6.

```
1.0000         0         0
     0    0.0000   -1.0000
     0    1.0000    0.0000
```

Figure 4 Rotation about x with 90 degrees

```
[cos(theta_two), -sin(theta_two), 0]
[sin(theta_two),  cos(theta_two), 0]
[             0,               0, 1]
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Figure 5 Rotation about z with theta two degrees

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[cos(theta_two), -sin(theta_two),  0, 0]
[             0,               0, -1, 0]
[sin(theta_two),  cos(theta_two),  0, 0]
[             0,               0,  0, 1]
```

Figure 6 Transformation from frame 1 to frame 2

The transformation from 2 to 3 has a transition as L1 and it is along the x2 axis, its implementation is in Figure 7. The rotation part in these two frames is complicated, firstly there is a -90° about the z2 axis then theta3 is implemented. The problem is that the given theta3 parameter does not match with the DH parameters, so there should be a mapping between the given theta 3 and the one used in the calculations. A trigonometric view of the problem is represented in Figure 7 and the mapping applied according to that configuration. The mapping is necessary, because the frame placement convention does not work for given theta3 value. In other words, the x2 unit vector is along the L1 link and the x3 is along the L2 link. So, the rotation about the z2 axis should be represented as Θ_3` in Figure 7. The mapping expresses the $\Theta_3$` as function of given Θ_2 Θ_3. The relationship between them is read from the trigonometric view is given as equation 1. When the transition and rotation is expressed in a transformation matrix after mapping is applied, the matrix is in Figure 8.

Lastly, there is a transition between frame 3 and the tip of the end effector. Since, the effector is accepted as a point there is no need to assign any frame on it, and rotation is unnecessary for this application. So, the rotation matrix is identity, and the transformation matrix is in Figure 9.

In the end, all these transformation matrixes are multiplied and the transformation from frame zero to frame 3 is found, and this is the forward kinematics of the robot. Since the equations are too long, showing the whole transformation matrix is not appropriate in this paper, but it can be observed in the code. The end effector position can be read from the first three entities of the last column, after theta values are given. One example is represented in Figure 10, for theta1= -1.214, theta2= -0.657, theta3= 2.314. So, the end effector positions are Px = 200.4648 mm, Py = 149.2269 mm, Pz = -65.2768 mm according to the given base frame.

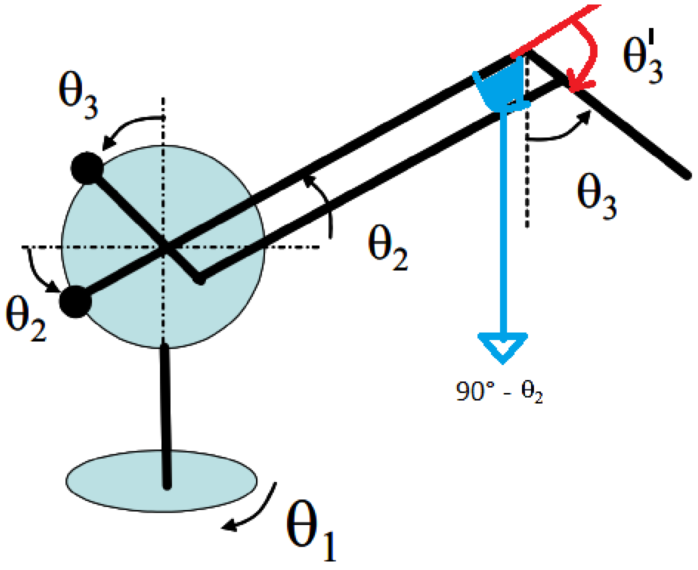

Figure 7 Mapping of theta 3

```
[ cos(theta2 - theta3), sin(theta2 - theta3), 0, 140]
[-sin(theta2 - theta3), cos(theta2 - theta3), 0,   0]
[                    0,                    0, 1,   0]
[                    0,                    0, 0,   1]
```

Figure 8 Transformation from frame two to three

1	0	0	0
0	1	0	-140
0	0	1	0
0	0	0	1

Figure 9 Transformation from frame 3 to tip frame

0.6899	-0.6340	-0.3493	200.4648
0.6767	0.7363	0	149.2269
0.2572	-0.2363	0.9370	-65.2768
0	0	0	1.0000

Figure 10 FK Matrix for Given Thetas

B. Inverse Kinematics

The inverse kinematics is calculation of the theta values for given cartesian coordinates. To do it, some equations for theta values are obtained trigonometrically, and represented in this paper. To solve inverse kinematics, top and side views of the robot are examined separately, then theta values are calculated with given parameters. Start with the top view of the robot, the top view is represented in Figure 11. In the view, as you can see the theta1 value is equal to inverse tangent of the Px and Pz + L1, and it is calculated in the code by using atan2 function to detect quadrant of the angle. Also, as discussed earlier the direction of the theta1 is opposite of the z1 axis, so the minus sign should be added after calculating the theta1 value. Then, atan2(Px, Pz +L1) is the first solution of the theta1. Intuitively, the exact opposite of this configuration has also the same position values, so first theta1 + 180° is also one of the two solutions. These are the possible solutions for the theta1. The longitude of the link along the x-z plane is also calculated as r1, it will be used in later calculations.

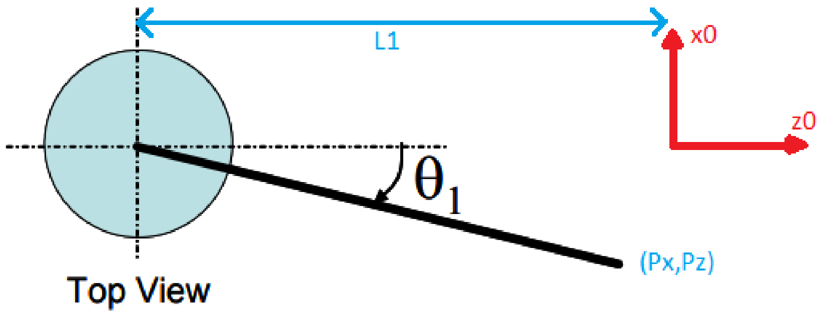

Figure 11 Top View of the Robot

To calculate theta2 value, it is separated into two parts. In Figure 12, the green angle is α and the red angle is β, they will be calculated separately and will be merged to calculate the theta2. Start with the calculation of the α, it can be calculated by using the Atan2 function, because it has determined sine and cosine values. The sin value of the angle is Py – L2 which is dashed red line in the figure, and the cosine value is r1 as calculated above. So, the α value is represented in equation 3.

Now, the β should be calculated, and cosine rule will be used for calculation. The rule is in equation 4, and obtained cosβ value from the rule is in equation 5. By using the cosine of the β, two different sinβ are obtained, one is positive and the other is negative roots of the equation 6. These two roots will represent elbow up and elbow down configurations of the robot. It means that for every theta1 value there are two different theta2 values. The two angles for the first theta1 configuration are obtained by the equations 7 and 8.

As mentioned earlier, there should be two more theta2 solutions for the second theta1 value. The solutions can be calculated after intuitive observation. Since the second theta1 is calculated by a 180° rotation, the theta2 configuration should be symmetric about the z axis of frame 1. After 180° of frame 1, link 1 must be mirrored according to middle vertical plane. Mathematically, the new angle of the theta2 should be 180° - theta2 to achieve this mirroring. So, the other two solutions are equation 9 and 10 for the theta2.

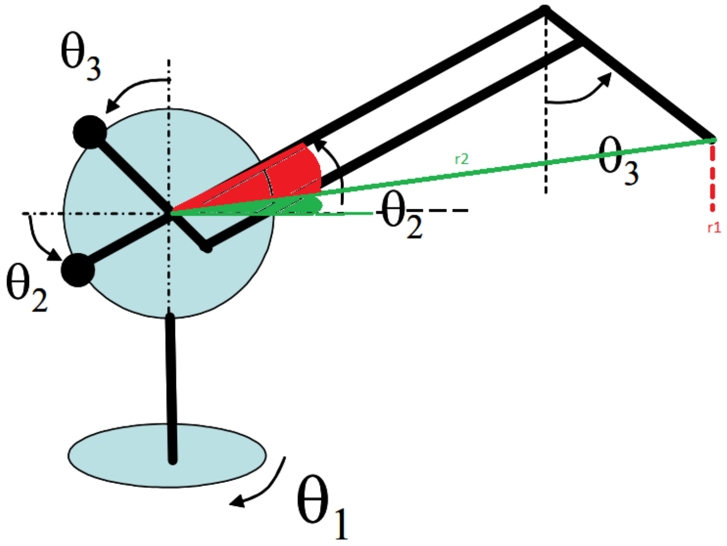

Figure 12 Side view for theta 2 calculation

The last part is the equations of the theta3. To obtain them cosine rule will be used again. In Figure 13, the blue shaded angle is the reference angle, and let represent it with the letter Ψ. To obtain Ψ, the calculation of r2 from Figure 12 is used. When the cosine rule is applied the result is equation 11. Then, from equation 6, there are two possible solutions Ψ. As discussed earlier in Figure 7, the blue angle is the summation of two angles, and the Ψ value is also dependent on the theta2. So, the relation between them is equation 12 and 13. In this sense, there should be 8 total solutions for theta3, but there are not. For every theta2 value two solutions for Ψ, but just one of them ensure the configuration. The evaluation is in the paragraph below.

As mentioned earlier theta2_1 is the solution for elbow up and theta2_2 is the elbow down. When these are evaluated, the Ψ value should be smaller than 180° in elbow up and the Ψ should be bigger than 180° in elbow down for the first two theta2s. This makes sinus of the Ψ positive for first solution and negative for the second solution. As the result, solutions are for theta3_1 and theta3_2 is in equation 14 and 15.

For the second theta1 configuration, the sensation is reversed. The Ψ value should be bigger than 180° in elbow up and should be smaller in elbow down. So, the equations are 16 and 17 for solutions of theta3_3 and theta3_4.

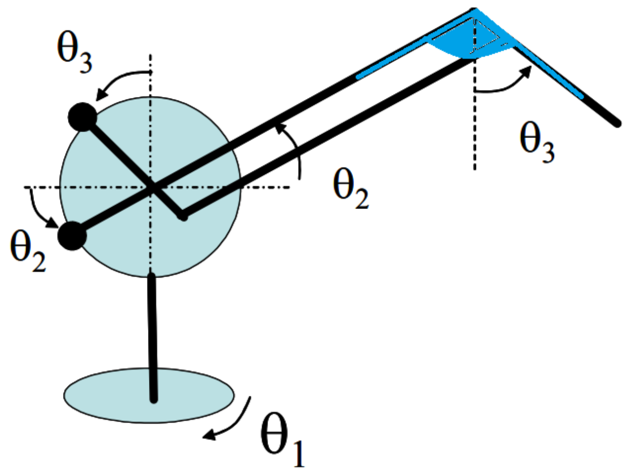

Figure 13 Theta 3 from Cosine Rule

C. *Equations*

$$-\Theta_3` + \Theta_3 + (90° - \Theta_2) = 180 \rightarrow \Theta_3` = \Theta_3 - \Theta_2 - 90° \quad (1)$$

$$r1 = sqrt(Px^2 + Pz^2) \quad (2)$$

$$\alpha = Atan2(Py-L2, r1) \quad (3)$$

$$(L_2)^2 = (r_2)^2 + (L_1)^2 - 2 r_2 L_1 \cos\beta \quad (4)$$

$$\cos\beta = ((r_2)^2 + (L_1)^2 - (L_2)^2) / (2 r_2 L_1) \quad (5)$$

$$\cos^2\beta + \sin^2\beta = 1 \quad (6)$$

$$theta2\_1 = \alpha + atan2(sqrt(1-\cos^2\beta), \cos\beta) \quad (7)$$

$$theta2\_2 = \alpha + atan2(sqrt(1-\cos^2\beta), \cos\beta) \quad (8)$$

$$theta2\_3 = 180° - theta2\_1 \quad (9)$$

$$theta2\_4 = 180° - theta2\_2 \quad (10)$$

$$\cos\Psi = ((L_1)^2 + (L_2)^2 - (r_2)^2) / (2 * L1 * L2) \quad (11)$$

$$theta3 = theta2 - pi/2 + atan2(sqrt(1- \cos^2\Psi), \cos\Psi) \quad (12)$$

$$theta3 = theta2 - pi/2 + atan2(-sqrt(1- \cos^2\Psi), \cos\Psi) \quad (13)$$

$$theta3\_1 = theta2\_1 - pi/2 + atan2(sqrt (1- \cos^2\Psi), \cos\Psi) \quad (14)$$

$$theta3\_2 = theta2\_2 - pi/2 + atan2(-sqrt (1- \cos^2\Psi), \cos\Psi) \quad (15)$$

$$theta3\_3 = theta2\_3 - pi/2 + atan2(-sqrt (1- \cos^2\Psi), \cos\Psi) \quad (16)$$

$$theta3\_4 = theta2\_4 - pi/2 + atan2(sqrt (1- \cos^2\Psi), \cos\Psi) \quad (17)$$

III. CONCLUSION

To sum up, the forward and inverse kinematics of the Phantom Model 1.0 are examined and represented in this paper. The forward kinematics is implemented by attaching frames on joints and using DH parameters. The inverse kinematics are examined under the trigonometric top and side views of the robot. The total number of solutions for the inverse kinematics is 4. By using these transformations in forward kinematics, and analytical solutions in the inverse kinematics desired values can be calculated for cartesian coordinates or for the joint angles.

REFERENCES

[1] J. J. Craig, Introduction to robotics: Mechanics and control. New York: Pearson, 2022.