Modern Stellar Astrophysics – Prof. Agüeros
Homework 2, Due Thursday, Oct 23

Feel free to write or type your answers, and to submit either a hard copy or electronic version of your answers. You may discuss the homework with your classmates, but please write up your homework independently.

Problem 1 - (15 points) - *Powering Jupiter by gravity*
As we saw in class, we can be confident that the Sun is in thermal equilibrium because its thermal timescale is short compared to its age. However, smaller objects need not be in thermal equilibrium, and their radiation can be powered entirely by gravity.

(a) Jupiter radiates more energy than it receives from the Sun by $8.7 \times 10^{-10}\ L_\odot$. Jupiter's radius is 7.1×10^9 cm and its mass is 1.9×10^{30} g. Compute its thermal timescale. Could gravitational contraction have powered this luminosity for Jupiter's entire lifetime of 4.5 Gyr? Is it safe to assume that Jupiter is in hydrostatic equilibrium?

(b) Use conservation of energy to estimate the rate at which Jupiter's radius is shrinking to power this radiation. You may ignore the factor of order unity that arises from Jupiter's unknown density distribution.

Problem 2 - (25 points) - *p-p I reaction rate*
In class I mentioned that the p-p I reaction rate is roughly

$$q \propto T^4.$$

For this problem you will derive the exact relationship. The reaction rate for the fusion of two nuclei A and B can be approximated as

$$R_{AB} = \frac{8}{\sqrt{3}} \frac{n_A n_B}{\pi \alpha Z_A Z_B m_r c} S(E_0) \left(\frac{E_G}{4kT}\right)^{2/3} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right].$$

Defining

$$\begin{aligned} \tau &= \left(\frac{27E_G}{4kT}\right)^{1/3} \\ C &= \frac{8}{9\sqrt{3}} \frac{n_A n_B}{\pi \alpha Z_A Z_B m_r c} S(E_0), \end{aligned}$$

we can re-express R_{AB} as

$$R_{AB} = C\tau^2 e^{-\tau}.$$

(a) Solve for

$$\frac{d\ln R_{AB}}{d\ln T}.$$

Hint: start by finding the relationship between $d\tau/\tau$ and dT/T.

(b) The reaction rate is an exponential. However, it can be approximated as a power law for temperatures T close to some reference temperature T_r. Show that this approximation is

$$R_{AB}(T) = R_{AB}(T_r)\left(\frac{T}{T_r}\right)^n$$

where

$$Pn = \frac{\tau - 2}{3}.$$

(c) Calculate the value of n for the p-p I chain at the temperature of the solar core. The rate-limiting reaction for this chain is the ${}^1_1\text{H}+{}^1_1\text{H}$ reaction, for which $E_G = 0.490$ MeV. How does the temperature dependence for q you find compare to $q \propto T^4$?

Problem 3 - (10 points) - Mean free paths

(a) Calculate how far you could see through the Earth's atmosphere if it had the opacity of the solar photosphere, $\kappa = 0.264$ cm^2 g^{-1}. Use $\rho = 1.2 \times 10^{-3}$ g cm^{-3} for the density of the Earth's atmosphere. Estimate roughly how this compares to how far you can actually see through the Earth's atmosphere.

(b) Crude estimates of the central density of the Sun yield $\rho_c = 162$ g cm^{-3} and a mean central opacity $\kappa = 1.16$ cm^2 g^{-1}. What is the mean free path of a photon in the center of the Sun?

(c) A photon with a mean free path l will travel a total distance $d = \sqrt{N}l$ in N random steps. Assume the value for l calculated in part (b) is appropriate from the center to the surface of the Sun. How many years will it take a photon to travel from the center to the surface?

Problem 4 - (20 points) - *Working with the Lane-Emden equation*
The analytic solution to the Lane-Emden equation of index $n = 1$ is

$$\theta = \frac{\sin \xi}{\xi}$$

where θ is the dimensionless density function ($\theta^n = \rho/\rho_c$) and ξ is the dimensionless radius ($\xi = r/\alpha$). At the surface of a star $\theta = 0$.

(a) Using this solution, derive an expression for the radius R of a star. Express your answer as a combination of constant quantities, including K.

(b) Calculate the value for K using the value for the solar radius $R_\odot$.

(c) For an isothermal star, which has a constant T throughout, and is also polytropic, the equation of state becomes $P \propto \rho$. What does this imply for the value of n? Could such a star exist?

Problem 5 - (30 points) - *STATSTAR stellar models*
`STATSTAR` is code modified from Carroll and Ostlie to run in the Python environment. For now we are just going to run the code to generate outputs. You can download the Enthought Python distribution for free from `http://www.enthought.com/products/epd.php` (look for the link to EPDFree, register, and off you go!). It is straightforward to install and run (at least on my Linux box...!).

I have posted `STATSTAR` and photocopies of a description of the code from Carroll and Ostlie to CourseWorks. If you have issues with any of this, please let me know!

If you would like to generate plots using Python, I recommend using the library matplotlib (you can find documentation for this easily on the web). Alternatively, you may use Excel, or any other plotting software/code with which you are comfortable.

(a) *run the code:* The code takes as input a star's mass, effective (surface) temperature (T_{eff}), luminosity, and composition (X and Z), and calculates a stellar model. It is actually extremely finicky and will fail for all but a small range of input conditions. To begin, run the code for a model that works: $M = 1.0\ M_{\odot}$, $L = 0.86071\ L_{\odot}$, $T_{eff} = 5500.2$ K, X = 0.70, Z = 0.008. Check that the output indicates that you have generated a good model and inspect the output file.

What are the central conditions that the code has calculated for this star? Rename the output file for this model run, because future runs will overwrite this file.

(b) *break the code:* Try running the code by modifying the inputs and see the different ways in which the code fails. Describe your inputs and the messages that the code has reported back. Try to get the code to fail in at least four different ways.

Even with the code's messages, it may not be obvious at this point exactly why the model fails. Speculate on why your input conditions may have caused the code to fail.

(c) *test another working model:* Try a model with 0.75 $M_{\odot}$, 0.1877 $L_{\odot}$, $T_{eff} = 3839.1$ K, X = 0.7, Z = 0.008. This model should also work. What are the central conditions for this star?

(d) *plot some results:* For the two working models in (a) and (c), use the output data and any available plotting program to plot ρ vs. R, T vs. R, and L vs. R. Where is most of the luminosity generated for these stars?

(e) *energy generation vs. T^n:* In the output file, eps (or ϵ) is the same as what we have defined as q. Plot eps/ρT^n vs. R for both stars, adopting n $\sim$ 4.5. What is the behavior of eps/ρT^n over the radii where most of the stars' luminosity is generated? Why might we expect this?

(f) *ρ vs. T:* Plot both models on a log ρ vs. log T plot. Does any part of the star appear to be in a regime where an ideal gas equation of state does not apply?