The Journal of the Astronautical Sciences, Vol. 54, Nos. 3 & 4, July–December 2006, pp. 415–431

Estimation Using Quaternion Probability Densities on the Unit Hypersphere[1]

Mark L. Psiaki[2]

Abstract

Techniques have been developed for carrying out manipulations of quaternion probability densities that are defined only on the unit hypersphere. The goal is to generalize standard probabilistic concepts so that Monte-Carlo-type methods can be used with quaternion attitude representations. The optimal quaternion estimate is not the simple expectation value. Instead, it is the unit-normalized minimum mean-squared error estimate, with squared error defined as the square of the sine of half of the total attitude error. This definition causes the optimal quaternion estimate to equal the normalized eigenvector that is associated with the largest eigenvalue of the second moment of the quaternion distribution. The other three eigenvectors and eigenvalues of the second moment model the covariance of the first three components of the multiplicative error quaternion. Thus, the quaternion second moment contains both the optimal quaternion estimate and its error covariance. This estimation technique is used to solve a simulated batch GPS attitude determination problem using a particle computation. Accuracies commensurate with traditional methods can be achieved by performing a Monte-Carlo calculation of the quaternion second moment. An additional contribution is a new quaternion probability density function whose formula is defined solely in terms of its second moment.

Introduction

Quaternion-based attitude estimation algorithms are useful because a unit-normalized quaternion provides a nonsingular attitude representation of minimal dimension. The four-element quaternion is often preferred over three Euler angles because they have a singularity and over the direction cosines matrix because it contains nine elements. The usefulness of quaternions is attested by the many quaternion-based attitude estimation methods that have been developed [1–6].

Two relatively new and powerful concepts in estimation are the sigma-points filter [7, 8] and the particle filter [9]. Both of these filters work with weighted

[1]Presented as paper AAS 05-461 at the AAS Malcolm D. Shuster Astronautics Symposium, State University of New York, Buffalo, New York, June 12–15, 2005.
[2]Professor, Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, N.Y. 14853-7501. E-mail: mlp4@cornell.edu. Tel: (607) 255-9100.

samples from *a priori* and a *posteriori* state probability distributions. Sigma-points filters, which are also known as unscented Kalman filters, make assumptions about the underlying distributions, and they use a number of samples that scales linearly with the dimension of the state space. Particle filters use Monte Carlo techniques. They make few, if any, assumptions about underlying distributions, which makes them potentially more accurate, but they can require many more samples than a sigma-points filter in order to function well. The advantage of using either of these new filters is that they both tend to deal with nonlinearities better than does an extended Kalman filter.

Attitude determination problems are fundamentally nonlinear, which makes this class of problems a natural application for a sigma-points filter or for a particle filter. Reference [6] applies a sigma-points filter to an attitude determination problem, and reference [10] applies a particle filter to do attitude determination. Both of these papers report improved results in comparison to an extended Kalman filter.

The quaternion's unit normalization constraint poses a challenge to the use of sample-based filters for attitude determination: The expectation value of a quaternion usually has a norm that is less than unity. This situation is illustrated in Fig. 1, which shows a two-dimensional projection of the unit hypersphere, a quaternion probability density on the hypersphere, and the resulting expected value of the quaternion, whose length is less than unity. For this reason, the expectation operation cannot be used to compute the quaternion estimate from the samples. Reference [6] deals with this difficulty by computing expectations using only the first three elements of a multiplicative correction to a nominal quaternion. This amounts to working in a local tangent space of the unit hypersphere. Reference [10] computes its quaternion estimate by minimizing the expected value of the Frobenius norm of the difference between the true direction cosines matrix and the direction cosines matrix associated with the quaternion estimate.

The goal of the present paper is to develop sensible quaternion probability calculations that work on the unit hypersphere. These will enable the use of

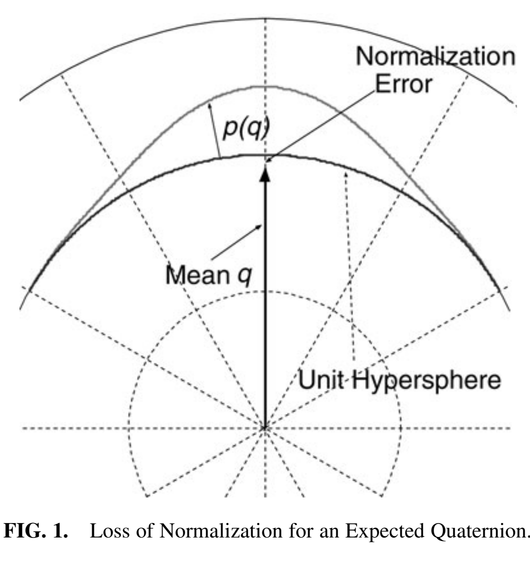

FIG. 1. Loss of Normalization for an Expected Quaternion.

sample-based estimation algorithms for attitude determination problems that use the quaternion representation. These calculations must avoid the restriction of working in the tangent space, as in reference [6], because it is not possible to represent all interesting distributions in that space. For example, a single tangent space cannot represent a total lack of attitude information because the corresponding probability density is equal at all points on the unit hypersphere, some of which do not map into the tangent space. Another reason for avoiding the restriction to the tangent space is that it seems unnecessarily limiting to presume that estimation algorithms cannot work directly with probability distributions that are defined on nonlinear manifolds. The new calculations seek to improve on the method of reference [10] by not resorting to direction cosines computations in order to compute a quaternion estimate and by enabling the computation of the quaternion estimation error covariance.

Other research related to the current effort includes a paper on uniform attitude distributions [11] and a paper that uses the direction cosines matrix to parameterize the attitude [12]. The former paper develops expressions for the probability density functions (PDFs) of uniform attitude distributions for various common representations, and it develops methods for sampling these distributions. Such distributions model the absence of attitude information. The latter paper uses ideas similar to those of reference [10] to develop an optimal estimate of the direction cosines matrix as a function of its probability distribution subject to its orthonormality and determinant constraints. It uses this idea and an approximation of the Fokker-Planck equation in order to develop an analytic attitude determination filter based on the direction cosines representation.

The present paper makes several contributions to the use of quaternion PDFs for attitude estimation. Its principal contribution is to develop an alternative to simple averaging for purposes of determining the quaternion estimate from its PDF. This alternative obeys the quaternion norm constraint and is shown to be equivalent to the constrained direction cosines estimates used in references [10] and [12]. Included with this contribution is a derivation of the covariance of the quaternion estimate's multiplicative error. Both the estimate and the error covariance are shown to be computable from the quaternion second moment. A second contribution is an exploration of the relationship between this new attitude estimate and that of the q-method. A third contribution is the development of a new quaternion PDF that is defined entirely in terms of its second moment. Also developed is a method for sampling this distribution. The final contribution is the solution of an example batch Global Positioning System (GPS) attitude determination problem using Monte-Carlo particle methods in conjunction with this paper's new ideas.

The remainder of this paper is divided into five sections plus conclusions. The second section develops and explains the new optimal unit-normalized quaternion calculation, and it derives an expression for the multiplicative error covariance. The third section equates the new quaternion estimate to that of the q-method for a suitably defined q-method PDF. The fourth section develops a new quaternion PDF. The fifth section solves a simulated batch GPS attitude determination problem in order to demonstrate how the new quaternion PDF calculations can be used in a practical particle-filter-type calculation. The sixth section discusses open issues for the use of quaternion probability distributions in estimation calculations, and the seventh section presents conclusions.

Optimal Quaternion Estimation Using a Quaternion Distribution

Optimal Estimation and the Expectation Value

The expectation value of the state vector is used as the state estimate because it minimizes the mean of the sum of the squared errors in the state components [13]. Suppose that $\hat{\mathbf{x}}$ is the as-yet-undetermined state estimate. Then the mean squared error cost function depends on $\hat{\mathbf{x}}$ as follows:

$$J_{MSE}(\hat{\mathbf{x}}) = E\{(\mathbf{x} - \hat{\mathbf{x}})^{\mathrm{T}}(\mathbf{x} - \hat{\mathbf{x}})\} = E\{\mathbf{x}^{\mathrm{T}}\mathbf{x}\} - 2\hat{\mathbf{x}}^{\mathrm{T}}E\{\mathbf{x}\} + \hat{\mathbf{x}}^{\mathrm{T}}\hat{\mathbf{x}} \tag{1}$$

where the expectations are implicitly understood as being conditioned on the measurements. The value of $\hat{\mathbf{x}}$ that minimizes this cost function is

$$\hat{\mathbf{x}}_{\mathrm{opt}} = \arg\min J_{MSE}(\hat{\mathbf{x}}) = E\{\mathbf{x}\} \tag{2}$$

The important point here is that the expectation value is used as the estimate only because it optimizes a certain cost function. There is nothing inherent to the expectation value that makes it the best estimate in all circumstances. If the simple expectation operation produces an estimate that is nonsensical, as in Fig. 1, then there is no reason to use it to compute an estimate from a conditional probability distribution.

Optimal Quaternion Estimation

A sensible quaternion estimate can be defined as being the solution to a constrained optimization problem. This approach starts by defining an error cost that is computed as a function of the quaternion estimate and that involves expectation operations. The most convenient cost is:

$$J_{MSQE}(\hat{\mathbf{q}}) = 1 - E\{(\mathbf{q}^{\mathrm{T}}\hat{\mathbf{q}})^2\} = 1 - \hat{\mathbf{q}}^{\mathrm{T}}E\{\mathbf{q}\mathbf{q}^{\mathrm{T}}\}\hat{\mathbf{q}} = 1 - \hat{\mathbf{q}}^{\mathrm{T}}P_{qq}\hat{\mathbf{q}} \tag{3}$$

This cost function equals the mean value of the square of the sine of half the total angular error between the quaternion estimate $\hat{\mathbf{q}}$ and the true quaternion $\mathbf{q}$. The 4×4 matrix $P_{qq} = E\{\mathbf{q}\mathbf{q}^{\mathrm{T}}\}$ is symmetric and positive definite and constitutes the second moment of the *a posteriori* quaternion probability density function. Its trace equals one because of the quaternion's unit normalization constraint.

The optimal quaternion estimate, $\hat{\mathbf{q}}_{\mathrm{opt}}$, is computed by minimizing $J_{MSQE}(\hat{\mathbf{q}})$ subject to the unit normalization constraint: $\hat{\mathbf{q}}^{\mathrm{T}}\hat{\mathbf{q}} = 1$. The form of the cost function on the extreme right-hand side of equation (3) can be used to show that $\hat{\mathbf{q}}_{\mathrm{opt}}$ is the unit-normalized eigenvector of P_{qq} that corresponds to its maximum eigenvalue $\lambda_{\max}$

$$P_{qq}\hat{\mathbf{q}}_{\mathrm{opt}} = \lambda_{\max}\hat{\mathbf{q}}_{\mathrm{opt}} \tag{4}$$

This solution has two interesting features. First, it resembles the solution of the q-method in that it is an eigenvector of a 4×4 symmetric matrix. Second, the solution is unique only up to a sign; i.e., if $\hat{\mathbf{q}}_{\mathrm{opt}}$ is a solution, then so is $-\hat{\mathbf{q}}_{\mathrm{opt}}$. This is desirable because $\hat{\mathbf{q}}_{\mathrm{opt}}$ and $-\hat{\mathbf{q}}_{\mathrm{opt}}$ both parameterize the same attitude.

Note that a general quaternion probability density function $p(\mathbf{q})$ that arises from an attitude determination problem should be an even function of $\mathbf{q}$, i.e., $p(\mathbf{q}) = p(-\mathbf{q})$, because $\mathbf{q}$ and $-\mathbf{q}$ both represent the same attitude. Unfortunately, problem modeling approximations may cause this symmetry to break down, as in Fig. 1. The definition of the quaternion estimate in equation (4) remains sensible even when the desired symmetry breaks down.

Note, also, that the second moment P_{qq} will not be a covariance matrix if $E\{\mathbf{q}\} \neq 0$. If the underlying quaternion distribution obeys $p(\mathbf{q}) = p(-\mathbf{q})$, then $E\{\mathbf{q}\} = 0$, and P_{qq} is a covariance matrix. In an estimation problem where approximations have caused symmetry to break down, as in Fig. 1, then P_{qq} normally will not be a covariance matrix. Nevertheless, the equation (4) definition of the quaternion estimate makes sense even if $E\{\mathbf{q}\} \neq 0$.

Equivalence to Optimal Frobenius Norm of Direction Cosines Error

References [10] and [12] define the attitude estimation error cost function in terms of the direction cosines matrix as

$$J_{MSAE}(\hat{A}) = E\{\|A - \hat{A}\|_F^2\} = 6 - 2E\{\mathrm{trace}(\hat{A}^T A)\} \tag{5}$$

where the matrix norm in the middle expression of equation (5) is the Frobenius norm. This cost function is minimized subject to the constraint that $\hat{A}$ be an element of SO(3). The resulting optimization problem is an orthogonal Procrustes problem, which the two references solve using the singular value decomposition.

The cost function in equation (5) can be re-expressed in terms of quaternions if one recognizes that $A = A(\mathbf{q})$ and $\hat{A} = \hat{A}(\hat{\mathbf{q}})$. Given the formula for $A(\mathbf{q})$ in reference [14], the cost function becomes

$$J_{MSAE}[\hat{A}(\hat{\mathbf{q}})] = 6 - 2E\{4(\hat{\mathbf{q}}^T\mathbf{q})^2 - 1\} = 8[1 - E\{(\hat{\mathbf{q}}^T\mathbf{q})^2\}] = 8 - 8\hat{\mathbf{q}}^T P_{qq}\hat{\mathbf{q}} \tag{6}$$

Thus, the direction cosines matrix cost function in equation (5) is equal to eight times the quaternion cost function in equation (3).[3] Reference [10] develops an equivalent quadratic form in the quaternion estimate $\hat{\mathbf{q}}$, but it does not recognize that the matrix in its quadratic form equals eight times the second moment of the quaternion probability distribution.

Covariance of the First Three Elements of the Multiplicative Error Quaternion

Given the posterior probability distribution, it is standard for an estimator to compute an error covariance for its estimate. The error of a quaternion estimate can be parameterized by the three-dimensional vector $\delta\mathbf{q}$ that contains the first three elements of the multiplicative error quaternion

$$\delta\mathbf{q} = \begin{bmatrix} -\hat{q}_4 & -\hat{q}_3 & \hat{q}_2 & \hat{q}_1 \\ \hat{q}_3 & -\hat{q}_4 & -\hat{q}_1 & \hat{q}_2 \\ -\hat{q}_2 & \hat{q}_1 & -\hat{q}_4 & \hat{q}_3 \end{bmatrix}\mathbf{q} = \hat{Q}_{me}\mathbf{q} \quad \text{so that} \quad \hat{\mathbf{q}} = \begin{bmatrix} \delta\mathbf{q} \\ \sqrt{1 - \delta\mathbf{q}^T\delta\mathbf{q}} \end{bmatrix} \otimes \mathbf{q} \tag{7}$$

where $\hat{q}_i$ is the ith element of $\hat{\mathbf{q}}$ and $\hat{Q}_{me}$ is defined to be the 3×4 matrix that multiplies $\mathbf{q}$ in the middle expression of the left-hand equation. The $\otimes$ symbol in the right-hand equation signifies quaternion multiplication such that $A(\mathbf{q}_a)A(\mathbf{q}_b) = A(\mathbf{q}_a \otimes \mathbf{q}_b)$ [14]. Note that $\hat{\mathbf{q}}$ in equation (7) refers to the optimal estimate $\hat{\mathbf{q}}_{\mathrm{opt}}$; the "opt" subscript is dropped from $\hat{\mathbf{q}}_{\mathrm{opt}}$ in the remainder of this paper because use of

[3]This equivalence was first pointed out in an unpublished derivation done by F. Landis Markley.

the optimal estimate is assumed in all but a single case, and the distinction is obvious in that case.

The 3×3 covariance of the multiplicative error is

$$P_{\delta q\,\delta q} = E\{\delta\mathbf{q}\,\delta\mathbf{q}^{\mathrm{T}}\} = E\{\hat{Q}_{me}\mathbf{q}\mathbf{q}^{\mathrm{T}}\hat{Q}_{me}^{\mathrm{T}}\} = \hat{Q}_{me}E\{\mathbf{q}\mathbf{q}^{\mathrm{T}}\}\hat{Q}_{me}^{\mathrm{T}} = \hat{Q}_{me}P_{qq}\hat{Q}_{me}^{\mathrm{T}} \qquad (8)$$

This covariance matrix constitutes the second moment of the projection of $\mathbf{q}$ onto the subspace that is perpendicular to $\hat{\mathbf{q}}$ because $\hat{Q}_{me}\hat{\mathbf{q}} = 0$.

Note that $P_{\delta q\,\delta q}$ is a true covariance only in the case when $E\{\delta\mathbf{q}\} = \hat{Q}_{me}E\{\mathbf{q}\} = 0$. $P_{\delta q\,\delta q}$ will be a true covariance when the quaternion probability distribution has the desired even symmetry because $E\{\mathbf{q}\} = 0$ in this case. $P_{\delta q\,\delta q}$ may fail to be a true covariance when approximations made by the estimation algorithm act to destroy the even symmetry. Despite this possibility, $P_{\delta q\,\delta q}$ will be referred to as a covariance for the remainder of this paper because it characterizes the second moment of an estimation error.

Equation (8) implies that the three eigenvalues of $P_{\delta q\,\delta q}$ are the three other eigenvalues of P_{qq} besides $\lambda_{\max}$. As proof of this assertion, consider the four-dimensional eigenvector q_{alt} of P_{qq} and the associated eigenvalue $\lambda_{alt} < \lambda_{\max}$. Then $P_{qq}q_{alt} = \lambda_{alt}q_{alt}$. Now consider $P_{\delta q\,\delta q}\hat{Q}_{me}\mathbf{q}_{alt} = [\hat{Q}_{me}P_{qq}\hat{Q}_{me}^{\mathrm{T}}]\hat{Q}_{me}\mathbf{q}_{alt} = \hat{Q}_{me}P_{qq}[\hat{Q}_{me}^{\mathrm{T}}\hat{Q}_{me} + \hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathrm{T}}]q_{alt}$. The $\hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathrm{T}}$ term in the latter expression can be included because $\hat{\mathbf{q}}^{\mathrm{T}}\mathbf{q}_{alt} = 0$ by virtue of the symmetry of P_{qq} and the fact that $\hat{\mathbf{q}}$ and $\mathbf{q}_{alt}$ are two distinct eigenvectors of P_{qq}. The formula for $\hat{Q}_{me}$ given in equation (7) implies that $\hat{Q}_{me}^{\mathrm{T}}\hat{Q}_{me} + \hat{\mathbf{q}}\hat{\mathbf{q}}^{\mathrm{T}} = I$, which allows the forgoing equations to be reduced to $P_{\delta q\,\delta q}\hat{Q}_{me}\mathbf{q}_{alt} = \hat{Q}_{me}P_{qq}\mathbf{q}_{alt} = \lambda_{alt}\hat{Q}_{me}\mathbf{q}_{alt}$. This latter equation proves that the three-dimensional vector $\hat{Q}_{me}\mathbf{q}_{alt}$ is an eigenvector of $P_{\delta q\,\delta q}$ whose eigenvalue λ_{alt} is also an eigenvalue of the original matrix P_{qq}. When coupled with the fact that $\mathrm{trace}(P_{qq}) = 1$, this equivalence of the eigenvalues implies that $J_{MSQE}(\hat{\mathbf{q}}) = \mathrm{trace}(P_{\delta q\,\delta q})$. Thus, the optimal $\hat{\mathbf{q}}$ minimizes the trace of its multiplicative error covariance matrix.

Note that the 4×4 quaternion second moment matrix P_{qq} contains all of the information needed in order to compute the optimal quaternion estimate and the covariance of its multiplicative error. The estimate is the normalized eigenvector corresponding to the maximum eigenvalue, and the error covariance is the projection of P_{qq} onto the subspace of the other three eigenvectors. Therefore, the matrix P_{qq} is a very important quaternion statistic.

Rejection of an Alternative Error Cost Function

Alternate error cost functions can be used to define sensible quaternion estimates. The original error function that was tried as part of the present study was

$$J_{MAQE}(\hat{\mathbf{q}}) = 1 - E\{|\mathbf{q}^{\mathrm{T}}\hat{\mathbf{q}}|\} \qquad (9)$$

Minimization of this function maximizes the mean absolute value of the cosine of half the total angular error between the true quaternion $\mathbf{q}$ and the quaternion estimate $\hat{\mathbf{q}}$. A low value of this cost function implies that $\hat{\mathbf{q}}$ is close to $\mathbf{q}$ in some average sense.

It is possible, in theory, to optimize this cost function subject to the unit normalization constraint $\hat{\mathbf{q}}^{\mathrm{T}}\hat{\mathbf{q}} = 1$. This cost function was rejected because the resulting optimization problem is difficult to solve. The difficulty arises because of the absolute value function used in equation (9). The cost function in equation (3) is preferred because it is easy to optimize by solving an eigenvalue problem.

Relationship Between Distribution-Based Quaternion Estimate and Q-Method Estimate

The q-method cost function can be written in the form

$$J_q(\mathbf{q}) = \frac{1}{2}\sum_{k=1}^{N} a_k \|\hat{\mathbf{w}}_k - A(\mathbf{q})\hat{\mathbf{v}}_k\|^2 = \sum_{k=1}^{N} a_k - \mathbf{q}^{\mathrm{T}} K \mathbf{q} \tag{10}$$

where the positive scalars $a_1, \ldots, a_N$ are cost weights, $\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_N$ are unit direction vectors that are measured in body coordinates, $\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_N$ are the corresponding known values of the unit direction vectors given in reference coordinates, and K is a symmetric 4×4 matrix that can be determined from the weights and the unit direction vectors [15].

The q-method cost function can be used to define an *a posteriori* quaternion probability density function. Suppose that each weight $a_k = 1/\sigma_k^2$, with σ_k being the per-axis measurement error standard deviation of the vector $\hat{\mathbf{w}}_k$. Then a sensible *a posteriori* quaternion PDF is

$$p_q(\mathbf{q}) = C_1 \exp\{-J_q(\mathbf{q})\} = C_2 \exp\{\mathbf{q}^{\mathrm{T}} K \mathbf{q}\} \tag{11}$$

if one assumes a uniform *a priori* distribution and if the density units are defined as probability per unit of hyper-area on the $\mathbf{q}^{\mathrm{T}}\mathbf{q} = 1$ hypersphere. The positive scalars C_1 and C_2 are related normalization constants.

The quaternion second moment can be computed from the distribution in equation (11), taking the form

$$P_{qq} = C_2 \int_{-\pi}^{\pi} d\alpha \int_{-\pi/2}^{\pi/2} d\beta \cos\beta \int_{-\pi/2}^{\pi/2} d\gamma \; \cos^2\gamma\, \mathbf{q}(\alpha,\beta,\gamma)\mathbf{q}^{\mathrm{T}}(\alpha,\beta,\gamma) \exp\{\mathbf{q}^{\mathrm{T}}(\alpha,\beta,\gamma) K \mathbf{q}(\alpha,\beta,\gamma)\} \tag{12}$$

where α, β, and γ are hyper-spherical coordinates that are used to define

$$\mathbf{q}(\alpha,\beta,\gamma) = \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma \\ \sin\alpha\cos\beta\cos\gamma \\ \sin\beta\cos\gamma \\ \sin\gamma \end{bmatrix} \tag{13}$$

It is straightforward to show that each eigenvector of K is also an eigenvector of P_{qq}. The derivation proceeds as follows: one first finds an orthonormal 4×4 transformation matrix V that diagonalizes K: $V^{\mathrm{T}}KV = \mathrm{diag}(\lambda_{K1}, \lambda_{K2}, \lambda_{K3}, \lambda_{K4})$—the eigenvalues of K are λ_{K1}, λ_{K2}, λ_{K3}, and λ_{K4}. The integral in equation (12) is carried out using hyper-spherical coordinates that are defined relative to this transformed coordinate system. The off-diagonal P_{qq} elements in the integral are zero in this coordinate system because of the following facts: The $(P_{qq})_{12}$, $(P_{qq})_{23}$, and $(P_{qq})_{24}$ integrands are odd functions of α, the $(P_{qq})_{13}$ and $(P_{qq})_{34}$ integrands are odd functions of β, and the $(P_{qq})_{14}$ integrand is an odd function of γ.

A formula for each eigenvalue of P_{qq} can be derived using this same transformed integral. The generic formula is

$$\lambda_{Pi} = C_2 \int_{-\pi}^{\pi} d\alpha \int_{-\pi/2}^{\pi/2} d\beta \cos\beta \int_{-\pi/2}^{\pi/2} d\gamma \cos^2\gamma \sin^2\gamma \exp\{\lambda_{Ki}\sin^2\gamma + \cos^2\gamma[\lambda_{Kj}\sin^2\beta + \cos^2\beta(\lambda_{Kk}\sin^2\alpha + \lambda_{Kl}\cos^2\alpha)]\} \tag{14}$$

where λ_{Pi} is the ith eigenvalue of P_{qq} and where the index set $\{i, j, k, l\}$ is any permutation of the set $\{1, 2, 3, 4\}$. No analytic expression for this integral has yet been found.

Although it has not been proven, it seems obvious that the maximum eigenvalue of K gives rise to the maximum eigenvalue of P_{qq} through equation (14). This implies that the same eigenvector of K that constitutes the q-method's attitude estimate also constitutes the optimal estimate that minimizes the cost function in equation (3) subject to the unit normalization constraint. Thus, the q-method and this paper's new quaternion estimation method produce identical attitude estimates for a sensibly defined quaternion PDF.

A New Quaternion Probability Density Function

It would be desirable to have an analytic quaternion PDF that is completely defined in terms of its quaternion second moment P_{qq}. This would allow P_{qq} to define this new distribution much as the mean and covariance serve to define a Gaussian distribution in R^n. This section develops a quaternion distribution of the desired form.

This new distribution is not unique in the sense of being the only possible quaternion distribution, nor is it necessarily the correct *a posteriori* distribution for a given estimation problem. This new distribution has been developed in the hope that it might serve as a useful standard distribution or approximating distribution for quaternions on the unit hypersphere much as the Gaussian distribution has served in R^n.

Derivation of the New Distribution

A new distribution that has the desired properties can be derived by starting with a four-dimensional Gaussian distribution whose mean is zero and whose covariance is P_{qq}: $\mathbf{x} \sim N(0, P_{qq})$. This arbitrary choice of a starting point is a mathematical convenience that leads quickly to a quaternion distribution that has the desired properties. The formula for the known second moment of the Gaussian distribution takes the form

$$P_{qq} = \int \mathbf{x}\mathbf{x}^{\mathrm{T}} \frac{\exp\left\{-\frac{1}{2}\mathbf{x}^{\mathrm{T}} P_{qq}^{-1}\mathbf{x}\right\}}{(2\pi)^2\sqrt{\det(P_{qq})}}\, \mathrm{d}\mathbf{x} \tag{15}$$

The trick to deriving the new distribution from this distribution is to rewrite this integral in hyper-spherical coordinates. Suppose that $\mathbf{x} = \rho\mathbf{q}(\alpha, \beta, \gamma)$. Then the integral can be rewritten in the form

$$\begin{aligned} P_{qq} &= \int_{-\pi}^{\pi} \mathrm{d}\alpha \int_{-\pi/2}^{\pi/2} \mathrm{d}\beta \cos\beta \int_{-\pi/2}^{\pi/2} \mathrm{d}\gamma \cos^2\gamma\, \mathbf{q}\mathbf{q}^{\mathrm{T}} \int_0^{\infty} \mathrm{d}\rho\, \rho^5 \frac{\exp\left\{-\frac{\rho^2}{2}\mathbf{q}^{\mathrm{T}} P_{qq}^{-1}\mathbf{q}\right\}}{(2\pi)^2\sqrt{\det(P_{qq})}} \\ &= \int_{-\pi}^{\pi} \mathrm{d}\alpha \int_{-\pi/2}^{\pi/2} \mathrm{d}\beta \cos\beta \int_{-\pi/2}^{\pi/2} \mathrm{d}\gamma \cos^2\gamma\, \mathbf{q}\mathbf{q}^{\mathrm{T}} \frac{2}{\pi^2\sqrt{\det(P_{qq})}\,(\mathbf{q}^{\mathrm{T}} P_{qq}^{-1}\mathbf{q})^3} \end{aligned} \tag{16}$$

where the dependence of $\mathbf{q}$ on α, β, and γ is implicit. The second line of this equation is derived from the first by performing the ρ integration. This operation amounts to a weighted projection of $\mathbf{x}$ onto the unit hypersphere in order to produce $\mathbf{q}$. An alternative second moment relationship also must be true, given by

$$P_{qq} = \int_{-\pi}^{\pi} \mathrm{d}\alpha \int_{-\pi/2}^{\pi/2} \mathrm{d}\beta \cos\beta \int_{-\pi/2}^{\pi/2} \mathrm{d}\gamma \cos^2\gamma \mathbf{q}\mathbf{q}^{\mathrm{T}} p_{\mathrm{new}}(\mathbf{q}) \tag{17}$$

Therefore, the new quaternion PDF resulting from equation (15) must be

$$p_{\mathrm{new}}(\mathbf{q}) = \frac{2}{\pi^2 \sqrt{\det(P_{qq})}\,(\mathbf{q}^{\mathrm{T}} P_{qq}^{-1} \mathbf{q})^3} \tag{18}$$

in order for the second moment expressions in equations (16) and (17) to be equivalent.

It is straightforward to show that this new distribution is unit normalized. One starts by taking the trace of both sides of equation (16). The trace of $\mathbf{q}\mathbf{q}^{\mathrm{T}}$ on the right-hand side is one due to the unit normalization constraint. The trace of P_{qq} is also one. These facts, when substituted into the resulting equation, prove the normalization of $p_{\mathrm{new}}(\mathbf{q})$ when integrated over the unit hypersphere. Note that this distribution has the property of even symmetry: $p_{\mathrm{new}}(\mathbf{q}) = p_{\mathrm{new}}(-\mathbf{q})$, which is desirable when $\mathbf{q}$ serves as an attitude representation, as mentioned previously.

Monte-Carlo Sampling of the New Quaternion Distribution

It would be useful to be able to draw random samples from the quaternion PDF defined in equation (18). One might want to do this in order to initialize a particle filter. Alternatively, one might want to use these samples in a sigma-points-type calculation that approximates the effects of nonlinear dynamics or measurement functions on sample statistics. The fifth section of this paper uses samples from this distribution to help it implement a Monte-Carlo calculation for another distribution.

It is efficient to develop a sampling algorithm on a transformed unit hypersphere. Let L be any real-valued 4×4 square-root of P_{qq} such that $LL^{\mathrm{T}} = P_{qq}$. A suitable L can be computed using Cholesky factorization. Let y be a point on the unit hypersphere of another four-dimensional space. One can use L to define the one-to-one mapping from this new unit hypersphere to the $\mathbf{q}$ unit hypersphere as

$$\mathbf{q} = \frac{L\mathbf{y}}{\sqrt{\mathbf{y}^{\mathrm{T}} L^{\mathrm{T}} L \mathbf{y}}} \tag{19a}$$

$$\mathbf{y} = \frac{L^{-1}\mathbf{q}}{\sqrt{\mathbf{q}^{\mathrm{T}} L^{-\mathrm{T}} L^{-1} \mathbf{q}}} \tag{19b}$$

where the notation $()^{-\mathrm{T}}$ indicates the transpose of the inverse.

The new probability density function in equation (18) can be mapped onto the $\mathbf{y}$ unit hypersphere to yield a corresponding probability density function for $\mathbf{y}$ as

$$p_{\mathrm{newy}}(\mathbf{y}) = \frac{2\mathbf{y}^{\mathrm{T}} L^{\mathrm{T}} L \mathbf{y}}{\pi^2} \tag{20}$$

The derivation of this formula is a straightforward, but tedious application of probability transformation. It includes an evaluation of the determinant of the transformation that maps local tangent coordinates on the $\mathbf{y}$ unit hypersphere to local tangent coordinates on the $\mathbf{q}$ unit hypersphere.

The $\mathbf{q}$ sampling method starts by sampling $\mathbf{y}$ from a uniform distribution on its unit hypersphere. This is accomplished by sampling the un-normalized vector $\bar{\mathbf{y}}$ from a four-dimensional Gaussian distribution with zero mean and an identity covariance, $\bar{\mathbf{y}} \sim N(0, I_{4\times4})$. The normalized $\mathbf{y} = \bar{\mathbf{y}}/(\bar{\mathbf{y}}^{\mathrm{T}}\bar{\mathbf{y}})^{0.5}$ is uniformly distributed over its unit hypersphere [11].

The final step of the sampling process modifies the $\mathbf{y}$ distribution to have the probability density given in equation (20). This is accomplished by using a probabilistic accept/reject test. The test samples a random scalar z from the uniform distribution between zero and one: $z \sim U(0, 1)$. It then compares z to the ratio $\mathbf{y}^{\mathrm{T}}L^{\mathrm{T}}L\mathbf{y}/(\lambda_{LTL})_{\max}$, where $(\lambda_{LTL})_{\max}$ is the maximum eigenvalue of the 4×4 symmetric positive definite matrix $L^{\mathrm{T}}L$. If the randomly sampled z is less than or equal to this ratio, then $\mathbf{y}$ is accepted, and the corresponding sample of $\mathbf{q}$ is computed using equation (19a). Otherwise, $\mathbf{y}$ is rejected, and the sampling calculation starts over by sampling a new un-normalized $\bar{\mathbf{y}}$ from the Gaussian distribution $N(0, I_{4\times4})$. This procedure creates the distribution in equation (20) because its accept/reject criterion starts with a probability distribution that is uniform on the $\mathbf{y}$ unit hypersphere and modifies it by proportionally scaling the relative probability of acceptance based on the factor $\mathbf{y}^{\mathrm{T}}L^{\mathrm{T}}L\mathbf{y}$.

The efficiency of this sampling procedure is measured by the average number of $\bar{\mathbf{y}}$ samples that must be drawn in order to produce each valid $\mathbf{q}$ sample. If the initial distribution is uniform, i.e., if $P_{qq} = 0.25\, I_{4\times4}$, then $L^{\mathrm{T}}L$ equals $0.25\, I_{4\times4}$, and the accept/reject test is always passed. The efficiency is 100% in this case, and the resulting algorithm degenerates into an algorithm found in reference [11]. The most inefficient case occurs when the largest eigenvalue of P_{qq} approaches one. In this case, the largest eigenvalue of $L^{\mathrm{T}}L$ is also nearly equal to one, but the average value of $\mathbf{y}^{\mathrm{T}}L^{\mathrm{T}}L\mathbf{y}$ is 0.25. This implies that the worst-case efficiency corresponds to a 25% acceptance rate. Given that this is a strange new distribution, it seems reasonable to sample at most an average of four four-dimensional Gaussian distributions and four scalar uniform distributions in order to produce each of its samples.

Solution of a Batch GPS Attitude Determination Problem Using the Quarternion 2nd Moment in a Particle-Based Algorithm

Problem Geometry and Measurements

The new quaternion PDF estimation techniques have been applied to a simulated example GPS attitude determination problem. The geometry of this problem is illustrated in Fig. 2. It depicts the antenna baseline vector $\mathbf{b}_i$ between antenna m and antenna n, which is known in body coordinates, the line-of-sight (LOS) vector $\hat{\mathbf{s}}_j$ from the body to tracked GPS satellite j, which is known in reference coordinates, and the two field-of-view (FOV) center vectors of the two antennas $\hat{\mathbf{a}}_m$ and $\hat{\mathbf{a}}_n$, which are known in body coordinates. The carat overstrikes (^) used here denote unit vectors, not estimates.

The GPS signals yield two types of attitude information. The single-differenced carrier phase for each antenna baseline and each tracked satellite is a high-accuracy interferometric measurement

$$\lambda\Delta\phi_{ij} = -\mathbf{b}_i^{\mathrm{T}}A(\mathbf{q})\hat{\mathbf{s}}_j + v_{ij} \tag{21}$$

where λ is the carrier wavelength (not an eigenvalue of a matrix), $\Delta\phi_{ij}$ is the measured carrier phase difference for baseline i and satellite j in cycles, and v_{ij} is the single-differenced carrier phase noise. This formula assumes that the two antennas are specified for the given baseline and that these two antennas' known geometry and carrier phase outputs are used to compute $\mathbf{b}_i$ and $\Delta\phi_{ij}$. It also assumes that carrier cycle ambiguities have been resolved and incorporated into the $\Delta\phi_{ij}$ measurement.

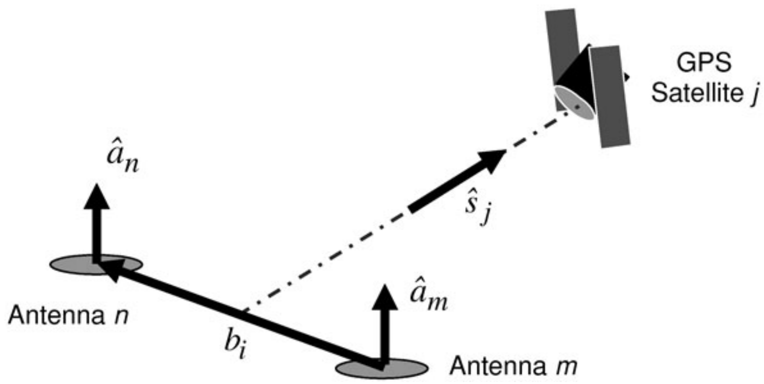

FIG. 2. Geometry of Carrier Phase GPS Attitude Determination Problem.

Additional information for the attitude determination problem comes from the visibility constraints. They take the form

$$\hat{\mathbf{a}}_m^{\mathrm{T}} A(\mathbf{q})\hat{\mathbf{s}}_j \geq c_{\min} \quad \text{and} \quad \hat{\mathbf{a}}_n^{\mathrm{T}} A(\mathbf{q})\hat{\mathbf{s}}_j \geq c_{\min} \tag{22}$$

where $c_{\min}$ is a minimum value for the cosine of the angle between the FOV center and the LOS vector that is required for a signal to be available. $c_{\min} = \cos(85°)$ is a typical value. These constraints provide coarse information that is used to preclude a gross 180 degree inversion of the attitude solution.

Monte-Carlo Solution of Single-Point Batch Problem

The attitude determination problem is to solve for the quaternion estimate $\hat{\mathbf{q}}$ that best obeys measurement equation (21) and FOV constraints (22) for all antenna baselines and all visible GPS satellites. This is a batch problem for estimating the attitude at a single point in time. It does not involve a dynamic propagation model, memory of past measurements, or filtering.

The approach to solving this problem is to use equations (21) and (22) in order to define a posterior quaternion PDF. This PDF is then used to compute the P_{qq} matrix. This computation uses Monte Carlo techniques in order to evaluate the necessary integral. The P_{qq} matrix is then used in the calculations of this paper's second section in order to deduce the quaternion estimate $\hat{\mathbf{q}}$ along with the covariance matrix of the multiplicative errors $P_{\delta q\,\delta q}$.

The posterior quaternion PDF takes the form

$$p(\mathbf{q}|\text{data}) = C\exp\{-J_{GPS}(\mathbf{q})\} \tag{23}$$

where C is a normalization constant and where the GPS measurement error cost function takes the form

$$\begin{aligned} J_{GPS}(\mathbf{q}) = {} & \frac{1}{2}\sum_{j=1}^{J}\begin{bmatrix}\lambda\Delta\phi_{1j} + \mathbf{b}_1^{\mathrm{T}}A(\mathbf{q})\hat{\mathbf{s}}_j \\ \vdots \\ \lambda\Delta\phi_{Ij} + \mathbf{b}_I^{\mathrm{T}}A(\mathbf{q})\hat{\mathbf{s}}_j\end{bmatrix}^{\mathrm{T}} R_j^{-1}\begin{bmatrix}\lambda\Delta\phi_{1j} + \mathbf{b}_1^{\mathrm{T}}A(\mathbf{q})\hat{\mathbf{s}}_j \\ \vdots \\ \lambda\Delta\phi_{Ij} + \mathbf{b}_I^{\mathrm{T}}A(\mathbf{q})\hat{\mathbf{s}}_j\end{bmatrix} \\ & + \frac{1}{2}\sum_{j=1}^{J}\sum_{m=1}^{M} J_{pen+}\left\{\frac{1}{\sigma_{pen}}\left[c_{\min} - \hat{\mathbf{a}}_m^{\mathrm{T}}A(\mathbf{q})\hat{\mathbf{s}}_j\right]\right\} \end{aligned} \tag{24}$$

for the J visible GPS satellites, the I independent antenna baseline pairs, and the M antennas. The symmetric positive definite matrix R_j is the covariance matrix of the jth satellite's measurement error vector $[v_{1j}, \ldots, v_{Ij}]^{\mathrm{T}}$. This matrix has nonzero off-diagonal elements because the single differencing operation gives rise to measurement error cross correlations. The penalty functions in the last summation of equation (24) penalize positive values of their input arguments in order to provide "soft" enforcement of the FOV constraints in equation (22) as

$$J_{pen+}(\Delta c) = \begin{cases} 0 & \text{if } \Delta c \leq 0 \\ \Delta c^2 & \text{if } 0 < \Delta c \end{cases} \tag{25}$$

The positive penalty weighting factor $1/\sigma_{pen}$ can be used to tune the penalty cost for a given level of constraint violation.

The brute-force Monte-Carlo solution to this problem computes P_{qq} approximately by using samples drawn from a uniform quaternion PDF. The computation takes the form

$$P_{qq} = \int_{hs} \mathbf{q}\mathbf{q}^{\mathrm{T}} p(\mathbf{q}|\text{data})\,\mathrm{d}\mathbf{q} \cong \frac{\sum_{k=1}^{N_{mc}} \mathbf{q}_{uk}\mathbf{q}_{uk}^{\mathrm{T}} p(\mathbf{q}_{uk}|\text{data})}{\sum_{k=1}^{N_{mc}} p(\mathbf{q}_{uk}|\text{data})} \tag{26}$$

where the integral is performed over the unit hypersphere and where each of the N_{mc} samples $\mathbf{q}_{uk}$ is drawn from the uniform quaternion distribution. These samples can be generated using techniques discussed in this paper's fourth section and in reference [11]. The value $p(\mathbf{q}_{uk}|\text{data})$ used in equation (26) is the *a posteriori* probability weight associated with sample $\mathbf{q}_{uk}$. This method of approximating P_{qq} is an application of particle filtering concepts [9].

The "Curse" of Dimensionality

A simple analysis shows that the calculations of equation (26) are impractical if the GPS attitude solution is likely to be accurate to about a degree or better. In order for the calculation in equation (26) to be accurate, it needs to produce a significant number of samples that fall within about one to three standard deviations of the optimal attitude estimate. If this does not happen, then only one $\mathbf{q}_{uk}$ sample will have a non-negligible weight $p(\mathbf{q}_{uk}|\text{data})$; it will be the sample that is statistically closest to the true attitude. All other samples will have negligible weights. The calculated P_{qq} matrix will have an eigenvalue that is near unity with the corresponding eigenvector pointing along the one non-negligible $\mathbf{q}_{uk}$ sample, and the $P_{\delta q\,\delta q}$ matrix will be negligibly small. This latter matrix will be a poor measure of the actual error in the resulting quaternion estimate. The true error will be on the order of half the average spacing between the $\mathbf{q}_{uk}$ samples.

A analysis of the unit hypersphere shows that the number of samples needed in order to achieve an average inter-sample spacing of θ degrees is

$$N_{mc} = \frac{16}{\pi}\left(\frac{180}{\theta}\right)^3 \tag{27}$$

For an average inter-sample spacing of $\theta = 1°$, this translates into $N_{mc} = 29.7 \times 10^6$ samples. This is a very large number of samples. Nevertheless, it is probably insufficient if the goal is to achieve a reasonable computational precision in the evaluation of equation (26) and if the accuracy of the GPS attitude solution is likely to be about 1 degree or better.

This difficulty is known as the "curse" of dimensionality. The "curse" is that the number of sample points needed for a given average density is exponential in the dimension of the underlying state space. The dimension of the quaternion unit hypersphere is three, which is why a cubed term appears in equation (27). This cubing operation leads to an astronomical growth in the required number of particles as the expected estimation error decreases.

Staged Monte-Carlo Calculations

The GPS problem's "curse" of dimensionality can be circumvented by employing a staged Monte-Carlo calculation. The idea of staging is to start with a coarse Monte-Carlo computation that yields a rough calculation of the optimal quaternion. This rough quaternion estimate is then used to generate a new set of quaternion samples that are centered about it and that have less dispersion than the previous set of samples. These new samples, in turn, are used in a refined calculation of P_{qq}.

Suppose that $\mathbf{q}_{rk}$ is a member of the new set of samples, and suppose that it is sampled from the distribution whose PDF is $p_r(\mathbf{q})$. $p_r(\mathbf{q})$ is called the re-sampling distribution. It will have been chosen in a way that tends to concentrate points relatively near the optimal quaternion estimate, which is where the posterior PDF $p(\mathbf{q}|\text{data})$ produces significant nonzero weights. The new samples can be used to recompute P_{qq} according to the formula

$$P_{qq} \cong \frac{\sum_{k=1}^{N_{mc}} \mathbf{q}_{rk}\mathbf{q}_{rk}^{\mathrm{T}} p(\mathbf{q}_{rk}|\text{data})/p_r(\mathbf{q}_{rk})}{\sum_{k=1}^{N_{mc}} p(\mathbf{q}_{rk}|\text{data})/p_r(\mathbf{q}_{rk})} \tag{28}$$

The PDF ratio $p(\mathbf{q}_{rk}|\text{data})/p_r(\mathbf{q}_{rk})$ acts as a modified posteriori probability weight in this equation; it serves a similar purpose to the value $p(\mathbf{q}_{uk}|\text{data})$ in equation (26). The use of such weights is dictated by Bayesian considerations and is known as importance sampling in the particle filtering literature [9].

The distribution $p_r(\mathbf{q}_{rk})$ is chosen to be an implementation of the new quaternion PDF that has been developed in the fourth section of this paper. The value of the quaternion second moment used to generate this distribution is

$$P_{qqpr} = Q_{pr}\begin{bmatrix} \sigma_{pr}^2 & 0 & 0 & 0 \\ 0 & \sigma_{pr}^2 & 0 & 0 \\ 0 & 0 & \sigma_{pr}^2 & 0 \\ 0 & 0 & 0 & (1 - 3\sigma_{pr}^2) \end{bmatrix} Q_{pr}^{\mathrm{T}} \tag{29}$$

where

$$Q_{pr} = \begin{bmatrix} \hat{q}_{pr4} & -\hat{q}_{pr3} & \hat{q}_{pr2} & \hat{q}_{pr1} \\ \hat{q}_{pr3} & \hat{q}_{pr4} & -\hat{q}_{pr1} & \hat{q}_{pr2} \\ -\hat{q}_{pr2} & \hat{q}_{pr1} & \hat{q}_{pr4} & \hat{q}_{pr3} \\ -\hat{q}_{pr1} & -\hat{q}_{pr2} & -\hat{q}_{pr3} & \hat{q}_{pr4} \end{bmatrix} \tag{30}$$

The quaternion $\hat{\mathbf{q}}_{pr}$ is the optimal quaternion estimate from the preceding Monte-Carlo calculation stage. The standard deviation σ_{pr} is a conservative estimate of the standard deviation of the worst of the first three multiplicative error components.

This choice of $p_r(\mathbf{q})$ concentrates its samples within $3\sigma_{pr}$ of the rough estimate $\hat{\mathbf{q}}_{pr}$. It is likely to use a limited number of samples better than does the uniform distribution of equation (26). Therefore, it is likely to generate an improved approximation of the true P_{qq} of the posterior PDF $p(\mathbf{q}|\text{data})$.

The staged approach can be applied recursively. The $\hat{\mathbf{q}}$ eigenvector of P_{qq} from one stage becomes the $\hat{\mathbf{q}}_{pr}$ of the re-sampling distribution for the succeeding stage. The value of σ_{pr} is reduced at successive stages if reduction is reasonable. The value of σ_{pr} should be larger than the expected per-axis error half angle between $\hat{\mathbf{q}}_{pr}$ and the true attitude, and it should be larger than the square root of the largest eigenvalue of $P_{\delta q\,\delta q}$ from the preceding stage. If the first stage uses a uniform quaternion distribution, then equation (27) can be used to estimate a reasonable σ_{pr} for the second stage. Given N_{mc} for the first stage, equation (27) can be solved for θ, which is expressed in degrees, and the value $\sigma_{pr} \cong \theta\pi/180$ should be used to generate the second stage's samples.

Simulation Results

Simulated data have been generated for several test cases, and the methods defined above have been used to calculate the attitude quaternion estimate and its error covariance. The new calculations have performed as expected in all cases. Consider a representative example that involves three GPS antennas that form two baselines with lengths 1.477 meters and 0.839 meters and that have an angular separation of 123.43 degrees. Each of the three antennas receives signals at the L1 frequency from the same set of four GPS satellites. The carrier phase measurement error standard deviation of each signal is $\lambda\sigma_\phi = 0.004$ meters.

This attitude determination problem has been solved in three Monte-Carlo stages. The first stage has used a uniform distribution on the unit hypersphere from which $N_{mc} = 10{,}000$ particles have been sampled. This corresponds to an average quaternion spacing of $\theta = 14.4$ degrees. The second stage has used the optimal $\hat{\mathbf{q}}$ from the first stage as its $\hat{\mathbf{q}}_{pr}$ guess, and it has used $\sigma_{pr} = 0.2315$ radians $= 0.9226\theta\pi/180$ to generate $N_{mc} = 100{,}000$ samples. This σ_{pr} value corresponds to a per-axis multiplicative error standard deviation of 26.53 degrees. The third stage has used $N_{mc} = 100{,}000$ samples drawn from a distribution centered at the optimal $\hat{\mathbf{q}}$ of the second stage and distributed with a σ_{pr} of 0.0524 radians, which corresponds to a per-axis error standard deviation of 6 degrees.

The resulting estimates and multiplicative error standard deviations are as follows. $\hat{\mathbf{q}}$ from the first stage has an attitude that is 13.4492 degrees away from the simulated truth value, but its computed per-axis multiplicative error standard deviations are all less then 10^{-14} degrees. This gross underestimation of the error on the part of $P_{\delta q\,\delta q}$ is the result of particle degeneracy [9] in which only one particle has a non-negligible weight, as discussed above. The second stage improves things significantly. Its $\hat{\mathbf{q}}$ has a total attitude error of only 2.8313 degrees, and its multiplicative error standard deviations are 0.0418 degrees, 0.0207 degrees, and 0.0084 degrees. Although these values are still low due to near-degeneracy of the samples, they are much more reasonable than the values from the first stage. The third stage's $\hat{\mathbf{q}}$ has a total attitude error that is only 0.2878 degrees, and its multiplicative error standard deviations are 0.3092 degrees, 0.2420 degrees, and 0.4237 degrees. These values are statistically reasonable. In fact, they are very near to the true optimal estimates that result in the limit as the number of samples N_{mc} approaches infinity.

No attempt has been made to minimize the numbers of Monte-Carlo samples that have been used at each stage of the calculations. The chosen N_{mc} values are probably conservative. They have been chosen to ensure reasonable calculation precision at the expense of extra computational cost because the main goal has been to demonstrate the theoretical correctness of the new calculations.

These example results demonstrate that the new quaternion estimation technique can be used with Monte-Carlo methods to solve attitude estimation problems. It is significant to note that the GPS attitude estimation problem solved in this section cannot be solved using the q-method because its measurements are cosine measurements rather than vector measurements. Of course, there exist more efficient methods to solve this particular problem, but the new technique's ability to solve this problem demonstrates how it might be applied.

Open Issues for Further Research

A number of interesting questions raised by the current results warrant further study. One issue concerns the development of alternate quaternion probability density functions that are defined by the quaternion second moment statistic P_{qq}. The new probability density function developed in this paper's fourth section may prove useful, but it has the undesirable property of not being in an exponential form. That is, its dependence on the quaternion does not enter through an exponential function. This fact complicates attempts to use this distribution as a prior or posterior PDF in the context of Bayesian estimation. A more attractive PDF would be exponential and would be fully defined by an analytic expression which depends only on $\mathbf{q}$ and P_{qq}.

This paper's third section develops a q-method PDF that has the desired exponential form. Although not yet completely proven, it is probably true that there is a one-to-one relationship between P_{qq} and the K matrix of the q-method. Therefore, the q-method PDF might provide an exponential PDF that is completely defined by the statistic P_{qq}. The one problem with defining this PDF in terms of P_{qq} is the inability to integrate the equation (14) relationship between the eigenvalues of P_{qq} and those of K. A closed-form solution of this integral would, therefore, be useful.

Another useful result would be a way to employ the P_{qq} statistic and an associated quaternion PDF in a quaternion sigma-points filter. The standard sigma-points filter is based on the assumption that the underlying prior and posterior PDFs are Gaussian [7, 8]. This assumption would have to be modified in order to use a different PDF and its P_{qq} in a sigma-points filter. The modification would need to develop a method for choosing quaternion sigma points and weights that were compatible with the new distribution. It also would need to develop a measurement update that was not based on the Gaussian assumption.

Perhaps the most interesting area for further research concerns the development of quaternion-based attitude determination particle filters. Reference [10] is a promising first attempt. It uses only 1,500 particles (i.e., samples) in a filter that estimates attitude and gyro biases using magnetometer and rate-gyro data. It incorporates a genetic algorithm in order to estimate the biases by non-particle methods. This is a useful way to reduce the "curse" of dimensionality, but other methods will likely be needed for different problems. This author has attempted to use a particle filter for an attitude estimation problem that uses magnetometer measurements alone to estimate quaternion and attitude-rate states. The filter uses the quaternion kinematics and Euler's equations for state dynamic propagation. Unfortunately, it consumes far too many computational resources even when its particle count is insufficient to overcome the "curse" of dimensionality. Such situations require the development of new strategies that speed execution while enabling the filter to carry out precise calculations using a limited number of particles.

Conclusions

A new method has been developed to perform optimal quaternion estimation by working directly with quaternion probability distributions defined on the unit hypersphere. The method solves a constrained optimization problem to determine its quaternion estimate. The optimal minimum multiplicative error estimate equals the eigenvector of the quaternion second moment matrix that points in the direction of the maximum eigenvalue. This new method's use of an eigenvalue/eigenvector solution makes it similar to the q-method. In fact, the q-method's quaternion estimate and the new method's estimate are identical for a suitably defined q-method posterior quaternion probability density function.

The new method has been used to solve a simulated batch GPS attitude determination problem. The solution algorithm operates on carrier phase data from multiple antennas to estimate attitude by using Monte-Carlo integration to determine the second moment of the posterior quaternion probability density function. This method produces accurate results starting with no idea of where the true quaternion lies, but it requires a staged execution that gradually narrows its Monte-Carlo sample region from the whole unit hypersphere to a region around the true attitude. A three-stage implementation has successfully estimated attitude to an accuracy of 0.3 degrees.

An additional result is a new quaternion probability density function that can be used as a nominal approximate distribution, much the same as a Gaussian distribution is used in an extended Kalman filter. An analytic expression for this function has been derived by projecting an unconstrained Gaussian distribution onto the unit hypersphere. This analytic expression takes the form of a rational function of the quaternion and its second moment. It has proved useful for selecting samples in the staged Monte-Carlo solution of the GPS attitude estimation problem, and it may prove useful for quaternion-based sigma-points filters and particle filters.

References

[1] SHUSTER, M. D. and OH, S. D. "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control,* Vol. 4, No. 1, 1981, pp. 70–77.

[2] LEFFERTS, E. J., MARKLEY, F. L., and SHUSTER, M. D. "Kalman Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control, and Dynamics,* Vol. 5, No. 5, 1982, pp. 417–429.

[3] BAR-ITZHACK, I. Y., DEUTSCHMANN, J., and MARKLEY, F. L. "Quaternion Normalization in Additive EKF for Spacecraft Attitude Determination," *Proceedings of the AIAA Guidance, Navigation, and Control Conference,* New Orleans, LA, August 1991, pp. 908–916.

[4] PSIAKI, M. L., THEILER, J., BLOCH, J., RYAN, S., DILL, R. W., and WARNER, R. E. "ALEXIS Spacecraft Attitude Reconstruction with Thermal/Flexible Motions Due to Launch Damage," *Journal of Guidance, Control, and Dynamics,* Vol. 20, No. 5, 1997, pp. 1033–1041.

[5] PSIAKI, M. L. "Attitude-Determination Filtering via Extended Quaternion Estimation," *Journal of Guidance, Control, and Dynamics,* Vol. 23, No. 2, March-April 2000, pp. 206–214.

[6] CRASSIDIS, J. L. and MARKLEY, F. L. "Unscented Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control, and Dynamics,* Vol. 26, No. 4, July-August 2003, pp. 536–542.

[7] JULIER, S., UHLMANN, J., and DURRANT-WHYTE, H. F. "A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators," *IEEE Transactions on Automatic Control,* Vol. AC-45, No. 3, 2000, pp. 477–482.

[8] WAN, E. A. and VAN DER MERWE, R. "The Unscented Kalman Filter," *Kalman Filtering and Neural Networks,* S. Haykin, ed., J. Wiley & Sons, New York, 2001, pp. 221–280.

[9] ARULAMPALAM, M. S., MASKELL, S., GORDON, N., and CLAPP, T. "A Tutorial on Particle Filters for Online Nonlinear/Non-Gaussian Bayesian Tracking," *IEEE Transactions on Signal Processing,* Vol. 50, No. 2, February 2002, pp. 174–188.

[10] OSHMAN, Y. and CARMI, A. " Estimating Attitude from Vector Observations Using a Genetic Algorithm-Embedded Quaternion Particle Filter," *Proceedings of the AIAA Guidance, Navigation, and Control Conference,* Paper No. AIAA-2004-5340, Providence, RI, Aug. 2004.

[11] SHUSTER, M. D. "Uniform Attitude Probability Distributions," *The Journal of the Astronautical Sciences,* Vol. 51, No. 4, Oct.-Dec. 2003, pp. 451–475.

[12] MARKLEY, F. L. "Attitude Filtering on SO(3)," *Proceedings of the Malcolm D. Shuster Astronautics Symposium,* Paper No. AAS 05-460, Buffalo, NY, June 2005.

[13] BAR-SHALOM, Y., LI, X.-R., and KIRUBARAJAN, T. *Estimation with Applications to Tracking and Navigation,* J. Wiley & Sons, New York, 2001, p. 99.

[14] WERTZ, J. R. (ed.) *Spacecraft Attitude Determination and Control,* D. Reidel Pub. Co., Boston, 1978, pp. 414–416, 758–759.

[15] SHUSTER, M. D. "In QUEST of Better Attitudes," *Advances in the Astronautical Sciences,* Vol. 108, Dec. 2001, pp. 2089–2117.