Graphs

A highly useful data structure for modeling of maps, networks, relationships, and so forth.

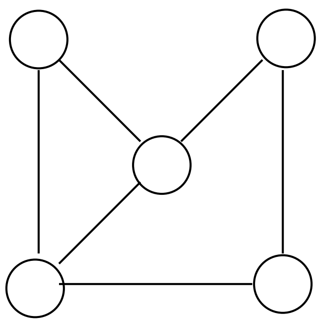
- Defined by two sets:
  - ☐ a set of nodes, also called vertices
  - ☐ a set of edges that are conections linking pairs of vertices
- Chapter topics:
  - ☐ Basic graph terminology
  - ☐ Graph implementations
  - ☐ Common graph traversal (search) algorithms
  - ☐ Common graph algorithms for shortest path
  - ☐ Spanning tree algorithms

Definitions

- A Graph **G** = (**V**,**E**) consists of a set of vertices **V** and a set of edges **E**, such that each edge in **E** is a connection between a pair of vertices in **V**.
  - ☐ The number of vertices is written $|\mathbf{V}|$ and the number of edges $|\mathbf{E}|$.
    - ◦ $|\mathbf{E}|$ may range from 0 up to $\Theta(|\mathbf{V}|^2)$.
    - ◦ A **sparse** graph is one with relatively few edges.
    - ◦ A **dense** graph is one with relatively many edges.
    - ◦ A **complete** graph is one with all possible edges.

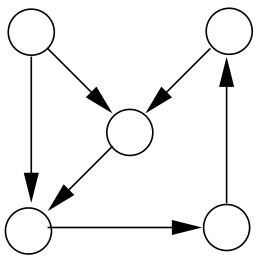
Definitions

- An **undirected graph** is a graph whose edges are not directed.
  - ☐ Example: an undirected graph

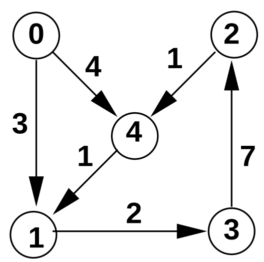


Definitions

- A **directed graph** or **digraph** is a graph whose edges are directed from one edge to another.
  - ☐ Example: a directed graph



  - ☐ Example: a labeled, weighted directed graph



Definitions

- **Adjacent**: two vertices joined by an edge. They are also called neighbors.

- **Incident**: an edge connecting vertices u and v, written as (u,v), is incident on u and v.

- **Path**: a path of length n-1 is formed by the sequence of vertices $v_1, v_2, \ldots, v_n$ if there exist edges from v_i to v_{i+1} for $1 \leq i < n$.
  - ☐ **Simple path**: all vertices on the path are distinct.
  - ☐ **Length of the path**: the number of edges it contains.
  - ☐ **Cycle**: path of length 3 or more connecting some vertex to itself.
  - ☐ **Simple cycle**: a cycle that is a simple path except for the first/last vertex.

Definitions

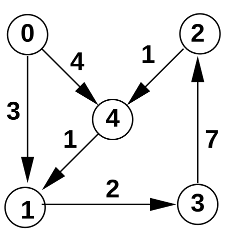
- Subgraph: a subgraph **S** = ($\mathbf{E}_s$,$\mathbf{V}_s$) is formed from graph **G** = (**V**,**E**) by selecting a subset $\mathbf{V}_s$ of **V** and a subset $\mathbf{E}_s$ of **E**.

- **Connected**: an undirected graph is connected if there is at least one path from any vertex to any other.

- **Acyclic**: a graph without cycles.
  - ☐ **Directed acyclic graph (DAG)**: a directed graph without cycles.
  - ☐ **Free tree**: a connected, undirected graph with no cycles.
  - ☐ **Free tree** (alternative): a connected, undirected graph with |**V**| - 1 edges.

Graph Representations

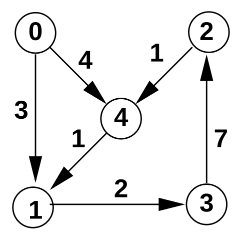
- Adjacency matrix:
  - ☐ If $|\mathbf{V}| = n$, then the matrix is an $n \times n$ array.
  - ☐ Rows are labeled 0 through $n-1$ corresponding to vertices v_0 to v_{n-1}.
  - ☐ Row i contains entries for vertex v_i.
  - ☐ The (i,j) entry represents whether there is an edge between v_i and v_j.
  - ☐ The (i,j) entry can be a single bit (1 for present, 0 for absent) or a weight (some number for 'present with weight x' or 0 for absent).
  - ☐ Space requirements: $\Theta(|\mathbf{V}^2|)$
  - ☐ Example: a directed graph

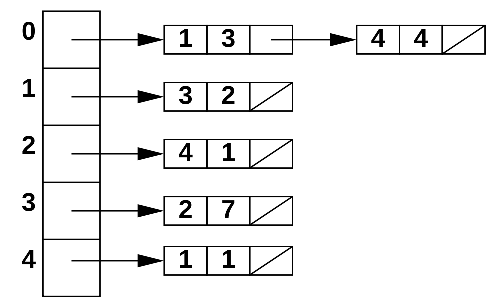


	0	1	2	3	4
0		3			4
1				2	
2					1
3			7		
4		1			

Graph Representations

- Adjacency list:
  - ☐ Represented by an array of linked lists.
  - ☐ If $|\mathbf{V}| = n$, then the array has n entries.
  - ☐ List i represents the list of vertices adjacent to v_i in a directed sense.
  - ☐ As with the matrix, an entry can be 0 or 1 for unweighted graphs or it can have another numeric value to represent a weight.
  - ☐ Space requirements: $\Theta(|\mathbf{V}| + |\mathbf{E}|)$
  - ☐ Example: a directed graph





  - ☐ In the linked-list node, the first field is the vertex label and second field a weight. The weight field is omitted if it is an unweighted graph.

Comparison of Representations

- Space efficiency: depends on the number of edges
  - ☐ Sparsely populated: adjacency list
  - ☐ Densely populated: adjacency matrix
- Time efficiency: often the adjacency list is better
  - ☐ Many algorithms require visiting of all neighbors...

Graph Implementations

- Graph abstract class

```
class Graph {
public:
  virtual int n() =0;
  virtual int e() =0;
  virtual int first(int) =0;
  virtual int next(int, int) =0;
  virtual void setEdge(int, int, int) =0;
  virtual void delEdge(int, int) =0;
  virtual int weight(int, int) =0;
  virtual int getMark(int) =0;
  virtual void setMark(int, int) =0;
};
```

The Edge Class

- Abstract class for graph edges

```
class Edge {
  int v1() =0;
  int v2() =0;
};
```


The Adjacency Matrix

- Adjacency Matrix Class Header:

```
class Graphm : public Graph {
private:
  int numVertex, numEdge;
  int **matrix;
  int *mark;
public:
  Graphm(int numVert) {
    int i, j;
    numVertex = numVert;
    numEdge = 0;
    mark = new int[numVert];
    for (i = 0; i<numVertex; i++)
      mark[i] = UNVISITED;
    matrix = (int**) new int*[numVertex];
    for (i = 0; i<numVertex; i++)
      matrix[i] = new int[numVertex];
    for (i = 0; i< numVertex; i++)
      for (int j = 0; j<numVertex; j++)
        matrix[i][j] = 0;
  }
  int first(int);
  int next(int, int);
  void setEdge(int, int, int);
  void delEdge(int, int);
  int weight(int, int);
  int getMark(int);
  void setMark(int, int);
}
```

The Adjacency Matrix

- Function Implementations

```
int first(int v) {
  int i;
  for (i = 0; i<numVertex; i++)
    if (matrix[v][i] != 0) return i;
  return i;
}

int next(int v1, int v2) {
  int i;
  for(i = v2+1; i<numVertex; i++)
    if (matrix[v1][i] != 0) return i;
  return i;
}
```


The Adjacency Matrix

- Function Implementations

```
void setEdge(int v1, int v2, int wgt) {
  Assert(wgt > 0, "Illegal weight value");
  if (matrix[v1][v2] == 0) numEdge++;
  matrix[v1][v2] = wgt;
}

void delEdge(int v1, int v2) {
  if (matrix[v1][v2] != 0) numEdge--;
  matrix[v1][v2] = 0;
}

int weight(int v1, int v2) {
  return matrix[v1][v2];
}

int getMark(int v) {
  return mark[v];
}

void setMark(int v, int val) {
  mark[v] = val;
}
```

The Adjacency List

- Adjacency List Class Header:

```
class Graphl : public Graph {
private:
  int numVertex, numEdge;
  List<Edge>** vertex;
  int *mark;
public:
  Graphl(int numVert) {
    int i, j;
    numVertex = numVert;  numEdge = 0;
    mark = new int[numVert];
    for (i = 0; i<numVertex; i++) mark[i] = UNVISITED;
    vertex = (List<Edge>**) new List<Edge>*[numVertex];
    for (i = 0; i<numVertex; i++)
      vertex[i] = new LList<Edge>();
  }

  int n();
  int e();
  int first(int);
  int next(int, int);
  void setEdge(int, int, int);
  void delEdge(int, int);
  int weight(int, int);
  int getMark(int);
  void setMark(int, int);
};
```


The Adjacency List

- Function Implementations:

```
int first(int v) {
  Edge it;
  vertex[v] -> setStart();
  if (vertex[v] -> getValue(it)) return it.vertex;
  else return numVertex;
}

int next(int v1, int v2) {
  Edge it;
  vertex[v1] -> getValue(it);
  if (it.vertex == v2) vertex[v1] -> next();
  else {
    vertex[v1] -> setStart();
    while (vertex[v1] -> getValue(it)
           && (it.vertex <= v2))
      vertex[v1] -> next();
  }
  if (vertex[v1] -> getValue(it)) return it.vertex;
  else return numVertex;
}
```

The Adjacency List

- Function Implementations:

```
void setEdge(int v1, int v2, int wgt) {
  Assert(wgt>0, "Illegal weight value");
  Edge it(v2, wgt);
  Edge curr;
  vertex[v1] -> getValue(curr);
  if (curr.vertex != v2)
    for (vertex[v1] -> setStart();
         vertex[v1] -> getValue(curr);
         vertex[v1] -> next())
      if (curr.vertex >= v2) break;
  if (curr.vertex == v2)
    vertex[v1] -> remove(curr);
  else numEdge++;
  vertex[v1] -> insert(it);
}

void delEdge(int v1, int v2) {
  Edge curr;
  vertex[v1] -> getValue(curr);
  if (curr.vertex != v2)
    for (vertex[v1] -> setStart();
         vertex[v1] -> getValue(curr);
         vertex[v1] -> next())
      if (curr.vertex >= v2) break;
  if (curr.vertex == v2) {
    vertex[v1] -> remove(curr);
    numEdge--;
  }
}
```


The Adjacency List

- Function Implementations:

```
int weight(int v1, int v2) {
  Edge curr;
  vertex[v1] -> getValue(curr);
  if (curr.vertex != v2)
    for (vertex[v1] -> setStart();
         vertex[v1] -> getValue(curr);
         vertex[v1] -> next())
      if (curr.vertex >= v2) break;
  if (curr.vertex == v2)
    return curr.weight;
  else
    return 0;
}

int getMark(int v) {
  return mark[v];
}
void setMark(int v, int val) {
  mark[v] = val;
}
```

Graph Traversals

It is often useful to visit the vertices in some specific order.

- Generic Traversal Function

```
void graphTraverse(const Graph* G) {
  for (v = 0; v < G -> n(); v++)
    G -> setMark(v, UNVISITED);
  for (v = 0; v < G -> n(); v++)
    if (G -> getMark(v) == UNVISITED)
      doTraverse(G,v);
```

- The `doTraverse(G,v)` function could be one of
  - ☐ Depth-first search
    - For a given vertex, recursively visit all neighbors.
    - Effect is to follow a branch through the graph to its conclusion.
  - ☐ Breadth-first search
    - For a given vertex, examine all neighbors before visiting vertices further away.
    - Effect is to visit "one hop away", "two hops away", ...
  - ☐ Topological sort
    - Laying out vertices of a DAG in a linear order (according to prerequisite relationships).

Graph Traversals

- Depth-First Search

```
void DFS(Graph* G, int v) {
  PreVisit(G, v);
  G -> setMark(v, VISITED);
  for (int w = G -> first(v);
       w < G -> n();
       w = G -> next(v,w))
    if (G -> getMark(w) == UNVISITED)
      DFS(G, w);
  PostVisit(G, v);
}
```

Graph Traversals

- Breadth-First Search

```
void BFS(Graph* G, int start, Queue<int>* Q) {
  int v, w;
  Q -> enqueue(start);
  G -> setMark(start, VISITED);
  while (Q->length() != 0) {
    Q->dequeue(v);
    PreVisit(G, v);
    for (w = G -> first(v);
         w < G -> n();
         w = G -> next(v,w))
      if (G -> getMark(w) == UNVISITED) {
        G -> setMark(w, VISITED);
        Q -> enqueue(w);
      }
    PostVisit(G, v);
  }
}
```


Graph Traversals

- Recursive Topological Sort

```
// Public function
void topsort(Graph* G) {
  int i;
  for (i = 0; i < G -> n(); i++)
    G -> setMark(i, UNVISITED);
  for (i = 0; i < G -> n(); i++)
    if (G -> getMark(i) == UNVISITED)
      tophelp(G, i);
}

// Private function
void tophelp(Graph* G, int v) {
  G -> setMark(v, VISITED);
  for (int w = G -> first(v);
       w < G -> n();
       w = G -> next(v,w))
    if (G -> getMark(w) == UNVISITED)
      tophelp(G, w);
  printout(v);
}
```

Graph Traversals

- Queue-Based Topological Sort

```
void topsort(Graph* G, Queue<int>* Q) {
  int Count[G -> n()];
  int v, w;
  for (v = 0; v < G -> n(); v++) Count[v] = 0;
  for (v = 0; v < G -> n(); v++)
    for (w = G -> first(v);
         w < G -> n();
         w = G -> next(v,w))
       Count[w]++;
  for (v = 0; v < G -> n(); v++)
    if (Count[v] == 0)
      Q -> enqueue(v);
  while (Q -> length() != 0) {
    Q -> dequeue(v);
    printout(v);
    for (w = G -> first(v);
         w < G -> n();
         w = G -> next(v,w)) {
      Count[w]--;
      if (Count[w] == 0)
        Q -> enqueue(w);
    }
  }
}
```


Shortest-Paths Problems

Sometimes it is useful to use a graph to find the shortest path from point A to B.

- Edges are labeled with real numbers representing weights, costs, distances, delay, etc.
- Goal is to find the smallest weighted path.
- Single-source shortest-paths problem:
  - ☐ Given a vertex *s* in graph **G**, find a shortest path from *s* to every other vertex in **G**.
- Approach 1:
  - ☐ Add vertices to a list **S** in order of distance from the source.
  - ☐ Given a vertex v_i not yet in **S**:
    - ∘ $d(s, v_i) = min_{u \in \mathbf{S}}(d(s, u) + w(u, v_i))$
    - ∘ Means: find the minimum combination of "short path from s to a vertex already in **S** plus a weight coming from a vertex in **S** to the new vertex x."

Single-Source Shortest-Paths Problem

- Dijkstra's algorithm:

```
void Dijkstra(Graph* G, int* D, int s) {
  int i, v, w;
  for (i = 0; i < G -> n(); i++) {
    v = minVertex(G, D);
    if (D[v] == INFINITY) return;
    G -> setMark(v, VISITED);
    for (w = G -> first(v);
         w < G -> n();
         w = G -> next(v,w))
      if (D[w] > (D[v] + G -> weight(v, w)))
        D[w] = D[v] + G -> weight(v, w);
  }
}

int minVertex(Graph* G, int* D) {
  int i, v;
  for (i = 0; i < G -> n(); i++)
    if (G -> getMark(i) == UNVISITED) {
      v = i;
      break;
    }
  for (i++; i < G -> n(); i++)
    if ((G -> getMark(i) == UNVISITED)
        && (D[i] < D[v]))
      v = i;
  return v;
}
```


Shortest-Paths Problems

- All-Pairs shortest-paths problem:
  - Find the shortest distance between all pairs of vertices in the graph.
  - That is, for every $u, v \in \mathbf{V}$, calculate $d(u, v)$
- Try 1: run Dijkstra's algorithm $|\mathbf{V}|$ times
  - Works well if the graph is sparse, but not if it is dense.
- Try 2:
  - Uses concept of **k-path**: any intermediate vertex on a path between vertices u and v must be labeled less than k.
  - Direct edge between u and v is a 0-path
  - $D_k(v, u)$ is the length of the shortest k-path from v to u.
  - If that shortest k-path is already known, then
    - The $(k+1)$-path goes through vertex k: the best path is the best k-path from v to k followed by the best k-path from k to u.
    - The $(k+1)$-path does not go through vertex k: keep the best k-path seen before.

All-Pairs Shortest-Paths Problem

- Floyd's Algorithm:

```
void Floyd(Graph* G) {
  int D[G -> n()][G -> n()];
  for (int i = 0; i < G -> n(); i++)
    for (int j = 0; j < G -> n(); j++)
      D[i][j] = G -> weight(i, j);
  for (int k = 0; k < G -> n(); k++)
    for (int i = 0; i < G -> n(); i++)
      for (int j = 0; j < G -> n(); j++)
        if (D[i][j] > (D[i][k] + D[k][j]))
          D[i][j] = D[i][k] + D[k][j];
}
```


Minimum-Cost Spanning Trees

- A **minimum-cost spanning tree** (**MST**) of **G** contains the vertices of **G** and a subset of its edges.

- Properties:

  1. has minimum total cost measured by summing values for all of the edges in the subset.

  2. keeps the vertices connected.

- Applications:

  - ☐ find the shortest set of wires connecting circuit components

  - ☐ Connecting a set of phones to use the least amount of wire

Prim's Algorithm

- Start with any vertex u
  - ☐ Pick the least-cost edge connected to u that doesn't create a cycle; assume that edge is (u, v).
  - ☐ Add vertex v and edge (u, v) to the graph
  - ☐ Repeat this until all vertices of the graph have been added.
- Finding a minimum-cost vertex:

```
int minVertex(Graph* G, int* D) {
  int i, v;  // Initialize v to any unvisited vertex;
  for (i = 0; i < G -> n(); i++)
    if (G -> getMark(i) == UNVISITED) {
      v = i;
      break;
    }
  for (i = 0; i < G -> n(); i++)
    if ((G -> getMark(i) == UNVISITED) && (D[i] < D[v]))
      v = i;
  return v;
}
```


Prim's Algorithm

- The algorithm:

```
void Prim(Graph* G, int* D, int s) {
  int V[G -> n()];
  int i, w;
  for (i = 0; i < G -> n(); i++) {
    int v = minVertex(G, D);
    G -> setMark(v, VISITED);
    if (v != s)
      AddEdgetoMST(V[v], v);
    if (D[v] == INFINITY)
      return;
    for (w=G -> first(v);
         w < G -> n();
         w = G -> next(v,w))
      if (D[w] > G -> weight(v,w)) {
        D[w] = G -> weight(v,w);
        V[w] = v;
      }
  }
}
```

Kruskal's Algorithm

- Partition the set of vertices into |**V**| equivalence classes
- Process edges in order of weight
  - ☐ An edge is added to MST (and two equivalence classes combined) if it connects two vertices in different equivalence classes.
  - ☐ Repeat until only one equivalence class exists.
  - ☐ Store edges in a min heap to process in order of weight.

Kruskal's Algorithm

- The algorithm:

```
void Kruskel(Graph* G) {
  Gentree A(G -> n());
  KruskElem E[G -> e()];
  int i;
  int edgecnt = 0;
  for (i = 0; i < G -> n(); i++)
    for (int w = G -> first(i);
         w < G -> n();
         w = G -> next(i,w)) {
      E[edgecnt].distance = G -> weight(i, w);
      E[edgecnt].from = i;
      E[edgecnt++].to = w;
    }

  minheap H(E, edgecnt, edgecnt);
  int numMST = G -> n();
  for (i = 0; numMST > 1; i++) {
    KruskElem temp;
    H.removemin(temp);
    int v = temp.from;
    int u = temp.to;
    if (A.differ(v, u)) {
      A.UNION(v, u);
      AddEdgetoMST(temp.from, temp.to);
      numMST--;
    }
  }
}
```