# Astronomy C3101
# Modern Stellar Astrophysics
# Homework 3

Rasmi Elasmar

Tuesday, November 18, 2014

**Problem 1 -** *Luminosity variation from stellar pulsation*

(a) Estimate the resulting change in luminosity $\delta L$.

$L = 4\pi R^2 \sigma T^4$
$L_0\big(1 + \frac{\delta L}{L_0}\big) = 4\pi\big[R_0\big(1 + \frac{\delta R}{R_0}\big)\big]^2\sigma\big[T_0\big(1 + \frac{\delta T}{T_0}\big)\big]^4$
$L_0\big(1 + \frac{\delta L}{L_0}\big) = 4\pi\big[R_0^2\big(\frac{\delta R}{R_0}^2 + 2\frac{\delta R}{R_0} + 1\big)\big]\sigma\big[T_0^4\big(\frac{\delta T}{T_0}^4 + 4\frac{\delta T}{T_0}^3 + 6\frac{\delta T}{T_0}^2 + 4\frac{\delta T}{T_0} + 1\big)\big]$
Discarding higher-order terms (or using the Taylor expansion for brevity),
$L_0\big(1 + \frac{\delta L}{L_0}\big) \approx 4\pi\big[R_0^2\big(1 + 2\frac{\delta R}{R_0}\big)\big]\sigma\big[T_0^4\big(1 + 4\frac{\delta T}{T_0}\big)\big]$
$\frac{\delta L}{L_0} \approx \big(1 + 2\frac{\delta R}{R_0}\big)\big(1 + 4\frac{\delta T}{T_0}\big) - 1$

(b) Derive a relationship between $\delta R$ and $\delta T$.

Gas is adiabatic, relating $\delta P$ to $\delta\rho$:
$P = K_a\rho^{\gamma_a}$
$P_0\big(1 + \frac{\delta P}{P_0}\big) = K_a\big[\rho_0\big(1 + \frac{\delta\rho}{\rho_0}\big)\big]^{\gamma_a}$
Taylor expanding,
$\approx K_a\rho_0^{\gamma_a}\big(1 + \gamma_a\frac{\delta\rho}{\rho_0}\big)$
But since
$\delta P = K_a\rho_0^{\gamma_a}\gamma_a\frac{\delta\rho}{\rho_0}$,
$\frac{\delta P}{P_0} = \gamma_a\frac{\delta\rho}{\rho_0}$

Gas is ideal, relating $\delta P$ to $\delta T$:
$P = \frac{R}{\mu}\rho T$
$P_0\big(1 + \frac{\delta P}{P_0}\big) = \frac{R}{\mu}\rho_0\big(1 + \frac{\delta\rho}{\rho_0}\big)T_0\big(1 + \frac{\delta T}{T_0}\big)$
$P_0 = \frac{R}{\mu}\rho_0 T_0$
$1 + \frac{\delta P}{P_0} = \big(1 + \frac{\delta\rho}{\rho_0}\big)\big(1 + \frac{\delta T}{T_0}\big)$
$1 + \frac{\delta P}{P_0} = 1 + \frac{\delta\rho}{\rho_0} + \frac{\delta T}{T_0} + \frac{\delta\rho}{\rho_0}\frac{\delta T}{T_0}$
Dropping higher-order terms,
$\frac{\delta P}{P_0} = \frac{\delta\rho}{\rho_0} + \frac{\delta T}{T_0}$
$\frac{\delta T}{T_0} = \frac{\delta P}{P_0} - \frac{\delta\rho}{\rho_0}$

Use homologous expansion relation to relate $\delta\rho$ to $\delta R$ (just like in Lecture 15):
$dm = 4\pi R^2\rho dR$
$= 4\pi\big[R_0\big(1 + \frac{\delta R}{R_0}\big)\big]^2\rho_0\big(1 + \frac{\delta\rho}{\rho_0}\big)dR_0\big(1 + \frac{\delta R}{R_0}\big)$
$= 4\pi R_0^2\rho_0 dR_0\big(1 + 3\frac{\delta R}{R_0} + \frac{\delta\rho}{\rho_0}\big)$
But since $dm = 4\pi R_0^2\rho_0 dR_0 = 0$,
$3\frac{\delta R}{R_0} + \frac{\delta\rho}{\rho_0} = 0$

$\frac{\delta\rho}{\rho_0} = -3\frac{\delta R}{R_0}$

Taking those three relations and substituting/combining,
$\frac{\delta T}{T_0} = \frac{\delta P}{P_0} - \frac{\delta\rho}{\rho_0}$
$\frac{\delta P}{P_0} = \gamma_a \frac{\delta\rho}{\rho_0}$
$\Rightarrow \frac{\delta T}{T_0} = \frac{\delta\rho}{\rho_0}(\gamma_a - 1)$
$\frac{\delta\rho}{\rho_0} = -3\frac{\delta R}{R_0}$
$\Rightarrow \frac{\delta T}{T_0} = -3\frac{\delta R}{R_0}(\gamma_a - 1)$
Peak luminosity

(c) Arrive at an estimate for the relationship between $\delta L$ and $\delta R$. Does the peak luminosity of a pulsating star with $\gamma_a = 5/3$ occur when its radius is at its maximum value or its minimum value?
$\frac{\delta L}{L_0} \approx \left(1 + 2\frac{\delta R}{R_0}\right)\left(1 + 4\frac{\delta T}{T_0}\right) - 1$
$\frac{\delta T}{T_0} = -3\frac{\delta R}{R_0}(\gamma_a - 1)$
Substituting,
$\frac{\delta L}{L_0} \approx \left(1 + 2\frac{\delta R}{R_0}\right)\left(1 + 4(-3\frac{\delta R}{R_0}(\gamma_a - 1))\right) - 1$
$\frac{\delta L}{L_0} \approx \left(1 + 2\frac{\delta R}{R_0}\right)\left(1 - 12\frac{\delta R}{R_0}(\gamma_a - 1)\right) - 1$
When $\gamma_a = 5/3$,
$\frac{\delta L}{L_0} \approx \left(1 + 2\frac{\delta R}{R_0}\right)\left(1 - 12\frac{\delta R}{R_0}(2/3)\right) - 1$
$\frac{\delta L}{L_0} \approx \left(1 + 2\frac{\delta R}{R_0}\right)\left(1 - 8\frac{\delta R}{R_0}\right) - 1$
$\frac{\delta L}{L_0} \approx \left(1 - 6\frac{\delta R}{R_0} - 16(\frac{\delta R}{R_0})^2\right) - 1$
Removing higher-order terms,
$\frac{\delta L}{L_0} \approx -6\frac{\delta R}{R_0}$
There is an inverse relationship, so the peak pulsating luminosity occurs when the radius is at its minimum value.

**Problem 2 -** *Generating the $T$ vs. $\rho$ diagram*

(a) Calculate $\mu_I, \mu_e$, and $\mu$.

$X = 0.75, Y = 0.25$
$\frac{1}{\mu_I} = X + \frac{1}{4}Y = 0.8125$
$\mu_I = 1.2308$
$\frac{1}{\mu_e} = \frac{1}{2}(1 + X) = 0.875$
$\mu_e = 1.1429$
$\frac{1}{\mu} = \frac{1}{\mu_I} + \frac{1}{\mu_e} = 1.6875$
$\mu = 0.5926$

(b) Solve for the boundary between the ideal gas zone and the non-relativistic degenerate gas zone.
$K_0\rho T = K_1\frac{1}{\mu_e^{5/3}}\rho^{5/3}$
$T = \frac{K_1}{K_0}\frac{1}{\mu_e^{5/3}}\rho^{2/3}$
$log(T) = log\left(\frac{K_1}{K_0}\frac{1}{\mu_e^{5/3}}\right) + log(\rho^{2/3})$
$\frac{2}{3}log(\rho) = log(T) - log\left(\frac{K_1}{K_0}\frac{1}{\mu_e^{5/3}}\right)$
$log(\rho) = \frac{3}{2}log(T) - \frac{3}{2}log\left(\frac{K_1}{K_0}\frac{1}{\mu_e^{5/3}}\right)$
$R = 8.3145 \times 10^7 erg/K \cdot mol$
$K_0 = \frac{R}{\mu} = 1.4031 \times 10^8 erg/K \cdot mol$
$K_1 = 1.0 \times 10^7 m^4 kg^{-2/3} s^{-2} = 1 \times 10^{13} cm^4 g^{-2/3} s^{-2}$
$\mu_e = 1.1429$
$log(\rho) = \frac{3}{2}log(T) - 7.1344$

(c) Solve for the boundary between the non-relativistic and relativistic degenerate gas zones.
$K_1 \frac{1}{\mu_e^{5/3}} \rho^{5/3} = K_2 \frac{1}{\mu_e^{4/3}} \rho^{4/3}$
$\rho^{1/3} = \frac{K_2}{K_1} \mu_e^{1/3}$
$log(\rho^{1/3}) = log\big(\frac{K_2}{K_1} \mu_e^{1/3}\big)$
$log(\rho) = 3log\big(\frac{K_2}{K_1} \mu_e^{1/3}\big)$
$K_1 = 1.0 \times 10^7 m^4 kg^{-2/3} s^{-2} = 1 \times 10^{13} cm^4 g^{-2/3} s^{-2}$
$K_2 = 1.24 \times 10^{10} m^3 kg^{-1/3} s^{-1} = 1.24 \times 10^{15} cm^3 g^{-1/3} s^{-1}$
$\mu_e = 1.1429$
$log(\rho) = 6.3383$

(d) Solve for the boundary between the ideal gas zone and the relativistic degenerate gas zone.
$K_0 \rho T = K_2 \frac{1}{\mu_e^{4/3}} \rho^{4/3}$
$T = \frac{K_2}{K_0} \frac{1}{\mu_e^{4/3}} \rho^{1/3}$
$log(T) = log\big(\frac{K_2}{K_0} \frac{1}{\mu_e^{4/3}} \rho^{1/3}\big)$
$\frac{1}{3} log(\rho) = log(T) - log\big(\frac{K_2}{K_0} \frac{1}{\mu_e^{4/3}}\big)$
$log(\rho) = 3log(T) - 3log\big(\frac{K_2}{K_0} \frac{1}{\mu_e^{4/3}}\big)$
$K_0 = \frac{R}{\mu} = 1.4031 \times 10^8 erg/K \cdot mol$
$K_2 = 1.24 \times 10^{10} m^3 kg^{-1/3} s^{-1} = 1.24 \times 10^{15} cm^3 g^{-1/3} s^{-1}$
$\mu_e = 1.1429$
$log(\rho) = 3log(T) - 20.607$

(e) Solve for the value of $(T, \rho)$ where the ideal gas pressure, non-relativistic degenerate gas pressure, and relativistic gas pressure are all equal.

(f) Solve for the boundary between the ideal gas zone and the radiation pressure taking the boundary as $P_{rad} = 10 P_{gas}$.
$10\big(K_0 \rho T\big) = \frac{1}{3} a T^4$
$K_0 \rho = \frac{1}{30} a T^3$
$log(\rho) = log\big(\frac{a}{30 K_0} T^3\big)$
$log(\rho) = 3log(T) + log\big(\frac{a}{30 K_0}\big)$
$R = 8.3145 \times 10^7 erg/K \cdot mol$
$K_0 = \frac{R}{\mu} = 1.4031 \times 10^8 erg/K \cdot mol$
$a = 7.6 \times 10^{-15} erg/cm^3/K^4$
$log(\rho) = 3log(T) - 23.7434$

(g) Plot the boundaries between the different zones for $logT(K)$ = 6–10 and $log\rho(g\, cm^{-3})$ = 0–10.

**Problem 3 -** *Convection*

(a) Show that the envelope of a star that has a Kramers Law opacity (with a = 1, b = -3.5) is stable against convection if the equation of state is that of an ideal gas with $\gamma = 5/3$.

Kramer's law: $\kappa = c \rho^a T^b$
$\kappa = c \rho T^{-3.5}$
Gas is adiabatic
Gas is ideal: $P = \frac{R}{\mu} \rho T$
$\rho = \frac{\mu}{R} \frac{P}{T}$, $T = \frac{\mu}{R} \frac{P}{\rho}$
As related in lecture 16,
$dP = \frac{R}{\mu} \big(\rho \frac{dT}{dr} + T \frac{d\rho}{dr}\big) dr$
$= \big(\frac{P}{T} \frac{dT}{dr} + \frac{P}{\rho} \frac{d\rho}{dr}\big) dr$

$P = K_a \rho^\gamma$
$dP = K_a \gamma \rho^{\gamma-1} \frac{d\rho}{dr} dr$
$= \gamma \frac{P}{\rho} \frac{d\rho}{dr} dr$
$\gamma \frac{P}{\rho} \frac{d\rho}{dr} = \frac{P}{T} \frac{dT}{dr} + \frac{P}{\rho} \frac{d\rho}{dr}$
$\frac{dT}{dr} = (\gamma - 1) \frac{T}{P} \frac{P}{\rho} \frac{d\rho}{dr}$
$= \left(\frac{\gamma-1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr}$
$\frac{dln(T)}{dln(P)} = \frac{PdT}{TdP} = \frac{\gamma-1}{\gamma} = \frac{5/3-1}{5/3} = \frac{2/3}{5/3} = \frac{2}{5}$

(b) Use this result to predict which main-sequence stars should have convective envelopes.

**Problem 4 -** (see attached)