REGULARITY OF WEAK SOLUTIONS TO UNIFORMLY ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS: AN EXPLORATION OF THE DE GIORGI-NASH-MOSER THEOREM

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ABSTRACT

Problems in partial differential equations are motivated by a wide range of research fields, including more mathematical ones like calculus of variations, or topics in applied sciences. Many of these problems require finding classical solutions to partial differential equations (PDEs). These solutions are often challenging to prove the existence of, unlike weak solutions which can be calculated through generalized methods. Proving that weak solutions are sufficiently regular to qualify as classical solutions changes the key problem: instead of trying to prove existence, the focus shifts to proving regularity. This process therefore provides a very effective method to find classical solutions. This thesis is an expository survey aimed at readers with an advanced undergraduate's level of familiarity with concepts in mathematical analysis and partial differential equations, and focuses on the regularity of weak solutions to uniformly elliptic partial differential equations of second order. Throughout this work, Sobolev spaces are utilized to explore the weak solutions, and the work culminates in the proof of the De Giorgi-Nash-Moser Theorem. This theorem proves that weak solutions are Hölder regular and streamlines the process of finding classical solutions to uniformly elliptic partial differential equations, including the minimal surface equation.

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CONTENTS

1

INTRODUCTION

Partial differential equations (PDEs) are essential tools for observing the behavior of multivariable functions. Functions that are constant are usually of no significant interest, but observing the specifics of those that change is of utmost importance in many areas. Not only are they essential tools when it comes to tackling physical problems or modeling evolving systems, but they can also teach us a lot about topics in mathematical analysis. First, we will define some key terms and collect basic notations to build the vocabulary necessary to establish the problem that lies at the heart of this thesis.

1.1 BACKGROUND

Operators and Equations

A *differential operator* is an operator that takes a differentiable function as its input, and produces a new function that incorporates the derivatives of the input function, usually in multiple variables. Differential operators can be used to define partial differential equations, in the form of

$$Lu = f. \tag{1.1}$$

If $f = 0$ everywhere, we call Equation 1.1 homogeneous. If not, we call it inhomogeneous. Functions u that satisfy this equation are called *solutions* to the equation. Operators can have different properties that their associated equations inherit. One of the most important properties an operator can have is linearity. For functions u, v and a constant c, we say that a differential operator L is linear if

$$\begin{cases} L(u+v) = Lu + Lv \\ L(cu) = c \cdot L(u) \end{cases} .$$

The order of a PDE is another key property, which corresponds to the highest derivative the operator that governs it contains. This thesis will focus on second order equations. Second-degree linear partial differential equations can belong to one of three groups: *elliptic*, *hyperbolic*, and *parabolic*.

Elliptic Partial Differential Equations

Elliptic partial differential equations will be the focus of this thesis. For these equations in the form $Lu = f$, if $Lu \geqslant f$, then u is a subsolution, and if the inequality is in the other direction, u is called a supersolution. The most famous of elliptic PDE's is the *Laplace equation*, which for a function u with a domain U is expressed as

$$\Delta u = \sum_{i}^{n} u_{ii} = f.$$

The solutions to this equation in the homogeneous case are called harmonic functions. The Laplace equation can be considered the blueprint for a more general class of equations, and is considered to be the fundamental elliptic partial differential equation. Often, derivatives of multiple variables are mixed together, and they have coefficients that depend on that appear in the equation, as can be seen below

$$Lu = \sum_{i,j}^{n} a^{ij} u_{ij}.$$

A convenient way to express these coefficients is in the form of a matrix, which we express as

$$A(x) = \left(a^{ij}\right)_{1 \leqslant i,j \leqslant n}.$$

For the Laplace equation, $A(x)$ is simply the identity matrix. If the matrix $A(x)$ is *positive definite*, we call the PDE *elliptic*. This thesis will focus on *uniformly elliptic* PDEs, which is when there exists positive real values λ and Λ such that

$$\lambda I \leqslant A(x) \leqslant \Lambda I.$$

Here, we use the classical definition of matrix inequalities, where we say that M is greater than N if the matrix $M - N$ is positive-definite, meaning if for every nonzero real column vector v, $v^T(M - N)v$ is positive. A partial differential equation expressed as

$$Lu = \sum_{i,j=1}^{n} \left(a^{ij}(x) u_i\right)_j + \sum_{i=1}^{n} b^i(x) u_i + c(x) u$$

is said to be in *divergence form*. In this thesis, we will focus on the case when $b^i = 0$ and $c = 0$ for all i, when the operator ends up looking like

$$\operatorname{div}\left(A(x) \nabla u\right).$$

From this point onward, L will be used to exclusively denote uniformly elliptic differential operators.

1.2 FUNDAMENTAL QUESTIONS IN PDES

Partial differential equations do not necessarily have solutions, and even if they do their properties differ from case to case. Therefore, some of the key problems in the field of PDEs relate to *existence*, *uniqueness*, and *regularity*. As the name implies, existence problems explore whether the equation has a solution, uniqueness problems determine whether these solutions are unique, and regularity problems explore the properties these solutions have and how regular they are.

Existence and Uniqueness Problems

One of the most famous forms of an existence problem is the Dirichlet problem, where a PDE is given describing the behavior of a function in the interior of a set alongside the value it takes on the boundary. The goal is to find a function that solves this equation. This type of boundary condition is called a Dirichlet condition. There are many techniques used to pursue existence, depending on the structure of the equation. Popular methods include variational techniques or the Perron method, which utilizes the maximum principle – a cornerstone result for elliptic PDEs. When it comes to applications of this problem, the boundary values are often dictated by the problem that motivates the question, such as in problems pertaining to electromagnetic fields, wave dynamics, or quantum mechanics. Finding classical solutions to these problems, that is solutions in C^2, can be quite challenging, and therefore a different notion of solution needs to be considered. This is what motivates the introduction of *weak solutions* which will be discussed in Section 1.3.

After finding a solution to a PDE it is important to determine whether it is unique or whether other solutions exist. For the Dirichlet problem in elliptic and parabolic differential equations, the linearity of solutions and the maximum principle can be used in conjunction to prove the uniqueness of solutions to this problem. In other types of equations, however, this process isn't as straightforward.

Regularity Problems

The regularity problem is about observing the properties of solutions to a problem. When it comes to second-order elliptic equations, the most famous of these are the Schauder estimates. These estimates state various bounds for the Hölder norms of sufficiently smooth solutions. They assume the existence of solutions, and are therefore called *a priori*. Another big motivations for regularity problems is determining whether weak solutions fulfill the requirements to be classical solutions, or at least have better properties than originally expected.

1.3 WEAK SOLUTIONS

We will now focus our attention to equations in divergence form as this thesis will be exclusively focusing on them. For equations that can be expressed as such over a domain U that is an open subset of $\mathbb{R}^n$, we can multiply both sides by any test function $\phi \in C_C^\infty(U)$ and integrate over U so that we are left with

$$\int_U A(x)\nabla u \cdot \nabla \phi \, dx = 0. \tag{1.2}$$

It is obvious to see that u need not be C^2 for this integral to be well-defined and other spaces can satisfy our requirements without being as restrictive. We are therefore led to a particular space of functions that satisfy our needs, which is the Sobolev space $H^1(U)$, a crucial tool for handling such problems. Once we define this space, we say that $u \in H^1(U)$ that satisfies Equation 1.2 is called a *weak solution*.

1.4 DE GIORGI-NASH-MOSER THEOREM AND MOTIVATION

The cornerstone theorem of this thesis, originally proven in [4], is as follows.

Theorem 1.I (De Giorgi-Nash-Moser). *Let* $u \in H^1(B_1)$. *If* $Lu = 0$ *in* B_1,

$$u \in C^{0,\alpha}(B_1) \text{ and } \|u\|_{C^{0,\alpha}(B_{1/2})} \leqslant C\|u\|_{L^2(B_1)}$$

where α *and* C *depend only on* n, λ, Λ.

The theorem signifies that weak solutions to such PDEs are continuous and provides a bound on the Hölder norm. This Hölder regularity has many useful applications in calculus of variations, which often has problems in the form of

$$\min_{\mathcal{A}} \int_{\Omega} F(\nabla u)dx \tag{1.3}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $\mathcal{A}$ is the space of admissible competitors. Minimizers of the functional in Equation 1.3 must satisfy the Euler-Lagrange equation:

$$\operatorname{div}(\nabla F(\nabla u)) = \sum_{i,j=1}^{n} F_{ij}(\nabla u)u_{ij} = 0,$$

where $F_{ij}(p) = \partial_i\partial_j F(p)$ and $p = (p^1, \dots, p^n)$. We then set a^{ij} by $D^2F(\nabla u) = A(x)$. We then define $v = u_k$, which leads to

$$\operatorname{div}(A(x)\nabla v) = 0. \tag{1.4}$$

If F is sufficiently nice and u is Lipschitz, then $A(x)$ is uniformly elliptic, and therefore we can apply the De Giorgi-Nash-Moser theorem to Equation 1.4. This gives us that v is Hölder continuous and therefore $u \in C^{1,\alpha}$. This allows for the weak solutions to be valid classical solutions for minimization problems using Schauder estimates. One of the most famous of these problems is the minimal surface equation. The area of a surface is determined by

$$A(u) = \int_U \sqrt{1 + |\nabla u|^2}dx \tag{1.5}$$

which has critical points when in the domain U

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{1.6}$$

Equation 1.5 is simply Equation 1.3 with $F(t) = \sqrt{1+t^2}$, and therefore the methodology outlined above can be used to prove the regularity of weak solutions to 1.6.

1.5 OUTLINE

In Chapter 2, we will engage in a discussion on Sobolev spaces, defining some key concepts that allow us to examine them, defining the space, and demonstrating various properties of functions that belong in such spaces, providing the proofs of these results when relevant. In Chapter 3, we will prove some lemmas using the results established throughout the thesis to prove the De Giorgi-Nash-Moser theorem. Appendix A will include some background information on topics such as Lebesgue and Hölder spaces. More advanced readers might be familiar with the results displayed and proven there, but it should serve as a helpful reminder to those readers, and be quite important for those with limited familiarity.

2

SOBOLEV SPACES

To understand the theorem in question, as well as the specific solutions we are looking for, we need to use certain tools that working in Sobolev Spaces will give us. In this section, we will be exploring Sobolev spaces and proving key theorems related to them. For further references, the reader is encouraged to refer to [1] and [2].

2.1 DEFINING SOBOLEV SPACES

In order to define what a Sobolev space is, we need to introduce the notion of weak derivatives, as well as some of their properties.

Definition 2.I. For $u, v \in L^1_{loc}(U)$ and a multi-index α, we say that v is the α^{th} *weak partial derivative* of u, represented by $D^\alpha u = v$, if for all test functions $\phi \in C_c^\infty(U)$

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx.$$

The weak derivative follows some rules that we would traditionally expect from derivatives.

Theorem 2.II (Uniqueness of weak derivatives)**.** *The α^{th} weak partial derivative of a function u is unique up to a set of measure zero assuming it exists.*

With the weak derivative defined, we can now define Sobolev spaces.

Definition 2.III. For $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{Z}^+$, the *Sobolev space* denoted by $W^{k,p}(U)$ is the set of all locally summable functions $u : U \to \mathbb{R}$ such that for all α multi-indices that satisfy $|\alpha| \leqslant k$, the weak derivative $D^\alpha u$ exists, and is in $L^p(U)$. We endow this space with the the following norm:

$$||u||_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha| \leqslant k} \int_U |D^\alpha u|^p \, dx\right)^{1/p}, & 1 \leqslant p < \infty \\ \sum_{|\alpha| \leqslant k} \operatorname{ess\,sup}_U |D^\alpha u|^p, & p = \infty \end{cases}.$$

With this definition, we can observe certain properties of the weak derivative.

Theorem 2.IV. *For $u, v \in W^{k,p}(U)$ and multi-indices α, β with $|\alpha| + |\beta| \leqslant k$,*

- *If* $D^\alpha u \in W^{k-|\alpha|,p}(U), D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$

- *For all* $\lambda, \mu \in \mathbb{R}, \lambda u + \mu v \in W^{k,p}(U)$ *and* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$
- *For an open subset* $V \subset U$, $u \in W^{k,p}(V)$
- *For* $\zeta \in C_c^\infty$, $\zeta u \in W^{k,p(U)}$ *and*
  $D^\alpha(\zeta u) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha - \beta} u$

Proof. Refer to [2] Chapter 5.2.3, Theorem 1. □

Now that the spaces are well-defined, here are the Sobolev-equivalents of some familiar notions.

Definition 2.V. For $\{u_m\}_{m=1}^\infty$, $u \in W^{k,p}(U)$, if

$$\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(U)} = 0$$

we say that u_m *converges* to u, which we denote by

$$u_m \to u \,\text{in}\, W^{k,p}(U).$$

If for each $V \subset\subset U$, $u_m \to u \,\text{in}\, W^{k,p}(V)$, we say that

$$u_m \to u \,\text{in}\, W^{k,p}_{loc}(U).$$

This definition of convergence allows us to define completeness in the context of Sobolev spaces, giving us the following result.

Theorem 2.VI. *For all positive integers* k *and* $1 \leqslant p \leqslant \infty$, $W^{k,p}(U)$ *is a Banach space.*

Proof. Refer to [2] Chapter 5.2.3, Theorem 2. □

In the next chapter we will be particularly interested in Sobolev spaces with $p = 2$, so the following notation is helpful.

Remark. The closure of $C_c^\infty(U)$ in $W^{k,p}(U)$ is denoted by $W_0^{k,p}(U)$. We denote $W^{k,2}(U)$ by $H^k(U)$, and similarly its closure is denoted by H_0^k.

2.2 APPROXIMATION THEOREMS AND EXTENSIONS

This section contains theorems that utilize mollifications as defined in A.XXVI to approximate Sobolev functions with continuous functions.

Theorem 2.VII (Local approximation by smooth functions)**.** *For* $u \in W^{k,p}(U)$ *and* $1 \leqslant p < \infty$, *let* $u^\epsilon = \eta_\epsilon * u$ *in* U_ϵ. u^ϵ *has the following properties.*

1. $u^\epsilon \in C^\infty(U_\epsilon), \epsilon > 0;$
2. $u^\epsilon \to u$ *in* $W^{k,p}_{loc}(U)$, *as* $\epsilon \to 0$.

Proof. The first statement of this theorem follows directly from Theorem A.XXVII.
For part 2, we let α be a multi-index with $|\alpha| \leqslant k$. We then pick $\phi(y) = \eta_\epsilon(x-y)$ to be the test function for our weak derivatives, and show that for $x \in U_\epsilon$,

$$\begin{aligned} D^\alpha u^\epsilon(x) &= D^\alpha\left(\int_U \eta_\epsilon(x-y)u(y)dy\right) = \int_U D^\alpha_x \eta_\epsilon(x-y)u(y)dy \\ &= (-1)^{|\alpha|}\int_U D^\alpha_y \eta_\epsilon(x-y)u(y)dy = (-1)^{|\alpha|}\int_U (-1)^{|\alpha|}\eta_\epsilon(x-y)D^\alpha u(y)dy \\ &= \int_U \eta_\epsilon(x-y)D^\alpha u(y)dy = [\eta_\epsilon * D^\alpha u](x) \end{aligned}$$

and therefore $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$. We can use Theorem A.XXVII to state that in $L^p(V)$, for each $|\alpha| \leqslant k$, as $\epsilon \to 0$

$$D^\alpha u^\epsilon \to D^\alpha u$$

and therefore for the same limit

$$||u^\epsilon - u||^p_{W^{k,p}(V)} = \sum_{|\alpha|\leqslant k} ||D^\alpha u^\epsilon - D^\alpha u||^p_{L^p(V)} \to 0.$$

□

We can go one step further and generalize this theorem in the following way, where we go from $W^{k,p}_{loc}$ to $W^{k,p}$

Theorem 2.VIII (Global approximation by smooth functions)**.** *For a bounded* U*, assume* u *is in* $W^{k,p}(U)$ *for some* $1 \leqslant p < \infty$*. Then, there exist functions* $u_m \in C^\infty(U) \cap W^{k,p}(U)$ *such that*

$$u_m \to u \text{ in } W^{k,p}(U).$$

Proof. Let $U = \bigcup_{i=1}^\infty U_i$ where

$$U_i = \{x \in U |\, \mathrm{dist}(x, \partial U) > 1/i\}.$$

Let $V_i = U_{i+3} - \bar{U}_{i+1}$. We then choose a $V_0 \subset\subset U$ such that we can rewrite U as $U = \bigcup_{i=0}^\infty V_i$. We then let $\{\xi_i\}_{i=0}^\infty$ be a smooth partition of unity subordinate to $\{V_i\}_{i=0}^\infty$. This means that $0 \leqslant \xi_i \leqslant 1$ for $\xi_i \in C^\infty_c(V_i)$, and $\sum_{i=0}^\infty \xi_i = 1$ on U. We then choose any $u \in W^{k,p}(U)$. Using Theorem 2.IV, we can see that $\xi_i u \in W^{k,p(U)}$ and $\mathrm{spt}(\xi_i u) \subset V_i$. We then fix a $\delta > 0$ and define ϵ_i so that with

$$u^i = \eta_{\epsilon_i} * (\xi_i u),$$

u^i satisfies the following properties

$$\begin{cases} ||u^i - \xi_i u||_{W^{k,p}(U)} \leqslant \frac{\delta}{2^{i+1}} & i \in \mathbb{Z}_{\geqslant 0} \\ \mathrm{spt}\, u^i \subset W_i & i \in \mathbb{Z}^+ \end{cases}$$

where $W_i = U_{i+4} - \bar{U}_i \supset V_i$ for $i = 1, \ldots$. We then define v as $v = \sum_{i=0}^{\infty} u^i$. For each $V \subset\subset U$, there are at most finitely many nonzero terms in the sum, and therefore $v \in C^\infty(U)$. For each $V \subset\subset U$, we can see that

$$\|v - u\|_{W^{k,p}(V)} \leqslant \sum_{i=0}^{\infty} \left|\left|u^i - \xi_i u\right|\right|_{W^{k,p}(U)} \leqslant \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \delta.$$

We can then take the supremum over $V \subset\subset U$ to see that $\|v - u\|_{W^{k,p}(U)} \leqslant \delta$ as can be seen in Theorem 3 in Chapter 5.3.3 of [2]. □

Remark. This theorem can be further extended to be valid for $u_m \in C^\infty(\bar{U})$ for a nice enough ∂U.

The ability to approximate Sobolev functions with continuous functions is a very useful tool. Another similarly useful tool allows us to extend a function to the whole space while preserving its Sobolev property.

Theorem 2.IX (Extension Theorem)**.** *For a bounded U with C^1 boundary, select a bounded set V such that $U \subset\subset V$. Then, there exists $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$, a linear operator such that for all $u \in W^{1,p}(\mathbb{R}^n)$*

1. $Eu = u$ *almost everywhere in* U;
2. Eu *has support within* V;
3. $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C \|u\|_{W^{1,p}(U)}$ *where* C *depends only on* p, U, V.

We call such Eu *an extension of* u *to* $\mathbb{R}^n$.

Remark. For this theorem, we will only prove the case where there exists a $x^0 \in \partial U$ such that ∂U is 'flat' near that point. If this assumption isn't met, a mapping can be used to 'straighten out' the boundary in order to allow us to use a similar approach, as can be seen in Theorem 1 in Chapter 5.4 of [2].

Proof. Fix $x^0 \in \partial U$ and assume that ∂U is flat near x^0, lying in $\{x_n = 0\}$. Then, we assume that there exists an open ball $B = B_r(x^0)$ such that

$$\begin{cases} B^+ = B \cap \{x_n \geqslant 0\} \subset \bar{U} \\ B^- = B \cap \{x_n \leqslant 0\} \subset \mathbb{R}^n - U \end{cases}.$$

Using the approximation theorems, we can assume that $u \in C^1(\bar{U})$ and define $\bar{u}$ as

$$\bar{u}(x) = \begin{cases} u(x), & x \in B^+ \\ -3u(x_1, \ldots, x_{n-1}, -x_n) + 4u(x_1, \ldots, x_{n-1}, -x_n/2), & x \in B^- \end{cases}. \tag{2.1}$$

We call this a higher order reflection of u from B^+ to B^-. We can see that letting $Eu = \bar{u}$ satisfies the first two properties of the theorem. We then define $u^- = \bar{u}|_{B^-}$ and $u^+ = \bar{u}|_{B^+}$. We can see that plugging $x_n = 0$ into Equation 2.1 gives us

$$u^-_{x_n}|_{\{x_n=0\}} = u^+_{x_n}|_{\{x_n=0\}}$$

and therefore

$$u^-_{x_i}|_{\{x_n=0\}} = u^+_{x_i}|_{\{x_n=0\}}$$

for $i \in \{1, \dots, n-1\}$. We can therefore see that

$$D^\alpha u^-|_{\{x_n=0\}} = D^\alpha u^+|_{\{x_n=0\}}$$

for $|\alpha| \leqslant 1$, and hence $\bar{u} \in C^1(B)$. We can then see that for a constant C,

$$\|u\|_{W^{1,p}(B)} \leqslant C \|u\|_{W^{1,p}(B^+)}$$

which concludes our proof. □

2.3 EMBEDDING THEOREMS

There are three different cases we will look at with regards to the relationship between p and n. For this thesis, the $p < n$ is the most important one, and the other two are provided below for the sake of completeness.

$1 \leqslant p < n$

The Sobolev embedding theorem is an incredibly important result in the study of these spaces, and it is as follows. Here, we use $p^* = \frac{np}{n-p}$.

Theorem 2.X (Sobolev embedding)**.** *For a bounded* U *with a* C^1 *boundary, assume* $u \in W^{1,p}(U)$*. Then,* $u \in L^{p^*}(U)$*, with*

$$\|u\|_{L^{p^*}}(U) \leqslant C \|u\|_{W^{1,p}(U)}$$

where C *only depends on* $p, n,$ *and* U*.*

To prove this, we need to establish the Gagliardo-Nirenberg-Sobolev inequality, whose result is a significant portion of the aforementioned embedding theorem.

Theorem 2.XI (Gagliardo-Nirenberg-Sobolev inequality)**.** *There exists a* $C = C(p, n)$ *such that for all* $u \in C^1_c(\mathbb{R}^n)$

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C \|Du\|_{L^p(\mathbb{R}^n)}.$$

Proof. Initially, we assume that $p = 1$. u has compact support, therefore for all $i \in \mathbb{Z}^+$ and $x \in \mathbb{R}^n$,

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, dy_i$$

$$|u(x)| \leqslant \int_{-\infty}^{\infty} |Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\, dy_i$$

$$|u(x)|^{\frac{n}{n-1}} \leqslant \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\, dy_i \right)^{\frac{1}{n-1}}$$

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1 &\leqslant \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\ &= \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\ &\leqslant \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dy_i \right)^{\frac{1}{n-1}}\end{aligned} \tag{2.2}$$

where the final inequality of Equation 2.2 is given by the Hölder inequality. Then, we define the following integrals

$$I_1 = \int_{-\infty}^{\infty} |Du|\, dy_1,\ I_i = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |Du|\, dx_1 \right) dy_i$$

for $i \in \mathbb{Z}_{\geqslant 3}$. We now integrate Equation 2.2 with respect to x_2 and use the Hölder inequality

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1\, dx_2 &\leqslant \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dy_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=1,\ i\neq 2}^{n} I_i^{\frac{1}{n-1}}\, dx_2 \\ &\leqslant \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dy_2 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dy_1\, dx_2 \right)^{\frac{1}{n-1}} \\ &\qquad \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dx_2\, dy_i \right)^{\frac{1}{n-1}}.\end{aligned}$$

We then continue integrating with respect to x_i, up to x_n, to arrive at

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, dx &\leqslant \prod_{i=1} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du|\, dx_1 \ldots dy_i \ldots dx_n \right)^{\frac{1}{n-1}} \\ &= \left(\int_{\mathbb{R}^n} |Du|\, dx \right)^{\frac{n}{n-1}}\end{aligned} \tag{2.3}$$

which concludes the proof for $p = 1$. For $1 < p < n$, we let $v = |u|^{\gamma}$ with $\gamma > 1$, and apply the estimate in Equation 2.3.

$$\begin{aligned}\int_{\mathbb{R}^n} |v|^{\frac{n}{n-1}}\, dx &\leqslant \left(\int_{\mathbb{R}^n} |Dv|\, dx \right)^{\frac{n}{n-1}} \\ \int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\, dx &\leqslant \left(\int_{\mathbb{R}^n} |D\, |u|^{\gamma}|\, dx \right)^{\frac{n}{n-1}}\end{aligned}$$

which allows us to conclude that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\, dx \right)^{\frac{n-1}{n}} &\leqslant \int_{\mathbb{R}^n} |D\, |u|^{\gamma}|\, dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1}\, |Du|\, dx \\ &\leqslant \gamma \left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}}\, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p\, dx \right)^{\frac{1}{p}}.\end{aligned} \tag{2.4}$$

By picking γ to satisfy $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}$, which can be achieved by picking

$$\gamma = \frac{n(p-1)}{n-p} + 1 = \frac{p(n-1)}{n-p} > 1,$$

we can let $p^* = \frac{np}{n-p} = \frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}$, which allows us to rewrite Equation 2.4 as

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx \right)^{\frac{n-1}{n}} \leqslant C \left(\int_{\mathbb{R}^n} |Du|^p \, dx \right)^{1/p}.$$

$\square$

With the this proof, we can now prove the Sobolev embedding theorem.

Proof. By Theorem 2.IX, we can find a $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\bar{u} = u$ in U, $\bar{u}$ has compact support, and $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C \|u\|_{W^{1,p}(U)}$. The compact support of $\bar{u}$ leads to the existence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ with $m \in \mathbb{Z}^+$ where

$$u_m \to \bar{u} \text{ in } W^{1,p}(\mathbb{R}^n)$$

by Theorem 2.VII. By Theorem 2.XI, we get that

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C \|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$$

for all $n, l \geqslant 1$. Therefore,

$$u_m \to \bar{u} \text{ in } L^{p^*}(\mathbb{R}^n).$$

Theorem 2.XI also implies that

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C \|Du_m\|_{L^p(\mathbb{R}^n)}.$$

Since u_m converges to $\bar{u}$ in these two spaces, we can get the bound

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C \|D\bar{u}\|_{L^p(\mathbb{R}^n)}.$$

The initial bound on $\bar{u}$ with respect to its $W^{1,p}(\mathbb{R}^n)$ norm, this statement concludes the proof. $\square$

We also have a similar results for functions in $W_0^{1,p}(U)$.

Theorem 2.XII. *Assume* U *is bounded and* $u \in W_0^{1,p}(U)$. *Then,*

$$\|u\|_{L^q(U)} \leqslant C \|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$, *and* $C = C(p, q, n, U)$.

Remark. This theorem is also sometimes referred to as Poincaré's inequality.

The following theorem allows us to prove how Sobolev spaces are compactly embedded in Lebesgue spaces as defined in A.XII.

Theorem 2.XIII (Rellich-Kondrachov). *For a bounded* U *with a* C^1 *boundary, and* $1 \leqslant q < p^*$,

$$W^{1,p}(U) \subset\subset L^q(U).$$

Proof. Fix $1 \leqslant q < p^*$. Since we assume the boundedness of U, Theorem 2.X implies

$$W^{1,p}(U) \subset L^q(U), \ \|u\|_{L^q(U)} \leqslant C\, \|u\|_{W^{1,p}(U)}\,. \tag{2.5}$$

Therefore, it suffices to prove that every bounded sequence in $W^{1,p}(U)$ has a subsequence that converges in $L^q(U)$. Let $\{u_m\}_{m=1}^{\infty}$ be such a sequence. Using Theorem 2.IX, we can assume $U = \mathbb{R}^n$ without loss of generality and that the functions in $\{u_m\}_{m=1}^{\infty}$ all have compact support in $V \subset \mathbb{R}^n$, where V is a bounded open set. We also assume that $\sup_m \|u_m\|_{W^{1,p}(V)}$ is finite. Now, we observe the mollifications of these functions.

$$u_m^{\epsilon} = \eta_\epsilon * u_m.$$

We assume that all of $\{u_m^\epsilon\}_{m=1}^{\infty}$ have support in V too.
If u_m are smooth,

$$\begin{aligned} u_m^\epsilon(x) - u_m(x) &= \int_{B_1} \eta(y)\,(u_m(x-\epsilon y) - u_m(x))\,dy \\ &= \int_{B_1} \eta(y) \left(\int_0^1 \frac{d}{dt}\,(u_m(x-\epsilon t y))\,dt \right) dy \\ &= -\epsilon \int_{B_1} \eta(y) \left(\int_0^1 Du_m(x-\epsilon t y)\cdot y\,dt \right) dy, \end{aligned}$$

thus

$$\begin{aligned} \int_V |u_m^\epsilon(x) - u_m(x)|\,dx &\leqslant \epsilon \left(\int_{B_1} \eta(y) \left(\int_0^1 \int_V Du_m(x-\epsilon t y)dx \right) dt \right) dy \\ &= \epsilon \int_V |Du_m(z)|\,dz. \end{aligned}$$

By the approximation theorem, this estimate holds when $u_m \in W^{1,p}(V)$. Using the fact that V is bounded, this gives us

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \leqslant \epsilon\,\|Du_m\|_{L^1(V)} \leqslant \epsilon C\,\|Du_m\|_{L^p(V)}\,.$$

We therefore see that $u_m^\epsilon \to u_m$ uniformly in m in $L^1(V)$ using the boundedness assumption of the supremum of the norm as well as Equation 2.5. Since $1 \leqslant q < p^*$, we can use Theorem A.XXV to see that by picking a θ that satisfies $\frac{1}{q} = \theta + \frac{1-\theta}{p^*}$, which is $\theta = \frac{p^*-q}{q(p^*-1)}$, we have

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leqslant \|u_m^\epsilon - u_m\|_{L^1(V)}^{\theta}\,\|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta}\,.$$

We now use Theorem 2.XI to get that

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leqslant C\,\|u_m^\epsilon - u_m\|_{L^1(V)}^{\theta}$$

which concludes that $u_m^\epsilon \to u_m$ uniformly in m as $\epsilon \to 0$ in $L^q(V)$. For $x \in \mathbb{R}^n$, and $m \in \mathbb{Z}^+$

$$|u_m^\epsilon(x)| \leqslant \int_{B_\epsilon(x)} \eta_\epsilon(x-y)\,|u_m(y)|\,dy \leqslant \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\,\|u_m\|_{L^1(V)} \leqslant \frac{C}{\epsilon^n}$$

$$|Du_m^\epsilon(x)| \leqslant \int_{B_\epsilon(x)} |D\eta_\epsilon(x-y)|\,|u_m(y)|\,dy \leqslant \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\,\|u_m\|_{L^1(V)} \leqslant \frac{C}{\epsilon^{n+1}}$$

which proves that the sequence $\{u_m^\epsilon\}_{m=1}^\infty$ is uniformly bounded and equicontinuous for each fixed $\epsilon > 0$. To conclude our proof, we fix a $\delta > 0$ and pick an $\epsilon > 0$ to satisfy

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leqslant \frac{\delta}{2} \tag{2.6}$$

using the convergence we proved. Since all u_m and therefore u_m^ϵ have support in some bounded $V \subset \mathbb{R}^n$, we use the properties of $\{u_m^\epsilon\}_{m=1}^\infty$ and Theorem A.XIX to obtain the subsequence $\left\{u_{m_j}^\epsilon\right\}_{j=1}^\infty \subset \{u_m^\epsilon\}_{m=1}^\infty$ that converges uniformly on V. This implies

$$\limsup_{j,k\to\infty} \left|\left|u_{m_j}^\epsilon - u_{m_k}^\epsilon\right|\right|_{L^q(V)} = 0.$$

We now focus our attention to the following limit.

$$\begin{aligned}\limsup_{j,k\to\infty} \left|\left|u_{m_j} - u_{m_k}\right|\right|_{L^q(V)} &\leqslant \limsup_{j,k\to\infty} \left|\left|u_{m_j}^\epsilon - u_{m_j}\right|\right|_{L^q(V)} + \left|\left|u_{m_k}^\epsilon - u_{m_k}\right|\right|_{L^q(V)} \\ &\leqslant \frac{\delta}{2} + \frac{\delta}{2} = \delta.\end{aligned}$$

We can then dyadically define δ to find a subsequence $\{u_{m_l}\}_{l=1}^\infty$ using Cantor diagonalization as shown in the proof for Theorem A.XIX, such that

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0$$

which concludes our proof. □

$p > n$

In the case when $p > n$, we see that the theorems in Section 2.3 do not hold. Therefore, in this section we will briefly state the results for $n < p < \infty$. The key theorem is the following Morrey's inequality.

Theorem 2.XIV (Morrey's inequality)**.** *Assume $n < p \leqslant \infty$ and let $\gamma = 1 - \frac{n}{p}$. There exists a constant C that depends only on p and n such that for all $u \in C^1(\mathbb{R}^n)$*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leqslant C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}\,.$$

Consequently, we obtain the following embedding theorem.

Theorem 2.XV. *Let U be bounded and have a C^1 boundary. Assume that $n < p \leqslant \infty$ and $u \in W^{1,p}(U)$. u has a version $u^* \in C^{0,\gamma}(\bar{U})$ with*

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leqslant C\,\|u\|_{W^{1,p}(U)}$$

where γ is defined as in Theorem 2.XIV and $C = C(p, n, U)$.

For the proofs of these theorems, refer to Chapter 5.6.2 in [2].

$p = n$

For this edge case, see Chapter 5.8.1 in [2] for corresponding results.

2.4 POINCARÉ'S INEQUALITY

The two versions of Poincaré's inequality presented below prove some useful properties in Sobolev spaces.

Theorem 2.XVI (Poincaré's inequality)**.** *For a bounded* U *with a* C^1 *boundary and* $1 \leqslant p \leqslant \infty$, *there exists a* C *that depends only on* n, p, U *such that for all* $u \in W^{1,p}(U)$,

$$\|u - (u)_U\|_{L^p(U)} \leqslant C\,\|Du\|_{L^p(U)}$$

where $(u)_U = \fint_U u\,dy$.

Proof. To prove the above theorem by contradiction, we assume that for all $k \in \mathbb{Z}^+$, we can find a function $u_k \in W^{1,p}(U)$ such that

$$\|u_k - (u_k)_U\|_{L^p(U)} > k\,\|Du_k\|_{L^p(U)}\,. \tag{2.7}$$

We want can use these $\{u_k\}_{k=1}^{\infty}$ to define a sequence of functions by

$$v_k = \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}}. \tag{2.8}$$

We can see that

$$\begin{aligned} (v_k)_U &= \fint_U \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}}\,dy = \frac{1}{\|u_k - (u_k)_U\|_{L^p(U)}} \left(\fint_U (u_k - (u_k)_U)\,dy \right) \\ &= \frac{1}{\|u_k - (u_k)_U\|_{L^p(U)}} \cdot 0 = 0 \end{aligned} \tag{2.9}$$

and

$$\|v_k\|_{L^p(U)} = \left\| \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}} \right\|_{L^p(U)} = \frac{\|u_k - (u_k)_U\|_{L^p(U)}}{\|u_k - (u_k)_U\|_{L^p(U)}} = 1. \tag{2.10}$$

It follows from Equation 2.7 that

$$\frac{1}{k} > \frac{\|Du_k\|_{L^p(U)}}{\|u_k - (u_k)_U\|_{L^p(U)}} = \|Dv_k\|_{L^p(U)} \tag{2.11}$$

which implies that $\{v_k\}_{k=1}^{\infty}$ are bounded in $W^{1,p}(U)$. By Theorem 2.XIII, $W^{1,p}(U) \subset\subset L^p(U)$ and we can find a subsequence $\{v_{k_j}\}_{j=1}^{\infty} \subset \{v_k\}_{k=1}^{\infty}$ as well as a function $v \in L^p(U)$ that satisfies

$$v_{k_j} \to v \text{ in } L^p(U).$$

For $i \in \{1, \ldots, n\}$ and a test function $\phi \in C_c^{\infty}(U)$, we have

$$\int_U v\phi_{x_i} dx = \lim_{k_j\to\infty} \int_U v_{k_j}\phi_{x_i} dx = -\lim_{k_j\to\infty} \int_U v_{k_j,x_i}\phi dx. \tag{2.12}$$

The condition outlined in Equation 2.11, when applied to the last integral in Equation 2.12 gives us that

$$\int_U v\phi_{x_i} dx = 0.$$

This leads to the result that $v \in W^{1,p}(U)$ and that $Dv = 0$ almost everywhere. Therefore, v is constant due to the fact that U is connected. However, Equations 2.9 and 2.10 imply that

$$(v)_U = 0 \tag{2.13}$$

$$\|v\|_{L^p(U)} = 1 \tag{2.14}$$

respectively. Since v is constant, Equation 2.13 leads to $v = 0$ almost everywhere, which contradicts Equation 2.14. Therefore, the initial assumption is incorrect, and the theorem holds. □

Theorem 2.XVII (Poincaré's inequality for a ball)**.** *For $1 \leqslant p \leqslant \infty$, there exists a constant C that only depends on n and p such that*

$$\|u - (u)_{x,r}\|_{L^p(B_r(x))} \leqslant Cr\,\|Du\|_{L^p(B_r(r))}$$

for each ball $B_r(x) \subset \mathbb{R}^n$ and every $u \in W^{1,p}(B_r^0(x))$, where $(u)_{x,r} = ⨍_{B_r(x)} u dy$.

Proof. For $u \in W^{1,p}(B_r^0(x))$, let

$$v(y) = u(x + ry), y \in B_1.$$

This leads to $v \in W^{1,p}(B_1^0)$, and by applying Theorem 2.XVI with the choice of $U = B_1^0$,

$$\|v - (v)_{0,1}\|_{L^p(B_1)} \leqslant C\,\|Dv\|_{L^p(B_1)}.$$

Changing v back to u, we acquire

$$\|u - (u)_{x,r}\|_{L^p(B_r(x))} \leqslant Cr\,\|Du\|_{L^p(B_r(x))}.$$

□

3

DE GIORGI-NASH-MOSER THEOREM

In this chapter, we will be focusing on the key theorem of this thesis, building up to it by exploring some theorems and lemmas that lead up to this result, concluding in the proof of the theorem itself.

As defined in Chapter 1, we are working with

$$Lu = \operatorname{div}(A(x)\nabla u)$$

with ellipticity constants λ, Λ that satisfy

$$\lambda I \leqslant A(x) \leqslant \Lambda I.$$

A weak solution to $Lu = 0$ is a function $u \in H^1(U)$ that satisfies

$$\int_U A(x)\nabla u \cdot \nabla\phi dx = 0 \tag{3.1}$$

for any test function $\phi \in H_0^1(U)$. During this chapter, we will be using C to denote a constant that will depend on λ, Λ, and n. Since the value of this constant is not important, it will be used throughout this chapter to denote any such constant and therefore will not have a fixed value.

Theorem 3.I (De Giorgi-Nash-Moser). *Let $u \in H^1(B_1)$. If $Lu = 0$ in B_1,*

$$u \in C^{0,\alpha}(B_1) \text{ and } \|u\|_{C^{0,\alpha}(B_{1/2})} \leqslant C\,\|u\|_{L^2(B_1)}$$

where α and C depend only on n, λ, Λ.

In order to prove this theorem, we first need to focus on the following theorems that will be used in the proof. This inequality will allow us to bound the L^2 norm of the gradient of a function using the L^2 norm of the function itself.

Theorem 3.II (Caccioppoli Inequality). *For a $u \in H^1(B_1)$ that satisfies $Lu = 0$ in B_1,*

$$\|\nabla u\|_{L^2(B_{1/2})} \leqslant C\,\|u\|_{L^2(B_1)}$$

where C only depends on n, λ, Λ.

Remark. This theorem also holds if u us a subsolution, i.e. $Lu \geqslant 0$.

Proof. Plugging $\phi = \eta^2 u$ into 3.1, we get

$$0 = \int_{B_1} A\nabla u \cdot \nabla\left(\eta^2 u\right) dx = \int_{B_1} A\nabla u \cdot \left(\eta^2 \nabla u + 2u\eta\nabla\eta\right) dx. \tag{3.2}$$

Since $A(x) \geqslant \lambda I$,

$$\lambda \int_{B_1} \eta^2 |\nabla u|^2 \, dx \leqslant \int_{B_1} \eta^2 A(x) \nabla u \cdot \nabla u dx$$

and by using Equation 3.2, we see that

$$\int_{B_1} \eta^2 A(x) \nabla u \cdot \nabla u dx = -2 \int_{B_1} u\eta A(x) \nabla u \cdot \nabla \eta dx.$$

By the Cauchy-Schwartz inequality,

$$-\int_{B_1} u\eta A(x)\nabla u \cdot \nabla\eta dx \leqslant \|A\|_\infty \left(\int_{B_1} \eta^2 |\nabla u|^2 \, dx\right)^{1/2} \left(\int_{B_1} u^2 |\nabla \eta|^2 \, dx\right)^{1/2}.$$

We now use the boundedness of A, provided by ellipticity, and Young's inequality to obtain

$$\|A\|_\infty \left(\int_{B_1} \eta^2 |\nabla u|^2 \, dx\right)^{1/2} \left(\int_{B_1} u^2 |\nabla \eta|^2 \, dx\right)^{1/2} \leqslant \Lambda \left(\epsilon \int_{B_1} \eta^2 |\nabla u|^2 \, dx + \frac{1}{\epsilon} \int_{B_1} u^2 |\nabla \eta|^2 \, dx\right)$$

which allows us to conclude that

$$\lambda \int_{B_1} \eta^2 |\nabla u|^2 \, dx \leqslant 2\Lambda \left(\epsilon \int_{B_1} \eta^2 |\nabla u|^2 \, dx + \frac{1}{\epsilon} \int_{B_1} u^2 |\nabla \eta|^2 \, dx\right).$$

By picking a small enough ϵ, we can absorb the first term into the left hand side. We also know that $\eta = 1$ on $B_{1/2}$, which allows us to conclude the proof of out theorem as we demonstrate that

$$\int_{B_{1/2}} |\nabla u|^2 \, dx \leqslant C(n, \lambda, \Lambda) \int_{B_1} u^2 dx.$$

□

Remark. It follows from the proof that

$$\int_{B_1} \eta^2 |\nabla u|^2 \, dx \leqslant C \int_{B_1} u^2 |\nabla \eta|^2 \, dx. \tag{3.3}$$

Remark. This theorem also holds for subsolutions of L.

With the Caccioppoli inequality proven, we can now focus on proving the De Giorgi-Nash-Moser theorem, which requires the two following lemmas.

Lemma 3.III. *If* $u \in H^1(B_1)$ *satisfies* $Lu = 0$ *in* B_1, *then for* $u^+ = \max\{u, 0\}$

$$\|u\|_{L^\infty(B_{1/2})} \leqslant C \left|\left|u^+\right|\right|_{L^2(B_1)}.$$

.

Proof. Let $\tilde{u} = c\,||u^+||^{-1}_{L^2(B_1)}\,u$. Whenever u is a solution for L, so is $\tilde{u}$, as

$$L\tilde{u} = L\left(c\,||u^+||^{-1}_{L^2(B_1)}\,u\right) = c\,||u^+||^{-1}_{L^2(B_1)} \cdot Lu = 0.$$

By definition, $\|\tilde{u}^+\|_{L^2(B_1)} \leqslant c$, making $\tilde{u}$ a bounded solution to L. To conclude our proof, we want to show that $\|\tilde{u}\|_{L^\infty(B_{1/2})} \leqslant 1$ by picking c small enough. Let η be as in the Caccioppoli inequality. Plugging $\tilde{u}$ into Equation 3.3 gives us

$$\int_{B_1} \eta^2 \left|\nabla\tilde{u}^+\right|^2 dx \leqslant C \int_{B_1} (\tilde{u}^+)^2 \left|\nabla\eta\right|^2 dx.$$

We know that

$$\left|\nabla(\eta\tilde{u}^+)\right|^2 \leqslant C\left(\eta^2\left|\nabla\tilde{u}^+\right|^2 + (\tilde{u}^+)^2\left|\nabla\eta\right|^2\right)$$

and therefore we can combine these two equations to get

$$\int_{B_1} \left|\nabla(\eta\tilde{u}^+)\right|^2 dx \leqslant C \int_{B_1} \left|\nabla\eta\right|^2 (\tilde{u}^+)^2 dx. \tag{3.4}$$

We then choose ρ and δ such that $\rho + \delta \leqslant 1$ and redefine η so that it has support in $B_{\rho+\delta}$. With this new definition, $0 \leqslant \eta \leqslant 1$ and $\eta = 1$ in B_ρ. Additionally, $|\nabla\eta| \leqslant C/\delta$. This means that Equation 3.4 holds for $B_{\rho+\delta}$ as well. Redefining η this way also allows us to write

$$\int_{B_\rho} (\tilde{u}^+)^2 dx = \int_{B_\rho} \eta^2 (\tilde{u}^+)^2 dx$$

which we can apply the Hölder inequality to in order to acquire

$$\begin{aligned}\int_{B_\rho} (\tilde{u}^+)^2 dx &\leqslant \left(\int_{B_\rho} ((\tilde{u}^+)^2)^{n/(n-2)} dx\right)^{(n-2)/n} \left|\{\tilde{u}^+ > 0\} \cap B_\rho\right|^{2/n} \\ &\leqslant ||\eta\tilde{u}^+||^2_{L^{2^*}(B_{\rho+\delta})} \left|\{\tilde{u}^+ > 0\} \cap B_\rho\right|^{2/n}.\end{aligned}$$

For simplicity we assume that $n \geqslant 3$. We then proceed with the following, where $C = C(n, \lambda, \Lambda)$ and utilize Theorem 2.X to go from the L^{2^*} norm of the function to the L^2 norm of its gradient

$$\begin{aligned}||\eta\tilde{u}^+||^2_{L^{2^*}(B_{\rho+\delta})} \left|\{\tilde{u}^+ > 0\} \cap B_\rho\right|^{2/n} &\leqslant C\,||\nabla(\eta\tilde{u}^+)||^2_{L^2(B_{\rho+\delta})} \left|\{\tilde{u}^+ > 0\} \cap B_\rho\right|^{2/n} \\ &\leqslant C\left(\int_{B_{\rho+\delta}} \left|\nabla\eta\right|^2 (\tilde{u}^+)^2 dx\right) \left|\{\tilde{u}^+ > 0\} \cap B_\rho\right|^{2/n}.\end{aligned}$$

The second inequality follows from Equation 3.4. We therefore get that

$$\int_{B_\rho} (\tilde{u}^+)^2 dx \leqslant C\frac{1}{\delta^2}\left|\{\tilde{u} > 0\} \cap B_\rho\right|^{2/n} \int_{B_{\rho+\delta}} (\tilde{u}^+)^2 dx. \tag{3.5}$$

For $t \geqslant 0$, Equation 3.5 holds for $(\tilde{u}-t)^+$ as well, as for $v = (\tilde{u}-t)^+$, $Lv \geqslant 0$ as long as $L\tilde{u} = 0$, allowing us to apply Equation 3.3 on it. Now we let $a_k = \int_{B_k} (\tilde{u}-t_k)^{+2}$, where

$$\begin{cases} B_k = B_{\frac{1}{2}+\frac{1}{2^k}} \\ t_k = 1 - \frac{1}{2^k} \\ A_k = \{\tilde{u} > t_k\} \cap B_k \\ \rho = \frac{1}{2} + \frac{1}{2^{k+1}} \\ \delta = \frac{1}{2^k} \end{cases}$$

We now apply Equation 3.5 to $(\tilde{u}-t_{k+1})^+$ which gives us

$$\begin{aligned} a_{k+1} &= \int_{B_{k+1}} ((\tilde{u}-t_{k+1})^+)^2 dx \\ &\leqslant C2^{2k} \left(\int_{B_k} ((\tilde{u}-t_{k+1})^+)^2 dx \right) |\{\tilde{u} > t_{k+1}\} \cap B_{k+1}|^{2/n} \\ &= C2^{2k} \left(\int_{B_k} ((\tilde{u}-t_{k+1})^+)^2 dx \right) |A_{k+1}|^{2/n}. \end{aligned}$$

Using this formula, we can also write a_k and explore how it behaves

$$a_k = \int_{B_k} ((\tilde{u}-t_k)^+)^2 dx \geqslant \int_{A_{k+1}} (t_{k+1}-t_k)^2 dx \geqslant 2^{-2k} |A_{k+1}| \tag{3.6}$$

which we can use to conclude that

$$|A_{k+1}|^{2/n} \leqslant (2^{2k} a_k)^{2/n}. \tag{3.7}$$

We can then combine Equations 3.6 and 3.7 results to acquire

$$a_{k+1} \leqslant C2^{2k+4k/n} a_k^{1+2/n} \leqslant C2^{4k} a_k^{1+2/n}. \tag{3.8}$$

In order to conclude the proof, it suffices to show that if $a_{k+1} \leqslant C2^{4k} a_k^{1+2/n}$ and $a_0 = \|\tilde{u}^+\|_{L^2(B_1)}$ is sufficiently small, $\lim_{k\to\infty} a_k = 0$. To prove this convergence, we use induction to prove that

$$a_k \leqslant 2^{-Mk-C_0} \tag{3.9}$$

as long as $a_0 \leqslant 2^{-C_0}$ where M and C_0 are large. To prove this statement via induction, we want to show that $a_{k+1} \leqslant 2^{-M(k+1)-C_0}$ whenever Equation 3.9 holds. Using Equations 3.8 and 3.9, we say that $C = 2^{\mathcal{C}}$ and write the following inequality

$$\begin{aligned} a_{k+1} &\leqslant 2^{4k+\mathcal{C}} a_k^{1+\frac{2}{n}} \\ &\leqslant 2^{4k+\mathcal{C}} 2^{(-Mk-C_0)(1+\frac{2}{n})}. \end{aligned} \tag{3.10}$$

We want to show that $a_{k+1} \leqslant 2^{-M(k+1)-C_0}$ to complete the proof by induction, which by Equation 3.10 we can show by proving that

$$4k + \mathcal{C} + (-Mk - C_0)(1 + \frac{2}{n}) \leqslant -M(k+1) - C_0.$$

Expanding both sides of the inequality gives

$$4k + \mathcal{C} - Mk - \frac{2Mk}{n} - C_0 - \frac{2C_0}{n} \leqslant -Mk - M - C_0.$$

which is equivalent to

$$4k - \frac{2Mk}{n} + \mathcal{C} - \frac{2C_0}{n} + M \leqslant 0.$$

If we prove that both $4k - \frac{2Mk}{n}$ and $\mathcal{C} - \frac{2C_0}{n} + M$ are less than or equal to zero, it suffices to prove the inequality above. These conditions are met if $M \geqslant 2n$ and $C_0 \geqslant \frac{n}{2}(\mathcal{C} + M)$. $\square$

Lemma 3.IV (Weak Harnack inequality for subsolutions)**.** *Let* $u \in H^1(B_1)$ *satisfy* $Lu \geqslant 0$ *in* B_1. *If there exists a* $\delta > 0$ *such that*

$$|\{u \leqslant 0\} \cap B_1| \geqslant \delta\, |B_1|,$$

then

$$u^+ \leqslant (1 - c(n, \lambda, \Lambda, \delta))\, ||u^+||_{L^\infty(B_1)} \text{ in } B_{1/2}.$$

Proof. Using the previous lemma's method of normalizing a solution, we can assume $||u^+||_{L^\infty(B_1)} = 1$ up to multiplication by a constant. Therefore, it suffices to show that $u^+ \leqslant 1 - c(n, \lambda, \Lambda, \delta)$ in $B_{1/2}$.
We see that in $B_{1/2}$

$$u^+ \leqslant C\, ||u^+||_{L^\infty(B_1)}\, |\{u > 0\} \cap B_1|^{1/2}$$

through Lemma 3.III, with the same value of C. If δ is large enough, we have

$$C\, ||u^+||_{L^\infty(B_1)}\, |\{u > 0\} \cap B_1|^{1/2} \leqslant C(1-\delta)^{1/2}\, |B_1|^{1/2} \leqslant \frac{1}{2}$$

in $B_{1/2}$, which concludes the proof, as it shows that $u^+ \leqslant \frac{1}{2}$.
If δ isn't large enough to allow us to use the property above directly, we can generate a new sub-solution for which we can use this result. We do so by defining

$$v_k = \frac{(u - t_k)^+}{1 - t_k}$$

where $t_k = 1 - 2^{-k}$. We can now see that

$$v_k \leqslant \frac{1}{2} \iff (u - t_k) \leqslant \frac{1}{2}(1 - t_k) \iff (u - t_k)^+ \leqslant 2^{-k-1}$$

thus proving that if there exists a k for which $v_k \leqslant 1/2$, this would lead to the desired conclusion. In view of the discussion above, we need to find k for which $|\{u > t_k\}|$ is sufficiently small, i.e.

$$|\{u > t_k\} \cap B_1| << |B_1|.$$

We proceed inductively. Since $t_0 = 0$, $|\{u > t_0\}| = |\{u > 0\}|$. Let $|\{u > 0\}|$ be sufficiently large but still bounded above by $(1-\delta)|B_1|$. If $|\{u > t_1\}| = \left|u > \frac{1}{2}\right|$ is also sufficiently large, then

$$\left|\left\{0 < u < \frac{1}{2}\right\}\right| \leqslant (1-\delta)|B_1| - C|B_1|.$$

Then, if $\left|\left\{u > \frac{3}{4}\right\}\right|$ is also sufficiently large, $\left|\left\{\frac{1}{2} < u < \frac{3}{4}\right\}\right|$ will be smaller. These iterations will take place until we get to an l such that $|\{t_l < u < t_{l+1}\}|$ is small, $|\{u > t_{l+1}\}|$ is large, and $|\{u \leqslant t_l\}|$ is bounded below. We therefore need to show that for $v = v_l$ as defined previously, there is a contradiction if $\left|\left\{0 < v < \frac{1}{2}\right\}\right|$ is small whereas $|\{v \leqslant 0\}|$ and $\left|\left\{v \geqslant \frac{1}{2}\right\}\right|$ are bounded by below.

By Theorem 3.II, we get that

$$\int_{B_{1-\tau}} |\nabla v|^2 \, dx \leqslant C(\tau) \int_{B_{1-\tau/2}} v^2 \, dx \leqslant C(\tau)$$

where we use that $\|v\|_\infty \leqslant 1$. We can also use the Hölder inequality to write

$$\int_{B_{1-\tau} \cap \{0 < v < 1/2\}} |\nabla v| \, dx \leqslant \left(\int_{B_{1-\tau}} |\nabla v|^2 \, dx\right)^{1/2} \left|\left\{0 < v < \frac{1}{2}\right\} \cap B_{1-\tau}\right|^{1/2}.$$

We can then combine these two in order to write

$$\left|\left\{0 < v < \frac{1}{2}\right\} \cap B_{1-\tau}\right| \geqslant C(\tau) \left(\int_{B_{1-\tau} \cap \{0 < v < 1/2\}} |\nabla v| \, dx\right)^2. \tag{3.11}$$

Let $\bar{v}$ be defined as

$$\bar{v} = \begin{cases} 0 & v \leqslant 0 \\ v & 0 < v < 1/2 \\ 1/2 & v \geqslant 1/2 \end{cases}.$$

With this, we have

$$\int_{B_{1-\tau} \cap \{0 < v < 1/2\}} |\nabla v| \, dx = \int_{B_{1-\tau}} |\nabla \bar{v}| \, dx$$

which by Theorem 2.XVI can be seen that

$$\int_{B_{1-\tau}} |\nabla \bar{v}| \, dx \geqslant \int_{B_{1-\tau}} |\bar{v} - a| \, dx \geqslant \int_{B_{1-\tau} \cap \{v \leqslant 0\}} |a| \, dx + \int_{B_{1-\tau} \cap \{v \geqslant 1/2\}} \left|\frac{1}{2} - a\right| dx \tag{3.12}$$

where $a = ⨍_{B_{1-\tau}} v dx$. If $a \geqslant \frac{1}{4}$, then the first term on the right hand side of Equation 3.12 is greater than or equal to $\frac{1}{4}|\{v \leqslant 0\}|$, and if $a < \frac{1}{4}$, then the second term of that right hand side is greater than or equal to $\frac{1}{4}\left|\left\{v \geqslant \frac{1}{2}\right\}\right|$. We combine this with our assumption that both of the sets we are integrating over have measures that are bounded below, and therefore

$$\int_{B_{1-\tau} \cap \{v \leqslant 0\}} |a| \, dx + \int_{B_{1-\tau} \cap \{v \geqslant 1/2\}} \left|\frac{1}{2} - a\right| dx \geqslant C$$

which, when combined with Equation 3.11 gives a lower bound on $\left|\left\{0 < v < \frac{1}{2}\right\}\right|$. This concludes that $\left|\left\{0 < v < \frac{1}{2}\right\}\right|$ cannot be arbitrarily small, completing our proof. □

With the lemmas proven, it is finally time to prove the De Giorgi-Nash-Moser theorem.

Proof. As a reminder, we want to prove that for $u \in H^1(B_1)$, if $Lu = 0$ in B_1,

$$u \in C^{0,\alpha}(B_1) \text{ and } \|u\|_{C^{0,\alpha}(B_{1/2})} \leqslant C \|u\|_{L^2(B_1)}.$$

We can normalize u by dividing it by its L^2 norm, allowing us to assume that $\|u\|_{L^2} = 1$. Therefore by Lemma 3.III, is bounded in $L^\infty(B_{1/2})$ by a constant C that only depends on n, λ, and Λ. We want to show that u is continuous at zero with Hölder modulus of continuity with a similar constant. We can prove this by proving that for all sufficiently small r,

$$\operatorname{osc}_{B_{r/2}} u \leqslant (1-c) \operatorname{osc}_{B_r} u \tag{3.13}$$

for some constant c. For $x \in B_1$, we can rescale u to define

$$u_r(x) = u(rx).$$

We similarly rescale A with $A_r = A(rx)$ and see that u_r is a solution to

$$\operatorname{div}(a_r(x) u_r(x)) = 0.$$

A_r has the same ellipticity constants as A, and therefore up to rescaling and multiplication by a constant we can assume that $r = 1$ and $|u| \leqslant 1$ in B_1. This reduces Equation 3.13 to

$$\operatorname{osc}_{B_{1/2}} u \leqslant 1 - c.$$

If $\{u \leqslant 0\}$ covers more than half of B_1, ie $|\{|u \leqslant 0|\} \cap B_1| \geqslant \frac{1}{2}|B_1|$, Lemma 3.IV shows that

$$u \leqslant u^+ \leqslant \left(1 - c\left(\frac{1}{2}\right)\right) \tag{3.14}$$

in $B_{1/2}$. If $\{u \leqslant 0\}$ covers less than half of B_1, then $|\{-u < 0\}| \geqslant |B_1|$ and applying Lemma 3.IV to $-u$ gives

$$-u \leqslant (-u)^+ \leqslant \left(1 - c\left(\frac{1}{2}\right)\right)$$

which is equivalent to

$$u \geqslant -1 + c\left(\frac{1}{2}\right) \tag{3.15}$$

in $B_{1/2}$. Rescaling once more and applying the same argument gives us

$$\operatorname{osc}_{B_{1/4}} u \leqslant (1-c)^2$$

and further iterations lead to the generalized result for $k \in \mathbb{N}$

$$\operatorname{osc}_{B_{2^{-k}}} u \leqslant (1-c)^k.$$

This argument can be repeated in every point of $B_{1/2}$ to demonstrate continuity and conclude the proof. $\square$

A

DEFINITIONS, AND BASIC RESULTS

This chapter of the Appendix will introduce the notation that will be used throughout the thesis, as well as defining some key terms and properties that will be used in other sections.

A.1 NOTATION

Definition A.I. U is used to denote an open subset of $\mathbb{R}^n$.

Definition A.II. A *ball* on a set U is an open subset denoted by $B_r(x_0)$, defined as

$$B_r(x_0) = \{x \in U | d(x, x_0) < r\}$$

where d is the metric associated with U. If $x_0 = 0$, we simply denote $B_r(x_0)$ by B_r.

Definition A.III. For a function u, we say that u^* is a *version* of u if $u = u^*$ almost everywhere.

Definition A.IV. A *multi-index* α is an n-tuple of non-negative integers of the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. $|\alpha|$ denotes $\sum_{i=1}^{n} \alpha_i$.

Definition A.V. For $\epsilon > 0$, U_ϵ is

$$U_\epsilon = \{x \in U | \operatorname{dist}(x, \partial U) > \epsilon\}$$

Definition A.VI. The operator osc denotes the magnitude of oscillations of a function in a given set. Specifically,

$$\operatorname{osc}_S f = \sup_S f - \inf_S f.$$

A.2 FUNCTIONAL ANALYSIS

Definition A.VII. For a real linear space X, a mapping $||\cdot|| : X \to [0, \infty)$ is called a *norm* if for all $u, v \in X$ and $\lambda \in \mathbb{R}$,

- $||u + v|| \leqslant ||u|| + ||v||$
- $||\lambda u|| = |\lambda| \, ||u||$
- $||u|| = 0$ if and only if $u = 0$.

For a similar X, a mapping $[\cdot] : X \to [0, \infty)$ is called a semi-norm if for all $u, v \in X$ and $\lambda \in \mathbb{R}$,

- $[u+v] \leqslant [u]+[v]$
- $[\lambda u] = |\lambda|\,[u]$.

If X is endowed with a norm, we say that X is a *normed space*.

Definition A.VIII. For a normed space X, a sequence of functions $\{u_k\}_{k=1}^{\infty} \subset X$ and a function $u \in X$, we say that u_k *converges to* u if

$$\lim_{k\to\infty} \|u_k - u\| = 0$$

which we also denote as $u_k \to u$.

Definition A.IX. We say that a sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is Cauchy if for all $\epsilon > 0$, there exists a $N > 0$ such that

$$\|u_k - u_l\| < \epsilon \text{ for all } k, l > N.$$

X is *complete* if every Cauchy sequence in X converges.

Definition A.X. A *Banach space* is a complete, normed linear space.

Definition A.XI. For two sets X, Y and a function $f : X \to Y$, if f is injective and structure-preserving, it is called an *embedding*. Embeddings from X to Y are denoted as $f : X \hookrightarrow Y$. The subset relationship, arguable the most famous of embeddings, is when f is the identity function.

Definition A.XII. For Banach spaces X and Y, we say that X is *compactly embedded* in Y, which we denote by

$$X \subset\subset Y$$

if

- there exists a constant C such that for all $u \in X$, $\|u\|_Y \leqslant C\,\|u\|_X$
- every bounded sequence in X is pre-compact in Y, i.e. the closure of X is compact in Y.

A.3 SPACES OF CONTINUOUS FUNCTIONS

Definition A.XIII. For $f : U \to \mathbb{R}^n$, $f \in C^k$ if the derivatives $f', f'', f''', \ldots, f^{(k)}$ exist and are continuous. If $f \in C^{\infty}$, we say that f is *smooth*. If $f \in C^k$ and has compact support, i.e. only takes non-zero values in a compact subset of the domain, we say that $f \in C^k_c$.

Definition A.XIV. For U, an open subset of $\mathbb{R}^n$ and $0 < \gamma \leqslant 1$, we say that a function $u : U \to \mathbb{R}$ is *Hölder continuous with exponent* γ if

$$\forall x, y \in U,\ |u(x) - u(y)| \leqslant C\,|x-y|^{\gamma}.$$

When $\gamma = 1$, we call these functions *Lipschitz continuous*, or simply Lipschitz.

Definition A.XV. For U, an open subset of $\mathbb{R}^n$, and a bounded and continuous function $u : U \to \mathbb{R}$,

$$\|u\|_{C(\bar{U})} = \sup_{x \in U} |u(x)| \, .$$

The γ^{th} *Hölder seminorm* of u is defined as

$$[u]_{C^{0,\gamma}(\bar{U})} = \sup_{x,y \in U x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x-y|^{\gamma}} \right\}$$

and the γ^{th} *Hölder norm* of u is defined as

$$\|u\|_{C^{0,\gamma}(\bar{U})} = \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})} \, .$$

Definition A.XVI. The *Hölder space* $C^{k,\gamma}(\bar{U})$ is the set of all functions $u \in C^k(\bar{U})$ for which

$$\|u\|_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha| \leqslant k} \|D^{\alpha} u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^{\alpha} u]_{C^{0,\gamma}(\bar{U})}$$

is finite. Equivalently, $C^{k,\gamma}(\bar{U})$ is the set of functions $u \in C^k(\bar{U})$ that are k times continuously differentiable, whose k^{th} partial derivative is bounded, and are Hölder continuous with exponent γ.

Theorem A.XVII. $C^{k,\gamma}(\bar{U})$ *is a Banach space.*

The following theorem is one of the crucial compactness theorems for continuous functions. Before this result, we want to remind the readers of an important concept used in this theorem.

Definition A.XVIII. A sequence of real valued functions $\{f_k\}_k = 1^l$ where each $f_k : X \to \mathbb{R}$ for a normed space X, is called *equicontinuous* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x, y \in X$ and $n \in N$,

$$d(x, y) < \delta$$

implies

$$|f_n(x) - f_n(y)| < \epsilon.$$

Theorem A.XIX (Arzela-Ascoli compactness criterion)**.** *Let* $\{f_k\}_{k=1}^{\infty}$ *be a sequence of real valued and bounded functions defined on* $\mathbb{R}^n$ *for* $k \in \mathbb{Z}^+$. *Assume that* $\{f_k\}_{k=1}^{\infty}$ *are uniformly equicontinuous. Then, there exists a subsequence* $\left\{f_{k_j}\right\}_{j=1}^{\infty} \subseteq \{f_k\}_{k=1}^{\infty}$ *and a continuous function* f *such that* $f_{k_j} \to f$ *uniformly on compact subsets of* $\mathbb{R}^n$.

Remark. This theorem also holds for a sequence of functions defined on any compact metric space.

Proof. Let $S = \mathbb{Q}^n$. We know that S is countable, as well as being dense in $\mathbb{R}^n$. Due to being countable, we can express S as $S = \{x_1, x_2, \ldots\}$. Since the functions in the sequence are bounded, we know that $\{f_k(x_1)\}_{k=1}^{\infty}$ is bounded. By the Bolzano-Weierstrass theorem, every infinite bounded sequence in $\mathbb{R}^n$ has a convergent subsequence, and therefore so does $\{f_k(x_1)\}_{k=1}^{\infty}$. We will write this convergent subsequence as $f_{1,k}{}_{k=1}^{\infty}$. With this definition, $\{f_{1,k}(x_2)\}_{k=1}^{\infty}$ is bounded and therefore has a convergent subsequence $\{f_{2,k}(x_2)\}_{k=1}^{\infty}$. Since $\{f_{2,k}\}_{k=1}^{\infty}$ is a subsequence for $\{f_{1,k}\}_{k=1}^{\infty}$ it converges at both x_1 and x_2. By repeating this process, we can generate a countable collection of subsequences that we can arrange as

$$\begin{matrix} f_{1,1} & f_{1,2} & f_{1,3} & \dots \\ f_{2,1} & f_{2,2} & f_{2,3} & \dots \\ f_{3,1} & f_{3,2} & f_{3,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

where the l-th row converges at $x_1, x_2, \dots, x_l$ and each row is a subsequence of the row above it. Therefore, the diagonal sequence $\{f_{k,k}\}$ is a subsequence of $\{f_k\}$ that converges in S.

Let's denote $f_{k,k}$ by g_k. For an $\epsilon > 0$, we can pick a $\delta > 0$ to suit the equicontinuity of our assumption such that $|g_k(x) - g_k(y)| < \epsilon/3$ for all $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}^+$. We can then fix a $M > 1/\delta$ such that S_M is δ-dense in $\mathbb{R}^n$. $\{g_k\}$ converges at each point of S_M and therefore there exists $N > 0$ such that for all $s \in S_M$

$$k, l > N \implies |g_k(s) - g_l(s)| < \epsilon/3. \tag{A.1}$$

For a fixed $x \in \mathbb{R}^n$, x is within δ distance of some of the $s \in S_M$, so for $k, l > M$

$$|g_k(x) - g_l(x)| \leqslant |g_k(x) - g_l(s)| + |g_k(s) - g_l(s)| + |g_k(s) - g_l(x)|. \tag{A.2}$$

Of the three terms in the right hand side of Equation A.2, the first and the third are $< \epsilon/3$ by our initial choice, and by choice of N the middle term is $< \epsilon/3$ by Equation A.1. Therefore, for $\epsilon > 0$, we can find an N such that for all $x \in \mathbb{R}^n$,

$$k, l > n \implies |g_k(x) - g_l(x)| < \epsilon$$

which concludes that $\{g_k\}$ is uniformly Cauchy and therefore uniformly convergent. □

A.4 LEBESGUE SPACES

Definition A.XX. For $1 \leqslant p < \infty$ and a measurable $f : U \to \mathbb{R}$, the L^p *norm* of f is

$$\|f\|_{L^p} = \|f\|_p = \left(\int_U |f|^p \, dx\right)^{1/p}.$$

The L^∞ *norm* of f is defined as

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C \geqslant 0 \,|\, |f(x)| \leqslant C \text{ for almost every } x\}.$$

Definition A.XXI. For $1 \leqslant p \leqslant \infty$, $L^p(U)$ is the set of measurable functions $f : U \to \mathbb{R}$ with finite L^p norms. L^p spaces are also called *Lebesgue spaces*. The space of locally integrable L^p functions is denoted as L^p_{loc}.

The L^p spaces have useful properties.

Theorem A.XXII. L^p *spaces are Banach spaces.*

Theorem A.XXIII (Hölder's inequality)**.** *For* $1 \leqslant p, q \leqslant \infty$ *that satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and $f, g : X \to \mathbb{R}$,

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q .$$

The proof of this theorem follows from Young's inequality, which is as follows.

Theorem A.XXIV (Young's inequality)**.** *For* $a, b \geqslant 0$, *with* $p, q > 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

Remark. For $p = q = 2$, Young's inequality simplifies to

$$2ab \leqslant a^2 + b^2$$

which is also known as the Cauchy-Schwartz inequality.

The Hölder inequality is useful for proving the following theorem.

Theorem A.XXV (Interpolation inequality for L^p norms)**.** *Let* $1 \leqslant s \leqslant r \leqslant t \leqslant \infty$, *and define* θ *such that*

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}.$$

If $u \in L^s(U) \cap L^t(U)$ *then* $u \in L^r(U)$ *and*

$$\|u\|_{L^r(U)} \leqslant \|u\|^{\theta}_{L^s(U)} \|u\|^{1-\theta}_{L^t(U)}$$

Approximating L^p functions with smooth functions is a useful tool that will be required in some of the theorems that are used in this thesis. Therefore, we need to introduce the concept of mollification.

Definition A.XXVI. For a locally integrable function $f : U \to \mathbb{R}$ and $\epsilon > 0$, the *mollification* of f is defined as

$$f^{\epsilon} = \eta_{\epsilon} * f.$$

The convolution can be expanded to write

$$f^{\epsilon}(x) = \int_U \eta_{\epsilon}(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_{\epsilon}(y)f(x-y)dy$$

in U_{ϵ}, where the standard $\eta \in C^{\infty}(\mathbb{R}^n)$ is defined as

$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1 \\ 0, & |x| \geqslant 1 \end{cases}$$

where C is chosen to satisfy

$$\int_{\mathbb{R}^n} \eta \, dx = 1.$$

η_ϵ is defined as

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right).$$

We can see that $\eta_\epsilon \in C^\infty$ and satisfies

$$\int_{\mathbb{R}^n} \eta_\epsilon \, dx = 1.$$

The support of η_ϵ is a subset of $B(0, \epsilon)$ by definition.

Theorem A.XXVII. *For $\epsilon > 0$ and the mollification f^ϵ,*

1. $f^\epsilon \in C_c(U_\epsilon)$*;*
2. $f^\epsilon \to f$ *almost everywhere as* $\epsilon \to 0$*;*
3. *If* $f \in C(U)$*,* $f^\epsilon \to f$ *uniformly on compact subsets of* U*;*
4. *If* $1 \leqslant p < \infty$ *and* $f \in L^p_{loc}$*, then* $f^\epsilon \to f$ *in* $L^p_{loc}(U)$*.*

Proof. First, we'll prove Statement 1. Fix $x \in U_\epsilon$, and $i \in \{1, \dots, n\}$, and a small h that satisfies $x + he_i \in U_\epsilon$.

$$\begin{aligned}\frac{f^\epsilon(x+he_i) - f^\epsilon(x)}{h} &= \frac{\eta}{h\epsilon^n} \int_U \left[\left(\frac{x+he_i-y}{\epsilon}\right) - \left(\frac{x-y}{\epsilon}\right)\right] f(y)dy \\ &= \frac{\eta}{h\epsilon^n} \int_V \left[\left(\frac{x+he_i-y}{\epsilon}\right) - \left(\frac{x-y}{\epsilon}\right)\right] f(y)dy\end{aligned}$$

for an open set $V \subset\subset U$.
$\frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right]$ converges uniformly to $\frac{1}{\epsilon}\frac{\partial \eta}{\partial x_i}\left(\frac{x-y}{\epsilon}\right)$ on V, and therefore

$$\frac{\partial f^\epsilon}{\partial x_i}(x) = \int_U \frac{\partial \eta_\epsilon}{\partial x_i}(x-y) f(y) dy.$$

Similarly, $D^\alpha f^\epsilon(x)$ exists for any multi-index α, and

$$D^\alpha f^\epsilon(x) = \int_U D^\alpha \eta_\epsilon(x-y) f(y) dy, \ (x \in U_\epsilon).$$

This concludes the proof for the first part of the theorem.

For Statement 2, Theorem A.XXVIII tells us that for almost every $x \in U$,

$$\lim_{r \to 0} ⨍_{B(x,r)} |f(y) - f(x)| \, dy = 0. \tag{A.3}$$

With fixed x that satisfies the above,

$$
\begin{aligned}
|f^\epsilon(x) - f(x)| &= \left| \fint_{B(x,\epsilon)} \eta^\epsilon(x-y)\,[f(y)-f(x)]\,dy \right| \\
&\leqslant \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{x-y}{\epsilon}\right) |f(y)-f(x)|\,dy \\
&\leqslant C \fint_{B(x,\epsilon)} |f(y)-f(x)|\,dy.
\end{aligned}
$$

As $\epsilon \to 0$, the right hand side of the equality goes to zero, implying that the left hand side also goes to zero, concluding the proof. For the following statement, we assume that $f \in C(U)$. Given $V \subset\subset U$, we choose a set W such that $V \subset\subset W \subset\subset U$. Therefore, Equation A.3 holds uniformly for $x \in V$ and by following the same steps, it implies that $f^\epsilon \to f$ uniformly on V.

For the final statement of the theorem, we assume $1 \leqslant p < \infty$ and $f \in L^p_{loc}(U)$. We then choose a $V \subset\subset U$ and pick an open W as above. For $x \in V$ and $1 < p < \infty$,

$$
\begin{aligned}
|f^\epsilon(x)| &= \left| \int_{B(x,\epsilon)} \eta_\epsilon(x-y) f(y)\,dy \right| \\
&\leqslant \int_{B(x,\epsilon)} \eta_\epsilon^{1-1/p}(x-y)\eta_\epsilon^{1/p}(x-y)\,|f(y)|\,dy \\
&\leqslant \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)\,dy \right)^{1-1/p} \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)\,|f(y)|^p\,dy \right)^{1/p}.
\end{aligned}
$$

We know that $\int_{B(x,y)} \eta_\epsilon(x-y)\,dy = 1$. Therefore, for a small enough $\epsilon > 0$,

$$
\begin{aligned}
\int_V |f^\epsilon(x)|^p\,dx &\leqslant \int_V \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)\,|f(y)|^p\,dy \right) dx \\
&\leqslant \int_W |f(y)|^p \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y)\,dx \right) dy \qquad \text{(A.4)} \\
&= \int_W |f(y)|^p\,dy
\end{aligned}
$$

which tells us that for such a small ϵ, $\|f^\epsilon\|_{L^p(V)} \leqslant \|f\|_{L^p(W)}$. We then fix $V \subset\subset W \subset\subset U$, $\delta > 0$ and choose a $g \in C(W)$ such that $\|f - g\|_{L^p(W)} < \delta$. We can then see that

$$
\begin{aligned}
\|f^\epsilon - f\|_{L^p(V)} &\leqslant \|F^\epsilon - g^\epsilon\|_{L^p(V)} + \|g^\epsilon - g\|_{L^p(V)} + \|g - f\|_{L^p(V)} \\
&\leqslant 2\,\|f - g\|_{L^p(W)} + \|g^\epsilon - g\|_{L^p(V)} \\
&\leqslant 2\delta + \|g^\epsilon - g\|_{L^p(V)}.
\end{aligned}
$$

Since $g^\epsilon \to g$ uniformly on V, we get that $\limsup_{\epsilon\to 0} \|f^\epsilon - f\|_{L^p(V)} \leqslant 2\delta$, concluding our proof.

If $p = 1$, then we can still show that $\|f^\epsilon\|_{L^1(V)} \leqslant \|f\|_{L^1(W)}$ holds, as equivalent to Equation A.4.

$$
\begin{aligned}
\|f^\epsilon\|_{L^1(V)} &= \int_V |f^\epsilon(x)|\,dx = \int_V \left| \int_{B(x,\epsilon)} \eta_\epsilon(x-y) f(y)\,dy \right| dx \\
&\leqslant \int_W |f(y)| \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y)\,dx \right) dy = \int_W |f(y)|\,dy = \|f\|_{L^1(W)}
\end{aligned}
$$

which allows for a similar argument to be used in the $p = 1$ case. □

Theorem A.XXVIII (Lebesgue's Differentiation Theorem)**.** *For* $f : \mathbb{R}^n \to \mathbb{R}$, *that satisfies* $f \in L^1_{loc}$ *the following hold:*

1. *For almost every* $x_0 \in \mathbb{R}^n$, $⨍_{B(x_0,r)} f dx \to f(x_0)$ *as* $r \to 0$
2. *For almost every* $x_0 \in \mathbb{R}^n$, $⨍_{B(x_0,r)} |f(x) - f(x_0)| \, dx \to 0$ *as* $r \to 0$

The proof will utilize some theorems that the reader is assumed to be familiar with. If not, the references to [8] should prove to be helpful.

Proof. For $x_0 \in \mathbb{R}^n$ and $r > 0$, let

$$(T_r f)(x_0) = \frac{1}{|B_r|} \int_{B(x_0,r)} |f(x) - f(x_0)| \, dx$$

and

$$(Tf)(x_0) = \limsup_{r \to 0} (T_r f)(x_0).$$

It suffices to prove that $Tf(x_0) = 0$ for almost every x_0. Let $y > 0$, and $k \in \mathbb{Z}^+$. Using the fact that $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ as proven in Theorem 3.14 in [8], we can find a function $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_{L^1} < \frac{1}{k}$. We then let $h = f - g$. Since g is continuous,

$$\begin{aligned} Tg(x_0) &= \limsup_{r \to 0} (T_r g)(x_0) \\ &= \limsup_{r \to 0} \frac{1}{|B_r|} \int_{B(x_0,r)} |g(x) - g(x_0)| \, dx = 0. \end{aligned}$$

We now look at $T_r h$.

$$(T_r h)(x_0) = \frac{1}{|B_r|} \int_{B(x_0,r)} |h(x) - h(x_0)| \, dx \leqslant \frac{1}{|B_r|} \int_{B(x_0,r)} |h(x)| \, dx + |h(x_0)| \, .$$

For Mf defined as

$$(Mf)(x_0) = \sup_{0 < r < \infty} \frac{1}{|B_r|} \int_{B(x_0,r)} |f(x)| \, dx,$$

we can say that

$$Th \leqslant Mh + |x|$$

and since $T_r f \leqslant T_r g + T_r h$, we get that

$$Tf \leqslant Mh + |h| \, .$$

This allows us to write the following relationship about these sets

$$\{Tf > 2y\} \subset \{Mh > y\} \cup \{|h| > y\} \, . \tag{A.5}$$

We can use Theorem 7.4 in [8] to say

$$|\{Mh > y\} \cup \{|h| > y\}| \leqslant \frac{3^n + 1}{yk}.$$

Since the left hand side of Equation A.5 is independent of k, we see that

$$\{Tf > 2y\} \subset \bigcap_{k=1}^{\infty} \{Mh > y\} \cup \{|h| > y\}.$$

We can see that the measure of the individual components being taken the intersection of decreases, and therefore the measure of the intersection itself is 0. This tells us that $\{Tf > 2y\}$ has measure zero, concluding the proof. $\square$

Remark. This theorem can be expanded for $1 \leqslant p < \infty$. If $f \in L^p_{loc}$, for almost every $x_0 \in \mathbb{R}^n$, we see that as $r \to 0$,

$$\fint_{B(x_0,r)} |f(x) - f(x_0)|^p \, dx \to 0.$$

BIBLIOGRAPHY

[1] Robert A. Adams and John J. F. Fournier. *Sobolev Spaces*. 2nd. Oxford: Elsevier, 2003.

[2] Lawrence C. Evans. *Partial Differential Equations*. 2nd. Providence, R.I.: American Mathematical Society, 2010.

[3] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. 1st. Berlin: Springer, 2001.

[4] Ennio De Giorgi. "Sulla Differenziabilità e L'Analiticà Delle Estremali Degli Integrali Multipli Regolari." In: *Memorie della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematicahe e Naturali.* 3 (1957), pp. 25–43.

[5] Qing Han and Fanghua Lin. *Elliptic Partial Differential Equations*. 1st. Providence, R.I.: American Mathematical Society, 2000.

[6] Peter J. Olver. *Introduction to Partial Differential Equations*. 1st. Cham: Springer, 2014.

[7] Walter Rudin. *Principles of Mathematical Analysis*. 3rd. New York: McGraw-Hill, 1976.

[8] Walter Rudin. *Real and Complex Analysis*. 3rd. New York: McGraw-Hill, 1987.

[9] Walter A. Strauss. *Partial Differential Equations*. 2nd. New Jersey: John Wiley & Sons, 2008.