Analysis of the pure logic of necessitation and its extensions

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(will be displayed again at the end)

Table of contents

What is The Pure Logic of Necessitation $\mathbf{N}$?

N, the pure logic of necessitation

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It is a non-normal modal logic

- without congruence! ($\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$)
- so it doesn't have a neighborhood semantics
- instead, it has a Kripke-like semantics

The rationale of N (1)

Fitting et al. (1992) read $\Box\varphi$ in $\mathbf{N}$ as "φ is already derived"

- We cannot say ψ is *already* derived even if φ and $\varphi \to \psi$ have been derived!
- This justifies the lack of the K axiom: $\Box\varphi \land \Box(\varphi \to \psi) \to \Box\psi$
- They used $\mathbf{N}$ to analyze non-monotonic reasoning

The rationale of N (2)

Kurahashi (2024) considered $\Box$ in $\mathbf{N}$ the simplest notion of provability, in terms of provability logic

- The most fundamental property of provability should be: "if something is proved, then it is provable"
- This justifies the presence of the necessitation rule: $\frac{\varphi}{\Box\varphi}$
- He identified that $\mathbf{N}$ is exactly the provability logic of all provability predicates

The Kripke-like semantics for $\mathbf{N}$

Without the K axiom, distinct $\Box$-formulas are hardly related

➡ The truth of $\Box\varphi$ must rely on its own accessibility relation

Definition (Fitting et al. (1992))

- Let $\mathscr{L}_\Box$ be the set of all modal formulas $(\bot, \wedge, \vee, \rightarrow, \Box)$
- An $\mathbf{N}$-*frame* consists of the set of worlds W, and an accessibility relation $\prec_\varphi$ over W, <u>for each $\varphi \in \mathscr{L}_\Box$</u>
- An $\mathbf{N}$-*model* consists of an $\mathbf{N}$-frame and a valuation $\Vdash$, where the truth of $\Box\varphi$ is determined <u>only by $\prec_\varphi$</u>:

$$w \Vdash \Box\varphi \ :\Longleftrightarrow\ \forall w' \in W\, (w \prec_\varphi w' \Rightarrow w' \Vdash \varphi)$$

Almost the same as Kripke semantics, with a twist on accessibility

Basic properties of $\mathbf{N}$

Theorem (Fitting et al. (1992))

$\mathbf{N}$ has the finite frame property (FFP) w.r.t. all $\mathbf{N}$-frames

Proof.

Routine, by constructing a finite model of $\mathbf{N}$. $\square$

Proposition

$\mathbf{N}$ is not locally tabular

Proof.

We have an infinite sequence of provably distinct formulas:

$$\Box p,\ \Box\neg\neg p,\ \Box\neg^4 p,\ \Box\neg^6 p,\ \ldots$$

$\square$

Extending N with an Axiom

$\Box^n\varphi \to \Box^m\varphi$

Several extensions of $\mathbf{N}$

Kurahashi considered several extensions that have a direct application in provability logic:

Theorem (Kurahashi (2024))

- $\mathbf{N4} := \mathbf{N} + \Box\varphi \to \Box\Box\varphi$ has FFP w.r.t. transitive $\mathbf{N}$-frames:
$$x \prec_{\Box\varphi} y \,\&\, y \prec_{\varphi} z \implies x \prec_{\varphi} z$$
- $\mathbf{NR} := \mathbf{N} + \frac{\neg\varphi}{\neg\Box\varphi}$ has FPP w.r.t. serial $\mathbf{N}$-frames:
$$\exists y \, (x \prec_{\varphi} y)$$

Like these, we can think of various $\mathbf{N}$ counterparts of normal modal logics, with similar frame conditions!

$\mathrm{Acc}_{m,n}$, the generalized transitivity axiom

Definition

- $x \prec_\varphi^k y$: "x can see y in k steps w.r.t. φ"

$$x \prec_{\Box^{k-1}\varphi} w_{k-1} \prec_{\Box^{k-2}\varphi} w_{k-2} \cdots w_2 \prec_{\Box\varphi} w_1 \prec_\varphi y$$

- (m,n)-accessibility: $x \prec_\varphi^m y \implies x \prec_\varphi^n y$
- $\mathrm{Acc}_{m,n} \coloneqq \Box^n\varphi \to \Box^m\varphi$

Here, transitivity is just $(2,1)$-accessibility, and the axiom $\Box\varphi \to \Box\Box\varphi$ is exactly $\mathrm{Acc}_{2,1}$. Now one may wonder:

Problem

Does $\mathbf{N} + \mathrm{Acc}_{m,n}$ have FFP w.r.t. (m,n)-accessible $\mathbf{N}$-frames?

Incompleteness of $\mathbf{N} + \mathbf{Acc}_{m,n}$

It turns out $\mathbf{N} + \mathrm{Acc}_{m,n}$ is not complete for some $m, n \in \mathbb{N}$:

Proposition

For $n \geq 2$, (1) $\neg\Box^{n+1}\bot$ is valid in all $(0, n)$-accessible $\mathbf{N}$-frames, but (2) $\mathbf{N} + \mathrm{Acc}_{0,n} \nvdash \neg\Box^{n+1}\bot$

Proof.

(1) Easy. (2) One can actually construct an $\mathbf{N}$-model where $\mathrm{Acc}_{0,n}$ is valid but $\neg\Box^{n+1}\bot$ is not. $\Box$

$\mathbf{N}$-models allow more subtle construction of countermodels as the accessibility relation $\prec_\varphi$ can be tweaked for each φ!

An additional rule to the rescue

Here, $\neg\Box^n\bot$ is provable in $\mathbf{N} + \mathrm{Acc}_{0,n}$ but $\neg\Box^{n+1}\bot$ is not

➡ adding the following rule would recover completeness:

$$\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$$

Proposition

This rule is admissible in every normal modal logic

Corollary

$\mathbf{N} + \mathrm{Acc}_{m,n} \;\subseteq\; \mathbf{N} + \frac{\neg\Box\varphi}{\neg\Box\Box\varphi} + \mathrm{Acc}_{m,n} \;\subseteq\; \mathbf{K} + \mathrm{Acc}_{m,n}$

The finite frame property of $\mathrm{N} + \frac{\neg\Box\varphi}{\neg\Box\Box\varphi} + \mathbf{Acc}_{m,n}$

Definition

$\mathbf{NA}_{m,n} := \mathbf{N} + \mathrm{Acc}_{m,n}$, and $\mathbf{N}^{+}\mathbf{A}_{m,n} := \mathbf{N} + \frac{\neg\Box\varphi}{\neg\Box\Box\varphi} + \mathrm{Acc}_{m,n}$

Theorem (K. & S.)

$\mathbf{N}^{+}\mathbf{A}_{m,n}$ has FFP w.r.t. (m, n)-accessible $\mathbf{N}$-frames

Proof.

We carefully construct a finite (m, n)-accessible countermodel for a non-theorem of $\mathbf{N}^{+}\mathbf{A}_{m,n}$. We note that the presence of $\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$ indeed contributes to the construction. $\Box$

Interpolation properties in $\mathrm{NA}_{m,n}$ and $\mathrm{N}^{+}\mathrm{A}_{m,n}$ (1)

The rule $\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$ seems to be only relevant when we consider the completeness theory w.r.t. the Kripke-like semantics.

The interpolation theorems hold with or without the rule:

Proposition

Both $\mathbf{NA}_{m,n}$ and $\mathbf{N}^{+}\mathbf{A}_{m,n}$ have cut-admissible sequent calculi

Corollary

Both $\mathbf{NA}_{m,n}$ and $\mathbf{N}^{+}\mathbf{A}_{m,n}$ enjoy CIP and LIP

Proof.

Just Use Maehara's Method™ □

Interpolation properties in $\mathrm{NA}_{m,n}$ and $\mathrm{N}^{+}\mathrm{A}_{m,n}$ (2)

We obtained an even stronger result:

Theorem

Both $\mathbf{NA}_{m,n}$ and $\mathbf{N}^{+}\mathbf{A}_{m,n}$ enjoy ULIP

Proof.

We embed both logics to the classical propositional logic $\mathbf{Cl}$, and reduce the problem to ULIP of $\mathbf{Cl}$, which is known. $\square$

Here, ULIP (uniform Lyndon —) is a strengthening of both UIP and LIP. See Kurahashi (2020) for details.

Bonus: a general method for proving ULIP

We also developed a general method for proving ULIP:

Theorem (S.)

For any logics $L \subseteq M$, if there is an embedding of M into L with certain properties, and L has ULIP, then so does M

Example

By the double negation embedding, ULIP of the intuitionistic propositional logic $\mathbf{Int}$ implies ULIP of $\mathbf{Cl}$.

No deep dive today. See Sato (2025) for details!

The Showdown (vs. $\mathbf{K} + \Box^n\varphi \to \Box^m\varphi$)

The showdown

Recall that:

$$\mathbf{NA}_{m,n} \ \subseteq \ \mathbf{N}^{+}\mathbf{A}_{m,n} \ \subseteq \ \mathbf{K} + \mathrm{Acc}_{m,n}$$

We shall compare the following properties of the above logics, which would highlight intriguing differences between them:

- Completeness
- The finite frame property
- The interpolation properties (CIP, LIP, UIP, ULIP)

Completeness: the hidden gems?

There is a classic result by Lemmon & Scott that $\mathbf{K} + \mathrm{Acc}_{m,n}$ is complete for every $m, n \in \mathbb{N}$

So it is interesting that $\mathbf{NA}_{m,n}$ is incomplete for some cases, and needs an extra rule $\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$ to fix it

- This rule is admissible in most logics, but seems to be very important for any logic with the necessitation rule

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Open Problem

Is there any other rule that is admissible in normal modal logics, but is essential for completeness of some logic extending $\mathbf{N}$?

The finite frame property: why so hard?

FFP of $\mathbf{K} + \mathrm{Acc}_{m,n}$ has been left unsolved* for decades, especially when $m < n$. Zakharyaschev (1997) referred to it as "one of the major challenges in completeness theory"

On the other hand, FFP of $\mathbf{N}^{+}\mathbf{A}_{m,n}$ is obtained, although not easily, by a direct construction of a finite countermodel!

*the cases when $m \geq 0$, $n = 1$ are solved by Gabbay (1972)

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Open Problem

Why is FFP of $\mathbf{K} + \mathrm{Acc}_{m,n}$ so hard to prove? Is there some logic between $\mathbf{N}^{+}\mathbf{A}_{m,n}$ and $\mathbf{K} + \mathrm{Acc}_{m,n}$ with the same difficulty?

*the cases when $m \geq 0$, $n = 1$ are solved by Gabbay (1972)

Interpolation properties: the K axiom to blame?

It is known that $\mathbf{K} + \mathrm{Acc}_{m,n}$ does not, in general, enjoy all of CIP, LIP, UIP, and ULIP:

- Bílková (2007) proved that $\mathbf{K4} = \mathbf{K} + \mathrm{Acc}_{2,1}$ lacks UIP
- Marx (1995) proved that $\mathbf{K} + \mathrm{Acc}_{1,2}$ lacks even CIP

However, for any m, n, $\mathbf{NA}_{m,n}$ and $\mathbf{N}^{+}\mathbf{A}_{m,n}$ enjoy all of them!

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Open Problem

To what extent the presence of the K axiom is *harmful* for a logic in terms of interpolation properties?

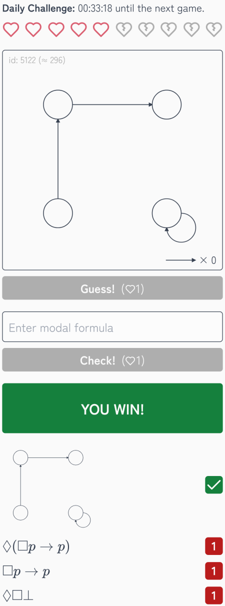
- Is there a logic between $\mathbf{N4}$ and $\mathbf{K4}$ that lacks UIP?
- Is there a logic between $\mathbf{N} + \mathrm{Acc}_{1,2}$ and $\mathbf{K} + \mathrm{Acc}_{1,2}$ that lacks CIP?

Thanks!

That's all!

The slides are available online, with the links to our papers:

Bonus: the Kripke game!



I made a Wordle-like game where you guess the shape of a Kripke frame, just with formulas. Give it a try!

cannorin.net/kripke

Appendix & References

The propositionalization method (1/2)

Propositionalization is a method that can be used to reduce ULIP of logic to that of a weaker one. It proceeds like this:

Given a logic X, let $\mathscr{L}_X$ designate the language of X.

Consider logics L and M s.t. $\mathscr{L}_L \subseteq \mathscr{L}_M$ and $L \subseteq M$.

Definition

Let L' be the same logic as L, but its propositional variables extended by adding a fresh one p_φ for every $\varphi \in \mathscr{L}_M$.

Definition

Let $\sigma : \mathscr{L}'_L \to \mathscr{L}_M$ be the substitution that replaces every p_φ with φ. It is easy to see that $L' \vdash \rho$ implies $M \vdash \sigma(\rho)$ for any $\rho \in \mathscr{L}'_L$.

The propositionalization method (2/2)

Definition

A pair of translations $\sharp, \flat : \mathscr{L}_M \to \mathscr{L}'_L$ is called a propositionalization of M into L if the following are met:

1. $M \vdash \varphi \to \psi$ implies $L' \vdash \varphi^\flat \to \psi^\sharp$;
2. $M \vdash \sigma(\varphi^\sharp) \to \varphi$ and $M \vdash \varphi \to \sigma(\varphi^\flat)$;
3. For $(\bullet, \circ) \in \{(+,-),(-,+)\}$ and $\natural \in \{\sharp, \flat\}$,
   $p \in v^\bullet(\varphi^\natural)$ implies $p \in v^\bullet(\varphi)$, and $p_\psi \in v^\bullet(\varphi^\natural)$ implies both $v^\bullet(\psi) \subseteq v^\bullet(\varphi)$ and $v^\circ(\psi) \subseteq v^\circ(\varphi)$.

Theorem (S.)

If L has ULIP, and there is a propositionalization of M into L, then M also has ULIP.

References

This talk is based on the papers indicated by $\star$.

- Melvin C. Fitting, V. Wiktor Marek, and Miroslaw Truszczyyński. The pure logic of necessitation. *Journal of Logic and Computation*, 2(3):349-373, 1992.
- Taishi Kurahashi. The provability logic of all provability predicates. *Journal of Logic and Computation*, 34(6):1108-1135, 2024.
- $\star$ Taishi Kurahashi and Yuta Sato. The Finite Frame Property of Some Extensions of the Pure Logic of Necessitation. *Studia Logica*, to appear.
- Taishi Kurahashi. Uniform Lyndon interpolation property in propositional modal logics. *Archive for Mathematical Logic*, 59(5-6):659-678, 2020.
- $\star$ Yuta Sato. Uniform Lyndon interpolation for the pure logic of necessitation with a modal reduction principle. Submitted. arXiv:2503.10176.

References

- Hitoshi Omori and Daniel Skurt. On Ivlev's Semantics for Modality. In *Many-valued Semantics and Modal Logics: Essays in Honour of Yuriy Vasilievich Ivlev*, 485:243-275, 2024.
- Michael Zakharyaschev. Canonical formulas for K4. III: The finite model property. *The Journal of Symbolic Logic*, 62(3):950-975, 1997.
- Dov M. Gabbay. A General Filtration Method for Modal Logics. *Journal of Philosophical Logic*, 1(1):29-34, 1972.
- Marta Bílková. Uniform Interpolation and Propositional Quanti fi ers in Modal Logics. *Studia Logica*, 85(1):1-31, 2007.
- Maarten Marx. Algebraic Relativization and Arrow Logic. ILLC Dissertation Series. Institute for Logic, Language and Computation, 1995.

Omori and Skurt rediscovered the same logic as $\mathbf{N}$, namely $\mathbf{M}^{+}$ in their paper. They also gave a non-deterministic many-valued semantics for $\mathbf{N}$.