Algorithm Analysis

An *algorithm* is a clearly specified set of simple instructions to be followed to solve a problem.

- What is a problem?
- What is a program, for that matter?
- Topics for analysis:
  - □ How to estimate the time required for a program
  - □ How to reduce the running time of a program
  - □ The consequences of careless use of recursion
  - □ Very efficient algorithms to compute:
    - ∘ x^y
    - ∘ gcd(a,b)

Estimation Techniques (Section 2.7)

- "Back of the envelope" or "back of the napkin" calculation
  1. Determine the problem's major parameters
  2. Derive an equation relating the parameters to the problem
  3. Select values for the parameters and apply the equation
- Example:
  - □ How many bookcases in the library are required to store 1 million pages?
  - □ Parameters:
    - ∘ length of a shelf in a case
    - ∘ number of shelves in a case
    - ∘ number of pages per unit of length
    - ∘
  - □ Estimate:
    - ∘ Pages per inch
    - ∘ feet per shelf
    - ∘ shelves per bookcase

Algorithm Efficiency

There are often many algorithms for a given problem. How do we choose the "best?"

- (Conflicting) goals of program design:
  - ☐ Algorithm is to be easy to understand, code, debug
  - ☐ Algorithm makes efficient use of computer's resources
- How do we measure an efficiency?
  - ☐ Empirical comparison (run the programs).
  - ☐ Asymptotic algorithm analysis.
  - ☐ Critical resources:
  - ☐ Factors affecting running time:
  - ☐ For most algorithms, running time depends on "size" of the input.
  - ☐ Running time is expressed as $T(n)$ for some function T on input size n

Examples of Growth Rate

- Example 1:

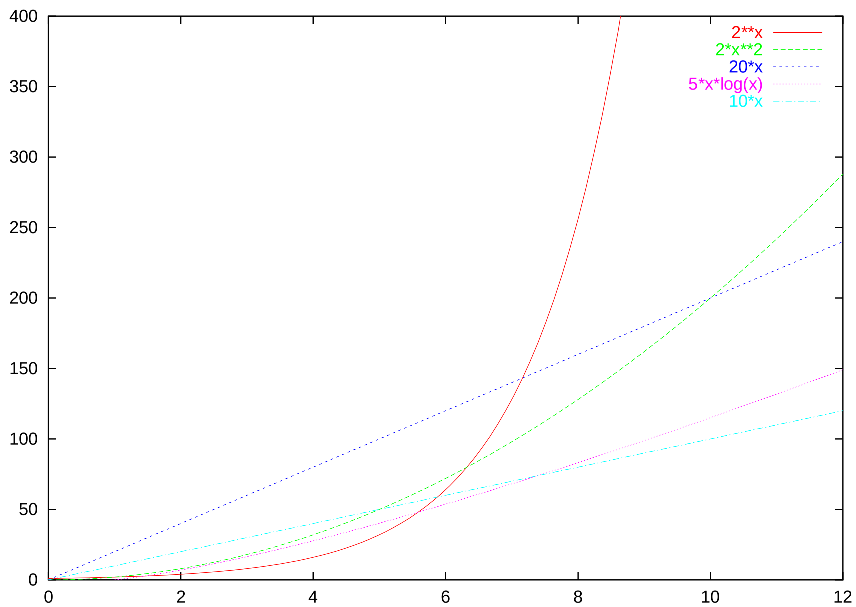
```
int largest (int *array, int n) {
  int currlarge = array[0];
  for (int i=1; i<n; i++)
    if (array[i] > currlarge)
      currlarge = array[i];
  return currlarge;
}
```

- Example 2: assignment statement
- Example 3:

```
sum = 0;
for (i=1; i<=n; i++)
  for (j=1; j<=n; j++)
    sum++;
```

Growth Rate Graph

- Certain high growth-rate functions may be more efficient at some locations



- Functions: $y = 2^x$, $y = 2x^2$, $y = 20x$, $y = 5x \log x$, $y = 10x$

Best, Worst, and Average Cases

- Not all inputs of a given size take the same time
- Sequential search for K in an array of n integers:
  - ☐ Best case:
  - ☐ Worst case:
  - ☐ Average case:
- While the average time seems to be the fairest measure, it may be difficult to determine
- When is the worst case time important?

Faster Computer or Algorithm?

- What happens when we buy a computer 10 times faster?

$T(n)$	n	n'	Change	n'/n
$10n$	$1,000$	$10,000$	$n' = 10n$	10
$20n$	500	$5,000$	$n' = 10n$	10
$5n \log n$	250	$1,842$	$\sqrt{10}n < n' < 10n$	7.37
$2n^2$	70	223	$n' = \sqrt{10}n$	3.16
2^n	13	16	$n' = n + 3$	—

n: Size of input that can be processed in one hour (10,000 steps)

n': Size of input that can be processed in one hour on the new machine (100,000 steps).

Asymptotic Analysis: $O(f(n))$

- The upper bound defined by "Big-Oh":

  Definition: For $T(n)$ a non-negatively valued function, $T(n) = O(f(n))$ if there exist two positive constants c and n_0 such that $T(n) \leq cf(n)$ for all $n \geq n_0$.

  Usage: "The algorithm is order of n^2 in [best, average, worst] case."

  Alternatively, "The algorithm is big-Oh of $n^2 \ldots$"

  Meaning: for all data sets big enough (i.e., $n \geq n_0$), the algorithm *always* executes in at most $cf(n)$ steps [in best, average, or worst case].

- Example: if $T(n) = 3n^2$ then $T(n)$ is ...
  - □ $T(n) = 3n^2$ is $O(n^3)$, also $O(n^4)$, $O(n^5)$, etc.
  - □ It is preferable to say that $T(n)$ is $O(n^2)$.
  - □ (We always wish to get the tightest upper bound.)

Big-Oh Examples

- Example 1: Finding value X in an array.
  - ☐ $T(n) = c_1 n/2$.
  - ☐ For all values of $n \geq 1$, $c_1 n/2 \leq c_1 n$.
  - ☐ Therefore, by the definition, $T(n)$ is $O(n)$ for $n_0 = 1$ and $c = c_1$.

- Example 2: $T(n) = c_1 n^2 + c_2 n$ in average case.
  - ☐ $c_1 n^2 + c_2 n \leq c_1 n^2 + c_2 n^2 \leq (c_1 + c_2) n^2$ for all $n \geq 1$.
  - ☐ $T(n) \leq cn^2$ for $c = c_1 + c_2$ and $n_0 = 1$.

- Example 3. $T(n) = c$. We say this is $O(1)$.

Pitfall

"The best case is for $n = 1$ since it runs most quickly."

- Big-oh refers to a **growth rate** as n grows to ∞.

- The best case is defined as which input of size n is the cheapest among all inputs of size n.

- Example: use insert sort to sort these lists
  - ☐ list 1: 1
  - ☐ list 2: 5, 9, 24, 1, 3, 12, 2, 16
  - ☐ list 3: 1, 2, 3, 5, 9, 12, 16, 24

Asymptotic Analysis: $\Omega(g(n))$

- The lower bound defined by "Big Omega":

  Definition: For $T(n)$ a non-negatively valued function, $T(n)$ is $\Omega(g(n))$ if there exist two positive constants c and n_0 such that $T(n) \geq cg(n)$ for all $n \geq n_0$.

  Usage: The algorithm is $\Omega(n^2)$ in [best, average, worst] case.

  Meaning: For *all* data sets big enough (that is, $n \geq n_0$), the algorithm *always* executes in at least $cg(n)$ steps.

- Example: $T(n) = c_1 n^2 + c_2 n$.
  - □ $c_1 n^2 + c_2 n \geq c_1 n^2$ for all $n \geq 1$.
  - □ $T(n) \geq cn^2$ for $c = c_1$ and $n_0 = 1$.
  - □ Therefore, $T(n)$ is $\Omega(n^2)$ by the definition.
  - □ We want the greatest lower bound.

Asymptotic Analysis: $\Theta(g(n))$

- The equality defined by "Big Theta":

  When big-Oh and big-Omega meet, we indicate this by using the big-Theta (i.e., Θ) notation.

  Definition: For $T(n)$ a non-negatively valued function, $T(n) = \Theta(g(n))$ if and only if $T(n)$ is $O(g(n))$ and $T(n)$ is $\Omega(g(n))$

  Usage: "The algorithm is $\Theta(g(n))\ldots$"

  Meaning: $g(n)$ provides the best (tightest) upper and lower bound for $f(n)$

- Example: $T(n) = c_1 n^2$.
  - □ Big-Oh:
    - ∘ $c_1 n^2 \leq c_1 n^2$ for all $n \geq 1$
    - ∘ Therefore, $T(n)$ is $O(n^2)$
  - □ Big-Omega:
    - ∘ $c_1 n^2 \geq c_1 n^2$ for all $n \geq 1$
    - ∘ Therefore, $T(n)$ is $\Omega(n^2)$
  - □ $T(n)$ is $O(n^2)$ and $\Omega(n^2)$, so $T(n)$ is $\Theta(n^2)$

Asymptotic Analysis: $o(g(n))$

- The strict upper bound defined by "little Oh"

  When a function is big-Oh but not big-Theta, then it is little-Oh

  Definition: For $T(n)$ a non-negatively valued function, $T(n) = o(g(n))$ if $T(n)$ is $O(g(n))$ and $T(n) \neq \Theta(g(n))$

  Usage: "The algorithm is $o(g(n)) \ldots$"

- Example: $T(n) = c_1 n^2$.
  - ☐ $c_1 n^2 \leq c_1 n^3$ for all $n \geq 1$, so $T(n)$ is $O(n^3)$
  - ☐ Generally, $c_1 n^2 \not\geq c_1 n^3$, so $T(n)$ is not $\Omega(n^3)$, thus not $\Theta(n^3)$
  - ☐ Therefore, $T(n)$ is $o(n^3)$

Simplifying Rules:

- Certain rules allow us to simplify the analysis

  1. If $f(n)$ is $O(g(n))$ and $g(n)$ is $O(h(n))$, then $f(n)$ is $O(h(n))$
  2. If $f(n)$ is $O(kg(n))$ for any constant $k > 0$ then $f(n)$ is $O(g(n))$
  3. If $T_1(n) = O(f(n))$ and $T_2(n) = O(g(n))$, then

     (a) $T_1(n) + T_2(n) = \max(O(f(n)), O(g(n)))$,

     (b) $T_1(n) \times T_2(n) = O(f(n) \times g(n))$
  4. If $T(n)$ is a *polynomial of degree* k, then $T(n) = \Theta(n^k)$
  5. $\log^k n = O(n)$ for any constant k.

- Points of style:
  - ☐ Do not include constants or low-order terms in a Big-Oh (or Ω or Θ)
  - ☐ It is bad to say $f(n) \leq O(g(n))$ (it's implied)
  - ☐ It is incorrect to say $f(n) \geq O(g(n))$ (it makes no sense)
  - ☐ L'Hôpital's rule can be applied, if necessary, to find relative growth rates

Typical Growth rates

- There is a terminology for certain growth rate functions:

Function	Name
c	Constant
$\log n$	Logarithmic
$\log^2 n$	Log-squared
n	Linear
$n \log n$	$n \log n$
n^2	Quadratic
n^3	Cubic
2^n	Exponential

Model

- Model of computation:
  - ☐ Normal computer
  - ☐ Sequential instruction execution
  - ☐ All simple instructions take one time unit
  - ☐ Fixed-size integers
  - ☐ No fancy operations that take one time unit
  - ☐ Infinite memory
- Weaknesses in the model:
  - ☐ Not all operations really take "one time unit"
  - ☐ Infinite memory allows us to ignore system realities

What Behavior to Analyze

- Two resources can generally be analyzed:
  - ☐ Time (we usually focus on this)
  - ☐ Space

- For a given input size, three running time functions can be defined:
  - ☐ $T_{\text{best}}(n)$: best case running time
  - ☐ $T_{\text{worst}}(n)$: worst case running time
  - ☐ $T_{\text{avg}}(n)$: average case running time
  - ☐ Of course, $T_{\text{best}}(n) \leq T_{\text{avg}}(n) \leq T_{\text{worst}}(n)$
  - ☐ Normally, worst and average case are of more interest:
    - ◦ Worst case represents a performance guarantee
    - ◦ Average case represents typical behavior, but it can be difficult to find
    - ◦ Best case can be useful for particular programming problems

Rules for Program Analysis

- Shortcuts used to analyze code:
  - ☐ **For loops**: running time is at most the running time of the statements inside times the number of iterations

    Example:

```
for (i = 0; i < n; i++) {  // n iterations *
  j = j + i;               // (1 statement
  k = k * i;               // + 1 statement)
}
```

  - ☐ **Nested loops**: analyze inside-out; total running time is the running time of the statements multiplied by the product of the sizes of all loops

    Example:

```
for (i = 0; i < n; i++)     // n iterations *
  for (j = 0; j < n; j++) { // n iterations *
    k++;                    // 1 statement
  }
```

Rules for Program Analysis (cont.)

- Shortcuts (cont.)
  - ☐ **Consecutive statements**: they are added; thus take the maximum

    Example:

```
for (i = 0; i < n; i++)     // n iterations
  a[i] = 0;                 // * one statement

for (i = 0; i < n; i++)     // n iterations
  for (j = 0; j < n; j++)   // * n iterations
    a[i] += a[j] + i + j;   // * 3 statements
```

  - ☐ **If/Then/Else clause**: running time of the test plus the higher complexity clause

    Example:

```
  if (x < 5) {
    for (i = 0; i < n*n; i++)
      j = j + i;
  }
  else
    j = 0;
```


Rules for Program Analysis (cont.)

- Shortcuts (cont.)
  - ☐ **While loop**: analyzed like a **for loop**

    Example:

```
i = 0;
While (i < n) {  // n iterations *
  j = j + i;     // (1 statement
  k = k * i;     // + 1 statement
  i++;           // + 1 statement)
}
```

  - ☐ **Switch statement**: take the complexity of most expensive case
  - ☐ **Subroutine call**: take the complexity of the subroutine

Program Fragment Analysis

- Example 1:
  - ☐ Assignment `a = b;`
  - ☐ This assignment takes constant time, so it is $\Theta(1)$.
- Example 2:

```
sum = 0;
for (i=1; i<=n; i++)
    sum += n;
```

- Example 3:

```
sum = 0;
for (j=1; j<=n; j++)
  for (i=1; i<=j; i++)
    sum++;
for (k=0; k<n; k++)
  A[k] = k;
```


Program Fragment Analysis

- Example 4:

```
sum1 = 0;
for (i=1; i<=n; i++)
  for (j=1; j<=n; j++)
    sum1++;

sum2 = 0;
for (i=1; i<=n; i++)
  for (j=1; j<=i; j++)
    sum2++;
```

- Example 5:

```
sum1 = 0;
for (k=1; k<=n; k*=2)
  for (j=1; j<=n; j++)
    sum1++;

sum2 = 0;
for (k=1; k<=n; k*=2)
  for (j=1; j<=k; j++)
    sum2++;
```

Maximum Subsequence Sum

The maximum subsequence sum (MSS) problem has many possible algorithms, whose complexity varies greatly

- Given (possibly negative) numbers $A_1, A_2, \ldots, A_n$, find the maximum value of $\sum_{k=i}^{j} A_k$.
- Example
  - ☐ Input is -1, 11, -4, 13, -5, -2
  - ☐ Answer is 20 (A_2 through A_4)
- Run times for several algorithms solving this problem:

Analysis of MSS Problem

- Cubic algorithm:

```
        int maxSubSum1(const vector<int> & a)
        {
/* 1*/    int maxSum = 0;

/* 2*/    for (int i = 0; i < a.size(); i++)
/* 3*/      for (int j = i; j < a.size(); j++)
            {
/* 4*/        int thisSum = 0;

/* 5*/        for (int k = i; k <= j; k++)
/* 6*/          thisSum += a[k];

/* 7*/        if (thisSum > maxSum)
/* 8*/          maxSum   = thisSum;
            }

/* 9*/    return maxSum;
        }
```

Analysis of MSS Problem

- Quadratic algorithm:

```
        int maxSubSum2(const vector<int> & a)
        {
/* 1*/    int maxSum = 0;

/* 2*/    for (int i = 0; i < a.size(); i++)
          {
/* 3*/      int thisSum = 0;
/* 4*/      for (int j = i; j < a.size(); j++)
            {
/* 5*/        thisSum += a[j];

/* 6*/        if (thisSum > maxSum)
/* 7*/          maxSum = thisSum;
            }
          }
/* 8*/    return maxSum;
        }
```


Analysis of MSS Problem

- $O(n \log n)$ algorithm:

```
        int maxSumRec(const vector<int> & a, int left,
                      int right) {
/* 1*/    if (left == right)  // Base case
/* 2*/      if (a[left] > 0)
/* 3*/        return a[left];
            else
/* 4*/        return 0;

/* 5*/    int center = (left + right) / 2;
/* 6*/    int maxLeftSum  = maxSumRec(a,left,center);
/* 7*/    int maxRightSum = maxSumRec(a,center+1,right);

/* 8*/    int maxLeftBorderSum = 0, leftBorderSum = 0;
/* 9*/    for (int i = center; i >= left; i--)
          {
/*10*/      leftBorderSum += a[i];
/*11*/      if (leftBorderSum > maxLeftBorderSum)
/*12*/          maxLeftBorderSum = leftBorderSum;
          }
/*13*/    int maxRightBorderSum = 0, rightBorderSum = 0;
/*14*/    for (int j = center + 1; j <= right; j++)
          {
/*15*/      rightBorderSum += a[j];
/*16*/      if (rightBorderSum > maxRightBorderSum)
/*17*/              maxRightBorderSum = rightBorderSum;
          }

/*18*/    return max3(maxLeftSum, maxRightSum,
/*19*/            maxLeftBorderSum + maxRightBorderSum);
        }
```

Analysis of MSS Problem

- Linear algorithm:

```
          int maxSubSum4(const vector<int> & a)
          {
/* 1*/        int maxSum = 0, thisSum = 0;

/* 2*/        for (int j = 0; j < a.size(); j++)
              {
/* 3*/            thisSum += a[j];

/* 4*/            if (thisSum > maxSum)
/* 5*/                maxSum = thisSum;
/* 6*/            else if (thisSum < 0)
/* 7*/                thisSum = 0;
              }

/* 8*/        return maxSum;
          }
```


Logarithms in Running Time

- General rules:
  - ☐ If it takes $O(1)$ time to cut problem size by a fraction, then the alorithm is $O(\log n)$
  - ☐ If it takes $O(1)$ time to cut problem size by a constant amount, then the alorithm is $O(n)$
- Remember: simply reading the input is $\Omega(n)$
- **binary search**: find an element in a sorted array (or vector)

```
int binary(int K, int* array, int left, int right) {
  int l = left-1;
  int r = right+1;
  while (l+1 != r) {
    int i = (l + r) / 2;
    if (K < array[i]) r = i;
    if (K == array[i]) return i;
    if (K > array[i]) l = i;
  }
  return UNSUCCESSFUL;
}
```

Analysis: how many elements can be examined in the worst case?

Logarithms in Running Time

- **Euclid's algorithm**: find the greatest common divisor (gcd)

```
        long gcd(long m, long n)
        {
/* 1*/    while(n != 0)
          {
/* 2*/      long rem = m % n;
/* 3*/      m = n;
/* 4*/      n = rem;
          }
/* 5*/    return m;
        }
```

- **Efficient exponentiation**: compute x^n

```
        long pow(long x, int n)
        {
/* 1*/    if(n == 0)
/* 2*/      return 1;
/* 3*/    if(n == 1)
/* 4*/      return x;
/* 5*/    if(isEven(n))
/* 6*/      return pow(x*x, n/2);
          else
/* 7*/      return pow(x*x, n/2) * x;
        }
```


Analyzing Problems

- Upper bound: upper bound of best known algorithm.
- Lower bound: lower bound for *every possible algorithm*.

- Example: sorting.
  - ☐ Cost of I/O: $\Omega(n)$
  - ☐ Bubble or insertion sort: $O(n^2)$
  - ☐ A better sort (Quicksort, Mergesort, heapsort, etc.): $O(n \log n)$

Multiple Parameters

Compute the rank ordering for all C pixel values in a picture of P pixels.

```
for (i = 0; i < C; i++)
  count[i] = 0;           // initialize counts
for (i = 0; i < P; i++)  // examine each pixel to
  count[value[i]]++;      // find freq of occurrence
sort(count);              // Sort the array
```

- If we use P as a measure, then the time is $\Theta(P \log P)$
- What is a more accurate measure? Why?

Space Bounds

Space bounds can also be analyzed with asymptotic complexity analysis.

- Algorithms: analyze time
- Data structures: analyze space
- **Space/Time Tradeoff Principle**:

  One can often achieve a reduction in time if one is willing to sacrifice space, or vice versa.

- Examples:
  - ☐ Encoding or packing information
  - ☐ Table lookup