Binary Trees

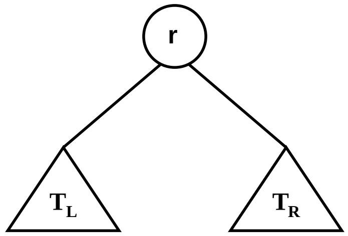
The linear access time of lists makes them prohibitive for large input sets.

- Tree structures:
  - ☐ Efficient access and update to large collections of data
  - ☐ Running time of many operations is $O(\log n)$ (or based on $\log n$)
  - ☐ Some types of trees can guarantee $O(\log n)$ in worst case
  - ☐ Binary trees are widely used, relatively easy to implement
- Uses for Trees
  - ☐ Arithmetic expression evaluation
  - ☐ Storing/searching data
  - ☐ Sorting, priority queues
  - ☐ Coding, Compression
  - ☐ File systems (general trees)
- Reading: all of Ch. 5

Definitions

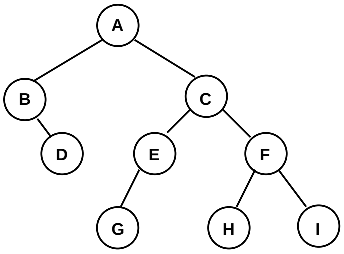
The definition of a binary tree is recursive:

- A **binary tree** is a collection of nodes.
  - ☐ The collection can be empty
  - ☐ Otherwise, a binary tree consists of
    - ◦ A distinguished node, **r**, called the **root**
    - ◦ Two binary trees, called the left and right **subtrees**, which may be empty or not
- Binary tree characteristics
  - ☐ The root of each subtree is a **child** of r
  - ☐ r is the **parent** of each subtree root.
  - ☐ Example using recursive definition:



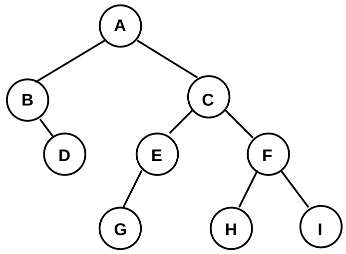
Definitions (cont.)

- Binary tree characteristics
  - ☐ **Path** from node n_1 to n_k: a sequence of nodes such that n_i is the parent of n_{i+1} for $1 \leq i < k$
  - ☐ **Length** of a path: the number of edges on the path ($k - 1$ using prior definition)
  - ☐ **Parent** of a node: immediate predecessor along the path from the root to that node
  - ☐ **Child** of a node: any immediate successor along the path from the root *through* that node
  - ☐ If there is a path from n_1 to n_2 then n_1 is an **ancestor** of n_2 and n_2 is a **descendant** of n_1
  - ☐ **Siblings**: nodes with the same parent



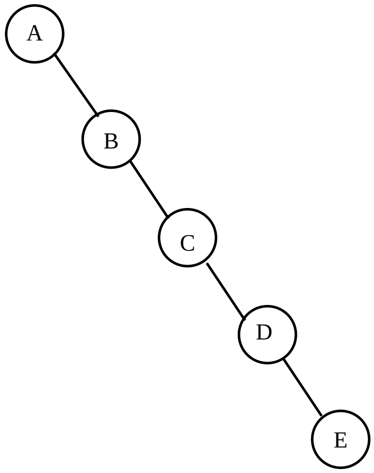
Definitions (cont.)

- Binary tree characteristics
  - ☐ **Leaf node**: a node with no children
  - ☐ **Internal node**: a node with at least one child
  - ☐ **Depth** of node n_i: length of the unique path from the root to n_i
  - ☐ **Height** of the tree: one more than the depth of the deepest node in the tree
  - ☐ All nodes of depth d are at **level** d in the tree
  - ☐ The root is at level 0 and has depth 0



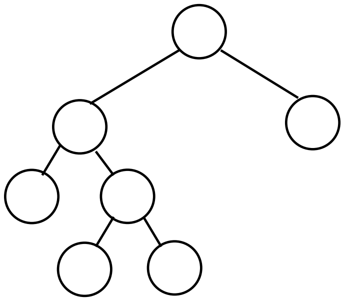
Binary Tree Depth

- Average depth
  - ☐ Normal binary trees: $O(\sqrt{n})$
  - ☐ Binary search trees: $O(\log n)$
- Worst-case binary tree is $O(n)$:



Full Binary Trees

- **Full binary tree**: each node is either a leaf or an internal node with exactly two nonempty children
  - ☐ Example:

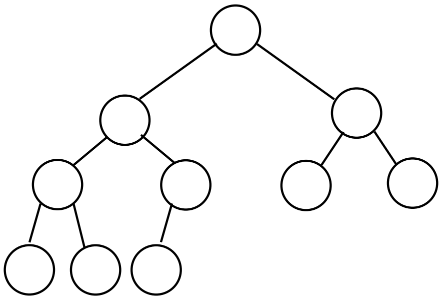


Full Binary Trees

- **Full Binary Tree Theorem**: The number of leaves in a non-empty full binary tree is one more than the number of internal nodes
  - ☐ Proof (by Mathematical Induction):
    - Base: a tree with one node has one leaf and no internal nodes.
    - Induction Hypothesis: Assume any FBT containing $n-1$ internal nodes has n leaves
    - Induction Step: Select an internal node whose children are both leaves and remove them...
- **Corollary to FBT Theorem**: The number of empty subtrees in a non-empty binary tree is one more than the number of nodes in the tree.
  - ☐ Proof: in an arbitrary tree, replace every empty subtree with a leaf node...

Complete Binary Trees

- **Complete binary tree**:
  - ☐ If the height of the tree is d, then all leaves except possibly level d are completely full.
  - ☐ The bottom level fills from left to right (all nodes are to the left)



Binary Tree Node ADT

- Abstract Base Class Node:

```
template <class Elem>
class BinNode {
  public:
    virtual Elem & val() = 0;
    virtual void setVal (const Elem &) = 0;
    virtual BinNode* left() const = 0;
    virtual void setLeft (BinNode*) = 0;
    virtual BinNode* right() const = 0;
    virtual void setRight (BinNode*) = 0;
    virtual bool isLeaf() = 0;
};
```


Binary Tree Traversals

A traversal is the act of visiting each node in the tree in some systematic fashion.

- A traversal that lists every node exactly once is called an **enumeration**
- Each visit involves some sort of work
- Traversals are usually defined recursively
- Three types:
  - ☐ Inorder: visit left subtree, then parent, then right subtree
  - ☐ Preorder: visit parent, then both subtrees
  - ☐ Postorder: visit both subtrees, then parent
  - ☐ Example:

```
template <class Elem>
void preorder(BinNode<Elem>* subroot) {
  if (subroot != NULL) {
    visit(subroot);
    preorder(subroot -> left());
    preorder(subroot -> right());
  }
}
```


Binary Tree Node Implementations

- Pointer-based nodes are most common
- How can we differentiate leaf and internal nodes? (And, should we?)
  - ☐ C++ Union construct
  - ☐ Use base class/subclass implementation
  - ☐ Don't
- Example: A simple node implementation (this one does not differentiate)

```
template <class Elem>
class BinaryNode {
  public:
    Elem element;
    BinaryNode *left;
    BinaryNode *right;
};
```

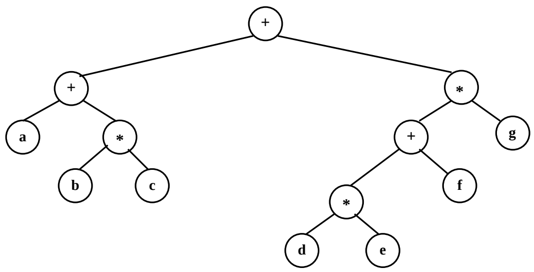

Union Implementation

- Primary problem is space inefficiency:

```
enum Nodetype {leaf, internal};
class VarBinNode {
  public:
    Nodetype mytype;
    union {
      struct {             // internal node
        VarBinNode *left;
        VarBinNode *right;
        operator opx;
      } intl;
    Operand var;           // leaf node
    }
};
```

Expression Trees

- Nodes:
  - ☐ Leaves are operands
  - ☐ Internal nodes are operators
- Expression tree for $(a + b * c) + ((d * e + f) * g)$



Constructing an Expression Tree

- Convert postfix expression to a tree
- Uses a stack to store the postfix expression
  - ☐ Pseudocode algorithm:

```
while (not end of postfix-expression) {
  read next symbol
  if symbol is an operand {
    create a one-node tree using operand
    push that tree onto the stack
  }
  else { // symbol is operator
    pop tree T1 from the stack
    pop tree T2 from the stack
    Form a new tree whose root is the operator
      T1 is right child
      T2 is left child
    push new tree onto stack
```


Using Inheritance (1)

- Create an abstract base class to differentiate
- Base class and Leaf node:

```
class VarBinNode {
  public:
    virtual bool isLeaf() = 0;
};

class LeafNode : public VarBinNode {
private:
  Operand Var;
public:
  LeafNode(const Operand & val) {
    var = val;
  }
  bool isLeaf() { return true; }
  Operand value() { return var; }
};
```


Using Inheritance (2)

- Internal Node:

```
class IntlNode : public VarBinNode {
private:
  VarBinNode *left;
  VarBinNode *right;
  Operator opx;
public:
  IntlNode(const Operator& op,
           VarBinNode *l, VarBinNode *l) {
    opx = op;
    left = l;
    right = r;
  }
  bool isLeaf() { return false; }
  VarBinNode *leftChild() { return left; }
  VarBinNode *rightChild() { return right; }
  Operator value() { return opx; }
};
```

Using Inheritance (3)

- Composite implementation
- Base class and Leaf node:

```
class VarBinNode {
  public:
    virtual bool isLeaf() = 0;
    virtual void trav() = 0;
};

class LeafNode : public VarBinNode {
private:
  Operand Var;
public:
  LeafNode(const Operand & val) {
    var = val;
  }
  bool isLeaf() { return true; }
  Operand value() { return var; }
  void trav() {
    cout << "Leaf: " << value() << endl;
  }
};
```


Using Inheritance (4)

- Composite Implementation
- Internal Node:

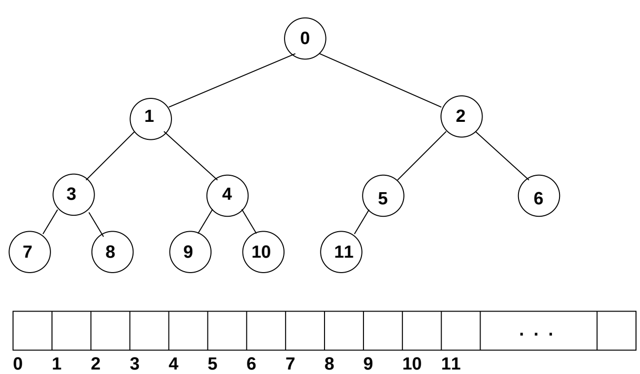
```
class IntlNode : public VarBinNode {
private:
  VarBinNode *left;
  VarBinNode *right;
  Operator opx;
public:
  IntlNode(const Operator& op,
           VarBinNode *l, VarBinNode *l) {
    opx = op;
    left = l;
    right = r;
  }
  bool isLeaf() { return false; }
  VarBinNode *leftChild() { return left; }
  VarBinNode *rightChild() { return right; }
  Operator value() { return opx; }
  void trav() {
    cout << "Internal: " << value() << endl;
    if (left() != NULL) left() -> trav();
    if (right() != NULL) right() -> trav();
  }

  void traverse(VarBinNode *root) {
    if (root != NULL) root -> trav();
  }
};
```


Space Overhead

- FBT Theorem:
  - ☐ (Roughly) half the pointers are NULL
  - ☐ If leaves store only data, then overhead depends whether the tree is full
  - ☐ Example: all nodes are the same, with two pointers to children
    - ∘ Overhead fraction: $o_f = o/t$
    - ∘ Total space: $t = n(2p + d)$
    - ∘ Overhead: $o = 2pn$
    - ∘ If $p = d$, then $o_f = 2p/(2p + d) = 2/3$
  - ☐ Eliminate pointers from leaf nodes:
    - ∘ Overhead
    $$o = \frac{n}{2}(2p)$$
    - ∘ Total space
    $$t = \frac{n}{2}(2p) + dn$$
    - ∘ Overhead fraction
    $$o_f = \frac{n/2(2p)}{n/2(2p) + dn} = \frac{p}{p + d}$$
    - ∘ This is 1/2 if $p = d$

Array Implementation

- Binary trees may be implemented with arrays
- Structure works best for complete binary trees



- Good example of logical vs. physical implementation
  - ☐ Complete binary tree is very limited
  - ☐ Space efficiency can be achieved

Array Implementation (cont.)

- Functions that may be necessary:
  - ☐ Parent(r) =
  - ☐ Leftchild(r) =
  - ☐ Rightchild(r) =
  - ☐ Leftsibling(r) =
  - ☐ Rightsibling(r) =

Binary Search Trees

BST property: all elements stored in the left subtree of a node whose value is K have values less than k. all elements stored in the right subtree of a node whose value is k have values $\geq$ k.

- The BST property allows elements to be ordered in a consistent manner
- Examples:
  - ☐ Insert 37, 24, 42, 7, 2, 40, 42, 32, 120
  - ☐ Insert 120, 42, 42, 7, 2, 32, 37, 24, 40

BST Node Class

- Code example uses friends and templates

```
template <class Comparable>
class BinarySearchTree;

template <class Comparable>
class BinaryNode
{
  Comparable element;
  BinaryNode *left;
  BinaryNode *right;

  BinaryNode(const Comparable &theElement,
             BinaryNode *lt, BinaryNode *rt)
    : element(theElement), left(lt), right(rt) { }
  friend class BinarySearchTree<Comparable>;
};
```

- Note: "Comparable" is used as a reminder
- No substantive difference from your basic binary node
- (See web site for book examples)

BST Operations

- Retrieving information:
  - ☐ `find`: return a pointer to that node
  - ☐ Options for failure of `find`:
    - throw an exception
    - return boolean as a reference or otherwise
    - return a special value
  - ☐ `findMin`: find the smallest element in a given (sub)tree
  - ☐ `findMax`: find the largest element in a given (sub)tree
- Operations that modify the tree
  - ☐ `insert`: insert according to BST property
  - ☐ `remove`: remove an element
  - ☐ `removeMin`: remove the smallest element (may be used by remove)
  - ☐ `removeAny`: remove the smallest element (used by Shaffer's dictionary ADT)
- See web site for Book's code examples

BST Insert and Remove

- Insert is relatively easy: preserve BST property
- Remove is the most difficult operation
- Three cases:
  - ☐ Remove a leaf
  - ☐ Remove a node that has one child
  - ☐ Remove a node that has two children
    - ∘ General strategy: replace node with smallest value of right subtree
- Examples:

Cost of BST Operations

- Find:
- Insert
- Remove

Heaps

- Sometimes, FIFO is not the best policy:
  - ☐ More important jobs may need to be processed first
  - ☐ Very long jobs may need to be processed last
  - ☐ Examples:
    - ∘ Print jobs
    - ∘ Multiuser OS process scheduling
- A **priority queue** can be used to satisfy this kind of environment.
  - ☐ A priority queue allows jobs to be ordered according to priority
  - ☐ Often, it supports only a small number of public operations:
    - ∘ `insert`: insert an element into the queue
    - ∘ `removeMin` removes the minimum value
    - ∘ Alternative: `removeMax` removes the maximum value

Heap Concept

A heap is a complete binary tree having the **heap property**

- Heap property:
  - ☐ In a min-heap, the value at a node is less than (or equal to) values at child nodes.
  - ☐ In a max-heap, the value at a node is greater than (or equal to) values at child nodes.
- Values in a heap are **partially ordered**
  - ☐ There is a relationship between the node and its children
  - ☐ There is **no** defined relationship between siblings (may be $>$ or $\geq$ or $<$ or $\leq$ in either direction)
- Heap representations are complete binary trees that (normally) use array-based implementations

Heap Operations

- Public operations that may be implemented:
  - ☐ `insert`: insert an element
  - ☐ `removeMax` or `removeMin`: remove the "first" element
  - ☐ `remove`: remove a specific element
  - ☐ `isEmpty`
  - ☐ `isFull`
  - ☐ `findMax` or `findMin`: find the "first" element
  - ☐ `buildHeap`: "heapify" the contents (may be a private operation)
  - ☐ `leftchild`, `rightchild`, `parent`, `isLeaf`: standard binary tree operations

Heap ADT

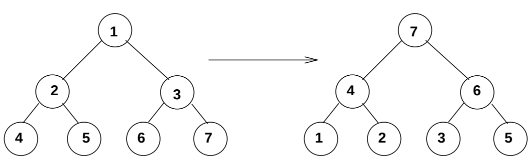
- Max heap that uses the binary tree array representation

```
template <Class Elem, class Comp>
class maxheap {
  private:
    Elem* heap;          // Pointer to heap array
    int size;            // Max heap size
    int n;               // Current no. elements stored
    void siftdown(int); // Put an element in its place

  public:
    maxheap(Elem*, int, int);
    int heapsize() const;
    bool isLeaf(int pos) const;
    int leftchild(int) const;
    int rightchild(int) const;
    int parent(int) const;
    void insert(const Elem&);
    bool removeMax(Elem&);
    bool remove(int, Elem&);
    void buildheap();
};
```

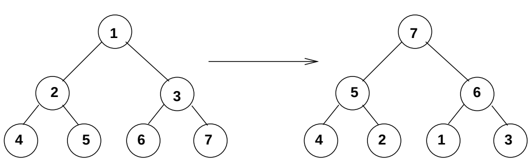

Building a Heap

What is the most efficient way to build a max heap?

- Example: exchange (4-2), (4-1), (2-1), (5-2), (5-4), (6-3), (6-5), (7-5), (7-6)



- Example: exchange (5-2), (7-3), (7-1), (6-1)

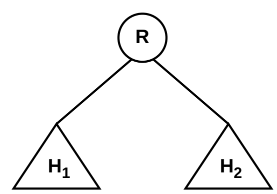


- It is undesirable to build a heap as you would a BST

How to Build a Max Heap

The process works in a fashion similar to an inductive proof.

- Given that H_1 and H_2 are already (max) heaps and R is element at root:



- Two possibilities exist:
  - ☐ $R \geq$ its two children: construction is complete
  - ☐ $R <$ one or both children: push R to its proper level as follows:
    - ◦ Exchange R with the greater-valued child
    - ◦ As long as R is out of place, descend through the tree with it until it reaches its proper place

Siftdown

- Siftdown accomplishes the "descend" process:

```
void maxheap::siftdown(int pos) {
  while (!isLeaf(pos)) {
    int newpos = leftchild(pos);
    int rc = rightchild(pos);
    if ((rc < n) && (heap[newpos] < heap[rc]))
      newpos = rc;
    if (heap[pos] < heap[newpos]) {
      swap(Heap,pos,newpos);
      pos = newpos;
    }
  }
}
```

- (See web site for templated version)
- Example(s):

Efficient Heap Build

For fast heap construction:

- Fill the array in input order
- Call `buildheap` procedure:
  - ☐ Works from high end of the array to low end
  - ☐ Calls `siftdown` for each item
  - ☐ Does not need to call `siftdown` for any leaf node.
- Example:
  - ☐ input file contains 42, 21, 33, 9, 12, 6, 7, 18, 72

Cost for Buildheap

- Given an unordered array, heap construction is very efficient:

$$\sum_{i=1}^{\log n} (i-1)n/2^i \approx n$$

- Idea:
  - ☐ Count the distance each element must go to reach final level
    - ∘ Only count downward moves
    - ∘ Once a node is processed, all nodes below it **must** be correct
  - ☐ Given a heap of height d, up to half the nodes are at depth $d \ldots$

Applications

- Selection Problem
- Event Simulation; ex: operation of a bank
  - ☐ Events:
    - ∘ customer arrival
    - ∘ customer departure
  - ☐ Simulation proceeds in "stages" based on events
  - ☐ Key idea is to advance the clock to next event at every stage:
    - ∘ When next customer in input file arrives
    - ∘ When a customer departs
  - ☐ Waiting line for customers is a queue
  - ☐ Waiting line for departures is a priority queue (heap)

Huffman Coding Trees

Using fixed length codes can waste space

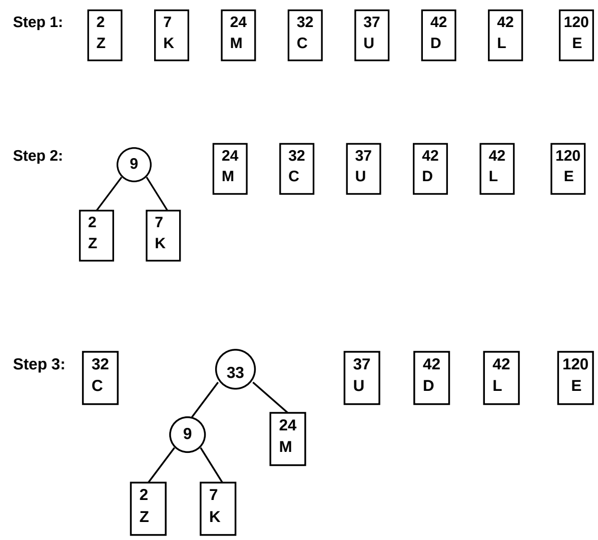
- Fixed length codes:
  - ☐ ASCII: 8 bits per character
  - ☐ Unicode: 16 bits per character
- Natural language does not have a uniform distribution of letters
  - ☐ Relative frequency of letters can be exploited
  - ☐ Variable length coding:

Z	K	F	C	U	D	L	E
2	7	24	32	37	42	42	120

  - ☐ Desire is to build the tree with **minimum external path weight**
    - ◦ **Weighted path length** of a leaf: weight of the leaf times its depth
    - ◦ The binary tree with minimum external path weight is the one with the minimum sum of weighted path lengths for a given set of leaves
    - ◦ Ex: a letter with high weight should have low depth to minimize its cost

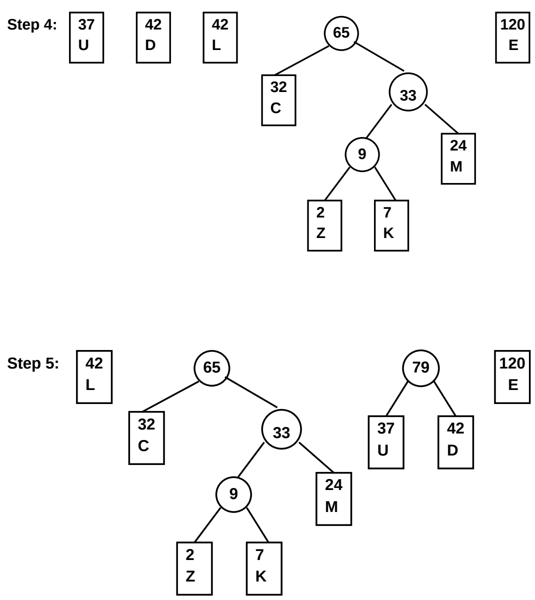
Huffman Tree Construction

- Create a list of nodes
  - ☐ Node contains letter/frequency pairs
  - ☐ Nodes are in increasing order of frequency
  - ☐ At each step, combine two smallest nodes into a binary tree and reorder as needed



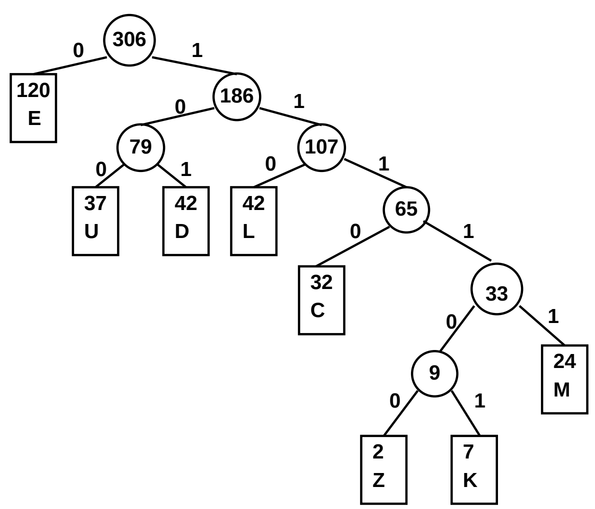
Huffman Tree Construction (cont.)

- Process continues until entire tree is built.



Assigning Codes

- Use the completed tree
  - ☐ Right branch assigns 1 bit
  - ☐ Left branch assigns 0 bit



Coding and Decoding

- A set of codes meets the **prefix property** if no code in the set is the prefix of another
- Examples:
  - ☐ Code for DEED:
  - ☐ Decode 1011001110111101
  - ☐ Expected cost per letter: